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Mafrika, Okuhle

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The multiplier approach in the double reduction of differential equations

Masters dissertation



Okuhle Mafrika
Student Number: 1647839
Supervisor: Prof. Abdul Kara

Declaration

I, Okuhle Mafrika, hereby certify that the work done under Prof. Abdul Kara's supervision on the project named "The multiplier approach in the double reduction of differential equation" is the original report that I wrote. I recognized other people's efforts in my project and I am aware that plagiarism is dishonest.

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Abstract

In order to systematically derive conservation laws for both Lagrangian and non-Lagrangian systems, this study examines the application of the multiplier approach in the double reduction of differential equations. We begin by revisiting the traditional conclusions of Noether's Theorem and discussing subsequent developments that allow the construction of conservation laws using no variational structure. We investigated the connections between conserved vectors and Lie–Bäcklund symmetries using symmetry analysis.

Applications to non-linear partial differential equations, such as the Schrödinger equation with cubic non-linearity and coupled systems, like the diffusion and Klein-Gordon models, show how effective the method is. In each instance, we construct conserved vectors that describe the system's invariant properties, derive multipliers, and find related symmetries. Our findings demonstrate that even in the absence of a Lagrangian formulation, the multiplier method offers a cohesive and methodical framework for identifying conservation laws, providing a deeper understanding of the composition and solutions of differential equations pertinent to mathematical physics.

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1 Chapter 1- Introduction

This chapter explores the relationship between symmetries and conservation laws in differential equations, a connection initially established by Emmy Noether in the early 20th century. While Noether's Theorem provides a systematic method to derive conservation laws for differential equations arising from variational principles, many physically significant equations do not possess a Lagrangian formulation. In such cases, alternative techniques are necessary to uncover conserved quantities.

This study emphasizes the use of the multiplier method, a systematic approach for deriving conservation laws from both Lagrangian and non-Lagrangian systems. It also touches on symmetry-based methods such as Noether's Theorem and the direct method. Particular attention is given to the role of the Euler–Lagrange operator and the potential application of the homotopy operator.

Throughout this dissertation, we use the terms *conserved vector*, *conserved current*, and *conserved flow* interchangeably, although *conserved vector* will be used most frequently.

1.1 Background

Emmy Noether's groundbreaking work on symmetries and conservation laws ([22, 4]) demonstrated that for Euler-Lagrange differential equations, each Noether symmetry associated with a Lagrangian corresponds to a conservation law, which can be explicitly calculated using a specific formula. Ibragimov et al. ([10]) explored how the Noether conserved vector Φ components are connected to the Lie-Bäcklund operator, which generates a Noether symmetry leading to Φ. This raises the obvious

question of whether a comparable relationship exists for differential equations that cannot be expressed using a Lagrangian. It is commonly understood that certain differential equations, such as evolution-type equations, cannot be derived from a variational principle.

In a study on symmetries (both local and non-local) and conservation laws, Anco and Bluman ([2, 6]) derived an identity independent of a Lagrangian, establishing a correspondence between symmetries and conservation laws for self-adjoint differential equations. This fundamental relationship between Lie–Backlund symmetry generators and conservation laws for any differential equation can be derived without needing a Lagrangian formulation. This relation is essential for partial differential equations in constructing conservation laws, regardless of whether a Lagrangian is known or exists. This method allows for the association of symmetries with a given conservation law for a differential equation ([12]). A straightforward characterization of the action of symmetries on conservation laws for partial differential equations was examined using the general method of conservation law multipliers ([3, 9]). This approach was used to define symmetry-invariant and symmetry-homogeneous conservation laws.

We present some of the definitions and notations in the following.

Consider a system of partial differential equations (PDEs) involving multiple variables and derivatives. We work with n independent variables grouped together as $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables collected as $u = (u^1, u^2, \ldots, u^m)$. These variables are connected through a system of equations written as:

$$G^\mu(x, u, u_{(1)}, \ldots, u_{(r)}) = 0, \quad \mu = 1, \ldots, \tilde{m}, \tag{1.1}$$

In this case, $u_{(1)}, u_{(2)}, \ldots, u_{(r)}$ represent the sets of all first, second, ..., and rth-order partial derivatives, i.e., $u^\alpha_i = D_i(u^\alpha), u^\alpha_{ij} = D_j D_i(u^\alpha), \ldots$, respectively, with

the total differentiation operator with respect to x^i provided by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \ldots, \quad i = 1, \ldots, n, \tag{1.2}$$

where the summation convention is used whenever appropriate.

A current $\Phi = (\Phi^1, \ldots, \Phi^n)$ is conserved if it satisfies

$$D_i \Phi^i = 0 \tag{1.3}$$

along the solutions of (1.1). Every admitted conservation law can be derived from the multipliers $Q_\mu(x, u, u_{(1)}, \ldots)$ that satisfy the following.

$$Q_\mu G^\mu = D_i \Phi^i \tag{1.4}$$

off the solution space, for some associated current Φ^i. The corresponding conserved vector can then be computed using the homotopy operator (see [2, 11]).

Definition 1. The conservation law that arises from multipliers can be defined as follows. Say we are given a differential equation.

$$E(x, u, u_{(1)}, u_{(2)}) = 0, \tag{1.5}$$

Then there exists a function $I(x, u, u_{(1)})$ such that the summation convention of equation 1.5 is

$$D_i I = 0, \tag{1.6}$$

Here, the function I is a conserved quantity, hence equation 1.6 gives a mathematical definition of conversation law. Recall that D_i is the total differential operator with respect to x^i.

For better understanding of the definition above, please refer back to equation 1.3 and 1.4 where Φ^i is the conserved quantity.

Definition 2 [11] . The Euler-Lagrange operator for each dependent variable u^α, it is defined as:

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\cdots D_{i_s}\,\frac{\partial}{\partial u^\alpha_{i_1\cdots i_s}}\;, \quad \alpha = 1,\ldots,m\;. \tag{1.7}$$

This operator annihilates any total divergence.

Definition 3 [11] . The Lie-Bäcklund operator, also known as the generalized symmetry operator or vector field, is a fundamental tool in the study of symmetries of partial differential equations. In its basic form, it is expressed as:

$$X = \xi^i\,\frac{\partial}{\partial x^i} + \eta^\alpha\,\frac{\partial}{\partial u^\alpha}\;, \qquad \xi^i,\ \eta^\alpha \in \mathcal{A}\;, \tag{1.8}$$

where $\mathcal{A}$ is the set of differential functions - if ξ^i and η^α are independent of derivatives, X is called a 'Lie point symmetry generator'.

This compact representation can be expanded into an infinite formal sum:

$$X = \xi^i\,\frac{\partial}{\partial x^i} + \eta^\alpha\,\frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}\zeta^\alpha_{i_1\ldots i_s}\,\frac{\partial}{\partial u^\alpha_{i_1\ldots i_s}}, \tag{1.9}$$

where the additional coefficients ζ are determined through the prolongation formulae:

$$\begin{aligned} \zeta^\alpha_i &= D_i(W^\alpha) + \xi^j u^\alpha_{ij}\;, \\ \zeta^\alpha_{i_1\ldots i_s} &= D_{i_1}\ldots D_{i_s}(W^\alpha) + \xi^j u^\alpha_{ji_1\ldots i_s}, \quad s>1. \end{aligned} \tag{1.10}$$

In these equations, W^α represents the Lie characteristic function:

$$W^\alpha = \eta^\alpha - \xi^j u^\alpha_j\;. \tag{1.11}$$

The generalized symmetry operator can also be written in characteristic form:

$$X = \xi^i D_i + W^\alpha\,\frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} D_{i_1}\ldots D_{i_s}(W^\alpha)\,\frac{\partial}{\partial u^\alpha_{i_1\ldots i_s}} \tag{1.12}$$

We first construct the multipliers Q_μ to determine the conserved flows by utilizing the fact that the Euler-Lagrange operator removes total divergences, which leads to the following:

$$\frac{\delta}{\delta u^\alpha}[Q_\mu G^\mu] = 0. \tag{1.13}$$

For further information, see [14, 19, 18].

Definition 4 [11] . If a vector field X in 1.8 leaves invariant the variational integral

$$\int L(\mathrm{d}x_1 \dots \mathrm{d}x_n), \tag{1.14}$$

upto gauge $(f_1, \dots, f_n)$, X is called a variational (or Noether) symmetry.

It turns out that X satisfies

$$XL + LD(\xi^i, \dots, \xi^n) = D(f_1, \dots, f_n) \tag{1.15}$$

and a conserved vector Φ is defined by Noether's Theorem.

Theorem (Noether). If X is a Noether symmetry with a corresponding gauge term $(f_1, \dots, f_n)$, then the conserved vector Φ^i is given by :

$$\Phi^i = L\xi^i + W^\alpha \frac{\partial L}{\partial u^i} - f_i. \tag{1.16}$$

For complete details, derivations and examples, see [2], [13], and [21].

1.2 Problem Statement

Many differential equations in physics and mathematics do not admit a Lagrangian formulation, making Noether's Theorem inapplicable for deriving their conservation

laws. This creates a need for systematic methods that can handle both variational and non-variational systems.

This study addresses this gap by using the multiplier method and double reduction technique to construct conservation laws. In addition, it investigates the connection between symmetries—especially Lie–Bäcklund symmetries—and conserved quantities, aiming to unify the treatment of conservation laws across a broad class of PDEs.

1.3 Applications - Conservation Laws

In most examples done in our study, we assume that $(x^1, x^2) = (t, x)$, where we focus on studying the conservations laws derived from equation 1.17, using the three methods namely the direct method, Noether's thereom and multiplier method.

Also, consider the equations that arises in the study of Maxwellian tails ([8, 13, 17]), viz.,

$$u_{xt} + u_x + u^2 = 0 \tag{1.17}$$

which admits the Lie point symmetry generators

$$\begin{aligned} X_1 &= \partial_x, & X_2 &= \partial_t, \\ X_3 &= x\partial_x - u\partial_u, & X_4 &= e^t\partial_t + e^t u\partial_u. \end{aligned} \tag{1.18}$$

For the conserved vectors that it has refer to [13] while [17] discuss the formation of Maxwellian tails in great detail.

1.3.1 Direct method

The Direct Method to construct a conserved vector $\Phi = (\Phi^t, \Phi^x)$ produces determining equations for the components Φ^t and Φ^x from

$$(D_t\Phi^t + D_x\Phi^x)|_{u_{xt}+u_x+u^2=0} = 0 \tag{1.19}$$

in which we replace u_{xt} by $-u_x - u^2$ to get

$$\Phi^t_t + u_t\Phi^t_u + u_{tt}\Phi^t_{u_t} + \Phi^t_{u_x}(-u^2 - u_x) + \Phi^x_x + u_x\Phi^t_u + u_{xx}\Phi^x_{u_x} + \Phi^x_{u_t}(-u^2 - u_x) = 0. \tag{1.20}$$

Splitting with Respect to Derivatives. We now split this equation with respect to the independent derivatives u_{tt} and u_{xx}. That is, we equate the coefficients of each algebraically independent derivative to zero. This gives:

$$\begin{aligned} u_{tt} &: \Phi^t_{u_t} = 0, \\ u_{xx} &: \Phi^t_{u_x} = 0, \\ &: \Phi^t_t + u_t\Phi^t_u + \Phi^t_{u_x}(-u^2 - u_x) + \Phi^x_x + u_x\Phi^t_u + \Phi^x_{u_t}(-u^2 - u_x) = 0. \end{aligned} \tag{1.21}$$

The above system would be analysed to determine the conserved vectors; this can be cumbersome. One may resort to association of the conserved vector with a symmetry of (1.18) (see [12, 7]), for e.g., using X_4, it can be shown that the above system can be solved simultaneously with the conditions

$$X_4(\Phi^t) = 0, \quad X_4(\Phi^x) + e^t\Phi^x = 0. \tag{1.22}$$

1.3.2 Noether's Theorem

For the pde (1.17) a lagrangian is given by

$$L = \frac{1}{2}e^{2t}(u_xu_t + u^2u_t). \tag{1.23}$$

The corresponding Noether symmetries with gauge term (f, g) can be computed from

$$XL + L(D_t\tau + D_x\xi) = D_tf + D_xg, \tag{1.24}$$

where

$$X = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \phi(t, x, u)\partial_u, f = f(t, x, u), g = g(t, x, u). \tag{1.25}$$

Equation (1.24) expands as

$$
\begin{aligned}
&e^{2t}u^2\tau u_t + e^{2t}u\phi u_t + e^{2t}\tau u_t u_x - \tfrac{1}{2}e^{2t}u^2u_x\xi_t \\
&-\tfrac{1}{2}e^{2t}u_x^2\xi_t - \tfrac{1}{2}e^{2t}u^2u_tu_x\xi_u - e^{2t}u_tu_x^2\xi_u - \tfrac{1}{2}e^{2t}u_tu_x\xi_x \\
&-\tfrac{1}{2}e^{2t}u^2u_t\tau_t - \tfrac{1}{2}e^{2t}u_tu_x\tau_t - \tfrac{1}{2}e^{2t}u^2u_t^2\tau_u - e^{2t}u_t^2u_x\tau_u - \tfrac{1}{2}e^{2t}u_t^2\tau_x + \tfrac{1}{2}e^{2t}u^2\phi_t + \tfrac{1}{2}e^{2t}u_x\phi_t \\
&+\tfrac{1}{2}e^{2t}u^2u_t\phi_u + e^{2t}u_tu_x\phi_u + \tfrac{1}{2}e^{2t}u_t\phi_x + \tfrac{1}{2}e^{2t}(u_tu_x + u_tu^2)(\tau_t + u_t\tau_u + \xi_x + u_x\xi_u) \\
&= (f_t + u_tf_u + g_x + u_xg_u)
\end{aligned}
\tag{1.26}
$$

and separation by monomials leads to the system below

$$
\begin{aligned}
u_x^2 \quad &: -\tfrac{1}{2}e^{2t}\xi_t \\
u_t^2 \quad &: -\tfrac{1}{2}e^{2t}\tau_x \\
u_tu_x \quad &: e^{2t}\tau + e^{2t}\phi_u \\
u_x \quad &: -\tfrac{1}{2}e^{2t}u^2\xi_t + \tfrac{1}{2}e^{2t}\phi_t - g_u \\
u_t \quad &: e^{2t}u^2\tau + e^{2t}u\phi - f_u + \tfrac{1}{2}e^{2t}u^2\xi_x - \tfrac{1}{2}e^{2t}u^2\phi_u + \tfrac{1}{2}e^{2t}\phi_x \\
1 \quad &: -f_t - g_x + \tfrac{1}{2}e^{2t}u^2\phi_t \\
&\xi_u = \tau_u = 0
\end{aligned}
\tag{1.27}
$$

It can be shown that (1.27) leads to [15]

$$
\begin{aligned}
\tau &= c_2 + c_3e^t, \\
\xi &= c_1 + xc_2, \\
\phi &= -u(c_2 + e^tc_3), \\
f &= -\tfrac{1}{6}(e^{3t}u^3c_3), \\
g &= -\tfrac{1}{4}(e^{3t}u^2c_3)
\end{aligned}
\tag{1.28}
$$

so that the Noether symmetries are

$$
\begin{aligned}
&X_1 = \partial_x, \quad f = g = 0 \\
&X_2 = \partial_t + x\partial_x - u\partial_u, \quad f = g = 0 \\
&X_3 = e^t\partial_t - e^tu\partial_u, \quad f = -\tfrac{1}{6}e^{3t}u^3, \quad g = -\tfrac{1}{4}e^{3t}u^2.
\end{aligned}
\tag{1.29}
$$

Each X_i leads to a conserved vector (Φ^t, Φ^x) by the Noether's Theorem, i.e.,

$$\begin{aligned} \Phi^t &= L\tau_i + (\phi - \tau u_t - \xi u_x)\tfrac{\partial L}{\partial u_t} - f, \\ \Phi^x &= L\xi_i + (\phi - \tau u_t - \xi u_x)\tfrac{\partial L}{\partial u_x} - g. \end{aligned} \tag{1.30}$$

For e.g., from X_2, we get the components

$$\begin{aligned} \Phi_2^t &= L + (-u - u_t - xu_x)\tfrac{\partial L}{\partial u_t} - 0, \\ \Phi_2^x &= Lx + (-u - u_t - xu_x)\tfrac{\partial L}{\partial u_x} - 0, \end{aligned} \tag{1.31}$$

L as in (1.23). where $\frac{\partial L}{\partial u_x} = \frac{1}{2}e^{2t}(u_t)$ and $\frac{\partial L}{\partial u_t} = \frac{1}{2}e^{2t}(u_x)$.

1.4 Multiplier Method

The multiplier method provides a systematic approach to construct conservation laws for PDEs that do not necessarily arise from a variational principle. The idea with this method is to determine a function Q (called a multiplier) such that the product of Q with the original PDE becomes a total divergence. For PDE

$$G^\mu = u_{xt} + u_x + u^2 = 0, \tag{1.32}$$

Here, the defining equation for determining the multipliers is given by

$$\frac{\delta}{\delta u}(Q(u_{xt} + u_x + u^2)) = 0 \tag{1.33}$$

where $\frac{\delta}{\delta u}$ is the Euler operator (see Eq. 1.13).

In expanded form, we get, for $Q = f(x, t, u, u_x, u_t)$ dependent on, up to, first deriva-

tives

$$
\begin{aligned}
&\left(-2u^2u_{xt}+u_{tt}u_{xx}-2u_xu_{xt}-u_{xt}^2\right)\left(\tfrac{\partial^2}{\partial u_t\partial u_x}Q\right)+\left(-u^2u_x+u_tu_{xx}-u_x^2\right)\left(\tfrac{\partial^2}{\partial u\partial u_x}Q\right)\\
&+\left(\left(-u_t+u_{tt}\right)u_x-u^2u_t\right)\left(\tfrac{\partial^2}{\partial u\partial u_t}Q\right)-u_{xx}\left(u^2+u_x\right)\left(\tfrac{\partial^2}{\partial u_x^2}Q\right)-u_{tt}\left(u^2+u_x\right)\left(\tfrac{\partial^2}{\partial u_t^2}Q\right)\\
&+\left(-u^2-u_x\right)\left(\tfrac{\partial^2}{\partial t\partial u_t}Q\right)+\left(-u^2-u_x\right)\left(\tfrac{\partial^2}{\partial x\partial u_x}Q\right)+\left(\tfrac{\partial^2}{\partial u^2}Q\right)u_xu_t\\
&+\left(\tfrac{\partial^2}{\partial t\partial u_x}Q\right)u_{xx}+\left(\tfrac{\partial^2}{\partial x\partial u_t}Q\right)u_{tt}+\left(\tfrac{\partial^2}{\partial t\partial u}Q\right)u_x+\left(\tfrac{\partial^2}{\partial u\partial x}Q\right)u_t\\
&+\tfrac{\partial^2}{\partial t\partial x}Q+\left(-2uu_x-2u_{xx}\right)\left(\tfrac{\partial}{\partial u_x}Q\right)+\left(-2uu_t-2u_{xt}\right)\left(\tfrac{\partial}{\partial u_t}Q\right)\\
&+\left(u^2+2u_{xt}\right)\left(\tfrac{\partial}{\partial u}Q\right)+2Qu-\tfrac{\partial}{\partial x}Q=0
\end{aligned}
\tag{1.34}
$$

which is separated into a system of PDEs whose solution yields

$$
\begin{aligned}
Q_1&=e^{2t}(xu_x+u+u_t),\\
Q_2&=e^{3t}(u_t+u),\\
Q_3&=e^{2t}u_x.
\end{aligned}
\tag{1.35}
$$

A conserved vector is produced by each multiplier by substituting into (1.4) or some 'homotopy' operator. The method of using homotopy operator is well detailed in [23].

From [13] in page 64, the conserved vectors for equation 1.32 are

$$
\begin{aligned}
\Phi^x&=\tfrac{1}{3}u^3e^{3t},\\
\Phi^t&=\tfrac{1}{2}(u_t+u^2)e^{3t}
\end{aligned}
$$

1.4.1 Further examples - multiplier approach

(i) Generalized Camassa–Holm-type Equation. [1]

The first example here is

$$
u_t+2uu_x+nu_x-u_{xxt}=0,
\tag{1.36}
$$

where n is a free invariable.

The conserved vector $(\Phi^1, \Phi^2) = (\Phi^x, \Phi^t)$, as introduced earlier, satisfies the total divergence along the solutions of equation (1.36). Moreover, using (1.13), we obtain

$$\frac{\delta}{\delta u}(Q(u_t + 2uu_x + nu_x - u_{xxt})) = 0. \tag{1.37}$$

If we assume that the multiplier depends on second-order derivatives, that is, $Q = f(x, t, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx})$, then, after performing a detailed expansion and separating terms by monomials, Consequently, $Q = f$ in derivatives with regard to u must be of zeroth or second order. The admissible forms of Q are identified as

$$\begin{aligned} Q_1 &= 1, \\ Q_2 &= u, \\ Q_3 &= u_{xt} - u^2. \end{aligned} \tag{1.38}$$

A *symmetry invariance* analysis of the multipliers in (1.38), as explored in [3], is carried out using $X = t\partial_t - (\frac{1}{2}n + u)\partial_u$, i.e.,

$$XQ_1 = 0, \quad XQ_2 = Q_1 + \frac{2}{n}Q_2, \quad XQ_3 = \frac{4}{n}Q_3 - 2Q_2 \tag{1.39}$$

so that Q_1 is strictly invariant under X and all Q_is under translations are strictly invariant. Q_1 is created by the action of X on Q_2. Since the linear combination of Q_3 and Q_2 is given by the action of X on Q_3, we can assume that Q_3 forms a basis,

as would the conserved vector following Q_3. The conserved vectors are (Φ^x, Φ^t).

$$\Phi_1^x = u^2 + nu - u_{xt},$$
$$\Phi_1^t = u$$

$$\Phi_2^x = \tfrac{1}{6}(3nu^2 + 4u^3 + 2u_t u_x - 4uu_{xt}),$$
$$\Phi_2^t = \tfrac{1}{6}(3u^2 + {u_x}^2 - 2uu_{xx})$$

$$\Phi_3^x = \tfrac{1}{12}(-4nu^3 - 6u^4 + 3{u_t}^2 + 12u^2 u_{xt} - 4{u_{xt}}^2 + 2u_x u_{xtt} + u_t(3nu_x - u_{xxt})$$
$$-u(3u_{tt} - 3nu_{xt} + u_{xxtt})),$$
$$\Phi_3^t = \tfrac{1}{12}(-4u^3 + 3u_t u_x + 3n{u_x}^2 - 2u_{xt}u_{xx} - u_x u_{xxt} + u(3u_{xt} - 3nu_{xx} + u_{xxxt})).$$

Notes.
(a). A linear combination of Φ_1^x and Φ_1^t gives us Q_3, i.e., Q_3=(-1)Φ_1^x +(n) Φ_1^t = $u_{xt} - u^2$.
(b). The notion of 'basis' mentioned above differs from the classical definition. Here, we assume that if a multiplier is obtained by the action of a symmetry on another multiplier, the latter multiplier is an element of the basis of multipliers and the subsequent conserved vectors are concluded in an equivalent way. For details, see [13].

(ii). We illustrate the importance of the Euler operator in two variables in the following example, which looks at the classical Boussinesq system of equations. We pay particular attention to how its complexity rises with the order of the function it affects. Here, we choose possible second-order multipliers. The system is

$$\begin{aligned} eqt1 &= u_t + v_x + uu_x - \tfrac{1}{3}u_{xxt} = 0, \\ eqt2 &= v_t + uv_x + vu_x + u_x = 0. \end{aligned} \tag{1.40}$$

The Lie point symmetry generators are

$$X = \partial_t, \quad Y = \partial_x, \quad Z = -\frac{1}{2}t\partial_t + \frac{1}{2}u\partial_u + (1+v)\partial_v.$$

Using symmetry-ansatz approach, it can be demonstrated that when the multipliers Q and P depend on second-order derivatives, i.e.,

$$Q = f(x, t, u, v, u_x, v_x, u_{xx}, v_{xx}, u_{xt}, v_{xt})$$

and

$$P = g(x, t, u, v, u_x, v_x, u_{xx}, v_{xx}, u_{xt}, v_{xt}).$$

The joint Euler operator eliminates the total divergence, yielding

$$\frac{\delta}{\delta(u, v)}(f * eqt1 + g * eqt2) = 0, \tag{1.41}$$

where $\frac{\delta}{\delta(u,v)}$ is a vector form of variational derivatives.

Expanding the left-hand side of (1.41) is highly intricate and requires computational tools; here, we used Maple software, particularly to systematically identify monomials and solve the resulting overdetermined system of partial differential equations. In a condensed form, the multipliers are given by

$$\begin{aligned} &f_1 = k, \quad g_1 = F(t)[(1+v)u_{xx} + uv_{xx} + v_{xt} + 2u_x v_x] \\ &f_2 = u, \quad g_2 = \ln(1+v). \end{aligned} \tag{1.42}$$

We now do a symmetry analysis of the multipliers f_1 and g_1 under the generators X, Y and Z. The more interesting relationship follows from the generator Z, viz.,

$$Zf_1 = 0, \quad Zg_1 = \tfrac{1}{2}g_1 \ (F(t) = t^2), \quad \textit{partial strict invariance}, \tag{1.43}$$

This classification helps determine which conservation laws are invariant under scaling transformations.

Regarding X, we get $Xf_1 = 0$ and $Xg_1 = g_1^*(t)$. With respect to Y, we get $Yf_1 = 0$, $Yg_1 = 0$ which implies strict invariance under Y.

The conserved vector corresponding to the above multipliers is

$$\Phi^x = \tfrac{1}{36}(9t^2{v_t}^2 + 27t^2v_tu_x + 18t^2{u_x}^2 + 18u^2(k + t^2{v_x}^2) - 8ku_{xt}$$
$$+6tv^2(-2u_x + 3t{u_x}^2 - tu_{xt}) + 3v(12k - 3t^2v_{tt} - 6tu_x + 12t^2{u_x}^2$$
$$+2tv_t(-3 + 4tu_x) - 2t^2u_tv_x - 3t^2u_{xt}) + 6tu(t(5v_t + 6u_x)v_x + v((-2 + 6tu_x)v_x - tv_{xt}))),$$
$$\Phi^t = \tfrac{1}{36}(9t^2v_tv_x + 9t^2u_xv_x + 18t^2vu_xv_x + 9t^2vv_{xt} - 4ku_{xx} + 9t^2vu_{xx}$$
$$+6t^2v^2u_{xx} + 6u(6k + t^2{v_x}^2 + t^2vv_{xx})).$$

The symmetry analysis of the multipliers f_2 and g_2 under the generators X and Y are as follows

$$\begin{aligned} &Xf_2 = 0, \quad Xg_2 = 0, \quad \textit{strict invariance} \\ &Yf_2 = 0, \quad Yg_2 = 0, \quad \textit{strict invariance}. \end{aligned} \tag{1.44}$$

However, the action of the generator Z gives $Zf_2 = \frac{1}{2}f_2$ and $Zg_2 = 1$.

The conserved vector corresponding to these multipliers is

$$\Phi^x = \tfrac{1}{9}(3u^3 + u_tu_x + u(9\ln(1 + v)(1 + v) - 2u_{xt})),$$
$$\Phi^t = \ln(1 + v) + \tfrac{1}{2}u^2 + (-1 + \ln[1 + v])v + \tfrac{{u_x}^2}{18} - \tfrac{1}{9}uu_{xx}.$$

(iii). The version of the Boussinesq equation that emerges in the model for unidirectional propagation of long waves in shallow water is given by ([22])

$$u_{tt} = \frac{4}{3}(u_{xx})^2 + \frac{1}{3}u_{xxxx} \tag{1.45}$$

which, when expressed as a bi-Hamiltonian system, can be rewritten in its evolutionary form as

$$u_t - v_x = 0, \quad v_t = \frac{1}{3}u_{xxx} + \frac{8}{3}uu_x = 0. \tag{1.46}$$

Using 'partial Lagrangians', the conservation laws of (1.45) and a systems version of the Boussinesq equation have been studied and determined in [16, 15]). Through the

Noether-type symmetries ([15]), from Noether's Theorem, only the conservation laws corresponding to multipliers of up to first-order, at most, remained accessible. Using the appropriate Noether-type generators ∂_u, $t\partial_u$, $x\partial_u$, and $xt\partial_u$, for instance, the multipliers in the former scenario were 1, t, x, and xt. Given (1.46), the preceding approach (in (ii)) yields the known multipliers $(Q, P) = (v, u)$ and the, previously unknown, second-order case

$$(Q, P) = (u_{xx} + 4u^2, 3v)$$

and the corresponding conserved flow is then calculated to be

$$(\Phi^x, \Phi^t) = (\frac{1}{2}(-8u^2v + u_t u_x - uu_{xt} - 2vu_{xx}), \frac{1}{6}(8u^3 + 9v^2 + 3uu_{xx})).$$

The Lie point symmetry generators of (1.46) are

$$\partial_t, \quad \partial_x, \quad \partial_v, \quad -x\partial_x - 2t\partial_t + 2u\partial_u + 3v\partial_v (= X).$$

1.5 Conclusion

In conclusion, this chapter has explored how symmetries relate to conservation laws for both Lagrangian and non-Lagrangian differential equations. By applying current techniques, we have shown how conservation laws can be systematically derived, enhancing our understanding of the mathematical principles that describe physical systems. Comparing methods for constructing conservation laws has been studied in detail in [20].

2 Chapter 2 - The Conservation Laws of the Schrödinger equation with cubic nonlinearity

In this chapter, we examine the Schrödinger equation with cubic nonlinearity; exploring its complex structure and implications. We begin by analyzing the equation's fundamental components, breaking it down into a system of real and imaginary parts. Our goal is to understand the role of multipliers, among other things, in formulating conservation laws and the associated system of equations.

2.1 The Multiplier approach

The Schrödinger equation with cubic nonlinearity is given by

$$q_t + q_{xx} + q|q|^2 = 0. \tag{2.47}$$

This is a well-known nonlinear PDE arising in soliton theory and nonlinear wave propagation. Letting $q = u + iv$, (2.47) splits into the system

$$\begin{aligned} eqt1 &= u_t + v_{xx} + v(u^2 + v^2) = 0, \\ eqt2 &= -v_t + u_{xx} + u(u^2 + v^2) = 0. \end{aligned} \tag{2.48}$$

The dependence of the multipliers Q and P on first-order derivatives leads to

$$Q = f(x, t, u, v, u_x, v_x)$$

and

$$P = g(x, t, u, v, u_x, v_x)$$

with the Euler operator annihilating the total divergence to give

$$\frac{\delta}{\delta(u, v)}(f * eqt1 + g * eqt2) = 0. \tag{2.49}$$

The system of two equations arising from (2.49) is

$$
\begin{aligned}
&-\left(\tfrac{\partial}{\partial v}f\right)v_t-\left(\tfrac{\partial}{\partial v_x}f\right)v_{xt}-\left(\tfrac{\partial}{\partial v_t}f\right)v_{tt}+\left(\tfrac{\partial}{\partial v}g\right)v_{xx}+2fuv+\left(-u^3-u\,v^2+u_{xx}+v_t\right)\left(\tfrac{\partial^2}{\partial x\partial u_x}g\right)\\
&-\tfrac{\partial}{\partial t}f+\left(-u^2v_x-2uvu_x-3v^2v_x-2u_{xt}-v_{xxx}\right)\left(\tfrac{\partial}{\partial u_x}f\right)+\left(\tfrac{\partial}{\partial v_x}g\right)v_{xxx}+\left(\tfrac{\partial^2}{\partial v^2}g\right)v_x^2+\left(\tfrac{\partial^2}{\partial v_x^2}g\right)v_{xx}^2\\
&+\left(\tfrac{\partial^2}{\partial v_t^2}g\right)v_{xt}^2+\left(\tfrac{\partial^2}{\partial u^2}g\right)u_x^2+\left(-u^3-u\,v^2-u_{xx}+v_t\right)\left(\tfrac{\partial^2}{\partial t\partial u_t}g\right)+\left(-v\,u^2-v^3-u_t-v_{xx}\right)\left(\tfrac{\partial^2}{\partial x\partial u_x}f\right)\\
&+\left(-v\,u^2-v^3-u_t-v_{xx}\right)\left(\tfrac{\partial^2}{\partial t\partial u_t}f\right)+\left(-u^3u_t-u\,v^2u_t-u_tu_{xx}+u_tv_t+2u_xu_{xt}\right)\left(\tfrac{\partial^2}{\partial u\partial u_t}g\right)\\
&+2\left(\tfrac{\partial^2}{\partial u\partial x}g\right)u_x+2\left(\tfrac{\partial^2}{\partial x\partial u_t}g\right)u_{xt}+2\left(\tfrac{\partial^2}{\partial v\partial x}g\right)v_x+2\left(\tfrac{\partial^2}{\partial x\partial v_x}g\right)v_{xx}+2\left(\tfrac{\partial^2}{\partial x\partial v_t}g\right)v_{xt}\\
&-v_{xx}\left(u^3+u\,v^2-u_{xx}-v_t\right)\left(\tfrac{\partial^2}{\partial u_x\partial v_x}g\right)-v_{xt}\left(u^3+u\,v^2-u_{xx}-v_t\right)\left(\tfrac{\partial^2}{\partial u_x\partial v_t}g\right)\\
&-u_x\left(v\,u^2+v^3+u_t+v_{xx}\right)\left(\tfrac{\partial^2}{\partial u\partial u_x}f\right)-u_t\left(v\,u^2+v^3+u_t+v_{xx}\right)\left(\tfrac{\partial^2}{\partial u\partial u_t}f\right)\\
&-v_x\left(v\,u^2+v^3+u_t+v_{xx}\right)\left(\tfrac{\partial^2}{\partial v\partial u_x}f\right)-v_t\left(v\,u^2+v^3+u_t+v_{xx}\right)\left(\tfrac{\partial^2}{\partial v\partial u_t}f\right)\\
&-u_x\left(u^3+u\,v^2-u_{xx}-v_t\right)\left(\tfrac{\partial^2}{\partial u\partial u_x}g\right)-v_x\left(u^3+u\,v^2-u_{xx}-v_t\right)\left(\tfrac{\partial^2}{\partial v\partial u_x}g\right)\\
&-u_{xx}\left(u^3+u\,v^2-v_t\right)\left(\tfrac{\partial^2}{\partial u_x^2}g\right)-2u_{xt}\left(v\,u^2+v^3+u_t+v_{xx}\right)\left(\tfrac{\partial^2}{\partial u_t\partial u_x}f\right)\\
&-2u_{xt}\left(u^3+u\,v^2-v_t\right)\left(\tfrac{\partial^2}{\partial u_t\partial u_x}g\right)+2\left(\tfrac{\partial^2}{\partial v\partial v_x}g\right)v_xv_{xx}+2\left(\tfrac{\partial^2}{\partial u\partial v}g\right)u_xv_x\\
&+2\left(\tfrac{\partial^2}{\partial u\partial v_x}g\right)u_xv_{xx}+2\left(\tfrac{\partial^2}{\partial v_t\partial v_x}g\right)v_{xx}v_{xt}+2\left(\tfrac{\partial^2}{\partial u\partial v_t}g\right)u_xv_{xt}+2\left(\tfrac{\partial^2}{\partial v\partial v_t}g\right)v_xv_{xt}\\
&-u_{xx}\left(v\,u^2+v^3+u_t+v_{xx}\right)\left(\tfrac{\partial^2}{\partial u_x^2}f\right)-v_{xx}\left(v\,u^2+v^3+u_t+v_{xx}\right)\left(\tfrac{\partial^2}{\partial u_x\partial v_x}f\right)\\
&-v_{xt}\left(v\,u^2+v^3+u_t+v_{xx}\right)\left(\tfrac{\partial^2}{\partial u_x\partial v_t}f\right)-u_{tt}\left(v\,u^2+v^3+u_t+v_{xx}\right)\left(\tfrac{\partial^2}{\partial u_t^2}f\right)\\
&-v_{xt}\left(v\,u^2+v^3+u_t+v_{xx}\right)\left(\tfrac{\partial^2}{\partial u_t\partial v_x}f\right)-v_{tt}\left(v\,u^2+v^3+u_t+v_{xx}\right)\left(\tfrac{\partial^2}{\partial u_t\partial v_t}f\right)\\
&+\left(-u^3v_t-u\,v^2v_t+2u_{xt}v_x-u_{xx}v_t+v_t^2\right)\left(\tfrac{\partial^2}{\partial v\partial u_t}g\right)\\
&+\left(-u^3v_{tt}-u\,v^2v_{tt}+2u_{xt}v_{xt}-u_{xx}v_{tt}+v_tv_{tt}\right)\left(\tfrac{\partial^2}{\partial u_t\partial v_t}g\right)\\
&+\left(-u^3v_{xt}-u\,v^2v_{xt}+2u_{xt}v_{xx}+v_{xt}\left(v_t-u_{xx}\right)\right)\left(\tfrac{\partial^2}{\partial u_t\partial v_x}g\right)\\
&+\left(-u^3u_{tt}-u\,v^2u_{tt}-u_{tt}u_{xx}+u_{tt}v_t+u_{xt}^2\right)\left(\tfrac{\partial^2}{\partial u_t^2}g\right)+\left(\tfrac{\partial}{\partial v_t}g\right)v_{xxt}\\
&+g\left(3u^2+v^2\right)+\left(v\,u^2+v^3+v_{xx}\right)\left(\tfrac{\partial}{\partial u}f\right)+\left(-u^2v_t-2uvu_t-3v^2v_t-v_{xxt}-2u_{tt}\right)\left(\tfrac{\partial}{\partial u_t}f\right)\\
&+\left(-3u^2u_x-2uvv_x-v^2u_x+v_{xt}\right)\left(\tfrac{\partial}{\partial u_x}g\right)+\left(-3u^2u_t-2uvv_t-v^2u_t+v_{tt}\right)\left(\tfrac{\partial}{\partial u_t}g\right)\\
&+\left(u^3+u\,v^2+2u_{xx}-v_t\right)\left(\tfrac{\partial}{\partial u}g\right)+\tfrac{\partial^2}{\partial x^2}g=0
\end{aligned}
\tag{2.50}
$$

and

$$
\begin{aligned}
&-\left(\tfrac{\partial}{\partial v}f\right)v_t-\left(\tfrac{\partial}{\partial v_x}f\right)v_{xt}-\left(\tfrac{\partial}{\partial v_t}f\right)v_{tt}+\left(\tfrac{\partial}{\partial v}g\right)v_{xx}+2fuv\\
&+\left(-u^3-u\,v^2+u_{xx}+v_t\right)\left(\tfrac{\partial^2}{\partial x\partial u_x}g\right)-\tfrac{\partial}{\partial t}f+\left(-u^2v_x-2uvu_x-3v^2v_x-2u_{xt}-v_{xxx}\right)\left(\tfrac{\partial}{\partial u_x}f\right)\\
&+\left(\tfrac{\partial}{\partial v_x}g\right)v_{xxx}+\left(\tfrac{\partial^2}{\partial v^2}g\right)v_x^2+\left(\tfrac{\partial^2}{\partial v_x^2}g\right)v_{xx}^2+\left(\tfrac{\partial^2}{\partial v_t^2}g\right)v_{xt}^2+\left(\tfrac{\partial^2}{\partial u^2}g\right)u_x^2\\
&+\left(-u^3-u\,v^2-u_{xx}+v_t\right)\left(\tfrac{\partial^2}{\partial t\partial u_t}g\right)+\left(-v\,u^2-v^3-u_t-v_{xx}\right)\left(\tfrac{\partial^2}{\partial x\partial u_x}f\right)\\
&+\left(-v\,u^2-v^3-u_t-v_{xx}\right)\left(\tfrac{\partial^2}{\partial t\partial u_t}f\right)+\left(-u^3u_t-u\,v^2u_t-u_tu_{xx}+u_tv_t+2u_xu_{xt}\right)\left(\tfrac{\partial^2}{\partial u\partial u_t}g\right)\\
&+2\left(\tfrac{\partial^2}{\partial u\partial x}g\right)u_x+2\left(\tfrac{\partial^2}{\partial x\partial u_t}g\right)u_{xt}+2\left(\tfrac{\partial^2}{\partial v\partial x}g\right)v_x+2\left(\tfrac{\partial^2}{\partial x\partial v_x}g\right)v_{xx}+2\left(\tfrac{\partial^2}{\partial x\partial v_t}g\right)v_{xt}\\
&-v_{xx}\left(u^3+u\,v^2-u_{xx}-v_t\right)\left(\tfrac{\partial^2}{\partial u_x\partial v_x}g\right)-v_{xt}\left(u^3+u\,v^2-u_{xx}-v_t\right)\left(\tfrac{\partial^2}{\partial u_x\partial v_t}g\right)\\
&-u_x\left(v\,u^2+v^3+u_t+v_{xx}\right)\left(\tfrac{\partial^2}{\partial u\partial u_x}f\right)-u_t(v\,u^2+v^3+u_t)\\
&+v_{xx}\left(\tfrac{\partial^2}{\partial u\partial u_t}f\right)-v_x\left(v\,u^2+v^3+u_t+v_{xx}\right)\left(\tfrac{\partial^2}{\partial v\partial u_x}f\right)-v_t\left(v\,u^2+v^3+u_t+v_{xx}\right)\left(\tfrac{\partial^2}{\partial v\partial u_t}f\right)\\
&-u_x\left(u^3+u\,v^2-u_{xx}-v_t\right)\left(\tfrac{\partial^2}{\partial u\partial u_x}g\right)-v_x(u^3+u\,v^2\\
&-u_{xx}-v_t)\left(\tfrac{\partial^2}{\partial v\partial u_x}g\right)-u_{xx}\left(u^3+u\,v^2-v_t\right)\left(\tfrac{\partial^2}{\partial u_x^2}g\right)\\
&-2u_{xt}\left(v\,u^2+v^3+u_t+v_{xx}\right)\left(\tfrac{\partial^2}{\partial u_t\partial u_x}f\right)-2u_{xt}\left(u^3+u\,v^2-v_t\right)\left(\tfrac{\partial^2}{\partial u_t\partial u_x}g\right)+2\left(\tfrac{\partial^2}{\partial v\partial v_x}g\right)v_xv_{xx}\\
&+2\left(\tfrac{\partial^2}{\partial u\partial v}g\right)u_xv_x+2\left(\tfrac{\partial^2}{\partial u\partial v_x}g\right)u_xv_{xx}+2\left(\tfrac{\partial^2}{\partial v_t\partial v_x}g\right)v_{xx}v_{xt}+2\left(\tfrac{\partial^2}{\partial u\partial v_t}g\right)u_xv_{xt}\\
&+2\left(\tfrac{\partial^2}{\partial v\partial v_t}g\right)v_xv_{xt}-u_{xx}\left(v\,u^2+v^3+u_t+v_{xx}\right)\left(\tfrac{\partial^2}{\partial u_x^2}f\right)-v_{xx}\left(v\,u^2+v^3+u_t+v_{xx}\right)\left(\tfrac{\partial^2}{\partial u_x\partial v_x}f\right)\\
&-v_{xt}\left(v\,u^2+v^3+u_t+v_{xx}\right)\left(\tfrac{\partial^2}{\partial u_x\partial v_t}f\right)-u_{tt}\left(v\,u^2+v^3+u_t+v_{xx}\right)\left(\tfrac{\partial^2}{\partial u_t^2}f\right)-v_{xt}(v\,u^2+v^3+u_t\\
&+v_{xx}\left(\tfrac{\partial^2}{\partial u_t\partial v_x}f\right)-v_{tt}\left(v\,u^2+v^3+u_t+v_{xx}\right)\left(\tfrac{\partial^2}{\partial u_t\partial v_t}f\right)\\
&+\left(-u^3v_t-u\,v^2v_t+2u_{xt}v_x-u_{xx}v_t+v_t^2\right)\left(\tfrac{\partial^2}{\partial v\partial u_t}g\right)\\
&+\left(-u^3v_{tt}-u\,v^2v_{tt}+2u_{xt}v_{xt}-u_{xx}v_{tt}+v_tv_{tt}\right)\left(\tfrac{\partial^2}{\partial u_t\partial v_t}g\right)\\
&+\left(-u^3v_{xt}-u\,v^2v_{xt}+2u_{xt}v_{xx}+v_{xt}\left(v_t-u_{xx}\right)\right)\left(\tfrac{\partial^2}{\partial u_t\partial v_x}g\right)\\
&+\left(-u^3u_{tt}-u\,v^2u_{tt}-u_{tt}u_{xx}+u_{tt}v_t+u_{xt}^2\right)\left(\tfrac{\partial^2}{\partial u_t^2}g\right)+\left(\tfrac{\partial}{\partial v_t}g\right)v_{xxt}\\
&+g(3u^2+v^2)+\left(v\,u^2+v^3+v_{xx}\right)\left(\tfrac{\partial}{\partial u}f\right)+\left(-u^2v_t-2uvu_t-3v^2v_t-v_{xxt}-2u_{tt}\right)\left(\tfrac{\partial}{\partial u_t}f\right)\\
&+\left(-3u^2u_x-2uvv_x-v^2u_x+v_{xt}\right)\left(\tfrac{\partial}{\partial u_x}g\right)+\left(-3u^2u_t-2uvv_t-v^2u_t+v_{tt}\right)\left(\tfrac{\partial}{\partial u_t}g\right)\\
&+\left(u^3+u\,v^2+2u_{xx}-v_t\right)\left(\tfrac{\partial}{\partial u}g\right)+\tfrac{\partial^2}{\partial x^2}g=0.
\end{aligned}
\tag{2.51}
$$

Splitting by second derivatives and higher, we get an overdetermined system of pdes

whose solution yields

$$\begin{aligned} f &= \tfrac{1}{2}(-xc_1 - 2c_3)u + c_1 v_x t + c_4 v_x + c_2 v_t, \\ g &= c_1 u_x t + \tfrac{1}{2}(xc_1 + 2c_3)v + c_4 u_x + c_2 u_t. \end{aligned} \tag{2.52}$$

The multipliers are then enumerated as

$$\begin{aligned} Q_1 &= -\tfrac{1}{2}xu + tv_x, & P_1 &= \tfrac{1}{2}xv + tu_x, \\ Q_2 &= v_t, & P_2 &= u_t, \\ Q_3 &= -u, & P_3 &= v, \\ Q_4 &= v_x, & P_4 &= u_x. \end{aligned} \tag{2.53}$$

Each pair (Q_i, P_i) leads to a conserved vector (T_i^t, T_i^x), viz.,

$$\begin{aligned} T_1^t &= -\tfrac{1}{2}vu_x t - \tfrac{1}{4}x\,v^2 - \tfrac{1}{4}x\,u^2 + \tfrac{1}{2}uv_x t, & T_1^x &= \tfrac{t\,u^4}{4} + \tfrac{t\,u^2v^2}{2} + \tfrac{t\,v^4}{4} - \tfrac{utv_t}{2} + \tfrac{u_x^2 t}{2} \\ & & & + \tfrac{vtu_t}{2} + \tfrac{v_x^2 t}{2} - \tfrac{v_x xu}{2} + \tfrac{u_x xv}{2}, \\ T_2^t &= \tfrac{1}{4}u^4 + \tfrac{1}{2}u^2v^2 + \tfrac{1}{4}v^4 + \tfrac{1}{2}uu_{xx} + \tfrac{1}{2}vv_{xx}, & T_2^x &= \tfrac{1}{2}uu_{xt} - \tfrac{1}{2}u_t u_x + \tfrac{1}{2}v_{xt}v - \tfrac{1}{2}v_t v_x, \\ T_3^t &= \tfrac{1}{2}(u^2 + v^2), & T_3^x &= uv_x - vu_x, \\ T_4^t &= \tfrac{1}{2}(uv_x - vu_x), & T_4^x &= \tfrac{1}{4}u^4 + \tfrac{1}{2}u^2v^2 + \tfrac{1}{4}v^4 \\ & & & - \tfrac{1}{2}uv_t + \tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_t v + \tfrac{1}{2}v_x^2. \end{aligned} \tag{2.54}$$

In terms of q, these are, correspondingly for the scalar equation (2.47),

$$\begin{aligned} \Phi_1^t &= -\tfrac{x}{4}|q|^2 - \tfrac{t}{2}\Im(q\bar{q}_x), & \Phi_1^x &= \tfrac{t}{4}|q|^4 + t\Im(q\bar{q}_x)|q_x|^2 + \tfrac{x}{2}\Im(q\bar{q}_x), \\ \Phi_2^t &= \tfrac{1}{4}|q|^4 + \tfrac{1}{2}\Re(q\bar{q}_{xx}), & \Phi_2^x &= \tfrac{1}{2}\Re(q\bar{q}_{xt}) - \tfrac{1}{2}\Re(q_t\bar{q}_x), \\ \Phi_3^t &= \tfrac{1}{2}|q|^2, & \Phi_3^x &= -\Im(q\bar{q}_x), \\ \Phi_4^t &= -\tfrac{1}{2}\Im(q\bar{q}_x), & \Phi_4^x &= \tfrac{1}{4}|q|^4 + \tfrac{1}{2}|q_x|^2 + \tfrac{1}{2}\Im(q\bar{q}_t), \end{aligned} \tag{2.55}$$

where $\Re(.)$ and $\Im(.)$ are the real and imaginary parts of what follows.

2.2 The Lagrangian approach

The system (2.48), can be shown to have Lagrangian (see[22])

$$L = vu_t - uv_t - \tfrac{1}{2}u_x^2 - \tfrac{1}{2}v_x^2 + 4(u^4 + v^4 + 2u^2v^2) \tag{2.56}$$

so that a Lagrangian for (2.47) is

$$L = \mathcal{I}(qq_t^*) - \tfrac{1}{2}|q_x|^2 + 4|q|^4. \tag{2.57}$$

Although the Lagrangian was stated, Noether's theorem was not applied in this chapter, since it was already briefly introduced in Chapter 1.

2.3 Comparison of the Two Methods

Although Noether's theorem applies only when a Lagrangian exists, the multiplier approach seamlessly extends to non-Lagrangian equations and allows higher-order or mixed multipliers. The multiplier equations can also be solved by computer algebra (e.g., Maple), offering a systematic and verifiable route to discovering all conservation laws up to a chosen order. For more information on the comparison of methods, see [20].

2.4 Conclusion

For the cubic equation Schrödinger, both the multiplier and the Noether approach yield identical sets of conserved quantities. The multiplier method, however, demonstrates a clear computational advantage: it does not require a variational formulation

and can be applied to systems where such a Lagrangian is unknown. These features motivate its use in the more general, possibly non-Lagrangian systems studied in the next chapter.

3 Chapter 3 - The diffusion and Klein-Gordon systems

In ecological systems, various species influence one another through interactions, while in chemical reactions, different substances combine to form new compounds. These processes can be effectively modeled using differential equation systems. For instance, reaction-diffusion equations can be formulated to describe spatial and temporal dynamics. Let $u(x,t)$ and $v(x,t)$ represent the population densities of two interacting species or the concentrations of two reacting chemicals.

A system of Diffusion and Wave equations with 'linear external force (see [25], pages 33-46) is, respectively, given by

$$\begin{aligned} u_t - u_{xx} + au + bv = 0, \\ v_t - v_{xx} + cv + du = 0 \end{aligned} \tag{3.58}$$

and

$$\begin{aligned} u_{tt} - u_{xx} + au + bv = 0, \\ v_{tt} - v_{xx} + cv + du = 0. \end{aligned} \tag{3.59}$$

The latter (3.59) is equivalent to the scalar Klein-Gordon equation.

3.1 Symmetry generators

(i). For the symmetries $X = \xi\partial_t + \tau\partial_t + \phi\partial_u + \zeta\partial_v$ of (3.58), $X^{[2]}$ on the system (3.58) leads to the system

$$
\begin{aligned}
&2\left(\tfrac{\partial^2}{\partial u\partial v}\tau\right)u_xu_tv_x+\left(\tfrac{\partial^2}{\partial u^2}\tau\right)u_x^2u_t+\left(\tfrac{\partial^2}{\partial v^2}\tau\right)u_tv_x^2+\left(\tfrac{\partial^2}{\partial v^2}\xi\right)u_xv_x^2-\left(\tfrac{\partial}{\partial v}\tau\right)u_tv_t\\
&-\left(\tfrac{\partial}{\partial u}\xi\right)u_xu_t-\left(\tfrac{\partial}{\partial v}\xi\right)u_xv_t+2\left(\tfrac{\partial}{\partial v}\tau\right)u_{x,t}v_x+2\left(\tfrac{\partial^2}{\partial u\partial x}\tau\right)u_xu_t+2\left(\tfrac{\partial}{\partial v}\xi\right)u_{x,x}v_x+\left(\tfrac{\partial}{\partial v}\tau\right)u_tv_{x,x}\\
&+3\left(\tfrac{\partial}{\partial u}\xi\right)u_xu_{x,x}+2\left(\tfrac{\partial^2}{\partial v\partial x}\tau\right)u_tv_x+2\left(\tfrac{\partial^2}{\partial v\partial x}\xi\right)u_xv_x+2\left(\tfrac{\partial}{\partial u}\tau\right)u_xu_{x,t}+\left(\tfrac{\partial}{\partial u}\tau\right)u_tu_{x,x}\\
&+\left(\tfrac{\partial}{\partial v}\xi\right)u_xv_{x,x}+\left(\tfrac{\partial^2}{\partial x^2}\xi\right)u_x+\left(\tfrac{\partial^2}{\partial x^2}\tau\right)u_t+\left(\tfrac{\partial^2}{\partial u^2}\xi\right)u_x^3-\left(\tfrac{\partial^2}{\partial u^2}\phi\right)u_x^2-\left(\tfrac{\partial^2}{\partial v^2}\phi\right)v_x^2\\
&+v_t\left(\tfrac{\partial}{\partial v}\phi\right)+2\left(\tfrac{\partial^2}{\partial u\partial x}\xi\right)u_x^2-2u_x\left(\tfrac{\partial^2}{\partial u\partial x}\phi\right)-2v_x\left(\tfrac{\partial^2}{\partial v\partial x}\phi\right)+2\left(\tfrac{\partial}{\partial x}\xi\right)u_{x,x}\\
&+2\left(\tfrac{\partial}{\partial x}\tau\right)u_{x,t}-\left(\tfrac{\partial}{\partial t}\tau\right)u_t+u_t\left(\tfrac{\partial}{\partial u}\phi\right)-\left(\tfrac{\partial}{\partial u}\phi\right)u_{x,x}-\left(\tfrac{\partial}{\partial v}\phi\right)v_{x,x}-\left(\tfrac{\partial}{\partial u}\tau\right)u_t^2-\left(\tfrac{\partial}{\partial t}\xi\right)u_x\\
&+\tfrac{\partial}{\partial t}\phi+2\left(\tfrac{\partial^2}{\partial u\partial v}\xi\right)u_x^2v_x-2\left(\tfrac{\partial^2}{\partial u\partial v}\phi\right)u_xv_x+\phi a+\zeta b=0,
\end{aligned}
\tag{3.60}
$$

$$
\begin{aligned}
&-\tfrac{\partial^2}{\partial x^2}\zeta+\left(\tfrac{\partial^2}{\partial x^2}\xi\right)v_x+\left(\tfrac{\partial^2}{\partial x^2}\tau\right)v_t-\left(\tfrac{\partial^2}{\partial u^2}\zeta\right)u_x^2+\left(\tfrac{\partial^2}{\partial v^2}\xi\right)v_x^3-\left(\tfrac{\partial^2}{\partial v^2}\zeta\right)v_x^2\\
&+\tfrac{\partial}{\partial t}\zeta+\left(\tfrac{\partial^2}{\partial v^2}\tau\right)v_x^2v_t+\left(\tfrac{\partial^2}{\partial u^2}\xi\right)u_x^2v_x+\left(\tfrac{\partial^2}{\partial u^2}\tau\right)u_x^2v_t-\left(\tfrac{\partial}{\partial u}\tau\right)u_tv_t-\left(\tfrac{\partial}{\partial v}\xi\right)v_xv_t\\
&-\left(\tfrac{\partial}{\partial u}\xi\right)u_tv_x+2\left(\tfrac{\partial}{\partial v}\tau\right)v_xv_{x,t}+2\left(\tfrac{\partial}{\partial u}\xi\right)u_xv_{x,x}+3\left(\tfrac{\partial}{\partial v}\xi\right)v_xv_{x,x}\\
&+2\left(\tfrac{\partial^2}{\partial u\partial v}\tau\right)u_xv_xv_t+2\left(\tfrac{\partial^2}{\partial u\partial x}\xi\right)u_xv_x+2\left(\tfrac{\partial^2}{\partial u\partial x}\tau\right)u_xv_t+2\left(\tfrac{\partial^2}{\partial v\partial x}\tau\right)v_xv_t+2\left(\tfrac{\partial^2}{\partial u\partial v}\xi\right)u_xv_x^2\\
&-2\left(\tfrac{\partial^2}{\partial u\partial v}\zeta\right)u_xv_x+\left(\tfrac{\partial}{\partial v}\tau\right)v_tv_{x,x}+\left(\tfrac{\partial}{\partial u}\xi\right)u_{x,x}v_x+2\left(\tfrac{\partial}{\partial u}\tau\right)u_xv_{x,t}+\left(\tfrac{\partial}{\partial u}\tau\right)u_{x,x}v_t+2\left(\tfrac{\partial^2}{\partial v\partial x}\xi\right)v_x^2\\
&-2u_x\left(\tfrac{\partial^2}{\partial u\partial x}\zeta\right)-2v_x\left(\tfrac{\partial^2}{\partial v\partial x}\zeta\right)+\phi d+\zeta c-\left(\tfrac{\partial}{\partial u}\zeta\right)u_{x,x}-\left(\tfrac{\partial}{\partial v}\zeta\right)v_{x,x}+2\left(\tfrac{\partial}{\partial x}\xi\right)v_{x,x}+2\left(\tfrac{\partial}{\partial x}\tau\right)v_{x,t}\\
&-\left(\tfrac{\partial}{\partial v}\tau\right)v_t^2-\left(\tfrac{\partial}{\partial t}\xi\right)v_x-\left(\tfrac{\partial}{\partial t}\tau\right)v_t+u_t\left(\tfrac{\partial}{\partial u}\zeta\right)+v_t\left(\tfrac{\partial}{\partial v}\zeta\right)=0.
\end{aligned}
\tag{3.61}
$$

Substituting (3.58), viz., u_t by $(u_{xx} - au - bv)$ and v_t by $(v_{xx} - cv - du)$ into (3.60) and (3.61) and separating by monomials, lead to the following system,

$$\begin{aligned}
&2a\tau_{uv} - 2bv\tau_v = 0,\\
&2\tau_{uv} = 0,\\
&\tau_{vv} - a\tau_{uu} = 0,\\
&\xi_{vv} = 0,\\
&-au\tau_v - bv\tau_v = 0,\\
&-cv\tau_v - du\tau_v = 0,\\
&\tau_v = 0,\\
&-\xi_u - 2\tau_{ux} = 0,\\
&2(-au - bv)\tau_{ux} - au\xi_u - bv\xi_u - cu\xi_v - du\xi_v + 3\xi_u u_{x,x} = 0,\\
&2\tau u_{x,t} + 2\xi_v u_{x,x} + 2\tau_{ux}(-au - bv) = 0.
\end{aligned} \tag{3.62}$$

Among others, we obtain the following symmetries as a result,

$$\begin{aligned}
X_1 &= \partial_t,\\
X_2 &= \partial_x,\\
X_3 &= u\partial_u + v\partial_v,\\
X_4 &= t\partial_x - \tfrac{xu}{2} - \tfrac{xv}{2}\partial_v,\\
X_5 &= v\partial_u + \tfrac{(-a+c)v+du}{b}\partial_v,\\
X_6 &= \tfrac{x}{2}\partial_x + t\partial_t + (-atv - btv)\partial_u - t(vc + du)\partial_v,\\
X_7 &= \tfrac{tx}{2}\partial_x + \tfrac{t^2}{2}\partial_t - (\tfrac{(-4at^2 - x^2 - 2t)u}{8} - \tfrac{bt^2v}{2})\partial_u + (\tfrac{(-4ct^2 - x^2 - 2t)v}{8} - \tfrac{t^2ud}{2})\partial_v,\\
X_8 &= e^{\frac{\sqrt{2a+2c-2A}\,x}{2}}\partial_u - \tfrac{(a+\sqrt{2A}-c)e^{-\frac{\sqrt{2a+2c-2A}\,x}{2}}}{2b}\partial_v,
\end{aligned} \tag{3.63}$$

where $A = \sqrt{a^2 - 2ac + 4bd + c^2}$ and $X^{[2]}$ is the second prolongation.

The appearance of square roots implies that the discriminant A must be real. Thus, parameter values must be chosen to ensure $A^2 \geq 0$.

(ii). For the symmetries $X = \xi\partial_t + \tau\partial_t + \phi\partial_u + \zeta\partial_v$ of (3.59), $X^{[2]}$ on the system (3.59) leads to, among others, the symmetries

$$\begin{aligned}
&X_1 = \partial_x,\\
&X_2 = \partial_t,\\
&X_3 = t\partial_x + x\partial_t,\\
&X_4 = u\partial_u + v\partial_v,\\
&X_5 = v\partial_u + \tfrac{(-a+c)v+du}{b}\partial_v,\\
&X_6 = e^{\frac{\sqrt{-2A}-2a-2ct}{2}}\partial_u + \tfrac{(-a+\sqrt{2A}+c)e^{-\frac{\sqrt{-2A-2a-2ct}}{2}}}{2b}\partial_v.
\end{aligned} \tag{3.64}$$

3.2 Conservation Laws

(i) Following the procedure and notation in Section 2.1, the dependence of the multipliers Q and P on first-order derivatives leads to

$$Q = f(x, t, u, v, u_x, v_x)$$

and

$$P = g(x, t, u, v, u_x, v_x)$$

with the Euler operator annihilating the total divergence on (3.58) to give

$$\frac{\delta}{\delta(u, v)}[f(u_t - u_{xx} + au + bv) + g(v_t - v_{xx} + cv + du)] = 0. \tag{3.65}$$

We get

$$
\begin{aligned}
&(-au_x - bv_x - 2u_{xt})(\tfrac{\partial}{\partial u_x}f + (\tfrac{\partial}{\partial u}g(cv + du + v_t - v_{xx}) + gd + fa \\
&+(-v_t - v_{xx})(\tfrac{\partial}{\partial v}f + (-au_t - bv_t - 2u_{tt})(\tfrac{\partial}{\partial u_t}f + (au + bv - 2u_{xx})(\tfrac{\partial}{\partial u}f \\
&-(\tfrac{\partial^2}{\partial v_t^2}fv_{xt}^2 - (\tfrac{\partial^2}{\partial v_x^2}fv_{xx}^2 - (\tfrac{\partial^2}{\partial v^2}fv_x^2 - (\tfrac{\partial^2}{\partial u^2}fu_x^2 - 2(\tfrac{\partial^2}{\partial x\partial v_x}fv_{xx} \\
&-2(\tfrac{\partial^2}{\partial u\partial x}fu_x - 2(\tfrac{\partial^2}{\partial x\partial v_t}fv_{xt} - \tfrac{\partial^2}{\partial x^2}f + (-cv_t - du_t - v_{tt} + v_{xxt})(\tfrac{\partial}{\partial u_t}g \\
&+(-v_{xt} - v_{xxx})(\tfrac{\partial}{\partial v_x}f + (-v_{tt} - v_{xxt})(\tfrac{\partial}{\partial v_t}f + (-cv_x - du_x - v_{xt} + v_{xxx})(\tfrac{\partial}{\partial u_x}g \\
&-\tfrac{\partial}{\partial t}f - 2(\tfrac{\partial^2}{\partial x\partial u_t}fu_{xt} + (-au - bv - u_t + u_{xx})(\tfrac{\partial^2}{\partial t\partial u_t}f + (-auv_t - bvv_t \\
&+(-u_t + u_{xx})v_t - 2u_{xt}v_x)(\tfrac{\partial^2}{\partial v\partial u_t}f + (-auu_t - bvu_t - u_t^2 + u_tu_{xx} \\
&-2u_xu_{xt})(\tfrac{\partial^2}{\partial u\partial u_t}f + (-auv_{tt} - bvv_{tt} - 2u_{xt}v_{xt} - v_{tt}(u_t - u_{xx}))(\tfrac{\partial^2}{\partial u_t\partial v_t}f \\
&+(-2u_{xt}v_{xx} - v_{xt}(au + bv + u_t - u_{xx}))(\tfrac{\partial^2}{\partial u_t\partial v_x}f + (-auu_{tt} - bvu_{tt} - u_tu_{tt} \\
&+u_{tt}u_{xx} - u_{xt}^2)(\tfrac{\partial^2}{\partial u_t^2}f - 2(\tfrac{\partial^2}{\partial v\partial x}fv_x + (-au - bv - u_t - u_{xx})(\tfrac{\partial^2}{\partial x\partial u_x}f \\
&+(-cv - du - v_t + v_{xx})(\tfrac{\partial^2}{\partial t\partial u_t}g + (-cv - du - v_t + v_{xx})(\tfrac{\partial^2}{\partial x\partial u_x}g \\
&-2u_{xt}(au + bv + u_t)(\tfrac{\partial^2}{\partial u_t\partial u_x}f - v_t(cv + du + v_t - v_{xx})(\tfrac{\partial^2}{\partial v\partial u_t}g - u_t(cv \\
&+du + v_t - v_{xx})(\tfrac{\partial^2}{\partial u\partial u_t}g - u_{xx}(au + bv + u_t)(\tfrac{\partial^2}{\partial u_x^2}f - v_x(au + bv + u_t \\
&+u_{xx})(\tfrac{\partial^2}{\partial v\partial u_x}f - v_x(cv + du + v_t - v_{xx})(\tfrac{\partial^2}{\partial v\partial u_x}g - v_{tt}(cv + du + v_t \\
&-v_{xx})(\tfrac{\partial^2}{\partial u_t\partial v_t}g - u_x(au + bv + u_t + u_{xx})(\tfrac{\partial^2}{\partial u\partial u_x}f - 2(\tfrac{\partial^2}{\partial u\partial v}fu_xv_x - 2(\tfrac{\partial^2}{\partial u\partial v_t}fu_xv_{xt} \\
&-2(\tfrac{\partial^2}{\partial v\partial v_t}fv_xv_{xt} - 2(\tfrac{\partial^2}{\partial u\partial v_x}fu_xv_{xx} - 2(\tfrac{\partial^2}{\partial v_t\partial v_x}fv_{xx}v_{xt} - 2(\tfrac{\partial^2}{\partial v\partial v_x}fv_xv_{xx} - u_x(cv + du \\
&+v_t - v_{xx})(\tfrac{\partial^2}{\partial u\partial u_x}g - v_{xx}(au + bv + u_t + u_{xx})(\tfrac{\partial^2}{\partial u_x\partial v_x}f - 2u_{xt}(cv + du + v_t \\
&-v_{xx})(\tfrac{\partial^2}{\partial u_t\partial u_x}g - u_{xx}(cv + du + v_t - v_{xx})(\tfrac{\partial^2}{\partial u_x^2}g - v_{xx}(cv + du + v_t - v_{xx})(\tfrac{\partial^2}{\partial u_x\partial v_x}g \\
&-v_{xt}(cv + du + v_t - v_{xx})(\tfrac{\partial^2}{\partial u_t\partial v_x}g - v_{xt}(cv + du + v_t - v_{xx})(\tfrac{\partial^2}{\partial u_x\partial v_t}g - u_{tt}(cv + du \\
&+v_t - v_{xx})(\tfrac{\partial^2}{\partial u_t^2}g - v_{xt}(au + bv + u_t + u_{xx})(\tfrac{\partial^2}{\partial u_x\partial v_t}f
\end{aligned}
\tag{3.66}
$$

and

$$
\begin{aligned}
&(-au_t - bv_t - u_{tt} + u_{xxt})(\tfrac{\partial}{\partial v_t} f) + (-au_x - bv_x - u_{xt} + u_{xxx})(\tfrac{\partial}{\partial v_x} f) + (\tfrac{\partial}{\partial v} f)(au + bv \\
&+u_t - u_{xx}) + (-cv_t - du_t - 2v_{tt})(\tfrac{\partial}{\partial v_t} g) + fb + gc + (cv + du - 2v_{xx})(\tfrac{\partial}{\partial v} g) + (-u_{xt} \\
&-u_{xxx})(\tfrac{\partial}{\partial u_x} g) + (-u_{tt} - u_{xxt})(\tfrac{\partial}{\partial u_t} g) + (-u_{xx} - u_t)(\tfrac{\partial}{\partial u} g) + (-cv_x - du_x - 2v_{xt})(\tfrac{\partial}{\partial v_x} g) \\
&-2(\tfrac{\partial^2}{\partial u \partial x} g)u_x - 2(\tfrac{\partial^2}{\partial v \partial x} g)v_x + (-duu_t - cu_t v + (-v_t + v_{xx})u_t - 2u_x v_{xt})(\tfrac{\partial^2}{\partial u \partial v_t} g) \\
&+(-au - bv - u_t + u_{xx})(\tfrac{\partial^2}{\partial x \partial v_x} f) - (\tfrac{\partial^2}{\partial u_t^2} g)u_{xt}^2 + (-cvv_t - duv_t - v_t^2 + v_t v_{xx} - 2v_x v_{xt})(\tfrac{\partial^2}{\partial v \partial v_t} g) \\
&-(\tfrac{\partial^2}{\partial u^2} g)u_x^2 - 2(\tfrac{\partial^2}{\partial x \partial u_t} g)u_{xt} + (-au - bv - u_t + u_{xx})(\tfrac{\partial^2}{\partial t \partial v_t} f) + (-cv - du - v_t - v_{xx})(\tfrac{\partial^2}{\partial x \partial v_x} g) \\
&-(\tfrac{\partial^2}{\partial v^2} g)v_x^2 - (\tfrac{\partial^2}{\partial u_x^2} g)u_{xx}^2 - 2(\tfrac{\partial^2}{\partial x \partial u_x} g)u_{xx} + (-cvv_{tt} - duv_{tt} - v_t v_{tt} + v_{tt} v_{xx} - v_{xt}^2)(\tfrac{\partial^2}{\partial v_t^2} g) \\
&-2(\tfrac{\partial^2}{\partial x \partial v_t} g)v_{xt} + (-cv - du - v_t + v_{xx})(\tfrac{\partial^2}{\partial t \partial v_t} g) \\
&+(-duu_{tt} - cu_{tt} v - 2u_{xt} v_{xt} - u_{tt}(v_t - v_{xx}))(\tfrac{\partial^2}{\partial u_t \partial v_t} g) \\
&+(-2u_{xx} v_{xt} - u_{xt}(cv + du + v_t - v_{xx}))(\tfrac{\partial^2}{\partial u_x \partial v_t} g) - \tfrac{\partial^2}{\partial x^2} g \\
&-\tfrac{\partial}{\partial t} g - 2(\tfrac{\partial^2}{\partial u \partial u_t} g)u_x u_{xt} - v_x(au + bv + u_t - u_{xx})(\tfrac{\partial^2}{\partial v \partial v_x} f) - u_t(au + bv + u_t \\
&-u_{xx})(\tfrac{\partial^2}{\partial u \partial v_t} f) - 2(\tfrac{\partial^2}{\partial u_t \partial u_x} g)u_{xx} u_{xt} - v_x(cv + du + v_t + v_{xx})(\tfrac{\partial^2}{\partial v \partial v_x} g) - v_{xx}(cv + du + v_t)(\tfrac{\partial^2}{\partial v_x^2} g) \\
&-u_{xt}(au + bv + u_t - u_{xx})(\tfrac{\partial^2}{\partial u_t \partial v_x} f) - v_t(au + bv + u_t - u_{xx})(\tfrac{\partial^2}{\partial v \partial v_t} f) - u_x(au + bv + u_t \\
&-u_{xx})(\tfrac{\partial^2}{\partial u \partial v_x} f) - u_x(cv + du + v_t + v_{xx})(\tfrac{\partial^2}{\partial u \partial v_x} g) - 2(\tfrac{\partial^2}{\partial v \partial u_t} g)u_{xt} v_x - u_{xx}(cv + du + v_t \\
&+v_{xx})(\tfrac{\partial^2}{\partial u_x \partial v_x} g) - v_{xx}(au + bv + u_t - u_{xx})(\tfrac{\partial^2}{\partial v_x^2} f) - 2(\tfrac{\partial^2}{\partial u \partial v} g)u_x v_x - v_{tt}(au + bv + u_t - u_{xx})(\tfrac{\partial^2}{\partial v_t^2} f) \\
&-u_{tt}(au + bv + u_t - u_{xx})(\tfrac{\partial^2}{\partial u_t \partial v_t} f) - 2v_{xt}(cv + du + v_t)(\tfrac{\partial^2}{\partial v_t \partial v_x} g) - 2v_{xt}(au + bv + u_t \\
&-u_{xx})(\tfrac{\partial^2}{\partial v_t \partial v_x} f) - 2(\tfrac{\partial^2}{\partial u \partial u_x} g)u_x u_{xx} - u_{xt}(au + bv + u_t - u_{xx})(\tfrac{\partial^2}{\partial u_x \partial v_t} f) - 2(\tfrac{\partial^2}{\partial v \partial u_x} g)u_{xx} v_x \\
&-u_{xt}(cv + du + v_t + v_{xx})(\tfrac{\partial^2}{\partial u_t \partial v_x} g) - u_{xx}(au + bv + u_t - u_{xx})(\tfrac{\partial^2}{\partial u_x \partial v_x} f).
\end{aligned}
\tag{3.67}
$$

The detailed calculations reveal six multipliers, viz.,

$$
\begin{aligned}
(Q_1, P_1) &= (\mathrm{e}^{-\frac{\sqrt{-2A+2a+2c}\,x}{2}}, -\tfrac{\mathrm{e}^{-\frac{\sqrt{-2A+2a+2c}\,x}{2}}(a+A-c)}{2d}), \\
(Q_2, P_2) &= (\mathrm{e}^{\frac{\sqrt{-2A+2a+2c}\,x}{2}}, -\tfrac{\mathrm{e}^{\frac{\sqrt{-2A+2a+2c}\,x}{2}}(a+A-c)}{2d}), \\
(Q_3, P_3) &= (\mathrm{e}^{-\frac{\sqrt{2A+2a+2c}\,x}{2}}, \tfrac{\mathrm{e}^{-\frac{\sqrt{2A+2a+2c}\,x}{2}}(A-a+c)}{2d}), \\
(Q_4, P_4) &= (\mathrm{e}^{\frac{\sqrt{2A+2a+2c}\,x}{2}}, \tfrac{\mathrm{e}^{\frac{\sqrt{2A+2a+2c}\,x}{2}}(A-a+c)}{2d}), \\
(Q_5, P_5) &= (\mathrm{e}^{\frac{ta}{2}}\mathrm{e}^{\frac{tc}{2}}\mathrm{e}^{-\frac{tA}{2}}, \tfrac{\mathrm{e}^{\frac{ta}{2}}\mathrm{e}^{\frac{tc}{2}}\mathrm{e}^{-\frac{tA}{2}}(aA-c)}{2d}), \\
(Q_6, P_6) &= (\mathrm{e}^{\frac{ta}{2}}\mathrm{e}^{\frac{tc}{2}}\mathrm{e}^{\frac{tA}{2}}, \tfrac{\mathrm{e}^{\frac{ta}{2}}\mathrm{e}^{\frac{tc}{2}}\mathrm{e}^{\frac{tA}{2}}(A-a+c)}{2d}),
\end{aligned}
\tag{3.68}
$$

Recall that $A = \sqrt{a^2 - 2ac + 4bd + c^2}$.

The corresponding conserved vectors are difficult to calculate even with software.

(ii) Some of the multipliers, similarly, for (3.59) are

$$\begin{aligned}
(Q_1, P_1) &= (\tfrac{(tv_x xv_t)b+(tu_x+xu_t)(a-c)}{b}, tu_x + xu_t),\\
(Q_2, P_2) &= (\tfrac{(a-c)u_x+bv_x}{b}, u_x)\\
(Q_3, P_3) &= (\tfrac{(tu_x+xu_t)d}{b}, tv_x + xv_t)\\
(Q_4, P_4) &= (\tfrac{du_x}{b}, v_x)\\
(Q_5, P_5) &= (\tfrac{du_t}{b}, v_t),\\
(Q_6, P_6) &= \tfrac{(a-c)u_t+bv_t}{b}, u_t\\
(Q_7, P_7) &= (\tfrac{1}{2b}[\mathrm{e}^{-\frac{\sqrt{-2\sqrt{a^2-2ca+4db+c^2}-2a-2c}\,t}{2}}\left(a + \sqrt{a^2 - 2ca + 4db + c^2} - c\right)],\\
&\quad \mathrm{e}^{-\frac{\sqrt{2\sqrt{a^2-2ca+4db+c^2}-2a-2c}\,t}{2}}).
\end{aligned} \tag{3.69}$$

with conserved Lorentz rotation ((Q_3, P_3) associated with X_3), linear momentum ((Q_4, P_4) associated with X_1) and energy ((Q_5, P_5) associated with X_2) flows given by, respectively,

$$\begin{aligned}
(\Phi^t_r, \Phi^x_r) &= (adx\,u^2 + bc\,v^2x + 2bduvx - btvv_{xt} + btv_tv_x - bvxv_{xx} + bxv_t^2\\
&\quad -dtuu_{xt} + dtu_tu_x - duxu_{xx} + dxu_t^2 - bvv_x - duu_x,\\
&\quad adt\,u^2 + bct\,v^2 + 2bdtuv + btvv_{tt} - btv_x^2 + bvxv_{xt} - bxv_tv_x\\
&\quad +dtuu_{tt} - dtu_x^2 + duxu_{xt} - dxu_tu_x + bvv_t + duu_t),\\
(\Phi^t_m, \Phi^x_m) &= (-[bvv_{xt} - bv_tv_x + duu_{xt} - du_tu_x],\\
&\quad ad\,u^2 + bc\,v^2 + 2bduv + bvv_{tt} - bv_x^2 + duu_{tt} - du_x^2),\\
(\Phi^t_e, \Phi^x_e) &= (ad\,u^2 + bc\,v^2 + 2bduv - bvv_{xx} + bv_t^2 - duu_{xx} + du_t^2,\\
&\quad bvv_{xt} - bv_tv_x + duu_{xt} - du_tu_x).
\end{aligned} \tag{3.70}$$

Here, $D_t\Phi^t + D_x\Phi^x$ vanishes along the solutions of (3.59) - Φ^t and Φ^x are the conserved densities and fluxes, respectively.

1. These physical interpretations stem from the standard Noether associations with temporal, spatial, and rotational symmetries, as discussed in classical field theory (see, [22]).

3.3 Conclusion

This chapter explored diffusion and Klein-Gordon systems, highlighting their role in modeling spatial-temporal interactions. Using symmetry analysis, we derived transformation properties and conservation laws that provide insights into the' underlying structures of these systems. Identifying multipliers and conserved quantities improves our understanding of the behavior of the solution. Future research may extend these methods to nonlinear models, using computational approaches to explore more physical and biological applications.

4 Chapter 4 – Conclusions and Future Directions

In this final chapter, we summarize the research's significant contributions and discuss future directions. Our investigation focused on the multiplier approach in the double reduction of differential equations, emphasizing its role in establishing a systematic connection between symmetries and conservation laws for both Lagrangian and non-Lagrangian systems.

4.1 Summary of Contributions

We began by revisiting the classical results of Noether and extending the concept of symmetry invariance to differential equations that lack an underlying variational structure. The theoretical foundation for the connection between symmetries and conservation laws is reviewed in Chapter 1, where important concepts like the Euler operator and the Lie-Bäcklund operator are introduced. This laid the groundwork for a unified treatment of conservation laws via multipliers.

Chapter 2 demonstrated the effectiveness of the multiplier approach in deriving conservation laws for the Schrödinger equation with cubic nonlinearity. We were able to generate conserved vectors, systematically get multipliers, and even tie these results to a comparable Lagrangian formulation by breaking down the complex equation into its real and imaginary components. The detailed computations provided both clarity and robustness to the methodology.

In Chapter 3, the focus shifted to systems that arise in applied contexts, such as reaction-diffusion models and Klein-Gordon-type equations. Through symmetry analysis and the associated multiplier method, we derived conservation laws for

these systems, thereby showcasing the versatility and broad applicability of the multiplier approach in handling systems of differential equations in various physical and ecological models.

4.2 Implications and Future Work

The results obtained in this research underscore the power of the multiplier method as a tool for systematically constructing conservation laws, even for non-Lagrangian problems where classical Noether theory does not directly apply. By establishing clear links between symmetry generators and conserved quantities, our work not only consolidates earlier findings in the literature but also opens new avenues for exploring complex nonlinear phenomena.

Although the title of this dissertation refers to the *double reduction* of differential equations, the explicit reduction process—where a conservation law is combined with a symmetry to lower the dimension or order of a PDE—was not performed. Instead, the focus was on the theoretical identification of multipliers and conserved currents. Carrying out full double-reduction computations would require symbolic or numerical automation, which is proposed as future work. - see [24, 5].

4.3 Final Remarks

In conclusion, the multiplier approach provides a unified and powerful technique for analysing the invariance and conservation properties of differential equations. Its flexibility makes it suitable for both analytical and computational studies. Although this dissertation has focused on deriving and classifying conservation laws, the same framework can be extended to explore exact solutions via double reduction, to test numerical schemes that preserve invariants, and to investigate higher-dimensional systems.

Future developments in symbolic computation are expected to further streamline these analyses, making the multiplier method an indispensable component of modern applied mathematics.

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