Saari and Simon showed that if a mechanism is required to be locally stable in the traditional sense (which does not require Lyapunov stability in response to perturbations of the demand function) at all regular equilibria and for some $N \in Gl(n)$, $D_z A(0, N)$ is non-singular, then no entries are ignorable. The following corollary is an analogue of their results as an application of theorem 7.8.

Corollary 7.22. A locally stable generalised Newton method admits no ignorable entries.

Proof. Given any entry ij, choose $N \in Gl(n)$ and $N'_{i,j} \in \mathbb{R}$ so that $\det N = -\det(N/N_{i,j})$. Then $\det i(N) = -\det i(N/N_{i,j})$ where $i(N) = -N^{-1}$. If M is locally stable, theorem 7.8 implies that $A(\cdot, N) \neq A(\cdot, N/N'_{i,j})$ when restricted to any neighbourhood of 0 in $\mathbb{R}^n$. Hence ij is not ignorable. □

Saari and Simon define a mechanism to be *effective* if there is a fixed open set of initial prices $V \subset Q$ such that for every regular demand function f, all solutions of

$$\dot{p} = A(f(p), Df(p)), \qquad p(0) \in V$$

converge to some equilibrium of f, and $D_z A(0, Df(p))$ is non-singular for some equilibrium p. They obtained the following restrictions on the ignorable entries for effective prices mechanisms

1. there cannot be two rc-distinct ingorable entries if the corresponding entries of $D_z A(0, N)$ are not identically zero on $Gl(n)$,
2. if an entire column of entries is ignored, no other entries can be ignored; and
3. if $n = 2$ or 3, there cannot be two rc-distinct ignorable entries; and if $n = 4$ there cannot be three ignorable entries two of which are rc-distinct.

If M is effective, it must be locally stable at p if $p \in V$ and f must be a regular demand function having p as its unique equilibrium. Using a result similar to proposition 7.16, Saari and Simon infer that all eigenvalues of the non-singular matrix $D_z A(0, N).N$ must have non-positive real parts whenever N is a hyperbolic matrix with $\det(-N) > 0$. If too many entries are ignored, they show that manipulating the ignored entries can create an eigenvalue with positive real part.

Corollary H applies the modified version of corollary 7.13 (modified as in the section on Stability at Unique Equilibria), that is

Corollary 7.23. If A is HLS at unique equilibria, then $D_z A(0, \cdot)$ is a homotopy equivalence. In particular, $j \circ D_z A(0, \cdot) \simeq 1_{\mathcal{M}^+_{p'}}$.

To show that if the characteristic root condition is strengthened by requiring that real parts be negative then rc-distinct entries cannot be ignored, a conclusion which implies the three conditions of Saari and Simon listed previously. The Newton method, A^*, described earlier indicates that this is the strongest possible restriction on the ignorable entries for mechanisms which are HLS at unique equilibria.

Theorem H. A generalised Newton method which is HLS at unique equilibria cannot ignore rc-distinct entries.

Proof. Suppose by way of contradiction that M is HLS at unique equilibria and has ignorable entries ij and km with $i \neq k$ and $j \neq m$. Then $n \geq 2$, and we will assume for the moment that $ij = 11$ and $km = 22$. Let $S^1 = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$, and define the function $\alpha : S^1 \to SO(n)$ by

$$\alpha(x,y) = \begin{bmatrix} x & -y & \vdots & \\ & & \vdots & 0 \\ y & x & \vdots & \\ \cdots & \cdots & \cdots & \cdots \\ & 0 & \vdots & I_{n-2} \end{bmatrix}$$

where I_{n-2} denotes the $(n-2) \times (n-2)$ identity matrix.

Define $\beta : S^1 \to Gl(n)^+$ by $\beta(x,y) = \alpha(2,y)$. Since 11 and 22 are ignorable entries,

$$D_z A(0, \alpha(x,y)) = D_z A(0, \beta(x,y)) \quad \text{for all } (x,y) \in S^1.$$

Define $H : S^1 \times [0,1] \to Gl(n)^+$ by $H(x,y : \lambda) = \alpha(2-\lambda, y(1-\lambda))$. Then H is a homotopy between β and the constant function $(x,y) \to I_n$, so $D_z A(0, \alpha(\cdot)) = D_z A(0, \beta(\cdot)) : S^1 \to Gl(n)$ is homotopic to the constant function $(x,y) \mapsto D_z A(0, I_n)$. However, α generates the fundamental group of $SO(n)$ (see Whitehead [40]) and thus the fundamental group of $Gl(n)^+$. Corollary 7.13 implies the map $N \mapsto D_z A(0, N)$ induces an isomorphism on the fundamental group of $Gl(n)^+$, so $D_z A(0, \alpha)$ cannot be homotopic to a constant map. This contradiction shows that 11 and 22 cannot be ignorable entries. For general ij and km, let $g : Gl(n)^+ \to Gl(n)^+$ be a linear isomorphism constructed by interchanging rows and columns so that entry ij goes to 11 and entry km goes to 22 and the above holds with α replaced with $g^{-1} \circ \alpha$, β replaced with $g^{-1} \circ \beta$ and H replaced with $H(g^{-1}(\cdot); \cdot)$. $\square$

8. Decentralisation in Economic Mechanisms

In this, the final section, we present the "state of the art" results in this subject due to Saari and Williams [22]. Before we proceed, we summarise the results we have seen thus far : According to Walras, market pressures (as captured by the market excess demand) drives the economy to equilibrium. §5 showed that convergence was not guaranteed for arbitrary economies. §6 and §7 investigated modified price mechanisms which were only dependent upon the properties of the aggregate excess demand function and which were locally stable. Here we saw that the informational requirements include all entries from the Jacobian matrix of the excess demand function. These results are discouraging because they imply that unless unrealistic requirements are imposed upon the system, standard economic assumptions need not lead to desired results. The instability with these economic systems must either be excepted or else the hypotheses behind them must be re-examined.

The approach of this section is resolve these problems by allowing information in addition to that contained in the excess demand function to be incorporated in the pricing mechanism. This extra information which is added to the price mechanism will involve information regarding the "economic environment" as a whole. Based on the recognition that the proofs seen thus far implicitly assume that the dynamic is a single consistent procedure for all environments, we disregard this assumptions and construct models based on a set of dynamical systems.

This situation can be viewed in terms of environmental coverage of a dynamical system : a given dynamical system is locally stable for certain economic environments, to obtain "global stability" the environment space must be a partitioned into several regions. In each of these regions a locally convergent dynamical system must be chosen. We are then presented with a set of dynamical systems. To use this set requires coarse information to identify which environmental region contains a particular environment. It may be that this additional information is "common knowledge" among agents. If not, it must be added to the system.

The main ideal here is to identify this additional information and to explain how it can be communicated in a decentralised fashion.

8.1. **Introduction.** The work in this section is based on the pioneering work done by Hurwicz in 1960 who introduced the abstract model of the collection and processing of information in a *decentralised economy*, that is, an economy in which decisions about production and consumption are carried out by the individual agents making up the economy rather than some central planning agency. According to Hurwicz, the two key aspects of an informationally decentralised economy are

1. information is dispersed among the agents; and
2. there are limits to communication, that is, agents must encode their information into a limited set of signals.

We keep these two points in mind when constructing new price mechanisms.

Consider an economy consisting l commodities, where each of the m agent's utility functions are concave and can be smoothly parametrised by a finite number of parameters. This hypothesis may seem overly restrictive, however, the results are still meaningful since many of the proofs in the theory of decentralisation involve a restriction to a subclass of environments which is Euclidean for example, Cobb-Douglas, quadratic or CES. The primary reason for working in this special setting, however, is to facilitate our analysis. Hopefully we will gain insight into more meaningful economic models by working first with this idealised setting.

The Euclidean space E_j of dimension k will denote the space of the j-th agent's *characteristics*. For instance, an element $e_j \in E_j$ might contain the exponents of the j-th agent's Cobb-Douglas utility function and also a vector which determines that agent's initial resources. This element defines the *private information* of the j-th agent. Let E be a subset of the space $E_1 \times \ldots \times E_l$ (which we assume has nonempty interior) and let $e = (e_1, \ldots, e_l) \in E$.

An agent's private information is encoded by communicating only his excess demand to the auctioneer or market. An important notion of models in this section is that of "privacy preserving". A mechanism will be called *privacy preserving* if each agent's excess demand and any additional information which the mechanism requires the agent to communicate, is based only upon his private information and the signals of the other agents. For this section, the excess demand function $f(e,p)$, which now depends on both the agent's characteristics and the price vector, will be a C^2 function and the function space of admissible f's is endowed with the strong topology. The price vector $p \in S$ is publicly observable. In addition, excess demand functions are assumed to satisfy the following "strong regularity" conditions.

Definition 8.1. An aggregate excess demand function $f(e,p)$ is *strongly regular* on E if

1. $|D_p f(e,p)| \neq 0$, whenever $f(e,p) = 0$; and
2. The zero sets of the determinants of the principal minors of $D_p f(\cdot, p)$ form a finite union of smooth, lower dimensional manifolds in E.

It is easy to show that this is a generic condition over the space of smooth, concave utility functions.

Assume the dynamics of the economy are

$$\dot{p} = f(e,p) = \sum_{i=1}^{m} f_i(e_i, p). \tag{8.1}$$

Notice that the form of these equations reflects the basic features of an informationally decentralised system : because the information is dispersed, each agent's messages are privacy preserving. Information is encoded in equation (8.1) to the extent that $f_j(e_j, p)$ conveys less information than the value of e_j itself.

We will consider modifications to the price mechanisms of the form

$$\dot{p} = M\left(f(e_1,p), h_1(e_1,p), \ldots, h_l(e_l,p)\right), \tag{8.2}$$

where M is a smooth function such that

$$M(0, \cdot, \ldots, \cdot) = 0 \tag{8.3}$$

and $h_j(e_j, p)$ corresponds to the added information from the j-th agent. The extremes vary from where it is reasonable to assume the j-th agent could completely reveal all of his private information to the case where this agent can communicate nothing more than the value of f_j.

The main result of this section is the following theorem which addresses the negative statements about the price mechanism.

Theorem I. Consider a pure exchange economy with $m \geq 2$ agents and $l \geq 2$ commodities. Assume that the space of environments, representing the initial endowments and the choice of utility functions, is a compact subset with nonempty interior of a Euclidean space. Furthermore, assume that the aggregate excess demand function is strongly regular on this space of environments. With the exception of an open set of environments with arbitrarily small Lebesgue measure, there exists a finite set of constant diagonal matrices $\{P\}$ and a privacy preserving selection function to choose the appropriate matrix so that $\dot{p} = P_j f$ is locally convergent.

8.2. **Decentralisation and Local Stability.** Local convergence is determined by the real parts of the eigenvalues of the Jacobian $D_p f$ when evaluated at equilibrium. The eigenvalues depend on the characteristic equations that in turn depends upon the determinants of the principal minors of $D_p f$. What we now do is use the signs of these minors to partition the space of environments into regions. Within each region, the original system can be modified to achieve local stability. This defines a set of dynamical systems, where each system is identified with a particular region in E. The first step on the way to proving Theorem I is theorem 8.4 which asserts that there are privacy preserving ways to select the appropriate modified system.

It is reasonably straight forward to find a privacy preserving mechanism that achieves this goal. A more meaningful objective would be to determine *all* smooth ways which such mechanisms can be constructed. This is particularity important when we recall that convergence is only one of the many criteria that an economic mechanism should satisfy. For instance it is conceivable that one particular method for attaining convergence does not satisfy certain desired incentive conditions while another one does. If a class of converging mechanisms is given, then it may be possible to select one of them that satisfies other desired constraints. The algorithm that constructs a class of smooth mechanisms constitutes the proof of theorem 8.4.

Certain environments and their corresponding equilibria are excluded by this procedure. These are identified in the following two definitions.

Definition 8.2. Let $B = (b_{ks})$ be a $d \times d$ matrix. Let S be a sequence of sets S_j, $j = 1 \ldots, d$, with the following properties :

1. S_j has j elements from $\{1, \ldots, d\}$; and
2. S_j is contained in S_{j+1}.

A *computational sequence* of minors, $\langle B, S\rangle$, is the sequence of principal minors $\{B(S_j)\}$ where $B(S_j)$ is the $j \times j$ matrix (b_{ks}) with indices k, s in S_j. A computational sequence is called a *stability sequence* if the determinant for each of the minors $\langle B, S\rangle$ is nonzero.

For the system (8.1), the computational sequence of f at (e^*, p^*) is $\langle D_p f(e^*, p^*), S\rangle$.

Definition 8.3. For a system (8.1) and a sequence S, let

$$E_S = \{(e, p) | f(e, p) = 0 \text{ and } \langle D_p f(e, p), S\rangle \text{ is } \textit{not} \text{ a stability sequence}\}.$$

Theorem 8.4. For a system (8.1), assume that for each e the equilibria are isolated. Then, for a given sequence S on $E\backslash E_S$ the following hold :

1. there exists a set D of dynamical systems, each of which satisfies equations (8.2) and (8.3); and
2. there exists a class of smooth, privacy preserving mechanisms that selects for each environment a locally converging system from D.

Proof. Initially, assume that for each e there is a unique equilibrium. According to the implicit function theorem, the (locally defined) function that assigns an equilibrium to each environment is smooth in a neighbourhood of e in $E\backslash E_S$.

The algorithm for constructing a mechanism is divided into two parts. The first part is to partition E into regions so that over each region equation (8.1) can be modified into a locally converging system of the form (8.2) and (8.3). This is based on the choice of stability sequence. The second part is to find privacy preserving ways to determine which region contains a particular environment. Finally, we handle multiple equilibria.

PART 1 : The new system will be of the form

$$A(e)f \tag{8.4}$$

where A is a diagonal matrix. Since A does not depend on p, the stability at the equilibria is given by the eigenvalues of $A(e)D_p f$. We show that the entries of $A(e)$ can be chosen so that the eigenvalues of (8.4) have negative real parts.

Let a sequence S be given. The Routh-Hurwitz[22] conditions for stability applied to $A(e)D_p f$ reduce to algebraic inequalities in the unknowns $A(e) = (a_{kk})$. The coefficients of these inequalities are polynomials in the determinants of the principal minors of $D_p f$. These inequalities can be solved in an iterative fashion to find a choice of $A(e)$ that is smooth in $E\backslash E_S$. This is illustrated below. Because $A(e)$ is defined in terms of algebraic inequalities, there is an infinite number of choices for this mapping that are smooth over E. Indeed, $A(e)$ can be found so that it is constant over any connected compact set in $E\backslash E_S$.

At the k-th stage of the iterative procedure to find $A(e)$, a_{kk} is selected. This term determines upper bounds on the remaining terms. We illustrate for the case $B = D_p f$

[22]See Appendix C

is a 3×3 matrix, the general case is similar. Let

$$S = \{\{1\}, \{1,2\}, \{1,2,3\}\};$$
$$b_1 = a_{11}|B(\{1\})| + a_{22}|B(\{2\})| + a_{33}|B(\{3\})|;$$
$$b_2 = a_{11}a_{22}|B(\{1,2\})| + a_{11}a_{33}|B(\{1,3\})| + a_{22}a_{33}|B(\{2,3\})|;$$
$$b_3 = a_{11}a_{22}a_{33}|B|.$$

According to the Routh-Hurwitz criterion, the algebraic inequalities that must be satisfied are

a. $b_1 < 0$;
b. $b_3 - b_1 b_2 > 0$; and
c. $b_3 < 0$.

To satisfy

a. : it is sufficient to choose $a_{11}(e)$ so that $a_{11}(e)|B(\{1\})| < 0$, (for example $a_{11} = -|B(\{1\})|$) and to impose upper bounds $-a_{11}|B(\{i\}| > 3\max\{a_{jj}|B(\{j\})|\}$ where $j = 1,2$;
b. : it suffices to choose $a_{22}(e)$ so that b_2 has a sign different from that of b_1 and to impose the following upper bound on the value of a_{33} : $b_3 - b_1 b_2 = a_{11}^2 a_{22}|B(\{1\})|\,|B(\{1,2\})|(1+z)$ where the magnitude of z is bounded above by $\frac{1}{2}$;
c. : choose a_{33} to differ in sign from $|B|$ and to satisfy the above inequalities.

Clearly, smooth functions can be found to satisfy these conditions in $E \backslash E_S$.

PART 2 : In general the function $A(e)$ is not in a privacy preserving form. To see this note that in equation (8.4), $a_{kk}(e)$ modifies the k-th component of f. The k-th component of f depends only on the private information of one agent, but in general, $a_{kk}(e)$ depends on the private information of several agents. Consequently, equation (8.4) is not in a privacy preserving form.

To solve this problem, partition the space of environments by the level sets of $A(e)$, that is, over each set in the partition, $A(e)$ is a *constant* diagonal matrix. A constant matrix is privacy preserving. So if a privacy preserving mechanism can be designed to determine which level set of $A(e)$ contains the particular environment, then the appropriate choice of the constant matrix $A(e)$ can be made. The resulting system is privacy preserving.

A trivial privacy preserving way to realise $A(e)$ is for each agent to reveal all of his private information. There are however more interesting ways to characterise all ways to realise $A(e)$ but these will not be presented here because they involve a significant amount of new machinery. The relevant references are Saari [20] in which a constructive approach to find the class of selection functions is described. This approach reduces the problem to a set of systems of algebraic inequalities whose solutions are differential forms. The solvability of these systems is analysed in [41]. The solutions define a class of mechanisms that realise $A(e)$

We now turn our attention to multiple equilibria. Consider the set of all equilibria and environments. By use of generic regularity conditions, the space of tuples (e,p) can be partitioned in such a way that each partition set contains no more than one equilibria for each e. For each set, an $A(e)$ and the privacy preserving mechanism are constructed as above. This means that e has many choices of $A(e)$ as it has equilibria. However, the initial condition selected for the system identifies a partition set and hence the appropriate choice of $A(e)$. This completes the proof. □

8.3. **A Discrete Approximation.** Local stability and decentralisation are compatible on $E\backslash E_S$. The algorithm constructed in theorem 8.4 provides an algorithm to construct a (locally effective) mechanism on $E\backslash E_S$. Now depending on the structure of E_S, the complexity of the computations in the algorithm can increase significantly. Simpler systems may be obtained be excising a small area of E_S. We demonstrate this for models with unique equilibria. Models with multiple equilibria are treated in the way described previously.

The method is called the "discrete approximation method" because the selection mechanisms that were used previously are replaced with step function approximations. Because a step function is constant over open sets, it is simpler to compute. For the same reason that it is easier to specify a step function than an arbitrary function, this approach can reduce the informational requirements of the modified system. These approximations are not defined for certain environments which form the set of excluded environments.

Again there are two parts to the proof. The first is to define a set of constant diagonal matrices to modify the dynamics. A privacy preserving selection mechanism is defined to choose the appropriate dynamic for each environment. For the first part we choose a computational sequence $\langle Df, S\rangle$. Let E_S be the set of environments where $\langle Df, S\rangle$ is not a stability sequence, choose $a > 0$, and let $S'(a)$ be the set of points within a distance a of E_S. The matrices $A(e)$ are defined over each component of $E\backslash S'(a)$ in the way described in theorem 8.4. Under appropriate conditions, $A(e)$ can be chosen to be consistent over each of the components of $E\backslash S'(a)$. $S(a)$ is a subset of our set of excluded environments.

The second step is to define the selection mechanism. To do this define a partition for E_j and assign an identifying symbol to each partition set, $j = 1, \ldots, l$. These partitions define a product partition over E where each partition set is identified by a unique l-tuple of symbols. Some of these are contained in level sets of $A(e)$, some are not. The union of the partitions sets of the latter type defines a set of environments that are excluded from this approach. The remaining partition sets can be viewed as approximating the interiors of the level sets of $A(e)$.

The selection mechanism is defined in the following way. For the j-th agent, e_j is contained in a unique partition set of E_j. The j-th agent communicates the symbol that is assigned to this partition set. The l-tuple of communicated symbols identifies a partition set of E. This partition set is in a level set of $A(e)$ and so this choice of $A(e)$ is the appropriate matrix for equation (8.4).

Part of the utility of the discrete approximation method depends on whether it can be defined so that the set of excluded environments is of arbitrarily small measure. The following theorem describes a condition where this is true.

Theorem 8.5. *Assume that E is a compact set in a finite dimensional space. Let $\epsilon > 0$. Then, generically, a discrete approximating approach exists for equation*

(8.1) where the excluded set of environments is contained in an open set of Lebesgue measure less that ϵ.

Because E is compact, it will follow that only a finite set of symbols is needed for the communication process.

Proof. Let $\langle Df, S \rangle$, E_S and $S'(a)$ be as before. Because E is a compact set, generically, for any $\epsilon > 0$, an $a > 0$ can be found so that the Lebesgue measure of $S'(a)$ is bounded above by ϵ. The compact set $E \backslash S'(2a)$ is contained in $E \backslash S'(a)$.

Because $E \backslash S'(a)$ is compact, on any component of $E \backslash S'(a)$, the values of the determinants of the elements of $\langle Df, S \rangle$ are bounded in magnitude, both above and away from zero. It now follows from the arguments given in Williams [42] that $A(\epsilon)$ can be chosen to be constant over each component of $E \backslash S'(a)$. Because $E \backslash S'(2a)$ is a proper subset of $E \backslash S'(a)$, each component of $E \backslash S'(2a)$ is contained in a level set of $A(\epsilon)$.

Choose partitions E_j, $j = 1, \ldots, N$, so that the diameter of the partition sets are strictly bounded above by $a/(N^{1/2})$. By the Pythagorean theorem, the diameter of any partition in E is strictly bounded above by a. Because the distance between the boundaries of $S'(2a)$ and $S'(a)$ is a, it follows that

1. the product partitions cover $E \backslash S'(2a)$; and
2. each partition set in E that meets $E \backslash S'(2a)$ is contained in $E \backslash S'(a)$.

The excluded set of environments are contained in the partition sets that are disjoint from $E \backslash S'(2a)$. By construction this set is contained in $S'(a)$, and this open set has Lebesgue measure bounded above by ϵ. Also, by construction, all remaining partition sets are contained in $E \backslash S'(a)$, and hence they are contained in the level sets of $A(\epsilon)$. □

Theorem I follows.

9. Conclusion

In summary, we have seen how the introduction of dynamical systems into general equilibrium theory has had a profound impact on the subject. Results once hypothesised have been proved false and this has some serious ramifications for theoretical economics. Even in the most ideal of models, pricing dynamics determined by Walras' law do not have to comply with Adam Smith's Invisible Hand Theory. Those universal pricing dynamics which are consistent with the Invisible Hand Theory require far to much information to be considered economically realistic as models.

The final section has offered a partial solution to our problems. By dropping the assumption of a universal pricing mechanism we have shown the existence of locally stable price mechanisms, however, even here there are problems. Firstly, we have made the rather strong assumption that excess demand is parametrised by finitely many parameters. While this may be an interesting start, there is no way as yet of extending this result to more realistic economies. Secondly, all the economies we have considered have been very idealised. The truth is that the economies in which we find ourselves are far more subtle and complex.

While the troubles facing theoretical economics are daunting, there is no doubt that the introduction of more sophisticated mathematical tools into the subject will be beneficial, even if only to rule out long held hypotheses. The breadth and depth of mathematics currently available in the subject pale in comparison to those used in physics and this is a cause for concern. With so many influential decisions made daily based upon gross oversimplifications or just inappropriate or inaccurate models, the motivation for the development of a sound theoretical base seems obvious (not to mention the inevitable monetary gains which accompany such improvements).

I hope that in my own small way I have at least highlighted a very small portion of an extremely important subject to an audience who would otherwise not have been aware of its existence.

Note : the following proposition depends heavily on the second countability of M. For this appendix a manifold will be viewed as a triple, consisting of an underlying point set, a second countable locally Euclidean topology on this set and a differentiable structure.

Proposition A.1. If $\psi : M \to N$ is a C^∞ map of manifolds which is also one-to-one, onto and everywhere non-singular, then ψ is a diffeomorphism.

Proof. ψ is a diffeomorphism if and only if $d\psi$ is surjective everywhere. If $d\psi$ is not surjective at some point and denoting $\dim M = m$ and $\dim N = n$, it must be that $m < n$.

Assume that $m < n$. Let (U, ϕ) be a co-ordinate system on N such that $\phi(U) = \mathbb{R}^n$. Since $\psi : M \to N$, the range of $\phi \circ \psi$ is all of $\mathbb{R}^n$. This as shall be shown leads to a contradiction since if $m < n$, the range of $\phi \circ \psi$ has measure zero in $\mathbb{R}^n$ (a set has measure zero in $\mathbb{R}^n$ if it can be covered by a sequence of balls, the union of whose volumes is arbitrarily small). That the range of $\phi \circ \psi$ has zero volume, follows from the second countability of M and the fact that a C^1 map from $\mathbb{R}^m$ to $\mathbb{R}^n$ has image of measure zero which in turn follows from the fact that a C^1 map from $\mathbb{R}^n$ to $\mathbb{R}^n$ takes sets of measure zero onto sets of measure zero. To prove this, observe that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1, $A \subset \mathbb{R}^n$ compact, then f has a Lipschitz constant K on A

$$\|f(x) - f(y)\| \leq K\|x - y\| \qquad x, y \in A$$

so that f magnifies the volume of balls in A by at most K^n and hence takes the sets of measure zero in A into sets of measure 0. $\square$

Appendix B. The Theorem of Mas-Colell

This appendix covers the proof of theorem D, which for convenience is restated here as :

Theorem B.1. Let f be an excess demand function on S. For any $\epsilon > 0$ there is a μ with $0 < \mu < \epsilon$ and an l consumer economy $(r, \succcurlyeq) = (r_1, \ldots, r_l, \succcurlyeq_1, \ldots, \succcurlyeq_l)$ with each $\succcurlyeq_i$ continuous, strictly convex and monotone, such that

1. $(r, \succcurlyeq)$ generates f on S_μ; and
2. $f^{-1}(0) \subset S_\mu$ is the set of equilibrium prices of $(r, \succcurlyeq)$.

Again for convenience, we restate the properties of an excess demand function.

- **W:** $p \cdot f(p) = 0$ for every $p \in S$.
- **B:** There is a $k \in \mathbb{R}$ such that for every $p \in S$ each component of $f(p)$ is bounded below by k.
- **D:** If the sequence $(p_j) \in S$ converges to $p_0 \in \partial S$, then $\|f(p_j)\| \to \infty$.

For every $p \in S$, define $g_e(p) = \epsilon - (p\epsilon)p$.

Lemma B.2. If f is an excess demand function there is a $\delta > 0$ such that $f(p)g_e(p) > 0$ whenever $p \in S \backslash S_\delta$

Proof. By (B) and (D) there is a $\delta > 0$ such that if $p \in S\backslash S_\delta$ then each component, $f^i(p)$, of the excess demand function $f(p)$ is greater than 1. But then

$$f(p)g_e(p) = \sum_{i=1}^{l} f^i(p) > 0$$

□

In view of this lemma, the proof of theorem B.1 is then a corollary of the following proposition.

Proposition B.3. For every $\epsilon > 0$ there is a $0 < \mu < \epsilon$ and a function $k : (0, \infty) \to (0, \infty)$ such that if $f : S \to \mathbb{R}^l$ is a continuous function satisfying $p \cdot f(p) = 0$ for every $p \in S$ and $f(p)g_e(p) > 0$ for $p \in S\backslash S_\epsilon$, then there exists a demand function f^* satisfying

1. $f^*|S_\epsilon = f|S_\epsilon$,
2. $f^*(p)g_e(p) > 0$ for all $p \in S\backslash S_\epsilon$,
3. f^* is generated by an economy $\{(\bar{P}, \succcurlyeq_i, r_i)\}_{i=1}^{l}$ such that for all i, $\|r_i\| \leq l(2 + k(x))$ for any $x \geq \max_{p \in S_\mu} \|f(p)\|$ and $\succcurlyeq_i$ continuous, monotone and strictly convex.

Among other things, this proposition makes it clear that the norm of the initial resources of the constructed economy depends only on the norm of the values of the excess demand (and not for example the norm of the derivatives were they to exist). No attempt is made to get a sharp bound on this value however.

We shall now spend some time defining two functions, u_a and v, which will be used extensively in the decomposition of f^* into l individual excess demand functions.

each generated by an agent $(\succcurlyeq, r, \bar{P})$. Throughout this appendix certain standard assumptions will be made on many functions we have cause to construct. These are stated now before beginning with the proof :

I. $w : P \to \mathbb{R}$ is a proper[23], strictly convex and C^2; moreover, for each sequence $\{p_n\} \in P$ such that $p_n \to p \in \partial P$, $\|Dw(p_n)\| \to \infty$; and there is an $x > 0$ such that if $\|p\| \leq 1$, then $pDw(p) \leq x$ and $Dw(p) \leq 0$.

II. v is the restriction to S of a function on P satisfying I

Note that the sum of functions which satisfy I (resp. II) satisfy I (resp. II). Also, if $v : S \to \mathbb{R}$ satisfies II then v is minimised in at most one point and for every $y \in v(S)$ the cone spanned by $v^{-1}((-\infty, y])$ is strictly convex[24]. Therefore, if v satisfies II, then for all $p \in S$, $Dv(p) \in T_pS$ is normal to the supporting hyperplane at p of the cone spanned by $v^{-1}((-\infty, v(p)])$ Hence, for $p, q \in S, p \neq q$, by elementary vector calculus $0 < pq < 1$ and it follows that $v(p) > v(q)$ implies $Dv(p)(q - (pq)p) < 0$.

Note that if $v : S \to \mathbb{R}$ satisfies II, then $p_n \to p \in \partial S$ implies $\|Dv(p_n)\| \to \infty$

As an example of a function which satisfies II consider

$$u_a(p) = \frac{1}{2\|a\|}\|p - a\|^2$$

where $a \in P$ is such that every component $a^j \geq 1$. Note here, that elementary vector calculus shows

$$-Du_a = \frac{1}{\|a\|}(a - (pa)p).$$

Lemma B.4. For any $a \in P$ with $a^j \geq 1$ for all $j = 1, \ldots, l$ and compact $K \subset S$ with $(a/\|a\|) \in K$, there is a function $v : S \to \mathbb{R}$, satisfying II and

1. $v|K = u_a$ and $v(p) \geq u_a(p)$ for all p; and
2. each component of $Dv(p)$ is bounded above by 2 for all $p \in S$

Proof. Define $\xi : P \to \mathbb{R}$ by

$$\xi(p) = -\frac{1}{l}\left(\sum_{i=1}^{l} \log p^i\right)$$

and $\hat{u}_a : P \to \mathbb{R}$ by

$$\hat{u}_a(p) = \frac{1}{2\|a\|}\|p - a\|^2$$

(so that $u_a = \hat{u}_a|S$). Both ξ and $\hat{u}_a$ satisfy I. Clearly, for $p \in S$ each component of

$$D(\xi|S)(p) = p - \frac{1}{l}\left(\frac{1}{p^1}, \ldots, \frac{1}{p^l}\right)$$

is bounded above by 1. Let $B \subset S$ be a neighbourhood of K with $\bar{B} \subset S$ and define $\gamma : P \to \mathbb{R}$ by $\gamma(p) = \max\{0, \xi(p) - s\}$ where s is chosen large enough so that $\gamma(p) = 0$ on a neighbourhood of $\bar{B}$. γ is convex and so we can smooth it out around $\gamma^{-1}(0)$ so

[23] A function is proper if inverse images of compact sets are compact.

[24] This can be seen by noting that if w defined on P satisfies I and $w(p), w(q) \leq y$ for $p, q \in S, p \neq q$, then for $0 < t < 1, \|tp + (1-t)q\| < 1$ and $w(tp + (1-t)q) < y$. Since w is decreasing on $\{q' : \|q'\| < 1\}$, $w\{(1/\|tp + (1-t)q\|)(tp + (1-t)q)\} < y$.

as to obtain a convex function $\hat{\gamma} : P \to \mathbb{R}$ which satisfies I (except strict convexity) and $\hat{\gamma}(p) \geq 0$ where each component of $D(\hat{\gamma}|S)(p)$ is less than 1 for $p \in S$, $\hat{\gamma}(p) = 0$ for $p \in B$. Now set

$$v = (\hat{u}_a + \hat{\gamma})|S = (u_a + \hat{\gamma})|S.$$

Clearly v satisfies II and each component of

$$Dv(p) = \frac{1}{||a||}((pa)p - a) + D(\hat{\gamma}|S)(p)$$

is less than 2. □

Having constructed our two most important functions u_a and v we now define the function k of proposition B.3. Let $\epsilon > 0$ be given. For each $\delta > 0$ there is a choice of linearly independent vectors $a_i \in P$, $i = 1, \ldots, l$ which satisfy

1. $a_i^j \geq 1$ for all $i, j = 1, \ldots, l$; and
2. $\dfrac{a_i^j}{||a_i||} = \delta$ for all $i \neq j$.

Let Γ denote the convex cone spanned by $\{a_i\}$. From now on, δ will be assumed to be small enough such that $S_{\epsilon/2} \subset \text{int}\Gamma$. Choose μ such that $\Gamma \cap S \subset S_\mu$.

Given $x > 0$, there exists a function $\theta(x) > 0$ such that for every $y \in B_x$, the ball in $\mathbb{R}^l$ with radius x, and $p \in S_{\epsilon/2}$, $y + \theta(x)p \in \Gamma$. Define

$$k(x) = \theta(x) + x$$

For each i pick a function $v_i : S \to \mathbb{R}$ satisfying the conditions of lemma B.4 with respect to a_i and $\Gamma \cap S$. The function $\sum u_{a_i}$ attains its maximum at a unique point $p = (1/\sqrt{l})e \in S$. Since $v_i \geq u_{a_i}$ and $v_i|S_\epsilon = u_{a_i}|S_\epsilon$ we get $\sum v_i(p) > \sum v_i((1/\sqrt{l})e)$ for all $p \neq (1/\sqrt{l})e$. Since $\sum v_i$ satisfies II we can conclude that for all $p \neq (1/\sqrt{l})e$

$$\sum Dv_i(p)(e - (pe)p) = \sum Dv_i(p)g_e(p) < 0.$$

Let f be an excess demand function satisfying $f(p)g_e(p) > 0$ for $p \in S \backslash S_\epsilon$. We now define the function f^*.

Pick $x' > \max_{p \in S_\mu} ||f(p)||$. Then for every $p \in S_{\epsilon/2}$, $f(p) + (k(x') - x')p \in \Gamma$. Since the vectors $\{a_i\}$ are linearly independent we can write, for $p \in S_{\epsilon/2}$

$$f(p) + (k(x') - x')p = \sum_{i=1}^{l} \beta_i(p)a_i$$

$\beta(p) > 0$ in a unique continuous way; moreover, for all i, $||\beta_i(p)a_i|| \leq ||f(p) + (k(x') - x')p|| \leq k(x')$ Since $-Du_{a_i}(p) = a_i - (pa_i)p$ is the orthogonal projection of a_i into T_p and therefore projecting into T_p, we get for $p \in S_{\epsilon/2}$

$$\sum_{i=1}^{l} \beta_i(p)(-Du_{a_i}(p))$$

and of course $||\beta_i(p)Du_{a_i}|| \leq ||\beta_i(p)a_i|| \leq k(x')$ for all i.

Let $\eta : S \to [0,1]$ be C^2 and such that

$$\eta(p) = \begin{cases} 1 & \text{for } p \in S \backslash S_{\epsilon/2} \\ 0 & \text{for } p \in S_\epsilon . \end{cases}$$

Define $f_i : S \to \mathbb{R}^l$ by

$$f_i = -(\eta(p) + (1 - \eta(p)\beta_i(p)) Dv_i(p)$$

and let $f^* = \sum f_i$. Note that f_i and f^* are excess demand functions.

Finally, we present a lemma which shows that every f_i can be generated by an agent $(\succcurlyeq, r, \bar{P})$ with the required properties.

Lemma B.5. Let $v : S \to \mathbb{R}$ satisfy II. Suppose that $f : S \to R^l$ is an excess demand function such that for all $p \in S$, the components of $f(p)$ bounded below by $\alpha > -\infty$ and $f(p) = -\beta(p)Dv(p)$ where $\beta(p)$ is positive and uniformly bounded away from zero. Then f is generated by $(\succcurlyeq, r, \bar{P})$ where $\succcurlyeq$ is a continuous, monotone, strictly convex preference relation on $\bar{P}$ and $r \in P, \|r\| \leq l|\alpha|$.

Proof. An outline of the proof is sketched here. r will be taken to be 0 and a continuous and monotone preference function $\succcurlyeq$ will be defined on $\mathbb{R}^l$. $\succcurlyeq$ could then be "convexified" using standard methods. In order to have $\succcurlyeq$ strictly convex, Debreu's techniques [5] would need to be applied in a second step.

There is a unique $p^* \in S$ for which $f(p^*) = 0$. Partition $\mathbb{R}^l$ into three regions

1. $\dot{P} = \bar{P} \backslash \{0\}$
2. $A = \{x | x \notin \dot{P}, p^* x > 0\}$
3. $B = \{x | p^* x \leq 0\}$

Let $u : S \to [0,1]$ be a continuous function satisfying $u(p^*) = 0$ and $u(p) \geq u(q)$ if and only if $v(p) \geq v(q)$.

For every $x \in A$ there is a unique $p(x) \in S$ such that x belongs to the positive ray spanned by $f(p(x))$. This follows since $Dv(p) \in T_p S$ is normal to the supporting hyperplane at p of the cone spanned by $v^{-1}(-\infty, v(p)]$ and $\beta(p)$ is positive and uniformly bounded away from zero.

So the function $x \mapsto p(x)$ is continuous. Let $x_n \in A$ be such that $x_n \to x$ where $x \not\in A$, then $p(x_n) \to \partial S$ if $x \in \dot{P}$ or $p(x_n) \to p^*$ if $x \in B$. It is also true that $p \in S, x \in A, px \leq 0$ and $x \notin \{tf(p) | t \geq 0\}$ implies that $u(p(x)) < u(p)$.

Define a function $\xi : \mathbb{R}^l \to \mathbb{R}$ as follows : if $a \in A$ write $x = tf(p(x))$

$$\xi(x) = \begin{cases} tu(p(x)) & x \in A \text{ and } t \leq 1 \\ u(p(x)) - \|x - f(p(x))\| & x \in A \text{ and } t > 1, \\ -\|x\| & x \in B \\ \min x^i & x \in \dot{P}. \end{cases}$$

It is not hard to verify that

1. ξ is continuous (this is a consequence of the fact that u is bounded, $u(p) = 0$, f is continuous and $\lim \|f(p_n)\| \to \infty$ as $p_n \to \partial S$.)
2. ξ has no local maximum