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Symmetry structures and conserved forms of systems of pdes

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School of Mathematics

October 6, 2019

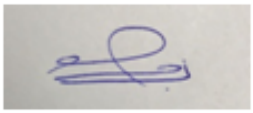
A thesis submitted to the Faculty of Science, University of the Witwatersrand,

Johannesburg, in fulfillment of the requirements for the degree of Doctor of Philosophy



DECLARATION

I hereby declare that except where due acknowledgement is made, this work has never been presented wholly or in part for the award of a degree at the university of the Witwatersrand or any other University.



Bader Mutair Alqurashi, October 6, 2019.

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DEDICATION

To my loving parents.

Abstract

We will study the symmetry, invariance properties and conservation laws of partial differential equations (pdes) that arise in a number of situations in mathematical physics. These will be range from Image Processing and noise removal algorithms to Timoshenko beam systems. Furthermore, we will study the invariance properties and approximate conservation laws of some nonlinear Schrödinger equation with **PT**-symmetric potentials with inhomogeneous nonlinearity and some nonlinear Schrödinger equation involving a spatially extended system consisting of two coupled elements.

Contents

Declaration i

Acknowledgements ii

Dedication iii

Abstract iv

1 Introduction 1

2 Image Processing and 'noise removal algorithms' - the pdes and their invariance properties & conservation laws 8

2.1 Introduction 8

2.2 The pdes of the 'denoising algorithm' 9

2.2.1 Parabolic 'denoising pde' 9

2.2.2 The 'denoising pde' with a linear source term 12

2.2.3 The time evolution 'denoising pde' 13

2.3 Conclusions 15

3 Higher order conservation laws and integrability of Timoshenko beam systems 16

3.1 Introduction 16

3.2 Multipliers and conserved flows 18

3.3 Discussion 23

4 Invariance and 'approximate' conservation laws of some nonlinear Schrödinger equation with PT-symmetric potentials and inhomogeneous nonlinearity 24

4.1 Introduction 24

4.2 Conservation laws 26

4.2.1 Case 1 26

4.2.2 Case 2 31

4.3 Conclusion 32

5 'Approximate' conservation laws of some nonlinear Schrödinger equation involving a spatially extended system consisting of two coupled elements 33

5.1 Introduction 33

5.2 Conservation laws 35

5.2.1 Case 1 36

5.2.2 Case 2 38

5.3 Results and discussion 40

6 Conclusion **41**

References **47**

Chapter 1

Introduction

Partial Defferential Equations (pdes) have been successfully studied using the methods of invariance, conservation laws and a combination of these. A large number of classes of pdes arise all the time in the literature of engineering, mathematical finance, mathematical physics, relativity, to name a few. We will be studying, inter alia, the following classes of pdes.

Noise removal algorithms and Schrodinger equations: a. **PT**-symmetric potentials, b. Externally driven nonlinear cases, c. Anti cubic nonlinearity. We study the results from the symmetries and conservation laws of pdes. Also the problem of how to find symmetries and conservation laws for a given system of partial differential equations.

In the study of pdes, determining conserved quantities and constants of motion lead to detecting integrability and linearizations, finding potentials and checking the accuracy of numerical solution methods used from conservation laws. This thesis contais of four published artical.

In chapter 2, we analyse the symmetry, invariance properties and conservation laws of the pdes and minimization problems (variational functionals) that arise in the analyses of some noise removal algorithms. This chapter has been published [1].

In chapter 3, we show that some classes of the Timoshenko system of pdes that model the vibrations of a beam are rich in symmetry and conservation laws. In fact, it admits an infinite hierarchy of conservation laws and higher order symmetries which have implications on, inter alia, the integrability of the system. This chapter has been published [2].

In chapter 4, we construct and analyse the conservation laws (conserved densities) of a model of a nonlinear Schrödinger equation with **PT**-symmetric potentials and inhomogeneity. Noether's theorem is not applicable to this model as it is not variational and the conserved forms are adapted in an 'approximate' and novel way due to the nature of the system. This chapter has been published [3].

In chapter 5, we construct and analyse the conservation laws of a model of a nonlinear Schrödinger equation involving a spatially extended system consisting of two coupled elements. In the first of two cases, we study the space-time system and, separately, the stationary model arising from a standard transformation. In the second case, we analyse a 'toy model **PT**-symmetric dimer' involving a 'gain-loss amplitude parameter' for which only approximate conserved forms exist. In this latter case, Noether's theorem is not applicable as it is not variational and the conserved forms are adapted in an 'approximate' and novel way due to the nature of the system. This chapter has been published [4].

We present some preliminaries that will be used in the analyses that follow.

2.1 Symmetries, multipliers and conservation laws. Consider an rth-order system of partial differential equations pdes of n independent variables $\underline{s} = (s_1, s_2, \ldots, s_n)$ and m dependent variables $u = (u_1, u_2, \ldots, u_m)$ viz.,

$$E(\underline{s}, u, u_{(1)}, \ldots, u_{(r)}) = 0, \qquad u = 1, \ldots, \tilde{m}, \tag{1.1}$$

where a locally analytic function $f(\underline{s}, u, u_1, \ldots, u_k)$ of a finite number of dependent variables $u, u_1, \ldots, u_k$ denote the collections of all first , second ,..., kth-order partial derivatives and s is a multivariable, that is

$$u_i^\alpha = D_i(u^\alpha), \qquad u_{ij}^\alpha = D_j D_i(u^\alpha), \ldots \tag{1.2}$$

respectively, with the total differentiation operator with respect to s^i given by,

$$D_i = \frac{\partial}{\partial s^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots \qquad i = 1, \dots, m. \tag{1.3}$$

In order to determine conserved densities and fluxes, we resort to the invariance and multiplier approach based on the well known result that the Euler-Lagrange operator annihilates a total divergence. Firstly, if $(T^{s1}, T^{s2}, \dots)$ is a conserved vector corresponding to a conservation law, then

$$D_{s1} T^{s1} + D_{s2} T^{s2} + \dots = 0 \tag{1.4}$$

along the solutions of the differential equation $E(\underline{s}, u, u_{(1)}, \dots, u_{(r)}) = 0$.

Moreover, if there exists a nontrivial differential function Q, called a 'multiplier', such that

$$Q(\underline{s}, u, u_{(1)} \dots) E(\underline{s}, u, u_{(1)}, \dots, u_{(r)}) = D_{s1} T^{s1} + D_{s2} T^{s2} + \dots, \tag{1.5}$$

for some (conserved) vector $(T^{s1}, T^{s2}, \dots)$, then

$$\frac{\delta}{\delta u}[Q(\underline{s}, u, u_{(1)} \dots) E(\underline{s}, u, u_{(1)}, \dots, u_{(r)})] = 0, \tag{1.6}$$

where $\frac{\delta}{\delta u}$ is the Euler operator. Hence, one may determine the multipliers, using (1.10) and then construct the corresponding conserved vectors; several approaches for this exists of which the better known one is the 'homotopy' approach.

When a nontrivial multiplier exists, the homotopy operator allows one to reduce the computation of

$$\Phi = Div^{-1}(Q(\underline{s}, u, u_{(1)} \dots) E(\underline{s}, u, u_{(1)}, \dots, u_{(r)}))$$

(see details in [5, 6]); the homotopy operator reduces the integration by parts to a single variable integral.

Definition We define the homotopy operator in 2D (with variables (x, t)) through its two components

$$\mathcal{H}^x_{u(x,t)}(QE) = \int_0^1 \sum I^x_{u_j}(QE)[\lambda u] \frac{\mathrm{d}\lambda}{\lambda},$$

where, for e.g.,

$$I^x_{u_j}(QE) = \sum_{i_x=0}^{\infty}\sum_{i_t=0}^{\infty}(\frac{1+i_x}{1+i_x+i_t})D_x^{i_x}D_t^{i_t}(u_j E^{(1+i_x,t)}_{u_j(x,t)}(QE)),$$

and $(QE)[\lambda u]$ means replacing all u's and their derivatives by λu, i.e., u_x is replaced by $(\lambda u)_x$, etc. Also, similarly, for the t case, Here, $E^{(1+i_x,t)}_{u_j(x,t)}$ is the specific Euler operator. Then, $\Phi = (T^x, T^t)$ is given by the following theorem.

Theorem If Φ is a divergence, then

$$\Phi = (T^x, T^t) = Div^{-1}(Q(\underline{s}, u, u_{(1)}\dots)E(\underline{s}, u, u_{(1)}, \dots, u_{(r)})) = (\mathcal{H}^x_{u(x,t)}(QE), \mathcal{H}^t_{u(x,t)}(QE)).$$

Example [6] Suppose $f = QE = u_x v_y - u_{xx}v_y - u_y v_x + u_{xy}v_x$, then

$$\begin{aligned} I^x_u f &= uE^{(1,0)}_{u(x,y)}(f) + D_x(uE^{(2,0)}_{u(x,y)}(f)) + \tfrac{1}{2}D_y(uE^{(1,1)}_{u(x,y)}(f)) \\ &= u(\tfrac{\partial f}{\partial u_x} - 2D_x\tfrac{\partial f}{\partial u_{xx}} - D_y\tfrac{\partial f}{\partial u_{xy}}) + D_x(uD_x\tfrac{\partial f}{\partial u_{xx}}) + \tfrac{1}{2}D_y(uD_x\tfrac{\partial f}{\partial u_{xy}}) \\ &= uv_y + \tfrac{1}{2}u_y v_x - u_x v_y + \tfrac{1}{2}uv_{xy} \end{aligned}$$

and we can show that

$$I^x_v f = u_y v + v u_{xy}.$$

Hence,

$$\begin{aligned} \mathcal{H}^x_{u(x,y)} f &= \textstyle\int_0^1 (I^x_u f[\lambda(u,v)] + I^x_v f[\lambda(u,v)])\frac{\mathrm{d}\lambda}{\lambda} \\ &= \tfrac{1}{2}uv_y + \tfrac{1}{4}u_y v_x - \tfrac{1}{2}u_x v_y + \tfrac{1}{4}uv_{xy} - \tfrac{1}{2}u_y v + \tfrac{1}{2}vu_{xy}. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathcal{H}^y_{u(x,y)} f &= \textstyle\int_0^1 (I^y_u f[\lambda(u,v)] + I^y_v f[\lambda(u,v)])\frac{\mathrm{d}\lambda}{\lambda} \\ &= -\tfrac{1}{2}uv_x - \tfrac{1}{4}uv_{xx} + \tfrac{1}{4}u_x v_x + \tfrac{1}{2}u_x v - \tfrac{1}{2}vu_{xx} \end{aligned}$$

and

$$\begin{aligned} \Phi &= (T^x, T^y) \\ &= (\tfrac{1}{2}uv_y + \tfrac{1}{4}u_y v_x - \tfrac{1}{2}u_x v_y + \tfrac{1}{4}uv_{xy} - \tfrac{1}{2}u_y v + \tfrac{1}{2}vu_{xy}, \\ &\quad -\tfrac{1}{2}uv_x - \tfrac{1}{4}uv_{xx} + \tfrac{1}{4}u_x v_x + \tfrac{1}{2}u_x v - \tfrac{1}{2}vu_{xx}). \end{aligned}$$

If the system of differential equations is derived from a variational principle, then the conserved vector components are obtainable from Noether's Theorem which requires, firstly, the construction of variational symmetries (vector fields) $X = \xi^{s^i}\frac{\partial}{\partial s^i} + \eta^{u^\alpha}\frac{\partial}{\partial u^\alpha}$ that

leave the action integral invariant. It is well known that the vector fields that leave the system of differential equations invariant (generators of Lie point symmetries) contain the algebra of variational symmetries, if the latter exists [7, 8, 9].

Conservation laws may be expressed as conserved forms [10]. For example, if $\underline{s} = (t, x)$, the conserved form would be

$$\omega = T^t \mathrm{d}x - T^x \mathrm{d}t$$

(where (T^t, T^x) is the conserved vector such that $D_t T^t + D_x T^x = 0$ on the solutions of the pde $E(x, t, u, u_{(1)}, \ldots, u_{(r)}) = 0$). Here, $T^t \mathrm{d}x$ leads to the 'conserved density' if t and x are time and space, respectively.

2.2 Approximative conservation laws. Often, one encounters systems that involve one or several parameters for which the system may be exactly conserved (closed) for all values, some values or no values of the parameters. In the latter two cases, dependent on the relative nature of the parameters, one may consider an alternative study of the conservation of the system. In particular, if a parameter is relatively 'small', one may be able to construct an 'approximately closed system' that, in the limit of the parameter, is exact. To this end, consider a system dependent on a parameter of first order ϵ, viz.,

$$E^\epsilon(\underline{s}, u, u_{(1)}, \ldots, u_{(r)}, \epsilon) = O(\epsilon), \qquad u = 1, \ldots, \tilde{m}. \tag{1.7}$$

We say that the system is approximately conserved up to order ϵ if there exists a vector $(T^{s1}, T^{s2}, \ldots)$ such that $T^{si} = T^{si}(\underline{s}, u, u_{(1)}, \ldots, u_{(r)}, \epsilon)$ and

$$D_{s1} T^{s1} + D_{s2} T^{s2} + \ldots = O(\epsilon) \tag{1.8}$$

along the solutions of the differential equation $E^\epsilon(\underline{s}, u, u_{(1)}, \ldots, u_{(r)}, \epsilon) = O(\epsilon)$. In this case, the multiplier approach would entail the existence of a differential funtion Q such that

$$Q(\underline{s}, u, u_{(1)} \ldots) E^\epsilon(\underline{s}, u, u_{(1)}, \ldots, u_{(r)}, \epsilon) = D_{s1} T^{s1} + D_{s2} T^{s2} + \ldots. \tag{1.9}$$

In this case, then,

$$E[Q(\underline{s}, u, u_{(1)} \ldots) E^\epsilon(\underline{s}, u, u_{(1)}, \ldots, u_{(r)}, \epsilon)] = O(\epsilon). \tag{1.10}$$

In the above, the exactness is recovered in the limit, i.e., when $\epsilon \to 0$.

Notes.

(i) If $q = u + iv$, then $q = u + iv$, $q_x = u_x + iv_x$, $q_t = u_t + iv_t$, $q_{xx} = u_{xx} + iv_{xx}$, etc., and $\bar{q} = u - iv$, $\bar{q}_x = u_x - iv_x$, $\bar{q}_t = u_t - iv_t$, $\bar{q}_{xx} = u_{xx} - iv_{xx}$, etc. so that, for e.g.,

$$\bar{q}q_x - q\bar{q}_x = 2iuv_x - 2ivu_x$$

and rearranging, we get

$$\begin{aligned} uv_x - vu_x &= \tfrac{1}{2i}(\bar{q}q_x - q\bar{q}_x) \\ &= I(\bar{q}q_x) \end{aligned}$$

Similarly,

$$\begin{aligned} uv_{xx} - vu_{xx} &= \tfrac{1}{2i}(\bar{q}q_{xx} - q\bar{q}_{xx}) \\ &= I(\bar{q}q_{xx}) \end{aligned}$$

and so on.

Also,

$$\bar{q}q_x + q\bar{q}_x = 2uu_x + 2vv_x$$

so that

$$\begin{aligned} uv_x + vu_x &= \tfrac{1}{2}(\bar{q}q_x + q\bar{q}_x) \\ &= \hat{R}(q\bar{q}_x) \end{aligned}$$

and

$$\begin{aligned} uv_{xx} + vu_{xx} &= \tfrac{1}{2}(\bar{q}q_{xx} + q\bar{q}_{xx}) \\ &= \hat{R}(q\bar{q}_x) \end{aligned}$$

(ii) For systems with two complex functions q and p, let $q = u + iw$ and $p = v + iz$. Then, for e.g.,

$$qp = uv + iuz + iwv - wz, \qquad \bar{q}\bar{p} = uv - iuz - iwv - wz,$$

and

$$q\bar{p} = uv - iuz + iwv + wz, \qquad \bar{q}p = uv + iuz - iwv + wz,$$

so that

$$q\bar{p} - \bar{q}p = -2iuz + 2ivw, \qquad q\bar{p} + \bar{q}p = 2uz + 2wz$$

and, hence,

$$-uz + vw = \frac{1}{2i}(q\bar{p} - \bar{q}p), \qquad uv + wz = \frac{1}{2}(q\bar{p} + \bar{q}p).$$

Furthermore, for e.g.,

$$uv_x + wz_x = \frac{1}{2i}(q\bar{p}_x + \bar{q}_x p), \qquad u_x v + w_x z = \frac{1}{2}(q_x\bar{p} + \bar{q}_x p).$$

Note. We have not included the details relating to symmetries; these are will known and appear in many of the works that are listed in the references.

Chapter 2

Image Processing and 'noise removal algorithms' - the pdes and their invariance properties & conservation laws

2.1 Introduction

Amongst others, Rudin et al [11] and Esedoglu & Osher [12] have discussed and modelled, in considerable detail, the notion of *denoising* that arises in images wherein the goal is 'to remove noise from a corrupted digital image without blurring object boundaries'. Other detailed works in this regard by Osher are [13, 14, 15]. Essentially, they propose to denoise images by minimizing the total variation norm of the estimated solution. To this end, they 'derive a constrained minimization algorithm in which the constraints are determined by the noise statistics'. In this chapter, we study a number of models that are developed and discussed in [11]. We study the symmetry, invariance properties and conservation laws of the differential equations and/or minimization problems (variational functionals) that

are presented therein.

A number of other works and alternative approaches in the area of 'denoising' are available in [16, 17, 18, 19], inter alia. The result below are published in [1].

2.2 The pdes of the 'denoising algorithm'

The cases studied below are the various nonlinear pdes that, in the first two cases, are variational and are pdes in space variables only. The first of these variational pdes is a parabolic type one that is independent of a 'source term' whilst the second contains a linear source term in the dependent variable. When the latter of these two contains a time variable as an evolution parameter, the resulting pde mimics a Klein-Gordon equation.

2.2.1 Parabolic 'denoising pde'

Let $u(x, y)$ denote the 'clean image' for $(x, y) \in \mathbb{R}$. Then, a constrained minimization problem is given by minimizing [11] $\int L_1 \mathrm{d}x\mathrm{d}y$, where the Lagrangian, L_1, is given by

$$L_1 = \sqrt{u_x^2 + u_y^2} \tag{2.1}$$

subject to some constraints. The corresponding Euler or Euler-Lagrange operator on L_1 is $\frac{\delta L_1}{\delta u} = -\frac{u_{xx}{u_y}^2 - 2\,u_x u_y u_{xy} + u_{yy}{u_x}^2}{({u_x}^2 + {u_y}^2)^{3/2}}$.

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} - D_y \frac{\partial}{\partial u_y} \tag{2.2}$$

$$D_x = \frac{\partial}{\partial u} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{xy} \frac{\partial}{\partial u_y} \tag{2.3}$$

In order to find the variational symmetries, we have the following operators. The total derivative is given by

$$D_y = \frac{\partial}{\partial u} + u_y \frac{\partial}{\partial u} + u_{yy} \frac{\partial}{\partial u_y} + u_{xy} \frac{\partial}{\partial u_x} \tag{2.4}$$

.

Now the action of $\frac{\delta}{\delta u}$ on L_1 is given by

$$\begin{aligned} \frac{\delta L_1}{\delta u} &= \frac{\partial L_1}{\partial u} - D_x \frac{\partial L_1}{\partial u_x} - D_y \frac{\partial L_1}{\partial u_y} \\ &= \frac{\partial L_1}{\partial u} - D_x \frac{2u_x}{2\sqrt{u_x^2 + u_y^2}} - D_y \frac{2u_y}{2\sqrt{u_x^2 + u_y^2}} \end{aligned} \tag{2.5}$$

which contains

$$\frac{\delta L_1}{\delta u} = \frac{u_{xx} u_y^2 - 2u_x u_y u_{xy} u_{yy} y_x^2}{(u_x^2 + u_y^2)^{3/2}}, \tag{2.6}$$

$$\begin{aligned} -D_x \frac{2u_x}{2\sqrt{u_x^2 + u_y^2}} &= \frac{u_{xx}(u_x^2 + u_y^2) - u_x(u_{xx} u_x + u_{xy} u_y)}{(u_x^2 + u_y^2)^{3/2}} \\ &= u_{xx} \frac{u_x^2 + u_y^2 - u_x^2}{(u_x^2 + u_y^2)^{3/2}} + u_{xy} \frac{-u_x u_y}{(u_x^2 + u_y^2)^{3/2}}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} -D_y \frac{2u_y}{2\sqrt{u_x^2 + u_y^2}} &= \frac{u_{yy}(u_x^2 + u_y^2) - u_y(u_{yy} u_y + u_{xy} u_x)}{(u_x^2 + u_y^2)^{3/2}} \\ &= u_{yy} \frac{u_x^2 + u_y^2 - u_y^2}{(u_x^2 + u_y^2)^{3/2}} + u_{xy} \frac{-u_x u_y}{(u_x^2 + u_y^2)^{3/2}}. \end{aligned} \tag{2.8}$$

The variational symmetries (vector fields) that leave the functional $\int L_1 \mathrm{d}x\mathrm{d}y$ invariant with the corresponding conserved forms $\omega = T^x \mathrm{d}x + T^y \mathrm{d}y$ (where $(T^x, -T^y)$ is the conserved flow/vector)) are given by

(i). $-x\partial_x - y\partial_y + \partial_u$:

$$\frac{-y{u_x}^2 + u_y + u_x x u_y}{({u_x}^2 + {u_y}^2)^{1/2}}\mathrm{d}x + \frac{{u_y}^2 - u_x - u_y y u_x}{({u_x}^2 + {u_y}^2)^{1/2}}\mathrm{d}y$$

(ii). ∂_x:

$$\frac{-u_x u_y}{({u_x}^2 + {u_y}^2)^{1/2}}\mathrm{d}x + \frac{-u_y^2}{({u_x}^2 + {u_y}^2)^{1/2}}\mathrm{d}y$$

(iii). ∂_y:

$$\frac{u_x^2}{({u_x}^2 + {u_y}^2)^{1/2}}\mathrm{d}x + \frac{u_x u_y}{({u_x}^2 + {u_y}^2)^{1/2}}\mathrm{d}y$$

(iv). $-y\partial_x + x\partial_y$:

$$\frac{u_x\left(u_x x + u_y y\right)}{({u_x}^2 + {u_y}^2)^{1/2}}\mathrm{d}x + \frac{u_y\left(u_x x + u_y y\right)}{({u_x}^2 + {u_y}^2)^{1/2}}\mathrm{d}y$$

Here, (ii) and (iii) represent the conservation of linear momentum in x and y, respectively, and (iv) is the angular momentum.

If we consider the Euler-Lagrange equation corresponding to L_1

$$-\frac{u_{xx}{u_y}^2 - 2\,u_x u_y u_{xy} + u_{yy}{u_x}^2}{({u_x}^2 + {u_y}^2)^{3/2}} = 0, \tag{2.9}$$

it turns out that its algebra of point symmetry generators include generalisations of the vectors fields above. That is,

$$\begin{aligned}&[\tfrac{1}{2}\,x f_4\left(u\right)y + f_7\left(u\right)y + x^2 f_2\left(u\right) + f_8\left(u\right)x + f_9\left(u\right)]\partial_x\\ &+[x f_2\left(u\right)y + x f_3\left(u\right) + \tfrac{1}{2}\,f_4\left(u\right)y^2 + f_5\left(u\right)y + f_6\left(u\right)]\partial_y + f_1(u)\partial_u,\end{aligned}$$

or

$$\begin{array}{lll} f_1\partial_u & f_2(x^2\partial_x + xy\partial_y) & f_3 x\partial_y \\ f_4(xy\partial_x + y^2\partial_y) & f_5 y\partial_y & f_6\partial_y \\ f_7 y\partial_x & f_8 x\partial_x & f_9\partial_x \end{array}$$

(where the f_i's are arbitrary functions of u).

2.2.2 The 'denoising pde' with a linear source term

With a linear and nonlinear constraint, we have the Euler-Lagrange equation

$$\lambda_1 + \lambda_2 u - \frac{u_{xx}{u_y}^2 - 2\,u_x u_y u_{xy} + u_{yy}{u_x}^2}{({u_x}^2 + {u_y}^2)^{3/2}} = 0 \tag{2.10}$$

with Lagrangian given by

$$L_2 = \lambda_1 u + \frac{1}{2}\lambda_2 u^2 + \sqrt{u_x^2 + u_y^2} \tag{2.11}$$

which lead to the following variational symmetries and corresponding conserved forms.

(i). ∂_x (linear momentum in x):

$$\frac{-u_x u_y}{({u_x}^2 + {u_y}^2)^{1/2}}\mathrm{d}x - \frac{1}{2}\frac{2\sqrt{{u_x}^2 + {u_y}^2}\lambda_1 u + \sqrt{{u_1}^2 + {u_y}^2}\lambda_2 u^2 + 2\,{u_y}^2}{({u_x}^2 + {u_y}^2)^{1/2}}\mathrm{d}y$$

(ii). ∂_y (linear momentum in y):

$$\frac{1}{2}\frac{2\sqrt{{u_x}^2 + {u_y}^2}\lambda_1 u + \sqrt{{u_1}^2 + {u_y}^2}\lambda_2 u^2 + 2\,{u_x}^2}{({u_x}^2 + {u_y}^2)^{1/2}}\mathrm{d}x + \frac{u_x u_y}{({u_x}^2 + {u_y}^2)^{1/2}}\mathrm{d}y$$

(iii). $-y\partial_x + x\partial_y$ (angular momentum):

$$\begin{array}{l}\frac{1}{2}\frac{2\,x\sqrt{{u_x}^2+{u_y}^2}\lambda_1 u + x\sqrt{{u_1}^2+{u_y}^2}\lambda_2 u^2 + 2\,x{u_x}^2 + 2\,u_y y u_x}{({u_x}^2+{u_y}^2)^{1/2}}\mathrm{d}x \\ +\frac{1}{2}\frac{2\,y\sqrt{{u_x}^2+{u_y}^2}\lambda_1 u + y\sqrt{{u_1}^2+{u_y}^2}\lambda_2 u^2 + 2\,y{u_y}^2 + 2\,u_x x u_y}{({u_x}^2+{u_y}^2)^{1/2}}\mathrm{d}y\end{array}$$

The Euler equation (2.10) generates the symmetries

$$\begin{array}{l}[-\frac{f_1(u)\lambda_2 x}{\lambda_1+\lambda_2 u} + f_2(u)\,y + f_3(u)]\partial_x \\ +[-x f_2(u) - \frac{f_1(u)\lambda_2\,y}{\lambda_1+\lambda_2\,u} + f_4(u)]\partial_y + f_1(u)\partial_u,\end{array}$$

or

$$-f_1(\frac{\lambda_2}{\lambda_1 + \lambda_2\,u}x\partial_x + \frac{\lambda_2}{\lambda_1 + \lambda_2\,u}y\partial_y + \partial_u) \qquad f_2(y\partial_x - y\partial_y) \qquad f_3\partial_x \qquad f_4\partial_y$$

(where the f_i's are arbitrary functions of u).

The rotation symmetry $-y\partial_x + x\partial_y$ lead to the reduced equation

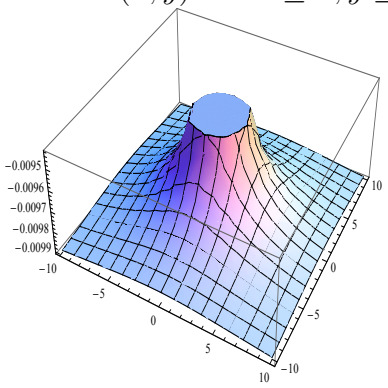
$$\lambda_1 + \lambda_2 u - \sqrt{2}/\sqrt{\alpha} = 0, \quad \alpha = x^2 + y^2$$

so that

$$u(x,y) = \frac{1}{\lambda_2}[\frac{\sqrt{2}}{\sqrt{x^2+y^2}} - \lambda_1]. \tag{2.12}$$

A profile of the solution for some choices of the parameters of (2.12) is given in Figure 1, i.e.,

Figure 2.1: $u(x,y) : -10 \leq x, y \leq 10$



The translation symmetries do not produce nontrivial solutions.

2.2.3 The time evolution 'denoising pde'

In the parabolic equation with 'time' as an evolution parameter, we obtain the evolution equation

$$u_t + \lambda_1 + \lambda_2 u - \frac{u_{xx}{u_y}^2 - 2\,u_x u_y u_{xy} + u_{yy}{u_x}^2}{({u_x}^2 + {u_y}^2)^{3/2}} = 0 \tag{2.13}$$

whose algebra of Lie symmetries is generated by

$$\begin{array}{lll} \partial_t, & \partial_x, & \partial_y, \\ y\partial_x - x\partial_y, & e^{-\lambda_2 t}\partial_u, & -x\partial_x - y\partial_y + \frac{\lambda_1+\lambda_2\,u}{\lambda_2}\partial_u, \\ e^{-\lambda_2 t}\partial_t - \lambda_2 u e^{-\lambda_2 t}\partial_u & & \end{array}$$

Here, the rotation symmetry lead to, again, $\alpha = x^2 + y^2$ and $u = v(t, \alpha)$ so that

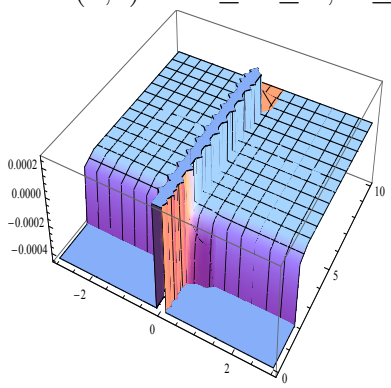
$$v_t + \lambda_1 + \lambda_2 v = \sqrt{2}/\sqrt{\alpha}$$

which is a first order linear pde. By the method of invariants, it can be shown that

$$u(x, y, t) = -\frac{1}{\lambda_2}[\lambda_1 - \frac{\sqrt{2}}{\sqrt{x^2 + y^2}}]e^{-\lambda_2 t}. \tag{2.14}$$

A projection of the solution in (u, x, t) $(y = 0)$ system for (2.13) given by (2.14) has the profile in Figure 2 below.

Figure 2.2: $u(x, t) : -3 \leq x \leq 3,\ 0 \leq t \leq 10$



The multiplier approach on (2.13) leads to a single multiplier $Q = e^{-\lambda_2 t}$ with the corre-

sponding conserved 2-form

$$
\begin{aligned}
&\tfrac{1}{t^2{\lambda_2}^3({u_x}^2+{u_y}^2)^{3/2}}[2\,u_{yy}{u_x}^2-4\,u_xu_yu_{xy}+2\,u_{xx}{u_2}^2-2\,\lambda_1\sqrt{{u_x}^2+{u_y}^2}{u_x}^2-2\,\lambda_1\sqrt{{u_x}^2+{u_y}^2}{u_y}^2\\
&-2\,\mathrm{e}^{\lambda_2 t}u_{yy}{u_x}^2-\mathrm{e}^{\lambda_2 t}{\lambda_2}^2t^2u_{yy}{u_x}^2+2\,\mathrm{e}^{\lambda_2 t}\lambda_2\,tu_{yy}{u_x}^2-4\,\mathrm{e}^{\lambda_2 t}\lambda_2\,tu_xu_yu_{xy}+2\,\mathrm{e}^{\lambda_2 t}{\lambda_2}^2t^2u_xu_yu_{1,2}\\
&+4\,\mathrm{e}^{\lambda_2 t}u_xu_yu_{xy}+2\,\mathrm{e}^{\lambda_2 t}\lambda_1\sqrt{{u_x}^2+{u_y}^2}{u_x}^2+2\,\mathrm{e}^{\lambda_2 t}\lambda_1\sqrt{{u_x}^2+{u_y}^2}{u_y}^2-\mathrm{e}^{\lambda_2 t}{\lambda_2}^2t^2u_{xx}{u_y}^2\\
&-2\,\mathrm{e}^{\lambda_2 t}u_{xx}{u_y}^2-2\,\mathrm{e}^{\lambda_2 t}\lambda_1\,t\lambda_2\sqrt{{u_x}^2+{u_2}^2}{u_x}^2-2\,\mathrm{e}^{\lambda_2 t}\lambda_1\,t\lambda_2\sqrt{{u_x}^2+{u_y}^2}{u_y}^2+2\,\mathrm{e}^{\lambda_2 t}\lambda_2\,tu_{xx}{u_y}^2\\
&+\mathrm{e}^{\lambda_2 t}\lambda_1\,t^2{\lambda_2}^2\sqrt{{u_x}^2+{u_y}^2}{u_x}^2+\mathrm{e}^{\lambda_2 t}\lambda_1\,t^2{\lambda_2}^2\sqrt{{u_x}^2+{u_y}^2}{u_y}^2+u\mathrm{e}^{\lambda_2 t}t^2{\lambda_2}^3\sqrt{{u_x}^2+{u_y}^2}{u_x}^2\\
&+u\mathrm{e}^{\lambda_2 t}t^2{\lambda_2}^3\sqrt{{u_x}^2+{u_y}^2}{u_y}^2]\mathrm{d}x\mathrm{d}y\\
&-\tfrac{y}{t^3{\lambda_2}^3({u_x}^2+{u_y}^2)^{3/2}}[2\,u_{yy}{u_x}^2-4\,u_xu_yu_{xy}+2\,u_{xx}{u_2}^2-2\,\lambda_1\sqrt{{u_x}^2+{u_y}^2}{u_x}^2\\
&-2\,\lambda_1\sqrt{{u_x}^2+{u_y}^2}{u_y}^2-2\,\mathrm{e}^{\lambda_2 t}u_{yy}{u_x}^2-\mathrm{e}^{\lambda_2 t}{\lambda_2}^2t^2u_{yy}{u_x}^2+2\,\mathrm{e}^{\lambda_2 t}\lambda_2\,tu_{yy}{u_x}^2\\
&-4\,\mathrm{e}^{\lambda_2 t}\lambda_2\,tu_xu_yu_{xy}+2\,\mathrm{e}^{\lambda_2 t}{\lambda_2}^2t^2u_xu_yu_{1,2}+4\,\mathrm{e}^{\lambda_2 t}u_xu_yu_{xy}+2\,\mathrm{e}^{\lambda_2 t}\lambda_1\sqrt{{u_x}^2+{u_2}^2}{u_x}^2\\
&+2\,\mathrm{e}^{\lambda_2 t}\lambda_1\sqrt{{u_x}^2+{u_y}^2}{u_y}^2-\mathrm{e}^{\lambda_2 t}{\lambda_2}^2t^2u_{xx}{u_y}^2-2\,\mathrm{e}^{\lambda_2 t}u_{xx}{u_y}^2-2\,\mathrm{e}^{\lambda_2 t}\lambda_1\,t\lambda_2\sqrt{{u_x}^2+{u_2}^2}{u_x}^2\\
&-2\,\mathrm{e}^{\lambda_2 t}\lambda_1\,t\lambda_2\sqrt{{u_x}^2+{u_y}^2}{u_y}^2+2\,\mathrm{e}^{\lambda_2 t}\lambda_2\,tu_{xx}{u_y}^2\\
&+\mathrm{e}^{\lambda_2 t}\lambda_1\,t^2{\lambda_2}^2\sqrt{{u_x}^2+{u_y}^2}{u_x}^2+\mathrm{e}^{\lambda_2 t}\lambda_1\,t^2{\lambda_2}^2\sqrt{{u_x}^2+{u_y}^2}{u_y}^2]\mathrm{d}x\mathrm{d}t\\
&+\tfrac{x}{t^3{\lambda_2}^3({u_x}^2+{u_y}^2)^{3/2}}[2\,u_{yy}{u_x}^2-4\,u_xu_yu_{xy}+2\,u_{xx}{u_2}^2-2\,\lambda_1\sqrt{{u_x}^2+{u_y}^2}{u_x}^2-2\,\lambda_1\sqrt{{u_x}^2+{u_y}^2}{u_y}^2\\
&-2\,\mathrm{e}^{\lambda_2 t}u_{yy}{u_x}^2-\mathrm{e}^{\lambda_2 t}{\lambda_2}^2t^2u_{yy}{u_x}^2+2\,\mathrm{e}^{\lambda_2 t}\lambda_2\,tu_{yy}{u_x}^2-4\,\mathrm{e}^{\lambda_2 t}\lambda_2\,tu_xu_yu_{xy}\\
&+2\,\mathrm{e}^{\lambda_2 t}{\lambda_2}^2t^2u_xu_yu_{1,2}+4\,\mathrm{e}^{\lambda_2 t}u_xu_yu_{xy}+2\,\mathrm{e}^{\lambda_2 t}\lambda_1\sqrt{{u_x}^2+{u_2}^2}{u_x}^2+2\,\mathrm{e}^{\lambda_2 t}\lambda_1\sqrt{{u_x}^2+{u_y}^2}{u_y}^2\\
&-\mathrm{e}^{\lambda_2 t}{\lambda_2}^2t^2u_{xx}{u_y}^2-2\,\mathrm{e}^{\lambda_2 t}u_{xx}{u_y}^2-2\,\mathrm{e}^{\lambda_2 t}\lambda_1\,t\lambda_2\sqrt{{u_x}^2+{u_2}^2}{u_x}^2-2\,\mathrm{e}^{\lambda_2 t}\lambda_1\,t\lambda_2\sqrt{{u_x}^2+{u_y}^2}{u_y}^2\\
&+2\,\mathrm{e}^{\lambda_2 t}\lambda_2\,tu_{xx}{u_y}^2+\mathrm{e}^{\lambda_2 t}\lambda_1\,t^2{\lambda_2}^2\sqrt{{u_x}^2+{u_y}^2}{u_x}^2+\mathrm{e}^{\lambda_2 t}\lambda_1\,t^2{\lambda_2}^2\sqrt{{u_x}^2+{u_y}^2}{u_y}^2]\mathrm{d}y\mathrm{d}t
\end{aligned}
$$

in which the term $T^t\mathrm{d}x\mathrm{d}y$ leads to the 'conserved density'.

2.3 Conclusions

We have shown that the pdes that arise in the various models of noise removal algorithms are abundant in symmetries that leave the systems invariant. These have applications in, inter alia, the reduction of the systems.

Chapter 3

Higher order conservation laws and integrability of Timoshenko beam systems

3.1 Introduction

The widely used Timoshenko beam model for the transverse vibration of a beam was developed in 1921 [20]. With 'shear deformation in the one-dimensional beam model, yields a model with two degrees of freedom, which drastically increased the applicability of the above beam model. The literature abounds with works on the Timoshenko model in various physical applications and also various subjects to boundary conditions [21, 22, 23, 24] [25]. In particular, we study the model

$$u_{tt} - u_{xx} + v_x = 0, \qquad \frac{1}{\alpha} v_{tt} - \frac{1}{\beta} v_{xx} + v - u_x = 0 \tag{3.1}$$

developed in Roux at al [26] and Labuschagne et al [27]. It turns out that the differential equations (3.1) and, indeed, many of the other versions that appear in the literature is rich in symmetry properties and the physically important concept of conservation laws. Thus, the methods and results here are applicable to the wide ranging Timoshenko models.

We, firstly, note that (3.1) is variational with the action integral $\int L\mathrm{d}x\mathrm{d}t$ where L is the Lagrangian

$$L = \frac{1}{2}[u_x^2 - u_t^2 + v_x^2 - v_t^2 uv_x - vu_x + v^2]. \tag{3.2}$$

One may use Noether's theorem [28] to construct conservation laws but we will show that alternative approaches yield a series of the standard and 'higher degree' conservation laws and symmetries.

The following statements on the wide ranging role of higher degree conservation laws are due to the extensive work of Ian Anderson, see [29, 30],[10] inter alia. 'Generalized vector fields first appeared as generalized or higher order symmetries of the KdV equation. For completely integrable equations like the KdV equation there are many schemes for generating infinite sequences of non-trivial conservation laws and it is usually not too difficult to show that (one may be able to construct) all possible classical conservation laws... Higher degree conservation laws reflect properties of partial differential equations pdes that cannot be detected by classical conservation laws alone. For example, it is well-known that the sine-Gordon and Liouville equations both admit infinitely many classical conservation laws. However, the Liouville equation is integrable by the method of Darboux while the sine-Gordon equation is not. This is reflected at the level of higher degree conservation laws by the fact that the sine-Gordon equation has no conserved forms of degree 4 and higher whereas the Liouville equation has infinitely many conserved forms of arbitrary degree. Higher degree conservation laws are also central to the theory of Hamiltonian systems ... and ... higher degree conservation laws play a pivotal role in the solution to the classical inverse problem of the calculus of variations'.

There are now well established works on the methods for constructing conservation laws, in general. We refer the reader to [31, 32, 33] [34, 35], to list a few.

The result below are published in [2].

3.2 Multipliers and conserved flows

We consider three cases of (3.1) based on the values of the physical, dimensionless parameters α and β.

We will denote $Q^\mu = (f, g)$, i.e., (1.9) becomes

$$f(x,t,u,v,u_x,\ldots)(u_{tt}-u_{xx}+v_x)+g(x,t,u,v,u_x,\ldots)(\frac{1}{\alpha}v_{tt}-\frac{1}{\beta}v_{xx}+v-u_x) = D_tT^t+D_xT^x. \tag{3.3}$$

Then, for a nontrivial conserved flow (T^t, T^x), the Euler operator annihilates

$$f(x,t,u,v,u_x,\ldots)(u_{tt}-u_{xx}+v_x)+g(x,t,u,v,u_x,\ldots)(\tfrac{1}{\alpha}v_{tt}-\tfrac{1}{\beta}v_{xx}+v-u_x).$$

(a) $\alpha = \beta = 1$

$$u_{tt} - u_{xx} + v_x = 0, \qquad v_{tt} - v_{xx} + v - u_x = 0 \tag{3.4}$$

(i). $f = u_x - v_{xx}, g = u_{xx}$:

$$\begin{aligned} T^t &= -1/2\,v_tu_{xx} - 1/2\,u_tu_x + 1/2\,u_tv_{xx} + 1/2\,vu_{xxt} + 1/2\,uu_{xt} - 1/2\,uv_{xxt}, \\ -T^x &= -1/2\,v_xu_{tt} - 1/2\,{v_x}^2 - {u_x}^2 + 1/2\,u_xv_{tt} + u_xv + 1/2\,vu_{xtt} - 1/2\,uv_{xtt} + 1/2\,uu_{tt} \end{aligned}$$

(ii). $f = -2\,tv_{xt} + xu_x - 2\,xv_{xx} - 1/2\,u - v_x + 2\,tu_t, g = 2\,xu_{xx} - xv_x - 1/2\,v + 2\,tu_{xt} + u_x$

$$\begin{aligned} T^t &= 1/2\,vtu_{xtt} + 1/2\,v_txv_x - v_ttu_{xt} - 1/2\,v_xtu_{xx} + u_txv_{xx} + 1/2\,v_xtu_{tt} - 1/2\,u_xtv_{tt} \\ &\quad +1/2\,vtu_{xxx} + 1/2\,u_xtv_{xx} - 1/2\,u_txu_x - 1/2\,vxv_{xt} + vxu_{xxt} - v_txu_{xx} - 1/2\,u_xtv + tu_tv_{xt} \\ &\quad -uxv_{xxt} + 1/2\,uxu_{xt} - 1/2\,vtv_{xx} - 1/2\,utv_{xtt} + 1/2\,utu_{xx} - 1/2\,utv_x - 1/2\,utv_{xxx} \\ &\quad +1/2\,t{u_x}^2 + 3/2\,vu_{xt} - 3/2\,uv_{xt} + u_tu - t{u_t}^2 - 1/2\,v_tu_x + 1/2\,t{v_x}^2 + 1/2\,u_tv_x, \end{aligned}$$

$$\begin{aligned} -T^x &= -1/2\,v_ttv_x + 1/2\,vtu_{ttt} - 1/2\,v_ttu_{tt} - 1/2\,utv_t + 1/2\,vtu_{xxt} + 1/2\,uxu_{tt} \\ &\quad -1/2\,vxv_{tt} + u_xtv_{xt} + vxu_{xtt} - 1/2\,vtv_{xt} - uxv_{xtt} + 1/2\,v_ttu_{xx} + xu_xv_{tt} - xv_xu_{tt} \\ &\quad -1/2\,utv_{xxt} + 1/2\,utu_{xt} - 1/2\,utv_{ttt} - v_xtu_{xt} - 3/2\,tu_tu_x + 2\,xu_xv - 3/2\,x{u_x}^2 \\ &\quad +1/2\,vu_{xx} - 1/2\,xv^2 - 1/2\,uv_{xx} + u_xu + vu_{tt} - vu - uv_{tt} - 1/2\,x{v_x}^2 \\ &\quad -1/2\,tu_tv_{xx} + 3/2\,tu_tv + 1/2\,tu_tv_{tt} \end{aligned}$$

(iii). $f = u_x, g = v_x$:

$$\begin{aligned} T^t &= -1/2\, v_t v_x - 1/2\, u_t u_x + 1/2\, v v_{xt} + 1/2\, u u_{xt}, \\ -T^x &= -1/2\, {v_x}^2 - 1/2\, {u_x}^2 + 1/2\, v v_{tt} + 1/2\, v^2 + 1/2\, u u_{tt} \end{aligned}$$

(iv). $f = -1/2\, u_t x + x v_{xt} + t v_{xx} + 1/2\, v_t, g = -x u_{xt} + 1/2\, v_t x - 1/2\, u_t - t u_{xx} + v_x t$:

$$\begin{aligned} T^t &= -1/4\, x v^2 + 1/4\, u v_{tt} + 1/2\, u v_{xx} - 1/4\, x {v_t}^2 + 1/2\, v_x v - 1/4\, x {v_x}^2 - 1/4\, x {u_x}^2 \\ &-1/4\, v u_{tt} - 1/2\, v u_{xx} + 1/2\, v_t t u_{xx} - 1/4\, v x u_{xtt} - 1/4\, v x u_{xxx} + 1/2\, u t v_{xxt} \\ &-1/4\, v_x x u_{tt} + 1/4\, u x v_{xxx} + 1/4\, u x v_{xtt} + 1/4\, v_x x u_{xx} - 1/2\, u_t t v_{xx} - 1/2\, v_t t v_x \\ &-1/2\, u_t x v_{xt} - 1/4\, u_x x v_{xx} + 1/2\, v t v_{xt} + 1/4\, x {u_t}^2 + 1/2\, v x v_{xx} - 1/2\, v t u_{xxt} \\ &+1/2\, v_t x u_{xt} + 1/4\, u_x x v_{tt} + 1/2\, u_x x v, \end{aligned}$$

$$\begin{aligned} -T^x &= -1/2\, u_x t v_{tt} + 1/2\, u_t x u_x - 1/2\, u_x x v_{xt} - 1/4\, v x u_{ttt} - 1/2\, u_t v x + 1/4\, u x v_{xxt} \\ &+1/4\, u x v_{ttt} - 1/2\, v t u_{xtt} + 1/2\, v x v_{xt} + 1/2\, v t v_{tt} - 1/4\, v_t x u_{xx} + 1/2\, u t v_{xtt} \\ &-1/4\, v_t u_x + 1/4\, u_t v_x + 1/2\, t {u_x}^2 - 3/4\, v u_{xt} + 3/4\, u v_{xt} + 1/2\, v v_t + 1/2\, v^2 t + 1/2 \\ &v_x x u_{xt} - 1/4\, u_t x v_{tt} - 1/4\, v x u_{xxt} - u_x t v + 1/4\, u_t x v_{xx} + 1/4\, v_t x u_{tt} + 1/2\, v_x t u_{tt} \end{aligned}$$

(v). $f = v \sin t - \sin t u_x + v_t \cos t, g = u_t \cos t + v_x \sin t$:

$$\begin{aligned} T^t &= -v_t u_t \cos t - 1/2\, v_t v_x \sin t - u_t v \sin t + 1/2\, u_t \sin t u_x + 1/2\, v \sin t v_{xt} \\ &+1/2\, v \cos t u_{xx} - 1/2\, u \sin t u_{xt} + 1/2\, u \cos t v_{xx}, \\ -T^x &= -1/2\, u_t v_x \cos t - 1/2\, {v_x}^2 \sin t - 3/2\, u_x v \sin t + 1/2\, \sin t {u_x}^2 - 1/2\, v_t u_x \cos t \\ &+1/2\, v \cos t u_{xt} + v^2 \sin t + 1/2\, v v_t \cos t + 1/2\, v \sin t v_{tt} \\ &-1/2\, u u_t \cos t - 1/2\, u v_x \sin t - 1/2\, u \sin t u_{tt} + 1/2\, u \cos t v_{xt} \end{aligned}$$

(vi). $f = v \cos t - \sin t v_t - u_x \cos t, g = v_x \cos t - u_t \sin t$:

$$\begin{aligned} T^t &= v_t u_t \sin t - 1/2\, v_t \cos t v_x - u_t \cos t v + 1/2\, u_t \cos t u_x + 1/2\, v \cos t v_{xt} \\ &-1/2\, v \sin t u_{xx} - 1/2\, u \cos t u_{xt} - 1/2\, u \sin t v_{xx}, \\ -T^x &= 1/2\, u_t v_x \sin t - 1/2\, \cos t {v_x}^2 - 3/2\, u_x \cos t v + 1/2\, \cos t {u_x}^2 \\ &+1/2\, v_t u_x \sin t - 1/2\, v \sin t u_{xt} + 1/2\, v \cos t v_{tt} + \cos t v^2 - 1/2\, v v_t \sin t \\ &-1/2\, u \cos t v_1 - 1/2\, u \sin t v_{xt} - 1/2\, u \cos t u_{tt} + 1/2\, u u_t \sin t \end{aligned}$$

(vii). $f = u_t, g = v_t$:

$$T^t = -1/2\, {v_t}^2 - 1/2\, {u_t}^2 + 1/2\, u_x v - 1/2\, v^2 + 1/2\, v v_{xx} + 1/2\, u u_{xx} - 1/2\, u v_x,$$
$$-T^x = -1/2\, v_t v_x - 1/2\, u_t u_x + 1/2\, v v_{xt} + 1/2\, u_t v + 1/2\, u u_{xt} - 1/2\, u v_t$$

(viii). $f = u_t - v_{xt}, g = u_{xt}$:

$$T^t = -1/2\, v_t u_{xt} + 1/4\, v_x u_{tt} - 1/4\, v_x u_{xx} + 1/4\, {v_x}^2 - 1/2\, {u_t}^2 + 1/2\, u_t v_{xt} - 1/4\, u_x v_{tt}$$
$$+1/4\, {u_x}^2 - 1/4\, u_x v + 1/4\, u_x v_{xx} + 1/4\, v u_{xtt} + 1/4\, v u_{xxx} - 1/4\, v v_{xx}$$
$$-1/4\, u v_{xtt} + 1/4\, u u_{xx} - 1/4\, u v_x - 1/4\, u v_{xxx},$$
$$-T^x = -1/4\, v_t u_{tt} + 1/4\, v_t u_{xx} - 1/4\, v_t v_x - 1/2\, v_x u_{xt} + 1/4\, u_t v_{tt}$$
$$-3/4\, u_t u_x + 3/4\, u_2 v - 1/4\, u_t v_{xx} + 1/2\, u_x v_{xt} + 1/4\, v u_{xxt} + 1/4\, v u_{ttt}$$
$$-1/4\, v v_{xt} + 1/4\, u u_{xt} - 1/4\, u v_t - 1/4\, u v_{xxt} - 1/4\, u v_{ttt}$$

(b) $\beta = 1$

$$u_{tt} - u_{xx} + v_x = 0, \qquad \frac{1}{\alpha} v_{tt} - v_{xx} + v - u_x = 0 \tag{3.5}$$

(i). $f = u_x, g = v_x$:

$$T^t = \tfrac{1}{2\alpha}[-v_t v_x - u_t u_x \alpha + v v_{xt} + u u_{xt} \alpha],$$
$$-T^x = \tfrac{1}{2\alpha}[-{v_x}^2 \alpha - {u_x}^2 \alpha + v v_{tt} + v^2 \alpha + u u_{tt} \alpha]$$

(ii). $f = u_t, g = v_t$:

$$T^t = \tfrac{1}{2\alpha}[-{v_t}^2 - {u_t}^2 \alpha + v\alpha\, u_x - v^2 \alpha + v\alpha\, v_{xx} + u\alpha\, u_{xx} - u\alpha\, v_x],$$
$$-T^x = -1/2\, v_t v_x - 1/2\, u_t u_x + 1/2\, v v_{xt} + 1/2\, u_t v + 1/2\, u u_{xt} - 1/2\, u v_t$$

(c)

$$u_{tt} - u_{xx} + v_x = 0, \qquad \frac{1}{\alpha} v_{tt} - \frac{1}{\beta} v_{xx} + v - u_x = 0 \tag{3.6}$$

(i). $f = u_x, g = v_x$:

$$T^t = \tfrac{1}{2\alpha}[-v_t v_x - u_t u_x \alpha + v v_{xt} + u u_{xt} \alpha],$$
$$-T^x = \tfrac{1}{2\alpha\beta}[-{v_x}^2 \alpha - {u_x}^2 \beta\, \alpha + v \beta\, v_{tt} + v^2 \alpha\, \beta + u u_{tt} \beta\, \alpha]$$

(ii). $f = u_t, g = v_t$:

$$T^t = \tfrac{1}{2\alpha\beta}[-{v_t}^2 \beta - {u_t}^2 \alpha\, \beta + v \alpha\, u_x \beta - v^2 \alpha\, \beta + v \alpha\, v_{xx} + u \alpha\, \beta\, u_{xx} - u \alpha\, \beta\, v_x],$$
$$-T^x = \tfrac{1}{2\beta}[-v_t v_x - u_t u_x \beta + v v_{xt} + v u_t \beta + u \beta\, u_{xt} - u \beta\, v_t]$$

It turns out that all the cases, including the general case (c) admit **infinitely** many conservation laws which are generated by the infinitely evolutionary (generalised) vector fields $X = f\partial_u + \gamma\partial_v$. In case (a), these have been computed independently via the multipliers that lead to the conserved flow. For e.g., in (a) (i), the evolutionary higher order vector field is $X = (u_x - v_{xx})\partial_u + u_{xx}\partial_v$ with an associated higher order conserved flow (T^t, T^x) given above. To show that X is a symmetry of (3.4), the required second prolongation of X is

$$\begin{aligned} X^{[2]} &= X + (u_{xx} - v_{xxx})\partial_{u_x} + (u_{xt} - v_{xxt})\partial_{u_t} + u_{xxx}\partial_{v_x} + u_{xxt}\partial_{v_t} + (u_{xxx} - v_{xxxx})\partial_{u_{xx}} \\ &\quad + (u_{xtt} - v_{xxtt})\partial_{u_{tt}} + u_{xxxx}\partial_{v_{xx}} + u_{xxtt}\partial_{v_{tt}} \end{aligned}$$

which on (3.4)b leads to $D_{xx}(u_{tt} - u_{xx} + v_x)$ and on (3.4)a is $u_{xtt} - v_{xxtt} + v_{xxxx} = -D_{xx}(v_{tt} - v_{xx} + v - u_x)$.

In (iii) and (vii) in case(a) and both listed in cases (b) and (c), the multipliers are not higher order but correspond to point symmetries ∂_x and ∂_t. In general, the infinitely many higher evolutionary vector fields can be obtained by a **recursion** process or utilising *recursion operators* discussed at length in [31].

We look at some illustrative examples.

In (a), a fourth multiplier is $(D_{tt}(u_x - v_{xx})), D_{tt}u_{xx}$ so that a fourth order evolutionary vector field of (3.4) is $X = (u_{xtt} - v_{xxtt})\partial_u + u_{xxtt}\partial_v$.

In case (c), corresponding to the multiplier with components $f = u_{xxt}$, $g = v_{xxt}$ with

evolutionary vector field $X = u_{xxt}\partial_u + v_{xxt}\partial_v$, we have the conserved vector/flow

$$\begin{aligned}
T^t \quad &= \tfrac{1}{6\alpha\beta}[-v_{xx}v_{tt}\beta + 2\,v_{xx}u_x\alpha\,\beta \\
&-2\,v_{xx}v\alpha\,\beta + {v_{xx}}^2\alpha - \alpha\,\beta\,u_{xx}u_{tt} + \alpha\,\beta\,{u_{xx}}^2 \\
&-2\,\alpha\,\beta\,u_{xx}v_1 - 3\,v_tv_{xxt}\beta + \alpha\,\beta\,{v_x}^2 \\
&-v_xv_{xxx}\alpha + v_xv_{xtt}\beta - 3\,u_tu_{xxt}\alpha\,\beta - u_x\alpha\,\beta\,u_{xxx} \\
&+u_x\alpha\,\beta\,u_{xtt} + vu_{xxx}\alpha\,\beta + vv_{xxxx}\alpha \\
&+2\,vv_{1,1,2,2}\beta + u\alpha\,\beta\,u_{xxxx} + 2\,u\alpha\,\beta\,u_{xxtt} - u\alpha\,\beta\,v_{xxx}], \\
-T^x \quad &= \tfrac{1}{6\alpha\beta}[v_tv_{xxx}\alpha - 3\,v_{xt}u_x\alpha\,\beta \\
&+4\,v_{xt}v\alpha\,\beta + 2\,u_{xt}\alpha\,\beta\,u_{tt} - 2\,u_{xt}\alpha\,\beta\,u_{xx} + 3\,u_{xt}\alpha\,\beta\,v_x \\
&+v_t\alpha\,\beta\,u_{xx} - 2\,v_t\alpha\,\beta\,v_x + u_t\alpha\,\beta\,u_{xxx} - u_t\alpha\,\beta\,u_{xtt} \\
&-u_t\alpha\,\beta\,v_{xx} - 2\,u_x\alpha\,\beta\,u_{xxt} - u_x\alpha\,\beta\,u_{ttt} + vu_{xxt}\alpha\,\beta + u\alpha\,\beta\,u_{xxxt} + 2\,u\alpha\,\beta\,u_{xttt} \\
&-u\alpha\,\beta\,v_{xxt} + 2\,v_{xt}v_{tt}\beta - 2\,v_{xt}v_{xx}\alpha - v_tv_{xtt}\beta - 2\,v_xv_{xxt}\alpha \\
&-v_xv_{ttt}\beta + vv_{xxxt}\alpha + 2\,vv_{xttt}\beta]
\end{aligned}$$

If $f = u_{xtt} - u_t$, $g = v_{xtt} - v_t$ and vector field $X = (u_{xtt} - u_t)\partial_u + (v_{xtt} - v_t)\partial_v$, the conserved flow is

$$\begin{aligned}
T^t \quad &= -\tfrac{1}{6\alpha\beta}[-2\,u_{xt}\alpha\,\beta\,u_{xx} + 2\,u_{xt}\alpha\,\beta\,u_{tt} \\
&+3\,u_{xt}\alpha\,\beta\,v_x + v_t\alpha\,\beta\,u_{xx} - 3\,u\alpha\,\beta\,v_x + 3\,u\alpha\,\beta\,u_{xx} + 3\,vu_x\alpha\,\beta \\
&-3\,{v_t}^2\beta - 3\,{u_t}^2\alpha\,\beta + 3\,vv_{xx}\alpha - 3\,v^2\alpha\,\beta - 3\,v_{xt}u_x\alpha\,\beta \\
&+4\,v_{xt}v\alpha\,\beta + 2\,v_{xt}v_{tt}\beta - 2\,v_{xt}v_{xx}\alpha + v_xv_{xxt}\alpha + 2\,v_tv_{xtt}\beta \\
&-v_xv_{ttt}\beta - 2\,vv_{xxxt}\alpha - vv_{xttt}\beta - 2\,v_t\alpha\,\beta\,v_x - 2\,u\alpha\,\beta\,u_{xxxt} \\
&+u_t\alpha\,\beta\,u_{xxx} + 2\,u_t\alpha\,\beta\,u_{xtt} - u_t\alpha\,\beta\,v_{xx} + u_x\alpha\,\beta\,u_{xxt} - u_x\alpha\,\beta\,u_{ttt} \\
&-2\,vu_{xxt}\alpha\,\beta - u\alpha\,\beta\,u_{xttt} + 2\,u\alpha\,\beta\,v_{xxt} + v_tv_{xxx}\alpha], \\
-T^x \quad &= -\tfrac{1}{6\alpha\beta}[-u_tu_{xxt}\alpha\,\beta + v_{tt}v_{xx}\alpha - 3\,v_x\alpha\,v_t \\
&-{u_{tt}}^2\alpha\,\beta + 3\,v_{xt}v\alpha - {v_{tt}}^2\beta - v_tv_{xxt}\alpha \\
&+{v_t}^2\alpha\,\beta - 2\,v\alpha\,v_{xxtt} + v_tv_{ttt}\beta + 3\,v_x\alpha\,v_{xtt} \\
&-vv_{tttt}\beta + \alpha\,\beta\,u_{xx}u_{tt} + 3\,u_x\alpha\,\beta\,u_{xtt} - 2\,u\alpha\,\beta\,u_{xxtt} \\
&-3\,u\alpha\,\beta\,v_t + 3\,u\alpha\,\beta\,u_{xt} + 2\,u\alpha\,\beta\,v_{xtt} - u\alpha\,\beta\,u_{tttt} \\
&+3\,vu_t\alpha\,\beta - 2\,v\alpha\,\beta\,u_{xtt} - 3\,u_x\alpha\,\beta\,u_t + u_t\alpha\,\beta\,v_{xt} + u_t\alpha\,\beta\,u_{ttt} \\
&-v_tu_{xt}\alpha\,\beta - u_{tt}\alpha\,\beta\,v_x - 2\,v_{tt}v\alpha\,\beta + v_{tt}u_x\alpha\,\beta]
\end{aligned}$$

3.3 Discussion

We have shown that the Timoshenko system of pdes that model the vibrations of a beam is rich in symmetry and conservation laws. In fact, it admits an infinite hierarchy of conservation laws which has implications on, inter alia, the integrability of the system. Finally, we note that the higher order symmetries may be obtained via 'recursion operators' - [31].

Chapter 4

Invariance and 'approximate' conservation laws of some nonlinear Schrödinger equation with PT-symmetric potentials and inhomogeneous nonlinearity

4.1 Introduction

In [36], Yan et al establish and discuss, in detail, a model of a nonlinear Scrödinger equation with **PT**-symmetric (parity-time) potentials and inhomogeneity. The literature describing **PT** models is vast, many of which are referred to in [36] and other works are [37, 38, 39] but we would allude the reader to the article on conservation laws and exact solution of certain **PT**-symmetric models in [40]. In short, the complexification of Hamiltonians which keeps the spectra real are referred to as **PT**-symmetry. This is a formalism of quantum mechanics which is often illustrated by the harmonic oscillator.

Here, Hamiltonian itself is Hermitian and commutes with the parity. In the class of models that satisfy a weaker condition the reality of the spectrum under certain circumstances lead to a condition called **PT**-symmetry. In particular, Yan et al consider 'how stable nonlinear modes can be excited in systems where the linear **PT** symmetry is broken. The idea is based on the possibility of switching on nonlinearity simultaneously with gain and dissipation. Such a possibility can be implemented, in particular, when the nonlinearity and gain-and-loss strength are characterized by a single parameter ϵ and disappear when this parameter becomes zero, i.e., $\epsilon = 0$. If at $\epsilon = 0$ the system is Hamiltonian (*variational*), it allows for stable propagation of the linear modes, and the only stability issue which has to be verified is the stability of the solution branch $\epsilon > 0$, bifurcating from $\epsilon = 0$'.

The model developed and discussed in [36] (with $\mu = \epsilon$) is

$$iq_t + \frac{1}{2}u_{xx} - U_\epsilon(x)q - G_\mu(x)|q|^2 q = 0, \tag{4.1}$$

where

$$U_\epsilon(x) = \Sigma_{j=0}^{2}\epsilon^{2j}V_j(x) + i\epsilon W(x), \qquad G_\mu = \mu^2 G(x). \tag{4.2}$$

We will show, in fact, that for $\epsilon > 0$, the model is not variational and, therefore, one cannot appeal to Noether's theorem to determine conservation laws. In fact, there are no conservation laws for this case. Yet, the system does display non trivial symmetry properties. This would lead us to a novel concept (and consequent procedure) of 'approximate/perturbed conservation laws'. That is, if a system is variational for a parameter $\epsilon = 0$ and not variational for $\epsilon > 0$ (the perturbed system) but shares the symmetry properties that lead to conservation laws in the former case (via Noether's theorem, for example), then one may construct, for ϵ close to zero, approximate conservation laws for the perturbed system that are exact conservation laws in the limiting case $\epsilon \to 0$ for the unperturbed case. The notions of approximate symmetries, approximate variational symmetries and associated conservation laws have been discussed in the past, for example, [32, 34, 35]. The approach of 'partial Lagrangians' [41] may also be a route to determining some sort of conservation laws. Here, however, the concepts and methodology are quite

different and rely on the result that the Euler operator annihilates total divergences. For our purposes, it would be almost zero, i.e., up to order ϵ.

The main results below have appeared [3].

4.2 Conservation laws

In what follows below, we construct conservation laws for cases that are modelled and listed in [36]. The significance of these cases are discussed therein.

4.2.1 Case 1

For most of the cases in [36], it turns out that we may write $U_\epsilon(x) = a(x) + i\epsilon b(x)$. Then, if $q = u + iv$, separation of the real and imaginary parts in (4.1) leads to the system

$$\begin{aligned} &u_t + \tfrac{1}{2}\, v_{xx} - \varepsilon b\,(x)\, u - a\,(x)\, v + 2\,\mu^2\sigma\,\mathrm{e}^{-\alpha\, x^2}\,(u^2+v^2)\, v = 0, \\ &-v_t + \tfrac{1}{2}\, u_{xx} - a\,(x)\, u + \varepsilon\, b\,(x)\, v + 2\,\mu^2\sigma\,\mathrm{e}^{-\alpha\, x^2}\,(u^2+v^2)\, u = 0 \end{aligned} \tag{4.3}$$

The pde (4.1) and, therefore, the system (4.3) is not derivable from a variational principle and the conservation laws, therefore, cannot be obtained via Noether's theorem as is the usually the case for Schrödinger type equations. It can be shown further that the respective systems have no conservation laws. However, the decoupled equivalent system (4.3) does admit two symmetry generators $X_1 = \partial_t$ and $X_2 = u\partial_v - v\partial_u$ which are usually associated with energy and charge conservation, respectively. The respective characteristics $Q_1 = (v_t, u_t)$ and $Q_2 = (u, -v)$ construed as multipliers do not satisfy (1.10), i.e.,

$$\begin{aligned} &\tfrac{\delta}{\delta(u,v)}[Q_i^1(u_t + \tfrac{1}{2}\, v_{xx} - \varepsilon b\,(x)\, u - a\,(x)\, v + 2\,\mu^2\sigma\,\mathrm{e}^{-\alpha\, x^2}\,(u^2+v^2)\, v) \\ &+Q_i^2(-v_t + \tfrac{1}{2}\, u_{xx} - a\,(x)\, u + \varepsilon\, b\,(x)\, v + 2\,\mu^2\sigma\,\mathrm{e}^{-\alpha\, x^2}\,(u^2+v^2)\, u)] \\ &\neq \mathbf{0} \end{aligned}$$

If we regard ϵ as a 'small' parameter, we may suppose (4.1) as a perturbation of some system with $\epsilon = 0$ in (4.1). A number of situations may be pursued, as a result. For example, the notions of approximate symmetries and approximate conservation laws (and their possible associations) as expounded in [32] and [34] may be studied. However, proceeding with the multiplier approach on constructing conservation laws, we revisit (4.2.1). Below, we show that in fact the Euler operator equals $\epsilon\mathbf{w}$ which goes to zero as ϵ goes to zero. Thus, we can construct conservation laws that are 'approximate' upto an order of ϵ. We enumerate, below, some of the cases (as in [36]) of (4.3).

(a). $a(x) = \frac{1}{2}x^2$, $b(x) = x$:

1. Energy

The Euler operator is

$$\begin{aligned}
&\tfrac{\delta}{\delta(u,v)}[v_t(u_t + \tfrac{1}{2}\,v_{xx} - \varepsilon b\,(x)\,u - a\,(x)\,v + 2\,\mu^2\sigma\,\mathrm{e}^{-\alpha\,x^2}\,(u^2+v^2)\,v)\\
&+u_t(-v_t + \tfrac{1}{2}\,u_{xx} - a\,(x)\,u + \varepsilon\,b\,(x)\,v + 2\,\mu^2\sigma\,\mathrm{e}^{-\alpha\,x^2}\,(u^2+v^2)\,u)]\\
&= (-2\epsilon x v_t, 2\epsilon x u_t) \to 0 \quad \text{as} \quad \epsilon \to 0
\end{aligned}$$

and the corresponding 'approximate' conserved form is

$$\begin{aligned}
\omega^1 \;&= [1/5\,t\varepsilon\,x v_t u - 1/5\,t\varepsilon\,x u_t v - \mu^2\sigma\,\mathrm{e}^{-\alpha\,x^2}v^2u^2 - 1/2\,\mu^2\sigma\,\mathrm{e}^{-\alpha\,x^2}v^4\\
&\quad -1/2\,\mu^2\sigma\,\mathrm{e}^{-\alpha\,x^2}u^4 - 1/4\,v v_{xx}\\
&\quad +1/4\,x^2v^2 - 1/4\,u u_{xx} + 1/4\,x^2u^2]\mathrm{d}x\\
&\quad +[-1/5\,x^2\varepsilon\,v_t u + 1/5\,x^2\varepsilon\,u_2 v + 1/4\,v_x v_t + 1/4\,u_x u_t\\
&\quad -1/4\,v v_{xt} - 1/4\,u u_{xt}]\mathrm{d}t
\end{aligned} \tag{4.4}$$

so that the the conserved density is

2. Charge

$$\begin{aligned}
&\tfrac{\delta}{\delta(u,v)}[v_t(u_t + \tfrac{1}{2}\,v_{xx} - \varepsilon b\,(x)\,u - a\,(x)\,v + 2\,\mu^2\sigma\,\mathrm{e}^{-\alpha\,x^2}\,(u^2+v^2)\,v)\\
&+u_t(-v_t + \tfrac{1}{2}\,u_{xx} - a\,(x)\,u + \varepsilon\,b\,(x)\,v + 2\,\mu^2\sigma\,\mathrm{e}^{-\alpha\,x^2}\,(u^2+v^2)\,u)]\\
&= (-2\epsilon x u, -2\epsilon x v) \to 0 \quad \text{as} \quad \epsilon \to 0
\end{aligned}$$

$$\begin{aligned}\omega^2 &= [1/5\,t\varepsilon\,xu^2 + 1/5\,t\varepsilon\,xv^2 - 1/2\,u^2 - 1/2\,v^2]\mathrm{d}x \\ &+[-1/5\,x^2\varepsilon\,u^2 - 1/5\,x^2\varepsilon\,v^2 + 1/2\,uv_x - 1/2\,vu_x]\mathrm{d}t\end{aligned} \tag{4.5}$$

Thus, the conserved density in complex functional form is

$$\Phi^t = (\frac{1}{5}t\epsilon x - \frac{1}{2})|q|^2.$$

(b). $a(x) = \frac{1}{2}x^2$, $b(x) = -\left(\alpha+3\right)x\mathrm{e}^{-1/2\,(\alpha+1)x^2}$:

In this case, we present only the conserved density as the complete conserved form is cumbersome; this is in fact clear from even the density alone.

1. Energy

The Euler operator is

$$\begin{aligned}&\tfrac{\delta}{\delta(u,v)}[v_t(u_t + \tfrac{1}{2}\,v_{xx} - \varepsilon b\left(x\right)u - a\left(x\right)v + 2\,\mu^2\sigma\,\mathrm{e}^{-\alpha\,x^2}\left(u^2+v^2\right)v) \\ &+u_t(-v_t + \tfrac{1}{2}\,u_{xx} - a\left(x\right)u + \varepsilon\,b\left(x\right)v + 2\,\mu^2\sigma\,\mathrm{e}^{-\alpha\,x^2}\left(u^2+v^2\right)u)] \\ &= (2\,v_t\varepsilon\,x\mathrm{e}^{-1/2\,(\alpha+1)x^2}\left(\alpha+3\right), -2\,u_t\varepsilon\,x\mathrm{e}^{-1/2\,(\alpha+1)x^2}\left(\alpha+3\right)) \to 0 \quad \text{as} \quad \epsilon \to 0\end{aligned}$$

and the conserved density, T^t is given by

$$
\begin{aligned}
T^t \;=& -\tfrac{1}{4x^4(\alpha+1)^{5/2}}\,[4\,\mu^2\sigma\,\mathrm{e}^{-\alpha\,x^2}v^4x^4\sqrt{\alpha+1}\alpha + 4\,\mu^2\sigma\,\mathrm{e}^{-\alpha\,x^2}u^4x^4\sqrt{\alpha+1}\alpha \\
&+2\,\mu^2\sigma\,\mathrm{e}^{-\alpha\,x^2}u^4x^4\sqrt{\alpha+1}\alpha^2 \\
&+18\,t\varepsilon\,\sqrt{\pi}\sqrt{2}\mathrm{erf}\left(1/2\,\sqrt{2}x\sqrt{\alpha+1}\right)v_tu + 4\,\mu^2\sigma\,\mathrm{e}^{-\alpha\,x^2}v^2u^2x^4\sqrt{\alpha+1} \\
&+12\,t\varepsilon\,u_tv\sqrt{\alpha+1}x^3\mathrm{e}^{-1/2\,(\alpha+1)x^2} \\
&-36\,t\varepsilon\,v_tu\sqrt{\alpha+1}x\mathrm{e}^{-1/2\,(\alpha+1)x^2} - 12\,t\varepsilon\,v_tu\sqrt{\alpha+1}x^3\mathrm{e}^{-1/2\,(\alpha+1)x^2} \\
&+36\,t\varepsilon\,u_tv\sqrt{\alpha+1}x\mathrm{e}^{-1/2\,(\alpha+1)x^2} \\
&-18\,t\varepsilon\,\sqrt{\pi}\sqrt{2}\mathrm{erf}\left(1/2\,\sqrt{2}x\sqrt{\alpha+1}\right)u_tv + 2\,\mu^2\sigma\,\mathrm{e}^{-\alpha\,x^2}v^4x^4\sqrt{\alpha+1}\alpha^2 \\
&+2\,\mu^2\sigma\,\mathrm{e}^{-\alpha\,x^2}v^4x^4\sqrt{\alpha+1} \\
&+2\,\mu^2\sigma\,\mathrm{e}^{-\alpha\,x^2}u^4x^4\sqrt{\alpha+1} + vv_{xx}x^4\sqrt{\alpha+1}\alpha^2 \\
&+2\,vv_{xx}x^4\sqrt{\alpha+1}\alpha + uu_{xx}x^4\sqrt{\alpha+1}\alpha^2 \\
&+2\,uu_{xx}x^4\sqrt{\alpha+1}\alpha - 12\,t\varepsilon\,v_tu\alpha\,\sqrt{\alpha+1}x\mathrm{e}^{-1/2\,(\alpha+1)x^2} \\
&-4\,t\varepsilon\,v_tu\alpha^2\sqrt{\alpha+1}x^3\mathrm{e}^{-1/2\,(\alpha+1)x^2} \\
&+4\,t\varepsilon\,u_2v\alpha^2\sqrt{\alpha+1}x^3\mathrm{e}^{-1/2\,(\alpha+1)x^2} + 16\,t\varepsilon\,u_tv\alpha\,\sqrt{\alpha+1}x^3\mathrm{e}^{-1/2\,(\alpha+1)x^2} \\
&+4\,\mu^2\sigma\,\mathrm{e}^{-\alpha\,x^2}v^2u^2x^4\sqrt{\alpha+1}\alpha^2 \\
&+12\,t\varepsilon\,u_tv\alpha\,\sqrt{\alpha+1}x\mathrm{e}^{-1/2\,(\alpha+1)x^2} + 6\,t\varepsilon\,\sqrt{\pi}\sqrt{2}\mathrm{erf}\left(1/2\,\sqrt{2}x\sqrt{\alpha+1}\right)\alpha\,v_tu \\
&-16\,t\varepsilon\,v_tu\alpha\,\sqrt{\alpha+1}x^3\mathrm{e}^{-1/2\,(\alpha+1)x^2} \\
&+8\,\mu^2\sigma\,\mathrm{e}^{-\alpha\,x^2}v^2u^2x^4\sqrt{\alpha+1}\alpha - 6\,t\varepsilon\,\sqrt{\pi}\sqrt{2}\mathrm{erf}\left(1/2\,\sqrt{2}x\sqrt{\alpha+1}\right)\alpha\,u_tv \\
&-x^6v^2\sqrt{\alpha+1} - x^6u^2\sqrt{\alpha+1} - x^6v^2\sqrt{\alpha+1}\alpha^2 - 2\,x^6v^2\sqrt{\alpha+1}\alpha \\
&-x^6u^2\sqrt{\alpha+1}\alpha^2 - 2\,x^6u^2\sqrt{\alpha+1}\alpha + vv_{xx}x^4\sqrt{\alpha+1} + uu_{xx}x^4\sqrt{\alpha+1}]
\end{aligned}
$$

2. Charge

Similarly, the conserved density, T^t is

$$\begin{aligned}
T^t &= \tfrac{1}{2x^4(\alpha+1)^{5/2}}[2\,\mathrm{e}^{-1/2\,(\alpha+1)x^2}\sqrt{\alpha+1}\alpha^2\epsilon\,tx^3u^2 \\
&+2\,\mathrm{e}^{-1/2\,(\alpha+1)x^2}\sqrt{\alpha+1}\alpha^2\epsilon\,tx^3v^2 \\
&+8\,\mathrm{e}^{-1/2\,(\alpha+1)x^2}\sqrt{\alpha+1}\alpha\,\epsilon\,tx^3u^2 \\
&+8\,\mathrm{e}^{-1/2\,(\alpha+1)x^2}\sqrt{\alpha+1}\alpha\,\epsilon\,tx^3v^2+6\,\mathrm{e}^{-1/2\,(\alpha+1)x^2}\sqrt{\alpha+1}\epsilon\,tx^3u^2 \\
&+6\,\mathrm{e}^{-1/2\,(\alpha+1)x^2}\sqrt{\alpha+1}\epsilon\,tx^3v^2 \\
&-\alpha^2x^4\sqrt{\alpha+1}u^2-\alpha^2x^4\sqrt{\alpha+1}v^2+6\,\mathrm{e}^{-1/2\,(\alpha+1)x^2}\sqrt{\alpha+1}\alpha\,\epsilon\,txu^2 \\
&+6\,\mathrm{e}^{-1/2\,(\alpha+1)x^2}\sqrt{\alpha+1}\alpha\,\epsilon\,txv^2 \\
&-3\,\epsilon\,t\sqrt{\pi}\sqrt{2}\mathrm{erf}\left(1/2\,\sqrt{2}x\sqrt{\alpha+1}\right)\alpha\,u^2 \\
&-3\,\epsilon\,t\sqrt{\pi}\sqrt{2}\mathrm{erf}\left(1/2\,\sqrt{2}x\sqrt{\alpha+1}\right)\alpha\,v^2-2\,\alpha\,x^4\sqrt{\alpha+1}u^2 \\
&-2\,\alpha\,x^4\sqrt{\alpha+1}v^2 \\
&+18\,\mathrm{e}^{-1/2\,(\alpha+1)x^2}\sqrt{\alpha+1}\epsilon\,txu^2+18\,\mathrm{e}^{-1/2\,(\alpha+1)x^2}\sqrt{\alpha+1}\epsilon\,txv^2 \\
&-9\,\epsilon\,t\sqrt{\pi}\sqrt{2}\mathrm{erf}\left(1/2\,\sqrt{2}x\sqrt{\alpha+1}\right)u^2 \\
&-9\,\epsilon\,t\sqrt{\pi}\sqrt{2}\mathrm{erf}\left(1/2\,\sqrt{2}x\sqrt{\alpha+1}\right)v^2 \\
&-u^2x^4\sqrt{\alpha+1}-v^2x^4\sqrt{\alpha+1}]
\end{aligned}$$

(c). $a(x)=\frac{1}{2}x^2$, $b(x)=-2\,\left(2\,\left(\alpha+3\right)x^2-\alpha-15\right)x\mathrm{e}^{-1/2\,(\alpha+1)x^2}$:

We only show here that the Euler operator is approximately zero so that one could obtain the approximate conservation law as above.

1. Energy

$$\begin{aligned}
&\tfrac{\delta}{\delta(u,v)}[u(u_t+\tfrac{1}{2}\,v_{xx}-\varepsilon b\left(x\right)u-a\left(x\right)v+2\,\mu^2\sigma\,\mathrm{e}^{-\alpha\,x^2}\left(u^2+v^2\right)v) \\
&-v(-v_t+\tfrac{1}{2}\,u_{xx}-a\left(x\right)u+\varepsilon\,b\left(x\right)v+2\,\mu^2\sigma\,\mathrm{e}^{-\alpha\,x^2}\left(u^2+v^2\right)u)] \\
&=(4\,\epsilon\,v_t\left(2\,\alpha\,x^2+6\,x^2-\alpha-15\right)x\mathrm{e}^{-1/2\,(\alpha+1)x^2} \\
&,-4\,\epsilon\,u_t\left(2\,\alpha\,x^2+6\,x^2-\alpha-15\right)x\mathrm{e}^{-1/2\,(\alpha+1)x^2}) \\
&\to 0\quad\text{as}\quad\epsilon\to 0
\end{aligned}$$

2. Charge

$$
\begin{aligned}
&\tfrac{\delta}{\delta(u,v)}[u(u_t+\tfrac{1}{2}\,v_{xx}-\varepsilon b\,(x)\,u-a\,(x)\,v+2\,\mu^2\sigma\,\mathrm{e}^{-\alpha\,x^2}\,(u^2+v^2)\,v)\\
&-v(-v_t+\tfrac{1}{2}\,u_{xx}-a\,(x)\,u+\varepsilon\,b\,(x)\,v+2\,\mu^2\sigma\,\mathrm{e}^{-\alpha\,x^2}\,(u^2+v^2)\,u)]\\
&=(4\,\epsilon\,(2\,\alpha\,x^2+6\,x^2-\alpha-15)\,x\mathrm{e}^{-1/2\,(\alpha+1)x^2}u,\\
&4\,\epsilon\,(2\,\alpha\,x^2+6\,x^2-\alpha-15)\,x\mathrm{e}^{-1/2\,(\alpha+1)x^2}v)\\
&\to 0\quad\text{as}\quad\epsilon\to 0
\end{aligned}
$$

4.2.2 Case 2

In the case of a double-well potential with **PT**-symmetry phases of the linear problem, we choose $a(x,g)=1/2\,x^2-1/2\,g^2\mathrm{e}^{-x^2}-2\,\sigma\,g^4\mathrm{e}^{-(\alpha+1)x^2}$ and $b(x)=-3/2\,x\mathrm{e}^{-x^2}$ so that the Euler operator on the the following cases again yield vectors that go to zero.

1. Energy

$$
\frac{\delta}{\delta(u,v)}[\ldots]=(-3\,v_t\epsilon\,x\mathrm{e}^{-x^2},-3\,u_t\epsilon\,x\mathrm{e}^{-x^2})\to 0\quad\text{as}\quad\epsilon\to 0
$$

$$
\begin{aligned}
T^t\;&=\tfrac{1}{16x^4}[8\,\mathrm{e}^{-\alpha\,x^2}\mu^2\sigma\,u^4x^4+16\,\mathrm{e}^{-\alpha\,x^2}\mu^2\sigma\,u^2v^2x^4\\
&+8\,\mathrm{e}^{-\alpha\,x^2}\mu^2\sigma\,v^4x^4+16\,u^2\sigma\,g^4\mathrm{e}^{-(\alpha+1)x^2}x^4\\
&+16\,v^2\sigma\,g^4\mathrm{e}^{-(\alpha+1)x^2}x^4+4\,u^2g^2\mathrm{e}^{-x^2}x^4+4\,v^2g^2\mathrm{e}^{-x^2}x^4\\
&-12\,\mathrm{e}^{-x^2}\epsilon\,tx^3uv_t+12\,\mathrm{e}^{-x^2}\epsilon\,tx^3u_tv\\
&-4\,u^2x^6-4\,v^2x^6+9\,\epsilon\,t\sqrt{\pi}\mathrm{erf}\,(x)\,v_tu-9\,\epsilon\,t\sqrt{\pi}\mathrm{erf}\,(x)\,u_tv\\
&-18\,\mathrm{e}^{-x^2}\epsilon\,txuv_t+18\,\mathrm{e}^{-x^2}\epsilon\,txu_tv\\
&+4\,uu_{xx}x^4+4\,vv_{xx}x^4],\\
T^x\;&=\tfrac{1}{16x^3}[-12\,\mathrm{e}^{-x^2}\epsilon\,x^3uv_t+12\,\mathrm{e}^{-x^2}\epsilon\,x^3u_tv+9\,\epsilon\,\sqrt{\pi}\mathrm{erf}\,(x)\,v_tu\\
&-9\,\epsilon\,\sqrt{\pi}\mathrm{erf}\,(x)\,u_tv\\
&-18\,v_t\epsilon\,x\mathrm{e}^{-x^2}u+18\,u_t\epsilon\,x\mathrm{e}^{-x^2}v-4\,u_{xt}ux^3+4\,u_tu_xx^3\\
&-4\,v_{xt}vx^3+4\,v_tv_xx^3]
\end{aligned}
$$

2. Charge

$$\frac{\delta}{\delta(u,v)}[\ldots] = (3\,\epsilon\, x\mathrm{e}^{-x^2}u, 3\,\epsilon\, x\mathrm{e}^{-x^2}v) \to 0 \quad \text{as} \quad \epsilon \to 0$$

$$\begin{aligned}
T^t &= \tfrac{1}{16x^4}[12\,\mathrm{e}^{-x^2}\epsilon\, tx^3u^2 + 12\,\mathrm{e}^{-x^2}\epsilon\, tx^3v^2 \\
&\quad +18\,\mathrm{e}^{-x^2}\epsilon\, txu^2 + 18\,\mathrm{e}^{-x^2}\epsilon\, txv^2 - 9\,\epsilon\, t\sqrt{\pi}\mathrm{erf}\,(x)\,u^2 \\
&\quad -9\,\epsilon\, t\sqrt{\pi}\mathrm{erf}\,(x)\,v^2 - 8\,u^2x^4 - 8\,v^2x^4], \\
T^x &= -\tfrac{1}{16x^3}[-12\,\mathrm{e}^{-x^2}\epsilon\, x^3u^2 - 12\,\mathrm{e}^{-x^2}\epsilon\, x^3v^2 \\
&\quad +9\,\sqrt{\pi}\mathrm{erf}\,(x)\,\epsilon\, u^2 + 9\,\sqrt{\pi}\mathrm{erf}\,(x)\,\epsilon\, v^2 \\
&\quad -18\,\epsilon\, x\mathrm{e}^{-x^2}u^2 - 18\,\epsilon\, x\mathrm{e}^{-x^2}v^2 + 8\,v_xux^3 - 8\,u_xvx^3]
\end{aligned}$$

The conserved density in complex function form is $\Phi^t = \frac{1}{16x^4}[12\,\mathrm{e}^{-x^2}\epsilon\, tx^3 + 18\,\mathrm{e}^{-x^2}\epsilon\, tx - 9\,\epsilon\, t\sqrt{\pi}\mathrm{erf}\,(x) - 8\,x^4]|q|^2$.

4.3 Conclusion

We have shown that even though the class of Schrödinger equation with **PT**-symmetric potentials and inhomogeneity do not display variational properties and are not conserved, one could construct quantities that are approximately conserved up to specified order of the 'perturbation'.

Chapter 5

'Approximate' conservation laws of some nonlinear Schrödinger equation involving a spatially extended system consisting of two coupled elements

5.1 Introduction

In [42], Barashenkov et al establish and discuss, in detail, a model of a nonlinear Scrödinger equation involving a spatially extended system consisting of two coupled elements, where the energy is gained in the first element and dissipated in the second one. The literature describing aspects of these and equivalent models are vast, many of which are referred to in [42] and some other works are [43, 44]. In a somewhat alternative approach, we would allude the reader to the article on conservation laws, invariance and exact solution of certain Scrödinger type systems [3, 40].

We will consider two types of systems studied in the above literature, in particular in [42]. In the first instance, the system under study

$$iq_t + q_{xx} - Q(x)q + g|q|^2 q = 0 \tag{5.1}$$

in which $Q(x) = A(x) + iB(x)$, the **PT**-symmetric potential, q is a 'dimensionless amplitude of the electric field' [42] displays conservation laws based on the form of $B(x)$. It turns out that the conservation of the corresponding stationary case

$$-r_{xx} + Q(x)r - g|r|^2 r = -\kappa^2 r \tag{5.2}$$

depends on similar forms of $B(x)$, where $q = r(x)e^{i\kappa^2 t}$.

In the second case, we study the conserved forms of the 'toy model **PT**-symmetric dimer' involving the 'gain-loss amplitude parameter' δ, viz.,

$$\begin{aligned} iq_t + p + g|q|^2 q &= i\delta q, \\ ip_t + q + g|p|^2 p &= -i\delta p, \end{aligned} \tag{5.3}$$

where q and p are 'complex amplitudes of the active and lossy modes' [42]. We will show, in fact, that for $\delta > 0$, the model is not variational and, therefore, one cannot appeal to Noether's theorem to determine conservation laws. In fact, there are no conservation laws for this case using the methods and approach of [32]. Yet, the system does display non trivial symmetry properties suggesting some sort of energy, momentum and charge conservation. This would lead us to a novel concept (and consequent procedure) of 'approximate/perturbed conservation laws'. That is, if a system is variational for a parameter $\delta = 0$ and not variational for $\delta > 0$ (the perturbed system) but shares the symmetry properties that lead to conservation laws in the former case (via Noether's theorem, for example), then one may construct, for δ close to zero, approximate conservation laws for the perturbed system that are exact conservation laws in the limiting case $\delta \to 0$ for the unperturbed case. The notions of approximate symmetries, approximate variational symmetries and associated conservation laws have been discussed in the past, for example, [32, 34, 35]. The approach of 'partial Lagrangians' [41] may also be a route to determining some sort of conservation laws. Here, however, the concepts and methodology are quite

different and rely on the result that the Euler operator annihilates total divergences. For our purposes, it would be 'almost' zero, i.e., up to order δ.

Often, one encounters systems that involve one or several parameters for which the system may be exactly conserved (closed) for all value/s, some value/s or no values of the parameters. In the latter two cases, dependent on the relative nature of the parameters, one may consider an alternative study of the conservation of the system. In particular, if a parameter is relatively 'small', one may be able to construct an 'approximately closed system' that, in the limit of the parameter, is exact. To this end, consider a system dependent on a parameter of first order ϵ, viz.,

$$E^{\epsilon}(\underline{s}, u, u_{(1)}, \ldots, u_{(r)}, \epsilon) = O(\epsilon), \qquad u = 1, \ldots, \tilde{m}. \tag{5.4}$$

We say that the system is approximately conserved up to order ϵ if there exists a vector $(T^{s1}, T^{s2}, \ldots)$ such that $T^{si} = T^{si}(\underline{s}, u, u_{(1)}, \ldots, u_{(r)}, \epsilon)$ and

$$D_{s1}T^{s1} + D_{s2}T^{s2} + \ldots = O(\epsilon) \tag{5.5}$$

along the solutions of the differential equation $E^{\epsilon}(\underline{s}, u, u_{(1)}, \ldots, u_{(r)}, \epsilon) = O(\epsilon)$. In this case, the multiplier approach would entail the existence of a differential funtion Q such that

$$Q(\underline{s}, u, u_{(1)} \ldots)E^{\epsilon}(\underline{s}, u, u_{(1)}, \ldots, u_{(r)}, \epsilon) = D_{s1}T^{s1} + D_{s2}T^{s2} + \ldots. \tag{5.6}$$

In this case, then,

$$\epsilon[Q(\underline{s}, u, u_{(1)} \ldots)E^{\epsilon}(\underline{s}, u, u_{(1)}, \ldots, u_{(r)}, \epsilon)] = O(\epsilon). \tag{5.7}$$

In the above, the exactness is recovered in the limit, i.e., when $\epsilon \to 0$.

5.2 Conservation laws

In what follows below, we construct conservation laws for cases that are modelled and listed in [42]. The significance of these cases are discussed therein. For purposes of

calculations and software usage, the conserved vectors are determined by splitting the respective equation/s into the real and imaginary parts and reconstructed into the complex form.

5.2.1 Case 1

In the case of the scalar equations time dependent (5.1) and time independent (5.2), we let and have the following conserved flows which follow from the multipliers.

(a). For (5.1), we the real and imaginary system is

$$\begin{aligned} &u_t + v_{xx} - A(x)v - B(x)u + g\left(u^2+v^2\right)v = 0, \\ &-v_t + u_{xx} - A(x)u + B(x)v + g\left(u^2+v^2\right)u = 0 \end{aligned} \tag{5.8}$$

(i) For $B = 0$, we obtain multipliers Q corresponding to conserved flows (T^t, T^x).

• $Q = (v_t, u_t)$ - Energy

$$\begin{aligned} T^t &= -\tfrac{1}{2}gv^2u^2 - \tfrac{1}{4}gv^4 - \tfrac{1}{4}gu^4 - \tfrac{1}{2}vv_{xx} + \tfrac{1}{2}A\left(x\right)v^2 - \tfrac{1}{2}uu_{xx} + \tfrac{1}{2}A\left(x\right)u^2 \\ T^x &= -[\tfrac{1}{2}v_xv_t + \tfrac{1}{2}u_xu_t - \tfrac{1}{2}vv_{xt} - \tfrac{1}{2}u_{xt}] \end{aligned} \tag{5.9}$$

The conserved density, in complex form, is

$$\Phi^t = -\frac{1}{2}|q|^4 + \frac{1}{2}A(x)|q|^2 - \frac{1}{4}\hat{R}(qq_{xx}).$$

• $Q = (u, -v)$ - Charge

$$\begin{aligned} T^t &= \tfrac{1}{2}(u^2+v^2) \\ T^x &= uv_x - vu_x \end{aligned} \tag{5.10}$$

The conserved density is

$$\Phi^t = \frac{1}{2}|q|^2.$$

(ii) When $B = \frac{A'(x)}{2\sqrt{k-A(x)}}$, we get an additional conservation law with components

$$\begin{aligned}
T^t &= \tfrac{1}{2\sqrt{k-A(x)}}[\tfrac{1}{2}v_x v_t + \tfrac{1}{2}u_x u_t - \tfrac{1}{2}vv_{xt} - \tfrac{1}{2}uu_{xt} - A(x)\,v^2 - vu_x\sqrt{k-A(x)} \\
&\quad +v^2k + u^2k - A(x)\,u^2 + uv_x\sqrt{k-A(x)}] \\
T^x &= -\tfrac{1}{4\sqrt{k-A(x)}}[-2\,gv^2\sqrt{k-A(x)}u^2 - gv^4\sqrt{k-A(x)} - gu^4\sqrt{k-A(x)} \\
&\quad -4\,uv_xk + 4\,v_xA(x)\,u - 2\,{v_x}^2\sqrt{k-A(x)} - 4\,u_xA(x)\,v - 2\,{u_x}^2\sqrt{k-A(x)} + 4\,vu_xk \\
&\quad +2\,A(x)\,v^2\sqrt{k-A(x)} - 2\,v\sqrt{k-A(x)}u_t + 2\,A(x)\,u^2\sqrt{k-A(x)} + 2\,uv_t\sqrt{k-A(x)}]
\end{aligned} \tag{5.11}$$

The conserved density is

(b) For the stationary system (5.2), we get the split system

$$\begin{aligned}
v_{xx} + Av + Bu - g\,(u^2+v^2)\,v - \kappa^2 v = 0, \\
u_{xx} + Au - Bv - g\,(u^2+v^2)\,u - \kappa^2 u = 0
\end{aligned} \tag{5.12}$$

(i) For $B = 0$, we obtain multipliers Q corresponding to conserved flows (flux) T^x.

- $Q = (u(uv_x - vu_x), -v(uv_x - vu_x))$

$$T^x = \tfrac{1}{2}u^2{v_x}^2 - v_xuvu_x + \tfrac{1}{2}v^2{u_x}^2 \tag{5.13}$$

The conserved flux is

- $Q = (u, -v)$ - Charge

$$T^x = uv_x - vu_x \tag{5.14}$$

The conserved flux is

(ii) When $B = \frac{A'(x)}{2\sqrt{k-A(x)}}$, we get an additional conservation law

$$\begin{aligned} T^x &= -\tfrac{1}{\sqrt{k-A(x)}}[-v_x uk + v_x A(x)\, u - \tfrac{1}{2}{v_x}^2\sqrt{k-A(x)} + u_x vk - u_x A(x)\, v \\ &-\tfrac{1}{2}{u_x}^2\sqrt{k-A(x)} + \tfrac{1}{2}A(x)\, v^2\sqrt{k-A(x)} - \tfrac{1}{2}\kappa^2 v^2\sqrt{k-A(x)} \\ &+\tfrac{1}{2}A(x)\, u^2\sqrt{k-A(x)} - \tfrac{1}{2}\kappa^2 u^2\sqrt{k-A(x)} - \tfrac{1}{4}gv^4\sqrt{k-A(x)} \\ &-\tfrac{1}{2}gv^2\sqrt{k-A(x)}u^2 - \tfrac{1}{4}gu^4\sqrt{k-A(x)}] \end{aligned} \tag{5.15}$$

The conserved flux is

5.2.2 Case 2

In this section we consider the complex system (5.3), in which we let $q = u + iw$ and $p = v + iz$, leading to system of four real pdes

$$\begin{aligned} &u_t + z + g(u^2+w^2)w = \delta u, \\ &w_t + v + g(u^2+w^2)u = -\delta w, \\ &v_t + w + g(v^2+z^2)z = -\delta v, \\ &-z_t + u + g(v^2+z^2)z = \delta z \end{aligned} \tag{5.16}$$

which is not variational. However, if we regard δ as a 'small' parameter, we may suppose (5.16) as a perturbation of some system with $\delta = 0$ in (5.3). Below, we show that in fact the Euler operator is zero in the limit, i.e., it goes to zero as δ goes to zero. Thus, we can construct conservation laws that are 'approximate' upto an order of δ. In this case, we present the conserved flows as one forms ω^i.

• Energy - $Q = (u_t, w_t, v_t, z_t)$

The Euler operator E is

$$\delta(-2w_t, 2u_t, 2z_t, -2v_t) \to 0 \quad \text{as} \quad \delta \to 0 \tag{5.17}$$

and the corresponding 'approximate' conserved form is

$$\begin{aligned}\omega^1 &= [\tfrac{1}{4}t\delta\, w_t u - \tfrac{1}{4}t\delta\, u_t w - \tfrac{1}{4}t\delta\, z_t v + \tfrac{1}{4}t\delta\, v_t z - \tfrac{1}{4}mz^4 - \tfrac{1}{2}mw^2u^2 \\ &\quad -\tfrac{1}{4}mw^4 - \tfrac{1}{2}mz^2v^2 - \tfrac{1}{4}mv^4 - \tfrac{1}{4}mu^4 - zw - vu]\mathrm{d}x \\ &\quad +[-\tfrac{1}{4}x\delta\ (uw_t - wu_t - vz_t + zv_t)]\mathrm{d}t\end{aligned} \tag{5.18}$$

so that the the conserved density is

• 'Charge' - $Q = (u, -w, v, -z)$

The Euler operator is

$$\delta(-2u, -2w, 2v, 2z) \to 0 \quad \text{as} \quad \delta \to 0 \tag{5.19}$$

so that the approximate conserved 'charge' is

$$\begin{aligned}\omega^2 &= [\tfrac{1}{4}t\delta\, u^2 - \tfrac{1}{4}t\delta\, v^2 + \tfrac{1}{4}t\delta\, w^2 - \tfrac{1}{4}t\delta\, z^2 - \tfrac{1}{2}z^2 - \tfrac{1}{2}w^2 - \tfrac{1}{2}v^2 - \tfrac{1}{2}u^2]\mathrm{d}x \\ &\quad -[\tfrac{1}{4}x\delta\ (u^2 - v^2 + w^2 - z^2)]\mathrm{d}t\end{aligned} \tag{5.20}$$

Thus, the conserved density in complex functional form is

$$\Phi^t = -\frac{1}{4}\delta x(|q|^2 - |p|^2).$$

• Momentum - $Q = (u_x, w_x, v_x, z_x)$

The operator E is

$$\delta(-2w_t, 2u_x, -2z_x, 2v_x) \quad \to 0 \quad \text{as} \quad \delta \to 0 \tag{5.21}$$

and the approximate conserved density is

$$\begin{aligned}\omega^3 &= [\tfrac{1}{4}t\delta\, zv_x + \tfrac{1}{4}t\delta\, uw_x - \tfrac{1}{4}t\delta\, vz_x - \tfrac{1}{4}t\delta\, wu_x + \tfrac{1}{2}zv_x + \tfrac{1}{2}wu_x - \tfrac{1}{2}vz_x - \tfrac{1}{2}uw_x]\mathrm{d}x \\ &\quad +[-\tfrac{1}{4}x\delta\, zv_x - \tfrac{1}{4}x\delta\, uw_x + \tfrac{1}{4}x\delta\, vz_x \\ &\quad +\tfrac{1}{4}x\delta\, wu_x + \tfrac{1}{2}mz^2v^2 + \tfrac{1}{4}mz^4 + \tfrac{1}{2}mw^2u^2 + \tfrac{1}{4}mw^4 + \tfrac{1}{4}mv^4 \\ &\quad +\tfrac{1}{4}mu^4 + \tfrac{1}{2}zv_t + zw - \tfrac{1}{2}uw_t - \tfrac{1}{2}vz_t + \tfrac{1}{2}wu_t + vu]\mathrm{d}t\end{aligned} \tag{5.22}$$

Thus, the conserved density in complex functional form is

5.3 Results and discussion

We constructed conservation laws of a model of a nonlinear Schrödinger equation involving a spatially extended system consisting of two coupled elements. Firstly, we studied the space-time system and, separately, the stationary model arising from a standard transformation. Secondly and independently, we analyse a 'toy model **PT**-symmetric dimer' involving a 'gain-loss amplitude parameter' for which only approximate conserved forms exist. Here, we have shown that even though the class of Scrödinger equation do not display variational properties and are not conserved, one could construct quantities that are approximately conserved up to a specified order of the 'perturbation'.

The result above appeared [4].

Chapter 6

Conclusion

We analyse the symmetry, invariance properties and conservation laws of the partial differential equations.

In chapter 2, we have shown that the pdes that arise in the different models of noise removal algorithms are abundant in symmetries that leave the systems invariant.

In chapter 3, we have shown that model the vibrations of a beam is rich in symmetry and conservation laws in Timoshenko system of pdes.

In chapter 4, we construct and analyse the conservation laws of a model of a nonlinear Schrödinger equation with **PT**-symmetric potentials and inhomogeneity.

In chapter 5, we constructed conservation laws of a model of a nonlinear Schrödinger equation involving a spatially extended system consisting of two coupled elements.

The construction of symmetries and conservation laws is now well established via a number of methods and the main method adopted here was the multiplier approach which subsequently resorted to the homotopy operator. This sequence can be utilised for a number of pdes that arise in mathematical physics and engineering that requires a straightforward and elegant method for constructing conservation laws especially if one wants to avoid the

Lagrangian and Noether's theorem approach for whatever reasons. We had gone further to adapt the method to pdes that contain a small parameter in what we called approximate conservation laws. The approach had not discussed an 'approximate Homotopy' operator but rather used the standard operator to finally construct the conserved vectors for such pdes. A future and interesting project would be to develop the notion an 'approximate Homotopy' operator and construct conserved flows following such an operator. The question would be 'how such a conserved flow would differ from what was obtained here'. Furthermore, how would such conservation laws be related to the established notions of 'approximate Lie symmetries'. In the future, we will attempt applications of our methods to problems and models in condensed matter, cosmology and high-energy physics.

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