THE CHROMATIC POLYNOMIAL OF A GRAPH

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ABSTRACT

In this work we study the chromatic polynomial $P(G,x)$ of a graph G of order p in the form $\sum_{i=0}^{p-1} a_i T_{p-i}$ where $T_{p-i} = x(x-1)^{p-i-1}$ is the chromatic polynomial of a tree of order p.

Fistly we express the chromatic polynomials of some graphs in tree form. We then study a special product that comes natural and is useful in the caculation of some chromatic polynomials. Next we use the tree form to study the chromatic polynomial of a graph obtained from a forest (tree) by "blowing up" or "replacing" the vertices of the forest (tree) by a graph. Then we give explicit expressions, in terms of induced subgraphs, for the first five coefficients of the chromatic polynomial of a connected graph. In the case of higher order graphs we develop some useful computational techniques to obtain some higher order coefficients. In the process we obtain some useful combinatorial identities, some of which are new. We discuss in detail the application of these combinatorial identities to some families of graphs. We also discuss pairs of graphs that are chromatically equivalent and graphs that are chromatically unique with special emphasis on wheels.

In conclusion, we mention some open questions and conjectures.

DECLARATION

I declare that the contents of this thesis are original except where due references have been made. It has not been submitted before for any degree to any other institution.

A A Adam

DEDICATION

To my Family and my Friends

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Professor Izak Broere's insight and knowledge of graph theory has been a source of inspiration for this thesis. He has been my mentor and guide throughout the course of this research. I am immensely grateful to him for his critical appraisal of the thesis and for his constant encouragement.

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CONTENTS

Chapter

ABSTRACT

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Fistly we express the chromatic polynomials of some graphs in tree form. We then study a special product that comes natural and is useful in the caculation of some chromatic polynomials. Next we use the tree form to study the chromatic polynomial of a graph obtained from a forest (tree) by "blowing up" or "replacing" the vertices of the forest (tree) by a graph. Then we give explicit expressions, in terms of induced subgraphs, for the first five coefficients of the chromatic polynomial of a connected graph. In the case of higher order graphs we develop some useful computational techniques to obtain some higher order coefficients. In the process we obtain some useful combinatorial identities, some of which are new. We discuss in detail the application of these combinatorial identities to some families of graphs.We also discuss pairs of graphs that are chromatically equivalent and graphs that are chromatically unique with special emphasis on wheels.

In conclusion, we mention some open questions and conjectures.

INTRODUCTION

In this thesis we write the chromatic polynomial of a graph in terms of the chromatic polynomial of a tree. We now introduce the concept of colouring the vertices of a graph which is the underlying theme of this thesis. Let a graph G, a natural number x and any set S with x elements (called colours) be given. We define an x-colouring of G as a mapping of the vertex-set $V(G)$ of G into the colour-set S, subject to the restriction that each edge of G must join vertices of two different colours. We denote the number of x-colourings of G by $P(G, x)$. In section 1.5 we prove that $P(G, x)$ is indeed a polynomial in x and henceforth we refer to it as the chromatic polynomial of a graph G.

In more detail the outline of this work is as follows:

Chapter 1 deals with preliminaries. In section 1.2 the notation and terminology in graph theory is given. In section 1.3 we give some intuitive results on the chromatic polynomials of complete graphs and trees. In section 1.4 we prove Whitney's Reduction Theorem and give a motivation for studying the chromatic polynomial of a graph in terms of chromatic polynomials of trees; we conclude this section by illustrating where Stirling numbers of the first and second kind are used. In section 1.5 we present several easily proved results on chromatic polynomials in terms of chromatic polynomials of trees.

Chapter 2 deals with operations on chromatic polynomials. In section 2.2 we introduce the special operations $\odot$, $\ominus$ and $\circledast$ and we use them to simplify calculations in computing some chromatic polynomials. We also use the special operation $\circledast$ in section 2.3 to go from normal form to tree form for some special family of graphs. In section 2.4 we illustrate combinatorial identities which make use of Stirling numbers of the first and second kind.

Chapter 3 is devoted to replacing vertices of a tree or forest with a connected graph. In section 3.2 we describe the chromatic polynomial of a graph obtained by replacing vertices of a tree with connected graphs using the special operations $\odot$ and $\ominus$. We generalize the above concept to forests in section 3.3.

Chapter 4 is concerned with the coefficients of the chromatic polynomial. In section 4.2 we begin by interpreting the first five coefficients of a chromatic polynomial in tree form; we then relate these coefficients to that given in normal form in section 4.3. We use these five coefficients to obtain some higher order coefficients in section 4.4. In section 4.5 we discuss the concept of connectivity and how the coefficients of a chromatic polynomial behave in tree form. We then present a new combinatorial identity in section 4.6 and use it in section 4.7 to interpret the coefficients of the chromatic polynomial of an m-gon-tree.

In Chapter 5 we discuss the chromatic equivalence of pairs of graphs. In section 5.2 we construct pairs of graphs that are chromatically equivalent. The theory of chordal graphs and their chromatic polynomials are discussed in section 5.3. In section 5.4 we describe some family of graphs that are chromatically unique and in section 5.5 we discuss the chromaticity of wheels.

Finally, we conclude with a chapter describing some open problems and conjectures.

CHAPTER 1
PRELIMINARIES

1.1 Introduction

The basic notions from graph theory used are all defined in section 1.2 – this makes this thesis self-contained. We continue to define and illustrate chromatic polynomials of graphs, the main issue at stake in this thesis, in section 1.3. In section 1.4 we give a motivation for studying the chromatic polynomial of a graph in terms of chromatic polynomials of trees, and in section 1.5 we discuss some elementary properties of chromatic polynomials.

1.2 Notation and Terminology

Most of the definitions in this section are taken from [4].

A *graph* G is a finite nonempty set $V(G)$ together with a (possible empty) set $E(G)$ (disjoint from $V(G)$) of two-element subsets of (distinct) elements of $V(G)$. The set $V(G)$ is called the *vertex set* of G and its elements are called *vertices*, while $E(G)$ is called the *edge set* of G and its elements are called *edges*. By an *element of a graph* we shall mean a vertex or an edge.

The edge $e = \{u, v\}$ is said to *join* u and v, and will henceforth be denoted by uv or vu. If $e = uv$ is an edge of a graph G, then u and v are *adjacent vertices*, while the edge e is *incident* with the two vertices u and v. Furthermore if e and f are distinct edges of G incident with a common vertex, then e and f are *adjacent edges*.

A graph G_1 is *isomorphic* to a graph G_2 if there exists a one-to-one mapping ϕ from $V(G_1)$ onto $V(G_2)$ such that two vertices u_1 and v_1 are adjacent in G_1 if and only if $\phi(u_1)$ and $\phi(v_1)$ are adjacent in G_2; it is written as $G_1 \cong G_2$.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(H) \subset V(G)$ or $E(H) \subset E(G)$ then H is a *proper subgraph* of G. A (p, q) - graph G has p vertices and q edges; p is referred to as the *order* of G and q as the *size* of G. Whenever a subgraph H of a graph G has the same order as that of G, then H is called a *spanning subgraph* of G.

If U is a nonempty subset of $V(G)$ then the subgraph $\langle U \rangle$ of G *induced* by U is the graph having vertex set U and whose edge set consists of those edges of G incident with two vertices in U. If $v \in V(G)$ and $V(G) \geq 2$ then $G - v$ denotes the subgraph induced by the set $V(G) - \{v\}$. If $e \in E(G)$, then $G - e$ denotes the subgraph with vertex set $V(G)$ and edge set $E(G) - \{e\}$.

The *degree* of a vertex v in G, denoted by deg$v$, is the number of edges of G incident with v. The *minimum degree* of a vertex in G is denoted by $\delta(G)$, and the *maximum degree* by $\Delta(G)$. A vertex of degree 0 is called an *isolated vertex* and a vertex of degree 1 is an *end-vertex*. The edge incident to such a vertex is called a *pendent edge*.

The *complement* of a graph G, denoted by $\bar{G}$, is the graph with vertex set $V(G)$ and such that two vertices are adjacent in $\bar{G}$ if and only if these vertices are not adjacent in G. A (p, q) - graph is *complete*, denoted by K_p, if every two of its vertices are adjacent. The complement $\overline{K_p}$ of the complete graph K_p has p vertices and no edges and is referred to as the *empty* graph of order p. The graph K_1 is called the *trivial* graph.

By a $u - v$ *walk* of a graph G is meant a finite, alternating sequence of vertices and edges of G, beginning with u and ending with v, such that every edge is immediately preceded and succeeded by the two vertices with which it is incident. A $u-v$ *trail* is a $u-v$ walk in which no edge is repeated. A $u-v$ *path* is a $u-v$ walk in which no vertex is repeated. A vertex u is said to be *connected* to a vertex v in a graph G if there exists a $u-v$ path in G. A graph G is *connected* if every two of its vertices are connected. A graph which is not connected is *disconnected*.

A *component* of a graph G is a connected subgraph of G not properly contained in any other connected subgraph of G. The *number* of components of a graph G is denoted by $k(G)$. Thus $k(G) = 1$ if and only if G is connected.

A nontrivial closed trail of G is referred to as a *circuit* of G. A circuit $v_1, v_2, \ldots, v_n, v_1$, $n \geq 3$, of a graph G whose n vertices v_i, $1 \leq i \leq n$, are distinct is called a *cycle* of G. A cycle of length n is denoted by C_n; it is unique up to isomorphism.

An *acyclic* graph has no cycles. A *tree* is any acyclic connected graph. A *forest* is an acyclic graph. Each component of a forest is a tree. A tree of order p is denoted by S_p; it is not unique up to isomorphism if $p \geq 4$.

A vertex v of a graph G is called a *cut-vertex* of G if $k(G-v) > k(G)$. An edge e of a graph G is called a *cut-edge* or *bridge* of G if $k(G-e) > k(G)$. A nontrivial connected graph G with no cut-vertices is called a *block*. A *block of a graph* G is a subgraph of G which is a block and which is not a proper subgraph of any subgraph of G with this property. The blocks of a graph G partition the edge set of G and two blocks have at most one vertex in common; this must then be a cut-vertex of G. An end-block of a graph G is a block of G which contains exactly one cut-vertex of G.

The *union* $G_1 \cup G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. It is usually formed if the graphs G_1 and G_2

are *disjoint*, that is, $V(G_1) \cap V(G_2) = \phi$. The *join* $G_1 + G_2$ of disjoint graphs G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 to each vertex of G_2.

The vertex-connectivity or simply *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal from G results in a disconnected graph or the trivial graph. The *edge-connectivity* $\kappa_1(G)$ of a graph G is the minimum number of edges whose removal from G results in a disconnected graph or the trivial graph. A graph G is said to be *n-connected*, $n \geq 1$, if $\kappa(G) \geq n$.

Let a graph G, a natural number n and any set S with n elements (called *colours*) be given. An *n-colouring* of G is a function f of $V(G)$ into S such that, for every edge $e = uv$ of G, $f(u) \neq f(v)$. If G has an n-colouring, it is called *n-colourable* - note that every graph has an n-colouring for some n (take $n = |V(G)|$ and f any one-to-one function for example). The minimum n for which a graph G is n-colourable is called the *vertex chromatic number* or simply the *chromatic number* of G and is denoted by $\chi(G)$. Note that a graph G is n-colourable if and only if $\chi(G) \leq n$; it is called *n-chromatic* if $\chi(G) = n$.

A graph G is *n-partite*, $n \geq 1$, if it is possible to partition $V(G)$ into n subsets V_1, V_2, ..., V_n (called *partite sets*) such that every element of $E(G)$ joins a vertex of V_i to a vertex of V_j, $i \neq j$. (Of course G is n-partite if and only if G is n-colourable; the notation of this definition is, however, convenient in what follows.) For $n = 2$, such graphs are called *bipartite graphs*. A *complete n-partite* graph G is an n-partite graph with partite sets $V_1, V_2, \ldots, V_n$ having the added property that if $u \in V_i$ and $v \in V_j$, $i \neq j$, then $uv \in E(G)$. A *complete bipartite* graph with partite sets V_1 and V_2, where $|V_1| = m$ and $|V_2| = n$, is then denoted by $K(m, n)$.

The *independence number* $\beta(G)$ of a graph G is the maximum number of mutually non-adjacent vertices in G. The *clique number* $\omega(G)$ of a graph G is the maximum number of vertices in any complete subgraph of G.

The join $K_1 + C_{n-1}$, $n \geq 3$, is called the *wheel* on n vertices and is denoted by W_n.

An *elementary subdivision* of a nonempty graph G is a graph obtained from G by the removal of some edge $e = uv$ and the addition of a new vertex w and edges uw and vw. A *subdivision* of G is a graph obtained from G by a succession of elementary subdivisions. A graph H is defined to be *homeomorphic from* G if either $H \cong G$ or H is isomorphic to a subdivision of G. A graph G_1 is *homeomorphic with* a graph G_2 if there exists a graph G_3 such that each of G_1 and G_2 is homeomorphic from G_3.

1.3 The chromatic polynomial of a graph

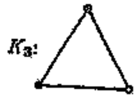
Let a graph G and a natural number n be given. Then it is possible to define $P(G,n)$ as *the number of n-colourings* of G. It is possible to prove (and we will do so in Lemma 1.5.1) that, for every graph G, $P(G,n)$ is a polynomial in n. Hence we will immediately change notation to $P(G,x)$, thinking of x as a complex variable (although x will almost always be real), and refer to this function as the *chromatic polynomial* of G. By way of illustration consider

G: 

The middle vertex can be coloured in any of the x colours. When this has been done, this colour is no longer available for colouring the end vertices. Hence the end vertices can be coloured independently each in $(x-1)$ ways. Thus

$$P(G,x) = x(x-1)^2 .$$

This result can be extended (see Lemma 1.5.3) to show that if G is any tree on p vertices then $P(G,x) = x(x-1)^{p-1}$ denoted by T_p. Next consider



There are x ways of colouring, say, the top vertex. There are then $(x-1)$ ways of colouring an adjacent vertex, and $(x-2)$ ways of colouring the remaining vertex. Thus

$$P(K_3, x) = x(x-1)(x-2).$$

This result can be extended (see Lemma 1.5.2) to

$$P(K_p, x) = x(x-1)(x-2)\dots(x-p+1),$$

denoted by $x^{\{p\}}$.

1.4 Motivation

In the remainder of this thesis we study the chromatic polynomial $P(G,x)$ of a graph G in its expression as a linear combination of the chromatic polynomials of trees, where $T_p = x(x-1)^{p-1}$ is the chromatic polynomial of a tree of order p (see section 1.5). In this section, we first answer two questions: **Why this can be done and how it can be done.**

In [14] the chromatic polynomial $P(G,x)$ of a graph G is often expressed as $\sum_i a_i x^i$ or as $\sum_i b_i x^{(i)}$ where $x^{(i)} = x(x-1)(x-2)\dots(x-i+1)$. This is always possible (see for example §4 of [6]) since the set $\{1, x, \dots, x^p, \dots\}$ as well as the set $\{1, x^{\{1\}}, \dots, x^{\{p\}}, \dots\}$ are bases for $Z[x]$, the Z-module of all polynomials in x over the ring of integers Z. Similarly, $\{1, T_1, T_2, \dots, T_p, \dots\}$, is a basis for this Z-module and hence can be exploited in a similar fashion. This answers the "why it can be done" question.

We now turn to the "how it can be done" question. In computing chromatic polynomials, one can make use of ***Whitney's Reduction Formula*** stated in the following theorem.

Theorem 1.4.1. Let G be a graph, and u, v be two non-adjacent vertices of G. Let G' be the graph obtained from G by joining u and v by an edge, and G'' be the graph obtained by identifying u and v, then

$$P(G,x) = P(G',x) + P(G'',x).$$

Proof. In any x-colouring of the vertices of G, either u and v have different colours or they have the same colour. The number of x-colourings in which u and v have different colours is unchanged if an edge joining u and v is added to G, and is therefore equal to $P(G',x)$. Similarly, the number of x-colourings in which u and v have the same colour is unchanged if u and v are identified, and is therefore equal to $P(G'',x)$. Hence

$$P(G,x) = P(G',x) + P(G'',x). \qquad \square$$

Repeated application of the above formula to any graph G will result in an expression of the form $P(G,x) = \sum_i b_i x^{(i)}$ which we shall refer to as the ***factorial form*** of $P(G,x)$.

The above recursive formula can also be used in the reverse process namely

$$P(G',x) = P(G,x) - P(G'',x)$$

where G is obtained from G' by deleting any edge uv and G'' is obtained from G by identifying u and v.

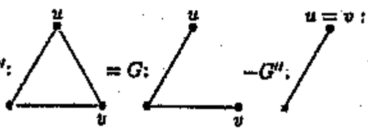
Repeated application of this formula to any graph G will result in an expression of the form $P(G,x) = \sum_i a_i x^i$ which we shall refer to as the *normal form* of $P(G,x)$.

If one is removing edges in chromatic reduction of a *connected* graph G, one might decide to stop the process once all the graphs involved are trees. This then results in an expression of the form $P(G,x) = \sum_i c_i T_i$, which we shall refer to as the *tree form* of $P(G,x)$. If on the other hand G is disconnected, we will later show how $P(G,x)$ can be obtained in tree form.

In the example which follows, we will (as is customary, see [14]) abuse notation by using a drawing of the graph for its chromatic polynomial.

Example 1.4.1.

We know that $P(K_3,x) = x(x-1)(x-2) = x^{(3)}$ in factorial form.

Also, $K_3 = G'$: 

by the recursive formula

$= T_3 - T_2$ in tree form

$= x^3 - 3x^2 + 2x$ in normal form.

We note, in conclusion of this section, that *Stirling numbers* are used to go from factorial form to normal form and vice versa. The particulars are as follows:

Stirling numbers of the *first kind*, denoted by $s(p,k)$, are used to convert a chromatic polynomial from factorial form to normal form:

$$\begin{aligned} x^{(p)} = p!\binom{x}{p} &= x(x-1)(x-2)\dots(x-p+1) \\ &= s(p,p)x^p + s(p,p-1)x^{p-1} + \cdots + s(p,0)x^{p-p} \\ &= \sum_{k=0}^{p} s(p,k)x^k \end{aligned}$$

where $s(0,0) = 1$ and $s(p,k) = s(p-1,k-1) - (p-1)s(p-1,k)$, $k = 1,2,\dots,p-1$.

Stirling numbers of the *second kind*, denoted by $S(p,k)$, are used to convert a chromatic polynomial from normal form to factorial form:

$$\begin{aligned} x^n &= S(p,p)x^{(p)} + S(p,p-1)x^{(p-1)} + \cdots + S(p,0)x^{(0)} \\ &= \sum_{k=0}^{p} S(p,k)x^{(k)} \end{aligned}$$

where $S(0,0) = 1$ and $S(p,k) = kS(p-1,k) + S(p-1,k-1)$, $k = 1,2,\dots,p-1$.

The reader is referred to [16] for more information about these numbers.

1.5 Basic Properties Of Chromatic Polynomials

The first thing we need to show is that $P(G,x)$ is indeed a polynomial in x. The following Lemma appears in [15]:

Lemma 1.5.1. If G is a graph on p vertices, then $P(G,x)$ is a polynomial of degree p.

Proof. For each nonnegative integer k, let $\alpha(G,k)$ denote the number of partitions of $V(G)$ into exactly k nonempty subsets, such that no edge of G joins two vertices in the same subset. From a set of x colours, there are $x(x-1)(x-2)\dots(x-k+1) = x^{(k)}$ ways of allocating a different colour to each of the subsets, and each of these gives a colouring of G. It follows that

$$P(G,x) = \sum_k x(x-1)(x-2)\dots(x-k+1)\alpha(G,k)$$

which is clearly a polynomial of degree p. □

Thus far we have calculated the chromatic polynomial of the empty graph on p vertices, denoted by $P(\bar{K}_p, x)$, to be x^p. We now establish a few more results on the chromatic polynomials of families of graphs beginning with the family of complete graphs which appears in [14].

Lemma 1.5.2. The complete graph K_p of order p has chromatic polynomial

$$P(K_p,x) = x(x-1)(x-2)\dots(x-p+1) = x^{(p)}\,.$$

Proof. Let K_p be the complete graph on p vertices. Choose any vertex of K_p and colour it; this can be done in x ways. Choosing another vertex we have $x-1$ colours with which it can be coloured, since it is adjacent to the first vertex. Choose another vertex; it is adjacent to both vertices already coloured, and can therefore be coloured in $x-2$ ways. We continue in this way; the last vertex can be given any of the remaining $x-(p-1)$ colours. Hence

$$P(K_p,x) = x(x-1)(x-2)\dots(x-p+1) = x^{(p)}.$$ □

We next consider the family of trees which also appears in [14].

Lemma 1.5.3. The chromatic polynomial of any tree S_p of order p is $P(S_p, x) = x(x-1)^{p-1}$.

Proof. We employ induction on p, the result being obvious for $p = 1$ and $p = 2$. Assume the chromatic polynomial of all trees with $p-1$ vertices is given by $x(x-1)^{p-2}$. Let v be an endvertex of G and suppose uv is the edge incident with v. By hypothesis, the tree $G' = G - v$ has $x(x-1)^{p-2}$ for its chromatic polynomial. The vertex v can be asigned any colour different from that assigned to u, so that v may be coloured in any of $x-1$ ways. Thus

$$P(G, x) = (x-1)P(G', x) = x(x-1)^{p-1}. \qquad \square$$

In section 4.5, we prove the converse of this lemma too. We will henceforth denote $P(S_p, x)$ by $T_p = x(x-1)^{p-1}$.

We next consider the family of cycles which appears in [14] in normal form as

$$P(C_p, x) = (x-1)^p + (-1)^p(x-1).$$

Lemma 1.5.4. For $p \geq 3$, the chromatic polynomial of the cycle C_p of order p is

$$P(C_p, x) = \sum_{i=0}^{p-2}(-1)^i T_{p-i}.$$

Proof. We employ induction on p. For $p = 3$ we have for any cycle on 3 vertices

$$P(C_3, x) = T_3 - T_2 = \sum_{i=0}^{3-2}(-1)^i T_{3-i},$$

by Example 1.4.1. Assume that for some fixed integer p, $p \geq 3$, the result is true for all cycles on n vertices, $n < p$. Let e be any edge of C_p. By Whitney's Reduction

Formula, applied to the edge e, we have:

$$\begin{aligned} P(C_p,x) &= P(C_p - e, x) - P(C_{p-1}, x) \\ &= P(S_p, x) - \sum_{i=0}^{p-1-2} (-1)^i T_{p-1-i} \quad \text{by the inductive hypothesis} \\ &= T_p - \sum_{i=0}^{p-3} (-1)^i T_{p-1-i} \quad \text{by Lemma 1.5.3.} \\ &= \sum_{i=0}^{p-2} (-1)^i T_{p-i}. \end{aligned}$$

Hence the lemma is true for all cycles. □

We note that for $p = 2$ this result gives the chromatic polynomial for the complete graph on 2 vertices, K_2. Our next result appears in [2]:

A *cascade of triangles* is defined recursively as any graph G_k that can be obtained by starting with $G_0 = K_2$ and, if G_i is defined, then G_{i+1} can be obtained by adding a new vertex v_i to G_i and joining v_i to any two adjacent vertices of G_i. Note that, in the terminology of [13] a cascade of triangles is $\{K_2\}$- *constructible* while in [21] it is called a *2-tree*.

We now consider the family of cascades of triangles which appears in [21] as

$$P(G,x) = x(x-1)(x-2)^{p-2}.$$

Lemma 1.5.5. If G is a cascade of triangles on p vertices then

$$P(G,x) = \sum_{i=0}^{p-2} (-1)^i \binom{p-2}{i} T_{p-i}.$$

Proof. We employ induction on p. For $p = 2$, the only cascade is K_2 with

$$P(K_2, x) = T_2 = \sum_{i=0}^{0}(-1)^i \binom{2-2}{i} T_{2-i}.$$

Assume that for some fixed integer p, $p \geq 3$, the formula holds for all cascades of triangles on n vertices, $n < p$. Now let G be any cascade of triangles on p vertices and let $e = uv$ be an edge of G with $\deg v = 2$. By Whitney's Reduction Formula, applied to the edge e we have

$$P(G, x) = P(G - e, x) - P(G - v, x)$$

where $G - e$ consists of a cascade of triangles on $p - 1$ vertices and a pendent edge, and $G - v$ is a cascade of triangles on $p - 1$ vertices. The vertex v can be assigned any colour different from that assigned to u, so that v may be coloured in any of $x - 1$ ways. Thus

$$\begin{aligned}
P(G, x) &= (x-1)P(G - v, x) - P(G - v, x) \\
&= (x-1)\sum_{i=0}^{p-3}(-1)^i \binom{p-3}{i} T_{p-1-i} - \sum_{i=0}^{p-3}(-1)^i \binom{p-3}{i} T_{p-1-i} \\
&\qquad \text{by the inductive hypothesis} \\
&= \sum_{i=0}^{p-3}(-1)^i \binom{p-3}{i} T_{p-i} - \sum_{i=0}^{p-3}(-1)^i \binom{p-3}{i} T_{p-1-i} \\
&= T_p - \binom{p-3}{1} T_{p-1} + \binom{p-3}{2} T_{p-2} - \binom{p-3}{3} T_{p-3} + \cdots + (-1)^{p-3} T_3 \\
&\quad - \left\{ T_{p-1} - \binom{p-3}{1} T_{p-2} + \binom{p-3}{2} T_{p-3} - \ldots + (-1)^{p-3}\binom{p-3}{p-3} T_2 \right\}
\end{aligned}$$

$$= T_p - \left\{ \binom{p-3}{1} + 1 \right\} T_{p-1} + \left\{ \binom{p-3}{2} + \binom{p-3}{1} \right\} T_{p-2} - \left\{ \binom{p-3}{3} \right.$$

$$\left. + \binom{p-3}{2} \right\} T_{p-3} + \cdots + (-1)^{p-3} \left\{ \binom{p-3}{p-3} + \binom{p-3}{p-4} \right\} T_3 + (-1)^{p-2} \binom{p-3}{p-3} T_2$$

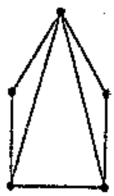
$$= T_p - \binom{p-2}{1} T_{p-1} + \binom{p-2}{2} T_{p-2} - \binom{p-2}{3} T_{p-3} + \cdots + (-1)^{p-2} \binom{p-2}{p-2} T_2$$

$$= \sum_{i=0}^{p-2} (-1)^i \binom{p-2}{i} T_{p-i} .$$

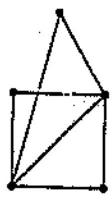
Hence the formula holds for all cascades. □

Example 1.5.1.

Consider G :



and H :



Then both G and H are cascades of triangles on 5 vertices, therefore

$$P(G, x) = P(H, x) = \sum_{i=0}^{3} (-1)^i \binom{3}{i} T_{5-i}$$

$$= T_5 - 3T_4 + 3T_3 - T_2.$$

We next consider a special class within the family of complete bipartite graphs; in factorial form these chromatic polynomials are described in [18] as

$$P(K(m,n), x) = \sum_{r=1}^{m} \sum_{s=1}^{n} S(m,r) S(n,s) x^{(r+s)} .$$

Lemma 1.5.6. The chromatic polynomial, $P(K(2,n),x)$, of the complete bipartite graph $K(2,n)$ on $n+2$ vertices is

$$P(K(2,n),x) = T_{n+1} + \sum_{i=0}^{n}(-1)^i \binom{n}{i} T_{n+2-i}.$$

Proof. Let u and v be the vertices in the class containing only two vertices. Then by joining u and v by an edge e and applying Whitney's Reduction Formula, Theorem 1.4.1, we have:

$$P(K(2,n),x) = P(G',x) + P(G'',x)$$

where $G' = K(2,n) + e$ is a cascade of triangles on $n+2$ vertices and $G'' = S_{n+1}$ is a tree on $n+1$ vertices. Hence by Lemma 1.5.5 and Lemma 1.5.3 we have:

$$P(K(2,n),x) = T_{n+1} + \textstyle\sum_{i=0}^{n}(-1)^i \binom{n}{i} T_{n+2-i}. \qquad \square$$

We now consider the family of wheels.

Lemma 1.5.7. The chromatic polynomial of the wheel W_p of order p is

$$P(W_p,x) = (-1)^{p-1}T_2 + \sum_{i=0}^{p-2}(-1)^i \binom{p-1}{i} T_{p-i}.$$

Proof. We employ induction on p. For $p = 4$ the formula is easily checked since $W_4 = K_4$. Assume that for some fixed integer p, $p \geq 5$, the result is true for all wheels on n vertices with $n < p$. Now let G be the wheel on p vertices and let uv be any edge on the rim of G. By Whitney's Reduction Formula, applied to the edge uv, we have:

$$P(W_p,x) = P(G',x) - P(G'',x)$$

where G' is a cascade of triangles on p vertices and G'' is a wheel on $p-1$ vertices. Hence by Lemma 1.5.5 and the inductive hypothesis we have:

$$\begin{aligned}
P(W_p,x) &= \sum_{i=0}^{p-2}(-1)^i\binom{p-2}{i}T_{p-i} - (-1)^{p-2}T_2 - \sum_{i=0}^{p-3}(-1)^i\binom{p-2}{i}T_{p-1-i} \\
&= (-1)^{p-1}T_2 + \sum_{i=0}^{p-2}(-1)^i\binom{p-2}{i}T_{p-i} - \sum_{i=0}^{p-3}(-1)^i\binom{p-2}{i}T_{p-1-i} \\
&= (-1)^{p-2}T_2 + \sum_{i=0}^{p-2}(-1)^i\binom{p-1}{i}T_{p-i}.
\end{aligned}$$

Hence the result is true for all wheels. □

This result can be compared to the result given in [19] as:

$$P(W_p,x) = x\{(x-2)^{p-1} + (-1)^{p-1}(x-2)\}.$$

The chromatic polynomial of more classes of graphs are computed elsewhere in the thesis.

CHAPTER 2
OPERATIONS ON GRAPHS AND CHROMATIC POLYNOMIALS

2.1 Introduction

We describe the operations $\times$, $*$, $\odot$, $\oplus$ and $\circledast$ on chromatic polynomials and use the join operation, $+$, on graphs to introduce some useful computational techniques whereby it will become easier to calculate chromatic polynomials of certain complex graphs. Finally we give useful formulae, for the chromatic polynomials of certain standard graphs, wherein the Stirling numbers of the first and second kind are the coefficients.

2.2 Computational Techniques

We begin with a basic result on trees which uses the operation $\times$.

Lemma 2.2.1. If T_p and T_q are the chromatic polynomials of trees on p and q vertices respectively, then $T_p \times T_q = T_{p+q} + T_{p+q-1}$.

Proof.

$$\begin{aligned}
T_p \times T_q &= x(x-1)^{p-1} \times x(x-1)^{q-1} \\
&= x^2(x-1)^{p+q-2} \\
&= x(x-1)^{p+q-2}\{(x-1)+1\} \\
&= x(x-1)^{p+q-1} + x(x-1)^{p+q-2} \\
&= T_{p+q} + T_{p+q-1}\,.
\end{aligned}$$

□

The next result appears as Theorem 3 in [14]:

Theorem 2.2.1. If a graph G consists of two subgraphs H_1 and H_2 which overlap in a complete graph on k vertices then

$$P(G,x) = \frac{P(H_1,x) \times P(H_2,x)}{x^{(k)}}.$$

Proof. The number of ways of colouring the common part is $x^{(k)}$. If we fix the colours of these k vertices there will be $\frac{P(H_1,x)}{x^{(k)}}$ ways of colouring the remaining vertices of H_1 and $\frac{P(H_2,x)}{x^{(k)}}$ ways of colouring the remaining vertices of H_2. Hence the total number of colourings is

$$x^{(k)} \times \frac{P(H_1,x)}{x^{(k)}} \times \frac{P(H_2,x)}{x^{(k)}} = \frac{P(H_1,x) \times P(H_2,x)}{x^{(k)}}. \qquad \square$$

To simplify some calculations, we use

Lemma 2.2.2. Any expression of the form $\frac{P(H_1,x) \times P(H_2,x)}{T_r}$ can be calculated as $\{P(H_1,x) \odot P(H_2,x)\} \ominus T_r$ where $\odot$ and $\ominus$ denote a type of multiplication and division in which $T_m \odot T_n = T_{m+n}$ and $T_m \ominus T_n = T_{m-n}$ respectively.

Proof. Let $P(H_1,x) = \sum_{i=0}^{p}(-1)^i a_i T_{m-i}$ and $P(H_2,x) = \sum_{j=0}^{q}(-1)^j b_j T_{n-j}$ where $m-p \geq r$ and $n-q \geq r$. Using Lemma 2.2.1 we have:

$$\frac{P(H_1,x) \times P(H_2,x)}{T_r}$$

$$= \frac{(\sum_{i=0}^{p}(-1)^i a_i T_{m-i}) \times (\sum_{j=0}^{q}(-1)^j b_j T_{n-j})}{T_r}$$

$$= [(T_m - a_1 T_{m-1} + a_2 T_{m-2} - \cdots + (-1)^{p-1} a_{p-1} T_{m-p+1} + (-1)^p a_p T_{m-p}) \times$$

$$(T_n - b_1 T_{n-1} + b_2 T_{n-2} - \cdots + (-1)^{q-1} b_{q-1} T_{n-q+1} + (-1)^q b_q T_{n-q})]/T_r$$

$$= [(T_{m+n} + T_{m+n-1}) - (a_1 + b_1)(T_{m+n-1} + T_{m+n-2}) + (a_2 + a_1 b_1 + b_2) \times$$

$$(T_{m+n-2}+T_{m+n-3})-(a_3+a_2b_1+a_1b_2+b_3)(T_{m+n-3}+T_{m+n-4})$$
$$+\cdots+(-1)^{p+q-1}(a_pb_{q-1}+a_{p-1}b_q)(T_{m+n-p-q+1}+T_{m+n-p-q})$$
$$+(-1)^{p+q}a_pb_q(T_{m+n-p-q}+T_{m+n-p-q-1})]/T_r$$

But

$$\begin{aligned}T_p+T_{p-1}&=x(x-1)^{p-1}+x(x-1)^{p-2}\\&=x(x-1)^{p-2}[x-1+1]\\&=x^2(x-1)^{p-2}\end{aligned}$$

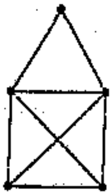
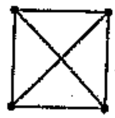
Hence

$$\frac{P(H_1,x)\times P(H_2,x)}{T_r}$$
$$=\left[x^2(x-1)^{m+n-2}-(a_1+b_1)x^2(x-1)^{m+n-3}+(a_2+a_1b_1+b_2)x^2(x-1)^{m+n-4}\right.$$
$$-(a_3+a_2b_1+a_1b_2+b_3)x^2(x-1)^{m+n-5}+\cdots+(-1)^{p+q-1}(a_pb_{q-1}$$
$$\left.+a_{p-1}b_q)x^2(x-1)^{m+n-p-q-1}+(-1)^{p+q}a_pb_qx^2(x-1)^{m+n-p-q-2}\right]/x(x-1)^{r-1}$$
$$=x(x-1)^{m+n-r-1}-(a_1+b_1)x(x-1)^{m+n-r-2}+(a_2+a_1b_1+b_2)x$$
$$(x-1)^{m+n-r-3}-(a_3+a_2b_1+a_1b_2+b_3)x(x-1)^{m+n-r-4}+\cdots+(-1)^{p+q-1}$$
$$(a_pb_{q-1}+a_{p-1}b_q)x(x-1)^{m+n-p-q-r}+(-1)^{p+q}a_pb_qx(x-1)^{m+n-p-q-r-1}$$
$$=T_{m+n-r}-(a_1+b_1)T_{m+n-r-1}+(a_2+a_1b_1+b_2)T_{m+n-r-2}$$
$$-(a_3+a_2b_1+a_1b_2+b_3)T_{m+n-r-3}+\cdots+(-1)^{p+q-1}(a_pb_{q-1}+a_{p-1}b_q)$$
$$T_{m+n-p-q-r+1}+(-1)^{p+q}a_pb_qT_{m+n-p-q-r}$$
$$=[T_m-a_1T_{m-1}+a_2T_{m-2}-\cdots+(-1)^{p-1}a_{p-1}T_{m-p+1}+(-1)^pa_p$$
$$T_{m-p}]\odot[T_n-b_1T_{n-1}+b_2T_{n-2}-\cdots+(-1)^{q-1}b_{q-1}T_{n-q+1}+(-1)^qb_qT_{n-q}]\ominus T_r$$
$$=\left[(\sum_{i=0}^{p}(-1)^ia_iT_{m-i})\odot(\sum_{j=0}^{q}(-1)^jb_jT_{n-j})\right]\ominus T_r$$

Hence

$$\frac{P(H_1,x)\times P(H_2,x)}{T_r} = [P(H_1,x)\odot P(H_2,x)]\ominus T_r\,. \qquad \Box$$

Example 2.2.1.

With G:  H_1:  H_2: 

and $T_r = T_2$:  , we have

$$P(G,x) = \frac{[P(K_3,x)\times P(K_4,x)]}{T_2} \quad \text{by Theorem 2.2.1.}$$

$$= [(T_3 - T_2)\odot(T_4 - 3T_3 + 2T_2)]\ominus T_2 \quad \text{by Lemma 2.2.2.}$$

$$= T_5 - 4T_4 + 5T_3 - 2T_2\,.$$

We stress that the operations $\odot$ and $\ominus$ can only be used in the type of expression specified in Lemma 2.2.2.

Corollary 2.2.1. If a graph G consists of two subgraphs H_1 and H_2 which overlap in a cut vertex then $P(G,x) = [P(H_1,x)\odot P(H_2,x)]\ominus T_1$. $\Box$

Another way of calculating these products is contained in our next result.

Lemma 2.2.3. $\sum_{i=0}^{m-2} a_i T_{m-i} \odot \sum_{j=0}^{n-2} b_j T_{n-j} = \sum_{i=0}^{m-2} a_i(\sum_{j=0}^{n-2} b_j T_{m+n-i-j})$.

Proof.

$$\sum_{i=0}^{m-2} a_i T_{m-i} \odot \sum_{j=0}^{n-2} b_j T_{n-j}$$

$$= (a_0 T_m + a_1 T_{m-1} + \cdots + a_{m-2} T_2) \odot (b_0 T_n + b_1 T_{n-1} + \ldots + b_{n-2} T_2)$$

$$= a_0 b_0 T_{m+n} + (a_0 b_1 + a_1 b_0) T_{m+n-1} + \cdots + a_{m-2} b_{n-2} T_4$$

$$= a_0 (b_0 T_{m+n} + b_1 T_{m+n-1} + \cdots + b_{n-2} T_{m+2}) - a_1 (b_0 T_{m+n-1}$$

$$+ b_1 T_{m+n-2} + \cdots + b_{n-2} T_{m+1})$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ a_{m-2} (b_0 T_{n+2} + b_1 T_{n+1} + \cdots + b_{n-2} T_4)$$

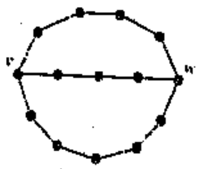
$$= \sum_{i=0}^{m-2} a_i \Big(\sum_{j=0}^{n-2} b_j T_{m+n-i-j}\Big). \qquad \square$$

Note that, since $\odot$ is a commutative product, it follows that

$$\sum_{i=0}^{m-2} a_i T_{m-i} \odot \sum_{j=0}^{n-2} b_j T_{n-j} = \sum_{j=0}^{n-2} b_j \Big(\sum_{i=0}^{m-2} a_i T_{m+n-i-j}\Big).$$

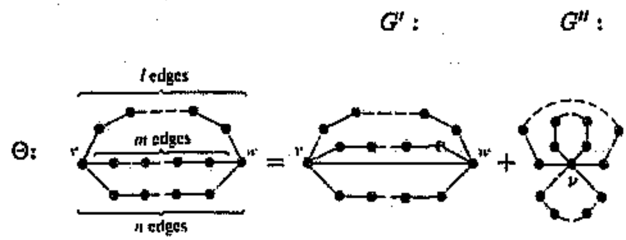
Lemma 2.2.2 can be effectively applied to determine the chromatic polynomials of certain special graphs. One such graph is the *theta graph*. A *theta graph*, Θ, has two vertices v and w of degree 3, between which are three paths whose other vertices all have degree 2. A typical theta graph, Θ, is shown below see [15].

Θ:



Lemma 2.2.4. The chromatic polynomial of the theta graph, Θ, for which the paths have l, m and n edges is $P(\Theta, x) = [P(C_{l+1}, x) \odot P(C_{m+1}, x) \odot P(C_{n+1}, x)] \ominus T_4 + [P(C_l, x) \odot P(C_m, x) \odot P(C_n, x)] \ominus T_2$.

Proof. By applying Whitney's Reduction Formula to the theta graph we have



$P(\Theta, x) = P(G', x) + P(G'', x)$, where G' is a graph in which the three cycles C_{l+1}, C_{m+1} and C_{n+1} overlap in a complete graph on two vertices and G'' is a graph in which the three cycles C_l, C_m and C_n overlap in a cut vertex v. Hence by repeated application of Theorem 2.2.1 and Corollary 2.2.1, we have by Lemma 2.2.2

$$P(\Theta, x) = [P(C_{l+1}, x) \odot P(C_{m+1}, x) \odot P(C_{n+1}, x)] \ominus T_4$$
$$+ [P(C_l, x) \odot P(C_m, x) \odot P(C_n, x)] \ominus T_2 . \qquad \square$$

Let $\circledast$ denote the operation on polynomials in which each term $a_i x^{p-i}$ in $P(x) = \sum_{i=0}^{p-1} a_i x^{p-i}$ is replaced by $a_i x(x-1)^{p-i}$, that is, $P^{\circledast}(x) = \sum_{i=0}^{p-1} a_i x(x-1)^{p-i}$.

Lemma 2.2.5. Suppose the polynomial $P(x)$ is a sum of terms of the form $ax(x-1)(x-2)\ldots(x-k)^i$. Then, for each such term, $P^{\circledast}(x)$ has $ax(x-1)\ldots(x-k)(x-k-1)^i$ as a term.

Proof. Since the opertion $\circledast$ satisfies $[P(x)+Q(x)]^{\circledast} = P^{\circledast}(x) + Q^{\circledast}(x)$, we need only to show it for one term, that is, $[ax(x-1)\ldots(x-k)^i]^{\circledast} = ax(x-1)\ldots(x-k)(x-k-1)^i$. This we do now by induction on k: For $k = 0$, our claim is nothing but the definition of ⊛. Hence suppose the result is true for $k-1$ and consider

$$\left[ax(x-1)\ldots(x-k+1)(x-k)^i\right]^{⊛} = \left[ax(x-1)\ldots(x-k+1)(\underline{x-k+1}-1)^i\right]^{⊛}$$

$$= \left[ax(x-1)\ldots(x-k+1)\sum_{l=0}^{i}\binom{i}{l}(x-k+1)^l(-1)^{i-l}\right]^{⊛}$$

$$= \left[\sum_{l=0}^{i}a\binom{i}{l}(-1)^{i-l}x(x-1)\ldots(x-k+1)^{l+1}\right]^{⊛}$$

$$= \sum_{l=0}^{i}a\binom{i}{l}(-1)^{i-l}x(x-1)\ldots(x-k+1)(x-k)^{l+1}$$

$$= ax(x-1)\ldots(x-k)\sum_{l=0}^{i}\binom{i}{l}(-1)^{i-l}(x-k)^l$$

$$= ax(x-1)\ldots(x-k)(x-k-1)^i .$$ □

The operation ⊛ will be used in section 2.3.

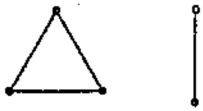
2.3 Operation on graphs and the chromatic polynomials.

We begin with a result which appears as Theorem 2 in [14]:

Theorem 2.3.1. If a graph has connected components $G_1, G_2, \ldots, G_k$, then $P(G,x) = P(G_1,x) \times P(G_2,x) \times \ldots \times P(G_k,x)$.

Proof. Since the components G_i, $1 \le i \le k$, are disjoint, the colouring of each component is independent of the colouring of the others. Hence the number of ways of colouring the whole graph G is simply the product, by the multiplication principle, of the numbers of colourings of the separate components. □

Example 2.3.1.

With G:  we have

$$P(G,x) = (T_3 - T_2) \times T_2$$
$$= T_5 + T_4 - (T_4 + T_3)$$
$$= T_5 - T_3$$

The next result appears as Theorem 4 in [14]:

Theorem 2.3.2. The chromatic polynomial of the join, $H_1 + H_2$, of two graphs H_1 and H_2 is $P(H_1,x) * P(H_2,x)$ where $*$ denotes a type of multiplication in which $x^{\{m\}} * x^{\{n\}} = x^{\{m+n\}}$.

Proof. Let the two graphs H_1 and have m and n vertices respectively. By repeated application of Whitney's Reduction Formula, $P(G,x) = P(G',x) + P(G'',x)$, to H_1 and H_2, separately, we express the chromatic polynomials of H_1 and H_2 in terms of the chromatic polynomials of complete graphs, namely

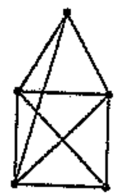
$$P(H_1,x) = P(K_m,x) + a_1P(K_{m-1},x) + a_2P(K_{m-2},x) + \cdots \quad (2.1)$$

and

$$P(H_2,x) = P(K_n,x) + b_1P(K_{n-1},x) + b_2P(K_{n-2},x) + \cdots \quad (2.2)$$

Also by repeated application of Whitney's Reduction Formula to $H_1 + H_2$, the H_1 and H_2 portions of the join $H_1 + H_2$ can be reduced in exactly the above way. In this process every vertex of each graph used in the reduction of H_1 will be adjacent to every vertex of each graph in the reduction of H_2. Thus we shall end up by expressing $P(H_1 + H_2, x)$ in terms of all possible joins of a complete graph from (2.1) and a complete graph from (2.2). But the join $K_p + K_q = K_{p+q}$, where K_p and K_q are complete graphs on p and q vertices respectively. Thus corresponding to a term $x^{(p)}$ of $P(H_1, x)$ and a term $x^{(q)}$ of $P(H_2, x)$, there will be a term $x^{(p+q)} = x^{(p)} * x^{(q)}$ in $P(H_1 + H_2, x)$. Hence $P(H_1 + H_2, x) = P(H_1, x) * P(H_2, x)$. □

Example 2.3.2.

With H_1:  , H_2:  , $H_1 + H_2$: 

$P(H_1, x) = x^{(2)}$ and $P(H_2, x) = x^{(3)} + x^{(2)}$, we have

$$
\begin{aligned}
P(H_1 + H_2, x) &= P(H_1, x) * P(H_2, x) \\
&= x^{(2)} * (x^{(3)} + x^{(2)}) \\
&= x^{(5)} + x^{(4)} \\
&= x(x-1)(x-2)(x-3)(x-4) + x(x-1)(x-2)(x-3) \\
&= x(x-1)(x-2)(x-3)^2 \\
&= x^5 - 9x^4 + 29x^3 - 39x^2 + 18x.
\end{aligned}
$$

We now consider a special case of Theorem 2.3.1.

Lemma 2.3.1. Let G_0 be any graph of order p with chromatic polynomial $P(G_0, x) = \sum_{i=0}^{p-1} (-1)^i a_i x^{p-i}$. If G_1 is the graph obtained from G_0 by adding a new vertex v

to G_0 and joining v to every vertex of G_0, then

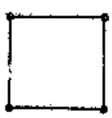
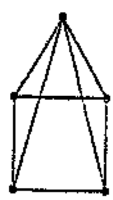
$$\begin{aligned} P(G_1, x) &= P^{\circledast}(G_0, x) \\ &= \sum_{i=0}^{p-1} (-1)^i a_i x (x-1)^{p-i}. \end{aligned}$$

Proof. Let $P(G_0, x) = a_0 x^p - a_1 x^{p-1} + \ldots + (-1)^{p-1} a_{p-1} x$. Since v is adjacent to every vertex of G_0 in G_1, each empty graph term x^k of $P(G_0, x)$, $1 \leq k \leq p$, is replaced by the polynomial of a tree on $k+1$ vertices, that is, by $x(x-1)^k$. Hence

$$\begin{aligned} P(G_1, x) &= \sum_{i=0}^{p-1} (-1)^i a_i x (x-1)^{p-i} \\ &= P^{\circledast}(G_0, x). \end{aligned}$$ □

Note that in Lemma 2.3.1, $P(G_1, x) = \sum_{i=0}^{p-1} (-1)^i a_i T_{p-i+1}$.

Example 2.3.3.

If G_0:  then G_1:  , where

$$\begin{aligned} P(G_0, x) &= T_4 - T_3 + T_2 \\ &= x(x-1)^3 - x(x-1)^2 + x(x-1) \end{aligned}$$

Hence

$$\begin{aligned} P(G_1, x) &= P^{\circledast}(G_0, x) \\ &= x(x-1)(x-2)^3 - x(x-1)(x-2)^2 + x(x-1)(x-2) \text{ by Lemma 2.2.5.} \end{aligned}$$

As a special case of Lemma 2.3.1 let G_0 be the cycle on p vertices. Then

$$\begin{aligned}P(G_0,x) &= P(C_p,x)\\ &= T_p - T_{p-1} + \cdots + (-1)^{p-2}T_2\\ &= x(x-1)^{p-1} - x(x-1)^{p-2} + \cdots + (-1)^{p-2}x(x-1)\end{aligned}$$

and

$$\begin{aligned}P(G_1,x) &= P^{\circledast}(G_0,x)\\ &= x(x-1)(x-2)^{p-1} - x(x-1)(x-2)^{p-2}\\ &+ \cdots + (-1)^{p-2}x(x-1)(x-2)\end{aligned}$$

which is the chromatic polynomial of the wheel on $p+1$ vertices. This proves

Corollary 2.3.1. $P(W_{p+1},x) = \sum_{i=0}^{p-2}(-1)^i x(x-1)(x-2)^{p-1-i}$. □

Since the wheel W_p of order p (which can also be considered as a pyramid) consists of the cycle C_{p-1} with an additional vertex joined to all the other vertices, we consider, in a similar way two other families of graphs.

The *biwheel* U_p of order p consists of the wheel W_{p-1} with an additional vertex joined to all the other vertices, that is $U_p = K_1 + W_{p-1}, \quad p \geq 5$.

The *bipyramid* B_p of order p is obtained from U_p by deleting the edge joining the two added vertices.

We note that U_5 is K_5 and that B_6 is the graph of the octahedron. We give equivalent formulae for U_p and B_p, (see [15]), in the following lemma:

Lemma 2.3.2.

$$\text{(a)}\quad P(U_p,x)=(-1)^{p-2}x(x-1)(x-2)+\sum_{i=0}^{p-3}(-1)^i\binom{p-2}{i}x(x-1)(x-2)^{p-2-i}$$

$$=\sum_{i=0}^{p-4}(-1)^i x(x-1)(x-2)(x-3)^{p-3-i}$$

$$\text{(b)}\quad P(B_p,x)=x(x-1)(x-2)^{p-2}+(-1)^{p-2}x(x-1)(x-2)$$

$$+\sum_{i=1}^{p-3}(-1)^i\left[\binom{p-2}{i}-1\right]x(x-1)(x-2)^{p-2-i}.$$

Proof. (a) Since $P(W_{p-1},x)=(-1)^{p-2}x(x-1)+\sum_{i=0}^{p-3}(-1)^i\binom{p-2}{i}x(x-1)^{p-2-i}$ by Lemma 1.5.7, we have from Lemma 2.3.1 (by adding a new vertex and joining it to all the vertices of W_{p-1}) that

$$P(U_p,x)=(-1)^{p-2}x(x-1)(x-2)+\textstyle\sum_{i=0}^{p-3}(-1)^i\binom{p-2}{i}x(x-1)(x-2)^{p-2-i}.$$

Similarly from Corollary 2.3.1 since $P(W_{p-1},x)=\sum_{i=0}^{p-4}(-1)^i x(x-1)(x-2)^{p-3-i}$ we have $P(U_p,x)=\sum_{i=0}^{p-4}(-1)^i x(x-1)(x-2)(x-3)^{p-3-i}$. We prove (b) using Whitney's Reduction Formula

$$P(B_p,x)=P(U_p,x)+P(W_{p-1},x)$$

$$=(-1)^{p-2}x(x-1)(x-2)+\sum_{i=0}^{p-3}(-1)^i\binom{p-2}{i}x(x-1)(x-2)^{p-2-i}$$

$$+\sum_{i=0}^{p-4}(-1)^i x(x-1)(x-2)^{p-3-i}$$

$$=(-1)^{p-2}x(x-1)(x-2)+x(x-1)(x-2)^{p-2}+$$

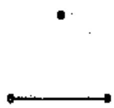
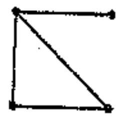
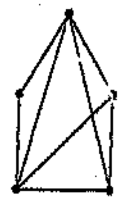
$$\sum_{i=1}^{p-3}(-1)^i\left[\binom{p-2}{i}-1\right]x(x-1)(x-2)^{p-2-i}\qquad\square$$

Our next result generalizes Lemma 2.3.1.

Theorem 2.3.3. Let G_0 be any graph of order p with chromatic polynomial $P(G_0,x) = \sum_{i=0}^{p-1}(-1)^i a_i x^{p-i}$. If G_m is the join of G_0 and K_m, for any $m = 1,2,\ldots,$ then $P(G_m,x) = \sum_{i=0}^{p-1}(-1)^i a_i x(x-1)(x-2)\ldots(x-m)^{p-i}$.

Proof. Repeated application of Lemma 2.3.1 and Lemma 2.2.5. □

Example 2.3.4.

With G_0:  , G_1:  and G_2: 

we have $P(G_0,x) = x^3 - x^2$,

$$\begin{aligned} P(G_1,x) &= P^{\circledast}(G_0,x) \\ &= x(x-1)^3 - x(x-1)^2 \\ &= T_4 - T_3 \end{aligned}$$

and

$$\begin{aligned} P(G_2,x) &= P^{\circledast}(G_1,x) \\ &= P^{\circledast\circledast}(G_0,x) \\ &= x(x-1)(x-2)^3 - x(x-1)(x-2)^2 \end{aligned}$$

Our next result is a corollary of Theorem 2.2.1 and is closely related to Lemma 2.3.1.

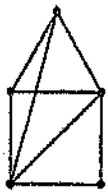
Lemma 2.3.3. Let G_0 be any graph with chromatic polynomial $P(G_0,x)$. If G_1 is a graph obtained from G_0 by adding a new vertex v to G_0 and joining v to every vertex of a complete subgraph of G_0 on n vertices, then $P(G_1,x) = (x-n)P(G_0,x)$.

Proof. Since v is a vertex on a complete graph on $n+1$ vertices which overlaps with G_0 in K_n we have by Theorem 2.2.1

$$\begin{aligned}
P(G_1, x) &= \frac{P(G_0, x) \times P(K_{n+1}, x)}{x^{(n)}} \\
&= \frac{P(G_0, x) \times x(x-1)(x-2)\dots(x-n+1)(x-n)}{x(x-1)(x-2)\dots(x-n+1)} \\
&= (x-n)P(G_0, x).
\end{aligned}$$

□

Example 2.3.5.

With G_0:  and G_1: 

we have

$$\begin{aligned}
P(G_1, x) &= (x-3)P(G_0, x) \\
&= (x-3)x(x-1)(x-2)^2 \\
&= x(x-1)(x-2)^2(x-3).
\end{aligned}$$

2.4 Combinatorial Formulae

Our next result is a corollary of Theorem 2.3.2.

Lemma 2.4.1. $P(K(m,n), x) = \sum_{r=1}^{m} S(m,r) \sum_{s=1}^{n} S(n,s) x^{(r+s)}$.

Proof. The chromatic polynomials of the empty graphs on m and n vertices, respectively, can be expressed in terms of the chromatic polynomials of the complete graphs, using Stirling numbers of the second kind, by

$$x^m = \sum_{r=1}^{m} S(m,r)x^{\{r\}}$$

and

$$x^n = \sum_{s=1}^{n} S(n,s)x^{\{s\}}$$

Now by Theorem 2.3.2 we have

$$\begin{aligned} P\big(K(m,n),x\big) &= P(x^m,x) * P(x^n,x) \\ &= \sum_{r=1}^{m} S(m,r)x^{\{r\}} * \sum_{s=1}^{n} S(n,s)x^{\{s\}} \\ &= \Big[S(m,1)x^{\{1\}} + S(m,2)x^{\{2\}} + \cdots + S(m,m)x^{\{m\}}\Big] \\ &\quad * \sum_{s=1}^{n} S(n,s)x^{\{s\}} \\ &= S(m,1)\sum_{s=1}^{n} S(n,s)x^{\{1+s\}} + S(m,2)\sum_{s=1}^{n} S(n,s)x^{\{2+s\}} \\ &\quad + \cdots + S(m,m)\sum_{s=1}^{n} S(n,s)x^{\{m+s\}} \\ &= \sum_{r=1}^{m} S(m,r)\sum_{s=1}^{n} S(n,s)x^{\{r+s\}}. \end{aligned}$$

□

A *cascade of K_m's* is defined recursively as any graph G_k that can be obtained by starting with $G_0 = K_{m-1}$ and, if G_i is defined, then G_{i+1} can be obtained by adding a new vertex v_i to G_i and joining v_i to any $m-1$ pairwise adjacent vertices of G_i

As a special case of Theorem 2.3.3, let G_0 be the empty graph, $\bar{K}_n$, on n vertices, then $G_m = K_m + \bar{K}_n$ which is a cascade of K_{m+1}'s. In this case $P(K_m + \bar{K}_n, x) = x(x-1)(x-2)\dots(x-m)^n$, which we denote by $x_n^{\{m\}}$, so that $x_n^{\{1\}} = x(x-1)^n = T_{n+1}$ which is the chromatic polynomial of a tree on $n+1$ vertices.

Lemma 2.4.2. $x_n^{\{m\}} = \sum_{s=1}^{n} S(n,s) x^{\{m+s\}}$.

Proof. Since $x_n^{\{m\}} = P(K_m + \bar{K}_n, x)$ is the chromatic polynomial of the join of K_m and $\bar{K}_n$ we have by Theorem 2.3.2

$$\begin{aligned} x_n^{\{m\}} &= P(K_m + \bar{K}_n, x) \\ &= P(K_m, x) * P(\bar{K}_n, x) \\ &= x^{\{m\}} * \sum_{s=1}^{n} S(n,s) x^{\{s\}} \\ &= \sum_{s=1}^{n} S(n,s) x^{\{m+s\}}. \end{aligned}$$

□

As a special case of Lemma 2.4.2 we can express the chromatic polynomial of a tree on $n+1$ vertices in terms of the complete graph basis, that is

$$\begin{aligned} x_n^{\{1\}} &= T_{n+1} \\ &= \sum_{s=1}^{n} S(n,s) x^{\{1+s\}}. \end{aligned}$$

We now apply Theorem 2.3.3 to bipartite graphs.

Let G_0 be an empty graph on n vertices with $P(\bar{K}_n, x) = x^n$, then $P(G_1, x) = P\big(K(1,n), x\big) = x(x-1)^n = T_{n+1}$, since G_1 is a tree on $n+1$ vertices. To compute $P\big(K(2,n), x\big)$ we apply Whitney's Reduction Formula: $P(G,x) = P\big(K(2,n), x\big) =$

$P(G', x) + P(G'', x)$ where G' is a cascade of triangles on $n+2$ vertices and G'' is a tree on $n+1$ vertices. In fact $G' \cong G_2$ and $G'' \cong G_1$. Therefore

$$\begin{aligned} P\big(K(2,n),x\big) &= x(x-1)(x-2)^n + x(x-1)^n \\ &= x_n^{\{2\}} + x_n^{\{1\}} \\ &= \sum_{s=1}^{n} S(n,s)x^{\{2+s\}} + \sum_{s=1}^{n} S(n,s)x^{\{1+s\}} \\ &= \sum_{r=1}^{2} S(2,r)x_n^{\{r\}} \end{aligned}$$

We now obtain a formula for the chromatic polynomial of the bipartite graph, $K(m,n)$, in terms of cascades:

Lemma 2.4.3. $P\big(K(m,n),x\big) = \sum_{r=1}^{m} S(m,r)x_n^{\{r\}}$

Proof.

$$\begin{aligned} P\big(K(m,n),x\big) &= \sum_{r=1}^{m} S(m,r) \sum_{s=1}^{n} S(n,s)x^{\{r+s\}} \quad \text{by Lemma 2.4.1.} \\ &= \sum_{r=1}^{m} S(m,r)x_n^{\{r\}}. \quad \text{by Lemma 2.4.2.} \end{aligned}$$

□

Example 2.4.1.

$$\begin{aligned} P\big(K(4,5),x\big) &= \sum_{r=1}^{4} S(4,r)x_5^{\{r\}} \\ &= S(4,1)x_5^{\{1\}} + S(4,2)x_5^{\{2\}} + S(4,3)x_5^{\{3\}} + S(4,4)x_5^{\{4\}} \\ &= x_5^{\{1\}} + 7x_5^{\{2\}} + 6x_5^{\{3\}} + x_5^{\{4\}} \end{aligned}$$

We now express the chromatic polynomial of the complete graph on m vertices in terms of the chromatic polynomial of cascades.

Lemma 2.4.4. $x^{(m)} = \sum_{i=1}^{m-n} s(m-n, i) x_i^{(n)}$.

Proof. By Theorem 2.3.3 and induction on m. □

As a special case of Lemma 2.4.4, with $n = 1$, we can express the chromatic polynomial of the complete graph on m vertices in terms of the chromatic polynomial of trees, that is

$$\begin{aligned} x^{(m)} &= \sum_{i=1}^{m-1} s(m-1, i) x_i^{(1)} \\ &= \sum_{i=1}^{m-1} s(m-1, i) T_{i+1}. \end{aligned}$$

CHAPTER 3

CHROMATIC POLYNOMIALS OF GRAPHS OBTAINED BY REPLACING A VERTEX WITH A GRAPH

3.1 Int·oduction

If T is a tree and v is a vertex of T and e is an edge containing v, then the *branch of T with respect to* (v, e) is the maximal subtree of T which contains e and has v as an end vertex. If v is incident to the edges $e_1, e_2, \ldots, e_k$, then there are k branches in T containing v as an end vertex. If B and H are graphs and v is a vertex of B then we say that B is *attached* to H *at* v if a vertex of H is identified to v. If H is a connected graph and T is a tree we use the notation $G(H,T)$ for any graph that results if we choose any vertex v of T of degree k and attach the k branches of T containing v to (any k vertices of) H. One may think of $G(H,T)$ as a graph obtained from T by replacing the vertex v with H or "blowing up" the vertex v to H. We start by studying the chromatic polynomial of a graph obtained by replacing the vertices of a tree by graphs in section 3.2 and extend this concept to forests in section 3.3.

3.2 Replacing the vertices of a tree by graphs

Most of the results of this section appear in [2,3] (see also [20]):

Theorem 3.2.1. If H is a connected graph on m vertices with $P(H,x) = \sum_{i=0}^{m-2}(-1)^i a_i T_{m-i}$ and T is any tree on r vertices and G is any connected $(p, p+n)$-graph of the form $G = G(H,T)$, then $P(G,x) = \sum_{i=0}^{m-2}(-1)^i a_i T_{p-i}$.

Proof. The result is clearly true if $r = 1$ and hence we assume T to be nontrivial. We employ induction on the number of edges $|E(G)|$ of G. If G is a tree then

$P(G,x) = T_p$. But then H is a tree and $P(H,x) = T_m$. Assume that for some fixed integer k, the result is true for all connected graphs G with $|E(G)| \leq k-1$. Now consider G with p vertices and $p+n = k$ edges and assume that $G = G(H,T)$ with $P(H,x) = \sum_{i=0}^{m-2}(-1)^i a_i T_{m-i}$. Let w be any end-vertex of T and let u be adjacent to w in G. Then, by Whitney's Reduction Formula applied to the edge uw, we have $P(G,x) = P(G',x) - P(G'',x)$ where G' is a $(p, p+n-1)$-graph and G'' is a $(p-1, p+n-1)$-graph. Now by Theorem 2.3.1 and the inductive hypothesis we have $P(G',x) = P(G'',x) \times T_1$ and $P(G'',x) = \sum_{i=0}^{m-2}(-1)^i a_i T_{p-1-i}$. Therefore

$$\begin{aligned}
P(G',x) &= \left[\sum_{i=0}^{m-2}(-1)^i a_i T_{p-1-i}\right] \times T_1 \\
&= \left[a_0 T_{p-1} - a_1 T_{p-2} + a_2 T_{p-3} - \cdots + (-1)^{m-2} a_{m-2} T_{p-1-m+2}\right] \times T_1 \\
&= a_0(T_p + T_{p-1}) - a_1(T_{p-1} + T_{p-2}) + a_2(T_{p-2} + T_{p-3}) - \ldots \\
&\quad + (-1)^{m-2} a_{m-2}(T_{p-m+2} + T_{p-1-m+2}) \quad \text{by Lemma 2.2.1}
\end{aligned}$$

and $P(G'',x) = a_0 T_{p-1} - a_1 T_{p-2} + a_2 T_{p-3} - \cdots + (-1)^{m-2} a_{m-2} T_{p-1-m+2}$. Therefore

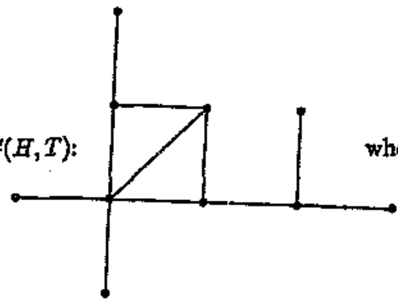
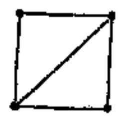
$$\begin{aligned}
P(G,x) &= P(G',x) - P(G'',x) \\
&= a_0 T_p - a_1 T_{p-1} + a_2 T_{p-2} - \cdots + (-1)^{m-2} a_{m-2} T_{p-m+2} \\
&= \sum_{i=0}^{m-2}(-1)^i a_i T_{p-i}.
\end{aligned}$$

Hence the result follows. □

Note that the result of this theorem can also be stated as:

$$P(G,x) = P(H,x) \odot T_{r-1}.$$

Example 3.2.1.

Consider $G = G(H, T)$:  where H: 

with $P(H, x) = T_4 - 2T_3 + T_2$.

Then $P(G, x) = T_{10} - 2T_9 + T_8$ with $p = 10$.

In the above example T is a tree on 7 vertices. We may replace any vertex of T with the graph H and obtain the same chromatic polynomial of G; that is we may attach the branches of T to H in any fashion.

We now consider graphs that overlap in a cut vertex.

Theorem 3.2.2. If G is any connected graph with blocks $H_1, H_2, \ldots, H_p$, then

$$P(G, x) = [P(H_1, x) \odot P(H_2, x) \odot \cdots \odot P(H_p, x)] \ominus T_{p-1}.$$

Proof. We employ induction on p. For $p = 2$ the result follows from Theorem 2.3.1 and Lemma 2.2.2. Assume that the result is true for all connected graphs H with k blocks, $k \leq p - 1$. Now consider any connected graph G with p blocks. Assume, without loss of generality, that H_p is an endblock of G, v is the cut-vertex on H_p and let $H = G - (V(H_p) - v)$. By Corollary 2.2.1 we have

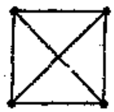
$$\begin{aligned} P(G, x) &= [P(H, x) \odot P(H_p, x)] \ominus T_1 \\ &= [\{(P(H_1, x) \odot P(H_2, x) \odot \ldots \odot P(H_{p-1}, x)) \ominus T_{p-2}\} \odot P(H_p, x)] \ominus T_1 \end{aligned}$$

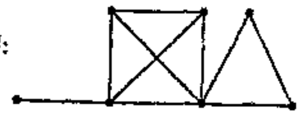
$$= [P(H_1, x) \odot P(H_2, x) \odot \ldots \odot P(H_{p-1}, x) \odot P(H_p, x)] \odiv T_{p-1}$$

Hence the result follows. □

Note that the result of Theorem 3.2.1 can be obtained from Theorem 3.2.2 by choosing $H_1 = H$ and $H_2 = H_3 = \ldots = H_p = K_2$.

Example 3.2.2.

then H_1 :  H_2 : 

If G:  and H_3 : 

Therefore

$$P(G, x) = [P(K_2, x) \odot P(K_4, x) \odot P(K_3, x)] \odiv T_2$$

$$= [T_2 \odot (T_4 - 3T_3 + 2T_2) \odot (T_3 - T_2)] \odiv T_2$$

$$= T_7 - 4T_6 + 5T_5 - 2T_4$$

Theorem 3.2.3. If H_1 is a 2-connected graph on m vertices with $P(H_1, x) = \sum_{i=0}^{m-2}(-1)^i a_i T_{m-i}$ and H_2 is a 2-connected graph on n vertices with $P(H_2, x) = \sum_{j=0}^{n-2}(-1)^j b_j T_{n-j}$ and G is any graph formed by connecting H_1 and H_2 by a cut edge e, then $P(G, x) = P(H_1, x) \odot P(H_2, x)$.

Proof. By Theorem 3.2.2 we have

$$P(G, x) = [P(H_1, x) \odot P(K_2, x) \odot P(H_2, x)] \odiv T_2$$

$$= [P(H_1, x) \odot T_2 \odot P(H_2, x)] \odiv T_2$$

$$= P(H_1, x) \odot P(H_2, x).$$

□

If G is a tree with vertex set $\{1,2,\ldots,p\}$ and $H_1, H_2, \ldots, H_p$ are connected graphs then a graph $G(H_1, H_2, \ldots, H_p)$ is formed by blowing up each vertex i of G to H_i while one vertex of H_i is adjacent to one vertex of H_j iff $ij \in E(G)$.

Theorem 3.2.4. If G is any tree of order p, then the chromatic polynomial of $G(H_1, H_2, \ldots, H_p)$ is

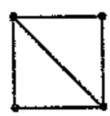
$$P(H_1, x) \odot P(H_2, x) \odot \cdots \odot P(H_p, x).$$

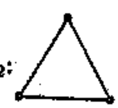
Proof. We employ induction on p. For $p = 2$ the result follows from Theorem 3.2.3. Assume that the result is true for all trees of order k with $k \leq p - 1$. Now consider a tree G of order p. Assume, without loss of generality, that an end vertex of G is blown up with H_p. Let $H = G(H_1, H_2, \ldots, H_p) - V(H_p)$. Then by Theorem 3.2.3 we have

$$\begin{aligned} P(G(H_1, H_2, \ldots, H_p), x) &= P(H, x) \odot P(H_p, x) \\ &= P(H_1, x) \odot P(H_2, x) \odot \ldots \odot P(H_{p-1}, x) \odot P(H_p, x) \end{aligned}$$

Hence the result follows. □

Example 3.2.3.

Here H_1: 

H_2: 

H: 

H_3: •

with G a tree on three vertices.

Therefore

$$P(H,x) = P(H_1,x) \odot P(K_3,x) \odot P(K_1,x)$$
$$= (T_4 - 2T_3 + T_2) \odot (T_3 - T_2) \odot T_1$$
$$= T_8 - 3T_7 + 3T_6 - T_5\,.$$

Theorem 3.2.4 can also be obtained as a special case of Theorem 3.2.2 by replacing the chromatic polynomial of each cut edge by T_2.

3.3 Replacing the vertices of a forest by graphs

The above concepts were confined to connected graphs, we now concentrate our efforts on the disconnected case. We begin by generalising Theorem 3.2.3.

Lemma 3.3.1. If H_1 and H_2 are disjoint graphs and G is any graph formed by connecting H_1 and H_2 by an edge e, then

$$P(G,x) = P(H_1,x) \odot P(H_2,x).$$

Proof. Let G_1 and G_2 be graphs formed by adding the edge e to H_1 and H_2 respectively. Now by Theorem 2.2.1 and Lemma 2.2.2 we have:

$$P(G_i,x) = \frac{P(H_i,x) \times T_2}{x^{(1)}}$$
$$= [P(H_i,x) \odot T_2] \ominus T_1$$
$$= P(H_i,x) \odot T_1 \quad \text{for } i = 1,2\,.$$

Also by Theorem 2.2.1 and Lemma 2.2.2 we have

$$\begin{aligned}
P(G,x) &= \frac{P(G_1,x)\times P(G_2,x)}{x^{(2)}} \\
&= [P(G_1,x)\odot P(G_2,x)]\ominus T_2 \\
&= \{[P(H_1,x)\odot T_1]\odot[P(H_2,x)\odot T_1]\}\ominus T_2 \\
&= \{[P(H_1,x)\odot P(H_2,x)]\odot T_2\}\ominus T_2 \\
&= P(H_1,x)\odot P(H_2,x).
\end{aligned}$$ □

The result of Theorem 2.3.1 can also be given in terms of $\odot$ and $\ominus$. We first consider the special case when $k=2$.

Lemma 3.3.2. If a graph G is the union of a 2-connected graph G_1 and a graph G_2, then

$$P(G,x) = P(G_1,x)\odot P(G_2,x) + [P(G_1,x)\odot P(G_2,x)]\ominus T_1.$$

Proof. By Whitney's Reduction Formula we have

$$P(G,x) = P(G',x) + P(G'',x)$$

where G' is any graph formed by connecting G_1 and G_2 by a cut edge uv and G'' is obtained from G by identifying u and v. Hence $P(G,x) = P(G_1,x)\odot P(G_2,x) + [P(G_1,x)\odot P(G_2,x)]\ominus T_1$ by Theorem 3.2.3 and Corollary 2.2.1. □

Theorem 3.3.1. If a graph G has connected components $G_1, G_2, \ldots, G_k$, then

$$P(G,x) = \sum_{j=0}^{k-1}\binom{k-1}{j}[P(G_1,x)\odot P(G_2,x)\odot\ldots\odot P(G_k,x)]\ominus T_j$$

where $T_0 = 1$.

Proof. We employ induction on k the number of components of G. For $\kappa = 2$ we have by Lemma 3.3.2 that

$$P(G,x) = P(G_1,x) \odot P(G_2,x) + [P(G_1,x) \odot P(G_2,x)] \oplus T_1$$

$$= \sum_{j=0}^{1} \binom{2-1}{j} [P(G_1,x) \odot P(G_2,x)] \oplus T_j$$

Assume that the result is true for all graphs H with at most $k-1$ connected components. Now consider G with connected components $G_1, G_2, \ldots, G_k$. By Whitney's Reduction Formula we have

$$P(G,x) = P(G',x) + P(G'',x)$$

where G' is a graph formed by connecting the graph H with $k-1$ components $G_1, G_2, \ldots, G_{k-1}$ to G_k by an edge uv and G'' is obtained from G by identifying u and v. Therefore

$$P(G,x) = P(H,x) \odot P(G_k,x) + [P(H,x) \odot P(G_k,x)] \oplus T_1$$

by Lemma 3.3.1 and Corollary 2.2.1

where $$P(H,x) = \sum_{j=0}^{k-2} \binom{k-2}{j} [P(G_1,x) \odot P(G_2,x) \odot \ldots \odot P(G_{k-1},x)] \oplus T_j$$

by the inductive hypothesis. Hence

$$P(G,x) = \left[\sum_{j=0}^{k-2} \binom{k-2}{j} \{P(G_1,x) \odot P(G_2,x) \odot \cdots \odot P(G_{k-1},x)\} \oplus T_j\right]$$

$$\odot P(G_k,x) + \left[\{\sum_{j=0}^{k-2} \binom{k-2}{j} \{P(G_1,x) \odot P(G_2,x) \odot \cdots\right.$$

$$\left. \odot P(G_{k-1},x)\} \oplus T_j\} \odot P(G_k,x)\right] \oplus T_1$$

$$= \left[\binom{k-2}{0} \{P(G_1,x) \odot P(G_2,x) \odot \ldots \odot P(G_{k-1},x)\} + \binom{k-2}{1} \{P(G_1,x)\right.$$

$$\odot P(G_2,x) \odot \cdots \odot P(G_{k-1},x)\} \oplus T_1 + \cdots + \binom{k-2}{k-2} \{P(G_1,x) \odot P(G_2,x)$$

$$\left. \odot \cdots \odot P(G_{k-1},x)\} \oplus T_{k-2} \right] \odot P(G_k,x)$$

$$+ \left[\binom{k-2}{0} \{P(G_1,x) \odot P(G_2,x) \odot \cdots \odot P(G_{k-1},x)\} \odot P(G_k,x) \right] \oplus T_1$$

$$+ \left[\binom{k-2}{1} \{\{P(G_1,x) \odot P(G_2,x) \odot \cdots \odot P(G_{k-1},x)\} \oplus T_1\} \right.$$

$$\left. \odot P(G_k,x) \right] \oplus T_1 + \cdots + \left[\binom{k-2}{k-2} \{\{P(G_1,x) \odot P(G_2,x) \odot \cdots \right.$$

$$\left. \odot P(G_{k-1},x)\} \oplus T_{k-2}\} \odot P(G_k,x) \right] \oplus T_1$$

$$= \binom{k-2}{0} \{P(G_1,x) \odot P(G_2,x) \odot \cdots \odot P(G_{k-1},x) \odot P(G_k,x)$$

$$+ \binom{k-2}{1} \{P(G_1,x) \odot (PG_2,x) \odot \cdots \odot P(G_{k-1},x) \odot P(G_k,x)\} \oplus T_1$$

$$+ \cdots + \binom{k-2}{k-2} \{P(G_1,x) \odot P(G_2,x) \odot \cdots \odot P(G_{k-1},x) \odot P(G_k,x)\} \oplus T_{k-2}$$

$$+ \binom{k-2}{0} \{P(G_1,x) \odot P(G_2,x) \odot \cdots \odot P(G_{k-1},x) \odot P(G_k,x)\} \oplus T_1$$

$$+ \binom{k-2}{1} \{P(G_1,x) \odot P(G_2,x) \odot \cdots \odot P(G_{k-1},x) \odot P(G_k,x)\} \oplus T_2 + \cdots$$

$$+ \binom{k-2}{k-2} \{P(G_1,x) \odot P(G_2,x) \odot \cdots \odot P(G_{k-1},x) \odot P(G_k,x)\} \oplus T_{k-1}$$

$$= \binom{k-1}{0}\{P(G_1,x) \odot P(G_2,x) \odot \cdots \odot P(G_{k-1},x) \odot P(G_k,x)\}$$

$$+ \binom{k-1}{1}\{P(G_1,x) \odot P(G_2,x) \odot \cdots \odot P(G_{k-1},x) \odot P(G_k,x)\} \ominus T_1 + \ldots$$

$$+ \binom{k-1}{k-1}\{P(G_1,x) \odot P(G_2,x) \odot \cdots \odot P(G_{k-1},x) \odot P(G_k,x)\} \ominus T_{k-1}$$

$$= \sum_{j=0}^{k-1} \binom{k-1}{j}[P(G_1,x) \odot P(G_2,x) \odot \cdots \odot P(G_k,x)] \ominus T_j$$

using the fact that $\binom{k-i}{i} + \binom{k-i}{i-1} = \binom{k-i+1}{i}$. Hence the result follows. □

Corollary 3.3.1. If each $G_i \cong K_1, 1 \leq i \leq k$, then $G \cong \bar{K}_k$ and

$P(\bar{K}_k, x) = \sum_{j=0}^{k-1} \binom{k-1}{j} T_{k-j}$. □

In the normal form this polynomial is known to be $P(\bar{K}_k, x) = x^k$.

Corollary 3.3.2. If each G_i is a tree on p_i vertices and $p = \sum_{i=1}^{k} p_i$, then

$P(G,x) = \sum_{j=0}^{k-1} \binom{k-1}{j} T_{p-j}$. □

In the normal form this polynomial is

$$\begin{aligned}
P(G,x) &= x^k(x-1)^{p-k} \\
&= x(x-1+1)^{k-1}(x-1)^{p-k} \\
&= x\left[\sum_{j=0}^{k-1}\binom{k-1}{j}(x-1)^{k-1-j}\right](x-1)^{p-k} \\
&= \sum_{j=0}^{k-1}\binom{k-1}{j}x(x-1)^{p-1-j} \\
&= \sum_{j=0}^{k-1}\binom{k-1}{j}T_{p-j}.
\end{aligned}$$

Example 3.3.1.

With G:  we have

G_1 G_2 G_3

$$P(G,x) = \sum_{j=0}^{3-1} \binom{3-1}{j} [P(K_3,x) \odot P(S_3,x) \odot P(S_2,x)] \oplus T_j$$

$$= \sum_{j=0}^{2} \binom{2}{j} [(T_3 - T_2) \odot T_3 \odot T_2] \oplus T_j$$

$$= \sum_{j=0}^{2} \binom{2}{j} [T_8 - T_7] \oplus T_j$$

$$= \binom{2}{0}[T_8 - T_7] \oplus T_0 + \binom{2}{1}[T_8 - T_7] \oplus T_1 + \binom{2}{2}[T_8 - T_7] \oplus T_2$$

$$= T_8 - T_7 + 2[T_7 - T_6] + [T_6 - T_5]$$

$$= T_8 + T_7 - T_6 - T_5 .$$

We now consider graphs obtained by replacing a vertex of a forest by a graph.

Theorem 3.3.2. If a graph G is the disjoint union of two subgraphs G_1, a connected graph on n vertices with $P(G_1,x) = \sum_{i=0}^{n-2}(-1)^i a_i T_{n-i}$, and G_2 a forest on $p-n$ vertices, $n < p$, and k components, then

$$P(G,x) = \sum_{j=0}^{n+k-2} \left[\sum_{i=0}^{n-2} (-1)^i a_i \binom{k}{j-i} \right] T_{p-j}.$$

Proof. By Whitney's Reduction Formula we have $P(G,x) = P(G',x) + P(G'',x)$

where G' is any graph formed by connecting G_1 and G_2 by an edge uv and G'' is obtained from G by identifying u and v. Therefore by Lemma 3.3.1 and Corollary 2.2.1 we have:

$$P(G,x) = P(G_1,x) \odot P(G_2,x) + [P(G_1,x) \odot P(G_2,x)] \oplus T_1$$

$$= \left[\sum_{i=0}^{n-2}(-1)^i a_i T_{n-i} \odot \sum_{j=0}^{k-1}\binom{k-1}{j} T_{p-n-j}\right]$$

$$+ \left[\sum_{i=0}^{n-2}(-1)^i a_i T_{n-i} \odot \sum_{j=0}^{k-1}\binom{k-1}{j} T_{p-n-j}\right] \oplus T_1$$

$$= \sum_{i=0}^{n-2}(-1)^i a_i T_{n-i} \odot \left[\sum_{j=0}^{k-1}\binom{k-1}{j} T_{p-n-j} + \sum_{j=0}^{k-1}\binom{k-1}{j} T_{p-n-j-1}\right]$$

$$= \sum_{i=0}^{n-2}(-1)^i a_i T_{n-i} \odot \left[T_{p-n} + \binom{k-1}{1} T_{p-n-1} + \cdots + \binom{k-1}{k-1} T_{p-n-k+1}\right.$$

$$\left. + \binom{k-1}{0} T_{p-n-1} + \cdots + \binom{k-1}{k-2} T_{p-n-k+1} + \binom{k-1}{k-1} T_{p-n-k}\right]$$

$$= \sum_{i=0}^{n-2}(-1)^i a_i T_{n-i} \odot \left[T_{p-n} + \binom{k}{1} T_{p-n-1} + \cdots + \binom{k}{k-1} T_{p-n-k+1}\right.$$

$$\left. + \binom{k}{k} T_{p-n-k}\right] \quad \text{since} \quad \binom{k-i}{i} + \binom{k-i}{i-1} = \binom{k-i+1}{i}$$

$$= \left[T_n - a_1 T_{n-1} + \ldots + (-1)^{n-3} a_{n-3} T_3 + (-1)^{n-2} a_{n-2} T_2\right]$$

$$\odot \left[T_{p-n} + \binom{k}{1} T_{p-n-1} + \cdots + \binom{k}{k-1} T_{p-n-k+1} + \binom{k}{k} T_{p-n-k}\right]$$

$$= T_p + \left[\binom{k}{1} - a_1\binom{k}{0}\right] T_{p-1} + \left[\binom{k}{2} - a_1\binom{k}{1} + a_0\binom{k}{0}\right] T_{p-2}$$

$$+\left[\binom{k}{3}-a_2\binom{k}{2}+a_1\binom{k}{1}-a_0\binom{k}{0}\right]T_{p-3}+\cdots+\left[(-1)^{n-3}\binom{k}{k}a_{n-3}\right.$$

$$\left.+(-1)^{n-2}\binom{k}{k-1}a_{n-2}\right]T_{p-n-k+3}+(-1)^{n-2}\binom{k}{k}a_{n-2}T_{p-n-k+2}$$

$$=\sum_{j=0}^{n+k-2}\left[\sum_{i=0}^{n-2}(-1)^i a_i\binom{k}{j-i}\right]T_{p-j}\,. \qquad \square$$

Corollary 3.3.3. If G is the disjoint union of two subgraphs G_1, a cycle on n vertices, and G_2 a forest on $p-n$ vertices, $n<p$, and k components, then

$$P(G,x)=\sum_{j=0}^{n+k-2}\left[\sum_{i=0}^{n-2}(-1)^i\binom{k}{j-i}\right]T_{p-j}. \qquad \square$$

Example 3.3.2.

With G:  we have

G_1 G_2

$P(G_1,x)=T_4-2T_3+T_2$; $n=4$; $k=2$; $p=9$; $a_0=1$; $a_1=2$; and $a_2=1$. Hence

$$P(G,x)=\sum_{j=0}^{4+2-2}\left[\sum_{i=0}^{4-2}(-1)^i a_i\binom{2}{j-i}\right]T_{9-j}$$

$$=\sum_{j=0}^{4}\left[\sum_{i=0}^{2}(-1)^i a_i\binom{2}{j-i}\right]T_{9-j}$$

$$= T_9 + \left[a_0\binom{2}{1} - a_1\binom{2}{0}\right]T_8 + \left[a_0\binom{2}{2} - a_1\binom{2}{1} + a_2\binom{2}{0}\right]T_7$$

$$+ \left[a_0\binom{2}{3} - a_1\binom{2}{2} + a_2\binom{2}{1}\right]T_6 + \left[a_0\binom{2}{4} - a_1\binom{2}{3} + a_2\binom{2}{2}\right]T_5$$

$$= T_9 + [2-2]T_8 + [1-4+1]T_7 + [0-2+2]T_6 + [0-0+1]T_5$$

$$= T_9 - 2T_7 + T_5 .$$

CHAPTER 4
THE COEFFICIENTS OF THE CHROMATIC POLYNOMIAL

4.1 Introduction

In this chapter we study the coefficients $a_0, a_1, \ldots, a_{p-1}$ of the chromatic polynomial $P(G,x) = \sum_{i=0}^{p-1}(-1)^i a_i T_{p-i}$ of a connected graph G of order p.

In the proof of Lemma 1.5.1 it was shown that $P(G,x) = \sum_k \alpha(G,k)x^{(k)}$ where the coefficient $\alpha(G,k)$ denotes the number of partitions of $V(G)$ into exactly k nonempty subsets, such that no edge of G joins two vertices in the same subset. This gives us an interpretation of the coefficients of $P(G,x)$ in factorial form. The interpretation of the coefficients in the normal form of the chromatic polynomial $P(G,x)$ requires the inclusion-exclusion principle; it is due to Whitney [23] (see also [14]): that $P(G,x) = \sum_{r=0}^{k}(-1)^r N(p,r)x^p$ where $N(p,r)$ is the number of subgraphs of G with p components and r edges, and k is the number of edges in G. This result was used by Farrell [9] to interpret the first five coefficients of the chromatic polynomial in normal form. In section 4.2 we start by interpreting the first five coefficients of $P(G,x)$ in tree form. The same results are obtained by using Farrell's results in section 4.3. In section 4.4 we discuss techniques of computing some higher order coefficients and in section 4.5 we consider the signs of the coefficients of $P(G,x)$ in tree form in terms of the connectivity of G. We present a new combinatorial identity in section 4.6 and apply it in section 4.7 to determine the coefficients of the chromatic polynomial of a cascade of graphs.

4.2 Interpretation of the coefficients

The results of this paragraph correspond to the results of Farrell in [9] in the sense that he obtains formulas for the coefficients b_0, b_1, b_2, b_3 and b_4 in a chromatic polynomial in the normal form $P(G,x) = \sum_{i=0}^{p-1}(-1)^i b_i x^{p-i}$. More than that our results can be deduced from his by equating $\sum_{i=0}^{p-1}(-1)^i a_i T_{p-i} = \sum_{i=0}^{p-1}(-1)^i a_i x(x-1)^{p-i-1}$ and $P(G,x) = \sum_{i=0}^{p-1}(-1)^i b_i x^{p-i}$ and expressing a_0, a_1, a_2, a_3 and a_4 in terms of the b_i's; we do so in the next section. We give, however, independent proofs in this section (see also [1]).

Throughout this section, let G be a connected $(p, p+n)$ – graph with

$$P(G,x) = \sum_{i=0}^{p-1}(-1)^i a_i T_{p-i}\,.$$

Theorem 4.2.1. $a_0 = 1$; the coefficient of T_p in $P(G,x)$ is $1 = \binom{n+1}{0}$.

Proof. By the nature of Whitney's Reduction Formula there is, at each stage, exactly one graph having p vertices. This is therefore true at the final stage when a tree on p vertices is reached. Hence the coefficient of T_p is 1. □

Theorem 4.2.2. $a_1 = n+1 = \binom{n+1}{1}$.

Proof. We employ induction on n. For $n = -1$ we have for every tree G of order p

$$P(G,x) = T_p - 0T_{p-1} + 0T_{p-2}\dots$$

$$\text{i.e.} \qquad a_1 = -1+1 = \binom{-1+1}{1} = 0\,.$$

Assume that for some fixed integer n, $n \geq 0$, the result is true for all connected $(p, p+n')$ – graphs with $n' \leq n-1$. Now consider G with p vertices and $p+n$ edges.

By Whitney's Reduction Formula

$$P(G,x) = P(G',x) - P(G'',x)$$

where G' has p vertices and $p+n-1$ edges and G'' has $p-1$ vertices and at most $p+n-1$ edges. Since the result is true for G' and G'' we can write

$$P(G',x) = T_p - a'_1 T_{p-1} + a'_2 T_{p-2} - a'_3 T_{p-3} + \cdots$$

and

$$P(G'',x) = \qquad T_{p-1} - a''_1 T_{p-2} + a''_2 T_{p-3} - \cdots .$$

Hence

$$P(G,x) = T_p - (a'_1 + 1)T_{p-1} + (a'_2 + a''_1)T_{p-2} - (a'_3 + a''_2)T_{p-3} + \cdots$$

Now by the inductive hypothesis

$$a'_1 = \binom{n}{1} = n .$$

Therefore

$$a_1 = a'_1 + 1 = n + 1 = \binom{n+1}{1}.$$

Since the result is true for all p and all connected $(p, p+n)$ - graphs, it is true for all connected graphs. □

Let c_n be the number of induced cycles of length n, $n \geq 3$.

Theorem 4.2.3. $a_2 = \binom{n+2}{2} - c_3$.

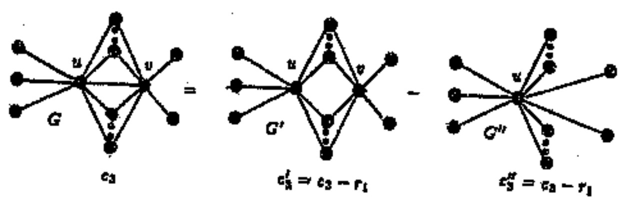
Proof: We employ induction on n. For $n = -1$ we have for every tree G of order p

$$P(G,x) = T_p - 0T_{p-1} + 0T_{p-2} - 0T_{p-3} + \cdots$$

i.e.

$$a_2 = \binom{-1+2}{2} - 0 = 0 - 0 = 0.$$

Assume that for some fixed integer n, $n \geq 0$, the result is true for all connected $(p, p+n') -$ graphs, with $n' \leq n-1$. Now consider G, with p vertices and $p+n$ edges. Let uv be an edge in G and suppose that uv is on r_1 triangles, $r_1 \leq c_3$. (Hence uv is a common edge of these triangles).



By Whitney's Reduction Formula, applied to the edge uv, we have

$$P(G,x) = P(G',x) - P(G'',x)$$

where G' has p vertices, $p+n-1$ edges and $c_3' = c_3 - r_1$ triangles; G'' has $p-1$ vertices, $p+n-1-r_1$ edges and $c_3'' = c_3 - r_1$ triangles. Since the result is true for G' and G'' we can write

$$P(G',x) = T_p - a_1' T_{p-1} + a_2' T_{p-2} - a_3' T_{p-3} + \cdots$$

and

$$P(G'',x) = T_{p-1} - a_1'' T_{p-2} + a_2'' T_{p-3} - \cdots$$

Hence

$$P(G,x) = T_p - (a_1'+1)T_{p-1} + (a_2'+a_1'')T_{p-2} - (a_3'+a_2'')T_{p-3} + \cdots$$

where

$$a_2 = a_2' + a_1''.$$

Now by the inductive hypothesis

$$a_2' = \binom{n+1}{2} - (c_3 - r_1)$$

and by Theorem 4.2.2

$$a_1'' = \binom{n - r_1 + 1}{1}.$$

Therefore

$$\begin{aligned} a_2 = a_2' + a_1'' &= \binom{n+1}{2} - (c_3 - r_1) + \binom{n - r_1 + 1}{1} \\ &= \binom{n+1}{2} - c_3 + r_1 + n - r_1 + 1 \\ &= \binom{n+1}{2} + \binom{n+1}{1} - c_3 \\ &= \binom{n+2}{2} - c_3 . \end{aligned}$$

Since the result is true for all p and all connected $(p, p+n)$ – graphs, it is true for all connected graphs. □

Let k_n be the number of complete subgraphs of order n in G, $n \geq 3$.

Theorem 4.2.4. $a_3 = \binom{n+3}{3} - c_3\binom{n+1}{1} - (c_4 - 2k_4)$.

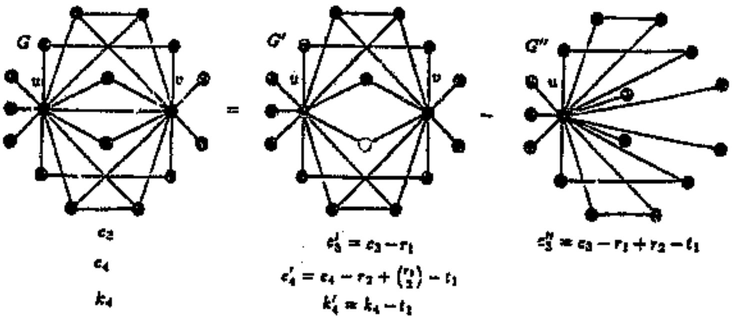
Proof: We employ induction on n. For $n = -1$ we have for every tree G of order p

$$P(G, x) = T_p - 0T_{p-1} + 0T_{p-2} - 0T_{p-3} + 0T_{p-4} - \cdots$$

$$\text{i.e.} \quad a_3 = \binom{-1+3}{3} - 0\binom{-1+1}{1} - (0 - 2(0)) = 0 .$$

Assume that for some fixed integer n, $n \geq 0$, the result is true for all connected $(p, p+n')$ – graphs, with $n' \leq n-1$. Now consider G, with p vertices and $p+n$

edges. Let uv be an edge in G and suppose that uv is on r_1 triangles, $r_1 \leq c_3$; r_2 induced cycles of length 4, $r_2 \leq c_4$; and t_1 complete subgraphs of order 4, $t_1 \leq k_4$.



By Whitney's Reduction Formula, applied to the edge uv, we have

$$P(G,x) = P(G',x) - P(G'',x).$$

Now G' has p vertices, $p+n-1$ edges, $c_3' = c_3 - r_1$ triangles, $c_4' = c_4 - r_2 + \binom{r_1}{2} - t_1$ induced cycles of length 4 and $k_4' = k_4 - t_1$ complete subgraphs of order 4. Also G'' has $p-1$ vertices, $p-1+n-r_1$ edges and $c_3'' = c_3 - r_1 + r_2 - t_1$ triangles. Since the result is true for G' and G'' we can write

$$P(G',x) = T_p - a_1' T_{p-1} + a_2' T_{p-2} - a_3' T_{p-3} + a_4' T_{p-4} - \cdots$$

and

$$P(G'',x) = \qquad T_{p-1} - a_1'' T_{p-2} + a_2'' T_{p-3} - a_3'' T_{p-4} + \cdots .$$

Hence

$$P(G,x) = T_p - (a_1'+1)T_{p-1} + (a_2' + a_1'')T_{p-2} - (a_3' + a_2'')T_{p-3} + (a_4' + a_3'')T_{p-4} - \cdots$$

where

$$a_3 = a_3' + a_2''.$$

Now by the inductive hypothesis

$$a_3' = \binom{n+2}{3} - (c_3 - r_1)n - \{c_4 - r_2 + \binom{r_1}{2} - t_1 - 2(k_4 - t_1)\}$$

and by Theorem 4.2.3

$$a_3'' = \binom{n - r_1 + 2}{2} - (c_3 - r_1 + r_2 - t_1).$$

Therefore

$$\begin{aligned}
a_3 &= a_3' + a_3'' \\
&= \binom{n+2}{3} - (c_3 - r_1)n - c_4 + r_2 - \binom{r_1}{2} + t_1 + 2k_4 - 2t_1 \\
&+ \binom{n - r_1 + 2}{2} - c_3 + r_1 - r_2 + t_1 \\
&= \binom{n+2}{3} - c_3\binom{n+1}{1} - (c_4 - 2k_4) + \binom{n - r_1 + 2}{2} - \binom{r_1}{2} + r_1 + r_1 n
\end{aligned}$$

where

$$\begin{aligned}
&\binom{n - r_1 + 2}{2} - \binom{r_1}{2} + r_1 + r_1 n \\
&= \frac{n^2 - 2nr_1 + 3n - 3r_1 + r_1^2 + 2}{2} - \frac{r_1(r_1 - 1)}{2} + \frac{2r_1}{2} + \frac{2r_1 n}{2} \\
&= \frac{n^2 - 2nr_1 + 3n - 3r_1 + r_1^2 + 2 - r_1^2 + r_1 + 2r_1 + 2r_1 n}{2} \\
&= \frac{n^2 + 3n + 2}{2} \\
&= \binom{n+2}{2}.
\end{aligned}$$

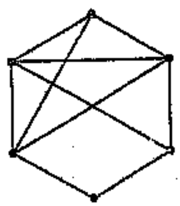
Therefore

$$a_3 = \binom{n+2}{3} + \binom{n+2}{2} - c_3\binom{n+1}{1} - (c_4 - 2k_4)$$
$$= \binom{n+3}{3} - c_3\binom{n+1}{1} - (c_4 - 2k_4).$$

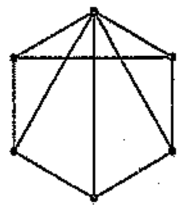
Since the result is true for all p and all connected $(p, p+n)$ – graphs, it is true for all connected graphs. □

Example 4.2.1. (6,10)-graphs

With G :



$c_3 = 5$
$c_4 = 2$
$k_4 = 1$ and H :
$w_5(1) = 1$



$c_3 = 5$
$c_5 = 1$

we note that both G and H are (6,10)-graphs containing 5 triangles. Therefore $a_0 = 1$, $a_1 = n+1 = 4+1 = 5$ and $a_2 = \binom{n+2}{2} - c_3 = \binom{4+2}{2} - 5 = 15 - 5 = 10$. Now for graph G

$$a_3 = \binom{n+3}{3} - c_3\binom{n+1}{1} - (c_4 - 2k_4)$$
$$= \binom{7}{3} - 5\binom{5}{1} - [2 - 2(1)]$$
$$= 35 - 25 - 0$$
$$= 10$$

and for graph H $a_3 = \binom{7}{3} - 5\binom{5}{1} = 10$. In fact, G and H have the same chromatic polynomial: $P(G,x) = P(H,x) = T_6 - 5T_5 + 10T_4 - 10T_3 + 4T_2$.
(Such graphs are called chromatically equivalent and are studied in Chapter 5).

A *broken wheel* $W_p(n)$ is a graph obtained from the wheel W_p on p vertices by deleting n consecutive spoke edges, $n \leq p-1$.

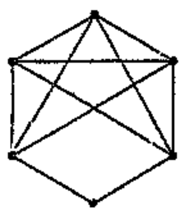
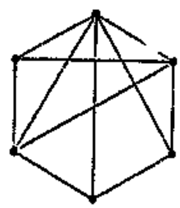
Let $k(2,3)$ be the number of induced subgraphs isomorphic to $K(2,3)$; $w_5(1)$ be the number of induced subgraphs isomorphic to $W_5(1)$ and w_5 be the number of induced subgraphs isomorphic to W_5.

Theorem 4.2.5.

$$a_4 = \binom{n+4}{4} - c_3\binom{n+2}{2} - (c_4 - 2k_4)\binom{n+1}{1} - \{c_5 - \binom{c_3}{2} - k(2,3)$$
$$- 2w_5(1) - 3w_5 + 3k_4 + 6k_5\}.$$

Proof. A tedious calculation which involves induction on p and n. □

Example 4.2.2. (6,11)-graphs

With G :  $c_3 = 7$, $c_4 = 3$, $k_4 = 2$ and H:  $c_3 = 7$, $c_4 = 1$, $k_4 = 1$, $w_5 = 1$

$w_5(1) = 3$

we note that both G and H are (6,11)-graphs containing 7 triangles. Therefore $a_0 = 1$, $a_1 = n+1 = 5+1 = 6$ and $a_2 = \binom{7}{2} - 7 = 21 - 7 = 14$. Also for G $a_3 = \binom{8}{5} - 7\binom{6}{1} - [3-2(2)] = 56-42+1 = 15$ and for H $a_3 = \binom{8}{5} - 7\binom{6}{1} - [1-2(1)] = 56-42+1 = 15$. Now for graph G $a_4 = \binom{5+4}{4} - 7\binom{5+2}{2} - (-1)\binom{5+1}{1} - [0 - \binom{7}{2} - 0 - 2(3) - 3(0) + 3(2) + 6(0)] = 126 - 147 + 6 - [-21 - 6 + 6] = 6$ and for graph H $a_4 = \binom{9}{4} - 7\binom{7}{2} + \binom{6}{1} - [0 - \binom{7}{2} - 0 - 2(0) - 3(1) + 3(1) + 6(0)] = 126 - 147 + 6 - [-21 - 3 + 3] = 6$. In fact $P(G,x) = P(H,x) = T_6 - 6T_5 + 14T_4 - 15T_3 + 6T_2$.

Also, for the graphs G and H from Example 4.2.1, we have for graph G $a_4 = \binom{4+4}{4} - 5\binom{4+2}{2} - (0)\binom{4+1}{1} - [0 - \binom{5}{2} - 0 - 2(1) - 3(0) + 3(1) + 6(0)] = 70 - 75 - 0 - [-10 - 2 + 3] = 70 - 75 + 9 = 4$ and for graph H $a_4 = \binom{4+4}{4} - 5\binom{4+2}{2} - (0)\binom{4+1}{1} - [1 - \binom{5}{2} - 0 - 2(0) - 3(0) + 3(0) + 6(0)] = 70 - 75 - [1 - 10] = 70 - 75 + 9 = 4$.

The above technique can be used to obtain expressions for $a_5, a_6, \ldots$ of $P(G,x)$ in terms of induced subgraphs of G. However, as seen above, the exercise would become more tedious and the expressions more complicated.

4.3 Relating Coefficients

We now consider the relationship between the coefficients of a connected graph in normal form with that of a connected graph in tree form.

Lemma 4.3.1. If G is a connected graph on p vertices with
$P(G,x) = \sum_{i=0}^{p-1} (-1)^i b_{p-i} x^{p-i}$ in normal form and
$P(G,x) = \sum_{i=0}^{p-1} (-1)^i a_{p-i} x(x-1)^{p-1-i}$ in tree form, then

$$b_{p-k} = \sum_{i=0}^{k} \binom{p-i-1}{k-i} a_{p-i} \quad \text{and}$$

$$a_{p-k} = \sum_{i=0}^{k} (-1)^{k-i} \binom{p-i-1}{k-i} b_{p-i}.$$

Proof. Since $\sum_{i=0}^{p-1} (-1)^i b_{p-i} x^{p-i} = \sum_{i=0}^{p-1} (-1)^i a_{p-i} x(x-1)^{p-1-i}$ we have

$$b_p x^p - b_{p-1} x^{p-1} + b_{p-2} x^{p-2} - b_{p-3} x^{p-3} + \cdots + (-1)^{p-1} b_1 x$$

$$= a_p x(x-1)^{p-1} - a_{p-1} x(x-1)^{p-2} + a_{p-2} x(x-1)^{p-3} - \cdots + (-1)^{p-1} a_1 x$$

$$= a_p \left[x^p - \binom{p-1}{1} x^{p-1} + \binom{p-1}{2} x^{p-2} - \cdots + (-1)^{p-1} x \right]$$

$$- a_{p-1}\left[x^{p-1} - \binom{p-2}{1}x^{p-2} + \binom{p-2}{2}x^{p-3} - \cdots + (-1)^{p-2}x\right]$$

$$+ a_{p-2}\left[x^{p-2} - \binom{p-3}{1}x^{p-3} + \binom{p-3}{2}x^{p-4} - \cdots + (-1)^{p-3}x\right]$$

$$- \cdots\cdots\cdots\cdots\cdots\cdots + (-1)^{p-1}a_1 x$$

$$= a_p x^p - \left[\binom{p-1}{1}a_p + \binom{p-2}{0}a_{p-1}\right]x^{p-1} + \left[\binom{p-1}{2}a_p + \binom{p-2}{1}a_{p-1}\right.$$

$$\left. + \binom{p-3}{0}a_{p-2}\right]x^{p-2} - \left[\binom{p-1}{3}a_p + \binom{p-2}{2}a_{p-1}\right.$$

$$\left. + \binom{p-3}{1}a_{p-2} + \binom{p-4}{0}a_{p-3}\right]x^{p-3} + \cdots$$

$$+ (-1)^{p-1}[a_p + a_{p-1} + a_{p-2} + \cdots + a_1].$$

Now on equating coefficients we have

$$b_p = a_p$$

$$b_{p-1} = \binom{p-1}{1}a_p + \binom{p-2}{0}a_{p-1}$$

$$b_{p-2} = \binom{p-1}{2}a_p + \binom{p-2}{1}a_{p-1} + \binom{p-3}{0}a_{p-2}$$

$$\vdots$$

So that in general

$$b_{p-k} = \binom{p-1}{k}a_p + \binom{p-2}{k-1}a_{p-1} + \binom{p-3}{k-2}a_{p-2} + \cdots$$

$$+ \binom{p-k}{1}a_{p-k+1} + \binom{p-k-1}{0}a_{p-k}$$

$$= \sum_{i=0}^{k} \binom{p-i-1}{k-i} a_{p-i}$$

Similarly

$$\begin{aligned}
a_p &= b_p \\
a_{p-1} &= b_{p-1} - \binom{p-1}{1} b_p \\
a_{p-2} &= b_{p-2} - \binom{p-2}{1} a_{p-1} - \binom{p-1}{2} a_p \\
&= b_{p-2} - \binom{p-2}{1} b_{p-1} + \binom{p-1}{2} b_p \\
&\vdots
\end{aligned}$$

In general $a_{p-k} = \sum_{i=0}^{k} (-1)^{k-i} \binom{p-i-1}{k-i} b_{p-i}$. □

We now prove Theorem 4.2.5 using Farrell's results. For completeness, we prove Theorems 4.2.1 to 4.2.4 too in this way.

Let $P(G,x) = \sum_i (-1)^i b_i x^{p-i}$. Then the following results appear in [9]:

$$\begin{aligned}
b_0 &= 1 \\
b_1 &= q, \quad \text{the number of edges} \\
b_2 &= \binom{q}{2} - c_3 \\
b_3 &= \binom{q}{3} - (q-2)c_3 - c_4 + 2k_4 \\
b_4 &= \binom{q}{4} - \binom{q-2}{2} c_3 - \binom{c_3}{2} - (q-3)c_4 + (2q-9)k_4 \\
&\quad - c_5 - 6k_5 + k(2,3) + 2w_5(1) + 3w_5.
\end{aligned} \tag{4.1}$$

If we write $P(G,x) = \sum_i (-1)^i a_i x(x-1)^{p-i-1} = \left(\sum_i (-1)^i a_i T_{p-i}\right)$ then on equating coefficients, as before, in the two expressions for $P(G,x)$, we have

$$b_0 = a_0 = 1$$

$$b_1 = \binom{p-1}{1} a_0 + a_1 = q$$

Since G is a $(p, p+n)$-graph we have

$$\begin{aligned} a_1 &= q - (p-1) \\ &= p + n - p + 1 \\ &= \binom{n+1}{1}. \end{aligned}$$

Also $b_2 = \binom{p-1}{2} a_0 + \binom{p-2}{1} a_1 + a_2 = \binom{q}{2} - c_3$. Hence

$$\begin{aligned} a_2 &= \binom{q}{2} - \binom{p-2}{1}\binom{n+1}{1} - \binom{p-1}{2} - c_3 \\ &= \binom{p+n}{2} - (p-2)(n+1) - \frac{(p-1)(p-2)}{2} - c_3 \\ &= \frac{(p+n)(p+n-1) - 2(pn + p - 2n - 2) - (p^2 - 3p + 2)}{2} - c_3 \\ &= \frac{n^2 + 3n + 2}{2} - c_3 \\ &= \binom{n+2}{2} - c_3. \end{aligned}$$

Also $b_3 = \binom{p-1}{3} a_0 + \binom{p-2}{2} a_1 + \binom{p-3}{1} a_2 + a_3 = \binom{q}{3} - (q-2)c_3 - c_4 + 2k_4$. Hence

$$a_3 = \binom{p+n}{3} - \binom{p-1}{3} - \binom{p-2}{2}\binom{n+1}{1} - \binom{p-3}{1}\left[\binom{n+2}{2} - c_3\right]$$

$$-(p+n-2)c_3 - c_4 + 2k_4$$

$$= \frac{(p+n)(p+n-1)(p+n-2)}{6} - \frac{(p-1)(p-2)(p-3)}{6}$$

$$- \frac{3(p-2)(p-3)(n+1)}{6} - \frac{3(p-3)(n+2)(n+1)}{6}$$

$$-c_3(p+n-2+p+3) - c_4 + 2k_4$$

$$= \frac{(p+n)(p^2+2pn-3p+n^2-3n+2) - (p-1)(p^2-5p+6)}{6} +$$

$$\frac{-3(n+1)(p^2-5p+6) - 3(p-3)(n^2+3n+2)}{6}$$

$$-c_3(n+1) - (c_4 - 2k_4)$$

$$= \frac{n^3+6n^2+11n+6}{6} - c_3(n+1) - (c_4 - 2k_4)$$

$$= \binom{n+3}{3} - c_3(n+1) - (c_4 - 2k_4).$$

Also

$$b_4 = \binom{q}{4} - \binom{q-2}{2}c_3 + \binom{c_3}{2} - (q-3)c_4 + (2q-9)k_4$$

$$- c_5 - 6k_5 + k(2,3) + 2w_5(1) + 3w_5$$

$$= \binom{p-1}{4}a_0 + \binom{p-2}{3}a_1 + \binom{p-3}{2}a_2 + \binom{p-4}{1}a_3 + a_4$$

Hence

$$a_4 = \binom{p+n}{4} - \binom{p-1}{4} - \binom{p-2}{3}\binom{n+1}{1} - \binom{p-3}{3}\left[\binom{n+2}{2} - c_3\right]$$

$$
\begin{aligned}
&-\binom{p-4}{1}\left[\binom{n+3}{3}-c_3\binom{n+1}{1}-(c_4-2k_4)\right]-\binom{p+n-2}{2}c_3+\binom{c_3}{2}\\
&-(p+n-3)c_4+(2p+2n-9)k_4-c_5-6k_5+k(2,3)+2w_5(1)+3w_5\\
&=\binom{p+n}{4}-\binom{p-1}{4}-\binom{p-2}{3}\binom{n+1}{1}-\binom{p-3}{2}\binom{n+2}{2}\\
&-\binom{p-4}{1}\binom{n+3}{3}-c_3\left[-\binom{n+1}{1}\binom{p-4}{1}-\binom{p-3}{2}+\binom{p+n-2}{2}\right]\\
&-\left[-\binom{p-4}{1}(c_4-2k_4)+(p+n-3)c_4-(2p+2n-9)k_4\right]\\
&-\left[c_5-\binom{c_3}{2}-k(2,3)-2w_5(1)-3w_5+6k_5\right]\\
&=\binom{n+4}{4}-c_3\binom{n+2}{2}-(c_4-2k_4)\binom{n+1}{1}\\
&-\left[c_5-\binom{c_3}{2}-k(2,3)-2w_5(1)-3w_5+3k_4+6k_5\right].
\end{aligned}
$$

The above expressions of the five coefficients can be used to compute chromatic polynomials of all the graphs of order at most six (since each of them has at most five non-zero coefficients). We now use other techniques to compute chromatic polynomials of some graphs of higher order.

4.4 Higher Order Coefficients

Lemma 4.4.1. Let G be a connected graph. If G contains a complete subgraph K_m then

$$P(K_m,x)|P(G,x) \quad \text{or} \quad x^{(m)}|P(G,x).$$

Proof. The number of ways of colouring K_m, using x colours, is $x^{(m)}$. If we fix the colouring of these m vertices then there will be $\frac{P(G,x)}{x^{(m)}}$ ways of colouring the remaining vertices of G. Hence $x^{(m)}|P(G,x)$. □

Remember that, by Lemma 2.4.4 with $n = 1$, the chromatic polynomial of the complete graph can be expressed as $P(K_m,x) = \sum_{i=0}^{m-1} s_{m-i}T_{i+1}$ where $s_{m-i} = s(m-1,i)$ are Stirling numbers of the first kind. This will be used in our next theorem. together with the fact given by Lemma 4.4.1: if the connected graph G contains K_m as a subgraph, then $P(G,x)$ can be written as $P(K_m,x) \times f(x)$ where $f(x)$ is a polynomial with integer coefficients.

Theorem 4.4.2. Let G be a connected $(p, p+n)$-graph with $P(G,x) = \sum_{i=0}^{p-2}(-1)^i a_i T_{p-i}$. If G contains a complete subgraph K_m and $P(G,x) = P(K_m,x) \times f(x)$ with $f(x) = \sum_{i=0}^{p-m} b_i T_{p-m-i}$, then

$$b_0 = 1,$$

$$b_1 = -\binom{a_1+s_2+1}{1},$$

$$b_2 = \binom{a_1+s_2+2}{2} - (c_3 - \binom{m}{3}),$$

$$b_3 = -\left[\binom{a_1+s_2+3}{3} - (c_3 - \binom{m}{3})\binom{a_1+s_2+1}{1} - (c_4 - 2(k_4 - \binom{m}{4}))\right]$$

and

$$b_4 = \binom{a_1+s_2+4}{4} - (c_3 - \binom{m}{3})\binom{a_1+s_2+2}{2}$$

$$- (c_4 - 2(k_4 - \binom{m}{4}))\binom{a_1+s_2+1}{1} - \left[c_5 - \binom{c_3 - \binom{m}{3}}{2}\right.$$

$$-k(2,3) - 2w_5(1) - 3w_5 + 3(k_4 - \binom{m}{4}) + 6(k_5 - \binom{m}{5})\Big]$$

where $T_0 = 1$.

Proof.

$$P(G,x) = P(K_m,x) \times f(x)$$

$$= (s_1T_m + s_2T_{m-1} + s_3T_{m-2} + s_4T_{m-3} + \cdots + s_{m-1}T_2)\times$$

$$(b_0T_{p-m} + b_1T_{p-m-1} + b_2T_{p-m-2} + b_3T_{p-m-3} + \cdots + b_{p-m}T_0)$$

$$= b_0s_1(T_p + T_{p-1}) + b_1s_1(T_{p-1} + T_{p-2}) + b_0s_2(T_{p-1} + T_{p-2})$$

$$+ b_2s_1(T_{p-2} + T_{p-3}) + b_1s_2(T_{p-2} + T_{p-3}) + b_0s_3(T_{p-2} + T_{p-3})$$

$$+ b_3s_1(T_{p-3} + T_{p-4}) + b_2s_2(T_{p-3} + T_{p-4}) + b_1s_3(T_{p-3} + T_{p-4})$$

$$+ b_0s_4(T_{p-3} + T_{p-4}) + \cdots + b_{p-m}s_{m-1}(T_2 + T_1)$$

$$= b_0s_1T_p + (b_0s_1 + b_1s_1 + b_0s_2)T_{p-1}$$

$$+ (b_1s_1 + b_0s_2 + b_2s_1 + b_1s_2 + b_0s_3)T_{p-2}$$

$$+ (b_2s_1 + b_1s_2 + b_0s_3 + b_3s_1 + b_2s_2 + b_1s_3 + \text{[illegible]})T_{p-3}$$

$$+ \cdots + b_{p-m}s_{m-1}T_1$$

$$= \sum_{k=0}^{p-2}(-1)^k a_k T_{p-k}$$

Hence

$$\left[\sum_{i=0}^{k-1} s_{i+1}b_{k-1-i} + \sum_{i=0}^{k} s_{i+1}b_{k-i}\right] T_{p-k} = (-1)^k a_k T_{p-k}$$

Now for

$k = 0$; we have $s_1b_0 = a_0 = 1$

Thus $b_0 = 1$ since $s_1 = s(m-1, m-1) = 1$

$k = 1$; we have $s_1 b_0 + s_1 b_1 + s_2 b_0 = -a_1$

$1 + b_1 + s_2 = -a_1$

Thus $b_1 = -(a_1 + s_2 + 1) = -\binom{a_1+s_2+1}{1}$

$k = 2$; we have $s_1 b_1 + s_2 b_0 + s_1 b_2 + s_2 b_1 + s_3 b_0 = a_2$

$b_1 + s_2 + b_2 + s_2 b_1 + s_3 = a_2$

Thus $b_2 = a_2 - s_3 + a_1 - s_2 b_1 + 1$ since $b_1 + s_2 = -a_1 - 1$.

But by Theorem 4.2.3 $a_2 = \binom{a_1+1}{2} - c_3$ and $s_3 = \binom{-s_2+1}{2} - \binom{m}{3}$.

Therefore

$$b_2 = \binom{a_1+1}{2} - c_3 - \binom{-s_2+1}{2} + \binom{m}{3} + a_1 - s_2(-a_1 - s_2 - 1) + 1$$

$$= \binom{c_1+1}{2} - \binom{-s_2+1}{2} + a_1 + a_1 s_2 + s_2^2 + s_2 + 1 - \left[c_3 - \binom{m}{3}\right]$$

$$= \frac{a_1(a_1+1) - (-s_2)(-s_2+1) + 2a_1 + 2a_1 s_2 + 2s_2^2 + 2s_2 + 2}{2} - \left[c_3 - \binom{m}{3}\right]$$

$$= \frac{a_1^2 + 3a_1 + s_2^2 + 3s_2 + 2a_1 s_2 + 2}{2} - \left[c_3 - \binom{m}{3}\right]$$

$$= \binom{a_1 + s_2 + 2}{2} - \left[c_3 - \binom{m}{3}\right]$$

$k = 3$; we have $s_1 b_2 + s_2 b_1 + s_3 b_0 + s_1 b_3 + s_2 b_2 + s_3 b_1 + s_4 b_0 = -a_3$

$b_2 + s_2 b_1 + s_3 + b_3 + s_2 b_2 + s_3 b_1 + s_4 = -a_3$

Thus $b_3 = -a_3 - s_4 - a_2 - s_3 b_1 - a_1 - s_2 b_2 - 1$ since $b_2 + s_2 b_1 + s_3 = a_2 + a_1 + 1$.

But by Theorem 4.2.4

$$a_3 = \binom{a_1+2}{3} - c_3(a_1) - (c_4 - 2k_4) \quad \text{and}$$

$$s_4 = -\binom{-s_2+2}{3} + \binom{m}{3}(-s_2) + \left[-2\binom{m}{4}\right].$$

Therefore

$$b_3 = -\binom{a_1+2}{3} + c_3a_1 + (c_4 - 2k_4) + \binom{-s_2+2}{3} + s_2\binom{m}{3} + 2\binom{m}{4}$$

$$-\binom{a_1+1}{2} + c_3 + \left[-\binom{-s_2+1}{2} + \binom{m}{3}\right][-a_1 - s_2 - 1]$$

$$-a_1 - s_2\left[\binom{a_1+s_2+2}{2} - c_3 + \binom{m}{3}\right] - 1$$

$$= \frac{-a_1(a_1+1)(a_1+2)}{6} + \frac{(-s_2)(-s_2+1)(-s_2+2)}{6}$$

$$-\frac{a_1(a_1+1)}{2} + \frac{(-s_2)(-s_2+1)(a_1+s_2+1)}{2} - a_1$$

$$-\frac{s_2(a_1+s_2+2)(a_1+s_2+1)}{2} - 1 + c_3a_1 + c_3 + c_3s_2$$

$$+\binom{m}{3}[s_2 - a_1 - s_2 - 1 - s_2] + \left[c_4 - 2k_4 + 2\binom{m}{4}\right]$$

$$= \frac{-a_1^3 - 3a_1^2 - 2a_1 - s_2^3 + 3s_2^2 - 2s_2 - 3a_1^2 - 3a_1 + 3a_1s_2^2 + 3s_2^3}{6} +$$

$$\frac{-3a_1s_2 - 3s_2 - 6a_1 - 3a_1^2s_2 - 6a_1s_2^2 - 9a_1s_2 - 3s_2^3 - 9s_2^2 - 6s_2 - 6}{6}$$

$$+\left[c_3 - \binom{m}{3}\right][a_1 + s_2 + 1] + \left[c_4 - 2(k_4 - \binom{m}{4})\right]$$

$$= -\left[\binom{a_1+s_2+3}{3} - (c_3 - \binom{m}{3})\binom{a_1+s_2+1}{1} - (c_4 - 2(k_4 - \binom{m}{4}))\right]$$

$k = 4$; we have

$s_1b_3 + s_2b_2 + s_3b_1 + s_4b_0 + s_1b_4 + s_2b_3 + s_3b_2 + s_4b_1 + s_5b_0 = a_4$

$b_3 + s_2b_2 + s_3b_1 + s_4 + b_4 + s_2b_3 + s_3b_2 + s_4b_1 + s_5 = a_4$

Thus $b_4 = a_4 - s_5 + a_3 - s_4b_1 + a_2 - s_3b_2 + a_1 - s_2b_3 + 1$ since $b_3 + s_2b_2 + s_3b_1 + s_4 = -a_3 - a_2 - a_1 - 1$

But by Theorem 4.2.5 $a_4 = \binom{a_1+3}{4} - c_3\binom{a_1+1}{2} - (c_4 - 2k_4)a_1 - [c_5 - \binom{c_3}{2} - k(2,3) - 2w_5(1) - 3w_5 + 3k_4 + 6k_5]$ and $s_6 = \binom{-s_2+3}{4} - \binom{m}{3}\binom{-s_2+1}{2} + 2\binom{m}{4}(-s_2) - [-\binom{\binom{m}{3}}{2} + 3\binom{m}{4} + 6\binom{m}{5}]$.

Therefore

$$
\begin{aligned}
b_4 = {} & \binom{a_1+3}{4} - c_3\binom{a_1+1}{2} - (c_4 - 2k_4)a_1 \\
& - \left[c_5 - \binom{c_3}{2} - k(2,3) - 2w_5(1) - 3w_5 + 3k_4 + 6k_5\right] - \binom{-s_2+3}{4} \\
& + \binom{m}{3}\binom{-s_2+1}{2} + 2s_2\binom{m}{4} - \binom{\binom{m}{3}}{2} + 3\binom{m}{4} + 6\binom{m}{5} \\
& + \binom{a_1+2}{3} - c_3a_1 - (c_4 - 2k_4) - \left[\binom{-s_2+2}{3} + \binom{m}{3}s_2\right. \\
& \left. + 2\binom{m}{4}\right][a_1 + s_2 + 1] + \binom{a_1+1}{2} - c_3 - \left[\binom{-s_2+1}{2} - \binom{m}{3}\right] \times \\
& \left[\binom{a_1+s_2+2}{2} - c_3 + \binom{m}{3}\right] + a_1 + s_2\left[\binom{a_1+s_2+3}{3}\right. \\
& \left. - c_3\binom{a_1+s_2+1}{1} + \binom{m}{3}\binom{a_1+s_2+1}{1} - c_4 + 2k_4 - 2\binom{m}{4}\right] + 1
\end{aligned}
$$

$$= \binom{a_1+3}{4} - \binom{-s_2+3}{4} + \binom{a_1+2}{3} - \binom{-s_2+2}{3}(a_1+s_2+1)$$

$$+ \binom{a_1+1}{2} - \binom{-s_2+1}{2}\binom{a_1+s_2+2}{2} + a_1 + s_2\binom{a_1+s_2+3}{3} + 1$$

$$- c_3\left[\frac{a_1^2+a_1}{2} + \frac{2a_1}{2} + \frac{2}{2} + \frac{s_2(-s_2+1)}{2} + a_1 s_2 + s_2^2 + s_2 + \binom{m}{3}\right]$$

$$- (c_4 - 2k_4)(a_1 + 1 + s_2) - 2s_2\binom{m}{4} + 2s_2\binom{m}{4}$$

$$- 2\binom{m}{4}(a_1+s_2+1) + \binom{m}{3}\left[\binom{-s_2+1}{2} - s_2(a_1+s_2+1) - \binom{-s_2+1}{2}\right.$$

$$\left. + \binom{a_1+s_2+2}{2} + s_2(a_1+s_2+1)\right] - \left[c_5 - \binom{c_3}{2} + \binom{\binom{m}{3}}{2} - k(2,3)\right.$$

$$\left. - 2w_5(1) - 3w_5 + 3k_4 - 3\binom{m}{4} + 6k_5 - 6\binom{m}{5}\right]$$

$$= \binom{a_1+s_2+4}{4} - c_3\binom{a_1+s_2+2}{2} - c_3\binom{m}{3} + \binom{m}{3}\binom{a_1+s_2+2}{2}$$

$$+ \binom{m}{3}\binom{m}{3} - \left(c_4 - 2\left[k_4 - \binom{m}{4}\right]\right)(a_1+s_2+1) - \left[c_5 - \binom{c_3}{2}\right.$$

$$\left. + \binom{\binom{m}{3}}{2} - k(2,3) - 2w_5(1) - 3w_5 + 3k_4 - 3\binom{m}{4} + 6k_5 - 6\binom{m}{5}\right]$$

$$= \binom{a_1+s_2+4}{4} - \left(c_3 - \binom{m}{3}\right)\binom{a_1+s_2+2}{2}$$

$$- \left(c_4 - 2\left[k_4 - \binom{m}{4}\right]\right)\binom{a_1+s_2+1}{1}$$

$$- \left[c_5 - \binom{c_3 - \binom{m}{3}}{2} - k(2,3) - 2w_5(1) - 3w_5 + 3\left(k_4 - \binom{m}{4}\right)\right.$$

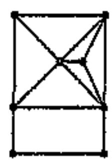
$$+6\left(k_5 - \binom{m}{5}\right)\Bigg] \quad \text{since}$$

$$-\binom{c_3}{2} + \binom{\binom{m}{3}}{2} - c_3\binom{m}{3} + \binom{m}{3}\binom{m}{3} = -\binom{c_3 - \binom{m}{3}}{2}. \qquad \square$$

Again the same technique can be repeated to find expressions for $b_5, b_6, \ldots$ in the situation of Theorem 4.4.2 but the details clearly become too involved.

Example 4.4.1.

With G :



$c_3 = 7$

$c_4 = 2$

$k_4 = 1$

$w_5 = 1$

we have that G is a (8,14)-graph that contains a complete graph on 4 vertices so that $P(K_4, x) = T_4 - 3T_3 + 2T_2$. Thus $a_1 = n + 1 = 6 + 1 = 7$, $s_2 = -3$ and $P(G, x) = (T_4 - 3T_3 + 2T_2) \times (b_0T_4 + b_1T_3 + b_2T_2 + b_3T_1 + b_4)$ where

$$b_0 = 1$$

$$b_1 = -(a_1 + s_2 + 1) = -(7 - 3 + 1) = -5$$

$$b_2 = \binom{a_1 + s_2 + 2}{2} - (c_3 - \binom{m}{3})$$

$$= \binom{7 - 3 + 2}{2} - (7 - \binom{4}{3}) = 15 - 3 = 12$$

$$b_3 = -\left[\binom{a_1 + s_2 + 3}{3} - (c_3 - \binom{m}{3})\binom{a_1 + s_2 + 1}{1} - (c_4 - 2\{k_4 - \binom{m}{4}\})\right]$$

$$= -\left[\binom{7 - 3 + 3}{3} - (7 - \binom{4}{3})\binom{7 - 3 + 1}{1} - (2 - 2\{1 - \binom{4}{4}\})\right]$$

$$= -(35 - 15 - 2) = -18$$

$$b_4 = \binom{a_1+s_2+4}{4} - (c_3 - \binom{m}{3})\binom{a_1+s_2+2}{2}$$

$$- \left(c_4 - 2\left[k_4 - \binom{m}{4}\right]\right)\binom{a_1+s_2+1}{1} - \left[c_5 - \binom{c_3 - \binom{m}{3}}{2} - k(2,3)\right.$$

$$\left. - 2w_5(1) - 3w_5 + 3\left(k_4 - \binom{m}{4}\right) + 6\left(k_5 - \binom{m}{5}\right)\right]$$

$$= \binom{7-3+4}{4} - \left(7 - \binom{4}{3}\right)\binom{7-3+2}{.2} - 2(7-3+1)$$

$$- \left[-\binom{7-4}{2} - 3(1) + 3(1-1)\right]$$

$$= \binom{8}{4} - 3\binom{6}{2} - 2(5) - [-3-3]$$

$$= 70 - 45 - 10 + 6 = 21$$

Hence $P(G,x) = (T_4 - 3T_3 + 2T_2) \times (T_4 - 5T_3 + 12T_2 - 18T_1 + 21)$.

Note that $s_{m-1}b_{p-m-1} + s_{m-1}b_{p-m} = (-1)^{p-2}a_{p-2}$, so that in the above example for $m = 4$ and $p = 8$ we have

$$s_3b_3 + s_3b_4 = (-1)^{8-2}a_6$$

$$(2)(-18) + 2(21) = a_6$$

$$a_6 = 42 - 36$$

$$= 6$$

If the clique number of a graph is high then we can use Theorem 4.4.2 to compute higher order coefficients.

Recall from Section 3.1 that $G(K_m, T)$ is a graph obtained from a tree T by replacing the vertex v with K_m. We will use this notation in our next theorem. Consider a graph G of order n wherein if the chromatic number of G $\chi(G) \geq m \geq 3$, $m < n$, then $P(G, x)$ can be written as a linear combination of $P(G(K_m, T), x)$.

Theorem 4.4.3. Let G be a graph of order n with $\chi(G) \geq m \geq 3$, and $m < n$. Then $P(G,x) = \sum_{i=0}^{n-2}(-1)^i a_i T_{n-i}$ can be written as
$\sum_{i=0}^{n-m}(-1)^i b_i (\sum_{k=1}^{m-1} s(m-1, m-k) T_{n-i-k+1})$ iff
$a_k = \sum_{i=0}^{m-2}(-1)^i b_{k-i} s(m-1, m-i-1)$ for $0 \leq k \leq n-2$
with $b_{n-j} = 0$ for $2 \leq j \leq m-1$.
Proof.

$$\sum_{i=0}^{n-2}(-1)^i a_i T_{n-i}$$

$$= \sum_{i=0}^{n-m}(-1)^i b_i \left(\sum_{k=1}^{m-1} s(m-1, m-k) T_{n-i-k+1}\right)$$

$$\iff a_0 T_n - a_1 T_{n-1} + \cdots + (-1)^{n-2} a_{n-2} T_2$$

$$= b_0(s(m-1, m-1)T_n + \quad -1, m-2)T_{n-1} + \cdots + s(m-1, 1)T_{n-m+2})$$

$$- b_1(s(m-1, m-1)T_{n-1} + s(m-1, m-2)T_{n-2} + \cdots + s(m-1, 1)T_{n-m+1})$$

$$+ \cdots + (-1)^{n-m} b_{n-m}(s(m-1, m-1)T_m + s(m-1, m-2)T_{m-1}$$

$$+ \cdots + s(m-1, 1)T_2)$$

$$\iff a_0 = s(m-1, m-1)b_0,$$

$$a_1 = s(m-1, m-1)b_1 - s(m-1, m-2)b_0,$$

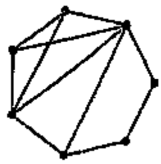
$$a_2 = s(m-1, m-1)b_2 - s(m-1, m-2)b_1 + s(m-1, m-3)b_0, \ldots$$

$$(-1)^{n-2} a_{n-2} = (-1)^{n-m} b_{n-m} s(m-1, 1)$$

$$\iff a_k = \sum_{i=0}^{m-2} (-1)^i b_{k-i} s(m-1, m-i-1)$$

for $0 \leq k \leq n-2$ with $b_{n-j} = 0$ for $2 \leq j \leq m-1$. □

Example 4.4.2. (7,11)-graph

With G:  we have that $\chi(G) = 4 = m, n = 4$

and $P(K_4, x) = T_4 - 3T_3 + 2T_2$; so that

$P(G(K_4, T), x) = T_n - 3T_{n-1} + 2T_{n-2}$ for $n = 7, 6, 5$ and 4.

Now

$$P(G, x) = b_0(T_7 - 3T_6 + 2T_5) - b_1(T_6 - 3T_5 + 2T_4)$$
$$+ b_2(T_5 - 3T_4 + 2T_3) - b_3(T_4 - 3T_3 + 2T_2)$$

where

$$1 = a_0 = s(m-1, m-1)b_0,$$

$$b_0 = 1$$

$$a_1 = s(m-1, m-1)b_1 - s(m-1, m-2)b_0 = b_1 - s(3, 2)$$

$$\binom{4+1}{1} = b_1 + 3,$$

$$b_1 = 2$$

$$a_2 = s(3,3)b_2 - s(3,2)b_1 + s(3,1)b_0,$$

$$\binom{4+2}{2} - 5 = b_2 - (-3)(2) + 2$$

$$b_2 = 2$$

$$a_3 = s(3,3)b_3 - s(3,2)b_2 + s(3,1)b_1,$$

$$\binom{4+3}{3} - 5(5) - (1-2) = b_3 - (-3)(2) + 2(2),$$

$$b_3 = 1$$

Therefore $P(G,x) = (T_7 - 3T_6 + 2T_5) - 2(T_6 - 3T_5 + 2T_4) + 2(T_5 - 3T_4 + 2T_3) - (T_4 - 3T_3 + 2T_2)$.

Theorem 4.4.4. Let G be a connected graph of order n with $P(G,x) = \sum_{i=0}^{n-1}(-1)^i a_i x_{n-i}^{\{m\}}$, $m \geq 1$. If $\chi(G) \geq m+2$ then $\sum_{i=0}^{n-1}(-1)^i a_i = 0$.

Proof.

$$\begin{aligned}
P(G,x) &= \sum_{i=0}^{n-1}(-1)^i a_i x_{n-i}^{\{m\}} \\
&= a_0 x_n^{\{m\}} - a_1 x_{n-1}^{\{m\}} + a_2 x_{n-2}^{\{m\}} + \cdots + (-1)^{n-1} a_{n-1} x_1^{\{m\}} \\
&= a_0 x(x-1)(x-2)\ldots(x-m)^n - a_1 x(x-1)(x-2)\ldots(x-m)^{n-1} \\
&+ a_2 x(x-1)(x-2)\ldots(x-m)^{n-2} \\
&+ \cdots + (-1)^{n-1} a_{n-1} x(x-1)(x-2)\ldots(x-m) \\
&= x(x-1)(x-2)\ldots(x-m)[a_0(x-m)^{n-1} - a_1(x-m)^{n-2} \\
&+ \cdots + (-1)^{n-1} a_{n-1}]
\end{aligned}$$

Now since $\chi(G) \geq m+2$ we have $P(G, m+1) = 0$. Therefore

$$P(G, m+1) = (m+1)(m)(m-1)\cdots(1)[a_0(1)^{n-1} - a_1(1)^{n-2} + \cdots + (-1)^{n-1}a_{n-1}] = 0.$$

Since $m \geq 1$ we must have $\sum_{i=0}^{n-1}(-1)^i a_i = 0$. □

As a special case of Theorem 4.4.4 let $m = 1$ then

$$P(G,x) = \sum_{i=0}^{n-1}(-1)^i a_i x_{n-i}^{(1)}$$

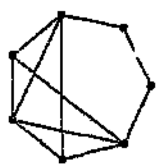
$$= \sum_{i=0}^{n-1}(-1)^i a_i x(x-1)^{n-i}$$

$$= \sum_{i=0}^{n-1}(-1)^i a_i T_{n+1-i}$$

and we have $P(G,x)$ expressed in terms of the chromatic polynomials of trees where if $\chi(G) \geq 1 + 2 = 3$ then $\sum_{i=0}^{n-1}(-1)^i a_i = 0$.

Example 4.4.3.

With G:



$c_3 = 4$
$c_4 = 1$
$c_5 = 3$
$w_5 = 1$

we know that G is a (7,11)-graph with $n = 4$ so that

$$a_1 = \binom{4+1}{1} = 5,$$

$$a_2 = \binom{6}{2} - 4 = 15 - 4 = 11,$$

$$a_3 = \binom{7}{3} - 4\binom{5}{1} - 1 = 35 - 21 = 14,$$

$$a_4 = \binom{8}{4} - 4\binom{6}{2} - 1\binom{5}{1} - \left[3 - \binom{4}{2} - 3(1)\right] = 70 - 60 - 5 - 3 + 6 + 3 = 11.$$

Now since $\chi(G) \geq 3$ we have $\sum_{i=0}^{6-1}(-1)^i a_i = 0 = a_0 - a_1 + a_2 - a_3 + a_4 - a_5$. Therefore $a_5 = a_0 - a_1 + a_2 - a_3 + a_4 = 1 - 5 + 11 - 14 + 11 = 4$. Hence $P(G,x) = T_7 - 5T_6 + 11T_5 - 14T_4 + 11T_3 - 4T_2$.

This example illustrates how we can calculate the chromatic polynomial of any graph G of order 7 with $\chi(G) \geq 3$ using the remark just before the example and the results of section 4.2.

4.5 The Coefficients and Connectivity

If $P(G,x)$ is the chromatic polynomial, in normal form, of a graph G on p vertices, then it is known, see [14], that the multiplicity of the root 0 in $P(G,x)$ is the number of connected components of G.

Furthermore it is known that the coefficients of $P(G,x)$ alternate in sign. In tree form, these matters are related by our next result (which appears in [3]).

Lemma 4.5.1. The coefficients of a chromatic polynomial $P(G,x) = \sum_{i=0}^{p-1} a_i T_{p-i}$ alternate in sign if and only if G is connected.

Proof. Let G be a connected $(p, p+n)$-graph. We employ induction on p and n. For $n = -1$, G must be a tree on p vertices and $p-1$ edges and hence $P(G,x) = T_p$. Assume that for some fixed integer $n, n \geq 0$, the result is true for all connected $(p', p'+n')$-graphs with $p' \leq p$ and $n' \leq n-1$. Now consider any connected graph

G with p vertices and $p+n$ edges. By Whitney's Reduction Formula applied to any edge uv, which is not a cut-edge (which exists because $n \geq 0$), we have:

$$P(G,x) = P(G',x) - P(G'',x)$$

where G' is a connected graph on p vertices and $p+n-1$ edges and G'' is a connected graph on $p-1$ vertices and less than $p+n-1$ edges. Since the result is therefore true for G' and G'' we can write

$$P(G',x) = T_p - a_1'T_{p-1} + a_2'T_{p-2} - a_3'T_{p-3} + \cdots \quad \text{and}$$
$$P(G'',x) = T_{p-1} - a_1''T_{p-2} + a_2''T_{p-3} - \cdots$$

where all the a_i', a_i'' are positive integers. Hence $P(G,x) = T_p - (a_1'+1)T_{p-1} + (a_2'+a_1'')T_{p-2} - (a_3'+a_2'')T_{p-3} + \ldots$ in which the coefficients alternate in sign. On the other hand, if G has $k \geq 2$ components and $P(G,x) = \sum_{i=0}^{p-1}(-1)^i b_i T_{p-i} = \sum_{i=0}^{p-1}(-1)^i b_i x(x-1)^{p-i-1}$ with each $b_i \geq 0$ and at least one b_i positive, then $x^k | P(G,x)$. But $\dfrac{P(G,x)}{x} = \sum_{i=0}^{p-1}(-1)^i b_i (x-1)^{p-1-i}$ and with $x = 0$ this is $\pm\sum_{i=0}^{p-1} b_i$ which is not zero. This contradiction proves the converse. □

We now aim at characterizing, in tree form, the chromatic polynomials of forests. The result (Theorem 4.5.1) extends the characterization of the chromatic polynomials of trees given in The[illegible]3 of [14].

Lemma 4.5.2. Let G be a graph with p vertices and $p-k$ edges. Then G is a forest if and only if G has k components.

Proof. If G is a forest with l components, p_i vertices and q_i edges in the ith component, then $q_i = p_i - 1$ for each i. Hence $p-k = \sum_{i=0}^{l} q_i = \sum_{i=0}^{l}(p_i - 1) = p-l$, that is, $l = k$. If G has k components but is not a forest, then, in the above notation,

$q_i \geq p_i - 1$ for each i with strict inequality for at least one i. Hence q, the number of edges of G, satisfies $q = \sum_{i=0}^{k} q_i > \sum_{i=0}^{k}(p_i - 1) = p - k$, a contradiction. □

Theorem 4.5.1. If G is a graph, then $P(G,x) = \sum_{j=0}^{k-1} \binom{k-1}{j} T_{p-j}$ if and only if G is a forest with p vertices and k components.

Proof. If G is a forest with p vertices and k components, then $P(G,x) = \sum_{j=0}^{k-1}\binom{k-1}{j}T_{p-j}$ by Corollary 3.3.2. Suppose $P(G,x) = \sum_{j=0}^{k-1}\binom{k-1}{j}T_{p-j}$. Then $P(G,x) = x^k(x-1)^{p-k}$ (as shown after Corollary 3.3.2). Hence G has p vertices (by Lemma 1.5.1), $p-k$ edges (by section 4.3) and k components and hence is a forest by Lemma 4.5.2. □

Corollary 4.5.1. G is a tree of order p if and only if $P(G,x) = T_p$. $(k=1)$ □

We note that forests are not the only graphs which have no negative coefficients. This can be seen from the following example:

Example 4.5.1. If G is a graph on $p = n + m + 1$ vertices with $G = C_n \cup \bar{K}_{m+1}$ where $m + 2 \geq n + 1 \geq 5$ and n even, then by Corollary 3.3.3 we have

$$P(G,x) = \sum_{j=0}^{p-2}\left[\sum_{i=0}^{n-2}(-1)^i\binom{m+1}{j-i}\right]T_{p-j}$$

$$= T_p + \left[\binom{m+1}{1} - \binom{m+1}{0}\right]T_{p-1} + \left[\binom{m+1}{2} - \binom{m+1}{1}\right.$$

$$\left. + \binom{m+1}{0}\right]T_{p-2} + \cdots + \left[\binom{m+1}{m} - \binom{m+1}{m+1}\right]T_3 + \binom{m+1}{m+1}T_2$$

$$= T_p + \binom{m}{1}T_{p-1} + \binom{m}{2}T_{p-2} + \cdots + \binom{m}{i}T_{m+1-i}$$

$$+\cdots+T_{n+1}+T_{m+2}+\binom{m}{1}T_{m+1}+\binom{m}{2}T_m+\cdots+T_2.$$

To illustrate the above general formula consider $n=6$ and $m+1=6$, then $p=n+m+1=12$ and $m=5$. Hence

$$\begin{aligned}P(G,x) &= T_{12}+\binom{5}{1}T_{11}+\binom{5}{2}T_{10}+\binom{5}{3}T_9+\binom{5}{4}T_8+\binom{5}{5}T_7\\ &\quad +T_7+\binom{5}{1}T_6+\binom{5}{2}T_5+\binom{5}{3}T_4+\binom{5}{4}T_3+\binom{5}{5}T_2\\ &= T_{12}+5T_{11}+10T_{10}+10T_9+5T_8+2T_7\\ &\quad +5T_6+10T_5+10T_4+5T_3+T_2.\end{aligned}$$

4.6 A New Combinatorial Identity

We prove a combinatorial identity which, to us, is new and apply it in section 4.7 to obtain the coefficients of a chromatic polynomial of some special family of graphs.

Theorem 4.6.1. For all nonnegative integers n and k with $k \leq n+1$ we have $\binom{n+1}{k}=\sum_{i=0}^{\lfloor \frac{k}{2}\rfloor}(-1)^i\binom{n+i}{i}\binom{n+k-2i}{k-2i}$, where $\lfloor x \rfloor$ is the greatest integer $\leq x$.

Proof. In order to prove this we note that the binomial coefficients $\binom{n+1}{k}$, $k=0,1,\ldots,n+1$ can be obtained as the unique solution of the recursion formula $f(n+1,k)=f(n,k)+f(n,k-1)$ with the boundary values $f(n+1,0)=f(n+1,n+1)=1$. Hence, if we define $f(n+1,k)=\sum_{i=0}^{\lfloor \frac{k}{2}\rfloor}(-1)^i\binom{n+1}{i}\binom{n+k-2i}{k-2i}$, we need only prove that f satisfies these equations. First, consider

$$f(n+1,k)=\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}(-1)^i\binom{n+1}{i}\binom{n+k-2i}{k-2i}$$

$$=\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}(-1)^i\binom{n}{i}\binom{n+k-2i}{k-2i}+\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}(-1)^i\binom{n}{i-1}\binom{n+k-2i}{k-2i}$$

$$\text{since}\binom{n+1}{i}=\binom{n}{i}+\binom{n}{i-1}$$

$$=\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}(-1)^i\binom{n}{i}\binom{n+k-1-2i}{k-2i}+\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}(-1)^i\binom{n}{i}\binom{n+k-1-2i}{k-1-2i}$$

$$+\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}(-1)^i\binom{n}{i-1}\binom{n+k-2i}{k-2i}$$

$$=f(n,k)+\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}(-1)^i\binom{n}{i}\binom{n+k-2-2i}{k-1-2i}$$

$$+\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}(-1)^i\binom{n}{i}\binom{n+k-2(i+1)}{k-2(i+1)}+\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}(-1)^i\binom{n}{i-1}\binom{n+k-2i}{k-2i}$$

$$=f(n,k)+f(n,k-1)\circledast+0^{\dagger}$$

$$=f(n,k)+f(n,k-1)$$

Notes:

$\circledast$ The second term of the previous step is $f(n,k-1)$ if $\lfloor\frac{k-1}{2}\rfloor=\lfloor\frac{k}{2}\rfloor$ and this happens only if k is odd. For k even, however, the last term of $\sum_{i=0}^{\lfloor\frac{k}{2}\rfloor}(-1)^i\binom{n}{i}\binom{n+k-2-2i}{k-1-2i}$ contains $\binom{n+k-2-2i}{k-1-2\lfloor\frac{k}{2}\rfloor}$ which is 0 since then $k-1-2\lfloor\frac{k}{2}\rfloor=-1<0$. Hence this summation effectively stops at $\lfloor\frac{k}{2}\rfloor-1=\lfloor\frac{k-1}{2}\rfloor$. † The terms of the last two summations in the previous step cancel in pairs except two, which are both zero. Finally

$$f(n+1,0) = (-1)^0 \binom{n+1}{0}\binom{n}{0} = 1 \quad \text{and}$$

$$\begin{aligned}
f(n+1,n+1) &= f(n,n+1) + f(n,n) \\
&= \binom{2}{0} f(n-1,n+1) + \binom{2}{1} f(n-1,n) + \binom{2}{2} f(n-1,n-1) \\
&= \binom{3}{0} f(n-2,n+1) + \binom{3}{1} f(n-2,n) + \binom{3}{2} f(n-2,n-1) \\
&+ \binom{3}{3} f(n-2,n-2) \\
&= \dots\dots\dots\dots\dots\dots\dots\dots \\
&= \binom{n}{0} f(1,n+1) + \binom{n}{1} f(1,n) + \cdots + \binom{n}{i} f(1,n+1-i) \\
&+ \cdots + f(1,1) \\
&= 0 + 0 + \cdots + 0 + \cdots + 0 + 1 \\
&= 1
\end{aligned}$$

since $f(1,k) = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^i \binom{1}{i}\binom{k-2i}{k-2i}$ has only non-zero terms where $i = 0$ and $i = 1$ and they cancel: $\binom{1}{0}\binom{k}{k} - \binom{1}{1}\binom{k-2}{k-2} = 0$ unless $k = 1$, in which case $i = 0$ produces the only term $\binom{1}{0}\binom{k}{k} = 1$. □

Since $f(n+1,n+1) = \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^i \binom{n+1}{i}\binom{2n+1-2i}{n+1-2i} = 1$ we have

$$1 = \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^i \binom{n+1}{i}\binom{2n+1-2i}{n} \qquad \text{since} \binom{n}{k} = \binom{n}{n-k}$$

$$= \binom{n}{0}\binom{2n+1}{n} - \binom{n+1}{1}\binom{2n-1}{n} + \cdots$$

$$+ (-1)^{\lfloor \frac{n+1}{2} \rfloor}\binom{n+1}{\lfloor \frac{n+1}{2} \rfloor}\binom{2n+1-2\lfloor \frac{n+1}{2} \rfloor}{n}$$

$$= \binom{n}{0}\binom{2n+1}{n} - \left[\binom{n}{1} + \binom{n}{0}\right]\binom{2n-1}{n}$$

$$+ \left[\binom{n}{2} + \binom{n}{1}\right]\binom{2n-3}{n} - \cdots + (-1)^{\lfloor \frac{n+1}{2} \rfloor}$$

$$\left[\binom{n}{\lfloor \frac{n+1}{2} \rfloor} + \binom{n}{\lfloor \frac{n+1}{2} \rfloor - 1}\right]\binom{2n+1-2\lfloor \frac{n+1}{2} \rfloor}{n}$$

$$= \binom{n}{0}\left[\binom{2n+1}{n} - \binom{2n-1}{n}\right] - \binom{n}{1}\left[\binom{2n-1}{n} - \binom{2n-3}{n}\right] + \cdots$$

$$+ (-1)^{\lfloor \frac{n+1}{2} \rfloor}\binom{n}{\lfloor \frac{n+1}{2} \rfloor}\left[\binom{2n+1-2\lfloor \frac{n+1}{2} \rfloor}{n} - \binom{2n-1-2\lfloor \frac{n+1}{2} \rfloor}{n}\right]$$

$$= \binom{n}{0}\left[\binom{2n}{n-1} + \binom{2n-1}{n-1}\right] - \binom{n}{1}\left[\binom{2n-2}{n-1} + \binom{2n-3}{n-1}\right]$$

$$+ \cdots + (-1)^{\lfloor \frac{n+1}{2} \rfloor}\binom{n}{\lfloor \frac{n+1}{2} \rfloor}\left[\binom{2n-2\lfloor \frac{n+1}{2} \rfloor}{n-1} + \binom{2n-1-2\lfloor \frac{n+1}{2} \rfloor}{n-1}\right]$$

$$= \binom{n}{0}\binom{2n}{n-1} - \binom{n}{1}\binom{2n-2}{n-1} + \cdots + (-1)^{\lfloor \frac{n+1}{2} \rfloor}\binom{n}{\lfloor \frac{n+1}{2} \rfloor}$$

$$\binom{2n-2\lfloor \frac{n+1}{2} \rfloor}{n-1} + \binom{n}{0}\binom{2n-1}{n-1} - \binom{n}{1}\binom{2n-3}{n-1}$$

$$+ \cdots + (-1)^{\lfloor \frac{n+1}{2} \rfloor}\binom{2n-1-2\lfloor \frac{n+1}{2} \rfloor}{n-1}$$

$$= \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor}(-1)^i\binom{n}{i}\binom{2n-2i}{n-1} + \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor}(-1)^i\binom{n}{i}\binom{2n-1-2i}{n-1}$$

$$= \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^i \binom{n}{i} \binom{2n-2i}{n-1} + 1$$

Hence

Corollary 4.6.1.

$$\sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^i \binom{n}{i} \binom{2n-2i}{n-1}$$

$$= \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^i \binom{n}{i} \binom{2n-2i}{n+1-2i} = 0 .$$ □

4.7 The coefficients of the chromatic polynomial of a cascade of graphs

We now use the combinatorial identity, presented in Theorem 4.6.1, which reminds us of the inclusion-exclusion principle to interpret the coefficients of the chromatic polynomial of a cascade of graphs. In Lemma 1.5.5 it was shown that if G is a cascade of triangles on p vertices and $(p+n)$-edges then

$P(G,x) = \sum_{i=0}^{p-2} (-1)^i \binom{p-2}{i} T_{p-i} = x(x-1)(x-2)^{p-2}$.

Since G is a $(p, p+n)$-graph then the number of triangles in G is $c_3 = p-2$ and $n = p-3$. Now if we write $P(G,x) = \sum_{i=0}^{p-2} (-1)^i a_i T_{p-i} = \sum_{i=0}^{p-2} (-1)^i \binom{p-2}{i} T_{p-i}$ then by equating the coefficients we have $a_k = \binom{p-2}{k} = \binom{c_3}{k} = \binom{n+1}{k}$. Hence we have:

Lemma 4.7.1. If G is a cascade of triangles on p vertices and $(p+n)$-edges then

$$a_k = \binom{n+1}{k} = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^i \binom{n+1}{i} \binom{n+k-2i}{k-2i}$$

where $\lfloor x \rfloor$ is the greatest integer $\leq x$.

Proof. By Theorem 4.6.7. $\square$

We note that if G is a $(p, p+n)$-graph which is a cascade of triangles then

$$a_k = \binom{p-2}{k} = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^i \binom{p-2}{i} \binom{p-3+k-2i}{k-2i}$$

since $n = p-3$ and

$$a_k = \binom{c_3}{k} = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^i \binom{c_3}{i} \binom{n+k-2i}{k-2i}$$

since $c_3 = n+1$.

Using this latter formula we have

$$a_0 = \binom{n}{0} = 1$$

$$a_1 = \binom{n+1}{1}$$

$$a_2 = \binom{n+2}{2} - c_3$$

$$a_3 = \binom{n+3}{3} - c_3\binom{n+1}{1}$$

$$a_4 = \binom{n+4}{4} - c_3\binom{n+2}{2} + \binom{c_3}{2}$$

$$a_5 = \binom{n+5}{5} - c_3\binom{n+3}{3} + \binom{c_3}{2}\binom{n+1}{1}$$

$$a_6 = \binom{n+6}{6} - c_3\binom{n+4}{4} + \binom{c_3}{2}\binom{n+2}{2} - \binom{c_3}{3} \ldots$$

If we now compare these coefficients with the general coefficients given in section 4.2 we note that some of these terms also appear in the general term, for example $\binom{n+4}{4} - c_3\binom{n+2}{2} + \binom{c_3}{2}$ also appears in the general term a_4 given in Theorem 4.2.5, that is,

$$a_4 = \binom{n+4}{4} - c_3\binom{n+2}{2} - c_4\binom{n+1}{1} + 2k_4\binom{n+1}{1} - c_5$$
$$+ \binom{c_3}{2} + k(2,3) + 2w_5(1) + 3w_5 - 3k_4 - 6k_5 .$$

In section 2.4 we defined the chromatic polynomial of a cascade of K_{m+1}'s as $x_n^{\{m\}} = x(x-1)(x-2)\dots(x-m)^n$. Now a cascade of triangles, K_3's, on p vertices and $(p+n)$-edges has the chromatic polynomial

$$\begin{aligned} x(x-1)(x-2)^{p-2} &= x_{p-2}^{\{2\}} \\ &= \sum_{i=0}^{p-2}(-1)^i a_i T_{p-i} \\ &= \sum_{i=0}^{p-2}(-1)^i a_i x_{p-1-i}^{\{1\}} \\ &= \sum_{i=0}^{p-2}(-1)^i a_i x(x-1)^{p-1-i} \end{aligned}$$

where

$$a_k = \binom{p-2}{k} = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor}(-1)^i\binom{k_3}{i}\binom{p-3+k-2i}{k-2i}.$$

If we now apply Lemma 2.3.3 by adding a new vertex and joining it to every vertex of a cascade of triangles on p vertices we obtain the chromatic polynomial of a cascade of K_4's $= x(x-1)(x-2)(x-3)^{p-2} = x^{\{3\}}_{p-2} = \sum_{i=0}^{p-2}(-1)^i a_i x^{\{2\}}_{p-1-i}$ which is on $(p+1)$-vertices. Hence a cascade of K_4's on p vertices has chromatic polynomial $\sum_{i=0}^{p-3}(-1)^i a_i x^{\{2\}}_{p-2-i}$ where $a_k = \binom{p-3}{k} = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor}(-1)^i \binom{k_4}{i}\binom{p-4+k-2i}{k-2i}$.

Repeated application of Lemma 2.3.3 gives:

Theorem 4.7.1. A cascade of K_m's on p vertices has chromatic polynomial $\sum_{i=0}^{p-m+1}(-1)^i a_i x^{\{m-2\}}_{p-m+2-i}$ where $a_k = \binom{p-m+1}{k} = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor}(-1)^i \binom{k_m}{i}\binom{p-m+k-2i}{k-2i}$.

We now generalize the combinatorial identity given in Lemma 4.7.1. Let m be a positive integer where $m \geq 3$. The graphs called *m-gon-trees* are defined by recursion. The smallest m-gon-tree is the m-cycle, C_m, which is the only two-connected graph containing m vertices and m edges. An m-gon-tree with $k+1$ m-gons (C_m's) is obtained from an m-gon-tree with k m-gons by adding a new m-gon which has one edge in common with any m-gon of an m-gon-tree with k m-gons. A 3-gon-tree is the same as a cascade of triangles.

We now consider the chromatic polynomial of an m-gon-tree G with k m-gons which appears in [22] in normal form as $P(G,x) = x(x-1)\{Q(C_m,x)\}^k$ where $Q(C_m,x) = P(C_m,x) \div x(x-1)$ with $P(C_m,x) = (x-1)^m + (-1)^m(x-1)$.

If G is an m-gon-tree with k m-gons then

$$P(G,x) = \{P(C_m,x) \ominus T_2\}^k \odot T_2$$

$$= \{P(C_m,x)\}^k \ominus T_{2k-2}$$

where $\{P(C_m,x)\}^k = P(C_m,x) \odot P(C_m,x) \odot \ldots \odot P(C_m,x)$, a product with k factors.

Now since $P(C_m, x) = \sum_{i=0}^{n-2}(-1)^i T_{m-i}$ we can write

$$
\begin{aligned}
P(G, x) &= \left\{ \sum_{i=0}^{m-2} (-1)^i T_{m-i} \right\}^k \oplus T_{2k-2} \\
&= \sum \binom{k}{k_1 k_2 \ldots k_{m-1}} \{T_m\}^{k_1} \{-T_{m-1}\}^{k_2} \{T_{m-2}\}^{k_3} \\
&\quad \{-T_{m-3}\}^{k_4} \ldots \{(-1)^{m-2} T_2\}^{k_{m-1}} \oplus T_{2k-2}
\end{aligned}
$$

where the sum is taken over all non-negative integers $k_1, k_2, \ldots, k_{m-1}$ with $k_1 + k_2 + \cdots + k_{m-1} = k$, and

$$
\binom{k}{k_1 k_2 \ldots k_{m-1}} = \frac{k!}{k_1! k_2! \ldots k_{m-1}!}
$$

by the Multinomial Theorem. For $m = 3$ we have

$$
\begin{aligned}
P(C_3, x) &= \left\{ \sum_{i=0}^{1} (-1)^i T_{3-i} \right\}^k \oplus T_{2k-2} \\
&= \sum \binom{k}{k_1 k_2} \{T_3\}^{k_1} \{-T_2\}^{k_2} \oplus T_{2k-2}
\end{aligned}
$$

where

$$
\begin{aligned}
\binom{k}{k_1 k_2} &= \frac{k!}{k_1! k_2!} \quad \text{with} \quad k = k_1 + k_2 \\
&= \binom{k}{k_1} \\
&= \binom{k}{k_2}
\end{aligned}
$$

which are the binomial coefficients. As an illustration consider the following example:

Example 4.7.1. Let $k = 3$ and $m = 4$ then

$$P(G,x) = \left\{\sum_{i=0}^{2}(-1)^i T_{4-i}\right\}^3 \ominus T_4$$

$$= \sum \binom{3}{k_1 k_2 k_3} \{T_4\}^{k_1}\{-T_3\}^{k_2}\{T_2\}^{k_3} \ominus T_4 .$$

Hence the coefficient of:

$$T_8 = \{T_4\}^3\{-T_3\}^0\{T_2\}^0 \ominus T_4 \quad \text{is} \quad \frac{3!}{3!0!0!} = 1$$

$$T_7 = \{T_4\}^2\{-T_3\}^1\{T_2\}^0 \ominus T_4 \quad \text{is} \quad \frac{3!}{2!1!0!} = 3$$

$$T_6 = \{T_4\}^2\{-T_3\}^0\{T_2\}^1 \ominus T_4 \quad \text{is} \quad \frac{3!}{2!0!1!} = 3 \quad \text{and}$$

$$= \{T_4\}^1\{-T_3\}^2\{T_2\}^0 \ominus T_4 \quad \text{is} \quad \frac{3!}{1!2!0!} = 3$$

$$T_5 = \{T_4\}^1\{-T_3\}^1\{T_2\}^1 \ominus T_4 \quad \text{is} \quad \frac{3!}{1!1!1!} = 6 \quad \text{and}$$

$$= \{T_4\}^0\{-T_3\}^3\{T_2\}^0 \ominus T_4 \quad \text{is} \quad \frac{3!}{0!3!0!} = 1$$

$$T_4 = \{T_4\}^0\{-T_3\}^2\{T_2\}^1 \ominus T_4 \quad \text{is} \quad \frac{3!}{0!2!1!} = 3 \quad \text{and}$$

$$= \{T_4\}^1\{-T_3\}^0\{T_2\}^2 \ominus T_4 \quad \text{is} \quad \frac{3!}{1!0!2!} = 3$$

$$T_3 = \{T_4\}^0\{-T_3\}^1\{T_2\}^2 \ominus T_4 \quad \text{is} \quad \frac{3!}{0!1!2!} = 3$$

$$T_2 = \{T_4\}^0\{-T_3\}^0\{T_2\}^3 \ominus T_4 \quad \text{is} \quad \frac{3!}{0!0!3!} = 1$$

Thus $P(G,x) = T_8 - 3T_7 + 6T_6 - 7T_5 + 6T_4 - 3T_3 + T_2$.

As can be seen from Example 4.7.1 it is tedious working out the coefficients of an m-gon-tree using the multinomial formula given above. We now generalize

Lemma 4.7.1 to give a useful computational technique to calculate these coefficients. We begin with the following lemma:

Lemma 4.7.2. If a graph H consists of two subgraphs: G of order p with $P(G,x) = \sum_{i=0}^{p-2}(-1)^i a_i T_{p-i}$ and the cycle C_m of order m and they overlap in a K_2 to form H. Then $P(H,x) = \sum_{i=0}^{p+m-4}(-1)^i b_i T_{p+m-2-i}$ with $b_k = a_k + a_{k-1} + \ldots + a_{k-m+2}$.

Proof. By Lemma 2.2.2 we have

$$P(H,x) = [P(G,x) \odot P(C_m,x)] \ominus T_2$$

$$= \left[(\sum_{i=0}^{p-2}(-1)^i a_i T_{p-i}) \odot (\sum_{i=0}^{m-2}(-1)^i T_{m-i})\right] \ominus T_2$$

$$= [(a_0 T_p - a_1 T_{p-1} + \ldots + (-1)^{p-2} a_{p-2} T_2) \odot$$

$$(T_m - T_{m-1} + \ldots + (-1)^{m-2} T_2)] \ominus T_2$$

$$= a_0 T_{p+m-2} - (a_1 + a_0) T_{p+m-3} + (a_2 + a_1 + a_0) T_{p+m-4}$$

$$- \ldots + (-1)^{p+m-4} a_{p-2} T_2$$

$$= \sum_{i=0}^{p+m-4} (-1)^i b_i T_{p+m-2-i}$$

with $b_k = \sum_{i=0}^{m-2} a_{k-i}$. □

Corollary 4.7.1. If H is as in Lemma 4.7.2 with $G \cong C_m$ then $\{b_k\}$ is the sequence of natural numbers $b_k = m-1-|k-m+2|$, that is, $1, 2, 3, \ldots, m-2, m-1, m-2, \ldots, 2, 1$ with a maximum coefficient of $m-1$.

We notice that the coefficients described in this corollary first increase in absolute value, and then decrease; two successive coefficients may be equal (as in Example 1.5.1 where $P(G,x) = T_5 - 3T_4 + 3T_3 - T_2$). A sequence of numbers with this property is called *unimodal*.

The Unimodal Conjecture. Let G be a *connected* graph with $P(G,x) = \sum_{i=0}^{p-2}(-1)^i a_i T_{p-i}$. Then the statement
$a_j > a_{j+1}$ and $a_{j+1} < a_{j+2}$
is false for every $j = 0, 1, \ldots, p-4$.

This conjecture asserts that one never finds a coefficient flanked by larger coefficients. The above conjecture can be replaced by a stronger assertion, see [15], namely:

Logarithmic Concavity. Let G be a *connected* graph with $P(G,x) = \sum_{i=0}^{p-2}(-1)^i a_i T_{p-i}$. Then the statement
$a_j a_{j+2} \leq a_{j+1}^2$ holds for all $j = 0, 1, \ldots, p-4$.

We remark that the above conjecture does not hold in general for the chromatic polynomials of disconnected graphs expressed in tree form; see Example 4.5.1.

Theorem 4.7.2. If G is a $(p, p+n)$-graph which is a m-gon-tree where $P(G,x) = \sum_{i=0}^{p-2}(-1)^i a_i T_{p-i}$ then

$$a_k = \sum_{i=0}^{\lfloor \frac{k}{m-1} \rfloor} (-1)^i \binom{n+1}{i} \binom{n+k-(m-1)i}{k-(m-1)i} \tag{4.1}$$

Proof. We employ induction on n. For $n = 0$ it is easy to check; for $n = 1$ it is harder to see and hence we do it. If $n = 1$, then G consists of two copies of C_m which overlap in a K_2. Hence, by Corollary 4.7.1 we have that $a_k = m - 1 - |k - m + 2|$. Now consider $\sum_{i=0}^{\lfloor \frac{k}{m-1} \rfloor}(-1)^i \binom{2}{i}\binom{1+k-(m-1)i}{k-(m-1)i}$ for $k = 0, 1, \ldots, 2m-4$

First: For $k = 0, 1, \ldots, m-2$, this sum is

$$\begin{aligned}
&\sum_{i=0}^{0}(-1)^i\binom{2}{i}\binom{1+k-(m-1)i}{k-(m-1)i} \\
&= (-1)^0\binom{2}{0}\binom{1+k}{k} \\
&= k+1;
\end{aligned}$$

that is, the terms are $1, 2, \ldots, m-1$.

Next: For $k = m-1, m, \ldots, 2m-4$ this sum is

$$\begin{aligned}
&\sum_{i=0}^{1}(-1)^i\binom{2}{i}\binom{1+k-(m-1)i}{k-(m-1)i} \\
&= (-1)^0\binom{2}{0}\binom{1+k}{k} + (-1)^1\binom{2}{1}\binom{1+k-(m-1)}{k-(m-1)} \\
&= k+1-2\binom{1+k-(m-1)}{1} \\
&= k+1-2(1+k-m+1) \\
&= 2m-k-3;
\end{aligned}$$

that is, the terms are $m-2, m-3, \ldots, 1$.

But these are exactly the values of a_k for $k = 0, 1, \ldots, 2m-4$. Assume the result is true for all m-gon-trees with p vertices and $p+n+1$ edges, that is, with at most $n+1$ cycles on m vertices ($c_m = n+1$). Now let H be an m-gon-tree on p vertices and $p+n+2$ edges, that is, with $n+2$ cycles where $P(H, x) = \sum_{i=0}^{p+m-4}(-1)^i b_i T_{p+m-2-i}$. By Lemma 4.7.2 we have that $b_k = a_k + a_{k-1} + \ldots + a_{k-m+2}$ where $a_{k-j} = \sum_{i=0}^{\lfloor \frac{k-j}{m-1} \rfloor}(-1)^i\binom{n+1}{i}\binom{n+k-j-(m-1)i}{k-j-(m-1)i}$ for $j = 0, 1, \ldots, m-2$.

Hence we need only prove that

$$\sum_{i=0}^{\lfloor \frac{k}{m-1} \rfloor} (-1)^i \binom{n+2}{i} \binom{n+1+k-(m-1)i}{k-(m-1)i}$$

$$= \sum_{i=0}^{\lfloor \frac{k}{m-1} \rfloor} (-1)^i \binom{n+1}{i} \binom{n+k-(m-1)i}{k-(m-1)i}$$

$$+ \sum_{i=0}^{\lfloor \frac{k-1}{m-1} \rfloor} (-1)^i \binom{n+1}{i} \binom{n+k-1-(m-1)i}{k-1-(m-1)i} + \cdots$$

$$+ \sum_{i=0}^{\lfloor \frac{k-(m-2)}{m-1} \rfloor} (-1)^i \binom{n+1}{i} \binom{n+k-(m-2)-(m-1)i}{k-(m-2)-(m-1)i}.$$

$$\text{Now} \quad \sum_{i=0}^{\lfloor \frac{k}{m-1} \rfloor} (-1)^i \binom{n+2}{i} \binom{n+1+k-(m-1)i}{k-(m-1)i}$$

$$= \sum_{i=0}^{\lfloor \frac{k}{m-1} \rfloor} (-1)^i \binom{n+1}{i} \binom{n+1+k-(m-1)i}{k-(m-1)i}$$

$$+ \sum_{i=0}^{\lfloor \frac{k}{m-1} \rfloor} (-1)^i \binom{n+1}{i-1} \binom{n+1+k-(m-1)i}{k-(m-1)i}$$

$$\text{since} \quad \binom{n+2}{i} = \binom{n+1}{i} + \binom{n+1}{i-1}$$

$$= \sum_{i=0}^{\lfloor \frac{k}{m-1} \rfloor} (-1)^i \binom{n+1}{i} \binom{n+k-(m-1)i}{k-(m-1)i}$$

$$+ \sum_{i=0}^{\lfloor \frac{k}{m-1} \rfloor} (-1)^i \binom{n+1}{i} \binom{n+k-(m-1)i}{k-1-(m-1)i}$$

$$+ \sum_{i=0}^{\lfloor \frac{k}{m-1} \rfloor} (-1)^i \binom{n+1}{i-1} \binom{n+1+k-(m-1)i}{k-(m-1)i}$$

$$= \sum_{i=0}^{\lfloor \frac{k}{m-1} \rfloor} (-1)^i \binom{n+1}{i} \binom{n+k-(m-1)i}{k-(m-1)i}$$

$$+ \sum_{i=0}^{\lfloor \frac{k}{m-1} \rfloor} (-1)^i \binom{n+1}{i} \binom{n+k-1-(m-1)i}{k-1-(m-1)i}$$

$$+ \sum_{i=0}^{\lfloor \frac{k}{m-1} \rfloor} (-1)^i \binom{n+1}{i} \binom{n+k-1-(m-1)i}{k-2-(m-1)i}$$

$$+ \sum_{i=0}^{\lfloor \frac{k}{m-1} \rfloor} (-1)^i \binom{n+1}{i-1} \binom{n+1+k-(m-1)i}{k-(m-1)i}$$

$$= \sum_{i=0}^{\lfloor \frac{k}{m-1} \rfloor} (-1)^i \binom{n+1}{i} \binom{n+k-(m-1)i}{k-(m-1)i}$$

$$+ \sum_{i=0}^{\lfloor \frac{k-1}{m-1} \rfloor} (-1)^i \binom{n+1}{i} \binom{n+k-1-(m-1)i}{k-1-(m-1)i}$$

$$+ \sum_{i} (-1)^i \binom{n+1}{i} \binom{n+k-2-(m-1)i}{k-2-(m-1)i}$$

$$+ \sum_{i} (-1)^i \binom{n+1}{i} \binom{n+k-3-(m-1)i}{k-3-(m-1)i}$$

$$+ \sum_{i=0}^{\lfloor \frac{k}{m-1} \rfloor} (-1)^i \binom{n+1}{i-1} \binom{n+1+k-(m-1)i}{k-(m-1)i}$$

$$= \sum_{i=0}^{\lfloor \frac{k}{m-1} \rfloor} (-1)^i \binom{n+1}{i} \binom{n+k-(m-1)i}{k-(m-1)i}$$

$$+ \sum_{i=0}^{\lfloor \frac{k-1}{m-1} \rfloor} (-1)^i \binom{n+1}{i} \binom{n+k-1-(m-1)i}{k-1-(m-1)i}$$

$$+\cdots+\sum_i (-1)^i \binom{n+1}{i}\binom{n+k-(m-2)-(m-1)i}{k-(m-2)-(m-1)i}$$

$$+\sum_i (-1)^i \binom{n+1}{i}\binom{n+k-(m-2)-(m-1)i}{k-1-(m-2)-(m-1)i}$$

$$+\sum_{i=0}^{\lfloor \frac{k}{m-1} \rfloor} (-1)^i \binom{n+1}{i-1}\binom{n+1+k-(m-1)i}{k-(m-1)i}$$

The terms of the last two summations cancel. Hence

$$\sum_{i=0}^{\lfloor \frac{k}{m-1} \rfloor} (-1)^i \binom{n+2}{i}\binom{n+1+k-(m-1)i}{k-(m-1)i}$$

$$= \sum_{i=0}^{\lfloor \frac{k}{m-1} \rfloor} (-1)^i \binom{n+1}{i}\binom{n+k-(m-1)i}{k-(m-1)i}$$

$$+ \sum_{i=0}^{\lfloor \frac{k-1}{m-1} \rfloor} (-1)^i \binom{n+1}{i}\binom{n+k-1-(m-1)i}{k-1-(m-1)i} + \dots$$

$$+ \sum_{i=0}^{\lfloor \frac{k-(m-2)}{m-1} \rfloor} (-1)^i \binom{n+1}{i}\binom{n+k-(m-2)-(m-1)i}{k-(m-2)-(m-1)i}. \qquad \square$$

As an illustration, let $m = 4$ and remember that $c_4 = n+1$. Then $a_k = \sum_{i=0}^{\lfloor \frac{k}{3} \rfloor} (-1)^i \binom{c_4}{i}\binom{n+k-3i}{k-3i}$ so that

$$a_0 = \binom{n+0}{0} = 1$$

$$a_1 = \binom{n+1}{1}$$

$$a_2 = \binom{n+2}{2}$$

$$a_3 = \binom{n+3}{3} - c_4$$

$$a_4 = \binom{n+4}{4} - c_4\binom{n+1}{1}$$

$$a_5 = \binom{n+5}{5} - c_4\binom{n+2}{2}$$

$$a_6 = \binom{n+6}{6} - c_4\binom{n+3}{3} + \binom{c_4}{2} \dots$$

Concerning Example 4.7.1 with $k = 3 = c_4$ and $n + 1 = 3$, we have

$$a_0 = \binom{n}{0} = \binom{2}{0} = 1$$

$$a_1 = \binom{2+1}{1} = 3$$

$$a_2 = \binom{2+2}{2} = 6$$

$$a_3 = \binom{2+3}{3} - 3 = 7$$

$$a_4 = \binom{2+4}{4} - 3\binom{2+1}{1} = 15 - 9 = 6$$

$$a_5 = \binom{2+5}{5} - 3\binom{2+2}{2} = 21 - 18 = 3$$

$$a_6 = \binom{2+6}{6} - 3\binom{2+3}{3} + \binom{3}{2} = 28 - 30 + 3 = 1$$

Hence $P(G, x) = T_8 - 3T_7 + 6T_6 - 7T_5 + 6T_4 - 3T_3 + T_2$.

The above coefficients can be calculated without recourse to the formula (4.1). For

$m = 3$ we obtain the recursion formula $b_k = a_k + a_{k-1}$, which is similar to the recursion formula $f(n+1,k) = f(n,k) + f(n,k-1)$ described in Theorem 4.5.1, and use **Pascal's triangle** to obtain the coefficients of a 3-gon-tree or a cascade of triangles. For $m = 4$ we obtain the recursion formula $b_k = a_k + a_{k-1} + a_{k-2}$ and use it to obtain the following array of numbers, starting with the coefficient of T_4 and the absolute value of the coefficients of $P(C_4, x) = T_4 - T_3 + T_2$ in the second row.

1
1 1 1
1 2 3 2 1
1 3 6 7 6 3 1
1 4 10 16 19 16 10 4 1
1 5 15 30 45 51 45 30 15 5 1

To illustrate how each entry is obtained in the above array we begin with the second row with $a_0 = 1$, $a_1 = 1$ and $a_2 = 1$. Using the recursion formula $b_k = a_k + a_{k-1} + a_{k-2}$ we obtain $b_0 = a_0 = 1$, $b_1 = a_1 + a_0 = 2$, $b_2 = a_2 + a_1 + a_0 = 3$, $b_3 = a_3 + a_2 + a_1 = a_2 + a_1 = 2$, and $b_4 = a_4 + a_3 + a_2 = a_2 = 1$ which are the entries of the third row. Now to illustrate how the coefficients can be calculated for Example 4.7.1 we begin with the third row with $a_0 = 1$, $a_1 = 2$, $a_2 = 3$, $a_3 = 2$ and $a_4 = 1$. Again using the recursion formula $b_k = a_k + a_{k-1} + a_{k-2}$ we obtain $b_0 = a_0 = 1$, $b_1 = a_1 + a_0 = 3$, $b_2 = a_2 + a_1 + a_0 = 6$, $b_3 = a_3 + a_2 + a_1 = 7$, $b_4 = a_4 + a_3 + a_2 = 6$, $b_5 = a_5 + a_4 + a_3 = a_4 + a_3 = 3$ and $b_6 = a_6 + a_5 + a_4 = a_4 = 1$, which are the entries of the fourth row. The above can be repeated for any m-gon-tree.

CHAPTER 5

CHROMATICALLY EQUIVALENT GRAPHS

5.1 Introduction

Two graphs G and H are said to be *chromatically equivalent* if they have the same chromatic polynomial, that is, $P(G,x) = P(H,x)$. A graph G is said to be *chromatically unique* if $P(G,x) = P(H,x)$ implies that G is isomorphic to H. In section 5.2 we dicuss families of graphs that are chromatically equivalent; in section 5.3 we discuss chordal graphs; in section 5.4 we discuss chromatically unique graphs and in section 5.5 we try to address the question: Which of the wheels $W_p, p \geq 4$, are chromatically unique?

5.2 Chromatic Equivalence

It is very clear that a chromatic polynomial does not generally belong exclusively to one graph. There are many pairs of graphs which are chromatically equivalent and nonisomorphic. For example two nonisomorphic trees of order $p \geq 4$ have the same chromatic polynomial – namely, $T_p = x(x-1)^{p-1}$; n-gon-trees with $n \geq 3$ are chromatically equivalent but need not be isomorphic if they have more than two n-gon cycles; cascades of K_m with $m \geq 3$ are chromatically equivalent but need not be isomorphic if they have more than two K_m's.

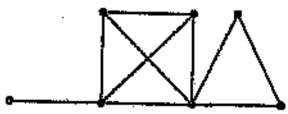
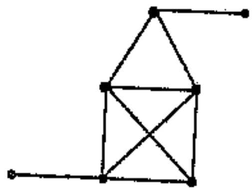
It seems natural to ask whether there is some way of proving that two graphs are chromatically equivalent without actually calculating the chromatic polynomial of either of them. For some classes of graphs the answers are at hand.

Suppose G is a connected graph with blocks $H_1, H_2, \ldots, H_p$ and G has two nontrivial blocks H_i and H_j which share a cut vertex v. Consider the graph F obtained from

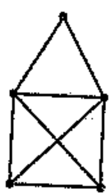
G by splitting v into two and pasting an edge of H_i to an edge of H_j and growing a new edge onto this graph which forms a new end-vertex. Then by Theorem 3.2.2 $P(G,x) = [P(H_1,x) \odot P(H_2,x) \odot \cdots \odot P(H_i,x) \odot P(H_j,x) \odot \cdots \odot P(H_p,x)] \oplus T_{p-1}$. Now in F we have that H_i and H_j overlap in K_2. Thus by Lemma 2.2.2 and Theorem 3.2.2 we have $P(F,x) = [P(H_1,x) \odot P(H_2,x) \odot \ldots \odot ((P(H_i,x) \odot P(H_j,x)) \oplus T_2) \odot T_2 \odot \ldots \odot P(H_p,x)] \oplus T_{p-1} = P(G,x)$ while $G \ncong F$ because F has one more end-vertex than G. Hence we have:

Lemma 5.2.1. If G is a connected graph with at least two nontrivial blocks then G is not chromatically unique. □

Example 5.2.1.

If G:  then F: 

By Example 3.2.2 $P(G,x) = T_7 - 4T_6 + 5T_5 - 2T_4$

Let H: 

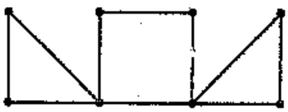
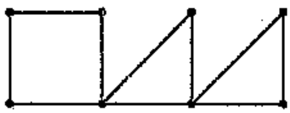
then by Example 2.2.1, $P(H,x) = T_5 - 4T_4 + 5T_3 - 2T_2$. Hence by Theorem 3.2.1 $P(F,x) = T_7 - 4T_6 + 5T_5 - 2T_4$, so that $P(G,x) = P(F,x)$ but $G \ncong F$.

A *cactus* is a connected graph in which any two cycles are edge disjoint. We shall show that all cacti with a given number of vertices, edges and cycles of each length $l = 3, 4, 5, \ldots$ are chromatically equivalent.

Lemma 5.2.2. All cacti with a fixed number of vertices, edges and cycles of each length $l = 3, 4, 5, \ldots$ are chromatically equivalent.

Proof. Apply Theorem 3.2.2 to the cacti and their blocks. □

Example 5.2.2.

If G:  and H: 

then by Theorem 3.2.2 we have

$$\begin{aligned} P(G,x) &= [P(C_3,x) \odot P(C_4,x) \odot P(C_3,x)] \oplus T_{3-1} \\ &= [(T_3 - T_2) \odot (T_4 - T_3 + T_2) \odot (T_3 - T_2)] \oplus T_2 \\ &= T_8 - 3T_7 + 4T_6 - 3T_5 + T_4 \\ &= P(H,x) \end{aligned}$$

We now consider the case of disconnected graphs.

Theorem 5.2.1. If a graph G has nontrivial 2-connected components $G_1, G_2, \ldots, G_{k+1}$ and m trivial components, then G is chromatically equivalent to each graph $H \cup F$ where H is constructed from these nontrivial components of G by overlapping a nontrivial G_i with a nontrivial $G_j, i \neq j$, in K_2 in any fashion until no more nontrivial components remain and F is any forest with $k + m$ components and

$2k+m$ vertices.

Proof. $P(H \cup F, x) = P(H,x) \odot P(F,x) + [P(H,x) \odot P(F,x)] \ominus T_1$ by Lemma 3.3.2. But $P(H,x) = [P(G_1,x) \odot P(G_2,x) \odot \cdots \odot P(G_{k+1},x)] \ominus T_{2k}$ by repeated application of Theorem 2.2.1 and Lemma 2.2.2, and $P(F,x) = \sum_{j=0}^{k+m-1} \binom{k+m-1}{j} T_{2k+m-j}$ by Corollary 3.3.2. Therefore

$$P(H \cup F, x) = \left[\left\{P(G_1,x) \odot P(G_2,x) \odot \ldots \odot P(G_{k+1},x)\right\} \ominus T_{2k}\right] \odot$$

$$\sum_{j=0}^{k+m-1} \binom{k+m-1}{j} T_{2k+m-j} + \left(\left[\{P(G_1,x) \odot P(G_2,x) \odot \ldots\right.\right.$$

$$\left.\left.\odot P(G_{k+1},x)\} \ominus T_{2k}\right] \odot \sum_{j=0}^{k+m-1} \binom{k+m-1}{j} T_{2k+m-j}\right) \odot T_1$$

$$= \sum_{j=0}^{k+m-1} \binom{k+m-1}{j} \left[P(G_1,x) \odot P(G_2,x) \odot \ldots\right.$$

$$\left.\odot P(G_{k+1},x)\right] \odot T_{m-j} + \left[\sum_{j=0}^{k+m-1} \binom{k+m-1}{j} \left\{P(G_1,x)\right.\right.$$

$$\left.\left.\odot P(G_2,x) \odot \ldots \odot P(G_{k+1},x)\right\} \odot T_{m-j}\right] \ominus T_1$$

$$= \sum_{j=0}^{k+m-1} \binom{k+m-1}{j} \left\{P(G_1,x) \odot P(G_2,x) \odot \ldots \odot\right.$$

$$\left.P(G_{k+1},x) \odot T_m\right] \ominus T_j + \sum_{j=0}^{k+m-1} \binom{k+m-1}{j}$$

$$\left[P(G_1,x) \odot P(G_2,x) \odot \ldots \odot P(G_{k+1},x) \odot T_m\right] \ominus T_{j+1}$$

$$= \sum_{j=0}^{k+m} \binom{k+m}{j} \Big[P(G_1, x) \odot P(G_2, x) \odot \ldots \odot P(G_{k+1}, x) \odot T_m \Big] \oplus T_j$$

since $\binom{k+m-1}{i} + \binom{k+m-1}{i-1} = \binom{k+m}{i}$

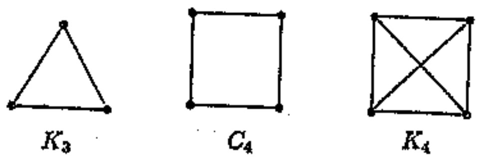
$$= P(G, x)$$

by Theorem 3.3.1. □

We remark that the condition that the G_i's in Theorem 5.2.1 are 2-connected is somewhat artificial. The same result can be proven without this condition but with a longer proof.

Example 5.2.3.

If G:



then

by Theorem 3.3.1, with $k = 5$,
we have:

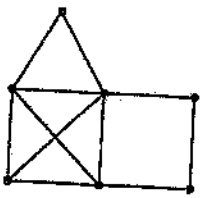
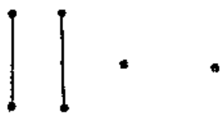
$$P(G, x) = \sum_{j=0}^{5-1} \binom{5-1}{j} \Big[P(K_3, x) \odot P(C_4, x) \odot P(K_4, x) \odot T_2 \Big] \oplus T_j$$

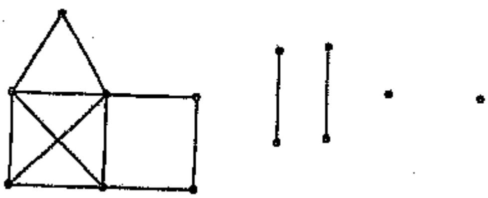
$$= \sum_{j=0}^{4} \binom{4}{j} \Big[(T_3 - T_2) \odot (T_4 - T_3 + T_2) \odot (T_4 - 3T_3 + 2T_2) \odot T_2 \Big] \oplus T_j$$

$$= \sum_{j=0}^{4} \binom{4}{j} \Big[T_{13} - 5T_{12} + 10T_{11} - 11T_{10} + 7T_9 - 2T_8 \Big] \oplus T_j$$

$$
\begin{aligned}
&= (T_{13} - 5T_{12} + 10T_{11} - 11T_{10} + 7T_9 - 2T_8) \\
&+ \binom{4}{1}(T_{12} - 5T_{11} + 10T_{10} - 11T_9 + 7T_8 - 2T_7) \\
&+ \binom{4}{2}(T_{11} - 5T_{10} + 10T_9 - 11T_8 + 7T_7 - 2T_6) \\
&+ \binom{4}{3}(T_{10} - 5T_9 + 10T_8 - 11T_7 + 7T_6 - 2T_5) \\
&+ \binom{4}{4}(T_9 - 5T_8 + 10T_7 - 11T_6 + 7T_5 - 2T_4) \\
&= T_{13} - T_{12} - 4T_{11} + 3T_{10} + 4T_9 - 5T_8 + 5T_6 - T_5 - 2T_4
\end{aligned}
$$

Now consider

H:  with F:  then

$H \cup F$:  with $k = 4$

where

$$P(H,x) = \left[(T_3 - T_2) \odot (T_4 - T_3 + T_2) \odot (T_4 - 3T_3 + 2T_2)\right] \ominus T_4$$

$$= [T_{11} - 5T_{10} + 10T_9 - 11T_8 + 7T_7 - 2T_6] \ominus T_4$$

$$= T_7 - 5T_6 + 10T_5 - 11T_4 + 7T_3 - 2T_2$$

and

$$P(F,x) = \sum_{j=0}^{4-1} \binom{4-1}{j} T_{6-j}$$

$$= \sum_{j=0}^{3} \binom{3}{j} T_{6-j}$$

$$= T_6 + 3T_5 + 3T_4 + T_3$$

Therefore

$$P(H \cup F, x) = P(H,x) \odot P(F,x) + \left[P(H,x) \odot P(F,x)\right] \ominus T_1$$

$$= (T_7 - 5T6 + 10T_5 - 11T_4 + 7T_3 - 2T_2) \odot (T_6 + 3T_5 + 3T_4 + T_3)$$

$$+ \left[(T_7 - 5T_6 + 10T_5 - 11T_4 + 7T_3 - 2T_2) \odot (T_6 + 3T_5 + 3T_4 + T_3)\right] \ominus T_1$$

$$= T_{13} - T_{12} - 4T_{11} + 3T_{10} + 4T_9 - 5T_8 + 5T_6 - T_5 - 2T_4$$

5.3 Chordal Graphs

If G is a graph with $\chi(G) = k+1$, then $P(G,x) = 0$ for $x = 0, 1, \ldots, k$. Hence the chromatic polynomial $P(G,x)$ can be written as

$$P(G,x) = x^{m_0}(x-1)^{m_1}(x-2)^{m_2}\ldots(x-k)^{m_k}\left[x^n + a_1x^{n-1} + \cdots + a_n\right]$$

$$= x^{m_0}(x-1)^{m_1}(x-2)^{m_2}\ldots(x-k)^{m_k} f(x)$$

where $m_0, m_1, \ldots, m_k, n$ are positive integers and $a_1, a_2, \ldots, a_n$ are integers with $m_0 + m_1 + \ldots + m_k + n = p$, the number of vertices of G and
$m_1 + 2m_2 + 3m_3 + \ldots + km_k + a_1 = q$, the number of edges of G.

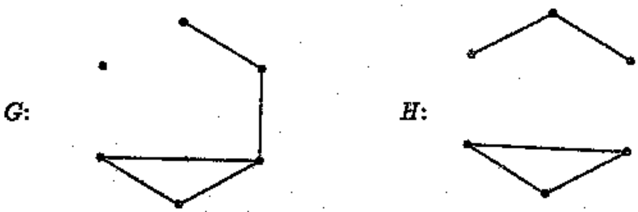
We now look at a class of graphs for which $f(x) = 1$, that is, $P(G, x)$ is completely factorable over the nonnegative integers.

A graph G is called *chordal* (or triangulated) if every cycle of G with length greater than 3 has a chord, that is, an edge joining two non-consecutive vertices of the cycle. It can be shown, see [20], that a graph is a chordal graph if and only if it can be built up step by step, starting with a single vertex, each new vertex being joined to every vertex of a complete subgraph of the existing graph; if the complete subgraph has n vertices, then $x - n$ colours are available to colour the new vertex. Hence each step of the construction contributes a factor $x - n$ to the chromatic polynomial, which must therefore be of the form:

$$P(G, x) = x^{m_0}(x-1)^{m_1}(x-2)^{m_2}\ldots(x-k)^{m_k},$$

for some integer k, where $m_0, m_1, \ldots, m_k$ are positive integers. Clearly, $k+1$ is the size of the largest clique in G and the chromatic number of G. Conversely, given a set of positive integers $m_0, m_1, \ldots, m_k$ we can usually construct several graphs having $P(G, x)$ as the chromatic polynomial.

Example 5.3.1.



G and H are two of the graphs that have chromatic polynomial $x^2(x-1)^3(x-2)$.

It is an open problem to characterize those graphs G whose chromatic polynomials $P(G,x)$ have the form

$$P(G,x) = x^{m_0}(x-1)^{m_1}(x-2)^{m_2}\dots(x-k)^{m_k}\,.$$

We now make some remarks on this problem:

After expanding the right hand side of $P(G,x)$, the highest exponent of x is $m_0+m_1+\cdots+m_k$ which is, by Lemma 1.5.1, the number of vertices p of G; that is $m_0+m_1+\cdots+m_k=p$. Also

$$\begin{aligned}
P(G,x) &= x^{m_0}(x-1)^{m_1}(x-2)^{m_2}\dots(x-k)^{m_k}\\
&= x^{m_0}\Bigg(\left[x^{m_1}-\binom{m_1}{1}x^{m_1-1}+\binom{m_1}{2}x^{m_1-2}(-1)^2+\cdots\right]\\
&\qquad\left[x^{m_2}-2\binom{m_2}{1}x^{m_2-1}+4\binom{m_2}{2}x^{m_2-2}+\dots\right]\\
&\qquad\left[x^{m_3}-3\binom{m_3}{1}x^{m_3-1}+9\binom{m_3}{2}x^{m_3-2}+\dots\right]\\
&\qquad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\qquad\left[x^{m_k}-k\binom{m_k}{1}x^{m_k-1}+k^2\binom{m_k}{2}x^{m_k-2}+\dots\right]\Bigg)
\end{aligned}$$

Therefore the coefficient of $x^{m_0+m_1+\dots+m_k-1}$ is $-(m_1+2m_2+3m_3+\ldots+km_k)$ which by equation (4.1) in section 4.3 is the number of edges q of G; that is

$$m_1+2m_2+3m_3+\ldots+km_k=q\,.$$

It is tempting to think, in fact it was conjectured, see [5], that $P(G,x)=x^{m_0}(x-1)^{m_1}(x-2)^{m_2}\dots(x-k)^{m_k}$ if and only if G is a chordal graph, but this is not so. We will now show how some non-chordal graphs whose chromatic polynomials have the same form as $P(G,x)$ described above can be constructed.

Lemma 5.3.1. Let H be an elementary subdivision of K_p. Then
$P(H,x) = x(x-1)(x-2)\dots(x-p+2)(x^2-px+2p-3)$.

Proof. Let v be the "new" vertex (of degree two) of H. By Whitney's Reduction Formula applied to any edge adjacent to v we have

$$P(H,x) = P(H_1,x) - P(H_2,x)$$

where $P(H_1,x) = \dfrac{P(K_{p-1},x).P(K_{p-1},x).(x-1)}{P(K_{p-2},x)}$ by Theorem 2.2.1

and $P(H_2,x) = P(K_p,x)$.

Therefore

$$P(H,x) = \frac{x(x-1)(x-2)\dots(x-p+2).x(x-1)(x-2)\dots(x-p+2).(x-1)}{x(x-1)(x-2)\dots(x-p+3)}$$

$$- x(x-1)(x-2)\dots(x-p+1)$$

$$= x(x-1)^2(x-2)\dots(x-p+2)^2 - x(x-1)(x-2)\dots(x-p+1)$$

$$= x(x-1)(x-2)\dots(x-p+2)[(x-1)(x-p+2)-(x-p+1)]$$

$$= x(x-1)(x-2)\dots(x-p+2)(x^2-px+2p-3).$$ □

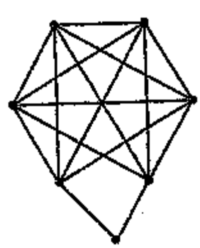
Example 5.3.2. For $p=6$ in Lemma 5.3.1 we have

$$P(H,x) = x(x-1)(x-2)(x-3)(x-4)(x^2-6x+9)$$

$$= x(x-1)(x-2)(x-3)^3(x-4)$$

Hence

H:



which is not chordal.

This example is given in [5]. We now generalize Lemma 5.3.1 to:

Theorem 5.3.1. Let G be a connected graph on p vertices with two complete subgraphs H_1, on $p-1$ vertices, and H_2, on $p-n$ vertices, which overlap in a complete graph on $p-n-m$ vertices, $p > n+m$. If H is a graph obtained from G by adding a new vertex v to G and joining v to two nonadjacent vertices of H_1 and H_2, then

$$P(H,x) = x(x-1)(x-2)\ldots(x-p+2)\big(x-(p-n-1)\big)\big(x-(p-n-2)\big)$$

$$\ldots\big(x-(p-n-m+1)\big)\Big[x^2-(p-n-m+2)x$$

$$+2(p-n-m+2)-3\Big].$$

Proof. By Whitney's Reduction Formula applied to any edge adjacent to v we have

$P(H,x) = P(G_1,x) - P(G_2,x)$ where

$P(G_1,x) = (x-1)P(C,x)$ and $P(G_2,x) = P(G+e,x)$

where e is the edge between the two nonadjacent vertices of H_1 and H_2. By Whitney's Reduction Formula applied to the edge e we have

$$P(H,x) = (x-1)P(G,x) - \left[P(G,x) - \frac{P(K_{p-1},x)P(K_{p-n},x)}{P(K_{p-n-m+1},x)}\right]$$

$$= \frac{(x-1)P(K_{p-1},x)P(K_{p-n},x)}{p(K_{p-n-m},x)} - \left[\frac{P(K_{p-1},x)P(K_{p-n},x)}{P(K_{p-n-m},x)}\right.$$

$$\left. - \frac{P(K_{p-1},x)P(K_{p-n},x)}{P(K_{p-n-m+1},x)}\right]$$

$$= \frac{(x-1)P(K_{p-1},x)P(K_{p-n},x)}{P(K_{p-n-m},x)}$$

$$- \frac{P(K_{p-1},x)P(K_{p-n},x)[x-(p-n-m+1)]}{P(K_{p-n-m+1},x)}$$

$$= x(x-1)^2(x-2)\dots(x-p+2)\big(x-(p-n-1)\big)\big(x-(p-n-2)\big)$$

$$\dots\big(x-(p-n-m)\big) - x(x-1)(x-2)$$

$$\dots(x-p+2)\big(x-(p-n-1)\big)\big(x-(p-n-2)\big)\dots\big(x-(p-n-m+1)\big)^2$$

$$= x(x-1)(x-2)\dots(x-p+2)\big(x-(p-n-1)\big)\big(x-(p-n-2)\big)$$

$$\dots\big(x-(p-n-m+1)\big)\{(x-1)(x-p+n+m)-(x-p+n+m-1)\}$$

$$= x(x-1)(x-2)\dots(x-p+2)\big(x-(p-n-1)\big)\big(x-(p-n-2)\big)\dots$$

$$\big(x-(p-n-m+1)\big)\{x^2-(p-n-m+2)x+2(p-n-m+2)-3\}. \qquad \square$$

Example 5.3.3.

For $p = 7$, $n = 2$ and $m = 1$ in Theorem 5.3.1 we have

$$P(H,x) = x(x-1)(x-2)\dots\big(x-(7-2-1+1)\big)\{x^2-(7-2-1+2)x$$

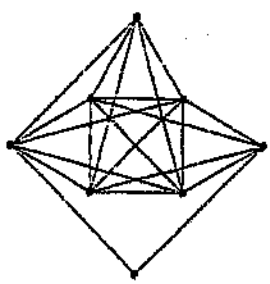
$$+ 2(7-2-1+2)-3\}$$

$$= x(x-1)(x-2)(x-3)(x-4)(x-5)\{x^2-6x+9\}$$

$$= x(x-1)(x-2)(x-3)^3(x-4)(x-5)$$

Hence

H :



which is not chordal.

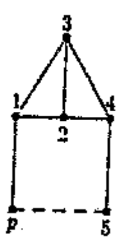
5.4 Chromatically Unique Graphs

We now consider graphs which have a chromatic polynomial all to themselves, that is if $P(G,x) = P(H,x)$ then $G \cong H$, such a graph is said to be chromatically unique. For example the empty graphs, the empty graphs with one edge, the complete graphs and the complete graphs without one edge are chromatically unique. In [7] it is shown that the cycle C_n on n vertices is chromatically unique. Later in [12] it is shown that all theta graphs are chromatically unique. Also in [17] it is shown that the bipartite graphs $K(m,n)$ are chromatically unique; and in [24] it is shown that the wheels, W_p on p vertices, are chromatically unique for odd p. The latest result in [8] is that the broken wheel, $W_p(1)$ on p vertices, is chromatically unique if p is even.

We now describe two families of graphs which are chromatically unique. For a proof of the uniqueness, see [7]. The graphs in these families can be obtained from the wheels by deleting some consecutive spoke edges, that is they are broken wheels (see section 4.2).

The first family consists of the graph $W_p(p-4)$, for $p \geq 4$, where the first graph in this family $W_4(0)$ is the wheel W_4 (or K_4).

$W_4(0)$:  $W_5(1)$:  $W_6(2)$: 

$W_p(p-4)$: 

By Whitney's Reduction Formula applied to the edge 13 we have

$$P(W_p(p-4),x) = (x-2)P(C_{p-1},x) - (x-2)P(C_{p-2},x)$$
$$= (x-2)\left[(x-1)^{p-1} - (x-1)^{p-2} + (-1)^{p-1}2(x-1)\right]$$

using $P(C_p,x) = (x-1)^p + (-1)^p(x-1)$ see section 1.5.

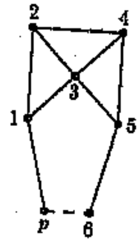
Also using $P(C_p,x) = \sum_{i=0}^{p-2}(-1)^i T_{p-i}$ we have

$$P(W_p(p-4),x) = (x-2)P(C_{p-1},x) - (x-2)P(C_{p-2},x)$$
$$= (x-2)\left[\sum_{i=0}^{p-3}(-1)^i T_{p-1-i} - \sum_{i=0}^{p-4}(-1)^i T_{p-2-i}\right]$$
$$= (x-2)\left[T_{p-1} - 2T_{p-2} + \ldots + (-1)^{p-3}2T_2\right] \qquad (5.1)$$
$$= (x-2)\left[x(x-1)^{p-2} - 2x(x-1)^{p-3} + \ldots + (-1)^{p-3}2x(x-1)\right]$$
$$= x(x-1)(x-2)\left[(x-1)^{p-3} - 2(x-1)^{p-4} + \ldots + (-1)^{p-3}2\right]$$

The second family consists of the graphs $W_p(p-5)$, for $p \geq 5$, where the first graph in this family $W_5(0)$ is the wheel W_5.

$W_5(0)$:  $W_6(1)$:  $W_7(2)$: 

$W_p(p-5)$:



By Whitney's Reduction Formula applied to the edge 24 we have

$$P\big(W_p(p-5),x\big) = (x-2)^2 P(C_{p-2},x) - P\big(W_{p-1}(p-5),x\big)$$
$$= (x-2)^2\big[(x-1)^{p-2} + (-1)^{p-2}(x-1)\big] - P\big(W_{p-1}(p-5),x\big)$$

Also

$$P\big(W_p(p-5),x\big) = (x-2)^2 \sum_{i=0}^{p-4}(-1)^i T_{p-2-i} - P\big(W_{p-1}(p-5),x\big)$$
$$= (x-2)^2\big[x(x-1)^{p-3} - x(x-1)^{p-4} + \cdots + (-1)^{p-4}x(x-1)\big]$$
$$- P\big(W_{p-1}(p-5),x\big)$$
$$= x(x-1)(x-2)\Big[\{(x-1)-1\}\{(x-1)^{p-4} - (x-1)^{p-5}$$
$$+ \cdots + (-1)^{p-4}\}\Big] - P\big(W_{p-1}(p-5),x\big)$$
$$= x(x-1)(x-2)\Big[(x-1)^{p-3} - 2(x-1)^{p-4} + \cdots$$
$$+ (-1)^{p-4}2(x-1) + (-1)^{p-3}\Big]$$
$$- x(x-1)(x-2)\Big[(x-1)^{p-4}$$
$$- 2(x-1)^{p-5} + \cdots + (-1)^{p-4}2\Big] \quad \text{by (5.1)}$$

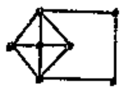
Hence

$$P\big(W_p(p-5),x\big)=x(x-1)(x-2)\big[x-1)^{p-3}-3(x-1)^{p-4}+4(x-1)^{p-5}-\ldots+(-1)^{p-3}3\big].$$

For $p \geq 5$ each $W_p(p-4)$ contains exactly two triangles and for $p \geq 6$ each $W_p(p-5)$ contains exactly three triangles. It is natural to consider the chromatic uniqueness of the corresponding family that contains exactly four triangles for $p \geq 7$. The following example shows that $W_p(p-6)$, for $p \geq 7$, is not chromatically unique.

Example 5.4.1.

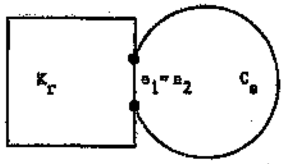
Consider

$W_7(1)$:  and G: 

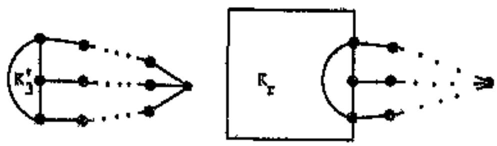
By applying Whitney's Reduction Formula, we find that
$P\big(W_7(1),x\big) = x(x-1)(x-2)\big[(x-2)^4 + (x-2)^2 - (x-2) + 1\big] = P(G,x)$.
Clearly $W_7(1) \not\cong G$.

We also describe two other classes of chromatically unique graphs. For a proof of the uniqueness, see [11].

Let r and s be integers such that $r, s \geq 3$. Let e_1 be an edge of a complete graph, K_r on r vertices, and e_2 be an edge of a cycle, C_s on s vertices. Denote by $G(r,s)$ the graph obtained by identifying e_1 and e_2 in the disjoint union of K_r and C_s. The graph $G(r,s)$ is chromatically unique for all $r, s \geq 3$.

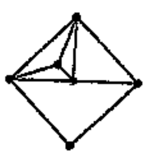
$G(r,s)$: 

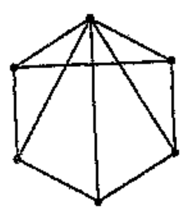
One may extend the structure of the graph $G(r,s)$ to obtain certain new graphs in the following way. Let K_3 be a triangle in K_r, $r \geq 3$, and let X be a K_4 homeomorph with a triangle K_3' as shown in (a) below. Denote by $\mathcal{G}_n(r), n \geq r+1$, the class of graphs of order n obtained by identifying K_3 and K_3' in the disjoint
and X. All graphs, <u>except two</u>, in the class $\mathcal{G}_n(r), n \geq r+1 \geq s$, are chromatically unique.



(a) (b)

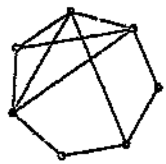
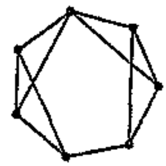
The only graph $\mathcal{G}_6(4)$ is the graph F.

F: 

W_6: 

It can be verified, see Lemma 5.4.1, that
$P(F,x) = x(x-1)(x-2)(x-3)(x^2-4x+5) = P(W_6,x)$ but $W_6 \not\cong F$.

The graph H is one of the two graphs in $\mathcal{G}_7(4)$.

H:  X: 

It can be verified that $P(H,x) = x(x-1)(x-2)^2(x-3)(x^2-3x+4) = P(X,x)$ but $X \not\cong H$.

5.5 Wheels

In [24] it is shown that the wheel W_p is chromatically unique for all odd integers $p \geq 5$. The proof technique involved the fact that odd wheels are uniquely 3-colourable graphs. We now consider the question whether the wheels W_p are chromatically unique for even integers $p \geq 4$? W_4 (which is isomorphic to K_4) is chromatically unique. We will use $P(W_{p+1}, x) = \sum_{i=0}^{p-2}(-1)^i x(x-1)(x-2)^{p-1-i}$ from Corollary 2.3.1 to show that neither W_6 nor W_8 is chromatically unique. The following result also appears in [7]:

Lemma 5.5.1. W_6 is not chromatically unique.

Proof.

$$P(W_6, x) = \sum_{i=0}^{3}(-1)^i x(x-1)(x-2)^{4-i}$$

$$= x(x-1)(x-2)^4 - x(x-1)(x-2)^3 + x(x-1)(x-2)^2$$

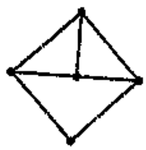
$$- x(x-1)(x-2)$$

$$= x(x-1)(x-2)\left[(x-2)^3 - (x-2)^2 + (x-2) - 1\right]$$

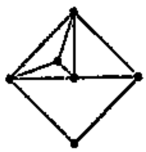
$$= x(x-1)(x-2)\left[(x-2)^2(x-3) + (x-3)\right]$$

$$= x(x-1)(x-2)(x-3)\left[(x-2)^2 + 1\right]$$

$$= x(x-1)(x-2)(x-3)(x^2 - 4x + 5)$$

Thus if we start with $W_5(1)$:  which

by Lemma 5.2.2 with $p = 4$ has chromatic polynomial
$P(W_5(1), x) = x(x-1)(x-2)(x^2 - 4x + 5)$, and add a new vertex and join it to three pairwise adjacent vertices we obtain the graph

F: 

which is not isomorphic to W_6 but satisfies $P(F, x) = (x-3)P(W_5(1), x)$ by Lemma 2.3.3. Also by Theorem 5.2.2 with $p = 5, n = 2$ and $m = 1$ we have

$$P(F, x) = x(x-1)(x-2)(x-3)(x^2 - 4x + 5).$$

Hence

$$P(W_6, x) = (x-3)P(W_5(1), x) = P(F, x)$$

where $W_6 \ncong F$. □

The following result also appears in [24]:

Lemma 5.5.2. W_8 is not chromatically unique.

Proof.

$$
\begin{aligned}
P(W_8, x) &= \sum_{i=0}^{5}(-1)^i x(x-1)(x-2)^{6-i} \\
&= x(x-1)(x-2)^6 - x(x-1)(x-2)^5 + \ldots - x(x-1)(x-2) \\
&= x(x-1)(x-2)\left[(x-2)^5 - (x-2)^4 + \ldots - 1\right] \\
&= x(x-1)(x-2)\left[(x-2)^4(x-3) + (x-2)^2(x-3) + (x-3)\right] \\
&= x(x-1)(x-2)(x-3)\left[(x-2)^4 + (x-2)^2 + 1\right] \\
&= x(x-1)(x-2)(x-3)\left[(x-2)^2 - (x-2) + 1\right]\left[(x-2)^2 + (x-2) + 1\right] \\
&= x(x-1)(x-2)(x-3)(x^2 - 5x + 7)(x^2 - 3x + 3)
\end{aligned}
$$

But

$$
\begin{aligned}
P(W_5, x) &= \sum_{i=0}^{2}(-1)^i x(x-1)(x-2)^{3-i} \\
&= x(x-1)(x-2)^3 - x(x-1)(x-2)^2 + x(x-1)(x-2) \\
&= x(x-1)(x-2)\left[(x-2)^2 - (x-2) + 1\right] \\
&= x(x-1)(x-2)(x^2 - 5x + 7)
\end{aligned}
$$

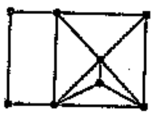
and

$$
\begin{aligned}
P(C_4, x) &= T_4 - T_3 + T_2 \\
&= x(x-1)^3 - x(x-1)^2 + x(x-1) \\
&= x(x-1)\left[(x-1)^2 - (x-1) + 1\right] \\
&= x(x-1)(x^2 - 3x + 3)
\end{aligned}
$$

Thus $P(W_8, x) = (x-3)P(W_5, x)(x^2 - 3x + 3)$. Let H_1 be the graph which is obtained from K_4 and W_5 by overlapping in a triangle

H_1:

so that $P(H_1, x) = (x-3)P(W_5, x)$ by Lemma 2.3.3, and F_1 be one of the graphs which are obtained from C_4 and H_1 by overlapping in an edge.



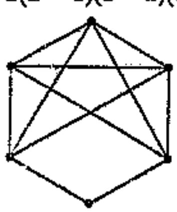
F_1:

Thus by Theorem 2.2.1 we have:

$$\begin{aligned} P(F_1, x) &= \frac{P(H_1, x)P(C_4, x)}{x(x-1)} \\ &= \frac{(x-3)P(W_5, x)x(x-1)(x^2-3x+3)}{x(x-1)} \\ &= (x-3)P(W_5, x)(x^2-3x+3) \\ &= P(W_8, x) \end{aligned}$$

Thus F_1 is chromatically equivalent to W_8. Clearly F_1 and W_8 are not isomorphic because they have different degree sequences. Also by Lemma 5.2.2 with $p = 5$ we have

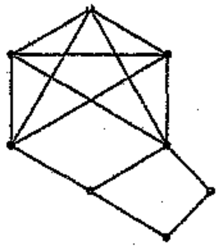
$P(H_2, x) = x(x-1)(x-2)(x-3)(x^2-5x+7)$ where



H_2:

Now if the graph F_2 is obtained from C_4 and H_2 by overlapping in an edge where

F_2:



then by Theorem 2.2.1 we have

$$\begin{aligned}
P(F_2, x) &= \frac{P(H_2, x) \times P(C_4, x)}{x(x-1)} \\
&= \frac{x(x-1)(x-2)(x-3)(x^2-5x+7) \times x(x-1)(x^2-3x+3)}{x(x-1)} \\
&= x(x-1)(x-2)(x-3)(x^2-5x+7)(x^2-3x+3) \\
&= P(W_8, x)
\end{aligned}$$

Clearly $F_2 \ncong W_8$. Hence W_8 is not chromatically unique. □

A computer investigation, see [15] revealed that W_8 is not chromatically unique since it has the same chromatic polynomial as 10 other graphs. It was conjectured in [24] that the wheels W_p are not chromatically unique for even $p \geq 6$.

We now consider W_{10} and apply the same techniques used in Lemma 5.5.1 and Lemma 5.5.2.

$$\begin{aligned}
P(W_{10}, x) &= \sum_{i=0}^{7} (-1)^i x(x-1)(x-2)^{8-i} \\
&= x(x-1)(x-2)^8 - x(x-1)(x-2)^7 + \cdots + (-1)^7 x(x-1)(x-2) \\
&= x(x-1)(x-2)\left[(x-2)^7 - (x-2)^6 + \cdots + (x-2) - 1\right]
\end{aligned}$$

$$= x(x-1)(x-2)\left[(x-2)^6(x-3)+(x-2)^4(x-3)\right.$$
$$\left.+(x-2)^2(x-3)+(x-3)\right]$$
$$= x(x-1)(x-2)(x-3)\left[(x-2)^6+(x-2)^4+(x-2)^2+1\right]$$
$$= x(x-1)(x-2)(x-3)\left[(x-2)^2+1\right]\left[(x-2)^4+1\right]$$
$$= P(W_6,x)\left[(x-2)^4+1\right]$$

There is no graph with polynomial $x(x-1)\left[(x-2)^4+1\right]$ or $x(x-1)(x-2)\left[(x-2)^4+1\right]$. Hence by this method we cannot obtain a graph that is chromatically equivalent to W_{10}. A computer investigation also in [15] revealed that W_{10} is indeed chromatically unique. It remains an unsolved problem to determine whether or not the wheels W_p are chromatically unique for even integers $p \geq 12$.

We used the above techniques to tackle thi problem for $p \geq 12$. Unfortunately we found no graphs that are chromatically equivalent to W_p. We now give the chromatic polynomials of some of these graphs in various forms.

$$P(W_{12},x) = x(x-1)(x-2)(x-3)\left[(x-2)^8+(x-2)^6+(x-2)^4+(x-2)^2+1\right]$$
$$= x(x-1)(x-2)(x-3)\left[(x-2)^4-(x-2)^3+\cdots+1\right]\left[(x-2)^4\right.$$
$$\left.+(x-2)^3+\cdots+1\right]$$
$$= P(W_7,x)(x-3)\left[(x-2)^4+(x-2)^3+(x-2)^2+(x-2)+1\right]$$

$$P(W_{14},x) = x(x-1)(x-2)(x-3)\left[(x-2)^{10}+(x-2)^8+\ldots+(x-2)^2+1\right]$$
$$= x(x-1)(x-2)(x-3)\left[(x-2)^4+(x-2)^2+1\right]\left[(x-2)^6+1\right]$$
$$= P(W_8,x)\left[(x-2)^2+1\right]\left[(x-2)^4-(x-2)^2+1\right]$$

$$P(W_{16},x) = x(x-1)(x-2)(x-3)\left[(x-2)^{12} + (x-2)^{10} + \cdots + (x-2)^2 + 1\right]$$

$$= P(W_9,x)(x-3)\left[(x-2)^6 + (x-2)^5 + (x-2)^4 + (x-2)^3 + (x-2)^2 + (x-2) + 1\right]$$

$$P(W_{18},x) = x(x-1)(x-2)(x-3)\left[(x-2)^{14} + (x-2)^{12} + \cdots + (x-2)^2 + 1\right]$$

$$= P(W_{10},x)\left[(x-2)^8 + 1\right]$$

$$P(W_{20},x) = x(x-1)(x-2)(x-3)\left[(x-2)^{16} + (x-2)^{14} + \cdots + (x-2)^2 + 1\right]$$

$$= P(W_{11},x)(x-3)(x^2-3x+3)\left[(x-2)^6 + (x-2)^3 + 1\right]$$

CONCLUSION

There are many unsolved problems relating to chromatic polynomials, see [14], [15] and [22]. We present a few in the following sections:

Characterization of Chromatic Polynomials

We begin with the question: What is a necessary and sufficient condition for a polynomial to be the chromatic polynomial of some graph? In other words, is there some way to determine whether a given polynomial belongs to a graph? We have derived various *necessary* conditions for a polynomial to be the chromatic polynomial of some graph – the polynomial must be monic, has terms that alternate in sign in normal form (for connected graphs in tree form), has zero constant term and so on - but these conditions taken together are not *sufficient*. For example the polynomial

$$T_7 - 5T_6 + 10T_5 - 10T_4 + 6T_3 - 2T_2 = x(x-1)(x-2)\{(x-2)^4 + 1\}$$

(see section 5.5 on wheels)

satisfies these conditions but is not the chromatic polynomial of any graph. From the given polynomial we can determine the number of vertices, edges, triangles etc. of a graph (if it exists) by looking at the degree of the polynomial, the first few coefficients and applying the knowledge described in Chapter 4. We then generate all such graphs and find their chromatic polynomials. This could work but it is a tedious method and certainly not an effective way to determine whether a given polynomial is chromatic.

Chromatic Equivalence and Chromatic Uniqueness

It is clear that two or more distinct graphs may have the same chromatic polynomial. This prompts the question: What is a necessary and sufficient condition for two or more graphs to have the same chromatic polynomial? It is easy to compute whether two graphs are chromatically equivalent, but to determine whether there are other graphs with the same chromatic polynomial is quite another problem. We have seen in Chapter 5 how we can take some graphs apart at cut-vertices and reassemble them by pasting edges to get a different graph with the same chromatic polynomial. This prompts speculation on the possibilty of an algorithm which will generate all the graphs with a given chromatic polynomial. This last problem is clearly an extension of the first problem in this section, that of characterizing chromatic polynomials.

The same can be said about chromatic uniqueness. Is there a method of recognizing whether a given graph is chromatically unique? The known results, see Chapter 5, were obtained by carefully wringing from a chromatic polynomial every drop of information about the graph, until it can be shown that there is only one graph. At present there seems to be no other way of resolving this problem.

The Unimodal Conjecture

A property that is very noticeable when one looks at lists of chromatic polynomials is that the coefficients first increase in absolute value and then decrease; two successive coefficients may be equal, but it seems that one never finds a coefficient flanked by larger coefficients, and it is natural to conjecture: If b_j are the coefficients of the chromatic polynomial in *normal form*, that the statement

$$b_j > b_{j+1} \quad \text{and} \quad b_{j+1} < b_{j+2}$$

is false for all j. This conjecture is replaced by the property of the *strong logarithmic concavity*, namely that

$$b_j b_{j+2} < b_{j+1}^2 \quad \text{holds for all } j\,.$$

The coefficients that arise when a chromatic polynomial is expressed in *factorial* *form* also seem to exhibit a unimodal distribution, and it has been conjectured that they too have the property of strong logarithmic concavity.

The coefficients that arise when a chromatic polynomial is expressed in *tree form* the situation is rather different. The unimodal property does not hold for many disconnected graphs, see Example 4.5.1, and examples are known of connected graphs for which three or more con...ive coefficients are equal, see appendix, thus contradicting the property of strong logarithmic concavity. However, no counter-examples are known to the slightly weaker conjecture: If a_j are the coefficients of the chromatic polynomial of a *connected* graph in *tree form*, that these coefficients have the property of (*weak*) *logarithmic concavity*, namely that:

$$a_j a_{j+2} \leq a_{j+1}^2 .$$

Chromatic Polynomials of Families of Graphs

There appears to be no connection between the chromatic polynomials of a graph and its complement. Hence the complements of any family of graphs can form a fertile ground for further research. For further information see [15].

Even within the area of finding general formulae for chromatic polynomials of families of graphs, it is remarkably easy to formulate exceedingly difficult problems. For example,

The $m \times n$ chessboard or lattice graph.

The graph $G_{m\times n}$ with vertices at the integer lattice points (u, v) with $1 \leq u \leq n$, $1 \leq v \leq m$, on the cartesian plane, and in which each vertex is joined to the four or fewer vertices at distance 1 from it, is called the $m \times n$ *chessboard or lattice graph.*

Find a general formula for the chromatic polynomial $P(G_{m\times n}, x)$ of the graph $G_{m\times n}$.

Arbitrary Trees of Polygons

Chao and Li, see [22], proved the following characterization of m-gon-trees: A graph G is an m-gon-tree with k m-gons (where $m \geq 3k \geq 1$) if and only if $P(G,x) = x(x-1)\{Q(C_m,x)\}^k$ where $Q(C_m,x) = P(C_m,x) \div x(x-1)$, see also section 4.7.

It is an open problem to generalize the above characterization to *arbitrary trees of polgons*. By arbitrary trees of polygons, we mean trees of polygons where the length of the polygons are not required to be the same.

Finally in section 5.3 we stated an open problem to characterize those graphs G whose chromatic polynomials $P(G,x)$ have the form

$$P(G,x) = x^{m_0}(x-1)^{m_1}(x-2)^{m_2}\dots(x-k)^{m_k}$$

and in section 5.5 it remains an unsolved problem to determine whether or not the wheels W_p are chromatically unique for even integer $p \geq 12$.

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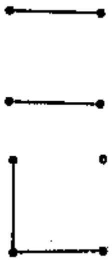
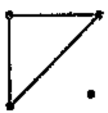
APPENDIX

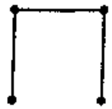
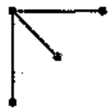
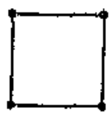
$T_p = x(x-1)^{p-1}$ Tree Form
$C_p = (x-1)^p + (-1)^p(x-1)$ Cycle Form
$x^{\{p\}} = x(x-1)(x-2)\ldots(x-p+1)$ Factorial Form

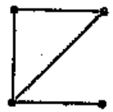
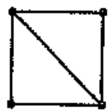
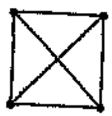
$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(1;0)$	x	T_1	C_1	$x^{\{1\}}$

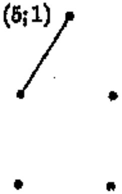
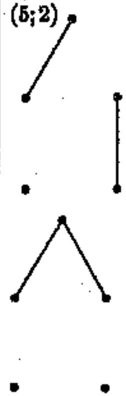
$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(2;0)$ 	x^2	$T_2 + T_1$	$C_2 + C_1$	$x^{\{2\}} + x^{\{1\}}$
$(2;1)$ 	$x^2 - x$ $= x(x-1)$	T_2	C_2	$x^{\{2\}}$

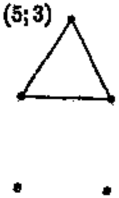
$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(3;0)$ 	x^3	$T_3+2T_2+T_1$	$C_3+3C_2+C_1$	$x^{(3)}+3x^{(2)}+x^{(1)}$
$(3;1)$ 	x^3-x^2 $=x^2(x-1)$	T_3+T_2	C_3+2C_2	$x^{(3)}+2x^{(2)}$
$(3;2)$ 	x^3-2x^2+x $=x(x-1)^2$	T_3	C_3+C_2	$x^{(3)}+x^{(2)}$
$(3;3)$ 	x^3-3x^2+2x $=x(x-1)(x-2)$	T_3-T_2	C_3	$x^{(3)}$

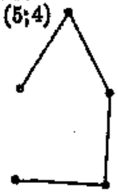
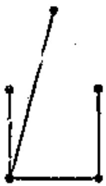
$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(4;0)$ 	x^4	$T_4+3T_3+3T_2+T_1$	$C_4+4C_3+6C_2+C_1$	$x^{\{4\}}+6x^{\{3\}}+7x^{\{2\}}+x^{\{1\}}$
$(4;1)$ 	x^4-x^3 $=x^3(x-1)$	$T_4+2T_3+T_2$	$C_4+3C_3+3C_2$	$x^{\{4\}}+5x^{\{3\}}+4x^{\{2\}}$
$(4;2)$ 	$x^4-2x^3+x^2$ $=x^2(x-1)^2$	T_4+T_3	$C_4+2C_3+C_2$	$x^{\{4\}}+4x^{\{3\}}+2x^{\{2\}}$
$(4;3)$ 	$x^4-3x^3+2x^2$ $=x^2(x-1)(x-2)$	T_4-T_2	$C_4+C_3-C_2$	$x^{\{4\}}+3x^{\{3\}}$

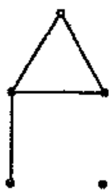
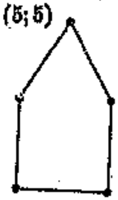
$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(4;3)$  	$x^4-3x^3+3x^2-x$ $=x(x-1)^3$	T_4	C_4+C_3	$x^{\{4\}}+3x^{\{3\}}+x^{\{2\}}$
$(4;4)$ 	$x^4-4x^3+6x^2-3x$ $=x(x-1)(x^2-3x+3)$	$T_4-T_3+T_2$	C_4	$x^{\{4\}}+2x^{\{3\}}+x^{\{2\}}$

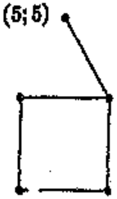
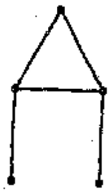
$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(4;4)$ 	$x^4-4x^3+5x^2-2x$ $=x(x-1)^2(x-2)$	T_4-T_3	C_4-C_2	$x^{\{4\}}+2x^{\{3\}}$
$(4;5)$ 	$x^4-5x^3+8x^2-4x$ $=x(x-1)(x-2)^2$	$T_4-2T_3+T_2$	$C_4-C_3-C_2$	$x^{\{4\}}+x^{\{3\}}$
$(4;6)$ 	$x^4-6x^3+11x^2-6x$ $=x(x-1)(x-2)(x-3)$	$T_4-3T_3+2T_2$	$C_4-2C_3-C_2$	$x^{\{4\}}$

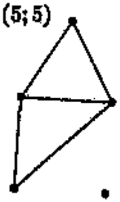
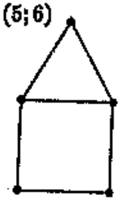
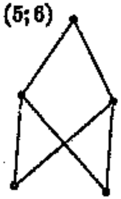
$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(5;0)$ 	x^5	$T_5+4T_4+6T_3+4T_2+T_1$	$C_5+5C_4+10C_3+10C_2+C_1$	$x^{(5)}+10x^{(4)}+25x^{(3)}+15x^{(2)}+x^{(1)}$
$(5;1)$ 	x^5-x^4 $=x^4(x-1)$	$T_5+3T_4+3T_3+T_2$	$C_5+4C_4+6C_3+4C_2$	$x^{(5)}+9x^{(4)}+19x^{(3)}+8x^{(2)}$
$(5;2)$ 	$x^5-2x^4+x^3$ $=x^3(x-1)^2$	$T_5+2T_4+T_3$	$C_5+3C_4+3C_3+C_2$	$x^{(5)}+8x^{(4)}+14x^{(3)}+4x^{(2)}$

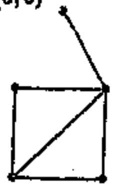
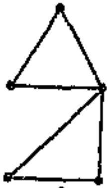
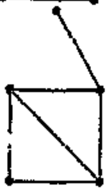
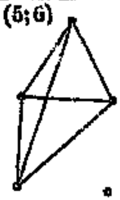
$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(5;3)$ 	$x^5 - 3x^4 + 2x^3$ $= x^3(x-1)(x-2)$	$T_5 + T_4 - T_3 - T_2$	$C_5 + 2C_4 - 2C_2$	$x^{\{5\}} + 7x^{\{4\}} + 9x^{\{3\}}$
$(5;3)$ 	$x^5 - 3x^4 + 3x^3 - x^2$ $= x^2(x-1)^3$	$T_5 + T_4$	$C_5 + 2C_4 + C_3$	$x^{\{5\}} + 7x^{\{4\}} + 10x^{\{3\}} + 2x^{\{2\}}$

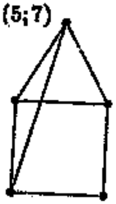
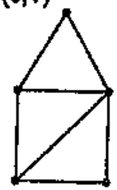
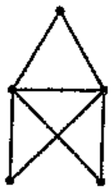
$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(5;4)$   	$x^5-4x^4+6x^3-4x^2+x$ $=x(x-1)^4$	T_5	C_5+C_4	$x^{\{5\}}+6x^{\{4\}}+7x^{\{3\}}+x^{\{2\}}$

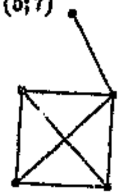
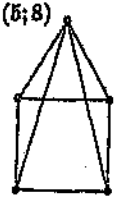
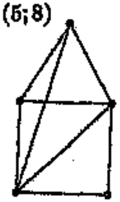
$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(5;4)$  	$x^5-4x^4+5x^3-2x^2$ $=x^2(x-1)^2(x-2)$	T_5-T_3	$C_5+C_4-C_3-C_2$	$x^{(5)}+6x^{(4)}+5x^{(3)}$
$(5;4)$ 	$x^5-4x^4+6x^3-3x^2$ $=x^2(x-1)(x^2-3x+3)$	T_5+T_2	$C_5+C_4+C_2$	$x^{(5)}+6x^{(4)}+7x^{(3)}+2x^{(2)}$
$(5;5)$ 	$x^5-5x^4+10x^3-10x^2+4x$ $=x(x-1)(x-2)(x^2-2x+2)$	$T_5-T_4+T_3-T_2$	C_5	$x^{(5)}+5x^{(4)}+5x^{(3)}$

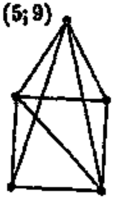
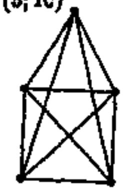
$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(5;5)$ 	$x^5 - 5x^4 + 10x^3 - 9x^2 + 3x$ $= x(x-1)^2(x^2 - 3x + 3)$	$T_5 - T_4 + T_3$	$C_5 + C_2$	$x^{(5)} + 5x^{(4)} + 5x^{(3)} + x^{(2)}$
$(5;5)$   	$x^5 - 5x^4 + 9x^3 - 7x^2 + 2x$ $= x(x-1)^3(x-2)$	$T_5 - T_4$	$C_5 - C_3$	$x^{(5)} + 5x^{(4)} + 4x^{(3)}$

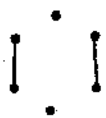
$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(5;5)$ 	$x^5 - 5x^4 + 8x^3 - 4x^2$ $= x^2(x-1)(x-2)^2$	$T_5 - T_4 - T_3 + T_2$	$C_5 - 2C_3$	$x^{\{5\}} + 5x^{\{4\}} + 3x^{\{3\}}$
$(5;6)$ 	$x^5 - 6x^4 + 14x^3 - 15x^2 + 6x$ $= x(x-1)(x-2)(x^2-3x+3)$	$T_5 - 2T_4 + 2T_3 - T_2$	$C_5 - C_4 + C_2$	$x^{\{5\}} + 4x^{\{4\}} + 3x^{\{3\}}$
$(5;6)$ 	$x^5 - 6x^4 + 15x^3 - 17x^2 + 7x$ $= x(x-1)(x^3-5x^2+10x-7)$	$T_5 - 2T_4 + 3T_3 - T_2$	$C_5 - C_4 + C_3 + 2C_2$	$x^{\{5\}} + 4x^{\{4\}} + 4x^{\{3\}} + x^{\{2\}}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(5;6)$   	$x^5 - 6x^4 + 13x^3 - 12x^2 + 4x$ $= x(x-1)^2(x-2)^2$	$T_5 - 2T_4 + T_3$	$C_5 - C_4 - C_3 + C_2$	$x^{(5)} + 4x^{(4)} + 2x^{(3)}$
$(5;6)$ 	$x^5 - 6x^4 + 11x^3 - 6x^2$ $= x^2(x-1)(x-2)(x-3)$	$T_5 - 2T_4 - T_3 + 2T_2$	$C_5 - C_4 - 3C_3 + C_2$	$x^{(5)} + 4x^{(4)}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(5;7)$ 	$x^5 - 7x^4 + 19x^3 - 23x^2 + 10x$ $= x(x-1)(x-2)(x^2-4x+5)$	$T_5 - 3T_4 + 4T_3 - 2T_2$	$C_5 - 2C_4 + C_3 + 2C_2$	$x^{(5)} + 3x^{(4)} + 2x^{(3)}$
$(5;7)$  	$x^5 - 7x^4 + 18x^3 - 20x^2 + 8x$ $= x(x-1)(x-2)^3$	$T_5 - 3T_4 + 3T_3 - T_2$	$C_5 - 2C_4 + 2C_2$	$x^{(5)} + 3x^{(4)} + x^{(3)}$

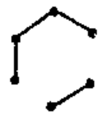
$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(5;7)$ 	$x^5 - 7x^4 + 17x^3 - 17x^2 + 6x$ $= x(x-1)^2(x-2)(x-3)$	$T_5 - 3T_4 + 2T_3$	$C_5 - 2C_4 - C_3 + 2C_2$	$x^{(5)} + 3x^{(4)}$
$(5;8)$ 	$x^5 - 8x^4 + 24x^3 - 31x^2 + 14x$ $= x(x-1)(x-2)(x^2-5x+7)$	$T_5 - 4T_4 + 6T_3 - 3T_2$	$C_5 - 3C_4 + 2C_3 + 3C_2$	$x^{(5)} + 2x^{(4)} + x^{(3)}$
$(5;8)$ 	$x^5 - 8x^4 + 23x^3 - 28x^2 + 12x$ $= x(x-1)(x-2)^2(x-3)$	$T_5 - 4T_4 + 5T_3 - 2T_2$	$C_5 - 3C_4 + C_3 + 3C_2$	$x^{(5)} + 2x^{(4)}$

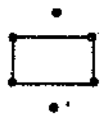
$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(5;9)$ 	$x^5 - 9x^4 + 29x^3 - 39x^2 + 18x$ $= x(x-1)(x-2)(x-3)^2$	$T_5 - 5T_4 + 8T_3 - 4T_2$	$C_5 - 4C_4 + 3C_3 + 4C_2$	$x^{(5)} + x^{(4)}$
$(5;10)$ 	$x^5 - 10x^4 + 35x^3 - 50x^2 + 24x$ $= x(x-1)(x-2)(x-3)(x-4)$	$T_5 - 6T_4 + 11T_3 - 6T_2$	$C_5 - 5C_4 + 5C_3 + 5C_2$	$x^{(5)}$

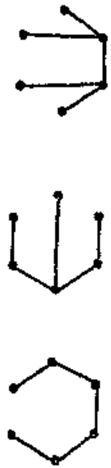
$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;0)$ 	x^6	$T_6 + 5T_5 + 10T_4 + 10T_3 + 5T_2 + T_1$	$C_6 + 6C_5 + 15C_4 + 20C_3 + 15C_2 + C_1$	$x^{\{6\}} + 15x^{\{5\}} + 65x^{\{4\}}$ $+90x^{\{3\}} + 31x^{\{2\}} \cdot x^{\{1\}}$
$(6;1)$ 	$x^6 - x^5$ $= x^5(x-1)$	$T_6 + 4T_5 + 6T_4 + 4T_3 + T_2$	$C_6 + 5C_5 + 10C_4 + 10C_3 + 5C_2$	$x^{\{6\}} + 14x^{\{5\}}$ $+55x^{\{4\}} + 65x^{\{3\}} + 16x^{\{2\}}$
$(6;2)$  	$x^6 - 2x^5 + x^4$ $= x^4(x-1)^2$	T [illegible] $3T_5 + 3T_4 + T_3$	$C_6 + 4C_5 + 6C_4 + 4C_3 + C_2$	$x^{\{6\}} + 13x^{\{5\}}$ $+46x^{\{4\}} + 46x^{\{3\}} + 8x^{\{2\}}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;3)$ 	$x^6 - 3x^5 + 3x^4 - x^3$ $= x^3(x-1)^3$	$T_6 + 2T_5 + T_4$	$C_6 + 3C_5 + 3C_4 + C_3$	$x^{(6)} + 12x^{(5)}$ $+38x^{(4)} + 32x^{(3)} + 4x^{(2)}$

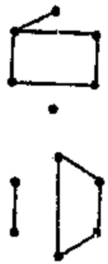
$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;3)$ 	$x^6 - 3x^5 + 2x^4$ $= x^4(x-1)(x-2)$	$T_6 + 2T_5 - 2T_3 - T_2$	$C_6 + 3C_5 + 2C_4 - 2C_3 - 3C_2$	$x^{(6)} + 12x^{(5)}$ $+37x^{(4)} + 27x^{(3)}$
$(6;4)$ 	$x^6 - 4x^5 + 6x^4 - 4x^3 + x^2$ $= x^2(x-1)^4$	$T_6 + T_5$	$C_6 + 2C_5 + C_4$	$x^{(6)} + 11x^{(5)}$ $+31x^{(4)} + 22x^{(3)} + 2x^{(2)}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;4)$   	$x^6-4x^5+6x^4-4x^3+x^2$ $=x^2(x-1)^4$	T_6+T_5	$C_6+2C_5+C_4$	$x^{\{6\}}+11x^{\{5\}}$ $+31x^{\{4\}}+22x^{\{3\}}+2x^{\{2\}}$

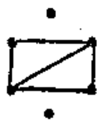
$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;4)$ 	$x^6-4x^5+6x^4-3x^3$ $=x^3(x-1)(x^2-3x+3)$	$T_6+T_5+T_3+T_2$	$C_6+2C_5+C_4+C_3+2C_2$	$x^{(6)}+11x^{(5)}$ $+31x^{(4)}+23x^{(3)}+4x^{(2)}$
$(6;4)$  	$x^6-4x^5+5x^4-2x^3$ $=x^3(x-1)^2(x-2)$	$T_6+T_5-T_4-T_3$	$C_6+2C_5-2C_3-C_2$	$x^{(6)}+11x^{(5)}$ $+30x^{(4)}+18x^{(3)}$

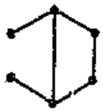
$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;5)$ 	$x^6-5x^5+10x^4-10x^3+5x^2-x$ $=x(x-1)^5)$	T_6	C_6+C_5	$x^{(6)}+10x^{(5)}$ $+25x^{(4)}+15x^{(3)}+x^{(2)}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;5)$   	$x^6 - 5x^5 + 10x^4 - 10x^3 + 5x^2 - x$ $= x(x-1)^5)$	T_6	$C_6 + C_5$	$x^{\{6\}} + 10x^{\{5\}}$ $+25x^{\{4\}} + 15x^{\{3\}} + x^{\{2\}}$
$(6;5)$ 	$x^6 - 5x^5 + 10x^4 - 10x^3 + 4x^2$ $= x^2(x-1)(x-2)(x^2-2x+2)$	$T_6 - T_2$	$C_6 + C_5 - C_2$	$x^{\{6\}} + 10x^{\{5\}}$ $+25x^{\{4\}} + 15x^{\{3\}}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;5)$ 	$x^6-5x^5+10x^4-9x^3+3x^2$ $=x^2(x-1)^2(x^2-3x+3)$	T_6+T_3	$C_6+C_5+C_4+C_2$	$x^{\{6\}}+10x^{\{5\}}$ $+25x^{\{4\}}+16x^{\{3\}}+2x^{\{2\}}$
$(6;5)$ 	$x^6-5x^5+9x^4-7x^3+2x^2$ $=x^2(x-1)^3(x-2)$	T_6-T_4	$C_6+C_5-C_4-C_3$	$x^{\{6\}}+10x^{\{5\}}$ $+24x^{\{4\}}+12x^{\{3\}}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;5)$   	$x^6 - 5x^5 + 9x^4 - 7x^3 + 2x^2$ $= x^2(x-1)^3(x-2)$	$T_6 - T_4$	$C_6 + C_5 - C_4 - C_3$	$x^{(6)} + 10x^{(5)}$ $+24x^{(4)} + 12x^{(3)}$

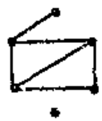
$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
(6;5) 	$x^6 - 5x^5 + 8x^4 - 4x^3$ $= x^3(x-1)(x-2)^2$	$T_6 - 2T_4 + T_2$	$C_6 + C_5 - 2C_4 - 2C_3 + C_2$	$x^{\{6\}} + 10x^{\{5\}}$ $+23x^{\{4\}} + 9x^{\{3\}}$
(6;6) 	$x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 5x$ $= x(x-1)(x^4 - 5x^3 + 10x^2 - 10x + 5)$	$T_6 - T_5 + T_4 - T_3 + T_2$	C_6	$x^{\{6\}} + 9x^{\{5\}}$ $+20x^{\{4\}} + 10x^{\{3\}} + x^{\{2\}}$
(6;6) 	$x^6 - 6x^5 + 15x^4 - 20x^3 + 14x^2 - 4x$ $= x(x-1)^2(x-2)(x^2 - 2x + 2)$	$T_6 - T_5 + T_4 - T_3$	$C_6 - C_2$	$x^{\{6\}} + 9x^{\{5\}}$ $+20x^{\{4\}} + 10x^{\{3\}}$

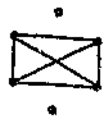
$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;6)$    	$x^6 - 6x^5 + 15x^4 - 19x^3 + 12x^2 - 3x$ $= x(x-1)^3(x^2 - 3x + 3)$	$T_6 - T_5 + T_4$	$C_6 + C_3$	$x^{\{6\}} + 9x^{\{5\}}$ $+20x^{\{4\}} + 11x^{\{3\}} + x^{\{2\}}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;6)$    	$x^6-6x^5+14x^4-16x^3+9x^2-2x$ $=x(x-1)^4(x-2)$	T_6-T_5	C_6-C_4	$x^{(6)}+9x^{(5)}$ $+19x^{(4)}+8x^{(3)}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;6)$   	$x^6 - 6x^5 + 14x^4 - 16x^3 + 9x^2 - 2x$ $= x(x-1)^4(x-2)$	$T_6 - T_5$	$C_6 - C_4$	$x^{(6)} + 9x^{(5)}$ $+19x^{(4)} + 8x^{(3)}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;6)$ 	$x^6 - 6x^5 + 15x^4 - 17x^3 + 7x^2$ $= x^2(x-1)(x^3 - 5x^2 + 10x - 7)$	$T_6 - T_5 + T_4 + 2T_3 - T_2$	$C_6 + 3C_3 + C_2$	$x^{\{6\}} + 9x^{\{5\}}$ $+20x^{\{4\}} + 13x^{\{3\}} + 2x^{\{2\}}$
$(6;6)$ 	$x^6 - 6x^5 + 14x^4 - 15x^3 + 6x^2$ $= x^2(x-1)(x-2)(x^2 - 3x + 3)$	$T_6 - T_5 + T_3 - T_2$	$C_6 - C_4 + C_3$	$x^{\{6\}} + 9x^{\{5\}}$ $+19x^{\{4\}} + 9x^{\{3\}}$
$(6;6)$ 	$x^6 - 6x^5 + 13x^4 - 12x^3 + 4x^2$ $= x^2(x-1)^2(x-2)^2$	$T_6 - T_5 - T_4 + T_3$	$C_6 - 2C_4 + C_2$	$x^{\{6\}} + 9x^{\{5\}}$ $+18x^{\{4\}} + 6x^{\{3\}}$

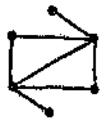
$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;6)$   	$x^6-6x^5+13x^4-12x^3+4x^2$ $=x^2(x-1)^2(x-2)^2$	$T_6-T_5-T_4+T_3$	$C_6-2C_4+C_2$	$x^{(6)}+9x^{(5)}$ $+18x^{(4)}+6x^{(3)}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
(6;6) 	$x^6-6x^5+11x^4-6x^3$ $=x^3(x-1)(x-2)(x-3)$	$T_6-T_5-3T_4+T_3+2T_2$	$C_6-4C_4-2C_3+3C_2$	$x^{\{6\}}+9x^{\{5\}}+16x^{\{4\}}$
(6;7) 	$x^6-7x^5+21x^4-34x^3+29x^2-10x$ $=x(x-1)(x-2)(x^3-4x^2+7x-5)$	$T_6-2T_5+3T_4-3T_3+T_2$	$C_6-C_5+C_4-2C_2$	$x^{\{6\}}+8x^{\{5\}}$ $+16x^{\{4\}}+7x^{\{3\}}$
(6;7) 	$x^6-7x^5+20x^4-30x^3+24x^2-8x$ $=x(x-1)(x-2)^2(x^2-2x+2)$	$T_6-2T_5+2T_4-2T_3+T_2$	$C_6-C_5-C_2$	$x^{\{6\}}+8x^{\{5\}}$ $+15x^{\{4\}}+5x^{\{3\}}$
(6;7) 	$x^6-7x^5+21x^4-33x^3+27x^2-9x$ $=x(x-1)(x^2-3x+3)^2$	$T_6-2T_5+3T_4-2T_3+T_2$	$C_6-C_5+C_4+C_3-C_2$	$x^{\{6\}}+8x^{\{5\}}$ $+16x^{\{4\}}+8x^{\{3\}}+x^{\{2\}}$

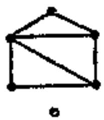
$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;7)$  	$x^6 - 7x^5 + 21x^4 - 32x^3 + 24x^2 - 7x$ $= x(x-1)^2(x^3 - 5x^2 + 10x - 7)$	$T_6 - 2T_5 + 3T_4 - T_3$	$C_6 - C_5 + C_4 + 2C_3 - C_2$	$x^{\{6\}} + 8x^{\{5\}}$ $+16x^{\{4\}} + 9x^{\{3\}} + x^{\{2\}}$
$(6;7)$  	$x^6 - 7x^5 + 20x^4 - 29x^3 + 21x^2 - 6x$ $= x(x-1)^2(x-2)(x^2 - 3x + 3)$	$T_6 - 2T_5 + 2T_4 - T_3$	$C_6 - C_5 + C_3 - C_2$	$x^{\{6\}} + 8x^{\{5\}}$ $+15x^{\{4\}} + 6x^{\{3\}}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;7)$  	$x^6 - 7x^5 + 20x^4 - 29x^3 + 21x^2 - 6x$ $= x(x-1)^2(x-2)(x^2-3x+3)$	$T_6 - 2T_5 + 2T_4 - T_3$	$C_6 - C_5 + C_3 - C_2$	$x^{(6)} + 8x^{(5)}$ $+15x^{(4)} + 6x^{(3)}$

$(p; q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;7)$    	$x^6 - 7x^5 + 19x^4 - 25x^3 + 16x^2 - 4x$ $= x(x-1)^3(x-2)^2$	$T_6 - 2T_5 + T_4$	$C_6 - C_5 - C_4 + C_3$	$x^{(6)} + 8x^{(5)}$ $+14x^{(4)} + 4x^{(3)}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;7)$   	$x^6 - 7x^5 + 19x^4 - 25x^3 + 16x^2 - 4x$ $= x(x-1)^3(x-2)^2$	$T_6 - 2T_5 + T_4$	$C_6 - C_5 - C_4 + C_3$	$x^{\{6\}} + 8x^{\{5\}}$ $+14x^{\{4\}} + 4x^{\{3\}}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;7)$   	$x^6 - 7x^5 + 19x^4 - 25x^3 + 16x^2 - 4x$ $= x(x-1)^3(x-2)^2$	$T_6 - 2T_5 + T_4$	$C_6 - C_5 - C_4 + C_3$	$x^{(6)} + 8x^{(5)}$ $+14x^{(4)} + 4x^{(3)}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;7)$ 	$x^6 - 7x^5 + 19x^4 - 23x^3 + 10x^2$ $= x^2(x-1)(x-2)(x^2-4x+5)$	$T_6 - 2T_5 + T_4 + 2T_3 - 2T_2$	$C_6 - C_5 - C_4 + 2C_3$	$x^{(6)} + 8x^{(5)}$ $+14x^{(4)} + 6x^{(3)}$
$(6;7)$  	$x^6 - 7x^5 + 18x^4 - 20x^3 + 8x^2$ $= x^2(x-1)(x-2)^3$	$T_6 - 2T_5 + 2T_3 - T_2$	$C_6 - C_5 - 2C_4 + 2C_3 + C_2$	$x^{(6)} + 8x^{(5)}$ $+13x^{(4)} + 3x^{(3)}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;7)$  	$x^6 - 7x^5 + 17x^4 - 17x^3 + 6x^2$ $= x^2(x-1)^2(x-2)(x-3)$	$T_6 - 2T_5 - T_4 + 2T_3$	$C_6 - C_5 - 3C_4 + C_3 + 2C_2$	$x^{(6)} + 8x^{(5)} + 12x^{(4)}$
$(6;8)$ 	$x^6 - 8x^5 + 28x^4 - 51x^3 + 47x^2 - 17x$ $= x(x-1)(x^4 - 7x^3 + 21x^2 - 30x + 17)$	$T_6 - 3T_5 + 6T_4 - 5T_3 + 2T_2$	$C_6 - 2C_5 + 3C_4 + C_3 - 3C_2$	$x^{(6)} + 7x^{(5)}$ $+13x^{(4)} + 7x^{(3)} + x^{(2)}$
$(6;8)$ 	$x^6 - 8x^5 + 28x^4 - 50x^3 + 44x^2 - 15x$ $= x(x-1)(x^4 - 7x^3 + 21x^2 - 29x + 15)$	$T_6 - 3T_5 + 6T_4 - 4T_3 + T_2$	$C_6 - 2C_5 + 3C_4 + 2C_3 - 3C_2$	$x^{(6)} + 7x^{(5)}$ $+13x^{(4)} + 8x^{(3)} + x^{(2)}$

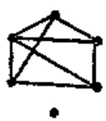
$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;8)$ 	$x^6 - 8x^5 + 27x^4 - 47x^3 + 41x^2 - 14x$ $= x(x-1)(x-2)(x^3 - 5x^2 + 10x - 7)$	$T_6 - 3T_5 + 5T_4 - 4T_3 + T_2$	$C_6 - 2C_5 + 2C_4 + C_3 - 3C_2$	$x^{(6)} + 7x^{(5)}$ $+12x^{(4)} + 5x^{(3)}$
$(6;8)$ 	$x^6 - 8x^5 + 26x^4 - 44x^3 + 39x^2 - 14x$ $= x(x-1)(x-2)(x^3 - 5x^2 + 9x - 7)$	$T_6 - 3T_5 + 4T_4 - 4T_3 + 2T_2$	$C_6 - 2C_5 + C_4 - 2C_2$	$x^{(6)} + 7x^{(5)}$ $+11x^{(4)} + 2x^{(3)}$
$(6;8)$ 	$x^6 - 8x^5 + 27x^4 - 48x^3 + 44x^2 - 16x$ $= x(x-1)(x-2)^2(x^2 - 3x + 4)$	$T_6 - 3T_5 + 5T_4 - 5T_3 + 2T_2$	$C_6 - 2C_5 + 2C_4 - 3C_2$	$x^{(6)} + 7x^{(5)}$ $+12x^{(4)} + 4x^{(3)}$

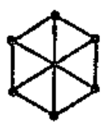
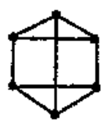
$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;8)$    	$x^6 - 8x^5 + 26x^4 - 43x^3 + 37x^2 - 12x$ $= x(x-1)(x-2)^2(x^2-3x+3)$	$T_6 - 3T_5 + 4T_4 - 3T_3 + T_2$	$C_6 - 2C_5 + C_4 + C_3 - 2C_2$	$x^{(6)} + 7x^{(5)}$ $+11x^{(4)} + 3x^{(3)}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;8)$   	$x^6 - 8x^5 + 26x^4 - 42x^3 + 33x^2 - 10x$ $= x(x-1)^2(x-2)(x^2-4x+5)$	$T_6 - 3T_5 + 4T_4 - 2T_3$	$C_6 - 2C_5 + C_4 + 2C_3 - 2C_2$	$x^{(6)} + 7x^{(5)}$ $+11x^{(4)} + 4x^{(3)}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;8)$    	$x^6-8x^5+25x^4-38x^3+28x^2-8x$ $=x(x-1)^2(x-2)^3$	$T_6-3T_5+3T_4-T_3$	$C_6-2C_5+2C_3-C_2$	$x^{(6)}+7x^{(5)}$ $+10x^{(4)}+2x^{(3)}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;8)$   	$x^6-8x^5+25x^4-38x^3+28x^2-8x$ $=x(x-1)^2(x-2)^3$	$T_6-3T_5+3T_4-T_3$	$C_6-2C_5+2C_3-C_2$	$x^{(6)}+7x^{(5)}$ $+10x^{(4)}+2x^{(3)}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;8)$   	$x^6 - 8x^5 + 24x^4 - 34x^3 + 23x^2 - 6x$ $= x(x-1)^3(x-2)(x-3)$	$T_6 - 3T_5 + 2T_4$	$C_6 - 2C_5 - C_4 + 2C_3$	$x^{(6)} + +7x^{(5)} + 9x^{(4)}$
$(6;8)$ 	$x^6 - 8x^5 + 24x^4 - 31x^3 + 14x^2$ $= x^2(x-1)(x-2)(x^2 - 5x + 7)$	$T_6 - 3T_5 + 2T_4 + 3T_3 - 3T_2$	$C_6 - 2C_5 - C_4 + 5C_3$	$x^{(6)} + 7x^{(5)}$ $+9x^{(4)} + 3x^{(3)}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;8)$ 	$x^6-8x^5+23x^4-28x^3+12x^2$ $=x^2(x-1)(x-2)^2(x-3)$	$T_6-3T_5+T_4+3T_3-2T_2$	$C_6-2C_5-2C_4+4C_3+C_2$	$x^{(6)}+7x^{(5)}+8x^{(4)}$
$(6;9)$ 	$x^6-9x^5+36x^4-75x^3+78x^2-31x$ $=x(x-1)(x^4-8x^3+28x^2-47x+31)$	$T_6-4T_5+10T_4-11T_3+5T_2$	$C_6-3C_5+6C_4-C_3-6C_2$	$x^{(6)}+6x^{(5)}$ $+11x^{(4)}+6x^{(3)}+x^{(2)}$
$(6;9)$ 	$x^6-9x^5+34x^4-67x^3+67x^2-26x$ $=x(x-1)(x-2)(x^3-6x^2+14x-13)$	$T_6-4T_5+8T_4-9T_3+4T_2$	$C_6-3C_5+4C_4-C_3-5C_2$	$x^{(6)}+6x^{(5)}$ $+9x^{(4)}+2x^{(3)}$
$(6;9)$ 	$x^6-9x^5+34x^4-66x^3+64x^2-24x$ $=x(x-1)(x-2)^2(x^2-4x+6)$	$T_6-4T_5+8T_4-8T_3+3T_2$	$C_6-3C_5+4C_4-5C_2$	$x^{(6)}+6x^{(5)}$ $+9x^{(4)}+3x^{(3)}$

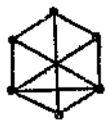
$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;9)$ 	$x^6-9x^5+34x^4-65x^3+61x^2-22x$ $=x(x-1)(x-2)(x^3-6x^2+14x-11)$	$T_6-4T_5+8T_4-7T_3+2T_2$	$C_6-3C_5+3C_4+2C_3-4C_2$	$x^{(6)}+6x^{(5)}$ $+9x^{(4)}+4x^{(3)}$
$(6;9)$ 	$x^6-9x^5+33x^4-62x^3+59x^2-22x$ $=x(x-1)(x-2)(x^3-6x^2+13x-11)$	$T_6-4T_5+7T_4-7T_3+3T_2$	$C_6-3C_5+3C_4-4C_2$	$x^{(6)}+6x^{(5)}$ $+8x^{(4)}+x^{(3)}$
$(6;9)$ 	$x^6-9x^5+33x^4-61x^3+56x^2-20x$ $=x(x-1)(x-2)^2(x^2-4x+5)$	$T_6-4T_5+7T_4-6T_3+2T_2$	$C_6-3C_5+3C_4+C_3-4C_2$	$x^{(6)}+6x^{(5)}$ $+8x^{(4)}+2x^{(3)}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;9)$  	$x^6 - 9x^5 + 33x^4 - 61x^3 + 56x^2 - 20x$ $= x(x-1)(x-2)^2(x^2-4x+5)$	$T_6 - 4T_5 + 7T_4 - 6T_3 + 2T_2$	$C_6 - 3C_5 + 3C_4 + C_3 - 4C_2$	$x^{\{6\}} + 6x^{\{5\}}$ $+8x^{\{4\}} + 2x^{\{3\}}$
$(6;9)$ 	$x^6 - 9x^5 + 32x^4 - 57x^3 + 51x^2 - 18x$ $= x(x-1)(x-2)(x-3)(x^2-3x+3)$	$T_6 - 4T_5 + 6T_4 - 5T_3 + 2T_2$	$C_6 - 3C_5 + 2C_4 + C_3 - 3C_2$	$x^{\{6\}} + 6x^{\{5\}} + 7x^{\{4\}}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;9)$   	$x^6 - 9x^5 + 32x^4 - 56x^3 + 48x^2 - 16x$ $= x(x-1)(x-2)^4$	$T_6 - 4T_5 + 6T_4 - 4T_3 + T_2$	$C_6 - 3C_5 + 2C_4 + 2C_3 - 3C_2$	$x^{(6)} + 6x^{(5)}$ $+7x^{(4)} + x^{(3)}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;9)$  	$x^6 - 9x^5 + 32x^4 - 56x^3 + 48x^2 - 16x$ $= x(x-1)(x-2)^4$	$T_6 - 4T_5 + 6T_4 - 4T_3 + T_2$	$C_6 - 3C_5 + 2C_4 + 2C_3 - 3C_2$	$x^{(6)} + 6x^{(5)}$ $+7x^{(4)} + x^{(3)}$
$(6;9)$  	$x^6 - 9x^5 + 32x^4 - 55x^3 + 45x^2 - 14x$ $= x(x-1)^2(x-2)(x^2 - 5x + 7)$	$T_6 - 4T_5 + 6T_4 - 3T_3$	$C_6 - 3C_5 + 2C_4 + 3C_3 - 3C_2$	$x^{(6)} + 6x^{(5)}$ $+7x^{(4)} + 2x^{(3)}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;9)$    	$x^6-9x^5+31x^4-51x^3+40x^2-12x$ $=x(x-1)^2(x-2)^2(x-3)$	$T_6-4T_5+5T_4-2T_3$	$C_6-3C_5+C_4+3C_3-2C_2$	$x^{(6)}+6x^{(5)}+6x^{(4)}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;9)$ 	$x^6-9x^5+29x^4-39x^3+18x^2$ $=x^2(x-1)(x-2)(x-3)^2$	$T_6-4T_5+3T_4+4T_3-4T_2$	$C_6-3C_5-C_4+7C_3$	$x^{(6)}+6x^{(5)}+4x^{(4)}$
$(6;10)$ 	$x^6-10x^5+42x^4-90x^3+95x^2-38x$ $=x(x-1)(x-2)(x^3-7x^2+19x-19)$	$T_6-5T_5+12T_4-14T_3+6T_2$	$C_6-4C_5+7C_4-2C_3-8C_2$	$x^{(6)}+5x^{(5)}$ $+7x^{(4)}+2x^{(3)}$
$(6;10)$ 	$x^6-10x^5+41x^4-85x^3+87x^2-34x$ $=x(x-1)(x-2)(x^3-7x^2+18x-17)$	$T_6-5T_5+11T_4-12T_3+5T_2$	$C_6-4C_5+6C_4-C_3-7C_2$	$x^{(6)}+5x^{(5)}$ $+6x^{(4)}+x^{(3)}$

$(p;q)$	Normal Form	Tree Form	Cyclo Form	Factorial Form
$(6;10)$  	$x^6-10x^5+41x^4-84x^3+84x^2-32x$ $=x(x-1)(x-2)^2(x^2-5x+8)$	$T_6-5T_5+11T_4-11T_3+4T_2$	$C_6-4C_5+6C_4-7C_2$	$x^{\{6\}}+5x^{\{5\}}$ $+6x^{\{4\}}+2x^{\{3\}}$
$(6;10)$  	$x^6-10x^5+40x^4-79x^3+76x^2-28x$ $=x(x-1)(x-2)^2(x^2-5x+7)$	$T_6-5T_5+10T_4-9T_3+3T_2$	$C_6-4C_5+5C_4+C_3-6C_2$	$x^{\{6\}}+5x^{\{5\}}$ $+5x^{\{4\}}+x^{\{3\}}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;10)$  	$x^6-10x^5+40x^4-80x^3+79x^2-30x$ $=x(x-1)(x-2)(x-3)(x^2-4x+5)$	$T_6-5T_5+10T_4-10T_3+4T_2$	$C_6-4C_5+5C_4-6C_2$	$x^{(6)}+5x^{(5)}+5x^{(4)}$
$(6;10)$  	$x^6-10x^5+39x^4-74x^3+68x^2-24x$ $=x(x-1)(x-2)^3(x-3)$	$T_6-5T_5+9T_4-7T_3+2T_2$	$C_6-4C_5+4C_4+2C_3-5C_2$	$x^{(6)}+5x^{(5)}+4x^{(4)}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;10)$  	$x^6-10x^5+39x^4-74x^3+68x^2-24x$ $=x(x-1)(x-2)^3(x-3)$	$T_6-5T_5+9T_4-7T_3+2T_2$	$C_6-4C_5+4C_4+2C_3-5C_2$	$x^{(6)}+5x^{(5)}+4x^{(4)}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;10)$  	$x^6 - 10x^5 + 38x^4 - 68x^3 + 57x^2 - 18x$ $= x(x-1)^2(x-2)(x-3)^2$	$T_6 - 5T_5 + 8T_4 - 4T_3$	$C_6 - 4C_5 + 3C_4 + 4C_3 - 4C_2$	$x^{\{6\}} + 5x^{\{5\}} + 3x^{\{4\}}$
$(6;10)$ 	$x^6 - 10x^5 + 35x^4 - 50x^3 + 24x^2$ $= x^2(x-1)(x-2)(x-3)(x-4)$	$T_6 - 5T_5 + 5T_4 + 5T_3 - 6T_2$	$C_6 - 4C_5 + 10C_3 - C_2$	$x^{\{6\}} + 5x^{\{5\}}$
$(6;11)$ 	$x^6 - 11x^5 + 49x^4 - 109x^3 + 118x^2 - 48x$ $= x(x-1)(x-2)(x-3)(x^2-5x+8)$	$T_6 - 6T_5 + 15T_4 - 18T_3 + 8T_2$	$C_6 - 5C_5 + 9C_4 - 3C_3 - 10C_2$	$x^{\{6\}} + 4x^{\{5\}} + 4x^{\{4\}}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;11)$  	$x^6 - 11x^5 + 49x^4 - 108x^3 + 115x^2 - 46x$ $= x(x-1)(x-2)(x^3 - 8x^2 + 23x - 23)$	$T_6 - 6T_5 + 15T_4 - 17T_3 + 7T_2$	$C_6 - 5C_5 + 9C_4 - 2C_3 - 10C_2$	$x^{(6)} + 4x^{(5)}$ $+4x^{(4)} + x^{(3)}$
$(6;11)$  	$x^6 - 11x^5 + 48x^4 - 103x^3 + 107x^2 - 42x$ $= x(x-1)(x-2)(x-3)(x^2 - 5x + 7)$	$T_6 - 6T_5 + 14T_4 - 15T_3 + 6T_2$	$C_6 - 5C_5 + 8C_4 - C_3 - 9C_2$	$x^{(6)} + 4x^{(5)} + 3x^{(4)}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;11)$   	$x^6-11x^5+47x^4-97x^3+96x^2-36x$ $=x(x-1)(x-2)^2(x-3)^2$	$T_6-6T_5+13T_4-12T_3+4T_2$	$C_6-5C_5+7C_4+C_3-8C_2$	$x^{(6)}+4x^{(5)}+2x^{(4)}$
$(6;11)$ 	$x^6-11x^5+45x^4-85x^3+74x^2-24x$ $=x(x-1)^2(x-2)(x-3)(x-4)$	$T_6-6T_5+11T_4-6T_3$	$C_6-5C_5+5C_4+5C_3-6C_2$	$x^{(6)}+4x^{(5)}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;12)$ 	$x^6-12x^5+58x^4-137x^3+154x^2-64x$ $=x(x-1)(x-2)(x^3-9x^2+29x-32)$	$T_6-7T_5+20T_4-25T_3+11T_2$	$C_6-6C_5+13C_4-5C_3-14C_2$	$x^{(6)}+3x^{(5)}$ $+3x^{(4)}+x^{(3)}$
$(6;12)$ 	$x^6-12x^5+57x^4-132x^3+146x^2-60x$ $=x(x-1)(x-2)(x-3)(x^2-6x+10)$	$T_6-7T_5+19T_4-23T_3+10T_2$	$C_6-6C_5+12C_4-4C_3-13C_2$	$x^{(6)}+3x^{(5)}+2x^{(4)}$
$(6;12)$  	$x^6-12x^5+56x^4-126x^3+135x^2-54x$ $=x(x-1)(x-2)(x-3)^3$	$T_6-7T_5+18T_4-20T_3+8T_2$	$C_6-6C_5+10C_4-C_3-11C_2$	$x^{(6)}+3x^{(5)}+x^{(4)}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
$(6;12)$ 	$x^6-12x^5+55x^4-120x^3+124x^2-48x$ $=x(x-1)(x-2)^2(x-3)(x-4)$	$T_6-7T_5+17T_4-17T_3+6T_2$	$C_6-6C_5+10C_4-11C_2$	$x^{(6)}+3x^{(5)}$
$(6;13)$ 	$x^6-13x^5+66x^4-161x^3+185x^2-78x$ $=x(x-1)(x-2)(x-3)(x^2-7x+13)$	$T_6-8T_5+24T_4-31T_3+14T_2$	$C_6-7C_5+15C_4-6C_3-16C_2$	$x^{(6)}+2x^{(5)}+x^{(4)}$
$(6;13)$ 	$x^6-13x^5+65x^4-155x^3+174x^2-72x$ $=x(x-1)(x-2)(x-3)^2(x-4)$	$T_6-8T_5+23T_4-28T_3+12T_2$	$C_6-7C_5+15C_4-5C_3-16C_2$	$x^{(6)}+2x^{(5)}$

$(p;q)$	Normal Form	Tree Form	Cycle Form	Factorial Form
(6; 14) 	$x^6-14x^5+75x^4-190x^3+224x^2-96x$ $=x(x-1)(x-2)(x-3)(x-4)^2$	$T_6-9T_5+29T_4-39T_3+18T_2$	$C_6-8C_5+20T_4-10C_3-21C_2$	$x^{(6)}+x^{(5)}$
(6; 15) 	$x^6-15x^5+85x^4-225x^3+274x^2-120x$ $=x(x-1)(x-2)(x-3)(x-4)(x-5)$	$T_6-10T_5+35T_4-50T_3+24T_2$	$C_6-9C_5+25C_4-15C_3-26C_2$	$x^{(6)}$