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Simple-Sum Giant Graviton Expansions for Orbifolds and Orientifolds

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We study giant graviton expansions of the superconformal index of 4D orbifold/orientifold theories. In general, a giant graviton expansion is given as a multiple sum over wrapping numbers. It is known that the expansion can be reduced to a simple sum for the $\mathcal{N} = 4$ $U(N)$ supersymmetric Yang–Mills (SYM) by choosing appropriate expansion variables. We find such a reduction occurs for a few examples of orbifold and orientifold theories: the $\mathbb{Z}_k$ orbifold and orientifolds with $O3$ and $O7$. We also argue that for a quiver gauge theory associated with a toric Calabi–Yau 3-fold the simple-sum expansion works only if the toric diagram is a triangle, i.e. the Calabi–Yau is an orbifold of $\mathbb{C}^3$.

Subject Index B10, B16, B21

1. Introduction

Anti-de Sitter/conformal field theory (AdS/CFT) correspondence [1–3] provides a useful window through which to investigate quantum gravity. Although we have not yet understood how to deal directly with quantum effects of gravity, we can obtain information on such effects through the correspondence from analyses in the boundary theories. The superconformal index [4–6] is an important and useful quantity for quantitative investigation of the correspondence. The index can be calculated on the gauge theory side as long as the theory is Lagrangian, and we are also able to calculate the index on the gravity side in an appropriate parameter region. In the strict large-N limit, which means N is much larger than the energy scale (or the order in the Taylor expansion of the index) which we are focusing on, the index obtained on the boundary side can be reproduced semiclassically on the gravity side as the contribution from massless fields living in the AdS background [5]. To access the quantum gravity effects via AdS/CFT correspondence, we should consider parameter regions beyond the strict large-N limit.

One interesting region is the one with the energy scale of order N^2 with large N. On the gravity side such a region is described by classical blackhole solutions, and it was found that the superconformal index of large-N gauge theory can correctly reproduce the Bekenstein-Hawking entropy by taking an appropriate limit of the index [7–9]. This discovery is important because it indicates that the boson–fermion cancellation does not occur for the majority of states in the

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Hilbert space and the index can be used as the thermal partition function by taking appropriate values of fugacities.

Another important parameter region, which we focus on in this work, is the one with energy comparable to N. In the q expansion of the index we find deviation from the large-N limit around this order. On the gauge theory side, this is related to the existence of additional operators or additional constraints due to the finiteness of the rank of the gauge group. On the gravity side, this can be interpreted as the contribution of extended branes. The first example of such a brane was found in the orientifold model, in which D3-branes wrapped on the topologically nontrivial three-cycle in $\boldsymbol{S}^5/\mathbb{Z}_2$ correspond to Pfaffian operators [10]. Because Pfaffian operators are BPS operators contributing to the index, the corresponding wrapped branes must also contribute to the index. In such an example it is natural to expect the finite-N corrections can be given in the form of expansion with respect to the wrapping numbers associated with nontrivial cycles. Even if there are no such nontrivial cycles, there exist stable extended brane configurations called giant gravitons [11–14]. They are BPS configurations, and should also contribute to the index.

Direct analyses of the contributions of wrapped branes to the superconformal index were carried out for $\mathcal{N}=4$ supersymmetric Yang–Mills (SYM) [15–17] and many other examples [18–23]. Essentially the same expansions were also studied on the gauge theory side in Refs. [24,25], and named giant graviton expansions. There exists a similar expansion of the superconformal index proposed by Murthy [26]. The relation between Murthy's expansion and the giant graviton expansion was studied in Refs. [27,28]. The contribution of giant gravitons to black hole entropy was studied in Refs. [29,30]. See also Refs. [5,31–34] for earlier works on the giant graviton contributions to indices and supersymmetric partition functions.

In the analysis of finite-N corrections to the superconformal index we need to include extended branes regardless of whether the brane wrapped on topologically nontrivial cycles. Although the term "giant gravitons" originally meant extended branes without topological wrappings, in this work we call general extended branes giant gravitons regardless of whether they have topological wrapping or not.

Let us consider the $\mathcal{N}=4$ $U(N)$ SYM, whose dual geometry is $AdS_5\times S^5$. The superconformal index is defined by

$$I=\mathrm{tr}[(-1)^F q^{J_1}p^{J_2}x^{R_x}y^{R_y}z^{R_z}], \tag{1}$$

where J_1 and J_2 are angular momenta and R_x, R_y, and R_z are R-charges. The fugacities for these generators, q, p, x, y, and z, are constrained by

$$qp=xyz, \tag{2}$$

to respect one of the supercharges. We can calculate the index $I_{U(N)}$ of the $\mathcal{N}=4$ $U(N)$ SYM by the localization formula

$$I_{U(N)}=\int_{U(N)}dg\,\mathrm{Pexp}\left(f_{\mathrm{vec}}\chi_{\mathrm{adj}}^{U(N)}(g)\right), \tag{3}$$

where Pexp is the plethystic exponential, $\int_{U(N)}dg$ is the gauge group integral with the Haar measure, $\chi_{\mathrm{adj}}^{U(N)}(g)$ is the $U(N)$ adjoint character,[1] and f_{vec} is the letter index of the $\mathcal{N}=4$ $U(1)$ vector multiplet

$$f_{\mathrm{vec}}(q,p,x,y,z)=i\left[\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{[0,1,0]}\right]=1-\frac{(1-x)(1-y)(1-z)}{(1-q)(1-p)}. \tag{4}$$

[1] See Appendix A for the explicit definitions of the character and the Haar measure.

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We use the notation $i[\mathcal{R}]$ for the index of an irreducible superconformal representation $\mathcal{R}$, and we adopt the notation in Ref. [35] for $\mathcal{R}$.

In the large-N limit, the $U(N)$ integral in Eq. (3) can be easily evaluated with the saddle point method, and the result is [5]:

$$I_{U(\infty)} = \mathrm{Pexp}\, f_{\mathrm{sugra}}, \tag{5}$$

where f_{sugra} is the letter index of the supergravity multiplet in $AdS_5 \times S^5$.

$$f_{\mathrm{sugra}} = \sum_{n=1}^{\infty} i\left[\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{[0,n,0]}\right] = \frac{x}{1-x} + \frac{y}{1-y} + \frac{z}{1-z} - \frac{q}{1-q} - \frac{p}{1-p}. \tag{6}$$

This is obtained by summing up contributions from modes in $AdS_5 \times S^5$ given in Refs. [36,37].

If N is finite, we have finite-N corrections, and these are given by the giant graviton expansion. Let us introduce three complex coordinates X, Y, and Z such that the S^5 is given by $|X|^2 + |Y|^2 + |Z|^2 = 1$. We take account of giant gravitons wrapped around three cycles $X = 0$, $Y = 0$, and $Z = 0$, and the giant graviton expansion of the index is given by the triple sum [15,17]:

$$\frac{I_{U(N)}}{I_{U(\infty)}} = \sum_{m_x,m_y,m_z=0}^{\infty} x^{m_x N} y^{m_y N} z^{m_z N} \mathrm{F}_{m_x,m_y,m_z}, \tag{7}$$

where m_x, m_y, and m_z are wrapping numbers associated with three three-cycles in S^5: $X = 0$, $Y = 0$, and $Z = 0$, respectively. For each set of wrapping numbers (m_x, m_y, m_z) the function F_{m_x,m_y,m_z} is the index of the theory realized on the system consisting of giant gravitons, and is N-independent because open strings on the giant gravitons do not couple with the background Ramond-Ramond (RR) flux. On each cycle a $U(m)$ gauge group is realized, and the gauge group of the theory on the giant graviton system is $G = U(m_x) \times U(m_y) \times U(m_z)$. We also have bi-fundamental fields coming from open strings attached on two D-branes wrapped around different cycles. The theory is a gauge theory with the triangle quiver diagram. We can calculate the functions F_{m_x,m_y,m_z} by a formula similar to Eq. (3):

$$F_{m_x,m_y,m_z} = \int_G dg\, \mathrm{Pexp}(i_x[m_x] + i_y[m_y] + i_z[m_z] + \cdots). \tag{8}$$

$\int_G dg$ is the integral over the gauge group $G = U(m_x) \times U(m_y) \times U(m_z)$ with the Haar measure. $i_x[m_x]$ ($i_y[m_y]$, $i_z[m_z]$) is the letter index of $U(m_x)$ ($U(m_y)$, $U(m_z)$) adjoint fields living on the cycle $X = 0$ ($Y = 0$, $Z = 0$), and the dots in the letter index represent the contribution of bi-fundamental fields living on the intersections.

Because the fields living on the cycle $X = 0$ belong to the $U(m_x)$ adjoint representation, $i_x[m_x]$ is given by $i_x[m_x] = f_{X=0}\chi^{U(m_x)}_{\mathrm{adj}}$ with the letter index $f_{X=0}$ of the $U(1)$ vector multiplet living on $X = 0$. Because the worldvolume of the giant graviton is $S^3 \times \mathbb{R}$, and is the same as the AdS boundary, the theory on the worldvolume is essentially the same as the $\mathcal{N} = 4$ SYM. An important difference is the action of symmetry generators, and the generators acting on the boundary and those acting on the cycle $X = 0$ are related by the involution map [15]:

$$\begin{aligned} \sigma_x : H \to H - 2R_x, \quad A \to -A, \quad J_1 \to R_y, \quad J_2 \to R_z, \\ R_x \to -R_x, \quad R_y \to J_1, \quad R_z \to J_2. \end{aligned} \tag{9}$$

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where A is the generator of $U(1)_R$ of type IIB supergravity normalized so that $A \in \mathbb{Z}/2$.[2] Correspondingly, we can obtain $f_{X=0}$ from $f_{\rm vec}$ by a simple variable change. This is also the case for the other two-cycles, and the variable changes to obtain the letter indices for three-cycles are given by Refs. [15,24]:

$$\begin{aligned}\sigma_x &: (q, p, x, y, z) \to (y, z, x^{-1}, q, p),\\ \sigma_y &: (q, p, x, y, z) \to (z, x, p, y^{-1}, q),\\ \sigma_z &: (q, p, x, y, z) \to (x, y, q, p, z^{-1}).\end{aligned} \tag{10}$$

(We use σ_I $(I = x, y, z)$ for both the involutions acting on the generators and variable changes for the fugacities.) With these variable changes, we can give $i_I[m_I]$ $(I = x, y, z)$ as follows:

$$i_I[m_I] = \sigma_I f_{\rm vec} \chi_{\rm adj}^{U(m_I)}. \tag{11}$$

The contribution from bi-fundamental fields denoted by dots in Eq. (8) can be obtained by directly analyzing the open string states. For example, the contribution from the intersection of cycles $X = 0$ and $Y = 0$ is $f_{xy}\chi^{(m_x,m_y)}$ with

$$f_{xy} = \left(\frac{z}{qp}\right)^{\frac{1}{2}} \frac{(1-q)(1-p)}{1-z} \tag{12}$$

and $\chi^{(m,m')}$ is the bi-fundamental character

$$\chi^{(m,m')} = \chi_{\rm fund}^{U(m)} \chi_{\overline{\rm fund}}^{U(m')} + \chi_{\overline{\rm fund}}^{U(m)} \chi_{\rm fund}^{U(m')}. \tag{13}$$

Although we can write down the integrand in Eq. (8), there is a difficulty in carrying out the gauge integral. To obtain the functions F_{m_x,m_y,m_z} that correctly reproduce the known index we have to carefully choose contours in the integrals and pick up the correct poles. Although a set of rules for the pole selection for $\mathcal{N} = 4$ $U(N)$ SYM was proposed in Ref. [17], its derivation and rules for more general theories have not yet become known. Although the rules for the functions associated with a single cycle like $\mathrm{F}_{m,0,0}$ are simple and natural, treatment of bi-fundamental fields is involved and calculation of F_{m_x,m_y,m_z} for intersecting giant gravitons is complicated. (See Ref. [25] for a proposal for integration contours and Ref. [38] for its application to the Schur index of $\mathcal{N} = 4$ SYM.)

We can avoid this problem if we can somehow remove the contributions from intersecting giant gravitons. Surprisingly, this is possible. Gaiotto and Lee [24] proposed a giant graviton expansion with a simple sum:

$$\frac{I_{U(N)}}{I_{U(\infty)}} = \sum_{m_x=0}^{\infty} x^{m_x N} F_{m_x,0,0}. \tag{14}$$

The reason for the reduction of the triple-sum expansion to the simple sum is explained with a special behavior of the functions F_{m_x,m_y,m_z}, which is referred to as "the wall-crossing" in Ref. [24]. Namely, functions F_{m_x,m_y,m_z} are not analytic on some walls in the space parametrized by the fugacities, and by choosing an appropriate chamber, some of the functions become identically zero, and they are decoupled from the index calculation. In other words, by choosing

[2]The $U(1)_R$ symmetry acts on the two three-form flux fields nontrivially, and is broken to $\mathbb{Z}_4$ generated by $e^{\pi i A}$ (for a generic value of the axiodilaton field) due to the flux quantization. Similar to angular momenta J_i and R-charges R_a, A is also related to the fermion number F by $e^{2\pi i A} = (-1)^F$. We use a convention with the quantum numbers $(J_1, J_2, R_x, R_y, R_z, A) = (-\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2})$ for the supercharge respected by the definition of the superconformal index.

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appropriate expansion variables, we can decouple some contributions and we can simplify the giant graviton expansion [39].

To clarify what is happening in functions F_{m_x,m_y,m_z}, let us first consider a simple toy model:

$$F(q) = \mathrm{Pexp}(q + q^2 + q^3 + \cdots). \tag{15}$$

If we Taylor expand this function around $q = 0$, this gives the following nontrivial expansion:

$$F(q) = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \cdots. \tag{16}$$

However, if we regard $F(q)$ as a function of $s = q^{-1}$ and perform the s-expansion (expansion around $q = \infty$), we obtain

$$F(q) = \prod_{k=1}^{\infty} \frac{1}{1-q^k} = \prod_{k=1}^{\infty} \frac{-s^k}{1-s^k} = s^{\infty} + \cdots = 0. \tag{17}$$

Indeed, the function $F(q)$ has a singular wall along the unit circle $|q| = 1$, and it is a nontrivial function inside the wall, while it is trivial outside the wall.

Let us return to the functions F_{m_x,m_y,m_z}. We can explain the relation between two expansions, one with the simple sum and the other with the triple sum, by different choices of the expansion variables. Now we have five fugacities, $q_I = (q, p, x, y, z)$, constrained by Eq. (2). To specify expansion variables we introduce four independent auxiliary variables t_i ($i = 1, \ldots, 4$) and write five fugacities in terms of t_i as follows:

$$q_I = \prod_{i=1}^{4} t_i^{d_{I,i}}. \tag{18}$$

Then, we carry out t_1-expansion first, and then sequentially perform t_2-, t_3-, and t_4-expansions in that order. This multiple expansion is specified by the set of constants $d_{I,i}$. Actually, we focus only on the first expansion specified by $d_{I,1}$. Let us denote t_1 by t and $d_{I,1}$ by d_I. The first expansion with respect to t ($= t_1$) is equivalently performed by the t-expansion after the replacement

$$q_I \to t^{d_I} q_I. \tag{19}$$

We can regard t as a fugacity for the operator

$$\sum_I d_I Q_I, \quad Q_I = (J_1, J_2, R_x, R_y, R_z). \tag{20}$$

In the following we call the constant d_I "the degree" assigned to the fugacity q_I and denote it by $d_I = \deg(q_I)$. The consistency with the constraint (2) requires the degrees satisfy

$$\deg(q) + \deg(p) = \deg(x) + \deg(y) + \deg(z). \tag{21}$$

In Ref. [17], the triple-sum expansion (7) with the degrees

$$\deg(q, p, x, y, z) = \left(\tfrac{3}{2}, \tfrac{3}{2}, 1, 1, 1\right) \tag{22}$$

was studied. Then the expansion variable t is the fugacity for the operator

$$\sum_I d_I Q_I = \frac{3}{2}(J_1 + J_2) + R_x + R_y + R_z = H + \frac{1}{2}(J_1 + J_2). \tag{23}$$

In this case all F_{m_x,m_y,m_z} give nontrivial contributions. The degrees adopted in Ref. [24], which proposed the simple-sum expansion (14), are

$$\deg(q, p, x, y, z) = (1, 1, 0, 1, 1) \tag{24}$$

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corresponding to the charge

$$\sum_I d_I Q_I = J_1 + J_2 + R_y + R_z = H - R_x. \tag{25}$$

With the degrees (24) the contributions with $m_y + m_z \geq 1$ decouple.

Let us see how the decoupling occurs with the degrees in Eq. (24). In the toy model with Eq. (17), the expansion with $\deg(q) = +1$ gives the nontrivial expansion (16), whereas $\deg(q) = -1$ gives the trivial one. This occurs as follows. Each term q^k in the letter index gives the factor $1/(1 - q^k)$ in the plethystic exponential, and if $d = \deg(q)$ is negative, the t-expansion of this factor starts with $-t^{|d|}q^{-1}$. Namely, each negative-degree term in the letter index gives a positive power of t in the plethystic exponential, and if we have infinitely many such terms, the result becomes $t^{+\infty} = 0$. Based on this, we obtain the following simple criterion for the decoupling:

- **The decoupling criterion:**
  If the letter index includes infinitely many negative-degree terms with positive coefficients, its plethystic exponential is trivial and the contribution decouples.[3]

Let us apply the criterion to F_{m_x,m_y,m_z} for $\mathcal{N} = 4$ $U(N)$ SYM and show the decoupling for $m_y + m_z \geq 1$. As we explained, F_{m_x,m_y,m_z} is given in Eq. (8) with the adjoint contributions (11). In particular, if $m_y \geq 1$, $i_y[m_y]$ includes

$$\sigma_y f_{\text{vec}} = 1 - \frac{(1 - y^{-1})(1 - q)(1 - p)}{(1 - z)(1 - x)}, \tag{26}$$

corresponding to the constant term (the Cartan part) in $\chi_{\text{adj}}^{U(m_y)}$. If we adopt the degrees in Eq. (24), this letter index contains infinitely many negative-degree terms of the form $x^k y^{-1}$ ($k = 0$, 1, 2, …). Therefore, the contributions with $m_y \geq 1$ decouple. This is also the case for $i_z[m_z]$ with $m_z \geq 1$. This does not happen to $i_x[m_x]$ because

$$\sigma_x f_{\text{vec}} = 1 - \frac{(1 - x^{-1})(1 - q)(1 - p)}{(1 - y)(1 - z)} \tag{27}$$

does not contain negative-degree terms and $F_{m_x,0,0}$ with $m_x \geq 1$ give nontrivial contributions.

An advantage of the simple-sum expansion is that we can calculate $F_{m_x,0,0}$ much more easily than general contributions from intersecting branes. By using the relation $f_{X=0} = \sigma_X f_{\text{vec}}$ we can relate $F_{m,\,0,\,0}$ and $I_{U(m)}$ by

$$\mathrm{F}_{m,0,0} = \int_{U(m)} dg\, \mathrm{Pexp}\left(\sigma_x f_{\text{vec}} \chi_{\text{adj}}^{U(m)}\right) = \sigma_x I_{U(m)}. \tag{28}$$

Therefore, the expansion (14) can be written as

$$\frac{I_{U(N)}}{I_{U(\infty)}} = \sum_{m=0}^{\infty} x^{mN} \sigma_x I_{U(m)}. \tag{29}$$

A purpose of this paper is to discuss generalization of the simple-sum expansions to orbifold and orientifold theories. We will not give comprehensive analysis. We demonstrate in a few examples that the decoupling occurs and the triple-sum expansion reduces to the simple-sum expansion. We use the decoupling criterion above as a main tool to check the decoupling and we numerically test that the simple-sum expansion actually gives the correct index.

An interesting point of the simple-sum GG expansion is that not only the left-hand side in Eq. (29) but also the right-hand side is given in terms of the superconformal index of 4D theories

[3] We have also to confirm that negative-degree terms with negative coefficients does not give the factor $t^{-\infty}$ canceling the $t^{+\infty}$. In the following examples we can easily confirm it.

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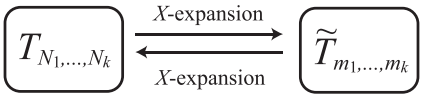

Fig. 1. $T_{N_1,\ldots,N_k}$ and $\widetilde{T}_{m_1,\ldots,m_k}$ are mutually related by the X-expansions.

labeled by the rank m of the gauge group. We can thus consider the large-m limit. In fact, the theory also has the holographic dual, and we can apply the giant graviton expansion to the theory again. We will show in some examples that the expansion of the "dual" theory gives the original theory. Namely, the relation is mutual and invertible. For $\mathcal{N}=4$ $U(N)$ SYM, this relation is "self-dual," but in general two theories may be different. In the following sections we demonstrate how we can obtain the "dual" theory from the original one.

Another interesting point of the orbifold and orientifold theories is that three fugacities x, y, and z may not be symmetric. As we mentioned above, the degrees (24) give a simple-sum expansion associated with the cycle $X=0$. Let us call such an expansion "the X-expansion." Similarly, we can also define the Y-expansion and the Z-expansion associated with $Y=0$ and $Z=0$, respectively. For $\mathcal{N}=4$ SYM this does not give anything new because of the symmetry among fugacities. However, in more general cases with fewer supersymmetries, the three simple-sum expansions may give different expansions for a single theory.

It is natural to ask if the simple-sum expansion works for more general examples like $AdS_5\times \mathrm{SE}_5$, where SE_5 is a Sasaki–Einstein fivefold. It is hard to believe that the simple-sum expansion works for such a case because the expansion is based on the analysis of fluctuation modes on branes wrapped around a specific supersymmetric three-cycle in SE_5, and the global structure of the manifold cannot be captured. We will discuss the decoupling of supersymmetric cycles for toric SE_5 based on the decoupling criterion above, and show that the simple-sum expansion works only for SE_5 whose toric diagram is a triangle. This means that the SE_5 needs to be an orbifold of S^5 for the simple-sum expansion to work.

This paper is organized as follows. In Sect. 2, we study giant graviton expansions for the $\mathbb{Z}_k$ orbifold with $\mathcal{N}=2$ supersymmetry. In Sect. 3, we discuss $O3$ orientifold models with $\mathcal{N}=4$ supersymmetry. In Sect. 4, we consider another orientifold projection with an $O7$-plane. In Sect. 5, we discuss extension to toric quiver gauge theories. Sect. 6 is devoted to discussion.

2. $S^5/\mathbb{Z}_k$

In this section we consider $\mathcal{N}=2$ quiver gauge theory realized on probe D3-branes in a $\mathbb{C}^2/\mathbb{Z}_k$ background, which we call $T_{N_1,\ldots,N_k}$. The AdS/CFT correspondence of these theories was studied in Refs. [40–42]. See Ref. [34] for analytic results for the Schur limit of the index.

We will find that the decoupling works for X-expansion. By the X-expansion we obtain a dual theory as the theory on GG, which we call $\widetilde{T}_{m_1,\ldots,m_k}$. We also find that the X-expansion of $\widetilde{T}_{m_1,\ldots,m_k}$ gives the original theory $T_{N_1,\ldots,N_k}$ (Fig. 1).

2.1. *Boundary theories:* $T_{N_1,\ldots,N_k}$

2.1.1. *Projection.* We consider the boundary theory obtained from the $\mathcal{N}=4$ $U(N)$ SYM by the $\mathbb{Z}_k$ orbifold projection with the generator

$$U_k=\exp\left(\frac{2\pi i}{k}(R_x-R_y)\right). \tag{30}$$

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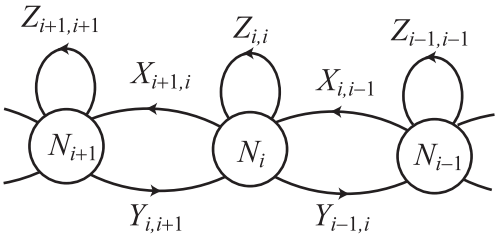

Fig. 2. A part of the circular quiver diagram of the $\mathbb{Z}_k$ orbifold theory.

The field contents are obtained from that of the $\mathcal{N}=4$ $U(N)$ SYM by picking up the $\mathbb{Z}_k$ invariant degrees of freedom [43]. The insertion of U_k in the trace of the superconformal index is realized by the following $\mathbb{Z}_k$ action on the fugacities:

$$(q, p, x, y, z) \to \left(q, p; \omega_k x, \omega_k^{-1} y, z\right), \quad \omega_k = e^{\frac{2\pi i}{k}}. \tag{31}$$

In addition, the action on the Chan–Paton factor is realized by the following $\mathbb{Z}_k$ action on the gauge fugacities:

$$\zeta_a \to \omega_k^{h_a} \zeta_a \quad (a = 1, \ldots, N), \tag{32}$$

where h_a are holonomy variables and each component takes value in $\mathbb{Z}/(k\mathbb{Z}) = \{1, \ldots, k-1, k\}$, and without losing generality we can assume $h_1 \le h_2 \le \cdots \le h_N$ (by using the Weyl group). In other words, holonomy variables are given by

$$\{h_a\} = \{1^{N_1}, 2^{N_2}, \ldots, k^{N_k}\}, \tag{33}$$

where N_i are nonnegative integers constrained by $N_1 + N_2 + \cdots + N_k = N$. The gauge group $U(N)$ is broken by the orbifolding to $G_{\rm UV} = U(N_1) \times \cdots \times U(N_k)$, and flows in the IR to $G_{\rm IR} = SU(N_1) \times \cdots \times SU(N_k)$. The resulting theory is the quiver gauge theory shown in Fig. 2, which we denote by $T_{N_1, N_2, \ldots, N_k}$.

In the IR the diagonal subgroups $U(1)_i \subset U(N_i)$ become global baryonic symmetries. We will put off the discussion of the baryonic charges, and we here focus on the sector with vanishing baryonic charges. In the index calculation this is equivalent to carrying out the gauge fugacity integral not for $G_{\rm IR}$ but for $G_{\rm UV}$ including $U(1)_i$.

In the following we first consider the case with $N_i = \overline{N}$ (and so $N = k\overline{N}$). The superconformal index of the nonbaryonic sector of $T_{\overline{N},\ldots,\overline{N}}$ is given by

$$I^0_{T_{\overline{N},\ldots,\overline{N}}} = \int_{G_{\rm UV}} dg \, \mathrm{Pexp}\left(P_k\left[f_{\rm vec}\chi^{U(N)}_{\rm adj}\right]\right), \tag{34}$$

where the superscript "0" indicates this is the index for the nonbaryonic sector and P_k denotes the projection associated with the $\mathbb{Z}_k$ action (31,32) on the fugacities:

$$P_k\left[f_{\rm vec}(x, y, z, q, p)\chi^{U(N)}_{\rm adj}(\zeta_a)\right] = \frac{1}{k}\sum_{\omega \in \mathbb{Z}_k} f_{\rm vec}(\omega x, \omega^{-1} y, z; q, p)\chi^{U(N)}_{\rm adj}(\omega^{h_a}\zeta_a). \tag{35}$$

Note that we do not remove the contribution from the k IR-free $U(1)$ vector multiplets in Eq. (34).

The large-N limit of the theory $T_\infty = \lim_{\overline{N}\to\infty} T_{\overline{N},\ldots,\overline{N}}$ is dual to $AdS_5 \times (S^5/\mathbb{Z}_k)$ where the $\mathbb{Z}_k$ action on S^5 is given by Eq. (30). The fixed locus of the orbifold is $AdS_5 \times S^1$. The index is

$$\begin{aligned} I_{T_\infty} &= \mathrm{Pexp}\left[P_k(f_{\rm sugra}) + (k-1) f^{AdS_5\times S^1}_{\rm tensor}\right] \\ &= \mathrm{Pexp}\left(\frac{1}{1-x^k} + \frac{1}{1-y^k} + k\frac{1}{1-z} - k\frac{q}{1-q} - k\frac{p}{1-p}\right). \end{aligned} \tag{36}$$

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The letter index consists of two parts: the contribution from the supergravity multiplet in the ten-dimensional bulk and the contribution from $k-1$ tensor multiplets living on the six-dimensional fixed locus. $f_{\text{tensor}}^{AdS_5\times S^1}$ is the letter index of a single tensor multiplet. See Appendix B.1 for a derivation of $f_{\text{tensor}}^{AdS_5\times S^1}$.

2.1.2. *Decoupling.* As in the S^5 case, the index of the theory on a system of giant gravitons with wrapping numbers m_x, m_y, and m_z is given by Eq. (8) with different $i_I[m_I]$ (and different terms represented by dots). In the orbifold case $i_I[m_I]$ are given by

$$i_I[m_I] = P_k\left(\sigma_I f_{\text{vec}}\chi_{\text{adj}}^{U(m_I)}\right), \tag{37}$$

where P_k is defined by Eq. (35). Let us focus on the constant term in the adjoint characters. For each cycle we obtain

$$\begin{aligned}
P_k[\sigma_x f_{\text{vec}}] &= 1-\frac{(1-x^{-1}y^{k-1})(1-q)(1-p)}{(1-y^k)(1-z)},\\
P_k[\sigma_y f_{\text{vec}}] &= 1-\frac{(1-y^{-1}x^{k-1})(1-q)(1-p)}{(1-x^k)(1-z)},\\
P_k[\sigma_z f_{\text{vec}}] &= 1-\frac{(1-z^{-1})(1-x^k y^k)(1-q)(1-p)}{(1-x^k)(1-y^k)(1-xy)}.
\end{aligned} \tag{38}$$

Let us first consider the Z-expansion defined with the degrees $\deg(q,p,x,y,z)=(1,1,1,1,0)$. We can easily see that none of the three functions in Eq. (38) contains negative-degree terms for $k\geq 2$. This means all three cycles give nontrivial contributions, and the triple-sum expansion does not reduce to the simple-sum expansion.

Next, let us consider X-expansion defined with the degrees $\deg(q,p,x,y,z)=(1,1,0,1,1)$. In this case, the functions $P_k[\sigma_y f_{\text{vec}}]$ and $P_k[\sigma_z f_{\text{vec}}]$ contain infinitely many negative-degree terms. Then, the contributions with $m_y+m_z\geq 1$ become trivial, and we obtain the simple-sum expansion.

As we will discuss in detail in the next subsection, the gauge group $U(m)$ on m coincident giant gravitons is broken down to $\widetilde{G}_{\text{UV}}=U(m_1)\times\cdots\times U(m_k)$ due to nontrivial holonomies on the giant gravitons. We denote the theory by $\widetilde{T}_{m_1,m_2,\ldots,m_k}$. The baryonic charges in $T_{\overline{N},\ldots,\overline{N}}$ are related to the ranks m_k of the unbroken gauge groups, and vanishing baryonic charges correspond to the gauge groups with equal ranks: $m_1=m_2=\cdots=m_k=:\overline{m}$. We denote such a theory by $\widetilde{T}_{\overline{m},\ldots,\overline{m}}$. This means that only the terms with wrapping number $m=k\overline{m}$ contribute to the index of the nonbaryonic sector. Therefore, the simple-sum expansion takes the form

$$\frac{I^0_{T_{\overline{N},\ldots,\overline{N}}}}{I_{T_\infty}}=\sum_{\overline{m}=0}^{\infty}x^{k\overline{m}\overline{N}}\sigma_x I^0_{\widetilde{T}_{\overline{m},\ldots,\overline{m}}}, \tag{39}$$

where the index appearing on the right-hand side is that for the nonbaryonic sector because the baryonic charges in $\widetilde{T}_{\overline{m},\ldots,\overline{m}}$ are related to the ranks N_i, and we consider the special case with all N_i being the same.

2.2. *Theories on giant gravitons:* $\widetilde{T}_{m_1,\ldots,m_k}$

As we discussed in Sect. 2.1.2 the X-expansion of the index $I_{T_{\overline{N},\ldots,\overline{N}}}$ gives $I_{\widetilde{T}_{\overline{m},\ldots,\overline{m}}}$, which is the index of the theory realized on the worldvolume of $m=k\overline{m}$ coincident giant gravitons with the unbroken gauge group $\widetilde{G}_{\text{UV}}=U(\overline{m})^k$.

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2.2.1. *Projection.* The theory on giant gravitons on the cycle $X=0$ is also $\mathbb{Z}_k$ orbifold theory, and the $\mathbb{Z}_k$ action is obtained from the action in the original theory by the operator map (9):

$$\widetilde{U}_k = \sigma_x U_k \sigma_x = \exp\left(-\frac{2\pi i}{k}J_1 - \frac{2\pi i}{k}R_x\right), \tag{40}$$

where U_k is the $\mathbb{Z}_k$ generator in Eq. (30). This $\mathbb{Z}_k$ generator contains J_1 nontrivially acting on the boundary coordinates, and the boundary becomes $\mathbb{R}_t \times S^3/\mathbb{Z}_k$, which has a fixed locus $S^1 \times \mathbb{R}_t$. The theory is locally the $\mathcal{N}=4$ $U(m)$ SYM and $\mathbb{Z}_k$ identification breaks the gauge symmetry to $\widetilde{G}_{\rm UV} = U(m_1) \times \cdots U(m_k)$.

The insertion of the operator $\widetilde{U}_k$ in the trace of the index is realized by the following $\mathbb{Z}_k$ action on the fugacities:

$$(q, p; x, y, z) \to \left(\omega_k^{-1} q, p; \omega_k^{-1} x, y, z\right). \tag{41}$$

The gauge fugacities are also transformed by

$$\zeta_a \to \omega_k^{\widetilde{h}_a} \zeta_a \quad (a = 1, \ldots, m), \tag{42}$$

where $\widetilde{h}_a$ are holonomy variables which take values in $\mathbb{Z}/(k\mathbb{Z}) = \{1, \ldots, k\}$. Without losing generality we assume

$$\{\widetilde{h}_a\} = \{1^{m_1}, 2^{m_2}, \ldots, k^{m_k}\}, \tag{43}$$

where m_i are nonnegative integers constrained by $m_1 + \cdots + m_k = m$. The holonomy breaks the $U(m)$ gauge symmetry to $\widetilde{G}_{\rm UV} = U(m_1) \times \cdots \times U(m_k)$. As we mentioned at the end of the previous subsection the vanishing baryonic charges in $T_{\overline{N},\ldots,\overline{N}}$ correspond to holonomies with $m_1 = \cdots = m_k =: \overline{m}$. With such a choice of holonomy, the index is given by

$$I^0_{\widetilde{T}_{\overline{m},\ldots,\overline{m}}} = \int_{\widetilde{G}_{\rm UV}} dg \, \mathrm{Pexp}\left[\widetilde{P}_k\left(f_{\rm vec}\chi^{U(m)}_{\rm adj}\right)\right], \tag{44}$$

where $\widetilde{P}_k$ is the projection associated with the $\mathbb{Z}_k$ actions Eqs. (41) and (42), and is explicitly defined by

$$\widetilde{P}_k f_{\rm vec}\chi^{U(m)}_{\rm adj} = \frac{1}{k}\sum_{\omega\in\mathbb{Z}_k} f_{\rm vec}(\omega^{-1}q, p, \omega^{-1}x, y, z)\chi^{U(m)}_{\rm adj}(\omega^{\widetilde{h}_a}\zeta_a). \tag{45}$$

The index of the large-$\overline{m}$ limit $\widetilde{T}_\infty = \lim_{\overline{m}\to\infty}\widetilde{T}_{\overline{m},\ldots,\overline{m}}$ is given by

$$\begin{aligned} I_{\widetilde{T}_\infty} &= \mathrm{Pexp}\left(\widetilde{P}_k(f_{\rm sugra}) + (k-1)f^{AdS_3\times S^3}_{\rm tensor}\right) \\ &= \mathrm{Pexp}\left(\frac{x^k}{1-x^k} + k\frac{y}{1-y} + k\frac{z}{1-z} - \frac{q^k}{1-q^k} - k\frac{p}{1-p}\right). \end{aligned} \tag{46}$$

On the gravity side this is reproduced as the contributions from the gravity multiplet in the ten-dimensional bulk and $k-1$ tensor multiplets on the six-dimensional fixed locus $AdS_3 \times S^3$. See Appendix B.2 for the explicit form of $f^{AdS_3\times S^3}_{\rm tensor}$.

2.2.2. *Decoupling.* Let us consider the triple-sum expansion of $I_{\widetilde{T}_{\overline{m},\ldots,\overline{m}}}$. The functions F_{m_x,m_y,m_z} appearing in the expansion take the form of Eq. (8) with $i_I[m_I]$ given by

$$i_I[m_I] = \widetilde{P}_k\left(\sigma_I f_{\rm vec}\chi^{U(m_I)}_{\rm adj}\right), \tag{47}$$

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which are similar to Eq. (37) but P_k is replaced by $\widetilde{P}_k$. Let us focus on the constant terms in the adjoint characters $\chi_{\text{adj}}^{U(m_I)}$. For each cycle we have

$$\begin{aligned}
\widetilde{P}_k[\sigma_x f_{\text{vec}}] &= 1 - \frac{(1+x^{-1}q)(1-p)}{(1-y)(1-z)}, \\
\widetilde{P}_k[\sigma_y f_{\text{vec}}] &= 1 - \frac{(1-y^{-1})(1-qx^{k-1})(1-p)}{(1-x^k)(1-z)}, \\
\widetilde{P}_k[\sigma_z f_{\text{vec}}] &= 1 - \frac{(1-z^{-1})(1-qx^{k-1})(1-p)}{(1-x^k)(1-y)}.
\end{aligned} \tag{48}$$

Let us first consider whether simple-sum Y-expansion works. With the degrees $\deg(q, p, x, y, z) = (1, 1, 1, 0, 1)$, only $\widetilde{P}_k\sigma_z f_{\text{vec}}$ contains infinitely many negative-degree terms, and the decoupling works only partially. Therefore, we cannot obtain simple-sum expansion associated with the cycle $Y = 0$ (and such is also the case for $Z = 0$).

On the other hand, with the degrees $\deg(q, p, x, y, z) = (1, 1, 0, 1, 1)$, $\widetilde{P}_k[\sigma_y f_{\text{vec}}]$ and $\widetilde{P}_k[\sigma_z f_{\text{vec}}]$ contain infinitely many negative-degree terms, and the corresponding cycles decouple. Therefore, the triple sum reduces to the simple sum in the form

$$\frac{I^0_{\widetilde{T}_{\overline{m},\ldots,\overline{m}}}}{I_{\widetilde{T}_\infty}} = \sum_{\overline{N}=0}^{\infty} x^{k\overline{m}\overline{N}} \sigma_x I^0_{T_{\overline{N},\ldots,\overline{N}}}. \tag{49}$$

On the right-hand side the index of the original theory $T_{\overline{N},\ldots,\overline{N}}$ appears because σ_x is an involution and applying σ_x on $\widetilde{U}_k$ gives the original $\mathbb{Z}_k$ generator U_k.

2.3. *Baryonic charges*

In the analysis in the previous subsections we focused on the nonbaryonic sector, and obtained Eqs. (39) and (49). Let us generalize the relations by including states with nonvanishing baryonic charges.

We can introduce baryonic charges on both sides of the relation. We use B_i and $\widetilde{B}_i$ to denote the charges in $T_{\overline{N},\ldots,\overline{N}}$ and those in $\widetilde{T}_{\overline{m},\ldots,\overline{m}}$, respectively. As we will explain shortly, baryonic charges on one side are related to the ranks of the unbroken gauge symmetries on the other side. Namely, B_i are identified with m_i and $\widetilde{B}_i$ are identified with N_i up to certain equivalence relations. In general, if SU factors had different ranks, the spectrum of gauge-invariant baryonic operators would be complicated. Therefore, in the following we turn on only one of B_i and $\widetilde{B}_i$ for simplicity.

Let us first discuss the baryonic charges B_i in $T_{\overline{N},\ldots,\overline{N}}$. The index of the baryonic sector with charges B_i can be obtained by the insertion of the background charges

$$\prod_{i=1}^{k} \left(\frac{\det g_i}{\det g_{i+1}}\right)^{B_i} = \prod_{i=1}^{k} \left(\frac{\zeta_{i,1}\cdots\zeta_{i,\overline{N}}}{\zeta_{i+1,1}\cdots\zeta_{i+1,\overline{N}}}\right)^{B_i} \tag{50}$$

in Eq. (34). These are defined so that the baryonic operators $\det X_{i+1,i}$ and $\det Y_{i,i+1}$ carry $B_i = +1$ and -1, respectively. Because Eq. (50) is invariant under the shift $B_i \to B_i + c$ with i-independent constant c, the k charges B_i are redundant, and the baryonic sectors are labeled by the equivalence classes $[B_1, \ldots, B_k]$ defined with the equivalence relation

$$B_i \sim B_i + c. \tag{51}$$

This means that only the differences $B_i - B_j$ are physical quantities.

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To extend the GG expansion (39), we should also modify $I^0_{\widetilde{T}_{\overline{m},\dots,\overline{m}}}$ on the right-hand side. The ranks m_i appearing in $\widetilde{T}_{m_1,\dots,m_k}$ are related to the baryonic charges B_i in $T_{\overline{N},\dots,\overline{N}}$. This is because giant gravitons correspond to baryonic operators and baryonic charges correspond to the numbers of giant gravitons with different holonomy variables on them. In the single wrapping sector with $m = 1$ there are k contributions with only one of m_i being 1. They correspond to k baryonic operators $\det X_{i+1,i}$ ($i = 1, \dots, k$). With this relation we identify m_i with the baryonic charges B_i up to the equivalence relation (51).

The giant graviton expansion for the baryonic sector is given by

$$\frac{I^{[B_1,\dots,B_k]}_{T_{\overline{N},\dots,\overline{N}}}}{I_{T_\infty}} = \sum_{(m)\in[B]} x^{(m_1+\cdots+m_k)\overline{N}}\sigma_x I^0_{\widetilde{T}_{m_1,\dots,m_k}}, \tag{52}$$

where $\sum_{(m)\in[B]}$ is the summation over nonnegative integers m_i belonging to the equivalence class $[B_i]$. The large-N limit appearing in the denominator on the left-hand side in Eq. (52) is the same as before. The inverse expansion of Eq. (52) is given by

$$\frac{I^0_{\widetilde{T}_{m_1,\dots,m_k}}}{I_{\widetilde{T}_\infty}} = \sum_{\overline{N}=0}^{\infty} x^{\overline{N}(m_1+\cdots+m_k)}\sigma_x I^{[m_1,\dots,m_k]}_{T_{\overline{N},\dots,\overline{N}}}. \tag{53}$$

We can also consider the baryonic charges $\widetilde{B}_i$ in $\widetilde{T}_{\overline{m},\dots,\overline{m}}$. Then the corresponding ranks N_i change. For arbitrary ranks m_i the spectrum of gauge-invariant baryonic operators is complicated. So, we consider the simple case with equal ranks $m_1 = m_2 = \cdots = m_k =: \overline{m}$. On the $T_{N_1,\dots,N_k}$ side this corresponds to restricting the states to the nonbaryonic sector. The GG expansion (49) becomes

$$\frac{I^{[\widetilde{B}_1,\dots,\widetilde{B}_k]}_{\widetilde{T}_{\overline{m},\dots,\overline{m}}}}{I_{\widetilde{T}_\infty}} = \sum_{(N)\in[\widetilde{B}]} x^{\overline{m}(N_1+\cdots+N_k)}\sigma_x I^0_{T_{N_1,\dots,N_k}}, \tag{54}$$

and its inverse expansion is

$$\frac{I^0_{T_{N_1,\dots,N_k}}}{I_{T_\infty}} = \sum_{\overline{m}=0}^{\infty} x^{\overline{m}(N_1+\cdots+N_k)}\sigma_x I^{[N_1,\dots,N_k]}_{\widetilde{T}_{\overline{m},\dots,\overline{m}}}. \tag{55}$$

2.4. *Numerical tests*

2.4.1. *X-expansion of* $I^0_{T_{\overline{N},\dots,\overline{N}}}$. First let us test the expansion (39) whose right-hand side is an infinite sum over the wrapping number $\overline{m}$. We introduce a cutoff $\overline{m}_{\max}$, and see how the difference

$$\Delta_{T_{\overline{N},\dots,\overline{N}}}(\overline{m}_{\max}) = \frac{I^0_{T_{\overline{N},\dots,\overline{N}}}}{I_{T_\infty}} - \sum_{\overline{m}=0}^{\overline{m}_{\max}} x^{k\overline{m}\overline{N}}\sigma_x I^0_{\widetilde{T}_{\overline{m},\dots,\overline{m}}} \tag{56}$$

changes as we increase $\overline{m}_{\max}$.

To realize the X-expansion we take the degrees $\deg(q, p, x, y, z) = (1, 1, 0, 1, 1)$. To reduce the computational cost, we take the unrefined parametrization

$$(q, p, x, y, z) = \left(tx^{\frac{1}{2}}, tx^{\frac{1}{2}}, x, t, t\right), \tag{57}$$

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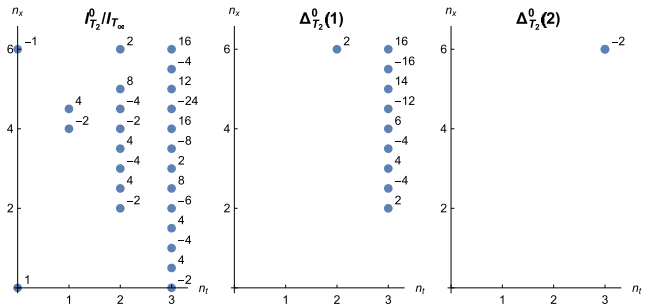

Fig. 3. $I_{T_{2,2}}/I_{T_\infty}$ and $\Delta_{T_{2,2}}(\overline{m}_{\max})$ with $\overline{m}_{\max} = 1, 2$ are shown as 2D plots. A term $t^{n_t}x^{n_x}$ with nonvanishing coefficient in the Taylor expansion is expressed as a dot at the coordinates (n_t, n_x), and the coefficient of the term is shown beside the dot.

and treat the index as a function of two variables t and x. We first expand with respect to t, and then we perform the x-expansion. For example, the ratio $I_{T_{2,2}}/I_{T_\infty}$ is

$$\begin{aligned} I^0_{T_{2,2}}/I_{T_\infty} &= \big(1 - x^6 + o(x^6)\big)t^0 \\ &\quad + \big(-2x^4 + 4x^{\frac{9}{2}} + o(x^6)\big)t^1 \\ &\quad + \big(-2x^2 + 4x^{\frac{5}{2}} - 4x^3 + 4x^{\frac{7}{2}} - 2x^4 - 4x^{\frac{9}{2}} + 8x^5 + 2x^6 + o(x^6)\big)t^2 \\ &\quad + \big(-2 + 4x^{\frac{1}{2}} - 4x + 4x^{\frac{3}{2}} - 6x^2 + 8x^{\frac{5}{2}} + 2x^3 - 8x^{\frac{7}{2}} + 16x^4 - 24x^{\frac{9}{2}} \\ &\quad + 12x^5 - 4x^{\frac{11}{2}} + 16x^6 + o(x^6)\big)t^3 + o(t^3). \end{aligned} \tag{58}$$

If we subtract $\overline{m} = 0$ and $\overline{m} = 1$ contributions many terms are canceled, and further subtraction of the $\overline{m} = 2$ contribution removes more terms appearing in Eq. (58):

$$\begin{aligned} \Delta_{T_{2,2}}(1) &= o(x^6)t^0 + o(x^6)t^1 + (2x^6 + o(x^6))t^2 \\ &\quad + \big(2x^2 - 4x^{\frac{5}{2}} + 4x^3 - 4x^{\frac{7}{2}} + 6x^4 - 12x^{\frac{9}{2}} + 14x^5 - 16x^{\frac{11}{2}} + 16x^6 + o(x^6)\big)t^3 \\ &\quad + o(t^3), \\ \Delta_{T_{2,2}}(2) &= o(x^8)t^0 + o(x^8)t^1 + o(x^8)t^2 + (-2x^6 + o(x^8))t^3 + o(t^3). \end{aligned} \tag{59}$$

We show the cancellation graphically by using 2D plots in Fig. 3.

2.4.2. *X-expansion of* $I^0_{\widetilde{T}_{\overline{m},\ldots,\overline{m}}}$. Let us numerically test the expansion (49) for $\widetilde{T}_{\overline{m},\ldots,\overline{m}}$. We introduce a cutoff $\overline{N}_{\max}$ and calculate the error

$$\Delta_{\widetilde{T}_{\overline{m},\ldots,\overline{m}}}(\overline{N}_{\max}) = \frac{I^0_{\widetilde{T}_{\overline{m}}}}{I_{\widetilde{T}_\infty}} - \sum_{\overline{N}=0}^{\overline{N}_{\max}} x^{k\overline{m}\overline{N}}\sigma_x I^0_{T_{\overline{N},\ldots,\overline{N}}}. \tag{60}$$

We show the ratio $I^0_{\widetilde{T}_{2,2}}/I_{\widetilde{T}_\infty}$ and the errors $\Delta_{\widetilde{T}_{2,2}}(\overline{N}_{\max})$ with $\overline{N}_{\max} = 1$ and 2 as 2D plots in Fig. 4.

2.4.3. *Baryonic sector.* Let us numerically test Eq. (52). As a simple case we consider $k = 2$ and $[B_1, B_2] = [1, 0]$. We define the error function

$$\Delta^{[1,0]}_{T_{\overline{N},\overline{N}}}(c_{\max}) = \frac{I^{[1,0]}_{T_{\overline{N},\overline{N}}}}{I_{T_\infty}} - \sum_{c=0}^{c_{\max}} x^{(2c+1)\overline{N}}\sigma_x I^0_{\widetilde{T}_{c+1,c}}. \tag{61}$$

Figure 5 shows the ratio $I^{[1,0]}_{T_{\overline{N},\overline{N}}}/I_{T_\infty}$ and the error function $\Delta^{[1,0]}_{T_{\overline{N},\overline{N}}}(c_{\max})$ for $c_{\max} = 1$, 2, and 3.

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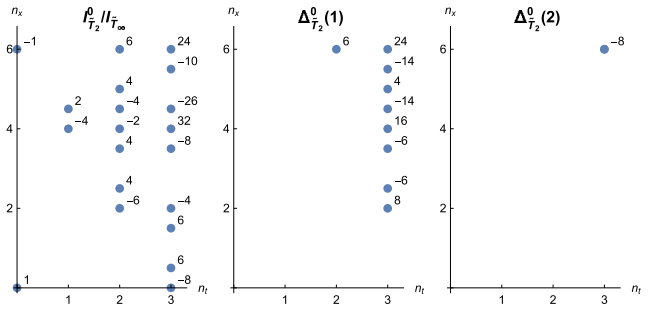

Fig. 4. The ratio $I^0_{\widetilde{T}_{2,2}}/I_{\widetilde{T}_\infty}$ and the errors $\Delta_{\widetilde{T}_{2,2}}(\overline{N}_{\max})$ with $\overline{N}_{\max} = 1$ and 2.

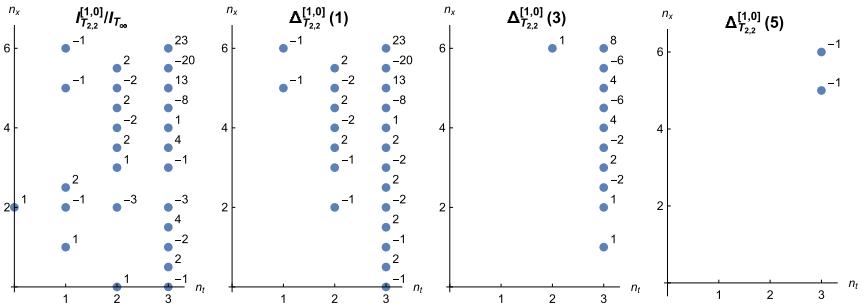

Fig. 5. The ratio $I^{[1,0]}_{T_{\overline{N},\overline{N}}}/I_{T_\infty}$ and the error function $\Delta^{[1,0]}_{T_{\overline{N},\overline{N}}}(c_{\max})$ for $c_{\max} = 1$, 2, and 3 are shown as 2D plots.

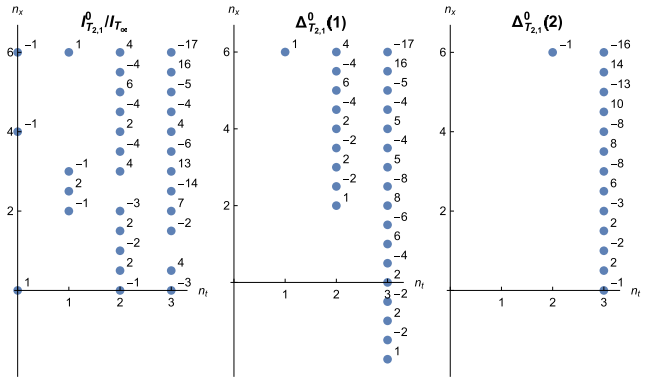

Fig. 6. The ratio $I^0_{T_{3,2}}/I_{T_\infty}$ and the errors $\Delta^0_{T_{3,2}}(\overline{m}_{\max})$ with $\overline{m}_{\max} = 1$ and 2.

We test the expansion (55) for $k = 2$ and $(N_1, N_2) = (3, 2)$. We define the error function

$$\Delta^0_{T_{3,2}}(\overline{m}_{\max}) = \frac{I^0_{T_{3,2}}}{I_{T_\infty}} - \sum_{\overline{m}=0}^{\overline{m}_{\max}} x^{5\overline{m}} \sigma_x I^{[1,0]}_{\widetilde{T}_{\overline{m},\ldots,\overline{m}}}. \tag{62}$$

Numerical results for $I^0_{T_{3,2}}/I_{T_\infty}$ and the errors $\Delta^0_{T_{3,2}}(\overline{m}_{\max})$ with $\overline{m}_{\max} = 1$ and 2 are shown in Fig. 6.

3. O3-D3 system

In this section we discuss $\mathcal{N} = 4$ SYM realized by O3-D3 systems. The AdS/CFT correspondence of the model was first studied in Ref. [10]. We will mainly discuss the case with an $O3^-$-plane, and we denote the corresponding $O(2N)$ SYM by $T_{O(2N)}$. We will find the X-expansion works for $T_{O(2N)}$, and the theory on giant gravitons, which we denote by $\widetilde{T}_{O(2m)}$, is another ori-

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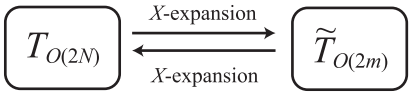

Fig. 7. $T_{O(2N)}$ and $\widetilde{T}_{O(2m)}$ are related by X-expansions.

Table 1. Extended directions of branes. Directions of D3 and $O3$ are shown by using the coordinates in the flat ten-dimensional spacetime. In the description of the directions for giant gravitons the time directions are treated as the radial direction in the XYZ space. The column O/Sp shows the gauge group realized on D-branes when the orientifold plane is $O3^-$. For $O3^+$ all the gauge groups become Sp.

	1	2	3	4	X		Y		Z		O/Sp
D3	✓	✓	✓	✓							O
$O3^-$	✓	✓	✓	✓							
$\mathrm{GG}_{X=0}$							✓	✓	✓	✓	O
$\mathrm{GG}_{Y=0}$					✓	✓			✓	✓	O
$\mathrm{GG}_{Z=0}$					✓	✓	✓	✓			O

entifold theory. The inverse X-expansion also works, and it gives the original theory $T_{O(2N)}$ as the theory on giant gravitons (Fig. 7).

3.1. *Boundary theories:* $T_{O(2N)}$

3.1.1. *Projection.* Let us consider $\mathcal{N}=4$ SYM with orthogonal and symplectic gauge groups realized by orientifolds. We first consider the case with an $O3^-$-plane, which gives the orthogonal gauge groups.

The orientifold is defined with the $O3$ flip operator

$$U_{O3} = e^{\pi i S}, \quad S = R_x + R_y + R_z + A, \tag{63}$$

commuting with all generators in the $\mathcal{N}=4$ superconformal algebra.

Table 1 shows the directions of D3-branes, $O3$-planes, and the worldvolume of giant gravitons. The gauge group (O or Sp) should be chosen according to the relative positions of D-branes and the $O3$-plane.

For the following analysis it is convenient to refine the superconformal index by introducing a $\mathbb{Z}_2$ fugacity $\eta = \pm 1$ for the operator S:

$$\widehat{I}(q,p,x,y,z,\eta) = \mathrm{tr}[(-1)^F q^{J_1} p^{J_2} x^{R_x} y^{R_y} z^{R_z} \eta^S]. \tag{64}$$

Then, the orientifold projection operator P_{O3} acting on letter indices is defined by

$$P_{O3}[\cdots] = [\cdots]_+, \tag{65}$$

where we introduced the notation

$$[\cdots]_\pm = \frac{1}{2}\left[(\cdots)|_{\eta=+1} \pm (\cdots)|_{\eta=-1}\right]. \tag{66}$$

The introduction of the $\mathbb{Z}_2$ fugacity η modifies the variable changes in Eq. (10) as follows [15]:

$$\begin{aligned}
\sigma_x(q,p,x,y,z,\eta) &= (y\eta, z\eta, x^{-1}, q\eta, p\eta, \eta),\\
\sigma_y(q,p,x,y,z,\eta) &= (z\eta, x\eta, p\eta, y^{-1}, q\eta, \eta),\\
\sigma_z(q,p,x,y,z,\eta) &= (x\eta, y\eta, q\eta, p\eta, z^{-1}, \eta).
\end{aligned} \tag{67}$$

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To describe the orientifold action on the Chan–Paton factor it is convenient to define the refined character $\widehat{\chi}^{U(N)}_{\rm adj}$ by

$$\widehat{\chi}^{U(N)}_{\rm adj} = \chi^{(N)}_{+} + \eta\chi^{(N)}_{-}, \tag{68}$$

where $\chi^{(N)}_{\pm}$ for $g \in O(N) \subset U(N)$ are

$$\chi^{(N)}_{+} = \chi^{O(N)}_{\rm adj}, \quad \chi^{(N)}_{-} = \chi^{O(N)}_{\rm sym}, \tag{69}$$

and for $g \in Sp(N/2) \subset U(N)$

$$\chi^{(N)}_{+} = \chi^{Sp(N)}_{\text{anti-sym}}, \quad \chi^{(N)}_{-} = \chi^{Sp(N)}_{\rm adj}. \tag{70}$$

Equation (70) makes sense only for even N. The letter index of the orientifold theory with $O3^-$ is the $\mathbb{Z}_2$-invariant part of the refined $U(N)$ letter index

$$P_{O3}\left[f_{\rm vec}\widehat{\chi}^{U(N)}_{\rm adj}\right] = f_{\rm vec}\chi^{O(N)}_{\rm adj}. \tag{71}$$

When we consider orientifold with $O3^+$ we should replace $\widehat{\chi}^{U(N)}_{\rm adj}$ by $\eta\widehat{\chi}^{U(N)}_{\rm adj}$ and then the orientifold projection gives $f_{\rm vec}\chi^{Sp(N/2)}_{\rm adj}$. With the letter index (71), the full index is given by

$$I_{T_{O(2N)}} = \int_{O(2N)} dg\,\mathrm{Pexp}\left[f_{\rm vec}\chi^{O(2N)}_{\rm adj}\right]. \tag{72}$$

See Appendix A for the explicit forms of the Haar measure and the character.

The large-N limit of the index $I_{T_{O(N)}}$ is given by

$$I_{T_{O(\infty)}} = \mathrm{Pexp}[P_{O3}\widehat{f}_{\rm sugra}]. \tag{73}$$

$\widehat{f}_{\rm sugra}$ is the $\mathbb{Z}_2$-refined index of supergravity Kaluza–Klein modes in $AdS_5 \times S^5$ defined by

$$\widehat{f}_{\rm sugra}|_{\eta=+1} = f_{\rm sugra}, \quad \widehat{f}_{\rm sugra}|_{\eta=-1} = \widetilde{f}_{\rm sugra}, \tag{74}$$

where the $\mathbb{Z}_2$ twisted index $\widetilde{f}_{\rm sugra}$ is given by

$$\begin{aligned}\widetilde{f}_{\rm sugra} &= \sum_{n=1}^{\infty}(-1)^n i\left[\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{[0,n,0](0,0)}\right] \\ &= \frac{1}{2}\left(\frac{(1-x)(1-y)(1-z)(1+q)(1+p)}{(1+x)(1+y)(1+z)(1-q)(1-p)} - 1\right).\end{aligned} \tag{75}$$

(Equation (75) is obtained by directly calculating the alternating sum over n.)

3.1.2. *Decoupling.* Let us discuss whether we can decouple some cycles by assigning appropriate degrees. The giant graviton contribution again takes the form of Eq. (8) with

$$i_I[m_I] = P_{O3}\left[\sigma_I f_{\rm vec}\widehat{\chi}^{U(m_I)}_{\rm adj}\right] = \sum_{\pm}[\sigma_I f_{\rm vec}]_{\pm}\chi^{(m_I)}_{\pm}. \tag{76}$$

Let us assign the degrees $\deg(q, p, x, y, z) = (1, 1, 0, 1, 1)$. The letter index $i_y[m_y]$ with $m_y \geq 1$ includes

$$[\sigma_y f_{\rm vec}]_{\pm} = \sum_{\pm}\left[1 - \frac{(1-y^{-1})(1-\eta q)(1-\eta p)}{(1-\eta y)(1-\eta z)}\right]_{\pm}. \tag{77}$$

We can easily check that for both signs $[\sigma_y f_{\rm vec}]_{\pm}$ contains infinitely many negative-degree terms. It is the case also for $[\sigma_z f_{\rm vec}]_{\pm}$ associated with the cycle $Z = 0$. Therefore, F_{m_x,m_y,m_z} with $m_y + m_z \geq 1$ decouple, and the GG expansion reduces to the simple sum.

If the gauge group is not $SO(2N)$ but $O(2N)$, Pfaffian operators are not gauge invariant. This means only contributions with an even wrapping number should be included. The simple-sum

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Table 2. Extended directions of branes.

	1	2	3	4	X		Y		Z		O/Sp
D3	✓	✓	✓	✓							O
$O3^-$							✓	✓	✓	✓	
$GG_{X=0}$							✓	✓	✓	✓	O
$GG_{Y=0}$					✓	✓			✓	✓	Sp
$GG_{Z=0}$					✓	✓	✓	✓			Sp

giant graviton expansion is given by

$$\frac{I_{T_{O(2N)}}}{I_{T_{O(\infty)}}} = \sum_{m=0}^{\infty} x^{2mN} \sigma_x I_{\widetilde{T}_{O(2m)}}, \tag{78}$$

where we denote the theory on m coincident giant gravitons by $\widetilde{T}_{O(m)}$.

3.2. *Theories on giant gravitions:* $\widetilde{T}_{O(2m)}$

3.2.1. *Projection.* The X-expansion of $I_{T_{O(2N)}}$ is the sum of the contributions from the theories realized on giant gravitons wrapped around $X=0$. Let $\widetilde{T}_{O(m)}$ be the theory realized on m giant gravitons and $I_{\widetilde{T}_{O(m)}}$ be its index. $\widetilde{T}_{O(m)}$ is defined as the orientifold of $\mathcal{N}=4$ $U(m)$ SYM with the orientifold flip operator

$$\widetilde{U}_{O3} = \sigma_x U_{O3} \sigma_x = e^{\pi i(-R_x+J_1+J_2-A)} = e^{\pi i(J_1+J_2+R_y+R_z-S)} \tag{79}$$

where S is defined in Eq. (63). The directions of the worldvolumes of D-branes and the O-plane are shown in Table 2.

Equation (79) nontrivially acts on the AdS boundary, and the orientifold gives a theory in $\mathbb{R}_t \times S^3/\mathbb{Z}_2$. It is locally the same as the $\mathcal{N}=4$ $U(m)$ SYM, but nontrivial holonomy breaks the gauge symmetry down to $O(m)$. The fixed locus is an $O3$-plane, which is localized at the center of AdS_5 and wrapped around $S^3 \subset S^5$ given by $X=0$.

The insertion of $\widetilde{U}_{O3}$ in the trace of the $\mathbb{Z}_2$-refined index (64) is equivalent to the sign flips for some fugacities

$$(q, p, x, y, z, \eta) \to (-q, -p, x, -y, -z, -\eta). \tag{80}$$

Therefore, the orientifold projection operator acting on letter indices is defined by

$$\begin{aligned} \widetilde{P}_{O3}\mathrm{F}(q, p, x, y, z, \eta) &= \frac{1}{2}(\mathrm{F}(q, p, x, y, z, +1) + \mathrm{F}(-q, -p, x, -y, -z, -1)) \\ &= [\mathrm{F}(\eta q, \eta p, x, \eta y, \eta z, \eta)]_+. \end{aligned} \tag{81}$$

With this projection, the index $I_{\widetilde{T}_{O(m)}}$ is given by

$$I_{\widetilde{T}_{O(m)}} = \int_{O(m)} dg\, \mathrm{Pexp}\left[\widetilde{P}_{O3}\left(f_{\mathrm{vec}}\widehat{\chi}_{\mathrm{adj}}^{U(m)}\right)\right]. \tag{82}$$

The large-m limit of the index $I_{\widetilde{T}_{O(m)}}$ is given by

$$I_{\widetilde{T}_{O(\infty)}} = \mathrm{Pexp}[\widetilde{P}_{O3}\widehat{f}_{\mathrm{sugra}}] = \mathrm{Pexp}\,\frac{f_{\mathrm{sugra}}(q, p, x, y, z) + \widetilde{f}_{\mathrm{sugra}}(-q, -p, x, -y, -z)}{2}. \tag{83}$$

3.2.2. *Decoupling.* The index for giant gravitons can be easily obtained by combining the variable changes and the $\widetilde{U}_{O3}$-projection. The functions F_{m_x,m_y,m_z} are given by Eq. (8) with

$$i_x[m_x] = \widetilde{P}_{O3}[\sigma_x f_{\mathrm{vec}}\widehat{\chi}_{\mathrm{adj}}^{U(m_x)}], \quad i_I[m_I] = \widetilde{P}_{O3}[\sigma_I f_{\mathrm{vec}}\eta\widehat{\chi}_{\mathrm{adj}}^{U(m_I)}] \quad (\text{for } I = y, z). \tag{84}$$

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These are explicitly given by

$$\begin{aligned} i_x[m_x] &= \left(1 - \frac{(1-x^{-1})(1-q)(1-p)}{(1-y)(1-z)}\right) \chi_{\text{adj}}^{O(m_x)}, \\ i_y[m_y] &= \sum_{\pm} \left[1 - \frac{(1-\eta y^{-1})(1-q)(1-p)}{(1-z)(1-\eta x)}\right]_{\pm} \chi_{\mp}^{(m_y)}, \\ i_z[m_z] &= \sum_{\pm} \left[1 - \frac{(1-\eta z^{-1})(1-q)(1-p)}{(1-y)(1-\eta x)}\right]_{\pm} \chi_{\mp}^{(m_z)}. \end{aligned} \tag{85}$$

We can easily check that both $i_y[m_y]$ with $m_y \geq 1$ and $i_z[m_z]$ with $m_z \geq 1$ contain infinitely many negative-degree terms provided we assign the degrees $\deg(q, p, x, y, z) = (1, 1, 0, 1, 1)$. Hence, the X-expansion reduces to the simple-sum expansion of the form

$$\frac{I_{\widetilde{T}_{O(2m)}}}{I_{\widetilde{T}_{O(\infty)}}} = \sum_{N=0}^{\infty} x^{2mN} \sigma_x I_{T_{O(2N)}}. \tag{86}$$

3.3. *Sp(N), SO(2N + 1), and SO(2N)*

If we replace $O3^-$ with $O3^+$, we obtain the theory with a symplectic gauge group. In the index calculation, this is realized by replacing $\widehat{\chi}_{\text{adj}}^{U(N)}$ in Eqs. (72) and (82) with $\eta\widehat{\chi}_{\text{adj}}^{U(N)}$. The decoupling again works, and instead of Eqs. (78) and (86) we obtain the expansion

$$\frac{I_{T_{Sp(N)}}}{I_{T_{Sp(\infty)}}} = \sum_{m=0}^{\infty} x^{2mN} \sigma_x I_{\widetilde{T}_{Sp(m)}}, \tag{87}$$

and the inverse expansion

$$\frac{I_{\widetilde{T}_{Sp(m)}}}{I_{\widetilde{T}_{Sp(\infty)}}} = \sum_{N=0}^{\infty} x^{2mN} \sigma_x I_{T_{Sp(N)}}. \tag{88}$$

The gauge group $O(2N)$ consists of two disconnected components. One of them is $SO(2N)$, and we denote the other component by $\overline{SO}(2N)$. Correspondingly, the index for $O(2N)$ splits into two parts:

$$I_{T_{O(2N)}} = \frac{1}{2}\left(I_{T_{SO(2N)}} + I_{T_{\overline{SO}(2N)}}\right). \tag{89}$$

The two contributions correspond to the two elements of $\mathbb{Z}_2 = O(2N)/SO(2N)$. This $\mathbb{Z}_2$ symmetry couples to Pfaffian operators. In $O(2N)$ gauge theory this $\mathbb{Z}_2$ is gauged, and Pfaffian operators do not contribute to the index. This is analogous to the baryonic $U(1)$ symmetries studied in Sect. 2. For the latter we can extract the contribution of states with nonvanishing baryonic charges by inserting the factor (50) in the gauge fugacity integral in Eq. (72). This is also possible for the $\mathbb{Z}_2$ charge. By inserting the factor $\det g$ ($g \in O(2N)$) in the integral (72) we obtain the index of the $\mathbb{Z}_2$-odd states:

$$I^{(-)}_{T_{O(2N)}} = \frac{1}{2}\left(I_{T_{SO(2N)}} - I_{T_{\overline{SO}(2N)}}\right). \tag{90}$$

Because a giant graviton which corresponds to a Pfaffian operator is $\mathbb{Z}_2$ odd, only configurations with odd wrapping numbers contribute to Eq. (90), and the expansion (78) is changed to

$$\frac{I^{(-)}_{T_{O(2N)}}}{I_{T_{O(\infty)}}} = \sum_{m=0}^{\infty} x^{(2m+1)N} \sigma_x I_{\widetilde{T}_{O(2m+1)}}. \tag{91}$$

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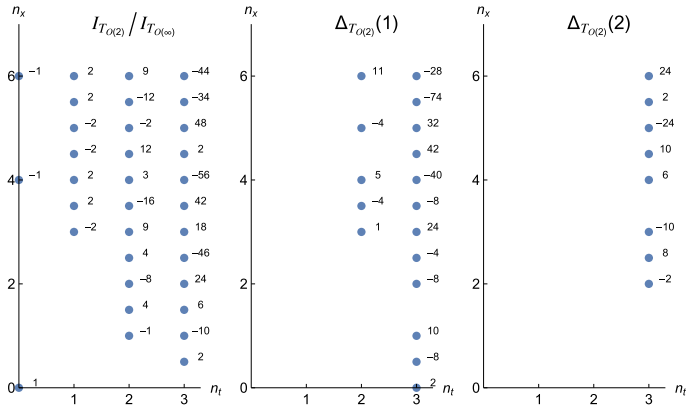

Fig. 8. The ratio $I_{T_{O(2)}}/I_{T_{O(\infty)}}$ and the errors $\Delta_{T_{O(2)}}(m_{\max})$ with $m_{\max} = 1$ and 2.

The sum of Eqs. (78) and (91) gives the expansion for the $SO(2N)$ gauge group

$$\frac{I_{T_{SO(2N)}}}{I_{T_{SO(\infty)}}} = \sum_{m=0}^{\infty} x^{mN} \sigma_x I_{\widetilde{T}_{O(m)}}. \tag{92}$$

The two expansions (78,91) are similar to the expansion (52) for the baryonic sector in the sense that the number of giant gravitons is constrained according to the value of the $\mathbb{Z}_2$ charge.

We can interchange the roles of the $\mathbb{Z}_2$ charge and the number of branes, and obtain the following expansion similar to Eq. (55):

$$\frac{I_{T_{O(2N+1)}}}{I_{T_{O(\infty)}}} = \sum_{m=0}^{\infty} x^{(2N+1)m} \sigma_x I^{(-)}_{\widetilde{T}_{O(2m)}}, \tag{93}$$

where $I^{(-)}_{\widetilde{T}_{O(2m)}}$ is defined with the $\det g$ insertion into Eq. (82), and given in a similar way to Eq. (90).

The left-hand side of Eq. (93) should be identical with the left-hand side of Eq. (87) due to the Montonen–Olive duality. The consistency requires a nontrivial relation between $I_{\widetilde{T}_{Sp(m)}}$ and $I^{(-)}_{\widetilde{T}_{O(2m)}}$ appearing in the expansions. In fact, we can numerically confirm the following relation holds:

$$I_{\widetilde{T}_{Sp(m)}} = x^{-m} I^{(-)}_{\widetilde{T}_{O(2m)}}. \tag{94}$$

This is the counterpart of the Montonen–Olive duality on the giant graviton side.

3.4. *Numerical tests*

3.4.1. *X-expansion of* $I_{T_{O(2N)}}$. Let us check the expansion of Eq. (78) for $T_{O(2N)}$. Here and in the following numerical tests, we again employ the unrefined parametrization (57). We introduce a cutoff $m_{\max}$, and calculate the error function below for $N = 1$,

$$\Delta_{T_{O(2N)}}(m_{\max}) = \frac{I_{T_{O(2N)}}}{I_{T_{O(\infty)}}} - \sum_{m=0}^{m_{\max}} x^{2mN} \sigma_x I_{\widetilde{T}_{O(2m)}}. \tag{95}$$

The results are shown in Fig. 8 as 2D plots.

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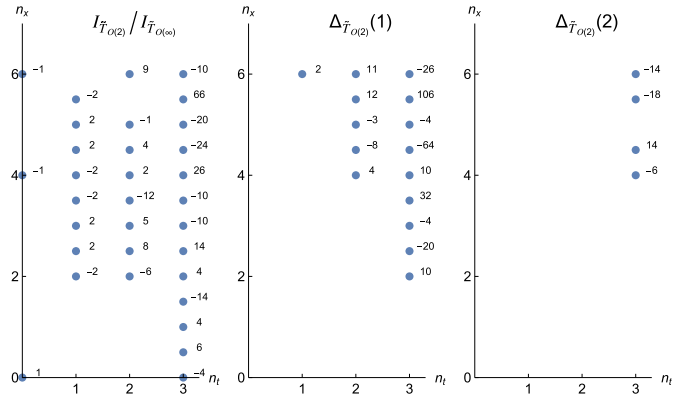

Fig. 9. The ratio $I_{\tilde{T}_{O(2)}}/I_{\tilde{T}_{O(\infty)}}$ and the errors $\Delta_{\tilde{T}_{O(2)}}(m_{\max})$ with $m_{\max} = 1$ and 2.

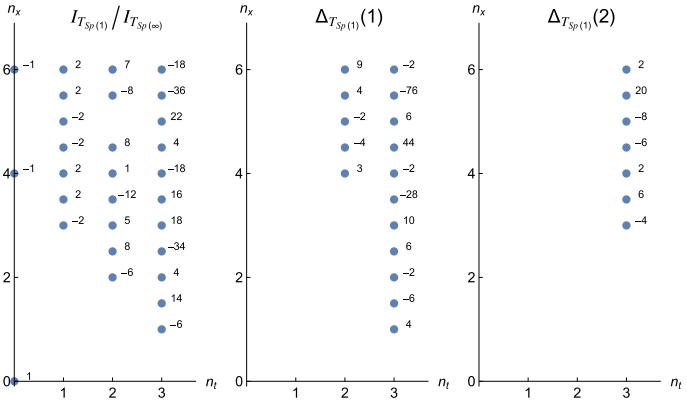

Fig. 10. The ratio $I_{T_{Sp(1)}}/I_{T_{Sp(\infty)}}$ and the errors $\Delta_{T_{Sp(1)}}(m_{\max})$ with $m_{\max} = 1$ and 2.

3.4.2. *X-expansion of* $I_{\tilde{T}_{O(2N)}}$. Let us check the inverse expansion for $T_{O(2N)}$, namely, the expansion of $I_{\tilde{T}_{O(2N)}}$ (86). We calculate the error function

$$\Delta_{\tilde{T}_{O(2N)}}(m_{\max}) = \frac{I_{\tilde{T}_{O(2N)}}}{I_{\tilde{T}_{O(\infty)}}} - \sum_{m=0}^{m_{\max}} x^{2mN}\sigma_x I_{T_{O(2m)}} \tag{96}$$

for $N = 1$ with cutoff $m_{\max} = 1$ and 2. The results are shown in Fig. 9.

3.4.3. *X-expansion of* $I_{T_{Sp(N)}}$, $I_{T_{SO(2N+1)}}$, *and* $I_{T_{SO(2N)}}$. Firstly, let us check the expansion of $I_{T_{Sp(N)}}$ (87) and its inverse expansion (88). We introduce cutoff $m_{\max}$ and calculate the error functions

$$\Delta_{T_{Sp(N)}}(m_{\max}) = \frac{I_{T_{Sp(N)}}}{I_{T_{Sp(\infty)}}} - \sum_{m=0}^{m_{\max}} x^{2mN}\sigma_x I_{\tilde{T}_{Sp(m)}}, \tag{97}$$

$$\Delta_{\tilde{T}_{Sp(N)}}(m_{\max}) = \frac{I_{\tilde{T}_{Sp(N)}}}{I_{\tilde{T}_{Sp(\infty)}}} - \sum_{m=0}^{m_{\max}} x^{2mN}\sigma_x I_{T_{Sp(m)}}, \tag{98}$$

for $N = 1$. The results are shown in Figs. 10 and 11.

Secondly, we check the expansion of Eq. (91). We calculate the error function

$$\Delta_{I^{(-)}_{T_{O(2N)}}}(m_{\max}) = \frac{I^{(-)}_{T_{O(2N)}}}{I_{T_{O(\infty)}}} - \sum_{m=0}^{m_{\max}} x^{(2m+1)N}\sigma_x I_{\tilde{T}_{O(2m+1)}} \tag{99}$$

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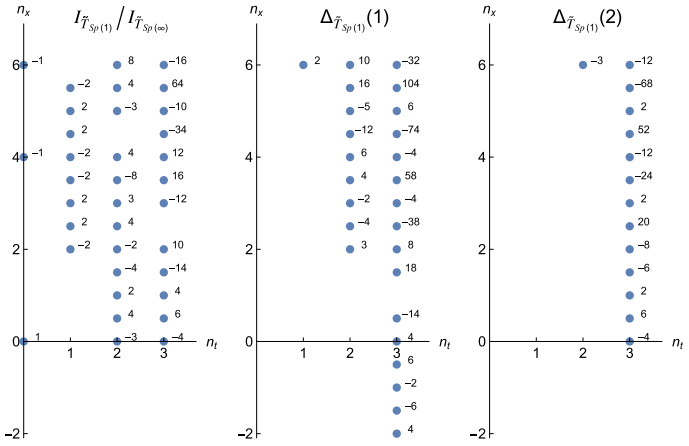

Fig. 11. The ratio $I_{\widetilde{T}_{Sp(1)}}/I_{\widetilde{T}_{Sp(\infty)}}$ and the errors $\Delta_{\widetilde{T}_{Sp(1)}}(m_{\max})$ with $m_{\max}=1$ and 2.

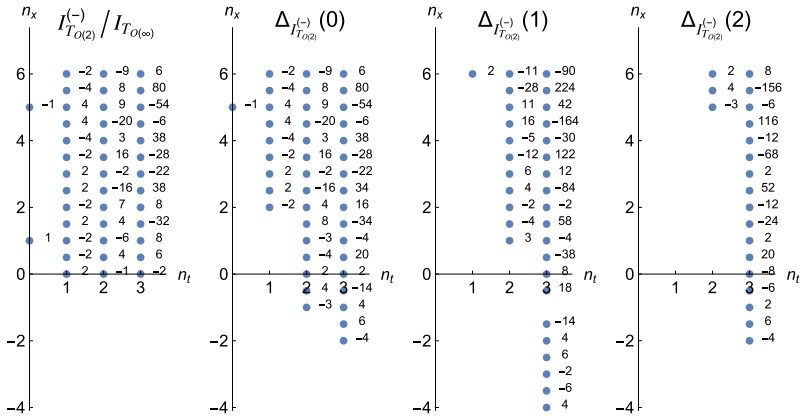

Fig. 12. The ratio $I^{(-)}_{T_{O(2)}}/I_{T_{O(\infty)}}$ and the errors $\Delta_{I^{(-)}_{T_{O(2)}}}(m_{\max})$ with $m_{\max}=0$, 1, and 2.

for $N=1$. The results are shown in Fig. 12.

Finally, we check the expansion of Eq. (93), by calculating the error function

$$\Delta_{T_{O(2N+1)}}(m_{\max})=\frac{I_{T_{O(2N+1)}}}{I_{T_{O(\infty)}}}-\sum_{m=0}^{m_{\max}}x^{(2N+1)m}\sigma_x I^{(-)}_{\widetilde{T}_{O(2m)}} \tag{100}$$

with $m_{\max}=1$ and 2 for $N=1$. The results are shown in Fig. 13.

4. O7-D3 system

In this section we discuss $\mathcal{N}=2$ superconformal field theory (SCFT) realized on D3-branes probing the $O7$-plane background, which we denote by $D_4[N]$. We will find all X-, Y-, and Z-expansions work. X-expansion (which is essentially the same as the Y-expansion up to the exchange $x\leftrightarrow y$) gives another orientifold theory denoted by $\widetilde{D}_4[m]$, whereas Z-expansion gives the original theory. The X-expansion works for $\widetilde{D}_4[m]$ and it gives the original theory $D_4[N]$ (Fig. 14).

4.1. *Boundary theories: $D_4[N]$*

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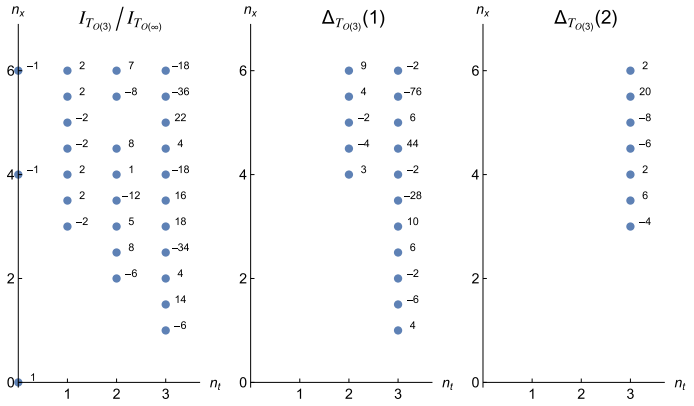

Fig. 13. The ratio $I_{T_{O(3)}}/I_{T_{O(\infty)}}$ and the errors $\Delta_{T_{O(3)}}(m_{\max})$ with $m_{\max} = 1$ and 2.

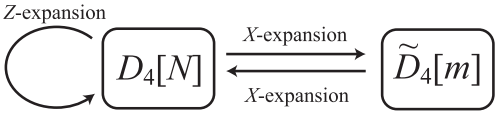

Fig. 14. GG expansions for $D_4[N]$ and $\widetilde{D}_4[m]$.

4.1.1. *Projection.* Let us consider the $\mathcal{N} = 2$ SCFT realized on N D3-branes in the background of an $O7^-$-plane defined with the orientifold flip operator

$$U_{O7} = e^{i\pi(R_z - A)} = e^{i\pi(2R_z + R_x + R_y - S)} = e^{i\pi(R_x - R_y - S)}. \tag{101}$$

The $O7^-$ worldvolume is space-filling in AdS_5 and wraps around $S^3 \subset S^5$ given by $Z = 0$. See Table 3. To keep the conformal invariance we need to introduce four D7-branes (and their mirror images) coincident with the $O7$-plane.

The orientifold projection with Eq. (101) breaks the $SO(6)_R$ symmetry down to $SU(2)_R \times U(1)_R \times SU(2)_F$, with Cartan operators $R_x + R_y$, R_z, and $R_x - R_y$. The theory realized on the worldvolume of N D3-branes is an $\mathcal{N} = 2$ $Sp(N)$ SCFT [44], which we call $D_4[N]$. It has $SO(8)$ flavor symmetry realized on the 7-branes, and we can refine the index by introducing $SO(8)$ fugacities.

The insertion of U_{O7} in the $\mathbb{Z}_2$ refined index (64) is equivalent to the variable change

$$(q, p, x, y, z, \eta) \to (q, p, -x, -y, z, -\eta). \tag{102}$$

Correspondingly, we define the projection operator P_{O7} acting on letter indices by

$$\begin{aligned} P_{O7}\mathrm{F}(q, p, x, y, z, \eta) &= \frac{1}{2}[\mathrm{F}(q, p, x, y, z, +1) + \mathrm{F}(q, p, -x, -y, z, -1)] \\ &= [\mathrm{F}(q, p, \eta x, \eta y, z, \eta)]_+. \end{aligned} \tag{103}$$

With this projection operator, the index of $D_4[N]$ is given by

$$I_{D_4[N]} = \int_{Sp(N)} dg\, \mathrm{Pexp}\left(P_{O7}\left[f_{\mathrm{vec}}\eta\widehat{\chi}^{U(2N)}_{\mathrm{adj}}\right] + f_{\mathrm{hyp}}\chi^{SO(8)}_{\mathbf{8}_v}\chi^{Sp(N)}_{\mathrm{fund}}\right). \tag{104}$$

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The first term in the letter index is the contribution from D3-D3 open strings, and is explicitly given by

$$P_{O7}\left[f_{\rm vec}\eta\widehat{\chi}^{U(2N)}_{\rm adj}\right] = \sum_{\pm}[f_{\rm vec}(q,p,\eta x,\eta y,z)]_{\pm}\chi^{U(2N)}_{\mp}$$
$$= \left(1-\frac{(1+xy)(1-z)}{(1-q)(1-p)}\right)\chi^{Sp(N)}_{\rm adj} + \frac{(x+y)(1-z)}{(1-q)(1-p)}\chi^{Sp(N)}_{\rm anti-sym}. \quad (105)$$

The second term in the letter index is the contribution of D3-D7 open strings. $f_{\rm hyp}$ is the letter index of the $\mathcal{N}=2$ hypermultiplet:

$$f_{\rm hyp} = \frac{(xy)^{\frac{1}{2}}(1-z)}{(1-q)(1-p)}. \quad (106)$$

See Appendix A for the explicit forms of the Haar measure and the characters.[4]

The holographic dual of $D_4[N]$ is $AdS_5\times(S^5/\mathbb{Z}_2)$, where $\mathbb{Z}_2$ action on S^5 is given by Eq. (101), and the worldvolume of the $O7$-plane and the four coincident D7-branes is $AdS_5\times S^3$ [45,46]. The large-N index can be calculated on the gravity side as the contribution from massless fields. There are two contributions:

$$I_{D_4[\infty]} = {\rm Pexp}\left[P_{O7}\widehat{f}_{\rm sugra} + f^{AdS_5\times S^3}_{\rm vector}\chi^{SO(8)}_{\mathbf{28}}\right]. \quad (107)$$

One is the contribution from the gravity multiplet:

$$P_{O7}\widehat{f}_{\rm sugra} = [\widehat{f}_{\rm sugra}(q,p,\eta x,\eta y,z,\eta)]_{+}$$
$$= \frac{1}{2}(f_{\rm sugra}(x,y)+\widetilde{f}_{\rm sugra}(-x,-y))$$
$$= \frac{1}{2}\left(\frac{x}{1-x}+\frac{y}{1-y}+\frac{z}{1-z}-\frac{q}{1-q}-\frac{p}{1-p}\right)$$
$$+\frac{1}{4}\left(\frac{(1+x)(1+y)(1-z)(1+q)(1+p)}{(1-x)(1-y)(1+z)(1-q)(1-p)}-1\right). \quad (108)$$

The other is the contribution from D7-D3 open strings. $f^{AdS_5\times S^3}_{\rm vector}$ is the letter index of the eight-dimensional vector multiplet in $AdS_5\times S^3$. See Appendix B.3 for a brief derivation of $f^{AdS_5\times S^3}_{\rm vector}$.

4.1.2. *Decoupling.* Again, the contribution from the system of giant gravitons with wrapping numbers m_x, m_y, and m_z is given by Eq. (8). We again focus only on the three terms $i_I[m_I]$. They are given by

$$i_x[m_x] = P_{O7}\left[\sigma_x f_{\rm vec}\widehat{\chi}^{U(m_x)}_{\rm adj}\right] = \sum_{\pm}\left[1-\frac{(1-\eta x^{-1})(1-\eta q)(1-\eta p)}{(1-y)(1-\eta z)}\right]_{\pm}\chi^{(m_x)}_{\pm},$$
$$i_y[m_y] = P_{O7}\left[\sigma_y f_{\rm vec}\widehat{\chi}^{U(m_y)}_{\rm adj}\right] = \sum_{\pm}\left[1-\frac{(1-\eta y^{-1})(1-\eta q)(1-\eta p)}{(1-x)(1-\eta z)}\right]_{\pm}\chi^{(m_y)}_{\pm},$$
$$i_z[m_z] = P_{O7}\left[\sigma_z f_{\rm vec}\eta\widehat{\chi}^{U(2m_z)}_{\rm adj}\right] = \sum_{\pm}\left[1-\frac{(1-z^{-1})(1-\eta q)(1-\eta p)}{(1-x)(1-y)}\right]_{\pm}\chi^{(2m_z)}_{\mp}. \quad (109)$$

With these letter indices, we can check that all the X-, Y-, and Z-expansions work.

[4]Note that the antisymmetric tensor representation of $Sp(N)$ is reducible and contains the singlet representation, which corresponds to the center of mass degrees of freedom in the O7-D3 system. In the following analysis we will not remove the singlet contribution, and $\chi^{Sp(N)}_{\rm anti-sym}$ includes the contribution from the singlet.

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Table 3. Extended directions of branes.See

	1	2	3	4	X		Y		Z		O/Sp
D3	✓	✓	✓	✓							Sp
$O7^-$	✓	✓	✓	✓	✓	✓	✓	✓			
$GG_{X=0}$							✓	✓	✓	✓	O
$GG_{Y=0}$					✓	✓			✓	✓	O
$GG_{Z=0}$					✓	✓	✓	✓			Sp

Table 4. Extended directions of branes.

	1	2	3	4	X		Y		Z		O/Sp
D3	✓	✓	✓	✓							O
$O7^-$	✓	✓			✓	✓	✓	✓	✓	✓	
$GG_{X=0}$							✓	✓	✓	✓	Sp
$GG_{Y=0}$					✓	✓			✓	✓	Sp
$GG_{Z=0}$					✓	✓	✓	✓			Sp

To obtain Z-expansion, we adopt the degrees $\deg(q, p, x, y, z) = (1, 1, 1, 1, 0)$. Then, we can show that both $i_x[m_x]$ with $m_x \geq 1$ and $i_y[m_y]$ with $m_y \geq 1$ contain infinitely many negative-degree terms, and they decouple. We obtain the simple-sum GG expansion

$$\frac{I_{D_4[N]}}{I_{D_4[\infty]}} = \sum_{m=0}^{\infty} z^{2mN} \sigma_z I_{D_4[m]}. \tag{110}$$

In this case, the theory on the m giant gravitons is again $D_4[m]$. This is because σ_z maps the orientifold operator U_{O7} to itself.

The X-expansion and the Y-expansion are essentially the same as each other and they are related by the Weyl reflection $x \leftrightarrow y$ of the $SU(2)_R$ symmetry. Let us consider the X-expansion for concreteness. To obtain X-expansion, we adopt the degrees $\deg(q, p, x, y, z) = (1, 1, 0, 1, 1)$. Then, we can easily confirm that both $i_y[m_y]$ with $m_y \geq 1$ and $i_z[m_z]$ with $m_z \geq 1$ contain infinitely many negative-degree terms, and the cycles $Y = 0$ and $Z = 0$ decouple. As the result, we obtain the simple-sum giant graviton expansion

$$\frac{I_{D_4[N]}}{I_{D_4[\infty]}} = \sum_{m=0}^{\infty} x^{mN} \sigma_x I_{\widetilde{D}_4[m]}, \tag{111}$$

where we denote the theory realized on giant gravitons by $\widetilde{D}_4[m]$. This will be defined in the next subsection.

4.2. *Theories on giant gravitons:* $\widetilde{D}_4[m]$

4.2.1. *Projection.* $I_{\widetilde{D}_4[m]}$ appearing on the right-hand side in Eq. (111) is the index of the orientifold theory $\widetilde{D}_4[m]$ defined with the orientifold flip operator

$$\widetilde{U}_{O7} = \sigma_x U_{O7} \sigma_x = e^{\pi i (J_2 + A)} = e^{i\pi (J_2 - R_x - R_y - R_z + S)}. \tag{112}$$

The fixed locus of $\widetilde{U}_{O7}$ is the $O7$-plane (together with four coincident D7-branes) along different directions from the one for U_{O7}. See Table 4. The worldvolume theory on D3-branes probing the 7-brane background is locally the $\mathcal{N} = 4$ $U(m)$ SYM, and the $\mathbb{Z}_2$ identification breaks the $U(m)$ gauge symmetry to $O(m)$.

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The insertion of $\widetilde{U}_{O7}$ in the $\mathbb{Z}_2$ refined index (64) is equivalent to the variable change

$$(q, p, x, y, z, \eta) \to (q, -p, -x, -y, -z, -\eta). \tag{113}$$

Correspondingly, the projection operator $P_{\widetilde{O7}}$ acting on letter indices is defined by

$$\begin{aligned} P_{\widetilde{O7}}F(q, p, x, y, z, \eta) &= \frac{1}{2}[F(q, p, x, y, z, +1) + F(q, -p, -x, -y, -z, -1)] \\ &= [F(q, \eta p, \eta x, \eta y, \eta z, \eta)]_+. \end{aligned} \tag{114}$$

With this projection operator, the index is given by

$$I_{\widetilde{D}_4[m]} = \int_{O(m)} dg \,\mathrm{Pexp}\left(P_{\widetilde{O7}}\left[f_{\mathrm{vec}}\widehat{\chi}^{U(m)}_{\mathrm{adj}}\right] + f_\psi \chi^{O(m)}_{\mathrm{vec}}\chi^{D_4}_{\mathbf{8}_v}\right), \tag{115}$$

where the first term in the letter index is the contribution of D3-D3 open strings, and is rewritten as

$$P_{\widetilde{O7}}\left[f_{\mathrm{vec}}\widehat{\chi}^{U(m)}_{\mathrm{adj}}\right] = \sum_{\pm}[f_{\mathrm{vec}}(q, \eta p, \eta x, \eta y, \eta z)]_\pm \chi^{(m)}_\pm. \tag{116}$$

The second term is the contribution of chiral fermions arising from D3-D7 open strings. They are living on the J_2-fixed locus in the boundary S^3, and quantum numbers carried by them are J_1 and $SO(8)$ flavor charges. f_ψ is given by

$$f_\psi = -\frac{q^{\frac{1}{2}}}{1-q}. \tag{117}$$

The index of the theory $\widetilde{D}_4[\infty]$ in the large-N limit is

$$I_{\widetilde{D}_4[\infty]} = \mathrm{Pexp}\left(P_{O7}[\widehat{f}_{\mathrm{sugra}}] + f^{AdS_3\times S^5}_{\mathrm{vector}}\chi^{SO(8)}_{\mathbf{28}}\right), \tag{118}$$

where

$$P_{\widetilde{O7}}[\widehat{f}_{\mathrm{sugra}}] = \frac{1}{2}[f_{\mathrm{sugra}} + \widetilde{f}_{\mathrm{sugra}}(q, -p, -x, -y, -z)] \tag{119}$$

is the contribution from the supergravity multiplet in the ten-dimensional bulk, and the second term is the contribution from D7-D7 open strings. $f^{AdS_3\times S^5}_{\mathrm{vector}}$ is the letter index of the eight-dimensional vector multiplet in $AdS_3 \times S^5$. See Appendix B.4.

4.2.2. *Decoupling.* In the triple-sum GG expansion each contribution with a specific set of wrapping numbers (m_x, m_y, m_z) takes the form of Eq. (8). The contribution from each cycle $i_I[m_I]$ is given by

$$i_I[m_I] = P_{\widetilde{O7}}[\sigma_I f_{\mathrm{vec}}\eta\widehat{\chi}^{U(2m_I)_{\mathrm{adj}}}]. \tag{120}$$

Because $P_{\widetilde{O7}}$ is symmetric under the permutations among x, y, and z, X-, Y-, and Z-expansions are essentially the same. Let us consider X-expansion with the degrees $\deg(q, p, x, y, z) = (1, 1, 0, 1, 1)$. We need to show the decoupling of the cycles $Y = 0$ and $Z = 0$. The letter index associated with $Y = 0$ is

$$i_y[m_y] = \sum_{\pm}\left[1 - \frac{(1-\eta y^{-1})(1-\eta q)(1-p)}{(1-x)(1-z)}\right]_\pm \chi^{(2m_y)}_\mp. \tag{121}$$

We can easily confirm that this contains infinitely many negative-degree terms if $m_y \geq 1$, and hence the cycle $Y = 0$ decouples. It is the case for the $Z = 0$ cycle, too, and the triple-sum expansion reduces to the simple-sum expansion of the form

$$\frac{I_{\widetilde{D}_4(m)}}{I_{\widetilde{D}_4(\infty)}} = \sum_{N=0}^{\infty} x^{mN}\sigma_x I_{D_4[N]}. \tag{122}$$

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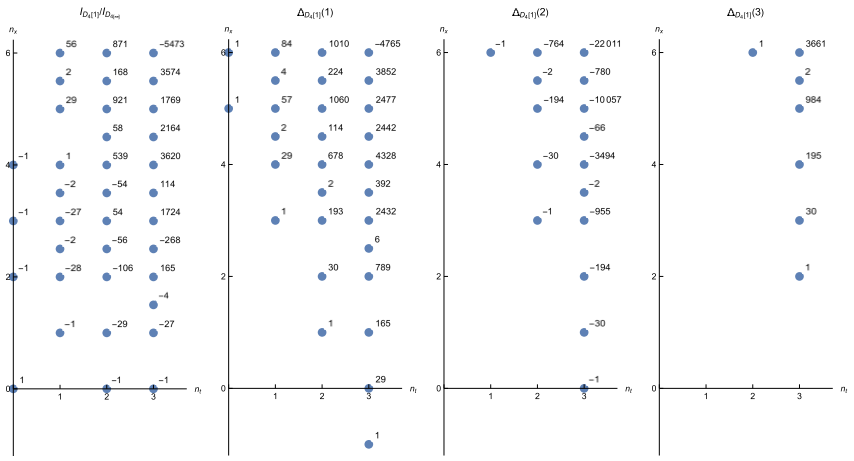

Fig. 15. The ratio $I_{D_4[1]}/I_{D_4[\infty]}$ and the errors $\Delta_{D_4[1]}(m_{\max})$ with $m_{\max} = 1$, 2, and 3 are shown as 2D plots.

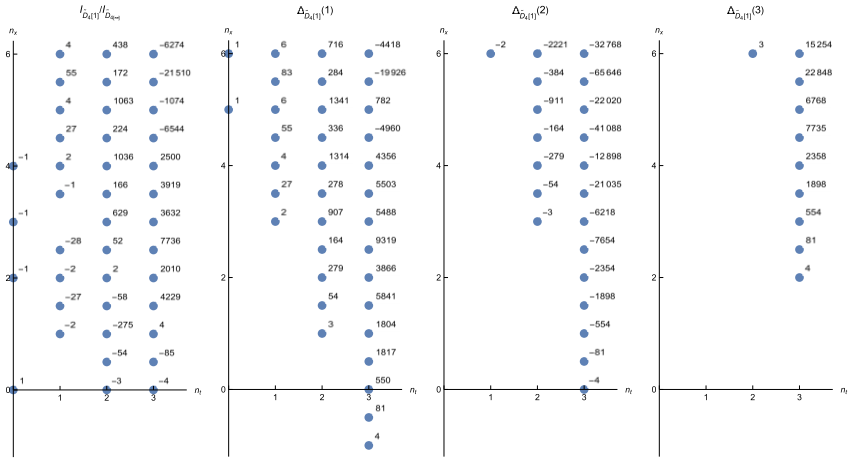

Fig. 16. The ratio $I_{\widetilde{D}_4(1)}/I_{\widetilde{D}_4(\infty)}$ and the errors $\Delta_{\widetilde{D}_4[1]}(N_{\max})$ with $N_{\max} = 1$, 2, and 3.

4.3. *Numerical tests*

4.3.1. *X-expansion of* $I_{D_4[N]}$. Let us numerically test the expansion in Eq. (111). We introduce a cutoff $m_{\max}$ and define the error function

$$\Delta_{D_4[N]}(m_{\max}) = \frac{I_{D_4[N]}}{I_{D_4[\infty]}} - \sum_{m=0}^{m_{\max}} x^{mN} \sigma_x I_{\widetilde{D}_4[m]}. \tag{123}$$

We calculate the ratio $I_{D_4[1]}/I_{D_4[\infty]}$ and the error $\Delta_{D_4[1]}(m_{\max})$ for the unrefined fugacities (57) and the results with $m_{\max} = 1$, 2, and 3 are shown in Fig. 15.

4.3.2. *X-expansion of* $I_{\widetilde{D}_4[m]}$. Let us test the expansion of $I_{\widetilde{D}_4[m]}$ in Eq. (122). We introduce a cutoff $N_{\max}$ and define the error function

$$\Delta_{\widetilde{D}_4[m]}(N_{\max}) = \frac{I_{\widetilde{D}_4(m)}}{I_{\widetilde{D}_4(\infty)}} - \sum_{N=0}^{N_{\max}} x^{mN} \sigma_x I_{D_4[N]}. \tag{124}$$

The ratio $I_{\widetilde{D}_4(1)}/I_{\widetilde{D}_4(\infty)}$ and the errors $\Delta_{\widetilde{D}_4[1]}(N_{\max})$ with $N_{\max} = 1$, 2, and 3 are calculated, and the results are shown in Fig. 16.

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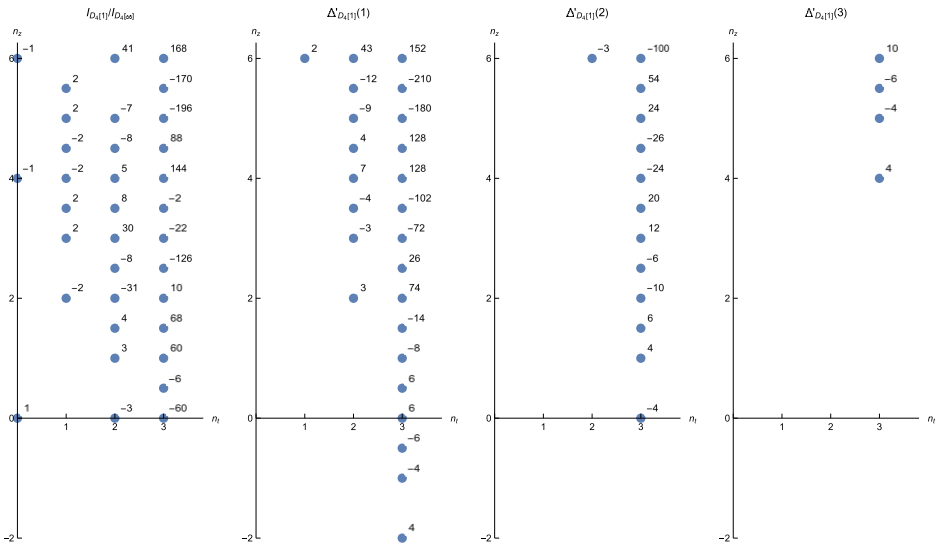

Fig. 17. The ratio $I_{D_4[1]}/I_{D_4[\infty]}$ and the errors $\Delta'_{D_4[1]}(m_{\max})$ with $m_{\max} = 1$, 2, and 3.

4.3.3. *Z-expansion of* $I_{D_4[N]}$. Let us numerically test the Z-expansion of $I_{D_4[N]}$ in Eq. (110). We introduce a cutoff $m_{\max}$ and define the error function

$$\Delta'_{D_4[N]}(m_{\max}) = \frac{I_{D_4[N]}}{I_{D_4[\infty]}} - \sum_{m=0}^{m_{\max}} z^{2mN} \sigma_z I_{D_4[m]}. \tag{125}$$

The ratio $I_{D_4[N]}/I_{D_4[\infty]}$ and the errors $\Delta'_{D_4[N]}(m_{\max})$ are calculated for unrefined fugacities

$$(q, p, x, y, z) = \left(tz^{\frac{1}{2}}, tz^{\frac{1}{2}}, t, t, z\right), \tag{126}$$

and the results for N=1 are shown in Fig. 17.

5. Toric quiver gauge theories

5.1. *General rules*

Let us consider a toric quiver gauge theory associated with a toric Calabi–Yau cone. There is a systematic prescription to determine the gauge theory from the toric data of the Calabi–Yau [47–54]. Let (x_I, y_I) $(I = 1, \ldots, d)$ be the lattice points on the boundary of the corresponding toric diagram labeled in the counterclockwise order. The gauge theory has d $U(1)$ symmetries corresponding to the d boundary points. Let R_I be the $U(1)$ generators normalized so that $R_I = \pm\frac{1}{2}$ for supercharges and $R_I = 0$ or $R_I = 1$ for scalar component fields in chiral multiplets. The action of R_I on the toric fibers of X is given by the vector $V_I = (x_I, y_I, 1)$. If $d > 3$ V_I are not linearly independent, $d - 3$ linear combinations of V_I vanish. Let B_a $(a = 1, \ldots, d - 3)$ be the corresponding linear combinations of R_I. B_a generate nongeometric symmetries, which are often called baryonic symmetries.

The superconformal index of the $\mathcal{N} = 1$ toric quiver gauge theory is defined by

$$I = \mathrm{tr}\left[(-1)^F q^{J_1} p^{J_2} \prod_{I=1}^{d} v_I^{R_I}\right], \quad qp = \prod_{I=1}^{d} v_I^{R_I}. \tag{127}$$

We are interested in the spectrum of eigenvalues of R_I. Let us denote the R_I charge of an operator $\mathcal{O}$ by $R_I[\mathcal{O}]$. For a BPS operator $\mathcal{O}$ contributing to the superconformal index $R_I[\mathcal{O}]$ are expressed as a lattice point in the d-dimensional lattice, and the Taylor expansion of the index can be regarded as the formal sum of weighted lattice points. The charges of an operator $\mathcal{O}$ are

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specified by giving the corresponding fugacity $v_{\mathcal{O}}$, and we also use the notation $R_I[v_{\mathcal{O}}] = R_I[\mathcal{O}]$. For example, the elementary fugacities v_I satisfy $R_I[v_J] = \delta_{IJ}$, and we can regard them as the dual basis of the R-charges R_I.

Supersymmetric three-cycles on which a D3-brane can wrap and which contribute to the multiple-sum giant graviton expansion can be defined for each I as the fixed locus of R_I, and we denote them by S_I. If a boundary point I of the toric diagram is an internal point of a side, the fixed locus S_I is a 1D circle, which can be interpreted as a shrinking three-cycle, and the cycle corresponding to a corner point has finite size. For this reason the distinction between the corner points and the others is important. We define the set of all boundary points $\mathbb{I} = \{1, \ldots, d\}$ and the subset $\mathbb{I}_C \subset \mathbb{I}$ corresponding to the corner points. The complement of $\mathbb{I}_C$ in $\mathbb{I}$ is denoted by $\mathbb{I}^*_C = \mathbb{I} \backslash \mathbb{I}_C$. As we will see below, wrapping numbers m_I for $I \in \mathbb{I}^*_C$ are related to holonomies on branes wrapped around finite-size three-cycles.

For a toric diagram with perimeter d, the giant graviton expansion is a d-ple sum:

$$\frac{I_N}{I_\infty} = \sum_{m_{\mathbb{I}}} v_{\mathbb{I}}^{m_{\mathbb{I}}} \mathrm{F}_{m_{\mathbb{I}}}, \tag{128}$$

where we used shorthand notation $m_{\mathbb{I}} = \{m_1, \ldots, m_d\}$, $v_{\mathbb{I}}^{m_{\mathbb{I}}} = \prod_{I \in \mathbb{I}} v_I^{m_I}$, and $\mathrm{F}_{m_{\mathbb{I}}} = \mathrm{F}_{m_1, \ldots, m_d}$. However, because the sum over $m_{\mathbb{I}^*_C}$ is the holonomy sum, it is natural to divide $m_{\mathbb{I}}$ into the genuine wrapping numbers $m_{\mathbb{I}_C}$ and holonomy variables $m_{\mathbb{I}^*_C}$, and we rewrite Eq. (128) as

$$\frac{I_N}{I_\infty} = \sum_{m_{\mathbb{I}_C}} v_{\mathbb{I}_C}^{m_{\mathbb{I}_C}} \left(\sum_{m_{\mathbb{I}^*_C}} v_{\mathbb{I}^*_C}^{m_{\mathbb{I}^*_C}} \mathrm{F}_{m_{\mathbb{I}}} \right). \tag{129}$$

We can regard the sum in the parentheses as the index of the theory on the GG system with wrapping numbers $m_{\mathbb{I}_C}$. In this sense the GG expansion is a $|\mathbb{I}_C|$-ple sum, and the GG expansion of the $\mathbb{Z}_k$ orbifold theory studied in Sect. 2 is triple-sum for generic degree assignment in this sense. We want to reduce the sum by an appropriate choice of the degrees.

The index $\mathrm{F}_{m_{\mathbb{I}}}$ is given by

$$F_{m_{\mathbb{I}}} = \int dg \, \mathrm{Pexp} \left(\sum_{I \in \mathbb{I}_C} P_I \left[f_I \chi_{\mathrm{adj}}^{U(m_I)} \right] + \sum_{I, I'} f_{II'} \chi^{(m_I, m_{I'})} \right). \tag{130}$$

$\chi^{(m_I, m_{I'})}$ are the bi-fundamental characters in Eq. (13). P_I and f_I in the first term of the letter index are the projection and the letter index associated with the finite-size cycle S_I, respectively. Note that P_I is the projection operator acting on both f_I and $\chi_{\mathrm{adj}}^{U(m_I)}$, and the action on $\chi_{\mathrm{adj}}^{U(m_I)}$ depends on the holonomy variables $m_{\mathbb{I}^*_C}$. $\sum_{I,I'}$ in the second term is the summation over pairs of adjacent corners (I, I'), and $f_{I,I'}$ is the letter index associated with the intersection $S_I \cap S_{I'}$. In the second term we should also take account of the holonomy dependence, which is omitted in the following for simplicity.

f_I and $f_{I,I'}$ can be obtained according to the toric structure encoded in the toric diagram as follows.

As we mentioned above, the Taylor expansion of the index can be expressed as the formal sum of a set of weighted lattice points in the d-dimensional lattice. In the large-N limit, operators with baryonic charges do not contribute, and $B_a[u] = 0$ are satisfied for all terms u in the Taylor expansion of the index. This defines the 3D sublattice in the d-dimensional lattice. In addition, BPS bounds guarantee $R_{\mathbb{I}}[u] \geq 0$. (Namely, $R_I[u] \geq 0$ for all $I \in \mathbb{I}$.) These inequalities define a cone in the 3D lattice.

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The large-N index I_∞ is given by [55,56]:[5]

$$I_\infty = \sum_{I=1}^{d} \frac{w_I}{1-w_I}, \tag{131}$$

where w_I for a specific I is the primitive fugacity satisfying

$$B_a[w_I] = R_I[w_I] = R_{I+1}[w_I] = 0, \quad R_{\mathbb{I}}[w_I] \geq 0. \tag{132}$$

The primitive fugacity is the one such that all fugacities satisfying the same conditions are given as positive powers of it.

For each corner point I let $(I_{-1}, I_0 = I, I_{+1})$ be the three consecutive corner points. We define the three nonbaryonic fugacities $w_{I,\alpha}$ $(\alpha = 0, \pm 1)$ for each I as the dual basis of R_{I_α} by the conditions

$$B_a[w_{I,\alpha}] = 0, \quad R_{I_\alpha}[w_{I,\beta}] = \delta_{\alpha\beta} \quad (\alpha, \beta = 0, \pm 1). \tag{133}$$

We can show $w_{I,-1}w_{I,0}w_{I,+1} = \prod_{I=1}^{d} v_I = qp$.

In the case of the orbifold S^5/Γ the toric diagram is a triangle, and there are three corner points corresponding to R_x, R_y, and R_z. If $R_I = R_x$, $R_{I_{+1}} = R_y$, and $R_{I_{-1}} = R_z$, the corresponding fugacities are $w_{I,0} = x$, $w_{I,+1} = y$, and $w_{I,-1} = z$. In fact, the letter index f_I for a giant graviton wrapped around a finite-size cycle S_I is obtained simply by replacing x, y, and z in $\sigma_x f_{\text{vec}}$ by $w_{I,0}$, $w_{I,+1}$, and $w_{I,-1}$, respectively:

$$f_I = (\sigma_x f_{\text{vec}})|_{x\to w_{I,0}, y\to w_{I,+1}, z\to w_{I,-1}} = 1 - \frac{\left(1 - w_{I,0}^{-1}\right)(1-q)(1-p)}{(1-w_{I,+1})(1-w_{I,-1})}. \tag{134}$$

If the lattice generated by the three vectors V_{I_α} is not the whole 3D lattice but its sublattice, then some of $R_J[w_{I,\alpha}]$ are fractional. Such terms must be removed, and this is realized by the projection P_I in Eq. (130).

The letter index of bi-fundamental fields can also be written with these fugacities. Let I and I' $(=I_{+1})$ be two consecutive corners. We introduce fugacities $w_{I,\alpha}$ and $w_{I',\alpha}$ according to Eq. (133). If one or both of the intersecting cycles S_I and $S_{I'}$ is orbifolded, we need to take account of the coupling with holonomies. Let us consider the simple case with neither S_I nor $S_{I'}$ being orbifolded. This is the case when both V_{I_α} and $V_{I'_\alpha}$ span the whole 3D lattice. Then, fugacities $w_{I,-1}$ and $w_{I',+1}$ are the same. Let us denote them by w. This is the fugacity associated with the $U(1)$ symmetry shifting the intersection. The letter index of the bi-fundamental fields along the intersection $S_I \cap S_{I'}$ is obtained from Eq. (12) by replacing z with w:

$$f_{II'} = \left(\frac{w}{qp}\right)^{\frac{1}{2}} \frac{(1-q)(1-p)}{1-w}. \tag{135}$$

Now, let us consider degree assignment. We assign d_q, d_p, and d_I to q, p, and v_I, respectively. This means that we introduce an auxiliary variable t by

$$(q, p, v_I) \to \left(t^{d_q} q, t^{d_p} p, t^{d_I} v_I\right) \quad d_q + d_p = \sum_{I=1}^{d} d_I, \tag{136}$$

and perform the t-expansion first. t is the fugacity for the charge

$$Q_t \equiv d_q J_1 + d_p J_2 + R_t, \quad R_t \equiv \sum_{I=1}^{d} d_I R_I, \tag{137}$$

[5] This gives the large-N index of quiver gauge theory with SU gauge groups. This is different from Eq. (36) by the contribution from the IR-free $U(1)$ vector multiplets included in Eq. (36).

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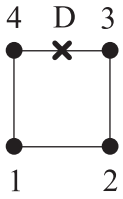

Fig. 18. The toric diagram of the conifold.

and the degree of a fugacity u is nothing but $\deg(u) = Q_t[u]$. R_t is a linear combination of R_I. Because we are not interested in the baryonic charges, it is enough to know its action on the toric fibers, which is expressed as the linear combination of V_I. With the normalization $\sum_{I=1}^{d} d_I = 1$, it can be expressed as a point $D = \sum_{I=1}^{d} d_I(x_I, y_I)$ in the toric diagram, and if the position of D is specified, degrees of nonbaryonic fugacities are fixed. In particular, the degrees of the fugacities $w_{I,\alpha}$ are determined by the relative positions of D and three corners I_α. If D is inside the triangle made by the three corners I_α, $\deg(w_{I,\alpha}) > 0$, whereas if D and a corner I_α are on the opposite sides of the line passing through the other two corners, then $\deg(w_{I,\alpha}) < 0$. Remark that D is always in the $|\mathbb{I}_C|$-gon of the toric diagram (including its boundary), and $\deg(w_{I,\pm})$ are always nonnegative whereas $\deg(w_{I,0})$ may be negative.

Let us apply the decoupling criterion. For decoupling of a finite-size cycle S_I, the letter index f_I with the projection P_I applied needs to include infinitely many terms with $Q_t < 0$. Because J_1 and J_2 (the exponents of q and p) are nonnegative for all terms in the expansion of Eq. (134), there must be infinitely many terms with $R_t < 0$. This can be true if the following conditions hold:

$$\deg(w_{I,0}) > 0 \quad \text{and} \quad (\deg(w_{I,+1}) = 0 \quad \text{or} \quad \deg(w_{I,-1}) = 0). \tag{138}$$

This means

$$D \in (I_{-1}, I_0] \cup [I_0, I_{+1}), \tag{139}$$

where $(I_{-1}, I_0]$ is the segment between I_{-1} and I_0 with I_{-1} excluded and I_0 included. $[I_0, I_{+1})$ is similarly defined. Note that Eq. (139) is a necessary condition, and to show the decoupling of a cycle S_I we need to confirm that infinitely many negative-degree terms remain after the projection P_I.

Obviously, the condition (139) is satisfied at most for two corners I, and the $|\mathbb{I}_C|$-ple sum giant graviton expansion at best reduces to $(|\mathbb{I}_C| - 2)$-ple sum expansion. For an orbifold S^5/Γ, the toric diagram is a triangle with $|\mathbb{I}_C| = 3$, and as we discussed in Sect. 2 it may be possible to obtain simple-sum expansion. Unfortunately, this is not possible for the case with $|\mathbb{I}_C| \geq 4$.

5.2. *Klebanov–Witten theory*

As the simplest example of nonorbifold toric Calabi–Yau, let us consider the conifold. The toric diagram is shown in Fig. 18. It is a square, and the four corners are labeled by $I = 1, 2, 3, 4$.

There are a set of prescriptions to read off information of the corresponding quiver gauge theory from the toric diagram [49,50]. It is convenient to use the bipartite graph associated with the toric diagram. The graph is drawn on the torus, and is called the brane tiling. In the brane tiling, faces, edges, and vertices correspond to gauge groups, bi-fundamental chiral multiplets, and terms in the superpotential, respectively.

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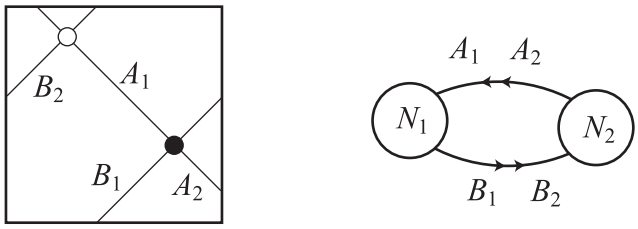

Fig. 19. The brane tiling and the quiver diagram of the Klebanov–Witten theory.

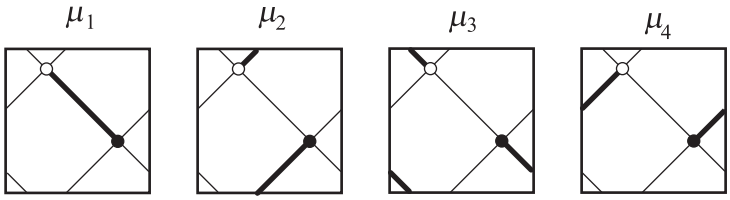

Fig. 20. The perfect matchings in the brane tiling for the conifold.

Table 5. Charge assignments for the chiral multiplets in the Klebanov–Witten theory. χ_Φ is the contribution of the scalar component of the chiral multiplet to the letter index. χ_{12} and χ_{21} are the characters for the bi-fundamental representations $(N_1, \overline{N}_2)$ and $(N_2, \overline{N}_1)$, respectively.

Φ	F_A	F_B	R_1	R_2	R_3	R_4	χ_Φ
A_1	1	0	1	0	0	0	$v_1\chi_{12}$
A_2	-1	0	0	0	1	0	$v_3\chi_{12}$
B_1	0	1	0	1	0	0	$v_2\chi_{21}$
B_2	0	-1	0	0	0	1	$v_4\chi_{21}$

The brane tiling and the quiver diagram for the conifold are shown in Fig. 19. The quiver gauge theory is called the Klebanov–Witten theory [57], which we denote by KW[N].

We can also read off charge assignment for R_I from the brane tiling by using perfect matchings. All perfect matchings in the tiling are shown in Fig. 20. Each perfect matching is associated with an internal or boundary lattice point of the toric diagram. Let μ_I be a perfect matching associated with a boundary point I. μ_I is a subset of edges, and defines a subset of the chiral multiplets. We assign $R_I = +1$ to them, and $R_I = 0$ to the others. There is a unique perfect matching for each $I \in \mathbb{I}_C$, and charge assignment for R_I ($I = 1, 2, 3, 4$) is uniquely determined as shown in Table 5.

The global symmetry of the theory is

$$G_{\mathrm{KW}_N} = U(1)_r \times SU(2)_A \times SU(2)_B \times U(1)_B. \tag{140}$$

The $U(1)_r$ charge r, the $SU(2)_A$ Cartan generator F_A, and $SU(2)_B$ Cartan generators F_B are given by

$$r = \frac{1}{2}(R_1 + R_2 + R_3 + R_4), \quad F_A = R_1 - R_3, \quad F_B = R_2 - R_4, \tag{141}$$

and $U(1)_B$ is the baryonic symmetry. Corresponding to the linear relation of four vectors $V_1 - V_2 + V_3 - V_4 = 0$, the $U(1)_B$ charge B is given by

$$NB = R_1 - R_2 + R_3 - R_4. \tag{142}$$

The normalization of B is chosen so that det A_1 carries $B = +1$.

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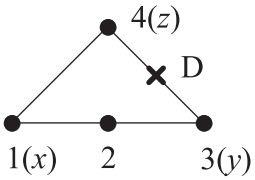

Fig. 21. The toric diagram of $\mathbb{C}^3/\mathbb{Z}_2$. There are four boundary points $\mathbb{I} = \{1, 2, 3, 4\}$, and three of them, $\mathbb{I}_C = \{1, 3, 4\}$, are corners. D is the point that specifies the degrees used for the X-expansion.

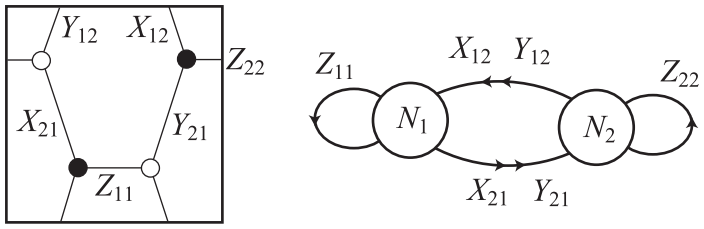

Fig. 22. The brane tiling and the quiver diagram of the $\mathbb{Z}_2$ orbifold theory.

The degrees for the point D shown in Fig. 18 are

$$\deg(v_1, v_2, v_3, v_4) = (0, 0, \tfrac{1}{2}, \tfrac{1}{2}) \tag{143}$$

up to the ambiguity for baryonic charge. With these degrees the cycles S_3 and S_4 decouple, and the expansion becomes double-sum associated with S_1 and S_2:

$$\frac{I_{\mathrm{KW}[N]}}{I_{\mathrm{KW}[\infty]}} = \sum_{m_1,m_2} v_1^{m_1 N} v_2^{m_2 N} \mathrm{F}_{m_1,m_2,0,0}. \tag{144}$$

The fugacities $w_{I,\alpha}$ are given by

$$w_{I,0} = v_I v_{I+2}^{-1}, \quad w_{I,\pm 1} = v_{I\pm 1} v_{I+2}. \tag{145}$$

The letter index f_I for a giant graviton wrapped on S_I is

$$f_I = 1 - \frac{(1 - v_{I+2}/v_I)(1-q)(1-p)}{(1 - v_{I+1}v_{I+2})(1 - v_{I-1}v_{I+2})}, \tag{146}$$

and the letter index for the intersection modes on $S_I \cap S_{I+1}$ is

$$f_{I,I+1} = \left(\frac{v_{I+2}v_{I+3}}{qp}\right)^{\frac{1}{2}} \frac{(1-q)(1-p)}{1 - v_{I+2}v_{I+3}}. \tag{147}$$

5.3. $\mathbb{Z}_2$ *orbifold*

Let us reconsider the $\mathbb{Z}_2$ orbifold theory $T_{N,N}$ discussed in Sect. 2.1 as an example of toric quiver gauge theories. The toric diagram is shown in Fig. 21. Corresponding to the three corners $\mathbb{I}_C = \{1, 3, 4\}$, there are three finite-size cycles S_1, S_3, and S_4, which are respectively referred to as $X = 0$, $Y = 0$, and $Z = 0$ cycles in Sect. 2, and R_I ($I \in \mathbb{I}_C$) are related to the R-charges used in Sect. 2 by

$$R_x = R_1, \quad R_y = R_3, \quad R_z = R_4. \tag{148}$$

The brane tiling and the quiver diagram for the $\mathbb{Z}_2$ orbifold theory are shown in Fig. 22. In terms of $\mathcal{N} = 1$ multiplets, $T_{N,N}$ consists of six chiral multiplets shown in Table 6 and two $U(N)$ vector multiplets V_{11} and V_{22}.

As in the previous example we can determine charge assignment of R_I by using perfect matchings. All perfect matchings for the $\mathbb{Z}_2$ theory are shown in Fig. 23. An important difference from

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Table 6. Chiral multiplets in the $\mathbb{Z}_2$ orbifold theory $T_{N,N}$ and their charges.

Φ	$R_x = R_1$	$R_y = R_3$	$R_z = R_4$	NB	R_2	χ_Φ
X_{12}	1	0	0	1	0	$xb^{\frac{1}{N}}\chi_{12} = v_1\chi_{12}$
X_{21}	1	0	0	-1	1	$xb^{-\frac{1}{N}}\chi_{21} = v_1v_2\chi_{21}$
Y_{12}	0	1	0	1	0	$yb^{\frac{1}{N}}\chi_{12} = v_3\chi_{12}$
Y_{21}	0	1	0	-1	1	$yb^{-\frac{1}{N}}\chi_{21} = v_3v_2\chi_{21}$
Z_{11}	0	0	1	0	0	$z\chi_{11} = v_4\chi_{11}$
Z_{22}	0	0	1	0	0	$z\chi_{22} = v_4\chi_{22}$

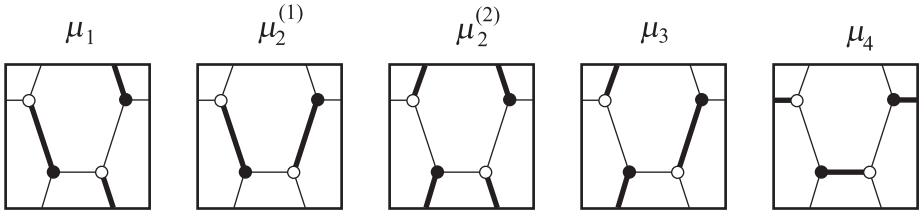

Fig. 23. Perfect matchings in the brane tiling of the $\mathbb{Z}_2$ orbifold theory.

the previous example is that we have the vertex $I = 2$ which is not a corner. In general, there are more than one perfect matchings associated with a point $I \in \mathbb{I}_C^*$, and we should choose one of them to define the corresponding charge R_I. This ambiguity affects the definition of the baryonic charges. In the case of the $\mathbb{Z}_2$ orbifold theory $T_{N,N}$, we have two perfect matchings $\mu_2^{(1)}$ and $\mu_2^{(2)}$ for vertex $I = 2$, and here we choose $\mu_2^{(1)}$. The resulting charges are shown in Table 6.

The global symmetry of the $T_{N,N}$ theory is

$$G_{T_{N,N}} = SU(2)_R \times U(1)_R \times SU(2)_F \times U(1)_B, \tag{149}$$

where $SU(2)_R \times U(1)_R$ is the R-symmetry of the $\mathcal{N} = 2$ superconformal algebra, $SU(2)_F$ is the flavor symmetry, and $U(1)_B$ is the baryonic symmetry. $SU(2)_R \times U(1)_R \times SU(2)_F$ is the geometric symmetry in the sense that it is realized as the isometry of the background geometry. The $SU(2)_R$ Cartan generator $R_{SU(2)}$, the $U(1)_R$ charge $R_{U(1)}$, and the $SU(2)_F$ Cartan generator F are given by

$$R_{SU(2)} = R_1 + R_3, \quad R_{U(1)} = R_4, \quad F = R_1 - R_3. \tag{150}$$

The baryonic charge is determined from the linear dependence $V_1 - 2V_2 + V_3 = 0$ as

$$NB = R_1 - 2R_2 + R_3. \tag{151}$$

The normalization of B is chosen so that $\det X_{12}$ carries $B = +1$.

The cross in the toric diagram in Fig. 21 shows the point D that gives the degrees

$$\deg(v_1, v_2, v_3, v_4) = (0, 0, \tfrac{1}{2}, \tfrac{1}{2}) \tag{152}$$

(up to ambiguity for the baryonic charge). These are the same as the degrees for the X-expansion used in Sect. 2 up to normalization. According to the rule (139) two cycles S_3 and S_4 decouple, and the giant graviton expansion becomes the double sum

$$\frac{I_{T_{N,N}}}{I_{T_\infty}} = \sum_{m_1=0}^{\infty} \sum_{m_2} v_1^{m_1} v_2^{m_2} \mathrm{F}_{m_1,m_2,0,0}. \tag{153}$$

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m_1 is the wrapping number for the finite-size cycle S_1, and runs over all nonnegative integers. m_2, the wrapping number for the shrining cycle S_2, can be regarded as the holonomy variable, as we will explain below.

In the analysis in Sect. 2 we consider the index $I^{[B_1,\ldots,B_k]}_{T_{N_1,\ldots,N_k}}$ for sectors with specific baryonic charges. For the comparison to Eq. (153) we should introduce the baryonic fugacities and define the index as the sum over all baryonic sectors. For $\mathbb{Z}_2$ orbifold theory $T_{N,N}$ we define the baryonic charge $B = B_1 - B_2$ and introduce a single fugacity b for B. Then, the index including all baryonic sectors is

$$I_{T_{N,N}} = \sum_{B=-\infty}^{\infty} b^B I^{[B,0]}_{T_{N,N}}. \tag{154}$$

By substituting Eq. (52) into $I^{[B,0]}_{T_{N,N}}$ we obtain

$$\frac{I_{T_{N,N}}}{I_{T_\infty}} = \sum_{m',m''=0}^{\infty} b^{m'-m''} x^{(m'+m'')N} \sigma_x I^0_{\widetilde{T}_{m',m''}}. \tag{155}$$

Let us confirm that Eqs. (153) and (155) are the same. The relation between two sets of fugacities (x, y, z, b) and (v_1, v_2, v_3, v_4) can be read off from $x^{R_x} y^{R_y} z^{R_z} b^B = v_1^{R_1} v_2^{R_2} v_3^{R_3} v_4^{R_4}$ as follows:

$$v_1 = x b^{\frac{1}{N}}, \quad v_2 = b^{-\frac{2}{N}}, \quad v_3 = y b^{\frac{1}{N}}, \quad v_4 = z. \tag{156}$$

With these relations we can rewrite Eq. (155) as follows:

$$\frac{I_{T_{N,N}}}{I_{T_\infty}} = \sum_{m',m''=0}^{\infty} v_1^{(m'+m'')N} v_2^{m''N} \sigma_x I^0_{\widetilde{T}_{m',m''}}. \tag{157}$$

By comparing the prefactors in Eqs. (153) and (157) we obtain

$$m' = m_1 - m_2, \quad m'' = m_2. \tag{158}$$

These relations imply that m_2 is the parameter specifying the symmetry-breaking pattern $U(m_1) \to U(m_1 - m_2) \times U(m_2)$. In other words, m_2 is the parameter specifying the holonomy

$$\{\widetilde{h}_a\} = \{1^{m_1-m_2}, 2^{m_2}\} \tag{159}$$

on the m_1 coincident giant gravitons. This interpretation requires $0 \le m_2 \le m_1$. The upper bound $m_2 \le m_1$ would be interpreted as a kind of s-rules.

We can also confirm that $F_{m_1,m_2,0,0}$ in Eq. (153) reproduces the result in Sect. 2. Because the cycles S_3 and S_4 decouple, Eq. (130) gives the letter index

$$P_1 f_1 \chi^{U(m_1)}_{\text{adj}}. \tag{160}$$

The fugacities $w_{1,\alpha}$ are given by

$$w_{1,0} = v_1 v_2^{\frac{1}{2}} = x, \quad w_{1,+1} = v_3 v_2^{\frac{1}{2}} = y, \quad w_{1,-1} = v_4 = z. \tag{161}$$

With these fugacities we can see $f_1 = \sigma_x f_{\text{vec}}$ and Eq. (160) agrees with $i_x[m_x]$ in Eq. (37) with $m_x = m_1$ and the holonomy (159).

5.4. *RG flow*

The orbifold theory $T_{N,N}$ flows to the Klebanov–Witten theory KW[N] by the deformation with the superpotential

$$W = m\Big(\operatorname{tr} Z_{11}^2 - \operatorname{tr} Z_{22}^2\Big). \tag{162}$$

Because the RG flow does not change the superconformal index we can relate $I_{T_{N,N}}$ and $I_{\text{KW}[N]}$ by the RG flow.

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The deformation breaks the global symmetry $G_{T_{N,N}}$ of the UV theory to $G_{\rm flow} = U(1)_r \times SU(2)_F \times U(1)_B$, and at the IR fixed point $G_{\rm flow}$ is enhanced to $G_{\rm KW}$. Therefore, $I_{T_{N,N}} = I_{{\rm KW}[N]}$ holds only for restricted values of fugacities corresponding to $G_{\rm flow}$. Let r, f, and b be the fugacities for $U(1)_r$, $SU(2)_F$, and $U(1)_B$ in $G_{\rm flow}$, respectively. The restriction on the fugacities of the UV theory is given by

$$x = fr^{\frac{1}{2}}, \quad y = f^{-1}r^{\frac{1}{2}}, \quad z = r, \quad b = b, \quad (r^2 = qp). \tag{163}$$

For the restricted values the fugacity for the mass terms (162), which carry $(R_1, R_2, R_3, R_4) = (-1, -1, -1, 1)$, is

$$\frac{v_4}{v_1 v_2 v_3} = \frac{z}{xy} = 1. \tag{164}$$

Concerning the symmetry $G_{\rm KW}$, only the diagonal subgroup of $SU(2)_A \times SU(2)_B$ is preserved in $G_{\rm flow}$. The restricted fugacities v'_I of the IR theory are

$$v'_1 = fr^{\frac{1}{2}}b^{\frac{1}{N}}, \quad v'_2 = fr^{\frac{1}{2}}b^{-\frac{1}{N}}, \quad v'_3 = f^{-1}r^{\frac{1}{2}}b^{\frac{1}{N}}, \quad v'_4 = f^{-1}r^{\frac{1}{2}}b^{-\frac{1}{N}}. \tag{165}$$

(In this subsection we use the primed variables v'_I for KW[N] for distinction from v_I for $T_{N,N}$.)

With the localization formula on the gauge theory side we can easily check that the two indices agree for the restricted values of fugacities. The letter index of the $\mathbb{Z}_2$ orbifold theory is given by

$$f_{\rm v}(\chi_{11}) + f_{\rm v}(\chi_{22}) + \sum_{\Phi} f_{\rm c}(\chi_\Phi) \tag{166}$$

with χ_Φ shown in Table 6, where $f_{\rm v}$ and $f_{\rm c}$ are the letter index for the vector multiplet and the chiral multiplet, respectively:

$$f_{\rm c}(\chi_R) = \frac{\chi_R - qp\overline{\chi}_R}{(1-q)(1-p)}, \quad f_{\rm v}(\chi_{\rm adj}) = \left(1 - \frac{1-qp}{(1-q)(1-p)}\right)\chi_{\rm adj}. \tag{167}$$

The letter index of the Klebanov–Witten theory is also given by Eq. (166) with χ_Φ shown in Table 5. Indeed, we can easily check

$$f_{X_{12}} = f_{A_1}, \quad f_{Y_{12}} = f_{A_2}, \quad f_{X_{21}} = f_{B_1}, \quad f_{Y_{21}} = f_{B_2}, \quad f_{Z_{11}} = f_{Z_{22}} = 0, \tag{168}$$

and the two indices agree at the level of the letter indices in the localization formula.

Finally, let us confirm the GG expansions in Eqs. (157) and (144) are the same. The two sets of degrees (152,143) are both consistent with the degrees

$$\deg(f, r, b) = (-\tfrac{1}{2}, 1, 0). \tag{169}$$

We can easily confirm that the prefactor of the orbifold theory in Eq. (153) and the prefactor of the Klebanov–Witten theory in Eq. (144) agree provided we identify (m', m'') on the $T_{N,N}$ side with (m_1, m_2) on the KW[N] side:

$$v_1^{(m'+m'')N} v_2^{m''N} = x^{(m'+m'')N} b^{m'-m''} = (fr^{\frac{1}{2}})^{(m'+m'')N} b^{m'-m''} = v_1'^{m'N} v_2'^{m''N}. \tag{170}$$

We can also confirm the agreement of the letter index for giant gravitons. In the GG expansion of $T_{N,N}$ the letter index for m_1 coincident giant gravitons on the $X = 0$ cycle with the holonomy (159) is

$$\begin{aligned} i_x[m] &= P_2\left[\sigma_x f_{\rm vec} \chi_{\rm adj}^{U(m)}\right] \\ &= \left(1 - \frac{(1-x^{-1}y)(1-q)(1-p)}{(1-y^2)(1-z)}\right)\left(\chi_{\rm adj}^{U(m')} + \chi_{\rm adj}^{U(m'')}\right) \\ &\quad + \frac{(x^{-1}-y)(1-q)(1-p)}{(1-y^2)(1-z)}\chi^{(m',m'')}. \end{aligned} \tag{171}$$

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For the Klebanov–Witten theory, the letter index is given by

$$f_1\chi_{\rm adj}^{U(m_1)}+f_2\chi_{\rm adj}^{U(m_2)}+f_{12}\chi^{(m_1,m_2)}, \tag{172}$$

where f_1 and f_2 are given in Eq. (146) and f_{12} is given in Eq. (147). The two letter indices (171,172) agree for the restricted values of fugacities.

6. Discussion

In this paper we discussed the reduction of multiple-sum giant graviton expansions to simple-sum expansions. For orbifold and orientifold examples the triple-sum expansion for generic degrees can be reduced to the simple-sum expansion by assigning appropriate degrees.

All theories studied in this paper are Lagrangian theories, and we can calculate the superconformal index directly by using the localization formula. This is in general not the case. For example, M2-brane theory (ABJM theory [58]) and M5-brane theory (six-dimensional (2,0) theory) are related by the simple-sum giant graviton expansions [39]. In this case, the calculation of the index of (2,0) theory is only possible with indirect methods [59–63]. We can use the giant graviton expansion as another convenient method to calculate the index of (2,0) theory by using the ABJM index [64]. The orbifold version of the relation between M2-theory and M5-theory is also interesting because it may enable us to calculate the index of six-dimensional (1,0) theories by using orbifolds of ABJM theory.

Another interesting class of non-Lagrangian theories includes Argyres Douglas and Minahan Nemeschanski theories realized on D3-branes probing 7-brane backgrounds with constant axiodilation. They are labeled by the type of 7-brane $G=H_0$, H_1, H_2, D_4, E_6, E_7, E_8 and the rank N. The AdS/CFT correspondence of these theories in the large-N limit was studied in Refs. [45,46], and it was confirmed in Ref. [22] that the multiple-sum expansion works well at least for the leading giant graviton contributions. If we can apply the simple-sum expansion to this class of theories it gives interesting relations among the indices. Actually, the $D_4[N]$ theory studied in Sect. 4 is a special case of the general theories $G[N]$. We confirmed that the Z-expansion of the $D_4[N]$ theory works and is self-dual. If this is also the case for general $G[N]$, the following relation should hold:

$$\frac{I_{G[N]}}{I_{G[\infty]}}=\sum_{m=0}^{\infty}z^{\Delta_G mN}\sigma_z I_{G[m]}, \tag{173}$$

with Δ_G being the dimension of the Coulomb branch operator of the $G[1]$ theory. It would be interesting to confirm whether this relation holds, and, if so, to what extent we can bootstrap the index $I_{G[N]}$ with the relation (173).

In the orbifold and orientifold cases, we can consider three expansions: X-, Y-, and Z-expansions. Although decoupling does not always occur, for some examples we can perform the expansion in more than one way, and we can consider "the web of GG expansions" by applying GG expansions repeatedly. In the case of $\mathcal{N}=4$ $U(N)$ SYM all three expansions work. In this case the giant graviton expansion is "self-dual" in the sense that the theory on the giant graviton is also $\mathcal{N}=4$ $U(N)$ SYM, and application of the giant graviton expansions gives just different frames. Each step of the giant graviton expansions is specified by one of the variable changes σ_x, σ_y, and σ_z, and a frame is specified by the composition of the variable changes. Combining three variable changes σ_x, σ_y, σ_z and permutations in (x, y, z) and (q, p) associated with the Weyl groups of $SU(4)_R$ and $SO(4)_{\rm spin}$ symmetries we can generate many frames. All these frames give $\mathcal{N}=4$ $U(N)$ SYM. However, in more general orbifolds and orientifolds, it

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may be possible to generate the web including different theories. By applying a projection P, we can obtain an orbifold theory, and in a frame specified by the variable change σ the projection is replaced by $\sigma P\sigma^{-1}$, and in general this is different from the original one. Therefore, the web consists of theories with different orbifold actions. Although in the orbifold case all theories in the web are Lagrangian theories, in more general cases (like $G[N]$ and orbifolds of M-brane theories) the web may contain both Lagrangian and non-Lagrangian theories, and then it will be convenient to analyze theories that are difficult to analyze directly. This is analogous to the S-duality of type IIB string theory. The duality takes the type IIB theory to the same theory (with the different coupling constant). Let us take $\mathbb{Z}_2$ projection with the worldsheet parity Ω, which gives type I theory. Then the other side of the duality is also $\mathbb{Z}_2$ projected theory. However, this $\mathbb{Z}_2$ is not Ω but $(-1)^{F_R}$, which gives the $SO(32)$ heterotic string. In this way, by applying a projection, we can generate new duality from the duality for the theories before the projection. We can consider a similar situation for the web of giant graviton expansions.

As we have emphasized, the simple-sum expansion is much easier to calculate because it does not have issues of integration contours. This is because the contribution from a single cycle is obtained by a simple variable change from the standard index, for which we can adopt the standard choice of the contours. In Sect. 5 we saw that the giant graviton expansions of the orbifold theory $T_{N,N}$ and the Klebanov–Witten theory KW[N] are related by the RG flow. In this relation, the simple-sum expansion of $T_{N,N}$ generates the double-sum expansion of KW[N]. This gives information about rules for pole selection in KW[N]. This kind of relation may be useful to find general rules for the pole selection for general multiple-sum giant graviton expansions.

In all the examples in this work the simple-sum expansions are invertible. Namely, if expansion of the index of T_N gives the index of $\widetilde{T}_m$, then the expansion of $\widetilde{T}_m$ gives T_N. It would be interesting if we could analytically prove the invertibility. Suppose that we have functions F_N ($N \in \mathbb{Z}$) which vanish for negative N, and they have giant graviton expansion of the form

$$F_N = F_\infty \sum_{m\in\mathbb{Z}} x^{mN}\sigma_x \widetilde{F}_m. \tag{174}$$

The invertibility means that a similar expansion of $\widetilde{F}_m$ gives the original functions F_N. Namely, if the functions F'_N appearing in the expansion of $\widetilde{F}_m$

$$\widetilde{F}_m = \widetilde{F}_\infty \sum_{N\in\mathbb{Z}} x^{mN}\sigma_x F'_N \tag{175}$$

are the same as F_N, then we can say the expansion is invertible. Unfortunately, we have not yet succeeded in proving (or disproving) this fact. We will only comment on what happens if we naively substitute Eq. (175) into Eq. (174). It gives

$$F_N = (F_\infty \sigma_x \widetilde{F}_\infty) \sum_{m\in\mathbb{Z}} \sum_{N'\in\mathbb{Z}} x^{m(N-N')} F'_{N'}. \tag{176}$$

Let us suppose x is a generic phase factor $x = e^{2\pi i\alpha}$ with irrational α. Then with an appropriate regularization we obtain

$$\sum_{m\in\mathbb{Z}} x^{m(N-N')} = \mathcal{N}\delta_{N,N'}, \tag{177}$$

where $\mathcal{N}$ is a large number depending on the regularization. Then we obtain

$$F_N = \mathcal{N}(F_\infty \sigma_x \widetilde{F}_\infty) F'_N. \tag{178}$$

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If the product of F_∞ and $\sigma_x\widetilde{F}_\infty$ gave $\mathcal{N}^{-1}$ we would obtain the expected result $F_N = F'_N$. Because the large-N indices F_∞ and $\widetilde{F}_\infty$ are given as the plethystic exponential of the letter indices f_∞ and $\widetilde{f}_\infty$ of the massless fields in AdS, this can be formally calculated as follows:

$$F_\infty\sigma_x\widetilde{F}_\infty = \mathrm{Pexp}(f_\infty + \sigma_x\widetilde{f}_\infty). \tag{179}$$

Interestingly, in all examples of pairs of theories related by the X-expansion, we find the relation

$$f_\infty + \sigma_x\widetilde{f}_\infty = -1 \tag{180}$$

holds. Therefore, naively, Eq. (179) becomes $\mathrm{Pexp}(-1) = 0$. This is nice because we want to obtain $\mathcal{N}^{-1}$ as the result of the regularization. Of course the above manipulation is so naive and formal that it does not make sense as it is. It would be nice if we could improve the derivation.

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Appendix A. Characters and Haar measures

In general, the Haar measure is given by the plethystic exponential of the negative of the adjoint character with constant terms removed.

For $G = U(N)$ the adjoint character is given by

$$\chi^{U(N)}_{\mathrm{adj}}(\zeta_a) = \chi^{U(N)}_{\mathrm{fund}}\chi^{U(N)}_{\overline{\mathrm{fund}}} = \sum_{1\le a,b\le N}\frac{\zeta_a}{\zeta_b}, \tag{A1}$$

where $\chi^{U(N)}_{\mathrm{fund}}$ and $\chi^{U(N)}_{\overline{\mathrm{fund}}}$ are the characters of the fundamental and the antifundamental representations, respectively:

$$\chi^{U(N)}_{\mathrm{fund}} = \sum_{a=1}^{N}\zeta_a, \quad \chi^{U(N)}_{\overline{\mathrm{fund}}} = \sum_{a=1}^{N}\frac{1}{\zeta_a}. \tag{A2}$$

In the sum in Eq. (A1) each term with $a = b$ gives 1, and in total the constant term in $\chi^{U(N)}_{\mathrm{adj}}$ is N. The Haar measure is given by

$$\begin{aligned}\int_{U(N)} d\mu &= \frac{1}{N!}\int\prod_{a=1}^{N}\frac{d\zeta_a}{2\pi i\zeta_a}\,\mathrm{Pexp}(-\chi^{U(N)}_{\mathrm{adj}}(\zeta_a) + N)\\ &= \frac{1}{N!}\int\prod_{a=1}^{N}\frac{d\zeta_a}{2\pi i\zeta_a}\prod_{a\neq b}\left(1-\frac{\zeta_a}{\zeta_b}\right).\end{aligned} \tag{A3}$$

The overall factor $1/N!$ can be fixed by the normalization condition $\int_G d\mu = 1$.

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The characters for the symplectic and orthogonal groups are obtained from $U(N)$ characters by appropriate restrictions and projections. To obtain $Sp(N)$ and $SO(2N)$ characters we start from $U(2N)$, and impose the following constraints on the $2N$ fugacities ζ_a ($a = 1, \ldots, 2N$):

$$\zeta_{a+N} = \frac{1}{\zeta_a} \quad (a = 1, \ldots, N). \tag{A4}$$

Then the fundamental and the antifundamental characters in Eq. (A2) (with N replaced by $2N$) become the same. The adjoint characters for $Sp(N)$ and $SO(2N)$ are given by taking the symmetric and antisymmetric products of two copies of the fundamental character:

$$\begin{aligned}
\chi_{\text{adj}}^{Sp(N)} &= \sum_{1\le a\le b\le 2N} \zeta_a\zeta_b = \sum_{1\le a\le b\le N} \left(\zeta_a\zeta_b + \frac{1}{\zeta_a\zeta_b}\right) + \sum_{1\le a,b\le N} \frac{\zeta_a}{\zeta_b}, \\
\chi_{\text{adj}}^{SO(2N)} &= \sum_{1\le a< b\le 2N} \zeta_a\zeta_b = \sum_{1\le a< b\le N} \left(\zeta_a\zeta_b + \frac{1}{\zeta_a\zeta_b}\right) + \sum_{1\le a,b\le N} \frac{\zeta_a}{\zeta_b}.
\end{aligned} \tag{A5}$$

Both these characters include constant term N, and the Haar measures are given by

$$\begin{aligned}
\int_{Sp(N)} d\mu &= \frac{1}{N!2^N} \int \prod_{a=1}^{N} \frac{d\zeta_a}{2\pi i\zeta_a} \operatorname{Pexp}\left(-\chi_{\text{adj}}^{Sp(N)}(\zeta_a) + N\right), \\
\int_{SO(2N)} d\mu &= \frac{1}{N!2^{N-1}} \int \prod_{a=1}^{N} \frac{d\zeta_a}{2\pi i\zeta_a} \operatorname{Pexp}\left(-\chi_{\text{adj}}^{SO(2N)}(\zeta_a) + N\right).
\end{aligned} \tag{A6}$$

The adjoint character and the Haar measure of $SO(2N + 1)$ are obtained in a similar way starting from $U(2N + 1)$ and imposing the constraints

$$\zeta_{a+N} = \frac{1}{\zeta_a} \quad (a = 1, \ldots, N), \quad \zeta_{2N+1} = 1. \tag{A7}$$

The adjoint character is

$$\chi_{\text{adj}}^{SO(2N+1)} = \sum_{1\le a< b\le 2N+1} \zeta_a\zeta_b = \chi_{\text{adj}}^{SO(2N)} + \sum_{a=1}^{N} \left(\zeta_a + \frac{1}{\zeta_a}\right), \tag{A8}$$

and the normalized Haar measure is

$$\int_{SO(2N+1)} d\mu = \frac{1}{N!2^N} \int \prod_{a=1}^{N} \frac{d\zeta_a}{2\pi i\zeta_a} \operatorname{Pexp}\left(-\chi_{\text{adj}}^{SO(2N+1)}(\zeta_a) + N\right). \tag{A9}$$

The orthogonal group $O(2N)$ consists of two disconnected components: the component $G_+ = SO(2N) \subset O(2N)$ containing the identity and the other component which we denote by $G_- \subset O(2N)$. We normalize the Haar measure for each component by $\int_{G_\pm} d\mu = 1$, and the normalized measure for $O(2N)$ is given by

$$\int_{O(2N)} d\mu = \frac{1}{2}\int_{G_+} d\mu + \frac{1}{2}\int_{G_-} d\mu. \tag{A10}$$

To obtain the adjoint character and the Haar measure for G_- we start from $U(2N)$ and impose the constraints

$$\zeta_{a+N} = \frac{1}{\zeta_a} \quad (a = 1, \ldots, N-1), \quad \zeta_N = 1, \quad \zeta_{2N} = -1. \tag{A11}$$

Remark that the value $\zeta_{2N} = -1$ should be set after the calculation of the plethystic exponential. For example, $\operatorname{Pexp}\zeta_{2N} = 1/(1 - \zeta_{2N}) = 1/2$. To avoid confusion with fermionic terms

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with negative coefficients, we introduce an auxiliary variable θ which is set to be -1 after the calculation of the plethystic exponential. The adjoint character is

$$\chi_{\text{adj}}^{G_-} = \sum_{1\le a<b\le 2N} \zeta_a\zeta_b = \chi_{\text{adj}}^{SO(2N-1)} + \theta\left[\sum_{a=1}^{N}\left(\zeta_a + \frac{1}{\zeta_a}\right) + 1\right], \tag{A12}$$

and the normalized Haar measure is

$$\int_{G_-} d\mu = \frac{1}{(N-1)!2^N}\int \prod_{a=1}^{N-1}\frac{d\zeta_a}{2\pi i\zeta_a}\left[\text{Pexp}(N-1+\theta-\chi_{\text{adj}}^{G_-}(\zeta_a))\right]\Big|_{\theta=-1}. \tag{A13}$$

Appendix B. Contributions from fixed loci

B.1. Tensor multiplet on $AdS_5 \times S^1$

The fixed locus of the $\mathbb{Z}_k$ generator (30)

$$U_k = \exp\left(\frac{2\pi i}{k}(R_x - R_y)\right) \tag{B1}$$

is $AdS_5 \times S^1$, and tensor multiplets live on it. All the component fields are U_k neutral, i.e. $R_x - R_y = 0$ for all fields. Let us express the other quantum numbers by the notation

$$\left[\tfrac{J_1+J_2}{2}, \tfrac{J_1-J_2}{2}\right]_H^{\frac{R_x+R_y}{2},R_z}. \tag{B2}$$

The modes of a single tensor multiplet on the fixed locus belong to $\mathcal{E}_{\ell(0,0)}$ representations ($\ell \ge 1$), and the conformal representations with the following primaries appear:

$$\underline{[0,0]_\ell^{0,+\ell}},\quad \underline{\left[0,\tfrac{1}{2}\right]_{\ell+\frac{1}{2}}^{\frac{1}{2},+\ell-\frac{1}{2}}},\quad [0,1]_{\ell+1}^{0,+\ell-1}\oplus\underline{[0,0]_{\ell+1}^{1,+\ell-1}},\quad \left[0,\tfrac{1}{2}\right]_{\ell+\frac{3}{2}}^{\frac{1}{2},+\ell-\frac{3}{2}},\quad [0,0]_{\ell+2}^{0,+\ell-2}. \tag{B3}$$

The primary states of the underlined representations contribute to the index:

$$\begin{aligned} i\left([0,0]_\ell^{0,+\ell}\right) &= \tfrac{z^\ell}{(1-q)(1-p)} && (\ell=1,2,\ldots),\\ i\left([0,\tfrac{1}{2}]_{\ell+\frac{1}{2}}^{+\frac{1}{2},+\ell-\frac{1}{2}}\right) &= -\tfrac{z^{\ell-1}(q+p)}{(1-q)(1-p)} + \delta_{\ell,1}\tfrac{qp}{(1-q)(1-p)} && (\ell=1,2,\ldots),\\ i\left([0,0]_{\ell+1}^{+1,+\ell-1}\right) &= \tfrac{z^{\ell-2}qp}{(1-q)(1-p)} && (\ell=2,3,\ldots). \end{aligned} \tag{B4}$$

The term including $\delta_{\ell,1}$ is the contribution from the equation of motion of the fermion. By summing up all contributions we obtain

$$f_{\text{tensor}}^{AdS_5\times S^1} = \frac{z}{1-z} - \frac{q}{1-q} - \frac{p}{1-p} \tag{B5}$$

for a single tensor multiplet.

B.2. Tensor multiplet on $AdS_3 \times S^3$

The fixed locus of Eq. (40)

$$\widetilde{U}_k = \exp\left(-\frac{2\pi i}{k}(J_1 + R_x)\right) \tag{B6}$$

is $AdS_3 \times S^3$, and (2,0) tensor multiplets live on the six-dimensional locus. All component fields are $\widetilde{U}_k$ neutral, i.e. $J_1 + R_x = 0$. Let us express other quantum numbers in the form

$$[J_2]_H^{\frac{R_y+R_z}{2},\frac{R_y-R_z}{2},\frac{R_x-J_1}{2}}. \tag{B7}$$

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The mode expansion of a single tensor multiplet on the fixed locus gives the conformal representations with the following primaries:

$$\begin{array}{ccccc} & & [0]^{\frac{\ell-2}{2},\frac{\ell-2}{2},0}_{\ell+2} & & \\ & \left[+\frac{1}{2}\right]^{\frac{\ell-1}{2},\frac{\ell-2}{2},\frac{1}{2}}_{\ell+\frac{3}{2}} & & \left[-\frac{1}{2}\right]^{\frac{\ell-2}{2},\frac{\ell-1}{2},\frac{1}{2}}_{\ell+\frac{3}{2}} & \\ \underline{[+1]^{\frac{\ell}{2},\frac{\ell-2}{2},0}_{\ell+1}} & & [0]^{\frac{\ell-1}{2},\frac{\ell-1}{2},0\oplus 1}_{\ell+1} & & [-1]^{\frac{\ell-2}{2},\frac{\ell}{2},0}_{\ell+1} \\ & \underline{\left[+\frac{1}{2}\right]^{\frac{\ell}{2},\frac{\ell-1}{2},\frac{1}{2}}_{\ell+\frac{1}{2}}} & & \left[-\frac{1}{2}\right]^{\frac{\ell-1}{2},\frac{\ell}{2},\frac{1}{2}}_{\ell+\frac{1}{2}} & \\ & & \underline{[0]^{\frac{\ell}{2},\frac{\ell}{2},0}_{\ell}} & & \end{array} \tag{B8}$$

The underlined components include BPS states contributing to the index. The contributions from these conformal representations are

$$\begin{aligned} i\left([0]^{+\frac{\ell}{2},\frac{\ell}{2},0}_{\ell}\right) &= \frac{(y^{\ell}+\cdots+z^{\ell})}{1-p} && (\ell=1,2,\ldots), \\ i\left(\left[+\tfrac{1}{2}\right]^{+\frac{\ell}{2},\frac{\ell-1}{2},\frac{1}{2}}_{\ell+\frac{1}{2}}\right) &= -\frac{(y^{\ell-1}+\cdots+z^{\ell-1})(yz+p)}{1-p} && (\ell=1,2,\ldots), \\ i\left([+1]^{+\frac{\ell}{2},\frac{\ell-2}{2},0}_{\ell+1}\right) &= \frac{(x^{\ell-2}+\cdots+y^{\ell-2})pyz}{1-p} && (\ell=2,3,\ldots). \end{aligned} \tag{B9}$$

Summing up all contributions with $\ell = 1, 2, \ldots$ we obtain

$$f^{AdS_3\times S^3}_{\text{tensor}} = \frac{y}{1-y} + \frac{z}{1-z} - \frac{p}{1-p}. \tag{B10}$$

B.3. Vector multiplet on $AdS_5 \times S^3$

The fixed locus of Eq. (101)

$$U_{O7} = e^{i\pi(R_x - R_y - S)} \tag{B11}$$

is $AdS_5 \times S^3$.

All components are U_{O7} neutral, i.e. $R_x - R_y = S$. Because S commute with all superconformal generators, the eigenvalue of S is common for all component fields in an irreducible superconformal representation. So is the generator $\frac{R_x - R_y}{2}$. Namely, this is the Cartan generator of the flavor symmetry $SU(2)_F$. Let us use the notation

$$\left[\tfrac{J_1+J_2}{2}, \tfrac{J_1-J_2}{2}\right]^{\frac{R_x+R_y}{2},\frac{R_x-R_y}{2},R_z}_{H} \tag{B12}$$

to represent the quantum numbers. We obtain the conformal representations with the following primaries:

$$\begin{array}{ccccc} & & [0,0]^{\frac{\ell-2}{2},\frac{\ell}{2},0}_{\ell+4} & & \\ & \left[0,\frac{1}{2}\right]^{\frac{\ell-1}{2},\frac{\ell}{2},+\frac{1}{2}}_{\ell+\frac{7}{2}} & & \left[\frac{1}{2},0\right]^{\frac{\ell-1}{2},\frac{\ell}{2},-\frac{1}{2}}_{\ell+\frac{7}{2}} & \\ [0,0]^{\frac{\ell}{2},\frac{\ell}{2},+1}_{\ell+3} & & \left[\frac{1}{2},\frac{1}{2}\right]^{\frac{\ell}{2},\frac{\ell}{2},0}_{\ell+3} & & [0,0]^{\frac{\ell}{2},\frac{\ell}{2},-1}_{\ell+3} \\ & \underline{\left[\frac{1}{2},0\right]^{\frac{\ell+1}{2},\frac{\ell}{2},+\frac{1}{2}}_{\ell+\frac{5}{2}}} & & \left[0,\frac{1}{2}\right]^{\frac{\ell+1}{2},\frac{\ell}{2},-\frac{1}{2}}_{\ell+\frac{5}{2}} & \\ & & \underline{[0,0]^{\frac{\ell+2}{2},\frac{\ell}{2},0}_{\ell+2}} & & \end{array} \tag{B13}$$

This is the representation $\widehat{\mathcal{B}}_{\frac{\ell+2}{2}} \otimes [\frac{\ell}{2}]_{SU(2)_F}$. The two conformal representations with underlines contribute to the index. The contributions are given by

$$\begin{aligned} i\left([0,0]^{\frac{\ell+2}{2},\frac{\ell}{2},0}_{\ell+2}\right) &= \frac{xy(x^{\ell}+\cdots+y^{\ell})}{(1-q)(1-p)} && (\ell=0,1,2,\ldots), \\ i\left(\left[\tfrac{1}{2},0\right]^{\frac{\ell+1}{2},\frac{\ell}{2},+\frac{1}{2}}_{\ell+\frac{5}{2}}\right) &= -\frac{qp(x^{\ell}+\cdots+y^{\ell})}{(1-q)(1-p)} && (\ell=0,1,2,\ldots). \end{aligned} \tag{B14}$$

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By summing up all contributions, we obtain

$$f_{\text{vector}}^{AdS_5\times S^3} = \frac{xy(1-z)}{(1-q)(1-p)(1-x)(1-y)}. \tag{B15}$$

B.4. Vector multiplet on $AdS_3 \times S^5$

The fixed locus of the $O7$-flip (112)

$$\widetilde{U}_{O7} = e^{\pi i(J_2+A)} \tag{B16}$$

is $AdS_3 \times S^5$. Let us use the notation

$$[J_1]_H^{R_{SU(4)},J_2} \tag{B17}$$

to represent the quantum numbers. $R_{SU(4)}$ is given by the Dynkin labels. The mode expansion in $AdS_3 \times S^5$ gives the conformal representations with the following primaries:

$$\begin{array}{ccccc} & & [-1]^{[0,\ell-2,0],0}_{\ell+3} & & \\ & \left[-\frac{1}{2}\right]^{[0,\ell-2,1],+\frac{1}{2}}_{\ell+\frac{5}{2}} & & \left[-\frac{1}{2}\right]^{[1,\ell-2,0],-\frac{1}{2}}_{\ell+\frac{5}{2}} & \\ [0]^{[0,\ell-1,0],+1}_{\ell+2} & & [0]^{[1,\ell-2,1],0}_{\ell+2} & & [0]^{[0,\ell-1,0],-1}_{\ell+2} \\ & \underline{\left[+\frac{1}{2}\right]^{[1,\ell-1,0],+\frac{1}{2}}_{\ell+\frac{3}{2}}} & & \left[+\frac{1}{2}\right]^{[0,\ell-1,1],-\frac{1}{2}}_{\ell+\frac{3}{2}} & \\ & & \underline{[+1]^{[0,\ell,0],0}_{\ell+1}} & & \end{array} \tag{B18}$$

Dynkin labels $[R_1, R_2, R_3]$ correspond to the following R-charges:

$$(R_x, R_y, R_z) = R_1\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right) + R_2(1,0,0) = R_3\left(\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}\right). \tag{B19}$$

The underlined conformal representations contribute to the index:

$$\begin{aligned} i\left([+1]^{[0,\ell,0],0}_{\ell+1}\right) &= \tfrac{q(x^\ell+\cdots+y^\ell+\cdots+z^\ell)}{1-q} && (\ell = 0,1,2,\ldots), \\ i\left(\left[+\tfrac{1}{2}\right]^{[1,\ell-1,0],+\frac{1}{2}}_{\ell+\frac{3}{2}}\right) &= -\tfrac{pq(x^{\ell-1}+\cdots+y^{\ell-1}+\ldots+z^{\ell-1})}{(1-q)} && (\ell = 1,2,3,\ldots). \end{aligned} \tag{B20}$$

Summing up all contributions, we obtain

$$f_{\text{vector}}^{AdS_3\times S^5} = \frac{q(1-p)}{(1-q)(1-x)(1-y)(1-z)}. \tag{B21}$$

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