Chapter 2

The classical Szegö theory, the Szegö function and Szegö asymptotics off and on τ

Notation

Throughout $d\mu$ will denote a positive, finite Borel measure on τ. $\varphi_n(z) = \varphi_n(d\mu, z)$ will be the corresponding orthonormal polynomial with respect to $d\mu$. If $d\mu$ is absolutely continuous we write $\mu' = w(\theta)$. $Re(f)$ and $Im(f)$ will denote the real and imaginary parts of a function f. We denote by $L^\infty(\tau)$ the space of essentially bounded functions with respect to "meas" on τ and by $L^\infty(d\mu)(\tau)$ the same class with respect to $d\mu$ on τ. We adopt the same convention for $L^P(\tau)$ and $L^P(d\mu)(\tau)$, $1 \leq P < \infty$. Also $\| \, \|_{L^\infty}$, $\| \, \|_{L^\infty(d\mu)}$, $\| \, \|_{L^P}$, $\| \, \|_{L^P(d\mu)}$ will denote the corresponding norms on these spaces. By $f \sim \Sigma$ we mean that f has a Fourier expansion Σ. $C_1, C_2 \ldots$ will denote constants independent of n, z, ξ, w and although numbered the same, they will differ from chapter to chapter. Finally, we explain some terminology which will prove important in what follows.

Remark 2.1

Let $(C_n) \subseteq \mathbf{C} \setminus (0)$ be a sequence and $r \neq 0$. Then

(a) $\lim_{n\to\infty} \dfrac{C_n}{r^n} = d \neq 0$ is called a Strong Asymptotic;

(b) $\lim_{n\to\infty} \dfrac{C_{n+1}}{C_n} = r$ is called a Ratio Asymptotic;

(c) $\lim_{n\to\infty} C_n^{\frac{1}{n}} = r$ is called a nth Root Asymptotic.

We note that it is straightforward to see that (a) implies (b) implies (c) but not conversely.

(d) By uniform convergence of a sequence of functions (f_n) to a function f we mean convergence in the L^∞ or $L^\infty(d\mu)$ norms, and by mean or weak convergence of (f_n) to f we mean convergence in $\| \|_{L^p}$ or $\| \|_{L^p(d\mu)}$.

Section 2A : The classical Szegö Theory

In this section we investigate $\lim_{n\to\infty} \gamma_n(d\mu)$ and give some mean asymptotics for φ_n on τ, under the assumption that $\log w \in L'[0, 2\pi]$. From now on if w is a weight function such that $\log w \in L'[0, 2\pi]$, then we will say that w satisfies Szegö's condition on τ and the corresponding asymptotic will be called a Szegō asymptotic. We note that we will give non Szegō analogues to the above results in Chapter 5.

Lemma 2.2

Let $d\mu(e^{i\theta}) = w(\theta)d\theta$, $w > 0$ $a.e.(meas)$. Then,

$$\lim_{n\to\infty} \gamma_n(d\mu) \le \exp\Big(-\frac{1}{4\pi}\int_0^{2\pi} \log w(\theta)d\theta\Big). \tag{2.1}$$

Moreover,

$$\int_0^{2\pi} \log w(\theta)d\theta < \infty. \tag{2.2}$$

Proof:

Let $\pi_n(z) = \prod_{j=1}^n (z - z_j)$ be a monic polynomial of degree n. Then using

Jensen's inequality with $f(t) = e^t$ we get

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \left|\pi_n(z)\right|^2 d\mu(z) \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left|\pi_n(e^{i\theta})\right|^2 w(\theta)d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f\left(\log\left[\left|\pi_n(e^{i\theta})\right|^2 w(\theta)\right]\right)d\theta \\
&\geq f\left(\int_0^{2\pi} \log\left[\left|\pi_n(e^{i\theta})\right|^2 w(\theta)\right]\frac{d\theta}{2\pi}\right) \\
&= f\left(2\sum_{j=1}^{n}\int_0^{2\pi} \log\left|e^{i\theta} - z_j\right|\frac{d\theta}{2\pi} + \frac{1}{2\pi}\int_0^{2\pi} \log w(\theta)d\theta\right) \\
&\geq f\left(\frac{1}{2\pi}\int_0^{2\pi} \log w(\theta)d\theta\right) \\
&= \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log w(\theta)d\theta\right)
\end{aligned}
$$

[where we used

$$
\int_0^{2\pi} \log\left|e^{i\theta} - z_j\right|d\theta = \begin{cases} 0, & |z_j| \leq 1, \\ 2\pi\log|z_j|, & |z_j| > 1. \end{cases}
$$

which is proved using the mean value theorem for harmonic functions]. Thus we have,

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \left|\pi_n(z)\right|^2 d\mu(z) \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left|\pi_n(e^{i\theta})\right|^2 w(\theta|d\theta \geq \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log w(\theta)d\theta\right). \qquad (2.3)
\end{aligned}
$$

Thus using (1.20) $\gamma_n^{-2} \geq \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log w(\theta)d\theta\right)$ which implies (2.1). To get (2.2) apply (2.3) with $\pi_n = 1$. ■

Remarks:

Throughout this chapter when deriving results for asymptotics of $(\varphi_n)_{n=0}^{\infty}$ we will require a Szegő condition on w. The above inequality will be shown to be an equality. However we still include the case that $\int_0^{2\pi} \log w(\theta)d\theta = -\infty$

that is $\lim_{n\to\infty}\gamma_n$ exists but may equal ∞. It will be shown later that in fact $\log w \in L'[0,2\pi]$ iff $\lim_{n\to\infty}\gamma_n = \gamma < \infty$ and it is this result which will serve as a starting point for us to study Szegō Asymptotics of $(\varphi_n)_{n=0}^{\infty}$ on and off τ. Later in studying non Szegō Asymptotics we shall remove the condition $\log w \in L'[0,2\pi]$ and merely require the weaker Erdōs Turan type condition $w > 0$ a.e. (meas) or the weaker condition on the reflection coefficients $\lim_{n\to\infty}|\Phi_n(0)| = 0$. We will also consider the alternative idea, again due to Paul Nevai, of investigating not the measure directly but the coefficients in the three term recursion formula generating the orthogonal polynomials.

Lemma 2.3

Let $T(\theta)$ be a trigonometric polynomial of degree m such that $T(\theta) \geq 0$, $0 \in [0,2\pi]$. Then there exists an algebraic polynomial π_m of degree m such that $T(\theta) = |\pi_m(e^{i\theta})|^2$, $\theta \in [0,2\pi]$ and such that all the roots of π_m lie in $\{z : 0 < |z| \leq 1\}$ or $\{z : |z| \geq 1\}$.

Proof:

We can write, $T(\theta) = \frac{c_0}{2} + \sum_{j=1}^{m}(c_j \cos j\theta + d_j \sin j\theta)$, where c_j, $d_j \in \mathbb{R}$, $c_m \neq 0$, $d_m \neq 0$. Then letting $z = e^{i\theta}$,

$$T(\theta) = \frac{c_0}{2} + \sum_{j=1}^{m}\left[\frac{c_j}{2}(z^j + (\bar{z})^{+j}) + \frac{d_j}{2i}(z^j - (\bar{z})^{+j})\right] = \bar{z}^m G(z)$$

where

$$G(z) = \frac{c_0}{2}z^m + \sum_{j=1}^{m}\left(\frac{c_j}{2}(z^{m+j} - z^{m-j}) + \frac{d_j}{2i}(z^{m+j} - z^{m-j})\right)$$

with leading coefficient $\frac{c_m}{2} + \frac{d_m}{2i} \neq 0$ and constant coefficient $\frac{c_m}{2} - \frac{d_m}{2i} \neq 0$. Then for $z = e^{i\theta}$, $z^{-m}G(z) = T(\theta) = \overline{T(\theta)} = z^m\overline{G((\bar{z})^{-1})}$. Thus if z_j is a zero of G, so is $(\bar{z}_j)^{-1}$. Next if $\theta_0 \in [0,2\pi]$ is a zero of $T(\theta)$, as $T(\theta) \geq 0$, θ_0

has even order ℓ say, so we can write $t(\theta) = (\theta - \theta_0)^\ell R(\theta)$ where R is analytic and non zero at θ_0. Set $z_0 = e^{i\theta_0}$. Then for $z = e^{i\theta}$,

$$G(z) = z^m T(\theta) = z^m (z - z_0)^\ell \left(\frac{\theta - \theta_0}{\ell^{i\theta} - \ell^{i\theta_0}} \right)^\ell R(\theta) = (z - z_0)^\ell H(z),$$

H analytic and non zero at z_0. So we have that every zero of T on $[0, 2\pi]$ is an even order zero corresponding to an even order zero of G. So we can write for some $C \neq 0$, k_j a non negative integer, $G(z) = C \prod_{|z_j|<1} (z - z_j)$ $(z - (\overline{z_j})^{-1}) \prod_{|z_j|=1} (z - z_j)^{2k_j}$ where there is a total of $2m$ factors. Then

$$\begin{aligned}
T(\theta) &= (z)^{-m} G(z) \\
&= C \prod_{|z_j|<1} (z - z_j)(1 - (z\overline{z}_j)^{-1}) \prod_{|z_j|=1} (z - z_j)^{k_j} (1 - z_j z^{-1})^{k_j} \\
&= \left(C \Big[\prod_{|z_j|<1} (\overline{z}_j)^{-1} \Big] \Big[\prod_{|z_j|=1} z_j^{k_j} \Big] \right) \left(\prod_{|z_j|<1} (z - z_j)(\overline{z}_j - \overline{z}) \right) \times \\
&\quad \times \left(\prod_{|z_j|=1} (z - z_j)^{k_j} (\overline{z}_j - \overline{z})^{k_j} \right) \\
&= d \left| \prod_{|z_j|<1} (z - z_j) \prod_{|z_j|=1} (z - z_j)^{k_j} \right|^2
\end{aligned}$$

where we used $|z| = 1$ implies $\overline{z} = (z)^{-1}$. Now as $T(\theta) \geq 0$, $d > 0$. So $T(\theta) = \left|\pi_m(e^{i\theta})\right|^2$ where

$$\pi_n(z) = \sqrt{d} \prod_{|z_j|<1} (z - z_j) \prod_{|z_j|=1} (z - z_j)^{k_j}$$

has degree m with all its zeros in $|z| \leq 1$. Then $\pi_m^*(z)$ has all its zeros on $|z| \geq 1$ and $T(\theta) = \left|\pi_m(e^{i\theta})\right|^2 = \left|\pi_m^*(e^{i\theta})\right|^2$. Also $G(0) \neq 0$ which implies $\pi_m(0) \neq 0$. ∎

Lemma 2.4

Let $d\mu(e^{i\theta}) = \frac{d\theta}{T(\theta)}$, $\theta \in [0, 2\pi]$ where $T(\theta)$ is a trigonometric polynomial of degree m with all its zeros in $|z| > 1$ such that $\pi_m(0) > 0$ and $T(\theta) = |\pi_m(e^{i\theta})|^2$, $\theta \in [0, 2\pi]$ as in Lemma 2.3. Then for $n \geq m$,

$$\varphi_n(z) = z^{n-m}\pi_m^*(z), \tag{2.4}$$

$$\gamma_n(d\mu) = \exp\Big(-\frac{1}{4\pi}\int_0^{2\pi} \log(T(\theta))^{-1} d\theta\Big) = \pi_m(0). \tag{2.5}$$

Proof:

Now $z^{n-m}\pi_m^*(z)$ is a polynomial of degree n with all its zeros in $|z| \leq 1$. Also for $z = e^{i\theta}$, $T(\theta) = |\pi_m(z)|^2 = \pi_m(z)\overline{\pi_m(z)} = \pi_m(z)\overline{\pi_m((\bar{z})^{-1})} = \pi_m(z)(z)^{-m}\pi_m^*(z)$. Thus if $S(z)$ is a polynomial of degree $k \leq n-1$ then,

$$\frac{1}{2\pi}\int_0^{2\pi} [z^{n-m}\pi_m^*(z)]\overline{S(z)}d\mu(z)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{z^{n-m}\pi_m^*(z)\overline{S(z)}}{T(\theta)} d\theta (z = e^{i\theta}) = \frac{1}{2\pi i}\int_{|z|=1} \frac{z^{n-k-1}S^*(z)}{\pi_m(z)} dz = 0$$

[by Cauchy's integral theorem as $K + 1 \leq n$ and $\pi_m(z)$ has all its roots in $|z| > 1$].

Thus we have,

$$\frac{1}{2\pi}\int_0^{2\pi} [z^{n-m}\pi_m^*(z)]\overline{S(z)}d\mu(z) = 0. \tag{2.6}$$

Also we have,

$$\frac{1}{2\pi}\int_0^{2\pi} [z^{n-m}\pi_m^*(z)]\overline{[z^{n-m}\pi_m^*(z)]}d\mu(\theta)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{|\pi_m^*(z)|^2}{T(\theta)} d\theta = \frac{1}{2\pi}\int_0^{2\pi} d\theta = 1. \tag{2.7}$$

So by (2.6), (2.7) and uniqueness from Theorem 1.2 $\varphi_n(z) = z^{n-m}\pi_m^*(z)$ provided $\pi_m(0)$ is normalised to be > 0. Next π_m^* has all its zeros in

$|z| < 1$ and has leading coefficient $\pi_m(0)$. Thus we may write $\pi_m^*(z) = \pi_m(0)\prod_{j=1}^m (z - z_j)$ say. Also by (2.4) $\pi_m(0) = \gamma_n(d\mu)$ thus we have

$$\frac{1}{2\pi}\int_0^{2\pi} \log(T(\theta))^{-1} d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log|\pi_m^*(e^{i\theta})|^{-2} d\theta$$
$$= \frac{1}{\pi}\int_0^{2\pi}\left[\log|\pi_m(0)|^{-1} + \sum_{j=1}^m \log|e^{i\theta} - z_j|^{-1}\right] d\theta = 2\log|\pi_m(0)|^{-1}$$

[where we used the mean value theorem as in Lemma 2.2]. Thus,

$$\frac{1}{2\pi}\int_0^{2\pi} \log(T(\theta))^{-1} = 2\log|\pi_m(0)| = 2\log(\gamma_n)^{-1}$$

and so

$$\gamma_n = \exp\left(-\frac{1}{4\pi}\int_0^{2\pi} \log(T(\theta))^{-1} d\theta\right). \blacksquare$$

We can now achieve equality in Lemma 2.2 and prove the famous

<u>Theorem 2.5 (Szegõ 1918 — 1920)</u>

Let $d\mu(e^{i\theta}) = w(\theta)d\theta$, $w > 0$ a.e. (meas) $\theta \in [0, 2\pi]$ then

$$\lim_{n\to\infty} \gamma_n(d\mu) = \exp\left(-\frac{1}{4\pi}\int_0^{2\pi} \log w(\theta) d\theta\right). \tag{2.8}$$

<u>Remarks:</u>

(a) If $\log w \notin L^1[0, 2\pi]$ we interpret the right-hand side as ∞.

(b) We observe that by Lemma 1.6 that not only do we know that $\gamma \leq \infty$ but we know what it is, although not necessarily finite.

<u>Proof:</u>

Let $\gamma = \lim_{n\to\infty} \gamma_n(d\mu)$, $\gamma \leq \infty$ and by (2.1) $\gamma \leq \exp\left(-\frac{1}{4\pi}\int_0^{2\pi} \log w(\theta) d\theta\right)$. Now if $\gamma = \infty$ the proof is complete. So we may assume without loss of generality that $\gamma < \infty$. By Lemma 2.3 choose $T(\theta)$ a trigonometric polynomial of degree $m > 0$ in $[0, 2\pi]$ such that $T(\theta) = |\pi_m(e^{i\theta})|^2$, $\theta \in [0, 2\pi]$, $\pi_m(0) > 0$. Then by Lemma 1.6,

$$\begin{aligned}
\gamma^{-2} \leq \gamma_m^{-2} &= \lambda_{m+1}(d\mu, 0) \\
&= \min_{\deg R \leq m} \frac{1}{2\pi}\int_0^{2\pi} \frac{|R(z)|^2 d\mu(z)}{|R(0)|^2} (z = e^{i\theta}) \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} \frac{|\pi_m(z)|^2 d\mu(z)}{|\pi_m(0)|^2} \\
&= |\pi_m(0)|^{-2} \frac{1}{2\pi}\int_0^{2\pi} T(\theta) w(\theta) d\theta.
\end{aligned}$$

Thus

$$\gamma \geq \exp\left(-\frac{1}{4\pi}\int_0^{2\pi} \log(T(\theta))^{-1} d\theta\right)\left(\frac{1}{2\pi}\int_0^{2\pi} Tw(\theta)d\theta\right)^{-1/2}.$$

So using (2.1) we obtain,

$$\begin{aligned}
\exp\left(-\frac{1}{4\pi}\int_0^{2\pi} \log w(\theta) d\theta\right) \geq \gamma \geq \exp\left(-\frac{1}{4\pi}\int_0^{2\pi} \log(T(\theta))^{-1} d\theta\right) \times \\
\times \left(\frac{1}{2\pi}\int_0^{2\pi} (Tw)(\theta) d\theta\right)^{-1/2}.
\end{aligned}$$

That is,

$$\begin{aligned}
&\exp\left(-\frac{1}{4\pi}\int_0^{2\pi} \log w(\theta) d\theta\right) \\
(2.9) \qquad &\geq \exp\left(-\frac{1}{4\pi}\int_0^{2\pi} \log(T(\theta))^{-1} d\theta\right)\left(\frac{1}{2\pi}\int_0^{2\pi} Tw(\theta)d\theta\right)^{-1/2}
\end{aligned}$$

holds for all positive trigonometric polynomials. So by Weierstrass' approximation theorem and dominated convergence we may extend (2.9) to hold for all T continuous, positive and 2π periodic. Next as the continuous functions are $L^1[0, 2\pi]$ dense in the simple functions (2.9) holds for all simple functions and using monotone convergence, for all positive Borel measurable functions T. Finally since any non negative Borel measurable function T is the limit of the decreasing sequence $\left(T - \frac{1}{n}\right)_{n=1}^{\infty}$ by monotone convergence (2.9) holds for T non negative Borel measurable. Thus if $\log w \in L'[0,2\pi]$, let $T = \frac{1}{w}$ in (2.9) and the result holds, also if $\log w \notin L'[0,2\pi]$ let $T = \frac{1}{w} + \frac{1}{n}$ in (2.9) and use monotone convergence to achieve the result again. ■

We end this section by giving some mean asymptotics for $\varphi_n(z)$ on τ where w satisfies a Szegö condition.

<u>Theorem 2.6</u> (Mean Asymptotics for $\varphi_n(z)$)

Let $d\mu(e^{i\theta}) = w(\theta)d\theta$, $\theta \in [0,2\pi]$, $w > 0$ *a.e.*, $\log w \in L^1[0,2\pi]$. Then for $z = e^{i\theta}$

(2.10) $$K(z) = \sum_{j=0}^{\infty} \varphi_j(z)\overline{\varphi_j(0)} \text{ is defined } a.e.\ (d\mu),\ \theta \in [0,2\pi];$$

(2.11) $$K(0) = \frac{1}{2\pi}\int_0^{2\pi} |K(z)|^2 d\mu(z) = \exp\left(-\frac{1}{2\pi}\int_0^{2\pi} \log w(\theta)d\theta\right) < \infty;$$

(2.12) $$\lim_{n\to\infty}\left\| \frac{K(z)}{(K(0))^{1/2}} - \varphi_n^*(z)\right\|_{L^2(d\mu)}^2 = 0;$$

(2.13) $$\lim_{n\to\infty}\left\| \frac{z^n\overline{K((\bar{z})^{-1})}}{(K(0))^{1/2}} - \varphi_n(z)\right\|_{L^2(d\mu)}^2 = 0.$$

<u>Proof</u>

Now by Lemma 1.6 and Theorem 2.5,

(2.14) $$\sum_{j=0}^{\infty}|\varphi_j(0)|^2 = \lim_{n\to\infty}\gamma_n^2 = \exp\left(-\frac{1}{2\pi}\int_0^{2\pi} \log w(\theta)d\theta\right).$$

Now let $S_\ell(z) = \sum_{j=0}^{\ell} \varphi_j(z)\overline{\varphi_j(0)}$, $\ell \geq 0$. Then if $m > \ell$

$$\begin{aligned}
\|S_m - S_\ell\|^2_{L^2(d\mu)} &= \frac{1}{2\pi}\int_0^{2\pi} |S_\ell(z) - S_m(z)|^2 d\mu(z) \\
&= \frac{1}{2\pi}\int_0^{2\pi} \Big|\sum_{j=\ell+1}^{m} \varphi_j(z)\overline{\varphi_j(0)}\Big|^2 d\mu(z) \\
&= \sum_{j=\ell+1}^{m} |\varphi_j(0)|^2 \quad \text{(by orthonormality)} \\
&\leq \sum_{j=\ell+1}^{\infty} |\varphi_j(0)|^2 \to 0 \quad \text{as } \ell, m \to \infty.
\end{aligned}$$

So $(S_n)_{n=0}^{\infty}$ is a Cauchy sequence in $L^2(d\mu)$ which is complete. So

$$\lim_{n\to\infty} S_n(z) = \sum_{j=0}^{\infty} \varphi_j(z)\overline{\varphi_j(0)}$$

exists in $L^2(d\mu)$ and hence a.e. $(d\mu)$. Now define $K(0)$ by (2.10) and then (2.14) implies (2.11). Next,

$$\begin{aligned}
&\left\|\frac{K(z)}{K(0)^{1/2}} - \varphi_n^*(z)\right\|_{L^2(d\mu)} \\
&= \left\|K(z)\left(\frac{1}{(K(0))^{1/2}} - \frac{1}{\gamma_n}\right) + \frac{1}{\gamma_n}(K(z) - \gamma_n\varphi_n^*(z))\right\|_{L^2(d\mu)} \\
&\leq \|K\|_{L^2(d\mu)}\left|\frac{1}{(K(0))^{1/2}} - \frac{1}{\gamma_n}\right| + \frac{1}{\gamma_n}\|K - S_n\|_{L^2(d\mu)}
\end{aligned}$$

(by 1.19) $\to 0$ as $n \to \infty$ by (2.14) and definition of K and S_n. Thus we have (2.12), and (2.13) follows by taking the star transform of (2.12) with $|z| = 1$. ■

Section 2B : The Szegö function

We begin with some results on conjugate functions and Poisson kernels which we will require for verifying properties of the Szegö function and for the later chapters. We then introduce the Szegö function and prove its intrinsic properties.

Definition 2.7 (Conjugate functions)

Let $f \in L^1[0,2\pi]$ then

$$\tilde{f} = -\frac{1}{2\pi} \lim_{\epsilon \to 0} \int_\epsilon^{2\pi} (f(\theta+t) - f(\theta-t)) \cot \frac{t}{2} dt \tag{2.15}$$

exists a.e. (meas) and is called the conjugate function of f.

Remarks:

The following statements and results are well known and can be found in ([28], [4], [3]).

(a) $f \in L^2[0,2\pi]$ implies $\tilde{f} \in L^2[0,2\pi]$.

(b) $f \sim \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos j\theta + b_j \sin j\theta$ implies $\tilde{f} \sim \sum_{j=1}^{\infty} -b_j \cos j\theta + a_j \sin j\theta$ and both Fourier series are a.e.(meas) Abel Summable to f and $\tilde{f}$ respectively. That is

$$\lim_{r \to 1-} \left(\frac{a_0}{2} + \sum_{j=1}^{\infty} \left(a_j \cos_j \theta_0 + b_j \sin j\theta_0 \right) r^j \right) = f(e^{i\theta_0}) a.e.(meas)$$

and

$$\lim_{r \to 1-} \sum_{j=1}^{\infty} (a_j \sin j\theta_0 - b_j \cos_j \theta_0) r^j = \tilde{f}(\theta_0) a.e.(meas)$$

for $\theta_0 \in [0,2\pi]$ fixed.

(c) The integral in (2.15) is a principle valued integral.

Lemma 2.8 (Fatou)

Let $h\colon [0,2\pi] \to \mathbb{R}$, $h \in L^1[0,2\pi]$, further let

$$Re\left(\frac{1+ze^{-it}}{1-ze^{-it}}\right) = P(r,\theta-t) = \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}, \quad r \in [0,1)$$

$\theta, t \in [0,2\pi]$ be the Poisson kernel of the unit disc. Then if

$$Q[h,r,\theta] = \frac{1}{2\pi} \int_0^{2\pi} h(s) P(r,\theta-s) ds, \; 0 < r < 1, \; \theta \in [0,2\pi]$$

then,

$$\lim_{r \to 1-} \frac{1}{2\pi} \int_0^{2\pi} \Big| Q[h, r, \theta] - h(\theta) \Big| d\theta = 0.$$

Proof:

It is well known that $P(r, s) \geq 0$, $r \in [0, 1)$ and

$$\frac{1}{2\pi} \int_0^{2\pi} P(r, \theta - s), \ ds = 1. \tag{2.16}$$

Also by writing

$$P(r, s) = \frac{1 - r^2}{(1 - r)^2 + 4r \sin^2\left(\frac{s}{2}\right)}$$

one can deduce that given $\epsilon \in (0, \pi/2)$

$$\max\Big\{ P(r, s), \ s \in [\epsilon, \pi] \Big\} \to 0 \ \text{ as } \ r \to 1- . \tag{2.17}$$

Step 1:

We first prove the result for $h: [0, 2\pi] \to \mathbb{R}$, 2π periodic and continuous. Let $w(h, \delta)$ denote the modulus of continuity of h that is,

$$w(h, \delta) = \max_{|x_1 - x_2| \leq \delta} \Big| h(x_1) - h(x_2) \Big|, \ \delta > 0.$$

(See ([2] p.86, [21] 75–79)). Now as h is uniformly continuous on $[0, 2\pi]$ given $\epsilon > 0$ there exists $\delta > 0$ such that

$$w(h, \delta) < \epsilon/2. \tag{2.18}$$

Also using (2.17) there exists r_0 such that if $r \in (r_0, 1)$

$$\|h\|_{L^\infty[-1,1]} \max\Big\{ P(r, s) : s \in [\delta, \pi] \Big\} < \epsilon/4. \tag{2.19}$$

Then for $r \geq r_0$ and $\theta \in [0, 2\pi]$,

$$
\begin{aligned}
&\left|Q[h,r,\theta] - h(\theta)\right| \\
&\quad = \left|\frac{1}{2\pi}\int_0^{2\pi} h(s)P(r,\theta-s)ds - \frac{h(\theta)}{2\pi}\int_0^{2\pi} P(r,\theta-s)ds\right| \\
&\quad = \left|\frac{1}{2\pi}\int_0^{2\pi} (h(s)-h(\theta))P(r,\theta-s)ds\right| \\
&\quad \leq \frac{1}{2\pi}\int_{s:\ |s-\theta|\leq\delta} w(h,\delta)P(r,\theta-s)ds + \\
&\quad\quad + \frac{1}{2\pi}\int_{s:\ |s-\theta|>\delta} 2\|h\|_{L^\infty[-1,1]}P(r,\theta-s)ds \\
&\quad < \epsilon/2 + \epsilon/2 = \epsilon
\end{aligned}
$$

by (2.18) and (2.19). So

$$
\max_{\theta\in[0,2\pi]}\left|Q[h,r,\theta] - h(\theta)\right| = 0 \text{ as } r \to 1-.
$$

<u>Step 2:</u>

Next we show if $h \in L^1[0,2\pi]$ then,

$$
\frac{1}{2\pi}\int_0^{2\pi}\left|Q[h,r,\theta]\right|d\theta \leq \frac{1}{2\pi}\int_0^{2\pi}\left|h(\theta)\right|d\theta.
$$

Well,

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}\left|Q[h,r,\theta]\right|d\theta \\
&\quad = \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{1}{2\pi}\int_0^{2\pi} h(s)P(r,\theta-s)ds\right|d\theta \\
&\quad \leq \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{1}{2\pi}\int_0^{2\pi}\left|h(s)\right|P(r,\theta-s)ds\right]d\theta \\
&\quad = \frac{1}{2\pi}\int_0^{2\pi}\left|h(s)\right|\cdot\left[\frac{1}{2\pi}\int_0^{2\pi}P(r,\theta-s)d\theta\right]ds \\
&\quad\text{(by Fubini's Theorem)} \\
&\quad = \frac{1}{2\pi}\int_0^{2\pi}\left|h(s)\right|ds \text{ (by (2.16))}.
\end{aligned}
$$

Step 3:

Let $C_c(X) = \{f : f : X \to \mathbb{C}$ continuous with compact support$\}$ and $C(X) = \{f : f : X \to \mathbb{C}$ continuous$\}$. Then it is immediate that if X is compact $C_c(X) = C(X)$. Further, the closure of $C_c(X)$ is $L^P(d\mu)$ see ([24] pp.68–69). So given $h \in L^1[0, 2\pi]$, choose $g : [0, 2\pi] \to \mathbb{R}$ continuous and 2π periodic such that

$$\frac{1}{2\pi}\int_0^{2\pi} \left|h(\theta) - g(\theta)\right| d\theta < \epsilon.$$

Then,

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \Big|Q[h, r, \theta] - h(\theta)\Big| d\theta \\
&\quad \le \frac{1}{2\pi}\int_0^{2\pi} \Big|Q[h - g, r, \theta] + Q[g, r, \theta] - g(\theta) + (g - h)(\theta)\Big| d\theta \\
&\quad \le \frac{1}{2\pi}\int_0^{2\pi} |h - g|(\theta) d\theta + \frac{1}{2\pi}\int_0^{2\pi} \Big|Q[g, r, \theta] - g(\theta)\Big| d\theta + \\
&\qquad + \frac{1}{2\pi}\int_0^{2\pi} |g - h|(\theta) d\theta < 3\epsilon \text{ as } r \to 1-
\end{aligned}$$

using Step 1 and Step 2. ∎

We can now prove Lemma 2.9 which will prove crucial in proving Theorem 6.5 of Rakhmanov in Chapter 6.

Lemma 2.9

Let $w : [0, 2\pi] \to [0, \infty)$, $w > 0$ a.e. (meas). Let $d\mu = w(\theta) d\theta$ and

$$F_\mu(z) = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{e^{i\theta} + z}{e^{i\theta} - z}\right) d\mu(\theta), \; |z| < 1, \; z = re^{i\alpha}.$$

Then a.e.

$$(2.20) \qquad \tilde{w}(z) = -\frac{1}{\pi}\int_0^{2\pi} \frac{w(e^{i\theta}) - w(z)}{|\xi - z|^2} Im(\bar{\xi} z) d\theta, \quad \xi = e^{i\theta}, \; |z| = 1;$$

$$(2.21) \qquad F_\mu(z) = \big(w(z) + i\tilde{w}(z)\big), \; |z| = 1.$$

Remark

It will be shown in Lemma 6.3 that $F_\mu(z)$ exists a.e. $(d\mu)$, for $|z| = 1$ and as $\tilde{w}$ exists a.e. (meas), (2.21) is understood to be true at those points where $F_\mu(z)$ exists.

Proof:

$$Re\big(F_\mu(z)\big) = \frac{1}{2\pi}\int_0^{2\pi} Re\Big(\frac{e^{i\theta}+z}{e^{i\theta}-z}\Big)\,d\mu(e^{i\theta}).$$

Now,

$$Re\Big(\frac{e^{i\theta}+z}{e^{i\theta}-z}\Big) = Re\Big(\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\Big) = \frac{1-|z|^2}{1+|z|^2-2r\cos(\alpha-\theta)}.$$

Thus,

$$Re\big(F_\mu(z)\big) = \frac{1}{2\pi}\int_0^{2\pi} P(r,\theta,\alpha)\,d\mu(e^{i\theta}) \to w(e^{i\theta})$$

a.e. as $r \to 1-$ by Lemma 2.8. Also,

$$\begin{aligned}
Im\big(F_\mu(z)\big) &= \frac{1}{2\pi}\int_0^{2\pi} Im\Big(\frac{e^{i\theta}+z}{e^{i\theta}-z}\Big)\,d\mu(e^{i\theta}) \\
&= \frac{1}{2\pi}\int_0^{2\pi} Im\Big(\frac{(e^{i\theta}+z)(\overline{e^{i\theta}-z})}{|e^{i\theta}-z|^2}\Big)\,d\mu(e^{i\theta}) = \frac{1}{\pi}\int_0^{2\pi}\frac{Im(\bar{\xi}z)w(\xi)}{|\xi-z|^2}\,d\theta \\
&= -\frac{1}{2\pi}\int_0^{2\pi} w(\xi)\cot\Big(\frac{\theta-\alpha}{2}\Big)\,d\theta
\end{aligned}$$

[as $|\xi - z|^2 = 4\sin^2\Big(\frac{\theta-\alpha}{2}\Big)$, $\xi = e^{i\theta}$, $z = e^{i\alpha}$, $Im(\bar{\xi}z) = \sin(\alpha-\theta)$].

Also

$$PV\frac{1}{2\pi}\int_0^{2\pi}\cot\Big(\frac{\theta-\alpha}{2}\Big)\,d\theta = 0,$$

thus

$$Im\big(F_\mu(z)\big) = -\frac{1}{2\pi}\int_0^{2\pi}\big(w(\xi)-w(z)\big)\cot\Big(\frac{\theta-\alpha}{2}\Big)\,d\theta = \tilde{w}(z)\ a.e.\ (meas)$$

$$= \frac{1}{\pi}\int_0^{2\pi}\frac{Im(\bar{\xi}z)w(\xi)}{|\xi-z|^2}\,d\theta \tag{2.22}$$

then we have $F_\mu(z) = w(z) + i\tilde{w}(z)$. ■

Remark:

We have seen that for weights w on the unit circle such that $w > 0$ a.e. (meas), $\lim \gamma_n = \gamma \leq \infty$. We now restrict ourselves to $\gamma < \infty$ and formulate the important result that this fact is equivalent to a Szegö condition on w. Under these assumptions, we can define the Szegö function which will be used to develop asymptotics on $|z| \geq R > 1$ and $|z| = 1$.

We begin with the important:

Theorem 2.10

Let $d\mu(e^{i\theta}) = w(\theta)d\theta$, $w > 0$ a.e., $\log w \in L^1[0, 2\pi]$. Then

(2.23) $$\log w \in L^1[0, 2\pi]$$

implies

(2.24) $$\lim_{n\to\infty} \gamma_n = \gamma < \infty.$$

There exists a subsequence of polynomials $\varphi^*_{n_\nu}(z)$ such that uniformly in $|z| \leq r < 1$

(2.25) $$\lim_{\nu\to\infty} \varphi^*_{n_\nu}(z) = \pi(z)$$

where $\pi(z)$ is analytic and non zero in $|z| < 1$. Moreover,

(2.26) $$\lim_{\nu\to\infty} \varphi^*_{n_\nu}(z) = (\gamma)^{-1} \sum_{j=0}^{\infty} \overline{\varphi_j(0)}\varphi_j(z) = \pi(z)$$

uniformly for $|z| \leq r < 1$.

Remarks:

It will be shown that

(2.27) $$\pi(z) = (\text{Szegö function})^{-1}.$$

Further it is shown in ([5], pp.14–18), that

(2.28) $\qquad$ (2.23) iff (2.24) iff (2.25) iff (2.26)

with the full sequence converging to $\pi(z)$.

Proof:

Suppose (2.23) holds. Then we have using orthonormality and (1.5) that

$$\begin{aligned}(\gamma_n)^{-2} &= \int_0^{2\pi}\left|\frac{\varphi_n^*(e^{i\theta})}{\gamma_n}\right|^2 d\mu(\theta) \geq \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\varphi_n^*(e^{i\theta})}{\gamma_n}\right|^2 w(\theta)d\theta \\ &\geq \exp\left(\frac{2}{2\pi}\int_0^{2\pi}\log\left|\frac{\varphi_n^*(e^{i\theta})}{\gamma_n}\right|d\theta\right)\exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log w(\theta)d\theta\right) \\ &\qquad \text{(by Jensen's inequality)} \\ &= \exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log w(\theta)d\theta\right)\end{aligned}$$

(by the mean value theorem and $\varphi_n^*(0) = \gamma_n$). So

$$\gamma^{-2} = \lim_{n\to\infty}\gamma_n^{-2} \geq \exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log w(\theta)d\theta\right) > 0. \tag{2.29}$$

Now by (1.37) $|\varphi_n^*(z)|^2 \geq (1-|z|^2)\gamma_0^2$. Thus for $|z| \leq r < 1$ the sequence $(\varphi_n^*(z))_{n=0}$ is uniformly bounded from below so we may apply the Vitali–Montel theorem of analytic functions to obtain a subsequence of polynomials $(\varphi_{n_\nu}^*(z))$ such that uniformly for $|z| \leq r < 1$,

$$\lim_{\nu\to\infty}\varphi_{n_\nu}^*(z) = \pi(z) \tag{2.30}$$

where $\pi(z)$ is analytic in $|z| < 1$ and be equal to ∞. But since $\varphi_n^*(z) \neq 0$, $|z| \leq 1$ (Lemma 1.8), by Hurwitz's theorem on the zeros of sequences of analytic functions ([7], p.205), $\pi(z) \neq 0$, $|z| < 1$. Now as $\varphi_{n_\nu}^*(0) = \gamma_{n_\nu}$ it follows that (2.23) implies $\pi(z) \neq \infty$ which implies it is analytic and non zero in $|z| < 1$. Finally, by Lemma 1.7 we have

$$\gamma_n\varphi_n^*(z) = \sum_{j=0}^{n}\varphi_j(z)\overline{\varphi_j(0)},\ |z| < 1.$$

Thus by (2.30) and (2.24) we have

$$\lim_{\nu\to\infty} \varphi^*_{n_\nu}(z) = (\gamma)^{-1} \sum_{j=0}^{\infty} \overline{\varphi_j(0)}\varphi_j(z) = \pi(z)$$

uniformly for $|z| \le r < 1$. ■

Corollary 2.11

$$\log w(\theta) \in L^1[0,2\pi] \text{ iff } \lim_{n\to\infty} \varphi^*_n(z) = (\gamma)^{-1} \sum_{j=0}^{\infty} \overline{\varphi_j(0)}\varphi_j(z) = \pi(z).$$

Proof:

Use Theorem 2.5, Theorem 2.10 with (2.28). ■

Definition 2.12

Let $w\colon [0,2\pi] \to [0,\infty)$ satisfy $w \in L^1[0,2\pi]$, $\log w \in L^1[0,2\pi]$ and let $|z| < 1$. Then the interior Szegõ function for w is defined as follows:

$$D(w,z) = \exp\left(\frac{1}{4\pi}\int_0^{2\pi} \log w(\theta)\left(\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\right)d\theta\right).$$

Remark:

We end this section by proving and formulating the main properties of D, and then extending the definition of D to $|z| > 1$ and even $|z| = 1$. A sufficient condition for the existence of D on $|z| = 1$ will also be given.

Lemma 2.13 (Properties of $D - (I)$)

$D(w,z)$ is analytic and non zero for $|z| < 1$ and

$$D(w,0) = \exp\left(\frac{1}{4\pi}\int_0^{2\pi} \log w(\theta)d\theta\right); \tag{2.31}$$

(2.32) $$D(fg,z) = D(f,z)D(g,z),\ |z| < 1;$$

(2.33) $$D(w^\alpha, z) = D(w,z)^\alpha,\ \alpha \in \mathbf{C};$$

(2.34) $$\left|D(w, re^{i\theta})\right| = \exp\left(\frac{1}{4\pi}\int_0^{2\pi} \log w(\theta) P(r, \theta - t)\, dt\right).$$

(where $P(r, \theta - t)$ is the Poisson kernel of the unit disc).

Proof:

(2.31) and (2.32) follow from Definition 2.12 and $|z| < 1$. (2.33) follows by taking a suitable choice of branches and (2.34) follows using $|e^z| = \exp(Re(z))$. ■

Lemma 2.14 (Properties of $D - (II)$)

Let $w\colon [0,2\pi] \to [0,\infty)$ satisfy $w \in L^1[0,2\pi]$ and $\log w \in L^1[0,2\pi]$. By the analyticity of D in $|z| < 1$, write for $|z| < 1$, $D(w,z) = \sum_{j=0}^{\infty} d_j z^j$. Then

(2.35) $$\lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} \left|\log\left|D(w, re^{i\theta})\right|^2 - \log w(\theta)\right| d\theta = 0;$$

(2.36) $$\lim_{r \to 1-} \frac{1}{2\pi} \int_0^{2\pi} \left|D(w, re^{i\theta})\right|^2 d\theta = \sum_{j=0}^{\infty} |d_j|^2 = \frac{1}{2\pi}\int_0^{2\pi} w(\theta)\, d\theta;$$

(2.37) $$\lim_{r \to 1-} \frac{1}{2\pi} \int_0^{2\pi} \left|\left|D(w, re^{i\theta})\right|^2 - w(\theta)\right| d\theta = 0.$$

If T is a trigonometric polynomial of degree m and positive in $[0,2\pi]$ and $T(\theta = \left|\pi_m(e^{i\theta})\right|^2$ with all the zeros of π_m in $|z| > 1$ and $\pi_m(0) > 0$ then

(2.38) $$D(T,z) = D(|\pi_m|^2, z) = \pi_m(z).$$

Let $\alpha_1, \dots, \alpha_\ell > 0$, $z_j = r_j e^{i\theta_j}$, $r_j > 1$, $\theta_j \in [0, 2\pi]$, $1 \le j \le \ell$, $z = e^{i\theta}$, $T(\theta) = \prod_{j=1}^{\ell} |e^{i\theta} - r_j e^{i\theta_j}|^{\alpha_j} = \prod_{j=1}^{\ell} |z - z_j|^{\alpha_j}$. Then,

$$(2.39) \quad D(T, z) = D\Big(\prod_{j=1}^{\ell} |z - z_j|, z\Big) = \exp\Big(-\frac{i}{2}\sum_{j=1}^{\ell} \theta_j \alpha_j\Big) \prod_{j=1}^{\ell} (z - z_j)^{\alpha_j/2}.$$

Proof:

By (2.34)

$$\log\big|D(w, re^{i\theta})\big|^2 = \frac{1}{2\pi}\int_0^{2\pi} \log w(t) P(r, \theta - t)\,dt = Q[\log w, r, \theta]$$

so

$$\frac{1}{2\pi}\int_0^{2\pi} \Big|\log\big|D(w, re^{i\theta})\big|^2 - \log w(\theta)\Big|\,d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \Big|Q\big[\log w, r, \theta\big] - \log w(\theta)\Big|\,d\theta \to 0 \text{ as } r \to 1-,$$

by Lemma 2.8. Next

$$\frac{1}{2\pi}\int_0^{2\pi} \big|D(w, re^{i\theta})\big|^2 d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \Big(\sum_{j=0}^{\infty} d_j r^j e^{ij\theta}\Big)\Big(\sum_{j=0}^{\infty} \overline{d_j}\, \overline{e^{ij\theta}}\, \overline{r^j}\Big)$$
$$= \sum_{j=0}^{\infty} |d_j|^2 r^{2j}.$$

So

$$(2.40) \quad \frac{1}{2\pi}\int_0^{2\pi} |D(w, re^{i\theta}|^2 d\theta = \sum_{j=0}^{\infty} |d_j|^2 r^{2j}.$$

As D was analytic in $|z| < 1$, the series in (2.40) is uniformly convergent for $|z| \le R < 1$. Also if

$$d\mu(t) = \frac{1}{2\pi} P(r, \theta - t)\,dt,\ t \in [0, 2\pi]$$

then $d\mu$ is a Borel measure on $[-\pi,\pi]$ and

$$\int_0^{2\pi} d\mu(t) = \frac{1}{2\pi}\int_0^{2\pi} P(r,\theta-t)dt = 1$$

(by 2.16). So

$$\begin{aligned}
|D(w,re^{i\theta})|^2 &= \exp\int_0^{2\pi} \log w(t)d\mu(t) \\
&\le \int_0^{2\pi} \exp\log w(t)d\mu(t) = \frac{1}{2\pi}\int_0^{2\pi} w(t)P(r,\theta-t)dt \\
&= Q[w,r,\theta]
\end{aligned}$$

by Jensen's inequality. Therefore

$$\begin{aligned}
\sum_{j=0}^{\infty}|d_j|^2 r^{2j} &= \frac{1}{2\pi}\int_0^{2\pi} |D(w,re^{i\theta})|^2 d\theta \\
&\le \frac{1}{2\pi}\int_0^{2\pi} Q[w,r,\theta]d\theta \to \frac{1}{2\pi}\int_0^{2\pi} w(\theta) \text{ as } r\to 1-
\end{aligned}$$

(by Lemma 2.8). Thus we have

$$\begin{aligned}
\sum_{j=0}^{\infty}|d_j|^2 &= \sum_{j=0}^{\infty}|d_j|^2 \lim_{r\to 1-} r^{2j} \\
&= \lim_{r\to 1-}\sum_{j=0}^{\infty}|d_j|^2 r^{2j} \le \frac{1}{2\pi}\int_0^{2\pi} w(\theta)d\theta < \infty.
\end{aligned}$$

Therefore we have,

$$\sum_{j=0}^{\infty}|d_j|^2 < \infty. \tag{2.41}$$

Now if we define

$$D(w,e^{i\theta}) = \sum_{j=0}^{\infty} d_j e^{ij\theta},\ \theta\in[0,2\pi]$$

and form the partial sums

$$S_n(z) = \sum_{j=0}^{n} d_j z^j$$

then for $m > n$,

$$\frac{1}{2\pi}\int_0^{2\pi}\left|S_m(e^{i\theta}) - S_n(e^{i\theta})\right|^2 d\theta = \sum_{j=n+1}^{m} |d_j|^2 \to 0 \text{ as } m,n \to \infty$$

(by 2.41). Therefore $\left(S_n(e^{i\theta})\right)_{n=1}^{\infty}$ is a Cauchy sequence in $L^2[0,2\pi]$ which is complete so it converges to $D(w,e^{i\theta})$ in $L^2[0,2\pi]$ and thus exists a.e. (meas). Also

$$\frac{1}{2\pi}\int_0^{2\pi}\left|D(w,e^{i\theta}) - D(w,re^{i\theta})\right|^2 d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{j=0}^{\infty} d_j e^{ij\theta}(1-r^j)\right|^2 d\theta = \sum_{j=0}^{\infty}|d_j|^2(1-r^j)^2 \to 0 \text{ as } r \to 1-.$$

So,

$$\frac{1}{2\pi}\int_0^{2\pi}|D(w,e^{i\theta}) - D(w,re^{i\theta})|^2 d\theta \to 0 \text{ as } r \to 1-. \tag{2.42}$$

Then

$$\frac{1}{2\pi}\int_0^{2\pi}\left||D(w,e^{i\theta})|^2 - |D(w,re^{i\theta})|^2\right| d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\Big|\Big(|D(w,e^{i\theta})| - |D(w,re^{i\theta})|\Big) \times$$
$$\times\Big(|D(w,e^{i\theta})| + |D(w,re^{i\theta})|\Big)\Big| d\theta$$
$$\leq I_1 \cdot I_2, \quad \text{(by the Cauchy Schwarz inequality),}$$

[where

$$I_1 = \left(\frac{1}{2\pi}\int_0^{2\pi}\left||D(w,e^{i\theta})| - |D(w,re^{i\theta})|\right|^2 d\theta\right)^{1/2}$$
$$I_2 = \left(\frac{1}{2\pi}\int_0^{2\pi}\left||D(w,e^{i\theta})| + |D(w,re^{i\theta})|\right|^2 d\theta\right)^{1/2}$$]

$$\leq I_3 \cdot I_4, \quad (\text{using } \big||a|-|b|\big| \leq |a-b|,\ (x+y)^2 \leq 2x^2+2y^2),$$

$$\Bigg[\text{where}$$

$$I_3 = \left(\frac{1}{2\pi}\int_0^{2\pi} |D(w,e^{i\theta}) - D(w,re^{i\theta})|^2 d\theta\right)^{1/2}$$

$$I_4 = \left(\frac{2}{2\pi}\int_0^{2\pi} \left(|D(w,e^{i\theta})|^2 + |D(w,re^{i\theta})|^2\right) d\theta\right)^{1/2}\Bigg]$$

$$= I_3 \cdot \left[2\left(\sum_{j=0}^{\infty} |d_j|^2 + \sum_{j=0}^{\infty} r^{2j}|d_j|^2\right)\right]^{1/2} \to 0 \text{ as } r \to 1-$$

by (2.42). So we have shown that

$$\frac{1}{2\pi}\int_0^{2\pi} \Big||D(w,e^{i\theta})|^2 - |D(w,re^{i\theta})|^2\Big| d\theta \to 0 \text{ as } r \to 1-. \tag{2.43}$$

We shall show

$$|D(w,e^{i\theta})|^2 = w(\theta) \text{ a.e. (meas) } \theta \in [0,2\pi],$$

then (2.37) follows from (2.43). So let $r \in (0,1)$, $\epsilon > 0$ and define,

$$G_{r,\epsilon} = \left\{\theta \in [0,2\pi] : \Big||D(w,e^{i\theta})|^2 - |D(w,re^{i\theta})|^2\Big| > \epsilon\right\}$$

$$H_{r,\epsilon} = \left\{\theta \in [0,2\pi] : \Big|\log|D(w,re^{i\theta})|^2 - \log w(\theta)\Big| > \epsilon\right\}.$$

Then by (2.43) meas $G_{r,\epsilon} \to 0$ as $r \to 1-$ and by (2.35), meas $H_{r,\epsilon} \to 0$ as $r \to 1-$. Choose $\epsilon \in (0,1)$. Then for $\theta \in [0,2\pi] \backslash H_{r,\epsilon} \bigcup G_{r,\epsilon}$,

$$\Big||D(w,e^{i\theta})|^2 - w(\theta)\Big| \leq I_5 + I_6$$

$$\Bigg[\text{where}$$

$$I_5 = \Big||D(w,e^{i\theta})|^2 - |D(w,re^{i\theta})|^2\Big|,$$

$$I_6 = \Big||D(w,re^{i\theta})|^2 - w(\theta)\Big|\Bigg]$$

$$\leq \epsilon + w(\theta)\Big|\exp\left(\log|D(w,re^{i\theta})|^2 - \log w(\theta)\right) - 1\Big|$$

$$\leq \epsilon + w(\theta)(\exp\epsilon - 1) \leq \epsilon(1 + ew(\theta))$$

[as $\exp \epsilon - 1 \le e\epsilon$, $\epsilon \in (0,1)$]. Now choose $r_n \in (0,1)$, $n \ge 1$ such that

$$meas\left(G_{r_n,\frac{1}{n}} \bigcup H_{r_n,\frac{1}{n}}\right) \le \frac{1}{n^2}.$$

Let

$$H = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \left(G_{r_n,\frac{1}{n}} \bigcup H_{r_n,\frac{1}{n}}\right).$$

Then for all $m \ge 1$

$$\begin{aligned} meas\, H &\le meas\left(\bigcup_{n=m}^{\infty} \left(G_{r_n,\frac{1}{n}} \bigcup H_{r_n,\frac{1}{n}}\right)\right) \\ &\le \sum_{n=m}^{\infty} \frac{1}{n^2} \to 0 \text{ as } m \to \infty. \end{aligned}$$

So meas $(H) = 0$. Also for $\theta \in [0,2\pi]\backslash H$,

$$\theta \notin \bigcup_{n=m}^{\infty} \left(G_{r_n,\frac{1}{n}} \bigcup H_{r_n,\frac{1}{n}}\right),$$

some m, which implies

$$\theta \notin \left(G_{r_n,\frac{1}{n}} \bigcup H_{r_n,\frac{1}{n}}\right)$$

for all $n \ge m$. This implies $\left|D(w,e^{i\theta})\right|^2 = w(\theta)$, $\theta \notin H$. So $\left|D(w,e^{i\theta})\right|^2 = w(\theta)$ $a.e.(meas)$ $\theta \in [0,2\pi]$. Finally, (2.37) implies

$$\begin{aligned} &\left|\frac{1}{2\pi}\int_0^{2\pi} |D(w,re^{i\theta})|^2 d\theta - \frac{1}{2\pi}\int_0^{2\pi} w(\theta) d\theta\right| \\ &\quad \le \frac{1}{2\pi}\int_0^{2\pi} \left||D(w,re^{i\theta})|^2 - w(\theta)\right| d\theta \to 0 \text{ as } r \to 1- \end{aligned}$$

by (2.36). Then part of (2.36) follows and (2.40) implies the rest. For the proofs of (2.38) and (2.39), see [4], p.211. ∎

<u>Lemma 2.15</u> (Properties of $D - (III)$)

(a) $D(w,e^{i\theta})$ can be formally defined as follows

(2.44) $$D(w,e^{i\theta}) = \lim_{r \to 1-} D(w,re^{i\theta})$$

which exists a.e. (meas).

(b) $D(w,z)$ is determined by the following conditions (not necessarily unique):

$$D(w,z) \in H_2,\ D(z) \neq 0,\ |z| < 1 \tag{2.45}$$

and $D(0)$ is real and positive.

(c) Define $w_n = \left(K_n(z,z)\right)^{-1}$. Then,

$$\lim_{n\to\infty} w_n(d\mu, z) = (1 - |z|^2)\left|D(w,z)\right|^2 \tag{2.46}$$

holds uniformly for $|z| \leq r < 1$.

(d) If f is analytic in $|z| < 1+\epsilon$, some $\epsilon > 0$, then writing $f(\theta) = f(e^{i\theta})$ we have

$$f(z) = C_1 D^2(f,z) \prod_{\sigma} \left(\frac{\bar{\sigma}z - 1}{z - \sigma}\right), |z| < 1 \tag{2.47}$$

where $|C_1| = 1$ and σ runs over the zeros of f with $|\sigma| < 1$ and $\left|\frac{\bar{\sigma}z - 1}{z - \sigma}\right| < 1$ for $|z| < 1$.

(e) There are many criteria for the coexistence of $D(w, e^{i\theta})$. One such criterion is the following. Let $w\colon [0, 2\pi] \to [0, \infty)$ and suppose that

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|w(t) - w(\theta)\right| \left|\cot\left(\frac{\theta - t}{z}\right)\right| dt$$

exists. Then

$$\lim_{r\to 1-} D(w, re^{i\theta}) = D(w, e^{i\theta}). \tag{2.48}$$

We remark that other criteria weaker than (e) can be found in ([28, p.90, p.252]).

Proof:

(2.44) follows immediately from Lemma 2.14. For the proof of (b) see ([4] pp.212–213; [27] pp.274–275). To see (c) note that we will prove in Theorem 2.17 that

$$(2.49)\qquad \left|\varphi_n(z) - z^n\overline{\left(D(w,(\bar{z})^{-1})\right)^{-1}}\right| \to 0;\ \varphi_n^*(z) \to D^{-1}(w,z)$$

uniformly for $|z| \le r < 1$ as $n \to \infty$, where D is defined with respect to w. Also from the definition of K_n (Lemma 1.7) we have,

$$(2.50)\qquad |\varphi_n^*(z)|^2 = (1-|z|^2)K_n(z,z) + |\varphi_n(z)|^2.$$

Hence uniformly for $|z| \le r < 1$ using (2.49) and (2.50)

$$\lim_{n\to\infty} K_n(z,z) = (1-|z|^2)^{-1}|D(w,z)|^{-2}$$

which implies (2.46). We also note that a proof of (e) can be found in ([27] pp.278–279). Finally, the proof of (d) follows from the fact that f belongs to Nevalinna's class N of analytic functions which will be explained as a remark after the proof. ■

Remarks: (Nevalinna's Class N)

The following definitions and results can be found in ([3] pp.16–17, [27] p.277).

(a) A function $f(z)$ analytic in $|z| < 1$ is said to be in Nevalinna's Class N or just $f \in N$ if

$$(2.51)\qquad \lim_{r\to 1-}\int_0^{2\pi} \log^+|f(re^{i\theta}|d\theta < \infty$$

where

$$\log^+(x) = \begin{cases} \log x, & x \ge 1, \\ 0, & 0 < x < 1. \end{cases}$$

(b) Given f analytic in $|z| < 1$. Then

$$(2.53) \qquad f \in N \quad \text{iff} \quad f = g \cdot (h)^{-1},$$

where g and h are bounded analytic functions in $|z| < 1$.

(c) If $f \in N$ then,

$$(2.53) \qquad f(z) = B(z) \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \left(\frac{e^{it} + z}{e^{it} - z}\right) d\lambda(t)\right)$$

where $\lambda(t)$ is a real valued function of bounded variation and $B(z)$ is the Blaschke product of f with zeros ξ_n such that $0 < |\xi_n| < 1$, $\Sigma(1 - |\xi_n|) < \infty$.

(d) Returning to (d) of Lemma 2.15, we can be assured that f is bounded in $|z| \leq 1 + \epsilon/2$ say. Thus

$$\lim_{r \to 1-} \int_0^{2\pi} |\log^+ f(re^{i\theta})| d\theta < C_2$$

say. Thus $f \in N$ and (2.54) applies with $\prod_\sigma = B$.

We end this section with:

Definition 2.16

Let $w : [0, 2\pi] \to [0, \infty]$ satisfy $w \in L^1[0, 2\pi]$, $\log w \in L^1[0, 2\pi]$ then,

(a) The interior Szegõ function is defined by

$$(2.54) \qquad D(w, z) = \begin{cases} \exp\left(\frac{1}{4\pi} \int_0^{2\pi} \left(\frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}}\right)\right) \log w(\theta) d\theta, & |z| < 1, \\ \lim_{r \to 1-} D(w, rz), & |z| = 1. \end{cases}$$

(b) The exterior Szegõ function is defined by

$$(2.55) \qquad D(w, z, ex) = \begin{cases} \exp\left(\frac{1}{4\pi} \int_0^{2\pi} \left(\frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}}\right)\right) \log w(\theta) d\theta, & |z| > 1, \\ \lim_{r \to 1+} D(w, rz), & |z| = 1, \end{cases}$$

where the limit for $|z| = 1$ is understood to exist a.e. (meas).