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Non-Planar AdS/CFT from Group Representation Theory

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Abstract

In this thesis we explore certain limits of the AdS/CFT correspondence for integrability. This is done by calculating the action of the dilatation operator on operators known as restricted Schur polynomials, which are AdS/CFT dual to D3-branes known as giant gravitons. We focus on operators in $\mathcal{N} = 4$ super-Yang-Mills theory, which is dual to type IIB string theory on an $AdS_5 \times S^5$ background. We find that, in various cases, this theory is integrable in a large N non-planar limit.

Declaration

I declare that this thesis is my own, unaided work. It is being submitted for the Degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

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Stephanie Smith

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Date

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I would like to thank Garreth James Kemp and Badr Awad Elseid Mohammed for their collaboration, and Robert de Mello Koch for being an excellent supervisor.

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1 Introduction

The AdS/CFT correspondence describes an equivalence between theories with gravity in $d+1$ dimensions and theories without gravity in d dimensions. The duality was motivated by the study of D-branes, which are extended objects in string theory on which open strings can end. The correspondence provides a concrete non-perturbative definition of quantum gravity, the theory that unifies quantum field theory and general relativity into a single consistent framework. In this work we study this duality, focusing on the duality between $\mathcal{N} = 4$ super Yang-Mills (SYM) theory and type IIB string theory on an $AdS_5 \times S^5$ background.

Our primary goal is to look for signals that there are limits of the theory in which it is integrable. For a system to be integrable, it needs to posses as many conserved quantities as the number of its degrees of freedom. The dynamics of an integrable system simplifies dramatically and hence such limits would provide important instances in which we can hope to gather detailed information about the AdS/CFT duality.

It has previously been shown that $\mathcal{N} = 4$ SYM is integrable in the planar limit. Initial studies of non-planar corrections lead to claims that integrability is a property only of the planar limit. This has proved to be false [17]. Indeed, certain large N but non-planar limits of the su(2) sector of the theory seem to enjoy integrability[1]. A major goal of this work is to further

[1]We do not disagree with the results of [17]. The study of [17] considered non-planar diagrams that provide the first sub-leading corrections to the large N limit. The non-

our understanding of non-planar integrability by both extending it to other sectors and to higher loops. In the following sections we shall present the background information and tools needed to do this.

1.1 The AdS/CFT Correspondence

In [15] Maldacena conjectured the existence of a correspondence between string theories in a bulk Anti de Sitter space and conformally invariant field theories on the boundary of the Anti-de Sitter space. The correspondence claims that gauge theories at strong coupling behave like weakly coupled string theories and that string theories at strong coupling correspond to weakly coupled gauge theories. The duality thus gives us the tools to do difficult calculations in gravity theories by looking at their corresponding conformal field theories, and vice versa. This provides a framework to study strongly coupled theories which cannot be solved perturbatively. An example of a calculation that can be performed to check the AdS/CFT correspondence is the calculation of correlation functions in the field theory. From this two point function we obtain a scaling 'dimension', which should match the energy of the dual state in the string theory.

As mentioned, if the theory enjoys integrability, it is possible to perform detailed and general studies and checks of the correspondence. There are powerful theorems which forbid integrability in $3+1$ dimensional quantum field theories. Here, when we talk of integrability in $\mathcal{N}=4$ super-Yang-Mills

planar diagrams we consider contribute at the leading order of the large N expansion.

theory, we are referring to the fact that the planar dilatation operator can be identified with the Hamiltonian of an integrable system. This will be discussed in detail later on. The fact that such a map exists allows us to compute the planar scaling dimensions to all loops.

In this work we focus on this version of integrability in the $\mathcal{N} = 4$ super Yang-Mills (SYM) theory in four dimensional Minkowski space. As a consequence of the fact that this gauge theory is AdS/CFT dual to type IIB string theory on the $AdS_5 \times S^5$ background, our studies may shed light on quantum gravity in a negatively curved space-time. Concretely we will focus on operators whose bare dimension grows as N in the large N limit. To capture the large N limit for these operators, we need to sum many non-planar diagrams, which corresponds to non-perturbative physics in the dual string theory. The particular set of gauge invariant operators that we focus on are known as restricted Schur polynomials. They are AdS/CFT dual to D3 branes with a spherical world volume, known as giant gravitons.

Giant gravitons are branes extended in either the AdS_5 or S^5 space of an $AdS_5 \times S^5$ background. They expand with a radius that is proportional to the square root of their angular momentum, with a maximum angular momentum proportional to N. They are classically stable due to the Lorentz-like force which grows with angular momentum, balancing their world-volume tension. Giant gravitons which wrap around an S^3 in AdS_5 or S^5 are known as AdS and sphere gravitons respectively. The sphere branes have a maximum size governed by the size of S^5. This translates into an upper bound

for their angular momentum.

One of the reasons why the correspondence between $\mathcal{N} = 4$ SYM and IIB string theory on $AdS_5 \times S^5$ is so simple, is that this gravity-gauge pair enjoys invariance under a large collection of symmetries. These include supersymmetry, R-symmetry and of course conformal symmetry. Supersymmetry transformations are generated by super charges Q^I_α and $Q^{\dagger\, I}_{\dot{\alpha}}$. It describes a symmetry between fermions and bosons, and also Poincare invariance. Conformal symmetry consists of Poincare transformations, dilatations (scaling transformations), and special conformal transformations, which consist of an inversion followed by a translation and another inversion. Finally, R-symmetry is a global symmetry which is generated by R-charge and rotates superspace coordinates. The theory also has a local $U(N)$ gauge invariance.

The bulk AdS_5 space is a maximally symmetric solution to the Einstein equations with a negative cosmological constant. The isometry group of this space is the same as the conformal symmetry group of $\mathcal{N} = 4$ SYM. According to the AdS/CFT correspondence, the $\mathcal{N} = 4$ SYM 'lives' on the conformal boundary of the bulk.

The metric of AdS_5 is given by

$$ds^2 = \frac{r^2}{R^2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \frac{R^2}{r^2}dr^2 \tag{1.1}$$

R is the radius of curvature of the space. As $r \to \infty$, the conformal boundary of the space is reached. The geometry of the conformal boundary is described by a collection of Minkowski metrics all related by conformal rescalings. The

fact that the space-time is negatively curved has dramatic consequences. Any particle released with any velocity, from the center of the space, will follow a geodesic which returns again to the center ($r = 0$) of the space. A crude but useful way to state things is that we are studying gravity in a box.

1.2 $\mathcal{N} = 4$ Super Yang-Mills Theory

$\mathcal{N} = 4$ super Yang-Mills theory is a gauge invariant supersymmetric conformal field theory. We will study the theory with gauge group $U(N)$. This theory enjoys the maximum possible supersymmetry allowed for a quantum field theory in $3 + 1$ dimensions. Since in addition this theory has an AdS/CFT dual, it is a simple toy model of strongly coupled field theory.

As a consequence of the conformal invariance enjoyed by the theory, the S-matix is not a good observable. We can still, however, compute correlation functions. These define scaling dimensions which can be observed. Indeed, the scaling dimensions are among the central observables of interest to us in this theory.

The Lagrangian of $\mathcal{N} = 4$ SYM is as follows

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}Tr(F^{\mu\nu}F_{\mu\nu}) - \frac{1}{2}Tr(D^{\mu}\phi^{i}D_{\mu}\phi_{i}) \\ &\quad -\frac{1}{4}g_{YM}^{2}Tr\left(\left[\phi^{i},\phi^{j}\right]\left[\phi_{i},\phi_{j}\right]\right) + Tr(\dot{\psi}_{\dot{\alpha}}^{a}\sigma_{\mu}^{\dot{\alpha}\beta}D^{\mu}\psi_{\beta a}) \\ &\quad -\frac{ig_{YM}}{2}Tr(\psi_{\alpha a}\sigma_{i}^{ab}\epsilon^{\alpha\beta}\left[\phi^{i},\psi_{\beta b}\right]) - \frac{ig_{YM}}{2}Tr(\dot{\psi}_{\dot{\alpha}}^{a}\sigma_{ab}^{i}\epsilon^{\dot{\alpha}\dot{\beta}}\left[\phi_{i},\dot{\psi}_{\dot{\beta}}^{b}\right])\end{aligned} \quad (1.2)$$

Present in this Lagrangian are Lorentz scalar fields, ϕ^i, gauge fields, A_μ

and fermion fields ψ and $\dot{\psi}$. The indices $\alpha,\ \dot{\alpha},\ \beta, \dot{\beta} = 1, 2$ are left and right handed spinor indices which run over the fermion states. $i, j = 1 \ldots 6$ run over the six scalar fields in the gauge theory. The scalar fields are $N \times N$ matrices, transforming in the adjoint representation of $U(N)$. We can obtain $\mathcal{N} = 4$ SYM theory as the dimensional reduction of $\mathcal{N} = 1$ SYM theory in ten dimensions. In the ten dimensional string theory, the index μ runs over space-time indices 0 to 3, and i and j run over the remaining six space dimensions, on which we reduce.

Physical observables in a gauge theory of matrix fields include the gauge invariant operators made up of products of traces of the fields. A basis for these gauge invariant operators that is particularly useful for the study of this thesis is provided by the restricted Schur polynomials. Restricted Schur polynomials are built out of the six real scalar fields ϕ_i of $\mathcal{N} = 4$ SYM. They are labelled by representations of both $U(N)$ the unitary group and S_n, the group of permutations, or the symmetric group. The unitary and symmetric groups are related by Schur-Weyl duality, which implies that Young diagrams label irreducible representations of both groups and further, that their characters can be related. This relation between their characters is the Schur polynomial. Schur-Weyl duality will play an important role in what follows.

In this work, the spectrum of mass scaling dimensions of restricted Schur polynomials will be studied, allowing us to determine whether integrability is present. Remarkably, we find the anomalous dimensions of these complicated operators are determined by a set of decoupled oscillators.

1.3 The Large N and Planar Limits

1.3.1 The Large N Limit

The large N limit refers to the limit of $U(N)$ gauge theories where $N \to \infty$. In [1], 't Hooft proposed that quantum fluctuations in matrix field theories should be of order $\mathcal{O}(\frac{1}{N^2})$. In standard quantum field theories $\hbar$ can be traded for the coupling constant which runs as the scale of the quantum field theory is changed. In contrast to this, $\frac{1}{N^2}$ does not run. When N is taken to be large, the fluctuations approach zero and we are left with a classical theory. The large N limit of $\mathcal{N} = 4$ SYM is, according to AdS/CFT, the classical IIB string theory on the $AdS_5 \times S^5$ background. $\hbar$ corrections in the quantum gravity are dual to $\frac{1}{N^2}$ corrections in the gauge theory.

The coupling parameters of IIB string theory and $\mathcal{N} = 4$ SYM are given by g_s and g_{YM} respectively. The relation between the string coupling and the Yang-Mills coupling is summarized by the following relations

$$\left(\frac{R}{l_s}\right)^4 = 4\pi g_{YM}^2 N = 4\pi g_s N \tag{1.3}$$

Here R is the radius of curvature of the bulk $AdS_5 \times S^5$ space and l_s is the string length.

The t'Hooft coupling is given by

$$\lambda = g_{YM}^2 N \tag{1.4}$$

Expansions can simultaneously be performed in both λ and $\frac{1}{N}$, which means we have a theory with two expansion parameters. In the gauge theory, a small value for λ indicates a weak coupling and a large value indicates strong coupling. For small g_{YM}, $N \to \infty$ and $\lambda \to \infty$ is known as the low energy or decoupling limit of the gravity theory. In this limit, open and closed strings decouple from each other, closed strings are non-interacting and branes are a boundary condition for open strings. A remarkable consequence of the AdS/CFT correspondence is that, by studying a classical string theory, we can learn about strongly coupled gauge theories that cannot be subjected to perturbative approximations.

An extremely important feature of the large N limit, as we explain below, is that it allows us to equate the expectation value of a product of operators to the product of the expectation values of the operators. This is a key idea.

Suppose we have a field theory containing some matrix field X. At large N, for a collection of n operators $\mathcal{O}_i$, each built from X's, the following relation can be shown

$$< \mathcal{O}_1 \mathcal{O}_2 \dots \mathcal{O}_n >=< \mathcal{O}_1 >< \mathcal{O}_2 > \dots < \mathcal{O}_n > \tag{1.5}$$

Here it is assumed that n is of order one and the number of matrix fields X in each operator is also of order one. We will return to this assumption below and see what the consequences of relaxing it are.

The equation quoted above arises from evaluating the expectation value of products of operators, each of which is a trace of a product of matrices X.

The expectation value of an operator can be rewritten as

$$< \mathcal{O}_1 >= \sum_i \mu_i \mathcal{O}_1(i) \tag{1.6}$$

Here i is the index running over possible values of the field $X = X(i)$, μ_i is the probability that X has the value $X(i)$ and $\mathcal{O}_1(i)$ is the value of operator $\mathcal{O}_1$ when $X = X(i)$. We also have the normalization condition $\sum_i \mu_i = 1$.

The correlation function of this operator can be calculated using the path integral method of QFT as follows

$$< \mathcal{O} >= \int [dX]\, \mathcal{O}\, e^{-S} \tag{1.7}$$

The integration measure is given by

$$[dX] = \prod_{i=1}^{N} dX_{ii} \prod_{i<j} d\Re e(X_{ij}) d\Im m(X_{ij}) \tag{1.8}$$

The first product represents the integrals over diagonal elements of matrix X, and the second and third represent the integrals over the real and imaginary parts of the upper right triangular matrix elements. We do not need to include those elements below the diagonal in the lower left triangle of the matrix since we have a hermitian matrix, $X = X^\dagger$. There are N diagonal elements, and $\frac{N(N-1)}{2}$ complex upper diagonal elements, giving a total of N^2 integrals contained in $[dX]$.

In this example, we shall use the action $S = \frac{1}{2} tr(X^2)$. This allows us to exploit Gaussian integration methods in order to evaluate $< \mathcal{O} >$. As a simple example, suppose we have $< \mathcal{O} >=< tr(X^2) >$

$$< tr(X^2) > \;\; = \;\; \frac{\int [dX]\, X_{ij} X_{ji} e^{-\frac{1}{2} tr(X^2)}}{\int [dX]\, e^{-\frac{1}{2} tr(X^2)}}$$

$$= \frac{d}{dJ_{ji}}\frac{d}{dJ_{ij}} I\left[J\right]\Big|_{J=0} \tag{1.9}$$

$I\left[J\right]$ is the generating function given by

$$I\left[J\right] = \frac{\int \left[dX\right] e^{-\frac{1}{2}tr(X^2)} e^{tr(JX)}}{\int \left[dX\right] e^{-\frac{1}{2}tr(X^2)}} \tag{1.10}$$

J is a source coupled to the field X. To evaluate I we use the method of completing the square and then integrate

$$\begin{aligned} \int \left[dX\right] e^{-\frac{1}{2}tr(X^2)} &= \sqrt{2\pi}^{N^2} \\ \int \left[dX\right] e^{-\frac{1}{2}tr(X^2)} e^{tr(JX)} &= e^{\frac{1}{2}tr(J^2)} \sqrt{2\pi}^{N^2} \end{aligned} \tag{1.11}$$

We have

$$I\left[J\right] = e^{\frac{1}{2}tr(J^2)}$$

This factor of $\sqrt{2\pi}^{N^2}$ cancels with the normalization factor in the original correlator, so we are left with the following calculation for the expectation value

$$\begin{aligned} < tr(X^2) > &= \frac{d}{dJ_{ji}}\frac{d}{dJ_{ij}} I\left[J\right]\Big|_{J=0} \\ &= \frac{d}{dJ_{ji}}\frac{d}{dJ_{ij}} \left[e^{\frac{1}{2}tr(J^2)}\right]\Big|_{J=0} \\ &= \frac{d}{dJ_{ji}} \left[J_{ji} e^{\frac{1}{2}tr(J^2)}\right]\Big|_{J=0} \\ &= \delta_{jj}\delta_{ii} \\ &= N^2 \end{aligned} \tag{1.12}$$

If we repeat the process for $< tr(X^2)tr(X^2) >$, we find that

$$< tr(X^2)tr(X^2) >= N^4 + 2N^2 \tag{1.13}$$

In general, in quantum field theory, the value of a correlation function is the classical value of the operator with quantum fluctuations added on. This is given by something of the form

$$\int [dX]\, \mathcal{O} e^{-S} = \mathcal{O}_{Classical} + (\#)\hbar + \cdots \tag{1.14}$$

When we say that the large N limit is a classical limit with $\frac{1}{N^2}$ playing the role of $\hbar$, we mean

$$\int [dX]\, \mathcal{O} e^{-S} = \mathcal{O}_{Classical} + (\#)\frac{1}{N^2} + \cdots \tag{1.15}$$

If the equation for $< tr(X^2)tr(X^2) >$ is factorized in this way, we have

$$< tr(X^2)tr(X^2) >= N^4(1 + \frac{2}{N^2})$$

Set N to be large, so that we only need the leading term

$$< tr(X^2)tr(X^2) > \sim N^4$$

We compare this to the previous result:

$$< tr(X^2) >< tr(X^2) >= N^4 =< tr(X^2)tr(X^2) > \tag{1.16}$$

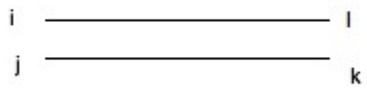
In general one can check that, at the leading order at large N

$$\sum_i \mu_i \mathcal{O}_1(i)\mathcal{O}_2(i)\cdots\mathcal{O}_n(i) = \sum_{i_1} \mu_{i_1}\mathcal{O}_1(i_1)\sum_{i_2}\mu_{i_2}\mathcal{O}_2(i_2)\cdots\sum_{i_n}\mu_{i_n}\mathcal{O}_n(i_n) \tag{1.17}$$

As mentioned, this formula assumes that n is $\mathcal{O}(1)$ and the number of X's in each operator is $\mathcal{O}(1)$. This equation is only satisfied if we set one of the probabilities, for example μ_{i^*} equal to 1, implying that $\mu_i = 0$ for all $i \neq i*$. This means that we have $X = X(i^*)$ with a probability of 1, so we know the state of the field X. Only one X configuration contributes to the expectation value, implying that the system is in a classical limit.

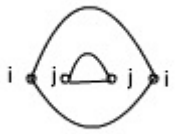
1.3.2 The Planar Limit

In matrix field theories, Feynman diagrams are drawn by joining pairs of parallel lines whose end points are labelled by matrix indices. These are known as ribbons. In order for these diagrams to reproduce correlators calculated using the path integral method in a particular theory, they must adhere to a specific set of rules. Suppose we look at the propagator of some theory of matrices: $< M_{ij}M_{kl} >= c\delta_{il}\delta_{jk}$. Here c is a factor associated with the propagator as calculated using the path integral method outlined in the previous section. For instance, if we had $S = \frac{\omega}{2}tr(X^2)$, where ω is a constant parameter, after normalization we would have $c = \omega$. Of course, the δ's quoted above also arise from the Gaussian integral calculation, and complete the Feynman rule for the propagator. We construct a Feynman diagram for this propagator by drawing two labelled dots for each matrix, then joining the dots with a ribbon:



This two point correlator is not physical since the operator is not constructed from a trace of a product of fields. Considering a gauge invariant product of operators, constructed from traces of products of fields, to compute a correlator we repeat the above process, joining like indices and joining ribbon ends to ribbon ends. In $U(N)$ and $SU(N)$ gauge theories, the ribbons must not twist. For example, consider the following correlator: $< tr(M^2) >= M_{ij}M_{ji} = c\delta_{ii}\delta_{jj}$. Here i and j are summed over, so $< tr(M^2) >= cN^2$, where N is again the rank of the gauge group. In graph

form this is:



It must be noted that all the different diagrams that can be formed by connecting ribbons in different ways must be summed. To do this, draw the dots labeling the matrices, connect the like indices to indicate a trace, and then connect ribbons in all possible ways in which they don't twist. This allows loop corrections to be added to the value of the correlation function. Diagrams that can be drawn on topologically planar surfaces, such as spheres, with no ribbons crossing are known as planar diagrams. For example, the diagram above can be drawn on a planar surface without any ribbons crossing each other. The planar diagrams come with the largest powers of N. Sub-leading diagrams do have ribbons crossing when they are drawn on a sphere. When this arises the diagrams are drawn on surfaces which have handles, such as toruses, and are known as nonplanar diagrams. On these new surfaces, the ribbons do not cross.

The relation between N and the surface on which the ribbon graph is drawn to ensure no ribbons cross is simple. We will develop this relation in the next few paragraphs. We count the number of closed loops in the diagram, and this gives us the order in N. Diagrams of leading order in N correspond to planar diagrams, and corrections which are not of leading order in N correspond to nonplanar diagrams. In the above planar diagram,

the only contribution to $< tr(M^2) >$, there is a factor of N^2 and the diagram consists of two closed loops. Here is an example of a nonplanar diagram



This is a sub-leading term in the large N expansion, of order N, contributing to the correlator $< tr(M^4) >$. This diagram can be drawn on a torus with a single hole.

Each Feynman diagram of the matrix model is a triangulation on some such surface. The order in $\frac{1}{N^2}$ is related to the topology of the surface that the ribbon graph triangulates, and performing a loop expansion corresponds to performing a $\frac{1}{N^2}$ expansion. The surfaces associated to the diagrams are very naturally interpreted as the world sheet of a string.

Each propagator in the gauge theory is represented by a pair of lines corresponding to indices of the fields represented. Whenever an index loop closes, the ribbon diagram picks up a factor of N from summing over the values each index can take. It is possible to show the relation between the large N expansion and the topological expansion of the surface on which the ribbon diagram is found by using Euler's theorem for polyhedrons [3]. This relates the number of vertices, edges and faces to the genus of the surface on which the polyhedron is drawn.

Consider a theory that has both cubic and quartic interactions. For a general diagram, there are P propagators, V vertices and I index loops. There are V_3 three point vertices, V_4 four point vertices, such that the total number of vertices is given by

$$V = V_3 + V_4$$

We will focus on vacuum diagrams. In this case each end of a ribbon must connect to a vertex so that

$$2P = \sum_n nV_n \tag{1.18}$$

If we then attach a small surface to every index loop in the diagram in such a way that it resembles a polyhedron, we can use Euler's theorem, which states

$$I - P + V = 2 - 2H \tag{1.19}$$

is a topological invariant of the resulting surface. Here H is the number of handles, or genus, of the surface on which the polyhedron is drawn. A sphere or plane has a value of $H = 0$, a torus has $H = 1$ etc.

It is now possible to show that diagrams with leading order in N correspond to planar diagrams, or diagrams drawn on surfaces with genus $H = 0$. Set each three point vertex to have a coupling factor of g, and each four point vertex to have a factor of g^2. Each index loop has a factor of N, giving us

$$g^{V_3} g^{2V_4} N^I = g^{2P-2V} N^I = (g^2 N)^{F+2H-2} N^{2-2H} \tag{1.20}$$

If we send $N \to \infty$ and keep $g^2 N$ fixed, it can be seen that the order in N associated to the diagram is given by N^{2-2H}. This means that diagrams

that are drawn on surfaces with genus $H = 0$ provide the leading order in N, which is N^2.

Nonplanar corrections in matrix theories are analogous to $\hbar$ corrections in string theories. We can think of surfaces with handles (such as toruses) as stringy quantum corrections, and each handle on a surface carries a factor of $\mathcal{O}(\frac{1}{N^2})$. The motivation for expansion in the parameter N arose in t'Hooft's studies of strongly coupled gauge theories, such as QCD [1], where there is no possibility for a perturbative expansion of the theory. He proposed this expansion in terms of the number of colors -or rank of the gauge group- as an alternative to expanding in terms of the coupling constant of the theory.

It was thought that the planar and large N limits coincide. In the case of operators with dimension of order one, sub-leading terms in N and nonplanar diagrams are suppressed in the same way, and the two do indeed coincide. It has more recently been shown [18] that the large N and planar limits are not necessarily the same. The correlation functions of operators with bare dimension $\mathcal{O}(\sqrt{N})$ or larger, have combinatoric factors associated with quantum corrections so large that the sub-leading diagrams are no longer suppressed. In the case of these operators, the classical limit that is obtained is now a large N but non-planar limit.

2 Introduction to Group Theory

In this chapter we will discuss some important group theoretical concepts that will be needed to understand the rest of this work. The concept of Schur-Weyl duality will play a central role in what follows. The Schur-Weyl duality is the duality between two subgroups of the general linear group on $(\mathbb{C}^p)^{\otimes n}$, namely the group action of the unitary group $U(p)$ and the canonical representation of the symmetric group S_n. It describes a one-to-one association between irreducible representations of $U(p)$ and S_n. In order to exploit the Schur-Weyl duality, we need an understanding of some S_n and $U(p)$ representation theory. Much of the theory presented in both chapters two and three was developed in [27], [22], [23], [25], [51], [13] and [14].

2.1 Some S_n Group Representation Theory

2.1.1 S_n and Young Diagrams

The symmetric group, S_n is the group isomorphic to permutations of n objects. This group contains $n!$ elements, one for each possible permutation of n objects. These elements can further be partitioned into conjugacy classes. For example, the group S_3 has the following elements:

$$S_3 = \{1, (12), (23), (13), (123), (132)\}$$

The full expression of a group element includes cycles of all lengths, including one cycles, for example $(12) = (12)(3)$. For simplicity the one cycles are excluded. The notation (123) indicates that the positions of the numbers

are swapped as follows:

$$1 \rightarrow 2 \rightarrow 3 \rightarrow 1$$

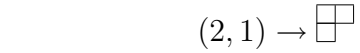
$(12), (23), (13)$ fall into the conjugacy class consisting of a one-cycle and a two-cycle. Elements in the same conjugacy class have identical cycle structure, that is, they have the same number of one cycles, two cycles, etc.

The complete set of irreducible representations of S_n can be labelled by Young diagrams with n boxes. Valid Young diagrams must strictly have the following properties:

- The left borders of all rows must be aligned with each other.
- The top borders of all columns must be aligned.
- Each row may not be longer than the one above,
- Each column may not be longer than the one to its left.

As stated above elements that have the same cycle structure, for example (123) and (132), belong to the same conjugacy classes. The cycle structure can be thought of as a partition of n. These different partitions of n are also represented by Young diagrams, with partitions of n corresponding to row structures. Thus, Young diagrams can be used to label both conjugacy classes and irreducible representations. Consider the group S_3. Denote $(1, 1, 1)$ as three one cycles, or the identity, $(2, 1)$ as the product of a two cycle and a one cycle and (3) as a three cycle. These partitions correspond to the following Young diagrams

$$(1, 1, 1) \rightarrow \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}$$

$(2,1) \rightarrow$ 

$(3) \rightarrow$ 

In general, we can write down the number of boxes in each row of the diagram using the notation $(r_1, r_2, \ldots, r_p)$. The shape of each diagram with some integer number of boxes n gives us a particular partition of n. In terms of the row lengths

$$n = \sum_{i=1}^{p} r_i \tag{2.1}$$

Young diagrams also label irreducible representations of $U(N)$. In this context, each box in a Young diagram can be labelled with a factor $N+i-j$, known as the weight of the box. i is the column number and j is the row number of the box. N is the factor associated with the box in the top left hand corner of the diagram; 1 is added to every box to the right in the horizontal direction and 1 is subtracted from each box as one moves down the columns. For example, if $N = 3$ the factors are

$$R = \begin{array}{|c|c|c|c|c|c|c|} \hline 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 2 & 3 & 4 & 5 & 6 \\ \cline{1-5} \end{array}$$

We denote the product of all factors in diagram R by f_R.

Another number associated with Young diagrams is the hooks of the diagram. This plays a role in both S_n and $U(N)$ group theory. To obtain

the hook length of a given box, draw a line starting from the box towards the bottom of the page out of the diagram. Draw another line from the box towards the right of the page until you exit the diagram. The length of the hook of the particular box is the number of boxes through which the lines pass. We concentrate on the factor found by taking the product of the hook lengths of all boxes in a particular diagram, denoted $hooks_R$. For the diagram above, the hooks would be filled in as follows:

$$R = \begin{array}{|c|c|c|c|c|c|c|}\hline 8&7&6&5&4&2&1\\\hline 5&4&3&2&1\\\cline{1-5}\end{array}$$

2.1.2 Basis Construction and Young's Orthogonal Representation

One can construct a basis for the carrier space of representation R of S_n, and the matrices representing group elements acting on this space, $\Gamma_R(\sigma)$, using Young diagrams. The resulting Young-Yamanouchi basis is an orthonormal basis in which each labeled diagram corresponds to a vector in the basis of the carrier state. It is constructed by labeling the boxes in a Young diagram from 1 to n in such a way that if all boxes labelled from 1 to k, $k < n$ are removed, a valid diagram remains. For example, allowed labels of a particular S_3 representation in this basis are

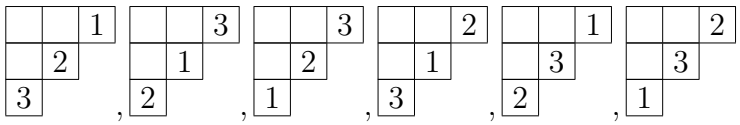
$$\begin{array}{|c|c|}\hline 3&1\\\hline 2\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 3&2\\\hline 1\\\cline{1-1}\end{array}$$

The labelings

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}, \qquad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}, \qquad \begin{array}{|c|c|}\hline 2&1\\\hline 3\\\cline{1-1}\end{array}, \qquad \begin{array}{|c|c|}\hline 2&3\\\hline 1\\\cline{1-1}\end{array}$$

are not allowed. Note that when a Young diagram is labeled, it is referred to as a Young tableau.

Before describing how to construct the matrices $\Gamma_R(\sigma)$, we will introduce another construction that will be utilized. This is the notion of partially labelled Young tableaux. This construction is useful when working with representations of $S_n \times S_m$ that have been subduced from representations of S_{n+m}. The irreducible representations of $S_n \times S_m$ are labeled by two Young diagrams (r, s), one for S_n and one for S_m. In this basis the m boxes from S_m are labelled with integers from 1 to m in such a way that removing the boxes leaves a valid diagram r. The same set of boxes is removed each time. Every valid labeling of these boxes removed constitutes a state in an $S_n \times S_m$ irreducible representation with r the Young diagram for S_n. For example, suppose we are working with S_{3+3}. The partially labelled Young tableaux for a particular representation would be



In general, diagrams with m boxes labelled are a collection of states that give us the basis for $S_n \times (S_1)^m$, from which we will build $S_n \times S_m$ representations later on. It is clear that the Young diagrams labeling representations of S_n are found by removing the labelled boxes.

Young's orthogonal representation is most easily described by a rule for constructing matrix representations for S_n elements. The rule can be stated by giving the action on Young-Yamanouchi states of adjacent two-cycle per-

mutations, given by $(i, i+1)$. Only the rule for S_n elements of this form is stated since these two-cycles generate all other elements in the group. Denote the factor of a box labelled i in Young-Yamanouchi basis c_i and let $\hat{R}$ be a Young tableau with the same shape as diagram R. Let $\hat{R}_{ij}$ be the tableau with boxes i and j swapped around. The action of group elements on this basis is then

$$\Gamma_R(i, i+1)|\hat{R}>=\frac{1}{c_i - c_{i+1}}|\hat{R}>+\sqrt{1-\frac{1}{(c_i - c_{i+1})^2}}|\hat{R}_{i,i+1}> \tag{2.2}$$

A very particular large N limit that we will define shortly allows us to work with a simplified version of Young's orthogonal representation. To motivate the simplification, consider the action of a group element of S_{n+m} on some partially labelled Young tableau

$$\Gamma_R(1,2)|\begin{array}{|c|c|c|c|}\hline & & & 1\\\hline 2\\\cline{1-1}\end{array}>=\frac{1}{c_1 - c_2}|\begin{array}{|c|c|c|c|}\hline & & & 1\\\hline 2\\\cline{1-1}\end{array}>+\sqrt{1-\frac{1}{(c_1 - c_2)^2}}|\begin{array}{|c|c|c|c|}\hline & & & 2\\\hline 1\\\cline{1-1}\end{array}> \tag{2.3}$$

The simplification occurs for Young diagrams with p rows (p is order 1) with the total number of boxes given by $n + m \sim \mathcal{O}(N)$. In a general diagram with long rows, the difference in length between two particular rows is $\mathcal{O}(N)$. In this case, any two boxes i and j not in the same row will have a factor difference $|c_i - c_j| \sim \mathcal{O}(N)$. This means that the above equation can be simplified into two parts, one for states with boxes i and j in the same row k, denoted $|k_i, k_j>$, and one for i and j in different rows k and l denoted $|k_i, l_j>$

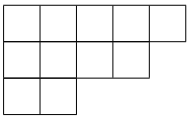
$$\Gamma_R(i, j)|k_i, k_j>=|k_i, k_j> \tag{2.4}$$

$$\Gamma_R(i,j)|k_i,l_j>=|k_j,l_i> \tag{2.5}$$

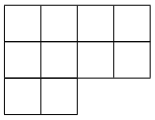
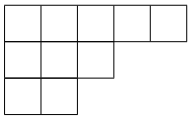
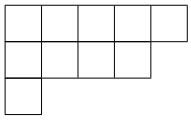
This simplification is a consequence of the fact that the factor $\frac{1}{c_i-c_j} \to 0$ at large N because $|c_i - c_j| \to \mathcal{O}(N)$.

2.1.3 Removing Boxes

A single box can be removed from an irreducible representation R of S_{n+m} in order to produce an irreducible representation of S_{n+m-1}, denoted R'_i. i labels the row from which the box is removed. This can be done by removing any box from R that still leaves a valid Young diagram. For example, suppose we have

$R =$ 

Boxes can be removed from this diagram in three possible ways:

$R'_1 =$  $R'_2 =$  $R'_3 =$ 

It is well known that if we restrict representation R to a S_{n+m-1} subgroup, representations R'_1, R'_2 and R'_3 are all subduced with unit multiplicity.

We denote $c_{RR'}$ as the factor of the box removed from R to obtain R'. For the example we are considering, the factors are

$$c_{RR'_1} = N + 4$$

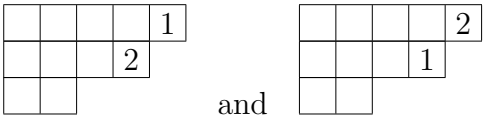
$$c_{RR'_2} = N + 2$$

$$c_{RR'_3} = N - 1$$

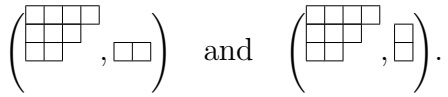
$$\sum_{R'} c_{RR'} = 3N + 5$$

We will need to consider the problem of determining which $S_n \times S_m$ irreducible representations (r, s) are subduced by a given S_{n+m} irreducible representation R. The only representations r that appear are those that can be obtained by removing boxes from R. There are as many distinct copies of r as there are valid ways to remove the m boxes from R to obtain r. The label s is a diagram built from the m boxes removed from R. Since there are many possible ways to get the label (r, s), we need a multiplicity label.

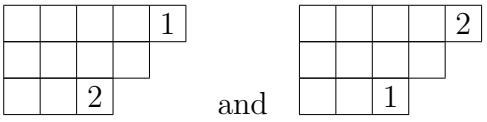
In the example above, suppose $m = 2$. Then possible labels (r, s) are obtained by removing two boxes from R in all possible ways, and arranging the removed boxes in all possible ways to obtain irreducible representations of S_2. Repeated shapes are distinguished by multiplicity labels. For the partially labelled Young diagrams



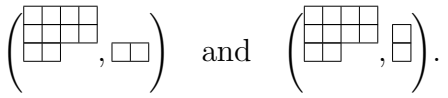
we can construct the following $S_n \times S_m$ states



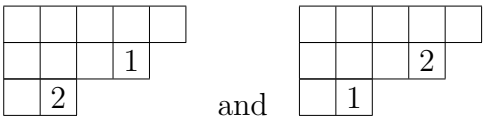
From the partially labelled Young diagrams



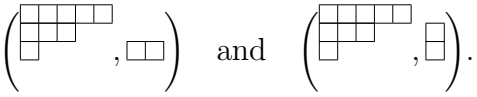
we can construct the following $S_n \times S_m$ states



From the partially labelled diagrams



we can construct the following $S_n \times S_m$ states



2.2 Constructing Restricted Schur Polynomials

There are now enough tools to explain the construction of the operators that will be used in all calculations of the action of the dilatation operator in this work. The operators are called restricted Schur polynomials.

We will study restricted Schur polynomials built out of the six scalar fields which take values in the $u(N)$ adjoint representation of $\mathcal{N} = 4$ SYM. These scalars are denoted ϕ_i, where $i = 1, \ldots, 6$. These six Hermittean matrix scalars are arranged into three complex valued fields as follows

$$\begin{aligned} Z &= \phi_1 + i\phi_2 \\ Y &= \phi_3 + i\phi_4 \\ X &= \phi_5 + i\phi_6 \end{aligned} \tag{2.6}$$

In this section, we shall focus on restricted Schur polynomials in the $su(2)$

sector, which are made up of the Z and Y fields.

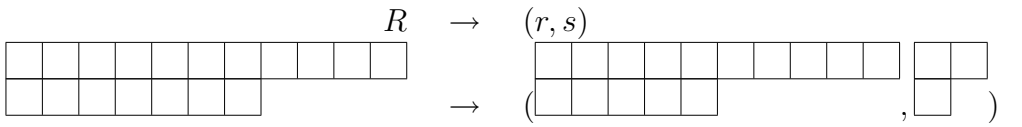
The general form of a restricted Schur polynomial is given by

$$\chi_{R,(r,s,)jk}(Z,Y) = \frac{1}{n!m!}\sum_{\sigma\in S_{n+m}} \chi_{R,(r,s)jk}(\sigma) Tr\left(\sigma Y^{\otimes m} Z^{\otimes n}\right) \tag{2.7}$$

This formula will be broken up into several parts in order to understand it

2.2.1 The Labeling $R,(r,s,)jk$

The label R is an irreducible representation of the symmetric group S_{n+m} in the form of a Young diagram with $n+m$ boxes. r and s are Young diagrams with n and m boxes respectively. r is an irreducible representation of the group S_n and s is an irreducible representation of S_m. The group S_{n+m} has a subgroup, $S_n \times S_m$, whose irreducible representations are labelled by (r,s). An irreducible representation of S_{n+m}, R, can subduce many different representations of $S_n \times S_m$. For a particular R, one such example follows



(2.8)

j and k label the multiplicity of (r,s). Since a particular representation of $S_n \times S_m$ can be subduced from R more than once, it is necessary to keep track of different copies of (r,s).

Finally, note that restricted Schur polynomials labelled by Young diagrams with long columns correspond to sphere giant gravitons, and long rows correspond to AdS giants.

2.2.2 $\chi_{R,(r,s)jk}(\sigma)$: The Restricted Character

The character of a group element is the trace over the matrix representation of the element. If $\Gamma_R(\sigma)$ is the matrix representing a symmetric group element σ in representation R, the character of the element σ in this representation is

$$\chi_R(\sigma) = Tr(\Gamma_R(\sigma)) \tag{2.9}$$

$\chi_{R,(r,s)jk}(\sigma)$ is known as the restricted character. It is given by

$$\chi_{R,(r,s)jk}(\sigma) = Tr_{(r,s)jk}\left(\Gamma_R(\sigma)\right) \tag{2.10}$$

To compute the restricted character one must trace the row index of $\Gamma_R(\sigma)$ over the sub- carrier space associated with the j^{th} copy of (r,s), and trace the column index over the subspace associated with the k^{th} copy of (r,s). In performing traces over the carrier space of (r,s), the row and column indices may come from different copies of (r,s). This is the reason that there are two multiplicity labels. It is clear that if $j \neq k$ the trace is not summing diagonal elements of the carrier space. The operators constructed by summing off diagonal elements are part of the basis of local operators, so they must participate in our discussion.

The restricted character can be written in terms of a symmetric group operator $P_{R\to(r,s)jk}$

$$\chi_{R,(r,s)jk}(\sigma) = Tr_{(r,s)jk}\left(\Gamma_R(\sigma)\right) = Tr\left(P_{R\to(r,s)jk}\Gamma_R(\sigma)\right) \tag{2.11}$$

If we concentrate on Young diagrams R with two rows, where there are no multiplicities, $P_{R\to(r,s)}$ is a projection operator. It projects from the carrier space of R to the subspace which carries (r,s). Note that if the number of rows p is greater than two, this operator is in general an intertwiner. It is constructed similarly to a projection operator. It is generally these operators that makes restricted Schur polynomials rather complicated to work with, since they are difficult to construct explicitly. One of the goals later in the work is to show an analytical construction of these operators using a novel version of the Schur-Weyl duality.

2.2.3 Y and Z Fields

The structure of the trace of products of the Y and Z fields can be described using symmetric group elements σ which permute the indices of the matrices. Written explicitly

$$Tr\left(\sigma Y^{\otimes m}Z^{\otimes n}\right) = Tr(Y^{i_1}_{i_{\sigma(1)}}Y^{i_2}_{i_{\sigma(2)}}\cdots Y^{i_m}_{i_{\sigma(m)}}Z^{i_{m+1}}_{i_{\sigma(m+1)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}) \tag{2.12}$$

For example let $\sigma = (2, m+1)$

$$Tr((2,m+1)Y^{i_1}_{j_1}\cdots Y^{i_m}_{j_m}Z^{i_{m+1}}_{i_{\sigma(m+1)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}) = Tr(ZY)Tr(Z)^{n-1}Tr(Y)^{m-1} \tag{2.13}$$

In general if we wish to write the restricted Schur polynomial for a specific representation $R, (r, s)$, the sum over σ tells us to sum over all possible permutations of matrix indices according to the element in S_{n+m} we have picked, giving a sum of all allowed trace structures. The position of each matrix according to its upper index i_i will be referred to as its slot. For example, i_m denotes the the m^{th} slot. The Y matrices are in the 1^{st} to m^{th} slots and the Z matrices are in the $(m+1)$th to $(m+n)$th slots.

2.3 $U(p)$ Representation Theory

The Schur-Weyl duality between the symmetric group and the unitary group is studied in order to simplify working with the subspaces employed by restricted Schur polynomials, that are irreducible representations of $S_n \times S_m$. In this section, we will explore the translation from the symmetric group labels $R, (r, s)$ to $U(p)$ formalism. In order to do so, we first discuss elementary $U(p)$ representation theory.

2.3.1 Lie Algebra

It is simpler to study the Lie algebra, $u(p)$, of the unitary group than it is to study the actual group $U(p)$. Many of the results obtained in doing so carry over to the group. One such example is that the Clebsch-Gordan coefficients of the two are identical.

Our first task is to construct a basis for the $U(p)$ Lie algebra. Set E_{ij} with

$1 \leq i,\ \ j \leq p$ to be a matrix with only one non zero element, $(E_{ij})_{rs} = \delta_{ir}\delta_{js}$. A basis for the Lie algebra can then be generated by

$$iE_{kk} \qquad 1 \leq k \leq p \tag{2.14}$$

$$\left.\begin{array}{l} i(E_{k,k-1} + E_{k-1,k}) \\ E_{k,k-1} - E_{k-1,k} \end{array}\right\} 1 < k \leq p \tag{2.15}$$

2.3.2 $u(p)$ Irreducible Representations

So far, we have studied Young diagrams as labels for S_n irreducible representations and the basis states of their carrier spaces. We now look at Gelfand-Tsetlin patterns, which provide a set of labels for the states of the carrier space of $u(p)$ irreducible representations. The carrier spaces of irreducible representations of the general linear group $GL(p, \mathbb{C})$ share a basis with those of its subgroup, $U(p)$. Further, each irreducible representation of $GL(p, \mathbb{C})$ always restricts to a unique irreducible representation of $U(p)$. Labelings for irreducible representations of the Lie algebra $gl(p, \mathbb{C})$ are thus good labels for irreducible representations of $u(p)$. Exploiting this, we shall study the construction of irreducible representations of the general linear group and apply them without modification to the unitary group.

An inequivalent irreducible representation of $GL(p, \mathbb{C})$ is constructed by specifying a sequence of p integers that satisfies $m_{kp} \geq m_{k+1,p}$ for $1 \leq k \leq p-1$,

$$\mathbf{m} = (m_{1p}, m_{2p}, \ldots m_{pp}) \tag{2.16}$$

This irreducible representation restricted onto $GL(p-1,\mathbb{C}) \supset GL(p,\mathbb{C})$ is a reducible representation which decomposes into a direct sum of irreducible representations of $GL(p-1,\mathbb{C})$ satisfying $m_{kp} \geq m_{k,p-1} \geq m_{k+1,p}$ for $1 \leq k \leq p-1$

$$\mathbf{m}' = (m_{1,p-1}, m_{2,p-1}, \ldots, m_{p-1,p-1}) \tag{2.17}$$

The carrier spaces of $GL(p,\mathbb{C})$ irreducible representations thus gives rise to many $GL(p-1,\mathbb{C})$ carrier spaces after restriction to the subgroup $GL(p-1,\mathbb{C})$. This process can be repeated up to the subgroup $GL(1,\mathbb{C})$, which has one dimensional carrier spaces. For each possible choice of the set of $m_{k,p}$ values we get a distinct basis state. This labeling exploits the sequence of subgroups of $GL(p,\mathbb{C})$ in order to label the basis states, which when suitably presented leads to the so called Gelfand-Tsetlin patterns.

The sequences of integers, $\mathbf{m}, \mathbf{m}' \ldots$, are referred to as the weights of their respective irreducible representations. Gelfand-Tsetlin patterns are obtained by arranging these weights into triangular arrays denoted $\mathbf{M}$, with the structure

$$\mathbf{M} = \begin{pmatrix} m_{1p} & m_{2p} & \cdots & \cdots & \cdots & m_{p-1,p} & m_{pp} \\ & m_{1,p-1} & & \cdots & & m_{p-1,p-1} & \\ & & & \cdots & & & \\ & & & \vdots & & & \\ & & & m_{12} \; m_{22} & & & \\ & & & m_{11} & & & \end{pmatrix} \tag{2.18}$$

The top row of the pattern is the weight specifying the irreducible representation of the state. The entries of the lower rows are subject to the condition $m_{kp} \geq m_{k,p-1} \geq m_{k+1,p}$ for $1 \leq k \leq p-1$, and give the sequence of irreducible representations of the state as we successively restrict from $GL(p, \mathbb{C})$ to $GL(1, \mathbb{C})$. The dimension of an irreducible representation with weight $\mathbf{m}$ is equal to the number of valid Gelfand-Tsetlin patterns with $\mathbf{m}$ in the top row.

Gelfand-Tsetlin patterns can be used to define Δ and Σ weights which we now introduce. These weights are not unique labels for states in the carrier spaces, as two different Gelfand-Tsetlin patterns may have the same weights. Define the sum of all integers in a particular row of the Gelfand-Tsetlin pattern by

$$\sigma_l(\mathbf{M}) = \sum_{k=1}^{l} m_{k,l} \tag{2.19}$$

The sequence of row sums defines the Σ weight of $(\mathbf{M})$

$$\Sigma(\mathbf{M}) = (\sigma_p(\mathbf{M}), \sigma_{p-1}(\mathbf{M}), \ldots, \sigma_1(\mathbf{M})) \tag{2.20}$$

The number of states $\overrightarrow{v}(\mathbf{M})$ in the carrier space that have the same Σ weight is called the inner multiplicity, $I(\Sigma)$, of the state. The inner multiplicity is used to determine how many restricted Schur polynomials can be defined. Using the row sums introduced above, the Δ weights are

$$\begin{aligned} \Delta(M) &= (\sigma_p(M) - \sigma_{p-1}(M), \sigma_{p-1}(M) - \sigma_{p-2}(M), \ldots, \sigma_1(M) - \sigma_0(M)) \\ &\equiv (\delta_p(M), \delta_{p-1}(M), \ldots, \delta_1(M)) \end{aligned} \tag{2.21}$$

Note that there is also an inner multiplicity, $I(\Delta)$, denoting the number of states with the same Δ weight. It is not difficult to argue that $I(\Delta) = I(\Sigma)$.

In the next section we will argue that the components of the Δ weight tell us how many boxes must be removed from each row of R to obtain r.

The number of restricted Schur polynomials that can be constructed is found by writing all valid Gelfand-Tsetlin patterns for the particular Young diagram s arranged using the boxes removed from R. Only states with the correct Δ weight for the way the boxes were removed from R are kept. The number of states is thus equal to the inner multiplicity or number of copies of the label (r, s) subduced from R. In constructing the projector, since we can use different copies for the row and column labels, the square of this multiplicity is summed over, to give the total number of projectors and thus restricted Schur polynomials for the given (r, s).

2.3.3 Relating Young Diagrams to Gelfand-Tsetlin Patterns

We now translate from Young diagrams $R, (r, s)$ labeling S_{n+m} irreducible representations to Young diagrams labeling $u(p)$ irreducible representations, in order to exploit the duality between the irreducible representations of these two groups.

We will start by explaining how Young diagrams can be used to label states in the carrier space of a $U(p)$ irreducible representation. Every Young diagram with at most p rows, including the diagram with no boxes, uniquely labels a $u(p)$ irreducible representation. These diagrams are labeled in such a way that each box contains an integer i, $1 \leq i \leq p$, with the rules that each integer label is greater than or equal to the one on its left, and each number is strictly larger than the one in the box above it. They are referred to as semi-standard Young tableaux. Basis states of a $u(p)$ irreducible representation that are identified by some Young diagram D can be uniquely labelled by the set of all semi-standard Young tableaux with the same shape as D. The dimension of the carrier space of D is equal to the number of valid semi-standard Young tableaux of the same shape.

Every Gelfand-Tsetlin pattern corresponds to a unique semi-standard Young tableau. Start with the unlabeled Young diagram D. The top row of the Gelfand-Tsetlin pattern tells you the shape of the Young diagram. For

example

$$\mathbf{M} = \begin{bmatrix} 4 & 3 & 1 & 1 \\ \cdots & \cdots & \cdots & \\ \cdots & \cdots & & \\ \cdots & & & \end{bmatrix} \quad \rightarrow \quad D = \ydiagram{4,3,1,1} \tag{2.22}$$

The last row of the Gelfand-Tsetlin tells us which boxes are labelled with 1. The number in the last row specifies the shape of a Young tableau of 1's to be superimposed over the original diagram, D. For example, if we have a 2 in the bottom row, this corresponds to the diagram $\square\square$, which when labelled and superimposed onto D gives us

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-3} \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c}{} \\ \cline{1-3} \multicolumn{1}{|c|}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \multicolumn{1}{|c|}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

The second last row of the Gelfand-Tsetlin pattern specifies which boxes of D are labelled with a two. This is superimposed over the previous partially labelled tableau. We repeat this process until we get to the first row, which indicates the boxes labelled with p's (here $p = 4$).

$$\mathbf{M} = \begin{bmatrix} 4 & 3 & 1 & 1 \\ 3 & 2 & 1 & \\ 3 & 2 & & \\ 2 & & & \end{bmatrix} \quad \rightarrow \quad \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 2 & 2 & 2 \\ \hline 2 & 2 \\ \cline{1-2} \end{array} \quad \begin{array}{|c|c|c|} \hline 3 & 3 & 3 \\ \hline 3 & 3 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|c|} \hline 4 & 4 & 4 & 4 \\ \hline 4 & 4 & 4 \\ \cline{1-3} 4 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \tag{2.23}$$

$$D = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 4 \\ \hline 2 & 2 & 4 \\ \cline{1-3} 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array}$$

The number of boxes in D containing the label l, where $1 \leq l \leq p$, in row k of the Young tableau is $m_{kl} - m_{k,l-1}$. If $k > l$, $m_{kl} = 0$. Two Gelfand-Tsetlin patterns with the same Δ weight have the same entries arranged in different ways. The inner multiplicity counts the different ways in which to arrange a fixed set of entries in the Young tableau. It is also clear that the i^{th} component of the Δ weight counts how many boxes in the semi-standard Young tableau are populated with i.

The Schur-Weyl duality thus allows us to relate irreducible representations of the symmetric group with irreducible representations of the unitary group using a corresponding set of labels, the Young diagrams.

2.4 Constructing $P_{R\rightarrow(r,s)jk}$

In this section, based on [56], the symmetric group operators, or projection operators, are calculated analytically for systems with less than five Y fields and order N Z fields. These operators grow more complicated as the number of Y's increase. As a consequence restricted Schur polynomials are difficult to work with.

We will overcome this difficulty, providing a simple construction of $P_{R\rightarrow(r,s)jk}$ by using the group theory tools outlined earlier in this section. We examine the subduction from S_{n+m} representations to $S_n \times (S_1)^m$ representations. This step simplifies the process of constructing the symmetric group operators which project from the carrier space of S_{n+m} representations to that of

$S_n \times S_m$ representations. We start by restricting the carrier space of the S_{n+m} irreducible representation, R, to the subgroup $S_n \times (S_1)^m$. This decomposes S_{n+m} into a direct sum of invariant subspaces, each of which is the carrier space of a particular irreducible representation of $S_n \times (S_1)^m$.

In order to specify irreducible representations of the subgroup $S_n \times (S_1)^m$ we need only include r, the irreducible representation of S_n. This is because there is only one irreducible representation of S_1, $\square$, so we need not include labels of the composite subgroup. The only diagrams r subduced by R are those obtained by removing boxes from R in different ways. The different orders of removing a set of m boxes from R leads to multiple subspaces which all carry the same irreducible representation r. This multiplicity is easily resolved by specifying the order of removing boxes from R. These partially labelled Young-tableaux represent a sub space carrying irreducible representations of $S_n \times (S_1)^m$. These diagrams can then be assembled in such a way that the resulting linear combinations carry an irreducible representation of $S_n \times S_m$.

We wish to construct operators $P_{R \to (r,s)}$ for Young diagrams with number of rows $p \sim \mathcal{O}(1)$. To do this we will make use of a novel Schur-Weyl duality, developed in the framework of Young diagrams with p rows containing $(n+m) \sim \mathcal{O}(N)$ boxes. It is also possible to study the case of diagrams with long columns. There is a simple transformation that relates the two so that we need only describe the case of long rows.

A general diagram consists of rows with length $\mathcal{O}(N)$. Suppose $m = \alpha N$, where $\alpha << 1$. If the m boxes in the partially labelled tableau are labelled $1 \leq i, j \leq m$ such that i and j are in different rows, we have $c_i - c_j \sim \mathcal{O}(N)$. This allows us to use the simplified form of Young's orthogonal representation, given in equations (2.4) and (2.5).

Consider elements of the S_m subgroup acting on the labelled boxes. A matrix representation of this action can be obtained by treating the partially labelled diagrams as Young-Yamanouchi states. For a diagram with p rows and m boxes labelled in all possible consistent ways, there are p^m possible tableaux. It is possible to associate a p-dimensional vector with every labelled box, giving m vectors $\overrightarrow{v}(i)$. Here i corresponds to the number labeling the particular box, and of course $i = 1 \ldots m$. The components of the vectors are denoted $\overrightarrow{v}(i)_n$ where $n = 1 \ldots p$. If box i is in the j^{th} row, $\overrightarrow{v}(i)_n = \delta_{nj}$.

For every labelled box there is a vector space, V_p. If we take the tensor product of these spaces, we can trade the set of partially labelled Young tableaux for a set of p^m dimensional vectors

$$\overrightarrow{v}(1) \otimes \cdots \otimes \overrightarrow{v}(m) \tag{2.24}$$

These vectors span a space denoted $V_p^{\otimes m}$. Using the language introduced above, we say that $\overrightarrow{v}(i)$ occupies the i^{th} 'slot'.

The action of subgroup S_m in the case where boxes i and j occupy different rows of R in the large N version of Young's orthogonal representation implies

the following action on vector space $V_p^{\otimes m}$

$$\sigma \cdot (\overrightarrow{v}(1) \otimes \cdots \otimes \overrightarrow{v}(m)) = \overrightarrow{v}(\sigma(1)) \otimes \cdots \otimes \overrightarrow{v}(\sigma(m)) \tag{2.25}$$

In other words, $\sigma \in S_m$ will move the vector from the i^{th} slot of $V_p^{\otimes m}$ into the slot positioned at $\sigma(i)$ without changing the actual value of the vector.

We define the matrix action of $U(p)$ on this vector space as follows: if $D(U)$ is the $p \times p$ unitary matrix representation of the element $U \in U(p)$, we have

$$U \cdot (\overrightarrow{v}(1) \otimes \cdots \otimes \overrightarrow{v}(m)) = D(U)\overrightarrow{v}(1) \otimes \cdots \otimes D(U)\overrightarrow{v}(m) \tag{2.26}$$

In this case, the action of $U \in U(p)$ is to change the value of the vector without changing its position. The action of $U(p)$ is exactly the same on every slot.

Assume that the subgroup referred to by the Gelfand-Tsetlin pattern are obtained by "freezing" the p^{th} component, and then the $(p-1)^{th}$ component, and so on. With this assumption it is now clear that the Δ weight tells us how to remove boxes from R to obtain r.

It is clear that the actions of S_m and $U(p)$ on $V_p^{\otimes m}$ as defined above, commute. Any two commuting operators can be simultaneously diagonalized. Indeed, suppose $\mathbf{A}$ and $\mathbf{B}$ are two commuting operators which are separately diagonalizable. We can write them in terms of their eigenvalues and projectors

$$\mathbf{A} = \sum_j \lambda_j \mathbf{P}_j \qquad \mathbf{B} = \sum_k \psi_k \mathbf{Q}_k \tag{2.27}$$

where λ and ψ are the eigenvalues and $\mathbf{P}$ and $\mathbf{Q}$ are the projectors. Since we have

$$[\mathbf{A}, \mathbf{B}] = 0 \qquad \Rightarrow \mathbf{P}_j \mathbf{Q}_k = \mathbf{Q}_k \mathbf{P}_j \tag{2.28}$$

Define $\mathbf{R}_{jk} = \mathbf{P}_j \mathbf{Q}_k$ which can be shown to be an hermittean projection operator. We also have

$$\sum_j \mathbf{R}_{jk} = \sum_j \mathbf{P}_j \mathbf{Q}_k = \mathbf{Q}_k \text{ and } \sum_k \mathbf{R}_{jk} = \sum_k \mathbf{P}_j \mathbf{Q}_k = \mathbf{P}_j \tag{2.29}$$

Thus

$$\mathbf{A} = \sum_{jk} \lambda_j \mathbf{R}_{jk} \qquad \mathbf{B} = \sum_{jk} \psi_k \mathbf{R}_{jk} \tag{2.30}$$

By definition, $\mathbf{A}$ and $\mathbf{B}$ are simultaneously diagonalizable.

Due to this fact and to consequences of the Schur-Weyl duality, we can organize the vector space in a way that is effective in the construction of $P_{R,(r,s)jk}$. Indeed using Schur-Weyl duality we can argue that

$$V_p^{\otimes m} = \oplus_s V_s^{U(p)} \otimes V_s^{S_m} \tag{2.31}$$

The sum runs over all diagrams s built from m boxes with at most p rows. The dimension of the vector space can be written as a producet of $U(p)$ and S_m dimensions as

$$p^m = \sum_s Dim(s) d_s \tag{2.32}$$

$Dim(s)$ is the dimension of s as an irreducible representation of $U(p)$ and d_s is the dimension of s as an irreducible representation of S_m.

In this construction, the identification of states with good $U(p)$ labels implies identifying states with good S_m labels. As a result of Schur-Weyl duality, the symmetric group operators $P_{R\to(r,s)jk}$ have good $U(p)$ labels and as such can be constructed using $U(p)$ group theory. In order to do this, a translation from the symmetric group labels $R,(r,s)$ to a set of $U(p)$ labels is required.

The first label is Young diagram s which labels an irreducible representation of $U(p)$. It is the exact same diagram that labels the representation of S_m in the original labels. The second label is r, the irreducible representation of S_n. The third label is a state labelled by a Gelfand-Tsetlin pattern from the carrier space of $U(p)$ irreducible representations . It has a Δ weight that describes the removal of boxes from R to produce r. From this Gelfand-Tsetlin pattern we can construct the semi-standard Young tableau that provides a $U(p)$ irreducible representation.

The following discussion illustrates a detailed example of this third label. For some Gelfand-Tsetlin pattern M, we know that each row corresponds to a number in the Young tableau, and a subgroup in the chain $U(1)\subset U(2)\subset \cdots \subset U(p)$

$$M=\begin{pmatrix} 5 & & 4 & & 3 \\ & 3 & & 2 & \\ & & 2 & & \end{pmatrix} \to \begin{matrix} 3 \\ 2 \\ 1 \end{matrix} \to \begin{array}{|c|c|c|c|c|} \hline 1&1&2&3&3 \\ \hline 2&2&3&3 \\ \cline{1-4} 3&3&3 \\ \cline{1-3} \end{array}$$

The numbers labeling this semi-standard tableau can be identified with the row from which a box has been removed from R. If both the Δ weight

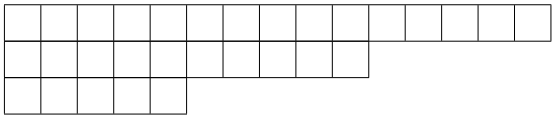
and r are known, it is possible to reconstruct the original label R, since the number of boxes labelled i is the number of boxes removed from row i of R to produce r, given by $\delta(i)$.

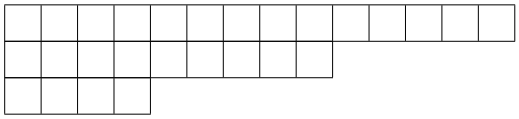
Certain multiplicities must be resolved when translating between S_{n+m} labels and the new set of labels. For instance, two states in the same $U(p)$ representation that have the same Δ weight correspond to the same single set of labels $R, (r, s)$. This is because when $p \geq 3$ it is possible to subduce the same (r, s) from R more than once. Previously, multiplicity labels j and k were mentioned. These indices are organized by $U(p)$ representations, and run from 1 to $I(\Delta(M))$, where $I(\Delta(M))$ is the inner multiplicity, or number of states with the same Δ weights. It must be noted that as a $U(p)$ representation, s also has a multiplicity. This is resolved by the states of the S_m representation s.

Using the new labels, it is simple to define the symmetric group operator $P_{R\rightarrow(r,s)jk}$

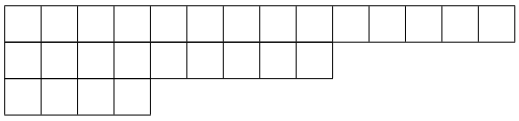
$$P_{R\rightarrow(r,s)jk} = \sum_{\alpha=1}^{d_s} |s, M^j, \alpha >< s, M^k, \alpha| \otimes \mathbf{I}_r \tag{2.33}$$

Here, α is the multiplicity for the $U(p)$ states given by s, organized by s as an irreducible representation of S_m. j and k select states with a particular Δ weight, and $\mathbf{I}_r$ is the identity matrix on the carrier space of r. The final example of this section demonstrates the relationship between the old labels and the new labels. Suppose $p = 3$. The old labels, $R, (r, s)$ are taken to be

$R =$ 

$r =$  $s =$ 

The new labels would be

$r =$  $s =$ 

with $\Delta = (1, 1, 1)$. This corresponds to one box being removed from each row of R. The final labels are Gelfand-Tsetlin patterns with the top row describing the shape of s. There are two of these labels, since it is possible to find two consistent patterns with the correct Δ weight:

$$M_1 = \begin{pmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{pmatrix} \qquad M_2 = \begin{pmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{pmatrix}$$

Since we have two multiplicity labels (M_1 and M_2) we could define four independent restricted Schur polynomials.

3 Action of the Dilatation Operator

In this chapter, by studying the action of the dilatation operator, we shall give insight into integrability in $\mathcal{N} = 4$ SYM. In this way we introduce the methodology that we use to further explore the problem. In this way we introduce a framework for calculating the spectrum of anomalous dimensions of restricted Schur polynomials. Our discussion will motivate the problems that we chose to study later in this work. We look to [56] for mathematical tools and a general explanation of integrability in terms of restricted Schur polynomials. The paper [56] contains further background information for the calculations performed in the following chapters.

The restricted Schur polynomials that we examine are built out of the scalar fields ϕ^i appearing in the Lagrangian of $\mathcal{N} = 4$ SYM. In general, restricted Schur polynomials can also be built out of gauge or fermion fields. They provide a basis to describe the scalar gauge invariant local operators of the theory and have duals in asymptotically $AdS_5 \times S^5$ backgrounds. If labelled by Young diagrams with order 1 rows or columns of length order N, they are identified with objects in the dual gravity theory known as giant gravitons. Depending on the labeling of the restricted Schur polynomial, these can be either sphere or AdS giant gravitons.

There are some useful characteristics of the restricted Schur polynomial operators that make them easy to work with. The restricted Schur basis is complete; multitrace operators or linear combinations of multitrace operators can be written as linear combinations of restricted Schur polynomials. The

two point function of the free theory has been computed exactly in [27]. It is known exactly as a function of N. Indeed, the computation of [27] sums all of the free field theory Feynman diagrams, not just the planar ones, allowing us to go beyond a planar limit. It is noteworthy that the restricted Schur basis diagonalizes the free two point function, and there is only weak mixing at the quantum level.

Consider restricted Schur polynomial operators with a classical dimension of order N. The correlation functions of these large operators generate large combinatoric factors associated with nonplanar ribbon diagrams. This means that nonplanar corrections cannot be suppressed as these large factors enhance these effects to the same order or even larger than the planar contributions. This is why the study of integrability in large N but nonplanar limits of $\mathcal{N} = 4$ SYM is a non-trivial project.

The study of integrability involves calculating the action of the dilatation operator. The eigenvalues of the dilatation operator produce the spectrum of anomalous dimensions of operators. In [14], diagonalization of the one-loop dilatation operator was performed numerically. The article concentrated on the action of the dilatation operator in decoupled sectors of the theory AdS/CFT dual to two sphere giant gravitons. It was shown that the spectrum of anomalous dimensions corresponds to a set of decoupled harmonic oscillators. The harmonic oscillator is a known integrable system; reproducing its spectrum indicates that the operators in question are integrable.

In [17] the one-loop dilatation operator acting on a two graviton system was diagonalized analytically. It was also shown that it was possible to compute general symmetric group operators, entering the construction of restricted Schur polynomials, using spin chains. This provides a feasible way to deal with systems with a large number of Y fields. In section 2.4, based on [56], a further simplification of the computation of these symmetric group operators was described. This used the Schur-Weyl duality, rather than spin chains, for the construction.

In this section methods for studying the dilatation operator acting on systems of $p > 2$ giant gravitons analytically is developed. This will allow us to construct restricted Schur polynomials corresponding to systems of p giant gravitons, labelled by Young diagrams with p rows or columns, by using $u(p)$ representation theory. We will also be able to organize the multiplicity of $S_n \times S_m$ representations subduced from a particular S_{n+m} irreducible representation by appealing to the inner multiplicity appearing in $u(p)$ representation theory. Finally, we shall have a method to evaluate the action of the dilatation operator using $u(p)$ Clebsch-Gordan coefficients.

3.1 The Dilatation Operator

Now that the symmetric group operator $P_{R\to(r,s)}$ has been constructed explicitly, we can calculate the action of the one loop dilatation operator in the $su(2)$ sector. Before this can be achieved, it is necessary to explain how the derivative of a restricted Schur polynomial is taken.

3.1.1 Derivatives

Calculating the action of the dilatation operator involves taking derivatives of a restricted Schur polynomial with respect to the Y and Z fields. As an example, study the the action of an operator $tr(\frac{d}{dY}\frac{d}{dZ})$ on a restricted Schur polynomial

$$\begin{aligned} & tr(\frac{d}{dY}\frac{d}{dY})\left[\chi_{R,(r,s,)jk}(Z,Y)\right] \\ & = \frac{1}{n!m!}\sum_{\sigma\in S_{n+m}}\chi_{R,(r,s)jk}(\sigma)\frac{d}{dY_j^i}\frac{d}{dZ_i^j}(Y^{i_1}_{i_{\sigma(1)}}\cdots Y^{i_m}_{i_{\sigma(m)}}Z^{i_{m+1}}_{i_{\sigma(m+1)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}) \end{aligned}$$

There are m possible fields for the Y derivative to act on and n possible fields for the Z derivative to act on. Swapping Z's between slots or Y's between slots is clearly a symmetry of the restricted Schur polynomial. Consequently we can act with the derivatives on a single Z and a single Y and simply multiply the result by mn. Thus, we obtain

$$\begin{aligned} = \quad & \frac{1}{(n-1)!(m-1)!}\sum_{\sigma\in S_{n+m}}\chi_{R,(r,s)jk}(\sigma) \\ & \times(\delta^{i_1}_j\delta^i_{i_\sigma(1)}Y^{i_2}_{i_{\sigma(2)}}\cdots Y^{i_m}_{i_{\sigma(m)}}\delta^{i_{m+1}}_i\delta^j_{i_\sigma(m+1)}Z^{i_{m+2}}_{i_{\sigma(m+2)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}) \end{aligned}$$

$$\begin{aligned} = \quad & \frac{1}{(n-1)!(m-1)!}\sum_{\sigma\in S_{n+m}}\chi_{R,(r,s)jk}(\sigma) \\ & \times(\delta^{i_1}_{i_{\sigma(m+1)}}\delta^{i_{m+1}}_{i_\sigma(1)}Y^{i_2}_{i_{\sigma(2)}}\cdots Y^{i_m}_{i_{\sigma(m)}}Z^{i_{m+2}}_{i_{\sigma(m+2)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}) \end{aligned} \tag{3.1}$$

3.1.2 One-Loop Dilatation Operator

In [2] and [8] the exact form of the one loop dilatation operator of $\mathcal{N}=4$ SYM was found to be

$$D_2 = -g_{YM}^2 tr\,[Y,Z]\,[\partial_Y,\partial_Z] \tag{3.2}$$

This multiplies out to four terms, each of which contain a derivative that acts on a Y field or Z field belonging to the restricted Schur polynomial. The derivative is taken as shown above. The convention that we adopt here, is to differentiate the first Y field or the first Z field. Thus, we act on

$$Y^{i_1}_{i_{\sigma(1)}} \quad \text{and} \quad Z^{i_{m+1}}_{i_{\sigma(m+1)}}$$

This is purely a convention and does not affect the final result.

We now give a detailed calculation of the action of one of the terms in the dilatation operator. The remaining terms are evaluated in a very similar way.

$$\begin{aligned}
& -g_{YM}^2 tr(YZ\partial_Y\partial_Z)\chi_{R,(r,s)jk}(Y,Z) \\
= \;& \frac{-g_{YM}^2}{n!m!}\sum_{\sigma\in S_{n+m}} tr_{(r,s)jk}\left(\Gamma_R(\sigma)\right)(YZ)^a_c(\partial_Y)^c_d(\partial_Z)^d_a Y^{i_1}_{i_{\sigma(1)}} Z^{i_{m+1}}_{i_{\sigma(m+1)}} \quad Y^{\otimes m-1}Z^{\otimes n-1} \\
= \;& \frac{-g_{YM}^2}{(n-1)!(m-1)!}\sum_{\sigma\in S_{n+m}} tr_{(r,s)jk}\left(\Gamma_R(\sigma)\right)(YZ)^{i_{m+1}}_{i_{\sigma(1)}}\delta^{i_1}_{i_{\sigma(m+1)}} Y^{\otimes m-1}Z^{\otimes n-1}
\end{aligned} \tag{3.3}$$

Make the substitution $\sigma \to \psi(1,m+1)$ in order to permute indices on the fields and the δ

$$(YZ)^{i_{m+1}}_{i_{\sigma(1)}}\delta^{i_1}_{i_{\sigma(m+1)}} \to (YZ)^{i_{m+1}}_{i_{\psi(m+1)}}\delta^{i_1}_{i_{\psi(1)}} \tag{3.4}$$

We are left with

$$\frac{-g_{YM}^2}{(n-1)!(m-1)!}\sum_{\psi\in S_{n+m}} tr_{(r,s)jk}(\Gamma_R((1,m+1)\psi)\delta^{i_1}_{i\psi(1)}Y^{i_2}_{i\psi(2)}\cdots Y^{i_m}_{i\psi(m)} \times(YZ)^{i_{m+1}}_{i\psi(m+1)}Z^{i_{m+2}}_{i\psi(m+2)}\cdots Z^{i_{m+n}}_{i\psi(m+n)}) \quad (3.5)$$

The same process is repeated for all four of the terms of the dilatation operator. Once all of the terms have been collected, we are left with the following

$$D\chi_{R,(r,s)jk}(Y,Z) =$$

$$\frac{-g_{YM}^2}{(n-1)!(m-1)!}\sum_{\psi\in S_{n+m}} tr_{(r,s)jk}(\Gamma_R((1,m+1)\psi-\psi(m+1,1))\times \delta^{i_1}_{i\psi(1)}Y^{i_2}_{i\psi(2)}\cdots Y^{i_m}_{i\psi(m)}\times(YZ-ZY)^{i_{m+1}}_{i\psi(m+1)}Z^{i_{m+2}}_{i\psi(m+2)}\cdots Z^{i_{m+n}}_{i\psi(m+n)}) \quad (3.6)$$

This sum runs over permutations of the indices on the fields where $1 = \psi(1)$ is fixed. Fixing this element is a restriction to the subgroup S_{n+m-1}. To carry out this constrained sum, one sums over cosets of S_{n+m-1}. An operator $\hat{O}_g$, which is the sum of all 2-cycles in the group S_g makes a natural appearance. Indeed, once we have rewritten the sum as a sum over S_{m+n-1} we find that we need to evaluate the following character [22].

$$\chi_{R,(r,s)jk}(\psi\left[N+\hat{O}_{n+m}-\hat{O}_{n+m-1}\right]) \quad (3.7)$$

The combination $\hat{O}_{n+m}-\hat{O}_{n+m-1}$ is known as a Jucys-Murphy element and it plays an important role in modern approaches to the representation theory of the symmetric group. The $\hat{O}_n$ are Casimirs of S_n with eigenvalue

$\hat{O}_n|R>=((number\,of\,row\,pairs\,in\,R)-(number\,of\,column\,pairs\,in\,R))|R>$

where row and column pairs are the number of distinct pairs we can form from the boxes in the row and column respectively and $|R>$ is any state in the carrier space of irreducible representation R. If we use this result in $\chi_{R,(r,s)jk}(\psi\left[N+\hat{O}_{n+m}-\hat{O}_{n+m-1}\right])$, we find that $N+\hat{O}_{n+m}-\hat{O}_{n+m-1}$ evaluates to a definite value on subspaces R' of R, which are carrier spaces of S_{n+m-1}. This value is $c_{RR'}$, the weight of the box pulled off of R to obtain R'. The final result is

$$\chi_{R,(r,s)ij}\left(\Gamma_{R'}(\sigma)\left[N+\hat{O}_{n+m}-\hat{O}_{n+m-1}\right]\right)=\bigoplus_{R'}Tr_{(r,s)ij}(\Gamma_{R'}(\psi))c_{RR'} \quad (3.8)$$

See [22] for further details. We now have

$$D\chi_{R,(r,s)jk}(Y,Z)=$$

$$\frac{-g_{YM}^2}{(n-1)!(m-1)!}\sum_{\psi\in S_{n+m-1}}\sum_{R'}c_{RR'}tr_{(r,s)jk}(\,[\Gamma_R((1,m+1),\psi]\times$$
$$Y^{i_2}_{i\psi(2)}\cdots Y^{i_m}_{i\psi(m)}\times[Y,Z]^{i_{m+1}}_{i\psi(m+1)}\,Z^{i_{m+2}}_{i\psi(m+2)}\cdots Z^{i_{m+n}}_{i\psi(m+n)}) \quad (3.9)$$

From our result above it is clear that the sum over R' is over all possible diagrams that can be obtained by removing a box from R. The permutation $\Gamma_R(1,m+1)$ mixes Y and Z slots.

We now use $\psi(1)=1$ and the relation

$$Y^{i_1}_{i_{\sigma(1)}}\cdots Y^{i_m}_{i_{\sigma(m)}}Z^{i_{m+1}}_{i_{\sigma(m+1)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}=tr\left(\sigma Y^{\otimes m}Z^{\otimes n}\right) \quad (3.10)$$

This gives

$$\frac{-g_{YM}^2}{(n-1)!(m-1)!}\sum_{\psi\in S_{n+m-1}}\sum_{R'}c_{RR'}tr_{(r,s)jk}(\,[\Gamma_R((1,m+1),\psi]\,)\times \\ tr\left((1,m+1)\psi-\psi(1,m+1)Y^{\otimes m}Z^{\otimes n}\right) \quad (3.11)$$

The next step exploits the identity found in [51] which expresses a general multi-trace operator in terms of restricted Schur polynomials

$$tr\left(\sigma Y^{\otimes m}Z^{\otimes n}\right)=\sum_{T,(t,u)}\frac{d_T n!m!}{d_t d_u(n+m)!}\chi_{T,(t,u)}(\sigma^{-1})\chi_{T,(t,u)}(Z,Y) \quad (3.12)$$

This leads to

$$D\chi_{R,(r,s)jk}(Y,Z)=\sum_{T,(t,u)lm}M_{R,(r,s)jk;T,(t,u)lm}\chi_{T,(t,u)ml}(Y,Z) \quad (3.13)$$

$$M_{R,(r,s)jk;T,(t,u)lm}= \\ g_{YM}^2\sum_{\psi\in S_{n+m-1}}\sum_{R'}\frac{c_{RR'}d_T n\ m}{d_t d_u(n+m)}\Big(\\ \times tr_{(r,s)jk}\left(\Gamma_R(1,m+1)\Gamma_{R'}(\psi)-\Gamma_{R'}(\psi)\Gamma_R(1,m+1)\right) \\ \times tr_{(t,u)lm}\left(\Gamma_{T'}(\psi^{-1})\Gamma_T(1,m+1)-\Gamma_T(1,m+1)\Gamma_{T'}(\psi^{-1})\right)\Big) \quad (3.14)$$

To simplify this further, use the fundamental orthogonality relation

$$\sum_{\psi\in S_{n+m-1}}\Gamma_{R'}(\psi)\Gamma_{T'}(\psi^{-1})=\frac{|S_{n+m-1}|}{d_{R'}}\delta_{R'T'} \quad (3.15)$$

to perform the sum over ψ. In this step we also rewrite the restricted character in the form given in equation (2.11). The final result is

$$M_{R,(r,s)jk;T,(t,u)lm}= \\ g_{YM}^2\sum_{R'}\frac{c_{RR'}d_T n\ m}{d_t d_u(n+m)}$$

$$\times \quad tr\left(\left[\Gamma_R(1,m+1), P_{R\to(r,s);jk}\right] I_{R'T'} \left[\Gamma_R(1,m+1), P_{T\to(t,u);ml}\right] I_{T'R'}\right) \tag{3.16}$$

Here $I_{R'T'}$ is an intertwiner. An intertwiner I_{AB} is the mapping from the carrier space of irreducible representation A to the carrier space of irreducible representation B. A detailed explanation of intertwiners can be found in Appendix A.1. $I_{R'T'}$ is only non-zero if R' and T' have the same shape.

Since the exact two point function of restricted Schur polynomials is known, we can rewrite restricted Schur polynomials in terms of normalized operators $O_{R,(r,s)jk}(Y,Z)$. The relation between the restricted Schur polynomials and the operators normalized to have unit two point function is

$$\chi_{R,(r,s)jk}(Y,Z) = \sqrt{\frac{f_R hooks_R}{hooks_r hooks_s}} O_{R,(r,s)jk}(Y,Z) \tag{3.17}$$

When studying the spectrum of the dilatation operator, it is convenient to work with the normalized operators. In terms of normalized operators, the eigenproblem of the dilatation operator becomes

$$DO_{R,(r,s)jk}(Y,Z) = \sum_{T,(t,u)lm} N_{R,(r,s)jk;T,(t,u)ml} O_{T,(t,u)ml}(Y,Z) \tag{3.18}$$

$$\begin{aligned} &N_{R,(r,s)jk;T,(t,u)ml} = \\ &\quad -g_{YM}^2 \sum_{R'} \frac{c_{RR'} d_T n\ m}{d_t d_u (n+m)} \sqrt{\frac{f_T hooks_T hooks_r hooks_s}{f_R hooks_R hooks_t hooks_u}} \times \\ &tr\left(\left[\Gamma_R(1,m+1), P_{R\to(r,s);jk}\right] I_{R'T'} \left[\Gamma_R(1,m+1), P_{T\to(t,u);ml}\right] I_{T'R'}\right) \end{aligned} \tag{3.19}$$

3.2 Calculating the Action of the Dilatation Operator

The action of the dilatation operator can now be found by evaluating the trace of a product of symmetric group operators $P_{R\to(r,s)jk}$ and $P_{T\to(t,u)ml}$, intertwiners and matrix representations of group elements. The constant factors appearing as the coefficient of the trace can also be simplified. Some tools for calculating the trace and the simplification of the factors appear in appendices A.2 and A.3. Using the methods outlined in the appendices we shall perform a general evaluation of the trace, and apply everything developed thus far to the example of a two giant graviton system, that is, restricted Schur polynomials labelled by Young diagrams with two long rows.

The next goal of this chapter is to calculate the spectrum of anomalous dimensions of the dilatation operator. This is achieved by evaluating and diagonalizing the general expression for $N_{R,(r,s)jk;T,(t,u)lm}$. We turn to this task in the next subsection.

3.2.1 General Trace Evaluation

The general form of the trace that was found in the previous section is given by

$$\mathcal{T} = tr\left(\left[\Gamma_R(1,m+1), P_{R\to(r,s)jk}\right] I_{R'T'} \left[\Gamma_T(1,m+1), P_{T\to(t,u)ml}\right] I_{T'R'}\right) \tag{3.20}$$

This can be rewritten as the product a trace over the m Y slots and one Z slot, and a trace over the other $n-1$ Z slots. The $n-1$ slots are the carrier space of r' and t'. Another notation for this r' is R^{m+1}, which emphasizes

the fact that r can be obtained by removing $m+1$ boxes from R. If the $m+1^{th}$ box is removed from row i, the dimension of this representation is denoted $d^i_{R^{m+1}}$. Tracing over the remaining $n-1$ Z slots gives a factor of $d^i_{R^{m+1}}$. Tracing over the m Y slots and $m+1^{th}$ Z slot is a trace over V_p^{m+1}.

We use the $u(p)$ basis matrices given for the E's from chapter two in order to find the action of group element representations $\Gamma_R(1, m+1)$ on the intertwiners. The methodology of this is described in detail in Appendix A.2. Let b be the row of R from which a box is removed to get R', and let a be the same for T, such that $R' = T'$. The trace then simplifies to

$$\begin{aligned}
&\mathcal{T} = \\
&\delta_{ab}\delta_{RT}\delta_{(r,s),(t,u)}\delta_{jm}\delta_{kl}d^b_{R^{m+1}}\left[tr_{V_p^{\otimes m}}\left(P_{R\to(r,s)lk}E^{(1)}_{bb}\right) + tr_{V_p^{\otimes m}}\left(P_{R\to(r,s)jm}E^{(1)}_{bb}\right)\right] \\
&+d^b_{R^{m+1}}tr_{V_p^{\otimes m}}\left(P_{R\to(r,s)lk}E^{(1)}_{bb}P_{T\to(t,u)lm}E^{(1)}_{aa}\right) \\
&+d^b_{R^{m+1}}tr_{V_p^{\otimes m}}\left(P_{R\to(r,s)lk}E^{(1)}_{aa}P_{T\to(t,u)lm}E^{(1)}_{bb}\right)
\end{aligned} \tag{3.21}$$

The first term appearing corresponds to the case where $R = T$ and the second two are the result of the $R \neq T$ case. It is useful to rewrite the symmetric group operator or projector in a bra-ket notation

$$P_{R\to(r,s)ij} = \sum_{a=1}^{d_s} |M^i_s, a >< M^j_s, a| \tag{3.22}$$

Here $1 \geq i, j \geq I(\Delta(M))$. M^i_s and M^j_s label states of a $U(p)$ irreducible representation s with a given Δ weight. a labels multiplicity and is organized by said Gelfand-Tsetlin patterns M^i_s and M^j_s. We can rewrite these states as

a linear combination of ways of removing a box from s in terms of Clebsch-Gordan coefficients

$$|M_s^i, a >= \sum_{M_{s'}, M_{\mathbf{10}}} C^{M_s^i}_{M_{s'}, M_{\mathbf{10}}} |M_{\mathbf{10}} > \otimes |M_{s'}, b > \tag{3.23}$$

A general $U(p)$ Clebsch-Gordan coefficient in terms of the states labeled by these Gelfand-Tsetlin patterns is

$$C^{M_s^i}_{M_{s'}, M_{\mathbf{10}}} = (< M_{\mathbf{10}} \otimes < M_{s'}, b|) \, |M_s^i, a > \tag{3.24}$$

Here b is a multiplicity organized by Gelfand-Tsetlin patterns of the $U(p)$ irreducible representation s' .

3.2.2 Two Graviton System

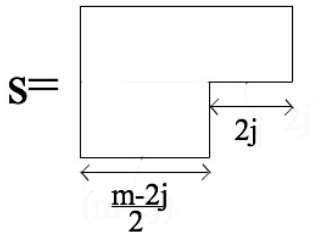
In this work we will explore the two graviton system in depth. For this reason, in this section, the two graviton case will be discussed in detail and diagonalized.

A system of two giant gravitons corresponds to a restricted Schur polynomial labelled by a Young diagram R with two rows of length $\mathcal{O}(N)$. Long rows correspond to AdS gravitons; it is also possible to consider the problem for which R has two long columns, corresponding to sphere gravitons. The two results are related in a very simple way, as we explain below. A restricted Schur polynomial built out of n Z fields and m Y fields where $p = 2$ correspond to a system of two giant gravitons with m strings or impurities attached.

The new set of labels that was constructed using the Schur-Weyl duality replaced Young diagram labels s and multiplicity i with Young diagram s and Gelfand-Tsetlin pattern M. In the case of two gravitons the Gelfand-Tsetlin patterns label states in the irreducible representations of $U(2)$. The row sum of the top row in M is equal to m, which is fixed. Both the Δ weight and s can be replaced by $SU(2)$ angular momentum labels, j and j^3, where

$$j = 0, \frac{1}{2}, 1, \frac{3}{2} \ldots \qquad\qquad -j \leq j^3 \leq j \tag{3.25}$$

Here, the label j is related to the shape, or number of boxes in each row, of diagram s such that



j^3 is associated with the Δ weight and describes how the removal of m boxes from R is performed. Concretely $j^3 = \frac{n_1 - n_2}{2}$, where n_1 and n_2 are the number of boxes removed from row one and row two of R respectively, to get (r, s). The $U(2)$ state of s can then be relabeled in terms of j and j^3

$$\begin{bmatrix} m_{12} & m_{22} \\ & m_{11} \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} m_{22} + 2j & m_{22} \\ & m_{22} + j + j^3 \end{bmatrix} \tag{3.26}$$

Note that m is conserved and $m = 2(m_{22} + j)$. There are four $U(2)$ Clebsch-Gordan coefficients that will be needed to evaluate the dilatation

operator

$$< j-\frac{1}{2}, j^3-\frac{1}{2}; \frac{1}{2}, \frac{1}{2}|j, j^3 >= \sqrt{\frac{j+j^3}{2j}} \tag{3.27}$$

$$< j-\frac{1}{2}, j^3+\frac{1}{2}; \frac{1}{2}, -\frac{1}{2}|j, j^3 >= \sqrt{\frac{j-j^3}{2j}} \tag{3.28}$$

$$< j+\frac{1}{2}, j^3-\frac{1}{2}; \frac{1}{2}, \frac{1}{2}|j, j^3 >= -\sqrt{\frac{j-j^3+1}{2(j+1)}} \tag{3.29}$$

$$< j+\frac{1}{2}, j^3+\frac{1}{2}; \frac{1}{2}, -\frac{1}{2}|j, j^3 >= \sqrt{\frac{j+j^3+1}{2(j+1)}} \tag{3.30}$$

The diagram s can be specified in terms of the number of columns with two boxes, given by $\frac{m-2j}{2}$, and the number of columns with one box, $2j$. When performing the trace in equation (3.21), a factor of $\sum_{s'} d_{s'}$ arises. In the two graviton case, the sum over R' in (3.19) accounts for two cases: a box is removed from row one of s or a box removed from row two of s. Our definition of j^3 implies that a box in the top row corresponds to a 'spin up' state and a box in the bottom row is a 'spin down' state. When removing boxes from either row, the number of columns containing one box changes. A box removed from row one produces state with $2j$ replaced by a value of $2j-1$ and a box removed from row two gives a state with value $2j+1$.

$$\text{Row 1 } M_{s'} = |j-\frac{1}{2}, j^3-\frac{1}{2}>$$
$$\text{Row 2 } M_{s'} = |j+\frac{1}{2}, j^3-\frac{1}{2}>$$

In terms of $SU(2)$ language we also relabel the Gelfand-Tsetlin patterns in

the projection operator

$$M_s \to |j, j^3>$$
$$M^1_{\mathbf{10}} \to |\frac{1}{2}, \frac{1}{2}>$$
$$M^2_{\mathbf{10}} \to |\frac{1}{2}, -\frac{1}{2}>$$

The r labels can also be specified by new numbers: b_0 is the number of columns with two boxes and b_1 is the number of columns with one box. Note that

- The length of the first row in R is $R_1 = b_0 + b_1 + n_1$
- The length of the second row in R is $R_2 = b_0 + n_2$
- The length of the first row in r is $r_1 = b_0 + b_1$
- The length of the second row in r is $r_2 = b_0$
- $m = n_1 + n_2$
- $2j^3 = n_1 - n_2$

We can describe our normalized operators using these numbers, that is, we replace $O_{R,(r,s)jk}(Y, Z) \to O(b_0, b_1; j, j^3)$. The diagonal terms of the dilatation operator are those for which j is conserved. Nine terms arise when the dilatation operator acts on $O(b_0, b_1; j, j^3)$. The complete details of the terms conserving j will be shown. The $j - 1$ and $j + 1$ terms quoted in the final result are found in the same manner so our discussion of these terms will be schematic. For $R = T$ the matrix element of the dilatation operator is

$$-2g_{YM}^2 \frac{r_j \; c_{RR'} \; m}{R_j \; d_s} \sum_{s'} d_{s'} \left[(C^{M_s}_{M_{s'},M^j_{\mathbf{10}}})^2 - (C^{M_s}_{M_{s'},M^j_{\mathbf{10}}})^4 \right] \delta_{jl}\delta_{im} \tag{3.31}$$

The Clebsch-Gordan coefficients can be written in terms of the states defined to replace the Gelfand-Tsetlin patterns. The coefficient for a box removed from row one of s is

$$C^{M_s}_{M_{s'},M^1_{\mathbf{10}}} \rightarrow < j - \frac{1}{2}, j^3 - \frac{1}{2}; \frac{1}{2}, \frac{1}{2} | j, j^3 > \tag{3.32}$$

In terms of the new labels, the other ingredients in equation (3.31) are

$$s_1 : \quad m\frac{d_{s'}}{d_s} = \frac{hooks_s}{hooks_{s'}} = \frac{2j}{2j+1}\frac{m+2j+2}{2} \tag{3.33}$$

$$s_2 : \quad m\frac{d_{s'}}{d_s} = \frac{hooks_s}{hooks_{s'}} = \frac{2j+2}{2j+1}\frac{m-2j}{2} \tag{3.34}$$

$$R_1 : \quad c_{RR'} = (N + b_0 + b_1 - 1) \sim (N + b_0 + b_1)\left(1 + \mathcal{O}(\frac{n_1}{N + b_0 + b_1})\right) \tag{3.35}$$

$$R_2 : \quad c_{RR'} = (N + b_0 + -2) \sim (N + b_0)\left(1 + \mathcal{O}(\frac{n_2}{N + b_0})\right) \tag{3.36}$$

$$\frac{r_1}{R_1} = 1 + \mathcal{O}(\frac{n_1}{b_0 + b_1}) \tag{3.37}$$

$$\frac{r_2}{R_2} = 1 + \mathcal{O}(\frac{n_2}{b_0}) \tag{3.38}$$

Thus for $R = T$ we find the following expression for the matrix element of the dilatation operator

$$-\frac{g_{YM}^2}{2}(2N + 2b_0 + b_1)\left(m - \frac{(m+2)(j^3)^2}{j(j+1)}\right) \tag{3.39}$$

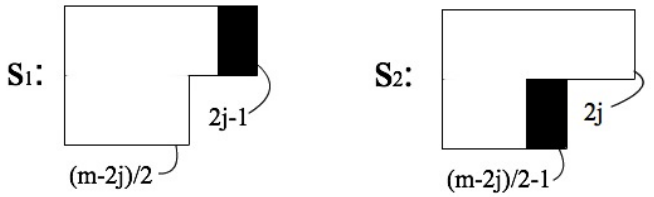


Figure 1: The diagram shows how the factors for the s labels arise when boxes are removed

When $R \neq T$, the dilatation operator matrix element in terms of Clebsch-Gordan coefficients is

$$2g_{YM}^2 \sqrt{c_{RR'}c_{TT'}} \sqrt{\frac{r_k t_l}{R_k T_l}} \frac{m}{d_s} \sum_{s'} d_{s'} (C^{M_s}_{\tilde{M}_{s'},M^2_{\mathbf{10}}})^2 (C^{M_s}_{M_{s'},M^1_{\mathbf{10}}})^2 \tag{3.40}$$

When a box is removed from the first row of s, the Clebsch-Gordan coefficients are given by

$$(C^{M_s}_{\tilde{M}_{s'},M^2_{\mathbf{10}}})^2 (C^{M_s}_{M_{s'},M^1_{\mathbf{10}}})^2 = < j-\frac{1}{2}, j^3-\frac{1}{2}; \frac{1}{2}, \frac{1}{2} | j, j^3 >^2 < j-\frac{1}{2}, j^3+\frac{1}{2}; \frac{1}{2}, -\frac{1}{2} | j, j^3 >^2 \tag{3.41}$$

For a box removed from the second row

$$(C^{M_s}_{\tilde{M}_{s'},M^2_{\mathbf{10}}})^2 (C^{M_s}_{M_{s'},M^1_{\mathbf{10}}})^2 = < j+\frac{1}{2}, j^3-\frac{1}{2}; \frac{1}{2}, \frac{1}{2} | j, j^3 >^2 < j+\frac{1}{2}, j^3+\frac{1}{2}; \frac{1}{2}, -\frac{1}{2} | j, j^3 >^2 \tag{3.42}$$

For a box removed from either row one or row two of R, the other factors that arise in this evaluation are:

$$\sqrt{c_{RR'}c_{TT'}} \approx \sqrt{(N+b_0+b_1)(N+b_0)}$$

$$\sqrt{\frac{r_k t_l}{R_k T_l}} \approx 1$$

$m\frac{d_{s'}}{d_s}$ is computed as for the $R = T$ case. In total, the matrix element evaluates to

$$\sqrt{(N+b_0+b_1)(N+b_0)}g_{YM}^2\left(m-\frac{(m+2)(j^3)^2}{j(j+1)}\right) \tag{3.43}$$

As far as the b_0, b_1 labels are concerned, it is useful to introduce

$$\begin{aligned}\Delta O(b_0,b_1;j,j^3) &= \sqrt{(N+b_0+b_1)(N+b_0)}\Big(O(b_0+1,b_1-2;j,j^3)\\ &\quad +O(b_0-1,b_1+2;j,j^3)\Big)-(2N+2b_0+b_1)O(b_0,b_1;j,j^3)\end{aligned} \tag{3.44}$$

Here Δ is an operator acting only on the r labels of our original restricted Schur polynomial. Finding the spectrum and eigenstates of this operator is one of the main goals of the work presented in Chapter 5. The action of this operator seen in the above equation, was found by labeling the weights of boxes removed from Young diagrams R and T using b_0 and b_1. When all nine terms generated by the acting with the dilatation operator on $O(b_0,b_1;j,j^3)$ are gathered together, we have

$$\begin{aligned}&DO(b_0,b_1;j,j^3) =\\ &g_{YM}^2\Big(-\frac{1}{2}\left(m-\frac{(m+2)(j^3)^2}{j(j+1)}\right)\Delta O(b_0,b_1;j,j^3)\\ &+\sqrt{\frac{(m+2j+4)(m-2j)}{(2j+1)(2j+3)}}\frac{(j+j^3+1)(j-j^3+1)}{2(j+1)}\Delta O(b_0,b_1;j+1,j^3)\\ &+\sqrt{\frac{(m+2j+2)(m-2j+2)}{(2j+1)(2j-1)}}\frac{(j+j^3)(j-j^3)}{2j}\Delta O(b_0,b_1;j-1,j^3)\Big)\end{aligned} \tag{3.45}$$

Note that for the case of two long columns, the action of the dilatation oper-

ator follows by simply replacing the factors $N + b_i$ with $N - b_i$. The number of rows with two boxes is given by b_0 and the number of rows with one box is b_1. In diagram s the number of rows with one box is $2j$ and the number of rows with two boxes is $\frac{m-2j}{2}$.

To solve for the spectrum of the dilatation operator there are two eigenproblems to be solved, namely one associated with the r label involving operator Δ, and one associated to the s label as seen in the above equation. The Δ eigenproblem will be studied in depth in Chapter 5.

Our final result, equation (3.45), found using the Schur-Weyl duality, is in agreement with the equation calculated for the one loop dilatation operator in [17]. This is a good check of our methods. We are now left with the task of diagonalizing the dilatation operator.

3.3 Diagonalization

In order to diagonalize the action found for $DO(b_0, b_1; j, j^3)$, we make the following ansatz for operators with a good scaling dimension

$$O_{p,n} = \sum_{b_1} f(b_0, b_1) O_{p,j^3}(b_0, b_1) = \sum_{j,b_1} C_{p,j^3}(j) f(b_0, b_1) O_{j,j^3}(b_0, b_1) \tag{3.46}$$

The eigenproblem to solve is

$$DO(p, n) = \kappa O(p, n) \tag{3.47}$$

where κ is the one-loop anomalous dimension. In order to solve this equation, we convert (3.47) into recursion relations for the coefficients $C_{p,j^3}(j)$ and $f(b_0, b_1)$. There are two such relations (see [17] for more details)

- $$-\alpha_{p,j^3} C_{p,j^3}(j) =$$

$$\sqrt{\frac{(m+2j+4)(m-2j)}{(2j+1)(2j+3)}}\frac{(j+j^3+1)(j-j^3+1)}{2(j+1)} C_{p,j^3}(j+1)$$
$$+\sqrt{\frac{(m+2j+2)(m-2j+2)}{(2j+1)(2j-1)}}\frac{(j+j^3)(j-j^3)}{2j} C_{p,j^3}(j-1)$$
$$-\frac{1}{2}\left(m-\frac{(m+2)(j^3)^2}{j(j+1)}\right) C_{p,j^3}(j) \tag{3.48}$$

- $$\kappa f(b_0, b_1) =$$

$$-\alpha_{p,j^3} g_{YM}^2 \Big(\sqrt{(N+b_0)(N+b-0+b_1)}(f(b_0-1, b_1+2)+f(b_0+1, b_1-2))$$
$$-(2N+2b_0+b_1) f(b_0, b_1)\Big)\kappa f(b_0, b_1) \tag{3.49}$$

[44] illustrates solutions to these recursion relations, in the form of Kravchuk and Hahn polynomials written in terms of the hypergeometric functions ${}_2F_1$ and ${}_3F_2$ respectively. We have

$$C_{p,j^3}(j) = (-1)^{\frac{m}{2}-p}(\frac{m}{2})! \sqrt{\frac{(2j+1)}{(\frac{m}{2}-j)!(\frac{m}{2}+j+1)!}} {}_3F_2\left(\begin{matrix} |j^3|-j, & j+|j^3|+1, & -p \\ |j^3|-\frac{m}{2}, & 1 & \end{matrix}; 1\right) \tag{3.50}$$

and

$$f(b_0,b_1) = (-1)^n(\frac{1}{2})^{N+b_0+\frac{b_1}{2}}\sqrt{\begin{pmatrix} 2N+2b_0+b_1 \\ N+b_0+b_1 \end{pmatrix}\begin{pmatrix} 2N+2b_0+b_1 \\ n \end{pmatrix}} \times {}_2F_1\begin{pmatrix} -(N+b_0+b_1), & n \\ -(2N+2b_0+b_1) & \end{pmatrix};2 \tag{3.51}$$

Here $|j^3| \leq j \leq \frac{m}{2}$ and $0 \leq p \leq \frac{m}{2} - |j^3|$. When one solves for the eigenvalues using these relations, one finds

$$-\alpha_{p,j^3} = -2p = 0, -2, -4, \ldots, -(m-2|j^3|) \tag{3.52}$$

and

$$\kappa = 4n\alpha_{p,j^3}g^2_{YM} = 8png^2_{YM} \text{ where } n = 0,1,2,\ldots \tag{3.53}$$

It is clear that the spectrum of anomalous dimensions described by the eigenvalues κ is evenly spaced with energy gap set by the coupling constant, g^2_{YM}.

3.3.1 The Continuum Limit

The systems studied thus far have consisted of objects labelled by Young diagrams with large row length differences. This allows us to study the recursion relations previously presented in a continuum limit, in which they become partial differential equations. In the equation for $f(b_0,b_1)$, replace

$$\frac{2b_1}{\sqrt{N+b_0}}$$

with the continuous variable, ρ such that

$$f(b_0, b_1) \to f(\rho)$$

Since $b_1 << N + b_0$, the following expansions can be performed:

$$\sqrt{(N + b_0 + b_1)(N + b_0)} = (N + b_0)\left(1 + \frac{1}{2}\frac{b_1}{N + b_0} - \frac{1}{8}\frac{b_1^2}{(N + b - 0)^2} + \cdots\right) \tag{3.54}$$

$$f(\rho - \frac{1}{\sqrt{N + b_0}}) = f(\rho) - \frac{1}{\sqrt{N + b_0}}\frac{\partial f}{\partial \rho} + \frac{1}{2(N + b_0)}\frac{\partial^2 f}{\partial \rho^2} \tag{3.55}$$

Using these expressions, we find that the recursion relations describing the action of the dilatation operator become partial differential equations

$$\kappa f(\rho) = 2\alpha_{p,j^3} g_{YM}^2 \left[-\frac{\partial^2}{\partial \rho^2} + \rho^2\right] f(\rho) \tag{3.56}$$

It is clear that this is the equation for an harmonic oscillator with a frequency of $2\alpha_{p,j^3} g_{YM}^2$. This allows us to conclude that in the large N nonplanar limit of $\mathcal{N} = 4$ SYM theory, the spectrum of anomalous dimensions of local operators in the form of restricted Schur polynomials, does indeed correspond to the energy spectrum of an harmonic oscillator. Solutions of the continuum differential equations and the discrete recursion relations agree with one another, see [17] for some detailed comparisons. The continuum case makes the assumption that $b_1 << N + b_0$, implying that the discrete case is the most general treatment.

3.4 Gauss Graphs

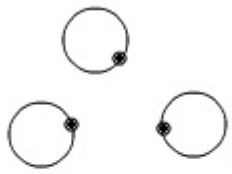
Excitations of D-branes such as giant gravitons can be described by attaching open strings to the branes. Giant gravitons have compact world volumes. A highly non-trivial consequence of a compact world volume is that states of the world volume theory must obey the Gauss law. Indeed, this law implies that the total charge on the world volume must be zero. This constraint governs the way in which strings can be attached to the gravitons, and the number of possible excitation states that can be found which obey this constraint corresponds to the number of operators found in [21], a fact that was proved in [78].

In the language of what has been presented in this chapter, excitations are the m impurities corresponding to Y fields. It is possible to write the action of the dilatation operators on restricted Schur polynomials and to explicitly verify the emergence of the Gauss law, a fact that will be explained below. We now introduce some diagrams, Gauss graphs, that simplify the translation between excited brane states and gauge theory operators greatly.

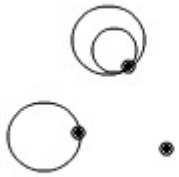
To make the discussion concrete, we will consider a specific example. Suppose we have a system of three giant gravitons with Δ weight $\Delta = (1, 1, 1)$. This describes a three graviton system with three impurities or open strings attached. The rule from the Gauss law tells us that the number of strings entering the graviton must be equal to the number of strings leaving it. We draw three dots and connect them with three strings in all possible ways that do not violate this law. The number of rows in the Young diagram R

labeling the restricted Schur polynomial corresponds to the number of dots or gravitons.

The Δ weight tells us how many strings start from each brane. Some examples are given below.



$\Delta = (1, 1, 1)$



$\Delta = (2, 1, 0)$



$\Delta = (2, 0, 1)$

Since we now have three rows, the operator Δ given above generalizes to three different operators. For p rows there would be $\frac{p(p-1)}{2}$ distinct operators, one for each pair of the p rows. The equation which generalizes (3.44), for

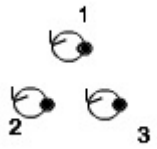
the Δ operator associated to rows i and j is

$$\Delta_{ij}O(r_i, r_j) = \sqrt{(N + r_i)(N + r_j)}\Big(O(r_i + 1, r_j - 1) + O(r_i - 1, r_j + 1)\Big) - (2N + r_i + r_j)O(r_i, r_j) \tag{3.57}$$

In this expression, r_l is the length of the l^{th} row of r. We have only explicitly displayed the dependence of our operators on rows i and j, suppressing all other variables. We do not distinguish between Δ_{ij} and Δ_{ji}.

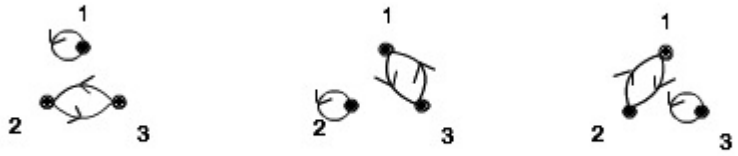
As explained above, we draw a diagram to describe each configuration of the giant graviton system. The action of the dilatation operator is then summarized by these diagrams. We explain this connection using our example below.

1. Strings entering and leaving the same graviton:



$$DO(b_1, b_2) = 0 \tag{3.58}$$

2. Strings connecting two gravitons in all possible ways:

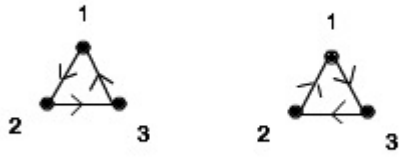


$$DO(b_1, b_2) = -2g_{YM}^2 \Delta_{23} O(b_1, b_2) \tag{3.59}$$

$$DO(b_1, b_2) = -2g_{YM}^2 \Delta_{31} O(b_1, b_2) \tag{3.60}$$

$$DO(b_1, b_2) = -2g_{YM}^2 \Delta_{12} O(b_1, b_2) \tag{3.61}$$

3. Strings connecting three gravitons in all possible ways:



For both configurations we have

$$DO(b_1, b_2) = -g_{YM}^2 (\Delta_{12} + \Delta_{23} + \Delta_{31}) O(b_1, b_2) \tag{3.62}$$

The full details of this calculation can be found in [56]. It was conjectured in [56] that the action of the dilatation operator found using the Gauss graph diagrams in general, is

$$DO(b_i) = -\sum_{ij} g_{YM}^2 n_{ij} \Delta_{ij} O(b_i) \tag{3.63}$$

This has been proved in [78].

3.5 Chapter Summary

This chapter showed the explicit calculation of the spectrum of anomalous dimensions for restricted Schur polynomials dual to systems of two gravitons and gave the basic framework for a system with $p > 2$. The article that was reviewed here, 'Giant Graviton Oscillators' [56], showed evidence that these systems are integrable for restricted Schur polynomials with $\mathcal{O}(1)$ long rows or columns. The main features of this chapter were the use of the Schur-Weyl duality, and the demonstration that integrability in $\mathcal{N} = 4$ SYM is a feature of not only the planar limit, but also a particular large N but non-planar limit.

In the next three chapters, we shall use the results of this chapter in order to further explore integrability in $\mathcal{N} = 4$ SYM theory. We shall study the two graviton problem for restricted Schur polynomials beyond the $SU(2)$ sector, we shall examine the eigenproblem of the Δ operators that act on the r labels, and we shall explore whether integrability of these systems holds beyond one loop.

4 Non Planar Integrability: Beyond the $SU(2)$ Sector

In this section we would like to provide further evidence that integrability is not only a property of the planar limit. Once again we study the large N limit of a set of operators whose bare dimension is of order N, the restricted Schur polynomials. For this class of operators, the planar approximation does not give an accurate description of the large N limit and one is forced to tackle the problem of summing an infinite number of non-planar corrections. The work presented in this chapter is novel. It was reported in Robert De Mello Koch, Badr Awad Elseid Mohammed, and Stephanie Smith, *Non Planar Integrability: Beyond the $SU(2)$ Sector* Int. J. Mod. Phys. A 26, 4553 (2011).

We will start with a quick review of studies considering related questions. In [11] BMN operators in an LLM background [12] were considered. In [13, 14] the spectrum of anomalous dimensions of operators AdS/CFT dual [15] to giant gravitons [16] was considered. In these cases, the operators considered all belonged to the $SU(2)$ sector of the theory. The resulting numerical spectra suggest that the dilatation operator reduces to a set of decoupled harmonic oscillators. As discussed in Section 3, [17] studied the class of restricted Schur polynomials with two rows/columns. This allowed an analytic demonstration that the spectrum of the dilatation operator reduces to that of a set of decoupled harmonic oscillators, once again in the $SU(2)$ sector. This has been extended to operators with any number of rows

or columns in [56, 78].

The main goal of this chapter is to extend the results of [13, 14, 17] beyond the $SU(2)$ sector. We show that the previous results generalize nicely and we can again give an analytic demonstration that the spectrum of the dilatation operator reduces to that of a set of decoupled harmonic oscillators.

We begin by deriving an analytic expression for the action of the one loop dilatation operator on restricted Schur polynomials built using three complex scalars. This is a new result and generalizes the result for the $SU(2)$ sector obtained in [14]. In section 4.2 we describe our construction of the projection operators needed to define the restricted Schur polynomials. Again, we focus on restricted Schur polynomials labeled by Young diagrams that have two rows/columns. The relevant projectors project from an irreducible representation of S_{n+m+p} to an irreducible representation of an $S_n \times S_m \times S_p$ subgroup. For two rows/columns a given irreducible $S_n \times S_m \times S_p$ representation is subduced at most once from a given S_{n+m+p} irreducible representation.

As discussed in [14] this simplifies the problem of computing the projectors significantly. Our construction trades the problem of constructing the projector for the eigenproblem of certain $S_m \times S_p$ Casimirs. This eigenproblem is then solved by translating it into a spin chain language, significantly generalizing the construction of [17]. Note that the original article presented here did not use Schur-Weyl duality in this construction, as discussed in previous sections of this work. The approach employing Schur-Weyl duality was

developed after its completion.

4.1 Action of the Dilatation Operator

In this section we will study the action of the one loop dilatation operator on restricted Schur polynomials built using three complex adjoint scalars. The main result of this section, which generalizes results known for the $SU(2)$ sector[14], is the simple formula (4.5) for the action of the dilatation operator.

Our operators are built using the six scalar fields ϕ^i, which take values in the adjoint of $u(N)$ in $\mathcal{N} = 4$ super Yang Mills theory. Assemble these scalars into the three complex combinations

$$Z = \phi_1 + i\phi_2, \qquad Y = \phi_3 + i\phi_4, \qquad X = \phi_5 + i\phi_6 \,.$$

The operators we consider are built using $O(N)$ of these complex scalar fields. These operators have a large $\mathcal{R}$-charge and consequently, non-planar contributions to the correlation functions of these operators are not suppressed at large N[18]. The computation of the anomalous dimensions of these operators is then a problem of considerable complexity. This problem has been effectively handled by new methods which employ group representation theory[19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31] allowing one to sum all diagrams (planar and non-planar) contributing. Indeed, the two point function of restricted Schur polynomials[21, 22, 23, 25] can be evaluated exactly in the free field theory limit[27]. The restricted Schur polynomials provide a basis for the local operators[61] which diagonalize the free two point function and which have highly constrained mixing at the quantum

level[23, 25, 13, 14, 17]. For the applications that we have in mind, this basis is clearly far superior to the trace basis. Mixing between operators in the trace basis with this large $\mathcal{R}$-charge is completely unconstrained even at the level of the free theory.

The restricted Schur polynomials are

$$\begin{aligned}\chi_{R,(r,s,t)}(Z^{\otimes n}, Y^{\otimes m}, X^{\otimes p}) &= \frac{1}{n!m!p!}\sum_{\sigma\in S_{n+m+p}} \mathrm{Tr}_{(r,s,t)}(\Gamma_R(\sigma)) X^{i_1}_{i_{\sigma(1)}}\cdots X^{i_p}_{i_{\sigma(p)}} \\ &\quad \times Y^{i_{p+1}}_{i_{\sigma(p+1)}}\cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}} Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}}\cdots Z^{i_{n+m+p}}_{i_{\sigma(n+m+p)}}\end{aligned} \tag{4.1}$$

We use n to denote the number of Zs, m to denote the number of Ys and p to denote the number of Xs. R is a Young diagram with $n+m+p$ boxes or equivalently an irreducible representation of S_{n+m+p}. r is a Young diagram with n boxes or equivalently an irreducible representation of S_n, s is a Young diagram with m boxes or equivalently an irreducible representation of S_m and t is a Young diagram with p boxes or equivalently an irreducible representation of S_p. The S_n subgroup acts on $m+p+1, m+p+2, ..., m+p+n$ and therefore permutes indices belonging to the Zs. The S_m subgroup acts on $p+1, p+2, ..., p+m$ and hence permutes indices belonging to the Ys. The S_p subgroup acts on $1, 2, ..., p$ and hence permutes indices belonging to the Xs.

Taken together (r, s, t) specify an irreducible representation of $S_n \times S_m \times S_p$. $\mathrm{Tr}_{(r,s,t)}$ is an instruction to trace over the subspace carrying the irreducible representation[2] (r, s, t) of $S_n \times S_m \times S_p$ inside the carrier space for

[2]In general, because (r, s, t) can be subduced more than once, we should include a

irreducible representation R of S_{n+m+p}. This trace is easily realized by including a projector $P_{R\to(r,s,t)}$ (from the carrier space of R to the carrier space of (r,s,t)) and tracing over all of R, i.e. $\mathrm{Tr}_{(r,s,t)}(\Gamma_R(\sigma)) = \mathrm{Tr}(P_{R\to(r,s,t)}\Gamma_R(\sigma))$.

The one loop dilatation operator, when acting on operators composed from the three complex scalars X, Y, Z, is[33, 34, 35, 36, 4, 37]

$$D = -g^2_{\mathrm{YM}}\mathrm{Tr}\,[Y,Z][\partial_Y,\partial_Z] - g^2_{\mathrm{YM}}\mathrm{Tr}\,[X,Z][\partial_X,\partial_Z] - g^2_{\mathrm{YM}}\mathrm{Tr}\,[Y,X][\partial_Y,\partial_X]. \tag{4.2}$$

The action of the dilatation operator on the restricted Schur polynomials belonging to the $SU(2)$ sector has been worked out in [13, 14]. In what follows, we will work with operators normalized to give a unit two point function. The two point functions for restricted Schur polynomials has been computed in [27]

$$\langle \chi_{R,(r,s,t)}(Z,Y)\chi_{T,(u,v,w)}(Z,Y)^\dagger\rangle = \delta_{R,(r,s,t)\,T,(u,v,w)} f_R \frac{\mathrm{hooks}_R}{\mathrm{hooks}_r\,\mathrm{hooks}_s\,\mathrm{hooks}_t} \tag{4.3}$$

In this expression f_R is the product of the factors[3] in Young diagram R and hooks_R is the product of the hook lengths of Young diagram R. The normalized operators $O_{R,(r,s,t)}(Z,Y)$ can be obtained from

$$\chi_{R,(r,s,t)}(Z,Y,X) = \sqrt{\frac{f_R\,\mathrm{hooks}_R}{\mathrm{hooks}_r\,\mathrm{hooks}_s\,\mathrm{hooks}_t}}\,O_{R,(r,s,t)}(Z,Y,X)\,. \tag{4.4}$$

multiplicity index. We will not write or need this index in this article. We will, in the next section, restrict our attention to restricted Schur polynomials that are labeled by Young diagrams with two rows or columns. A huge simplification that results is that all possible representations (r,s,t) are subduced exactly once.

[3]The term weights is also frequently used. The factor/weight of a box in the i^{th} row and j^{th} column is $N+j-i$.

The computation of the dilatation operator is a simple extension of the analysis presented in [14] so that we will only quote the final result. In terms of the normalized operators

$$DO_{R,(r,s,t)}(Z,Y,X) = \sum_{T,(u,v,w)} N_{R,(r,s,t);T,(u,v,w)} O_{T,(u,v,w)}(Z,Y,X) \tag{4.5}$$

$$\begin{aligned} &N_{R,(r,s,t);T,(u,v,w)} = \\ &-\sum_{R'} \frac{c_{RR'} g_{YM}^2 d_T}{d_{R'} d_u d_v d_w (n+m+p)} \sqrt{\frac{f_T \, \text{hooks}_T \, \text{hooks}_r \, \text{hooks}_s \text{hooks}_t}{f_R \, \text{hooks}_R \, \text{hooks}_u \, \text{hooks}_v \, \text{hooks}_w}} \times \Big[nm \text{Tr} \Big(\\ &\Big[\Gamma_R((p+m+1,p+1)), P_{R\to(r,s,t)}\Big] I_{R'\,T'} \Big[\Gamma_T((p+m+1,p+1)), P_{T\to(u,v,w)}\Big] I_{T'\,R'}\Big) \\ &+np\text{Tr}\Big(\Big[\Gamma_R((1,p+m+1)), P_{R\to(r,s,t)}\Big] I_{R'\,T'} \Big[\Gamma_T((1,p+m+1)), P_{T\to(u,v,w)}\Big] I_{T'\,R'}\Big) \\ &+ mp\text{Tr}\Big(\Big[\Gamma_R((1,p+1)), P_{R\to(r,s,t)}\Big] I_{R'\,T'} \Big[\Gamma_T((1,p+1)), P_{T\to(u,v,w)}\Big] I_{T'\,R'}\Big)\Big] \end{aligned} \tag{4.6}$$

$c_{RR'}$ is the factor of the corner box removed from Young diagram R to obtain diagram R', and similarly T' is a Young diagram obtained from T by removing a box. This factor arises after using the reduction rule of [80, 22]. The intertwiner I_{AB} is a map from the carrier space of irreducible representation A to the carrier space of irreducibe representation B. Consequently, by Schur's Lemma, A and B must be Young diagrams of the same shape. The intertwiner operators relevant for our study have been discussed in detail in [14].

4.2 Projection Operators

The goal of this section is to construct the projection operators needed to define the restricted Schur polynomials we study in this article. This con-

struction clearly defines the class of operators being considered. The approximations being employed in this construction are carefully considered.

The class of operators $\chi_{R,(r,s,t)}(Z,Y,X)$ we will study in this article are labeled by Young diagrams that each have 2 rows or columns. We further take n to be order N and m,p to be αN with $\alpha \ll 1$. Thus, there are a lot more Z fields than there are Ys or Xs. The mixing of these operators with restricted Schur polynomials that have $n \neq 2$ rows or columns (or of even more general shape) is suppressed at least by a factor of order $\frac{1}{\sqrt{N}}$[4]. Thus, at large N the 2 row or column restricted Schur polynomials do not mix with other operators, which is a huge simplification. This is the analog of the statement that for operators with a dimension of $O(1)$, different trace structures do not mix at large N. The fact that the two column restricted Schur polynomials are a decoupled sector at large N is expected: these operators correspond to a well defined stable semi-classical object in spacetime (the two giant graviton system).

Note that as a consequence of the fact that there are a lot more Zs than Ys and Xs, contributions to the dilatation operators coming from interactions between Zs and Ys or between Zs and Xs will over power the contribution coming from interactions between Xs and Ys. Consequently we can simplify the action of the dilatation operator to

$$N_{R,(r,s,t);T,(u,v,w)} = -\sum_{R'} \frac{c_{RR'} g_{YM}^2 d_T n}{d_{R'} d_u d_v d_w (n+m+p)}$$

[4]Here we are talking about mixing at the quantum level. There is no mixing in the free theory[27].

$$\times \sqrt{\frac{f_T \,\mathrm{hooks}_T \,\mathrm{hooks}_r \,\mathrm{hooks}_s \mathrm{hooks}_t}{f_R \,\mathrm{hooks}_R \,\mathrm{hooks}_u \,\mathrm{hooks}_v \,\mathrm{hooks}_w}} \times \Big[m\mathrm{Tr}\Big(\tag{4.7}$$
$$\Big[\Gamma_R((p+m+1,p+1)), P_{R\to(r,s,t)}\Big] I_{R'\,T'} \Big[\Gamma_T((p+m+1,p+1)), P_{T\to(u,v,w)}\Big] I_{T'\,R'}\Big)$$
$$+ p\mathrm{Tr}\left(\Big[\Gamma_R((1,p+m+1)), P_{R\to(r,s,t)}\Big] I_{R'\,T'} \Big[\Gamma_T((1,p+m+1)), P_{T\to(u,v,w)}\Big] I_{T'\,R'}\right)\Big] \,.$$

We will obtain an analytic expression for the above operator in this chapter.

4.2.1 Two Rows

We will make use of Young's orthogonal representation for the symmetric group. This representation is most easily defined by considering the action of adjacent permutations (permutations of the form $(i, i+1)$) on the Young-Yamonouchi states. The permutation $(i, i+1)$ when acting on any given Young-Yamonouchi state will produce a linear combination of the original state and the state obtained by swapping the positions of i and $i+1$ in the Young-Yamonouchi symbol. The precise rule is most easily written in terms of the axial distance between i and $i+1$. If i appears in row r_i and column c_i of the Young-Yamonouchi symbol and $i+1$ appears in row r_{i+1} and column c_{i+1} of the Young-Yamonouchi symbol, then the axial distance between i and $i+1$ is

$$d_{i,i+1} = c_i - r_i - (c_{i+1} - r_{i+1}) \,. \tag{4.8}$$

In terms of this axial distance, the action of $(i, i+1)$ is

$$(i, i+1)\,|\text{state}\rangle = \frac{1}{d_{i,i+1}}\,|\text{state}\rangle + \sqrt{1 - \frac{1}{d_{i,i+1}^2}}\,|\text{swapped state}\rangle \tag{4.9}$$

where the Young-Yamonouchi symbol of $|\text{swapped state}\rangle$ state is obtained from the Young-Yamonouchi symbol of $|\text{state}\rangle$ by swapping the positions of

i and $i+1$. See [39] for more details.

The reason why we use Young's orthogonal representation is that it simplifies dramatically for the operators we are interested in. To construct the projectors $P_{R\to(r,s,t)}$ we will imagine that we start by removing $m+p$ boxes from R to produce r. We label the boxes in the order that they are removed. Of course, after each box is removed we are left with a valid Young diagram; this is a nontrivial constraint on the allowed numberings. Thus, after labeling these boxes we have a total of 2^{m+p} partially labeled Young diagrams, each corresponding to a subspace r of the subgroup $S_n \times (S_1)^{m+p}$ of the original S_{n+m+p} group. We now need to take linear combinations of these subspaces in such a way that we obtain the correct irreducible representation (s,t) of the $S_m \times S_p$ subgroup that acts on the labeled boxes. For the class of operators that we consider, the number of boxes that we remove $(= m+p)$ is much less that the number of boxes in R $(= m+n+p \approx n)$. In the figure below we show R and the boxes that must be removed from R to obtain r. It is clear that the axial distance $d_{i,i+1}$ is 1 if the boxes are in the same row so that

$$(i, i+1)\,|\text{state}\rangle = |\text{state}\rangle \qquad \text{for boxes in the same row}\,. \tag{4.10}$$

It is also clear that $d_{i,i+1}$ is $O(N)$ for boxes in different rows. At large N we can simply set $(d_{i,i+1})^{-1} = 0$ so that

$$(i, i+1)\,|\text{state}\rangle = |\text{swapped state}\rangle \qquad \text{for boxes in different rows}\,. \tag{4.11}$$

The representation that we have obtained is very similar to a representation

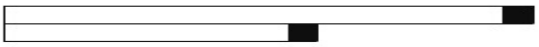
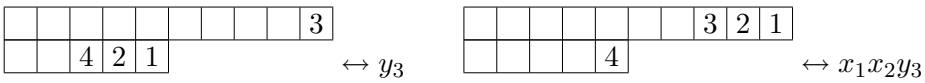


Figure 2: Shown above is the Young diagram R. The boxes that are to be removed from R to obtain r are colored black.

which has already been studied in the mathematics literature [40]. Motivated by this background, define a map from a labeled Young diagram to a monomial. Our Young diagram has $m + p$ boxes labeled and the labels are distributed between the upper and lower rows. Ignore the boxes that appear in the lower row. For boxes labeled i in the upper row include a factor of x_i in the monomial if $1 \leq i \leq p$ and a factor of y_i if $p + 1 \leq i \leq p + m$. If none of the boxes in the first row are labeled, the Young diagram maps to 1. Thus, for example, when $m = 2$ and $p = 2$



The symmetric group acts by permuting the labels on the factors in the monomial. Thus, for example, $(12)x_1y_3 = x_2y_3$. This defines a reducible representation of the group $S_m \times S_p$. It is clear that the operators[5]

$$d_1 = \sum_{i=1}^{p} \frac{\partial}{\partial x_i} \qquad d_2 = \sum_{i=p+1}^{p+m} \frac{\partial}{\partial y_i} \tag{4.12}$$

commute with the action of the $S_m \times S_p$ subgroup. These operators generalize closely related operators introduced by Dunkl in his study of intertwining functions [41]. They act on the monomials by producing the sum of terms

[5]It may be helpful (and it is accurate) for the reader to associate the x_i, y_j of these operators with the $X^i_{\sigma(i)}, Y^j_{\sigma(j)}$ appearing in the definition of the restricted Schur polynomials.

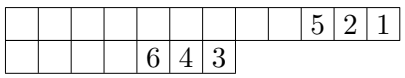
that can be produced by dropping one x factor for d_1 or one y factor for d_2 at a time. For example

$$d_1(x_1x_2y_3) = x_2y_3 + x_1y_3, \qquad d_2(x_1x_2y_3) = x_1x_2\,.$$

The adjoint[6] produces the sum of monomials that can be obtained by appending a factor, without repeating any of the factors (this is written for $m = 2 = p$ impurities but the generalization to any m is obvious)

$$d_1^\dagger(y_3) = x_1y_3 + x_2y_3, \qquad d_1^\dagger(x_1y_3) = x_1x_2y_3, \qquad d_2^\dagger(x_1y_3) = x_1y_3y_4\,.$$

The fact that d_1 and d_2 commute with all elements of $S_m \times S_p$, implies that $d_1^\dagger$ and $d_2^\dagger$ will too. Thus, $d_1^\dagger d_1$ and $d_2^\dagger d_2$ will also commute with all the elements of the $S_m \times S_p$ subgroup and consequently their eigenspaces will furnish representations of the subgroup. These eigenspaces are irreducible representations - consult [40] for useful details and results. This last fact implies that the problem of computing the projectors needed to define the restricted Schur polynomials can be replaced by the problem of constructing projectors onto the eigenspaces of $d_1^\dagger d_1$ and $d_2^\dagger d_2$. This amounts to solving for the eigenvectors and eigenvalues of $d_1^\dagger d_1$ and $d_2^\dagger d_2$. This problem is most easily solved by mapping the labeled Young diagrams into states of a spin chain. The spin at site i can be in state spin up $(+\frac{1}{2})$ or state spin down $(-\frac{1}{2})$. The spin chain has $m+p$ sites and the box labeled i tells us the state of site i. If box i appears in the first row, site i is in state $+\frac{1}{2}$; if it appears in the second row site i is in state $-\frac{1}{2}$. For example,



$$\leftrightarrow \left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle$$

[6]Consult Appendix B.2 for details on the inner product on the space of monomials.

Both $d_1^\dagger d_1$ and $d_2^\dagger d_2$ have a very simple action on this spin chain: Introduce the states

$$\left|\frac{1}{2}\right\rangle = \begin{bmatrix} 1 \\ \\ 0 \end{bmatrix} \qquad \left|-\frac{1}{2}\right\rangle = \begin{bmatrix} 0 \\ \\ 1 \end{bmatrix}$$

for the possible states of each site and the operators

$$\sigma^+ = \begin{bmatrix} 0 & 1 \\ \\ 0 & 0 \end{bmatrix} \qquad \sigma^- = (\sigma^+)^\dagger = \begin{bmatrix} 0 & 0 \\ \\ 1 & 0 \end{bmatrix}$$

which act on these states

$$\sigma^+ \left|-\frac{1}{2}\right\rangle = \left|\frac{1}{2}\right\rangle, \qquad \sigma^+ \left|\frac{1}{2}\right\rangle = 0, \qquad \sigma^- \left|\frac{1}{2}\right\rangle = \left|-\frac{1}{2}\right\rangle, \qquad \sigma^- \left|-\frac{1}{2}\right\rangle = 0\,.$$

We can write any of the states of the spin chain as a tensor product of the states $|\frac{1}{2}\rangle$ and $|-\frac{1}{2}\rangle$. For example

$$\left|-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle = \left|-\frac{1}{2}\right\rangle \otimes \left|-\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}\right\rangle \otimes \left|-\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}\right\rangle$$

for a system with 6 lattice sites. Label the sites starting from the left, as site 1, then site 2 and so on till we get to the last site, which is site 6. The operator σ^- acting at the third site (for example) is

$$\sigma_3^- = 1 \otimes 1 \otimes \sigma^- \otimes 1 \otimes 1 \otimes 1\,.$$

We can then write

$$d_1^\dagger d_1 \quad = \sum_{\alpha=1}^{p} \sum_{\beta=1}^{p} \sigma_\alpha^+ \sigma_\beta^-\,, \tag{4.13}$$

$$d_2^\dagger d_2 \quad = \sum_{\alpha=p+1}^{p+m} \sum_{\beta=p+1}^{p+m} \sigma_\alpha^+ \sigma_\beta^- \tag{4.14}$$

This is a long ranged spin chain. In terms of the Pauli matrices

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4.15}$$

we define the following "total spins" of the system

$$J^1 = \sum_{\alpha=1}^{p} \frac{1}{2}\sigma_\alpha^1\,, \quad J^2 = \sum_{\alpha=1}^{p} \frac{1}{2}\sigma_\alpha^2\,, \quad J^3 = \sum_{\alpha=1}^{p} \frac{1}{2}\sigma_\alpha^3\,, \tag{4.16}$$

$$\mathbf{J}^2 = J^1J^1 + J^2J^2 + J^3J^3\,,$$

and

$$K^1 = \sum_{\alpha=p+1}^{p+m} \frac{1}{2}\sigma_\alpha^1\,, \quad K^2 = \sum_{\alpha=p+1}^{p+m} \frac{1}{2}\sigma_\alpha^2\,, \quad K^3 = \sum_{\alpha=p+1}^{p+m} \frac{1}{2}\sigma_\alpha^3\,, \tag{4.17}$$

$$\mathbf{K}^2 = K^1K^1 + K^2K^2 + K^3K^3\,.$$

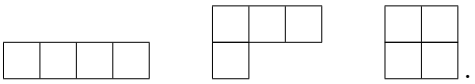
We use capital letters for operators and little letters for eigenvalues. In terms of these total spins we have

$$d_1^\dagger d_1 = \mathbf{J}^2 - J^3(J^3+1)\,, \qquad d_2^\dagger d_2 = \mathbf{K}^2 - K^3(K^3+1)\,.$$

Thus, eigenspaces of $d_1^\dagger d_1$ can be labeled by the eigenvalues of $\mathbf{J}^2$ and eigenvalues of J^3, and the eigenspaces of $d_2^\dagger d_2$ can be labeled by the eigenvalues of $\mathbf{K}^2$ and eigenvalues of K^3. Consequently, the labels $R, (r,s,t)$ of the restricted Schur polynomial can be traded for these eigenvalues. Indeed, consider the restricted Schur polynomial $\chi_{R,(r,s,t)}(Z,Y,X)$. The $\mathbf{K}^2 = k(k+1)$ quantum number tells you the shape of the Young diagram s that organizes the impurities: if there are N_1 boxes in the first row of s and N_2 boxes in the second, then $2k = N_1 - N_2$. The $\mathbf{J}^2 = j(j+1)$ quantum number tells you the shape of the Young diagram t that organizes the impurities: if there are N_1 boxes

in the first row of t and N_2 boxes in the second, then $2j = N_1 - N_2$. The $J^3 + K^3$ eigenvalue of the state is always a good quantum number, both in the basis we start in where each spin has a sharp angular momentum or in the basis where the states have two sharp "total angular momenta". The $j^3 + k^3$ quantum number tells you how many boxes must be removed from each row of R to obtain r. Denote the number of boxes to be removed from the first row by n_1 and the number of boxes to be removed from the second row by n_2. We have $2j^3 + 2k_3 = n_1 - n_2$. This gives a complete construction of the projection operators we need.

To get some insight into how the construction works, lets count the states which appear for the example $m = p = 4$. There are three possible Young diagram shapes which appear

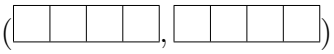
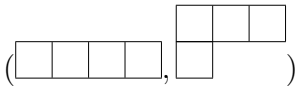
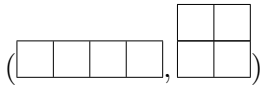
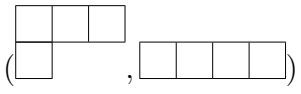
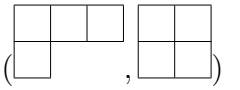
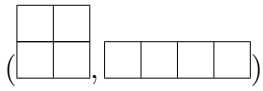
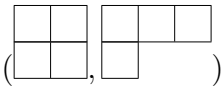
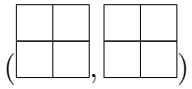
.

These correspond to a spins of $2, 1, 0$ respectively. As irreducible representations of S_4 they have a dimension of 1, 3 and 2 respectively. Coupling four spins we have

$$\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} = \mathbf{2} \oplus \mathbf{31} \oplus \mathbf{20}\,.$$

These results illustrate that each state of a definite spin labels an irreducible representation of the symmetric group and further that for our 8 spins we

find the following organization of states

$S_m \times S_p$ irrep	**K** irrep	**J** irrep	dimension
	$k = 2$	$j = 2$	25
	$k = 2$	$j = 1$	45
	$k = 2$	$j = 0$	10
	$k = 1$	$j = 2$	45
	$k = 1$	$j = 1$	81
	$k = 1$	$j = 0$	18
	$k = 0$	$j = 2$	10
	$k = 0$	$j = 1$	18
	$k = 0$	$j = 0$	4

The last column is obtained by taking a product of the dimension of the $S_m \times S_p$ irreducible representation by the dimension $(2k + 1)(2j + 1)$ of the associated spin multiplets. Summing the entries in the last column we obtain 256 which is indeed the number of states in the spin chain. For a detailed example of how the construction works see Appendix B.1.

Summary of the Approximations made:

- We have neglected mixing with restricted Schur polynomials that have $n \neq 2$ rows. These mixing terms are at most $O(\frac{1}{\sqrt{N}})$ so that this approximation is accurate at large N.

- The terms arising from an interaction between the Xs and Ys have been neglected. Since there are a lot more Zs than Xs and Ys the one loop dilatation operator will be dominated by terms arising from an interaction between Zs and Xs and between Zs and Ys.

- In simplifying Young's orthogonal representation for the symmetric group we have replaced certain factors $(d_{i,i+1})^{-1} = O(N^{-1})$ by $(d_{i,i+1})^{-1} = 0$. This is valid at large N. The fact that $d_{i,i+1} = O(N)$ is a consequence of the fact that we have Young diagrams with two rows, that we consider an operator whose bare dimension grows parametrically with N and that there are a lot more Zs than Xs and Ys. Thus boxes in different rows, corresponding to Xs and Ys, are always separated by a large axial distance at large N.

4.2.2 Two Columns

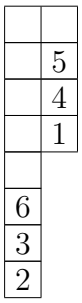
To treat the case of two columns, we need to account for the fact that Young's orthogonal representation simplifies to

$$(i, i+1)\,|\text{state}\rangle = -\,|\text{state}\rangle \qquad \text{for boxes in the same column}\,, \tag{4.18}$$

$$(i, i+1)\,|\text{state}\rangle = |\text{swapped state}\rangle \qquad \text{for boxes in different columns}\,. \tag{4.19}$$

Note the minus sign on the first line above. We can account for this sign, generalizing [17], by employing a description that uses Grassmann variables. To describe the first p boxes, introduce the $2p$ variables x_i^+, x_i^-, where $i = 1, 2, ..., p$. To describe the next m boxes, introduce the $2m$ variables y_j^-, y_j^+, where $j = p+1, p+2, ..., p+m$. Each labeled Young diagram continues to

have an expression in terms of a monomial. Boxes in the right most column have a superscript +; boxes in the left most column have a superscript −. Each monomial is ordered with (i) xs to the left of ys and (ii) within each type (x or y) of variable, variables with a − superscript to the left of variables with a + superscript. Finally within a given type and a given superscript the variables are ordered so that the subscripts increase from left to right. Thus, for example, when $m = 3 = p$ we have



$$\leftrightarrow x_1^- x_2^+ x_3^+ y_4^- y_5^- y_6^+ \ .$$

If we now allow $S_m \times S_p$ to act on the monomials by acting on the subscripts of each variable without changing the order of the variables, we recover the correct action on the labeled Young diagrams.

It is a simple matter to show that

$$d_1 = \sum_{i=1}^{p} x_i^+ \frac{\partial}{\partial x_i^-}, \qquad d_2 = \sum_{i=p+1}^{p+m} y_i^+ \frac{\partial}{\partial y_i^-}, \tag{4.20}$$

both commute with the symmetric group. It is again simple to show that[7]

$$d_1^\dagger = \sum_{i=1}^{p} x_i^- \frac{\partial}{\partial x_i^+}, \qquad d_2^\dagger = \sum_{i=p+1}^{p+m} y_i^- \frac{\partial}{\partial y_i^+}. \tag{4.21}$$

[7]Assuming we only consider monomials that are ordered as we described above, the inner product of two identical monomials is 1 and of two different monomials is 0.

We can again define two $S_m \times S_p$ Casimirs as $d_1^\dagger d_1$ and $d_2^\dagger d_2$. In terms of the spin variables

$$\tilde{\sigma}_n^i = (\sigma_n^3)^n \sigma_n^i (\sigma_n^3)^n$$

we have

$$d_1^\dagger d_1 = \tilde{\mathbf{J}}^2 - \tilde{J}^3(\tilde{J}^3+1)\,, \qquad d_2^\dagger d_2 = \tilde{\mathbf{K}}^2 - \tilde{K}^3(\tilde{K}^3+1)\,. \tag{4.22}$$

Thus, the eigenspaces of $d_1^\dagger d_1$ can be labeled by the eigenvalues of $\tilde{\mathbf{J}}^2$ and eigenvalues of $\tilde{J}^3$, and the eigenspaces of $d_2^\dagger d_2$ can be labeled by the eigenvalues of $\tilde{\mathbf{K}}^2$ and eigenvalues of $\tilde{K}^3$. Consequently, the labels $R, (r,s,t)$ of the restricted Schur polynomial can again be traded for these eigenvalues. The remaining discussion is now identical to that of two rows and is thus not repeated.

4.3 Evaluation of the Dilatation Operator

In this section we will argue that all of the factors in the dilatation operator have a natural interpretation as operators acting on the spin chain. This allows us to explicitly evaluate the action of the dilatation operator. Our final formula for the dilatation operator is given as the last formula in this section.

The bulk of the work involved in evaluating the dilatation operator comes from evaluating the traces

$$\mathrm{Tr}\left(\Big[\Gamma_R((p{+}m{+}1,p{+}1)), P_{R\to(r,s,t)}\Big] I_{R'\,T'} \Big[\Gamma_T((p{+}m{+}1,p{+}1)), P_{T\to(u,v,w)}\Big] I_{T'\,R'}\right), \tag{4.23}$$

and

$$\mathrm{Tr}\left(\left[\Gamma_R((1,p+m+1)),P_{R\to(r,s,t)}\right]I_{R'\,T'}\left[\Gamma_T((1,p+m+1)),P_{T\to(u,v,w)}\right]I_{T'\,R'}\right). \tag{4.24}$$

When we evaluate the second trace above, the intertwiners can be taken to act on the first site of the spin chain. This term corresponds to an interaction between a Z and X field. The first p sites of the spin chain correspond to X fields so that the intertwiner could have acted on any of the first p sites of the chain. When we evaluate the first trace above, the intertwiners can be taken to act on the $(p+1)$th site of the spin chain. This term corresponds to an interaction between a Z and Y field. The last m sites of the spin chain correspond to Y fields so that the intertwiner could have acted on any of the last m sites of the chain. Consider an intertwiner which acts on the first site of the chain. If the box from row i is dropped from R and the box from row j is dropped from T, the intertwiner becomes

$$I_{R'T'}=E_{ij}\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}\,,\qquad I_{T'R'}=E_{ji}\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}\,, \tag{4.25}$$

where E_{ij} is a 2×2 matrix of zeroes except for a 1 in row i and column j. We will use a simpler notation according to which we suppress all factors of the 2×2 identity matrix and indicate which site a matrix acts on by a superscript. Thus, for example

$$I_{R'T'}=E^{(1)}_{ij},\qquad I_{T'R'}=E^{(1)}_{ji}\,. \tag{4.26}$$

Next, consider $\Gamma_R((p+m+1,p+1))$ which acts on a slot occupied by a Z and a slot occupied by a Y and $\Gamma_R((1,p+m+1))$ which acts on a slot occupied by a Z and a slot occupied by an X. To allow an action on the Z slot, enlarge

the spin chain by one extra site (the Z site). The projectors and intertwiners all have a trivial action on this $(m+p+1)$th site. $\Gamma_R((p+m+1,p+1))$ will swap the spin in the $(m+p+1)$th site with the spin in site $p+1$. Thus, we have

$$\begin{aligned} I_{R'T'}\Gamma_R((p+m+1,p+1)) &= \sum_{k=1}^{2} E_{ij}^{(p+1)} E_{kk}^{(m+p+1)} \Gamma_R((p+m+1,p+1)) \\ &= \sum_{k=1}^{2} E_{ik}^{(p+1)} E_{kj}^{(m+p+1)}, \end{aligned}$$

$$\Gamma_R((p+m+1,p+1))I_{R'T'} = \sum_{k=1}^{2} E_{kj}^{(p+1)} E_{ik}^{(m+p+1)},$$

$$\Gamma_R((p+m+1,p+1))I_{R'T'}\Gamma_R((p+m+1,p+1)) = E_{ij}^{(m+p+1)}\,.$$

Since $\Gamma_R((1,p+m+1))$ will swap the spin in the $(m+p+1)$th site with the spin in site 1, very similar arguments give

$$I_{R'T'}\Gamma_R((1,p+m+1)) = \sum_{k=1}^{2} E_{ik}^{(1)} E_{kj}^{(m+p+1)},$$

$$\Gamma_R((1,p+m+1))I_{R'T'} = \sum_{k=1}^{2} E_{kj}^{(1)} E_{ik}^{(m+p+1)},$$

$$\Gamma_R((1,p+m+1))I_{R'T'}\Gamma_R((1,p+m+1)) = E_{ij}^{(m+p+1)}\,.$$

Our only task now is to evaluate traces of the form

$$\begin{aligned} &\mathrm{Tr}\left(\Gamma_R((1,p+m+1))P_{R\to(r,s,t)}I_{R'\,T'}\Gamma_T((1,p+m+1))P_{T\to(u,v,w)}I_{T'\,R'}\right) \\ &= \sum_{k,l=1}^{2} \mathrm{Tr}\left(E_{ik}^{(1)}E_{kj}^{(m+p+1)}P_{R\to(r,s,t)}E_{jl}^{(1)}E_{li}^{(m+p+1)}P_{T\to(u,v,w)}\right) \end{aligned} \tag{4.27}$$

To perform this final trace, our strategy is always the same two steps. For the first step, evaluate the trace over the $(n+p+1)$th slot. It is clear that the trace over the $p+m+1$th slot factors out and further that

$$\mathrm{Tr}\,(E_{kj}^{(m+p+1)}E_{li}^{(m+p+1)}) = \delta_{jl}\delta_{ik} \tag{4.28}$$

so that we obtain

$$\mathrm{Tr}\left(E^{(1)}_{ii}P_{R\to(r,s,t)}E^{(1)}_{jj}P_{T\to(u,v,w)}\right) \tag{4.29}$$

To evaluate this final trace we will rewrite the projectors a little. Notice that $E^{(1)}_{kk}$ only has a nontrivial action on the first site of the spin chain. Thus, we rewrite the projector, separating out the first site. As an example, consider

$$P_{R\to(r,s,t)} = \sum_{\alpha=1}^{d_t} \left|j,j^3,\alpha\right\rangle\left\langle j,j^3,\alpha\right| \otimes \sum_{\beta=1}^{d_s} \left|k,k^3,\beta\right\rangle\left\langle k,k^3,\beta\right| . \tag{4.30}$$

To make sense of this formula recall that the labels j,k,j^3,k^3 can be traded for the r,s,t labels. In going from the LHS of this last equation to the RHS we have translated labels and we assure you that nothing is lost in translation. In figure 2 we remind the reader of how the translation is performed. We will refer to the Young diagram corresponding to spin j, built with p blocks as s^p_j in what follows. The piece of the projector that acts on the first p sites is

$$P_{\to t} \equiv \sum_{\alpha=1}^{d_t} \left|j,j^3,\alpha\right\rangle\left\langle j,j^3,\alpha\right| . \tag{4.31}$$

If we couple the spins at sites $2,3,...,p$ together, we obtain the states $|j\pm\frac{1}{2},j^3\pm\frac{1}{2},\alpha\rangle$ with the degeneracy label α running from 1 to the dimension of the irreducible S_{p-1} representation associated to spin $j\pm\frac{1}{2}$. This irreducible representation is labeled by the Young diagram $s^{p-1}_{j\pm\frac{1}{2}}$. The Clebsch-Gordan coefficients

$$\left\langle j-\frac{1}{2},j^3-\frac{1}{2};\frac{1}{2},\frac{1}{2}|j,j^3\right\rangle = \sqrt{\frac{j+j^3}{2j}}\,, \tag{4.32}$$

$$\left\langle j+\frac{1}{2},j^3-\frac{1}{2};\frac{1}{2},\frac{1}{2}|j,j^3\right\rangle = -\sqrt{\frac{j-j^3+1}{2(j+1)}}\,, \tag{4.33}$$

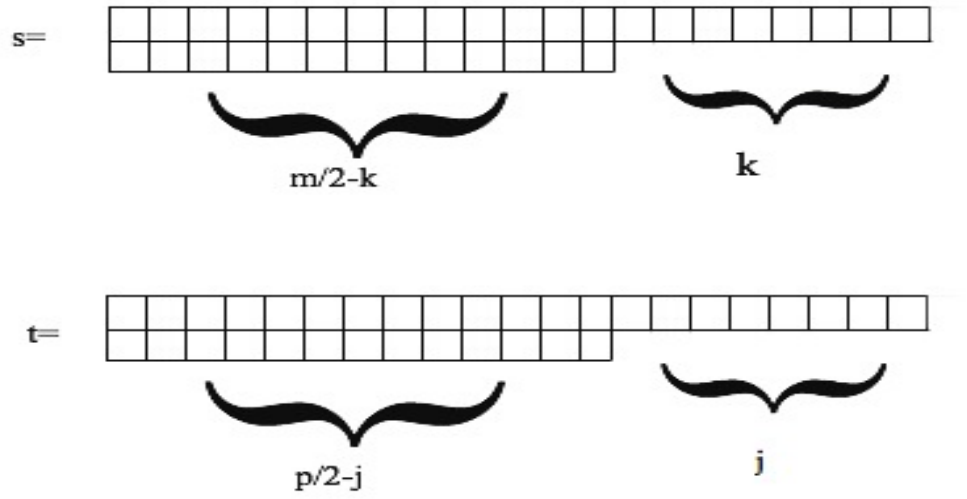


Figure 3: How to translate between the j, k and the s, t labels.

$$\left\langle j-\frac{1}{2}, j^3+\frac{1}{2}; \frac{1}{2}, -\frac{1}{2} | j, j^3 \right\rangle = \sqrt{\frac{j-j^3}{2j}}\,, \tag{4.34}$$

$$\left\langle j+\frac{1}{2}, j^3+\frac{1}{2}; \frac{1}{2}, -\frac{1}{2} | j, j^3 \right\rangle = \sqrt{\frac{j+j^3+1}{2(j+1)}}\,. \tag{4.35}$$

tell us how to couple the first site with the remaining spins to obtain the projector (4.31). Thus, we finally have ($s1 = s^{p-1}_{j-\frac{1}{2}}$, $s2 = s^{p-1}_{j+\frac{1}{2}}$)

$$|\phi, \alpha\rangle = \sqrt{\frac{j+j^3}{2j}} \left| \frac{1}{2}, \frac{1}{2}; j-\frac{1}{2}, j^3-\frac{1}{2}, \alpha \right\rangle + \sqrt{\frac{j-j^3}{2j}} \left| \frac{1}{2}, -\frac{1}{2}; j-\frac{1}{2}, j^3+\frac{1}{2}, \alpha \right\rangle\,, \tag{4.36}$$

$$\begin{aligned} |\psi, \beta\rangle \quad &= -\sqrt{\frac{j-j^3+1}{2(j+1)}} \left| \frac{1}{2}, \frac{1}{2}; j+\frac{1}{2}, j^3-\frac{1}{2}, \beta \right\rangle \\ &+ \sqrt{\frac{j+j^3+1}{2(j+1)}} \left| \frac{1}{2}, -\frac{1}{2}; j+\frac{1}{2}, j^3+\frac{1}{2}, \beta \right\rangle\,, \end{aligned} \tag{4.37}$$

$$P_{\rightarrow t} = \sum_{\alpha=1}^{d_{s1}} |\phi, \alpha\rangle \langle \phi, \alpha| + \tag{4.38}$$

We could of course perform exactly the same manipulations on the projector $P_{\rightarrow s}$ that acts on the last m sites of the spin chain. Now, using the obvious identities

$$E_{11}^{(1)} |\phi, \alpha\rangle = \sqrt{\frac{j+j^3}{2j}} \left|\frac{1}{2}, \frac{1}{2}; j-\frac{1}{2}, j^3-\frac{1}{2}, \alpha\right\rangle , \tag{4.39}$$

$$E_{22}^{(1)} |\phi, \alpha\rangle = \sqrt{\frac{j-j^3}{2j}} \left|\frac{1}{2}, -\frac{1}{2}; j-\frac{1}{2}, j^3+\frac{1}{2}, \alpha\right\rangle , \tag{4.40}$$

$$E_{11}^{(1)} |\psi, \beta\rangle = -\sqrt{\frac{j-j^3+1}{2(j+1)}} \left|\frac{1}{2}, \frac{1}{2}; j+\frac{1}{2}, j^3-\frac{1}{2}, \beta\right\rangle , \tag{4.41}$$

$$E_{22}^{(1)} |\psi, \beta\rangle = \sqrt{\frac{j+j^3+1}{2(j+1)}} \left|\frac{1}{2}, -\frac{1}{2}; j+\frac{1}{2}, j^3+\frac{1}{2}, \beta\right\rangle , \tag{4.42}$$

it becomes a simple matter to evaluate the above traces.

Finally, in the limit that we consider, the coefficients of the traces appearing in the dilatation operator are easily evaluated using

$$\frac{c_{RR'} d_T d_{r'} n}{d_{R'} d_u d_v d_w (n+m+p)} \sqrt{\frac{f_T \text{hooks}_T \text{hooks}_r \text{hooks}_s \text{hooks}_t}{f_R \text{hooks}_R \text{hooks}_u \text{hooks}_v \text{hooks}_w}}$$

$$= \frac{\sqrt{c_{RR'} c_{TT'}} \sqrt{\text{hooks}_s \text{hooks}_t \text{hooks}_v \text{hooks}_w}}{m! p!} . \tag{4.43}$$

In the above expression, r' is obtained by removing a box from r. The box that must be removed from R to obtain R' and the box that must be removed from r to obtain r' are both removed from the same row. Putting

things together we find

$$
\begin{aligned}
DO_{j,j^3}(b_0,b_1) = g_{YM}^2 \Bigg[& -\frac{1}{2}\left(m - \frac{(m+2)(j^3)^2}{j(j+1)}\right)\Delta O_{j,j^3,k,k^3}(b_0,b_1) \\
& + \sqrt{\frac{(m+2j+4)(m-2j)}{(2j+1)(2j+3)}}\,\frac{(j+j^3+1)(j-j^3+1)}{2(j+1)}\Delta O_{j+1,j^3,k,k^3}(b_0,b_1) \\
& + \sqrt{\frac{(m+2j+2)(m-2j+2)}{(2j+1)(2j-1)}}\,\frac{(j+j^3)(j-j^3)}{2j}\Delta O_{j-1,j^3,k,k^3}(b_0,b_1) \\
& -\frac{1}{2}\left(p - \frac{(p+2)(k^3)^2}{k(k+1)}\right)\Delta O_{j,j^3,k,k^3}(b_0,b_1) \\
& + \sqrt{\frac{(p+2k+4)(p-2k)}{(2k+1)(2k+3)}}\,\frac{(k+k^3+1)(k-k^3+1)}{2(k+1)}\Delta O_{j,j^3,k+1,k^3}(b_0,b_1) \\
& + \sqrt{\frac{(p+2k+2)(p-2k+2)}{(2k+1)(2k-1)}}\,\frac{(k+k^3)(k-k^3)}{2k}\Delta O_{j,j^3,k-1,k^3}(b_0,b_1)\Bigg]
\end{aligned}
\tag{4.44}
$$

where

$$
\begin{aligned}
\Delta O(b_0,b_1) \quad &= \sqrt{(N+b_0)(N+b_0+b_1)}(O(b_0+1,b_1-2)+O(b_0-1,b_1+2)) \\
&\quad -(2N+2b_0+b_1)O(b_0,b_1).
\end{aligned}
\tag{4.45}
$$

Above, we have explicitly carried out the discussion for two long rows. To obtain the result for two long columns, replace

$$\sqrt{(N+b_0)(N+b_0+b_1)} \to \sqrt{(N-b_0)(N-b_0-b_1)},$$

$$(2N+2b_0+b_1) \to (2N-2b_0-b_1)$$

in the expression for $\Delta O(b_0,b_1)$. This completes our evaluation of the dilatation operator.

4.4 Diagonalization of the Dilatation Operator

In this section we reduce the eigenvalue problem for the dilatation operator to the problem of solving a five term recursion relation. The explicit solution of this recursion relation allows us to argue that the dilatation operator reduces to a set of decoupled oscillators. Thus, the problem we are studying is indeed integrable.

We make the following ansatz for the operators of good scaling dimension

$$\sum_{b_1} f(b_0,b_1)\, O_{pq,j^3,k^3}(b_0,b_1) \;=\; \sum_{j,k,b_1} C_{pq,j^3,k^3}(j,k)\, f(b_0,b_1) O_{j,j^3,k,k^3}(b_0,b_1)\,. \tag{4.46}$$

Inserting this ansatz into (4.44) we find that the $O_{pq,j^3,k^3}(b_0,b_1)$'s satisfy the recursion relation

$$\begin{aligned}
&-\alpha_{rq,j^3,k^3} C_{rq,j^3,k^3}(j,k) = \\
&\sqrt{\frac{(m+2j+4)(m-2j)}{(2j+1)(2j+3)}}\,\frac{(j+j^3+1)(j-j^3+1)}{2(j+1)} C_{rq,j^3,k^3}(j+1,k) \\
&+\sqrt{\frac{(m+2j+2)(m-2j+2)}{(2j+1)(2j-1)}}\,\frac{(j+j^3)(j-j^3)}{2j} C_{rq,j^3,k^3}(j-1,k) \\
&-\frac{1}{2}\left(m-\frac{(m+2)(j^3)^2}{j(j+1)}\right) C_{rq,j^3,k^3}(j,k) \\
&+\sqrt{\frac{(p+2k+4)(p-2k)}{(2k+1)(2k+3)}}\,\frac{(k+k^3+1)(k-k^3+1)}{2(k+1)} C_{rq,j^3,k^3}(j,k+1) \\
&+\sqrt{\frac{(p+2k+2)(p-2k+2)}{(2k+1)(2k-1)}}\,\frac{(k+k^3)(k-k^3)}{2k} C_{rq,j^3,k^3}(j,k-1) \\
&-\frac{1}{2}\left(p-\frac{(p+2)(k^3)^2}{k(k+1)}\right) C_{rq,j^3,k^3}(j,k)\,.
\end{aligned} \tag{4.47}$$

Exploiting the $j^3 \to -j^3$ and $k^3 \to -k^3$ symmetries of this equation, we need only solve for the $j^3 \geq 0$ and $k^3 \geq 0$ cases. The ranges for j and k are

$$0 \leq |j^3| \leq j \leq \frac{m}{2} \qquad 0 \leq |k^3| \leq k \leq \frac{p}{2}\,.$$

From the form of the recursion relation, it is natural to make the "separation of variables" ansatz

$$C_{rq,j^3,k^3}(j,k) = C_{r,j^3}(j) C_{q,k^3}(k)\,. \tag{4.48}$$

Our five term recurrence relation now reduces to two three term recurrence relations

$$\begin{aligned}
&-\alpha_{r,j^3} C_{p,j^3}(j,) = \\
&\sqrt{\frac{(m+2j+4)(m-2j)}{(2j+1)(2j+3)}}\,\frac{(j+j^3+1)(j-j^3+1)}{2(j+1)} C_{r,j^3}(j+1) \\
&+\sqrt{\frac{(m+2j+2)(m-2j+2)}{(2j+1)(2j-1)}}\,\frac{(j+j^3)(j-j^3)}{2j} C_{r,j^3}(j-1) \\
&-\frac{1}{2}\left(m - \frac{(m+2)(j^3)^2}{j(j+1)}\right) C_{r,j^3}(j)\,,
\end{aligned} \tag{4.49}$$

$$\begin{aligned}
&-\alpha_{q,k^3} C_{q,k^3}(k) = \\
&\sqrt{\frac{(p+2k+4)(p-2k)}{(2k+1)(2k+3)}}\,\frac{(k+k^3+1)(k-k^3+1)}{2(k+1)} C_{q,k^3}(k+1) \\
&+\sqrt{\frac{(p+2k+2)(p-2k+2)}{(2k+1)(2k-1)}}\,\frac{(k+k^3)(k-k^3)}{2k} C_{q,k^3}(k-1) \\
&-\frac{1}{2}\left(p - \frac{(p+2)(k^3)^2}{k(k+1)}\right) C_{q,k^3}(k)
\end{aligned} \tag{4.50}$$

These are identical to the three term recursion relations that appear in [17]. To solve these recurrence relations, introduce the Hahn polynomial[42]

$$Q_n(x;\alpha,\beta,N) \equiv {}_3F_2\left(\begin{matrix}-n,n+\alpha+\beta+1,-x\\ \alpha+1,-N\end{matrix}\Big|1\right) \tag{4.51}$$

From the recurrence relation obeyed by Hahn polynomials (see equation (1.5.3) in [42]) we have

$$\begin{aligned} & r\,{}_3F_2\left(\begin{matrix}|j^3|-j,j+1+|j^3|,-r\\ 1,|j^3|-\frac{m}{2}\end{matrix}\Big|1\right) = \\ & \\ & \frac{(j+j^3+1)(j-j^3+1)(m-2j)}{2(j+1)(2j+1)}{}_3F_2\left(\begin{matrix}-1+|j^3|-j,j+2+|j^3|,-r\\ 1,|j^3|-\frac{m}{2}\end{matrix}\Big|1\right) \\ & -\left(\frac{m}{2}-\frac{(m+2)(j^3)^2}{2j(j+1)}\right){}_3F_2\left(\begin{matrix}|j^3|-j,j+1+|j^3|,-r\\ 1,|j^3|-\frac{m}{2}\end{matrix}\Big|1\right) \\ & +\frac{(j+j^3)(j-j^3)(m+2j+2)}{2j(2j+1)}{}_3F_2\left(\begin{matrix}1+|j^3|-j,j+|j^3|,-r\\ 1,|j^3|-\frac{m}{2}\end{matrix}\Big|1\right) \end{aligned}$$

Consequently, our recursion relation is solved by

$$C_{r,j^3}(j) = (-1)^{\frac{m}{2}-p}\left(\frac{m}{2}\right)!\sqrt{\frac{(2j+1)}{\left(\frac{m}{2}-j\right)!\left(\frac{m}{2}+j+1\right)!}}{}_3F_2\left(\begin{matrix}|j^3|-j,j+|j^3|+1,-r\\ |j^3|-\frac{m}{2},1\end{matrix}\Big|1\right) \tag{4.52}$$

$$|j^3| \le j \le \frac{m}{2}, \qquad 0 \le r \le \frac{m}{2}-|j^3|$$

and

$$C_{q,k^3}(k) = (-1)^{\frac{p}{2}-q}\left(\frac{p}{2}\right)!\sqrt{\frac{(2k+1)}{\left(\frac{p}{2}-k\right)!\left(\frac{p}{2}+k+1\right)!}}{}_3F_2\left(\begin{matrix}|k^3|-k,k+|k^3|+1,-q\\ |k^3|-\frac{p}{2},1\end{matrix}\Big|1\right) \tag{4.53}$$

$$|k^3| \le k \le \frac{p}{2}, \qquad 0 \le q \le \frac{p}{2}-|k^3|\,.$$

The associated eigenvalues are

$$-\alpha_{rq,j^3,k^3} = -2(r+q) = 0,-2,-4,...,-(m-2|j^3|+p-2|k^3|)\,. \tag{4.54}$$

Our eigenfunctions are essentially the Hahn polynomials. It is a well known fact that the Hahn polynomials are closely related to the Clebsch-Gordan coefficients of $SU(2)$ [43].

The eigenproblem of the dilatation operator now reduces to solving

$$\lambda \sum_{b_1} f(b_0, b_1)\, O_{rq,j^3,k^3}(b_0, b_1) = -\alpha_{rq,j^3,k^3} \sum_{b_1} f(b_0, b_1)\, \Delta O_{rq,j^3,k^3}(b_0, b_1)\,. \tag{4.55}$$

This eigenproblem implies $f(b_0, b_1)$ satisfy the recursion relation

$$\begin{aligned} &-\alpha_{rq,j^3,k^3} g_{YM}^2 [\, \sqrt{(N+b_0)(N+b_0+b_1)}(f(b_0-1, b_1+2) + f(b_0+1, b_1-2)) \\ &-(2N+2b_0+b_1) f(b_0, b_1)] = \lambda f(b_0, b_1) \end{aligned} \tag{4.56}$$

Since we work at large N, we can replace (4.56) by

$$\begin{aligned} &\lambda f(b_0, b_1) = -\alpha_{rq,j^3,k^3} g_{YM}^2 [\, \sqrt{(N+b_0)(N+b_0+b_1+1)} f(b_0-1, b_1+2) \\ &+ \sqrt{(N+b_0+1)(N+b_0+b_1)} f(b_0+1, b_1-2) - (2N+2b_0+b_1) f(b_1, b_1)] \end{aligned} \tag{4.57}$$

This recursion relation is precisely the recursion relation of the finite oscillator [44]! In the continuum limit (which corresponds to the large N limit) we recover the usual description of the harmonic oscillator, demonstrating rather explicitly that the eigenproblem of the dilatation operator reduces to solving a set of decoupled harmonic oscillators. The solution to (4.56) is [44]

$$f(b_0, b_1) = (-1)^n (\frac{1}{2})^{N+b_0+\frac{b_1}{2}} \sqrt{\tbinom{2N+2b_0+b_1}{N+b_0+b_1} \tbinom{2N+2b_0+b_1}{n}}\, {}_2F_1\left(\begin{matrix} -n, -(N+b_0+b_1) \\ -(2N+2b_0+b_1) \end{matrix} \middle| 2\right). \tag{4.58}$$

These solutions are closely related to the symmetric Kravchuk polynomial $K_n(x, 1/q, p)$ defined by

$${}_2F_1\left(\begin{matrix} -n, -x \\ -p \end{matrix}; q\right) = K_n(x, 1/q, p)\,. \tag{4.59}$$

The corresponding eigenvalue is $\lambda = 2n\alpha_{rq,j^3,k^3} g_{YM}^2$. Recall that $b_1 \geq 0$ so that only half of the wavefunctions are selected (those that vanish when $b_1 = 0$) and consequently the eigenvalue λ level spacing is $4\alpha_{rq,j^3,k^3} g_{YM}^2 = 8(p+q) g_{YM}^2$.

4.5 Summary

In this chapter we have studied the action of the dilatation operator on restricted Schur polynomials $\chi_{R,(r,s,t)}(Z,Y,X)$, built from three complex scalars X, Y and Z and labeled by Young diagrams with at most two rows or two columns. The operators have $O(N)$ fields of each of the three flavors, but there are many many more Zs than Xs or Ys. Our main result is that the dilatation operator reduces to a set of decoupled oscillators and is hence an integrable system. If we have m Ys and p Xs with p, m both even, we obtain a set of oscillators with frequency ω_{ij} and degeneracy d_{ij} given by

$$\omega_{ij} = 8(i+j) g_{YM}^2, \qquad d_{ij} = (2(m-i)+1)\,(2(p-j)+1)\,, \tag{4.60}$$

$$i = 0, 1, ..., m, \qquad j = 0, 1, ..., p\,.$$

If p is even and m is odd we have

$$\omega_{ij} = 8(i+j) g_{YM}^2, \qquad d_{ij} = 2\,(m-i+1)\,(2(p-j)+1)\,, \tag{4.61}$$

$$i = 0, 1, ..., m, \qquad j = 0, 1, ..., p\,.$$

If m is even and p is odd we have

$$\omega_{ij} = 8(i+j) g_{YM}^2, \qquad d_{ij} = 2\,(2(m-i)+1)\,(p-j+1)\,, \tag{4.62}$$

$$i = 0, 1, ..., m, \qquad j = 0, 1, ..., p\,.$$

If both p and m are odd we have

$$\omega_{ij} = 8(i+j)g_{YM}^2, \qquad d_{ij} = 4\,(m-i+1)\,(p-j+1)\,, \tag{4.63}$$

$$i = 0, 1, ..., m, \qquad j = 0, 1, ..., p\,.$$

4.6 Discussion

The oscillators corresponding to a zero frequency are BPS operators built using three complex scalars X, Y and Z.

The form of the dilatation operator (4.44) is intriguing: it looks like the sum of two of the dilatation operators computed in [17], with one acting on the Ys (with quantum numbers k, k^3) and one acting on the Xs (with quantum numbers j, j^3). With the benefit of hindsight, could we have anticipated this structure? The bulk of our effort involved evaluating traces like this one

$$\mathrm{Tr}\left(\Big[\Gamma_R((p{+}m{+}1,p{+}1)), P_{R\to(r,s,t)}\Big] I_{R'\,T'} \Big[\Gamma_T((p{+}m{+}1,p{+}1)), P_{T\to(u,v,w)}\Big] I_{T'\,R'}\right).$$

Notice that both $\Gamma_R((p+m+1,p+1))$ and $I_{R'\,T'}$ do not act on the first p sites of the spin chain. Further, our projector factorizes into a projector acting on the first p sites times a projector acting on the remaining m sites. Consequently, the trace over the first p sites gives $\delta_{tw}d_w$. The trace that remains is exactly of the form considered in [17], explaining our final answer (4.44).

5 From Large N Nonplanar Anomalous Dimensions to Open Spring Theory

The results presented in this chapter are novel and were first reported in Robert de Mello Koch, Garreth Kemp, Stephanie Smith. *From Large N Nonplanar Anomalous Dimensions to Open Spring Theory* Phys.Lett. B711 (2012) 398-403

In this chapter we will consider the diagonalization of the one loop dilatation operator when acting on restricted Schur polynomials $\chi_{R,(r,s)}(Z,Y)$ built from n Z fields and m Y fields, with $m \ll n$ and m,n both order N, as in [56]. For a system of p sphere giant gravitons, R is a Young diagram with p columns and $m+n$ boxes, r is a Young diagram with p columns and n boxes and s is a Young diagram with at most p columns. After diagonalizing on the s label, [56] finds that the resulting equations for the action of the dilatation operator can be labeled by configurations of open strings that are consistent with the Gauss Law, as well as labels specifying the Young diagram r, as defined in Figure 4. Using these configurations we now diagonalize on the r label.

For the configuration C with n_{ij} open strings stretching between branes i and j the one loop dilatation operator is given by

$$DO_C(\{s_i\}) = -g_{YM}^2 \sum_{\alpha\beta} n_{\alpha\beta}\Delta_{\alpha\beta}\, O_C(\{s_i\}) \tag{5.1}$$

where the operator Δ_{ij} acts as follows (Δ_{ij} only changes the values of s_i and s_j so that these are the only two variables that we display in the next

equation)

$$\Delta_{ij} O_C(s_i, s_j) = -(c_i + c_j) O_C(s_i, s_j) + \sqrt{c_i c_j}(O_C(s_i+1, s_j-1) + O_C(s_i-1, s_j+1)) \,. \tag{5.2}$$

In this last equation c_a is the factor of the last box in column a. Recall that a box in row i and column j has a factor $N - i + j$. The primary goal of this article is to explain how to diagonalize (5.1). This is achieved by mapping the operators $O_C(s_i, s_j)$ into states in the carrier space of a specific $U(N)$ irreducible representation. The dilatation operator is mapped into a $u(n)$ valued operator and, as a result, can easily be diagonalized. We then go on to show that the resulting spectrum is reproduced by a classical model of springs between masses.

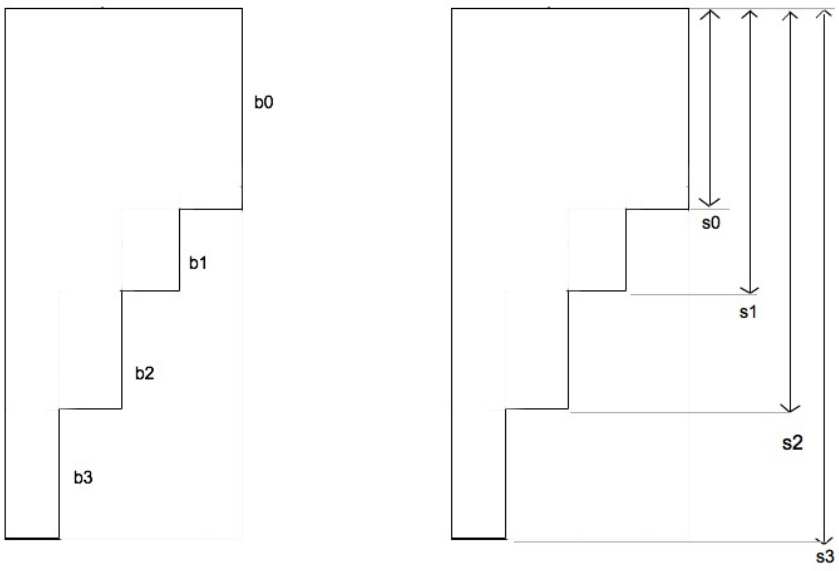


Figure 4: Definition of the b_is and s_is in terms of a Young diagram for $c = 4$ columns. The relation between the s_i and the b_i is easily read from the figure. For example, $s_2 = b_0 + b_1 + b_2$. Columns are ordered so that column length increases. They are then numbered starting from 0. For the Young diagram shown, the right most column is column 0 and the left most is column 3. The generalization to any c should be obvious.

5.1 Nonplanar Dilatation Operator

To start we will review a few elementary facts, familiar from angular momentum in quantum mechanics, that will play an important role later. The fundamental representation of $u(N)$ represents the elements of the Lie alge-

bra as $N \times N$ matrices. The generators can be taken as

$$(E_{kl})_{ab} = \delta_{ak}\delta_{bl}, \qquad k, l, a, b = 1, 2, ..., N\,. \tag{5.3}$$

We will study the operators (the labeling is such that $i > j$ i.e. Q_{ij} is not defined if $i < j$)

$$Q_{ij} = \frac{E_{ii} - E_{jj}}{2}, \qquad Q^+_{ij} = E_{ij}, \qquad Q^-_{ij} = E_{ji}\,, \tag{5.4}$$

which obey the familiar algebra of angular momentum raising and lowering operators

$$[Q_{ij}, Q^+_{ij}] = Q^+_{ij}, \qquad [Q_{ij}, Q^-_{ij}] = -Q^-_{ij}, \qquad [Q^+_{ij}, Q^-_{ij}] = 2Q_{ij}\,. \tag{5.5}$$

Although these commutators have been computed making use of the fundamental representation, we know that they would be the same if they had been computed in any representation and they define the representation independent Lie algebra.

General representations of these $su(2)$ subalgebras can be labeled with the eigenvalue of

$$L^2_{ij} \equiv Q^-_{ij}Q^+_{ij} + Q^2_{ij} + Q_{ij} = Q^+_{ij}Q^-_{ij} + Q^2_{ij} - Q_{ij} \tag{5.6}$$

and states in the representation are labeled by the eigenvalue of Q_{ij}

$$Q_{ij}|\lambda, \Lambda\rangle = \lambda|\lambda, \Lambda\rangle\,, \qquad L^2_{ij}|\lambda, \Lambda\rangle = (\Lambda^2 + \Lambda)|\lambda, \Lambda\rangle\,, \qquad -\Lambda \leq \lambda \leq \Lambda\,. \tag{5.7}$$

Recall that

$$Q^+_{ij}|\lambda, \Lambda\rangle = c_+|\lambda + 1, \Lambda\rangle\,, \qquad c_+ = \sqrt{(\Lambda + \lambda + 1)(\Lambda - \lambda)} \tag{5.8}$$

and

$$Q^-_{ij}|\lambda,\Lambda\rangle = c_-|\lambda-1,\Lambda\rangle\,, \qquad c_- = \sqrt{(\Lambda+\lambda)(\Lambda-\lambda+1)} \tag{5.9}$$

The N operators E_{ii} commute so that we can always choose a basis in which they are simultaneously diagonal. Recall the definition of b_i $i = 0, 1, ..., c-1$ for a Young diagram with c columns, given in Figure 4. The restricted Schur polynomials labeled by the Young diagram shown is identified with the state with $E_{ii} = 2(\,N - s_i\,)$. The advantage of identifying the restricted Schur polynomials with states of a $U(N)$ representation is that we can now write the dilatation operator as a $u(N)$ valued operator. In particular, the operators Δ_{ij} are

$$\Delta_{ij} = -\frac{1}{2}(E_{ii}+E_{jj}) + Q^-_{ij} + Q^+_{ij}\,. \tag{5.10}$$

For simplicity we will now focus on the case $c = 2$. In this case, identify

$$c_- = \sqrt{(N-b_0)(N-b_0-b_1+1)}, \qquad c_+ = \sqrt{(N-b_0+1)(N-b_0-b_1)} \tag{5.11}$$

so that

$$\Lambda = \frac{1}{2}b_{1,\max}, \qquad \lambda = \frac{1}{2}b_1\,.$$

We will focus on $b_{1,\max}$ even, so that Λ is integer. Not all states of the irreducible representation participate: because $b_1 \geq 0$ we have $\lambda \geq 0$. Thus, of the $2b_{1,\max}+1$ states, only $b_{1,\max}+1$ of them remain. Finally, we are interested in the limit $b_{1,\max} \sim \sqrt{N}$ with $N \to \infty$. It is only in this limit that (5.1) holds. Away from this limit (5.1) picks up corrections of order $1/b_{1,\max}$[56]. There is an obvious extension of this discussion for $c > 2$.

5.2 Strings between 2 giants

Consider a system of p-giants with p arbitrary except that we fix it to be $O(1)$. The Young diagrams relevant for these states have p columns. Consider the situation for which we have $2n_{ij}$ strings stretching between giants i and j. See Figure 5 for an example of the label C when $p = 6$ and $2n_{ij} = 4$. The results of this section are also directly applicable to the case that pairs of mutually distinct branes have strings stretching between them. In this case, the action of the dilatation operator is given by a sum of terms which commute and can each be diagonalized using the same method.

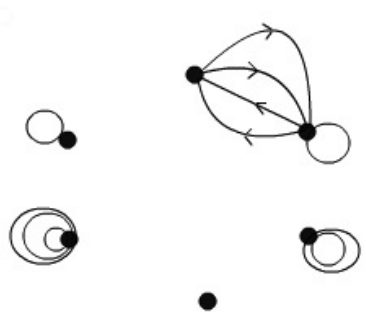


Figure 5: The label C for a system of 6 giants. $2n_{12} = 4$ strings stretch between branes 1 and 2. There is one more string attached to brane 2. Two strings are attached to brane 3, 3 strings to brane 5 and a single string to brane 6. The dilatation operator action depends only on the strings stretching between different branes[56].

Construction of Creation and Annihilation Operators: In this case

$$D = -2n_{ij} g_{YM}^2 \Delta_{ij} \,. \tag{5.12}$$

For a creation operator we want

$$[D, A^\dagger] = \alpha A^\dagger \tag{5.13}$$

with $\alpha > 0$. Make the ansatz

$$A^\dagger = aE_{ii} + bE_{jj} + cE_{ij} + dE_{ji}\,. \tag{5.14}$$

It is straight forward to verify that (5.13) implies

$$A^\dagger = \frac{1}{2}(E_{ii} - E_{jj}) + \frac{1}{2}E_{ij} - \frac{1}{2}E_{ji} \tag{5.15}$$

and $\alpha = 4n_{ij}g_{YM}^2$. To implement the condition $b_1 > 0$ we need to require that the oscillator wave function has a node at the origin - thus only odd parity (i.e. odd under $b_1 \to -b_1$) states are kept. This implies that half the states are kept so that we land up with a frequency of $8n_{ij}g_{YM}^2$. For $n_{ij} = 1$ this is in complete agreement with spectrum computed in [13, 54]. Thus, the spectrum of the dilatation operator is

$$\lambda = (8n_{ij}g_{YM}^2)n \tag{5.16}$$

with n a not negative integer. This is in complete agreement with the spectrum computed in [17]. There is a simple algebra obeyed by the creation and annihilation operators of this oscillator

$$[A, A^\dagger] = \frac{1}{2}(E_{ii}+E_{jj})+\Delta_{ij} = 2N-2b_0-b_1-\frac{D}{2g_{YM}^2} = b_{1,\max}-\frac{D}{2g_{YM}^2}\,. \tag{5.17}$$

If we introduce the oscillators $A = \sqrt{b_{1,\max}}a$ we find, for any state of finite energy in the $b_{1,\max} \to \infty$ limit

$$[a, a^\dagger] = 1 - \frac{D}{2b_{1,\max}g_{YM}^2} = 1\,. \tag{5.18}$$

Connection to Continuum Limit: We can ask how this compares to the frequencies computed after we have taken the continuum limit of the Δ_{ij}, described in appendix H of [56]. From that appendix, we find

$$D = -2g_{YM}^2 n_{ij} M_{ab} \left(\frac{\partial}{\partial x_a} \frac{\partial}{\partial x_b} - \frac{x_a x_b}{4} \right) \tag{5.19}$$

with

$$M_{11} = M_{22} = 1, \qquad M_{ij} = M_{ji} = -1.$$

The two frequencies are $4n_{ij}g_{YM}^2$ and 0. The zero frequency corresponds to the motion of the center of mass ($x_{cm} \propto x_i + x_j$). Fix this center of mass motion because the system of giants is fixed. The nonzero frequency reproduces what we found above, again after dropping half the states. Clearly then, the continuum limit catches the complete large $b_{1,\max}$ dynamics.

Classical Model: The operators we study are nearly supersymmetric so that it is natural to expect that they correspond to fast moving strings on the D-brane. It is thus natural to associate them with null trajectories in $AdS_5 \times S^5$ that are contained in the D-brane worldvolume. This analysis has been performed in [58]. See [59, 60] for additional relevant and useful discussion. The resulting null trajectory leads to a pp-wave and the light cone Hamiltonian is related to the anomalous dimension

$$H_{\text{light cone}} = \frac{1}{P^+} H_\perp = \Delta - n_Z - n_Y = D \tag{5.20}$$

where $H_\perp$ describes string oscillations in the perpendicular (to string motion) directions and $n_Z(n_Y)$ are the number of Zs (Ys) in the operator. See also [85] which is relevant to our discussion. What should we use for $H_\perp$? When we change the number of Z's in the giant we change the radius of the circle

on which it is orbiting; this corresponds to the direction transverse to the giants direction of motion - i.e. the oscillator that we have diagonalized above is describing oscillations in the perpendicular (to string motion) directions.

The Gauss Law picture of [56] suggests that the configuration we study consists of $2n_{12}$ strings stretching between the two giants. Each string is a single Y - so these are short strings that we will model as two endpoints. The spring constant for springs connected in parallel is the sum of the individual spring constants. Thus, the configuration we study will have $k \propto n_{12}$. The scale of the anomalous dimension is set by g_{YM}^2. Under AdS/CFT the anomalous dimension maps to an energy, so that g_{YM}^2 naturally sets the energy scale. To ensure that the scale of the potential energy is set by g_{YM}^2 we will choose the spring constant $k \propto g_{YM}^2$. Making a choice of a constant that will prove to be convenient below, we set $k = 4g_{YM}^2 n_{12}$. Adding a kinetic energy for the string endpoints, the Lagrangian describing this system is

$$L_\perp = \frac{1}{2}\dot{x}_i^2 + \frac{1}{2}\dot{x}_j^2 - \frac{1}{2}(4g_{YM}^2 n_{ij})(x_i - x_j)^2 \,. \tag{5.21}$$

The equations of motion (assuming the center of mass is at rest at the origin) are solved by

$$x_i = -x_j = A\sin(\sqrt{8g_{YM}^2 n_{ij}}t + \phi_0) \,. \tag{5.22}$$

The energy of this solution is given by

$$E_\perp = \frac{1}{2}\dot{x}_i^2 + \frac{1}{2}\dot{x}_j^2 + \frac{1}{2}(4g_{YM}^2 n_{ij})(x_i - x_j)^2 = A^2(8g_{YM}^2 n_{ij}) \tag{5.23}$$

which matches the anomalous dimensions.

5.3 Strings between 3 giants

In this section we consider the situation for which we have n_{ij} strings stretching between giants i and j, n_{jk} strings stretching between giants j and k and n_{ik} strings stretching between giants i and k. See Figure 6 for an example of the label C when $p = 5$ and $n_{ij} = 4$, $n_{jk} = 2$ and $n_{ik} = 0$. The results of this section are also directly applicable to the case that any number of pairs and/or triples of mutually distinct branes have strings stretching between them. Just like in the last section, in this case the action of the dilatation operator is given by a sum of terms which commute and can each be diagonalized using the same method.

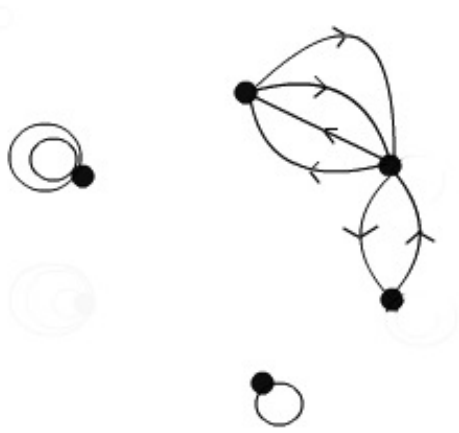


Figure 6: The label C for a system of 5 giants. $n_{12} = 4$ strings stretch between branes 1 and 2 and $n_{23} = 2$ strings stretching between branes 2 and 3. A string is attached to brane 4 and two strings are attached to brane 5.

Construction of Creation and Annihilation Operators: In this case, to be general, we should introduce the parameters n_{ij}, n_{ik} and n_{jk} (repeated indices are not summed)

$$D = -g_{YM}^2(n_{ij}\Delta_{ij} + n_{ik}\Delta_{ik} + n_{jk}\Delta_{jk})\,. \tag{5.24}$$

For any label C, the Gauss Law implies that $n_{ij} + n_{ik}$ is even, $n_{ij} + n_{jk}$ is even and $n_{ik} + n_{jk}$ is even. For a creation operator we again want (5.13). Make the ansatz

$$A^\dagger = aE_{ii} + bE_{ij} + cE_{ik} + dE_{ji} + eE_{jj} + fE_{jk} + gE_{ki} + hE_{kj} + iE_{kk}\,. \tag{5.25}$$

Then (5.13) gives 3 $A^\dagger$s. There is a nice analytic formula for the frequencies of these operators $\Omega_i = 2g_{YM}^2\omega_i$ where

$$\omega_1 = 2\gamma, \qquad \omega_2 = n_{ik} + n_{ij} + n_{jk} + \gamma, \qquad \omega_3 = n_{ik} + n_{ij} + n_{jk} - \gamma, \tag{5.26}$$

where

$$\gamma = \sqrt{n_{ij}^2 + n_{ik}^2 + n_{jk}^2 - n_{ij}n_{ik} - n_{jk}n_{ik} - n_{ij}n_{jk}}\,. \tag{5.27}$$

This proves that the spectrum of three giant system is indeed that of a set of oscillators. For the frequency ω_1 we find

$$\begin{aligned} A_1 \;=\; & \mathcal{N}_1\Big[(n_{ij} - n_{ik})(n_{ik} - n_{ij} - \gamma)E_{ii} \\ & + \left((n_{ik} - n_{ij})(n_{ik} - n_{ij} - \gamma) + (n_{ik} - n_{jk})(n_{ik} - n_{jk} - \gamma)\right)E_{ij} \\ & -(n_{ik} - \gamma - n_{jk})(n_{ik} - n_{jk})E_{ik} - (n_{ij} - n_{jk} + \gamma)(n_{ij} - n_{jk})E_{ji} \\ & +(n_{ij} - n_{ik} + \gamma)(n_{ij} - n_{jk})E_{jj} + (n_{ij} - n_{jk})(n_{ik} - n_{jk})E_{jk} \\ & + \left((n_{jk} - n_{ij})(n_{jk} - n_{ij} - \gamma) + (n_{ik} - n_{ij})(n_{ik} - n_{ij} - \gamma)\right)E_{ki} \\ & -((n_{ij} - n_{ik})(n_{ij} - n_{ik} + 2\gamma) + \gamma^2)E_{kj} \\ & -(n_{ij} - n_{ik} + \gamma)(n_{ik} - n_{jk})E_{kk}\Big] \end{aligned} \tag{5.28}$$

where

$$\begin{aligned}
&\mathcal{N}_1^{-2} = \\
&(n_{ij}-n_{ik})^2(n_{ik}-n_{ij}-\gamma)^2 + ((n_{ik}-n_{ij})(n_{ik}-n_{ij}-\gamma) \\
&+(n_{ik}-n_{jk})(n_{ik}-n_{jk}-\gamma))^2 + (-n_{ik}+n_{jk}+\gamma)^2(n_{ik}-n_{jk})^2 \\
&+(-n_{ij}+n_{jk}-\gamma)^2(n_{ij}-n_{jk})^2 + (n_{ij}-n_{ik}+\gamma)^2(n_{ij}-n_{jk})^2 \\
&+(n_{ij}-n_{jk})^2(n_{ik}-n_{jk})^2 + ((n_{jk}-n_{ij})(-n_{ij}+n_{jk}-\gamma) \\
&+(n_{ik}-n_{ij})(n_{ik}-n_{ij}-\gamma))^2 + (-(n_{ij}-n_{ik})(n_{ij}-n_{ik}+2\gamma)-\gamma^2)^2 \\
&+(n_{ik}-n_{ij}-\gamma)^2(n_{ik}-n_{jk})^2
\end{aligned} \tag{5.29}$$

For the frequency ω_2 we find

$$\begin{aligned}
A_2 = \mathcal{N}_2\Big((n_{jk}-n_{ij}-\gamma)(E_{ii}+E_{ji}+E_{ki}) + (n_{ij}-n_{ik}+\gamma)(E_{ij}+E_{jj}+E_{kj}) \\
+(n_{ik}-n_{jk})(E_{ik}+E_{jk}+E_{kk})\Big)
\end{aligned} \tag{5.30}$$

where

$$\mathcal{N}_2^{-1} = \sqrt{6\gamma(2\gamma+2n_{ij}-n_{jk}-n_{ik})}\,. \tag{5.31}$$

For the frequency ω_3 we find

$$\begin{aligned}
A_3 = \mathcal{N}_3\Big((n_{jk}-n_{ij}+\gamma)(E_{ii}+E_{ji}+E_{ki}) + (n_{ij}-n_{ik}-\gamma)(E_{ij}+E_{jj}+E_{kj}) \\
+(n_{ik}-n_{jk})(E_{ik}+E_{jk}+E_{kk})\Big)
\end{aligned} \tag{5.32}$$

where

$$\mathcal{N}_3^{-1} = \sqrt{6\gamma(2\gamma-2n_{ij}+n_{jk}+n_{ik})}\,. \tag{5.33}$$

These oscillators close the following algebra

$$[A_2, A_2^\dagger] = \frac{4}{3}\,(3N-3b_0-2b_1-b_2) + \frac{1}{3}(\Delta_{ij}+\Delta_{ik}+\Delta_{jk}) - P_2,$$

$$
\begin{aligned}
&[A_2, A_3^\dagger] = -A_1 \\
&[A_2, A_1^\dagger] = A_3, \\
&[A_2, A_3] = 0 = [A_1, A_2] \\
&[A_3, A_3^\dagger] = \frac{4}{3}\left(3N - 3b_0 - 2b_1 - b_2\right) + \frac{1}{3}(\Delta_{ij} + \Delta_{ik} + \Delta_{jk}) - P_3 \\
&[A_3, A_1] = A_2 \\
&[A_3, A_1^\dagger] = 0, \\
&[A_1, A_1^\dagger] = P_3 - P_2\,,
\end{aligned}
\tag{5.34}
$$

where

$$
\begin{aligned}
(4\gamma^2 - 2\gamma(n_{jk} + n_{ik} - 2n_{ij})P_2 &= \left(n_{ij} - n_{jk} + \gamma\right)^2 E_{ii} + \left(n_{ij} - n_{ik} + \gamma\right)^2 E_{jj} \\
+ \left(n_{jk} - n_{ik}\right)\left(n_{ij} - n_{jk} + \gamma\right)\left(E_{ik}{+}E_{ki}\right) &+ \left(n_{jk} - n_{ij} - \gamma\right)\left(n_{ij} - n_{ik} + \gamma\right)\left(E_{ji}{+}E_{ij}\right) \\
+ \left(n_{ij} - n_{ik} + \gamma\right)\left(n_{ik} - n_{jk}\right)\left(E_{jk} + E_{kj}\right) &+ \left(n_{ik} - n_{jk}\right)^2 E_{kk}
\end{aligned}
$$

and

$$
\begin{aligned}
(4\gamma^2 + 2\gamma(n_{jk} + n_{ik} - 2n_{ij})P_3 &= \left(n_{ij} - n_{jk} - \gamma\right)^2 E_{ii} + \left(n_{ij} - n_{ik} - \gamma\right)^2 E_{jj} \\
+ \left(n_{ij} - n_{ik} - \gamma\right)\left(n_{jk} - n_{ij} + \gamma\right)\left(E_{ij}{+}E_{ji}\right) &+ \left(n_{jk} - n_{ik}\right)\left(-n_{jk} + n_{ij} - \gamma\right)\left(E_{ik}{+}E_{ki}\right) \\
+ \left(n_{ik} - n_{jk}\right)\left(n_{ij} - n_{ik} - \gamma\right)\left(E_{jk} + E_{kj}\right) &+ \left(n_{ik} - n_{jk}\right)^2 E_{kk}\,.
\end{aligned}
$$

Note also that

$$
\begin{aligned}
[A_2, A_2^\dagger] &= [A_3, A_3^\dagger] + [A_1, A_1^\dagger] \\
[P_2, A_2^\dagger] = A_2^\dagger, \qquad [P_3, A_3^\dagger] &= A_3^\dagger\,.
\end{aligned}
\tag{5.35}
$$

Thus, if we set

$$
A_1 = \sqrt{3N - 3b_0 - 2b_1 - b_2}\,\sqrt{\frac{4}{3}}a_1, \qquad A_2 = \sqrt{3N - 3b_0 - 2b_1 - b_2}\,\sqrt{\frac{4}{3}}a_2,
$$

$$A_3 = \sqrt{3N - 3b_0 - 2b_1 - b_2}\sqrt{\frac{4}{3}}a_3 \tag{5.36}$$

and consider the limit in which $\sqrt{3N - 3b_0 - 2b_1 - b_2} \sim \sqrt{N} \to \infty$ we find

$$[a_1, a_1^\dagger] = 0, \qquad [a_2, a_2^\dagger] = 1, \qquad [a_3, a_3^\dagger] = 1$$

and all other commutators vanish. Thus, we only have 2 oscillators. After keeping only the states that have a node at $b_1 = 0$, we find that these oscillators have a frequency $4g_{YM}^2\omega_2$ and $4g_{YM}^2\omega_3$.

Connection to Continuum Limit: We can again ask how this compares to the frequencies computed after we have taken the continuum limit of the Δ_{ij}, described in appendix H of [56]. From that appendix, we find

$$D = -g_{YM}^2 M_{ab}\left(\frac{\partial}{\partial x_a}\frac{\partial}{\partial x_b} - \frac{x_a x_b}{4}\right) \tag{5.37}$$

with

$$M = \begin{bmatrix} n_{ij} + n_{ik} & -n_{ij} & -n_{ik} \\ -n_{ij} & n_{ij} + n_{jk} & -n_{jk} \\ -n_{ik} & -n_{jk} & n_{ik} + n_{jk} \end{bmatrix}. \tag{5.38}$$

The three frequencies are $\Lambda_i = 2g_{YM}^2\lambda_i$, where

$$\lambda_1 = 0, \qquad \lambda_2 = n_{ik} + n_{ij} + n_{jk} + \gamma, \qquad \lambda_3 = n_{ik} + n_{ij} + n_{jk} - \gamma, \tag{5.39}$$

and γ is defined as above. The zero frequency again corresponds to the center of mass, which we fix. Only the states with a node at $b_1 = 0$ will be retained, which doubles the above frequencies. Notice that the continuum limit has caught the full large b_1 spectrum.

Classical Model: Arguing exactly as we did in the last section leads to

$$L_\perp = \frac{1}{2}\dot{x}_i^2 + \frac{1}{2}\dot{x}_j^2 + \frac{1}{2}\dot{x}_k^2 - \frac{1}{2}(2g_{YM}^2 n_{ij})(x_i - x_j)^2$$

$$-\frac{1}{2}(4g_{YM}^2 n_{ik})(x_i - x_k)^2 - \frac{1}{2}(2g_{YM}^2 n_{jk})(x_j - x_k)^2 \,. \tag{5.40}$$

The equations of motion are

$$\frac{d^2 x_i}{dt^2} = -2g_{YM}^2 n_{ij}(x_i - x_j) - 2g_{YM}^2 n_{ik}(x_i - x_k)\,, \tag{5.41}$$

$$\frac{d^2 x_j}{dt^2} = 2g_{YM}^2 n_{ij}(x_i - x_j) - 2g_{YM}^2 n_{jk}(x_j - x_k)\,, \tag{5.42}$$

Again, fix the center of mass motion (the giant system is not moving anywhere). It is easy to solve these equations; there are two normal modes. The energy of the solution with both modes excited, with amplitudes A_1 and A_2, is given by

$$E_\perp = A_1^2 8 g_{YM}^2 (n_{ik} + n_{ij} + n_{jk} + \gamma) + A_2^2 8 g_{YM}^2 (n_{ik} + n_{ij} + n_{jk} - \gamma) \tag{5.43}$$

which again matches the anomalous dimensions.

5.4 Strings between 4 giants

The methods that we have outlined above work generally for any configuration C of open strings. However, not surprisingly, it becomes increasingly difficult to obtain simple analytic expressions. Obviously its a simple matter to get explicit numerical results for any C. In this section we will simply write the equations one needs to obtain in the case that strings stretch in an arbitrary way between four giant gravitons.

Construction of Creation and Annihilation Operators: In this case, to be general, we should introduce the parameters n_{ij}, n_{ik}, n_{il}, n_{jk}, n_{jl} and n_{kl}

$$D = -g_{YM}^2 (n_{ij}\Delta_{ij} + n_{ik}\Delta_{ik} + n_{il}\Delta_{il} + n_{jk}\Delta_{jk} + n_{jl}\Delta_{jl} + n_{kl}\Delta_{kl}) \tag{5.44}$$

For any C, $n_{ij}+n_{ik}+n_{il}$ is even, $n_{ij}+n_{jk}+n_{jl}$ is even, $n_{ik}+n_{jk}+n_{kl}$ is even and $n_{il}+n_{jl}+n_{kl}$ is even. For a creation operator we again want (5.13). This leads us to the eigenproblem of a 16×16 matrix. For general parameters we get 6 $A^\dagger$s. Only three of these survive in the large $b_{1,\max}$ limit. The frequencies of the oscillators which survive are roots of

$$\begin{aligned}
&x^3-2(n_{ij}+n_{jl}+n_{ik}+n_{jk}+n_{kl}+n_{il})x^2\\
&+(3n_{ik}n_{kl}+4n_{ij}n_{kl}+3n_{ij}n_{il}+3n_{ik}n_{il}+3n_{ik}n_{jk}\\
&+3n_{il}n_{jl}+3n_{kl}n_{il}+4n_{ik}n_{jl}+4n_{jk}n_{il}\\
&+3n_{jk}n_{jl}+3n_{kl}n_{jl}+3n_{ij}n_{jk}+3n_{ij}n_{jl}+3n_{ij}n_{ik}+3n_{jk}n_{kl})x\\
&-4n_{ij}n_{kl}n_{jl}-4n_{ij}n_{jk}n_{kl}-4n_{ij}n_{ik}n_{kl}-4n_{ij}n_{kl}n_{il}-4n_{jk}n_{ik}n_{kl}\\
&-4n_{jk}n_{ik}n_{il}-4n_{jk}n_{ik}n_{jl}-4n_{jl}n_{ik}n_{kl}-4n_{jl}n_{ik}n_{il}-4n_{jk}n_{kl}n_{il}-4n_{ij}n_{jk}n_{il}\\
&-4n_{ij}n_{jk}n_{jl}-4n_{ij}n_{ik}n_{jl}-4n_{ij}n_{ik}n_{il}-4n_{jl}n_{jk}n_{il}-4n_{jl}n_{kl}n_{il}=0\,.
\end{aligned} \tag{5.45}$$

It is now straight forward to construct the algebra of the resulting oscillators as well as their large b_1 limit. We again find that this result is consistent with both the continuum limit of D (as outlined in appendix H of [56]) and the classical model of masses and springs. This computation (as well as the extension to situations in which strings interconnect more than 4 giants) is straight forward but a little tedious.

5.5 Summary

In summary, two things have been achieved in this chapter. The continuum limit of the dilatation operator was obtained in appendix H of [56]. What

is the relation between the study of [56] and our result here? In [56] the large $b_{1,\max}$ limit was taken and the resulting eigenvalue problem was solved. Here we have first solved the eigenvalue problem and have then taken the large $b_{1,\max}$ limit. Our result is in perfect agreement with the continuum limit obtained in [56], and justifies the use of the simple Harmonic oscillator Hamiltonian obtained there. In particular, in the continuum limit the variables s_i become continuous coordinates and the operators of a good scaling dimension are obtained by summing restricted Schur polynomials with coefficients given by the harmonic oscillator wave functions. The second thing we have achieved is that the values of the anomalous dimensions have been reproduced by the normal mode frequencies of a coupled system of open strings. This provides non-trivial support for their interpretation in the dual theory as excited giant gravitons.

6 Nonplanar Integrability at Two Loops

6.1 Introduction and Questions

The work of this chapter is novel and first appeared in Robert de Mello Koch, Garreth Kemp, Badr Awad Elseid Mohammed, Stephanie Smith *Nonplanar integrability at two loops* JHEP 1210 (2012) 144]. In this chapter we will study the large N limit of the two loop anomalous dimensions of restricted Schur polynomials [21, 27]. Here, we will use the representation theory of symmetric and unitary groups developed earlier in this thesis. In the half-BPS sector a complete set of operators is given by the Schur polynomials $\chi_R(Z)$[71]. They are labeled with Young diagrams R. As mentioned, operators with R having order one rows of length order N or order one columns of length order N are dual to giant gravitons[72, 73, 74]. If R has $O(N^2)$ boxes the corresponding operator is dual to an LLM geometry [12]. The problem of diagonalizing the free field inner product for multi-matrix operators, while preserving global symmetries, was solved in [75, 76].

In this section our focus will be on restricted Schur polynomials and on the su(2) sector of the theory. In the su(2) sector one considers a restricted Schur polynomial built mainly from one type of matrix field Z, doped with impurities Y. Recall the restricted Schur polynomial in the $SU(2)$ sector is given by

$$\chi_{R,(r,s)\alpha\beta}(Z,Y)=\frac{1}{n!m!}\sum_{\sigma\in S_{n+m}}\mathrm{Tr}_{(r,s)\alpha\beta}\left(\Gamma^{(R)}(\sigma)\right)\mathrm{Tr}(\sigma Z^{\otimes n}\otimes Y^{\otimes m}) \quad (6.1)$$

Study of this chapter was motivated by the work on the nonplanar limit and

the diagonalization of the dilatation operator as outlined below.

Initial numerical studies showed, remarkably, that the spectrum of the dilatation operator is that of a set of decoupled oscillators. Early studies computed the exact action of the dilatation operator and then took the large N limit as a final step. These computations are quite involved and it is not easy to obtain general results. Indeed, [13] focused on two impurities while [14] considered 3 or 4 impurities. By working in the displaced corners approximation, [17] was able to directly implement the simplifications of the large N limit allowing the computation of results for an arbitrary number of impurities but under the constraint that R, r, s have at most two columns or rows.

This was then extended beyond the su(2) sector in [55] and to an arbitrary number of rows in [56]. This extension used a novel Schur-Weyl duality [56, 77] that emerges at large N in the displaced corners approximation. Using this novel Schur-Weyl duality, the states $|s\,\mu_1\,;\,i\rangle$ appearing in (B.1) are states of a U(p) representation where p is the number of rows or columns of the restricted Schur polynomial. This allows us to trade the pair $s\,\mu_1$ for a Gelfand-Tsetlin pattern if we wish. These results have a direct application to the sl(2) sector[70]. In the displaced corners approximation the action of the dilatation operator has an interesting structure. The eigenproblem of the anomalous dimensions factors into a product of two problems, one for the Zs involving Young diagram r and one for the Ys involving Young diagram s. In [56], based on numerical results, a conjecture for the solution to the

eigenproblem involving the s label was given. This conjecture has now been proven in [78].

The starting point of [78] is a proof that the number of excited giant graviton states as constrained by the Gauss Law, matches the number of restricted Schur polynomials in the gauge theory. The proof proceeds by associating excited giant graviton states to elements of a double coset involving permutation groups. Making heavy use of the ideas and methods of [75, 76], Fourier transformation on the double coset suggests an ansatz for the operators of a good scaling dimension. The operators obtained in this way, denoted $O_{R,r}(\sigma)$, are labeled by an element of a double coset σ and by the Young diagrams R and r.

In [78] it was proven that this ansatz indeed provides the conjectured diagonalization. Further, since the double coset structure is determined entirely by the Gauss Law which holds at all loops, these results suggest that the operators constructed in [78], may be relevant at higher loops. This is an issue we will manage to probe in this chapter. The eigenproblem on the r label has been considered in [79]. It is written in terms of a difference operator. The basic observation of [79] is to realize that this difference operator is an element of the Lie algebra of $U(p)$ when r has p rows or columns. Exploiting this insight [79] argued that the eigenproblem on the r label is related to a system of p particles in a line with 2-body harmonic oscillator interactions.

We can now give the set of questions that motivated this study. As

just discussed, the dilatation operator of $\mathcal{N} = 4$ super Yang-Mills theory is integrable in the large N displaced corners approximation of the su(2) sector at one loop[17, 56, 78]. The first question we wish to address is

1 **Is the dilatation operator integrable in the large N displaced corners approximation at higher loops?**

Although we are not able to give a complete answer to this question, we will test integrability at two loops. We can sharpen the above question. As described above, the action of the dilatation operator factorizes into an action on the Young diagram associated with the Zs and an action on the Young diagram associated with the Ys. The eigenproblem associated with the Ys appears (see [78]) to be determined by the Gauss Law constraint, which should hold at all higher loops. This motivates the question

2 **Do the $O_{R,r}(\sigma)$ of [78] continue to solve the Y eigenproblem at higher loops?**

The Z eigenproblem was solved in [79] by mapping it to a system of p particles in a line with 2-body harmonic oscillator interactions. The basic observation was to show that the operator to be diagonalized is an element of the Lie algebra of $U(p)$ when r has p rows or columns. Our third question is

3 **Can the two loop Z eigenproblem be mapped to a system of p particles, again using the Lie algebra of $U(p)$?**

The one loop spectrum of anomalous dimensions has some interesting features. One would have expected the eigenvalues of the one loop dilatation operator to be a function of the 't Hooft coupling. We find they are given

by an integer times g^2_{YM}. It is not completely clear how this should be interpreted. By computing the two loop correction to the anomalous dimension and requiring that it is small compared to the leading term, we hope to gain insight into both the interpretation of our results and in the precise limit that should be taken to get a sensible perturbative expansion. This motivates our fourth question

4 **Does the two loop correction to the anomalous dimension determine the precise limit that should be taken to get a sensible perturbative expansion?**

These questions are all answered in the discussion section. We will find that this limit of the theory continues to be integrable at two loops, that the one loop operators with a good scaling dimension are not modified at two loops and finally, that our perturbative expansion is sensible in the conventional 't Hooft limit.

6.2 Two Loop Dilatation Operator

Our goal is to evaluate the action of the two loop dilatation operator[8]

$$\begin{aligned} D_4 = & - 2g^2 : \mathrm{Tr}\left(\left[[Y,Z], \frac{\partial}{\partial Z}\right]\left[\left[\frac{\partial}{\partial Y}, \frac{\partial}{\partial Z}\right], Z\right]\right) : \\ & - 2g^2 : \mathrm{Tr}\left(\left[[Y,Z], \frac{\partial}{\partial Y}\right]\left[\left[\frac{\partial}{\partial Y}, \frac{\partial}{\partial Z}\right], Y\right]\right) : \\ & - 2g^2 : \mathrm{Tr}\left([[Y,Z], T^a]\left[\left[\frac{\partial}{\partial Y}, \frac{\partial}{\partial Z}\right], T^a\right]\right) : \end{aligned} \tag{6.2}$$

$$g = \frac{g^2_{YM}}{16\pi^2} \tag{6.3}$$

on restricted Schur polynomials. The normal ordering symbols here indicate that derivatives within the normal ordering symbols do not act on fields inside the normal ordering symbols. For the operators we study, $n \gg m$ so that only the first term in D_4 will contribute. We have in mind a systematic expansion in two parameters: $\frac{1}{N}$ and $\frac{m}{n}$. In Appenidx C.2 we show that keeping only the first term in D_4 corresponds to the computation of the leading term in this double expansion. The evaluation of the action of the one loop dilatation operator was carried out in [14]. The two loop computation uses many of the same techniques but there are a number of subtle points that must be treated correctly. The computation can be split into the evaluation of two types of terms, one having all derivatives adjacent to each other (for example $\mathrm{Tr}(ZYZ\partial_Z\partial_Y\partial_Z)$) and one in which only two of the derivatives are adjacent (for example $\mathrm{Tr}(YZ\partial_Z Z\partial_Y\partial_Z)$). We will deal with an example of each term paying special attention to points that must be treated with care.

First Term: Start by allowing the derivatives to act on the restricted Schur polynomial

$$\mathrm{Tr}(ZYZ\partial_Z\partial_Y\partial_Z)\chi_{R,(r,s)\alpha\beta}(Z,Y) =$$

$$\frac{mn(n-1)}{n!m!}\sum_{\psi\in S_{n+m}}\mathrm{Tr}_{(r,s)\alpha\beta}(\Gamma^{(R)}((1,m+2)\psi(m+1,m+2)))$$
$$\times\delta^{i_1}_{i_{\psi(1)}}Y^{i_2}_{i_{\psi(2)}}\cdots Y^{i_m}_{i_{\psi(m)}}(ZYZ)^{i_{m+1}}_{i_{\psi(m+1)}}\delta^{i_{m+2}}_{i_{\psi(m+2)}}Z^{i_{m+3}}_{i_{\psi(m+3)}}\cdots Z^{i_{m+n}}_{i_{\psi(m+n)}} \tag{6.4}$$

The two delta functions will reduce the sum over S_{n+m} to a sum over an S_{n+m-2} subgroup. This sum is most easily evaluated using the reduction rule of [80, 22]. The reduction rule rewrites the sum over S_{n+m} as a sum

over S_{n+m-2} and its cosets. This is most easily done by making use of Jucys-Murphy elements whose action is easily evaluated. To employ the same strategy in the current computation, the action of the Jucys-Murphy element will only be the simple one if we swap the delta function from slot $m+2$ to slot 2. This gives

$$\frac{mn(n-1)}{n!m!}\sum_{\psi\in S_{n+m-2}}\mathrm{Tr}_{(r,s)\alpha\beta}(\Gamma^{(R)}((1,m+2)(2,m+2)\psi(2,m+2)\hat{C}(m+1,m+2))$$
$$\times Y^{i_3}_{i_{\psi(3)}}\cdots Y^{i_m}_{i_{\psi(m)}}(ZYZ)^{i_{m+1}}_{i_{\psi(m+1)}}Y^{i_{m+2}}_{i_{\psi(m+2)}}Z^{i_{m+3}}_{i_{\psi(m+3)}}\cdots Z^{i_{m+n}}_{i_{\psi(m+n)}} \tag{6.5}$$

where $\hat{C}=(N+J_2)(N+J_3)$ with J_i a Jucys-Murphy element

$$J_i=\sum_{k=i}^{n+m}(i-1,k) \tag{6.6}$$

Since we sum over the S_{n+m-2} subgroup, we can decompose $R\vdash m+n$ into a direct sum of terms which involve the irreps $R''\vdash m+n-2$ of the subgroup[8]. As usual [80, 22], for each term in the sum, $\hat{C}$ is equal to the product of the factors of the boxes that must be removed from R to obtain R''. To rewrite the result in terms of restricted Schur polynomials, note that

$$\begin{aligned} Y^{i_3}_{i_{\psi(3)}} \quad & \cdots Y^{i_m}_{i_{\psi(m)}}(ZYZ)^{i_{m+1}}_{i_{\psi(m+1)}}Y^{i_{m+2}}_{i_{\psi(m+2)}}Z^{i_{m+3}}_{i_{\psi(m+3)}}\cdots Z^{i_{m+n}}_{i_{\psi(m+n)}} \\ &= \mathrm{Tr}\left(\psi(2,m+1,1)Y\otimes Z\otimes Y^{\otimes m-2}\otimes Z\otimes Y\otimes Z^{\otimes n-2}\right) \\ &= \mathrm{Tr}\left((2,m+2)\psi(2,m+1,1)(2,m+2)Y^{\otimes m}\otimes Z^{\otimes n}\right) \end{aligned} \tag{6.7}$$

[8]In general if R denotes a Young diagram, then R' denotes a Young diagram that can be obtained from R by removing one box, R'' denotes a Young diagram that can be obtained from R by removing two boxes etc.

and make use of the identity[61]

$$\mathrm{Tr}(\sigma Z^{\otimes n} \otimes Y^{\otimes m}) = \sum_{T,(t,u)\beta\alpha} \frac{d_T n! m!}{d_t d_u (n+m)!} \mathrm{Tr}_{(t,u)\alpha\beta}(\Gamma^{(T)}(\sigma^{-1})) \chi_{T,(t,u)\beta\alpha}(Z,Y) \tag{6.8}$$

After this rewriting the sum over S_{n+m-2} can be carried out using the fundamental orthogonality relation. The result is

$$\begin{aligned} \sum_{T,(t,u)\gamma\delta} \sum_{R'',T''} & \frac{d_T n(n-1)m}{d_t d_u d_{R''}(n+m)(n+m-1)} c_{RR'} c_{R'R''} \;\; \chi_{T,(t,u)\gamma\delta}(Z,Y) \\ & \times \mathrm{Tr}\Big(I_{T''R''}(2, m+2, m+1) P_{R,(r,s)\alpha\beta}(1, m+2, 2) I_{R''T''} \\ & \qquad \times (2, m+2) P_{T,(t,u)\delta\gamma}(m+2, 2, 1, m+1) \Big) \end{aligned} \tag{6.9}$$

The intertwiner $I_{R''T''}$ is a map (see Appendix D of [56] for details on its properties) from irrep R'' to irrep T''. It is only non-zero if R'' and T'' have the same shape. Thus, to get a non-zero result R and T must differ at most, by the placement of two boxes. We make further comments relevant for this trace before equation (6.10) below.

Second Term: Evaluation of the second term is very similar. In this case however, taking the derivatives produces a single delta function, which will reduce the sum over S_{n+m} to a sum over S_{n+m-1}. The delta function should be in slot 1. The reader wanting to check an example may find it useful to verify that

$$: \mathrm{Tr}(YZ\partial_Z Z \partial_Y \partial_Z) : \chi_{R,(r,s)\alpha\beta}(Z,Y) =$$

$$\sum_{T,(t,u)\gamma\delta} \sum_{R',T'} \frac{d_T n(n-1)m}{d_t d_u d_{R'}(n+m)} c_{RR'}$$

$$\times \mathrm{Tr}(I_{T'R'}(1,m+2,m+1)P_{R,(r,s)\alpha\beta}I_{R'T'}(1,m+1)P_{T,(t,u)\delta\gamma})\chi_{T,(t,u)\gamma\delta}(Z,Y) \tag{6.10}$$

The intertwiner $I_{R'\,T'}$ is a map from irrep R' to irrep T'. It is only non-zero if R' and T' have the same shape. Thus, to get a non-zero result R and T must differ at most, by the placement of a single box. It is perhaps useful to spell out explicitely the meaning of the trace above. The above trace is taken over the reducible S_{n+m} representation $R \oplus T$. In addition, the projectors within the trace allow us to rewrite the permutations appearing in the trace as

$$\mathrm{Tr}\left(I_{T'R'}\Gamma^{(R)}\Big((1,m+2,m+1)\Big)P_{R,(r,s)\alpha\beta}I_{R'T'}\Gamma^{(T)}\Big((1,m+1)\Big)P_{T,(t,u)\delta\gamma}\right) \tag{6.11}$$

The final result for the action of the dilatation operator is (this includes only the first term in (6.2) since $n \gg m$)

$$\begin{aligned} D_4\chi_{R,(r,s)\alpha\beta}(Z,Y) = & \\ & -2g^2 \sum_{T,(t,u)\gamma\delta} \sum_{R'\,T'} \frac{d_T n(n-1)mc_{RR'}}{d_t d_u d_{R'}(n+m)} M^{(b)}_{R,(r,s)\alpha\beta\ T,(t,u)\gamma\delta}\chi_{T,(t,u)\delta\gamma}(Z,Y) \\ & -2g^2 \sum_{T,(t,u)\gamma\delta} \sum_{R''\,T''} \frac{d_T n(n-1)mc_{RR'}c_{R'R''}}{d_t d_u d_{R''}(n+m)(n+m-1)} M^{(a)}_{R,(r,s)\alpha\beta\ T,(t,u)\gamma\delta}\chi_{T,(t,u)\delta\gamma}(Z,Y) \end{aligned} \tag{6.12}$$

where

$$M^{(a)}_{R,(r,s)\alpha\beta\ T,(t,u)\gamma\delta} = \mathrm{Tr}\left(I_{T''R''}(2,m+2)P_{R,(r,s)\alpha\beta}C_1 I_{R''T''}(2,m+2)P_{T,(t,u)\gamma\delta}C_1\right)$$

$$+\mathrm{Tr}\left(I_{T''R''}C_2 P_{R,(r,s)\alpha\beta}(2,m+2)I_{R''T''}C_2 P_{T,(t,u)\gamma\delta}(2,m+2)\right) \tag{6.13}$$

$$C_1 = [(m+2,2,1),(1,m+1)] \qquad C_2 = -C_1^T = [(m+2,1,2),(1,m+1)] \tag{6.14}$$

and

$$M^{(b)}_{R,(r,s)\alpha\beta\ T,(t,u)\gamma\delta} = \mathrm{Tr}\left(I_{T'R'}C_3 I_{R'T'}[(1,m+1),P_{T,(t,u)\gamma\delta}]\right) + \mathrm{Tr}\left(I_{T'R'}C_4 I_{R'T'}[(1,m+1),P_{T,(t,u)\gamma\delta}]\right) \quad (6.15)$$

$$C_3 = [(1,m{+}2,m{+}1),P_{R,(r,s)\alpha\beta}] \qquad C_4 = [(1,m{+}1,m{+}2),P_{R,(r,s)\alpha\beta}] \quad (6.16)$$

This formula is correct to all orders in $1/N$. Denote the number of rows in the Young diagram R labeling the restricted Schur polynomial by p. This implies that, since R subduces $S_n \times S_m$ representation (r,s) and $n \gg m$ that r has p rows and s has at most p rows. Now we will make use of the displaced corners approximation. To see how this works, recall that to subduce $r \vdash n$ from $R \vdash m+n$ we remove m boxes from R. Each removed box is associated with a vector in a p dimensional vector space V_p. Thus, the m removed boxes associated with the Ys define a vector in $V_p^{\otimes\, m}$. In the displaced corners approximation, the trace over $R \oplus T$ factorizes into a trace over $r \oplus t$ and a trace over $V_p^{\otimes m}$. The structure of the projector (B.1) makes it clear that the bulk of the work is in evaluating the trace over $V_p^{\otimes m}$. This trace can be evaluated using the methods developed in [56]. Introduce a basis for the fundamental representation of the Lie algebra u(p) given by $(E_{ij})_{ab} = \delta_{ia}\delta_{jb}$. Recall the product rule

$$E_{ij}E_{kl} = \delta_{jk}E_{il} \quad (6.17)$$

which we use extensively below. If a box is removed from row i it is associated to a vector v_i which is an eigenstate of E_{ii} with eigenvalue 1. The intertwining maps can be written in terms of the E_{ij}. For example, if we remove two boxes from row i of R and two boxes from row j of T, assuming that R'' and T''

have the same shape, we have

$$I_{T''R''} = E^{(1)}_{ji} E^{(2)}_{ji} \tag{6.18}$$

A big advantage of realizing the intertwiners in this way is that it is simple to evaluate the product of symmetric group elements with the intertwiners. For example, using the identification (for background, see for example [81])

$$(1, 2, m+1) = \mathrm{Tr}(E^{(1)} E^{(2)} E^{(m+1)}) \tag{6.19}$$

we easily find

$$(1, 2, m+1) I_{T''R''} \;=\; E^{(1)}_{kl} E^{(2)}_{lm} E^{(m+1)}_{mk} E^{(1)}_{ji} E^{(2)}_{ji} \;=\; E^{(1)}_{ki} E^{(2)}_{ji} E^{(m+1)}_{jk} \tag{6.20}$$

This is now enough to evaluate the traces appearing in (6.13) and (6.15).

We will consider the action of the dilatation operator on normalized restricted Schur polynomials. The two point function for restricted Schur polynomials is [27]

$$\langle \chi_{R,(r,s)\alpha\beta}(Z,Y) \chi_{T,(t,u)\gamma\delta}(Z,Y)^\dagger \rangle = \delta_{R,(r,s)\,T,(t,u)} \delta_{\alpha\gamma} \delta_{\beta\delta}\, f_R \frac{\mathrm{hooks}_R}{\mathrm{hooks}_r\, \mathrm{hooks}_s}$$

where f_R is the product of the factors in Young diagram R and hooks_R is the product of the hook lengths of Young diagram R. The normalized operators are thus given by

$$\chi_{R,(r,s)}(Z,Y) = \sqrt{\frac{f_R\, \mathrm{hooks}_R}{\mathrm{hooks}_r\, \mathrm{hooks}_s}} O_{R,(r,s)}(Z,Y)\,.$$

The components m_i of the vector $\vec{m}(R)$ record the number of boxes removed from row i of R to produce r. In the su(2) sector, both the one loop dilatation operator[56] and the two loop dilatation operator conserve $\vec{m}(R)$, recorded in

the factor $\delta_{\vec{m}(R)\vec{m}(T)}$ in (6.21) below. In terms of these normalized operators the dilatation operator takes the form

$$D_4 O_{R,(r,s)\mu_1\mu_2} = -2g^2 \sum_{u\,\nu_1\,\nu_2} \delta_{\vec{m}(R)\vec{m}(T)} M^{(ij)}_{s\mu_1\mu_2\,;\,u\nu_1\nu_2} \left(\Delta^{(1)}_{ij} + \Delta^{(2)}_{ij}\right) O_{R,(r,u)\nu_1\nu_2} \tag{6.21}$$

where

$$\begin{aligned} M^{(ij)}_{s\mu_1\mu_2\,;\,u\nu_1\nu_2} = \\ \frac{m}{\sqrt{d_u d_s}} \Big(\Big\langle \vec{m}, s, \mu_2\,;\, a | E^{(1)}_{ii} | \vec{m}, u, \nu_2\,;\, b \Big\rangle \Big\langle \vec{m}, u, \nu_1\,;\, b | E^{(1)}_{jj} | \vec{m}, s, \mu_2\,;\, a \Big\rangle \\ + \Big\langle \vec{m}, s, \mu_2\,;\, a | E^{(1)}_{jj} | \vec{m}, u, \nu_2\,;\, b \Big\rangle \Big\langle \vec{m}, u, \nu_1\,;\, b | E^{(1)}_{ii} | \vec{m}, s, \mu_2\,;\, a \Big\rangle \Big) \end{aligned} \tag{6.22}$$

To spell out the action of the operators $\Delta^{(1)}_{ij}$ and $\Delta^{(2)}_{ij}$ we will need a little more notation. Denote the row lengths of r by r_i. The Young diagram r^+_{ij} is obtained by deleting a box from row j and adding it to row i. The Young diagram r^-_{ij} is obtained by deleting a box from row i and adding it to row j. In terms of these Young diagrams define

$$\Delta^0_{ij} O_{R,(r,s)\mu_1\mu_2} = -(2N + r_i + r_j) O_{R,(r,s)\mu_1\mu_2} \tag{6.23}$$

$$\Delta^+_{ij} O_{R,(r,s)\mu_1\mu_2} = \sqrt{(N + r_i)(N + r_j)}\, O_{R^+_{ij},(r^+_{ij},s)\mu_1\mu_2} \tag{6.24}$$

$$\Delta^-_{ij} O_{R,(r,s)\mu_1\mu_2} = \sqrt{(N + r_i)(N + r_j)}\, O_{R^-_{ij},(r^-_{ij},s)\mu_1\mu_2} \tag{6.25}$$

We can now write

$$\Delta^{(1)}_{ij} = n(\Delta^+_{ij} + \Delta^0_{ij} + \Delta^-_{ij}) \tag{6.26}$$

$$\Delta^{(2)}_{ij} = (\Delta^+_{ij})^2 + \Delta^0_{ij}\Delta^+_{ij} + 2\Delta^+_{ij}\Delta^-_{ij} + \Delta^0_{ij}\Delta^-_{ij} + (\Delta^-_{ij})^2 \tag{6.27}$$

This completes the evaluation of the dilatation operator.

Our result for $\Delta_{ij}^{(2)}$ deserves a comment. The intertwiners $I_{T''R''}$ appearing in (6.13) only force the shapes of T and R to agree when two boxes have been removed from each. One might imagine removing a box from rows i, j of R to obtain R'' and from rows k, l of T to obtain T'', implying that in total four rows could participate. We see from $\Delta_{ij}^{(2)}$ that this is not the case - the mixing is much more constrained with only two rows participating. We discuss this point further in Appendix C.3.

6.3 Spectrum

An interesting feature of the result (6.21) is that the action of the dilatation operator has factored into the product of two actions: $\Delta_{ij}^{(1)} + \Delta_{ij}^{(2)}$ acts only on Young diagram r i.e. on the Zs, while $M^{(ij)}_{s\mu_1\mu_2\,;\,u\nu_1\nu_2}$ acts only on Young diagram s, i.e. on the Ys. This factored form, which also arises at one loop, implies that we can diagonalize on the $s\mu_1\mu_2; u\nu_1\nu_2$ and the $R, r; T, t$ labels separately. The diagonalization on the $s\mu_1\mu_2; u\nu_1\nu_2$ labels is identical to the diagonalization problem which arises at one loop. The solution was obtained analytically for two rows in [17] and then in general in [78]. Each possible open string configuration consistent with the Gauss Law constraint can be identified with an element of a double coset. A very natural basis of functions, constructed from representation theory, is suggested by Fourier transformation applied to this double coset. In this way [78] constructed an explicit formula for the wave function which solves the $s\mu_1\mu_2; u\nu_1\nu_2$ diagonalization. The resulting Gauss graph operators are labeled by elements of

the double coset. The explicit solution obtained in [78] is

$$O_{R,r}(\sigma) = \frac{|H|}{\sqrt{m!}} \sum_{j,k} \sum_{s\vdash m} \sum_{\mu_1,\mu_2} \sqrt{d_s} \Gamma^{(s)}_{jk}(\sigma) B^{s\to 1_H}_{j\mu_1} B^{s\to 1_H}_{k\mu_2} O_{R,(r,s)\mu_1\mu_2} \tag{6.28}$$

where the group $H = S_{m_1} \times S_{m_2} \times \cdots \times S_{m_p}$ and the branching coefficients $B^{s\to 1_H}_{j\mu_1}$ provide a resolution of the projector from irrep s of S_m onto the trivial representation of H

$$\frac{1}{|H|} \sum_{\sigma\in H} \Gamma^{(s)}_{ik}(\sigma) = \sum_{\mu} B^{s\to 1_H}_{i\mu} B^{s\to 1_H}_{k\mu} \tag{6.29}$$

The action of the dilatation operator on the Gauss graph operator is

$$D_4 O_{R,r}(\sigma) = -2g^2 \sum_{i<j} n_{ij}(\sigma) \left(\Delta^{(1)}_{ij} + \Delta^{(2)}_{ij}\right) O_{R,r}(\sigma) \tag{6.30}$$

The numbers $n_{ij}(\sigma)$ can be read off of the element of the double coset σ. Each possible Gauss operator is given by a set of m open strings stretched between p different giant graviton branes. As an example, consider $p = 4$ with $m = 5$. Two possible configurations are shown in Figure 7. Label the open strings with integers from 1 to $m = 5$ for our example. The double coset element can then be read straight from the open string configuration by recording how the open strings are ordered as closed circuits in the graph are traversed. For the graphs shown, (a) corresponds to $\sigma = (1245)(3)$ and (b) corresponds to $\sigma = (12)(34)(5)$. The numbers $n_{ij}(\sigma)$ tell us how many strings stetch between branes i and j. The branes themselves are numbered with integers from 1 to p, as shown in Figure 8 for our example. Thus, for (a) the non-zero n_{ij} are $n_{12} = 1$, $n_{23} = 1$, $n_{34} = 1$, and $n_{14} = 1$. Notice that we don't record strings that emanate and terminate on the same brane - string 3 in (a) or string 5 in (b), in this example. For (b) the non-zero n_{ij} are $n_{12} = 2$ and $n_{34} = 2$. For the details, see [78].

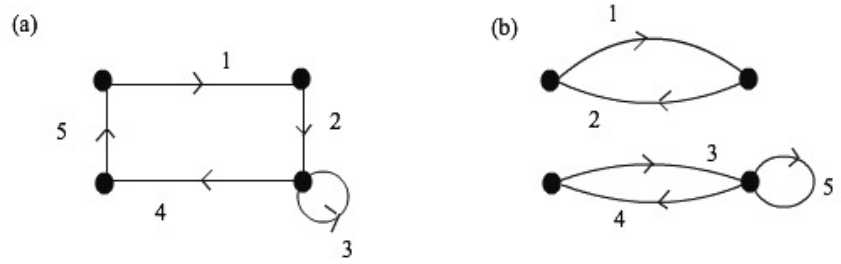


Figure 7: Two possible configurations for operators with $p = 4$ and $m = 5$.



Figure 8: Labeling of the giant graviton branes.

To obtain the anomalous dimensions, inspection of (6.30) shows that we now have to solve the eigenproblem of $\Delta_{ij}^{(1)}$ and $\Delta_{ij}^{(2)}$. The operator $\Delta_{ij}^{(1)}$ is simply a scaled version of the operator which plays a role in the one loop dilatation operator. The corresponding operator which participates at one loop was identified as an element of u(p) [79]. It is related to a system of p particles in a line with 2-body harmonic oscillator interactions[79]. The operator $\Delta_{ij}^{(2)}$ is new. Following [79], a useful approach is to study the continuum limit of $\Delta_{ij}^{(1)}$ and $\Delta_{ij}^{(2)}$. Towards this end, introduce the variables

$$y_j = \frac{r_{j+1} - r_1}{\sqrt{N + r_1}}, \qquad j = 1, 2, 3, ..., p-1 \tag{6.31}$$

which become continuous variables in the large N limit. We have numbered

rows so that $r_1 < r_2 < \cdots < r_p$. In the continuum limit our Gauss graph operators become functions of y_i

$$O_{R,r}(\sigma) \equiv O^{\vec{m}(R)}(\sigma, r_1, r_2, \cdots, r_p) \to O^{\vec{m}(R)}(r_1, y_1, \cdots, y_{p-1}) \tag{6.32}$$

Using the expansions

$$\sqrt{(N+r_i)(N+r_j)} = N + r_1 + \frac{y_i + y_j}{2}\sqrt{N+r_1} - \frac{(y_i - y_j)^2}{8} + O\left(\frac{1}{\sqrt{N+r_1}}\right) \tag{6.33}$$

and

$$\begin{aligned} O^{\vec{m}(R)}\ (r_1, y_1, \cdots, y_i + \frac{1}{\sqrt{N+r_1}}, \cdots, y_j - \frac{1}{\sqrt{N+r_1}}, \cdots, y_{p-1}) = \\ O^{\vec{m}(R)}(r_1, y_1, \cdots, y_{p-1}) + \frac{1}{\sqrt{N+r_1}}\frac{\partial}{\partial y_i} O^{\vec{m}(R)}(r_1, y_1, \cdots, y_{p-1}) \\ -\frac{1}{\sqrt{N+r_1}}\frac{\partial}{\partial y_j} O^{\vec{m}(R)}(r_1, y_1, \cdots, y_{p-1}) \\ +\frac{1}{2(N+r_1)}\left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial y_j}\right)^2 O^{\vec{m}(R)}(r_1, y_1, \cdots, y_{p-1}) \end{aligned} \tag{6.34}$$

we find that in the continuum limit

$$\Delta^{(1)}_{i+1\,j+1} O_{R,r}(\sigma) \to n\left[\left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial y_j}\right)^2 - \frac{(y_i - y_j)^2}{4}\right] O^{\vec{m}(R)}(r_1, y_1, \cdots, y_{p-1}) \tag{6.35}$$

$$\Delta^{(1)}_{1\,i+1} O_{R,r}(\sigma) \to n\left[\left(2\frac{\partial}{\partial y_i} + \sum_{j\neq i}\frac{\partial}{\partial y_j}\right)^2 - \frac{y_i^2}{4}\right] O^{\vec{m}(R)}(r_1, y_1, \cdots, y_{p-1}) \tag{6.36}$$

and

$$\Delta^{(2)}_{i+1\,j+1} O_{R,r}(\sigma) \to 2(N+r_1)\left[\left(\tfrac{\partial}{\partial y_i} - \tfrac{\partial}{\partial y_j}\right)^2 - \tfrac{(y_i - y_j)^2}{4}\right] O^{\vec{m}(R)}(r_1, y_1, \cdots, y_{p-1}) \tag{6.37}$$

$$\Delta^{(2)}_{1\,i+1} O_{R,r}(\sigma) \to 2(N+r_1)\left[\left(2\frac{\partial}{\partial y_i} + \sum_{j\neq i}\frac{\partial}{\partial y_j}\right)^2 - \frac{y_i^2}{4}\right] O^{\vec{m}(R)}(r_1, y_1, \cdots, y_{p-1}) \tag{6.38}$$

Remarkably, in the continuum limit both $\Delta_{ij}^{(1)}$ and $\Delta_{ij}^{(2)}$ have reduced to scaled versions of exactly the same operator that appears in the one loop problem. In the Appendix C.1 we argue for the same conclusion without taking a continuum limit. This implies that the operators that have a good scaling dimension at one loop are uncorrected at two loops.

It is now straight forward to obtain the two loop anomalous dimension for any operator of interest. An instructive and simple example is provided by $p = 2$ with[9] $n_{12} = n_{12}^{+} + n_{12}^{-} \neq 0$. In this case, the anomalous dimension $\gamma(g^2)$ which is the eigenvalue of

$$D = D_2 + D_4 \tag{6.39}$$

with[10]

$$D_2 = -2g : \mathrm{Tr}\left([Y, Z]\left[\frac{\partial}{\partial Y}, \frac{\partial}{\partial Z}\right]\right) : \tag{6.40}$$

and D_4 given in (6.2), is

$$\gamma = 16qn_{12}^{+}(g + (2N + 2r_1 + n)g^2) \tag{6.41}$$

$$q = 0, 1, 2, .., M \quad n_{12} = 0, 1, 2, ... \tag{6.42}$$

where the upper cut off M is itself a number of order N. Clearly, if the g^2 term is to be a small correction to the leading term, we must hold $\lambda_g \equiv gN$

[9]The number n_{12}^{+} counts the number of open strings stretching from giant graviton 1 to giant graviton 2; the number n_{12}^{-} counts the number of open strings stretching from giant graviton 2 to giant graviton 1. The Gauss Law constraint forces $n_{12}^{+} = n_{12}^{-}$. See [78] for more details.

[10]The normalization for both D_2 and D_4 follows [8]. This normalization for D_2 is a factor of 2 larger than the normalization used in [13, 14, 17, 70, 55, 56, 78, 79].

fixed, which corresponds to the usual 't Hooft limit. The fact that the usual 't Hooft scaling leads to a sensible perturbative expansion in this sector of the theory was already understood in [86]. We then find

$$\gamma = \frac{16qn_{12}}{N}(\lambda_g + (2 + 2\frac{r_1}{N} + \frac{n}{N})\lambda_g^2) \tag{6.43}$$

For a given open string plus giant system (i.e. a given n_{12}), in the large N limit, $x = \frac{q}{N}$ varies continuously from 0 to $x = \frac{M}{N}$ implying that the spectrum of anomalous dimensions

$$\gamma = 16xn_{12}(\lambda_g + (2 + 2\frac{r_1}{N} + \frac{n}{N})\lambda_g^2) \tag{6.44}$$

is itself continuous. At finite N this spectrum is discrete. Notice that since both n and r_1 are of order N, all three terms multiplying λ_g^2 in (6.44) are of the same size. Note that the value for γ (6.44) will recieve both $\frac{1}{N}$ corrections and $\frac{m}{n}$ corrections.

6.4 Summary and Answers

We can now return to the questions we posed in the introduction to this chapter.

1 **Is the dilatation operator integrable in the large N displaced corners approximation at higher loops?**

We don't know. We have however been able to argue that the dilatation operator is integrable in the large N displaced corners approximation at two loops. This requires both sending $N \to \infty$ and keeping $m \ll n$ to ensure the validity of the displaced corners approximation. At large N with $m \sim n$

we do not know how to compute the action of the dilatation operator and hence integrability in this situation is an interesting open problem. It seems reasonable to hope that integrability will persist in the large N displaced corners approximation at higher loops.

2 **Do the $O_{R,r}(\sigma)$ of [78] continue to solve the Y eigenproblem at higher loops?**

Yes, the Gauss graph operators do indeed solve the Y eigenproblem at two loops. The Y eigenproblem at two loops is identical to the Y eigenproblem at one loop, so that even the eigenvalues (given by $n_{ij}(\sigma)$ in (6.30)) are unchanged. The fact that the Gauss operators continue to solve the Y eigenproblem does not depend sensitively on the coefficients of the individual terms in the two loop dilatation operator (see Appendix C).

3 **Can the two loop Z eigenproblem be mapped to a system of p particles, again using the Lie algebra of U(p)?**

We have indeed managed to map the Z eigenproblem to the dynamics of p particles (in the center of mass frame). The two loop problem again has a very natural phrasing in terms of the Lie algebra of U(p). The one loop and two loop problems are different: they share the same eigenstates but have different eigenvalues. The fact that the eigenstates are the same does depend sensitively on the coefficients of the individual terms in the two loop dilatation operator (see Appendix C.3).

4 **Does the two loop correction to the anomalous dimension determine the precise limit that should be taken to get a sensible perturbative expansion?**

Yes - requiring that the two loop correction in (6.43) is small compared to the one loop term clearly implies that we should be taking the standard 't Hooft limit. Our result then has an interesting consequence: at large N, $x = q/N$ becomes a continuous parameter and we recover a continuous energy spectrum. This is clearly related to [82]. At any finite N the spectrum is discrete.

7 Conclusions

In this thesis, we set out to study the AdS/CFT correspondence beyond the planar limit, through the study of integrability. To do so we exploited the duality between $\mathcal{N} = 4$ super-Yang-Mills theory and type IIB string theory.

Our study involved calculating the action of the dilatation operator on a class of operators with bare dimension $\mathcal{N}$, known as restricted Schur polynomials. These operators are AdS/CFT dual to giant gravitons. We found that in the cases we studied, the anomalous dimensions of the restricted Schur polynomials corresponded to a set of decoupled harmonic oscillators, showing that the systems were in fact integrable.

In Chapter 4, we studied the action of the one loop dilatation operator on restricted Schur polynomials with three fields, X, Y and$Z$. This was an extension of [17] in that it studied integrability of operators beyond those in the $SU(2)$ sector.

An important new feature we have found is that before making approximations, such as the large row length difference of the Young diagrams labeling the restricted Scur polynomials, the spectrum of the dilatation operator is not equivalent to a collection of harmonic oscillators. This is similar to what one finds in the sector of operators with a bare dimension of order $O(1)$: in the large N limit (which in this case is the planar limit) one obtains an integrable system. Adding $1/N$ corrections seems to spoil the integrability [6, 7].

Apart from computing the spectrum of the dilatation operator, we have managed to compute the associated eigenstates. For two giant graviton systems these states are given in terms of Kravchuk polynomials and Hahn polynomials. The Hahn polynomials are closely related to the wave functions of the one dimensional harmonic oscillator[44] while the Hahn polynomials are closely related to the wave functions of the 2d radial oscillator [17]. The argument of these polynomials are given by j, k or b_1, which have a direct link to the Young diagrams labeling the operators, as summarized for example in figure 2[11]. Thus, the "space" on which the wave functions are defined comes from the Young diagram itself.

Based on our experience with the half BPS sector, it is natural to associate each one of the rows of the Young diagram with each one of the giant gravitons. Recalling that $Y = \phi_3 + i\phi_4$ we know that the number of Ys in each operator tells us the angular momentum of the operator in the 3-4 plane. Similarly, the number of Xs in each operator tells us the angular momentum of the operator in the 5-6 plane and the number of Xs in each operator tells us the angular momentum of the operator in the 1-2 plane. Giving an angular momentum to the giant gravitons will cause them to expand as a consequence of the Myers effect [45]. Thus, for example, the separation between the two gravitons in the 3-4 plane will be related to the difference in angular momenta of the two giants.

[11]The Young diagram r is not shown in figure 2. The number of columns with a single box is given by b_1.

Consequently, the quantum number k is acting like a coordinate for the radial separation between the two giants in the 3-4 plane. Thus, we see very concretely the emergence of local physics from the system of Young diagrams labeling the restricted Schur polynomial. This is strongly reminiscent of the 1/2 BPS case where the Schur polynomials provide wave functions for fermions in a harmonic oscillator potential and further, these wave functions very naturally reproduce features of the geometries and the phase space [12].

For the matrix model we are studying here it is not true that the matrices Z,Y,X commute, we can't simultaneously diagonalize them and there is no analog of the eigenvalue basis that is so useful for the large N dynamics of single matrix models. For the subsystem describing the BPS states however [46] has argued that the matrices might commute in the interacting theory and hence there may be a description in terms of eigenvalues. The argument uses the fact that the weak coupling and strong coupling limits of the BPS sector agree and the fact that at strong coupling we can be confident that the matrices commute. If this is the case, the eigenvalue dynamics should be the dynamics in an oscillator potential with repulsions preventing the collision of eigenvalues.

We have described a part of the BPS sector (as well as non-BPS operators) among the operators we have studied. We do indeed find the dynamics of harmonic oscillators. In the case of a single matrix it is possible to associate the rows of the Young diagram labeling a Schur polynomial with the eigenvalues of the matrix [47]. This provides a connection between the

eigenvalue description and the Schur polynomial description for single matrix models. Our results suggest this might have a generalization to multimatrix models.

The operators we have considered are dual to giant gravitons. A connection between the geometry of giant gravitons and harmonic oscillators was already uncovered in [48, 49, 50]. This work quantizes the moduli space of Mikhailov's giant gravitons so that one is capturing a huge space of states. It is this huge space of states that connects to harmonic oscillators. Our study is focused on a two giant system. Consequently, the oscillators that we have found are associated to this two giant system and excitations of it. It is natural to think that our oscillators arise from the quantization of the possible excitation modes of a giant graviton.

Chapter 5 consisted of calculating the anomalous dimensions of restricted Schur polynomials by mapping these polynomials into states of $U(N)$ irreducible representations. In chapter 4, we had concentrated on diagonalizing the dilatation operator on the Young diagram label s. In chapter five we used the open string configurations consistent with the Gauss law 3.4 in order to diagonalize on the r label. In doing so, we managed to find the form of the Δ operators seen in equation (4.45) in chapter 4.

We found that at one loop, mapping to $SU(2)$ angular momentum operators allowed us to describe the systems of giant gravitons connected by open strings using creation and annihilation operators. There were two main

results in this chapter.

The first is that we found agreement with [56]. In [56], the problem was tackled by taking the large $b_{1,max}$ limit and then solving the eigenvalue problem related to diagonalization on the r label. We did this the other way around, first solving the eigenvalue problem and then taking the limit, and the results were the same.

The second important result is that we found it possible to reproduce values of anomalous dimensions corresponding to the normal mode frequencies of a coupled system of for systems that were compliant with the Gauss law, thus allowing us to use Gauss graphs to greatly simplify our calculations.

In Chapter 6 we studied the action of the two loop dilatation operator on restricted Schur polynomials consisting of two fields. Our discussion has been developed for operators with a label R that has p long rows, which are dual to giant gravitons wrapping an $S^3 \subset AdS_5$. Operators labeled by an R that has p long columns are dual to giant gravitons wrapping an $S^3 \subset S^5$.

The anomalous dimensions for these operators are easily obtained from our results in this chapter (see section D.6 of [56] for a discussion of this connection). At two loops, they were found to have additive corrections to what was found at one loop for these operators, indicating that integrability in this system is still present when further corrections are included.

The $\Delta_{ij}^{(1)}$ for this case is obtained by replacing the $r_i \to -r_i$ and $r_j \to -r_j$ in (6.35) and (6.36), while $\Delta_{ij}^{(2)}$ for this case is obtained by replacing the $r_i \to -r_i$ and $r_j \to -r_j$ in (6.37) and (6.38). The result (6.30) is unchanged when written in terms of the new $\Delta_{ij}^{(1)}$ and $\Delta_{ij}^{(2)}$.

Finally, the fact that our operators are not corrected at two loops is remarkable. It is natural now to conjecture that they are in fact exact and will not be corrected at any higher loop. This is somewhat reminiscent of the BMN operators[11]. In that case it is possible to determine the exact anomalous dimensions as a function of the 't Hooft coupling λ_g[83]. Can we use similar methods to achieve this for the operators discussed in this thesis?

A Appendix: Computational details supporting Chapter 3

A.1 Intertwiners

Intertwiners are matrices which provide mappings between two isomorphic spaces. In the case of S_n representation theory, they provide a mapping between the carrier spaces of irreducible representations. If S_n acts on a vector space $V^{\otimes n}$, with $n > 1$, we have a reducible representation of the group. Suppose this representation includes irreducible representations of R and S, and represent the action of an element in S_n using a matrix:

$$\Gamma(\sigma) = \begin{bmatrix} \Gamma_R(\sigma) & 0 & \cdots \\ 0 & \Gamma_S(\sigma) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \tag{A.1}$$

Restriction to the subgroup S_{n-1} subuces a number of irreducible representations R' and S', for example $R \to R'_1$ and R'_2 and $S \to S'_1$ and S'_2.

$$\Gamma(\sigma) = \begin{bmatrix} \Gamma_{R'_1}(\sigma) & 0 & 0 & 0 & \cdots \\ 0 & \Gamma_{R'_2}(\sigma) & 0 & 0 & \cdots \\ 0 & 0 & \Gamma_{S'_1}(\sigma) & 0 & \cdots \\ 0 & 0 & 0 & \Gamma_{S'_1}(\sigma) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{A.2}$$

Suppose $R'_1 = S'_1$. According to the fundamental orthogonality relation

(equation (3.15) in Chapter 3)

$$\sum_{\sigma\in S_{n-1}} \begin{bmatrix} \Gamma_{R'_1}(\sigma) & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \end{bmatrix}_{ij} \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & \Gamma_{S'_1}(\sigma) & \\ & & & \ddots \end{bmatrix}_{ab}$$

$$= \frac{(n-1)!}{d_{R'_1}} \delta_{R'_1 S'_1} \begin{bmatrix} 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}_{ib} \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}_{aj}$$

$$= \frac{(n-1)!}{d_{R'_1}} \delta_{R'_1 S'_1} (I_{R'_1 S'_1})_{ib} (I_{S'_1 R'_1})_{aj} \tag{A.3}$$

It is clear that intertwiners are matrices with the dimension of the carrier space of R'_1.

A.2 Calculating Traces

For the purpose of finding the spectrum of anomalous dimensions, this section of the appendix will be dedicated to showing how the traces in equation (3.14) are evaluated. In order to perform these calculations we shall begin by studying the action of intertwiners on vector space $V^{\otimes n}$. The intertwiners $I_{R'T'}$ and $I_{T'R'}$ act on the first slot of V, in correspondence with the first box being removed from R or T.

In general, suppose that $R \neq T$. To get R' and T' remove a box from row i of R and j of T respectively in such a way that $R' = T'$. The intertwiners

involved in this operation can be written in terms of the basis of the Lie algebra $U(p)$ as

$$I_{R'T'} = E_{ij} \otimes \mathbf{1} \cdots \otimes \mathbf{1} \tag{A.4}$$

$$I_{T'R'} = E_{ji} \otimes \mathbf{1} \cdots \otimes \mathbf{1} \tag{A.5}$$

Here E is in the first slot and there are $m-1$ $p \times p$ identity matrices in the 2^{nd} to m^{th} slots. In the case where $R = T$, boxes must be removed in the same row of R and T to get $R' = T'$. In this case, intertwiners are written

$$I_{R'T'} = I_{T'R'} = E_{kk} \otimes \mathbf{1} \cdots \otimes \mathbf{1} \tag{A.6}$$

In the general form of the traces considered in chapter three, the matrices $\Gamma_R(1, m+1)$ act on intertwiners. This can also be calculated using the known action of the symmetric group as follows

- Left acting:

$$\begin{aligned} \Gamma_R(1,m+1) I_{R'T'} &= \Gamma_R(1,m+1) E_{ij} \otimes \mathbf{1} \cdots \otimes \mathbf{1} \\ &= \Gamma_R(1,m+1) \sum_{k=1}^{p} E_{ij} \otimes \mathbf{1} \cdots \otimes \mathbf{1} \otimes E_{kk} \\ &= \sum_{k} E_{kj} \otimes \mathbf{1} \cdots \otimes E_{ik} \end{aligned} \tag{A.7}$$

- Right acting:

$$\begin{aligned} I_{R'T'} \Gamma_R(1,m+1) &= E_{ij} \otimes \mathbf{1} \cdots \otimes \mathbf{1} \Gamma_R(1,m+1) \\ &= \sum_{k=1}^{p} E_{ij} \otimes \mathbf{1} \cdots \otimes \mathbf{1} \otimes E_{kk} \Gamma_R(1,m+1) \\ &= \sum_{k} E_{ik} \otimes \mathbf{1} \cdots \otimes E_{kj} \end{aligned} \tag{A.8}$$

- We often streamline our notation as follows

$$\begin{aligned} &\sum_{k=1}^{p} E_{ij} \otimes \mathbf{1} \cdots \otimes \mathbf{1} \otimes E_{kk} \rightarrow \sum_{k=1}^{p} E_{ij}^{(1)} E_{kk}^{(m+1)} \\ &\Rightarrow \Gamma_R(1,m+1) E_{ij}^{(1)} E_{kk}^{(m+1)} = E_{kj}^{(1)} E_{ik}^{(m+1)} \end{aligned} \tag{A.9}$$

A.3 The Factors

Another step in evaluating the dilatation operator involves simplifying the constant coefficient that multiplies the traces,

$$-g_{YM}^2 \sum_{R'} \frac{c_{RR'} d_T nm}{d_{R'} d_t d_u (n+m)} \sqrt{\frac{f_T hooks_T hooks_r hooks_s}{f_R hooks_R hooks_t hooks_u}} \tag{A.10}$$

Note that when the trace is evaluated, it produces a factor of $d_{R^{m+1}}$ which multiplies this coefficient. The notation R^{m+1} refers to Young diagram R with $m+1$ boxes removed, or equivalently diagrams r or t with one box removed to give us r' (or equivalently t', since $r' = t'$). Thus, $d_{R^{m+1}}$ can also be written $d_{r'}$. R_i is the length of row i in diagram R. R and T differ by the placement of at most one box in a different row. When they do differ we have

$$R_i = T_i \qquad a \neq b$$

$$R_a = T_a + 1$$

and

$$R_b = T_b - 1$$

$R' = T'$ is the diagram found by pulling one correctly chosen box off of either R or T.

The product includes the factors

- $$\frac{d_T}{d_{R'}} \sqrt{\frac{hooks_T}{hooks_R}} \frac{1}{n+m}$$

  and

  $$n \sqrt{\frac{hooks_r}{hooks_t}} \frac{d_{R^{m+1}}}{dt}$$

- $$\frac{m}{d_s}$$

  This combines with $\sum_{s'} d_{s'}$ from the trace to give us some constants which combine with Clebsch-Gordan coefficients in the evaluation of the trace.

- $$c_{RR'}\sqrt{\frac{f_T}{f_R}} = \sqrt{c_{RR'}c_{TT'}}$$

  since all weights in $\frac{f_T}{f_R}$ cancel except for that of the box removed from each diagram.

- $$\left(\frac{d_T}{d_{R'}}\sqrt{\frac{hooks_T}{hooks_R}}\frac{1}{n+m}\right)\left(n\sqrt{\frac{hooks_r}{hooks_t}}\frac{d_{R^{m+1}}}{d_t}\right)$$

The simplification of the above factors uses the following ideas

- $$d_R = \frac{hooks_R}{(n+m)!}$$

  where $(n+m)!$ is the order of the symmetric group S_{n+m} whose irreducible representations are labelled by diagram R.

- After evaluating the trace we learn

$$r_a = t_a + 1 \quad and \quad r_b = t_b - 1$$

- The lengths of rows in diagrams R, T, r and t are all of order $\mathcal{O}(N)$. As such, the differences between corresponding row lengths in R, r and T, t is negligible; for instance $R_i \sim r_i$.

Evaluating

$$\frac{d_T}{d_{R'}} \sqrt{\frac{hooks_T}{hooks_R}} \frac{1}{n+m}$$

we have

$$\begin{aligned} \frac{hooks_{R'}}{\sqrt{hooks_T hooks_R}} &= \sqrt{\frac{hooks_{R'}}{hooks_R}} \sqrt{\frac{hooks_{T'}}{hooks_T}} \\ &= \frac{1}{\sqrt{R_b T_a}} \end{aligned}$$

$$n \sqrt{\frac{hooks_r}{hooks_t}} \frac{d_{R^{m+1}}}{dt} = n \sqrt{\frac{hooks_r}{hooks_t}} \frac{d_{r'}}{dt} = \sqrt{r_b t_a}$$

This leaves a factor of

$$g_{YM}^2 \frac{\sqrt{c_{RR'} c_{TT'}} m \sqrt{r_b t_a}}{\sqrt{R_b T_a} d_s} \tag{A.11}$$

B Computational details supporting Chapter 4

B.1 Example Projector

In the section we will consider the case that $m = p = 3$. Towards this end, we couple the states of 3 spin $\frac{1}{2}$-particles to obtain

$$\left|-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\rangle = \left|\frac{3}{2},-\frac{3}{2}\right\rangle ,$$

$$\left|-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle^{A} + \frac{1}{\sqrt{6}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle^{B} + \frac{1}{\sqrt{3}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle ,$$

$$\left|-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle = -\frac{1}{\sqrt{2}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle^{A} + \frac{1}{\sqrt{6}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle^{B} + \frac{1}{\sqrt{3}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle ,$$

$$\left|-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle = +\sqrt{\frac{2}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle^{B} + \frac{1}{\sqrt{3}}\left|\frac{3}{2},\frac{1}{2}\right\rangle ,$$

$$\left|\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\rangle = -\sqrt{\frac{2}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle^{B} + \frac{1}{\sqrt{3}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle ,$$

$$\left|\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}}\left|\frac{1}{2},\frac{1}{2}\right\rangle^{A} - \frac{1}{\sqrt{6}}\left|\frac{1}{2},\frac{1}{2}\right\rangle^{B} + \frac{1}{\sqrt{3}}\left|\frac{3}{2},\frac{1}{2}\right\rangle ,$$

$$\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle = -\frac{1}{\sqrt{2}}\left|\frac{1}{2},\frac{1}{2}\right\rangle^{A} - \frac{1}{\sqrt{6}}\left|\frac{1}{2},\frac{1}{2}\right\rangle^{B} + \frac{1}{\sqrt{3}}\left|\frac{3}{2},\frac{1}{2}\right\rangle ,$$

$$\left|\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle = \left|\frac{3}{2},\frac{3}{2}\right\rangle$$

The spin $\frac{3}{2}$ representation is organized by S_3 irreducible representation $\ydiagram{3}$, which is one dimensional, so that the spin $\frac{3}{2}$ multiplet is not degenerate. The spin $\frac{1}{2}$ representation is organized by S_3 irreducible representation $\ydiagram{2,1}$ which is two dimensional. Consequently, the spin $\frac{1}{2}$ occurs with degeneracy

2. A and B label the two multiplets. Thus, picking a particular state, A and B should label the two states in the S_3 irreducible representation which is labeled by the Young diagram $\begin{array}{|c|c|}\hline \ \ & \ \ \\ \hline \ \ \\ \cline{1-1}\end{array}$. From the results above we easily find

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle^A = \frac{1}{\sqrt{2}}\left|\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle - \frac{1}{\sqrt{2}}\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle$$

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle^B = -\frac{1}{\sqrt{6}}\left|\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle - \frac{1}{\sqrt{6}}\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle$$

Taking the direct product with another such multiplet arising from coupling a further three spins, we should obtain the four states of the $S_3 \times S_3$ irreducible representation labeled by the pair of Young diagrams ($\begin{array}{|c|c|}\hline \ \ & \ \ \\ \hline \ \ \\ \cline{1-1}\end{array}$, $\begin{array}{|c|c|}\hline \ \ & \ \ \\ \hline \ \ \\ \cline{1-1}\end{array}$). These four states are easily constructed

$$\begin{aligned}|1,1\rangle = \quad & \frac{1}{2}\left|\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle - \frac{1}{2}\left|\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle \\ & -\frac{1}{2}\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle + \frac{1}{2}\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle\end{aligned}$$

$$\begin{aligned}|1,2\rangle = \quad & -\frac{1}{2\sqrt{3}}\left|\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle - \frac{1}{2\sqrt{3}}\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle \\ & +\frac{1}{\sqrt{3}}\left|\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle + \frac{1}{2\sqrt{3}}\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle \\ & +\frac{1}{2\sqrt{3}}\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle - \frac{1}{\sqrt{3}}\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle\end{aligned}$$

$$\begin{aligned}|2,1\rangle = \quad & -\frac{1}{2\sqrt{3}}\left|\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle - \frac{1}{2\sqrt{3}}\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle \\ & +\frac{1}{\sqrt{3}}\left|-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle + \frac{1}{2\sqrt{3}}\left|\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle \\ & +\frac{1}{2\sqrt{3}}\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle - \frac{1}{\sqrt{3}}\left|-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle\end{aligned}$$

$$
\begin{aligned}
|2,2\rangle = \quad & \frac{1}{6}\left|\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle + \frac{1}{6}\left|\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle \\
& -\frac{1}{3}\left|\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle + \frac{1}{6}\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle \\
& +\frac{1}{6}\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle - \frac{1}{3}\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle \\
& -\frac{1}{3}\left|-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle - \frac{1}{3}\left|-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle \\
& +\frac{2}{3}\left|-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle
\end{aligned}
$$

It is rather simple to check that these four states do indeed span the carrier space of the $S_3 \times S_3$ representation labeled by ($\ydiagram{2,1}$, $\ydiagram{2,1}$). As an example, (12) has a matrix representation

$$
\Gamma(12) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \Gamma_{\ydiagram{2,1}}((12)) \otimes \Gamma_{\ydiagram{2,1}}(\mathbf{1}) \ .
$$

Given a basis of the required carrier space, it is now trivial to construct the associated projector.

B.2 The Space $L(\Omega_{m,p})$

In this Appendix we discuss the representation theory relevant for the construction developed in chapter 4. We highly recommend the article [40] for related background material. Consider the group $S_p \times S_m$. Define

$$
\Omega_{k,l} = (S_p / S_{p-l} \times S_l) \times (S_m / S_{m-k} \times S_k) \tag{B.1}
$$

to be the space of all pairs of k, l subsets, where the k subsets are subsets of $\{1, 2, ..., p\}$ and the l subsets are subsets of $\{p+1, p+2, ..., p+m\}$. If $p = 2$ and

$m = 2$ then $\Omega_{1,1} = \{\{1;3\},\{1;4\},\{2;3\},\{2;4\}\}$ and $\Omega_{2,2} = \{\{1,2;3,4\}\}$ etc. You can identify a k,l subset with a monomial. For example, we'd identify $\{1;3\}$ with x_1y_3 and $\{1,2;4\}$ with $x_1x_2y_4$. Thus, we can consider $\Omega_{k,l}$ to be the space of distinct monomials in two types of variables (x_i and y_i) with $k+l$ factors and no factor repeats. Ordering of the factors is not important so that $x_1x_2y_4$ and $y_4x_1x_2$ are exactly the same element of $\Omega_{2,1}$. Our main interest is in $L(\Omega_{k,l})$ which is the space of complex valued functions on $\Omega_{k,l}$. The group $S_p \times S_m$ has a very natural action on $L(\Omega_{k,l})$: we can define this action by defining it on each monomial. The symmetric group $S_m \subset S_p \times S_m$ acts by permuting the labels on the x_i factors in the monomial and the symmetric group $S_p \subset S_p \times S_m$ acts by permuting the labels on the y_i factors in the monomial. Thus, for example, for $m = 3 = p$

$$(12)x_1x_2y_4 = x_1x_2y_4 \qquad (45)x_1x_2y_4 = x_1x_2y_5\,.$$

There is a natural inner product under which the monomials are orthonormal, so that, for example

$$\langle x_1x_2y_4, x_1x_2y_4\rangle = 1, \qquad \langle x_1x_2y_4, x_1x_3y_4\rangle = 0 = \langle x_1x_2y_4, x_1x_2y_5\rangle\,.$$

$L(\Omega_{k,l})$ furnishes a reducible representation of the group $S_m \times S_p$. The relevance of $L(\Omega_{k,l})$ for us here is that the projectors acting in $L(\Omega_{k,l})$ projecting onto an irreducible representation of $S_p \times S_m$ are precisely the projectors we need to define the restricted Schur polynomials. Consider the operator

$$d_1 = \sum_{i=1}^{p} \frac{\partial}{\partial x_i}\,. \tag{B.2}$$

It maps from $L(\Omega_{k,l})$ to $L(\Omega_{k-1,l})$. Further, it commutes with the action of $S_p \times S_m$. Because of this, elements of the kernel of d_1 form an invariant

$S_p \times S_m$ subspace. Similarly,

$$d_2 = \sum_{i=p+1}^{p+m} \frac{\partial}{\partial y_i}\,, \tag{B.3}$$

maps $L(\Omega_{k,l})$ to $L(\Omega_{k,l-1})$ and it also commutes with the action of $S_p \times S_m$. Thus, the elements of the kernel of d_2 will also form an invariant $S_p \times S_m$ subspace. Using results from [40] it follows that the intersection of the kernel of d_1, the kernel of d_2 and $L(\Omega_{k,l})$ is an irreducible representation of $S_p \times S_m$.

An example will help to make this discussion concrete. For $m = 3 = p$ the intersection of the kernel of d_1, the kernel of d_2 and $L(\Omega_{1,1})$ is clearly spanned by the polynomials

$$\phi_1 = \frac{x_1 - x_2}{\sqrt{2}}\frac{y_4 - y_5}{\sqrt{2}}, \qquad \phi_2 = \frac{x_1 - x_2}{\sqrt{2}}\frac{y_4 + y_5 - 2y_6}{\sqrt{6}},$$
$$\phi_3 = \frac{x_1 + x_2 - 2x_3}{\sqrt{6}}\frac{y_4 - y_5}{\sqrt{2}}, \qquad \phi_4 = \frac{x_1 + x_2 - 2x_3}{\sqrt{6}}\frac{y_4 + y_5 - 2y_6}{\sqrt{6}}\,.$$

It is easy to check that

$$(12)\phi_1 = -\phi_1, \qquad (12)\phi_2 = -\phi_2, \qquad (12)\phi_3 = \phi_3, \qquad (12)\phi_4 = \phi_4,$$
$$(23)\phi_1 = \frac{1}{2}\phi_1 + \frac{\sqrt{3}}{2}\phi_3, \qquad (23)\phi_2 = \frac{1}{2}\phi_2 + \frac{\sqrt{3}}{2}\phi_4,$$
$$(23)\phi_3 = -\frac{1}{2}\phi_3 + \frac{\sqrt{3}}{2}\phi_1, \qquad (23)\phi_4 = -\frac{1}{2}\phi_4 + \frac{\sqrt{3}}{2}\phi_2,$$
$$(45)\phi_1 = -\phi_1, \qquad (45)\phi_2 = \phi_2, \qquad (45)\phi_3 = -\phi_3, \qquad (45)\phi_4 = \phi_4,$$
$$(56)\phi_1 = \frac{1}{2}\phi_1 + \frac{\sqrt{3}}{2}\phi_2, \qquad (56)\phi_2 = -\frac{1}{2}\phi_2 + \frac{\sqrt{3}}{2}\phi_1,$$
$$(56)\phi_3 = \frac{1}{2}\phi_3 + \frac{\sqrt{3}}{2}\phi_4, \qquad (56)\phi_4 = -\frac{1}{2}\phi_4 + \frac{\sqrt{3}}{2}\phi_3,$$

Thus, we have the following group elements

$$\Gamma\left((12)\right) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\Gamma\left((23)\right) = \begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

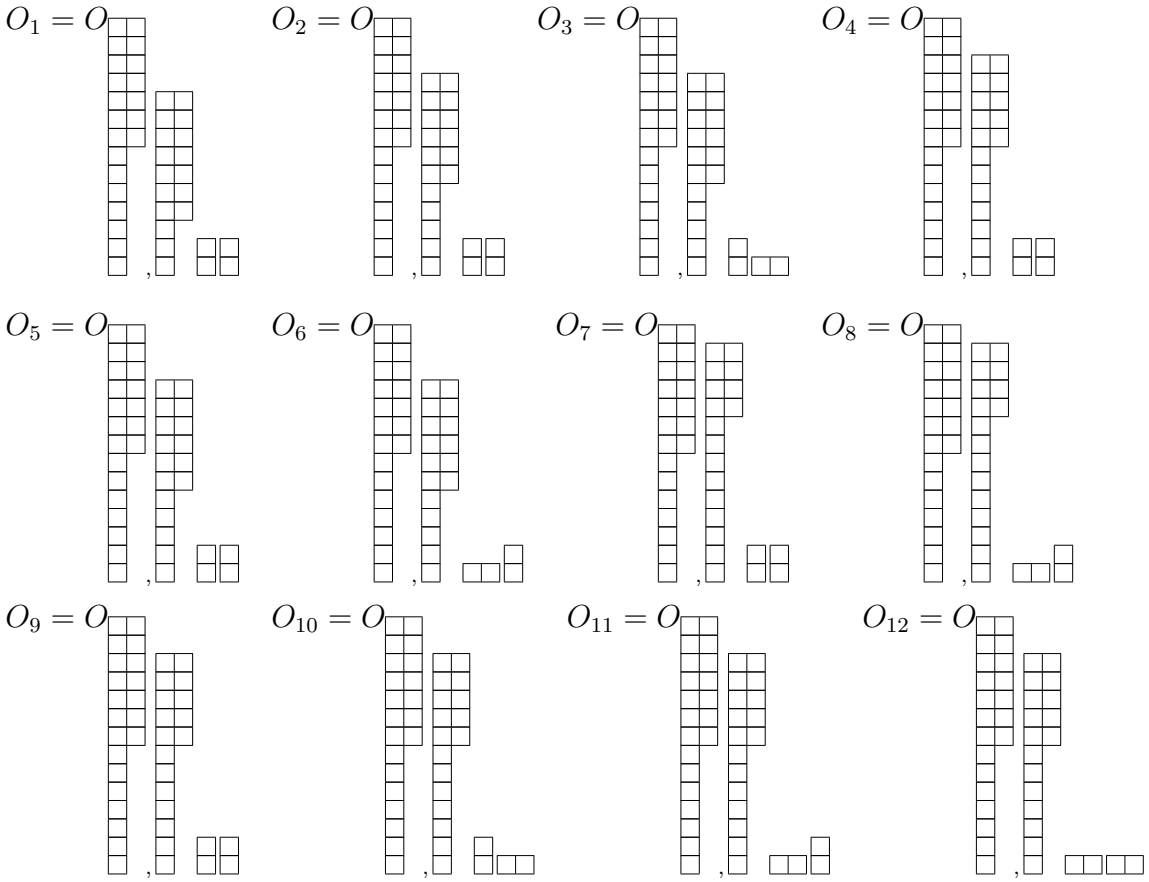
$$\Gamma\left((45)\right) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

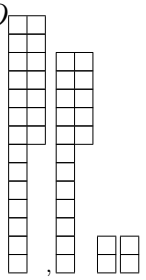
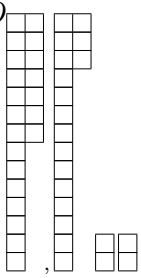
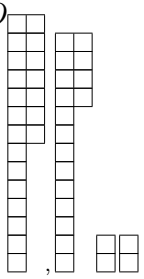
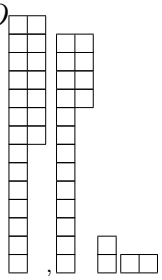
$$\Gamma\left((56)\right) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.$$

Using these matrices it is possible to compute all elements of the group now, and then to compute characters. In this way, it is a simple matter to identify this as the ($\square\square$, $\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$) irreducible representation of $S_3 \times S_3$.

B.3 Explicit Evaluation of the Dilatation Operator for $m = p = 2$ and Numerical Spectrum

We have explicitly evaluated the dilatation operator (4.7) for the case $m = p = 2$. There are a total of 16 operators that can be defined. Our notation for these operators is $O_{R,(r,s,t)} = O_i(b_0, b_1)$. The labels b_0 and b_1 specifies the second label of the restricted Schur polynomial: r has b_0 rows with two boxes and b_1 rows with a single box. The label $i = 1, ..., 16$ tells you what the labels s, t are and it tells you how the boxes are removed from R to obtain r. These labels are defined as



$O_{13} = O$  $O_{14} = O$  $O_{15} = O$  $O_{16} = O$  .

When computing the dilatation operator, we assume that $b_1 \ll b_0$, $b_0 = O(N)$ and $b_1 = O(N)$. The spectrum of the dilatation operator that we obtain, when diagonalized numerically, does not reproduce the spectrum of a set of decoupled oscillators. We do obtain a set of energy levels that is very well approximated by a linear spectrum $E_n = \omega n$ with ω given by the average (over n) of $E_{n+1} - E_n$. However, $E_{n+1} - E_n$ is not exactly equal to $8g^2_{YM}$ - it fluctuates around this value. We have also numerically verified that after invoking the approximations spelled out at the end of section 3.1, we do indeed obtain equation (4.44) and hence with these approximations the spectrum of the dilatation operator is again reproduced by a collection of decoupled oscillators. Thus, it is only after invoking the approximations of section 3.1 that we definitely obtain an integrable system.

The same conclusion is reached by studying the simpler system $m = 2$, $p = 1$, which involves 8 operators.

C Computational details supporting Chapter 6

C.1 $\Delta_{ij}^{(2)}$ as an element of $\mathbf{u}(p)$

In this appendix we will argue that, at large N, the eigenstates of $\Delta_{ij}^{(1)}$ are also eigenstates of $\Delta_{ij}^{(2)}$. We focus on the case that $p = 2$. Towards this end we will recall some details from chapter 5. Recall that in the fundamental representation of $u(N)$ the generators can be taken as

$$(E_{kl})_{ab} = \delta_{ak}\delta_{bl} \qquad k, l, a, b = 1, 2, ..., N \tag{C.1}$$

Introduce the operators (the labeling is such that $i > j$ i.e. Q_{ij} is not defined if $i < j$)

$$Q_{ij} = \frac{E_{ii} - E_{jj}}{2} \qquad Q_{ij}^{+} = E_{ij}, \qquad Q_{ij}^{-} = E_{ji} \tag{C.2}$$

which obey the familiar algebra of angular momentum raising and lowering operators

$$[Q_{ij}, Q_{ij}^{+}] = Q_{ij}^{+} \qquad [Q_{ij}, Q_{ij}^{-}] = -Q_{ij}^{-} \qquad [Q_{ij}^{+}, Q_{ij}^{-}] = 2Q_{ij} \tag{C.3}$$

Irreps of these su(2) subalgebras can be labeled with the eigenvalue of

$$L_{ij}^2 \equiv Q_{ij}^{-}Q_{ij}^{+} + Q_{ij}^2 + Q_{ij} = Q_{ij}^{+}Q_{ij}^{-} + Q_{ij}^2 - Q_{ij} \tag{C.4}$$

and states in the representation are labeled by the eigenvalue of Q_{ij}

$$Q_{ij}|\lambda, \Lambda\rangle = \lambda|\lambda, \Lambda\rangle \qquad L_{ij}^2|\lambda, \Lambda\rangle = (\Lambda^2 + \Lambda)|\lambda, \Lambda\rangle \qquad -\Lambda \leq \lambda \leq \Lambda \tag{C.5}$$

The restricted Schur polynomials can be identified with particular states in a definite irrep. The reader may consult chapter 5 for the details. Identifying

the restricted Schur polynomials with states of a $U(p)$ representation allows us to write $\Delta_{ij}^{(1)}$ as a $u(p)$ valued operator

$$\Delta_{ij}^{(1)} = n\left(-\frac{1}{2}(E_{ii}+E_{jj}) + Q_{ij}^{-} + Q_{ij}^{+}\right) \equiv n\Delta_{ij} \tag{C.6}$$

Note that

$$\mathcal{C} = E_{ii} + E_{jj} \tag{C.7}$$

commutes with all elements (C.2) of the su(2) algebra and hence defines a Casimir of this algebra. It is simply a constant times the identity in a given u(p) irrep. It is not difficult to check using chapter 5 that $\Delta_{ij}^{(1)}$ defines a discrete oscillator with creation operator given by

$$A^{\dagger} = \frac{1}{2}(E_{ii}-E_{jj}) + \frac{1}{2}E_{ij} - \frac{1}{2}E_{ji} \qquad [\Delta_{ij}, A^{\dagger}] = -2A^{\dagger} \tag{C.8}$$

As pointed out in chapter 5, a correctly normalized creation operator is given by $a^{\dagger}$ with $A^{\dagger} = \sqrt{M}a^{\dagger}$, where M is introduced in (6.42). It is straight forward to verify that $\Delta_{ij}^{(2)}$ is given by

$$\Delta_{ij}^{(2)} = (Q^{+})^2 - \frac{\mathcal{C}}{2}Q^{+} + 2Q^{+}Q^{-} - \frac{\mathcal{C}}{2}Q^{-} + (Q^{-})^2 \tag{C.9}$$

and hence that

$$[\Delta_{ij}^{(2)}, A^{\dagger}] = -4(\Delta_{ij} + \frac{\mathcal{C}}{4})A^{\dagger} - 4Q^{+} - 4Q \tag{C.10}$$

In terms of a correctly normalized operator at large N we have (the last two terms in (C.10) can be dropped in the limit)

$$[\Delta_{ij}^{(2)}, a^{\dagger}] = -4(\Delta_{ij} + \frac{\mathcal{C}}{4})a^{\dagger} \tag{C.11}$$

There are two things worth noting at this point. First, when acting in the basis of energy eigenstates, it is clear that $a^\dagger$ is indeed a creation operator but, due to the appearance of Δ_{ij}, with a "state dependent frequency". Said differently, $a^\dagger$ continues to move us to higher eigenstates but the energies of these states are not equally spaced. Second, we can show that this result is in perfect agreement with section 6.3. To make a comparison with section 6.3 we need to restrict attention to states for which the eigenvalue of Δ_{ij} is finite, so that on this subspace we can replace $\Delta_{ij} + \frac{\mathcal{C}}{4} \to \frac{\mathcal{C}}{4}$. Using the value for $\mathcal{C}$ computed in chapter 5, for any state of finite energy, we have

$$[\Delta_{ij}^{(2)}, a^\dagger] = -2\,(2N + 2r_1)\,a^\dagger \tag{C.12}$$

in perfect agreement with section 6.3.

C.2 Simplifications of the $m \ll n$ limit

In this Appendix we will explain why keeping the first term in (6.2) corresponds to computing the leading term in a systematic expansion of the anomalous dimension in a series expansion in $\frac{1}{N}$ and $\frac{m}{n}$. Notice that the first term in (6.2) contains two derivatives with respect to Z and one derivative with respect to Y, whilst the second term contains one derivative with respect to Z and two derivatives with respect to Y. Since the number of Zs (given by n) is much greater than the number of Ys (given by m) we should expect the leading contribution to come from the first term in (6.2). In this Appendix we will demonstrate that this is indeed the case.

It is simplest to consider the expression (6.21). The factor $M^{(ij)}_{s\mu_1\mu_2\,;\,u\nu_1\nu_2}$

includes

$$\begin{aligned}&\left\langle \vec{m}, s, \mu_2 \,;\, a|E^{(1)}_{ii}|\vec{m}, u, \nu_2 \,;\, b\right\rangle \left\langle \vec{m}, u, \nu_1 \,;\, b|E^{(1)}_{jj}|\vec{m}, s, \mu_2 \,;\, a\right\rangle \\ +&\left\langle \vec{m}, s, \mu_2 \,;\, a|E^{(1)}_{jj}|\vec{m}, u, \nu_2 \,;\, b\right\rangle \left\langle \vec{m}, u, \nu_1 \,;\, b|E^{(1)}_{ii}|\vec{m}, s, \mu_2 \,;\, a\right\rangle \end{aligned} \tag{C.13}$$

which involves traces over interwiners acting in $V^{\otimes m}$. It has no dependence on the representation r of the Zs and hence, has no dependence on n. Thus, all n dependence comes from the coefficient multiplying the above term (C.13). We will therefore study the coefficient of this term. As a consequence of the fact that the first term in (6.2) contains two derivatives with respect to Z and one derivative with respect to Y, this term will have a coefficient which includes the factor

$$\frac{d_T n(n-1)m d_{r''}}{d_t d_u d_{R''}(n+m)(n+m-1)} \tag{C.14}$$

Recall that r'' is obtained by removing two boxes from r. The factor of $d_{r''}$ is produced when we take two derivatives with respect to Z. In the limit that $m \ll n$ we now find

$$\frac{d_T n(n-1)m d_{r''}}{d_t d_u d_{R''}(n+m)(n+m-1)} = \frac{m}{d_u}\left[1 + O\left(\frac{m}{n}\right)\right] \tag{C.15}$$

For the second term in (6.2), the corresponding factor is now

$$\frac{d_T m(m-1)n d_{r'}}{d_t d_u d_{R''}(n+m)(n+m-1)} \tag{C.16}$$

The Young diagram r' is obtained by removing one box from r. The factor of $d_{r'}$ is produced when we take a single derivative with respect to Z. In the limit that $m \ll n$ we now find

$$\frac{d_T m(m-1)n d_{r'}}{d_t d_u d_{R''}(n+m)(n+m-1)} = \frac{m(m-1)}{n d_u}\left[1 + O\left(\frac{m}{n}\right)\right] \tag{C.17}$$

Notice that (C.17) is smaller than (C.16) by a factor of $\frac{m}{n}$ as we expected. The second term in (6.2) will thus contribute at higher order in a systematic $\frac{m}{n}$ expansion.

Finally, performing the sum over the Lie algebra index in the third term in (6.2) gives a term that is identical to the one loop dilatation operator, except that it is supressed by a power of N. Thus, it does not contribute to the leading order in a large N expansion.

Thus, to summarize, keeping only the first term in D_4 in (6.2) corresponds to the computation of the leading term in the double expansion in the parameters $\frac{1}{N}$ and $\frac{m}{n}$.

C.3 On the action of the Dilatation Operator

In this Appendix we want to discuss how sensitively integrability depends on the coefficients of the individual terms appearing in D_4. We will start by making a few comments on the structure of $\Delta_{ij}^{(2)}$ that we obtained in (6.27).

Recall that we argued

$$\begin{aligned}
&\mathrm{Tr}(ZYZ\partial_Z\partial_Y\partial_Z)\chi_{R,(r,s)\alpha\beta}(Z,Y)\\
&=\sum_{T,(t,u)\gamma\delta}\sum_{R'',T''}\frac{d_T n(n-1)m}{d_t d_u d_{R''}(n+m)(n+m-1)}c_{RR'}c_{R'R''}\ \ \chi_{T,(t,u)\gamma\delta}(Z,Y)\\
&\quad\times\mathrm{Tr}(I_{T''R''}(2,m+2,m+1)P_{R,(r,s)\alpha\beta}(1,m+2,2)I_{R''T''}\\
&\qquad\times(2,m+2)P_{T,(t,u)\delta\gamma}(m+2,2,1,m+1))
\end{aligned}\tag{C.18}$$

in section 6.2. Focus on the trace appearing in the second line above. Assume that we obtain R' from R by dropping a box from row i and that we obtain R'' from R' by dropping a box from row j. Further, assume that we obtain T' from T by dropping a box from row k and that we obtain T'' from T'

by dropping a box from row l. Clearly then, we are allowing four rows of the Young diagram to participate when the dilatation operator acts. With these assumptions, we easily find (see (6.18), (6.19) and (6.20) as well as the discussion around these equations)

$$I_{R''T''} = E^{(1)}_{ik} E^{(2)}_{jl} \qquad I_{T''R''} = E^{(1)}_{ki} E^{(2)}_{lj} \tag{C.19}$$

and

$$(m+2,2,1,m+1) I_{T''R''}(2,m+2,m+1) = E^{(1)}_{li} E^{(m+1)}_{kj} \tag{C.20}$$

$$(1,m+2,2) I_{R''T''}(2,m+2) = E^{(1)}_{jk} E^{(m+2)}_{il} \tag{C.21}$$

In obtaining these results we have made heavy use of the simplifications in the action of the symmetric group that arise in the displaced corners approximation. It is now a simple matter to find

$$\mathrm{Tr}(I_{T''R''}(2,m+2,m+1) P_{R,(r,s)\alpha\beta}(1,m+2,2) I_{R''T''}(2,m+2) P_{T,(t,u)\delta\gamma}(m+2,2,1,m+1))$$

$$= \mathrm{Tr}(E^{(1)}_{li} E^{(m+1)}_{kj} P_{R,(r,s)\alpha\beta} E^{(1)}_{jk} E^{(m+2)}_{il} P_{T,(t,u)\delta\gamma}) \tag{C.22}$$

Since the projectors $P_{R,(r,s)\alpha\beta}$ and $P_{T,(t,u)\delta\gamma}$ have a trivial action on slots $m+1$ and $m+2$, the above result is only non-zero when $i = l$ and $k = j$ - so that only two rows participate.

This reduction from four possible rows participating to two rows participating is determined by (C.20) and (C.21). These equations are corrected when going beyond the displaced corners approximation and, in that case, all four rows do indeed enter. For all of the terms appearing in the first line

of D_4, we find this reduction to two rows for each term separately. Further, we find that each trace is individualy proportional to $M^{(ij)}_{s\mu_1\mu_2\,;\,u\nu_1\nu_2}$ defined in (6.22). This implies that the answer to question 2 that we posed in the introduction is completely insensitive to the precise coefficients of the terms appearing in D_4[12].

At this point it is natural to ask if the reduction of the dilatation operator to a set of decoupled oscillators (and thus the observed integrability) is likewise also insensitive to the detailed coefficients. We will see that this is not the case - the emergence of an oscillator does depend sensitively on the precise values of the coefficients of the terms appearing in D_4.

Consider equation (6.27). Individual terms appearing in (6.27) can be traced back to particular terms appearing in D_4. For example, the terms proportional to $(\Delta^+_{ij})^2$ and $(\Delta^-_{ij})^2$ come from the terms $\mathrm{Tr}(ZZY\partial_Z\partial_Z\partial_Y)$ and $\mathrm{Tr}(YZZ\partial_Y\partial_Z\partial_Z)$. Notice that these two terms are related by daggering. Similarly, the terms $\Delta^0_{ij}\Delta^+_{ij}$ and $\Delta^0_{ij}\Delta^-_{ij}$ come from the terms $\mathrm{Tr}(ZYZ\partial_Z\partial_Z\partial_Y)$ $\mathrm{Tr}(ZZY\partial_Z\partial_Y\partial_Z)$, $\mathrm{Tr}(ZYZ\partial_Y\partial_Z\partial_Z)$ and $\mathrm{Tr}(YZZ\partial_Z\partial_Y\partial_Z)$ which are again related by daggering. Changing the relative weights of terms appearing in D_4 will change the relative weight of terms appearing in (6.27).

To explore the effect of these changed coefficients on integrability, imagine we assign coefficient α to the terms $\mathrm{Tr}(ZZY\partial_Z\partial_Z\partial_Y)$ and $\mathrm{Tr}(YZZ\partial_Y\partial_Z\partial_Z)$

[12]If one includes the remaining (subleading) terms in D_4 that we have discarded in the $m \ll n$ limit, the dilatation operator starts to mix different Gauss graph operators. This suggests that the integrability we study here is a property of the large N limit and of the displaced corners approximation (i.e. $m << n$) and may not survive when subleading corrections are included.

in D_4. We now find $\Delta_{ij}^{(2)}$ is replaced by

$$\Delta_{ij}^{\alpha(2)} = \alpha(\Delta_{ij}^{+})^2 + \Delta_{ij}^{0}\Delta_{ij}^{+} + 2\Delta_{ij}^{+}\Delta_{ij}^{-} + \Delta_{ij}^{0}\Delta_{ij}^{-} + \alpha(\Delta_{ij}^{-})^2 \qquad \text{(C.23)}$$

It is straight forward to check, using the approach of [79] that this operator does not admit creation and annihilation operators and hence does not define an oscillator. A very instructive way to get some insight into what is going on, is to consider the continuum limit of section 6.3. We find

$$\Delta_{ij}^{\alpha(2)} O_{R,r}(\sigma) \to 2N^2(\alpha-1)O_{R,r}(\sigma) + 2(r_i+r_j)N(\alpha-1)O_{R,r}(\sigma) + O(N) \qquad \text{(C.24)}$$

Compare this to (6.37) and (6.38). Even the scaling with N of the eigenvalues of $\Delta_{ij}^{\alpha(2)}$ and $\Delta_{ij}^{(2)}$ disagree. Indeed, with $\alpha = 1$ we have a delicate cancelation of the leading order terms - as we clearly see in (C.24). It is the subleading terms that combine to produce an oscillator. Note that all of the terms in (6.27) contribute at the leading order. Thus, the sensitive dependence we see on the coefficient of the terms $\mathrm{Tr}(ZZY\partial_Z\partial_Z\partial_Y)$ and $\mathrm{Tr}(YZZ\partial_Y\partial_Z\partial_Z)$ extends to the other terms in D_4 too.

This last point deserves explanation. The terms in $\Delta_{ij}^{(2)}$ can be collected into three groups which are each hermittian: $(\Delta_{ij}^{+})^2 + (\Delta_{ij}^{-})^2$, $\Delta_{ij}^{0}\Delta_{ij}^{+} + \Delta_{ij}^{0}\Delta_{ij}^{-}$ and finally $2\Delta_{ij}^{+}\Delta_{ij}^{-}$. The relative coefficients of the terms producing these pieces is fixed by hermitticity. For example $\mathrm{Tr}(ZZY\partial_Z\partial_Z\partial_Y) + \beta\mathrm{Tr}(YZZ\partial_Y\partial_Z\partial_Z)$ is only hermittian if $\beta = 1$ and in this case the terms sum to $(\Delta_{ij}^{+})^2 + (\Delta_{ij}^{-})^2$. The particular coefficients of the terms that appear in $\Delta_{ij}^{(2)}$ ensure that when we take the continuum limit (i) the terms proportional to N^2 cancel, (ii) the terms proportional to $(r_i + r_j)N$ cancel and (iii) the surviving terms sum to produce an operator that admits exactly the same

creation and annihilation operators as the one loop dilatation operator does. The integrability we have studied here depends on a careful fine tuning of the terms appearing in D_4.

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