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VARIATION IN TEACHERS' CHOICE AND USE OF EXAMPLES WITHIN A MATHEMATICS DEPARTMENT: AFFORDANCES IN THE INTRODUCTION OF FUNCTIONS

A Research Report

submitted to the Faculty of Education, *University of the Witwatersrand*,
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Abstract

In mathematics classrooms, the examples that teachers choose and use impact the affordances for learning that are offered to their students. This study investigates to what extent such affordances for learning might differ within different teachers' classes within one mathematics department at a school in Johannesburg. In the department that was analysed, teachers have the agency to choose their own examples and to structure the teaching of mathematical topics themselves. In a case study design, three teachers from this department were observed teaching their two introductory lessons on Grade 10 functions. All of the examples used in their lessons were extracted and analysed using variation theory. The examples were first analysed in and of themselves, and then with the teachers' mediation. The teachers were also interviewed to provide insight into their intended object of learning for the lessons. Of interest in the analysis was how the multiple representations of functions were integrated in the lessons. The analysis indicates that the affordances for learning across the three classes differed substantially, both regarding the sequencing of objects of learning within the topic of functions, and in the aspects and features that were opened up for discernment. These findings raise questions regarding how much agency teachers within one department should have in structuring the teaching of objects of learning and in selecting examples to use in their lessons.

Epigraph

"[P]eople may see the world differently, simply because things – or certain aspects of things – that are visible to some people are invisible to others."

(Marton, 2015, p.59)

"When a red ivory ball, seen for the first time, has been withdrawn, it will leave a mental representation of itself, in which all that it simultaneously gave us will indistinguishably co-exist. Let a white ball succeed to it; now and not before, will an attribute detach itself, and the colour, by force of contrast, be shaken out into the foreground. Let the white ball be replaced by an egg, and this new difference will bring the form into notice from its slumber. And thus, that which began by being simply an object cut out from the surrounding scene becomes for us first a red object, and then a red round object; and so on."

(Martineau, as cited by Gibson, 1969, in Marton, 2015, p.59)

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"For unto whomsoever much is given, of her shall be much required" – Luke 12:48

I have sincerely strived to produce good work.

Plagiarism Declaration

I declare that this Research Report is my own, unaided work, except as indicated in the acknowledgments, the text and the references. It is being submitted for the Degree of Master of Education at the *University of the Witwatersrand*, Johannesburg. It has not been submitted before for any degree or examination at any other University.

Kathryn Anne McLachlan 15 March 2023

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Chapter 1: Introduction

Introduction, Background and Rationale

I have often wondered, in my capacity as a high school mathematics teacher, how teaching and learning differ across classes in a mathematics department. Although a mathematics department functions to some extent as a unit, this does not guarantee that teaching across the different department members' classes is conducted in a similar fashion. In the mathematics department in which I teach, we spend much time together discussing our students, assessments, and the order in which we teach topics, and occasionally share resources. However, we do not plan our day-to-day lessons at example or task level. Furthermore, notwithstanding that all students and teachers have access to the same prescribed textbook, this textbook is seen as *a* rather than *the* resource, and it is not required that we base our lessons on this textbook content. As qualified professionals, we are given much freedom to teach in a way that 'works for us' and to sequence required content as we see fit within each topic of the curriculum. As such, students across a grade in different classes in my department may encounter different content, tasks and examples in their day-to-day lessons depending on which teacher's class they are in and how this teacher chooses and uses examples.

Examples are generally positioned as a crucial and influential aspect of mathematics teaching (Arzarello, Ascari & Sabena, 2011; Bills, Dreyfus, Mason, Tsamir, Watson & Zaslavsky, 2006; Goldenberg & Mason, 2008; Zaslavsky & Zodik, 2007; Zaslavsky, 2010; Zaslavsky, 2019). This influence is understandable as examples serve as cultural mediating tools that connect students with abstract mathematical ideas while also functioning as an important element of mathematical communication between students and teachers, and students with themselves (Goldenberg & Mason, 2008). Exposure to examples develops (personal) example spaces, namely the collection of examples that a student or teacher can draw on in response to a task, prompt or situation (Bills et al., 2006). Such spaces can be rich and varied, providing multiple tools with which, and perspectives from, a novel task can be tackled; or meagre, providing limited resources with which to make sense of a new situation. The quality and variety of examples to which students are exposed can thus be instrumental to their mathematical understanding. From this perspective, learning mathematics can be seen as gaining access to new examples and creating richer (inter)connections within and between example spaces (Goldenberg & Mason, 2008). As a mathematics teacher is a primary means by which students experience examples, how teachers choose and use examples is a crucial aspect of their classroom teaching practice.

Ference Marton's theory of variation (2006; 2015; amongst others) puts forward how students learn from examples and suggests that how examples are structured and sequenced by teachers influence student learning. His theory suggests that humans naturally notice variation in objects that are experienced close together (physically and/or temporally) and that examples that are juxtaposed and sequenced to show variation will lead to the discernment of critical aspects of an object of learning, and ultimately generalisation across differences. Such generalisation "comes about through appreciating dimensions of (possible) variation and ranges of permissible change, by seeing generality through the particular" (Goldenberg & Mason, 2008, p.190). Once more than one aspect of an object of learning has been discerned, 'fusion' is reached when students are able to notice multiple aspects within one example and to discern which are critical to the task at hand and which are not. Variation theory can be used to analyse and/or plan lessons (Watson, 2017). It enables one to consider how example choice and use allows for the discernment and then the generalisation of critical aspects of the lesson's objects of learning, so that these can eventually be simultaneously discerned and manipulated. Although developed separately from the notion of example spaces, variation theory (VT) allows for the analysis of example spaces in order to consider the affordances for learning that are embedded in the sequencing and juxtaposition of the chosen examples.

It was against this backdrop of example choice and use that I began an investigation into how teaching differs across classes in the mathematics department in which I teach. As an entry-point to this broad question, I decided to examine and compare different teachers' choice and use of examples on the same topic to juxtapose the affordances for learning created by these example choices. My study, however, remained small and took on the form of a case study where I observed three teachers' two introductory lessons to Grade 10 functions to gather data as a means of comparing the affordances for learning on offer across the different classes. 'Functions' was selected as the topic of investigation because of its mathematical significance as one of the crucial unifying ideas in mathematics education (Knuth, 2000). The topic represents an important step in the development of abstract mathematical knowledge (Piaget, Grize, Szeminska & Bang, 1968) and, due to functions' multiple representations, is one of the initial points in a student's mathematical journey where one symbolic system is used in the expression and understanding of another symbolic system (Leinhardt, Zaslavsky & Stein, 1990). 'Grade 10' was chosen as the level of investigation because it is students' first in-depth encounter with the study of functions in the South African school system. 'Functions' remains a large topic in Grade 11 and Grade 12 and makes up the joint-largest section (from a mark's perspective) of the Paper 1 Grade 12 Mathematics Matric examination. The two 'introductory lessons' were selected for observation as these were the students' initial lessons of encounter with the topic of functions in the Further Education and Training (FET) phase. It was thus of interest to consider how the observed teachers choose to launch the study of this topic and on what they focused in setting initial knowledge foundations of functions.

In my study, I focused on the affordances for learning that were offered to students through the teachers' selection, sequencing, and mediation of examples on functions. As working with multiple representations is a key aspect of this topic, this was one aspect of the affordances for learning that I examined in particular. Adler (2017) describes that "what is taught is not synonymous with what is learned, but what is taught – made available to learn – is critical" (p. xii). By examining and juxtaposing the affordances for learning in the three teachers' lessons through their choice and use of examples, I was thus able to consider how similar or different the affordances for learning were in the three different classes within this mathematics department.

It is important to note from the start that my study did not aim to compare the affordances for learning created by different teachers from the position of someone aiming to critique or pass judgement. Rather, I hoped to learn from these teachers' skill and to consider the possible implications of the similar or different affordances for learning experienced by their students. Additionally, I hoped to ultimately consider what this study's findings may mean practically for my mathematics department and for mathematics departments that operate under similar *modus operandi*.

Research Questions

This study thus intended to investigate the following principal research question and associated sub-questions:

Principal research question:

What affordances for learning are made available through the choice and use of examples in the introduction of Grade 10 functions by three teachers in one mathematics department, and how are these similar or different?

Sub-questions:

1. Which 'objects of learning' of the notion of functions did the teachers open up through the use of patterns of variation?

2. How did the teachers engage with the multiple representations of functions within their choice and use of examples?

Conclusion

In Chapter 1, I described the background and rationale that inspired this study of teachers' choice and use of examples and explained what the study aimed to achieve. I also defined the research questions that guided the study's implementation.

The subsequent chapters of this research report are structured as follows to communicate and justify how the study was implemented:

- In Chapter 2, I position my study in the literature, considering literature on examples in the teaching and learning of mathematics and the teaching and learning of functions. I also describe the study's theoretical framework: Marton's (2015) variation theory.
- In Chapter 3, I engage with the study's research design. I discuss the data collection process and the associated ethical considerations, as well as how the data was analysed. Finally, I consider the validity and reliability of the study.

- In Chapter 4, I present the data analysis and findings of the study. Here, I first analyse the teachers' choice and use of examples vertically, and then juxtapose the affordances for learning created through their choice and use of examples horizontally.
- In Chapter 5, I conclude by discussing the study's findings as well as considering the limitations and implications of the study.

Chapter 2: Literature and Theoretical Framework

Introduction

In this chapter, I engage with the literature within which my study is situated, and which provides the study's theoretical foundations. First, I consider literature on examples in the teaching and learning of mathematics, including the definition of example used in this study. Next, I present work on functions and their multiple representations. I also engage with South African literature on exemplification and the teaching and learning of functions. Finally, the chapter concludes with a discussion of the study's theoretical framework: Marton's (2015) variation theory.

The Literature

Examples in the Teaching and Learning of Mathematics

The study of examples in mathematics education has gained attention and momentum within the last two decades (see, for example, Bills et al., 2006, and Zaslavsky, 2019). Examples have been studied in various ways, aspects of which include how experienced and novice teachers select examples (Bills & Bills, 2005); how teachers' choices and use of mathematical examples can afford or impede student learning (Zaslavsky & Zodik, 2007); the role examples play in the development of students' mathematical concepts (Arzarello et al., 2011); examples as an aspect of lesson design and analysis (Adler & Ronda, 2015; Adler & Pournara, 2019); and example use in multilingual teacher education classrooms (Essien, 2021b), to name but a few.

Prior to this recent interest, Rissland Michener's (1978) seminal paper on understanding mathematical understanding positioned examples as crucial to mathematical learning and an element of expert knowledge. She differentiates four types of examples, namely *start up examples*, *reference examples*, *model examples* and *counter-examples*. Start up examples provide a simple, stand-alone introduction to a topic that can however "be lifted to the general case" (p.366); reference examples are referred to frequently due to their wide applicability and their ability to link various concepts; model examples are generic and explain the general case; while counter-examples illustrate that a statement is false while they sharpen and clarify distinctions between concepts. These examples all form part of what Rissland Micheler describes as "major epistemological classes of Examples-space" (p.368) for mathematical knowledge.

Watson and Mason (2005) also consider the notion of example spaces as foundational to mathematical learning and knowledge, but from a different angle. They argue that

> [M]athematics is learned by becoming familiar with examples that manifest and illustrate mathematical ideas and by constructing generalisations from examples. (p.24)

As such, they suggest that acquiring mathematical knowledge occurs by developing 'example spaces', namely collections of examples that create understanding of a mathematical topic. Learning mathematics thus occurs by "exploring, rearranging, gaining fluency with and extending [one's] examples spaces, and the relationships between and within them" (p.6). Watson and Mason (2005) describe ways in which example spaces can be developed: experiencing new examples that appear unconnected, or weakly connected, to prior knowledge; experiencing new examples that appear connected to prior knowledge; by experiencing counter-examples; by restructuring existing example spaces so that 'forgotten' examples becomes more clearly integrated with examples that are regularly used; by realising that existing knowledge (examples) can be used in new ways; and by merging existing example spaces in news ways. Such development relies on the experiencing of new examples, or on tasks that promote the restructuring of existing example spaces. In the mathematics classroom, learning opportunities can thus largely be dependent on the examples and tasks that teachers make available to their students.

The definition of 'example' used in this study is drawn from Watson and Mason's (2005) work. The authors describe an example as being a particular of a larger topic and "anything from which a learner may generalise" (p.3) about this topic. This includes illustrations, representations and manipulatives of concepts and procedures; tasks, as worked examples or exercises tackled by students; as well as mathematical material used for inductive reasoning; and contexts used from which to draw mathematics. All such content, as presented by teachers in their practice, thus contributes towards how their students make sense of the subject.

Historically, Shulman (1986) also advanced the importance of examples, by including example-use as a crucial aspect of a teacher's pedagogical content knowledge (PCK), the intersection of their knowledge of pedagogy and their knowledge of subject content matter. Shulman suggests that this aspect of PCK includes knowledgably utilising:

> The most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others. Since there are no single most powerful forms of representation, the teacher must have at hand a veritable armamentarium of alternative forms of representation. (p. 9)

The necessity for teachers to have at their disposal such an 'armamentarium' of alternate examples to teach is echoed by Leinhardt (2001) who argues that multiple examples are needed for learning to occur, as one example alone cannot encapsulate and make clear the critical features of the greater class of which it is an example. In Zaslavsky and Zodik's (2007) discussion on how experienced teachers' use examples and how these afford or limit learning opportunities, they also stress the necessity of teachers having access to multiple examples. In their work, they highlight that mismatches frequently occur between teachers' intended message through their choice of examples and what students perceive. Many of the cases of teacher example-use that they discuss involve teachers flexibly modifying their initial examples based on student responses to the example. Accordingly, in a later study, Zodik and Zaslavsky (2008) differentiate between a teacher's *pre-planned examples* and *spontaneous examples* that occur impromptu in the lesson due to an unforeseen occurrence. Zaslavsky and Lavie (2005) engage with teachers' mediation of examples, and use the terms 'transparent' and 'opaque' regarding what different examples make clear. For instance, they consider the following set of three different representations of the same quadratic function and explain the following:

> 1. $y = (x+1)(x-3)$; 2. $y = (x-1)^2 - 4$; 3. $y = x^2 - 2x - 3$
>
> Each example is transparent to some features of the function and opaque with respect to others. E.g., the first example conveys the roots of the function (-1 and 3); the second one communicates straightforward the vertex of the parabola $(1; -4)$; and the third example transmits the y-intercept $(0;\ -3)$. (p.3)

This extract highlights how different representations of a concept create different affordances for learning. Additionally, the authors add that the links between the different representations may, however, not be obvious to the student without the teacher providing guidance through mediation. As such, Zaslavsky and Zodik (2007) suggest that it is both the *examples* and the *shifted focus of attention* by the teacher that influence what students notice and hence, what they learn. Essien (2021b), in the context of teacher education, suggests that teacher discourse and teacher moves are critical in this process, and argues that "what is possible to discern from a set of examples that are enacted in class, is a function of the interactional process and teacher moves of the teacher educator around the example space" (p.11). Therefore, it appears that a teacher's ability to create learning opportunities (from an examples perspective) relies on them having a large example space at their disposal, being able to engage flexibly and spontaneously with their example space, and then also being able to direct their students' attention through talk and action to the particular elements of their selected examples that communicate their learning goals.

The literature discussed so far makes clear the importance of examples to the teaching and learning of mathematics, and accordingly, the possible impact created by a teacher's use of examples in their teaching. Bills and Watson (2008) express this succinctly by stating that examples are "at the heart of the learners' mathematical experiences" (p.77) and argue further that any mathematical theory of learning that does not consider how teachers and students engage with examples is likely be incomplete. Ball et al. (2008), who investigate what knowledge teachers need to teach mathematics, attest to this literature, describing that using examples is part of a teacher's 'knowledge of content and students', explaining that in choosing examples and tasks, teachers need to anticipate student interest and motivation, as well as whether this content will be experienced as easy or difficult. Furthermore, they assert that working with examples is part of teacher 'knowledge of content and teaching', describing that teachers "sequence particular content for instruction. They choose which examples to start with and which examples to use to take students deeper into the content" (p.401). Such sequencing of examples adds an additional layer to teachers' example choices, as *how* teachers choose to sequence their examples can impact the possible learning opportunities afforded to their students.

Variation theory (VT), which will be discussed in greater detail as the theoretical framework of this study, posits that learning implies discernment or "differentiation rather than accumulation" of knowledge (Gibson & Gibson cited in Kullberg et al., 2017). In relation to the above literature on examples, this theory thus suggests that it is not purely an accumulation of examples that creates learning, but that the sequencing of an example set – allowing critical aspects of the content to become clear – creates powerful affordances for learning. Watson and Mason (2005) link these ideas to the notion of example spaces, in that example spaces gain structure by their 'owner' understanding the 'dimensions of possible variation' as well as the 'range of permissible change' that are allowed within a concept's example space. Goldenberg and Mason (2008) describe this as the discernment of possible features of an example without losing 'examplehood' of a concept. In the language of VT, when students engage with examples, they must first experience *contrast* to become aware of critical aspects of a concept, before experiencing *generalisation* to see how this aspect behaves in different scenarios. Finally, once generalisation has occurred for various critical aspects of a concept, *fusion* can take place, where multiple critical aspects can be discerned simultaneously. As such, VT suggests that it is the patterns and sequencing of examples, which make clear variation and invariance of a concept, that are of critical importance in creating affordances for learning (Kullberg et al., 2017). However, as described earlier by Zaslavsky and Lavie (2005), it is crucial that a teacher additionally draws attention to the variation within the selected example set to highlight the intended aspect under discussion (Kullberg, 2012).

Studies on Exemplification in Mathematics Education in South Africa

In the South African context, exemplification has been studied by Adler and others in their work on the Mathematical Discourse in Instruction framework (see, for example, Adler & Ronda, 2015; Alder and Venkat, 2014) and in the context of teacher professional development work using the Mathematical Discourse in Instruction framework (Adler & Pournara, 2019). Essien (2021a; 2021b), on the other hand, examines exemplification within the context of multilingual teacher education classrooms, where both mathematical and linguistic complexities become part of example selection and use. Additional South African studies, that involve exemplification in combination with the study of functions, are discussed later under "Studies on the Teaching and Learning of Functions in South Africa".

In conclusion, teachers' use of examples is a critical aspect of teacher knowledge which has much impact on the learning opportunities created in their classrooms. The types of examples teachers choose; the way they combine, sequence, and adapt examples; and how they draw attention to critical aspects through mediation, all influence their students' potential learning and resulting example spaces.

Functions and their Multiple Representations

The notion of function is one of fundamental significance in mathematics and thus also in mathematics education. The complex nature of functions, however, makes it a challenging topic to teach, with a particular difficulty being a function's characteristic of having multiple representations (Niss, 2020). A representation in mathematics is some sign or symbol that stands for or embodies an idea or relationship (Goldin, 2014). Duval (1999; see also Janvier 1987) points out mathematics' unique reliance on semiotic representation to document thinking. These semiotic representations are not the concept itself – they merely represent the idea. Crucially, the same concept can be represented in different semiotic systems, and students of mathematics need to be able to translate between these systems (Duval, 2006).

Leinhardt et al. (1990) argue that the topic of 'functions' is one of the earliest points in a student's mathematical pathway in which they use "one symbolic system to expand and understand another" (p.2). Although students may have previously dealt with concrete representations as a steppingstone to more abstract ideas, functions incorporate multiple abstract, symbolic systems that shed light on one another. Focusing on algebraic functions and their graphs, the authors suggest that neither can be treated as an isolated concept but that they define and construct the concept of 'function' together. Each representation communicates the concept of function differently, and contributes to the organisation of mathematical ideas. This characteristic is demanding on students, and additionally requires new understanding of notation and symbolic correspondences.

A function can be represented in a large variety of ways, including verbal, algebraic, graphic, diagrammatic, and tabular forms. Each of these highlights different aspects of the function (Nitsch et al., 2015), allowing some features to be more transparent and others more opaque (Zaslavsky & Lavie, 2005). Niss (2020) is of a similar opinion, but adds that this attribute "may obscure the underlying commonality – the core – of the concept across its different representations, especially as translating from one representation to another may imply loss of information" (p.304). In understanding functions, it is the synthesis between representations in which rich understanding lies. Understanding the 'bigger picture' of functions is not achieved by understanding one or multiple representations in isolation. Rather, it requires students to juggle between the representations to achieve real understanding (Goldenberg, 1995). Multiple representations thus play a dual role, initially being a potential obstacle to learning but ultimately also potentially creating rich conceptual understanding (Dreher & Kuntze, 2015).

Overall, the study of function's multiple representations is highly compatible with the study of example use and VT, as all three concepts complement one other in the construction of example spaces and students' resulting mathematical learning. I now briefly consider South African work on the teaching and learning of functions.

Studies on the Teaching and Learning of Functions in South Africa

The teaching and learning of functions has been investigated to some extent within the South African context. In several articles centred around 'mathematics for teaching', Grade 10 functions was the mathematical topic under discussion although not the main focus of the research (Adler & Pillay, 2007; Kazima, Pillay & Adler, 2008; Pournara, Hodgen, Adler & Pillay, 2015). Mudaly and Mpofu (2019) examined Grade 11 students' views on asymptotes from a commognitive approach. They found that the students' mathematical discourse was not coherent, with "[d]ifferent expressions of the same mathematical object elicit[ing] different responses" (p.734). Most relevant to my research is the work of Mbhiza (2022) and Pillay, Adler and Runesson Kempe (2022). Mbhiza (2022) investigates the selection, sequencing and variation within examples in Grade 10 function lessons through the lens of Sfard's theory of commognition. His study found that the teachers' example selection and sequencing could either constrain or enable the development of the narratives about function parameters that were endorsed by the teacher. Pillay et al. (2022), on the other hand, present research on how a teacher's sequencing and pairing of examples can open affordances for learning. They also discuss how teacher and student action in the mediation of the examples can bring the critical aspect of the object of learning into focus. Their study found that deliberately using variation in the midst of invariance, *in combination with* the teacher's mediation of pre-planned and learner-generated examples, was crucial in creating affordances for learning. This brings to mind the work of Essien (2021b) which creates a framework that combines 'variation theory' with 'meaning making as a dialogic process', and emphasises that in examining what is to be discerned in a set of examples comes about jointly through the selected examples and the teacher's mediation thereof.

In the next section, I present the theoretical framework of my study.

The Theoretical Framework

Variation Theory

This study uses Ference Marton's (2015) variation theory of learning (VT) as the foundation of its theoretical and analytical frameworks. Put simply, this theory asserts that variation is a required aspect of learning so that students can notice and discern what is to be learnt. VT's focus on considering how learning episodes create affordances for learning – including how this occurs through the choice and use of examples – makes it a lens from which I can meaningfully analyse my data and answer my research questions. In the next paragraphs, I unpack my understanding of Marton's (2015) version of VT and explain key terms and how he uses them.

The Object of Learning in Variation Theory

Marton (2015) argues that to understand learning, one needs to start by considering what it is that needs to be learnt in a particular learning situation. He calls this the *object of learning*. He argues that the object of learning can be described in three ways. Firstly, most broadly, it can be described from a *content* perspective – in the sense of a syllabus. Secondly, more specifically, the object of learning can be described from an *educational objectives* perspective. Educational objectives describe what it is that needs to be done with the content. Finally, and most specifically, the object of learning can be described in terms of its *aspects*. The aspects describe what it is that a student must be able to discern and know to successfully enact the educational objectives (Marton, 2015).

For example, in mathematics, an object of learning could have to do with quadratic functions. Here the *content* is 'quadratic functions' while the *educational objective* might be to 'sketch a quadratic function'. To achieve this, various *aspects* of quadratic functions would need to be discerned and known. These might include the shape of a parabola, knowing what are the crucial points for sketching (the turning point, the y-intercept, and the potential x-intercepts) and how to find them, and how these all work together in a sketch of a parabola.

Marton (2015) suggests that successful learning "enables [students] to deal with future, novel situations in powerful ways" (p.26). Such learning requires a student to be able to consider and discern multiple aspects of the object of learning simultaneously. The aspects of the object of learning that need to be noticed to deal with an activity, but that a student is not yet able to notice, are the *critical* aspects for learning (Marton, 2015).

Aspects and Their Features

As explained above, Marton (2015) argues that an object of learning can be described as being made up of multiple *aspects* – keys that allow the object of learning to be understood in different ways. Aspects describe the possible dimensions of an object of learning that a student must be able to discern. Contrastingly, Marton describes that *features* are the specifics that occur within the dimensions of an aspect.

For example, when learning to describe the collection of geometric figures in Figure 1, one possibility is to use the aspect of 'colour', and another possibility is to describe the aspect of 'shape'. Here, the aspect of 'colour' has the features green, blue, yellow and red, while the aspect of 'shape' has the features square, circle, triangle and star. Both aspects are necessary to accurately describe the geometric figures, for example 'a blue circle'.

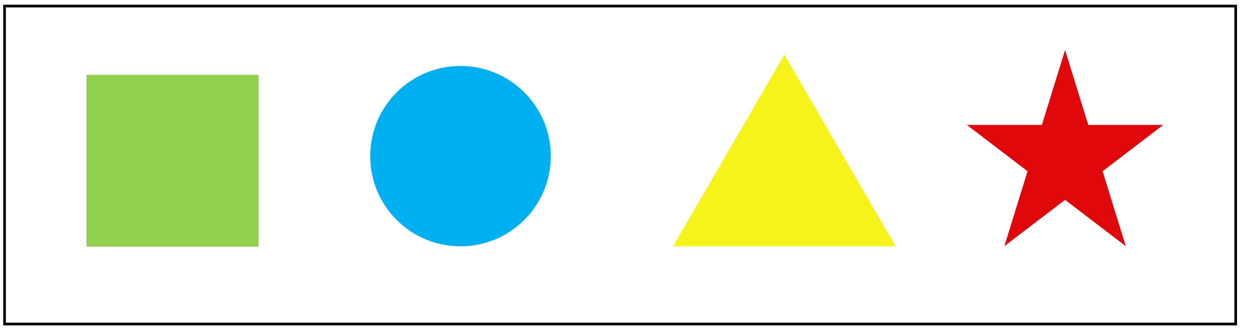


Figure 1 **A collection of geometric figures**

Marton (2015) explains that aspects and features cannot exist without each other. Discerning an aspect requires the knowledge of a minimum of two possible features. For example, to be able to discern the aspect colour, one needs to be able to discern a minimum of two features, e.g. green and blue. Marton (2015) explains that the "aspect (colour) is the experience of difference, and the features (colours) are those things that differ" (p.47).

However, what is a feature in one scenario, can become an aspect in another scenario (Marton, 2015). For example, consider the different shades of green in Figure 2. Here, green becomes an aspect (the colour green) of which many different shades exist, for example dark green, light green, lime green and mint green. Green could thus be a feature of the aspect 'colour'; or green could be the aspect, with different shades of green being the features.

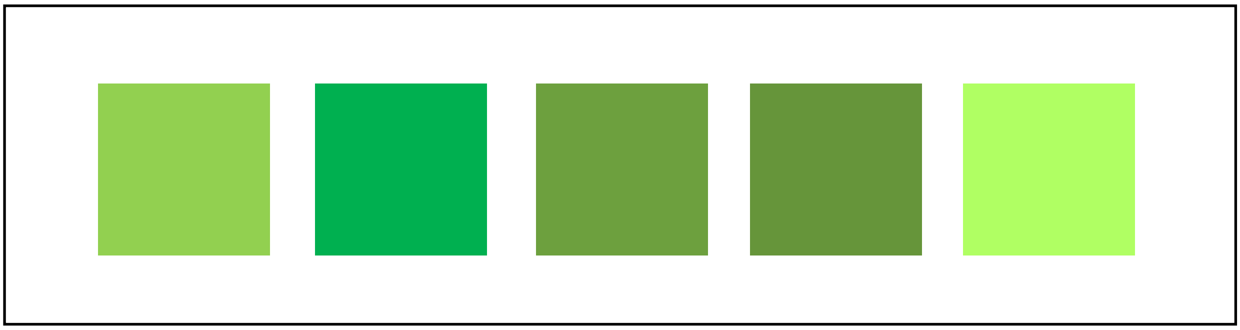


Figure 2 **Different shades of green**

Importantly, successful learning requires the discernment of aspects of the object of learning which are crucial to the object of learning, and the discernment of other aspects which can be disregarded. In the following example, Marton (2015) explains which aspects of a triangle need to be discerned and disregarded in order to recognise a triangle:

> In order to recognize a triangle, you must discern its three corners, but you have to disregard its rotational orientation, its size, and so on... Because every triangle will have a rotational orientation, you have to learn to discern (notice) this in order to be able to disregard the orientation and recognize the form as a triangle. The same goes for size, which is an inherent feature of any triangle. By discerning the size, it can be disregarded when comparing the angles of two triangles, for example. The learner thus must discern (open up) the necessary aspects of the object of learning in order to handle it in a powerful way. But she must also discern (open up) the non-necessary aspects of the object of learning in order to separate them and generalise across them. (p.33)

Marton (2015) further describes that '*necessary*' aspects and features are those aspects and features required to be understood in order to successfully engage with an object of learning. In Figure 1, two such necessary aspects were 'shape' and 'colour' in order to describe the geometric figures. Contrastingly, Marton explains that *critical* aspects and *critical* features are necessary aspects and features for achieving an educational objective that a student *does not yet know*, and therefore needs to learn to discern. They are thus critical to achieving the new educational objective.

Aspects and features thus provide an interesting way of engaging with the idea of misconceptions in learning. From this perspective, misconceptions occur when, in a new object of learning, only some of the necessary aspects and features are discerned while others remain unperceived, thus becoming critical aspects and features. These will now need to be discerned in order to remove the misconception (Marton, 2015).

The Intended, The Enacted and The Lived Object of Learning

A final distinction that needs to be made regarding the object of learning is between what is intended by the teacher, what happens in the classroom, and what is discerned by the student. Marton (2015) uses the term *intended* object of learning to describe what is aimed to be taught in a learning situation to achieve an educational objective. The *enacted* object of learning is what is made available to be learnt in a learning situation. Finally, the *lived* object of learning is what an individual student takes away from a learning situation. The lived object of learning can differ for each student in a learning situation, and will not necessarily align fully with the intended and enacted object of learning. Similarly, what is enacted in the classroom may be different to the teacher's intended object of learning.

Learning from the Variation Theory Perspective: Patterns of Variation and Invariance

Marton's (2015) variation theory posits that learning occurs as a student becomes able to discern critical features and aspects of an object of learning. As an object of learning will have multiple critical aspects and features, which all contribute to the concept at hand, ultimately a student will need to be able to discern multiple necessary aspects and features simultaneously.

However, to begin to know an object of learning, one first needs to encounter individual critical features and aspects before working with multiple aspects at once. Marton (2015) distinguishes between two types of learning with regards to features and aspects: firstly, the opening up (discernment) of new aspects and secondly, the discernment of new features within a known aspect. Returning to the earlier example regarding colour, learning colour as a new *aspect* would require the concept of colour to be opened up for the first time when at least two colours have been discerned. On the other hand, learning a new colour as a *feature* within a known aspect is when the aspect of colour is already established and a new colour is encountered. Marton (2015) describes the learning of a new aspect as the encounter of a new *dimension of variation* in the world. The features of the aspect can be described as the *values* that occur within that dimension of variation. In the language of Watson and Mason (2005) and Goldberg and Mason (2008), the former is described as "opening possible dimensions of variation" and the latter "expanding the permissible range of change".

But how are new critical features and aspects discerned? Marton (2015) describes this process broadly as follows:

> … in order to discern a certain feature and a certain aspect, the learner must experience *a certain difference (variation) against a background of sameness* in other respects; in order to bring together two different aspects, the learner must experience *variation in both, simultaneously*. (p.165; emphasis added)

A key principle of VT in general is that students see difference before sameness in their learning process (Marton & Pang, 2013, as cited in Kullberg et al., 2017). In the learning of features and aspects, this is described as *contrast*. Contrast is a *pattern of variation* where two or more aspects are considered and manipulated such that the desired/focused aspect of the object of learning varies while the other aspect(s) remains invariant.

For example, in the study of quadratic functions of the form $y = ax^2 + q$, an aspect to be mastered is the vertical shift (q). In illustrating this using contrast, the VT perspective might suggest to introduce q as a varying feature while keeping the aspect of 'shape' (a) the same. This is illustrated in Figure 3, where all three graphs have the same shape with only the vertical shift varying (thus opening up the dimension of variation of the vertical shift). This is what Marton (2015) refers to when he states that to come to know a new feature or aspect it should be experienced as "difference against a background of sameness" (p. 165), using variation to foreground the desired feature and invariance to background other aspects.

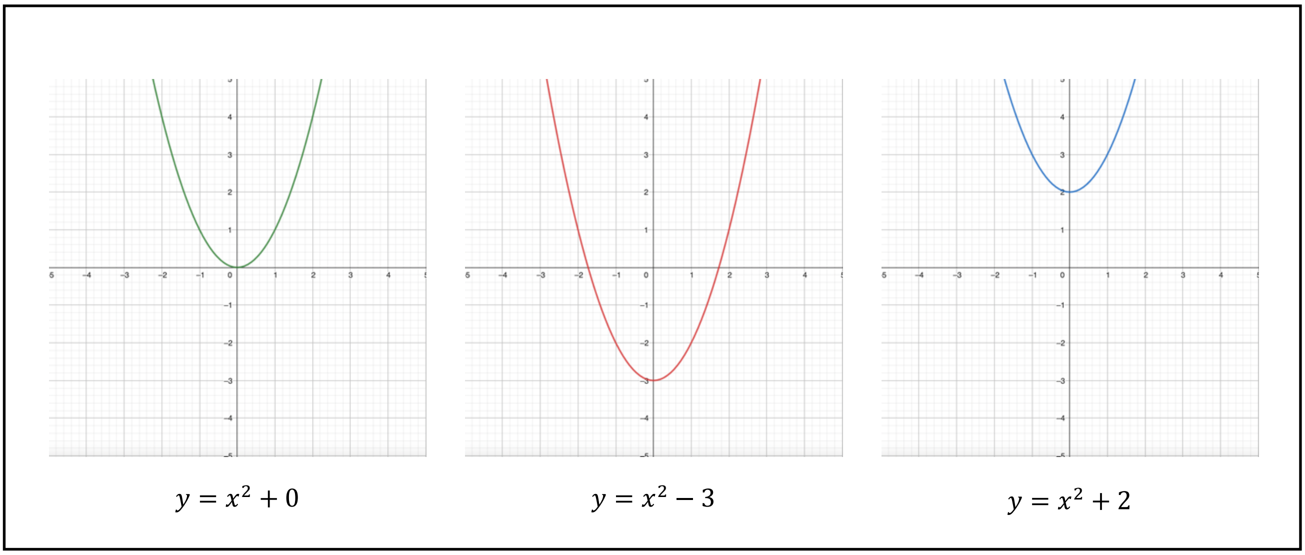


Figure 3 **Contrast: Varying the vertical shift of quadratic functions, while keeping 'shape' invariant**

Figure 4 provides a counter-example of contrast. Here, the desired object of learning (vertical shift) is less clear to discern as both the aspect of vertical shift (q) and the aspect of shape (a) vary simultaneously.

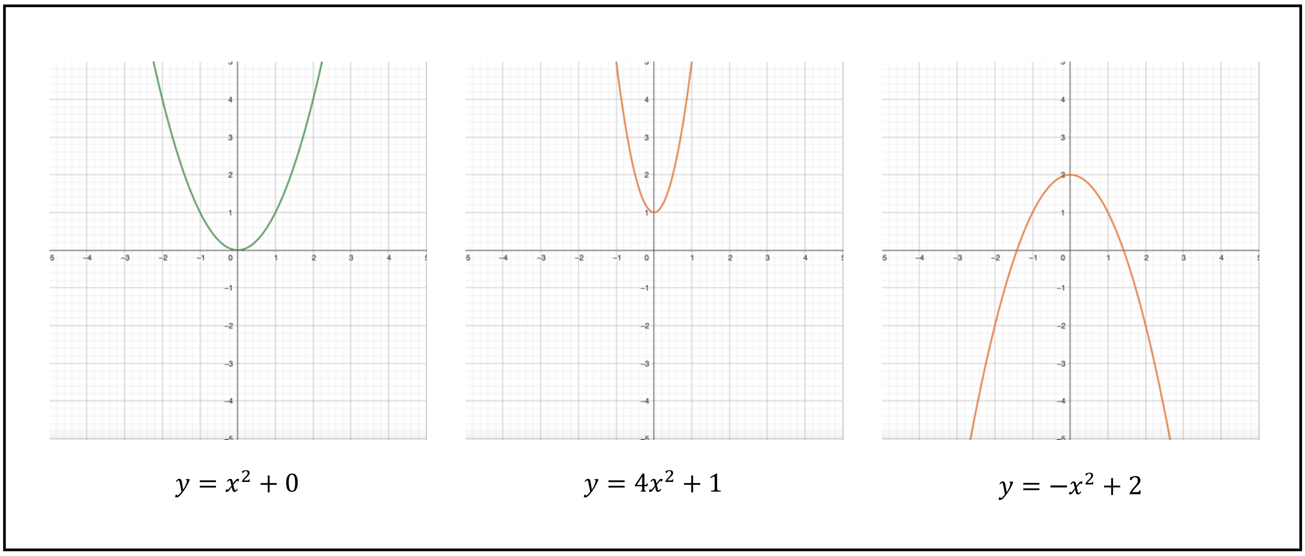
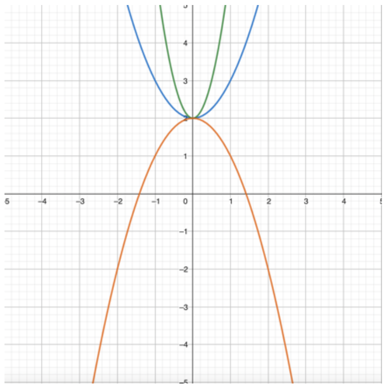


Figure 4 **Not contrast: varying the vertical shift, while varying shape simultaneously**

Once an aspect has been opened up through contrast, the next *pattern of variation* is that of *generalisation*. In generalisation, the critical aspect that was varied during contrast must be understood to maintain its newfound meaning even if other aspects vary (these aspects had previously remained invariant in the contrast phase). A possible example of contrast is exhibited in Figure 5. Here, the highlighted critical aspect (vertical shift) remains invariant while the shape varies. In other words, (q) remains invariant while (a) varies.



$$y = x^2 + 2$$

$$y = 4x^2 + 2$$

$$y = -x^2 + 2$$

Figure 5 **Generalisation: the vertical shift is now invariant while shape varies**

Here, during generalisation, the varying aspect of shape aids the understanding that a change of shape does not impact the vertical shift. In other words, by now varying the aspect that is not the main object of learning, while keeping the focused aspect invariant, the student learns what can be disregarded when considering the highlighted aspect of learning.

The patterns of variation *contrast* and *generalisation* separate features and aspects "from what they are aspects of and from each other" (Marton, 2015, p.51). These patterns allow individual necessary aspects of an overall object of learning to be appreciated and discerned. However, the object of learning must ultimately be understood as a whole – as a combination of particular aspects and features. This final pattern of variation is known as *fusion*, and it occurs when multiple aspects vary simultaneously (as occurred in Figure 4 when both vertical shift and shape varied simultaneously). Here, it becomes experienced how the varying of multiple aspects together interact as a whole. Interestingly, the situation in Figure 4 would not be desired when using contrast, but is desired when aiming for fusion, once contrast has been achieved.

In VT, learning is thus seen to occur by means of experienced patterns of variation and invariance, combined in different ways. In this way, an object of learning transforms from being an undifferentiated whole to being understood in parts (through contrast and generalisation) that become differentiated and integrated (through fusion).

Teacher Actions to Produce Patterns of Variations

Marton (2015) describes how, from a classroom teaching perspective, patterns of variation can be produced in three ways by the teacher to create affordances for learning. Firstly, this can be done through a *combination of instances* that occurs when examples are combined during a teaching episode. Here, the different examples can be used to create patterns of contrast, generalisation, or fusion. Secondly, the discernment of variation and invariance can be achieved by the teacher deliberately *shifting focus of attention* to different aspects and features of an object of learning that students are unlikely to notice spontaneously. Here, a teacher draws attention to a particular aspect of an example to create contrast, generalisation, or fusion. Finally, by using *differences as changes*, the teacher manipulates and changes a single instance (e.g., an example) rather than combining a series of different instances. A typical way that differences as changes can be achieved is through digital (graphing) software, where a single example can easily be manipulated into different forms. These three techniques all contribute towards creating affordances for learning during the enacted object of learning. Although Marton (2015) describes these techniques mainly as individual approaches to lesson design, they can be combined within a single learning episode.

Summary of Variation Theory

Marton's (2015) VT posits that sequenced patterns of variation and invariance within a set of examples allow for the critical features and aspects of an object of learning to become discerned, and ultimately combined. In this process, aspects are brought to light through contrast (focus on differences against a background of sameness), are then generalised (focus on similarities against a background of difference), and ultimately can become part of fusion (focus simultaneously on differences and similarities). Successful learning thus results in an integrated understanding of an object of learning, where individual aspects (and their features), and the mutual influence of these aspects, can be noticed, discerned, and understood. Marton (2015) suggests that seeing an object of learning in such a manner allows for future situations that involve the object of learning to be powerfully dealt with.

Visualisation of Key Aspects of Variation Theory as used in This Study

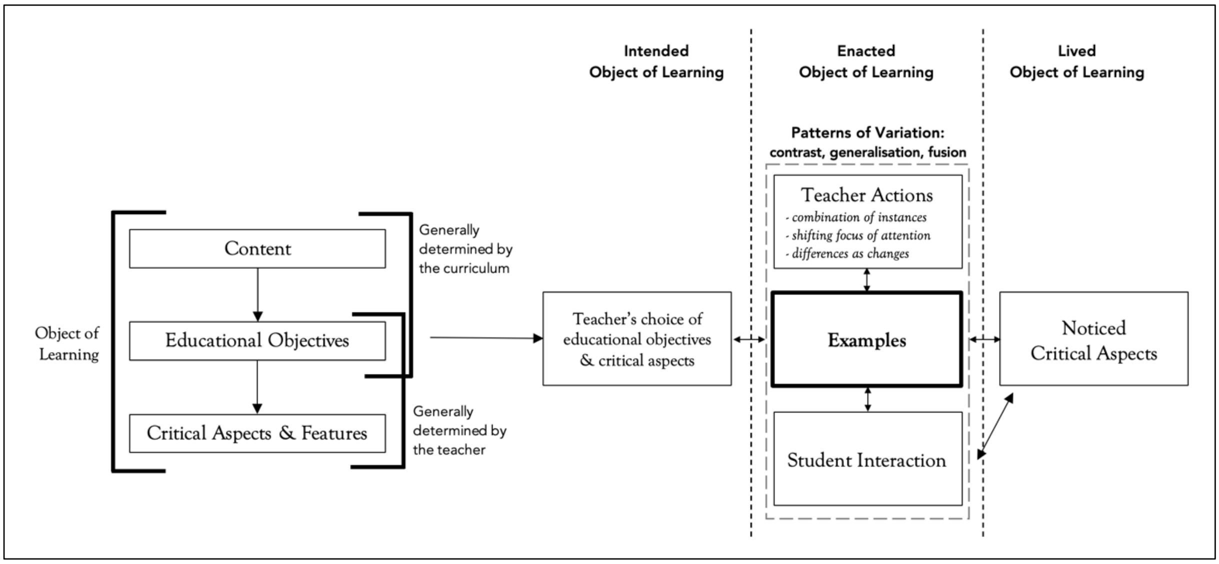


Figure 6 **A Visualisation of Variation Theory as Used in This Study**

Figure 6 presents a diagram that visualises my understanding of Marton's (2015) variation theory of learning in relation to this study. A learning episode begins with the object of learning. The object of learning can be described in terms of the content, the educational objectives, and the critical aspects and features. The content and educational objectives are generally determined by the curriculum. However, a teacher can influence how, and in what order, educational objectives are presented. The teacher also determines (deliberately or not) which critical aspects and features of the object of learning to highlight in a learning episode. Next, from the overall object of learning, the teacher selects an <u>intended object of learning</u> for a learning episode. The teacher then enters the next phase of <u>enacting the object of learning</u>. Here, there is an interplay between the selected examples and the teacher actions (combination of instances, shifting focus of attention, and differences as changes) used in their mediation. These operate together in creating patterns of variation. Although it is not a focus of Marton's work, I have included 'student interaction' as having an influence on patterns of variation, as student questions or comments could lead to the creation of *spontaneous examples* or *student generated examples* (as described in the literature on examples). Finally, the noticed critical aspects make up the <u>lived object of learning</u>.

In the diagram, the lines separating the intended, enacted and lived objects of learning are not solid, as these interact with each other during the course of a lesson and are not clear-cut. Additionally, all the arrows within the intended, lived and enacted objects of learning are bidirectional. This is because these connections are not linear and can influence each other. For example, in the middle of a lesson, a student comment (student interaction) may indicate the 'lived object of learning' to the teacher, which may inspire her to change her next 'intended object of learning', and thus her subsequent example choice.

In this study, I considered the objects of learning that occurred in the three teachers' lessons. I analysed the examples they chose regarding how these created opportunities for contrast, generalisation and fusion to occur. I also considered how the teachers' mediation of these examples (teacher action) and student interaction combined with the examples to create patterns of variation and thus affordances for learning in the six observed lessons.

Conclusion

In this chapter, I have engaged with the literature that framed my study. I considered literature on the use of examples in the teaching and learning of mathematics, and on the teaching and learning of functions and their multiple representations. Thereafter, I presented Marton's (2015) variation theory of learning, which is the theoretical framework of this study. In Chapter 3, I engage with the study's research design and discuss how the data were analysed.

Chapter 3: Research Design and Methodology

Introduction

In this chapter, I present the methodological design of the study. I discuss the study's setting, how participants were selected, the data collection process (classroom observations and interviews) and the associated ethical factors, and how the data were analysed. Finally, I discuss considerations regarding the validity and reliability of my data and findings.

Methodology

This research report considers, in-depth, three teachers' choice and use of examples. It is thus situated within the qualitative paradigm and takes the form of a comparative case study. Case studies are characterised by examining "[w]hat can be learned from the particular case" (Cohen, Manion & Morrison, 2007, p.85) yet hope to shed light on a general principle. In my study, I aimed to examine three teachers' choice and use of examples, who all worked within one specific mathematics department, but hoped to arrive at findings that might have applicability in other departments which operate according to similar *modus operandi*. Of relevance to my research is that case studies chronologically narrate events and blend the description of these events with an analysis thereof (Hitchcock & Hughes, 1995 as cited in Cohen et al., 2007). As such, a case study suited my research questions well as it allowed for the teachers' choice and use of examples in the lessons to be described and analysed in consecutive episodes as this had occurred in the classroom.

The Setting

This research study was undertaken at an all-boys' private high school in Gauteng, South Africa. The school is highly privileged, and the clear majority of students come from affluent socio-economic backgrounds. The language of learning and teaching at the school is English, and English is also the first language of most learners. If not the first language, then students are almost always highly proficient in English. I am a teacher at this school and thus observed my colleagues' lessons. Being a teacher-researcher at my own school came with a variety of advantages and disadvantages. For the former, the school and my colleagues were highly supportive of my research. Also, as I am a familiar school staff member, I believe that this aided in receiving consent from students and their parents/guardians for me to observe and video-record lessons. However, there were also disadvantages. I was concerned that my colleagues might feel threatened and/or judged by me observing and analysing their lessons (as this is not something that occurs frequently in our department). I was also aware that my longstanding work relationships with all three observed teachers might add an unintentional aspect of bias to my analyses. However, our longstanding work relationships might have also allowed the teachers to share more freely in the interviews.

Sampling

To answer my research questions, I decided to observe three teachers and their classes. This number of teachers made up more than half of the five Grade 10 classes at my school, and thus allowed for a relatively in-depth assessment of the teacher's choice and use of examples within this grade.

The Students

At the observed school, Grade 10 classes are streamed in three levels of ability-and-pace-based sets. In the year that this study was undertaken, there was one 'top' set, one 'second set', and three 'third' sets. The two 'higher' sets were designed to be taught at a faster pace and to spend more time on complex and/or extension-level questions than the three 'lower' sets. These three lower sets were designed to be taught content at a slower pace and with a greater emphasis on routine procedure than extension work. The students in the school are given the opportunity to request which set they would like to be in, and thus, even in a third set, these can include students who achieve a wide range of results. Generally, however, the assessment results are lower in the 'third' sets than in the higher sets. For my study, I opted to observe the lessons of the three 'third' sets as I wanted to compare the teachers' example choice and use without the complexity of considering different set-levels in the analysis. Each of the selected classes was made up of about 20 boys, aged 15-16 years old.

The Teachers

The teachers in this study were selected by the situation of their teaching one of the selected 'third' set Grade 10 classes. This worked seamlessly for two of the observed teachers (Teacher Y and Teacher Z). However, a complexity arose as I (the researcher) was the teacher of the final 'third' set class. As I could not observe myself, I thus asked another teacher in the department (who was available during the Grade 10 periods and willing to take part in the research study) to teach my class for the lesson observations in the role of a substitute teacher (Teacher X). Having teachers in our department substitute for each other is not unusual, and as the 'substitute' teacher was another teacher in the department, acting in a 'typical' manner of teaching content when the usual teacher was not available, observing him did not go against the intentions of my study: Substitute teachers also choose and use examples. I furthermore believed that this would provide for better data for my research questions than the possible alternative of observing a Grade 10 class that was a higher set. I also did not want to observe only two teachers, as this made my sample smaller. All three observed teachers (Teacher X, Teacher Y, and Teacher Z) had more than 15 years of teaching experience and had worked at the school for 8 years or more. To preserve anonymity, the three teachers were referred to as Teacher X, Teacher Y and Teacher Z, and the pronoun 'he' was used throughout the study for all three teachers.

None of the observed teachers had been formally trained in using variation theory as a teaching method. However, earlier in the year, the whole department had received a lecture from a visiting academic that included variation theory as a topic. All three teachers were thus aware of the basic principles of variation theory but had not been trained in its use.

Research Methods

The study made use of two data collection processes: classroom observations and teacher interviews. These two processes are described below. I decided to observe two lessons per teacher as this provided me the opportunity to observe how they moved from one lesson to the next lesson regarding their example choice and use. All classroom observations (6 lessons), and the post-observation teacher-interviews were recorded and transcribed.

Classroom Observations

I observed six lessons (two per teacher) as a known non-participant observer. I sat quietly at the back of the classroom during the lessons, taking notes while I observed the lessons. I additionally video-recorded the lessons (image and sound) from the back of the classroom. The camera was on a tripod and focused on the board and the teacher. These videos allowed me to clearly document the examples that were used and mediated by the teacher, whilst also recording student verbal contributions and/or student work on the board. A benefit of using classroom observation is that it "offers an investigator the opportunity to gather 'live' data from naturally occurring social situations" (Cohen et al., 2007, p.396). This added to the validity and authenticity of the data. Each observed lesson included between 45-50 minutes of instructional time. Thus, per teacher, I recorded between 90-100 minutes of their lessons.

Interviews

After the lesson observations had taken place, I met with each teacher for a one-on-one, face-to-face, semi-structured interview. Semi-structed interviews were selected so that I could cover the same important content in relation to my research questions in each interview (which contributed to the validity of the data) while having "the freedom to clarify [the teachers'] understanding and to ask follow-up questions to explore a viewpoint, to determine knowledge or to open up other explanations and answers to questions that were not foreseen when the research questions were determined" (Newby, 2014, p.356). This contributed to the authenticity of the data. The interviews were all voice-recorded to aid accuracy and to allow for transcription. In these interviews, each teacher shared their intended objects of learning for the lessons, and how they went about selecting and sequencing examples for the lessons. I also enquired into their collaboration with other teachers in the department regarding example choice. Before the interview, I had extracted every example used in their two lessons and had collated these into a document that we could refer to during the interview. We then discussed each example together, which provided further insight into the intended (and enacted) objects of learning.

Ethical Considerations

As this research project involved observing and video recording live participants, including students under the age of 18, ethical considerations were of high importance. My private school gave permission for the research to take place and I was then able to apply for ethical clearance. Ethical clearance for my study was granted by the *University of the Witwatersrand* (Protocol Number: 2022ECE020M; see Appendix B). Once ethical clearance had been granted, I approached the selected participants (teachers and students) and carefully explained the background and aims of my research, and how data would be collected.

For the students, this involved explaining the research project to them in person, answering questions, and giving them participant information sheets and consent forms to sign. I also provided their parents/guardians with participant information sheets on the boys' behalf and gave the parents/guardians a consent forms to sign. I furthermore made myself available for a *Google Meet* where any parent/guardian could ask questions. For a boy to be part of the study, both the boy and their parent/guardian had to give consent.

For the teachers, I explained the research, answered questions, and provided participant information sheets. As the teachers were involved in two types of data collection (lesson observations with video, and a semi-structured in-person interview that was voice recorded) they signed two consent forms – one for each type of data collection.

It was made clear to all the students and the teachers involved in this research that they could withdraw from the study at any point.

Data Analysis

The Analytical Process

The analytical task of this research project was to examine what was made available to learn regarding functions in the three classrooms based on the teachers' example choice and use. I began by analysing the classroom observations, and then moved on to the teacher interviews. In analysing the data, I used the language of Marton's (2015) VT, which framed this study both theoretically and analytically.

The analysis had two parts, visualised in Figure 7. In Part 1, I analysed each teacher's choice and use of examples vertically, that is, on a teacher-by teacher basis. This is indicated by the green arrows. Thereafter, in part 2, I analysed the findings of part 1 horizontally by juxtaposing my findings to observe similarities and differences in the affordances for learning in the different classes. This is indicated by the blue arrow.

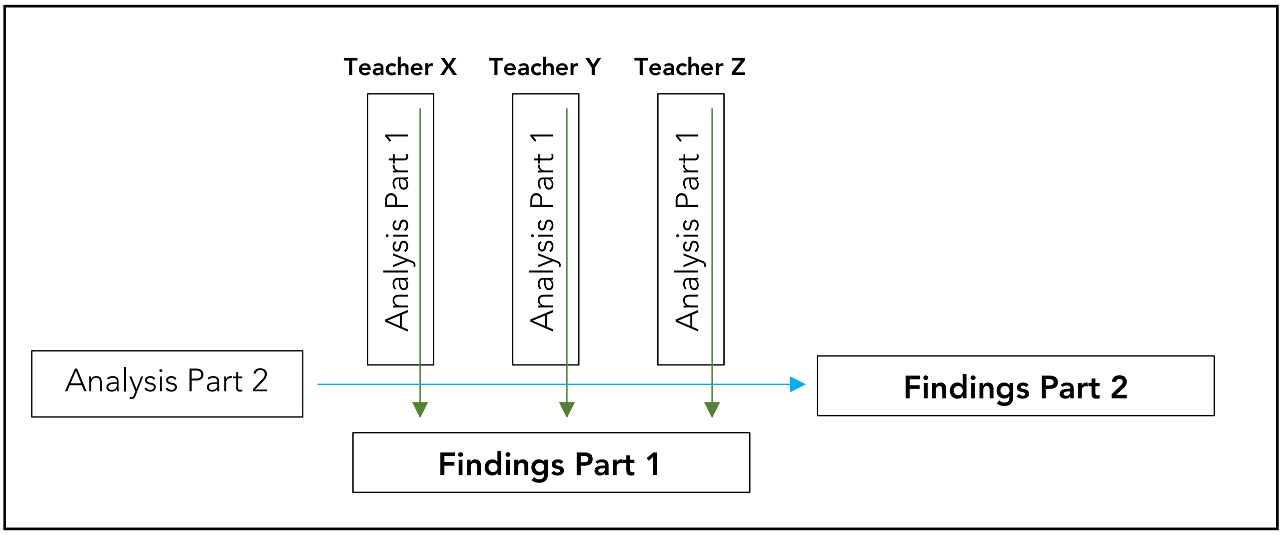


Figure 7 **The Analytical Process**

Analysis Part 1

Classroom Observations

<u>Step 1: Creating Detailed Lesson Descriptions</u>

For each teacher, I began my analysis by creating Detailed Lesson Descriptions in table form. Each lesson underwent its own lesson description. The aim of these descriptions was to create, in textual modality, a description of what occurred in the lessons. These lesson descriptions furthermore allowed me to organise my data. I thus documented (1) every example used in the lessons, in the order in which they occurred; (2) the representations of functions included in these examples; (3) a description of patterns of variation that occurred in the lessons, including the examples, teacher actions, and student interactions; (4) the object of learning, including content, educational objectives, and aspects and features. In this step, I also split the lessons into episodes. A new episode occurred during the lesson when a new object of learning came into focus through a set of connected examples that appeared to work together towards communicating the selected object of learning. When I began the analysis, I expected that each episode would only have one object of learning. However, as I worked through the six lessons, in a few instances, a particular set of examples that were grouped together were used to communicate two different but connected objects of learning. An example of such a situation was in Teacher X's second lesson (episode 2) where, by sketching using a table, the different types of functions were introduced (straight line, parabola, exponential, and hyperbola). This episode thus had two connected objects of learning through one set of examples: different types of functions, and sketching using a table. Similarly, in Teacher Y's second lesson, in episode 1, one set of examples was used to communicate knowledge on the connected notions of 'functions and relations' and 'domain and range', while in Teacher Z's first lesson, episode 1, a connected set of examples was used to communicate content regarding 'functions and rules' and 'inputs and outputs'. In my analysis, no episode had more than two objects of learning, and most had only one object of learning. In all cases, an episode was made up of a connected group of examples which was used to communicate one, or two connected, object(s) of learning, and a clear shift to a different object of learning marked the start of a new episode.

The tables were organised as follows: In column 1, I wrote down the episode number. In column 2, I documented the examples used in the lesson. Here, in black text, I replicated the example as presented to the students (generally as it was written on the board, but also any verbal examples that were mentioned). In grey text, I added what was written on the board as the teacher engaged with/solved/documented student contributions for the example. A lack of grey text indicated that this example occurred but was not engaged with further in written form. In column 3, I noted the representations that occur in the example. Here, I drew on both the example in itself and in its mediation. In column 4, I described patterns of variation (contrast, generalisation, fusion) that occurred through teacher actions (combination of instances, shifting focus of attention, differences as changes) and talk, or student interaction. Finally, in column 5, I described the object of learning of each episode. This was done using the terminology of Marton's (2015) exposition of VT: *content*, *educational objectives*, and *aspects and features*.

The creation of this table had various benefits: it allowed me to sequentially compare the examples used by the teacher, and to document how the examples were used. By documenting the examples in this way, patterns emerged that were harder to notice when watching live or video lesson content. However, as with all transfer of information from one mode to another (here, from video to textual form) some information and detail that occurred live in the classroom is necessarily lost. This tabular means of presenting a lesson summary was inspired by the work of Pillay et al. (2022). All the detailed lesson descriptions are available in Appendix A. I have included these descriptions to add to the transparency of my analysis and findings.

Step 2: Analysing the Examples (Innate Affordances)

As this research project is about exemplification, I began the formal analysis by considering the examples. In this step of the analysis, I considered the affordances for learning that were made available in the examples in and of themselves, that is, without mediation. I describe such affordances as the 'innate' affordances of the example space[1]. In relation to my Detailed Lesson Descriptions, I considered the data in Column 2 only. Here, I considered the patterns of variation that came about through the juxtaposition and sequencing of the examples, using Marton's (2015) notions of contrast, generalisation, and fusion to describe how the affordances for learning occurred. Additionally, I described the aspects and features of the object of learning that became discernible through these patterns of variation. The teacher's mediation of the examples was not included in this step of the analysis. How 'Step 2' relates to the theoretical framework is shown in Figure 8 (the examples were analysed and linked to the object of learning).

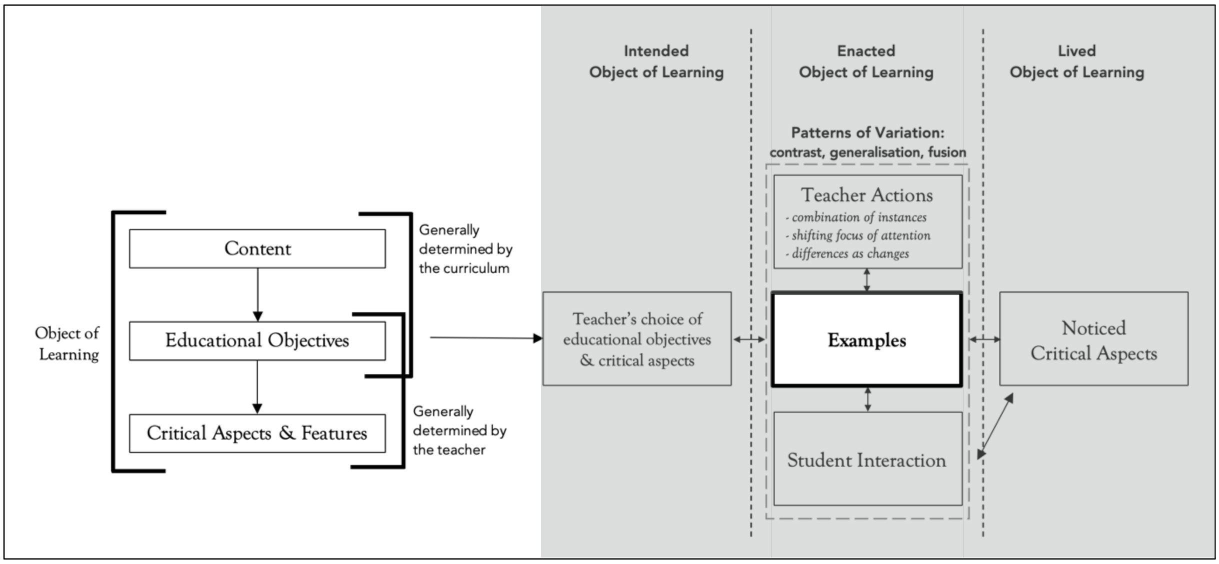


Figure 8 **Step 2: Focusing on the Examples' Innate Affordances for Learning**

Step 3: Analysing the Examples (Enacted and Lived Affordances)

After completing an analysis of the examples in and of themselves, I moved on to considering how the examples were enacted in the classroom. Here, I drew on the teacher actions (combination of instances, shifting focus of attention and differences as changes) and talk to

[1] The term "'innate' affordances" is my own. Having engaged with the literature I could not find an existing term to summarise the notion of 'affordances for learning that exist in an example space without a teacher's mediation' and after careful consideration used the term 'innate' affordances to do so.

describe how the teachers brough the examples to life. Furthermore, I considered student interaction, which sometimes allowed the 'lived object of learning' to become apparent. In relation to my Detailed Lesson Descriptions, I considered the data in Column 2 and Column 4. How 'Step 3' relates to the theoretical framework is shown in Figure 9 (the examples and their mediation, and any apparent 'lived' objects of learning, were analysed and linked to the object of learning).

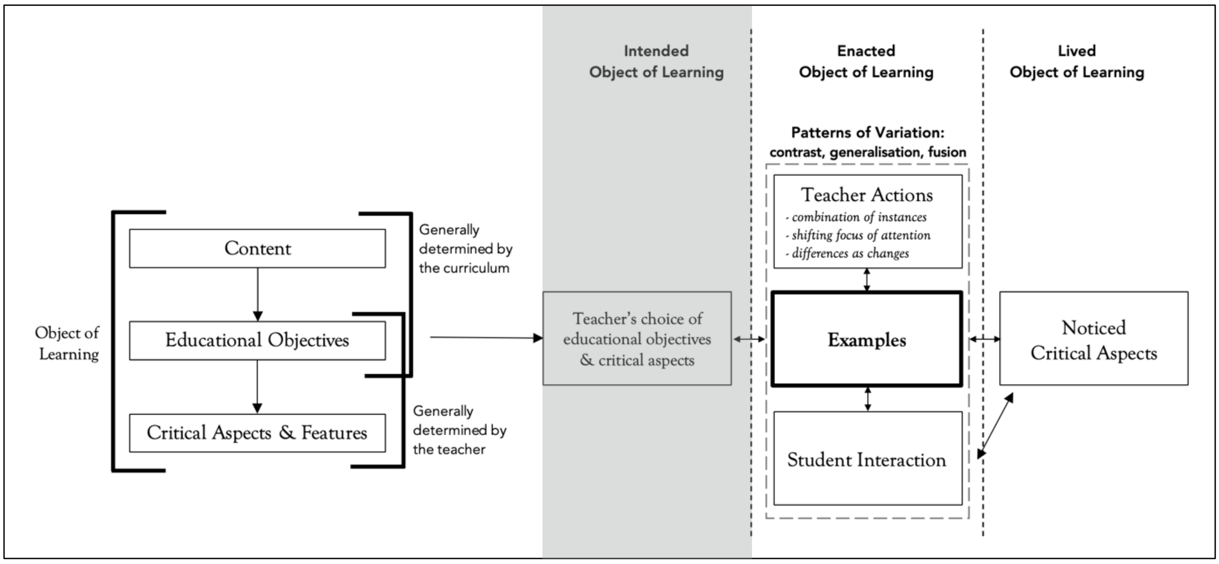


Figure 9 **Step 3: Considering the Examples' Mediation (The Enacted and Lived Objects of Learning)**

Teacher Interviews

Step 4: Intended Object of Learning

After completing the analysis of the classroom observations, I moved on to examining the teacher interviews. Here, I focused on how the teachers described their intentions for the lessons, which provided information on the intended object of learning. I also engaged with the teachers on their enactment of their selected examples, to provide further information regarding the intentions behind the enacted object of learning. How 'Step 4' relates to the theoretical framework is shown in Figure 9 (the examples were analysed and linked to the object of learning, including their mediation, through the lens of the teachers' intended object of learning).

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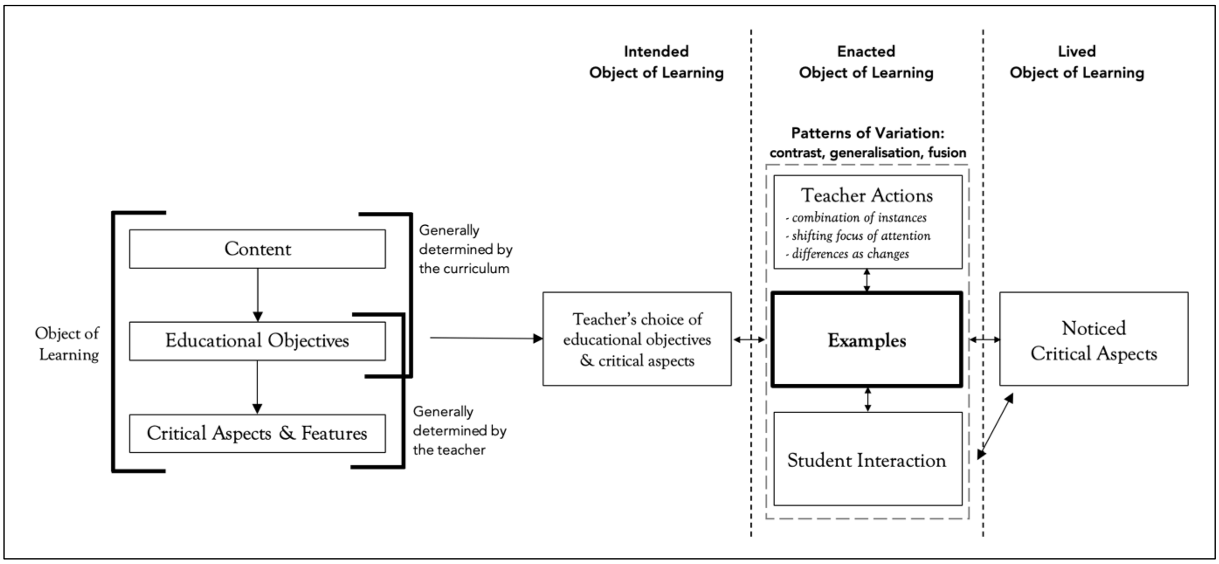


Figure 10 **Step 4: The Intended Object of Learning with the Enacted and Lived Objects of Learning**

Analysis Part 2

In the second part of the analysis, I collated the findings from Part 1 of the analysis to juxtapose the affordances for learning in the three teachers' lessons. Here, I considered the number of examples and the types of teacher actions used by the teachers. I then created a table which collated the objects of learning for the teachers' lessons using the 'content' description. This allowed for general similarities and differences across covered content to become apparent. Next, I engaged in greater detail with similarities and differences across the teachers' lessons regarding 'content', and how this content was enacted and mediated by the different teachers. Finally, I examined how the different teachers engaged with the multiple representations of functions.

Validity and Reliability

Cohen et al. (2007) write that "[t]hreats to validity and reliability can never be erased completely; rather the effects of these threats can be attenuated by attention to validity and reliability throughout a piece of research" (p.133). Throughout the creation and enactment of this project, I thus strived to collect, describe and interpret data in a manner that was as valid an reliable as possible. The following points were considered in this endeavour:

Validity

Validity considers the accuracy of collected data in describing what it claims to describe. From the perspective of internal validity, using video recording (sound and image) of the observed lessons contributed to my ability to document the teachers' choice and use of examples with accuracy, and to transcribe their and their students' words. These video recordings were complemented by my lesson observation notes. Similarly, voice recording of the interviews with the teachers allowed me to revisit and transcribe their exact words. Having these two sources of data furthermore contributed to the validity of the data, as they allowed me to examine the same phenomenon "the teachers' choice and use of examples" from two different perspectives: the lesson observations, and the teachers' own views.

Once I had completed the analysis of the three teachers' lessons, I furthermore used member checking to strive for internal validity. The three teachers were each given the analysis of their lessons (as provided in Analysis Part 1) and asked to inform me if they believed that my descriptions and analysis were inaccurate.

External validity became a consideration when working with the findings and implications of the study. Here, I strived to not make "inferences and generalisations beyond the capability of the data to support such statements" (Cohen et al., 2007, p.145). To do so, I had to keep in mind that my investigation was a case study and could not be generalised to a large degree. However, it provided valuable information for my department and other departments that function according to similar *modus operandi*.

Reliability

A study's reliability considers "the extent to which another researcher would arrive at similar results if they studied the same case using exactly the same procedures as the first researcher" (R.K. Yin, as cited in Gall et al., 2007, p.477). To strive for reliability in this study, I used an established theory (Marton's (2015) variation theory) as the basis of my data analysis. This theory has a set of predetermined and consistent criteria by which the data could be analysed (e.g. contrast / generalisation / fusion; and combination of instances / shifting focus of attention / differences as changes). This aided me in trying to ensure that as little bias as possible occurred in me analysing my colleagues' work. Additionally, I included all the examples used in all lessons in the respective analyses of the teachers' choice and use of examples. This was done to create transparency of analysis claims and interpretations. Finally, by using two sources of data, namely (1) the lesson observations and (2) the teacher interviews, I was able to consider the teachers' choice and use of examples from two perspectives. This *methodological triangulation* (Cohen et al., 2007) – using different research methods on the same object of study – aimed to further enhance the reliability of this research report.

How the Analysis and Findings are Presented

In chapter 4, I provide the analysis and findings of my investigation. I separate my analysis and findings into two parts, in line with Part 1 and Part 2 of the analysis. In Part 1, I provide the results of each teacher's analysis regarding the affordances for learning that were created in their lessons. I begin by presenting a summary of the objects of learning that came about in the lessons, to set the tone for what occurred in the lessons (similar to the *suspense structure* of reporting the analysis of case studies (Cohen et al., 2007)). Thereafter, I first present the analysis of the innate affordances for learning from the example set, and then engage with the affordances for learning created through the enactment of the examples (considering the intended, enacted, and lived objects of learning). Here, I draw on both the lesson observation and interview data. Finally, I describe how the teachers engaged with the key aspect of 'the multiple representations of functions'. In Part 2, I engage with the juxtaposition of the findings from Part 1, and describe the similarities and differences between lessons that become apparent.

Conclusion

In Chapter 3, I have described the details of the research design and methodology of this study. I have engaged with the setting and sampling of the research, and the methods of data collection. I also describe the ethical considerations that needed to be addressed for this study to take place. The data analysis process was then described, and the validity and reliability of the study considered. Finally, I describe how the analysis and findings of the data analysis are presented in Chapter 4.

Chapter 4: Analysis and Findings

Introduction

In this chapter, I provide my analysis of the three teachers' introductory lessons on Grade 10 functions. As described in Chapter 3, this analysis has two parts. In Part 1, I undertake a vertical teacher-by-teacher analysis where I engage with the affordances for learning that were created in the three teachers' classrooms. Subsequently, in Part 2, I engage in a horizontal analysis where I collate the findings from Part 1 to consider similarities and differences across the three teachers' lessons.

Analysis Part 1

Teacher X

Teacher X taught two contrasting lessons from an episode and example perspective. In the first lesson, there were 5 episodes, made up of between two to five examples per episode (total: 18 examples). Most of these examples were discussed by the teacher for a similar length of time. In the second lesson, there were 2 episodes, with Episode 1 made up of 15 (quick) examples, and Episode 2 made up of five (mainly lengthy) examples (total: 20 examples). Please keep in mind that Teacher X is a teacher in the department who was functioning as a substitute teacher for this class.

Objects of Learning: Summary

Table 1 documents, per episode, the objects of learning that occurred in Teacher X's lessons. The objects of learning are described in increasing detail by 'Content', 'Educational Objectives', and 'Aspects and Features', as detailed by Marton (2015). In this table, the examples' innate affordances for learning, and the affordances created through mediation and classroom interaction, are considered.

Table 1 **Teacher X Objects of Learning**

Lesson 1 Episode	**Content**	**Educational Objectives**	**Aspects and Features**
1	Finding the equation of straight line graphs	- finding the equation of a straight line graph from two given coordinates	- aspect: gradient features: m=3; m=1

		- using inspection to find the y-intercept - using substitution to find the y-intercept - using a graph 'drawn to scale' to find the y-intercept	- aspect: find the y-intercept features: by inspection, by drawing to scale, by substitution - aspect: the $(x; y)$ coordinate features: satisfy an equation, x as an input
2	Rules and relationships between x and y	- describe a relationship between x and y with words - seeing an equation in the form of $y = mx + c$ as a relationship between x and y (not just the equation of a graph) that can be expressed in words	- aspect: relationships/rules features: equations, words, coordinates - aspect: parts of a rule features: coefficients that multiply the input, constants that add to the input - aspect: the $(x; y)$ coordinate new feature: the coordinate $(x;\ y)$ is a representation of the rule to which it belongs
3	Different rules in different representations	- describing rules in different ways: words, equations, flow diagram, tables, ordered pairs - understanding the equivalence between different representations of the same function - understanding that rules can have different powers of x	- aspect: different representations of a rule features: flow diagram, ordered pairs, table, words, equation - aspect: degree of a rule features: linear, quadratic
4	Definition of key terms to do with functions	- understand the meaning of the terms: · function, domain, range, input, output - solving for unknown inputs and outputs	- aspect: input features: x, domain, horizontal axis, left position in a coordinate pair - aspect: representation of the input features: x, t - aspect: output features: y, range, vertical axis, right position in a coordinate pair
5	Function notation	- understand the usefulness of function notation - represent a function with function notation - find missing input and output values using function notation - use function notation in coordinate form - use function notation with fluency - know how to pronounce function notation	- aspect: function notation features: f(x), $g(x)$, $h(x)$ - aspect: using function notation features: input → output, output → input - aspect: the $(x; y)$ coordinate new feature: the coordinate $(x;\ y)$ as $(x;\ f(x))$ - aspect: degree of a rule features: linear, quadratic
Lesson 2 Episode	**Content**	**Educational Objective(s)**	**Aspects & Features**
1	- Function notation - Domain and range	- gain fluency in the use of function notation - identify domain and range from given points - find inputs and outputs of a function	- aspect: function notation features: f(x), $g(x)$ - aspect: using function notation features: input → output, output → input - aspect: degree of a rule features: linear, quadratic, cubic - aspect: representations of a function features: $f(x)$ equation, function map, set of ordered pairs, straight line

			- <u>aspect</u>: the $(x; y)$ coordinate <u>new feature</u>: the x-coordinate belongs to the domain of the function; the y-coordinate belongs to the range of the function
2	- Different types of functions - Sketching using a table	- understanding that different types of functions exist - using the technique of an input-output table to sketch different types of functions - using negative exponents - understanding the necessary accuracy of a sketch graph - using different representations to translate a graph equation to a graph sketch (equation → table → coordinate pairs → sketch).	- <u>aspect</u>: type of functions <u>features</u>: straight line, parabola, exponential, hyperbola - <u>aspect</u>: using a table <u>features</u>: inputs $-2;\ -1;\ 0;\ 1;\ 2;\ 3$ - <u>aspect</u>: an 'accurate' sketch graph <u>features</u>: regular axis increments; irregular axis increments - <u>aspect</u>: representations of a function <u>features</u>: equation; table; coordinate pairs; graph - <u>aspect</u>: degree of a function <u>features</u>: linear, quadratic, hyperbolic, (also exponential) - <u>aspect</u>: the $(x; y)$ coordinate <u>new feature</u>: a tool to sketch a graph

In the first lesson, Teacher X began with finding the equation of two straight line graphs (prior knowledge from the previous topic of 'analytical geometry'). Then, he moved on to focusing on equations as rules and relationships, which can be expressed in different forms and with different representations. Next, he introduced the concepts of function, domain and range, and ended with the object of learning of function notation. In Lesson 2, Episode 1 was a digital *Kahoot*[2] quiz game that went over content from Lesson 1 (function notation, and domain and range). Thereafter, he went over sketching four different types of functions (straight line, parabola, exponential and hyperbola) using the table method.

Example Space Used in the Introduction of Functions

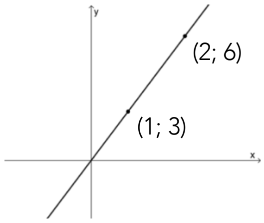
All the examples used by Teacher X across both lessons are documented in Table 2 and Table 3. The examples are provided per episode (as in Table 1), arranged horizontally in the sequence in which the examples occurred in each lesson. In the section that follows, as indicated previously, the examples are analysed in two ways: firstly, in and of themselves, through the lens of Marton's (2015) VT, without considering their enactment in the classroom. This analysis considers the

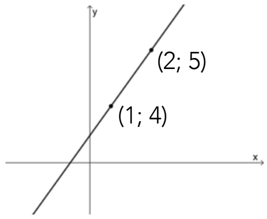
[2] In a *Kahoot* quiz, students join the game through an online platform where they enter a game pin. Then, the questions are projected on the board by the teacher (generally multiple choice and/or true-false questions) and the students select the correct solution to the question on their individual electronic device. Each question has a time limit. Points are awarded for selecting the correct answer, and also for how quickly the correct answer is selected. Incorrect solutions get no points. This adds a competitive element as the (fluctuating) top 5 positions are announced after every question. At the end of the Kahoot, there is a podium with the top 3 students' names shown. The teacher manages the Kahoot, and has control over starting each question. This allows for questions and solutions to be discussed for each individual question before the next question is projected.

affordances for learning that are innate in the examples based on how they were sequenced. In other words, it considers *what* was possible to learn considering purely the examples. In the second analysis, I engage with *how* and *why* the examples were brought to life in the classroom. Here, I considered the opportunities for learning that were created by the teacher actions and student interaction that mediated the affordances of the examples in the enacted lesson. I also draw on data from the interview with Teacher X regarding his intended object of learning. Please note that a table combining the Objects of Learning (as presented in Table 1) and the examples themselves (as presented in Table 2 and Table 3), as well as descriptions of Teacher X's mediation of the examples, is available in Appendix A.

Table 2 **Teacher X Lesson 1 Examples**

<table>
<tr><td>Episode</td><td colspan="4"></td></tr>
<tr><td>1</td><td>1(a)

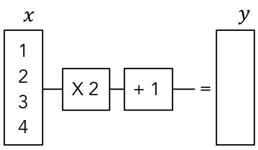
Find the equation of this graph:
</td><td>1(b)

Find the equation of this graph:
</td><td></td><td></td></tr>
<tr><td>2</td><td>2(a) – a definition

Rule which is linking an x-value to a y-value</td><td>2(b)

Express the rule of example 1(a) in words.</td><td>2(c)

Express the rule of example 1(b) in words.</td><td></td></tr>
<tr><td>3</td><td>3(a)

<u>Review</u>
a) Determine y.
b) Determine the rule as an equation and in words.
</td><td>3(b)

Determine the rule as an equation and in words.

{(1; 1), (2; 4), (4; 9); (4; 16)}</td><td>3(c)

Determine the rule as an equation and in words.

<table><tr><td>x</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td></tr><tr><td>y</td><td>3</td><td>5</td><td>7</td><td>9</td><td>11</td><td>13</td></tr></table></td><td></td></tr>
<tr><td>4</td><td>4(a) – a definition

<u>Function</u>
Is a rule where each input value (usually x) is linked to a unique output value (usually y)</td><td>4(b) – a definition

Set of input values – DOMAIN

Set of output values – RANGE</td><td>4(c) –table of connected terms

<table><tr><td>x</td><td>input</td><td>Domain</td><td>Horizontal</td></tr><tr><td>y</td><td>output</td><td>Range</td><td>Vertical</td></tr></table></td><td>4(d)

Let $y = -2x + 3$
1) evaluate y if $x = 2$
2) evaluate x if $y = 2$</td></tr>
<tr><td>5</td><td>5(a)

If $y = 4x + 2$ and $y = x^2$
Find y if $x = 2$</td><td>5(b) – a definition

<u>Function notation</u>
When we have multiple y values we let $y = f(x)$ or $y = g(x)$ or $y = h(x)$ etc
'f of x'</td><td>5(c)

If $y = 3x - 4$
let $f(x) = 3x - 4$

• evaluate y at $x = 2$
• find $f(2)$</td><td>5(d) – a deduction of function notation

$(x;\ y) \rightarrow (x;\ f(x))$</td></tr>
<tr><td></td><td>5(e)

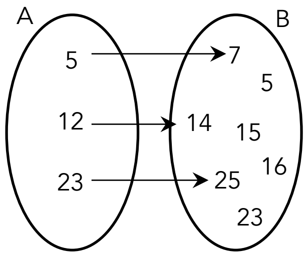
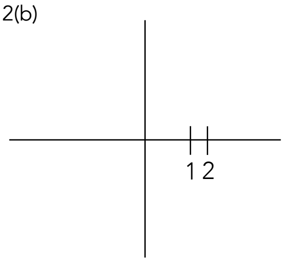
$f(x) = 2x + 3$
$g(x) = -x - 6$

Find 1) $f(1)$
2) $g(3)$
3) $g(-4)$</td><td>5(f)

$f(x) = 3x - 1$

Find:
1) $f(a)$
2) $f(\star)$
3) $f(0)$
4) $f\left(\frac{1}{3}\right)$
5) x if $f(x) = 8$
6) $f(elephant)$</td><td></td><td></td></tr>
</table>

Table 3 **Teacher X Lesson 2 Examples**

Episode				
1	1(a) – *Kahoot Quiz* If $f(x) = 3x - 7$, then the value of $f(2) =$ *Multiple choice options:* $13;\ 25;\ -1$	1(b) – *Kahoot Quiz* $g(x) = x^2 + 2x - 3$ find the value of $g(1)$ *Multiple choice options:* $0;\ -3;\ 19$	1(c) – *Kahoot Quiz* $f(x)$ is a fancy way to write… *Multiple choice options:* x; y-intercept; slope; y	1(d) – *Kahoot Quiz* The figure represents a mapping of x to $f(x)$. Determine $f(5)$ *Multiple choice options:* $f(5) = 7, f(5) = 5, f(5) = 12,$ $f(5) = 14$ 
	1(e) – *Kahoot Quiz* If $f(x) = 4x + 2$, find $f(2)$ *Multiple choice options:* $8;\ 10;\ 6;\ 2$	1(f) – *Kahoot Quiz* If $f(x) = x^2 - 3x$, find $f(-8)$ *Multiple choice options:* $-40;\ 4;\ -88;\ .88$	1(g) – *Kahoot Quiz* If $f(x) = x^3 - 1$, determine $f(2)$ *Multiple choice options:* $f(2) = 5;\ f(2) = 7;\ f(2) = 6;$ $f(2) = 8$	1(h) – *Kahoot Quiz* $f(x) = x^2 - 3x + 9$. Determine $f(0)$. *Multiple choice options* $f(0) = 9;\ f(0) = -3;$ $f(0) = 6; f(0) = 3$
	1(i) – *Kahoot Quiz* For what value of x would $g(x) = x + 10$ be 20? *Multiple choice options:* $10;\ 20;\ 30;\ -10$	1(j) – *Kahoot Quiz* If $g(x) = x + 7$ and $g(x) = 10$, then $x = 4$ *True or False?*	1(k) – *Kahoot Quiz* If $f(x) = x + 4$ and $f(x) = 20$ then $x =$ *Multiple choice options:* $2;\ 10;\ 0;\ 16$	1(l) – *Kahoot Quiz* The input value for a function is part of the DOMAIN *True or False?*
	1(m) – *Kahoot Quiz* $\{(-2;4),(-1;1),(0;0);\ (1;1)\}$ then the domain is: *Multiple choice options:* Domain = $\{-2;\ -1;\ 0\}$; Domain= $\{2\}$; Domain= $\{4;\ 1;\ 0\}$; Domain= $\{-2,-1,0,1\}$	1(n) – *Kahoot Quiz* The range of $\{(-2;4),(-1;1),$ $(0;0),(1;1),(2;4)\}$ is $\{0;1;4\}$ *True or False?*	1(o) – *Kahoot Quiz* All functions will be straight lines if we draw them on a Cartesian plane *True or False?*	
2	2(a) Sketch $y = 2x - 3$	2(b) 	2(c) Sketch $y = x^2 + 1$	2(d) Sketch $y = 2^x$
	2(e) Sketch $y = \frac{6}{x}$			

Examples: Innate Affordances

Lesson 1

In Episode 1, the two examples are similar in that the task is the same (find the equation of a straight line that is of the form $y = mx + c$), and both graphs have a positive gradient, although the exact gradients differ. In the provided coordinates, the x-coordinates of the two points are the same. The examples differ in that the y-coordinates of the provided points are different. Additionally, in example 1(a), the y-intercept is clearly the origin, while in example 1(b) it is above the origin. This juxtaposition of features thus highlights finding the y-intercept as a key aspect that is contrasted between the two examples. Finally, the equation of example 1(a) is $y = 3x$ and the equation of example 1(b) is $y = x + 3$. Here, the m-value of 1(a) is the same as the c-value of 1(b). The similarity of the 3s highlights the different 'positions' that they take in the two equations: In 1(a), the coefficient of x, here the gradient, and in 1(b), the constant added to x, here the y-value of the y-intercept. This allows for the discernment and contrast of the two key aspects of a linear function: m and c.

In Episode 2, example 2(a) introduces the idea of an equation as a rule that links an x-value to a y-value. Examples 2(b) and 2(c) are again the same task: this time, expressing the equations found in Episode 1 in words. Here, there are two relationships between examples to consider: variation and similarity between examples 2(b) and 2(c) within Episode 2, and variation and similarity between 1(a) and 2(b), and 1(b) and 2(c) across Episodes 1 and 2. Within Episode 2, the expression of both rules in words ('y is three times x' , and 'y is three more than x' respectively) again draws attention to the two aspects of m and c as this is what is different in the two rules. Across episodes, the relations of x and y remain constant from 1(a) to 2(b) and from 1(b) to 2(c). Here, with the relationships remaining constant, it is the aspect of the representation itself (equation versus words) and equivalence across representation (a rule in equation form remains the same rule when it is expressed in words) that becomes discerned.

In Episode 3, examples 3(a), 3(b) and 3(c) are the same in that they all involve the task of expressing a rule linking x and y in equation form and word form. Additionally, 3(a) and 3(b) are the same rule. The examples differ as follows: In example 3(a) the y-values first need to be found, while the y-values are provided in examples 3(b) and 3(c). Additionally, 3(a) and 3(c) are linear while 3(b) is a quadratic rule. Finally, the starting representations are different in all examples: 3(a) is a flow diagram, 3(b) is a set of ordered pairs, and 3(c) is a table. The similarity across the three examples regarding the task allows the different starting representations to be contrasted and discerned as different possible features of this aspect of rules. Additionally, it allows for the rules of different degree to be contrasted and discerned (feature of linear versus quadratic), highlighting 'degree' as an aspect of a rule. Specifically, between examples 3(a) and 3(c), the fact that the rule is the same between both examples contrasts the difference, yet fundamental equivalence, between their starting representations, namely flow diagram and table. The similarity across tasks, however, also allows for generalisation to become clear: the repeated exercise of expressing a rule from different starting representations in equation and word form highlights that regardless of what the starting representation is, the rule can be expressed in equation and word form. Thus, equivalence of a rule across different representations, and the fact that the same rule can be equivalently expressed in different representations is shown to be an aspect of rules.

In Episode 4, examples 4(a) and 4(b) are definitions. 4(c) is a table to help contrast and generalise meaning across key terms to do with functions. Here, within the example, contrast is clear in each column as related words to do with input and output are juxtaposed, while generalisation of meaning is clear in the two rows: all the terms in a row are features of the same aspect (input at the top, output at the bottom) in different contexts. Finally, in example 4(d), a function is provided. The two sub-questions within the exercise show similarity in their structure: "find $\square$ if $\square = 2$", with the x and y-variables swapping position in the two sub-questions. Here, in solving, contrast is created in that the value of the provided variable (which is the same in both cases) must be substituted in different places (allowing for the discernment of input versus output). These varying substitutions also result in different solving strategies. Finally, as the resulting coordinates in both sub-questions are different, namely $(2;\ -1)$ and $\left(\frac{1}{2};2\right)$, yet there is a 2 in both coordinates in different positions, this emphasises that an input value of 2 and an output value of 2 do not create the same coordinates. The difference between input and output values of a function can now be discerned.

Finally, in Episode 5, example 5(a) provides two functions of the form $y = \cdots$. The functions differ in that one is linear and the other is quadratic. For both, the instruction remains the same, to find y if $x = 2$. The similarity of instruction and input-value allows the different function equations, each function's degree, and the resulting different mathematical procedures to be discerned. Example 5(b) is a definition of function notation which includes features $f(x)$, $g(x)$ and $h(x)$. This aspect of function notation is thus discernible. Example 5(c) involves much similarity: the function $3x-4$ is expressed in y-form and $f(x)$-form, and the similar instructions of 'evaluate y at $x=2$' and 'find $f(2)$' are given. The resulting solutions will thus be the same, too. Through this similarity, contrast between y and $f(x)$ is created (positioning these as features of this aspect), and also in the two ways the same instruction can be given. Generalisation is also possible through these examples as their similarity of form and solution emphasise the equivalent meanings of the two exercises. Example 5(d) is a deduction of function notation and contrasts two ways that a coordinate pair can be written. In each coordinate, the abscissa is the same, while the ordinates differ. This contrasts the y and $f(x)$, while also emphasising their similar meanings, and adds a new feature to the aspect of $f(x)$: it can be used in coordinate form too. In example 5(e), two functions are given. These functions are the same in that they are both presented in function notation and are both linear. However, the function notation differs, and in one function the gradient and y-intercept are both positive while in the other function the gradient and y-intercept are both negative. This creates contrast between the 'names' of the functions, their gradients and y-intercepts. For the sub questions, two positive inputs are provided while the third is negative. This difference creates contrast in the procedure of substitution into the provided functions. Finally, in 5(f), one function is given, and the output for five provided inputs is desired. These inputs vary substantially, including numeric (whole number and fraction), algebraic, pictural and 'word' form. This creates generalisation across the procedure of substituting with function notation, while the contrast also brings in new features to the aspect of input-value. However, one exercise provides an output, and an input is desired. This juxtaposition brings attention to the difference between finding an output from an input and finding an input from an output as an aspect of working with functions.

To summarise, in Lesson 1, key aspects of functions that the examples bring to light innately are: the aspects 'm and c of linear functions'; the aspect 'representations of functions' including the features: graph, words, flow diagram, set of ordered pairs, table, and function notation; the aspect that 'a rule can be equivalently expressed across these representations'; the aspects of 'input and output'; and the aspect of using function notation to evaluate a function (input → output and output → to input).

Lesson 2

In Lesson 2, Episode 1 has 15 examples from the online *Kahoot* quiz. All *Kahoot* questions are either in 'multiple choice' or 'true or false' form. All questions were given with tight time pressure: approximately 10-20 seconds per question. I analysed these examples according to questions on the same objects of learning.

Examples 1(a) – 1(k) all have to do with function notation. The degree of the provided functions differs, including linear, quadratic, and cubic equations, which highlights contrasting features of this aspect. The function notation varies between $f(x)$ and $g(x)$, which allows for this aspect to be discerned. Provided input-values for the functions differ as they include positive and negative whole numbers, and the input-value zero. In most questions, the input-value is provided: 1(a), 1(b) and 1(d)-1(h) all require the output to be found. This creates generalisation of procedure across these questions. 1(i), 1(j) and 1(k), however, create contrast in comparison to the prior questions as 1(i)-1(k) provide the output and desire the input. Contrast is also created in example 1(d) which uses the representation of mapping in conjunction with function notation.

Examples 1(l), 1(m) and 1(n) are centred around domain and range. In examples 1(m) and 1(n), sets of coordinate pairs are provided from which the domain and range need to be found respectively. The two questions differ in that the domain of 1(m) has no repeated values whilst the range of 1(n) does have repeated values. This creates contrast when listing number sets, and highlights not listing the same value twice.

Finally, the last example of Episode 1 creates a link to Episode 2, asking if all graphs sketched on the Cartesian plane are straight lines. This sets the scene for Episode 2, where four different types of functions (2(a) – linear, 2(c) – quadratic, 2(d) – exponential, 2(e) – hyperbola) are sketched showing contrast of graphic form. Thus, the aspect of 'shape' of functions is opened up with four possible features. With these four functions, three are continuous while the hyperbola is discontinuous with two separate arms, and thus the feature of (visual) continuity is discernible. The hyperbola also leads to an undefined output for an input (at $x = 0$) – which is the first time this occurs across the two lessons. Thus, a new feature is added to the concept of output: it might be undefined. Finally, two graphs have (different) vertical shifts (2(a) downwards by 3 units and 2(b) upwards by 1 unit) while the other two do not. This contrast opens up the dimension of vertical shift, including the features 'upward shift' and 'downward shift' and 'no shift'. Example

2(b) cannot easily be understood without context of the lesson. This was a spontaneous counter-example to show how not to draw increments on the x-axis. This was contrasted to the regular increments used for the four sketched graphs.

To summarise, in Lesson 2, key aspects of functions that the examples bring to light innately are: the aspect of using 'function notation to evaluate a function' (features: input → output and output → to input); the aspect of seeing input (x) and output (y) values of a coordinate as part of the domain and range of the function to which the coordinate belongs'; and the aspect that 'different types of functions exist and have different shapes' (features: linear, parabola, exponential, hyperbola).

Examples: Enacted Affordances

Teacher X's Intended Objects of Learning

In the interview, Teacher X communicated the following intentions for the two lessons. In Lesson 1, he wanted to establish some idea regarding the students' levels of proficiency regarding prior knowledge (as he was a substitute teacher); provide a level of comfort regarding functions as students often have negative preconceptions regarding this object of learning; and finally, stretch them to the level of Grade 10 content. For Lesson 2, he chose to use a *Kahoot* as he judged this to provide an element of fun while quickly consolidating concepts from the previous lesson. Additionally, he stated that with its fast pace, it allowed for the development of fluency and agile use of mental arithmetic (rather than a calculator), when using function notation. The *Kahoot* also helped him to gauge the students' level of prior knowledge regarding functions: in the last *Kahoot* example (Lesson 2, example 1(o); Figure 11), he asked:

All functions will be straight lines if we draw them on a Cartesian plane True or False?

Figure 11 **Teacher X, Lesson 2, Example 1(o)**

Only 14/21 students correctly selected false. This was a misconception that Teacher X felt needed to be immediately addressed, and so in that lesson, he subsequently went on to draw a straight line, a parabola, an exponential function and a hyperbola to show the students examples of functions that were not just straight lines. He communicated that if (almost) all students had known that other function graphs exist, the next episode of the lesson would have been different. Teacher X emphasised that he tries to be responsive to student feedback during a lesson and to adjust the lesson, and subsequent examples, accordingly. He said that: "no matter what lesson plan you do, you always allow for some kind of a tangential, if you feel that you are pushing them [the students] too quickly". Teacher X's choice and use of examples during these lessons was thus not a fully linear process and he allowed his example choice regarding his intended object of learning to be guided by observations of the lived object of learning (student feedback) during the enacted object of learning.

<u>The Enactment of Some Key Aspects from the Chosen Examples</u>

In Lesson 1 and Lesson 2, two of the key aspects that were innately afforded by the example space were: input and output-values of a function, and relatedly, using function notation to evaluate functions (input → output and output → input). I now discuss how Teacher X enacted these aspects by drawing on specific examples from the lessons. In these enactments, I have furthermore chosen examples that show how Teacher X used a *combination of instances* in conjunction with *shifting focus of attention*. This was typical of his practice across both observed lessons. In the first example, this is done to mainly create contrast. In the second example, this is mainly done to create generalisation. Details of his enactment of other examples are available in Appendix A.

In example 4(d) from Lesson 1 (Figure 12) a *combination of instances* was used to create contrast between input and output. How the task was solved by the teacher is communicated in grey text.

Let $y = -2x + 3$
1) evaluate y if $x = 2$ input
$y = -2(2) + 3$
$y = -1$
$(2; -1)$
2) evaluate x if $y = 2$ output
$2 = -2x + 3$
$\frac{1}{2} = x$
$\left(\frac{1}{2}; 2\right)$

Figure 12 **Teacher X, Lesson 1, Example 4(d)**

In the exercise, for the same rule, Teacher X asked for y to be evaluated if $x = 2$, and then for x to be evaluated if $y = 2$. In the lesson, Teacher X deliberately drew attention to the variation between the two exercises by *shifting focus of attention*. He said:

> "What is the distinction between the first example and the second example? What is our focus of our activity here [pointing at 1)]? Substitute in the x in order to find the y. What is the difference here [pointing at 2)]? Substitute in the y in order to find out what the x would have been. Here you have the input [pointing at 1)]. Here you have the output [pointing at 2)].

Finally, he also added the words 'input' and 'output' to 1) and 2) respectively (as indicated), strengthening the contrast between these two exercises. The manner in which Teacher X deliberately brought attention to what he wanted his students to notice brings to mind the work of Zaslavsky and Zodik (2007), who argue that teachers need to deliberately highlight what they want students to notice to aid potential learning.

In the interview after the lesson, Teacher X stated regarding this question that "I didn't make it x is 2 and y is 5, for example, so that they [the students] could see that although the value was a 2 [for both], the 2 got inserted into a completely different place and it landed up with completely different techniques to solve for the missing value." This statement indicates that Teacher X has an awareness of the principle of contrast (Marton, 2015) although he has not used this term to describe his intended object of learning. By keeping the value of 2 invariant, the contrasting meaning of 2 as input and output, and the contrasting associated operations, thus became visible in doing these two exercises. In the enactment, it thus appears that the intended object of learning was achieved.

Teacher X also used a *combination of instances* and *shifting focus of attention* to create *generalisation* of procedure for the use of function notation. In Lesson 1, example 5(f) (Figure 13) he used the following exercise (the teacher's working out is in grey):

$f(x) = 3x - 1$

Find:

1) $f(a) = 3a - 1$
2) $f(\star) = 3 \star - 1$
3) $f(0) = 3(0) - 1 = -1$
4) $f\left(\frac{1}{3}\right) = 3\left(\frac{1}{3}\right) - 1 = 0$
5) x if $f(x) = 8$
6) f(elephant)= 3(elephant) − 1

5) x if $f(x) = 8$ output

$$3x - 1 = 8$$
$$3x = 9$$
$$x = 3$$

Figure 13 **Teacher X, Lesson 1, Example 5(f)**

During the lesson, Teacher X again made his teaching point explicit by *shifting focus of attention*. He said:

> "I'm trying to push a point home here... sometimes our input-values are not that pretty: they could be fractions, they could be negatives, they could be stars, they could be as they could be ps and in fact you could take something like really overboard [adding question 6) to the board] the $f(elephant)$ would be what? 3 times $elephant$ minus 1 [as he writes this down]. Just to push a point home. It doesn't matter what's in that x-bracket – that is your input value that you replace wherever you saw an x before".

After the lesson, he described this as exercise as follows:

> "Sometimes to land a concept you have to be ridiculous, don't you?... So $f(\star)$ or $f(elephant)$ ... it is just really to say, 'you know what, it doesn't matter what I put in there, it just means that's your input amount'."

.

In this example, and Teacher X's discussion thereof, it appears that he is attempting to illustrate what Watson and Mason (2005) describe as 'dimensions of possible variation' for input values, and the associated procedure, of function notation.

In exercise 5) of this example, however, Teacher X included one question where the output was given: $f(x) = 8$. With this exercise (which he did last – that is, he did exercise 6) before 5)) he reminded the class of the contrast of receiving a provided input or output. He deliberately drew attention to this contrast during the lesson by saying: "Is the x 8 this time? ... No, what is 8? The

$f(x)$ is 8. So what am I going to do – the input or the output? ...The output! ... Can you see how this little tweak can easily catch us?" This contrast was powerful as it was juxtaposed just after the generalisation of using a provided input had been done. It thus clarified when it is correct to substitute a given input for x, and when it is correct to substitute a given output for y. Again, in this example, Teacher X thus displays an awareness of the principle of contrast (Marton, 2015) although he does not describe it as such.

In his enactment of examples, Teacher X thus combined 'combination of instances' with 'shifting focus of attention', whist also deliberately telling the boys which aspects/features he wanted them to notice from examples. This occurred both in instances when he was focusing on developing contrast, and when he was focusing on creating generalisation of an idea.

When the Enactment goes Beyond what is Discernible in the Chosen Examples

In analysing Teacher X's examples innately, it did not appear clear to me that the function representation 'coordinate pair' would be an aspect of functions that would be woven between examples in many episodes across both lessons, and thus accumulate many features.

In Lesson 1, Episode 1 the coordinate was positioned as a point on a graph that satisfies an equation, with x as the input-value. In Episode 2, a new feature was added: now, the coordinate was positioned as a representation of its graph's rule, that is $(input; rule\ output)$ for example, Teacher X showed that the points (1; 3) and (2; 6) of Lesson 1, example 1(a)/2(b) show "the y value is triple the x-value" rule – in other words, $1 \times 3 = 3$ and $2 \times 3 = 6$. Throughout the lesson, Teacher X also provided the solution to input-output questions as coordinates (as shown in Figure 12), emphasising the feature of a coordinate $(input; output)$. When function notation was introduced, he furthermore showed the coordinate as $(x;\ y) \rightarrow (x;\ f(x))$, thus creating a connection between the coordinate and function notation. In Lesson 2, Episode 1, the coordinate featured as $(domain; range)$, that is, the x-value of a coordinate belongs to a function's domain while its y-value belongs to the function's range. Finally, in Lesson 2 Episode 2, the coordinate was generated from a table and used as a tool to sketch a function. Thus, by the teacher's use of *combination of instances* and *shifting focus of attention*, the aspect 'coordinate' gained multiple features during the course of the lesson and connected it to many other aspects of functions. Goldenberg (1995) discusses that it is crucial for students to be able to manipulate between representations of functions in order to develop real understanding of this object of learning. Teacher X's work here displays a linked but different approach to this standpoint: in his case, he has integrated a particular representation with various different aspects of functions. This handling of the representation affords the creation of connections between different aspects within the example space of the object of learning 'function'.

The Lived Object of Learning

In this analysis, no students were interviewed or formally tested, so any observations regarding the lived object of learning came about through the lesson observations. There was little student-driven student contribution in Teacher X's lessons. This was likely because the teacher was in a substitute role – not the regular teacher of the class – and there was not yet an established classroom culture. When asked direct questions, some boys provided correct solutions whilst others provided incorrect solutions that either the teacher or another boy corrected. The *Kahoot* provided the most clear lived object of learning. Here, the number of correct solutions per question ranged from 2/21 (the first question when the boys were not prepared for the quick pace) and 19/21. However, this quiz was mainly designed to be a formative assessment: to revise, stretch thinking, and promote fluency, so it cannot be used as a summative judgement on what was learnt. It appears that the most important use of the student responses and the lived object of learning that did occur in these lessons was how Teacher X used student responses to try and pitch the lesson and examples at an accessible level for the boys in the class.

Overall, Teacher X's enactment of the example space of his lessons often matched the innate affordances of the examples and sometimes went beyond what was apparent in the innate affordances of the example space. He also used examples to provide student feedback which was used to adjust his intended object of learning.

Examples: Use of Multiple Representations

Table 4 documents Teacher X's use of the multiple representations of functions used within an episode in column 2, and within individual examples per episode in column 3. Teacher X consistently incorporated multiple representations of the same function within episodes and within individual examples.

Table 4 **Teacher X Use of Multiple Representations**

Lesson 1 Episode	**Multiple Representations per episode**	**Combination of Representations per example**
1	graph, coordinate pair, equation	1(a) graph, coordinate pair, equation 1(b) graph, coordinate pair, equation
2	words, graph, coordinate pair	2(a) - 2(b) words, graph, coordinate pair 2(c) words, graph, coordinate pair
3	flow diagram, equation, words, set of ordered pairs, table	3(a) flow diagram, equation, words 3(b) set of ordered pairs, equation, words 3(c) table, equation, words
4	Cartesian plane (graph), equation, coordinate pair	4(a) - 4(b) - 4(c) Cartesian plane (graph) 4(d) equation, coordinate pair
5	equation, coordinate pair, function notation, words	5(a) equation, coordinate pair 5(b) - 5(c) equation, function notation, coordinate pair 5(d) coordinate pair, function notation 5(e) equation, function notation, coordinate pair, words 5(f) equation, function notation
Lesson 2 Episode	**Multiple Representations per episode**	**Combination of Representations per example**
1	Function notation, mapping, equation, set of ordered pairs, set notation, Cartesian plane (graph), graph	1(a) function notation 1(b) function notation 1(c) function notation 1(d) function notation, mapping 1(e) function notation 1(f) function notation 1(g) function notation 1(h) function notation 1(i) function notation 1(j) function notation, equation 1(k) function notation 1(l) - 1(m) set of ordered pairs, set notation 1(n) set of ordered pairs, set notation, Cartesian plane 1(o) graph, Cartesian plane
2	equation, table, coordinate pair, graph	2(a) equation, table, coordinate pair, graph 2(b) Cartesian plane 2(c) equation, table, coordinate pair, graph 2(d) equation, table, coordinate pair, graph 2(e) equation, table, coordinate pair, graph

In Lesson 1, Episode 3, and in Lesson 2, Episode 2, Teacher X emphasised the multiple representations most 'deliberately' as an object of learning in itself. In the former episode, he gave three functions in different starting representations (flow diagram, set of ordered pairs, table) and translated each into words and an equation. In the latter episode, for a straight line, parabola, exponential graph and hyperbola, he translated each given equation into a table, then coordinates, and ultimately a graph. However, the multiple representations of functions appear to be deeply ingrained in the way that he taught these two lessons beyond being its own object of learning. In Lesson 1, 10 of the 18 used examples integrate at least three different representations (within other objects of learning) while in Lesson 2, 5 of the 20 examples incorporate at least three different representations (4 of which incorporate four different representations). It can thus be argued that 'functions having multiple different possible representations' is a key aspect that students in this classroom were afforded the opportunity to learn.

Teacher X also expressed in the interview that he considers the multiple representations of functions when teaching this section. He stated that he is very deliberate about putting a rule into words because he finds that students "see functions as being in isolation… but the rule is the picture, it's the algebraic equation, it's the words". This view echoes the work of Leinhardt et al. (1990) which suggests that different representations define the object of learning 'functions' together. When asked by me about his consistent practice of providing solutions to input/output questions in coordinate form, Teacher X stated that "I just think that when you're in functions, you need to keep on referring to 'what does this look like on the graph'". This awareness was reflected in his use of the multiple representations across the two lessons (as highlighted in Table 4).

Final Remarks on Teacher X

Teacher X's two lessons included 38 examples on functions that covered the content: finding the equation of a straight line; rules and relationships between x and y; rules with different representations; definitions of function, domain, and range; function notation; and sketching different functions from a table. His example space included many inherent affordances for learning, and his enactment of the examples used the teacher actions *combination of instances* and *shifting focus of attention* to create contrast and generalisation. These actions enhanced the patterns of variation that were innate in the example set and thus opened up further learning opportunities. Throughout his lessons, the 'multiple representations of functions' was an aspect that was repeatedly emphasised, both as an object of learning in itself, and within other objects of learning.

Teacher Y

Teacher Y's two lessons were made up of 24 examples in 3 episodes, and 17 examples in 2 episodes respectively. Both lessons incorporated theory notes being projected on the board which were read and explained by the teacher. In both lessons there was a strong focus on understanding the difference between a relation and a function, and on domain and range. Function notation was briefly introduced at the end of the second lesson.

Objects of Learning: Summary

Table 5 documents, per episode, the objects of learning that occurred in Teacher Y's lessons. The objects of learning are described in increasing detail by 'Content', 'Educational Objectives', and 'Aspects and Features', as described by Marton (2015). In this table, the examples' innate affordances for learning, and the affordances created through mediation and classroom interaction, are considered.

Table 5 **Teacher Y Objects of Learning**

Lesson 1 Episode	Content	Educational Objectives	Aspects and Features
1	Rules and sequences	- recognise ordered sets of numbers that form a sequence - find the rule of a linear pattern - use the rule of a linear pattern to generate outputs from inputs - recognise that the rule of a pattern relates the input and the output values - become aware of the concept 'relation'	- aspect: sets of numbers features: ordered and unordered sets; ordered sets can be generated by a rule while unordered sets cannot - aspect: representations of a rule features: set notation, general term, flow diagram, set of ordered pairs - aspect: meaning of a rule features: used to generate terms of a pattern, something that relates input and output values - aspect: a relation features: where inputs and outputs are connected by a rule; a linear flow diagram; a set of ordered pairs
2	Key concepts / terminology to do with relations and functions	- understand what is a relation - understand what is a function - understand domain and range - understand the link between domain and range and defining a function - classify a set of ordered pairs as being a function or not	- aspect: relation features: a rule relating two variables; a number pattern; a circle graph; functions are relations - aspect: function features: function versus relation; every input has only one output; every function is a relation but not every relation is a function; a linear flow diagram; not a circle graph - aspect: representations of functions features: flow diagram; number pattern rule; set of ordered pairs - aspect: domain

			features: the inputs for a function; the x-values of coordinate pairs; the set of x-values for which a function is defined; the inputs of a flow diagram - aspect: range features: the outputs of a function; the y-values of coordinate pairs; the set of values that a function can produce; the outputs of a flow diagram - aspect: listing domain and range as a set features: listed in ascending order, repeated values are only listed once, set notation
3	Functions and relations	- identifying functions from mapping and coordinate pairs on a cartesian plane - understanding the difference between one input having many outputs and multiple inputs having the same outputs (when classifying functions) - being aware of one-to-one and one-to-many functions - understanding the importance of mathematical definitions in classifying mathematical objects	- aspect: representations of functions new features: mapping; points on the cartesian plane - aspect: mapping features: visualise a connected domain and range; the number of elements in each set can vary; can represent a relation that is or is not a function; links to one-to-one and many-to-one functions - aspect: cartesian plane features: visualise a connected domain and range from a point; the number of points on the graph can vary; can represent a relation that is or is not a function; links to one-to-one and many-to-one functions - aspect: function new features: can be many-to-one (visualised in mapping or coordinate form) or one-to-one (visualised in mapping or coordinate form); functions can be determined in different representations
Lesson 2 Episode	**Content**	**Educational Objective(s)**	**Aspects & Features**
1	- Functions and relations - Range and domain	- know the definition of a relation - know the definition of a function - identify a function from a sketch of a graph (informal vertical line test) - give the domain of a function/ relation from a sketch of a graph - give the range of a function/ relation from a sketch of a graph - use inequality notation	- aspect: relation new feature: graphical representation of relations; infinite relations; relations with restricted domains; arrows on graph arms - aspect: function new feature: graphical representation of functions; infinite functions (arrows on graph notation); functions with restricted domains - aspect: domain new feature: read off a graph; infinite; finite - aspect: range new feature: read off a graph; infinite; finite - aspect: types of graphs (sketch only) features: cubic; parabola; circle; straight line; absolute value inverse; cubic inverse - aspect: notation of domain and range features: set notation (previous lesson); inequality notation - aspect: inequality notation features: values in ascending order; direction of inequality sign; inclusive or exclusive inequality sign; curly brackets
2	Function notation	- understand the need for function notation	- aspect: function notation features: $f(x)$, $g(x), h(x)$, $k(x)$ - aspect: using function

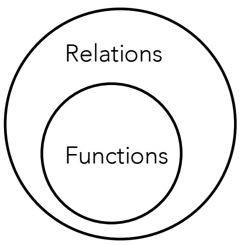
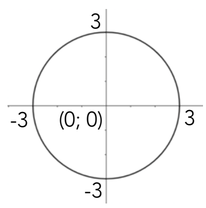
		- represent a function with function notation - understand the advantages of using function notation - evaluate functions using function notation (given input only) - know how to pronounce function notation	notation features: see independent/dependent variables; evaluate the function for a given input; - aspect: degree of a rule features: linear, quadratic

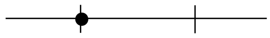
In Lesson 1, Teacher Y began by discussing number patterns and used this as a platform to shift the focus to the relationship between x and y. Next, he went through notes on theory to do with a relation and a function, domain, and range. This theory was then applied in doing exercises involving two sets of ordered pairs. Thereafter, he handed out a worksheet where functions were identified from maps and points on the Cartesian plane. In Lesson 2, more than half of the lesson was used to go through homework exercises that the students had found difficult. These exercises focused on identifying functions, domain, and range from sketches of various graphs. Thereafter, function notation was introduced.

Example Space Used in the Introduction of Functions

All the examples used by Teacher Y across both lessons are documented in Table 6 and Table 7. As with Teacher X, the examples are arranged horizontally in the sequence in which they occurred in each lesson. Thereafter, I discuss the examples' innate affordances for learning (using Marton's (2015) VT) before discussing their enactment in the classroom. When discussing their enactment, I consider teacher actions and student interaction as well as information provided in the interview with Teacher Y regarding his intended object of learning. As with Teacher X, a table combining the Objects of Learning (as presented in Table 5) and the examples themselves (as presented in Table 6 and Table 7), and descriptions of Teacher Y's mediation of the examples, is available in Appendix A.

Table 6 **Teacher Y Lesson 1 Examples**

Episode				
1	1(a) What do you notice about these numbers? $A = \{5; 7; 9; 11; 13; \ldots\}$	1(b) What do you notice about these numbers? $Z = \{1; 17; 18; 20; 27\}$	1(c) Write down a rule to generate the numbers of set A.	1(d) $T_n = 2n + 3$ variables
	1(e) Input → $\boxed{2n+3}$ → Output 1 → 5 2 → 7 3 → 9			
2	2(a) – *Projected Notes* "Relations and Functions" **are the most important topics in algebra.** Relations and functions – these are the two different words having different meanings mathematically. You might get confused about their difference. Before we go deeper, let's understand the difference between both with a simple example. An ordered pair is represented as (INPUT, OUTPUT). The relation shows the relationship between INPUT and OUTPUT. Whereas a function is a relation which derives one OUTPUT for each given INPUT. Note: All functions are relations but not all relations are functions.  	2(b) – *verbal, real world example* "All boys are human being but not all human beings are boys"	2(c) – *Projected Notes* What is a Function? A function is a relation that describes that there should be only one output for each input (or) we can say that a special kind of relation (a set of ordered pairs), which follows a rule i.e., every X-value should be associated with only one y-value is called a function. What is a Function? In mathematics, a function is a set of inputs with a single output in each case. Every function has a domain and range. The domain is the set of independent values of the variable x for a relation or a function defined. In simple words, the domain is a set of x-values that generate real values of y when substituted in the function. On the other hand, the range is a set of all possible values that a function can produce. The range of a function can be expressed in interval notation or in form of inequalities.	2(d) $T_n = 2n + 3$ $y = 2x + 3$
	2(e) List the domain and range of the relation: $(2; 1); (-3; 4); (0; 5)$	2(f) Is the relation in 2(e) a function?	2(g) – *a relation that is not a function*  0 → 3 0 → -3	2(h) a) List the domain and range b) Determine whether the relation is a function. $\{(1; 3); (-2; 4); (3; -2); (-2; 7)\}$

<table>
<tr>
<td></td>
<td>2(i)

</td>
<td>2(j) – verbal, real world example

“It is because, here, this is the same input [pointing at the two -2s], it is the same input. It's just one [Teacher Y][3] that you are inputting. So you don't have two [Teacher Ys], it's one [Teacher Y], although you are saying this and that [moving his body in two different directions for ‘this’ and ;that’] but it is one [Teacher Y]…so we write it once.</td>
<td>2(k) – verbal, real world example

“It's like saying to you that you must have one girlfriend not two”</td>
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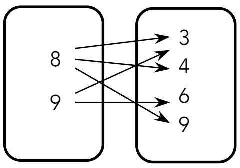
<tr>
<td>3</td>
<td>3(a)

Determine if this relation is a function:

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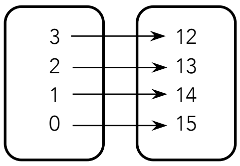
<td>3(b)

Determine if this relation is a function:

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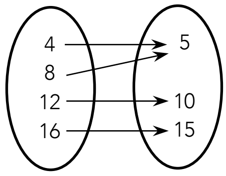
<td>3(c)

Determine if this relation is a function:

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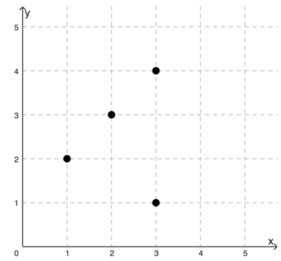
<td>3(d)

Determine if this relation is a function:
</td>
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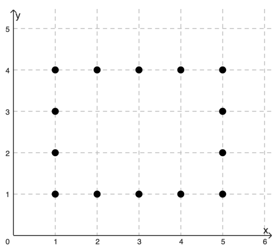
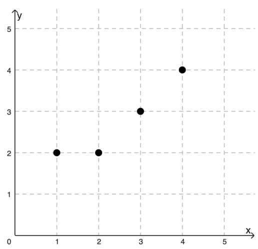
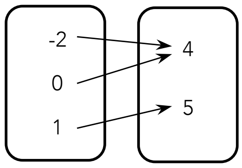
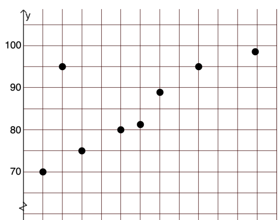
<tr>
<td></td>
<td>3(e)

Determine if this relation is a function:
</td>
<td>3(f)

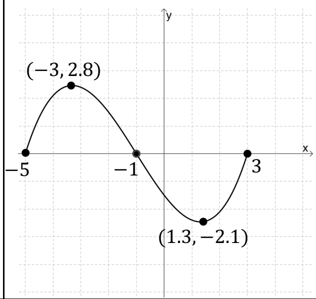
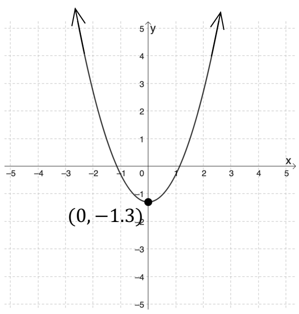
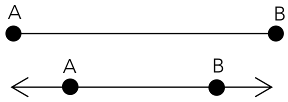
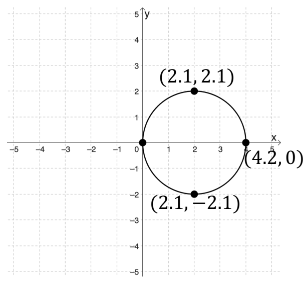
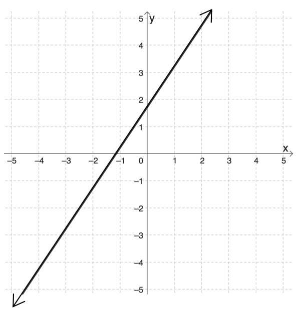
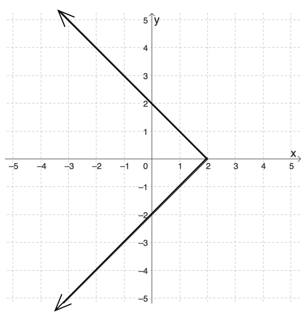
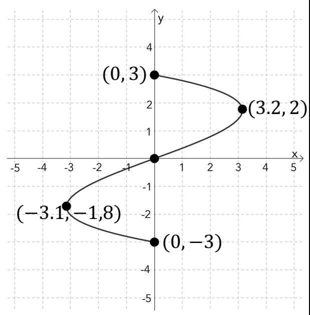
Determine if this relation is a function:
</td>
<td>3(g) – an example of a function

</td>
<td>3(h)

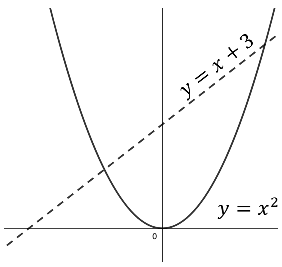
The graph shows the relationship between the hours Rachel studied and the exam grades she earned.

1) Is the relationship a function? Justify your answer. Use the Words “input” and “output” in your explanation, and connect them to the context represented by the graph.

2) Rachel plans to study 2 hours for her next exam. How might plotting her grade on the same graph change your answer to Exercise 11? Explain your reasoning.</td>
</tr>
</table>

[3] Here, the teacher used his own surname, but this has been replaced with ‘Teacher Y’. e.g., “It's just one McLachlan that you are inputting”.

Table 7 **Teacher Y Lesson 2 Examples**

Episode				
1	1(a) – *verbal definition by the teacher* Definition of a relation: A set of input and output values; a set relating the input and output values (given verbally by the teacher)	1(b) – verbal *definition by a student* Definition of a function: A relation that derives one output for each given input.	1(c) Find the domain and range of the relation. Is the relation a function? 	1(d) – *student solution process on the board* Solution to 1(c) domain and range: 1. Domain: $\{-5;\ -3;\ -1;$ 2. Domain: $\{-5 \leq x \geq 3, x \in R\}$ 3. Domain: $\{-5 \leq x \leq 3, x \in R\}$ 4. Range: $\{-2{,}1 \leq y \leq 2{,}8; y \in R\}$
	1(e) Find the domain and range of the relation. Is the relation a function? 	1(f) 	1(g) – *student solution process on the board* Solution to 1(e) domain and range: 1. Domain: $\{\infty \leq x < \underline{\infty};\ x \in R\}$ 2. Domain: $\{-\infty \leq x < \underline{\infty};\ x \in R\}$ 3. Domain: $\{-\infty < x < \infty;\ x \in R\}$ 4. Range: $\{-\infty \leq y \leq 1.3;\ y \in R\}$ 5. Range: $\{-1.3 \leq y \leq \infty;\ y \in R\}$ 6. Range: $\{-1.3 \leq y < \infty;\ y \in R\}$	1(h) Find the domain and range of the relation. Is the relation a function? 
	1(i) $x \in R$	1(j) Find the domain and range of the relation. Is the relation a function? 	1(k) Find the domain and range of the relation. Is the relation a function? 	1(l) Find the domain and range of the relation. Is the relation a function? 

2	2(a) 	2(b) – *Projected Notes* What is a Function Notation? Notation can be defined as a system of symbols or signs that denote elements such as phrases, numbers, words etc. Therefore, function notation is a way in which a function can be represented using symbols and signs. Function notation is a simpler method of describing a function without a lengthy written explanation. The most frequently used function notation is f(x) which is read "f of x". In this case, the letter x, placed within the parentheses and the entire symbol f(x) stand for domain set and range set respectively. Although f is the most popular letter used when writing function notation, any other letter of the alphabet can also be used either in upper or lower case.	2(c) $f(x) = x^2$ $f(1) = (1)^2 = 1$ $f(3) = (3)^2 = 9$	2 (d) – *Projected Notes* Advantages of using function notation • Since most functions are represented with various variables such as a, f, g, h, k etc, we use $f(x)$ in order to avoid confusion as to which function is being evaluated. • Function notation allows to identify the independent variable with ease. • Function notation also helps us to identify the element of a function which has to be examined. Consider a linear function $y = 3x + 7$. To write such function in function notation, we simply replace the variable y with the phrase $f(x)$ to get; $f(x) = 3x + 7$. This function $f(x) = 3x + 7$ is read as the value of f at x or a f of x.
	2(e) – *Projected Notes* How to Evaluate Functions? Function evaluation is the process of determining output values of a function. This is done by substituting the input values in the given function notation. Example 1 Write $y = x^2 + 4x + 1$ using function notation and evaluate the function at $x = 3$. <u>Solution</u> Given, $y = x^2 + 4x + 1$ By applying function notation, we get $f(x) = x^2 + 4x + 1$ Evaluation: Substitute x with 3 $f(3) = 3^2 + 4 \times 3 + 1 = 9 + 12 + 1$ $= 22$			

Examples: Innate Affordances

<u>Lesson 1</u>

In Lesson 1, Episode 1, number sets A (example 1(a)) and Z (example 1(b)) are similar in that they are both sets of numbers with 5 provided elements. This allows for contrast of the relationships between the numbers in each set to be discerned: In set A, the numbers form a linear sequence starting at 5 with a difference of 2. In set Z, the numbers appear to be random. Additionally, in set A, the numbers go on infinitely whereas in Z they do not. Example 1(c) then asks for the rule of the sequence in 1(a) to be given. This creates further contrast between the sets A and Z as there does not appear to be a rule that would produce the numbers in 1(b).

Example 1(d) is the rule of the number pattern in 1(a). Here, it is shown that T_n and n are variables. Finally, in example 1(e) the first three numbers from set A of example 1(a) are shown as outputs of a flow diagram with the rule from 1(d) for input values 1, 2 and 3. This now creates a relationship of similarity between examples 1(a), 1(c), 1(d) and 1(e) as they all have to do with the same rule. Firstly, this creates contrast and allows the representation of the number sequence to be discerned (set of numbers, number pattern rule, flow diagram). However, it also generalises in the sense that it shows that the rule can be shown in different representations and still be the same rule. In 1(e), the flow diagram representation also connects the rule (from 1(d)) with the input and output variables of the pattern. Thus, the numbers of set A are now able to be discerned as the output-values of the number pattern in 1(d). In Episode 1, the key aspect that comes to be discerned is that a number pattern can be represented as a set of numbers, a rule, and a flow diagram and that these representations are all acceptable to show the same number pattern.

In Episode 2, example 2(a) consists of notes on relations and function. The notes cover theory to do with relations and functions using input and output values as explanations. Using the terms 'input' and 'output' in both explanations creates a contrast of relationship for relations and functions. At the bottom of the notes, there is a Venn diagram showing functions as a subset of relations. Example 2(b) is a verbal, real world example that shows a similar 'set-subset' relationship, namely "all boys are human beings but not all human beings are boys". The similarity of relationship between this concrete real world example of humans and boys, and the abstract relationship between relations and functions could make the relations–functions relationship more discernible as an aspect of functions. In examples 2(c) more notes are provided. Here, domain and range are defined and positioned as being aspects of functions. In example 2(d) the number patterns rule from example 1(e) is rewritten using x and y for input and output. The similar structure of these two rules allows the contrast of T_n and y, and n and x to be discerned respectively. The similarity of structure, however, also highlights that these two equations will produce similar outputs. This is because in both cases the input is doubled and 3 is then added. Example 2(e) then provides three coordinate pairs and asks for the domain and range. This example makes it discernible that the domain is made up of the x-values of the coordinates, and the range of the y-values of the coordinates, highlighting this as a feature of domain and range. In 2(f) it is then asked if the three coordinates from 2(e) represent a function or not – it is, in fact, a function. 2(g) then uses a circle graph on the cartesian plane with a diagram underneath showing that input 0 has two outputs: 3 and -3. This example of a 'relation which is not a function' could be used to contrast with the function relationship of 2(e) to allow the aspects of function and relation to be discerned more clearly. Example 2(h) then creates clearer contrast with example 2(e). Example 2(h) is not a function as input -2 has two outputs: 4 and 7. However, this contrast is harder to immediately discern as example 2(e) has 3 coordinate pairs while 2(h) has four coordinate pairs, and all the provided coordinates have different combinations of x and y values. Examples 2(g) and 2(h) seen together show two examples of relations that are not functions. As they both are not functions, it becomes discernible that both 'graphical' and 'set of ordered pairs' representations can be relations that are not functions: what is similar in both cases is that at least one input has two different outputs. Example 2(i) shows a number line with a dot on -2 . This was the input value in 2(h) that had two output values. Examples 2(j) and 2(k) are both spontaneous, verbal, real-world examples that do not make much sense without the context of enactment in the classroom. 2(i) and 2(j) were both spontaneous examples addressing the fact that the -2 from example 2(h) would only be listed once in the domain of the relation of 2(h). In 2(i), it was shown that on the number line, -2 is the same value, and in 2(j), the teacher makes an analogy to himself as being unique: Regardless that -2 is used as in input in two coordinates, it is a unique value that can only be listed once, as the teacher is a unique individual who only exists once. Finally, example 2(k) is again a real world example that made an analogy that in a romantic relationship, one should have one girlfriend and not two. This is a reference to a function only being allowed one output per input, and if one input has two outputs it is not a function. In Episode 2, the key aspects that can be discerned from the inherent examples are that a function is a subset of a relation, and that a function may only have one output per input value while a relation can have multiple outputs per input value.

In Episode 3, examples 3(a), 3(b) and 3(c) can be examined as a set. Here, each shows a mapping representation of a relation, with the task to determine which are functions. The task and representation is thus similar across examples. The examples differ in that example 3(a) and example 3(b) are rectangular maps, while 3(c) uses oval maps. In example 3(a) there are 2 inputs and 4 outputs, while in 3(b) there are 4 inputs and 4 outputs, and in example 3(c) there are 4 inputs and 3 outputs. 3(a) is not a function (one-to-many) while 3(b) is a function (one-to-one) and 3(c) is a function (many-to-one). The mapping representation allows it to be discerned that one input to many outputs breaks the function definition (3(a)), while, even if two inputs link to the same output it is a function (3(c)). Examples 3(d), 3(e) and 3(f) also work together as a set. As in the example set 3(a)-3(c), there are three similar representations (points on the Cartesian plane) with the task to determine if the examples are functions or not. This example provides a third way of discerning input-output relationships: now as dots on the Cartesian plane. Example 3(d) is not a function as two different points have the same input of 3. 3(e) is not a function because multiple different points have input 1 and input 5. Finally, 3(f) is a function although two different points share an output of 2. Example 3(f) combined with example 3(d) create contrast of which coordinate values may be repeated in a function: in 3(d) the points $(3; 1)$ and $(3; 4)$ break the function definition (thus a relation) while in 3(f) the points $(1; 2)$ and $(2; 2)$ do not break the function definition. Seen holistically, the examples 3(a)-3(c) and 3(d)-3(f) and 2(e) and 2(h) provide multiple opportunities for contrast of function and relation to be discerned. Finally, Example 3(h) provides a real-world example of determining a function from a scatterplot. Here, the function-relation relationship again becomes discernible in 3(h).2 where a coordinate is added that changes the function to become a relation. In Episode 3, the key aspect of functions that can be discerned is 'what is a function' versus 'what is not a function'.

In summary, in Teacher Y's Lesson 1, the key aspects of a function that can be discerned are the aspect of multiple representations of the same rule; the definition of a function (and its position as a subset of a relation); identifying a function across multiple representations; and the aspects of domain and range.

Lesson 2

In Lesson 2, Episode 1, examples 1(a) and 1(b) revise definitions of relations and functions from the previous lesson. Here, the position of a function as a subset of a relation is apparent. Examples 1(c), 1(e), 1(h), 1(j), 1(k) a 1(l) can be examined together as they were from the same homework worksheet. Here, 6 different graphs are provided for which the domain and range must be determined. It must also be determined if the graph is a function. Within this example set, 1(c), 1(e) and 1(j) are functions; whilst 1(h), 1(k) and 1(l) are not. 1(e), 1(j) and 1(k) have arrows at the end of the arms and continue infinitely, while graphs 1(c), 1(h) and 1(l) end at specific points.

Examples 1(j) and 1(k) are made of straight lines, while the other graphs have curved lines. Through these differences, the aspect 'function versus not a function' can be discerned in graph form, while, for domain and range, 'specific end-points' versus 'infinite domains and ranges' can be discerned as an aspect. Examples 1(d) and 1(g) occurred during the lesson from students' work. Here, the notation was refined in an iterative process as features of inequality notation were noticed and understood (however, the feature of curly brackets { } remained an issue throughout). Example 1(f) contrasts a line segment and a line. This creates contrast between the domain and range of 1(c) that have clear endpoints, and the domain and range of 1(e) that are infinite. Example 1(f) adds a further feature to the aspect of 'infinite domain'. In Episode 1, the key aspects that were discerned were thus: identifying a function from a graph; domains and ranges that are infinite versus having finite end points; and inequality notation.

In Episode 2, example 2(a) shows two graphs (one linear and one quadratic) that intersect at two points. Their equations are given, both in the form $y = \cdots$. Here, there is little similarity between the graphs. Contrast can be observed in the graphs' shapes and the degree of their equations. The linear graph also has a vertical shift while the parabola does not. Examples 2(b), 2(d) and 2(e) are notes on function notation. In example 2(d) it is shown that $y = 3x + 7$ becomes $f(x) = 3x + 7$. Here, the similarity of the $3x + 7$ allows the contrast between y and $f(x)$ to become apparent. This occurs similarly in example 2(e) where $y = x^2 + 4x + 1$ is written as $f(x) = x^2 + 4x + 1$. Again, the similarity of equations allows the contrast of y and $f(x)$ to be discerned. The difference between the equations in 2(d) and 2(e) show that function notation can be used in both linear and quadratic equations. Finally, in example 2(c) and 2(e), function notation is used to show evaluation of a function for a given input. In 2(c), $f(x) = x^2$ is used to find $f(1)$ and $f(3)$. These similar processes allow generalisation of procedure to become apparent. In 2(e), $f(3)$ is again found but now for $f(x) = x^2 + 4x + 1$. The similarity of input 3 in these two cases allows for the substitution into the two different graphs to be discerned. In Episode 2, the key features are thus the aspect of a function being able to be written in the form $f(x) = \cdots$ instead of $y = \cdots$. It can also be discerned how to evaluate a function for a given input-value.

In summary, in Lesson 2, the aspects of 'function versus not a function' in graph form were discernible, as were the aspects 'expressing domain and range in inequality notation'. Finally, ‘function notation’ was introduced and was shown to be a different way of writing an equation while proving useful to evaluate functions for a given input.

Examples: Enacted Affordances

Teacher Y’s Intended Objects of Learning

In the interview, Teacher Y indicated that he views ‘functions’ as a substantial object of learning that is covered throughout the FET phase and at university. He therefore values setting firm foundations for the object of learning for the boys in his class. In particular, he views the concepts of ‘what is a function’, ‘domain’, and ‘range’ as foundational for a deep understanding of this object of learning. He said that in his experience, students only have a vague understanding of these terms, seeing functions as ‘graphs’, and domain and range as ‘x-values’ and ‘y-values’ in general. He explained that, however, not all graphs are functions, and also that the domain is not all x-values but specifically those which belong to the particular function at hand (and similarly for range). To him, the school’s prescribed textbook does not develop this type of foundational knowledge but rather focuses on “memorisation of terms without understanding”. Although ‘identifying functions from relations’ goes beyond what is required of Grade 10s in the curriculum and is thus not assessed, Teacher Y said that he believes it is worth using the two introductory lessons to set firm foundations regarding the definition of function, and developing conceptual understanding of domain and range.

The Enactment of a Key Aspect from the Chosen Examples

In Teacher Y’s lessons, a key aspect that was emphasised was finding the domain and range of provided functions and relations. I now discuss how Teacher Y enacted a feature of domain by drawing on specific examples from the lessons. In this enactment, I have furthermore chosen examples that show how he used a *combination of instances* and *shifting focus of attention*, as this was typical of his practice. Additionally, I have selected an instance which shows two further characteristics of Teacher Y’s enactment of examples: creating new, spontaneous examples to aid the explanation of mathematical ideas, and having much interaction with the students. Details of his enactment of other examples are available in Appendix A.

In Lesson 1, Episode 2, the class was given the two following examples to solve:

List the domain and range of the relation:

$(2; 1); (-3; 4); (0; 5)$

Figure 14 **Teacher Y, Lesson 1, Example 2(e)**

a) List the domain and range of the relation.

b) Determine whether the relation is a function.

$\{(1; 3); (-2; 4); (3; -2); (-2; 7)\}$

Figure 15 **Teacher Y, Lesson 1, Example 2(h)**

In my discussion, I focus on the writing down of the domain of the two relations, with particular emphasis on example 2(h) (Figure 15). In Figure 14, the domain was $\{-3;\ 0;\ 2\}$. Here, the teacher had shifted the focus of attention to the feature of listing the domain in ascending order. In example 2(h), the students then for the first time encountered the same input (-2) being linked to two different outputs (4 and 7). Thus, contrast of listing domain values was encountered through a combination of instances. To the boys, the -2 created a conundrum: should it be listed twice in the domain or only once? Boy A asked: “If there are two of the same domain, must we list them both?” The teacher asked the class to respond. Boy B then said: “Yes, because it’s not the same, it may have the same value but the output is different.” Other members of the class did not seem convinced with this argument, and the teacher then told the class that the -2 would only be listed once, i.e. a domain of $\{-2; 1; 3\}$. However, a few boys were still unhappy that the -2 was only listed once. Teacher Y then drew spontaneous example 2(i) on the board (Figure 16), a number line with -2 indicated.

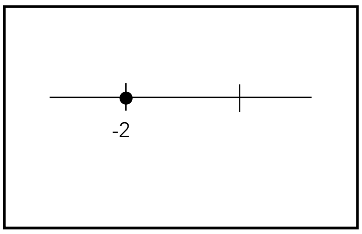


Figure 16 **Teacher Y, Lesson 1, Example 2(i)**

Teacher Y explained that each of the -2s in the domain would be indicated with the same dot on a number line and that the two dots would lie on top of each other. Thus, the value -2 is the same in both cases, and it should only be listed once in the domain. Boy B was still unhappy. He said: "each -2, sure I understand that it's the same value, and it's one domain, but then the domain has a different range, it has a different output. Therefore, why is it that you only write down one [-2] when the output of the two different ones [the two -2s] are going to be different?"

To this, Teacher Y responded with another spontaneous example referencing himself as a unique input. He said:

> "It is because, here, this is the same input [pointing at the two -2s], it is the same input. It's just one [Teacher Y][4] that you are inputting. So you don't have two [Teacher Ys], it's one [Teacher Y]. Although you are saying this [moving his body to the right] and that [moving his body to the left] but it is one [Teacher Y]...so we write it once."

Boy B, who was finding the listing of -2 only once difficult to understand then chuckled, and said confidently: "Yes, Sir, I get you".

This set of examples illustrates how Teacher Y engaged with the class's questions, allowed the students to interact with each other, and created spontaneous examples to aid explanations. Such interaction-heavy talk, and the creation of spontaneous examples (both written down and spoken) was typical of Teacher Y's practice across both lessons. His use of spontaneous examples, alongside his pre-planned examples, brings to mind Zaslavsky and Zodik's (2007) description of teachers needing to be able to flexibly engage with their own example set to create opportunities for learning for their students.

<u>When the Enactment goes Beyond what is Discernible in the Chosen Examples</u>

An aspect of function that Teacher Y enacted to a greater degree than was apparent in the innate affordances of the example space was the use of the formal definition of a function. The formal

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[4] Here, the teacher used his own surname, but this has been replaced with 'Teacher Y'. e.g., "It's just one McLachlan that you are inputting".

definition of a function was introduced in Lesson 1, Episode 2 (examples 2(a) and 2(c)) and revised in Lesson 2, Episode 1, example 1(b). However, beyond these examples (where the definition of a function was the aspect of focus), Teacher Y referred to the definition on multiple other occasions during the lesson.

For example, in Lesson 1, Episode 3, the following example was given (Figure 17):

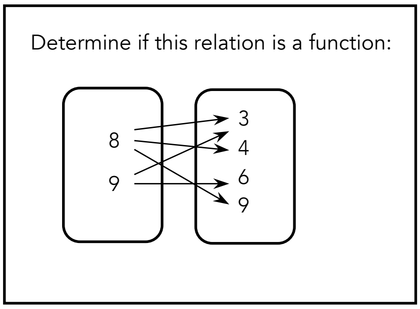


Figure 17 **Teacher Y, Lesson 1, Example 3(a)**

By this point in the lesson, the class was generally confident that this relation was not a function. When asked to justify their responses, Boy A said that it "… is not a function because 8 and 9 have more than one range", while Boy B explained that it was a relation because "…there are multiple ranges". Teacher Y was not happy with how these two boys expressed their reasoning. He responded by saying:

> "Let me advise you – whenever you define, you must define it in terms of the definition. It is there in your book also. Whenever you give a reason, don't just say 'it's one, more than one', because you will not get a mark… So always go back to the definition… We see here that this one input, this input 8, … has got 1, 2, 3 different outputs [pointing at 3, 4 and 9 as he says this]. So, according to the definition of a function, a relation is said to be a function if each and every input has got only one output. That is the definition of a function so for that reason, this is not".

Teacher Y thus emphasised the importance of the definition and referred to it when modelling how to answer questions about identifying functions. It was a key aspect of functions emphasised across both lessons, which is in line with his intended object of learning. Additionally, his use of a counter-example of 'function' is a means that Watson and Mason (2005) describe as affording his students the opportunity to develop their example space.

The Lived Object of Learning

This study did not gather data directly from students to investigate the lived object of learning. Student responses during the lesson were, however, documented and sometimes indicated the lived object of learning. Throughout Teacher Y's lessons, there was a high level of student interaction which indicated whether individual students were understanding the work or not. In Lesson 1, Episode 3, the following question was provided (Figure 18, example 3(c)):

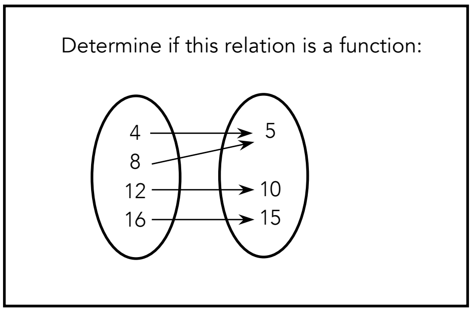


Figure 18 **Teacher Y, Lesson 1, Example 3(c)**

This was the first many-to-one function that the students encountered. During the lesson, this example was initially incorrectly classifies by Teacher Y as <u>not</u> being a function. A short while later, the students were then discussing example 3(f) (Figure 19) which is also a many-to-one function:

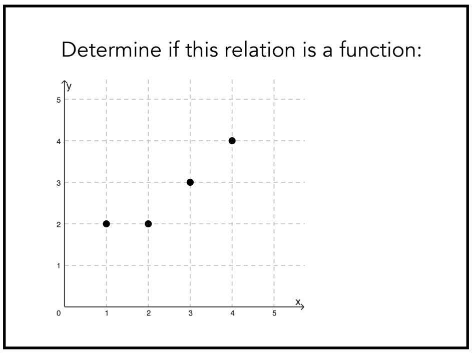


Figure 19 **Teacher Y, Lesson 1, Example 3(f)**

Here, Boy C expressed that this is not a function because points (1; 2) and (2; 2) share the same output on the y-axis. Boy C's solution and reasoning were incorrect; however, it convinced Boy B. Boy B, who had initially said that this was a function, then exclaimed: "He's right, he's correct!... because the output is shared by two inputs!". At this point, the teacher intervened, stating that the boys were not getting the definition right. He explained that multiple inputs can share the same output, as long as each input has only one output. After hearing this, Boy B then burst out: "So then, Sir, we were wrong for [example 3(c)] as well!"

To explain many-to-one functions, the teacher first created spontaneous example 3(g) and then went back to example 3(c) (Figure 18) and corrected the mistake, explaining that 3(c) is, indeed, a function. This conversation made clear that these two boys in the class were still progressing in their understanding. However, Boy B's identification of the mistake indicated that he was making progress in understanding the definition of a function. Teacher Y mentioned this interaction specifically in the interview as an interaction he thought showed development in the boys' understanding of what a function is.

Examples: Use of Multiple Representations

Table 8 documents Teacher Y's use of the multiple representations of functions used within an episode in column 2, and within individual examples per episode in column 3.

Table 8 **Teacher Y Use of Multiple Representations**

Lesson 1 Episode	**Multiple Representations per episode**	**Combination of Representations per example**
1	Set notation, number pattern, flow diagram	1(a) set notation 1(b) set notation 1(c) number pattern 1(d) number pattern 1(e) flow diagram
2	Coordinate pair, equation, set of ordered pairs, set notation, graph	2(a) coordinate pair 2(b) – 2(c) – 2(d) equation 2(e) set of ordered pairs; set notation 2(f) – 2(g) graph 2(h) set of ordered pairs; set notation 2(i) – 2(j) – 2(k) – 2(l) –
3	Mapping, graph	3(a) mapping 3(b) mapping 3(c) mapping 3(d) graph 3(e) graph 3(f) graph 3(g) mapping 3(h) graph
Lesson 2 Episode	**Multiple Representations per episode**	**Combination of Representations per example**
1	Graph, coordinate pairs	1(a) – 1(b) – 1(c) graph, coordinate pairs 1(d) – 1(e) graph, coordinate pairs 1(f) –

		1(g) – 1(h) graph, coordinate pairs 1(i) – 1(j) graph, coordinate pairs 1(k) graph, coordinate pairs 1(l) graph, coordinate pairs
2	Graph, equation, function notation	2(a) graph, equation, function notation 2(b) – 2(c) equation, function notation 2(d) function notation 2(e) equation, function notation

Teacher Y's lessons incorporated various features of function representations: set notation, number pattern, flow diagram, coordinate pair, equation, set of ordered pairs, graph, mapping, and function notation. Lesson 1, Episode 1, to the greatest extent highlighted that the same function can be represented in different ways, and that these different representations can show the same function. Here, the teacher moved from the representation set notation (with a sequence of numbers) to the corresponding number pattern, and then to the corresponding flow diagram to begin to develop the object of learning of a 'relation' (from which the object of learning 'function' was then derived). The ordered set in set notation was used to shift the focus of attention to an 'ordered sequence of numbers', the number pattern rule to shift the focus of attention to how 'inputs create outputs', and the flow diagram was used to show that 'the rule creates a relationship between input and output'. Here, the teacher was manipulating the affordances of the different representations of functions and relations (which allow particular aspects to be more transparent or more opaque) to highlight the desired object of learning (Nitsch et al., 2015; Zaslavsky & Lavie, 2005).

In the rest of the lesson, the different representations were generally used as different starting points to classify functions and/or find the domain and range and were not overtly linked to each other. Considering individual examples, most included one or two representations of functions, and one example, 2(a), included three representations. In these two lessons, the aspect of 'functions of having multiple possible representations' was thus present, but the connections between the different representations was not overtly highlighted.

Final Remarks on Teacher Y

Teacher Y's two lessons included 41 examples on functions that covered: rules and sequences; the definitions of relation, function, domain and range; identifying functions and relations;

identifying domain and range, and function notation. His examples space was mediated using the teacher actions *combination of instances* and *shifting focus of attention*, which he combined with the creation of multiple spontaneous examples to aid his explanations. His lessons were furthermore characterised by lively interaction with his students. Finally, in his two lessons, the multiple representations of functions were present, but this aspect of functions occurred implicitly rather than being a focus of his attention.

Teacher Z

Teacher Z used 11 examples in his first lesson (across 3 episodes), and 17 examples in his second lesson (across 2 episodes). In both lessons, he showed patterns of working at the board to explain a new concept, and then allowed the students some time to work independently on an example set, before going through the solutions together on the board. During these periods, he walked around and helped students individually, and occasionally addressed common questions and/or misconceptions on the board for everyone to see. These periods of individual work resulted in fewer examples being included in Teacher Z's lessons than were included in the lessons of Teacher X and Teacher Y.

Objects of Learning: Summary

Table 9 documents, per episode, the objects of learning that occurred in Teacher Z's lessons. The objects of learning are described horizontally in increasing detail under 'Content', 'Educational Objectives', and 'Aspects and Features', as outlined by Marton (2015). In this table, the examples' innate affordances for learning, and the affordances created through mediation and classroom interaction, are considered.

Table 9 **Teacher Z Objects of Learning**

Lesson 1 Episode	Content	Educational Objectives	Aspects and Features
1	- Functions and rules - Inputs and outputs	- understand a function as a rule - find outputs from given inputs - see a flow diagram and a table as different representations of the same rule/function	- aspect: function features: a rule; a flow diagram; an input-output table; an equation; related to number patterns; a number-crunching machine - aspect: flow diagram features: a function; creates outputs

			from inputs; similar to a table - aspect: table features: a function; creates outputs from inputs; similar to a flow diagram
2	Function notation	- understand function notation as a change of notation - understand how input and output relate to function notation - learn how to use function notation (evaluate an output from a given input).	- aspect: function notation features: $f(x)$; $g(x)$; a new way of expressing an equation; a representation of input and output; an instruction to evaluate the output from a given input
3	Sketching linear functions using a table	- using the technique of an input-output table to sketch linear functions - understanding the necessary accuracy of a sketch graph - using different representations to translate a graph equation to a graph sketch (equation → table → coordinate pairs → graph)	- aspect: function new features: relationship between an equation and its graph - aspect: function notation new features: can be used to label graphs - aspect: linear graph features: gradient; y-intercept; $y = mx + c$ - aspect: gradient features: steep (above 1); less steep (below 1) - aspect: representations of a function features: equation; table; coordinate pair; graph; function notation - aspect: an 'accurate' sketch graph features: regular axis increments; irregular axis increments - aspect: sketching graphs features: placing appropriate x-values and y-values on the axes; coordinate values may be decimals; knowing when a point can be excluded from a sketch
Lesson 2 Episode	**Content**	**Educational Objective(s)**	**Aspects & Features**
1	Parabola functions	- know the standard form equation of a parabola $y = ax^2 + q$ - understand the meaning of parameters a and q in the equation of a parabola - know the most basic parabola $y = x^2$ - know of parabola shapes in the real world - sketch a parabola using a table - know of the symmetric nature of parabola graphs and how this links to output values around the origin	- aspect: parabola features: $y = ax^2 + q$; has a U-shape; is symmetrical about the y-axis; basic equation is $y = x^2$ - aspect: a-value features: positive $a \rightarrow \cup$ (smiley face); negative $a \rightarrow \cap$ (frowny face) - aspect: q-value features: y-intercept; could be represented as c in Grade 10 - aspect: parabolas in the real world features: *McDonald's* M; throwing a board marker; throwing a basketball - aspect: sketching a parabola features: use a table, substitute negative inputs in brackets; even number spacing on the axes; plot coordinate pairs and connect; have a round shape not a sharp point at the turning point; arrows at the end of the arms

2	Parabola Sketch Graphs	- know the steps to draw a parabola without a Table - understand x-intercepts - solve a quadratic equation (to find x-intercepts) - understand the effects of parameters a and q in the equation of a parabola on a sketch graph - use online graphing tool *Desmos* - become aware of standard form of a parabola equation	- aspect: a-value new features: increasing positive a-values move the graph arms closer together; decreasing positive a-values move graph arms apart; $a = 0$ is a straight line not a parabola; negative a-values moving away from 0 move the arms closer together, negative a-values moving towards 0 move the arms further apart - aspect: q-value new features: can affect the number of x-intercepts - aspect: x-intercepts new features: where the graph cuts the x-axis; a parabola can have one or two; $y = 0$ always - aspect: y-intercept features: a parabola will have one, cannot have two - aspect: sketch graphs features: find and label key points clearly; get shape correct; less accuracy acceptable - aspect: standard form features: $y = ax^2 + q$ not $y - q = ax^2$

In Lesson 1, Teacher Z provided an informal definition of a function which he linked to a flow diagram. From there, he then expressed the flow diagram as an equation and a table. Next, he introduced function notation and provided some examples for the students to practice. Finally, he covered sketching linear functions using a table. In Lesson 2, Teacher Z focused on the parabola. He introduced the standard form $(y = ax^2 + q)$ and explained the effects of parameters a and q. He discussed some real-world examples of parabolas and then moved on to sketching a parabola using a table. This process lead to a discussion of the symmetry of a parabola graph. Finally, Teacher Z moved on to creating sketch graphs of parabolas using only their important features: x-intercept(s), y-intercept, turning point and shape. Finally, Teacher Z showed the class an online graphing tool to investigate parabolas digitally.

Example Space Used in the Introduction of Functions

All of the examples that Teacher Z used in his two lessons are documented in Table 10 and Table 11. As with the previous teachers, the examples are arranged horizontally in the sequence in which they occurred in each lesson, separated by episodes. I subsequently engage in an examination of the innate opportunities for learning that were present in the example sets. Thereafter, I consider the affordances for learning created during the mediation of the examples in the classroom. When discussing the enacted affordances, I also draw on the interview with

Teacher Z regarding his intended objects of learning. A table combining the Objects of Learning (as presented in Table 9) and the examples themselves (as presented in Table 10 and Table 11), as well as descriptions of Teacher Z's mediation of the examples, is available in Appendix A.

Table 10 **Teacher Z Lesson 1 Examples**

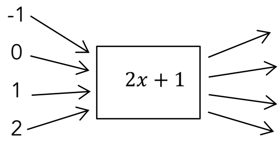
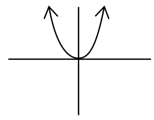
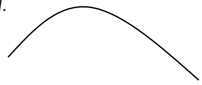
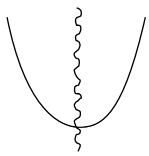
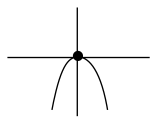
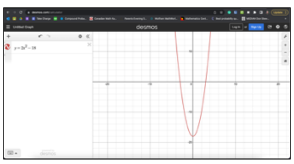
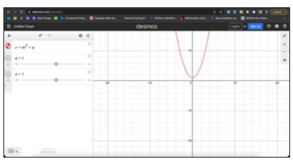
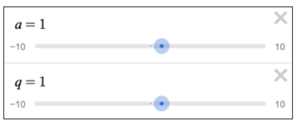
Episode				
1	1(a)- *verbal definition by the teacher* "A function is basically a rule – almost like a spider diagram"	1(b) Complete the flow diagram: 	1(c) Complete the table: $y = 2x + 1$ x | -1 | 0 | 1 | 2 y | | | |	
2	2(a) – *a notation change* $y = 2x + 1$ $\updownarrow$ $f(x) = 2x + 1$	2(b)- *a verbal example by the teacher* [New] notation is something that can trip you up but it shouldn't because it's not something new, it just looks new. **It's like when they put a new paint job on something. It's still the same thing.**	2(c)- *function notation* $f(0) = 1$ $\uparrow \quad \uparrow$ input output	2(d) Evaluate: $f(x) = 2x + 1$ & $g(x) = 5x$ 1) $f(3)$ 2) $g(2)$ 3) $f(g(2))$
3	3(a) Sketch: $f(x) = 2x + 1$	3(b) Sketch: $g(x) = 3x + 1$ Create a new table but draw your graph on the same Cartesian plane as where you sketched $f(x)$.	3(c) Sketch: $h(x) = \frac{1}{2}x + 1$ Create a new table but draw your graph on the same Cartesian plane as where you sketched $f(x)$.	3(d) – verbal analogy "…when you go on the hike [on Grade 10 camp], there are going to be some gradients that are steeper than others. When you are walking uphill but it's not terrible, that's a gradient of ½ or ¼ or something. When the gradient is bigger than 1, it's quite a steep gradient. When the gradient is less than 1, it's quite flat."

Table 11 **Teacher Z Lesson 2 Examples**

Episode				
1	1(a) <u>Parabola</u> <u>Equation:</u> $y = ax^2 + q$ <u>a-value</u> $a = +$  $a = -$  <u>q-value</u> y-int of graph	1(b)– *real world example* If the a-value is a positive number, then the graph has this happy face look to it…and if the a-value is a minus, then the graph has got this unhappy look to it.	1(c) <u>basic</u> $y = x^2$ 	1(d)– *real world example* Things, like you know, the *McDonalds* M, that's actually a parabola shape.
	1(e)– *real world example* *Teacher Z throws the board marker through the air to show the parabolic shape of the arc marker's trajectory and draws the shape on the board.* 	1(f)– *real world example* "When you throw basketballs, they get modelled by a parabola equation"	1(g) Sketch, using a table: $y = x^2 - 4$	1(h) – *substituting -3* $y = x^2 - 4$ $y = -3^2 - 4$ $y = (-3)^2 - 4$
	1(i) – *real world example* "Symmetry…do you remember like a butterfly and symmetry, what happens on the one side of the butterfly looks the same as the other side."	1(j) – sketch of parabola symmetry 		
2	2(a) <u>How to sketch</u> 1. x-int ($y = 0$) 2. y-int (q-value) 3. check a-value 4. Draw	2(b) Sketch, using key features: $y = -x^2 + 9$	2(c)- *a parabola with one x-intercept* 	2(d) Sketch, using key features: 1. $y = 2x^2 - 18$ 2. $y = -x^2 + 1$ 3. $y = 3x^2 - 3$
	2(e)- *Desmos online tool*  Equation: $y = -x^2 + 9$	2(f) - *Desmos online tool*   Equation: $y = ax^2 + q$ Sliders[5]: a-value; q-value	2(g) Sketch, using key features: $y - 25 = -x^2$	

[5] 'Sliders' are tools where the value of a variable in the graph equation can be changed by dragging a marker along a number line. The effect of the changing variable is shown in real time on the graph. The slider tools for the a-value and q-value are shown.

Examples: Innate Affordances

Lesson 1

In Lesson 1, Episode 1, an informal definition of a function is provided in example 1(a). This definition describes a function as a rule, and similar to a flow diagram. Example 1(b) then contains a flow diagram. This allows the definition to be immediately 'enacted'. The rule of the flow diagram is $2x + 1$, and the provided inputs are $-1; 0; 1; 2$. In example 1(c) the same rule is provided as an equation in the form $y = \cdots$ with a table showing inputs x and y. The inputs of the table are the same as the inputs for the flow diagram. The similarity of rule between the two representations allows the representation (flow diagram; equation with table) to be contrasted. At the same time, the similarity of inputs and outputs across both rules shows that regardless of representation, the rule produces the same inputs and outputs. In Episode 1, the key aspects that are highlighted are thus that a function is a rule, and that this rule can be represented in different representations while creating the same input-output sets.

In Episode 2, the equation from 1(c) is then represented using function notation in 2(a). Here, the similarity of rule $(2x + 1)$ allows the contrast of y and $f(x)$ to be discerned. However, as the y and $f(x)$ are located in the same position in the equation, and connected with a bidirectional arrow, their similar meaning is also discernible. This similar meaning is highlighted by the subsequent verbal analogy in 2(b). This explanation describes new notation as "a new paint job on something" which makes something look new although it is not new. In example 2(c), the input-output pair of $(0; 1)$ (generated previously in both 1(b) and 1(c) from the rule $y = 2x + 1$ as part of the flow diagram and table respectively) in expressed in function notation. This is now the third type of representation linked to the equation $y = 2x + 1$. In 2(c) for $f(0) = 1$, it is furthermore made clear with arrows and labels that the 0 is the input value while the 1 is the output value. Finally, in example 2(d), two functions are given with which to practise function notation. Again, the one function is $f(x) = 2x + 1$ while $g(x) = 5x$ has been added. This combination of these two equations now creates contrast between features $f(x)$ and $g(x)$, thus making this aspect of function notation discernible. $f(x)$ also has a c-value while $g(x)$ does not, which allows this aspect of a linear rule to be discerned. In the exercises of 2(d), the task is to evaluate $f(3)$, then $g(2)$, and finally $f(g(2))$. The input-values of 3 and 2 are simple, thus allowing for straightforward substitution and evaluation (without having to work with other potential aspects such as integers or fractions). This simplicity allows the procedure of 'evaluating a function using function notation' to be clear. Contrastingly, exercise 3, $f(g(2))$, is more complex. However, as the inner function $g(2)$ has already been found, this more complex procedure becomes more accessible. In this final exercise, it is emphasised that $g(2)$'s value of 10 can become the input of another function. In Episode 2, the key aspects that become discernible are thus 'function notation' and 'evaluating functions using function notation'. It is also made clear that function notation is just a different type of representation for a rule.

In Episode 3, there is a change of focus from function notation to sketching. Now, in example 3(a), the instruction is to sketch the graph of $f(x) = 2x + 1$. This now adds yet another representation to the rule of $y = 2x + 1$ that has featured in all three episodes of the lesson. In example 3(b), the instruction is to sketch $g(x) = 3x + 1$ on the same set of axes as 3(a), and in 3(c) to sketch a final graph of $h(x) = \frac{1}{2}x + 1$ on the same set of axes. From a rule perspective, these three equations are all linear and have the same c-value of 1. This allows contrast of function notation (features $f(x)$, $g(x)$ and $h(x)$) to be discerned, as well as the different m-values of 2, 3 and $\frac{1}{2}$ (which are all positive). From a sketch perspective, it will become clear that all graphs are straight lines and cut the y-axis at the same point of 1. The different steepness of the three different positive gradients will also become discernible. Thus, contrast of gradient will be visible due to the different steepness of the graphs, as will generalisation of the y-intercept of 1 , which will remain constant despite the change of gradients. In example 3(d), the steepness of gradients is highlighted with a real-world analogy to hiking up a mountain. This might aid the contrast of gradient created by the sketches as it is being linked to an imaginable, real-world situation. In Episode 3, the main aspect emphasised is thus 'sketching linear graphs', with a focus on the feature of gradient.

In summary, in Lesson 1, functions were positioned as rules and were shown to have multiple possible representations. The multiple representations were made discernible by keeping one linear rule constant throughout all episodes (although it is contrasted with other linear rules at different points). The aspect of function notation was also introduced, and the representation of 'graph sketch' was investigated, with a focus on the feature of gradient.

Lesson 2

In Lesson 2, Episode 1, the parabola is the aspect of attention. In 1(a) the equation is given as $y = ax^2 + q$ and the features of a and q are highlighted. Contrast is created by comparing what the graph looks like when a is positive or negative. In example 1(b), a verbal analogy is provided that describes the a-value as having the shape of a smile when $a > 0$ and the shape of a frown when $a < 0$. This creates a more accessible way of thinking about this feature than just from the theoretical description in 1(a). In example 1(c), the basic parabola of $y = x^2$ is given with a sketch. Here, the content of 1(a) and 1(b) is shown in graphical form for the first time, as a specific example of the general equation $y = ax^2 + q$. In examples 1(d), 1(e) and 1(f), three real-world examples are discussed to help communicate the shape of a parabola: the *McDonald*'s M, the actual physical throwing of a board marker across the classroom (paired with a rough sketch of the shape on the board), and the description of throwing a basketball. These are all examples that are applicable within the students' everyday lives and can be imagined. Additionally, all these three examples have negative a-values, which creates contrast to the graph of $y = x^2$ in 1(c). In 1(g), the graph of $y = x^2 - 4$ is sketched using a table. This is the same method used to sketch linear graphs in Lesson 1, and the method of 'using a table' is thus generalised by being used to sketch functions of different degree. This graph has the same a-value as the basic parabola of example 1(c) but with a q-value of -4. These two examples thus create contrast of q-value. Additionally, this sketch will have two distinct x-intercepts – in contrast to the basic parabola in 1(c) – and thus this feature will be opened up for discernment. In example 1(h) the equation from 1(g) is used and it is shown how -3 should be substituted into the quadratic function's x^2 term. This is done by contrasting a counter-example (the incorrect method) with how it should be done correctly with brackets. Here, this aspect of creating outputs for a quadratic function is discerned. Finally, examples 1(i) and 1(j) deal with symmetry. First, another verbal analogy is used that references a butterfly's symmetry. This is paired with a rough sketch of a parabola and its axis of symmetry in 1(j). Both were spontaneous examples. These two examples create generalisation by giving two examples that both have symmetric shapes. In Episode 1, the parabola graph is the key aspect that is opened up for investigation. Its features of shape, a-value, q-value and symmetry are opened up. Additionally, the skills of sketching and substituting negative values into the function are highlighted.

In Episode 2, example 2(a), the method of sketching a parabola using key features is presented. This creates contrast to the method of 'sketching by using a table' used in Episode 1. In example 2(b), the graph of $y = -x^2 + 9$ is sketched using the method of 2(a), thus creating an example of the theory in 2(a). In example 2(c) a parabola with one x-intercept is presented (this was a spontaneous example). This graph has a negative a-value like the graph in 2(b), which creates contrast of q-value ($q = 9$ versus $q = 0$). Additionally, contrast is created with the graph of 2(b), which has two x-intercepts. In Episode 1, example 1(g), the sketched graph also had two x-intercepts. Thus, the feature of x-intercepts can be more clearly discerned. In example 2(d), three parabolas are sketched using the strategy of example 2(a). These examples now show fusion, as multiple combinations of positive and negative a and q-values are used in these graphs ($y = 2x^2 - 18; y = -x^2 + 1; y = 3x^2 - 3$). Additionally, these more 'complex' values will furthermore affect the difficulty of solving the quadratic equations to find the x-intercepts. Thus, the level of fusion of this example set 'increases' as more complex aspects of quadratic equations now need to be drawn upon, in addition to the a and q-values. Example 2(g) is also a sketch question, for the function $y - 25 = -x^2$. Here, the form of the equation becomes discernible, as in all previous examples the equation was presented in the form $y = ax^2 + q$. Finally, in examples 2(e) and 2(f), digital graphing software is used. First, in 2(e), the same graph as was sketched in example 2(b) is drawn digitally, creating contrast to the hand-drawn version. Then, in example 2(f), the equation $y = ax^2 + q$ is plotted with sliders that manipulate the graph immediately as the a-value and q-value are changed on the slider number-line. Here, the features of a and q are thus presented in a more dynamic manner than is possible in a static, hand-drawn sketch. Thus, in Episode 2, 'sketching parabolas using key features' in the main aspect that is highlighted. In doing so, the features of a, q and x-intercepts become increasingly discernible.

In Lesson 2, the parabola function is thus the feature of functions that is opened up through the exploration of a variety of its features (a-value; q-value; shape, x-intercepts; symmetry). Additionally, this graph is sketched using three methods: table, by key features, and by using a digital graphing tool.

Examples: Enacted Affordances

Teacher Z's Intended Objects of Learning

Teacher Z mentioned various intended objects of learning for his lessons. In Lesson 1, he wanted to focus on linear functions. Regarding 'functions', he wanted to use a linear function to bring attention to the structure of input and output and help the students realise that this is something they already know from prior knowledge (e.g. flow diagrams and tables). In this process, Teacher Z also wanted to show the different representations of functions. Finally, he wanted to introduce function notation and then investigate the shape of linear function graphs with a focus on the feature of gradient. For Lesson 2, he worked with parabolas. Here, he deliberately started with simple notes and the basic graph of $y = x^2$ to use as a reference example. Later in the lesson, when he moved to sketch graphs using key features, he provided 4 procedural steps. In this case, there was an emotional aspect behind the choice of this object of learning: Teacher Z said that he normally gives students brief procedural steps when this is possible because, in his experience, he has seen that it makes students "feel safe". He indicated further that he does not really like procedural steps as he rather wants his students "to have a good understanding but … if they feel a bit safer [with the procedural steps, he is] ok with that". Finally, regarding his general selection and sequencing of examples in lessons, Teacher Z indicated that although he has a clear goal for his lessons, he is always highly adaptive in his example choice, explaining: "[a]s I view responses and I see where [the students'] understanding is then I realise, ok, we should do an example like this so that they can learn whatever they need to learn". This adaptive manner of selecting examples can be interpreted as Teacher Z trying to determine the students' existing example spaces, and to then to provide examples to develop these towards being able to discern the desired object of learning (Watson & Mason, 2005).

The Enactment of a Key Aspect from the Chosen Examples

A characteristic of Teacher Z's practice was that his individual lessons had a sense of continuity. For example, in Lesson 1, he used the same linear function $y = 2x + 1$ within every episode of the lesson within different objects of learning. In Lesson 2, the aspect of a parabola's 'shape' occurred throughout the lesson in different ways. In his teaching of the parabola's shape, teacher Z used the three teacher actions of *combination of instances*, *shifting focus of attention* and *differences as changes*. I will now draw on various examples from Lesson 2 in order to discuss his mediation of this aspect. Details of Teacher Z's enactment of other examples are available in Appendix A.

In the first example in Lesson 2, 1(a) (Figure 20), Teacher Z shifted the focus of attention to 'shape' being an aspect of a parabola. In this example, the parameters a (and q) were introduced.

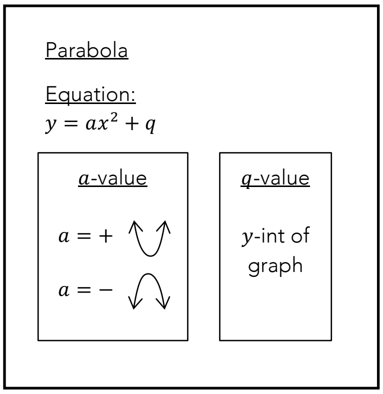


Figure 20 **Teacher Z, Lesson 2, Example 1(a)**

When mediating this example, Teacher Z said:

> "In that equation [pointing at $y = ax^2 + q$], the number a...and the number q are normally numbers. The x and y are always in an equation. Like with the straight line which you saw yesterday, the x and the y are always part of the equation and then the other parts mean something. Like in the straight line, the gradient is the m-value and it means something. But in this case the a-value and the q-value both mean something specific.

Here, the teacher is creating a combination of instances by drawing on a key aspect of the previous lesson (gradient and parameter m) to help facilitate the understanding of a parabola's new parameters a and q. Here, the teacher is creating a possibility of generalisation that parameters effect a graph's shape. Additionally, the teacher is creating an opportunity for students to develop and connect their own example spaces regarding straight line graphs and parabolas.

In the next example, 1(b) Teacher Z dealt more specifically with the effect that the a-value has on the graph. In this verbal analogy he said: "If the a-value is a positive number, then the graph has this happy face look to it...and if the a-value is a minus, then the graph has got this unhappy look to it". Here, he drew on a real-world analogy to help understand the graph's two possible shapes.

A bit later in the lesson, Teacher Z mentioned three further real word examples to help his students to visualise a parabola's shape. In examples 1(d), 1(e) and 1(f), he mentions the *McDonald*'s M and throwing a basketball, and also physically throws a board marker across the classroom to show a parabolic shape (which he sketches briefly on the board). Here, generalisation of shape becomes possible through this combination of instances.

The aspect of shape was also embedded in the exercises to sketch different parabola graphs in the lesson, and in the discussion of the function's aspect of symmetry. However, it was again overtly discussed towards the end of the lesson when Teacher Z used the teacher action 'differences as changes' by introducing a digital graphing calculator. Here, in example 2(f) he used the website desmos.com to plot the graph of $y = ax^2 + q$, and introduced sliders for a and q. It is difficult to capture the dynamic motion of the graph as the sliders were changed, but I attempt to describe this as best possible.

First, Teacher Z deliberately set $a = 1$ and $q = 0$ (Figure 22) and from there, changed the q-value up and down (while keeping $a = 1$). This allowed for the effect of the q-value to be contrasted and the a-value to be generalised. He specifically highlighted $q = 7$ (Figure 21) and $q = -1$ (Figure 23) as specific examples of a positive and negative q-value. Teacher Z then shifted the focus of attention to how the changing q-value resulted in a changing y-intercept and how it looked like the graph was being moved up and down while the shape remained the same.

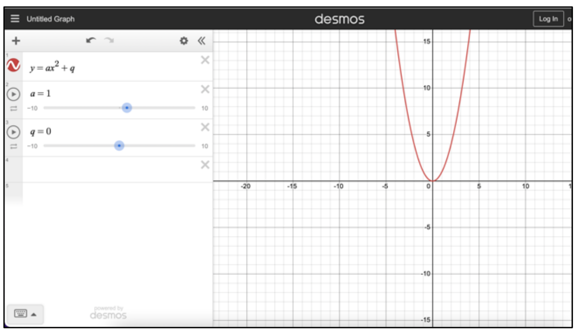


Figure 21 **Teacher Z, Lesson 2, Example 2(f) still 1**

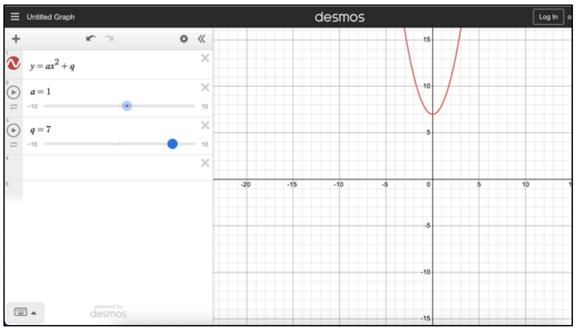


Figure 22 **Teacher Z, Lesson 2, Example 2(f) still 2**

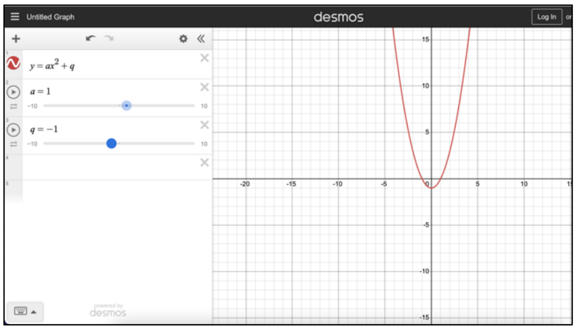


Figure 23 **Teacher Z, Lesson 2, Example 2(f) still 3**

Next, he set $q = -1$ and started to changing the a-value. Here, he first focused on positive a-values, showing that if the a-value increases, the graph's arms move closer together (Figure 24) while as the a-value decreases, the arms move further apart (Figure 25). Next, he moved to $a = 0$ which created a horizontal line (Figure 26). Teacher Z discussed this and described that "$a = 0$ kills the x^2, so it [the line] becomes $y = 0$ and it's a horizontal line". Finally, Teacher Z moved on to negative a-values and again showed that as the values become bigger negative numbers (moving away from 0), the arms move closer together (Figure 27) but as the a-values become smaller negative values (moving towards 0), the graph's arms move further apart (Figure 28).

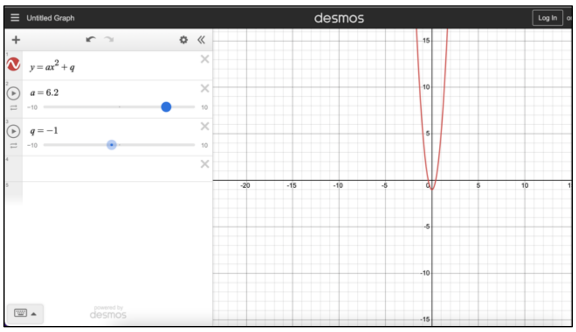


Figure 24 **Teacher Z, Lesson 2, Example 2(f) still 4**

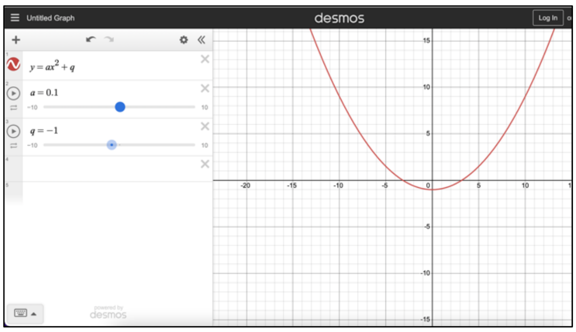


Figure 25 **Teacher Z, Lesson 2, Example 2(f) still 5**

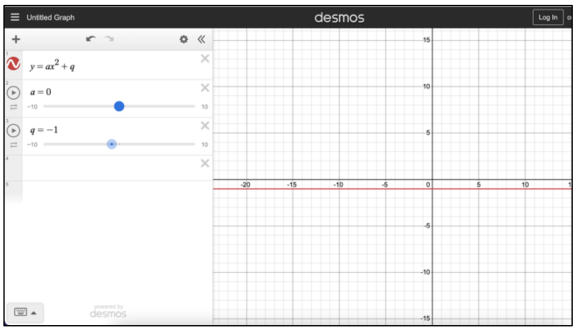


Figure 26 **Teacher Z, Lesson 2, Example 2(f) still 6**

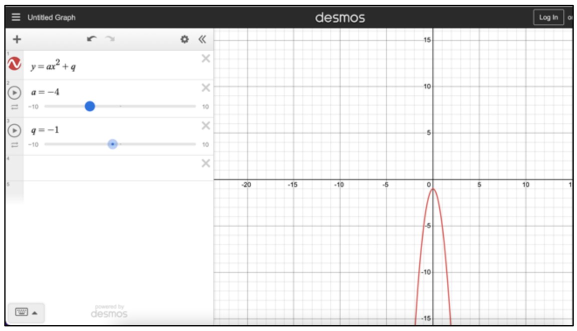


Figure 27 **Teacher Z, Lesson 2, Example 2(f) still 7**

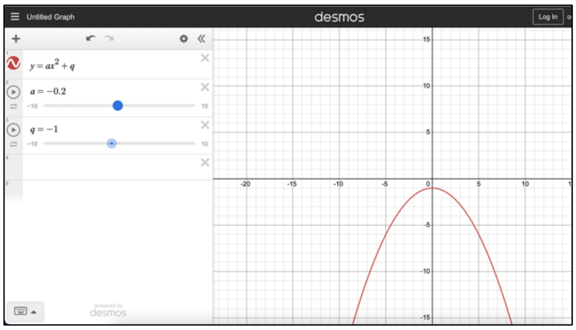


Figure 28 **Teacher Z, Lesson 2, Example 2(f) still 8**

Thus, throughout the lesson, Teacher Z worked with the graph's shape in different ways and in a manner that allowed for the effects of parameters a (and q) to be discerned. Teacher Z's use of dynamic graphic software additionally created an affordance for seeing an algebraic function and its graph as connected objects of learning which define and construct the object of learning of 'function' together (Leinhardt et al., 1990).

<u>When the Enactment goes Beyond what is Discernible in the Chosen Examples</u>

The boys in Teacher Z's class asked a number of questions throughout the lessons. In Lesson 2, Episode 2, example 2(b) the graph of $y = -x^2 + 9$ was sketched. This example had two x-intercepts and prompted Boy A to ask whether it is possible for a parabola to have only one x-intercept. In response, Teacher Z sketched spontaneous example 2(c) on the board (Figure 29).

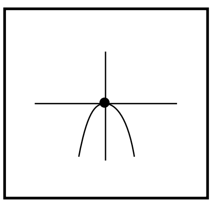


Figure 29 **Teacher Z, Lesson 2, Example 2(c)**

This example satisfied Boy A regarding how a parabola can have one x-intercept, but prompted Boy B to ask the following question: "Sir, can you not also have two y-intercepts?... Because your parabola can be reflected to have two y-intercepts…". Here, it appears that Boy B is referencing the inverse graph of a parabola. Teacher Z then explained that this is not a parabola and that a parabola cannot have two y-intercepts.

In this interaction, Goldenberg and Mason's (2008) description of "appreciating dimensions of (possible) variation and ranges of permissible change" (p.190) comes to mind . Through this interaction, it has been made clear that, yes, a parabola may have one or two x-intercepts and remain a parabola. However, a parabola may only have one y-intercept. If it is a graph with two y-intercepts, it is no longer a parabola (although still a possible graph). Having two y-intercepts causes the reflected parabola-shape to lose 'examplehood' of being a parabola (Goldenberg and Mason, 2008). Thus, from the student interaction, an affordance for learning was created that went beyond what was perceivable as an affordance for learning in the innate example space.

The Lived Object of Learning

In this analysis, students were not interviewed regarding the lived object of learning. Any observations regarding the lived object of learning came about through student contributions during the lessons. In one such interaction, a boy's misconception became clear. In Lesson 1, Episode 2, example 2(d), the class had been given $f(x) = 2x + 1$ and were asked to find $f(3)$. To this, a boy confidently stated that the solution was 1 (although the correct solution is 7). On interrogating the student response, Teacher Z realised that this boy had misunderstood the features of 'input' and 'output' positions in function notation: he was solving the question $f(x) = 3$, to which the answer is indeed $x = 1$. Through the lens of Marton's (2015) work, here the aspects 'input' and 'output' are beginning to become discerned but have not yet been fully discerned and linked to prior knowledge. Additionally, this interaction brings to mind that certain representations of functions (e.g. flow diagrams) are inherently more transparent regarding which value is the input and which value is the output than other representations (here, function notation) (Zaslavsky & Lavie, 2005). However, this student interaction allowed Teacher Z to discern the student's lived object of learning (a misconception) and address it, by shifting the focus of attention to how input and output relate to function notation. By the end of the interaction, the student indicated that he understood why his initial solution was incorrect and that the correct solution was 7.

Examples: Use of Multiple Representations

Table 12 documents Teacher Z's use of the multiple representations of functions. In column 2, it is indicated which representations occurred during each episode, while in column 3, it is given which representations were used in individual examples.

Table 12 **Teacher Z Use of Multiple Representations**

Lesson 1 Episode	**Multiple Representations per episode**	**Combination of Representations per example**
1	Flow diagram, table, equation, coordinate pair	1(a) – 1(b) flow diagram 1(c) table, equation, coordinate pair
2	Equation, function notation	2(a) equation, function notation 2(b) – 2(c) function notation 2(d) function notation
3	Equation, function notation, table, coordinate pairs, graph	3(a) equation, function notation, table, coordinate pairs, graph 3(b) equation, function notation, table, coordinate pairs, graph 3(c) equation, function notation, table, coordinate pairs, graph 3(d) –
Lesson 2 Episode	**Multiple Representations per episode**	**Combination of Representations per example**
1	Equation, graph, table, coordinate pairs	1(a) equation 1(b) – 1(c) equation, graph 1(d) – 1(e) – 1(f) – 1(g) equation, table, coordinate pairs, graph 1(h) equation 1(i) – 1(j) graph
2	Equation, coordinate pairs, graph	2(a) – 2(b) equation, coordinate pairs, graph 2(c) graph 2(d) equation, coordinate pairs, graph 2(e) equation, graph 2(f) equation, graph 2(g) equation, coordinate pairs, graph

In the table, it can be seen that Teacher Z frequently used more than one representation in individual examples. There are even three examples (Lesson 1, 3(a), 3(b), 3(c)) where five representations were used within the same exercise (to sketch a graph). The representations that Teacher Z used were: flow diagram, table, equation, function notation, coordinate pair, and graph. However, what particularly stood out about Teacher Z's use of the multiple representations was that in Lesson 1, he consistently used one function ($y = 2x + 1$) across all three episodes such that this one function was shown in flow diagram, table, equation, and function notation form, and as a graph. This similarity of rule across representations thus allowed the representations to be contrasted and discerned. Goldenberg (1995) describes that it is crucial to understand the synthesis between different representations of functions, rather than understanding individual representations separately. Teacher Z's use of one rule across multiple representation thus afforded the development of such a synthesised view of the object of learning 'representations of a function'.

Final Remarks on Teacher Z

Teacher Z's two lessons included 28 examples on functions. These examples covered objects of learning regarding: functions and rules; inputs and outputs; function notation; sketching linear functions from a table (with an emphasis on gradient); parabola functions; and parabola graphs. In his enactment of the examples, Teacher Z included the teacher actions *combination of instances*, *shifting focus of attention*, and *differences through changes* which created contrast and generalisation of aspects and features. Student interaction helped him to identify misconceptions. In his lessons, he emphasised the different representations of functions as a key aspect of the object of learning.

Analysis Part 2

How Did the Lessons Differ, How Were They the Same?

The teachers' lessons represent a snapshot of their students' first two lessons of encounter with the object of learning of FET 'functions'. In this second part of the analysis, I consider firstly how many examples were used by the teachers, and which teacher actions were used in their mediation. Thereafter, I examine similarity and differences in the objects of learning.

The analysis of the teachers' lessons showed differences and similarities regarding the number of examples that were used, and which teacher actions were used. This information is summarised in Table 13:

Table 13 **Juxtaposition of Number of Examples and Teacher Actions in the Three Teachers' Lessons**

	Teacher X	Teacher Y	Teacher Z
Number of Examples Across Both Lessons	41	38	28
Teacher Actions	- Combination of instances - Shifting focus of attention	- Combination of instances - Shifting focus of attention	- Combination of instances - Shifting focus of attention - Differences as Changes

The table shows that Teacher X and Teacher Y used a similar number of examples, but that Teacher Z used noticeably fewer. All three teachers frequently made use of the teacher actions *combination of instances* and *shifting focus of attention*, while only Teacher Z used *differences as changes* (in one example in his Lesson 2).

Marton (2015) suggests that the object of learning of teaching episodes can be described as 'content', 'educational objectives' and 'aspects and features', which each increase in the level of detail of description. Content is the broadest description and describes the object of learning in the sense of a syllabus. The educational objectives describe the necessary skills related to the content, while the aspect and features describe the parts of the content that must be discerned to achieve the desired educational objectives. I begin my analysis of the similarities and differences in the affordances for learning created by the teachers by considering the 'content' of the lessons. Table 14 summarises the objects of learning considering the 'content' afforded in the teachers' lessons. Similar content has been indicated with similar colours.

Table 14 **Juxtaposition of the Three Teachers' Main Content of Object of Learning Across Both Lessons**

Teacher X	Teacher Y	Teacher Z
Finding the equation of straight line graphs	Rules and sequences	Functions and rules
Rules and relationships between x and y	Key concepts / terminology to do with relations and functions	Inputs and outputs
Different rules in different representations	Functions and relations (Lesson 1)	Function notation
Definition of key terms to do with functions	Functions and relations (Lesson 2)	Sketching linear functions using a table
Function notation (Lesson 1)	Domain and Range	Parabola functions
Function notation (Lesson 2)	Function notation	Parabola Sketch Graphs
Domain and range		
Different types of functions		
Sketching using a table		

When reading this table, it is important to bear in mind that examining the object of learning by 'content' loses detail of how this content was taught (regarding educational objectives and discernible aspects and features). For example, 'inputs and outputs' only appear as 'content' in Teacher Z's column (as this was a key object of learning in his lessons), although this aspect of functions did occur to a lesser extent in both Teacher X and Teacher Y's lessons within other content (for example, for Teacher X, as an aspect of function notation, and for Teacher Y, in the definition of relations and functions). Nonetheless, using 'content' to consider the afforded objects of learning shows that there were two main similarities across all three teachers lessons: presenting a function as a 'rule' (indicated in blue), and 'function notation' (indicated in yellow). In contrast, a noticeable difference in the teachers' lessons was how they chose to interact with the content 'definition of a function' (indicated in orange), 'domain and range' (indicated in green), and 'sketching graphs' (indicated in grey). Finally, the order in which content was sequenced varied across the three classes.

The Teachers' Handling of Similar Content

In this section, I engage with the *educational objectives* and *aspects* covered by the three teachers on the content 'seeing a function as a rule' and 'function notation'.

Seeing a Function as a Rule

All three teachers, early on in their first lessons, made references to the object of learning 'rules'. For this content, Teacher X included multiple *educational objectives*, including 'describing rules with words'; 'seeing the equation of a straight-line graph as a rule'; 'describing rules using different representations'; and 'understanding that rules can have different powers of x'. Key *aspects* of rules that could be discerned were that rules can have different representations; that rules have different component parts; and that rules can have different degrees. The specific features of these aspects can be seen in Table 1. Contrastingly, Teacher Y included the *educational objectives* of 'finding the rule of a linear pattern', 'using a rule to generate input and outputs', and 'recognising that a rule relates inputs and outputs'. Key *aspects* of rules that could be discerned were that rules can have different representations; and that rules have different meanings in different contexts. The specific features of these aspects can be seen in Table 5. Finally, Teacher Z included the *educational objectives* of 'understanding a function as a rule' and 'understanding that a flow diagram and a table can be different representations of the same rule/function'. In Teacher Z's lesson, *aspects* of rules were not unpacked, rather, rules were explored as a feature of the aspect 'function'. One noticeable commonality between all three teachers' teaching about 'rules' was that the feature of 'flow diagram' occurred in all three classes.

Function Notation

All three teachers included function notation as a key object of learning in their lessons. Teacher X included *educational objectives* regarding 'understanding the usefulness of function notation'; 'representing a function using function notation'; 'finding inputs and outputs using function notation'; 'using function notation in coordinate form'; 'using function notation with fluency'; and 'know how to pronounce function notation'. Key *aspects* included the notation of function notation (e.g. $f(x)$, $g(x)$), and the procedures of finding inputs from outputs and outputs from inputs. The specific features of these aspects can be seen in Table 1. Teacher Y included the *educational objectives* 'understand the need for function notation'; 'represent a function using function notation'; 'understand the advantages of using function notation'; 'evaluating functions using function notation (given input only)'; and 'know how to pronounce function notation'. Key *aspects* included the notation of function notation, and using function notation. The specific features of these aspects can be seen in Table 5. Finally, Teacher Z included the *educational objectives*: 'understand function notation as a change of notation'; 'understand how input and output values relate to function notation'; and 'evaluating functions using function notation (given input only)'. He also included the aspect of the notation of function notation. The specific features of these aspects can be seen in Table 9.

In this closer analysis of similar content, it thus becomes clear that within this similar content the teachers often focus on different educational objectives and aspects.

Differences in Content Covered

In this section, I engage with noticeable differences in the three teachers' content regarding the objects of learning of their lessons.

Definition of Function

The three teachers differed noticeably in how they worked with the definition of a function. Teacher X provided a relatively formal definition of "a rule where each input value (usually x) is linked to a unique output value (usually y)" (Teacher X, Lesson 1, example 4(a)). In the interview, Teacher X brought up this definition and said:

> "Now, can you remember that although I gave them what a rule was before, it was just linking an x value to a y value. Then here I threw it in but didn't emphasise that [for a function] it's a 'unique' y value because, I mean you can go into a whole lesson about vertical line tests and all that kind of stuff, but the only time that they [the students] are ever going to need that again is in Grade 12 when we are doing inverses. So if you introduce this in Grade 10, in a big way, about vertical line tests... do we ever look at it again until we do inverses? No. So I dropped it in there – 'unique' – but didn't emphasise it."

Contrastingly, Teacher Y spent much time on the definition of functions, and provided in-depth definitions of a relation and a function and how these relate to each other (Teacher Y, Lesson 1, examples 2(a) and 2(c)). He also included multiple examples on identifying functions from relations across both of his lessons. Teacher Y indicated in his interview that he believes having a deep understanding of the definition of a function is foundational to developing meaningful understanding of this object of learning from Grade 10-12 (although he is aware that this goes beyond the Grade 10 curriculum). Finally, Teacher Z did not engage formally with the definition of a function. He described a function quite informally as "basically a rule – almost like a spider diagram" and as a "number crunching machine". Teacher Z did not comment on the definition of function in his interview. Thus, when it comes to the definition of a function, these three teachers took very different approaches, including a very informal definition, a formal definition that was not interrogated, and a formal definition that was explored and practiced at length. Students in the different classes will thus have had very different affordances for learning regarding this aspect of functions in their two introductory lessons.

Domain and Range

Domain and range differed similarly to 'definition of a function'. Teacher X provided definitions for domain and range in his first lesson (linking this to input and output, x and y, and the horizontal and vertical axes respectively). He also included three questions on domain and range in the *Kahoot* quiz (Teacher X, Lesson 2, Examples 1(l)-1(n)). Teacher Y, however, indicated in the interview that he regarded the aspects of domain and range as foundational knowledge of functions, and thus included definitions of domain and range, and multiple examples on domain and range throughout both lessons. His examples on domain and range were often linked to 'definition of a function' questions. Finally, Teacher Z did not teach domain and range at all in his two lessons. When asked about this in his interview, he explained that he prefers to teach domain and range after he has taught the four Grade 10 functions (straight line, parabola, hyperbola, and exponential). He explained that he then teaches it as an object of learning after the four types of graphs are known. Teacher Z's comments are a good reminder that the analysis in this research report is only of the two introductory lessons and does not cover the teaching of the remainder of the object of learning 'functions'. Nevertheless, students in these three teachers' classes have been offered very different affordances for learning regarding domain and range in their two introductory lessons on functions.

Sketching Graphs

A further difference in the lessons was how 'sketching graphs' was included. Teacher X showed the use of a table to sketch four functions: straight line, parabola, exponential, and hyperbola. This occurred at the end of his second lesson, as part of developing the students' awareness that not all graphs are straight lines. In Teacher Y's lessons, sketching functions was not covered at all. Although he utilised multiple graphs in his lessons, none of these were drawn from scratch. Finally, Teacher Z sketched straight lines and a parabola using a table, and also included sketching parabolas using key aspects (rather than a table). He provided his students with exercises where they sketched graphs independently during both lessons. Thus, after these two lessons, the students in the different classes were offered very different affordances for learning regarding the sketching of functions.

Discussion of Similar and Different Content

Watson and Mason (2005) argue that "mathematics is learned by becoming familiar with examples that manifest and illustrate mathematical ideas" (p.24). Further, they suggest that acquiring new mathematical knowledge (through examples) leads to the development of 'example spaces' on mathematical objects of learning. Crucially, they argue that students "make sense of mathematics from what is available to them" (p.52).

My analysis of the similar and different content taught in the three classes indicated that, in general, different content was covered regarding the object of learning 'functions'. Two similarities across the classes were examined, namely content relating to seeing functions as rules, and function notation. A closer analysis of the educational objectives and aspects covered by the three teachers on this similar content suggested that, although there was some overlap, these were often different and thus different affordances for learning were made available to the students. Additionally, content across the three classes was sequenced in different order. Thus, the students in the different classes' emergent example spaces might well include different content, from which students can draw different educational objectives, and in which students can discern different aspects and features. The structure and inter-connection of their emergent example spaces could also differ markedly.

The Teachers' Use of the Multiple Representations of Functions

The 'multiple representations' are a crucial aspect in the object of learning 'functions'. Goldenberg (1995) argues that a rich understanding of functions relies on an integrated understanding of the different possible representations and how they link to each other. Table 15 shows which representations of functions were included in the different teachers' lessons.

Table 15 **Which Function Representations were used by Each Teacher**

Representation	Teacher X	Teacher Y	Teacher Z
Coordinate pair	✓	✓	✓
Set of ordered pairs	✓	✓	
Equation	✓	✓	✓
Flow diagram	✓	✓	✓
Function notation	✓	✓	✓
Graph	✓	✓	✓
Mapping	✓	✓	
Number Pattern		✓	
Set notation	✓	✓	
Table	✓		✓
Words	✓		

Leinhart et al. (1990) additionally explain that different representations of a function advance different aspects of the notion of 'function' which all contribute to the object of learning. My analysis in Part 1 indicated that all three teachers' lessons included the multiple representations of functions. However, these were handled differently in the three classes. In Teacher X's lessons, the multiple representations of functions were most apparent as an aspect of a rule, and then as a tool with which to sketch functions (equation → table → coordinate pairs → graph). Many of his examples included multiple representations within individual examples. Thus, the multiple representations of functions were a strong aspect in his lessons. In Teacher Y's lessons, the individual examples included fewer representations than in Teacher X's lessons. However, many different representations of functions were included during the course of the lessons. In Teacher Y's examples, the multiple representations of functions were encountered mainly in his discussion of rules, and within questions where functions needed to be identified, and the domain and range provided. Finally, in Teacher Z's lessons, in Lesson 1, the multiple representations of functions were made apparent by his use of one rule that was used throughout all episodes of that lesson. The rule was manipulated from flow diagram → table with equation → function notation → sketch. Additionally, in Lesson 2, the multiple representations were also used as a tool by which to sketch a graph (as discussed for Teacher X). Thus, students in the three classes would have experienced different affordances for learning regarding the different representations of functions. However, although mediated in different ways and to different extents, the multiple representations will thus likely be part of the students' initial, foundational contact with the object of learning of functions. Considering the literature, this is positive for the students in these three classes.

The Three Teachers' Mediation of Examples

A final finding from my study was that all three of the teachers' mediation of their example spaces created affordances for learning that were not necessarily visible in the innate affordances of the example space. In all three classes, their mediation of the examples was thus crucial. This finding is in line with the work of Essien (2021a, 2021b), Kullberg et al. (2017), Pillay et al. (2022) and Zaslavsky and Zodik (2007) who all argue that the mediation of examples is crucial in the creation of affordances for learning. Within my theoretical and analytical framework, 'teacher actions' were the focus in examining how examples were mediated. In future work, I believe that it would be beneficial to extend the framework to also examine the student interaction in a lesson to a greater depth.

Conclusion

In this chapter, I have engaged with the data collected from the three teachers' two introductory lessons on functions, and from the semi-structured interviews. I first analysed the data vertically, for each teacher, and engaged with their selected examples through the lens of Marton's (2015) VT to identify the potential affordances for learning that were innate in the example space. Next, I considered how affordances for learning were created in the mediation of the examples in the classroom. It was clear that in all classes, the teachers' mediation created additional affordances for learning than were available in the innate example space. Thereafter, I engaged in a horizontal analysis where I collated the findings of the vertical analysis to look for similarities and differences in the affordances for learning that were on offer in the three teachers' classes. The analysis made clear that, although there were some similarities across the three teachers' lessons, the students in the three different classes experienced many distinct affordances for learning functions in these two introductory lessons. This included differences in affordances regarding content, educational objectives and aspects and features of the analysed objects of learning. Additionally, although all students were afforded the opportunity to engage with the multiple representations of functions, this was also enacted differently in the three classes. In Chapter 5, the findings of the analysis are summarised and discussed, and potential implications are considered.

Chapter 5: Discussion of Findings and Conclusions

Introduction

In this research report, I have detailed my investigation into the opportunities for learning created in different classes within the same mathematics department on the same object of learning. My study has examined three teachers' two introductory lessons on Grade 10 functions. This has been done by analysing their choice and use of examples through the lens of Marton's (2015) variation theory. Although this research report is a case study, and thus limited in scope, it has provided some useful insight into the different affordances for learning that are offered to the students of different classes in this department. To reiterate, this study aimed to answer the following principal research question:

> What affordances for learning are made available through the choice and use of examples in the introduction of Grade 10 functions by three teachers in one mathematics department, and how are these similar or different?

In doing so, I was guided by the following two sub-questions:

1. Which 'objects of learning' of the notion of function did the teachers open up through the use of patterns of variation?
2. How did the teachers engage with the multiple representations of functions within their choice and use of examples?

In this concluding chapter, I answer my research questions and discuss the findings that became evident through the analysis. Thereafter, I consider the limitations and implications of these findings. Finally, I reflect on how undertaking this study has impacted my own teaching practice.

Summary of Findings

To answer my research questions, I present a summary of the study's findings. The analysis of the data indicated that although all three teachers were teaching the same overall object of learning of 'Grade 10 functions', their choice of objects of learning within functions (regarding 'content', 'educational objectives' and 'aspects and features') differed substantially in their introductory lessons. Key similarities between the three teachers' lessons were the objects of learning regarding 'functions and rules' and 'function notation'. However, even the level of engagement with these objects of learning differed (as indicated in Table 1, Table 5, and Table 9). Key differences between the lessons occurred in the covering (or not covering) of the objects of learning 'definition of a function', 'domain and range' and 'sketching graphs'. Exact details of the objects of learning covered in the three teachers' lessons are available in Table 1, Table 5, and Table 9. Considering specifically the aspect of the 'multiple representations of functions', the way that the teachers used examples in this regard also differed (exact details are available in Table 4, Table 8, and Table 12). The teachers also differed regarding how these representations were mediated. As students make sense of mathematics through the examples that are made available to them (Watson and Mason, 2005) and the manner in which these examples are sequenced and mediated to create patterns of variation (Marton, 2015) it can thus be argued that the students in the different classes were afforded the opportunity to construct examples spaces including different content, which might be structured and interconnected differently.

Implications for The Mathematics Department

The findings of this research study make clear that in my mathematics department, it can be hypothesised that mathematical objects of learning are taught differently by the different teachers. Although the curriculum does provide clear guidelines on what content needs to be taught within an object of learning, my study indicates that different teachers might use different examples, in a different order, and mediated in different ways, within this content. This would afford students in different classes different example spaces that make clear different 'dimensions of possible variation' and 'ranges of permissible change' (Goldenberg and Mason, 2008; Watson and Mason, 2005) regarding the educational objectives and aspects of features of this content.

This observation has various implications. I firstly consider the implications for individual teachers. At my school, most students have different mathematics teachers in Grades 8, 9, 10 and 11 of their high school journeys. Thus, generally, the students in a teacher's class every year have been taught by another teacher the year before. My research study has created an awareness that the students might have been taught prior knowledge in a manner that differs to how their current teacher would have taught it. They may therefore have covered the expected content but have discerned different aspects of that content than their current teacher expects. Alternatively, students' example spaces on the object of learning might be structured quite differently to what their new teacher expects. Thus, teachers in my department, and departments that are similarly structured, should be aware that their students' prior knowledge cannot be assumed. This is, of course, a general truth in teaching, but my findings have given some indication of how the prior knowledge might differ.

Secondly, in relation to the above, one needs to consider what would be best for the students' learning regarding how much agency and autonomy is good for individual teachers to have within a department. My study thus raises the questions: Is experiencing quite different teaching from different teachers beneficial or detrimental to students' mathematical learning journeys? Do teachers provide better or worse instruction when given full autonomy to structure their teaching of mathematical objects of learning and to choose their own examples and tasks? How might one assess this? Should departments provide a basic level of structure for teachers to follow within a mathematical object of learning that needs to be adhered to? Should departments provide rigid planning regarding the selection and sequencing of examples that need to be covered within a mathematical object of learning? How much communication should occur between teachers in a department regarding the manner in which they sequence their content, and which examples they use? To what extent should teachers have the freedom to be responsive to their students 'lived object of learning' within lessons? I believe that none of these questions are easy to answer (and fall far beyond the scope of my study), but that they are intriguing to consider.

Limitations

There are various limitations that should be kept in mind when interpreting the results of this study. Firstly, one must consider instrument reactivity (the effect that the video recording and voice recording might have had on the data) and furthermore, that my presence as a researcher in the observed lessons might have brought about different behaviours in the participants than under 'normal', non-observed circumstances.

Additionally, as this is a case study, it is important to remember that the results cannot be generalised. However, the findings nonetheless provide valuable insight into the potentially variable opportunities for learning created in different classes of mathematics departments that function according to similar *modus operandi* as my school; that is, where teachers have the agency to plan the implementation of a mathematical object of learning themselves, and the department does not fix the order in which an object of learning is taught and/or the examples and tasks that are used within this object of learning.

Implications for Future Research

Beyond seeking answers to the questions that I have raised under 'Implications for the Mathematics Department', I believe that it would be useful to extend my study to observe all teachers in a grade for an entire mathematical object of learning. Such a study would provide more insight than my case study.

Additionally, both Shulman (1986) and Ball et al. (2008) include using examples as a crucial aspect of a teacher's pedagogical content knowledge. They indicate that individual teachers need to know how to select examples, how to choose useful representations of ideas, how to sequence content, and how to predict how students might respond to the chosen examples. I believe that my study highlights the need to go beyond individual teachers' pedagogical content knowledge regarding example choice and use, and to examine a department's 'collective pedagogical content knowledge' regarding example choice and use. Such a study could provide useful information to improve teaching and learning across a department and, hopefully, the learning of multiple students.

Concluding Remarks

This study has investigated the choice and use of examples by three teachers in one mathematics department. By examining the examples in their two introductory lessons on Grade 10 functions, it has emerged that the students in the different classes were afforded different opportunities for learning. This involved differences in selected and sequenced content, the educational objectives regarding this content, and the aspects and features that were opened up for possible discernment. The differences observed in this case study raise questions regarding how much agency and autonomy is good for teachers to have in planning their lessons and in structuring how to teach an object of learning. However, beyond the immediate context of this study, analysing and investigating teacher's choice and use of examples has shown me that this allows insight into more than just their example worlds. In many ways, it also provides glimpses of their views and philosophies regarding mathematics and how it should be taught and learnt.

Reflections Regarding My Teaching Practice

As I am a part-time student, I have been teaching full-time during the undertaking of this study. I have experienced that my in-depth engagement with the theoretical and analytical framework – Marton's (2015) variation theory – and my analysis of my colleagues' work, has started to impact how I teach. I have begun to select and sequence my examples with much greater care than I did before, and I have begun to deeply consider how I might create 'contrast' and then 'generalisation' through my example use, and to not reach 'fusion' too quickly. Doing this research has furthermore made me aware of the deep learning that occurs when one observes another teacher. It stimulates reflection of different ways to structure one's teaching, which examples are beneficial, and how best to sequence them. In future, I hope to watch my colleagues teach more frequently, and will invite them into my classroom.

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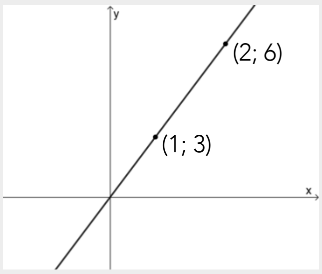
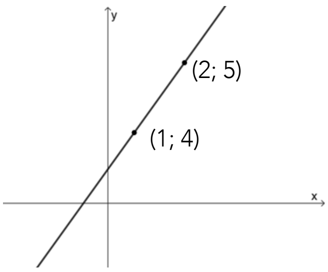
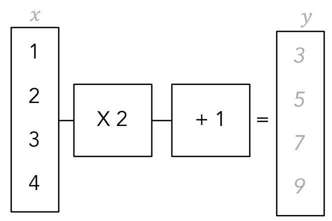
Appendix A: Detailed Lesson Descriptions

Teacher X: Detailed Lesson Descriptions

Table 16 **Teacher X Lesson 1 Detailed Description**

Episode	Example	Representations	Patterns of Variation (Examples / Teacher Actions / Student Interactions)	Object of Learning (Content / Educational Objectives / Aspects & Features)
1	1(a) Find the equation of this graph:  $m = \frac{6-3}{2-1}$ $m = 3$ $y = 3x$	- graph - coordinate pair - equation	Teacher X here (as a substitute teacher) is starting with examples that link to the section that has just been completed before he took on the class (analytical geometry) to functions, the new topic. This is to establish proficiency of prior knowledge. He then uses these examples as a base to draw on in the next episode (2). Examples 1(a) and 1(b), which appear sequentially in this episode are invariant in that they give the same task (find the equation) but vary in their gradient and y-intercept. Additionally, in 1(a), the y-intercept is the origin, while in 1(b) it is not. Teacher X makes use of a *combination of instances* with these examples. In comparing them, he creates <u>contrast</u> by discussing different ways to find the y-intercept of a line. In 1(a) the y-intercept is given as 0 as the graph goes through the origin and is thus found by inspection. In 1(b), one boy suggests a strategy of sketching the graph to scale in order to read off the y-intercept. Another boy suggests the method of finding the y-intercept by the substitution of a point on the line [indicated in grey in column 1 for 1(b)]. The teacher indicates that the latter is more efficient and focuses on this method. In this *combination of instances,* <u>generalisation</u> occurs in finding the gradient because both examples are solved by the same method of using the gradient formula $m = \frac{y_2 - y_1}{x_2 - x_1}$. Teacher X also uses *shifting focus of attention* with 1(b). Point (2; 5) is used as part of the substitution to find the y-intercept of the line. When using this point, he focuses on : - the $(x;\ y)$ setup of a coordinate - how and where to substitute $x = 2$ and $y = 5$ into the equation to find the y-intercept [he draws an arrow when describing where the 2 goes, and a second arrow for where the 5 goes. The arrows are included in the working out of (b)]	**Content:** - finding the equation of straight line graphs **Educational Objective(s):** - finding the equation of a straight line graph from two given coordinates - Using inspection to find the y-intercept - Using substitution to find the y-intercept - Using a graph 'drawn to scale' to find the y-intercept **Aspects & Features:** - <u>aspect:</u> gradient <u>features:</u> m=3; m=1 - <u>aspect:</u> find the y-intercept <u>features:</u> inspection, drawn to scale, substitution - <u>aspect:</u> the $(x; y)$ coordinate <u>features:</u> satisfy an equation, x as an input
	1(b) Find the equation of this graph:  $m = \frac{5-2}{4-1}$ $m = 1$ $y = 1x + c$ Sub (2; 5) $5 = 1(2) + c$ $c = 3$ $y = x + 3$	- graph - coordinate pair - equation		

			- that this point (on the line) satisfies the equation of the line - that x is the input of the point. By focusing on these different ways to consider the coordinate, he creates <u>contrast</u>.	
2	2(a) Rule which is linking an x-value to a y-value 2(b) Express 1(a) in words: *y value is triple (or 3 ×) the x value* 2(c) Express 1(b) in words: *y value is 3 more than the x value*	 - words - graph - coordinates - words - graph - coordinates	In Episode 2 there are three examples. Firstly 2(a), the statement that a rule links an x −value to a y −value and secondly 2(b) and 2(c), the instruction to express the rules (equations) found in examples 1(a) and 1(b) from Episode 1 in words. Teacher X here *shifts the focus of attention.* He places examples 1(a) and 1(b) in a new light in 2(b) and 2(c) – they are not just an equation of a straight line, but a rule linking x and y. This creates <u>contrasting</u> ways of seeing the equation $y = mx + c$: an equation of a graph, a straight line graph sketch, and a relationship that can be expressed in words. By expressing these relationships in words, a crucial variation between example 1(a) and 1(b) becomes more apparent. In 1(a)/2(b), the input variable x is being multiplied by 3 and there is no constant being added ($y = 3x$). In example 1(b)/2(c) the variable x is not being multiplied by a factor (other than 1) but a constant (3) is now being added to the input ($y = x + 3$). This creates <u>contrast</u> between the different elements of a linear rule: the input can be multiplied by a factor, and a constant can be added to the input. Finally, Teacher X uses *shifting focus of attention* to create <u>contrast</u> by adding a new feature to the aspect 'coordinate' that came up in Episode 1. He shows that the points (1; 3) and (2; 6) are examples of "the y value is triple the x-value" (from example 2(b)), while (1; 4) and (2; 5) are examples of "y is 3 more than the x value", from example 2(c). While doing so, he also creates a link between a rule, and a coordinate that belongs to this rule.	**Object of Learning:** - Rules and relationships between x and y **Educational Objective(s):** - describe a relationship between x and y with words - Seeing an equation in the form of $y = mx + c$ as a relationship between x and y (not just a graph equation) that can be expressed in words **Aspects & Features:** - <u>aspect:</u> relationships/rules <u>features:</u> equations, words, coordinates - <u>aspect:</u> parts of a rule <u>features:</u> coefficients that multiply the input, constants that add to the input - <u>aspect:</u> the $(x; y)$ coordinate <u>new feature:</u> the coordinate $(x;\ y)$ is a representation of the rule to which it belongs
3	3(a) <u>Review</u>  $y = 2x + 1$ 3(b) {(1; 1), (2; 4), (4; 9); (4; 16)} $(x;\ y)$ $y = x^2$	- flow diagram - equation - words - set of ordered pairs - equation	In Episode 3, there are 3 examples. 3(a) reviews the prior knowledge of flow diagrams by asking the boys to generate outputs from the provided inputs, 3(b) considers a set of ordered pairs, and 3(c) is a relationship shown in table form. In each case, the teacher and the boys discuss and find the rule that is represented [indicated in grey below the starting representation in each example]. These three examples vary in that they are three different starting representations of a rule (flow diagram, ordered pairs, table). The type of function also varies, with 3(a) and 3(c) being linear while 3(b) is quadratic. The examples are invariant in that examples 3(a) and 3(b) represent the same rule.	**Object of Learning:** - Different rules in different representations **Educational Objective(s):** - describing rules in different ways: · words · equations · flow diagrams · tables · ordered pairs/coordinates - understanding the equivalence between

<table>
<tr>
<td></td>
<td>3(c)
<table><tr><td>Input</td><td>x</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td></tr><tr><td>Output</td><td>y</td><td>3</td><td>5</td><td>7</td><td>9</td><td>11</td><td>13</td></tr></table>
$y = 2x + 1$</td>
<td>- words

- table
- equation
- words</td>
<td>In this episode, the teacher uses a combination of instances to create <u>contrast</u> regarding possible representations of a rule: flow diagram, ordered pairs, and table. This is made particularly clear in the contrast between examples 3(a) and 3(c) where the same relationship is represented in two different ways that lead to the same rule.

Secondly, <u>contrast</u> is created by this combination of instances that shows that rules can have different degrees (here, linear in 3(a) and 3(c) and quadratic in 3(b)).

This combination of instances also leads to <u>generalisation</u> in that rules linking x and y can be represented in different ways: flow diagram, ordered pairs, table, equation, words. The teacher deliberately shifts the focus of attention by showing that each example shows how x must be manipulated to generate y.</td>
<td>different representations of the same function

- understanding that rules can have different powers of x

Aspects & Features:
- <u>aspect:</u> different representations of a rule
<u>features:</u> flow diagram, ordered pairs, table, words, equation

- <u>aspect:</u> degree of a rule
<u>features:</u> linear, quadratic</td>
</tr>
<tr>
<td>4</td>
<td>4(a)
<u>Function</u>
Is a rule where each input value (usually x) is linked to a unique output value (usually y)

4(b)
Set of input values – DOMAIN
Set of output values – RANGE

4(c)
<table><tr><td>x</td><td>input</td><td>Domain</td><td>Horizontal</td></tr><tr><td>y</td><td>output</td><td>Range</td><td>Vertical</td></tr></table>
4(d)
Let $y = -2x + 3$
1) evaluate y if $x = 2$ input
$y = -2(2) + 3$
$y = -1$
$(2; -1)$
2) evaluate x if $y = 2$ output
$2 = -2x + 3$
$\frac{1}{2} = x$
$\left(\frac{1}{2}; 2\right)$</td>
<td>

- Cartesian plane

- equation
- coordinate pair</td>
<td>In Episode 4, there are four examples. A definition of function (4(a)), definitions of domain and range (4(b)) using input and output, a diagram showing linked vocabulary (4(c)), and an exercise using input and output (4(d)), which has two sub-questions.

Before defining the term function in 4(a), Teacher X shifts the focus of attention with examples from Episode 3. He says “All of these things [examples 3(a), 3(b), 3(c)] are talking about an x value being your input value and your y value as your output value. So x is the thing that goes into the number machine [pointing at the input of the flow diagram – example 3(a)] and y is dependent on what x was [pointing at the output of the flow diagram]”. He also adds the words “input” and “output” to the top and bottom rows of the table of example 3(c) in Episode 3 (as indicated). With these words and actions, the teacher is showing similarity across examples, and creates an opportunity for <u>generalisation</u> of the relationship between input and output.

He then provides the definitions of function, domain and range.

In example 4(c), Teacher X uses both variation and invariance in his table. The four columns show words with contrasting meanings, while the two rows show words with similar meanings. This allows for both <u>contrast</u> and <u>generalisation</u> to be apparent in the same example: contrast in the columns, generalisation across the rows.</td>
<td>Object of Learning:
- Definition of key terms to do with functions

Educational Objective(s):
- understand the meaning of the terms:
· function
· domain
· range

- solving for unknown inputs and outputs

Aspects & Features:
- <u>aspect:</u> input
<u>features:</u> x, domain, horizontal axis, left position in a coordinate

- <u>aspect:</u> representation of the input
<u>features:</u> x, t

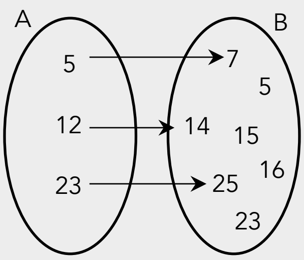
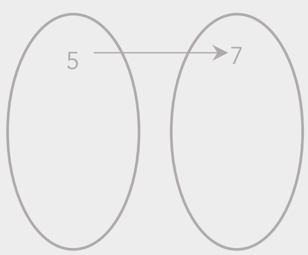
- <u>aspect:</u> output
<u>features:</u> y, range, vertical axis, right position in a coordinate</td>
</tr>
</table>

			When discussing the table, Teacher X also says: "So an input value, normally called x, they might come along and call it t just to mess with our heads - but it's normally an x - and the output value is normally y". This verbal statement opens up a new aspect of variation using <u>contrast</u>, namely the variable that can be ascribed to an input (he mentions t). In the final example in this episode – 4(d) – Teacher X very deliberately creates <u>contrast</u> between using input and output values. He does this by using a *combination of instances* and also *shifting focus of attention*. He says: "What is the distinction between the first example and the second example? What is our focus of our activity here [pointing at 1)]? Substitute in the x in order to find the y. What is the difference here [pointing at 2)]? Substitute in the y in order to find out what the x would have been. Here you have the input [pointing at 1)]. Here you have the output [pointing at 2)]". He also adds the words input and output to 1) and 2) respectively (as indicated), strengthening the contrast between these two exercises. For both 4d).1 and 4d).2, Teacher X writes the final solution as a coordinate, creating <u>generalisation</u> that regardless if you are given an input or an output as your starting point, your solution will lead to a coordinate pair of the function.	
5	5(a) If $y_1 = 4x + 2$ and $y_2 = x^2$ Find 1) y if $x = 2$ $y_1 = 4(2) + 2$ $y_1 = 10$ $(2;\ 10)$ $y_2 = (2)^2$ $y_2 = 4$ $(2;\ 4)$	- equation - coordinate pair	Episode 5 has six examples. 5(a) is used to introduce the need for function notation while contrasting a linear and quadratic function; 5(b) is a definition of function notation; 5(c) juxtaposes two different ways of asking for an output: once with words, and once with function notation; 5(d) shows a coordinate using function notation; 5(e) shows function notation being used with two different rules, and with positive and negative inputs; and 5(f) is an exercise that practises function notation for six different given inputs and one given output. In example 5(a), Teacher X initially provides two equations – *a combination of instances* – in the forms of $y =$ ... (one linear and one quadratic) and the instruction to find y if $x = 2$. This combination of linear and quadratic functions echoes Episode 3 where a linear and a quadratic rule are also combined. Now, however, attention is not shifted to the difference in degree. Teacher X rather *shifts the focus of attention* to show that having two equations in the form $y =$ ... can lead to confusion as to which calculated output belongs to which of the two functions. He says: "Now for me this is very confusing because I have to be very specific in ensuring	**Object of Learning:** - function notation **Educational Objective(s):** - understand the need for function notation - represent a function with function notation - find missing input and output values using function notation - use function notation in coordinate form - use function notation with fluency - know how to pronounce function notation **Aspects & Features:** - <u>aspect:</u> function notation <u>features:</u> $f(x)$, $g(x), h(x)$
	5(b) <u>Function notation</u> When we have multiple y values we let $y = f(x)$ or $y = g(x)$ or $y = h(x)$ etc 'f of x'			
	5(c) If $y = 3x - 4$ let $f(x) = 3x - 4$ • evaluate y at $x = 2$ $y = 3(2) - 4$ $y = 2$ $(2; 2)$ • find $f(2) = 3(2) - 4$ $= 2$ $(2; 2)$	- equation - function notation - coordinate pair		
	5(d)	- coordinate pair		

	$(x;\ y) \rightarrow (x;\ f(x))$	- function notation	that whoever is going to be looking at my answers knows which one [which rule] I'm working with". He goes on to introduce a subscript next to the two ys (y_1 and y_2) to differentiate the two functions. In doing this combination of examples, contrast is used in an unusual way to deliberately create a confusing situation and set up the usefulness of function notation. In 5(b), Teacher X defines function notation, immediately showing contrast through a *combination of instances* that show that function notation can use different variables (f, g, h). He also tells the boys how to 'say' function notation as "f of x". In 5(c) there is careful use of a *combination of instances* that allows for both contrast and generalisation to occur. Contrast is used to show that the same function can be represented in two ways ($y = 3x - 4$ and $f(x) = 3x - 4$) and that these two different representations belong to different instructions (evaluate y at $x = 2$ and find $f(2)$ respectively). Teacher X makes this point explicitly, while *shifting focus of attention*: "If I was working with this [pointing at $y = 3x - 4$] I would have to say "evaluate y at x is 2". If I wanted you to evaluate it at 2, I would have to use those words…but if I use this notation [pointing at $f(x) = 3x - 4$] I can just tell you to find the $f(2)$. I read it as "find f of 2". Find the rule's value when x is replaced specifically with an input value of 2. Can you see how it is a lovely shortcut?" However, the invariance of this *combination of instances* also allows for generalisation to occur. This is because one can see that the process of substituting 2, even from the two different instructions, leads to the same mathematical procedure and ultimate coordinate of $(2; 2)$. It thus becomes clear that the mathematical meaning is consistent, even if the notations differ. In 5(d), Teacher X again focuses on the $(x; y)$ coordinate and uses a *combination of instances* to show contrast in notation rewriting $(x; y)$ as $(x;\ f(x))$. However, this *combination of instances* also leads to generalisation of understanding that even in coordinate form – not just equation form – the variable y can be successfully replaced with $f(x)$. In example 5(e) contrast is used in a *combination of instances*. Firstly, contrast in notation (having an $f(x)$ and a $g(x)$). Secondly, contrast occurs in the function rule, with $f(x)$ having a positive m and a positive c value, while $g(x)$ has a negative m and a	- aspect: using function notation features: input → output, output → input - aspect: the $(x; y)$ coordinate new feature: the coordinate $(x;\ y)$ as $(x;\ f(x))$ - aspect: degree of a rule features: linear, quadratic
	5(e) $f(x) = 2x + 3$ $g(x) = -x - 6$ Find 1) $f(1) = 2(1) + 3$ $= 5$ $(1;\ 5)$ 2) $g(3) = -(3) - 6$ $= -9$ $(3;\ -9)$ 3) $g(-4) = -(-4) - 6$ $= -2$ $(-4;\ -2)$	- equation - function notation - coordinate pair - words		
	5(f) $f(x) = 3x - 1$ Find: 1) $f(a) = 3a - 1$ 2) $f(\star) = 3\star -1$ 3) $f(0) = 3(0) - 1 = -1$ 4) $f\left(\frac{1}{3}\right) = 3\left(\frac{1}{3}\right) - 1 = 0$ 5) x if $f(x) = 8$ 6) f(elephant)= 3(elephant) $- 1$ 5) x if $f(x) = 8$ output $3x - 1 = 8$ $3x = 9$ $x = 3$	- equation - function notation		

			negative c value. Finally, contrast is brought about by the input values (positive and negative numbers) and how this affects the substitution and calculations. Through this <u>contrast</u>, however, <u>generalisation</u> shines through as it becomes clear that regardless of the variable used for the function notion, and regardless of the signs of m, c and input values, finding an output from a given input has the same procedure with function notation. Finally, in 5(f), a *combination of instances* is used to show strong <u>generalisation</u> of the substitution procedure when finding an output from a given input in function notation. Teacher X states this very deliberately after doing sub-questions 1-4: "I'm trying to push a point home here… sometimes our input values are not that pretty: they could be fractions, they could be negatives, they could be stars, they could be as they could be ps and in fact you could take something like really overboard [adding q6) to the board] the $f(elephant)$ would be what? 3 times $elephant$ minus 1 [as he writes this down]. Just to push a point home. It doesn't matter what's in that x-bracket - that is your input value that you replace wherever you saw an x before". In this example set, <u>contrast</u> occurs in the combination of questions 1-4, and 6, with question 5. In 1-4 and 6, an input value is provided and an output is desired. In question 5, an output is given, and the input is desired. This could be seen as the creation of <u>fusion</u> of various ideas covered in the lesson: output values versus input values and the associated procedures, as well as gleaning the necessary information through function notation. Again, Teacher X highlights this explicitly when discussing question 5, saying "Is the x 8 this time? ... No, what is 8? The $f(x)$ is 8. So what am I going to do - the input or the output? The output! ... Can you see how this little tweak can easily catch us?" Here, he is combining various ideas into one instance, necessitating the boys to draw on various ideas simultaneously. This was the end of the first lesson. No homework was given.	

Table 17 **Teacher X Lesson 2 Detailed Description**

Episode	Example	Representations	Patterns of Variation (Examples / Teacher Actions / Student Interactions)	Object of Learning (Content / Educational Objectives / Aspects & Features)
1	Kahoot Exercise: 1(a) If $f(x) = 3x - 7$, then the value of $f(2) =$ *Multiple choice options:* $13;\ 25;\ \mathbf{-1}$ $f(x) = 3x - 7$ $f(2) = 3(2) - 7$ $= 6 - 7$ $= -1$ 1(b) $g(x) = x^2 + 2x - 3$ find the value of $g(1)$ *Multiple choice options:* $\mathbf{0};\ -3;\ 19$ $g(x) = x^2 + 2x - 3$ $g(1) = (1)^2 + 2(1) - 3$ $= 0$ 1(c) $f(x)$ is a fancy way to write… *Multiple choice options:* x; y-intercept; slope; $\boldsymbol{y}$ 1(d) The figure represents a mapping of x to $f(x)$. Determine $f(5)$ *Multiple choice options:* $\boldsymbol{f(5) = 7}, f(5) = 5, f(5) = 12,$ $f(5) = 14$   1(e) If $f(x) = 4x + 2$, find $f(2)$ *Multiple choice options:*	- function notation - function notation - function notation - function notation - mapping - function notation	In Episode 1 of this lesson, there are 15 examples. These examples were presented in a gamified way, namely through the online platform *Kahoot*. In a *Kahoot* quiz, students join the game through an online platform where they enter the game pin. Then, the questions are projected on the board by the teacher (generally multiple choice and/or true-false questions) and the students select the correct solution to the question on their individual electronic device. Each question has a time limit. Points are awarded for selecting the correct answer, and also for how quickly the correct answer is selected. Incorrect solutions get no points. This adds a competitive element as the (fluctuating) top 5 positions are announced after every question. At the end of the Kahoot, there is a podium with the top 3 students' names shown. The teacher manages the Kahoot, and has control over starting each question. This allows for questions and solutions to be discussed for each individual question before the next question is projected. Teacher X set up this *Kahoot* to have examples that were in either multiple choice or true-false form. Each example had a timer with limited time (the time limited varied slightly depending on the complexity of the question). Most of the examples tested content from Teacher X's Lesson 1, with a focus on gaining fluency in the use of function notation, and also revising the ideas of domain and range. 21 students took part in the *Kahoot*. The students in the class were initially not prepared for the fluency required for this game. For the first example (1(a)), they were only given 10 seconds to answer, and only 2 students in the class got the correct solution. After this poor result, the teacher engaged with the question, briefly revising how function notation works and how this particular question should have been solved. This explanation seems to have paid off, as for example 1(b), 15 students got the correct answer. Here, they were given 20 seconds, so there was less time pressure. <u>Contrast</u> of function name, $f(x)$ compared to $g(x)$, occurred between 1(a) and 1(b) through a *combination of instances* but this did not seem to confuse most students. This could	**Object of Learning:** - function notation - domain and range **Educational Objective(s):** - gain fluency in the use of function notation - identify domain and range from given points - find inputs and outputs of a function **Aspects & Features:** - <u>aspect:</u> function notation <u>features:</u> f(x), $g(x)$ - <u>aspect:</u> using function notation <u>features:</u> input → output, output → input - <u>aspect:</u> degree of a rule <u>features:</u> linear, quadratic, cubic - <u>aspect:</u> representations of a function <u>features:</u> $f(x)$ equation, function map, set of ordered pairs, straight line - <u>aspect</u>: the $(x; y)$ coordinate <u>new feature</u>: the x-coordinate belongs to the domain of the function; the y-coordinate belongs to the range of the function

	$8;\ \mathbf{10};\ 6;\ 2$ 1(f) If $f(x) = x^2 - 3x$, find $f(-8)$ *Multiple choice options:* $-40;\ 4;\ -88;\ .\mathbf{88}$ $f(-8) = (-8)^2 - 3(-8)$ $= (-8)^2 - 3(-8)$ $= 64 + 24$ $= 88$ 1(g) If $f(x) = x^3 - 1$, determine $f(2)$ *Multiple choice options:* $f(2) = 5;\ \boldsymbol{f(2) = 7};\ f(2) = 6;$ $f(2) = 8$ 1(h) $f(x) = x^2 - 3x + 9$. Determine $f(0)$. *Multiple choice options* $\boldsymbol{f(0) = 9};\ f(0) = -3;\ f(0) = 6;$ $f(0) = 3$ 1(i) For what value of x would $g(x) = x + 10$ be 20? *Multiple choice options:* $\mathbf{10};\ 20;\ 30;\ -10$ 1(j) If $g(x) = x + 7$ and $g(x) = 10$, then $x = 4$ True or **False**? $x + 7 = 10$ $x = 3$ 1(k) If $f(x) = x + 4$ and $f(x) = 20$ then $x =$ *Multiple choice options:* $2;\ 10;\ 0;\ \mathbf{16}$ 1(l) The input value for a function is part of the DOMAIN **True** or False? 1(m) $\{(-2;4), (-1;1), (0;0);\ (1;1)\}$ then the domain is: *Multiple choice options:* Domain = $\{-2;\ -1;\ 0\}$; Domain= $\{2\}$; Domain= $\{4;\ 1;\ 0\}$; **Domain**= $\{\mathbf{-2, -1, 0, 1}\}$	- function notation - function notation - function notation - function notation - function notation - equation - function notation - Set of ordered pairs - set notation	indicate that some learning and generalisation has occurred in that finding $f(x)$ and $g(x)$ indicate the same mathematical process. The teacher again did the question on the board after the students submitted their solutions. In doing so, he *shifted the focus of attention* to show contrast of degree between 1(a) – linear – and 1(b) – quadratic – and how this affected calculating the output. For 1(c), 12 students got the question correct. Here, in the explanation, Teacher X focused on how for slope (gradient) we have a symbol m and likewise for y, we have a symbol $f(x)$. For 1(d), the teacher for the first time introduced a function mapping representation (linked to $f(x)$). This representation did not occur in the previous lesson. However, unfortunately, the image of the function map was very difficult to read when projected on the board which made the question difficult to access. Only 4 boys got the question correct. The teacher redrew the mapping to explain how this representation works. Example 1(e) showed much similarity to 1(a), in terms of $f(x)$ notation, linear form, the input value 2, and a 10 second time limit. Here, however, 19 students got the question correct, compared to 2 in 1(a). This indicates that understanding and/or fluency had improved as the game progressed. In 1(f) there was unfortunately a typo in the correct solution, which led to many boys being confused. Here, for a substitution of a negative input into a quadratic function, the teacher *shifted the focus of attention* to highlight the importance of substituting the -8 correctly (as indicated in his working out). In 1(g), the teacher introduced the first function of degree 3 – a cubic. Here, 11 boys got the correct solution. This example created further <u>contrast</u> of degree through a *combination of instances*, compared to the linear and quadratic examples with which they had engaged so far in this episode. In 1(h), the boys were presented with a quadratic and asked to find $f(0)$. They were only given 10 seconds. 17 boys got this correct. However, the teacher picked up a comment from a student regarding the high time pressure. He then stated: "One of you said 'why is the time so fast?'. So gentlemen, in mathematics there are certain processes, alright, which we are not meant to take a long time over, alright. So for example, if I know it's the $f(0)$, I don't even have to substitute	

<table>
<tr>
<td></td>
<td>1(n)
The range of $\{(-2;4),(-1;1),$ $(0;0),(1;1),(2;4)\}$ is $\{0;1;4\}$
True or False?

<table><tr><td>x</td><td>input</td><td>Domain</td><td>Horizontal</td></tr><tr><td>y</td><td>output</td><td>Range</td><td>Vertical</td></tr></table>

1(o)
All functions will be straight lines if we draw them on a Cartesian plane True or False?</td>
<td>- Set of ordered pairs
- set notation
- Cartesian plane</td>
<td>into there [pointing at the first two terms, x^2 and $-3x$]. I know that I can cover those up because they're not going to contribute any value [covering the two terms with his hands that become 0 when substituting the input]. So the $f(0)$ being 9, we're not going to take time going 0 squared minus 3 times 0...so the more that you practice those, those kind of things are where you can pick up the pace, with accuracy, in a test. I don't want you to have to pick up your calculator and to actually substitute in the 0 squared there. Because this is where you can gain a bit of an advantage just being a little bit more mathematically agile in terms of working out the answers."

Here, the teacher is shifting focus of attention in order to overtly highlight the need for fluency.

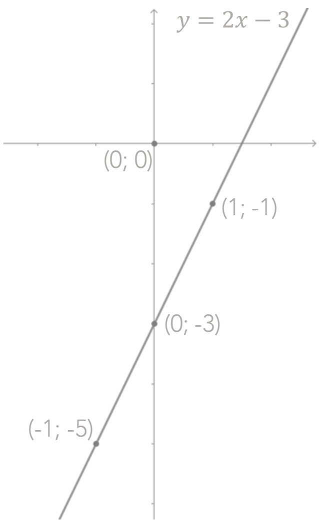
The combination of instances that occur from example 1(a) to example 1(h) as a whole promote <u>generalisation</u> of procedure to find the output when an input is provided with function notation. Although the examples vary in aspects such as degree, the sign of the input value, and the name of the function (f versus g), as described above, the procedure of substituting the provided input for x and computing the resulting output value of the expression remains consistent throughout.

In examples 1(i), 1(j), and 1(k), there is a shift from providing the input of a function and asking for the output as presented in previous examples, to providing the output and asking for the input. This creates <u>contrast</u> of input and output through a combination of instances. Brief discussion by the teacher is provided after the 1(i) and 1(j) and the number of students getting these questions correct increases from 15, to 18, to 19 respectively which indicates increasing understanding.

The question content then shifted to domain and range for examples 1(l), 1(m) and 1(n). In particular, 1(m) and 1(n) created <u>contrast</u> of how to list domain and range from ordered pairs regarding repetition of identical values. In 1(m), there are no repeated values so to list the domain one just lists the inputs. However, in 1(n), the outputs of 1 and 4 are repeated and only need to be listed once. The teacher focuses the attention on these <u>contrasting</u> situations, using a deliberate combination of instances. Additionally, he revisits example 4(c) from lesson 1 to remind the boys of the connections between x and y, domain and range, and input and output respectively.</td>
<td></td>
</tr>
</table>

<table>
<tr><td></td><td></td><td></td><td>Finally, in example 1(o), the teacher sets up the rest of the lesson by asking if all functions are straight lines. Only 15 boys knew that this is incorrect. The teacher then explains that he will be introducing other types of functions that are not straight lines in the rest of the lesson.</td><td></td></tr>
<tr><td>2</td><td><u>Look at Functions</u>

2(a)
Sketch
$y = 2x - 3$

<u>Table</u>
<table><tr><td>x</td><td>1</td><td>-1</td><td>0</td></tr><tr><td>y</td><td>-1</td><td>-5</td><td>3</td></tr></table>
</td><td>- equation
- table
- coordinate pair
- graph</td><td>Episode 2 differs quite substantially to episode 1. In episode 1, 15 examples occurred at quite a fast pace. In episode 2, only four sketch examples are tackled, and to a greater depth. In example 2(a), a linear graph is sketched using a table. In example 2(b) an inaccurate Cartesian plane is drawn, In 2(c) a quadratic graph is sketched using a table, in example 2(d), an exponential graph is sketched using a table, and in example 2(e), a hyperbola is sketched using a table.

In example 2(a), a straight line graph equation is given. Sketching such a graph should be existing knowledge as it occurs in analytical geometry, the topic that was taught prior to functions. Here, the teacher emphasised that having more points leads to a more accurate graph, and suggested 3 points, rather than just 2, to draw an accurate straight line. After filling in the table and plotting the three resulting coordinates, Teacher X focused on how a sketch graph does not need ‘perfect’ accuracy but that consistent increments along the axes are important. He then drew a counter-example to show what unequal increments might look like (example 2(b)). Finally, he engaged with a student’s question as to which inputs to use to draw a straight line. He suggested using $-1;\ 0;$ and 1 as inputs. He also stated that these points are useful because they indicate “what's happening round-about the origin”.

An important statement from Teacher X to come out of this example was the <u>generalising</u> statement that “one of the ways that we can draw a straight line is just with a table. And in fact for any function we can use a table” (emphasis added). In the next 3 sketch examples, he goes on to show this to be true.

Example 2(c) created a combination of instances with example 2(a) that shows <u>contrast</u> regarding a function’s degree and that this results in a different graph shape. In undertaking to sketch the graph, Teacher X immediately goes to his suggested method of using a table. He says: “Let's put a table together. And if you take my advice, let's just use x-values of $-1, 0$ and 1. Minimum 3 points. If that isn't conclusive for me, then I can add more x-values into my domain and create more y-values”.</td><td>Object of Learning:
- different types of functions
- sketching using a table

Educational Objective(s):
- understanding that different types of functions exist

- using the technique of an input-output table to sketch different types of functions

- using negative exponents

- understanding the necessary accuracy of a sketch graph

- using different representations to translate a graph equation to a graph sketch (equation → table → coordinate pairs → sketch)

Aspects & Features:
- <u>aspect:</u> type of functions
<u>features:</u> straight line, parabola, exponential, hyperbola

- <u>aspect:</u> using a table
<u>features:</u> inputs:
$-2;\ -1;\ 0;\ 1;\ 2;\ 3$

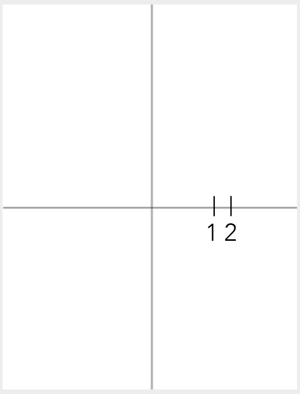
- <u>aspect:</u> an ‘accurate’ sketch graph
<u>features:</u> regular axis increments; irregular axis increments

- <u>aspect:</u> representations of a function
<u>features:</u> equation; table; coordinate; sketch

- <u>aspect:</u> degree of a function
<u>features:</u> linear, quadratic, hyperbolic, (also exponential)

- <u>aspect</u>: the $(x; y)$ coordinate
<u>new feature</u>: a tool to sketch a graph</td></tr>
</table>

2(b)



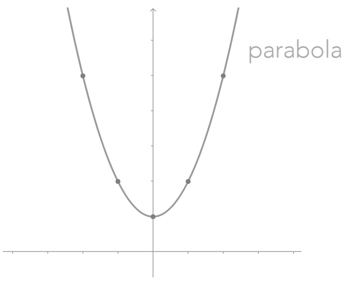
- cartesian plane

2(c)

$y = x^2 + 1$

Table

x	-1	0	1	2	-2
y	2	-1	2	5	5



- equation
- table
- coordinate pair
- sketch

2(d)

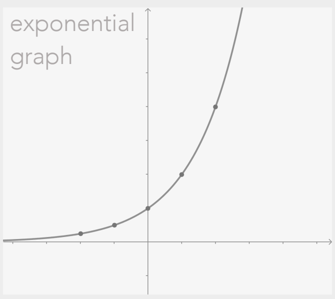
$y = 2^x$

x	-2	-1	0	1	2
y	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4

$\boxed{2^{-1} = \frac{1}{2}}$

$2^1 = \frac{2}{1}$

$2^{-2} = \frac{1}{4}$



- equation
- table
- coordinate pairs
- sketch

After plotting the three initial points, it becomes clear that this function is behaving differently to the straight line. Teacher X says: "Can you see there is a squared [pointing at 2(c)], whereas this was of the form $y = mx + c$ – straight line [pointing at example 2(a)]. So this is of a different form. It's got a squared in it [pointing at the x^2]. And then, can you see that your 3 points are definінately indicating to you that it is not a straight line [using his hand to pointing out the upward curve indicated by the three plotted points $(-1; 3)$, $(0; 1)$ and $(1; 3)$] . Now, if I want confirmation, I might want to put in 2 more points, and again, I would choose two more points which are still around here [around the origin], not out there somewhere [far away from the origin], because then I'm going to be plotting out here [far away from where the other points are].

Teacher X is here carefully *shifting focus of attention* in this *combination of instances* between 2(a) and 2(b) to create <u>contrast</u> of degree and shape. However, with his consistent use of the table method, he is starting to create <u>generalisation</u> of plotting method and strategy.

After plotting the final two points $(2; 5)$ and $(-2; 5)$ and connecting the 5 points with a smooth curve to draw the graph, the teacher communicates that the graph of example 2(b) is called a parabola, and adds this to the sketch as a label.

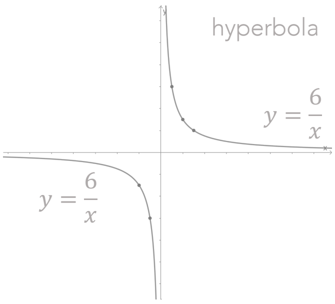
Moving on to example 2(c), Teacher X again very deliberately creates a *combination of instances* where he can *shift the focus of attention* to create <u>contrast</u> of form of the graph equation. After writing down $y = 2^x$, he says the following: "So because it's different to this formula [pointing at straight line] we expect it's not going to be a straight line. Because it's different to this formula [pointing at the parabola] we expect it's not going to be a parabola. So let's have a look at what we think it's going to look like.

He then again uses the table strategy to generate coordinate pairs. Now, the negative exponent creates a new issue that needs to be dealt with, as the boys are not confident what the outputs of inputs $x = -1$ and the $x = -2$ will be. This leads to a brief recap of the negative exponent rule. Here, as in example 2(b), Teacher X includes input values of -2 and 2 to help understand the form of the graph. Once he has plotted the points and drawn the sketch, he says the following: "This is called an exponential graph. Obviously taking its name from the fact that it is exponents. So it's the placement of the x

2(e)

$y = \frac{6}{x}$

x	-2	-1	0	1	2	3
y	-3	-6	und	6	3	2



- equation
- table
- coordinate pair
- graph

which is so important. It's in the exponent of the "power" and that's why it's an exponential graph.

In this last statement, the teacher is deliberate *shifting focus of attention* to how the graph's name is connected to the placement of the input x.

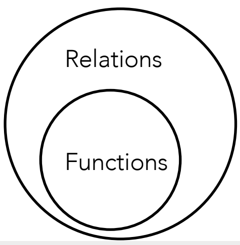
In the final example of the lesson, Teacher X writes down the equation of a hyperbola (example 2(d)). He immediately creates the table with input values $-2;\ -1;\ 0;\ 1;\ 2$. He starts with $x = 0$, which creates an undefined output. He explains this as that there is no corresponding x-value at $y = 0$. He then moves on to generating outputs for the positive inputs $x = 1$ and $x = 2$ and decides to add an input $x = 3$ to the table to help with understanding the shape of the graph. The resulting coordinates are plotted. After that, he moves on to the negative inputs, and plots the resulting coordinates. Finally, he sketches the graph by connecting the points in quadrant 1 and in quadrant 3. This graph, being discontinuous, differs from the first three graphs which are continuous. He highlights this, describing that for the hyperbola, "what you've got … is two completely separate graphs but making up one function", and goes on to label both arms of the graph as $y = \frac{6}{x}$. This function is not discussed in as much detail as the others as the lesson is drawing to a close. However, again, generalisation of strategy has occurred as this graph has also been drawn by generating points from a table.

At this end-point of the lesson, examples 2(b), 2(c) and 2(d) are all simultaneously shown on the board, which allows for visual comparison. The differences in equation and form are thus juxtaposed, allowing for contrast and variation to be seen. However, the same method is shown throughout, which also highlights invariance of sketching method: table, coordinate, plot, connect.

Teacher Y: Detailed Lesson Descriptions

Table 18 **Teacher Y Lesson 1 Detailed Description**

Episode	**Example**	**Representations**	**Patterns of Variation (Examples / Teacher Actions / Student Interactions)**	**Object of Learning (Content / Educational Objectives / Aspects & Features)**
1	1(a) $A = \{5; 7; 9; 11; 13; \ldots\}$ Ordered set sequence 1(b) $Z = \{1; 17; 18; 20; 27\}$ 1(c) $\boxed{T_n = 2n + 3}$ Rule $T_3 = 2(3) + 3 = 9$ $T_5 = 2(5) + 3 = 13$ 1(d) $\boxed{T_n} = 2n + 3$ variables 1(e) Input $\rightarrow$ $\boxed{2n+3}$ $\rightarrow$ Output 1 5 2 7 3 9 $\{(1; 5); (2; 7); (3; 9)\}$	- set notation - set notation - number pattern - number pattern - flow diagram	In episode 1, there are five examples. Firstly, 1(a) is an ordered set of numbers; then 1(b) is a random set of numbers (unordered); 1(c) is a rule of the linear pattern from 1(a); 1(d) is again the rule from 1(a); and finally 1(e), a flow diagram of the rule from 1(c) which leads to a set of ordered pairs. The lesson begins with Teacher Y putting 1(a) on the board and asking the boys what they notice. They are unsure and mention prime numbers (incorrect). Teacher Y then adds 1(b) to the board, thus creating a *combination of instances*. He asks: "what can you tell me about these two sets?" In the two sets, each has five elements but variation occurs in the relationship between the numbers in the sets. This creates <u>contrast</u> of the relationship between the numbers in each set (one ordered, one not ordered). After 1(b) is added, a boy (Boy B) now says that he notices a pattern in set A. Teacher Y revoices his contribution, saying "the difference between this one [pointing at 1(a)] and that one [pointing at 1(b)], is that this one [1(a)] is an ordered set…a sequence". The teacher then asks the boys to write the general term for the pattern and they work towards finding this together (example 1(c)). Teacher Y then describes this general term as a rule, and uses the rule to find T_3 and T_5. He then says: "Do you see that this rule is working for all [terms] in this sequence [1(a)], not in that one [1(b)]?, again *shifting focus of attention* to the <u>contrasting</u> sets of numbers. Next, he rewrites the rule in 1(d) in order to *shift the focus of attention* regarding the rule. In 1(c), he emphasised the rule as a general term that can create the elements of the ordered set in 1(a). Now, he focuses on n and T_n being variables, and further *shifts the focus of attention* to the fact that the two variables (n and T_n) are related by a rule ($2n + 3$). To emphasise this point, he draws the flow diagram of 1(e), and shows how input 1 becomes an output 5, input 2 creates an output of 7, and input 3 creates an output of 9. He then states that the input and the output are related, and that they form a relation. He then goes on to write the related inputs and	**Content:** - rules and sequences **Educational Objective(s):** - recognise ordered sets of numbers that form a sequence - find the rule of a linear pattern - use the rule of a linear pattern to generate outputs from inputs - recognise that the rule of a pattern relates the input and the output values - become aware of the concept 'relation' **Aspects & Features:** - <u>aspect</u>: sets of numbers <u>features</u>: ordered and unordered sets; ordered sets can be generated by a rule while unordered sets cannot - <u>aspect</u>: representations of a rule <u>features</u>: set of numbers, general term, flow diagram, set of ordered pairs - <u>aspect</u>: meaning of a rule <u>features</u>: used to generate terms of a pattern, something that relates input and output values - <u>aspect</u>: a relation <u>features</u>: where inputs and outputs are connected by a rule; a linear flow diagram; a set of ordered pairs

			outputs as a set of ordered pairs, which he also calls a relation.	
2	2(a) Notes *[projected on board, read out loud]* "Relations and Functions" **are the most important topics in algebra.** Relations and functions – these are the two different words having different meanings mathematically. You might get confused about their difference. Before we go deeper, let's understand the difference between both with a simple example. An ordered pair is represented as (INPUT, OUTPUT). *[this sentence is not read aloud]* The relation shows the relationship between INPUT and OUTPUT. Whereas a function is a relation which derives one OUTPUT for each given INPUT. Note: All functions are relations but not all relations are functions.  2(b) "All boys are human being but not all human beings are boys" 2(c) What is a Function? A function is a relation that describes that there should be only one output for each input (or) we can say that a special kind of relation (a set of ordered pairs), which follows a rule i.e., every X-value should be associated with only one y-value is called a function. What is a Function? In mathematics, a function is a set of inputs with a single output in each case. Every function has a domain and range. The domain is the set of independent values of the variable x for a relation or a function defined.	- coordinate pair	In episode 2, there are eleven examples. 2(a) and 2(c) are notes projected on the board. 2(b), 2(d), and 2(g) are spontaneous examples used in an attempt to elucidate the content of the notes. 2(e), 2(f) and 2(h) are exercises on functions, domain and range, 2(i) is a spontaneous example to help explain 2(h), and 2(j) and 2(k) are verbal examples used while explaining the exercises. Teacher Y begins by projecting example 2(a) on the board and reading the text aloud. When he reaches the explanation of a relation, he walks to the board where the examples from Episode 1 are still written. He says: "this set [pointing at the set of ordered pairs from 1(e)], it is a relation, it is relating two variable based on this rule [pointing at the rule Tn=2n+3 from 1(e)], so we call it a relation". By returning to work from Episode 1, Teacher Y creates a *combination of instances* of theory (his notes) and practice (an example of the theory). Drawing on the notes, he again *shifts the focus of attention* (as he did in Episode 1) that a number pattern rule can be seen as a relation. He then returns to the notes. When he reaches the "Note" in 2(a) which works with the Venn diagram below it, teacher Y then verbally adds example 2(b) to explain the relationship between relations and functions. This *combination of instances* uses a real-world example to create <u>generalisation</u> of relationship so that the boys can transfer their understanding of the real-world scenario (boys and humans) to the lesson topic: functions and relations. Teacher Y then returns to the notes in example 2(c). Here, he again creates a *combination of instances* by referring back to the work on the board from Episode 1. This time, he uses the flow diagram from 1(e) to emphasise that if a relation has only one output for each input (as occurs in the flow diagram) the relation is a function. Here, he *shifts the focus of attention* to show the flow diagram as a function (rather than a flow diagram or a representation of a number pattern). Teacher Y then returns to the notes in 2(c). When he has read the content on domain and range, Teacher Y again returns to the written work on the board. He goes to example 1(c) and rewrites it from being a number pattern to being a rule between x and y (example 2(d)). This creates a *combination of instances* between the $T_n = 2n + 3$ and $y = 2x + 3$. This	**Content:** - Key concepts/terminology to do with relations and functions **Educational Objective(s):** - understand what is a relation - understand what is a function - understand domain and range - understand the link between domain and range and defining a function - classify a set of ordered pairs as being a function or not **Aspects & Features:** - <u>aspect</u>: relation <u>features</u>: a rule relating two variables; a number pattern; a circle graph; functions are relations - <u>aspect</u>: function <u>features</u>: function versus relation; every input has only one output; every function is a relation but not every relation is a function; a linear flow diagram; not a circle graph - <u>aspect</u>: representations of functions <u>features</u>: flow diagram; number pattern rule; set of ordered pairs - <u>aspect</u>: domain <u>features</u>: the inputs for a function; the x-values of coordinate pairs; the set of x-values for which a function is defined; the inputs of a flow diagram - <u>aspect</u>: range <u>features</u>: the outputs of a function; the y-values of

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<td>In simple words, the domain is a set of x-values that generate real values of y when substituted in the function.

On the other hand, the range is a set of all possible values that a function can produce. The range of a function can be expressed in interval notation or in form of inequalities.</td>
<td></td>
<td rowspan="8">creates <u>contrast</u> between the variables T_n and y, and n and x respectively. However, it also shows <u>generalisation</u> of the linear form of the rule, and the coefficient 2 and constant +3. Additionally, he then goes to the flow diagram 1(e) and shifts the focus of attention for the flow diagram by describing the input values as the domain and describing the output values as the range.

Next, Teacher Y provides example 2(e), which the boys attempt independently. They find this difficult, so the teacher does it together with the class on the board. He emphasises that in listing a domain and range, the values must be listed in ascending order.

Next, he moves on to example 2(f) and, returning to the definition of a function, establishes that the relation is a function. However, the boys still seem to be struggling to understand. Teacher Y then draws example 2(g) on the board. In discussing this circle graph, a counter-example of a function, the boys seem to make progress with one student audibly saying: “it makes sense!”

Teacher Y then moves on to example 2(h). Here, example 2(e)/(f) it is a combination of instances with example 2(h). The questions are similar in that the same task is given for two sets of ordered pairs respectively. However, the examples vary in that in 2(h), the input value -2 occurs in two different points. This was not the case in 2(e), which was a function. This <u>contrast</u> is noticed creates an animated discussion in the classroom when the boys try to list the domain. They cannot decide if the value of -2 should be listed twice or only once.

Boy A asks: “If there are two of the same domain, must we list them both?” Teacher Y asks the class to respond. Boy B says: “Yes, because it’s not the same, it may have the same value but the output is different.” Other boys disagree with this, and Teacher Y then tells the class that the -2 would only be listed once. However, Boy C and Boy B are still unhappy that it has only been listed once. Teacher Y thus creates another spontaneous example in 2(i). He explains that if one were to put a dot on -2, to represent -2 on the number line, the two dots of the two -2s would lie on top of each other as it is the same value. Boy B is still unhappy. He says: “each negative 2, sure I understand that it’s the same value, and it’s one domain, but then the domain has a different range, it has a different output. Therefore, why is it than you only write down one [-2] when the output of the two</td>
<td rowspan="8">coordinate pairs; the set of values that a function can produce; the outputs of a flow diagram

- <u>aspect</u>: listing domain and range as a set
- <u>features</u>: listed in ascending order, repeated values are only listed once, set notation</td>
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<td>2(d)

$T_n = 2n + 3$

$y = 2x + 3$</td>
<td>- equation</td>
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<td>2(e)
1) List the domain and range of each relation:
$(2;1); (-3;4); (0;5)$

Domain: $\{-3;0;2\}$
Range: $\{1;\ 4;\ 5\}$</td>
<td>- set of ordered pairs
- set notation</td>
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<td>2(f) Is the relation in 2(e) a function?</td>
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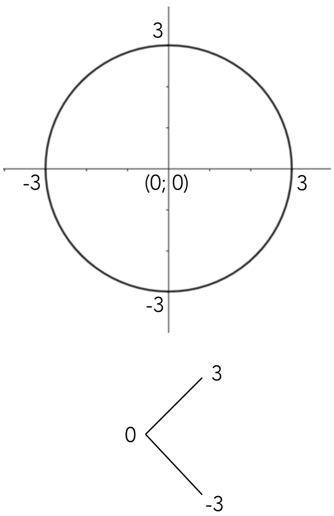
<td>2(g)

</td>
<td>- graph</td>
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<td>2(h)
a) List the domain and range
b) Determine whether the relation is a function.
$\{(1;3); (-2;4); (3;-2); (-2;7)\}$

Domain: $\{-2;1;3\}$
Range: $\{-2;\ 3;\ 4;7\}$</td>
<td>- set of ordered pairs
- set notation</td>
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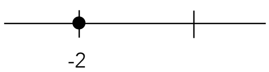
<td>2(i)

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<td>2(j)
“It is because, here, this is the same input [pointing at the two -2s], it is the</td>
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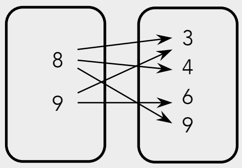
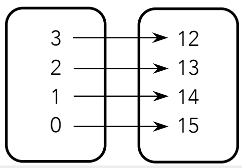
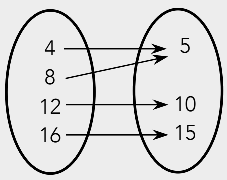
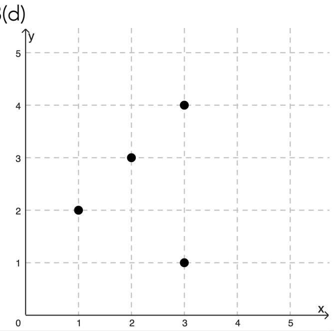
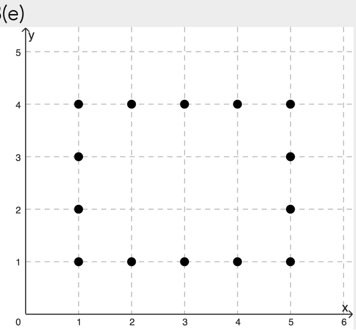
	same input. It's just one [Teacher Y][6] that you are inputting. So you don't have two [Teacher Ys], it's one [Teacher Y], although you are saying this and that [moving his body in two different directions for 'this' and ;that'] but it is one [Teacher Y]…so we write it once. *[the boys chuckle at this]*		different ones [the two −2s] are going to be different?" Teacher Y responds using another spoken 'real-world 'example. He creates spoken example 2(j), referencing himself as a 'unique' input that could only be listed once. Boy B, who was unhappy, laughs and says "Yes, Sir, I get you".	
	2(k) "It's like saying to you that you must have one girlfriend not two"		Boy D then asks a question: "What if you like have to find…use a rule…to find the output from that domain?" Teach Y: So this one [pointing at −2] if we're using the rule, we can't use the rule, it's not all relations where you can use the rule ... The rule is only applicable if it is a function. In answering the next part of the question, example 2(h) b), Boy B appears to be making progress with his understanding. He explains that the set of points is not a function, saying: "Because, Sir, the set input don't have a single output…I think." Teacher Y responds: It doesn't matter how many inputs you have - but each and every one of those must have only one output. Boy B (with more confidence): "And −2 has two outputs, therefore it is not a function." Teacher Y: "It's like saying to you that you must have one girlfriend not two…This negative 2, it's like it's having two girlfriends." Boy B: "It's cheating!" Teacher Y: Yes, it's cheating. It has 4 and this one [pointing at output 7] also. Boy A: "A player, Sir!" Teacher Y: "and that's not on, hey?"	

[6] Here, the teacher used his own surname, but this has been replaced with 'Teacher Y'. e.g., "It's just one McLachlan that you are inputting".

3	3(a)  3(b)  3(c)  3(d)  3(e)  3(f)	- mapping - mapping - mapping - graph - graph - graph	In episode 3, there are 8 examples. 7 of these come from a provided worksheet, while one (example 3(g)) is a spontaneous example. In all cases, the aim is to determine if the provided relationship is a function. The boys are given time to work on the examples and they are then discusses on the board. On the worksheet, 3(a), 3(b) and 3(c) (all mapping) are positioned in a row, and 3(d), 3(e), and 3(f) (all points on a cartesian plane) are positioned in a row. As exercises, examples 3(a)-3(c) and 3(d)-3(f) form a *combination of instances*. In 3(a)-3(c), the function representation remains the same while the relationships between inputs and outputs vary. Additionally, the number of inputs vary being 2, 4 and 4 per map, and the number of outputs vary being 4, 4, and 3 respectively per map. 3(a) is not a function (being one-to-many). 3(b) is a function (being one-to-one) and 3(c) is a function (being many-to-one). Therefore, there is very little that remains invariant in these three examples other than the representation. However, <u>contrast</u> regarding what is and what is not a function is created between these examples. In 3(d)-3(f), the function representation also remains the same (coordinates on a cartesian plane) while the relationships between coordinates vary, as do the number of coordinates per graph. 3(d) has 4 coordinates, with $(3;\ 1)$ and $(3;\ 4)$ rendering it not a function. Example 3(e) has 14 coordinates, with multiple coordinates vertically aligned (therefore also not a function). Finally, 3(f) has 4 coordinates and is a function, although two points share a y-value $(1; 2)$ and $(2; 2)$. Here, in this *combination of instances*, very few aspects stay the same across examples. However, the three examples do provide <u>contrast</u> regarding the effect of points vertically or horizontally arranged on the example being a function or not. Finally, in example 3(h), a real-world context is brought in, and the question asks for reasoning to be explained using the correct vocabulary. Here, the two sub questions are a *combination of instances* that will create <u>contrast</u> of 'function' versus 'not a function'. This is because these two exercises make clear that the addition of one coordinate (vertically in line with another) can change a function relationship to not being a function relationship . I now return to the lesson. While working on the worksheet, Boy B asks the teacher a	**Content:** - functions and relations **Educational Objective(s):** - identifying functions from mapping and coordinates on a cartesian plane - understanding the difference between one input having many outputs and multiple inputs having the same outputs (when classifying functions) - being aware of one-to-one and one-to-many functions - understanding the importance of mathematical definitions in classifying mathematical objects **Aspects & Features:** - <u>aspect</u>: representations of functions <u>new features</u>: mapping; points on the cartesian plane - <u>aspect</u>: mapping <u>features</u>: visualise a connected domain and range; the number of elements in each set can vary; can represent a relation that is or is not a function; links to one-to-one and many-to-one functions - <u>aspect</u>: cartesian plane <u>features</u>: visualise a connected domain and range from a point; the number of points on the graph can vary; can represent a relation that is or is not a function; links to one-to-one and many-to-one functions - <u>aspect</u>: function <u>features</u>: can be many-to-one (visualised in mapping or coordinate form) or one-to-one (visualised in mapping or

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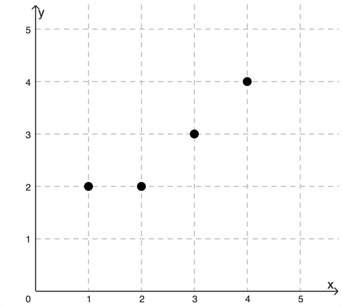
<td>
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<td rowspan="3">question: "Sir, can two domains share one range?... Ah, it's not a function, never mind." At this point, the teacher does not provide any comment on Boy B's contribution.

For example 3(a), the boys seem confident that it is not a function. Boy A says: "…because 8 and 9 have more than one range" while Boy B justifies with: "...there are multiple ranges."

In response, Teacher Y tries to improve the way that they are expressing their reasoning: "Let me advise you - whenever you define, you must define it in terms of the definition. It is there in your book also. Whenever you give a reason, don't just say it's one, more than one, because you will not get a mark... So always go back to the definition... We see here that this one input, this input 8, has got more than one inputs... [he means output, the boys correct him]... it has got 1, 2, 3 different outputs [pointing at them as he says this]. So according to the definition of a function a relation is said to be a function if each and every input has got only one output. That is the definition of a function so for that reason this is not. It has got 3 girlfriends, this one, this one this one [pointing at each output connected to 8]. This one [9] has got two [pointing at the two outputs linked to 9]. It's no good.

When discussing 3(b) – which is a function – Teacher Y briefly introduces the idea of one-to-one mapping, and mentions many-to-one mapping too.

Question 3(c) is determined to not be a function (although this is incorrect). It is only later in the lesson that this error is spotted.

Question 3(d) and 3(e) are classified as not being a functions.

3(f) brings about more discussion. Boy C does not think that it is a function because he sees that points $(1;\ 2)$ and $(2;\ 2)$ share the same output on the y-axis. After he states this, Boy B decides that Boy C is correct (although he initially thought it was a function), stating that: "He's right, he's correct!... because the output is shared by two inputs!".

At this point, the teacher cuts in, stating that the boys are not getting the definition right. He explains that multiple inputs can share the same output, so long as each input has only one output. After hearing this, Boy B then exclaims: "So then, Sir, we were wrong for number 3 as well."

Teacher Y first feels that he needs to explain why 3(f) is a function before addressing Boy B's</td>
<td rowspan="3">coordinate form)</td>
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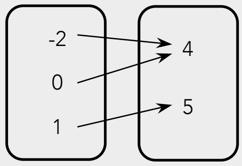
<td>3(g)

</td>
<td>- mapping</td>
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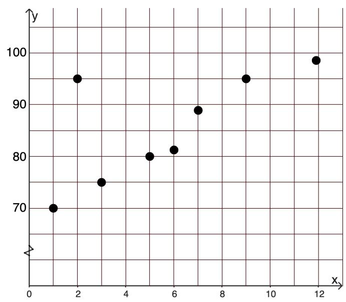
<td>3(h)
The graph shows the relationship between the hours Rachel studied and the exam grades she earned.

1) Is the relationship a function? Justify your answer. Use the words "input" and "output" in your explanation, and connect them to the context represented by the graph.


2) Rachel plans to study 2 hours for her next exam. How might plotting her grade on the same graph change your answer to Exercise 11? Explain your reasoning.</td>
<td>- graph</td>
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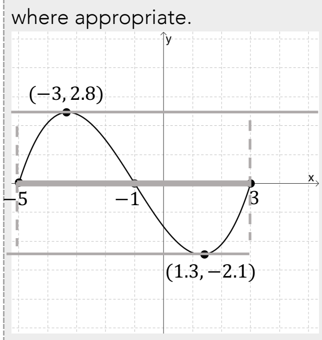
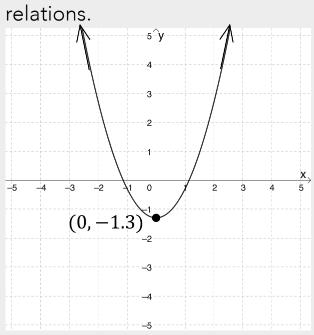
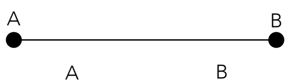
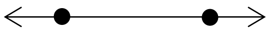
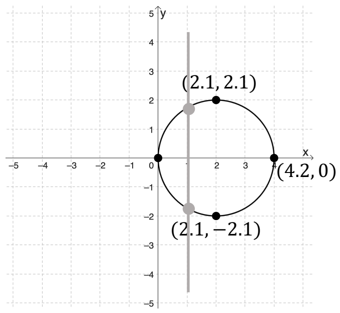
			concern. He draws example 3(g) on the board and carefully shows that although it is ‘one-to-many’, each and every input has only one output: $(-2;\ 4)$, $(0;\ 4)$ and $(1;\ 5)$. Boy B then wants to know what type or rule might create such a function, and Teacher Y says that this will be discussed at a later stage – for now, he wants to clarify what is a function and what is not a function. At this point, teacher Y returns to example 3(c) – previously incorrectly classified as not a function – and agrees with the protesting boys that it is, indeed, a function. Teacher Y then moves on to example 3(h) and asks the boys what they think. Boy B states: “Yes, it is a function. Because every output has one input. Eish. I swapped them around... Every input has one output, therefore making it a function.” Teacher Y then asks another boy, Boy D, what he thinks about the question. He agrees, stating “Yes, I say it's a function. It's a function because the input (which is hours studied) has one output (which is a grade). Through examples 2(e), 2(f) and 2(h) from Episode 2, and then examples 3(a)-3(g) from Episode 3, the process of identifying functions is consistently repeated, including examples of functions and counter-examples of functions (in different representations). This repetition of the same process allows for *generalisation* of what the definition of a function means, to shine through the variation of representation, number of inputs/outputs, and one-to-one, many-to-one, and one-to-many relations. Here, homework is given, and the lesson ends.	

Table 19 **Teacher Y Lesson 2 Detailed Description**

Episode	**Example**	**Representations**	**Patterns of Variation (Examples / Teacher Actions / Student Interactions)**	**Object of Learning (Content / Educational Objectives / Aspects & Features)**
1	1(a) Definition of a relation: A set of input and output values; a set relating the input and output values (given verbally by the teacher)		Lesson 2, Episode 1 has 12 examples. Of these, 1(a) and 1(b) were the verbal revision of definitions of a relation and a function from the previous lesson's work. Examples 1(c), 1(e), 1(h), 1(j), 1(k), and 1(l) are all exercises from a worksheet given for homework. Finally, examples 1(d), 1(f),1(g) and 1(i) were created spontaneously to help explain the worksheet examples. After going over definitions verbally, Teacher Y starts the homework worksheet. For example 1(c) Teacher Y first asks if the graph is a function (before finding the domain and range). Boy C says yes, because "all the points are in different areas". Teacher Y is unhappy with this explanation and says the following: "Remember, I advised you yesterday that whenever you are asked questions like that, you must always refer to the definition. In your mind, you must go to the definition. If I'm asking you whether this is a function or not, you must go to the definition. How would we know whether a relation is a function or not…each input is associated with only one output." He then goes to the graph and *shifts the focus of attention* to individual points, showing that at particular input x-values there is always just one output. This creates <u>contrasting</u> ways of seeing the graph – simply as a graph but also as a collection of input and output coordinates. The graph is then classified as a function and the class moves on to finding the domain and range of the function. Boy A, who suggests a solution to writing down the domain of example 1(c), starts listing a set of x-values that he appears to be reading off the given points of the graph (example 1(d).1). This is similar to how the domain of coordinate points was listed in the previous lesson. However, Teacher Y soon cuts in by saying: "…what about -4? [pointing at $x = -4$ on the graph] and what about between -4 and -5? How many numbers do we have between -5 and -4? Infinite... So now if you write it this way [pointing at Boy A's solution that Teacher Y has started to write on the board, example 1(d).1] you are excluding what? You are excluding -4 and other... so it can't be the correct one, don't you think so? Remember that the x-values between -5 and 3 [he walks to the graph and draws vertical dashed lines at $x = -5$ and $x = 3$,	**Content:** - functions and relations - range and domain **Educational Objective(s):** - know the definition of a relation - know the definition of a function - identify a function from a sketch of a graph (informal vertical line test) - give the domain of a function/relation from a sketch of a graph - give the range of a function/relation from a sketch of a graph - use inequality notation **Aspects & Features:** - <u>aspect</u>: relation <u>new feature</u>: graphical representation of relations; infinite relations; relations with restricted domains; arrows on graph arms - <u>aspect</u>: function <u>new feature</u>: graphical representation of functions; infinite functions (arrows on graph notation); functions with restricted domains - <u>aspect</u>: domain <u>new feature</u>: read off a graph; infinite; finite - <u>aspect</u>: range <u>new feature</u>: read off a graph; infinite; finite - <u>aspect</u>: types of graphs (graph only) <u>features</u>: cubic, parabola, circle, straight line, absolute value inverse, cubic inverse
	1(b) Definition of a function: A relation that derives one output for each given input (given verbally by Boy A)			
	1(c) – homework exercise Find the domain and range of the relations. Use interval notation where appropriate. 	- graph - coordinates		
	1(d) 1. $\{-5; -3; -1;$ 2. Domain: $\{-5 \le x \ge 3, x \in R\}$ 3. Domain: $\{-5 \le x \le 3, x \in R\}$ 4. Range: $\{-2{,}1 \le y \le 2{,}8;\ y \in R\}$			
	1(e) – homework exercise Find the domain and range of the relations. 	- graph - coordinate pairs		
	1(f) 			



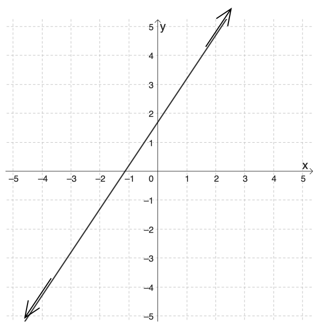
1(g) 1. Domain: $\{\infty \leq x < \underline{\infty};\ x \in R\}$ 2. Domain: $\{-\infty \leq x < \underline{\infty};\ x \in R\}$ 3. Domain: $\{-\infty < x < \infty;\ x \in R\}$ 4. Range: $\{-\infty \leq y \leq 1.3;\ y \in R\}$ 5. Range: $\{-1.3 \leq y \leq \infty;\ y \in R\}$ 6. Range: $\{-1.3 \leq y < \infty;\ y \in R\}$		
1(h) – homework Find the domain and range of the relations.	- graph - coordinate pairs	



Domain: $\{0 \leq x \leq 4{,}2;\ \ x \in R\}$

Range: $\{-2{,}1 \leq y \leq 2{,}1;\ y \in R\}$

1(i) $x \in R$		
1(j) – homework Find the domain and range of the relations.	- graph - coordinate pairs	



$x \in R$
$y \in R$

1(k) – homework Find the domain and range of the relations.	- graph - coordinate pairs	

and then draws a horizontal line along the x-axis between $x = -5$ and $x = 3$ – this has been indicated in example 1(c) in grey], if I ask you how many are they, you can't tell me that there are one, two, three... it's a finite. How many? How many between -5 and 3? Infinite. So how do we represent that?"

Another boy now comes forward and writes down how he expressed the domain for his homework (example 1(d).2). This has some errors in notation so Teacher Y corrects these creating example 1(d).3.

Thereafter, the teacher adds the following to interpret the meaning of the notation: "What you are actually saying is that all the x-values, all of them, even the ones that you don't know, even the ones that you haven't met, all of them, as long as they are between what? -5 and 3, including -5 and 3. Ok? So that would be the domain for that particular relation that you say is a function, and you are correct."

The teacher then moves on to finding the range. No student is keen to come up and write down a solution on the board and some boys indicate that they do not yet understand. The same boy who wrote down example 1(d).2 then comes to the board to write down his homework solution for the range. After some discussion regarding notation, example 1(d).4 is fixed on as the correct solution for the range. Teacher Y then interprets the inequality notation, and draws horizontal lines at the two turning points of the graph (indicated on the graph 1(c) in grey). This appears to help some students understand the range, as "oooooh" can be heard from a few boys in the class. The teacher is thus using a *combination of instances* – namely the inequality notation and the sketched horizontal lines on the graph – to provide two contrasting means of expressing the idea of the range. The meaning (range) that is invariant in both representations thus becomes clear through the variation of representation, allowing for generalisation of the idea of range.

The teacher then moves on to example 1(e). In *the combination of instances* between 1(c) and 1(e), the two graphs vary in the following features: a finite domain and an infinite domain; a finite range and an infinite range; and cubic graph and parabolic graph. The parabola also has arrows at the end of its arms (indicating that it goes on infinitely), unlike the cubic function; however, the two graphs are invariant in that they are both functions on the cartesian plane that have a domain and a range.

- aspect: notation of domain and range
- features: set notation (previous lesson); inequality notation

- aspect: inequality notation
- features: values in ascending order; direction of inequality sign; inclusive or exclusive inequality sign; curly brackets

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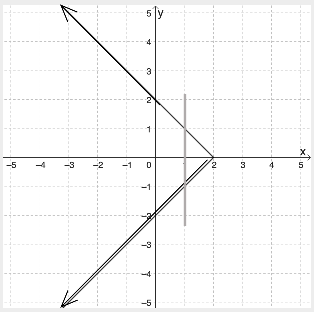

$-\infty < x \leq 2;\ x \in R$

$y \in R$
$-\infty < y < \infty;\ y \in R$
</td>
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When the teacher starts discussing the example, there appears to be confusion regarding the feature of the two upward arrows at the end of the arms of the parabola. Teacher Y therefore draws example 1(f) before engaging with the domain and range of 1(e).

In example 1(f), he attempts to explain the difference between a fixed line segment from A to B, and a line that goes infinitely beyond A and B. Focusing on the arrows on both ends of the bottom line, he asks: “what do these arrows mean?... this is not a line segment. It [the line] does not start there [at A] it starts from [pointing far to his right], we don’t know where…and goes to infinity [pointing far to his left].”
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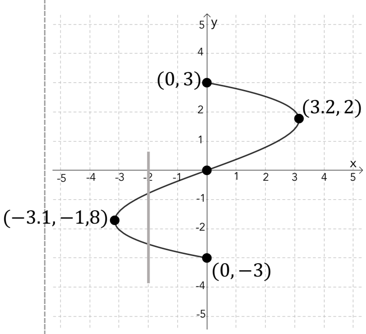
<td>
1(l) – homework
Find the domain and range of the relations.



$-3{,}1 \leq x \leq 3{,}2;\ \ x \in R$
$-3 \leq x \leq 3;\ \ y \in R$
</td>
<td>
- graph
- coordinate pairs
</td>
<td>
Boy B interjects: “Ja, to Limpopo.”

To this, Teacher Y responds:
“So when you mention Limpopo it means that there is an end point!” [general laughter]

Teacher Y then goes to the graph in 1(e) and explains that the two arrows pointing upwards on the arms of the graph also mean that the graph goes on in the same pattern forever. He then asks the class what is the domain. A large class discussion ensues, and the boys decide the domain must involve infinity. One boy says that he does not know how to write it in the correct notation to which another boy says: “just copy that thing” [meaning example 1(d).3]. This comment indicates that generalisation of process has been spotted: to write down the domain of 1(e), one can use the domain of 1(d).3 as a guide.

A boy then goes up and writes down 1(g).1 which is corrected by the teacher to 1(g).2. Then, in a similar process, one boy writes down the range as 1(g).4 which is corrected by a boy to 1(g).5.

Suddenly, Teacher Y notices that there is a consistent error in these expressions of domain and range: infinity is being included. He did not notice this error in 1(g).1 and it has been consistently reproduced in all following contributions. This shows a process of generalisation, as the boys are taking what they see as a correct solution and replicating it in subsequent work.

After correcting the notation, Teacher Y then asks if example 1(e) is a function or not. The boys still seem uncertain on how to tell if the graph is a function or not.

The teacher then moves on to example 1(h). Here, he begins by asking if this graph is a function. The class still seem unsure. They know the definition but are unsure how to apply it.
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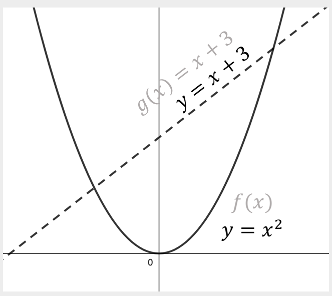
			Teacher Y then draws a vertical line at $x = 1$, showing that it cuts the graph at two points. Therefore, input value 1 has two outputs and the graph is not a function. Next, the teacher notices that he has not fully shown the class how the domain can be written for example 1(e) – an infinite domain. He then writes example 1(i) on the board, which he positions as an equivalent statement. In other words, $\{-\infty < x < \infty;\ x \in R\}$ can be written as just $x \in R$. Example 1(j) is covered quite quickly, with the teacher writing down the solution. For example 1(k), the boys are asked if it is a function. They are still unsure. One boy, however, is making progress. He says, with certainty that it is not a function because "if you look at the graph, Sir, the things on the x-axis have more than one output, Sir. I don't know if I worded that right." The teacher draws a vertical line at $x = 1$ to show the two outputs, and corrects the explanation, again referring to the definition. The final example discussed from the homework is 1(l). Here, the boys are beginning to sound more confident in their response that the graph is not a function. Teacher Y draws a vertical line at $x = -2$ to emphasise this. Considering the worksheet examples together, all are continuous graphs on the cartesian plane yet show contrast in many ways: infinite versus finite domains and ranges; different types of graphs; some are functions and some are relations; some have turning points while some do not.	
2	2(a)  When $x = 1$, $y = 1^2 = 1$ $f(1) = 1$ 2(b) What is a Function Notation? Notation can be defined as a system of symbols or signs that	- graph - equation - function notation	In Episode 2, there are five examples. Example 2(a) is a sketch of two functions; 2(b) is projected notes on function notation; 2(c) is an example of using function notation; 2(d) is more projected notes on function notation; and finally 2(e) is more notes, including a worked example of function notation being used to evaluate a function at a given input value. To begin, Teacher Y draws example 2(a) on the board. He first draws and labels $y = x^2$ and discusses evaluating the graph at $x = 1$. Next, he draws and labels the line $y = x + 3$. Using a *combination of instances* he then shifts the *focuses attention* to the difficulties that arise when two graph equations of the form $y = \cdots$ are provided. With this, he is using similarity between examples to show the need for the <u>contrast</u> that can be provided with function notation. He says: "If I draw another one [graph], $y = x + 3$ do you see that here I have y [pointing at $y = x + 3$] and even here, I have y [pointing at $y = x^2$].	**Content:** - function notation **Educational Objective(s):** - understand the need for function notation - represent a function with function notation - understand the advantages of using function notation - evaluate functions using function notation (given input) - know how to pronounce function notation **Aspects & Features:** - <u>aspect:</u> function notation

	denote elements such as phrases, numbers, words etc. Therefore, function notation is a way in which a function can be represented using symbols and signs. Function notation is a simpler method of describing a function without a lengthy written explanation. The most frequently used function notation is f(x) which is read "f of x". In this case, the letter x, placed within the parentheses and the entire symbol f(x) stand for domain set and range set respectively. Although f is the most popular letter used when writing function notation, any other letter of the alphabet can also be used either in upper or lower case. $f(x) = fx$ f of x		And if I can draw more, I can get confused which y I am talking about." He then introduces the idea of function notation and reads aloud the notes of example 2(b). He specifically highlights that $f(x)$ means 'f of x' not f times x ($f \times x$) as the notation might have been interpreted in Grade 8. While reading the notes, he breaks away to write down example 2(c) and illustrates the use of function notation from the graph provided in example 2(a). He says: "When I use this notation I can write here $f(1)$, so here I'm saying that when I input 1 in this function [pointing at $y = x^2$ written on the board]… I get $(1)^2$ which is 1. And if I input 3, so I write $f(3) = (3)^2 = 9$. So we are saying that the value of this function f (we call it f), when x, when the input value is 3, is 9. Teacher Y then moves back to the notes in example 2(b). When he reaches the statement that says "[a]lthough f is the most popular letter used when writing function notation, any other letter of the alphabet can also be used", he illustrates this by adding $g(x) = x + 3$ and $f(x)$ to the graphs of 2(a). Next, he moves to example 2(d). When he reaches the point concerning '[f]unction notation allows to identify the independent variable with ease", he *shifts the focus of attention* to create <u>contrast</u> between the expression $y = x^2$ and $f(3) = (3)^2 = 9$ to show that in the first expression it is difficult to identity the independent variable while in the latter, *3* can easily be identified as the independent variable. He again creates <u>contrast</u> using a combination of instances by writing $f(1) = 1$ under 'When $x = 1$, $y = 1^2 = 1$' of example 2(a) to show that the independent and dependent variables are clearer in function notation. He then returns to the notes. In the notes, the term 'linear' function appears which has not yet been discussed. Teacher Y then says: "My goal yesterday and today was to introduce you to an idea of a function; what is a function. We are going to do several functions, like a linear functions and quadratic functions, etc" Finally, Teacher Y reaches example 2(e) and quickly reads this aloud. Homework is given and the lesson ends.	<u>features:</u> f(x), $g(x), h(x)$, $k(x)$ - <u>aspect:</u> using function notation <u>features:</u> see independent / dependent variables; evaluate the function for a given input; - <u>aspect:</u> degree of a rule <u>features:</u> linear, quadratic
	2(c) $f(x) = x^2$ $f(1) = (1)^2 = 1$ $f(3) = (3)^2 = 9$	- equation - function notation		
	2(d) Advantages of using function notation • Since most functions are represented with various variables such as a, f, g, h, k etc, we use $f(x)$ in order to avoid confusion as to which function is being evaluated. • Function notation allows to identify the independent variable with ease. • Function notation also helps us to identify the element of a function which has to be examined. Consider a linear function $y = 3x + 7$. To write such function in function notation, we simply replace the variable y with the phrase $f(x)$ to get; $f(x) = 3x + 7$. This function $f(x) = 3x + 7$ is read as the value of f at x or a f of x.	- function notation		
	2(e)	- equation - function		

	How to Evaluate Functions? Function evaluation is the process of determining output values of a function. This is done by substituting the input values in the given function notation. Example 1 Write $y = x^2 + 4x + 1$ using function notation and evaluate the function at $x = 3$. Solution Given, $y = x^2 + 4x + 1$ By applying function notation, we get $f(x) = x^2 + 4x + 1$ Evaluation: Substitute x with 3 $f(3) = 3^2 + 4 \times 3 + 1 = 9 + 12 + 1$ $= 22$	notation		

Teacher Z: Detailed Lesson Descriptions

Table 20 **Teacher Z Lesson 1 Detailed Description**

<table>
<tr><th>Episode</th><th>Example</th><th>Representations</th><th>Patterns of Variation
(Examples / Teacher Actions / Student Interactions)</th><th>Object of Learning
(Content /
Educational Objectives /
Aspects & Features)</th></tr>
<tr><td rowspan="3">1</td><td>1(a)
Verbal definition:
“A function is basically a rule – almost like a spider diagram”</td><td></td><td rowspan="3">In Episode 1, there are 3 examples. 1(a) is a verbal definition of a function; 1(b) is a flow diagram; and 1(c) is a table with an equation. The function rule remains the same between examples 1(b) and 1(c).

Teacher Z starts the lesson by asking the class what they know about functions. After their vague responses, he provides a verbal definition, example 1(a). He then draws a flow diagram on the board (which the boys recognise immediately) and with the class finds the outputs for the provided inputs (example 1(b)). A boy then asks if functions are like number patterns. To this, Teacher Z responds: “I hear you. But it's not about the number pattern. That was a good question. Functions relate to number patterns but it’s also not where our investigation is going to be”.

He then refers back to example 1(b) and says: “So gents, this machine thing [pointing at the box with 2x+1], this number crunching thing, that is kind of what a function does. It takes something from one side and pops something out on the other side.”

Next, Teacher Z makes a link to prior knowledge from Grade 9 and represents the flow diagram as a table with a provided equation. Here, the rule remains the same and the representation changes. Teacher Z then explains that “[t]his question is exactly the same as the previous question” although the representation is different.

Teacher Z has thus used a combination of instances to create <u>contrast</u> of representation (flow diagram and table with equation) by keeping the function the same between example 1(b) and example 1(c).</td><td rowspan="3">Content:
- functions and rules
- inputs and outputs

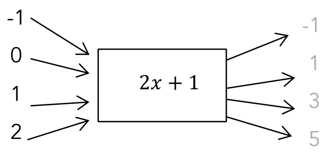
Educational Objective(s):
- understand a function as a rule

- find outputs from given inputs

- see a flow diagram and a table as different representations of the same rule

Aspects & Features:
- <u>aspect</u>: function
<u>features</u>: a rule; a flow diagram; an input-output table; an equation; related to number patterns; a number-crunching machine

- <u>aspect</u>: flow diagram
<u>features</u>: a function; creates outputs from inputs; similar to a table

- <u>aspect</u>: table
<u>features</u>: a function; creates outputs from inputs; similar to a flow diagram</td></tr>
<tr><td>1(b)

</td><td>- flow diagram</td></tr>
<tr><td>1(c)
<table><tr><td>x</td><td>-1</td><td>0</td><td>1</td><td>2</td></tr><tr><td>y</td><td>-1</td><td>1</td><td>3</td><td>5</td></tr></table>$y = 2x + 1$
$(0; 1)$</td><td>- table
- equation
- coordinate pairs</td></tr>
<tr><td rowspan="2">2</td><td>2(a)
$$y = 2x + 1$$
$\updownarrow$
$$f(x) = 2x + 1$$</td><td>- equation
- function notation</td><td rowspan="2">In Episode 2, there are four examples. 2(a) changes the notation of the equation in 1(c) to function notation; 2(b) is a verbal analogy regarding new notation; 2(c) is an example of function notation showing input and output; and 2(d) is an exercise to use function notation.

Episode 2 begins with Teacher Z saying the following:</td><td rowspan="2">Content:
- function notation

Educational Objective(s):
- understand function notation as a change of notation

- understand how input and</td></tr>
<tr><td>2(b)
[New] notation is something that can trip you up but it shouldn’t because it’s not something new, it</td><td></td></tr>
</table>

	just looks new. It's like when they put a new paint job on something. It's still the same thing.		"The first thing that I want to introduce that's new is – we're going to change our notation a little bit – and notation is something that can trip you up but it shouldn't because it's not something new it just looks new. It's like when they put a new paint job on something. It's still the same thing. Here, he is using a real-world example to position notation as a new representation of an idea thar remains unchanged. Teacher Z then explains that :"There's really only one reason we use this notation. It's because it's a bit more mathematical and we can use it in a bit more of an advanced way." Here, he is telling the class of the advantages of function notation. Next, he returns to example 1(c) and reuses the table to expand on what function notation is. He says: "Previously, if I wanted to talk about this $y = 1$ linked to the $x = 0$ [pointing at these values in the table] it's tricky to tell you that I want to talk about 'if the input is 0 then the output is 1'…In function notation you can actually do it very quickly. I can rewrite that as $f(0) = 1$ [he writes down example 2(c)] Now that looks like little written down but mathematically it says a lot. It means if I sub 0 into my machine then the output is a 1." He then moves on to the example 2(d) to practice using function notation. He provides two rules $f(x)$ and $g(x)$ (thus opening up a now aspect of function notation by creating <u>contrast</u> between the letters f and g). Both rules are linear with positive m-values. Teacher Z deliberately *shifts the focus of attention* to the <u>contrast</u> between the letters. He says: "So $f(x) = 2x + 1$ and $g(x) = 5x$. Now that might sound weird. I'm using two different letters. And when I use two different letters like a $f(x)$ and a $g(x)$ I'm talking about two different functions that we have in this questions. So I could refer to f – then I'm referring to this rule [points at $f(x)$], or if I refer to g, I'm referring to this rule points at $g(x)$],… f is like one of these number crunching machines [pointing at example 1(b)] and g is its own one." Here, by *shifting focus of attention* between the $f(x) = 2x + 1$ and the flow diagram in example 1(b) – which share the same rule – he is highlighting the equivalence of the representations and showing that <u>generalisation</u> can occur regarding the meaning of the two representations. He then starts to do the exercise with the class. For example 2(d).1, a boy says that the answer is 1. In interacting with the boy, Teacher Z	output relate to function notation - learn how to use function notation (evaluate an output from a given input). **Aspects & Features:** - <u>aspect</u>: function notation <u>features</u>: $f(x)$; $g(x)$; a new way of expressing an equation; a representation of input and output; an instruction to evaluate the output from a given input
	2(c) $f(0) = 1$ $\uparrow \quad \uparrow$ input output	- function notation		
	2(d) $f(x) = 2x + 1$ & $g(x) = 5x$ 1) $f(3) = 2(3) + 1$ $f(3) = 7$ 2) $g(2) = 5(2)$ $= 10$ 3) $f\big(g(2)\big)$ $= f(10) = 2(10) + 1$ $= 21$	- function notation		

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<td>established that the input and the output are being confused. The boy thought that the task was to find which value of x would create 3, rather than substitute 3 into the function. Teacher Z then immediately addresses this by adding the labels input and output to example 2(c) and explains that the value in the brackets of function notation is the provided input.

The class then continues to move through the exercises of example 2(d). A boy then asks about how these equations relate to graphs. Teacher Z then says that "all of these rules have associated graphs, and that is where we're actually heading".</td>
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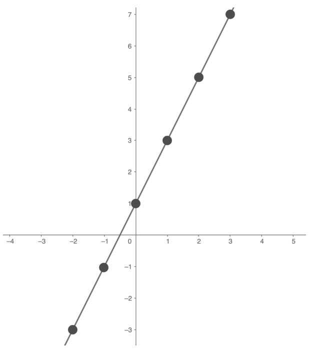
</tr>
<tr>
<td rowspan="3">3</td>
<td>3(a)

Sketch:
$f(x) = 2x + 1$
<table><tr><td>x</td><td>-3</td><td>-2</td><td>-1</td><td>0</td><td>1</td><td>2</td><td>3</td></tr><tr><td>y</td><td>-5</td><td>-3</td><td>-1</td><td>1</td><td>3</td><td>5</td><td>7</td></tr></table>$\downarrow$
$(-3; -5)$
</td>
<td>- equation
- function notation
- table
- coordinate pairs points
- graph</td>
<td rowspan="3">In Episode 3, there were three connected examples to sketch a graph and one verbal example regarding gradient. 3(a) was an example to sketch a graph that Teacher Z did as a worked example on the board. 3(b) and 3(c) were then given to the students to work on independently. Finally, 3(d) was a verbal real-world example using hiking up a mountain to discuss the steepness of different gradients.

Teacher Z began this section by saying: "Every time I give you an equation like this [$f(x) = 2x + 1$ and $g(x) = 5x$, from example 2(d)] it has its own graph… So every equation, if we put a bunch of input and output-values like we did over here [pointing at the table from example 1(c)]… this table creates a connection between x-values and y-values…every time you got an x-value and a y-value… it creates a coordinate. The 0 and the 1 [pointing at these values] will create a coordinate of $(0;\ 1)$ which is a coordinate on the cartesian plane that we can plot [he writes down coordinate $(0;\ 1)$ under the table in example 1(c)]."

He then continues:
"Now what functions is actually about is understanding how does the equation that I give you relate to the graph that it gives you."

He then writes down examples 3(a)-3(c) and tells the class that he will do 3(a) with them and that they will do 3(b) and 3(c) themselves.

While Teacher Z is busy working on the table for 3(a) a boy interjects: "Sir, I just noticed… the $f(x) = 2x + 1$ isn't that basically $y = mx + c$?"

To this, Teacher Z responds: "Yes, well done. Do you guys know what he just said there? He's tied this to what we did in analytical geometry… So we know what type of graph is this."</td>
<td rowspan="3">Content:
- sketching linear functions using a table

Educational Objective(s):
- using the technique of an input-output table to sketch linear functions

- understanding the necessary accuracy of a sketch graph

- using different representations to translate a graph equation to a graph sketch (equation → table → coordinate pairs → graph)

Aspects & Features:
- <u>aspect</u>: function
<u>new features</u>: relationship between an equation and its graph

- <u>aspect</u>: function notation
<u>new features</u>: can be used to label graphs

- <u>aspect</u>: linear graph
<u>features</u>: gradient; y-intercept; $y = mx + c$

- <u>aspect</u>: gradient
<u>features</u>: steep (above 1); less steep (below 1)

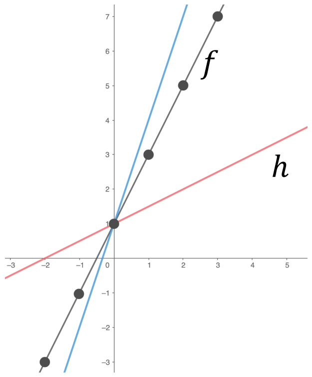
- <u>aspect</u>: representations of</td>
</tr>
<tr>
<td>3(b)

Sketch:
Create a new table but draw your graph on the same Cartesian plane as where you sketched $f(x)$.

$g(x) = 3x + 1$</td>
<td>- equation
- function notation
- table
- coordinate pairs points
- graph</td>
</tr>
<tr>
<td>3(c)
Create a new table but draw your graph on the same Cartesian plane as where you sketched $f(x)$.

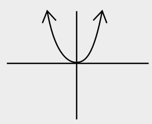
Sketch:
$h(x) = \frac{1}{2}x + 1$

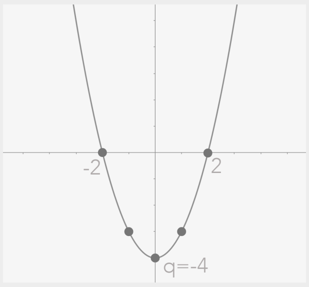
g</td>
<td>- equation
- function notation
- table
- coordinate pairs points
- graph</td>
</tr>
</table>

	 3(d) "…when you go on the hike, there are going to be some gradients that are steeper than others. When you are walking uphill but it's not terrible, that's a gradient of ½ or ¼ or something. When the gradient is bigger than 1, it's quite a steep gradient. When the gradient is less than 1, it's quite flat." (verbal example given by the teacher)		The class then establishes that it is a straight line graph with a gradient of 2 and a y-intercept of 1, although they continue to sketch from the table. When sketching the graph, Teacher Z emphasises the necessary accuracy of the axes of the Cartesian plane , explaining that the number line spacing must be constant. After he has sketched the graph, the boys then start working on 3(b) and 3(c). While doing this, Teacher Z walks around and answered queries. These included: - discussing how many x- and y-values to plot on the axes by looking at the respective values on one's table. - explaining that if a point is generated in a table that is beyond the $x-$ and $y-$values on the pre-existing axes, that this point can be left out and the graph can be drawn without it. - that it is acceptable to get decimal values as part of coordinates and that it is easier to use decimal form than fraction form to know where to plot a point. - that although the class has previously learnt do draw straight line graphs using only two points, using a table will serve a purpose as they will use this method to sketch more complex graphs. - that the difference between functions and number patterns is that functions can have non-Natural number inputs such as halves or quarters (unlike number patterns). Once the boys have sketched the two functions in their books, Teacher Z went through the work on the board. He sketched the two new graphs on the same Cartesian plane from 3(a). He then colour-coded the graphs and underlined their respective equation in the same colour. Next, he then explained that function notation allows one to label graphs using the function's letter (here: f, g, h) and did so. Then, Teacher Z asked the question: "What in these equations changed, and what in these equations stays the same?" To this, a student observed that "you did something to the x and … the 1 stayed the same…and the patterns of the ys changed." From this observation, Teacher Z went on to *shift the focus of attention*, and illustrates that the y-intercepts are all the same but that the gradients change. By keeping the y-intercept the same, it allowed the different gradients to be <u>contrasted</u> while <u>generalising</u> the $y-$intercept.	a function <u>features</u>: equation; table; coordinate; graph; function notation - <u>aspect</u>: an 'accurate' sketch graph <u>features</u>: regular axis increments; irregular axis increments - <u>aspect</u>: sketching graphs <u>features</u>: placing appropriate x-values and y-values on the axes; coordinates may be decimals; knowing when a point can be excluded from a sketch

			In engaging with the gradients, Teacher Z used a real world example. He mentioned their upcoming Grade 10 camp that would involve hikes. He said: "…when you go on the hike, there are going to be some gradients that are steeper than others. When you are walking uphill but it's not terrible, that's a gradient of $\frac{1}{2}$ or $\frac{1}{4}$ or something. When the gradient is bigger than 1, it's quite a steep gradient. When the gradient is less than 1, it's quite flat." At this point, the lesson ended. No homework was given.	

Table 21 **Teacher Z Lesson 2 Detailed Description**

Episode	Example	Representations	Patterns of Variation (Examples / Teacher Actions / Student Interactions)	Object of Learning (Content / Educational Objectives / Aspects & Features)
1	1(a) Parabola Equation: $y = ax^2 + q$ a-value $a = +$ $a = -$ q-value y-int of graph	- equation	In Lesson 2, Episode 1, there are 10 examples. Example 1(a) covers theory of parabolas; 1(b) is a real-world reference example; 1(c) is a basic parabola; 1(d) is a real world example; 1(e) is an action of throwing a board marker; 1(f) is a real world example; 1(g) sketches a parabola using a table; 1(h) discusses the substitution of negative values into a quadratic function; 1(i) is a real world example on symmetry; and 1(j) shows a rough sketch of a parabola's line of symmetry. Teacher Z begins the lesson by writing down standard form of a parabola in Grade 10: $y = ax^2 + q$. He says: "In that equation, the number a…and the number q are normally numbers. The x and y are always in an equation. Like with the straight line which you saw yesterday, the x and the y are always part of the equation and then the other parts mean something. Like in the straight line, the gradient is the m-value and it means something. But in this case the a-value and the q-value both mean something specific. By saying this, Teacher Z is *shifting focus of attention* to the different variables in the equation. He also creates contrast of the a-value of the parabola and the m-value of the straight line. In describing how the shape of a parabola links to the a-value, Teacher Z uses example 1(b) and references a smiling or frowning face. Next, he draws a basic parabola 1(c), highlighting that the a-value is 1 and the q-value is 0. He also *shifts the focus of attention* to putting arrows on the end of the arms of the graph as the graph will keep on going. In examples 1(d), 1(e), and 1(f), Teacher Z references real world examples of parabolas, mentioning the *McDonald's* M, throwing a board marker through the air (which he does physically) and throwing a basketball. This *combination of instances* allows for generalisation of the shape to occur. Before moving to 1(g), Teacher Z explains that the students will need to learn how to sketch a parabola, and how to find the equation of a parabola. In example 1(g), Teacher Z sketches a parabola using a table. While doing so, he uses example 1(h) to *shift the focus of attention* regarding how	**Content:** Parabola functions **Educational Objective(s):** - know the standard form equation of a parabola $y = ax^2 + q$ - understand the meaning of parameters a and q in the equation of a parabola - know the most basic parabola $y = x^2$ - know of parabola shapes in the real world - sketch a parabola using a table - know of the symmetric nature of parabola graphs and how this links to output values around the origin **Aspects & Features:** - aspect: parabola features: $y = ax^2 + q$; has a U-shape; is symmetrical about the y-axis; basic equation is $y = x^2$ - aspect: a-value features: positive $a \rightarrow \cup$ (smiley face); negative $a \rightarrow \cap$ (frowny face) - aspect: q-value features: y-intercept; could be represented as c in Grade 10 - aspect: parabolas in the real world features: *McDonald's* M; throwing a board marker; throwing a basketball - aspect: sketching a parabola features: use a table, substitute negative inputs in brackets; even number
	1(b) If the a-value is a positive number, then the graph has this happy face look to it…and if the a-value is a minus, then the graph has got this unhappy look to it.			
	1(c) basic $y = x^2$ 	- equation - graph		
	1(d) Things like you know the *McDonalds* M, that's actually a parabola shape.			
	1(e)- *action* *Teacher Z throws the board marker through the air to show the parabolic shape of the arc marker's trajectory and draws the shape on the board* 			
	1(f) "When you throw basketballs, they get modelled by a parabola equation"			

<table>
<tr>
<td rowspan="4"></td>
<td>1(g)
e.g.1 Sketch $y = x^2 - 4$ → q
<table><tr><td>x</td><td>-3</td><td>-2</td><td>-1</td><td>0</td><td>1</td><td>2</td><td>3</td></tr><tr><td>y</td><td>5</td><td>0</td><td>-3</td><td>-4</td><td>-3</td><td>0</td><td>5</td></tr></table>

</td>
<td>- equation
- table
- coordinate pairs
- graph</td>
<td rowspan="4">to incorrectly (counter-example) and then correctly substitute a negative x-value into a quadratic function. After the class have generated all the outputs in the table, Teacher Z shifts the focus of attention to the output values. He says:
Can you guys see what's happened here? Look at the numbers. They went 5; 0; -3; -4 and then -4 is kind of in the middle and then -3; 0; 5 [pointing at these values]. So what's happened on the one side, on the negative side, is the same as on the positive side. And that's because it [the parabola] has this shape that has the same y-values on both ends.

Teacher Z then sketches the coordinate pairs, reminding the students to have good spacing on the x-axis and y-axis of their graphs.

When connecting the coordinates, Teacher Z shifts the focus of attention to the shape. He says:
"Parabolas have this tendency to look like a V, if you're not careful they do not turn and then they land up with a sharp point. Try your best to have something round at the bottom…and when you draw put arrows on the end because it does carry on forever."

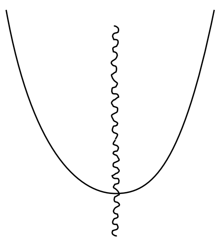
Once the shape of the graph is clear, a boy says: "So basically, the gist of a parabola is it just goes up and down, so it repeats".

Teacher Z responds explaining that the shape has to do with the symmetry of a parabola graph. This discussion around symmetry then leads to example 1(i) which references a butterfly to explain symmetry, and the sketch in 1(j) to show the approximate location of the graph's axis of symmetry.

Thereafter, Teacher Z shifts the focus of attention to the graph's q-value (-4) and the graph's y-intercept (-4), showing how these share the same value. A boy then asks why the y-intercept is c for a straight line, and q for a parabola. This leads to a discussion that in Grade 10, parabolas are always centred on the y-axis where the y-intercept and the turning point share a value. He explains that in Grade 11, when the graph shifts off the y-axis, that c will be the y-intercept and q will then have to do with the turning point.

Finally, another boy asks what the a-value of the graph is, and Teacher Z explains that it is 1.</td>
<td rowspan="4">spacing on the axes; plot coordinates and connect; have a round shape not a sharp point at the turning point; arrows at the end of the arms</td>
</tr>
<tr>
<td>1(h)

$y = x^2 - 4$
$y = -3^2 - 4$
$y = (-3)^2 - 4$</td>
<td>- equation</td>
</tr>
<tr>
<td>1(i)

"Symmetry…do you remember like a butterfly and symmetry, what happens on the one side of the butterfly looks the same as the other side."</td>
<td></td>
</tr>
<tr>
<td>1(j)
</td>
<td>- graph</td>
</tr>
<tr>
<td>2</td>
<td>2(a)

<u>How to sketch</u>
1. x-int ($y = 0$)
2. y-int (q-value)
3. check a-value</td>
<td></td>
<td>In Episode 2, there are seven examples. 2(a) gives procedural steps to sketch a parabola using only key points. 2(b) is a worked example to draw a parabola using these steps. 2(c) is an example of a parabola with only one x-intercept. 2(d) is an exercise for the students to</td>
<td>Content:
Parabola Sketch Graphs

Educational Objective(s):
- know the steps to draw a parabola without a table</td>
</tr>
</table>

<table>
<tr><td rowspan="4"></td><td>4. Draw</td><td></td><td rowspan="4">do to draw three parabolas. 2(e) sketches one of the exercises from 2(d) using online graphing software Desmos. 2(f) uses Desmos to create sliders to show the effects of a and q on a parabola. Finally, 2(g) gives a parabola equation to sketch which is not in standard form.

Teacher Z started Episode 2 by giving steps to draw a parabola without a table in example 2(a). The students' past experience of x-intercepts has only been for straight lines, so the teacher provides some explanation. He says: "…x-intercepts are the points where the graph crosses the x-axis;.. and we know that all points on the x-axis have a y-value of 0…because of that, we basically know what the y-value of the coordinate is on the x-axis, and therefore when we make $y = 0$ we are solving an equation which will then give us the x-value."

With this explanation, Teacher Z is shifting focus of attention to a feature of x-intercepts: their y-coordinate is always 0.

Teacher Z then does a worked example on the board to sketch the graph $y = -x^2 + 9$ in 2(b). When solving for the x-intercepts, he shifts the focus of attention to the fact that, for a parabola, this results in a quadratic equation. The class is rusty on solving these, and so he revises this briefly.

Once the x-intercept have been found, Teacher Z shifts the focus of attention to the fact that there are two x-intercept: $(-3;\ 0)$ and $(3;\ 0)$. A boy then asks if it is possible for a parabola to have just one x-intercept. In response, the teacher shows that for a negative a-value (that occurs in the example they are discussing) the q-value needs to be adjusted to create only one x-intercept. He draws example 2(c) in his explanation. Another boy then asks if it is possible for a parabola to have two y-intercepts. The teacher tells him, no, not for a parabola (although such graphs do exist).

The teacher also shifts the focus of attention to the necessary accuracy of a sketch graph. He explains that this is different to plotting from a table and that as long as the intercepts and turning point are approximately correctly positioned and correctly labelled, and if the graph's shape is correct, this is acceptable accuracy. However, due to the 'lack' of accuracy, he cautions against drawing two sketch-graphs on the same Cartesian plane.

Teacher Z then gives the class the set of exercises of example 2(d) to sketch. Many of the boys struggle with the factorisation required to find the x-intercepts. Teacher Z goes through how to do 2(d).1 but does not do the other exercises on the board.</td><td rowspan="4">- understand x-intercepts
- solve a quadratic equation (to find x-intercepts)

- understand the affects of parameters a and q in the equation of a parabola on a sketch graph

- use online graphing tool Desmos

- become aware of standard form of a parabola equation

Aspects & Features:

- <u>aspect</u>: a-value
<u>new features</u>: increasing positive a-values move the graph arms closer together; decreasing positive a-values move graph arms apart; $a = 0$ is a straight line not a parabola; negative a-values moving away from 0 move the arms closer together, negative a-values moving towards 0 move the arms further apart

- <u>aspect</u>: q-value
<u>new features</u>: affects the number of x-intercepts

- <u>aspect</u>: x-intercepts
<u>new features</u>: where the graph cuts the x-axis; a parabola can have one or two; $y = 0$ always

- <u>aspect</u>: y-intercept
<u>features</u>: a parabola will have one, cannot have two

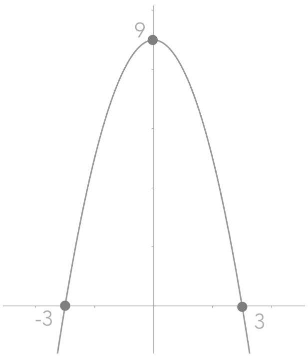
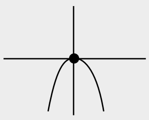
- <u>aspect</u>: sketch graphs
<u>features</u>: find and label key points clearly; get shape correct; less accuracy acceptable

- <u>aspect</u>: standard form
<u>features</u>: $y = ax^2 + q$ not $y - q = ax^2$</td></tr>
<tr><td>2(b)

Sketch $y = -x^2 + 9$

1. x-int ($y = 0$)
$0 = -x^2 + 9$
$x^2 - 9 = 0$
$(x - 3)(x + 3) = 0$
$x = 3$ or $x = -3$

y-int (q-value)
$y = 9$

$a = -$
</td><td>- equation
- coordinate pairs
- graph</td></tr>
<tr><td>2(c)
</td><td>- graph</td></tr>
<tr><td>2(d)

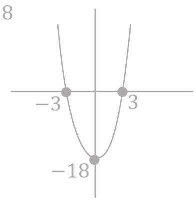
Sketch:
1. $y = 2x^2 - 18$
2. $y = -x^2 + 1$
3. $y = 3x^2 - 3$

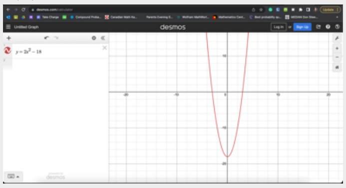
Solution to 1:

<u>x-int ($y = 0$)</u>

$0 = 2x^2 - 18$
$0 = 2(x^2 - 9)$
$0 = 2(x - 3)(x + 3)$
$x - 3 = 0$ or $x + 3 = 0$
$x = 3$ or $x = -3$

$x \cdot y = 0$
$2 \cdot x \cdot y = 0$

<u>y-int</u>
$y = 18$
</td><td>- equation
- coordinate pairs
- graph</td></tr>
</table>

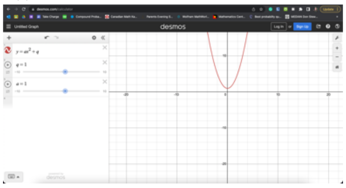
<table>
<tr><td></td><td>2(e)
</td><td>- equation
- graph</td><td rowspan="3">Next, Teacher Z introduces the online graphing website Desmos as a tool with which to digitally sketch and investigate parabolas. In example 2(e) he uses the programme to digitally sketch example 2(d).1 which has just been drawn by hand on the board.

Next, in example 2(f) he introduces the 'sliders' option, that show the effects of parameters a and q on the graph $y = ax^2 + q$. This is a differences as changes teacher action.

First, Teacher Z deliberately sets $a = 1$ and $q = 0$ and from there, changes the q-value up and down (while keeping $a = 1$). This allows for the effect of the q-value to be <u>contrasted</u>. He specifically highlights $q = 7$ and $q = -1$ as specific examples of a positive and negative q-value. Teacher Z then shifts the focus of attention to how the changing q-value results in a changing y-intercept and how it looks like the graph is being moved up and down.

Next, he sets $q = -1$ and moves to changing the a-value. Here, he first focuses on positive a-values, showing that if the a-value increases, the graph's arms move closer together while as the a-value decreases, the arms move further apart. Next, he moves to $a = 0$ which creates a horizontal line. Teacher Z discusses this and describes that "$a = 0$ kills the x^2, so it [the line] becomes $y = 0$ and it's a horizontal line". Finally, Teacher Z moves on to negative a-values and again shows that as the values become bigger negative numbers (moving away from 0), the arms move closer together but as the a-values become smaller negative values (moving towards 0), the graph's arms move further apart. In this second part of the exerice, the aspect "a" is explored and its various features (positive, zero, negative) are <u>contrasted</u>.

Finally, the last example of the lesson is given. Teacher Z asks the boys to sketch example 2(g). He shifts the focus of attention to the form of the parabola not being standard and that graphs are usually defined as $y = \cdots$. The boys are asked to draw this and it will be discussed next lesson.

This is the end of the second lesson.</td><td rowspan="3"></td></tr>
<tr><td></td><td>2(f)
</td><td>- equation
- graph</td></tr>
<tr><td></td><td>2(g)

Sketch:
$y - 25 = -x^2$</td><td>- equation
- coordinate pairs
- graph</td></tr>
</table>

Appendix B: Ethics Clearance Certificate

UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG

WSoE Wits School of Education

SCHOOL OF EDUCATION ETHICS COMMITTEE

CONSTITUTED UNDER THE UNIVERSITY HUMAN RESEARCH ETHICS COMMITTEE (NON-MEDICAL)

CLEARANCE CERTIFICATE — **PROTOCOL NUMBER: 2022ECE020M**

PROJECT TITLE	"Exemplification Across a Mathematics Department: Affordances for Learning in Introducing Grade 10 Functions"
INVESTIGATOR	**Kathryn Anne McLachlan**
SCHOOL/DEPARTMENT OF INVESTIGATOR	WSoE
DATE CONSIDERED	20 MAY 2022
DECISION OF THE COMMITTEE	Approved unconditionally
RISK LEVEL	NO RISK
EXPIRY DATE	Date of submission of the Research Report
ISSUE DATE OF CERTIFICATE	**CHAIRPERSON** Mbokane
	Dr. Batseba Mofolo-Mbokane

cc: Prof. Anthony Essien

DECLARATION OF INVESTIGATOR

To be completed in duplicate and **ONE COPY** returned to the Chairperson of the School/Department ethics committee.

I fully understand the conditions under which I am are authorized to carry out the abovementioned research and I guarantee to ensure compliance with these conditions. Should any departure to be contemplated from the research procedure as approved I/we undertake to resubmit the protocol to the Committee.

KMcLachlan

Signature

Date 30 / 05 / 2022

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