Economic Equilibrium Analysis with Riesz Geometry and Measure Theory

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Abstract

Equilibrium is a concern of each individual in the society, of an organisation, and of the entire world. This Masters dissertation consider the analysis of economic equilibrium for an infinite dimensional economy. Two mathematic models are used to conduct the analysis: the continuum economics model and the Riesz dual system $\langle \mathcal{L}_w, \mathcal{L}'_w \rangle$ model. Continuum economics show how an economy with many economic actors reach both an equilibrium of Walras and a core allocation. The second model, based on the paper by Aliprantis, Burkinshaw, Brown, explore the equilibrium properties embodied in the Edgeworth box. Three more equilibrium's concepts(extended Walras equilibrium, the quasiequilibrium and the approximate quasiequilibrium) as well as the connection between them are presented in the last chapter of this dissertation.

0.1 DECLARRATION

I declare that this dissertation is my own, unaided work.
It is being submitted in partial fulfilment of the requirements for the degree of Master of Science by dissertation in the University of the Witwatersrand, Johannesburg.
It has not been submitted before for any degree or examination in any other university.

Francis Kalonji Mudibu

0.2 PREFACE(Acknowledge)

There is no economy where is social inequality. It has been a nice moment to deal with such a subject as an economic equilibrium, particularly when it has to be assessed with the tools of Riesz Geometry and theoretic measure. We cannot succeed alone, we need available and good peoples. Destiny led me to Professor B.A. Watson who not only suggested the subject, red the whole dissertation, corrected my English and made technical and helpful suggestions to improve the presentation of this dissertation. Professor Watson receive my deep thanks. To you God Almighty, without you many things would not be possible, let your invisible hand continue to lead us.

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Chapter 1

Introduction

Economics is about finding a way society can distribute its scarce resources fairly among its members and maximize the want to each member. Markets are a natural mechanism that allocate resource through the decentralized decisions of the millions of households and firms in their act of buying and selling. Competitive markets work in such an ordering manner that the British economist Adam Smith in his 1776 book, "The Wealth of Nations", tried to understand the way that the *invisible hand* could lead economic agents to interact successfully in the market place while being solely driven by their own self interest [31]. In dealing with a general equilibrium of an economy, one looks at the way under which the reconciliation of economic agents (consumers, producers, landlords,...) acting solely on behalf of their own interest, with conflicting interests, is effected. Dating back from Adam Smith and continuing through Menger, Jevons and Walras(1874), there has been a net perception of the existence of a specific price system that plays the crucial coordinating and equilibrating role. In the system presented by Leon Walras, goods prices and the prices of factors used for production are interconnected. Leon Walras took into consideration the natural existing link between different sectors of the economy. He clearly showed how mathematics could be used in service of economics. In 1954, Arrow and Debreu showed how an economy with finite numbers of commodities and economic agents admits a Walrasian equilibrium. It was the first materialization of Walras prediction. The necessity to account for natural aspects of the world like uncertainty, in particular the need for models that are both stochastic and dynamic for the study of financial markets led researchers to extend the framework in which the existence of an equilibrium could be established.

During the 1970's, equilibrium existence results were obtained for economies with infinite numbers of agents and commodities. A number of economists (Bewley , Podczeck and Zame) provided the existence of an equilibrium of an economy whose commodity space is represented by $\mathcal{L}_\infty$ and the price system modeled by $\mathcal{L}_1$. In 1986 , Mas-colell [23] presented a general result of equilibrium existence by replacing the assumption of interior

point with the condition of uniform properness.
In this Master dissertation we are concerned with the problem of analyzing the conditions under which an economy with a large number of agents and commodities admits an equilibrium. Two approaches will be used for the analysis : the framework of locally solid Riesz spaces and that of continuum economy.
A competitive market is a market in which every participants behave solely as a price taker. Mechanically, this is conceivable only if the market involves infinitely many participants with every participants an infinitesimal fraction of the whole.
The idea of describing "pure competition" as a limit of a sequence of a pure competitive economies originate from Edgeworth who used a sequence of replica-economies. Using a continuum framework, Aumann (1966) proposed an economic model with infinite number of agents whereby he model the space of agents by the continuum [0,1]. Following Aumann's work, Hildenbrand generalized the concept of continuum economy to measure space. Ostroy, Zame, and Medecin used a continuous representation of a purely competitive sequence of economies to show the existence of equilibria for an economy described by a correspondence between a measure space of economic agents and their characteristics. By means of a discretization of measurable correspondence, Filipe Martins Da -Rocha (2002) provided an existence equilibria for an economy populated by an infinite numbers of economics agents whose preferences are not ordered and for which the commodity space is modeled with commodities that are differentiated. His discretization method consisted in using both the continuum and the Riesz theory approaches.

In the second chapter of this dissertation, we show how a model assuming a continuum of agents and a finite number of commodities reaches an equilibrium of Walras (Walrasian equilibrium) and a cooperative equilibrium (core). Atomless economy(or continuum economy) has been put forward as being an appropriate mathematical formulation of the usual notion of pure competition. This model relies on the distribution of economic actor' characteristics (preferences, wealth) as an appropriate description of an economy. In assessing a general equilibrium of an atomless economy, one is interested in the demand of all consumers in a large economy. The mean demand is then used as an approximation of the overall demand of the economy. The concept of the consumption sector of the economy is therefore introduced and the demand analysis of an atomless economy in such context amount to the analysis of the mean demand of the consumption sector of the economy.

As a first step, we construct the space of economic actor's characteristics: $\mathcal{Q} \times \mathcal{W}$ with $\mathcal{Q}$ denoting the collection of economic actor's tastes and $W \in \Re$ the consumer's wealth(income). The mean demand is shown to depend on the vector price and on the income-tastes distribution. One characteristic of a large economy is the induced convexity by the size of its consumption sector on the mean-demand. That is when consumer's tastes are non convex, the assumption of many participants induce convexity on the mean-demand. In section 4.4 we present the economic analysis of an atomless exchange economy *i.e.*, the way that the economy redistribute its resources among economic actors. The analysis focus on two categories of redistribution: the core allowance and the Walras allowance. The essential result of the analysis is the identity of the two outcomes of the atomless economic barter. It is shown that in the economy where the weight of each economic agents is insignificant, every equilibrium of Walras belongs to the core allowance and the core allowance belongs to the set of Walras allowance. As said earlier, one of the purpose of a continuum economy is to provide a mathematical model of a pure competition. Section 4.4.3 show how a continuum economy can be considered as resulting from a converging sequence of a pure competitive economies with economic actor's tastes and endowments contained in the set $\mathcal{Q} \times \mathbb{W}$. An equilibrium of Walras is about a price vector and an allowance. Equilibrium analysis of a competitive equilibrium amount to an equilibrium prices analysis and an allowance equilibrium analysis. To investigate the equilibrium prices, one define the mean excess demand function $\mathcal{Z}(\mho)$: the gap between the mean demand $\Phi(\varepsilon, \mho)$ and the supply $\int e\, d\eta$ of the economy ε for every price vector $\mho$. The properties of the mean excess demand allows one to state the fundamental mathematical result for economic equilibrium which point out the existence of a price vector $\mho^* \gg> 0$ that equates the quantity commodity good demanded and supplied in the economic barter. The properties of the mean excess demand and the above result are used to show that a convex economic barter with positive potential admits a Walras allowance and the core allowance. Thereafter we explore the properties of equilibrium prices. To do this, one define the Equilibrium prices correspondence $\mathcal{T}(\varepsilon)$. The set of normalized equilibrium prices for each economic barter is represented by $\mathcal{T}(\varepsilon)$: the set of vector prices having norm one and for which the total potential $\int \varepsilon\, d\eta$ belongs to the demand set. It is shown that the Equilibrium prices correspondence $\mathcal{T}(\varepsilon)$ is closed at every convex economic barter $\varepsilon : (\mathcal{A}, \beta, \eta) \longmapsto \mathcal{Q}_{mo} \times \mathbb{W}$ with $\int \varepsilon\, d\eta >> 0$ and only depends on the preferences-endowment distribution μ. To investigate the equilibrium allocation, the set of Walras allowance is defined. It is shown that the Walras allowance correspondence $\mathcal{L}(\varepsilon)$ is compact valued and upper hemicontinuous in distribution at each economic barter $\varepsilon : (\mathcal{A}, \beta, \eta) \longmapsto \mathcal{Q}_m^{sco} o \times \mathcal{W}$ with $\int e\, d\eta >> 0$. In fact, the above result is obtained under the assumption of strongly convex consumers' tastes. Therefore, there is a need to investigate the case of non strongly convex consumers'tastes. When economic actors' tastes are strongly convex, the set $\mathcal{L}(\varepsilon)$ of Walras allowance is uniquely determined equilibrium allowance. In the contrary case, one may have two economies ε_1 and ε_2 with the same distribution μ of economic actor's characteristics but with different distribution of Walras allowance. This situation naturally lead one to investigate the extent to which the tastes and the income of economics actors cause the realisation of the allowances that are Walrasian for the economy ε.

One then define the set of Walras measures of allowance's distribution, $\mathcal{WM}(\varepsilon)$, on the commodity space $\mathbb{R}^l_+$ and show that the closure with respect to the weak convergence of the set $\mathcal{WM}(\varepsilon 1)$ and $\mathcal{WM}(\varepsilon 2)$ of distribution of Walras allowance of two atomless economies $\varepsilon 1$ and $\varepsilon 2$ on $\mathcal{Q}_{mo} \times \mathbb{W}$ with the same distributions $\mu_\varepsilon 1 = \mu_\varepsilon 2$, is the same.

Further, one define the standard representation of the distribution μ *i.e.*, the mapping ε^{μ} which is nothing but a projection on the extended space of economic's agents tastes: $\varepsilon^{\mu}(\preceq, e, \xi) = (\preceq, e)$ for each economic actor's characteristics $(\preceq, e, \xi) \in \mathcal{Q} \times \mathbb{W} \times [0,1]$. One also define an atomless measure space of agents'characteristics $\mathcal{P} \times \mathbb{R}^l \times [0,1]$ by adding the interval $[0,1]$ on the usual space of agents' characteristics. We show that for any measure μ on $\mathcal{Q}_{mo} \times \mathbb{R}^l_+$ with $\int e\, d\mu \gg 0$ the set $\mathcal{WM}(\varepsilon^{\mu})$ of distribution of Walras allocation of the standard representation ε^{μ} is closed.
The two previous results enable one to define the Walras equilibrium distribution in term of income-tastes distribution μ.
It is known that the mean demand of a simple economy whose consumer's tastes are not convex, is not convex. However, a large numbers of economic participants induce convexity on the mean demand of the economy. With non convex tastes, the mean demand set may be non convex. This characteristic of a large economy is known as an induced convexity on the mean demand. Under this situation, one obtain an approximate equilibrium prices rather than an equilibrium prices. The more agents that participate in the economy the better the approximation. We show, by mean of a proposition, that if an approximate prices prevails rather than equilibrium prices, not every citizens will maximize their utility but at least the majority of them will get what they want.
The chapter end with the famous limiting theorem on the core of an economy that started with Francis Edgeworth in 1881 and enhanced by Debreu and Scarf (1972). In this framework of atomless economy, the infinite dimensional version of the Debreu-Scarf core equivalence is presented.
Knowledge of the conditions likely to lead to an economic equilibrium is an important thing for the society. But of particular attention is the properties related to that equilibrium.
The marvel of the competitive market lies on its equilibrium's properties. In fact, its equilibrium is such that every economic actors bow to the prevailing price. The fairness of its equilibrium's price is such that a banana's seller has incentive to wake up under rain early morning to supply the market and citizens have what they want.
That features of the competitive equilibrium has been subject of intense researches. Particularly appealing the Lausane School (Pareto and Edgeworth) and the Cambridge School(Alfred Marshall and Pigou).
We can learn from the limit theorem of Francis Edgeworth that he perceived equilibrium as being revealed when the number of consumers increase to infinity. At the infinity, the set of Walrasian allowances and Edgeworth allowances are identical. Even though Francis Edgeworth discussed only a few case of two dimensions, his geometrical picture holds for higher (infinite) dimensions. The power of Riesz theory for economic analysis is nothing but its ability to use the Edgeworth-box in infinite dimensions.
Riesz theory, specially locally convex-solid topology, happen to be appropriate framework for studying and analysing economic equilibrium's properties. The central concern of this

Master's dissertation is precisely to show how Riesz theory allow a constructive examination of the properties related to an economic equilibrium.
Economic barter have a decisive advantage in that they help illustrate the redistribution of the total potential to each of the economic actors.
Here we use tools provided by Riesz theory to conduct topological analysis of the Edgeworth-box, a territory containing all of the outcomes engendered by an economic barter, so as to locate the Pareto set.
In year 1906, Vilfredo Pareto looked at Walrasian equilibrium and saw it being performed not only by market mechanism but in addition by human attitude. He realized that an economic barter can reach the same equilibrium if each market participants accepts to give up a portion of what they cherish in order to get what they desire the most. He thus stated the condition of a possible optimum. Mas-Colell used Riesz theory and expressed this equilibrium condition in term of uniform properness. Economic literature call it the first welfare theorem for it asserts the equivalence between the allocations of the economy that are competitive with those that are Pareto. The second welfare theorem state on the converse. Uniform properness allow one to establish the equivalence of the first welfare theorem and the second in the infinite dimensional space.
The problem behind the equivalence between a competitive allowance and Pareto allowance amount to relate an optimal supported price allowance to an optimal free allowance. Mas-colell solved this problem by replacing the interiority assumption of the positive cone by the condition of uniform properness. Uniform properness is of great importance in showing the existence of a vector price capable of converting a free allowance to a price supporting allowance.
Classical analysis use a finite dimensional Edgeworth-box. However, when the number of economic participants increase, one need appropriate topology for infinite dimensional configuration (spaces). These spaces resemble to the $\mathbb{L}_1$-space, the $\mathbb{L}_2(\mu)$-space, the Dedekind Riesz space of real valued sequences, $\mathbb{R}_\infty$. The locally convex-solid topology, in conjunction with the structure of order of the solid Riesz spaces engenders remarkable properties: The Lesbesgue and Fatou properties but also metrizability, Frechet and the B-topologies. The locally convex-solid topology serves characterize compact subsets of $\mathscr{Z}_c^\sim$. Weak compactness are essential assumption for the existence of equilibrium. In particular, economic models of individual behavior generally assume that the reason for a consumer's choice is to maximize his utility. In this case, one need compactness in order to find optimal commodity bundles. Riesz theory solve compactness problem in infinite dimension by building the Riesz dual pair $\langle \mathbb{Z}_\Omega, \mathbb{Z}'_\Omega \rangle$ which in fact contain the Edgeworth-box *i.e.*, the order interval $[0, \Omega]$. The weak compactness of the order interval $[0, \Omega]$ enables solving optimality problem. In particular, the search of equilibrium (optimal allocations) is moved from the original Riesz economy $\langle \mathbb{Z}, \mathbb{Z}' \rangle$ (infinite dimension) to the Riesz dual pair $\langle \mathbb{Z}_\Omega, \mathbb{Z}'_\Omega \rangle$ (finite dimension). Then after the computation of the optimal allocations, the results are sent back to the original Riesz economy $\langle \mathbb{Z}, \mathbb{Z}' \rangle$. In section 5.8.2 of chapter

5, we present the construction of the Riesz dual pair.

An utility function carries consumer's tastes to a real value. It help obtain information about the continuity of the behaviour responsive of the consumer. Likewise, a price system attach market value to a consumer's preferences. We need, thus, to provide the utility function and the vector price with appropriate topology.

An investigation of the utility function shows that they belong to the class of linear functional called the topological dual, $\mathbb{Z}'$, of the vector space $\mathbb{Z}$ (with $\mathbb{Z}$ the commodity space). This class of Riesz spaces is our main focus. We aim to provide the utility function and the price system with an appropriate topology for infinite dimension analysis.

1.0.1 Locally convex topology

The distance (metric), the semi-norm and the norm (on the commodity set, or a subset of the commodity space) are also linear functions capable of engendering a locally convex topology ξ.

As first step, the commodity space (Riesz space) is endowed with a locally convex topology, ξ. Thereafter, we equip the locally convex space $(\mathbb{S}, \xi)$ with two topologies: the weak topology $\delta(\mathbb{S}, \mathcal{S}')$ and the Mackey topology $\delta(\mathbb{S}, \mathbb{S}')$ on $\mathbb{S}$. The first topology allows for continuity on $\mathbb{S}'$ (the topological dual of the Riesz space $\mathbb{S}$) and pointwise convergence on $\mathbb{S}$, while the second topology facilitates uniform convergence on the subsets of $\mathbb{S}'$ that are convex, balanced and $\delta(\mathbb{S}', \mathbb{S})$ - compact. Likewise, we equip the topological dual $\mathbb{S}'$ with the weak topology $\delta(\mathbb{S}', \mathbb{S})$ for pointwise convergence on $\mathbb{S}'$ and the strong topology $\Gamma(\mathbb{S}', \mathbb{S})$ for uniform convergence on the $\delta(\mathbb{S}, \mathbb{S}')$ - bounded subsets of $\mathbb{S}$.

1.0.2 Locally convex-solid topology

A locally convex-solid topology is a linear topology ξ on a Riesz space that possess both characteristics of being locally solid and convex. Locally convex-solid topologies provide us with the means for defining the absolute weak topology and the strong topology $\Gamma(\mathbb{S}', \mathbb{S})$, which is in fact a Hausdorff locally convex-solid topology on $\mathbb{S}'$.

Riesz theory sees the Edgeworth box as an order interval. In this Master's dissertation, locally convex-solid topology has been used to depict the optimal commodity bundle,the Pareto efficient allocation, from the set of allocations of the Edgeworth economy.

Chapter 2

Genesis of General Economic Equilibrium

2.1 Walras' General Equilibrium System

The essential of Walras' contribution is his recognition of the link between various sectors of the economy: households, firms, government and foreign sector; could be understood and communicated most efficiently through the use of mathematics. The way that Walras formulated the functioning of a market economy resulted in a system of simultaneous equations. For simplification we consider an economy with only two sectors: firms and households, and ignore the government and foreign sector. We further assume constant technology, that a household's preference remains constant through the analysis, that there is no transaction among firms, that the economy is at full employment, and that all industries are perfectly competitive. Assume there are n factors of production:

T,T',T'',... are different kinds of land.

P,P',P'',... are different kinds of labor.

K,K',K'',... are different kinds of capital goods.

There are m final goods represented by A, B, C,...

Prices for final goods are written as p_a, p_b, p_c,...(since good A is considered as numeraire, its price is equal to one i.e. $p_a = 1$).

The demands of households for final goods are represented by d_a, d_b, d_c,...

The quantities of factors of production offered by households are written as o_t, o_p, o_k,...(where o_t = land offered, o_p = labor offered, o_k = capital offered) and are assumed positive when households offer factors and are negative when they demand factors.

In equilibrium the flow of income to the households will be equal to the expenditure of households. The flow of income from the sale of land is measured by the amount of land offered (o_t) times the price of land (p_t). The income from the sale of others factors is obtained the same way.

The expenditure of a household for a given final goods is the product of its price times the quantities consumed, or demanded in equilibrium. We then obtain the equilibrium condition for a household or individual by setting income equal to expenditure.

$$o_t p_t + o_p p_p + o_k p_k + \cdots = d_a + d_b p_b + d_c p_c + \ldots \tag{2.1}$$

The above equation involves the prices of factors used in the economy for production as well as the price of final goods that households faces in the market. It also contains the quantities of the n factors offered by households (which need to be determined) and the quantities of the m final goods demanded by households. Since there are m final goods and n factors, the number of unknowns is $m + n$. There are m final goods, but since final good A is the numeraire, only m - 1 equations represent the equilibrium of the household in the market for final goods. The household's demand function for final goods depend on their utility, on their income (which is reflected in the prices of factors used by the economy for production), on the price of the final goods and on the price of all other final goods. There are m - 1 of these equations. The demand for good A can then be derived from equation (1). We thus have:

$$\begin{aligned} d_b &= f_b(p_t, p_p, p_k, \ldots, p_b, p_c, p_d, \ldots) \\ d_c &= f_c(p_t, p_p, p_k, \ldots, P_b, p_c, p_d, \ldots) \\ &\vdots \end{aligned} \tag{2.2}$$

The household's supply function of the factors used in the economy for production depend on the utility of retained factors of production, on the price of the factor, on the price of others factors and on the prices of final goods. There are thus n of these equations.

$$\begin{aligned} o_t &= f_t(p_t, p_p, p_k, \ldots, p_b, p_c, p_d, \ldots) \\ o_p &= f_p(p_t, p_p, p_k, \ldots, p_b, p_c, p_d, \ldots) \\ &\vdots \end{aligned} \tag{2.3}$$

The technical coefficients: a_t, a_p and a_k of production express the quantities of land, labor and capital which are needed for the production of one unit of a given product. Thus, a_t, a_p, a_k represent the quantities of land, labor, and capital necessary to produce one unit of good A.

We represent by capital letters the market demand and supply. That is the market demand for good A is written as D_a and is obtained by summing the demands of all the households for good A, *i.e.*, ($D_a = \sum d_a$). Likewise market supply of factors is derived and written as ($O_t = \sum o_t$).

In considering the equilibrium of households, we may take as given the prices of the final goods and the prices of the factors used for production. It is then possible to derive the households supply functions of factors of production (2) and the household's demand function for final goods (3). In analyzing general market equilibrium, it is not permissible to assume that either final or factor prices are given; they become unknowns.

Summing up, we have m-1 unknowns for the prices of final goods $(p_b, p_c, p_d, \dots)$, n unknowns for the prices of factors $(p_t, p_p, p_k, \dots)$, m quantities of final goods demanded (D_a, D_b, D_c,...) and n quantities of factors offered (D_t, D_p, D_k,...). In total we have $2m$ + $2n$ - 1 unknowns.

2.2 General Market Equilibrium Equations

The quantities of factors supplied in the market are functions of the prices of the factors and final goods. There are n equations of the following type.

$$\begin{aligned} O_t &= F_t(p_t, p_p, p_k, \dots, p_b, p_c, p_d, \dots) \\ O_p &= F_p(p_t, p_p, p_k, \dots, p_b, p_c, p_d, \dots) \\ &\vdots \end{aligned} \tag{2.4}$$

The quantities of final goods demanded in the market are also function of the price of factors and final goods. There are m-1 of these equations, and one equation expressing the demand for good A, the numeraire, for a total of m equations.

$$\begin{aligned} D_b &= F_b(p_t, p_p, p_k, \dots, p_b, p_c, p_d, \dots) \\ D_c &= F_c(p_t, p_p, p_k, \dots, p_b, p_c, p_d, \dots) \\ &\vdots \\ D_a &= O_t p_t + O_p p_p + \dots - (D_b p_b + D_c p_c + \dots) \end{aligned} \tag{2.5}$$

The quantities of factors used by firms must, in equilibrium, equal the quantity offered. These are n equations of this types.

$$\begin{aligned} O_t &= a_t D_a + b_t D_b + c_t D_c + \dots \\ O_p &= a_p D_a + b_p D_b + c_p D_c + \dots \\ &\vdots \end{aligned} \tag{2.6}$$

Final costs of production must, in equilibrium, equal prices. There are m equations of the type.

$$\begin{aligned} 1 &= a_t p_t + a_p p_p + a_k p_k + \dots \\ p_b &= b_t p_t + b_p p_p + b_k p_k + \dots \\ &\vdots \end{aligned} \tag{2.7}$$

These four systems of equations (2.4), (2.5), (2.6), and (2.7), and equation (2.1) provide a total of $2m + 2n$ equations. Equation (2.7) is not an independent equation in that it does not supply new information. Eliminating this equation we are left with $2m + 2n$ -1 equations to find an equal number of unknowns.

Thus, Walras' general equilibrium system lead to the following observations: households' equilibrium and the equilibrium in the markets for final goods is consistent with equilibrium for firm and equilibrium in factor markets.

2.3 Extension of Walras Finding

Leon Walras was the first to address the issue of economic equilibrium subject to a large economic markets populated by a number of economic agents, interacting among them with different interests. Can such markets be in an equilibrium?

His merit was to state it in a system of simultaneous equation. Walras conceptualization of the interdependence of the sectors of a market economy set him on the high place of the history of economic theory.

However, Walras did not provide any conclusive arguments relative to the existence of the solution to his system of simultaneous equations. This fact raised questions among economists about a possible solution of Walras' general equilibrium system. Is the system determinate or undeterminate? Is the equilibrium stable or unstable?

It took fifty years for the problem to be satisfactory solved principally by Kenneth Arrow and Gerard Debreu. This pioneering theory of general equilibrium laid the building stone of the concept of equilibrium in mechanic, economics and mathematics. During the conjectural period, the Walrasian's view of equilibrium has been considerably extended to Edgeworth's equilibrium and Pareto optimum. Research by a large number of authors has steadily extended the framework in which the existence of an equilibrium can be established. This Masters dissertation utilizes the framework of locally solid Riesz spaces to analyze the existence of equilibrium for an economy consisting of an infinite number of participants and commodities.

2.4 Genesis of General Economic Equilibrium

In 1838, Augustin Cournot attempted to present a mathematical formulation of the interdependence of the various part of the economic system. Thirty-six years later Walras built a model of general equilibrium in equation form which indicated the interrelatedness of the whole economy. In 1930, the American economist(Noble Price), Wassily Leontief presented an empirical general equilibrium model using input - output analysis. In a three-sector model, including agriculture, manufacturing and service, for example, the output of any sector, say grain produced by agriculture, is used as an input by all three sector of the economy: in agriculture for cattle feed, in manufacturing as a row product for cereal and as food for peoples in the society. Leontief expressed these relationship in mathematical form in a matrix which gave empirical content to the general equilibrium. Leontief's empirical general equilibrium model can provide much information about the interrelationship between different sectors of the economy. The question of what additional resources are needed in the copper, glass, steel, and rubber sectors of the economy if automobile production is increased 100.000 units, can properly be answered in an input-output model consisting of 200 sectors(i.e. 200 prices, 200 markets). Thus the complex relationship between the various sectors of an economy, which can be theoretically expressed in a Walrasian general equilibrium model, were given empirical content in Leontief's input-output analysis.

General equilibrium is concerned with the problem of finding prices system capable of coordinating the activities of economic's agents(consumers and producers).

An actual economy is composed of numerous agents, markets and institutions. Those atomistic decisions-making units such as individuals producers(firms), owners of productions factors(workers, landlords, owners of capital), consumers(households), etc...,who make decisions on their own initiatives and who seek to satisfy their own maximal interests, are likely to conflict with each others. competitive market mechanism reconcile individuals conflicts and coordinate individuals activities.

How this coordination takes place has been a central preoccupation of economic theory since Adam Smith and relayed by Jevons and Menger in 1870's and above all by Walras in 1874. By borrowing and adapting various tools from many fields within mathematics, mathematical economics has made it possible to state economic theory concisely and precisely. A number of mathematicians, Cournot, J.H Von Thunen, Jevons has worked at applying mathematical economic to various parts of economic structure.

Augustin Cournot was the first pioneer in stating hypotheses in mathematical form. His 1838's book "Researches into Mathematical Principles of the Theory of Wealth", has been seen as an effective attempt to bring mathematical into economics.

Leon Walras in his general equilibrium theory in 1874 was responsible for the most significant early application of mathematics to economics. Mathematical economics saw its apogee with Paul Samuelson(1915-) who made a serious effort to reduce the literary theoretical structure into mathematical form. Economics had become a social science with a fairly well developed theoretical structure and considerable internal consistency.

In the 1930s two mathematicians, Abram Wald(1902-1950) and John Von Newmann (1903-1957) turned their attention to economic theory and the study of equilibrium condition in static and dynamic model.

Chapter 3

Classical Analysis of Economic Equilibrium

3.1 Description of Economic System

The principal work of the economy is to allocate production and consumption among consumers. The use of a pure exchange economy is to illustrate how a competitive market better allocate the fixed amount of goods (endowments) initially available among consumers. The task of General Economic Equilibrium in such context is to set the consumer's demand.
In Debreu's Theory of value, the following are considered as primitive concepts: the commodity space, the price system and the consumption units (individual preferences and endowments).
A market is essentially characterized by the good that it is exchanged. A good is distinguished not only by its physical aspect but also by its quality, by the time - interval during which it is available and by its location: place at which it is available.
A *commodity* is a good (food, housing, and clothing) or a service (labor). The choice of the Euclidian plan $\mathbb{R}^l_+$ as a commodity space have double implications. It firstly mean that there is no upper-bound (no satiation) in the individual consumption. Secondly, it means that goods g are perfectly divisible.
The *economic system* can be described by the consumer's activity. Economy is a set of economic agents, each of which being characterized by his consumption set X_i, his endowments w_i and his preference relation ,$\succeq_i$. The consumer's consumption set is interpreted as a set of commodity plans. The consumer's consumption plan is the set of consumption vectors a priori available to him *i.e.* including all consumption vectors among which the individual could conceivably choose without considering his pocket(his budget constraints) Every consumption plan can be represented by a vector $x \in \mathbb{R}^G_+$.
Consumer's preferences characterize his behavior, it reflect his tastes for the goods and

services, his personal perception of the different events. Consumer's preference is a more basic concept in demand theory. However, it is a mathematical tool a bit difficult to handle. For this reason, it needed to find conditions which guarantee the representation of a preference by an utility function that is continuous since all utility functions are not necessary continuous. Debreu(1952) provided, in the form of theorems, conditions under which a complete preference order,$\succeq$, can be represented by a numerical function. In the second section of chapter three, we present these conditions as well as the definition of the utility function. The objective function(utility function) is used to obtain information about the continuity of the behavioral function(demand function).

3.2 Demand Function

The consumer's demand function is a solution to the utility maximization problem. When preferences are strictly quasi-concave, the solution to the utility maximization problem is called Walrasian or Marshallian demand function. When preferences are not concave but rather quasi-concave, the utility maximization problem does not admit a unique solution, we then speak about demand correspondence.
A price system is considered as the set of rates at which goods are exchanged. The fundamental economic problem of the consumers is to select the best affordable bundle. The compactness of the budget set allow one to use the classical Weierstrass' Theorem to find the best (optimal) commodity bundles. Each consumer determine his effective demand *i.e* his optimal quantity of good, his desire subject to his initial endowment in goods. For a given price vector p and a consumer's initial endowment w^i, individuals demand $d^i(p, w^i)$ for the i^{th} consumer is derived from the utility maximization hypothesis according to which consumer maximize his utility by choosing the best commodity bundle that agree with his income and his preference.

Continuity of the demand function is an important property in that when individual demand is not continuous or when consumer's preferences are not convex, then one cannot have the means of assessing individual demand following a price change. Therefore, one cannot make any inference about consumer's behavior. In case of strictly convexes, monotones and continuous consumer's preferences, the demand function is continue. Jevons and Menger showed how individual demand is a function of economic's agents utility. A strictly quasi-concave, monotone and twice derivable utility function lead to a demand function that is derivable with respect to the price. Strictly quasi-concavity of preferences as well the convexity of consumer's preferences are an essential condition for the evaluation of a Walrasian equilibria.
The excess demand $Z^i = d^i(p, p.w^i) - w^i$ for the i^{th} consumer is defined as the difference between his Walrasian demand $d^i(p, w^i)$ for good g at a price p and his revenue $p.w^i$ as well as his endowment w^i. The aggregate demand of an economy is made of the sum of all

individuals demand for the economy. Likewise, the aggregate excess demand function of the economy is constituted by the sum of all individuals excess demand: $Z(p) = \sum_i z^i(p)$. A pure - exchange economic use only internal partners there is no external exchange. It is thus a closed economic system. Evaluation of an equilibria in such an economy amount to find a set of specific prices that makes all commodities excess demand to be equal to zero. This special vector prices, equilibrium price vector, belong to the simplex:

$$\mathcal{S}^{G-1} = \{p \in R_+^G \mid \sum_{g=1}^{G} p_g = 1\} \tag{3.1}$$

$p = (p_1, ..., p_G) \in \mathbb{R}_+^G$.

Under convexity assumptions(of preferences), the existence proof of a Walrasian equilibrium lead to the use of theorems from topology. The natural one being the fixed point theorem for we need a function which is continuous and that map the set of prices into itself and which in addition is appealed to mimic the dynamic of the demand and supply (Walras tatonnement). In fact, the key mathematical tools in proving a competitive equilibrium were provided by algebraic topology in the form of separation theorem and fixed point theorem. This theorem was advanced by Browser, an eminent Dutch mathematician, at the beginning of twentieth century. The Browser's fixed point theorem was later reformulated firstly by John Newmann (1937) in connection with his model of economic growth stability and thereafter by S.Kakutani (1941). The convexity of the excess Aggregate demand allow one to use the standard fixed point argument so as to obtain the Walras'law: $p.\mathcal{Z}^i(p) = 0$. Walras'law tell us that for each consumer i and for each good g in the market, the value of the total consumption available should be equal to the value of the endowment (owned wealth). So far, general economic equilibrium has used three methods for the proof of a competitive equilibrium in a finite exchange economy:

- - The fixed point argument in line with the Arrow-Debreu (1954).
- - The method used by Negishi(1960) for a price supporting a free allocation such as a Pareto and
- - The core equivalence, the Debreu-Scarf (1963) characterization of a Walrasian equilibrium as Edgeworth equilibrium.

3.3 Classical Treatment of Economic Equilibrium

An allocation for an exchange economy is a redistribution of commodities bundle of the economy among its consumers.

Equilibrium of a competitive market is something that combine social and individual. It

guaranty a certain global coherence by respecting and satisfying the most desire expressed by consumers.
Walrasian equilibrium can simply be defined as being a pair of a price system and an associated allocation. Price system contain all information. It is a signal that is needed by consumers to set his consumption plans. At the equilibrium, the demand for all consumers are equals. A competition equilibrium of an economy is a vector price p and an allocation x such that:

- - market clair $i.e \sum_i x_i = \sum_i w_i$(demand equal supply).
- - given the vector price p, the allocation x maximize the utility of all consumers under budget constraint.

In the next chapter, we show how an infinite dimensional economy admit a competitive equilibrium.

3.4 Classical presentation of welfare theorem

Standing in the front rank of agrarian socialists, Leon Walras point out that in the interest of justice, he want to establish a regime of absolutely free competition. He attempted to show that free competition is the only one capable of providing society with a maximum satisfaction subject to a minimum of sacrifice (cost). Walras concept of equilibrium gave a vibrant debate among economists. One of the question of particular interest was the socioeconomic nature of the competitive equilibrium.
In his celebrated principle of invisible hand, Smith pointed out that every individual, in selfishly pursuing only his or her personal goal, is driven by an invisible hand, whilst providing the best good for all. Society therefore can expect a competitive equilibrium to bring that best good for all.
It is thus interesting to know whether the existing competitive equilibrium is equity, a complete social system. Mathematical economic played a great role in the attempt to specify the nature of an economic optimum and the condition necessary to achieve it. Particularly appealing the Lausane School (Pareto and Edgeworth) and the Cambridge School(Alfred Marshall and Pigou). Since all economists concurs in the belief that perfectly competitive market lead to an optimum allocation of resources, the solution would amount to correct the defects of the competitive market system.
Walras' general equilibrium provided the conditions necessary for an optimum allocation of resources. Pareto extended Walras'work into the area of welfare economics.
Considering the situation of a pure exchange economy between two consumers, Pareto showed that exchange will cease once the marginal rate of consumption of the two consumers will be equals. Pareto acknowledged that there is many way to reach that optimum and thus there is an infinity of Pareto optimum along the equilibrium path. That line

is the locus of Pareto efficient allocations. This line is known as the *(*contract curve), it describe the situation where none of the two consumers is willing to achieve a preferred situation - there are in equilibrium(Pareto equilibrium). It is a situation of duality. The current dissertation uses Riesz dual system to find an equilibrium under such a situation. Pareto showed that every competitive equilibrium is optimal. Efficient allocations are effected not by the price system as it is the case in Walrasian equilibrium but by means of appropriate reallocations of initial resources of consumers.
Pareto set forth the conditions that must exists to maximize welfare. It could now be shown that a perfectly competitive market would lead to maximum economic welfare.

In the year 1881, the British economist Francis Edgeworth provided a framework which brought about a new consideration of equilibrium [25, 12]. By means of a geometrical picture describing the exchange of two commodities that might arise in an economy with two types of consumers each of whom initially endowed and voluntary engaged in exchange. He observed that both agents will desire any outcome that fall beyond their indifference curves and will refuse any outcome that fall under their indifference curves. The region situated between the two indifference curves is the equilibrium region, or the region of agreement or the core allocations. Any exchange resulting in an outcome that fall in that region is blocked either by agent1 or agent2. The core of this exchange economy is thus localized in that region.

The notion of the core allocation was essentially originated by Edgeworth (1881)[17]. In fact, as stated above, Edgeworth'view is that an economy generally admits not only a competitive equilibrium but also a non competitive equilibrium(Edgeworth equilibrium). The term competitive involve many economic participants whilst the core equilibrium may result from an exchange economy between only two economic agents. Edgeworth's contributions played important role in the future development of general equilibrium analysis.

Chapter 4

Equilibrium Analysis in the framework of Atomless Economy

4.1 Introduction

Aumann (1964) observed that the mathematical models of the time were not in a position to mimic the real dynamic of the market structure called in economy the perfect competition system. According to him, the weight of an individual participants on the market mechanism can be mathematically ignored only in an economic system populated by an infinite numbers of economic participants. However, traditional mathematical model that pretended to represent perfect competition structure of the market economic, were not accounting for this mechanical aspect of the dynamic. He thus suggested a mathematical model capable of integrating a continuum of participants. He then introduced the concept of an atomless economy.
In fact, competitive equilibrium is consistently effected in a market structure populated by an infinitely many participants with every participant an infinitesimal fraction of the whole. Each market participant behaves as a price-taker. Even though Walras speaks only about *equilibre* and not competitive equilibrium, one can truly imagine that Walras as a mathematician was aware of the behavioral assumption on which that concept (equilibre) is based. One of the features of competitive markets is the equality between price and marginal utility and marginal cost. In a market with few sellers, there will be inadequate checks and balances to assure that price is determined by costs. And in such a circumstance the truth (magic) in the invisible hand doctrine may vanish. Therefore, the number of both the markets and market participants seems to be an important parameter. In his analysis of the working of competitive markets, Adam Smith regarded competition as intrinsically involving many economic participants [16]. In competitive equilibrium analysis, price system reconciles economic agents with mutually conflicting consumption and production plans. Beyond that, another essential parameter that enables price to

accomplish this task (of the invisible hand) is the number of market participants.

Robert Aumann (1964) showed that there is always a competitive equilibrium in a market system with infinitely many participants, even when preferences are not convex. Bewley (1972) [7] showed that an economy with infinitely many commodities achieves equilibrium under non convexity and increasing returns to scale. The above infinite economies thus conveys meaningful features. We learn from the law of large numbers that the measure of any random variable tends to its natural characteristic as the size of the sample under consideration increases.

Aumann (1966) [6] proposed an economic model with a continuum of economic agents. He firstly considered economic agents as a continuum [0,1]. Later on Hildenbrand [18] generalized Aumann's model to a measurable space. This way of modelings the set of economic agents provides a rigorous mathematical formalization of economic concepts according to which each agent have a negligible influence on global economic activities. Hildenbrand [18] generalized Aumann's result to an economy with production and complete and transitive preference. The notion of a continuum of agents dates back to Allen and Bewley (1939) and Houthakker (1955). The continuum concept was introduced in game theory as a continuum of players by Shapley (1953,1961) and Milnor and Shapley (1961), Shapley and Shapiro (1960), David (1961), and Peleg (1963).

Generally, individual behavior is described by a simple valued function(demand function). However, in economics and game theory, there are situations where behavior does not manifest itself in uniquely determined actions. Atomless economy is concerned with aggregate behavior, if individual behavior is not uniquely determined then one need instead to use another mathematical tool: multivalued function or correspondence.
Continuum economy intensely make use of correspondence relation from which it derive topological properties of the consumption set, the demand set, the equilibrium price correspondence and the Walras allocation correspondences. The first part of this chapter is devoted to the construction of the space of agents'characteristics: $\mathcal{L} \times \mathbb{R}$ with $\mathcal{L}$ denoting the set of consumer's preferences and W the real value of the consumer's wealth(income). An economic system without production and in which economic agents are only consumers is called a pure exchange economy. This economic model is aimed to show how a competitive economy efficiently allocates amounts of the goods among the consumers. The task of General Economic Equilibrium in such a context is to set the consumer's demand.

Part I

The Mathematic Of Continuum Economic

In demand analysis one is concerned with the existing similarity and disparity between economic actor's tastes. This fact lead one to consider distance separating those of the consumers who are alike in term of preference to those who have different taste. The extension by Maurice Frechet of the principals properties of the Euclidean space that could be be defined by means of the distance, allowed Felix Hausdorff to develop the metric space. This naturally lead to the notion of neighborhood, continuity, convergence, and limit. Therefor topology. These notions happen to form the main tools of the demand analysis.

4.2 Metric Space

4.2.1 Distance

Given a set $\mathcal{D}$, an application χ of the cartesian product $\mathcal{D} \times \mathcal{D}$ into the set of positive real numbers, $\mathbb{R}^+$, behaving in the way that:

1. $\star$ the distance between two given points x and y of the set $\mathcal{D}$, $\chi(x, y)$, is null whenever point x is equal to the point y (separability property).

2. $\star$ the distance from the point x to the point $y, \chi(x, y)$ is the same as the distance from the point y to the point $x, \chi(y, x)$ (symmetry property).

3. $\star$ For the points x, y, z of the set $\mathcal{D}$, the distance from the point x to the y is less or equal to the sum of the distance of (x, z) and (z, y). (triangular inequality).

Examples: For the set $\mathcal{D} = \mathbb{R}$ or $\mathbb{C}$, let g be the linear transformation assigning a pair (x, p) of the point of $\mathcal{D}^2$ to its image $s - p$ of D and the Euclidean norm $h(x) = |x|$. Then the function $h \circ g$ define a distance between two points s and p in $\mathcal{D}$: $\chi(s, p) = |s - p| =< s - p, s - p >^1 /2$. $\star$ In the set $\mathcal{D}$, consider the application defined by $\chi(f, g) = sup_x \in \mathcal{D}|f(x) - g(x)|$ for any function f and g of the bounded application of $\mathcal{D}$ into $\mathbb{R}$ or $\mathbb{C}$. Then χ is a distance.

4.2.2 Balls and Sphere

Given a metric space $(\mathcal{D}, \chi)$, for every element y of the set $\mathcal{D}$ and a positive real χ.

Definition 4.1. The set of points y whose distance from the center b is less than a positive real η called radius is defined by: $\{y \in \mathcal{D}$ for which $\chi(y, b) < \eta\}$. We call this set an open ball. An closed ball with center b and radius η is the set: $\{x \in \mathcal{D}$ for which $\chi(x, b) \leq \eta\}$. A sphere of center b and radius η is the set: $\{x \in \mathcal{D}$ for which $\chi(x, b) = \eta\}$

4.2.3 Metric Topology Of a Metric Space

Given a metric space $(\mathcal{D}, \chi)$, the set $\mathcal{T}_\chi$ of the subset ϑ of the metric space $(\mathcal{D}, \chi)$ for which there exists an open ball that is contained in ϑ, constitute a base for the topology derived from the metric χ on the set $\mathcal{D}$.

Definition 4.2. A subset E of a metric space $(\mathcal{D}, \chi)$ is said to be an open subset of the set $\mathcal{D}$ if it is empty or if it contain an open ball having a non zero positive radius and as center an element x of the subset E.

Let us now consider the set of these open subsets of the metric space $(\mathcal{D}, \chi)$ which are closed for a countable intersections and (even infinite)unions. We denote by $\mathfrak{T}$ that set and we observe that it contain the empty set as well as the space $\mathcal{D}$.

Definition 4.3. The set $\mathfrak{T}$ constitute a topology in the set $\mathcal{D}$. It is the topology derived from the metric χ on the set $\mathcal{D}$.

In case the class of open subsets $\mathfrak{T}$ as defined above have the separability property then the derived topology is said to be separable. We need also to notice that the set $\mathcal{D}$ may possess others topologies than T_χ. In the event these topologies coincide with T_χ then the space $(\mathcal{D}, \mathrm{T}_\chi)$ shall be called a metrizable topological space.

Proposition 4.4. *Every metric space $(\mathcal{D}, \chi)$ is normal and also separable. A subset of a metrizable space is also metrizable.*

From the derived topology, we can now define others topological properties of the metric space $(\mathcal{D}, \chi)$.

Definition 4.5. A point $\tilde{a}$ is said to be an adherent point to the subset B of the metric space $(\mathcal{D}, \chi)$ each of its neighborhood meet the subset E. A subset E of the metric space (D, χ) is dense if its adherence points is equal to E.

Characterization of the compact metric space

Let $(\mathcal{D}, \chi)$ be a metric space such that every sequence (x_n) of points in $(\mathcal{D}, \chi)$ admit at least one value of adherence. Then for every open covering $(\theta)_i$ of the space $(\mathcal{D}, \chi)$, there might exist a positive real η such that for all element x in the space (D, χ), there exist an index of I for which $B(x, \eta) \subset \theta)_i$.

4.2.4 The space of subsets of Metric Space

Earlier, we noticed the interesting role of open subsets in inducing a metric topology in a metric space. Here we consider a space whose elements shall be subsets of $(\mathcal{D}, \chi)$. Our first concern in this space is the convergence of a sequence of elements in $(\mathcal{D}, \chi)$. Let us then consider a sequence (T_n) of subsets of $(\mathcal{D}, \chi)$.

Definition 4.6. We say that a point x of $(\mathcal{D}, \chi)$ belong to the subset called The TOPOLOGICAL LIMES INFERIOR in short LI(T_n) if and only if for every neighborhood $\aleph$ of x there do exists an integer p for which the intersection of (T_n) with the neighborhood $\aleph$ is non empty for n greater or equal to p. The TOPOLOGICAL LIMES SUPERIOR in short LS(T_n) is the subset of the metric space $(\mathcal{D}, \chi)$ whose elements behave in a way that the intersection of each of its neighborhood with every element (T_n) of the sequence is non empty for an infinitely many n.

We need to notice that the number of elements x in the metric space (D, χ) that satisfy the condition of belonging to LS(T_n) is greater than these of elements likely to belong to LI(T_n). It follows that for every sequence (T_n) in the metric space $(\mathcal{D}, \chi)$, LI(T_n) $\subset LS(T_n)$. Whenever LI(T_n) will be equal to LS(T_n), we shall say that the sequence (T_n) of subsets of the metric (D, χ) admit a closed limit. We then speak of the closed convergence the sequence (T_n).

Proposition 4.7. *Every sequence (T_n) of subsets of a separable metric space admit a converging subspace.*

Our second step consist in equipping the space of subsets of the metric space with a metric(distance). Given two distinct subsets E and F of $(\mathcal{D}, \chi)$, let $V_\zeta(E)$ denote an ζ - neighborhood of the subset $E.i.e.$ the set of elements x of the metric space whose distance $\chi(x, E)$ with the subset E is less or equal ζ. Likewise we define $V_\zeta(F)$ as the set of elements x of the metric space whose distance $\chi(x, F)$ does not exceed ζ. Now, we denote by $ClS(\mathcal{D})$ the collection of all closed subsets of the metric space $(\mathcal{D}, \chi)$ and define the HAUSDORFF distance between the subset E and F in the following way: $\gamma(E, F)$ $:= \inf\{\varsigma \in [0, \infty]\}$ for which $E \subset V_\zeta(F)$ and $F \subset V_\zeta(E)$. The function γ associating two distinct subsets of $ClS(D)$ into an element ς of $[0, \infty]$ is characterized in the following way:

1. The Hausdorff distance between two subsets of $ClS(\mathcal{D})$ of $(\mathcal{D}, \chi)$ is null for two equal subsets of $ClS(\mathcal{D})$.
2. The Hausdorff distance $i.e.\gamma(E, F) = \gamma(F, E)$ two subsets E and F of $ClS(\mathcal{D})$.
3. The Hausdorff distance obey the law of triangular inequality.

It follows that the Hausdorff distance so defined is indeed a metric on $ClS(\mathcal{D})$. It might thus induce a topological metric.

Construction of the base of the topology of closed convergence

Let us first consider the sets of the form $[H : H_1]$ and assume that it is being non-empty. In this event, it might be an element $J \in [H : H_1]$ but this mean that $J \subset H$ and the intersection of J and H_1 is non-empty. Let then the element z belong to this intersection. This amount then to say that there exist an open ball of center z and positive radius ρ that is contained in H_1. Also there exist an open ball with center J and positive radius ρ that is contained in H. It follows that an open ball with center J and radius $\rho/2$ might be contained in the set $[H : H_1]$. This then show that $[H : H_1]$ is an open set. By an inductive argument, we can see that for a subset G of $ClS(D)$, one can find an open ball with center G and positive radius that is contained in a set of the $[H : H_1, ..., H_p]$. The characteristic of a compact set in a metric space, allows one to deduce the existence of an open recovering which can be of the form $[H : H_1, ..., H_p]$.

Proposition 4.8. *An open set of the space $(ClS(\mathcal{D}), \gamma)$ of the closed subsets of the metric space of the form $[H : H_1, ..., H_p]$ with $H, H_1, ..., H_p$ the open sets of $(ClS(\mathcal{D}), \gamma)$, defined by: $[H : H_1, ..., H_p] := \{J \in ClS(\mathcal{D})$ such that $J \subset HandJ \cap H_i \neq \emptyset\}$ for $i := 1, ..., p$. constitute a base of open sets of the topology of the Hausdorff distance.*

Definition 4.9. Let $\Im \equiv [H_1, ..., H_p]$ be a finite family of non-empty open subsets in the metric space (D, χ). The class of the closed subsets of $ClS(\mathcal{D})$ of the form: $[T, \Im] := \{J \in ClS(\mathcal{D})$ for which $J \cap T = \emptyset$ and $J \cap H \neq \emptyset, H \in \Im\}$ where T is a compact subset of the metric space $(\mathcal{D}, \chi)$. This class of subsets constitute a base for a topology of closed convergence on $(ClS(\mathcal{D})$. If the metric space happen to be a locally compact then the topology of closed convergence is separable.

Theorem 4.10. *The set $(ClS(\mathcal{D})$ of all closed subsets of the metric space equipped with the topology of closed convergence is a compact metrizable space whenever the metric space $(\mathcal{D}, \chi)$ is locally compact and separable. In this case, a sequence T_n converge to the subset T if and only if $LI(T_n = T = LS(T_n)$.*

4.3 CONVEXITY

Most assumption regarding the existence of economic equilibrium center on convexity. Finding from demand theory reveal that human being prefer mixture than extremes. Consumer's preferences are said to be naturally convex. Convex sets and convex functions play important role in the study of economic equilibrium. A common sort of extremum problem that one have to deal with in economic is that of maximizing a given function (utility function) over a convex set(demand set). Here we provide some characteristics of convexity.

4.3.1 convex sets

Definition 4.11 (Affine set)**.** Any line containing a linear combination of two of its points x, y belonging to $\mathbb{R}^{\ltimes}$ is called an affine set.

Examples of manifolds are the empty set and the space $\mathbb{R}^{\ltimes}$.

Theorem 4.12. *A subspace of $\mathbb{R}^{\ltimes}$ is any affine set that contain the origin. Every non-empty affine set is parallel to a unique subspace. Its dimension is determined by the dimension of the subspace parallel to it.*

Definition 4.13. An $(n-1)$ dimensional affine set in $\mathbb{R}^{\ltimes}$ is called an hyperplane. The empty set has dimension -1 by convention. An hyperplane can be characterize as being the set of elements y of $\mathbb{R}^{\ltimes}$ for which the inner product $< y, \breve{a} >= \gamma$ with $\breve{a}$ a vector perpendicular to the hyperplane.

Theorem 4.14. *An affine subset E of $\mathbb{R}^{\ltimes}$ is contained in a unique smallest set which in fact is the intersection of the collection of affine set containing E. It is called the affine hull of E and it is denoted by affE. A subset M of $\mathbb{R}^{\ltimes}$ is said to be convex if it contains a linear combination of its elements.*

Examples of convex affine sets are the empty set and the space $\mathbb{R}^{\ltimes}$ itself. But also the open half *i.e.* the set of the elements x of $\mathbb{R}^{\ltimes}$ having their inner product with a vector $\breve{a}$ perpendicular to the hyperplane,$< x, \breve{a} >$ not exceeding a real γ.

Definition 4.15. Every subset C of $\mathbb{R}^{\ltimes}$ that contain a positive scalar multiple of its elements is called cone. It is said to be a convex cone whenever it contain a scalar multiple of its elements and also a linear combination of its elements.

4.3.2 convex functions

Let g be a function from a subset M of $\mathbb{R}^{\ltimes}$ to $\mathbb{R}$ or $\pm\infty$. The epigraph of the function g is the set of pair (y, γ) for which its value $f(y)$ does not exceed γ for $y \in M$. It is denoted by *epig*. The convexity of the function g on the subset M depend on the convexity of its epigraph. The convex hull of family of the function $\{g_j\}_j \in J$ on $\mathbb{R}^{\ltimes}$ is denoted by $conv\{g_j\}_j \in J$. It is obtained by considering, on $\mathbb{R}^{\ltimes}$,the greatest convex function characterized by $g(y) \leq g_j(y)$ for all elements y of $\mathbb{R}^{\ltimes}$ and for all indices j.

Definition 4.16. Given a convex subset S of $\mathbb{R}^{\ltimes}$. If one consider the points z of $affS$ which are located in a distance $\chi(z, y) \leq \eta$ to the point y of S, for a real positive η. Then one obtain a relative interior of the convex set S. We have the following inclusion $riS \subset S$.

Convexity induce nice properties such as lower and upper semi-continuity.

Definition 4.17. Let (x_n) be a sequence of points in the subset B of $\mathbb{R}^{\ltimes}$ converging to a point x of B. Consider a real valued function g defined on the subset B of $\mathbb{R}^{\ltimes}$ whose limit exist in $\bar{\mathbb{R}}$. Then the function g is said to be lower semi-continuous at a point x of B if for every converging sequence (x_i) to a point x of B, we have that $g(x) \leq \lim_{\to} \infty g(x_i)$

The function g is said to be upper semi-continuous if $g(y) = \limsup_y \to \infty g(y)$

Theorem 4.18 (separation). *Two non-empty sets P_1 and P_2 in $\mathbb{R}^{\ltimes}$ are said to be separated by an hyperplane if there exist a vector $\aleph$ perpendicular to the hyperplane in a way that:*

1. *the infimum of the set of elements z belonging to P_1 for which the inner product $< z, \breve{a} >$ is greater or equal to the supremum of the set of elements $< y, \breve{a} >$ with y belonging to P_2.*

2. *the supremum of the set of elements $< z, \breve{a} >$ for which z belong to P_1 is strictly greater than the infimum of the set of elements $< z, \breve{a} >$ with z belonging to P_2.*

4.4 Measure Theory and Integration

Definition 4.19. A family $\aleph$ of the subsets of a set A is said to be a σ - algebra on the set A if that family contains both the empty set and the whole set, and if in addition it is stable under a countable union and intersection of its elements. It is also assume to be closed under the operation of complementarity.

Definition 4.20. A measurable space is a pair $(A, \aleph)$ of a set A and a σ-algebra $\aleph$ of subsets of A. The Borel σ-algebra on the metric space is the σ-algebra on A generated by the open subsets of A. It is denoted by $\beta(A)$.

Proposition 4.21 (Inverse Image of σ - algebra). *Given two measurable spaces $(\Upsilon_1, \aleph_1)$ and $(\Upsilon_2, \aleph_2)$ and an application $\hbar$ of Υ_1 into $(\Upsilon_2$. Then:*

1. *The family $\{\hbar^{-}1(B_2)$ with $B_2 \in \aleph_2\}$ is a σ -algebra. It is called inverse - image of $\aleph_2$ and is denoted by $\hbar^{-}1(\aleph_2)$.*

2. *For every σ - algebra $\aleph_2$ of Υ_2, $\hbar^{-}1(\sigma(B_2)) = \sigma(\hbar^{-}1(B_2))$.*

Measurable Mapping

Given $(\Upsilon_1, \aleph_1)$ and $(\Upsilon_2, \aleph_2)$ two measurable spaces. A function $\hbar$ from $(\Upsilon_1, \aleph_1)$ to $(\Upsilon_2, \aleph_2)$ is called a measure function if the inverse - image of the $\aleph_2$ by $\hbar$ is contained in the σ-algebra $\aleph_1$ *i.e.* $\hbar^{-}1(\aleph_2) \subset \aleph_1$, $\forall B_2 \in \aleph_2, \hbar^{-}1(B_2) \in \aleph_1$. For a Borel σ - algebra, we shall speak about Borel functions.

Measure Space

Definition 4.22. Given a measurable space $(\Upsilon, \aleph)$, an application η of the σ-algebra $\aleph$ to $\bar{\mathbb{R}_+}$ behaving in a way that:

1. the measure of the empty set is null.

2. the application η have the σ - additivity property *i.e.* for a sequence $(A_n)_n \in N$ of a pairwise disjoints elements of $\aleph$, one has $\eta(\bigcup_n \in NA_n) = \sum_{n\in N} \eta(A_n)$

The defined application η is called a measure on the measurable space $(\Upsilon, \aleph)$. A measurable space $(\Upsilon, \aleph)$ equipped with a measure η, become a measure space $(\Upsilon, \aleph, \eta)$

A subset B of the space A is said to be a null subset if it is contained in a subset D whose measure $\eta(D)$ is null. The Lesbesgue extension of a measure η is the family η^* of the sets of form $D \bigcup Q$ where D is an element of the σ - algebra $\aleph$ and Q a null subset of $\aleph$. There exist a unique measure η on $(\mathbb{R}, \beta(\mathbb{R}))$ having the following characteristics:

1. $\eta([0, 1]) = 1$

2. For every element B of $\beta(\mathbb{R})$ and for $a \in \mathbb{R}$, the function defined by $\eta : [a + x|x \in B] = \eta(B)$ is the Lesbesgue measure.

If an assertion concerning the elements of a given subset is true except for the elements of a null set then we say that the assertion is true almost everywhere.

Integral of Step functions

Let for $j \in 1, .., N$, α_j be a positive real and $(B_j)_j \in 1, .., N$ a measurable finite partition of the set Ω. Then $\Upsilon = \sum_{j=1}^{N} \alpha_j 1_B^j$ is a positive step function. The integral of Υ with respect to the measure η is defined as the number $\int \Upsilon \, d\eta = \sum_{j=1}^{N} \alpha_j \eta(B_j)$.

Integral of positive measurable functions

Definition 4.23. Let f be a positive real valued measurable function. The integral of f with respect to the measure η is the number $\int_\Omega f \, d\eta = \sup\{\int \Upsilon \, d\eta\}$ with Υ a positive step function and $\Upsilon \leq f$.

Theorem 4.24 (Beppo - Levi:Monotone convergence)**.** *Let* (g_n) *be an increasing(decreasing) sequence of measurable functions whose* $\lim g_n(x)$ *exist and is finite for each* $x \in \Omega$*. Then* $\lim_n g_n$ *is a measurable function and* $\lim_n \int g_n \, d\eta = \int \lim_n g_n$

Functions η - Integrable

Definition 4.25. A measurable function $\hbar$ is said to be η - integrable if $\int_{\Omega} |\hbar| \, d\eta$ is less than infinity. We denote by $L^1_{\eta}\,(\Omega, \eta, \mathbb{R})$ the set of η - integrable functions with values in $\mathbb{R}$.

Theorem 4.26 (Dominated convergence)**.** *Let (g_n) be a sequence of real valued measurable functions. Then if the sequence $(\hbar_n)$ converge η - almost every where to a measurable function $\hbar$ and if further there exist a measurable function g such that for every natural $n \in N$, one have $|\hbar_n| \leq g$ η - almost everywhere. Then $\hbar$ is η - integrable and $\int_{\Omega} \hbar \, d\eta = \lim_n \int_{\Omega} \hbar_n \, d\eta$.*

Distributions

Definition 4.27. Consider a measurable function $\hbar$ from a measure space $(\Omega, \aleph, \eta)$ to a metric space (D, χ) and for a subset A of the Borelian $\beta(D)$, let ν be a real positive measure defined on $\beta(D)$ by $\nu(A) := \delta\{w \in \Omega$ for which $\hbar(w) \in A\}$. The function $\delta \circ \hbar^{-}1$ is the distribution of the function $\hbar$.

Weak Convergence of Measure

The sequence (η_n) of measures defined on a metric space D is said to be weakly convergent to a measure η on the metric space D if for every real valued bounded and continuous function $\hbar$ we have that $\int \hbar \, d\eta_n$ converge to $\int \hbar \, d\eta$.

Convergence in Distribution

Assume that for every natural $n, \hbar_n$ be a measurable function from a measure space $\Omega_n, \aleph_n, \eta_n)$ into a metric space (D, χ). Then the sequence $(\hbar_n)$ will be said to converge in distribution to a measurable function in the metric space (D, χ) if the sequence (η_n) of distributions of $(\hbar_n)$ converges weakly to the distribution η of the function $\hbar$.

4.5 Correspondences

Generally, in science, one use existing association between facts and patterns to describe the nature or a system. Likewise in economic one need to analyse the existing association between economic actors and their demands. This kind of association is called in mathematic language: a relation or in particular a function.

Definition 4.28. Given two sets A and B, a correspondence from the set A to the set B is a mapping Ψ associating every element of the set A to an element of the family of non-empty subsets of the set $B i.e. 2^B$ or $\wp(B)$. Examples of correspondences are demand set correspondences, consumption set relation and budget set correspondence. The graph of a correspondence Ψ of H into T is the set $Gr(\Psi)$ of elements (x, y) of the cartesian product $H \times T$ for which y belongs to the subset $\Psi(x)$.

Definition 4.29 (Upper Hemi-Continuity)**.** A correspondence of a metric space H into the metric space T is said to be upper hemi-continuous at the point b of the set H if $\Psi(b)$ is a non empty subset of T and for every neighborhood v of $\Psi(b)$, there might exist a neighborhood N of the point b such that every element x of that neighborhood N have its image $\Psi(x)$ contained in the neighborhood v. In case the correspondence Ψ is Hemi-continuous at every point of the set H then we shall say that the correspondence Ψ is upper hemi-continuous.

Definition 4.30. A correspondence Ψ of a metric space H into the metric space T is said to be closed at a point b of the set H if every sequence (b_n, y_n) considered in the the product metric space $H \times T$ behave in a way that it converge to an element (b, y) of that product metric space and if for a natural n, y_n belongs to the set $\Psi(b_n)(n = 1, ...$ then the element y belong to the set $\Psi(b)$.

Remark 4.31. The closedness of a correspondence at a given point does not necessarily lead to an upper hemi-continuity of the correspondence. But in the event of a compact range of the correspondence , the closedness of the correspondence shall lead to the upper hemi-continuity of the correspondence at the given point.

Definition 4.32 (Lower Hemi-continuous Relations)**.** Let Ψ be a correspondence of a metric space H into the metric space T. The correspondence Ψ is said to be lower hemi-continuous at a point b if the set $\Psi(b)$ is non empty and for every open set F of the metric space T having non empty intersection with the set $\Psi(b)$, there might be a neighborhood $\aleph$ of the point b such that for every element y of $\aleph$, the intersection of the set $\Psi(b)$ with that neighborhood $\aleph$ is non empty.

Definition 4.33. A relation Υ of a metric space H into the metric space T is said to be continuous at a given point, if it is both upper hemi-continuity and lower hemi-continuous at that point. It will be said to be continuous if it is continuous at every point of the domain in which the relation is defined.

Of concern in applied economic is a correspondence between a measure space (space of economic actors) and a general Banach space (commodity space). Two specifications are commode in economic literature: a collective specification of K.Vind based on Liapunov's theorem and the individual point of view of R.J AUMANN. The mathematical tool of measure theory has helped to provide a proper mathematical formulation of the concept of competitive economy *i.e.* an economic system where the outcome cannot be influenced by individual but possibly by a group of organized economic agents (coalitions).

L.S. Shapley (1961) considered a game with a measure space of players. R.J AUMANN (1964) used a positive finite atomless measure space to show the identity of a competitive allocations and the core allocations. In the measure theoretic frame the domain of the correspondence, the set of economic actors is modeled as a measure space: $(\xi, \digamma, \zeta)$ with ξ representing the set of economic actors and $\digamma$ a σ - field of a coalition of economic actors, and ζ a positive counting measure defined on the σ - field $\digamma$. It provide with the percentage of the totality of economic actors having a given characteristic. This lead to the following correspondence: $\Psi : (\xi, \digamma, \zeta)$ into the real Banach space : $\mathbb{R}^{\lessdot}$ the graph of the above correspondence Ψ is the set of elements (ω, y) for which y belongs to $\Psi(w)$. The correspondence Ψ is said to have a measurable graph (or is a Bore - measurable) if its graph belong to the product algebra $\digamma \bigotimes \beta(\mathbb{R}^{\lessdot})$.

In the context of continuum the sum $\sum_a \in \digamma \Psi(a)$ is in fact $\int_\digamma \Psi \, d\zeta$ of the correspondence Ψ. The function $\hbar$ from $(\xi, \digamma, \zeta)$ to $\mathbb{R}^{\lessdot}$ is called a selection of the correspondence Ψ if $\hbar(\omega)$ belong to the set $\Psi(\omega)$ almost everywhere in ξ. If in addition the mapping $\hbar$ is measurable, that is if for any subset$B$ of the Borelian $\beta(\mathbb{R}^{\lessdot})$, the inverse image $\hbar^-1(B)$ belong to $\digamma$, then the function $\hbar$ will be said to be a measurable selection.

4.5.1 The Integral of a Correspondence

We denote by $\mathcal{L}_\Psi$ the set of ζ - integrable function $\hbar : (\xi, \digamma, \zeta)$ to $\mathbb{R}^{\lessdot}$ behaving in a way that $\hbar(\omega)$ belong to the set $\Psi(\omega)$ almost everywhere in ξ. The function in $\mathcal{L}_\Psi$ are called integral selection of the correspondence Ψ.

Theorem 4.34. *A correspondence Ψ of $(\xi, \digamma, \zeta)$ into a complete separable metric space $\mathbb{R}^{\lessdot}$ with a measurable graph admit a function $\hbar$ behaving in a way that $\hbar(\omega)$ belong to the set $\Psi(\omega)$ almost everywhere in ξ.*

Richter(1963), Aumann(1965), Hildenbrand(1974), have showed that a correspondence Ψ defined on an atomless measure space into a general Banach space have a non null convex set $\int \Psi \, d\zeta$ in $\mathbb{R}^{\lessdot}$. If in addition Ψ is a closed valued correspondence and integrably bounded with measurable graph then, not only its $\int \Psi \, d\zeta$ is non null but it is also a compact set of $\mathbb{R}^{\lessdot}$. However these results, based on Liapunov's theorem on the convexity of the range of an $\mathbb{R}^{\lessdot}$ - valued atomless vector measure, are not applicable to a correspondence with infinite dimensional range. Rustichini, Yannelis(1991) and Sun(1997) observed that the non - atomicity feature of the measure space is not enough as a property for the integration of a correspondence having infinite dimensional range. Konrad Podczeck(2006)[29] suggested the notion of super - atomless measure space so as to strength the condition of the non - atomicity. This dissertation present only the main result of His work.

4.5.2 Super-atomless measure space

A finite measure space $(\xi, \digamma, \zeta)$ is a measure space with $\zeta(\xi)$ less than infinity.

Construction of the measure algebra of a measure space

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Let A be a subset of the space ξ. Denote by $\mathcal{F}_A$ the trace - algebra on $Ai.e.$ a set whose elements result from the intersection of the subset A and the subset of the σ - field $\mathcal{F}$. We represent by ζ_A the subspace measure on the subset A defined by $\zeta_A(E) = \zeta^*(E)$ for a subset E of the trace σ - algebra $\mathcal{F}_A$ and ζ^* is the outer measure induced by ζ. We define by $Z(\zeta)$ the σ - ideal of null sets in the space ξ and by the quotient boolean $\hat{\xi} \equiv \frac{\mathcal{F}}{\zeta) \bigcap \mathcal{F}}$. Let $\sim$ be the equivalence relation on the σ - field $\mathcal{F}$ defined in the following way: Two subsets A and B are said to be in relation whenever their symmetric difference $A \triangle B$ belong to $Z(\zeta)$. The operations of intersections, unions, difference symmetric, and partial ordering are also defined. we shall, for instance, say that two subsets A^* and B^* of the algebra $\hat{\xi}$, determined by the subsets A and B of the σ - field $\mathcal{F}$, are such that A^* is contained in B^* ($A^* \subset B^*$) whenever $A \setminus B$ belong to $Z(\zeta)$ and $A^* \bigcap B^* = (A \bigcap B)^*$. The functional $\hat{\zeta}$ of the algebra $\hat{\xi}$ into $[0, \infty]$ is called the measure algebra and is denoted by $\hat{\zeta}(B^*) = \zeta(B)$ for any element of the σ - field $\mathcal{F}$ determining B^*. The pair $(\hat{\xi}, \hat{\zeta})$ is called the measure algebra of the measure space $(\xi, \mathcal{F}, \zeta)$. Any subset B of the algebra $\hat{\xi}$ that is closed with the operations of algebra (union,intersection,...) constitute a sub-algebra $\hat{B}$ of the algebra $\hat{\xi}$. A subset A of the algebra $\hat{\xi}$ is said to completely generate the algebra $\hat{\xi}$ if it happen to be the only one smallest order closed subalgebra that is included in $\hat{\xi}$ and that generate $\hat{\xi}$. The least cardinal number of any subset in the algebra $\hat{\xi}$ capable of completely generating the algebra $\hat{\xi}$ is called the Maharam type of the measure space $(\xi, \mathcal{F}, \zeta)$ or of the measure ζ. If the Maharam type of the measure of a subset E of the σ - field $\mathcal{F}$ with positive measure ζ_E is the same as the Maharam type of the measure ζ then the measure ζ is said to be of Maharam homogeneous. It follows that if a subset E of a measure space $(\xi, \mathcal{F}, \zeta)$ is an atom then $\hat{\xi}_E$ contains no proper subalgebra and thus the Maharam type of ζ_E shall be zero. A measure space $(\xi, \mathcal{F}, \zeta)$ is atomless if and only if the Maharam type of the subspace measure ζ_E of any subset E belonging to the σ-field $\mathcal{F}$ with positive measure ζ_E is infinite. This mean that the Maharam type of the subspace measure ζ_E for any subspace E of σ-field $\mathcal{F}$ is uncountable. A measure with such characteristic is called super-atomless.

Definition 4.35. If the Maharam type of the subspace measure ζ_E of any subset E of the σ-field of the measure space $(\xi, \mathcal{F}, \zeta)$ is uncountable then the measure space $(\xi, \mathcal{F}, \zeta)$ shall be said to be super-atomless.

Reinforcement of the condition of the non atomicity

Definition 4.36. Consider a correspondence Ψ of the measure space $(\xi, \mathcal{F}, \zeta)$ into the Banach space $\mathbb{R}^{\lessdot}$ and a non empty subset E of the σ-field $\mathcal{F}$. A function g of ξ into $\mathbb{R}^{\lessdot}$ is called the Bochner integral of g over E, denoted by $\int_E g \, d\zeta$. The Bochner integral, $\int_\xi \Psi \, d\zeta$, of the correspondence Ψ is the set $\{\int_\xi g\, d\zeta$ for which g is a Bochner integral (almost every where) selection of the correspondence $\Psi\}$

Theorem 4.37. *Given a finite measure space $(\xi, \digamma, \zeta)$ and an infinite dimensional Banach space $\mathbb{R}^{\lessdot}$. Then the measure ζ is said to be super-atomless if and only if the Bochner integral of the correspondence Ψ of the measure space $(\xi, \digamma, \zeta)$ into the Banach space $\mathbb{R}^{\lessdot}$, $\int_\xi \Psi\, d\zeta$ is convex.*

Theorem 4.38. *Given a finite measure space $(\xi, \digamma, \zeta)$ and an infinite dimensional Banach space $\mathbb{R}^{\lessdot}$ (which need not be separable). Then the measure ζ will be said to be super-atomless if and only if the Bochner integral of the correspondence Ψ of the measure space $(\xi, \digamma, \zeta)$ into the Banach space $\mathbb{R}^{\lessdot}$, $\int_\xi \Psi\, d\zeta$ is convex and weakly compact for every correspondence Ψ of ξ into the Banach space $\mathbb{R}^{\lessdot}$.*

4.6 Mathematical Description of Economic System

A market is essentially characterized by the goods that are exchanged. A *commodity* is a good (food, housing, and clothing) or a service (labor). The *economic system* can be described by the consumer's activity. A good is distinguished not only by its physical aspect but also by its quality, by the time - interval during which it is available and by its location: place at which it is available. A commodity is assumed to be infinitely divisible, we admit the existence of a finite number of distinguishable commodities indexed by an integer q running from 1 to l. If we identify every unit vector $1_q = (0, ..., 1, ..., 0)$ of the linear space $\mathbb{R}^l$ with one unit of the commodity $q(= 1, ..., l)$, one obtains a correspondence between the Euclidean space $\mathbb{R}^l$ and the universe of commodity. The obtained isomorphism leads us to designate the linear space $\mathbb{R}^l$ as the commodity space. However we shall designate it by $\mathbb{S}^l$ in this dissertation. Every bundle of commodities has therefore a vectorial representation in the universe of commodity $\mathbb{S}^l$ and their quantity can be expressed by a number. Let $\mathcal{C}$ be a set of consumption units (consumers), typically families or individuals but including also institutional consumers. Each consumers is characterized by a subscript i (i = 1,...,m), its activity is represented by a consumption vector X_i of the l - dimensional Euclidean commodity space $\mathbb{S}^l$.

Definition 4.39. *A consumption plan* is defined as being the amount of service and commodity that can be placed at the consumer's disposal. A consumption set of each consumer $\mathcal{C}$ in the economy is the collection of, a priori(without accounting with his buying power), possible consumption plans.

Compactness is a desirable property for a consumer's demand set. Economic models of individual behavior generally assume that the reason for a consumer's choice is to maximize his utility. As such, compactness allow one to use the classical Weierstrass

Theorem in order to infer that the choice made by a consumer effectively leads to the highest level of satisfaction.

To ensure compactness of the consumption set, one account with two facts. First the impossibility for a consumer to supply several types of labor within the 24 hours of a day. Secondly the impossibility for the consumer to combine all existing commodities. These considerations allow one to restrict the range of consumer's consumption set by considering the existence of a universal lower bound, say b, for all possible consumption plan of every economic agent. Finally, one consider the fact that every individual can survive if his consumption plan were restricted to be in every commodity less than a certain number k. Analytically, one can ignore the potential range of the consumption set. Under the above considerations, the consumption set can formally be defined as follows:

A consumption set $\mathcal{C}$ is a non empty subset of the commodity space which is closed, convex and bounded from below. The collection of all consumption sets having a non empty intersection with a given compact subset $K \leq \mathcal{C}^l$ and that are bounded from below, $b \leq \mathcal{C}^l$, for a given vector b of the space of commodity $\mathbb{S}^l$ is denoted by $\Im$. That is the quantities of commodities consumed by economic actors must enable him to survive. It turn out that the obtained consumption set is convex and compact.

4.7 Atomless Economic Approach to the Demand Analysis

Demand Analysis is concerned with individual's market behavior, the way a consumer go about choosing his consumption plan. Tastes are individual primitive, it describe individual's preferences. Consumer's choice stem on his preference. The economically relevant data for a consumer's demands are tastes and wealth (income). We assume that consumer can (subjectively) order his consumption set. He thus possess a pre-order relation, $\succeq$, which allow him to prefer the consumption plan u to the consumption plan z each time that he is given the opportunity of making a choice between the consumption plan u and z. There is thus on the consumption set $\mathcal{C}$ an order that is complete with respect to economic actor's tastes. Furthermore, our consumer is assumed to be rational *i.e*, his tastes is represented by an order relation which is complete (we expect our consumer to be able to compare all commodity bundle), and transitive (we require a certain coherence in the choice of the consumer). One can say that the way of choosing the commodity bundles by a consumer reveals a certain consistency of choice if it is described by a transitive preference relation. Lastly we require continuity so as to provide the concept of preference relation its full analytical power. In fact by assuming that commodity bundle are infinitely divisible one can infer, by way of continuity, that if $\{u_n\}$ and $\{z_n\}$ are two nets converging respectively to the commodity bundle u and z, if a consumer prefers the commodity bundle u_n to z_n for all n, then he must prefer the bundle u to z.

4.7.1 Space of Agents'characteristics

Definition 4.40 (Preferences Relations)**.** The existence of an economic actor's consumption set $\mathcal{C}$ and a relation $\succ \subset \mathcal{C} \times \mathcal{C}$ allows one to define *A preference relation*, a pair $(\mathcal{C}, \succ)$. The relation $\succ \subset \mathcal{C} \times \mathcal{C}$ is assumed to be transitive, irreflexive and open in $\mathcal{C} \times \mathcal{C}$.

Let $\mathcal{Q}$ be the collection of all preference relations such that the consumption set is an element of $\Im$. We shall designate by $\mathbb{W}$, the set of economic actor's wealth. Closedness and denseness are useful properties for large sets. Since we are concerned with the way in which the characteristics (tastes and wealth) of economic actors are distributed over a large set, our strategy shall consist in obtaining a way(possibility) for continuity, convergence and closedness. This consideration lead one to establish a correspondence between each preference relation $(\mathcal{C}, \succ)$ in the set $\mathcal{Q}$ and the set $\mathcal{T}$ of commodities pairs (u, z) which are such that the commodity bundle u is not preferred to the commodity bundle z.From the above, one observe that the defined set $\mathcal{T}$ is reflexive, negative transitive and a closed subset of $\mathcal{S}^l \times \mathcal{S}^l$. Furthermore, the collection of commodities bundle u that imply the existence of another commodity bundle z such that the pair (u, z) is an element of the set $\mathcal{T}$, belong to the set $\Im$.

Now,One can observe that the set $\mathcal{T}$, allows for the obtention of the following preference relation $\succ := \mathcal{C} \times \mathcal{C} \setminus (T)$, built by considering the portion of the consumption set $\mathcal{C}$ of commodity bundles $u \in \mathbb{S}^l$ for which the pair (u, u) of commodity bundles belong to the set $\mathcal{T}$. We represent by $\mathcal{Q} \times \mathbb{W}$ the space of economic actor's identifications since in this framework, economic actor is identified by means of his taste and his wealth (or income).

Topology of closed convergence

In demand analysis, one is interested in difference as well as in the similarity of consumer's tastes. Economically, consumers with similar tastes are assumed to have the same demand in similar wealth-price situations. Intuitively, similarity involves distance i.e., a metric. Mathematic economics use the notion of topology to clarify this intuitive concept of similar tastes.

Debreu (1970) used the Hausdorff distance to topologize the set $\mathcal{Q}^*$ of tastes. Kannai (1970) defined a topology on $\mathcal{Q}^*_{mo}$. He first represented every taste by an utility function and defined a metric on $\mathcal{Q}^*_{mo}$ in terms of these utility functions. Mertens (1970) proposed a topology similar to the topology of closed convergence introduced by Hildenbrand (1974) and which is presented here.

Earlier we described every preferences relation $(\mathcal{C}, >)$ in the set $\mathcal{Q}$ by the closed subset $\mathcal{T}$ of commodity bundle $(u, z) \in \mathcal{C} \times \mathcal{C}$ such that the commodity bundle u is not preferred to the commodity bundle z. We considered these pairs of commodity in the space $\mathbb{S}^{2l}$ of commodity bundles. As a result, one can consider the set $\mathcal{Q}$ of preference relation as a portion of $\mathcal{Q}(\mathbb{S}^{2l})$. The preceding set is made of subsets of $\mathbb{S}^{2l}$ that are closed. Hence, a natural candidate for a topology on the set of preference relation $\mathcal{Q}$ is the topology Σ_C of closed convergence.

We thus consider the coarsest topology on that set of preference relation. The coarset topology ensure the closedness of the following set, the set $(\mathcal{C}, \succ, u, z)$ of elements of $\mathcal{Q} \times \mathbb{S}^l \times \mathbb{S}^l$ for which the commodities u, z are such that the commodity u is not preferred to the commodity z.

Definition 4.41. Whenever the inferior topological limes $LI(T)_n$ is equal to the superior topological limes$LS(T)_n$ for a subset $\mathcal{T}$ of a metric space, we say that $\mathcal{T}$ is the closed limit of the sequence $\mathcal{T}_n$.

Theorem 4.42.

1. *The topology Σ_C of closed convergence guarantees the compactness and the metrizability of the set $\mathcal{Q}$ of preference relations.*
2. *The equality between the inferior topological limes $LI(T)_n$ and the superior topological limes$LS(T)_n$ imply the convergence of the sequence $(\mathcal{C}_n, \succ_n)$ of preference relations converges to $(\mathcal{C}, >)$ in the topological space $(\mathcal{Q}, \Omega_c)$.*

In a large economy, one cannot expect all consumers to be alike in terms of tastes. To account for the diversity of the characteristics of all economic actors, the set, $\mathcal{Q}$, of preference relation can be considered as an union of subsets of different tastes: monotonic, convex, strictly convex and non-satiated. We shall by now consider all these subsets of $\mathcal{Q}$.

Topological Properties of the Preferences Relations

Local Non-satiated

A *locally non-satiated* preference relation is a preference relation for which every commodity good u in the commodity set $\mathcal{C}$ and for every subset $\mathcal{V}$ in the vicinity of the commodity u, one can find another commodity good z that belong to the intersection of $\mathcal{V}$ and $\mathcal{C}$ such that z is preferred to the commodity good u.

Monotonicity

Whenever consumer consider two positive and distinct commodities good $u \neq z$ which are such that $0 \preceq u \preceq z$. If this fact lead one to infer that that $y \succ x$ then one assert that the preference relation under consideration is a monotonic relation. The collection of such preference relation is represented by $\mathcal{Q}_{mo}$.

Transitive Indifference

A *negatively transitive* preference relation is a preference relation such that for every commodity good u,z,w that are consumed by individual, whenever his behavior reveals that $u \not\succeq z$ and $z \not\succeq w$ it might follows that $x \not\succeq z$.

The collection of of all negatively transitive preference relation in $\mathcal{Q}$ is represented by $\mathcal{Q}^*$. As we shall see, this subset of $\mathcal{Q}$ will constitute the base from which some properties (convexity, indifference relation) will be built.

When individual does not have specific opinion on whether to allocate his income on the commodity good u or on the commodity good z, we say that economic actor is on the indifference curve relatively to his way of choosing. That is $u \sim z$ in which condition $u \not\succ z$ and $z \not\succ u$.

An indifference relation is defined by the consideration of a preference relation in the set of negative preference relation $\mathcal{Q}^*$. The indifference relation $\sim$ on $\mathcal{C}$ is characterized by its reflexive, transitive, and symmetric's properties .

Convexity

Naturally, human being have a taste for a mixture. When a consumer is indifferent between consuming two different commodity good u and v then for a real η strictly comprise between zero and one, one have that the mixture of the two commodity good $\eta u+(1-\eta)v$ shall be preferred than one of the two commodity good. A preference relation is termed strongly convex if the mixture of the two commodity good is strictly preferred that one of the two commodity good. We represent by $\mathcal{Q}^*_{co}$ and $\mathcal{Q}^*_{sco}$ respectively the set of all convex and strongly convex preference relations. In this framework, the convexity of preferences is only defined for preferences relations in $\mathcal{Q}^*$ since for relation in $\mathcal{Q}$ but not belonging to $\mathcal{Q}^*$, one cannot make use of the convexity property. We also consider the following subsets:

1. $\mathcal{Q}^*_{mo} := \mathcal{Q}^* \cup \mathcal{Q}_{mo}$

Lemma 4.43. *The subsets $\mathcal{Q}_m o$, $\mathcal{Q}^*$, $\mathcal{Q}^*_{co}$, and $\mathcal{Q}^*_{sco}$ are countable intersection of open sets in $\mathcal{Q}$.*

The subsets $\mathcal{Q}_{mo}$, $\mathcal{Q}^*$ and $\mathcal{Q}_{lns}$ of $\mathcal{Q}$ are not closed, and the closure of the subset $\mathcal{Q}^*$ relatively to the topology of closed convergence is not the same as that of the set of preference relation $\mathcal{Q}$. In the following, each time the preferences relation will be restricted to one of its subsets for technical reason (theoretical measure) we shall to account with the fact that these subsets are in fact Borel subsets of the compact space $\mathcal{Q}$. By then, every measure will be tight.

The defined topology Σ_C of closed convergence allow us to define the consumption set correspondence, the budget set and the demand set relation. Generally, individual behavior is described by a simple valued function (demand function). However, in economics and game theory, there are situations where behavior does not manifest itself in uniquely determined actions. Atomless economy is concerned with aggregate behavior, if individual behavior is not uniquely determined then one need instead to use another mathematical tool: multivalued function or correspondence.

Individual Demand Analysis.

Now, we use the notion of the correspondence to investigate the topological properties of the consumer consumption set, the budget set and individual demand.

Corollary 4.44. *The correspondence of the set of preference relation $\mathcal{Q}$ into the commodity set $\mathbb{S}^l$ is known as the consumption set relation for the consumer constitute his consumption set by selecting commodity good from the commodity set on the basis of his taste. This correspondence is closed and lower hemi-continuous.*

Now we extend the space of economic actor's identification $\mathcal{Q} \times \mathbb{W}$ by adding the commodity set as well as the price system, for it help capture consumer's behavior following a change in the price system. We thus have $\mathcal{Q} \times \mathbb{W} \times \mathbb{S}^l$ and we let $\Theta = \mathcal{Q} \times \mathbb{W} \times \mathbb{S}^l$. Furthermore, we consider the set of elements $(\mathcal{C}, \Im, \mho)$ (with $\mho$ the ruling price in the economy and $\Im$ the consumer'income or wealth) of Θ where the budget set is compact and the set of commodity good selected by the consumer are such that its market value is less than consumer's income $\Im$ i.e., $inf \mho.\mathcal{C} < \Im$. we denote by $\mathcal{B}$ the set of such points.

Price System

In this framework, price vector is taken as an element of the space of commodity $\mathbb{S}^l$ for it associates to every commodity h a real number $\mho_h$ its market value. If the price system $\mho$ prevails, the commodity bundle $x \in \mathbb{S}^l$ can be exchanged into the commodity bundle $x_1 \in \mathbb{S}^l$, if their values of exchange are the same. That is if $\mho.x = \mho.x_1$. The price of a commodity may be positive(scarce number), it is zero for a free commodity and negative for noxious commodity.

Budget set correspondence

Definition 4.45. Let $\mathcal{A}$ be an economic agent with consumption set $\mathcal{C}_c$ and wealth $\Im_c$. If the price $\mho$ prevails, then we denote by $\Gamma(\mathcal{C}, \Im_c, \mho)$ the amount of commodity good consumer is planning to buy under the ruling price and his current income. That is: $\Gamma(\mathcal{C}, \Im_c, \mho) = \{u \in \mathcal{C} \mid \mho.u \leq \Im\}$

For a given price space $\triangle^{G-1}$, there is a correspondence: $\triangle^{G-1} \longmapsto \mathbb{S}^l$ defined by $\Gamma_C(\mho, \Im) = \{u \in \mathcal{C} \mid \mho.u \leq \Im\}$ with $\mathcal{C}$ a subset of $\mathbb{S}^l$ which is compact. We call it the budget set correspondence and we claim that $\Gamma_C(\mho, \Im)$ is well defined.

Proposition 4.46. *The budget set relation Γ of Θ into $\mathbb{S}^l$ is closed and Γ is lower hemi-continuous at each point of $\mathcal{B}$.*

At prices $\mho$ economic agent chooses a commodity bundle that maximize his utility from the budget set.

Definition 4.47. For a consumer $\mathcal{A}$ with income $\Im_c$ and taste represented by $(\mathcal{C}, \succ)$. If the price system $\mho$ prevails, we define his *demand set* as:

$$\mathcal{D}(\mathcal{C}, \succ, \Im, \mho) := \{z \in \Gamma(\mathcal{C}, \Im, \mho)\}$$

such that there is no

$$\{u \in \Gamma(\mathcal{C}, \Im, \mho) with \quad u \succ z\}$$

Continuity properties of Individual Demand

In the description of individual behavior it is always important to know whether individual's chosen actions (consumption plan) respond continuously to changes in some parameter (prices or wealth). The discontinuity of consumer demand function shall prevent one from properly assessing consumer response following any change in price. The continuity of the choice correspondence of $\mho \in \mathbb{S}^l$ into the consumption set is therefore important for our assessment of individual behavior.

Theorem 4.48. *At each element of $\mathcal{B}$, the demand relation ψ of $\mathcal{Q} \times \mathbb{W} \times \mathbb{S}^l$ into $\mathbb{S}^l$ admit at least one element and is upper hemi-continuous, and its range is compact.*

Let us now see what can happen to the demand relation ψ at points where the budget set $\Gamma(\mathcal{C}, \Im, \mho)$ is not necessary compact or where the condition $\inf \wp.u < \Im$ may not be satisfied.

Proposition 4.49. *The set $\{((\mathcal{C}, \succ, \Im), \mho, u) \in \mathcal{Q} \times \mathbb{W} \times \mathbb{S}^l \times \mathbb{S}^l \mid u \in \psi(\mathcal{C}, \succ, \Im, \mho)\}$ is a Borel subset in $\mathcal{Q} \times \mathbb{W} \times \mathbb{S}^l$.*

4.7.2 Mean Demand and Consumption Sector

Mean Demand

In order to investigate the mean demand of an atomless economy, an uncountable infinite set of consumers, each consumer in the economy is identified with his economically relevant data (his taste and his income). As first step, we consider an economy populated by a limited number of economic actor say in the set $\mathcal{E}$, each of them having income $\Im$ and a taste $\succeq \in \mathcal{Q}$.

We further consider a correspondence ϵ of their collection $\mathcal{E}$ into their identifications $\mathcal{Q} \times \mathbb{W}$ (their income and taste).

We are particularly interested on the way these characteristics are distributed over the space $\mathcal{Q} \times \mathbb{W}$. For this purpose, we define a distribution measure ζ and shall call the space $\mathcal{Q} \times \mathbb{W}$, the space of economic actor's identifications.

We define a normalized counting measure θ on the set $\mathcal{E}$ of economic actors such that the image measure $\theta(\epsilon^{-}1(G))$, for each subset G of $\mathcal{Q} \times \mathbb{W}$, provide us with the number of economic actors whose identity are contained in the space $\mathcal{Q} \times \mathbb{W}$. The measure $\delta = \theta(\epsilon^{-}1(G))$ can be viewed as the economic actor's distribution. In this framework, a single economic actor is considered as being weightless with respect to the whole set of economic actors of the economy in a way that individual action or behavior can in anyway impact on the collective behavior. That is why the measure distribution δ is often considered in the literature as an atomless distribution. May the reader allows us to call it weightless distribution over the space $\mathcal{Q} \times \mathbb{W}$ of economic actors.

In general equilibrium analysis one is interested in the total demand of all consumers in a large economy. In this case it seems convenient to approximate the total demand by the mean demand. That is one can select a simple random sample of n consumers from infinite population of consumers and use the central limit theorem so as to infer that the sampling distribution of the sample mean follow the normal probability distribution. Empirical applications use the Pareto or the log normal distribution. Consideration of atomless distribution is motivated both from a large number (the consumption decision of a typical individual consumer on total demand have only small influence) and also from the diversification of consumer's identities as it is assumed that consumers are not all alike.

If the price system $\mho$ prevails, the demand set of consumer $\mathcal{A}$ with identity $\in \mathcal{Q} \times \mathbb{W}$ is denoted by $\psi(\mathcal{A}, \mho)$.

By the continuity of the individual demand function $\psi(\mathcal{A}, \mho)$ and the definition of the measure δ one can express the total demand as: $\Phi(\mathcal{E}, \mho) = \int_{\mathcal{E}} \psi(\mathcal{A}, \mho)\, d\theta$.

Consumption Sector

We now introduce the notion of *consumption sector* which we define as being a measurable correspondence $\mathcal{G}$ of a collection of economic actors $(\mathcal{E}, \gamma, \eta)$, a measure space into the space of their identities of demand $(\mathcal{Q}, \mathbb{W})$. We make the requirement that the mean income of economy $\int w \circ \mathcal{G}\, d\eta$ be finite. Three sorts of consumption sector can be distinguish in a weightless economy according to the type of measure space $(\mathcal{E}, \gamma, \eta)$: - a *Simple* consumption sector (for a simple measure space), a weightless (atomless) consumption sector (for a weightless measure space) and a *Convex* convex consumption sector (for a measure space where each economic actor have a convex taste.)

Topological Properties of the Mean Demand

Properties of the mean demand are crucial for equilibrium analysis. For instance when preferences are strongly convex, the mean demand of a set of consumers is unique and Walras allocation is uniquely determined by an equilibrium price. For a topologically large subset (dense and open) of economic actor's identities, the diameter of the demand set is small for a strongly convex economic actor's tastes. However as we shall see, atomless economies have the advantage of having the mean demand convex whenever individual demand are convex.

Convexity and Compactness of the Mean Demand

If the individual demand $\psi(\epsilon(A), \mho)$ is convex, the mean demand $\Phi(\mathcal{G}, \mho)$ only depends on the distribution of economic actor's identities (taste-income) $\delta = \theta \circ \mathcal{G}^{-}1$.
In this case, the mean demand can be expressed as :
$\Phi(\mathcal{G}, \mho) = \int_{\mathcal{Q} \times \mathbb{W}} \psi(\epsilon(A), \mho)\, d\delta$.

Proposition 4.50. *1. If the consumption sector is convex for every price $\mho \gg 0$ then the mean demand set $\Phi(\mathcal{G}, \mho)$ is convex. In fact, the measure space $(\mathcal{A}, \beta, \eta)$ can be expressed as a finite union of a different part of the unite i.e., an atom and a non-atom part. On the atom part, economic actor's taste is convex, and thus the demand set is convex. Hence, the mean demand is convex.*

2. If individual economic actor's budget sets are compact then the mean demand set $\Phi(\mathcal{G}, \mho)$ have at least one element and is compact.

The induced convexity on the mean demand can also be supported from Liapunov's Theorem which state on the convexity of the integral of a correspondence that is measurable on a measure space that is atomless.

Proposition 4.51. *Let $\mathcal{G}_n$ be a correspondence of $\mathcal{E}_n \to \mathcal{Q} \times \mathbb{W}$, a sequence of simple consumption sector such that:*

1. *$\#\mathcal{E}_n \longmapsto \infty$;*
2. *the sequence $(\delta_n^{\Im})$ of wealth distributions converges weakly to a distribution $\delta^{\Im}$,*
3. *and the sequence $(\int_{\Re} \xi, d\delta_n^{\Im}(\xi))$ of mean wealths converges to the mean $(\int_{\Re} \xi, d\delta^{\Im}(\xi))$.*

Then for each element of the bounded and closed set of price $\Delta = \{\mho | \mho > 0\}$ and for any real $\epsilon > 0$, there exist a linear functional $\mho \in \Delta$ and an integer K which is such that each time $n \geq K$, the convex hull of the mean demand $\Phi(\mathcal{G}, \mho)$ is contained in $\Gamma_\epsilon[\Phi(\mathcal{G}, \mho)]$.

Closeness of the Mean Demand relation.

The consumption demand in the society depend mainly on two economic parameters: income and the price vector. The consumer's demand correspondence describes how the consumer's consumption plan depend upon change in prices vector. Here we investigate the dynamic of the mean demand in relation to the price change and the distribution of income-tastes.

Theorem 4.52. *For each consumption sector $\mathcal{G} : (\mathcal{E}, \gamma, \delta) \longmapsto \mathcal{Q} \times \mathbb{W}$ and a strictly positive linear functional $\mho$, where $\inf \mho.w_C \leq \mathcal{W}_C$, the mean quasi - demand $\Phi(\mathcal{G}, \mho) = \int \psi(\widehat{G}(.), \mho)\, d\eta$ have at least one element and is compact, and its convex hull $\mathcal{CO}\Phi(\mathcal{G}, \mho)$ is closed at $(\mathcal{G}, \mho)$, i.e. for each sequence $\mathcal{G}_n$ of consumption vectors that converge to the consumption sector $\mathcal{G}$ and for each sequence $(\mho_n)$ of price vectors that converge to the linear functional $\mho$, one have $\limsup \mathcal{CO}\Phi(\mathcal{G}_n, \mho_n) \subset \mathcal{CO}\Phi(\mathcal{G}, \mho)$.*

If in addition the consumption sector $\mathcal{G}$ is convex then the mean demand is closed at $(\mathcal{G}, \mho)$.

Examples 1.Consider a consumption sector with income-tastes distribution δ and an independent sample of size n drawn from the distribution $\delta(n = 1, 2, ...)$. Then one obtain a sequence of consumption sector $\mathcal{G}_n$ that converges to the consumption sector $\mathcal{G}$ with probability one.

2.Consider the mapping of $\mathcal{Q}^*_{mo}$ into $\mathcal{Q}^*_{mo}$ defined by $\succeq \longmapsto \widehat{\succeq}$.
If $\mathcal{G} : (\mathcal{E}, \beta, \delta) \longmapsto \mathcal{Q}^*_{mo} \times \mathbb{W}$ is an atomless consumption sector then:
$\mathcal{G}_1 : (\mathcal{E}, \beta, \delta) \longmapsto \mathcal{Q}^*_{mo} \times \mathbb{W}$ defined by $\mathcal{G}_c := (\widehat{\succeq}_c, W_c)$, where $c = \mathcal{G}(a)$, is a consumption sector and for each linear functional $\mho \in \mathbb{S}^l$, $\Phi(\mathcal{G}, \mho) = \Phi(\mathcal{G}_1, \mho)$ the mean demand of the convex consumption sector.

3.Let the sequence $\mathcal{G}_n : (\mathcal{E}_n, \gamma_n, \delta_n) \longmapsto \mathcal{Q} \times \mathbb{W}$ of the consumption sectors be convergent to the consumption sector $\mathcal{G}_{:}(\mathcal{E}, \gamma, \delta) \longmapsto \mathcal{Q} \times \mathbb{W}$ such that the number of economic actors increase $\#\mathcal{E}_n \longmapsto \infty$ and $\mathrm{Supp}(\delta_n) = \mathrm{Supp}(\delta)(n = 1, ...)$. Then for every vector $\mho >> 0$ and every commodity good that belong the demand set $u \in \Phi(\mathcal{G}, \mho)$ there exists a sequence of commodity good $u_n \to u$ with u_n belong to the demand set $\Phi(\mathcal{G}_n, \mho)$.
The individual demand set $\psi(., \mho)$ for large (type) consumption sector is Lower hemi - continuous.

4.8 Economic Analysis of an Atomless Exchange Economy

In the framework of atomless economy, economic actors are represented by the element of the space of economic actors $\mathcal{Q} \times \mathbb{W}$. The focus stem on the way the economy allocate (quantitatively and qualitatively) the amount of good to the economic actors. In case the economy is made of limited (small) number of economic actors, one can make use of an application from the set of economic actors to their identities. However, when the set of economic actors is large *i.e.* (more than thousand), it is convenient to replace the individualistic description of economic actor by a more collectivistic view.
For the time being, the set $\mathcal{E}$ is taken to be infinite. This result by the total potential becoming infinitely large. Therefore we shall introduce the concept of mean potential.
In 1966, Aumann [6], proposed an economic model with a continuum of economic actors. He first considered economic actors as a continuum [0,1]. Later on Hildenbrand [18] generalized Aumann's model to a measurable space. This way of modelings the set of economic actors gives a rigorous mathematical formalization of economic concepts according to which each actor has a negligible influence on global economic activities. Aumann showed in, [6], that such an economy with complete and transitive taste can reach an equilibrium. Hildenbrand [18] generalized Aumann's result to an economy with production and complete transitive taste. The notion of a continuum of economic actors

dates back to Allen and Bewley (1939) and Houthakker (1955). Continuum concept has been introduced in game theory as a continuum of players by Shapley, Milnor, Shapiro, David and,Peleg. An economic barter amounts to the way of dividing up the total potential of the economy among economics actors. The focus of the present analysis stem on the qualitative outcome of the economic actors *i.e.* the one leading to an equilibrium state of an atomless economic barter. Two equilibrium concepts are considered here: the non-cooperative equilibrium and the cooperative equilibrium (equilibrium of Edgeworth). Our concern in this chapter is with the equilibrium's analysis of an atomless economic barter (where the influence of economic participant can be neglected).

Definition 4.53 (Economic Barter)**.** An atomless economic barter ε is a correspondence between a collection of economic actors that we assume as being measurable $(\mathcal{E}, \gamma, \delta)$ into the universe of their taste and their potential (their initial quantities of good in their disposal) $\mathcal{Q} \times \mathbb{S}^l$.

With the requirement that the average potential of the economy (the mean potential) be finite and strictly positive just because, an economic barter does not allow for production and thus all economic actors are pure consumers. As such, the above assumption is nothing but a minimum (necessary) condition for the existence of such an economy.

It follow from the above definition that the choice of each economic actor is dictated by his taste and his potential. The economy assign a portion of the total potential to each of the economic actor. That assignment, allowance, is assumed to be attainable *i.e.*, the sum of all the allowance must be equal to the total potential of the economy. That assignment of the set of economic actors $\mathcal{E}$ into the commodity set $\mathbb{S}^l$ has the following requirement: - the consumption vector, $C(e)$, which is this case is the allowance, the grant receive by each economic actors through that redistribution of the economic barter. For an atomless economy, such an assignment is an integrable mapping (function). It may be simple (when the set of the collection of economic actors is simple), as it may be atomless (when the set of economic actors is weightless), and convex when the measurable collection of economic actors is constitute by participants with convex taste.

The way in which the redistribution of the total potential is performed can be measured. We denote by θ such a measure on $\mathcal{Q} \times \mathbb{W}$.

4.8.1 Assessment of the Equilibrium of an Atomless Exchange Economy

The Core

There are two principal way of helping an organization or an economy to achieve an aspiration or an equilibrium. A cooperative way is often made from a regroupment of its members, usually named coalitions. Such a regroupment can change for the better upon apportion of the total potential. When a regroupment is no more in need of any apportion to change each member for the better, we say that such a regroupment has found what is needed. The set of such allowances that can bring the regroupment to this level of aspiration is called a *core* of an economy ε and is one of the focus of this section.

Walras allocation

A state of the economy is said to be an *equilibria* if every commodity has a price $\mho$ that is taken as given by each economic actors. Then agentℓ with identity $(\ell_C, \succ_C)$ and potential e_c considers only commodity vectors in his budget set $\{u \in \mathcal{C}_c \mid \mho.u \leq \mho.e\}$ and choose the one that is likely to maximize his utility.

When an economy ε produce such a commodity (allowance) and associate price vector $\mho \in \mathbb{S}^l$ *i.e.*, a vector price capable of enabling the economy to provide economic actors with such an allowance, we say that the state of the economy is an equilibria and the pair of that allowance and that price vector is called a *Walras Equilibrium* for the economy ε. We denote by $(a, \mho)$ that Walras equilibrium and by $\mathcal{L}(\varepsilon)$ the set of all Walras allowance produced by an economy ε.

The following proposition shows that in an atomless economy, a competitive allowance belongs to the core allowance.

Proposition 4.54. *For a weightless economy, the set of Walras allowance $\mathcal{L}(\varepsilon)$ belongs to the set of core allowance $\mathcal{EG}(\varepsilon)$ for every economic barter ε.*

Proof. Let us first consider a Walras allowance a and assume that it does not belong to the core of allowance of the economy $\mathcal{EG}(\varepsilon)$. This means that there exist a regroupment of economic actors $\mathcal{S} \in \gamma$ with positive measure, which is not satisfied with the allowance a and for which there exist an allowance b such that $b(\ell) > a(\ell)$ almost every where and $\int b\, d\nu = \int \varepsilon\, d\nu$. Therefore, the fact that $a(e)$ is a Walras allowance and $b(\ell) > a(\ell)$ almost every where in the regroupment $\mathcal{S}$ implies that $\mho.pot(\ell) < \mho.b(\ell)$ (where pot designate economic actor potential) almost every where with $\mho$ denoting an equilibrium price vector associated with the allowance a. So $\mho \int_S \varepsilon\, d\nu < \int_S b\, d\nu$ which is a contradiction. □

A mechanism of a system economic is say to be an equilibrium level if it has engendered a price level capable of equating the entire production of good and services to the need of the citizen of the society. If an economy is such that the influence of each single economic actor cannot account on the collective dynamic then we can expect these economic actors to adapt themselves to the prevailing price systems. This is the characteristic of an atomless economy. In such an economy the core of an allowance might coincide with the set of all Walras allowances. Francis Edgeworth observed the relationship between the core and Walras allowances. The following theorem guarantees that the locus of Walrasian allowance coincides with the locus of core allowance.

Theorem 4.55. *For an atomless economy ε whose potential mean is strictly positive and economic actor's tastes monotone, we have that:* $\mathcal{L}(\varepsilon) \equiv \mathcal{EG}(\varepsilon)$.

Unlike monopoly and oligopoly, pure competition is an ideal market structure where the weight of a single market participants can be neglected and thus allow for price - adaptation conditions.
As said earlier, atomless economic has the pretention of providing a mathematical model of pure competition. Here the concept of pure competition is formulate within the framework of economic barter.
In fact, one can base the idea on the dynamic of large numbers by considering a growing market with only one economic actor at the start. The first participant in the market is then joined by the second economic actor, then by the third, by the forth and so on. Now emphasizing only on the dynamic of the propagation of the noise in the market rather than in the dynamic of the price's evolution. One easily realize that at the start the voice of different economic actors are easily distinguishable. However, as the size of economic actors increases in the market, the voice of different economic actors are harmonized to result to one voice.
Hence, as first step in our formulation, we need to consider a sequence with increasing numbers of economic actors tending to infinity *i.e.*, $\#A_n \to \infty$.
Secondly, we need these millions of economic actors to reach one normalized voice. That is the potential-income distribution of the economy should be approximately the same. Formally, we want the sequence μ_n of potential-income distribution to be weakly convergent.
Finally, we need to make sure of the non-existence of some economic actors with no singular market position (monopoly or oligopoly). Dominant position in atomless economy is expressed from potential point of view. Formally we have the following conditions: if $E_n \subset A_n (n = 1, ...)$ and $(\#E_n/\#A_n) \longmapsto 0$ then $1/\#A \sum_{E_n} e \circ E_n \longmapsto 0$. In other words, we require that for every commodity h, the sequence $(1/\#A \sum_{\#A_n} e^h \circ E_n)$ of potential mean converges and that the limit is different from zero.

Definition 4.56 (Purely Competitive Sequences of Economies)**.** Let $\mathcal{Q}$ be any subset of $\mathcal{Q} \times \mathbb{S}^l$. A sequence ε_n of simple economies with identities in $\mathcal{Q}$ is termed purely competitive on $\mathcal{Q}$ if:

1. the number, $\#\mathcal{A}_n$, of economic actors in the economy ε_n become large;
2. the succession (δ_n) of taste - potential distributions converges weakly on $\mathcal{Q}$ to (δ);
3. $\lim \int e \, d\delta_n = \int e \, d\delta$;
4. $\int e \, d\delta \gg 0$.

The above definition provides a link between a purely competitive sequence and an atomless economy. Most of the results for atomless economies can be obtained via limits of purely competitive sequences.

Links between a purely competitive sequence with an atomless economy.

Definition 4.57 (Continuous Representation of a Purely Competitive Sequence)**.** Let ε_n be a succession of purely competitive simple economy $\mathcal{E}_n \longmapsto \mathcal{Q} \times \mathbb{S}^l$. A continuous representation of the sequence ε_n is defined by an atomless economy: $\varepsilon : (\mathcal{E}, \beta, \nu) \longmapsto \mathcal{Q} \times \mathbb{S}^l$ and the sequence $\mathcal{E}_n$ of measurable mappings $\mathcal{E}_n : \mathcal{E} \longmapsto \mathcal{E}_n$ which have the following properties.

1. $\nu(\mathcal{E}_n^{-1}(B)) = (\#B/\#\mathcal{A}_n)$ for every $B \subset \mathcal{E}_n(n = 1, ...)$ and
2. $\widehat{\varepsilon_n} := \varepsilon_n \circ \mathcal{A}_n \longmapsto \varepsilon$ almost every where in $\mathcal{E}$.

Thus, the atomless economy $\widehat{\varepsilon_n}$ and the simple economy ε_n have the same distribution δ_n of economic actor's identities and the sequence $(\varepsilon \circ \widehat{\varepsilon_n})$ of potential. A continuous representation of a purely competitive sequence is a powerful analytical tool in proving asymptotic properties of $\mathcal{EG}(\varepsilon_n)$ or $\mathcal{L}(\varepsilon_n)$.

Dreze - Gepts - Gabszewicz,(1969) considered the following simple economy $\varepsilon : \mathcal{E} \longmapsto \mathcal{Q}^*_{mo} \times \mathbb{S}^l_+$ with $\int e\, d\nu \gg 0$ and showed that $a \in \mathcal{L}(\varepsilon)$ if and only if $a_n \in \mathcal{EG}(\varepsilon_n)(n = 1, ...)$ where a_n and $\mathcal{C}(\varepsilon_n)$ denote the n - fold replication of the allowance a and the economy ε respectively.

In the end of this chapter, we provide the Debreu - Scarf versus of an atomless economy by considering the following simple economy $\varepsilon : \mathcal{E} \longmapsto \mathcal{Q}^*_{mo} \times \mathbb{S}^l_+$ and its n - fold replica ε_n and an allocation $a \in \mathcal{EG}(\varepsilon_n)$. Then for economic actors C and C' with the same identities we have $a(C) = a(C')$. That is $\varepsilon_n(C) = \varepsilon_n(C')$

4.8.2 Incidence of Consumption Sector's size on Equilibrium Price

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Using the classical assumption of convex tastes, Arrow - Debreu and Mckenzie showed that a Walrasian economy can reach equilibrium. For an economy whose economic actor's taste are convex, the mean demand is convex and therefore one can apply the standard fixed point argument theorem. However, when an economic actor's tastes are not convex, the standard fixed point is no longer applicable. For economies with many participants, one need to distinguish the situations where economic actor's tastes are strongly convex and the case of non-convex tastes. When the tastes of the economic actors under consideration are strongly convex, we show the existence of an equilibrium prices and a Walras allowance is uniquely determined by an equilibrium price vector.

For an atomless economy whose economic actor's tastes are not convex, there is no reason to expect the mean demand of the economy to be convex.However, the large size of the economy impact on the convexity of the mean demand and thus allow the existence of an approximate equilibrium prices with the degree of approximation depending on the size of the economy.
To investigate the equilibrium existence of an atomless economy, we first define the mean excess-demand correspondence. The proprieties of the mean excess-demand are used to state the fundamental mathematical result for economic equilibrium analysis.

Equilibrium Prices Correspondence

Definition 4.58 (Mean excess-demand correspondence)**.** We consider a weightless economy ε of a measure space $(\mathcal{E}, \gamma, \theta)$ of economic actors into the space of economic identities $\mathcal{Q} \times \mathbb{S}^l_+$. For a price vector $\mho$, the correspondence $\mho \longmapsto \mathcal{H}(\mho) := \Phi(\varepsilon, \mho) - \int \varepsilon \, d\theta$ is called the mean excess-demand of the economy.
A price vector $\mho$ likely to bring in balance the mean demand of the economy and its supply, is called the price of equilibrium. In others words if 0 belong to the set $\mathcal{H}(\mho)$ then $\mho$ is the price of equilibrium.

The following proposition present the properties of the mean excess - demand:

Proposition 4.59.

1. *An economy ε admit a price vector $\mho^*$ as a price of equilibrium if zero belong to $\mathcal{H}(\mho^*)$*

2. *The correspondence $\mathcal{H}$ is such that for a real η greater than zero and a strictly positive price $\mho$ one has $\mathcal{H}(\mho) = \mathcal{H}(\eta.\mho)$.*

3. *the correspondence $\mathcal{H}(\mho)$ satisfy Walras laws.*

4. *The correspondence $\mathcal{Z}$ is compact - valued, bounded from below and upper hemi-continuous.*

5. *the convergence of a succession of a strictly positive vectors price $\mho_n$ to a vector price $\mho$ allows the existence of the set:*
   $\inf\{\sum_{s=1}^{l} h^s$ *such that* h *belong to* $\mathcal{H}(\mho_n)\} > 0$ *for* n *tending to infinity.*

The following lemma provide an existence theorem for equilibrium prices in terms of the correspondence $\mathcal{H}(\mho)$.

Lemma 4.60. *If $\mathcal{H}$ is a correspondence of Δ into the space of commodity $\mathbb{S}^l$ satisfying the above properties. Then the atomless economy $(\mathcal{E}, \gamma, \theta) \longmapsto \mathcal{Q}_{mo} \times \mathbb{S}^l_+$ admit a vector $\mho^*$ such that zero belong to the convex hull $\mathcal{CO}(Z(\mho^*))$.*

Proof. Let Δ_n be the set of normalized vector price *i.e.*, for which the sum $\sum_{s=1} \mho_s$ is one and each vector price $\mho$ is greater or equal to $1/n$ for $s = 1, ..., l$ and n greater or equal to one. Now applying the fixed point theorem to the correspondence $\mho \longmapsto \mathcal{CO}(H(\mho))$ of Δ, into the commodity set $\mathbb{S}^l$, one has that the convex hull $\mathcal{CO}(H(\mho))$ is $u.h.c$ and hence there exist a vector $\mho_n \in \Delta_n$ and $\mathcal{H}_n \in \mathbb{S}^l$ such that

1. $\mathcal{H}_n \in \mathcal{CO}(H(\mho_n))$ and
2. $\mho.\mathcal{H}_n \leq 0$ for every $\mho \in \Delta_n (n \geq l)$.

The last step of our proof consist in showing that for each n, the correspondence $\mathcal{H}_n$ is zero. For this purpose, we consider the following succession of the price vector $(\mho_n)$ in Δ and we pretend that the given sequence converge to a strictly positive vector price. Otherwise,it shall follow from the above properties that the sum of the correspondence $\mathcal{H}_n$ is not zero but greater than zero. In the other hand, the second property tell us that the same sum is less than or equal to zero. We then deduce that the correspondence H_n equal to zero for n large enough. Now, assume the existence of an integer n^1 for which the vector price $\mho_n^1$ is an interior point of Δ_n^1. It then follow from Walras law that $\mho_n.H_n = 0$ for n tending to infinity. Therefore, for large n enough, H_n is equal to zero. □

Theorem 4.61. *The preceding proposition coupled with the convexity of the mean demand $\Phi(\varepsilon, \mho^*)$ lead to the following theorem. A convex economic barter with positive potential $\varepsilon : (\mathcal{E}, \gamma, \nu) \longmapsto \mathcal{Q}_{mo} \times \mathbb{S}^l_+$ admit an equilibrium that is a equilibrium of Walras $(a^*, \mho^*)$ with positive price equilibrium $\mho^*$.*

Corollary 4.62. *A convex economic barter with positive potential $\varepsilon : (\mathcal{E}, \gamma, \nu) \longmapsto \mathcal{Q}_{mo} \times \mathbb{S}^l_+$ admit a cooperative equilibrium (a core allocation).*

Definition 4.63 (Equilibrium Prices Correspondence)**.** For each economic barter ε we let $\Omega(\varepsilon)$ represent the collection of normalized prices equilibrium. That is $\Omega(\varepsilon)$ is the collection of linear functional $\mho$ with norm one and that belong to the space of commodity $\mathbb{S}^l$ and for which the mean potential $\int \varepsilon \, d\nu$ belong to the mean demand $\Phi(\varepsilon, \mho)\}$.

In fact, we know that for a convex economy $\varepsilon : (\mathcal{E}, \gamma, \nu) \to \mathcal{Q}_{mo} \times \mathbb{S}^l_+$ with positive potentials $\int \varepsilon \, d\nu$ the set $\Omega(\varepsilon)$ is non empty and depends only on the potential-income distribution δ_ε. However we cannot yet confirm its uniqueness. The following proposition and its corollary show the compactness of the equilibrium prices correspondence and its continuity on the distribution δ_ε of economic actor's income-tastes

Proposition 4.64 (Properties of the Equilibrium Price Correspondence)**.** *The equilibrium price correspondence $\Omega(\varepsilon)$ is closed at every convex economic barter $\varepsilon : (\mathcal{E}, \gamma, \nu) \to \mathcal{Q}_{mo} \times \mathbb{S}^l_+$ with $\int \varepsilon \, d\nu \gg 0$. That is for each sequence $\varepsilon_n)$ of economic barter with $\delta_n \to \delta$ and $\int e \, d\delta_n \to \int e \, d\delta$. It follows that $LS(\Omega(\varepsilon_n)) \subset \Omega(\varepsilon)$.*

Proof. Consider $\mho_n \in \Omega(\varepsilon_n)$. From the compactness of Δ, one can assume the existence of a vector price $\mho$ such that $\mho := \lim \mho_n$ and $\mho \in \Omega(\varepsilon)$. Further the monotonicity of tastes ensures the positivity of each of the prices $\mho_n \gg 0$. As by assumption, $\int e\, d\delta_n \in \Omega(\varepsilon_n, \mho_n)$ and $\int e\, d\nu_n \to \int e\, d\nu$, it follows that $\mho \gg 0$ and $\int e\, d\delta \in \mathcal{CO}(\widehat{\Phi(\varepsilon, \mho)})$. Hence, $\mho \gg 0$ and the fact that the consumption set $\mathcal{X}(C)$ is in this case the positive cone $\mathbb{R}^l_+$, implies $\widehat{\Phi(\varepsilon, \mho)} = \Phi(\varepsilon, \mho)$. Also from the convexity of the economy one infer that $\mathcal{CO}(\widehat{\Phi(\varepsilon, \mho)}) = \Phi(\varepsilon, \mho)$, and consequently $\int \varepsilon\, d\delta \in \Phi(\varepsilon, \mho)$. This means that $\mho \in \Pi(\varepsilon)$. □

The above result is obtained under the assumption of strongly convex economic actor's tastes and it provides only a weakly continuity property. However, the weak continuity property may not be of help if one need to compare equilibrium prices for different values of the tastes - potential distributions. In this case (stability analysis of equilibrium prices) one needs a continuity property that allows a dependence on different potential-tastes δ_n, *i.e.* a strong continuity property. This is provided by the following corollary where the space $\mathcal{Q}$ of economic actors is restricted to the set of convex and monotonic tastes $\mathcal{Q}^*mo_{sco}$ for theoretic measure reason while the space of commodity is restricted to a bounded subset of $\mathbb{S}^l_+$.

Corollary 4.65. *Let Υ denote a set of potential-tastes distribution δ on $\mathcal{Q}^*mo_{sco} \times T$ with $\int e\, d\nu >> 0$ where Q denotes a bounded subset in $\mathbb{S}^l_+$. Then for every $\epsilon 0$ there exists an open and dense subset $G \subset \Upsilon$ (with respect to the weak convergence) such that the correspondence of price equilibrium $\delta \to \Omega(\delta)$ is ϵ -continuous at every distribution δ in G. That is for every $\delta \in G$ there is a neighborhood V_δ such that: the supremum over the set $\{\delta' \in V_\delta\}$ of the distance $\zeta(\Omega(\delta), \Omega(\delta'))$ is less than ϵ.*

Walras equilibrium is about a price vector and an allowance effected by that equilibrium prices. The first step in the analysis of Walras equilibrium consist in the determination of the equilibrium prices then one assesses the allowance engendered by this equilibrium price. We now turn to the investigation of the Walras allowance correspondence.

Corollary 4.66. *Each succession of Walras allowances (a_n), admit a subsequence $(a_q), q = 1, ...,$ converging in distribution to an allowance a that belong to $\mathcal{L}(\varepsilon)$.*

Proof. Let (a_n, p_n) be an equilibrium of Walras for the economy ε_n. We know from the properties of the equilibrium prices correspondence that there exist a subsequence $(p_s)(s = 1, ...)$ such that $\underset{s}{lim} p_s =: p \gg 0$ and $p \in \Pi(\varepsilon)$. Since in the case of strongly convex tastes a Walras allowance is uniquely determined by an equilibrium price vector, one can thus say that the price vector $p \in \Pi(\varepsilon)$ determines an allowance a which is assumed to be in the mean-demand sets, *i.e.*, a commodity vector $c(a) \in \Phi(\varepsilon(a), \mho)$ with economic agents belonging in the set $\mathcal{A}$. We have now to show that the sequence (a_n) converges in distribution to a. To do this, we need first to consider the case where tastes

in the economies ε_n are strongly convex. We know that in this case the allowance is in the mean-demand sets *i.e.*, $a_n(c) \in \Phi(\varepsilon_n(a), \mho)$ with $a \in \mathcal{E}$. Therefore, showing that the sequence $(a_n(c))$ converges, amounts to show that the measure $\nu \circ a_n^{-1}$ converges or the function $\Phi(., \mho)$ converges with respect to the measure δ_n *i.e.*, the measure $\delta_n \circ \Phi^{-1}(., \delta_n)$. That is, we have to show that the sequence of measures $\delta_n \circ \Phi^{-1}(., \mho_n)$ on $\mathbb{S}^l$ weakly converges to the measure $\delta \circ \Phi^{-1}(., \mho)$. But since by assumption the succession, δ_n converges weakly to δ and the sequence $\Phi(., \mho_n)_(n = 1, ...)$ of function converges uniformly on compact sets to a continuous function $\Phi(., \mho)$. Since the individual demand function $(\preceq, e, \mho) \longmapsto \Phi(\preceq, \mho.e, \mho)$ of $\mathcal{S}^*_{mo} \times \Delta$ into $\mathbb{S}^l$ is known to be continuous, it is therefore uniformly continuous on every compact set.
For the second case of non strongly convex preferences, one has to use Skorokhod's Theorem so as to obtain a representation of the economies ε_n as well as all Walras allowances a_n with respect to a common measure and then we show that the sequence of Walras allowances converges almost everywhere. □
Once we depart from the assumption of strongly convex economic actor's tastes , an equilibrium price $\mho \in \Omega(\varepsilon)$ does no longer determine uniquely the equilibrium allowances. In this case, one may have two economies ε_1 and ε_2 having the same distribution δ of economic actors identities but with different distributions of Walras allowances. This situation naturally leads one to investigate the extent to which the distribution of economic actor's identities determines the set of distribution of Walras allowances produced by the economy ε. To do this, we represent by $\mathcal{DM}(\varepsilon)$ the collection of measures of Walras distribution allocations on the commodity space $\mathbb{S}^l$ and we observe that for a simple economy ε_1 and an atomless economy ε_2, the measure $\mathcal{DM}(\varepsilon_1)$ is contained in the measure $\mathcal{DM}(\varepsilon_2)$. But of particular interest is whether the set of measures of equilibrium distributions of two large sets are equal.

Proposition 4.67. *Consider two atomless economies ε_1 and ε_2 having similar distribution $\mu^1_\varepsilon = \mu^2_\varepsilon$ on $\mathcal{Q}_{mo} \times \mathbb{S}^l_+$. Then the sets $\mathcal{DM}(\varepsilon_1)$ and $\mathcal{DM}(\varepsilon_2)$ of distribution of Walras allocations of the economies ε_1 and ε_2 respectively, possess common closure relative to the weak convergence.*

As the collection of Walras allowances distributions of two atomless economies with the same distribution of economic actor's identities possess similar closure, it is interesting to investigate the closure of a the collection of Walras allowance distributions for an atomless economy $\mathcal{DM}(\varepsilon)$.

Before that, one need to notice that the assumption of atomless economy is not enough to guarantee that the distribution of economic actor's identities is sufficiently dispersed over the space of economic actors. What is needed is an economy with many agents of every identities in the support of the measure μ.
We therefore consider a distribution μ of economic actor's identities from which we define an atomless economy ε^μ with distribution μ which we shall call the standard representation

of μ. The mapping ε^μ is the projection. That is $\varepsilon^\mu(\preceq, e, \xi) = (\preceq, e)$ for every $(\preceq, e, \xi) \in \mathcal{Q} \times \mathbb{S}^l \times [0,1]$ and an atomless measure space of economic actor's identities given by $\mathcal{Q} \times \mathbb{S}^l \times [0,1]$.

Proposition 4.68. *Let μ be any measure on $\mathcal{Q}_{mo} \times \mathbb{S}^l_+$ with positive endowment $\int e\, d\mu \gg 0$. Then the set $\mathcal{DM}(\varepsilon^\mu)$ of distributions of Walras allowances of the standard representation ε^μ is closed.*

Proof. We consider a sequence (a_n) of Walras allowances for the economy ε^μ such that for each $n, a_n \in \mathcal{L}(\epsilon^\mu)$ has distribution $\mathcal{DM}a_n$. A sequence (a_n) is thus associated with the sequence $(\mathcal{DM}a_n)$ and we assume that the sequence $(\mathcal{DM}a_n)$ converges to some measure say δ. We want to show that sequence (a_n) converges to $f \in \mathcal{M}(\epsilon^\mu)$ with distribution $\mathcal{DM}a = \delta$. To do this, we consider the natural mapping which is concerned with allowances (a_n) and allocation's distributions (ε^μ). That is the mapping ε^μ and (a_n). We now regard ξ_n as the joint distribution of the mapping (ε^μ, a_n) as well as investigate the existence of a converging subsequence of ξ_n. In fact, the sequence (μ_n) is clearly tight and the set $\Pi(\varepsilon^\mu)$ of equilibrium prices is closed and strictly positive. Since, by assumption, the mean endowment $\int e\, d\mu$ exist and is strictly positive, we infer that the sequence (a_n) of allocations is bounded by an integrable function. It follows that the sequence $\mathcal{DM}a_n$ is tight and the tightness of the sequence ξ_n follows from the tightness of $\mathcal{DM}a_n$. We therefore conclude that the sequence ξ_n of the joint distribution is weakly convergent to ξ. Now, using Skorokhod's Theorem, one can deduce the existence of a measurable space $(\mathcal{E}, \beta, \nu)$ and a measurable mapping $(\varepsilon^1_n, a^1_n)(n = 1, ...)$ and (ε^1, a^1) of $\mathcal{E}$ into $\mathcal{Q}_{mo} \times \mathbb{S}^l_+ \times \mathbb{S}^l_+$ such that $(\varepsilon^1_n, a^1_n) \to (\varepsilon^1, a^1)$ almost everywhere in $\mathcal{E}$ and the distribution of (ε^1_n, a^1_n) and (ε^1, a^1) are ξ_n and ξ, respectively.

We now show that any allowance a that belong in $\mathcal{L}(\epsilon^\mu)$ has distribution $(\mathcal{DM}f) = \delta$. As the two mappings (ε^1_n, a^1_n) and (ε^μ, a_n) have the same joint distribution ξ_n, we have that the allocation a_n belong to $\mathcal{L}(\epsilon^\mu)$ and thus $a^1_n \in \mathcal{L}(\epsilon^1_n)$. The convergence of the sequence $(\varepsilon^1_n, f^1_n) \to (\varepsilon^1, f^1)$ implies that $f^1 \in \mathcal{L}(\epsilon^1)$. Besides, we know that the marginal distributions of ξ are μ and δ respectively. We now show that the economy ε^μ admit an allowance a for which the joint distribution of (ε^μ, a) is equal to ξ. To do this, we construct a measurable mapping of the standard representation $\mathcal{Q}_{mo} \times \mathbb{S}^l_+ \times [0,1]$ into $\mathcal{Q}_{mo} \times \mathbb{S}^l_+ \times \mathbb{S}^l_+$ whose distribution is ξ. We consider a probability measure δ, which in fact is a conditional probability distribution for $(t, \beta) \to \beta$ (the probability of having the allocation β in the commodity space $\mathbb{S}^l_+$ given that we have observe the distribution measure ξ. We have that δ is a measure of probability over the space of commodity $\mathbb{S}^l_+$ and for every Borel set G on the space of commodity $\mathbb{S}^l_+$ we have the correspondence $t \to \delta_t(G)$ and $\tau(U \times G) = \int_U \delta_t(G)\, \mu dt$ for Borel sets U in $\mathcal{Q}_{mo} \times \mathbb{S}^l_+$ and G in $\mathbb{S}^l_+$. Every measure δ_t on $\mathbb{S}^l_+$ can be obtained as the image of the Lebesgue measure of the mapping $f(t, .)$ of the interval $[0,1] \to \mathbb{S}^l_+$. It follows that the measurable mapping of the standard representation $\mathcal{Q}_{mo} \times \mathbb{S}^l_+ \times [0,1]$ into $\mathcal{Q}_{mo} \times \mathbb{S}^l_+ \times \mathbb{S}^l_+$ has the form $(t, \xi) \to [t, \omega]$ where

$\omega \equiv f(t, \xi)$. In fact, one can consider the cumulative distribution function $F(t, .)$ on δ in a way that $F(t, \beta) = \delta_t[0, \beta]$ and realise that $F(t, .)$ is a right - continuous function of $[0, 1]$ into $[0, 1]$ for every t and thus measurable for every allowance. $\square$
With the two preceding proposition on hand, one can now provide the definition of a Walras equilibrium distribution in terms of distributions of economic actor's identities.
Let E_p be a subset of $\mathcal{Q} \times \mathbb{S}^l_+ \times \mathbb{S}^l_+$ for every price vector $p \in \mathbb{S}^l_+$:
$E_p := \{(t, x) \in \mathcal{S} \times \mathbb{S}^l | x \in \phi(t, p)\}$. That is for a prevailing vector price p, an economic agent with characteristic $(\preceq, e)$ react by choosing the commodity bundle x which then is retrieved in his demand set $\phi(t, p)$. Let $T \subset \mathcal{Q} \times \mathbb{S}^l$. As before ξ represent a joint distribution measure on the space $T \times \mathbb{S}^l$ with its marginal distribution μ on T and δ on $\mathbb{S}^l$.

Definition 4.69. Given a distribution δ of the types of economic actors in the universe $\mathcal{Z} \subset \mathcal{Q} \times \mathbb{S}^l$. The given distribution δ admit a Walras distribution equilibrium which is in fact a measure ι on the universe $\mathcal{Z}$ having the following properties:

1. the distribution δ on the universe $\mathcal{Z}$ is nothing but the distribution marginal of the measure ι.
2. The mean demand $\int_{\mathcal{T}} e \, d\mu$ is equal to the mean supply $\int_{\S}^{l} x \, \delta(dx)$;
3. there exists in the space of the commodity $\mathcal{S}^l$ a non zero price vector $\mho$ for which the measure of probability $\iota(\mathcal{B}_{\mho})$ is one; with $\mathcal{B}_{\mho}$ the collection of economic actor's identities (c, u) in $\mathcal{Q} \times \mathbb{S}^l$ with u a commodity good in $\phi(c, \mho)$ and c consumer's tastes.

From the above definition one deduce that to each Walras equilibrium distribution δ we can associate a Walras allowance a for the standard representation ϵ^{μ} in a way that the joint distribution of (ϵ^{μ}, f) is nothing but the measure ξ.

Theorem 4.70. *Each distribution δ of economic actor's identities in $\mathcal{Q}_{mo} \times \mathbb{S}^l_+$ with positive endowment admit a Walras equilibrium distribution $\mathcal{DM}(\delta) \neq 0$. The correspondence of δ into $\mathcal{DM}(\delta)$ behave in the way that a succession (δ_n) of economic actor's identities on $\mathcal{Q}_{mo} \times \mathbb{S}^l_+$ is weakly convergent to a distribution δ in a manner that the mean endowment $\lim \int e \, d\delta_n$ admit a positive limit $\int e \, d\delta$. This result by each succession $\xi_n \in \mathcal{DW}(\mu)$, admitting a weakly convergent subsequence whose limit belongs to $\mathcal{DM}(\delta)$.*

Proof. In fact, we know from the preceding theorem that each atomless economy ϵ with distribution δ admits a Walras allowance a and the application (ϵ, a) admit a common distribution which is but a distribution of Walras equilibrium $\delta \in \mathcal{DM}(\delta)$. Now let ξ be that joint distribution and consider $\xi_n \in \mathcal{DM}(\mu_n), (n = 1, ...),$. We first show that the correspondence $\delta \longmapsto \mathcal{DM}(.)$ is continuous. But this amount to show that the sequence ξ_n is relatively compact and that it admits a convergent subsequence in $\mathcal{DM}(\delta)$.

Besides, the above definition allows us to say that for every distribution's measures $\xi_n \in \mathcal{DM}(\mu_n), n = 1, ...,$ there exist a strictly positive equilibrium price vector $p_n, n = 1, ...$ such that $\lim p_n := p \gg 0$.
The restriction of the agents' characteristics on the compact set $T \subset \mathcal{Q} \times \mathbb{S}^l$ guarantees the existence of a compact set K in the space of commodity $\mathbb{S}^l$ in a way that the demand set $\phi(t, p_n) \subset K$ for each $n = 1, ...$ and $t \in T$. Hence each measure $\xi_n, n = 1, ...,$ on the compact set $T \times K$ is concentrated. It follows that the sequence ξ_n is relatively compact and $\tau_n \to \tau$ and also $p_n \to p$. By definition, we have that the mean demand equals to the supply for each n. That is $\int_{\mathcal{T}} e\, d\mu_n = \int_{\Re}^{l} x\, \delta_n(dx)$. Besides, we know that the measures μ and δ are both marginal distributions of ξ on T. The convergence of the sequence (τ_n) induces that of the two others distributions. Therefore $\int_{\mathcal{T}} e\, d\mu_n \to \int_{\mathcal{T}} e\, d\mu$ and $\int_{\Re}^{l} x\, \delta_n(dx) \to \int_{\Re}^{l} x\, \delta(dx)$. It follows that $\int_{\mathcal{T}} e\, d\mu = \int_{\Re}^{l} x\, \delta(dx)$. $\square$

Approximate Equilibrium Prices

One of the characteristics of larger economy is that the size of its consumption sector has a convexity effect on the mean demand. When consumer's preferences are non-convex, the assumption of many participants induces a convexing effect on the mean demand. In this case, there are an approximate equilibrium prices rather than an equilibrium prices. The degree of approximation depends on the size of the consumption sector. The larger the consumption sector, the better the approximation to the equilibrium.

Theorem 4.71. *Let the sequence ε_n of simple economies with characteristics in $\mathcal{Q}_{mo} \times \mathbb{S}_{+}^{l}$ be purely competitive on $\mathcal{Q}_{mo} \times \mathbb{S}_{+}^{l}$. Then each economy ε_n admit a positive vector price $\mho_n$ which lead the distance of the mean endowment to the demand set in the economy ε_n to become arbitrary small for sufficiently large n i.e.dist$[\int_{\mathcal{T}} e\, d\mu_n, \phi(t, \mho)] \underset{n}{\to} 0$*

Proof. For every economy ε_n, we denote by $\varepsilon_n^{\mho}$ the consumption sector associated with each economy ε. It follows from the equilibrium prices theorem, the existence of a strictly positive price vector $\mho_n$ which is such that $|\mho_n| = 1$ and the mean demand $\int e\, d\mu_n$ belong to the convex hull $\mathcal{CO}(\phi(\epsilon_n, \mho_n))$ for n large enough. The convergence of the sequence $(\mho_n) \to \mho$ together with the convergence of the sequence ε_n in distribution lead to the convergence of the associated consumption sector $\varepsilon_n^{\mho}$. Finally since $\int e\, d\mu_n \to \int e\, d\mu$ it follows that $\lim \mho_n := \mho \gg 0$. $\square$

Walrasian equilibrium is both a mechanical and a social phenomenon. Equilibrium prices generates a state of the economy that enables every citizens to choose the most desirable commodity bundles of his budget set and the existence of a market where citizens can buy or sell any amount of each commodity. Now the natural question is whether an approximate equilibrium price can can provide as much as an equilibrium prices *i.e.*, an attainable allocations. The following proposition show that if an approximate prices prevails rather than equilibrium prices, not every citizens will maximize his utility but at least the majority of them will get what they want.

Proposition 4.72. *Consider a succession ε_n of a purely competitive economy having identities in $\mathcal{Q}_{mo} \times \mathbb{S}^l_+$. For every economy ε_n there is an allowance a_n and a positive price vector $\mho_n$ such that:*

1. $\sum_{C \in \mathcal{A}_n} (a_n(C) - e(\varepsilon_n(C))) = 0$;
2. $\mho_n.a_n(C) = \mho_n.e(\varepsilon_n(C))$ *for every agent* C *in economy* ε_n;
3. $\frac{1}{\#\mathcal{E}_n} \cdot \#\mathcal{C} \in \mathcal{E}_n | a_n(c) \not{i}n \phi(\varepsilon_n(c), \mho_n)| \to 0$;
4. $\max_{C \in \mathcal{E}_n} dist[a_n(c), \phi(\varepsilon_n(c), \mho_n)] \to 0$.

The debate about approximate equilibrium price lead us to the famous limit theorem on the core of an economy that started by Francis Edgeworth (1881) and enhanced by Debreu and Scarf (1963). Here we present the infinite dimensional analogue of Debreu-Scarf core equivalent.

4.8.3 Welfare Theorem in The framework of Atomless Economy

Francis Edgeworth (1871) was the first to realize the link between Walras equilibrium and the core allowance of an economic barter. He presented in the form of proposition the famous limit theorem or the well known Edgeworth proposition which state that as the size of economic actors tends to infinity, the allowances that belong to the core are gaining the characteristic of the Walras allowances. Debreu and Scarf (1963) provided a lucid general treatment to Edgeworth approach. In 1970, Peleg and Yaari extended the Debreu - Scarf argument to the space of real-valued sequences $\Re_\infty$. Brown-Robinson(1974) and Khan(1973) presented a very different approach of identifying the core and the set of Walras allowances. Hildenbrand(1974) provided its analysis in the framework of atomless economy. In an atomless economy,we can expect the natural law of the great number to confirm and enhance the first limit theorem on the core formulated by Edgeworth. The framework of Atomless economy possess a powerful analytical tool in proving asymptotic properties. As it can be seen through the following theorem, the continuous representation of a purely competitive sequence permit the establishment of the core equivalence. The message from atomless economic's framework is that the increasing change of the number of economic actors, bring with a competitive economy that generate a special price, the equilibrium price, the one likely to bring the commodity vector $h(a)$ in the space of commodity $\mathbb{S}^l$ as close as possible to the demand set. The more agents that participate to the economy, the closer the commodity vector will be to the demand set. This is what the next theorem express in the language of atomless economic.

We know that for an economy $\varepsilon : \mathcal{E} \to$ a subset T of a universe of consumer's identities, the set $\Gamma(\varepsilon)$ of its prices equilibrium depends only on the distribution μ of potential-tastes distribution and it contain at least one element. We shall thus write $\Gamma(\mu)$ instead of $\Gamma(\varepsilon)$. We present the first limit theorem with strongly convex assumptions.

Theorem 4.73. *Let the sequence ε_n of simple economies $\varepsilon_n : \mathcal{E}_n \to \mathcal{Q}^m o^*_{sco} \times \mathbb{S}^l$ be a purely competitive on $\mathcal{Q}^m o^*_{sco} \times \mathbb{S}^l$. Then:*

1. *to each real $\epsilon > 0$ and $\eta > 0$ one can associate an integer n_1 in a way that for each $n \geq n_1$ and for each allocation f in the core $\mathcal{C}(\varepsilon_n)$ there might exist a price vector $\mho$ in $\Gamma(\mu)$ with the property $\frac{1}{\#\mathcal{E}_n} \cdot \#\{(C) \in \mathcal{E}_n || f(c)$ - $\phi(\varepsilon_n(c), \wp)| \geq \eta| \leq \epsilon$ or sequentially;*

2. *For each succession of allowance $h_n \in \mathcal{C}(\varepsilon_n)$ there might be a price vector $\mho_n \in \Gamma(\mu)$ in a way that the sequence $(h_n(.) - \phi(\varepsilon_n(.), \wp))$ converge to zero i.e.$\frac{1}{\#\mathcal{E}_n} \cdot \#\{\mathcal{C} \in \mathcal{E}_n || f_n(c)$ - $\phi(\varepsilon_n(c), \mho))| \geq \eta\} \to 0$ for every $\eta > 0$.*

The proof of this theorem requires two technical results. These are presented without proof, one as a lemma and a proposition.

Lemma 4.74. *For each allocation in the core of an economy $\varepsilon \to \mathcal{Q} \times \mathbb{S}^l$ and for each $\tau > 0$ and $\epsilon > 0$, there exist a price vector in a way that the size of economic actors in the economy that can't maximize their utility under that price is less or equal to $(\frac{2\tau}{\epsilon})^2$ and is independent to the total number of actors in the economy.*

The coming proposition says that in a core allocation, the commodity vector $h(a)$, which is allocated to a small coalition B_n is indeed small compared to the total size of actors in the economy.

Proposition 4.75. *Let the sequence (ϵ_n) of simple economy with characteristic in $\mathcal{Q}_{mo} \times \mathbb{S}^l_+$ be purely competitive. If $h_n \in \mathcal{C}(\epsilon_n), n = 1, ...,$ then the sequence (f_n) is uniformly integrable. The sequence $(h_n), h_n \in \mathcal{C}(\epsilon_n)$ is uniformly integrable if and only if for every sequence (B_n) with $B_n \subset A_n$ and $\lim(\frac{\#B_n}{\#E_n}) = 0$ it follows that $\lim \frac{1}{\#E_n} \sum_{a \in \mathcal{E}_n} h_n(a) = 0$.*

Here is the sketch of the proof: The first step of the proof utilises the continuous representation of a simple economies and show that a subsequence of (ϵ_n) admits a continuous representation that is defined by an atomless economy and the sequence of measurable mapping $\psi : E \to E_n$ such that the corresponding sequence of allowances $\bar{h_n} = h_n \circ \psi$ converges almost every where to an allowance $\bar{h}$ of the limit economy ϵ. Now, denoting by μ the income-tastes distribution and by f the allowance of the limit economy (which is as we know an atomless economy) we have from the properties of the continuous representation that the measurable mapping $\psi_n : E \to E_n$ is such that $\nu(\psi^{-1}(S)) = \frac{\#S}{\#E_n}$ for every $S \subset E_n$ and $(\bar{\epsilon_n}, \bar{h_n}) \to (\epsilon, h)$ almost every where on A, where $\bar{\epsilon} = \epsilon_n \circ \psi_n$ and h_n

$= h_n \circ \psi_n (n = 1, ..,)$. We then use the above lemma to show that the limit allocation $\lim h_n = h$ is a Walras allocation for the limit economy ϵ.
The second part of the proof consist in showing that the equilibrium price p which belongs to $\Gamma(\mu)$ has the required properties. That is they are the supporting prices for the Walras equilibrium. We consider a succession h_n of allowance which are in the way that for each n, h_n belong to the core allocation $\mathcal{C}(\epsilon_n)$, and (for each n) there exist a price vector $\mho \in \Gamma(\mu)$ for which the subsequence $(h_n(.) - \phi(\epsilon_n(.), p))_{n \in Q}$ converges in measure to zero. The main idea stem on the fact that in a replica - economy where consumer's tastes are strongly, a core allowance assign the same commodity vector to agents with the same identities *i.e.* with the same distribution μ. Since the real question is about finding conditions under which core-allowance can be supported by a suitably chosen prices for every agents in the economy, we are led to consider a compact subset of economic actor's identities $\mathcal{Q}^m o^*_{sco} \times \mathbb{S}^l_+$. In fact it is a way of assuming that the economic identities (income-tastes) are not considerably different. In addition, the increase of the number of economic actors affect the distribution of economic actor's identities μ_n. We thus have to ensure that the distribution tend to a uniformity (similarity) in the sense that for every economic actor in the economy ϵ_n there are many others agents who have similar identity. That is $\lim \sup(\sup(\mu_n)) \subset \sup(\mu)$ (which is once again an expression of the law of the great number . $\square$

Chapter 5

Equilibrium Analysis in the setting of Riesz Dual System.

5.1 Introduction

General Economic Equilibrium is concerned with the study of the conditions under which the reconciliation of economic agents (consumers, producers, landlords,...) acting solely on behalf of their own interest, with conflicting interests, is effected. Dating back to Adam Smith and continuing through Menger, Jevons (1871) and Walras (1874), there has been a net perception of the existence of a specific price system that plays the crucial coordinating and equilibrating role. Leon Walras (1874) mathematically formulated a model of general economic equilibrium. In the early 1954's Kenneth Arrow, Gerard Debreu, and McKenzie set up a mathematical way of showing that a finite dimensional convex economy admit a competitive equilibrium. Nowadays, it is known in the literature as the Arrow-Debreu-McKenzie model for simple economies. Competitive equilibrium is about a collection of prices, one for each commodity, which would equate supply and demand for each commodity markets, and an allocation effected by these prices. Society is made of individuals. One particular interest of the economy is the production and distribution of the resources of the society. Mathematics has provided tools that help society to reach some of its objectives, notably its welfare. Given that economy admits many equilibriums states (some with pollution, inequalities and terrorist attacks and not desirable for the society) We need thus a criterion that shall allow one to depict a economic equilibrium likely to lead the society to a social optimum. After Walras' publication of the general equilibrium, two mathematicians: Vilfredo Pareto (1906) and Francis Edgeworth (1881), embarked on the investigation of the nature of Walrasian equilibrium. In his mathematical Psychis, Francis Edgeworth(1881) provided a framework which brought a new consideration of equilibrium. He presented a geometrical picture describing the exchange of two commodities that might arise in an economy with two types of consumers each of whom initially have endowment. He came to illuminating result; that exchange by redistribution of total endowment of both consumers, improves their welfare. He thus presented a cooperative, a socioeconomic view of equilibrium. Even though he discussed only a two dimension case, he was able to draw a useful conclusions : the contract curve and the of allowance are in exactly the same place; and further, the upward change of the number of economic actors in an exchange, is accompanied with a change of the core allocation becoming the Walrasian allocation. Edgeworth's finding played a leading role in the way of tackling general equilibrium question. His work has been considerably extended by a number of authors. Martin Shubick (1959) used n-person game theory to analyse Edgeworth optimality of exchange; Scarf (1962), Debreu (1963) used the theory of many person games, notably the concept of the core and put it in a form applicable to market economies. They showed that the collection of core allocations is contained in the collection of Walras allocations but they could not establish the reverse. Later on Aumann (1964) [6] returned to this question and showed the equivalence between Walras allowance and the core allowance for an atomless economy. One of the reasons of these attempts to extend Edgeworth's findings to a more general setting shall be the limiting theorem, whereby Edgeworth said that the time though long enough can make globalization preferable than protectionism. It is thus worth trying to discover what happens to the core allowance and the collection of Walras equilibrium when the number of economic participants increases. Edgeworth box is glutted with rich features that need to be explored. To do this, we need an appropriate mathematic tools. In 1995 Mas-Colell observed that the Edgeworth box was a precious tool for economic analysis. He pointed that the Edgeworth box, simple as it is, was remarkably powerful. There is virtually no phenomena or properties of general equilibrium exchange economic that cannot be depicted in it. This Master's dissertation use the Riesz dual system $\langle \mathcal{L}_w, \mathcal{L}'_w \rangle$, to show how efficiently the Edgeworth box helps depict equilibrium properties of an infinite dimensional economy. In year 1906, Vilfredo Pareto refined Edgeworth's analysis. His major contribution, along with the American economist Irving Fisher was the discovery of new analytical tool: the indifference curve, a characteristic of consumer's preferences. In this framework, he was able to observe that at the equilibrium, the indifference curves representing the preferences of the two consumers are tangent, meaning that their marginal rate of substitution are equal at this equilibrium point. This tangency condition is no more than an equilibrium condition since to reach that equilibrium, both consumers have to give up a portion of what they cherish in order to get that agreement. Pareto showed graphically that any competitive equilibrium is an optimum but he could not prove it for the mathematical tools required were at this period not so rich. In year 1950, Kenneth Arrow and Gerard Debreu, using the hyperplane's theorem for convex sets, mathematically established the identification of Walras allowance and Vilfredo allocations for convex finite economies. In chapter 2 we used a continuum economy, *i.e.* economy modeled by a measure space of economic agents, to extend the Arrow-Debreu-Mckenzie framework to an infinite dimensional economy. We see that an infinite dimensional economy with positive mean endowment and convex preferences reaches an equilibrium and that its Walras equilibrium distribution belong to the core of exchange economy as the core allocation belong to the set of Walrasian allocation. However, it is not possible to depict the equilibrium properties under this framework. In 1986 , Mas-Colell [23] replaced the assumption of interior point of the positive cone with the condition of uniform properness. This new property enable one to deal with some of the problems arising in general equilibrium, specially in the setting of infinite dimensional Riesz spaces with empty interior cones. In particular, it has allowed the infinite dimensional version of the general state of prosperity for the society. By making an Edgeworth box to function in infinite dimensions, Riesz theory provides a striking evidence of the existence of a social dimension in a competitive equilibrium.

5.2 Model Setting

5.2.1 Problem Statement

Debreu's profound insight was geometric. His use of the separation hyperplane for convex sets allowed him to establish the attainability of Pareto optimum allocation by a competitive price mechanism from the original geometric argument of Pareto (1906) [8]. In his 1954 paper, *Valuation Equilibrium and Pareto Optimum*, Debreu needed to extend the welfare proposition from finite dimensions to general linear topological spaces. That topology of the commodity spaces permitted him to prove the welfare proposition under the same general assumptions as it is possible in the finite dimension. The important point is that the choice of the topology played a crucial role. Earlier we defined a commodity as things that have to sold and which have physical characteristics, date and a place to be delivered are specified. This is to say that delivery of the commodity is subject to an exogenous event, simply speaking uncertain events determine consumption sets. Thus, to account for uncertainty one has to extend the classical economic theories of equilibrium and optimality to economic structure involving infinite dimensional states of nature and no natural terminal period i.e. infinite time horizon. Debreu [8] defined a good as being a contract that ensures the delivery of a physical good or service at a date t, at a place l, and in dependance of the realization of certain events that may occurs at date t. This type of model, presented in the classical monograph: "Theorie de la valeur"(theory of value), is known as the Arrow - Debreu - Mckenzie's model. Models that are dynamic are not able to capture some important aspects of the economy. Uncertainty, for instance, is a natural aspect that involve an infinite state of the nature. Thus, to account for uncertainty one has to extend the classical economic theories of equilibrium and optimality to model of economy that account for infinite time horizon [4]. We are thus conveyed to a model with infinite dimension. If we need to consider an economy with infinite horizon *i.e.* an economy for which the characteristic set (time, space or physical quality) of goods is infinite, or economy under uncertainty with infinite number of states of nature then we will have to consider a model that account for the production an the trading of many commodities.

5.2.2 Modeling with Infinity.

Debreu [8] is the first mathematician to model the commodity space by a topological vector space and to use a linear functional (continuous) to represent a function price on the commodity space. In his model, Debreu needed to use the assumption of an interior point on the production set. However this assumption is only satisfied under free disposal on production and a cone that is positive and whose interior have at least one element. The set of all measurable functions essentially bounded, endowed with the supremum norm happened to be a good example of a space satisfying this assumption. Radner in [30] showed that $\mathcal{L}_\infty$, the set of all essentially bounded functions, is appropriate to an economic interpretation. He suggested to model the price system by $\mathcal{L}_\infty$. Bewley, [7] used the universes $\mathcal{L}_\infty$ and $\mathcal{L}_\infty$ to represent the universe of commodity and that of price. He showed that such an economy modeled in this way, reach an equilibrium. However Bewley's model posed a problem for it could not be generalized to an ordered vectorial space for which the cone is positive and whose interior have no element. In 1986 , Mas-Colell [23] presented a general result of equilibrium existence by replacing the assumption of interior point with the condition of uniform properness. During the 1970s, equilibrium existence results were obtained for economies with infinite numbers of agents and commodities. Bewley [14] and Podczack [45] and Zame provided equilibrium existence results for economies with commodity spaces modeled by $\mathcal{L}_\infty$ and price system by $\mathcal{L}_1$. For economic models with inter temporal allocations, we need to take into consideration the infinite date in the constitution of the consumptions plan. In this case the commodity space can be modeled by $(l^\infty, \mathcal{S}^\infty[0,H], \varpi, \theta)$ or $\mathcal{S}_\infty([0,+\infty], \varpi, \theta)$ where ϖ is a tribute and θ. a measure of probability. If we consider a set $\mathcal{X}$ as a set of states of the nature then a function on $\mathcal{X}$ is a state contingent variables. For an economic model involving allocation under uncertainty, consumption depends on the states of the world (with possibly infinite states), with commodity and price space represented by $\mathcal{L}^2(\Omega, \Sigma, \mu)$.

Consideration of these realities require an appropriate mathematical theory susceptible to accompanying these models. Knowledge of the conditions likely to lead to an economic equilibrium is an important thing for the society. But of particular attention are the properties related to that equilibrium.

The limiting theorem tell us that Francis Edgeworth perceived equilibrium's property as being revealed at the infinity when the number of consumers increase. Walrasian allocation and Edgeworth allocation are identical in the limit and optimal allocations whether Pareto efficient, quasi-equilibrium, approximate quasi-equilibrium, extended Walrasian equilibrium are all equal. This is to say that although Francis Edgeworth discussed only a few case of two dimensions, he has designed his geometrical picture for infinite dimensions. The power of Riesz geometry for economic analysis is exactly its ability to use the Edgeworth-box in infinite dimensions.

Frederick Riesz is the mathematician who saw geometry in many concepts of mathematic. He firstly exploited the notion of distance introduced by Maurice Frechet(1906) and that of metric space introduced by Felix Hausdorff. He extended these notion to the space $\mathcal{C}[a,b]$ of Lebesgue integral by defining the distance $d(f,g)$ between two functions of $\mathcal{C}[a,b]$ as being equal to $\sup_{a\leq b}|f_1(x)-f_2(x)|$. It then became possible to define the uniform convergence of the functions. Riesz has been the first mathematician to seize the deepest and the power of the notion of the distance. He extended this concept on the space $\mathcal{L}_2[a,b]$ by defining the distance $d(f_1,f_2)=(\int_a^b(f_1(x)-f_2(x))^2\,dx)^1/2$ as well as the notion of quadratic mean (convergence in mean). Riesz provided a geometrical form to Hilbert theory by introducing the language of orthogonality, norm and Hilbert basis. He discovered the existing isometry between many classical functional spaces and the space l_2 or the vector subspace of l_2. This observation allowed some mathematician like Von Newmann to provide an axiomatic presentation of pre-Hilbertian spaces and a profound study of endomorphism of Hilbert spaces. Riesz generalized Schwartz inequality to $|\int f(x)g(x)\,dx|\leq(\int|f(x)|^p\,dx)^{\frac{1}{p}}(\int|h(x)|^{\frac{p}{p-1}}\,dx)^{p-1}p$. He solved the problem of the moment which consisted in finding a function g for which $\int_a^b g(x)h_i(x)\,dx=c_i$ for all i, for a sequence c_i of number and a sequence (h_i) of the function defined on $[a,b]$. The problem of the moment has been considered by Stieltjes(1894). Riesz considered a compleat, orthogonal system of the function in the space $\mathcal{L}^2[a,b]$ and showed that a necessary and sufficient condition for the existence of such function(a solution) in $\mathcal{L}^2[a,b]$ is that $\sum c_i^2<+\infty$ in a way that $g=\sum c_ih_i$. In this case, the strong convergence establish an isomorphism of l_2 and $\mathcal{L}^2[a,b]$. According to the method suggested by David Hilbert one need to consider the Fourier series of the function g and h_i. By so doing, one end with a system of linear system of equations with infinite unknown values. Once again the geometric view of Frederick Riesz made him to move the interval $[a,b]$ to the interval $[0,2\pi]$ and instead of considering the Fourier series of the function g and h_i, he rather consider the Fourier series of their coefficients (x_k) and (ai_k) respectively, with (ai_k) a compleat orthogonal system in l_2. This led to the following system of equation $\sum ai_kx_k=c_i$ which admit a unique solution. Frederick Riesz provided mathematic with powerful tools for functional analysis. He provided conditions under which a sequence of measurable functions defined on a measurable subset of $\mathbb{R}$ could converge. He set up the completeness of the space $C[a,b]$ and $\mathcal{L}^p[a,b](p>1)$ and of their dual. He treated the question of weak compactness as well as of the weak convergence in the dual space. The frame of spectral theory provided him with opportunity to develop the theory compact operators which in turn allowed the notion of integral equations to play important role in the elaboration of principal concepts of modern analysis in mathematic physic and now in economic. In 1996 Aliprantis, Burkinshaw and Brawn used Riesz geometry, specially Riesz dual system, to analyse the equilibrium of a competitive economy with infinite commodities and economic actors.

Riesz theory, specially locally convex-solid topology, happen to be appropriate framework for studying and analysing economic equilibrium's properties. The central concern of this Master's dissertation is precisely to show how Riesz theory allow a constructive examination of the properties related to an economic equilibrium.

5.2.3 Model Assumptions

Now, $\mathcal{Z}$ is regarded as a higher commodity universe of an economy with m economic actors. The number $m \in \mathcal{N}$ may be a billion as so long as it is a better approximation of the world population. We thus assume the existence of m consumption units indexed by $i \in \mathcal{N} = \{1, ..., m\}$, voluntarily engaged in exchange economic. Market exchange shall take place only if someone is benefiting from the exchange. The role of each economic agent consists in choosing from a given set of alternative consumption vectors. A bundle of commodities is a vector $x \in \mathcal{L}_+$. Each economic agents has an endowment $w^i \in \mathcal{L}_+$. Consumer endowments and preferences constitutes the primitives of the exchange economy. Which is why we require $w^i > 0$ for each i (otherwise there is no market). The motivation behind a consumer's choice is to maximize his satisfaction. This choice is made in accordance with a preference scale. Preferences characterize consumer's behavior, the criterion he uses in making his choice. It is most convenient in mathematical economics to represent preferences by a continuous (numerical) functions. Debreu (1952) provides conditions under which a mathematical function can properly describe a complete order $\leq$. As we shall see, Riesz spaces, by their structure, naturally fill those conditions.

Theorem 5.1. *If the sets $\{k \leq k'\}$ and $\{k' \leq k\}$ of elements of a connected, separable, and completely ordered space $\mathcal{X}$ are closed then there might exist on $\mathcal{X}$ a real, and order-preserving functions that is continuous.*

An economic agent in this economy is identified by his resources (endowments), his consumption set X (a partially ordered set) and by his preferences, represented by a binary relation having the following properties: For each commodity good u, v and $z \in X$ we have

1. the binary relation is *Reflexive*: meaning that each commodity good is good as itself.

2. *Transitive*: if the commodity good u preferred than the commodity good v and the commodity good v is preferred than the commodity good z then the commodity good u might be preferred to the commodity good z. A preference relation that is transitive, is an absolute consistency of choices.

3. *Complete*: Any commodity good is comparable to another commodity good.

Transitivity and complete order ensure that economic agents are effectively in the dynamic of maximizing satisfaction. This is a strong assumption for the existence of economic equilibrium.

The convexity of the preference relation reflects the consumer's taste for the mixture rather than the extremes. It is a mathematical expression of a consumer's attitude for which the set of commodity bundles weakly preferred from a convex set. Equilibrium is our central concern in this analysis. As said earlier, equilibrium involves global demand and demand depends on consumer's preferences (utility function) and consumer's income. Each value of price system generates a distribution of income $(p.w_1, ..., p.w_n)$. For this reason, equilibrium's analysis depends on preferences and prices analysis. As we shall see, most assumptions regarding equilibrium existence are about preferences (utility functions) and prices properties. Topological analysis of consumer's preferences has a great bearing on equilibrium's analysis. Consumer demand, which is the solution to the utility maximization problem, uniquely exists when consumer preferences are strictly quasi-concave. A continuous demand functions allow one to infer consumer market behavior following a price change. The continuity of the demand function is thus one of our desirable assumption. A strictly convex, monotone and continuous preference (strictly quasi-convex utility function) results in a continuous demand function. The above assumptions on preferences as well as the compactness of the Edgeworth economy will appear over and over in most of equilibrium theorems.

5.2.4 Topological analysis of consumer's preferences

Let $\mathcal{Y}$ be a consumption set and $\succeq$, a preference relation defined on $\mathcal{Y}$. The collection of commodity good u that are preferred than the commodity good v are said to be the no worse - than - x while, the collection of commodity good that are less preferred than the commodity good z are called the no - better - than z. The above two definitions are now used to present the continuity notion of the preference relations. Whenever the collection of the no worse than the commodity good u is closed, we said that the preference relation possessed by the consumer is *upper semi-continuous*; whilst, when the collection of the no better than a commodity good v of a consumer's consumption set is closed, one say that the preferable relation is *lower semi-continuous*. A continuous preference relation is a preference relation satisfying both properties: upper and lower semi-continuous. A preference relation is called convex if it is defined on a convex consumption set and if the collection of the no worse than any commodity good of the consumption set is convex. Whenever the quantity of a commodity good u consumed by an economic actor is greater or equal than the quantity of the commodity good v, lead to the observation that economic actor prefer the commodity good u than the commodity good v; we say that the preference relation is monotone.

Definition 5.2. If an economic actor prefer the commodity good u than any others commodity good of his (her) consumption set then the commodity good u is said to be a maximal commodity good for the economic actor's preference relation. If in addition, his preference relation is total and reflexive then the maximal commodity good is the greatest commodity good.

Un utility function is a mathematical (numerical) function that describe the consumer's taste (preference relation). A continuous utility function is desirable for economic analysis.

As our last assumption, is that markets are complete, *i.e.* each commodity good has a market where it can be traded. Utility functions as well as the price system are continuous linear functionals. The algebraic and topological dual of a topological vector space are important components (classes) of Riesz spaces from which Riesz theory has built remarkable topologies.

5.2.5 BANACH SPACE

Definition and Properties

Let $\mathcal{E}$ be a vector space on the field $K = \mathbb{R}$ or $\mathbb{C}$. A real valued function of $\mathcal{E}$, denoted $||.||$ is said to be a norm if it behave in the following way:

1. For every element x of $\mathcal{E}$, $||x||$ is greater than zero and $||x|| = 0$ if and only if $x = 0$.
2. The norm of the sum of two elements of $\mathcal{E}$ is less or equal than the sum of their norms (triangular inequality).
3. The function norm is homogeneous *i.e.* for an element x of $\mathcal{E}$ and for a real α, we have $\alpha x = |\alpha|.||x||$.

The vector space $\mathcal{E}$ equipped with the function norm is called a normed vector space. It is a particular case of a metric space. This can be seen by defining the distance $\delta(x, y)$ of two elements of $\mathcal{E}$ as being equal to $||x - y||$. The topology on $\mathcal{E}$ derived from the distance is the same as the topology of the norm. A sequence $x_{nn\geq 1} \subset \mathcal{E}$ is said to converge(strongly) to an element x of $\mathcal{E}$ if $||x_n - x|| \to 0$ whenever $n \to \infty$.

Definition 5.3. A sequence $x_{nn\geq 1} \subset \mathcal{E}$ is said to be of Cauchy if $\lim_{n,m\to\infty} ||x_n - x_m|| = 0$. A vector space is said to be of Banach if is a compleat space as a metric space.

Any convergent sequence in a normed space $(\mathcal{E}, ||.||)$ is a Cauchy squence.It is bounded and admit a limit.

Definition 5.4. Let $x_{nn\geq 1} \subset \mathcal{E}$ be a sequence in a normed vector space $\mathcal{E}$. We denote by $\sum_{n\geq 1} x_n$ its series and by $S_n := \sum_{j=1}^{n} x_n$ the sequence of partial sum. The series is said to converge if there exist an element s of $\mathcal{E}$ such that $S_n \to s$ whenever $n \to \infty$. In this case we have $\sum_{n\geq 1} x_n = s$. The series is said to be absolutely convergent if $\sum_{n\geq 1} ||x_n|| < \infty$.

Theorem 5.5 (Characterization of Banach space)**.** *For a normed vector space $\mathcal{E}$, the following assertions are equivalent:*

1. *$\mathcal{E}$ is a Banach space.*

2. *Every absolutely convergent series is (strongly) convergent.*

3. *Any sequence $x_{nn} \geq 1$ satisfying $||x_n|| \leq e^n$ with $0 < e < 1$ the series $\sum_{n\geq 1} x_n$ is (strongly) convergent.*

Proposition 5.6. *A subspace M of a Banach space $\mathcal{E}$ is also a Banach space if and only if M is a closed subspace of $\mathcal{E}$. A normed vector space that is not compleat is isomorphe to a dense subspace of a Banach space.*

Fundamental Examples of Banach Spaces.

1. The vector space of continuous functions over the interval $[0,1]$, $\mathcal{C}([0,1],K)$, endowed with the norm $||x||_\infty := \sup\{|x(t)| : t \in [0,1]\}$ is a separable Banach space.

2. $K^d = \mathbb{R}^d$ or $\mathbb{C}^d$ endowed with the norm $||x||_\infty := \max\{|u_j| : j = 1, 1, ...d\}$ and $||x||_p := (\sum_{j=1}^d |u_j|^p)^{\frac{1}{p}}, p \geq 1$ are separable Banach space, denoted by $l_p^d, p \in [1,\infty]$.

3. The vector space of the class of functions that are almost every where defined on $[0,1]$, measurable and p - integrable on $[0,1]$ with values in $K = \mathbb{R}$ or $\mathbb{C}$, $\mathcal{L}^p([0,1],K)$ with $p \in [1,\infty]$, endowed with $||x||_p = (\int_0^1 |x(s)|^p \, ds)^{\frac{1}{p}}$, is a separable Banach space.

4. $\mathcal{L}^\infty([0,1],K)$ the vector space of the class of the function essentially bounded, equipped with the norm $||x||_\infty := ess\sup(|x|) = \inf\{\sup_t \in B|x(t) :$ for all $B \subset [0,1]$ with measure null $\}$ is a no separable Banach space.

5.2.6 Hilbert Spaces

Pre-Hilbertian Spaces

Definition 5.7. An inner product on a vector space is a scalar function $< .,. >$ on $E \times E$ such that:

1. For every elements y of $\mathcal{E}$, the application $x \longrightarrow < x, y >$ have a linear property. That is for x_1, x_2 in $\mathcal{E}$ and for α, γ in K we have: $< \alpha.x_1 + \gamma.x_2, y > = \alpha < x_1, y > + \gamma < x_2, y >$.

2. The function has the positive define property: $< x, x >$ is greater than zero for all element x of $\mathcal{E}$ and $< x, x >= 0$ if and only if $x = 0$.

3. The function have the symmetric property $< x, y > = < y, x >$ for all x, y of $\mathcal{E}$.

Theorem 5.8. *The application $||.|| : \mathcal{E} \to \mathbb{R}$ defined by $||x|| := \sqrt{< x, x >}$ is a norm in the vector space $\mathcal{E}$. For every elements x and y of $\mathcal{E}$, $| < x, y > | \leq \sqrt{< x, x >}\sqrt{< y, y >}$: the Cauchy - Schwartz inequality.*

Definition 5.9. A Banach space is said to be a Hilbert space if there exist a scalar product $< ., . >$ on $\mathcal{E} \times \mathcal{E}$ for which the norm of the Banach space satisfy $||x|| := \sqrt{< x, x >}$.

Example

Let (Ω, β, δ) be a measurable space. The space $\mathcal{L}_2(\delta)$ equipped with the scalar product $< f, g >:= \int_\Omega f\bar{g}\, d\delta$ is a Hilbert space.

Definition 5.10. Let $\mathcal{H}$ be a Hilbert space and x, y elements of $\mathcal{H}$. The element x is said to be orthogonal to y if $< x, y >= 0$. It is denoted by $x \perp y$. For a subset M of $\mathcal{H}$, an element x is said to be orthogonal to Mif x is orthogonal to every elements of M. We denote by $M^\perp := \{x \in \mathcal{H} \mid x \perp M\}$ the orthogonal complement of M.

Theorem 5.11 (Riesz Decomposition)**.** *Let M be a closed subspace of the Hilbert space $\mathcal{H}$. Then every element x of $\mathcal{H}$ can be decomposed in a unique way as $x = m + p$ where $m \in M$ and $p \in M^\perp$. The element m is the orthogonal projection of x on M and it can be denoted by $P_M(x)$ while P_M will be said to be the operator of projection on M. We can then have $x = P_M(x) + P_M^\perp(x)$.*

Definition 5.12. $l^2(N)$ is the space of the sequence $u = (u_k)_k \in N$ of complex number such that $||u||_2^2 := \sum_{k \in N} |u_k|^2 < +\infty$. It is a Hilbert space for the scalar product $< u, v >:= \sum_{k \in N} u_k \bar{v}$.

Theorem 5.13 (Riesz - Fisher)**.** *Any separable Hilbert space of infinite dimension is isomorphe to $l^2(N)$. For a bounded linear function f and a Hilbert space $\mathcal{H}$, there exist a unique vector y_f of $\mathcal{H}$ such that for any element x of $\mathcal{H}$, $f(x) =< x, y_f >$ and $||f|| = ||y_f||$.*

We now present a theorem that provide an extension of a linear form satisfying some norm conditions. The Hahn - Banach theorem happen to be an important tool of functional analysis in that it introduce weak topologies, topology derived from the concept of duality. We shall rather present its geometrical form than its analytical form just to underline the geometrical feature of the convex sets.

Definition 5.14. A topological vector space is said to be locally convex if and only if every neighborhood of its origin contain a convex neighborhood.

Proposition 5.15. *By associating every convex neighborhood of the origin to a semi-norm p_α), we can say that a topological vector space is locally convex if and only if its topology τ is defined by a family of semi-norms p_α) behaving in a way that $p_\alpha(x) = 0 \Rightarrow x = 0$ for all α.*

A subset B of the topological vector space is said to be open in the event of the existence of an open ball having as center x an element of B and as radius a positive real ς, that is contained in the subset B. Normed vector spaces are locally convex.

Theorem 5.16 (Hahn - Banach)**.** *Given a locally convex vector space $\mathcal{G}$ (with semi-norms p_α)) and two disjoint non empty convex subsets E and F of $\mathcal{G}$. Then if E is an open subset, it might exist a linear form f of $\mathcal{G} \to \mathbb{R}$ that is continuous, non null behaving in a way that $\sup_E f \leq \inf_F f$. In the event of F being a closed subset and E a compact subset of $\mathcal{G}$, it might exist a linear form f of $G \to \mathbb{R}$ that is continuous, non null, behaving in a way that $\sup_E f \leq \inf_F f$.*

Topological Dual

For two vector spaces $\mathcal{E}$ and F and a bilinear form of $\mathcal{E} \times \mathcal{F}$ on $\mathbb{R}$ such that $< y, x >= 0$ for every element y of F imply that x equal zero. And for every element x of $\mathcal{E}$ such that $< y, x >= 0$ imply that y equal zero; one can define a topology of locally convex vector space on $\mathcal{E}$ by considering the following semi-norm: $p_B(x) = \sup_y \in B| < y, x > |$ where B is a subset of $\mathcal{F}$ $(1.a)$ and $p_A(y) = \sup_x \in A| < y, x > |$ where A is a subset of $\mathcal{E}$. $(1.b)$ If $\mathcal{E}$ is a separable locally convex vector space for which its topology τ_E is generated by its open subsets, one can consider the following bilinear form: $(x, y) \in \mathcal{E} \times \mathcal{E}^* \to < y, x >= g(x) \in \mathbb{R}$. According to Hahn - Banach theorem, the above crochet of duality satisfy the crochet defined previously in $(1.a)$ and $(1.b)$. One can then define a new topology on the vector space E, denoted by $\sigma(E, E^*)$, and a topology $\sigma(E^*, E)$ on the dual space E^*. We do have the following characteristics:

1. The topology $\sigma(E, E^*)$ is poor than the topology τ_E. In others words it have less open subsets, more compacts subsets and less continuous functions.

2. For a Banach space $(\mathcal{E}, ||.||)$, the topology induced by the norm $||.||$ is called the strong topology on $\mathcal{E}$ whilst the topology $\sigma(E, E^*)$ is said to be the weak topology on $\mathcal{E}$. The topology $\sigma(E^*, E)$ on E^* is also called weak topology.

3. For a sequence (f_n) of the function defined on the Banach space $(\mathcal{E}, ||.||)$, we shall say that if it converge weakly in the dual space $\mathcal{E}^*$) to a function f then f_n is bounded and $||f|| \leq \lim_n \to \infty \inf ||f_n||$.

4. When the vector space $\mathcal{E}$ is of infinite dimension, the weak topology happen to be distinct to the strong topology. There are subsets that are closed for the strong

topology but that are not closed for the weak topology. It is for instance the case with the unit sphere.

Price System

Debreu was the first to model a price system by a continuous linear form $\mathcal{P}$ on the universe of commodity. He considered a list of strictly positive number as a system of prices in which the i^{th} coordinate represent the amount required to the one unit of the i^{th} good. Thus the value of that unit is $p.x$. When normalized, vector price belong to the simplex $\mathcal{P} = \{p \in \mathbb{R}_+^\infty \mid \sum_{t=1}^{\infty} p_t = 1\}$. We shall now express all the above in Riesz terminology.

5.3 Overview Of Riesz Space

Definition 5.17 (Vector Space)**.** Let $\mathcal{S}$ be an space and $\mathcal{M}$ a collection of mathematical object of the space $\mathcal{S}$. If $\mathcal{M}$ is non empty and closed under the algebraic operations of addition and multiplication by the scalar then $\mathcal{M}$ is said to be a vector space over a field $\mathbb{R}$ or $\mathbb{C}$.

Definition 5.18 (Partially ordered vector space)**.** If now we define on the vector space $\mathcal{M}$ a binary relation, $\succeq$,possessing the following characteristics:

1. Each elements of $\mathcal{M}$ is allowed to be comparable to itself. That is for all $a \in \mathcal{M}$ we have $a \succeq a$
2. The defined binary relationship is assumed to be antisymmetric. That is for $a, b \in \mathcal{M}$, if $a \succeq b$ and $b \succeq a$ we shall have $a = b$.
3. We require the binary relationship to be transitive. That is for $a, b, c \in \mathcal{M}$, if $a \succeq b$ and $b \succeq c$ we shall have that $a \succeq c$.

In this event, the defined binary relationship is said to be a partially ordering relationship. A vector space equipped with such a a partially ordering relationship is called a partially order vector space($\mathcal{M}, \succeq$).

Definition 5.19 (Lattice)**.** Consider a non empty subset $\mathcal{F}$ of the vector space $\mathcal{M}$ and an element a of $\mathcal{F}$ such that for each element x of $\mathcal{F}$ we have $x \succeq a$. The element a is called an upper bound of $\mathcal{F}$. In the event $a \succeq s$, the element a is less or equal to any other upper bound s of the subset $\mathcal{F}$, a is termed the supremum or the least upper bound of $\mathcal{F}$. An element b of $\mathcal{F}$ which is such that $b \succeq x$ for all $x \in \mathcal{F}$, we say that b is a lower bound of $\mathcal{F}$ and it is said to be an infimum or a greatest lower bound if b is greater or equal to any others lower bound of $\mathcal{F}$. A partially ordered vector space in which two elements taken together admit a unique least upper bound (a supremum $a \bigvee b$) and a unique greatest lower bound (an infimum $a \bigwedge b$) is said to possess the structure of lattice.

Examples of vector lattice are $(\mathbb{R}, \succeq)$; $(\mathbb{C}, \succeq)$; the space of all real-valued functions on non-empty set $\mathbb{X}$ endowed with the pointwise ordering: $f \succeq g$ if and only if $f(x) \succeq)g(x)$. The existence of a lattice structure on a vector space $\mathcal{F}$ allow one to define a positive cone $i.e$ a subset of all positive elements of $\mathcal{F}$ for which the addition and scalar multiplication is closed.

A vector space in which the ordering possess the lattice structure is called Riesz space. Since that space of Riesz is first a vector space, it is natural to investigate the existence of its subspace likely to possess a required structure for a Riesz space. Any subset $\mathbb{M}$ of a Riesz space that is stable under vector addition and scalar multiplication is naturally a vector subspace of the Riesz. If in addition any pairs of elements r and s taken in the vector subspace $\mathbb{M}$ admit a supremum with respect to the defined ordering then the vector subspace is said to be a Riesz subspace. A vector subspace of a vector lattice that is stable under lattice operations ($a \bigvee b$ and $a \bigwedge b$) is termed a vector sublattice. A set of a Riesz subspace that is bounded both above and below is said to be an order interval. A vector subspace $\mathcal{F}$ of a Riesz space $\mathcal{S}$ is said to be solid if for an element s of $\mathcal{F}$ and r in $\mathcal{S}$ we have that $r \in \mathcal{F}$ whenever $|r| \leq |s|$. A solid vector subspace of a Riesz space is called an ideal. An ideal of a Riesz space that is generated by a single element say e is termed principal ideal. If $\mathcal{I}$ is such an ideal, we then have $\mathcal{I} = \cup_{n \in \mathcal{N}}[-ne, ne]$ and $\mathcal{I}_e = \mathcal{S}$. The element e of $\mathcal{S}$ is called an order unit or a strong order unit of the Riesz space $\mathcal{S}$. The real-valued function $||.||_e$on the principal ideal $\mathcal{I}_e$: $||u||_e =$ the infimum of the set of the positive real ζ for which $|u|$ is less or equal to $\zeta.e$ define a norm on the Archimedean vector lattice $\mathcal{S}$. Any subset $\mathcal{B}$ of the ideal $\mathcal{I}$ which is such that its supremum q belong to the ideal $\mathcal{I}$ is called a band. A band that is generated by a single element is termed principal band. A Dedekind complete Riesz space is a Riesz space for which every subset of it that is bounded from above, admit a supremum.

Definition 5.20. Consider the real - valued function π defined on the vector space $\mathcal{K}$ satisfying the following three properties:

1. For every element u of $\mathcal{K}$, $\pi(u)$ is greater or equal to zero.

2. For every elements u, v of $\mathcal{K}$, $\pi(u+v) \leq \pi(u) + \pi(v)$

3. For an element u of $\mathcal{K}$ and for a real α, we have $\pi(\alpha.u) = |\alpha|.\pi(u)$.

The defined function π is a semi-norm on the vector space $\mathcal{K}$. In the event the elements u, v belonging to the Riesz space $\mathcal{S}$ are such that $|u| \leq |v|$ in the Riesz space implies $\pi(u) \leq \pi(v)$,the semi-norm π will be termed a Riesz semi-norm. Also whenever $\pi(u) = 0 \Leftrightarrow u = 0$, the semi-norm π will be a norm on the Riesz space $\mathcal{S}$.

A normed vector space $\mathcal{E}$ that in addition possess a lattice structure such that for $x, y \in \mathcal{E}$ we have $|x| \leq |y| \Longrightarrow ||x|| \leq ||y||$ is said to be a Banach lattice.

Examples of Banach spaces are spaces l_p, c_o, c with their usual norms; $\mathcal{L}^p(\mu)$ with the point wise order almost everywhere.

It is worth noticing that the two fundamentals functions of the analysis of consumers behavior: the utility function and the price system, are also part of the class of Riesz spaces defined above. As we shall see in the next section, Riesz theory build a topological structure from the linear functionals. The resultant topologies: the locally convex, the locally solid, the locally convex-solid, the weak and the absolute weak topologies; in conjunction with the structure of order of the solid Riesz spaces generates remarkable properties: The Lebesgue and the Fatou properties but also metrizability, Frechet and B-topologies. Those properties will be in use in the application to general economic equilibrium.

5.4 Locally convex Riesz topology

5.4.1 Linear topology

Definition 5.21. A topology η on a vector space $\mathcal{K}$ susceptible of making continuous the algebraic operation of addition and multiplication scalar is called linear topology. By equipping the vector space $\mathcal{K}$ with a linear topology η, one obtain a topological vector space $(\mathcal{K}, \eta)$.

5.4.2 Locally convex topology

Definition 5.22. A locally convex topology on a vector space is a linear topology that has a neighborhood base at zero consisting of convex sets. Consider a linear functional, π, on the vector space $\mathcal{K}$ (a semi-norm), for any real ζ greater than zero, the collection of the element x of the vector space $\mathcal{K}$ for which the value (the image) $\pi(x)$ is less or equal than ζ, is a convex subset of $\mathcal{K}$ (the ζ - closed ball of π) . The collection of these linear forms (semi-norms) $\{\pi_\gamma\}_{\gamma \in D}$ on the vector space $\mathcal{K}$ provides us with a topology on $\mathcal{K}$ that is locally convex. We denote by ξ such topology. The collection of all linear functionals on the vector space $\mathcal{K}$ forms the algebraic dual $\mathcal{K}^*$ and the collection of all linear functionals on the vector space that are ξ-continuous , indicated by $\mathcal{K}'$, is called the topological dual of the topological vector space $(\mathcal{K}, \eta)$.

Riesz theory see the utility function and the price system as a linear functional on the consumption set. The Riesz representation theorem allows one to identify the vectors spaces $\mathcal{K}$ and $\mathcal{K}'$. If now we provide the Riesz space $\mathcal{Z}$ with a topology ξ that is locally convex, one obtain a locally convex Riesz space $(\mathcal{Z}, \xi)$. The existence of the topological dual allows us to introduce a new topology, the *weak topology* on the vector space $\mathcal{K}$.

The locally convex Riesz space can be equipped with two topologies: the weak topology $\theta(\mathcal{Z},\mathcal{Z}')$ and the Mackey topology $\varrho(\mathcal{Z},\mathcal{Z}')$ on $\mathcal{Z}$.

The topological dual $\mathcal{Z}'$ can in turn be endowed with the weak and the strong topology.

The weak topology $\theta(\mathcal{Z}',\mathcal{Z})$ is the locally convex topology on the vector space $\mathcal{K}'$ generated by the family of semi-norms $\{\pi_x : x \in \mathcal{K}\}$ where $\pi_x(g) = |g(x)|$ for $g \in \mathcal{K}'$. It is convenient for point wise convergence on $\mathcal{Z}'$. That is $g_\pi \xrightarrow{\theta(\mathcal{Z}',\mathcal{Z}} g \Longleftrightarrow g_\pi(x) \longrightarrow g(x)$.

The strong topology $\gamma(\mathcal{Z}',\mathcal{Z})$ on $\mathcal{Z}'$ is the locally convex topology generated by the family of semi-norm $\{\pi_B : B \in \mathcal{F}\}$ where $\pi_B(g) = sup\{|g(x)| : x \in F\}$ and $\mathcal{F}$ is the collection of all $\theta(\mathcal{Z},\mathcal{Z}')$ - bounded subsets of $\mathcal{Z}$. *The strong topology* $\gamma(\mathcal{Z}',\mathcal{Z})$ is the topology of uniform convergence on the $\theta(\mathcal{Z},\mathcal{Z}')$ - bounded subsets of $\mathcal{Z}$.

5.4.3 Locally solid topology

Definition 5.23. A linear topology ξ defined on a Riesz space $\mathcal{Z}$ is said to be locally solid if it has a neighborhood base at zero consisting of solid sets.

The lattice structure of the Riesz space has provided with nice properties such as Archimedean, Dedekind complete, Lebesgue, Fatou norm,...

Definition 5.24. A locally solid Riesz space $(\mathcal{Z},\xi)$ is said to satisfy the *Fatou property* (or that ξ is Fatou topology) if ξ has a base at zero consisting of solid and order closets sets. A locally solid Riesz space $(\mathcal{Z},\xi)$ satisfies the ξ - *Lebesgue property* if $\{x\}_\beta \downarrow 0$ in $\mathcal{Z}$ imply $\{u\}_\beta \xrightarrow{\xi} 0$. A locally solid topology ξ is a *pre - Lebesgue* topology if $0 \leq x_\beta \uparrow \leq x$ imply x_β is a ξ - Cauchy sequence. A locally solid topology ξ satisfy the *Lebesgue* property if $x_\beta \downarrow 0$ in $\mathcal{Z}$ implies $x_\beta \xrightarrow{\xi} 0$.

5.4.4 Locally convex-solid topology

Definition 5.25. A linear topology ξ on a Riesz space $\mathcal{Z}$ that satisfy both the locally convex and locally solid properties is known as a *locally convex - solid topology* ξ.

Theorem 5.26 (Luxemburg)**.** *For a Riesz space $\mathcal{Z}$ endowed with a locally convex - solid topology, the band generated by $\mathcal{Z}'$ in the Dedekind complete Riesz space $\mathcal{Z}^\sim$ is constitute of the subsets of linear functional that are ξ - continuous on the order intervals of $\mathcal{Z}$.*

Definition 5.27. The topology engender by the subset $\mathbb{B}$ of the Dedekind complete Riesz space $\mathcal{Z}^\sim$, the absolute weak topology, on the Riesz space $\mathcal{Z}$ is denoted by $|\theta|(\mathcal{Z},\mathbb{B})$. It is a locally convex topology that is engendered by the family of the Riesz semi-norms π_ψ with ψ belonging to the subset $\mathbb{B}$ and π_ψ is defined as follows: $\pi_\psi(x) = |\psi|(|x|)$.

Earlier we provided the dual topology (of a Hausdorff locally convex Riesz space $(\mathcal{Z},\xi)$) with the strong topology $\gamma(\mathcal{Z}',\mathcal{Z})$. The bidual $\mathcal{Z}''$ of $\mathcal{Z}$ is the dual topology of the Hausdorff locally convex space $(\mathcal{Z}', \gamma(\mathcal{Z}', \mathcal{Z}))$ where the strong topology $\gamma(\mathcal{Z}', \mathcal{Z})$ on $\mathcal{Z}'$ is engendered by the collection of semi-norms $\{\pi_B : B \in \mathcal{F}\}$ with $\mathcal{F}$ the collection of all ξ - bounded subsets of $\mathcal{Z}$. For $(\mathcal{Z}, \xi))$ a Hausdorff locally convex - solid Riesz space, the strong topology $\gamma(\mathcal{Z}', \mathcal{Z})$ is engendered by considering the collection of all ξ - bounded solid subsets of $\mathcal{Z}$ instead of the ξ - bounded subsets of $\mathcal{Z}$. More precisely,$\gamma(\mathcal{Z}', \mathcal{Z})$ is engendered by the family of the Riesz semi-norms $\{\pi_B : B \in \mathcal{F}\}$ with $\mathcal{F}$ a collection of all ξ - bounded solid subsets of $\mathcal{Z}$. By so doing, one obtains a Hausdorff locally convex-solid topology on $\mathcal{Z}'$ such that $\mathcal{Z}'' = (\mathcal{Z}', \gamma(\mathcal{Z}', \mathcal{Z}))$. It follows from Luxemburg's theorem that $\mathcal{Z}''$ is an ideal of $\mathcal{Z}'^{\sim}$ and the natural embedding of $\mathcal{Z}$ into its bidual $\mathcal{Z}''$ assigning each element x of $\mathcal{Z}$ to an element $x^{\sim}$ of $\mathcal{Z}''$ is such that $x^{\sim}(\psi) = \psi(x)$ with $\psi \in \mathcal{Z}'$. The embedding $x \to x^{\sim}$ is an algebraic isomorphism that is also an isomorphism. One need to observe that for a Riesz space equipped with norm, the strong topology $\gamma(\mathcal{Z}', \mathcal{Z})$ on $\mathcal{Z}'$ is engendered by the Riesz norm in a way that the natural embedding of $\mathcal{Z}$ into $\mathcal{Z}''$ is norm preserving. This is one of the characteristic we will exploit in this analysis. We exports the analysis from the Riesz commodity space $\mathcal{Z}$ to its bidual $\mathcal{Z}''$ so as to take advantage of the properties offered by the structure (environment) of $\mathcal{Z}''$. We thereafter transfer back the results to the original commodity space.

In a complete Riesz space (Banach Lattice) where the norm is generated by an inner product, the Riesz representation theorem help identify the Riesz space $\mathcal{Z}$ with its topological dual $\mathcal{Z}'$. This theorem states that the mapping: $\mathcal{Z} \to \mathcal{Z}'$ that assign to each element y in $\mathcal{Z}$, the real or complex number $\psi_x(y) =< y, x >$ is an isometry and thus identify $\mathcal{L}$ with $\mathcal{Z}'$.

Definition 5.28. For a Riesz norm π and a real number p belonging to the interval $1 < p < \infty$, the norm π is termed p - additive if for all elements u, v of the Riesz space $\mathcal{Z}$, satisfying infimum of $(u, v) equal 0$ we have $\pi^p(u + v) = \pi^p(u) + \pi^p(v)$. In the event the space $\mathcal{Z}$ is a Banach space then for a real number p belonging to the interval $1 \leq p \leq \infty$, $\mathcal{Z}$ is termed an abstract $\mathcal{Z}_p$ space. When the value of the real number p is equal to one, we have an abstract $\mathcal{Z}_1$ space or an $\mathcal{AL}$ space. For p equal to infinity, we talk about an abstract $\mathcal{M}$ - space or an $\mathcal{AM}$ space.

Definition 5.29. A Banach Lattice $\mathcal{Z}$ is said to be an $\mathcal{AL}$ - space whenever $||x + y|| = ||x|| + ||y||$ for all $x, y \in \mathcal{Z}$. When $||x \vee y|| = ||x|| \vee ||y||$ for every $x, y \in \mathcal{Z}$, the Banach Lattice $\mathcal{Z}$ will said to be an $\mathcal{AM}$ space.

AM - spaces and *AL* - spaces are mutually dual in the sense that in a Riesz space $\mathcal{Z}$ if the dual topological $\mathcal{Z}'$ of $\mathcal{Z}$ is an $\mathcal{M}$ - space then the Riesz space is an $\mathcal{L}$ - space. Likewise, if the dual topological $\mathcal{Z}'$ of the Riesz space $\mathcal{Z}$ is an $\mathcal{L}$ - space then the Riesz space is an $\mathcal{M}$ - space.

An $\mathcal{AM}$ space with unit is an $\mathcal{M}$ - Riesz space with an order unit element say e. In this case, one can defines a Riesz norm by considering the norm $||x||_\infty$ as the infimum of the set of strictly positive elements $\{\lambda > 0\}$ for which $|x| \leq \lambda.e$

For an arbitrary measure space $(\mathcal{X}, \Sigma, \mu)$, $\mathcal{L}_p(\mathcal{X}, \mu)$ is an abstract $\mathcal{L}_p$ - space. The set $\mathcal{C}(X)$ of continuous functions on a compact topological space X is a typical $\mathcal{AM}$ space with order unit.

Kakutami - Bohnenblust and Skrein in [3]showed that for a Banach lattice $\mathcal{L}$ to be an $\mathcal{AM}$ space with unit e, it must be a lattice isometric to some $\mathcal{C}(X)$ - space in a manner that the unit e is identified with a constant function 1 on X(up to homeomorphism). A normed Riesz space $\mathcal{Z}$ is an $\mathcal{M}$ space if and only if it is a lattice isometric to a Riesz subspace of a $\mathcal{C}(X)$ - space.

Every abstract $\mathcal{Z}_p$ - space is lattice isometric to a concrete Banach lattice $\mathcal{Z}_p(X, \Sigma, \mu)$. We shall intensely make use of these information in the setting of the equilibrium's model of the Riesz dual system $\langle \mathcal{Z}_\Omega, \mathcal{Z}'_\Omega \rangle$.

5.5 Riesz Dual Pair

Definition 5.30 (Riesz Pairs)**.** A dual pair $\langle \mathcal{Z}, \mathcal{Z}' \rangle$ of Riesz spaces is termed Riesz dual pair if the dual topological $\mathcal{Z}'$ is an ideal of the order dual $\mathcal{Z}^\sim$

5.5.1 Examples of Riesz dual pair.

Bewley (1972) [7] introduced the dual pair $\langle \mathcal{Z}_\infty(\delta), \mathcal{Z}_1(\delta) \rangle$. In this specification, the universe of commodity is indicated by $\mathcal{Z}_\infty(\delta)$ while the universe of price is indicated by $\mathcal{Z}_1(\delta)$, and δ is a σ-finite measure on the underlying space. *Victor Filipe Martins Da Rocha* [15] used two specifications of the Riesz dual system:

(i)The first specification considers $\left\langle \mathcal{L}_\infty(R_0^+, T, \zeta), \mathcal{Z}_1(R_0^+, T, \zeta) \right\rangle$. In this specification, Martins Da Rocha consider ζ as the Lebesgue measure, the set T as the admissible coalitions of consumers.

(ii) The second specification considers the commodity space as the random measures $M(V)$ on a metric compact space V and the price space as the set $C(V)$ of continuous real functions on V. In this case, the natural duality pairing can be specified as $\langle q, y \rangle = \int_V q(v)\, dy(v)$. Each point of V being an entire description of the types of some physical goods.

Jones, and Ostroy and Zame, assumed that all goods are perfectly divisible. If f is an element of the space price *i.e.*, a continuous real function on V and v an element of V then $f(v)$ can be interpreted as a unit value of a good for which the type is v

5.5.2 Exchange Economies under Riesz dual pair.

In this presentation, an economic barter is considered, a pure exchange where economic actors have initial potential (endowments) of goods that they will bring to the market and trade with each others. Production is therefore exogenous and it is included in the initials resources (endowments). Since endowments and preferences are the primitives of the economic barter, we shall write:$\mathcal{E} = \{\langle \mathcal{Z}, \mathcal{Z}' \rangle, (\mathcal{X}_1, \succeq_1, w_1), ..., (\mathcal{X}_q, \succeq_q, w_q)$ where as before the duality between the price and the goods that are sold in the market is expressed by the dual pair $\langle \mathcal{Z}, \mathcal{Z}' \rangle$. Here $\mathcal{Z}$ is the universe of commodity and $\mathcal{Z}'$ is the space of price. The economy is populated by q economic actors, each of them being characterized by their needs, their resources and their tastes (preferences). Economic actor's needs are indicated by a non-empty Riesz subspace $\mathcal{S}_i$ of the space of commodity $\mathcal{Z}$ of the economy. Individual's potential are described by points ω_i and individual's taste by the total pre-order, $\succeq_i$, on the consumption set $\mathcal{S}_i$. The dual pair $\langle \mathcal{Z}, \mathcal{Z}' \rangle$ is the socle (system) of the economy. In their dynamic, economic actors are solely driven by their tastes and by the ruling prices. Hence all the equilibrium concepts will be specified relatively to the dual pair system $\langle \mathcal{Z}, \mathcal{Z}' \rangle$.

We therefore designate an economic barter as a *Riesz economy* whenever the duality between sold goods and price in the market is indicated by a Riesz pair $\langle \mathcal{Z}, \mathcal{Z}' \rangle$ and each consumer's consumption set is a positive cone $\mathcal{Z}_+$. The duality indicated by a lattice pair, shall be termed a *lattice economy.* Likewise, for the space of commodity specified by a Riesz space, the dual pairing goods-price economy $\langle \mathcal{Z}, \mathcal{S}' \rangle$ shall be called a *Riesz commodity economy.* Economic barter is termed a *convex economy* whenever consumer's taste as well as their consumption sets are convex. A convex Riesz economy is called finite dimensional if its Riesz pair is described by $\langle \mathbb{R}^l, \mathbb{R}^l \rangle$ and its total potential are strictly positive. The size of the total potential $\Omega = \sum_{g=1}^{q} \omega_g$ determines the dimension of the economy. We are mainly concerned with the way in which the economic barter redistributes the total potential to each of the economic actors. Economic barter have a decisive advantage in that they help illustrate the dynamic(mechanism) of that redistribution. We claim that the outcome of that redistribution (for a competitive mechanism) is both a competitive equilibrium and a Pareto optimum. The initial values (parameters) of the process are the consumer's potential, consumer's tastes and the good's prices. Equilibrium is the last step in the iteration of the process. At each iteration, economic barter engender an allowance *i.e.*, an q-uple $(y_1, ..., y_q)$ of vectors such that $y_i \in \mathcal{S}_i$ for each economic actors and $\sum_{g=1}^{q} y_g = \mathcal{E}$. This market redistribution (outcome) may or not be the one desired by society. In the course of the process, market mechanism (the invisible hand) will result by producing a specific price to which every economic actors will consider as given (price-adaptation). This state of economic barter is about a prices and allowances such that each economic actors maximizes its satisfaction and the market of each goods satisfies the clearing conditions (the demand of each good equal its supply). This level of the economy is what is called equilibrium or equilibrium of Walras. In year 1906, Vilfredo Pareto [25] looked at that equilibrium (Walrasian) and saw it being performed not only by market mechanism but in addition by human attitude. He realized that an economic barter can reach the same equilibrium if each market participant accepts to give up a portion of what their need in order to get what they desire the most. Pareto thus stated the condition of a possible optimum. He defined an equilibrium as a state of an economy where it is good to make someone better off without making anyone else worse off. He called this point of equilibrium an optimum (Pareto optimum). Another mathematician, Francis Edgeworth (1881), saw an equilibrium of Walras as a social dimension. For him, cooperation among economic participants can poor social dimension on an equilibrium of Walras. Economics literature call it cooperative equilibrium. Even though there is always a price to be paid (a sacrifice), Pareto and Edgeworth equilibria concepts are defined without reference to prices. Kenneth Arrow and Gerard Debreu (1950) mathematically stated the equivalence between competitive equilibrium and Pareto optimum. Economic literature call it the welfare theorem. Using Riesz theory, we present the infinite dimensional version of the welfare theorem: the equivalence between competitive equilibrium and Pareto allocation. In addition, we show the capacity of Riesz theory to affine what Edgeworth has perceived at 1881 when he stated the limit theorem. That is at the infinity there is only one optimal allocations: Walrasian allocations, Edgeworth allocations, Pareto allocations, approximate quasi-equilibrium and extended Walrasian equilibrium are all equals at the infinity.

5.6 Topological Analysis of Edgeworth box.

An economy may admit many equilibrium states. Not all of them are desirable for society. A criterion is thus needed to ensure which equilibrium allocations is likely to lead society to a social optimum(equity). The benchmark has been provided by Pareto(1906) and is still relevant in our present days of globalization. An allowance is termed Pareto efficient, or Pareto optimal, if there is no other feasible allowance in the Edgeworth economy for which none of economic participants is at least as well off and no one is strictly better off. The locus of points that are Pareto optimal given tastes and potential called a Pareto set. Here we use tools provided by Riesz theory to conduct topological analysis of the Edgeworth box, a territory containing all of the outcomes engendered by an economic barter, so as to locate the Pareto set. To quote Mas Colell et all 1995, the Edgeworth box is remarkably powerful, there is virtually no phenomena or properties of general economic equilibrium that cannot be depicted in it. It is needed a mathematical tool, like locally convex-solid Riesz spaces. Any point inside this box represents a feasible allocation. Among these allowances, we have to locate the one likely to bring highest satisfaction to society. Classical economic analysis solves this problem by setting the Lagrangian. Riesz theory sees the Edgeworth box as an order interval, we can therefore use topological structure (locally convex-solid topology) to depict the greatest element of the set of allowances. To depict the Pareto set, we consider the set of feasible allowances for the given economic barter. We let $\mathcal{B} = \{(y_1, ..., y_q) \in S^q : y_i \in S_i\}$ with the requirement that the totality of allowances be equal to the total potential of the economy. That is $\sum_{i=1}^{q} y_i = \mathcal{E}$ where S_i

is the consumer's consumption set. The condition $\sum_{i=1}^{q} y_i = \mathcal{E}$ is the non-wasteful, the feasible condition, *i.e.* the total amount of all commodities in the economy must not be greater than the total capacity (endowment) of the economy. In Riesz theory, the space of commodity of the economy is model as an order interval $[0, \Omega] = \{y \in S : 0 \leq y \leq \Omega\}$. In this case, the space of commodity is termed *the Edgeworth box*. Now, following Pareto's thinking, an outcome of an economic barter will be termed:

1. *Pareto efficient (or Pareto optimal)* if the there is no others allocations that are better or equal than the one provided by the economic barter, while there are some which are far better.

2. *weakly Pareto efficient* if there is no allocation better than the one provided by the economic barter.

3. *Individually rational* if the allocations received by each economic participants is better than his (her) initial endowment. That is $y_i \succeq_i e_i$ for each i.

We shall indicated by $\mathcal{B}_r$ the collection of all individually rational allocations, *i.e.*$\mathcal{B}_r = \{(y_1, ..., y_q) \in \mathcal{B} : y_i \succ_i e_i\}$.

Likewise, the collection of weakly Pareto efficient allocations is indicated by $\mathcal{B}_{wp} = \{(y_1, ..., y_q) \in \mathcal{B}\}$ such that there is no other allocation in the collection $\mathcal{B}$ that are better than the y_i allocations and this for each i.

As a first step, the topological properties of the Edgeworth box are used to show the existence of a maximal element of the set $\mathcal{B}$ which in fact, is a Pareto allocation.

5.6.1 Existence of a Pareto allocation.

Theorem 5.31. *For an economy in which the space of commodity is modeled as a Riesz space whilst consumer's consumption set are assumed to belong to the cone that is positive, and with a weakly closed order interval* $[0, \Omega]$ *(Edgeworth box) and being populated by a minimum of two economic actors. Then the collection* $\mathcal{B}$ *of all allowances engendered by the economic barter is weakly compact if and only if the Edgeworth box* $[0, \Omega]$ *is weakly compact.*

Proof. In fact, since the initial endowment e_i of each consumers i belong to the commodity set $\mathcal{Z}$, it follows that the allocations $(e_1, ..., e_q)$ belong to $\mathcal{Z}^q$. Now, by definition of the set $\mathcal{B}$, we have that $\mathcal{B} \subset [0, E]^q$ (1) where $\sum_{i=1}^{q} e_i = \mathcal{E}$ the total endowment of the economy. Besides, since $\delta(\mathcal{Z}^q, (S')^q) = \prod_{i=1}^{q} \delta(\mathcal{Z}, \mathcal{S}')$, we have that $\mathcal{B}$ is a subset of $\mathcal{Z}^q$ that is $\delta(\mathcal{Z}^q, (S')^q)$ - closed. In case the Edgeworth is $\delta(\mathcal{Z}, \mathcal{S}')$ - compact, One can assert that the order interval which is the space of commodity $[0, E]^q$ is $\delta(\mathcal{Z}^q, (S')^q)$ - compact and so the subset $\mathcal{B}$ of $[0, e]^q$ which is $\delta(\mathcal{Z}^q, (S')^q)$ closed, is also $\delta(\mathcal{Z}^q, (S')^q)$ - compact. □*Now we show that a Pareto efficient allocations indeed exist.*

Theorem 5.32. *For an economy in which the space of commodity is modeled as a Riesz space whilst consumer's consumption set are assumed to belong to the cone that is positive, and with a weakly compact order interval* $[0, \Omega]$ *(Edgeworth box) and consumer's tastes are assumed to be weakly upper semi-continuous. Then one can assert the existence of allowances that are individually rational Pareto efficient.*

Proof. The first step in the prove of this theorem consist in defining an equivalence relation on the set $\mathcal{B}_r$(the weakly compact set of all individually rational allocation) $i.e(\mathcal{B}_r/\sim)$. Thereafter one have to apply the Zorn's lemma on the set of all equivalence classes $(\mathcal{B}_r/\sim, \succeq)$ embody with the partially ordered relation, so as to find the maximal element. As it is well known Zorn's lemma involve the existence of a net that might converge to some limit point, this should not be a problem since the continuity and the convergence conditions are guaranty by the assumption. The maximal element is therefore a Pareto efficient allocation. □

5.7 Equivalence between Competitive equilibrium and Pareto allocation

Using the Edgeworth box, Pareto showed graphically that every competitive equilibrium allocations are optimal. Later on in 1952, Gerard Debreu and Kenneth Arrow [8] independently and simultaneously used convex analysis (specially the separation theorem for convex sets) and mathematically established the equivalence between competitive allocations and Pareto allocations. This equivalence is nowadays known in the literature as the welfare theorem. Here we use Riesz theory to present the infinite dimensional version of the welfare theorem. However before doing so, it is worth remembering that competitive market mechanism tries to achieve an allocation by setting up an equilibrium price system under which the individual allocation of consumption and production are performed, whereas Pareto optimum are effected by means of appropriate reallocation of initial endowments. Pareto optimum is defined without reference to a price system. It is a free price allocation. Therefore, a move from a free price allocation to a competitive allocation (a price supporting allocation) requires the existence of a supporting price.

5.7.1 First welfare Theorem

The first welfare theorem poses a simple question: can we obtain any Pareto optimum from market mechanism? The answer is affirmative under the conditions provided by the following theorem.

Theorem 5.33 (First Welfare Theorem)**.** *Consider an economic barter in which economic participant's characteristics are such that their consumption sets are cones and* *their tastes are lower semi-continuous relatively to some topology that is assumed linear. Under the above condition, each allowance of Walras can be considered as a Pareto efficient allowance.*

Efficiency is when there is no need for government intervention (regulation) for a competitive mechanism to engender optimum allowances. Competitive markets naturally have everything that it is needed to bring optimum allowances to the society. All is about showing that every allowance that belong to the collection $\mathcal{B}_{wp}$ defined above, of Pareto efficient allowance can be attached to a vector price capable of preserving their dominance in the consumer's budget set. To do this, one need a version of the separation hyper-plan theorem appropriate to unlimited (algebraic and topological) structure. A.Mass Colell solved this problem by suggesting the condition of uniform properness. In fact, as it is known in finite dimensional economic analysis, Pareto equilibrium is reached when the marginal rate of substitution of economic agents are equals. Uniform properness is the infinite dimensional version of this condition. The cone condition was for the first time introduced in mathematical literature by Viklee. A.Mas-Colell (1986) [23] extended the concept in economics. Since then the concept has been subject of intense use economics. This notion allowed significant progress in general equilibrium, in particular for infinite dimensional Riesz spaces with empty interiors cones. As we shall see, this notion allow one to obtain the new form of the theorem relative to society prosperity in a space of higher measurement, and further that notion permit the existence of the quasi-equilibria.

Definition 5.34 (Properness)**.** Consider a Riesz space $\mathcal{Z}$ that possess a linear topology ξ, a strictly positive vector l, and a ξ- open neighborhood T of zero. Assume that for a commodity good u of the cone positive of $\mathcal{Z}$ and for a positive real β greater than zero, one have that the commodity good $u - \beta.l + s$ remain as preferred as u lead to the conclusion that the vector u does not belong to the neighborhood $\beta.T$. In this case, one assert that the defined taste is (l, T) - proper at a point u of the cone positive $\mathcal{Z}^+$. If the fact that $u - \beta.l + s$ is better or the same as u in the positive cone $\mathcal{Z}^+$, lead to the conclusion that u does not belong to the neighborhood $\beta.T$, one can then assert that the defined taste on the positive cone $\mathcal{Z}^+$ is uniformly (l, T) - proper.

We are now in the position of providing the second theorem regarding the prosperity of the society.

5.7.2 Second welfare Theorem

The second welfare theorem state on the ability of the market system to convert any weakly Pareto efficient allowance to a Walrasian allowance. Mas-Colell and Richard provided conditions under which a weakly Pareto efficient allowance can be supported by a linear functional and thus become a Walrasian allowance. Basically the key of these conditions stem on the properness conditions of consumer's preferences, the existence of a consistent locally convex topology on the commodity space and the possibility of the consumer's preference to be represented by a monotone, continuous and quasi-concave utility functions.The following lemma and theorem highlight these conditions.

Lemma 5.35 (Mass Colell - Richard)**.** *Consider an economy populated by economic actors with monotone and convex tastes, the price space is modeled as a Riesz subspace of the order dual. Further assume the existence of a locally convex topology on the space of commodity making consumer's tastes continuous and uniformly proper. Under theses assumptions, one can assert the existence of a non empty convex subset B of the cone positive in the dual topological $\mathcal{Z}'_+$ that is $\delta(\mathcal{Z}', \mathcal{Z})$-compact and which embody linear functional q, termed price in a way that it assigns each weakly Pareto efficient allowance u of the cone positive to a non empty convex subset $\kappa(u)$ of a cone positive of the dual topological $\mathcal{Z}'_+$. These linear functions q (price functions) are such that they reach their maximum at the point u (the Pareto efficient allowance), over the collection of all allocations engender by the economic barter. They are price supporting of these Pareto allowances.*

Proof. The proof of this lemma can be found in [23] and [3]$p.230$

Theorem 5.36 (The Mas-Colell Second Welfare Theorem)**.** *Consider a lattice economy populated by economic actors whose tastes are monotone, convex, uniformly Ω - proper and continuous for a consistent locally convex topology ζ on the space of commodity $\mathcal{Z}$. Under these assumptions, each weakly Pareto efficient allowance can become a competitive allowance.*

Proof. In fact let us consider the following allowance $(s_1, s_2, ..., s_m)$ and assume it to be a weakly Pareto efficient. Since by assumption consumer's tastes are monotone and continuous for a locally convex topology ξ on the commodity space $\mathcal{Z}$, there might exists on the dual space $\mathcal{Z}'_+$ of the commodity space $\mathcal{Z}$ a nonempty convex and $\delta(\mathcal{Z}', \mathcal{Z})$-compact subset of nonzero linear functional (prices vector) q attaining their maximum on the set of weakly Pareto efficient allowance $(s_1, s_2, ..., s_m)$. From the lattice structure of the economy, one can infer the existence of the supremum of these prices $\zeta = \bigvee_{i=1}^{m} q_i > 0$ and for each weakly Pareto allowance efficient $(s_1, s_2, ..., s_m)$ we have $q * s > 0$. Likewise the market value of the endowment Ω of the economy is such that $\zeta.\Omega = \sum_{i=m} \zeta.s_i$ is greater or equal to $\sum_{i=m} q_i.s_i = \zeta.\Omega = q.s > 0$. It follows that each allowance s_i is such that $\zeta.s_i = q_i.s_i$; clearly the assumed weakly Pareto efficient $(s_1, s_2, ..., s_m)$ happen to be supported by the prices vector q_i and as well by their supremum ζ. To convince ourself, consider the no worse than allowance $y \geq s_i$ as well as its market value $\zeta.y \geq q_i.yq_i = \zeta.s_i$. This really show that the assumed weakly Pareto is supported by the prices q and therefore it is an allowance of Walras. □

5.8 Equilibrium Analysis

5.8.1 Construction of the Riesz dual system $\langle \mathcal{Z}_\Omega, \mathcal{Z}'_\Omega \rangle$

Definition 5.37. A semi-norm ζ on a space of Riesz $\mathcal{Z}$ is termed a Riesz semi-norm if for two elements s and z of $\mathcal{Z}$ such that $|s| \leq |z|$ in $\mathcal{Z}$ implies $\zeta(s) \leq \zeta(z)$. Whenever the Riesz semi-norm fill the condition of being a norm, the Riesz semi-norm will be termed a Riesz norm. A linear functional (a semi-norm) on the space of Riesz $\mathcal{Z}$ is termed p-additive with $p \in [1, \infty]$ if the image $|\zeta(s+z)|^p$ is equal to $|\zeta(s)|^p + |\zeta(z)|^p$ and this for each element s, z of $\mathcal{Z}^+$ for which the infimum is zero.

Definition 5.38. An ideal of the space of the Riesz $\mathcal{Z}$ that is engendered by only one element $\{|d|\}$ is termed a principal ideal. Here $\mathcal{Z}_d$ is the set of elements h of $\mathcal{Z}$ for which there exists a positive real η such that $|h|$ is less or equal than $\eta|d|$. An element $e > 0$ of the space of the Riesz is termed an order unit (or a strong unit) if $\mathcal{Z}_e = \mathcal{Z}$.

We define on the principal ideal $\mathcal{Z}_\Omega$ (generated by the entire potential of the economy Ω) the real-valued function $||.||_\infty$, by considering for each commodity good u, $||u||_\infty$ as being the infimum of the set of the positive real ζ for which $|u|$ is less or equal to $\zeta.x$. In this case, the space of the Riesz $(\mathcal{Z}_\Omega, ||.||_\infty)$ is a Riesz semi-norm. In case the commodity space $\mathcal{Z}$ is a Banach lattice then $(\mathcal{Z}_\Omega, ||.||_\infty)$ is an space AM with unit Ω. We then obtain the commodity space $\mathcal{Z}_\Omega$ which in fact is a subspace of the universe of the commodity $\mathcal{Z}$. Its dual space is obtained using the Kakutani-Bohnenblust and Krein's theorem which assert the isometry between $(\mathcal{Z}_\Omega, ||.||_\infty)$ and some $C(\Theta)$-space where Θ is an Hausdorff and compact. The norm dual $(\mathcal{Z}'_\Omega, ||.||_\infty)$ of $(\mathcal{Z}_\Omega, ||.||_\infty)$ is known to be an space AL and a lattice isometric to some concrete $\mathcal{Z}_1(\delta)$-space.

5.8.2 Properties of a prices supporting optimal allocations.

Since the Edgeworth $[0, \Omega]$ lies entirely in the Riesz subspace $\mathcal{Z}_\Omega$, one observe that the essentials of the equilibrium analysis can be conducted in the Riesz system: $\langle \mathcal{Z}_\Omega, \mathcal{Z}'_\Omega \rangle$. All of the equilibrium concepts shall therefore be defined with respect to that system. All questions concerning optimality (whether the given allocation is Pareto optimal or an Edgeworth equilibrium or it belongs to the core allocation) will be addressed in the framework $\langle \mathcal{Z}_\Omega, \mathcal{Z}'_\Omega \rangle$.

Later in this chapter, we will need to relate a free optimal allowance with a price supporting allowance. Properties of price supporting an allowance shall guide in such a process. We shall further observe that most of these properties are about a linear functional on the order interval $[0, \Omega]$ as the equilibrium analysis is moved from the original economy to the Riesz dual system $< \mathcal{Z}_\Omega, \mathcal{Z}'_\Omega >$.

Definition 5.39 (Price supporting optimal allocation)**.** Consider a Riesz space of commodity $\mathcal{Z}$ and a Riesz subspace vector T of $\mathcal{Z}$. Now define a linear functional q from T into $\mathbb{R}$ assigning an allowance u in T that is engendered by an economic barter to its market value $q.u$. If a commodity good v that belong to the intersection of the space of commodity and the subspace T is such that v is preferred to the commodity good u and if the linear functional defined as above preserve that dominance then one can say that q support the allowance v.

Aliprantis-Brown and Burkinshaw in [3] considered a Riesz economy in which the total potential of the economy is a desired good for every consumer, the consumption set of the economic actors is modelled as the positive cone of the commodity space $\mathcal{Z}$ and there exists a locally convex-solid topology ξ on the universe of commodity $\mathcal{Z}$, rendering economic actor's tastes lower semi-continuous. They showed that under these assumptions any prices functions q defined on the principal ideal $\mathcal{Z}_\Omega$ and supporting the dominance of the allowance $(s_1, ...s_q)$ over the budget set, is made continuous on the order interval $[0, \Omega]$ by the locally convex topology ξ. If further the order interval is weakly compact then the defined linear functional is order continuous on $[0, \Omega]$.

Theorem 5.40 (N.C.Yannelis and W.R.Zama)**.** *If the consumption set of an economy is modelled as a positive cone $\mathcal{Z}^+$ whilst the space of the commodity $\mathcal{Z}$ possess a locally convex-solid topology ξ making uniformly ξ-proper each consumer's taste defined on the Riesz subspace $\mathcal{Z}_\Omega$ then any linear functional q that assigns each commodity good u of $\mathcal{Z}_\Omega$ to its market value $q * u$ in a way of preserving the dominance of an optimal allowance over the budget set, possess the ξ - continuity property on the principal ideal $\mathcal{Z}_\Omega$.*

Proof. Let $(s_1, ...s_m)$ be an allowance produced by a Riesz economy endowed with a locally convex-solid ξ consistent topology on the commodity space $\mathcal{Z}$. Assume further that the nonzero linear functional q defined on the dual space $\mathcal{Z}'_\Omega$ support the allowance $(s_1, ...s_m)$.It follows then from the result mentioned above the linear functional q is ξ - continuous on the Edgeworth box $[0, \Omega] \subset \mathcal{Z}_\Omega$. In addition the linear functional q is bounded by a multiple of Ω and Ω can therefore be said to be a vector of uniform properness. As such, one can infer the existence of a convex - solid ξ neighborhood V of zero in a way that for a real $\zeta > 0$ and a multiple $\zeta.w$ we have that $u - \zeta.w + s \succeq u$ in $\mathcal{Z}_\Omega$, implying that $s \notin \zeta.V$. However, since the Edgeworth box $[0, \Omega]$ incorporates all the economic activities of a Riesz commodity, the commodity good s might belong to the Edgeworth box $[0, \Omega]$ and so $0 \leq s \leq \Omega = \sum_{i=1}^m u_i$ and from the Riesz decomposition we can express s as $s = \sum i = 1^m s_i$ where for each i, s_i belong to $[0, u_i]$. Now defining a Riesz semi-norm $\Pi : \mathcal{Z} \mapsto \mathbb{R}$ defined from the commodity space $\mathcal{Z}$ to the real value in way that for each commodity good v we have $\Pi(v) = \inf\{\eta > 0 : v \in \eta.V\}$.It follows that the defined Riesz semi-norm Π is ξ - continuous on the commodity space $\mathcal{Z}$ and for each $i, s_i \in \mathcal{Z}$ imply $\Pi(s_i) \in \mathbb{R}$. Since the vector Ω happen to be a vector of uniform properness, we might have for a fixed real $\epsilon > 0$ and $\P(s_i) = \delta_i \ v_i = u_i + (\delta_i + \epsilon).w - s_i$ is greater or equal than zero. From that it clear;y appear that $u_i = v_i - (\delta_i + \epsilon).w + s_i$ is greater or equal than zero. In this event then if the commodity good $u_i = v_i - (\delta_i + \epsilon).w + s_i$ is as good as the commodity u_i then by uniform ξ - properness of the consumer's tastes on $\mathcal{Z}_\Omega$ we might have that $s_i \notin (\delta_i + \epsilon).V$ which lead to a contradiction since $\P(s_i) = \delta_i$. Consequently the commodity good v_i is strictly preferred to the commodity good $v_i - (\delta_i + \epsilon).w + s_i = u_i$ and since the linear functional q is a price supporting the allowance $(s_1, ...s_m)$ on $\mathcal{Z}_\Omega$, we shall have $q.v_i \geq q.u_i = q.[v_i - (\delta_i + \epsilon).w + s_i] = q.v_i - (\delta_i + \epsilon)q.w + q.s_i$.

Still in the aim of showing the ξ - continuous of the the linear functional q, we show that the Riesz semi-norm Π dominate the linear functional q. For each i and for any real $\epsilon > 0$ we have $q.s_i \leq (\delta_i + \epsilon)q.w$ and so $q.s_i \leq \delta_i.q.w = (q.w)\Pi(s_i)$. This result for all commodity good s in the interval $[0, w]$ by $q.s$ being equal to $\sum_{i=1}^{m} q.s_i \leq [\sum i = 1^m q.w]\Pi(s) = m(q.w)\Pi(s)$. As any commodity good $s \in \mathcal{Z}_\Omega$ is such that $|s| \leq \lambda.w$, we do have $|q.s| \leq q.|s| = \lambda.q.(\frac{1}{\lambda}|s|) = m(q.w)\Pi(s)$. We have thus showed that the price q is dominated by the semi-norm Π and since q is define on the order interval $[0, w]$, q is itself ξ - continuous. $\square$

5.8.3 Optimal Allocations

Definition 5.41 (Walrasian equilibrium and quasi-equilibrium)**.** An allocation $(u_1, ..., u_q)$ is termed:

- a *competitive equilibrium* if firstly there is a market mechanism susceptible of producing a vector price, that we call an equilibrium price. And secondly, all economic actors take this vector price as given, they adjust to that equilibrium price. Lastly then the equilibrium price engender an allowance that respond to economic actors expectation. Then we say the economy is in the situation of equilibrium.

- An allowance $(u_1, ..., u_q)$ is termed a *quasi-equilibrium* if one have a linear functional q in a way that a market value of any allowance $(v_1, ..., v_q)$ that is preferred to the allowance $(u_1, ..., u_q)$ might exceed the buying power of the economic actor.

Quasi-equilibrium

Here we provide conditions under which an exchange economy can admit an equilibrium (quasi-equilibrium in this case). Generally speaking, most equilibrium existence conditions stem from consumers behavior being expressed by the topological properties of monotone and quasi-concave preferences. These are the well known equilibrium conditions in classical analysis under linear topologies. What is new here is the fact that these properties are now depicted by the locally convex-solid topology.

Mass-Colell in [23] has established that an economy is such that with a consumption set made of a cone positive $\mathcal{Z}^+$ and a commodity space $\mathcal{Z}$ possessing a locally convex-solid topology ξ that render uniformly ξ-proper each consumer's taste defined on the Riesz subspace: the principal ideal $\mathcal{Z}_\Omega$ then such an economy reach an equilibrium that is a quasi-equilibrium in the following event:

1. the numerical function representing consumer's tastes in the economy is both quasi-concave and monotone, and thus ensure that economic actors in the economy are able to maximize their utility and the economy is utility closed.

2. the numerical function (utility function) defined on the consumption set (as in the assumption) is made continuous by the locally convex-solid topology ξ defined on the space of the commodity

3. the space of the commodity $\mathcal{Z}$ admit a locally convex-solid topology ξ that render uniformly Ω-proper each consumer's preference

Using the lattice structure of the Riesz space S.F.Richard and Mass-Colell in [24] showed that a lattice economy modeled in a way that the consumption set of economic actors is a positive cone whilst its price space is a subspace of the order dual $\mathcal{Z} \sim$, any linear function defined on the subspace $\mathcal{Z}_\Omega$ is order continuous specially when the order interval $[0, \Omega]$ is weakly compact. Such an economy is said to possess an equilibrium in the following event:

1. the numerical function representing consumer's tastes in the economy (the utility function) can reach a maximum over the consumption set. That is it must both be quasi-concave and monotone, and thus ensure that economic actors in the economy are able to maximize their utility.

2. the space of the commodity $\mathcal{Z}$ admit a locally convex-solid topology ξ that render uniformly Ω-proper each consumer's taste and the utility function defined on the consumption set (cone positive) are made continuous by that topology.

An economy experience many equilibrium state, all of them are not alike. The most desirable outcome from a given game(exchange economy) that is expected by an economic participant is the individual Pareto efficient allocation, simply because that allocation provides individual with more than his initial endowment. Globalisation seems to raise many questions. Apart from security issue, the question on everybody minds is whether it is really a remedy to the social global inequality. Is the actual global economic situation the final step of equilibrium or is it just one of the intermediary equilibrium state? So far,there is no sign of it providing the majority of global population with individual Pareto allocation. As such, it may be wise to investigate all possible equilibrium states of the economy in order to see if whether they belong to the same equivalence class in term of welfare. In this case, decision making of the society can see which outcome is possible from the resources in their disposal. One can thus decide whether it is appropriate to go back to a protectionism policy or to wait and see how globalisation will pound all its mechanism.

5.9 Optimality Analysis in $\langle \mathcal{L}_w, \mathcal{L}'_w \rangle$'s framework.

Since there is a way to identify the dimension of an economy from its total potential, Ω, we are then admit to say that Ω is a unit element of the commodity space. That is the principal ideal $\mathcal{Z}_\Omega$ made of the commodity good u of the space of commodity $\mathcal{Z}$ for which there exists a positive real η in a way that $|u|$ is less or equal to $\eta \cdot \Omega$. The principal ideal $\mathcal{Z}_\Omega$ is thus identical to the space of commodity $\mathcal{Z}$. Besides, we knows that the Edgeworth box $[0, \Omega]$ is a subset of the interval $[-\Omega, \Omega]$ that possess at least one element. By saying that, i mean exactly that the collection of realizable allocations of the economic barter $\mathcal{B}$ belongs to the ideal $\mathcal{Z}_\Omega$. As such, we can conduct meaningful optimality analysis in term of finding optimal allowances in $\mathcal{Z}_\Omega$ just as we derived the existence of individually rational allocations from the weak compactness of the Edgeworth box.

5.9.1 Model Setting

As a first step, we equip the ideal $\mathcal{Z}_\Omega$ with the lattice norm $||.||_\infty$. The defined norm engender on $\mathcal{Z}_\Omega$ a topology that is locally convex-solid and thus making $(\mathcal{Z}_\Omega, ||.||_\infty)$ a locally convex-solid Riesz space. Earlier we showed that $(\mathcal{Z}_\Omega, ||.||_\infty)$ is a AM space (and hence a Banach lattice). Besides, it is known that a typical AM-space with an order unit is the classical Riesz space $C(X)$ where X is a compact Hausdorff topological space, endowed with the sup norm $||.||_\infty$. However, the theorem by Kakutani-Bohnenblust and Krein tell us that a Banach lattice $\mathcal{Z}$ is an AM-space with unit e if and only if $\mathcal{Z}$ is a lattice isometric to some $C(X)$- space.

By then we deduce that $(\mathcal{Z}_w, ||.||_\infty)$ is a Riesz isometric to some $C(H)$, with H a Hausdorff compact space. The duality between the AM and the AL-spaces lead one to deduce that $\mathcal{Z}'_\Omega$, the norm dual of $(\mathcal{Z}_\Omega, ||.||_\infty)$, is an AL -space which is lattice isometric to some concrete $\mathcal{L}_1(\delta)$-spaces. Now, for the ideal $\mathcal{Z}_\Omega$ to obtain economic configuration, it is needed economic actor's preferences and endowments. To help $\mathcal{Z}_\Omega$ obtain these primitives of the economy, we first consider the following obvious inequality $|x| \leq ||x||_\infty$, from which we infer that for any lattice semi-norm η on $\mathcal{Z}$ such that $\eta(x) \leq ||x||_\infty$, we have that η is $||.||_\infty$-continuous on $\mathcal{Z}_\Omega$. Moreover, the utility function is as a linear functional, a norm continuous on the original economy $\mathcal{Z}$. Its restriction to the sub-economy $\mathcal{Z}_\Omega$ is also $||.||_\infty$-continuous. We have that the utility function carries consumer's tastes and endowment into the ideal $\mathcal{Z}_w$ and therefore making it a new commodity space. Now, $(\mathcal{Z}_w, ||.||_\infty)$ being a Riesz isometric to some $C(H)$-space, possess adequate structure for a commodity space. Like wise, the isometry of $(\mathcal{Z}'_\Omega, ||.||_\infty)$ to some concrete $\mathcal{L}_1(\delta)$-spaces provides it with appropriate structure of a price space. Once again as said Mas Colell, there is virtually no phenomena or property of general equilibrium exchange economic that cannot be depicted in this framework. As such, any equilibrium analysis can profitably be conducted in $\langle \mathcal{Z}_\Omega, \mathcal{Z}'_\Omega \rangle$ just like firms use to relocate to a country with cheaper cost of production and send back the goods produced to the origin country. As we shall see, there is an advantage to conduct analysis in the Riesz pair $\langle \mathcal{Z}_\Omega, \mathcal{Z}'_\Omega \rangle$ in lieu of the original economy $\langle \mathcal{Z}, \mathcal{Z}' \rangle$.

Modeling in infinite dimensional is a bit challenging. For instance, when the space of the commodity is of infinite dimension, the compactness of the budget set is no longer guarantee. Mass-Colell(1986) used the space of utility to avoid this dimensionality problem. Likewise, Peleg and Yaari (1970) considered non transferable utility. Both utility space and non transferable are independent of the dimension of the commodity space. However, these technical problem are well solved from Edgeworth box. Since the compactness of the Edgeworth box is in most case guaranty and also the fact that $[0, \Omega] \subset \mathcal{Z}_\Omega$, the setting of the Riesz dual $\langle \mathcal{Z}_\Omega, \mathcal{Z}'{}_\Omega \rangle$ provide better alternative to this technical problem.

One more features of the ideal $\mathcal{Z}_\Omega$ is that all existing relationship between optimal allocations that holds in finite dimensional exchange economies over finite dimensional commodity spaces still holds in the $\langle \mathcal{Z}_\Omega, \mathcal{Z}'_\Omega \rangle$ setting. This fact has also been confirmed by Gretsky and Ostroy (1985). They realized that the core equivalence, the Debreu-Scarf characterization of Walrasian allocations as Edgeworth allowances, for economic barter with an infinite number of economic actors and infinite dimensional space of commodity, that holds in finite dimensional economy, also holds in $\mathcal{Z}_\Omega$. Here the Riesz pair setting $\langle \mathcal{Z}_\Omega, \mathcal{Z}'_\Omega \rangle$ is used to relate the Edgeworth equilibria to the Walrasian equilibria and the quasi equilibria. We thereafter export the results to the original economy $\langle \mathcal{Z}, \mathcal{Z}' \rangle$. By so doing, Aliprantis, Brown and Burkinshaw came to the definition of extended Walrasian equilibrium and approximate quasi-equilibrium. Recall that the Walrasian equilibrium and the quasi-equilibrium are allocations which are supported by prices while others optimal allocations like the Edgeworth equilibrium and the Pareto efficient allocation are free price allocation. So a move from a price free allocations to a price supporting allocations can not be processed the same manner than a move from a price supporting allocations to a price free allocations. The following approximate condition is a key characteristic in setting up core equivalence for equilibrium.
Consider the following allowance $(v_1, ..., v_q)$, and define two sets L_i and T_i that contain at least one element. The first set L_i contain the consumer's commodity good that are preferred to the given allocation $(v_1, ..., v_q)$ while the second set T_i contain the same commodity good as L_i does but without the individual potential of each of the economic actors. More precisely $L_i = \{x \in X_i : x \succeq_i x_i\}$ and $T_i = L_i - \omega_i = \{s \in X : s + \omega_i \succeq_i v_i\}$. Now, since the commodities good that are contained in both sets L_i and T_i are taken from the consumer's consumption set that are assumed to be convex, we can assert that both sets L_i and T_i are convex sets. From there, we deduce that the convex hull T of the union of the T_i is given by: $T = co(\bigcup_{i=1}^q T_i) = \{ \sum_{i=1}^q \eta t_i : t_i \in T_i, \eta_i \geq 0\}$ for all i and $\sum_{i=1}^q \eta_i = 1\}$.

The following result(in the form of lemma), proved first by Peleg and Yaari and renewed by Aliprantis, Brown and Burkinshaw in [3] is used in the sketching of the proof of the subsequent theorem.

Lemma 5.42. *Let $\langle \mathcal{Z}, \mathcal{Z}' \rangle$ be a Riesz economy in which the order interval $[0, \Omega]$ is weakly compact, economic actor's taste are convex and weakly upper semi-continuous, and each economic actor desire the total endowment Ω. Under these assumptions, we have that zero cannot belong to the set $\zeta.\Omega + \bar{T}$ for each real ζ greater than zero and for a given Edgeworth equilibrium allocation $(v_1, ..., v_q)$.*

Theorem 5.43. *Consider a Riesz economy,$\langle \mathcal{Z}, \mathcal{Z}' \rangle$ in which economic actor's taste are convex and its subspace of commodity $\mathcal{Z}^+_\Omega$ possess a topology that is linear and strongly monotone. Under these assumptions, an allowance $(v_1, ..., v_q)$ produced by the above economic barter $\langle \mathcal{Z}, \mathcal{Z}' \rangle$ have the following equivalent characteristics: if the generated allowance $(v_1, ..., v_q)$ is an equilibrium of Edgeworth then it is also an approximate quasi-equilibrium relative to the sub-economy $\left\langle \mathcal{Z}_\Omega, \mathcal{Z}^+_\Omega \right\rangle$, and also an equilibrium of Walras relative to sub-economy $\left\langle \mathcal{Z}_\Omega, \mathcal{Z}^+_\Omega \right\rangle$.*

As first step, we shall show that if the allowance $(v_1, ..., v_q)$ produced by an economic barter is an equilibrium of Edgeworth then it is indeed a quasi-equilibrium. For this reason, we need first to notice that if an allowance $(v_1, ..., v_q)$ is an equilibrium of Edgeworth then it will hold this characteristic in each economic barter for which the space of commodity is a subspace Riesz of $\mathcal{Z}$ that contain the principal ideal $\mathcal{Z}_\Omega$. In this case, the given allowance $(v_1, ..., v_q)$ is obviously an equilibrium of Edgeworth relatively to the paired Riesz $\langle \mathcal{Z}_\Omega, \mathcal{Z}'_\Omega \rangle$.

In addition, the linearity and the strong monotonicity of economic actor's characteristics, particularly their tastes, allow these characteristics to be preserved by the norm $||.||_\infty$ which as we know are continuous on $\mathcal{Z}'_\Omega$. Let us now consider the sets L_i containing the consumer's commodity good that are preferred to the given allowance $(v_1, ..., v_q)$ while the second set T_i contain the same commodity good as L_i does but without the individual endowment of each of the economic actors. More precisely we have $L_i = \{s \in X_i : s \succeq_i v_i\}$ and the set $T_i = L_i - \omega_i$. Now, since the commodities good that are contained in both sets L_i and T_i are taken from the consumer's consumption set that are assumed to be convex, we can assert that both sets L_i and T_i are convex sets. From there, we are allow to deduce that the convex hull T of the union of the T_i is given by: $T = co(\bigcup_{i=1}^q T_i) = \left\{ \sum_{i=1}^q \eta t_i : t_i \in T_i, \eta_i \geq 0 \ \forall i and \sum_{i=1}^q \eta_i = 1 \right\}$.

Assume that an integer $n \in N$ is such that $n * \omega > v_i$ then since consumer's characteristic are strongly monotone on $\mathcal{Z}^+_\Omega$, the following shall happen :$n\omega \succ_i v_i$.

Besides, we know that the endowment Ω that engender the ideal $\mathcal{Z}_\Omega$ is an $||.||_\infty$ interior point of $\mathcal{Z}^+_\Omega$ and the same hold for the element $n.\Omega$. The $||.||_\infty$-continuity of the consumer's tastes,allow one to deduce that commodity good $n.\Omega$ is preferred to each of the commodity good v_i and thus the commodity good $n.\Omega$ is an $||.||_\infty$ interior point of the set L_1. It result from the above lemma that for a real ζ, one have $\zeta.\Omega + L_i - \omega_i \subseteq \zeta.\omega + G$. It come that the $||.||_\infty$ interior point of $\zeta.\Omega + G$ is not empty. As the allocation $(v_1, ..., v_q)$ is an equilibrium of Edgeworth relatively to the Riesz subspace $\mathcal{Z}_\Omega$, we have that $0 \notin \zeta.\Omega + \bar{T}$. In this case the use of the classical separation theorem allows to think about a positive linear functional q, defined on the Riesz subspace $\mathcal{Z}_\Omega^+$ which behave in the way that $q.(\zeta.\Omega + t)$ is strictly greater than zero for all commodity good t in T. If now we rewrite q as $\frac{q}{q} \cdot \Omega$, we shall have $q \cdot \Omega = 1$. Suppose now that there exist a commodity good h that is preferred to the commodity v_i for each economic actor i. Looking back at the definition of the set T, we shall realize that the commodity $h - \omega_i$ belong to the set T and so the market value of the commodity $q \cdot (\zeta \cdot \omega + h - \omega_i)$ is greater or equal to zero. We thus deduce that $q \cdot h$ is greater or equal to $q \cdot \omega_i - \zeta$. We have then showed that the allowance $(v_1, ..., v_q)$ is an approximative quasi-equilibrium relatively to the Riesz pair $\langle \mathcal{Z}_\Omega, \mathcal{Z}'_\Omega \rangle$.

Before moving on, let us bring some clarification on the notion of price free allocation for Edgeworth equilibrium. In fact, saying that the Edgeworth equilibrium is a price free allocations amount to say that an Edgeworth allocation is an specific characteristic of the space of the commodity. In case the space of commodity $\mathcal{Z}$ is a Riesz space, then the allowance shall keep on being an equilibrium of Edgeworth in each economic barter for which the space of commodity is a Riesz subspace of $\mathcal{Z}$ and hold with the principal ideal $\mathcal{Z}_\Omega$

From there, we see that any equilibrium analysis can appropriately be restricted to the Riesz system $< \mathcal{Z}_\Omega, \mathcal{Z}'_\Omega >$. Under that system, every equilibrium of Walras is automatically an equilibrium of Edgeworth. By then, we deduce that that an equilibrium of Edgeworth is in fact the allowance that are an equilibrium of Walras relatively to the paired $< \mathcal{Z}_\Omega, \mathcal{Z}'_\Omega >$. Since an equilibrium of Walras need a linear functional to support its dominance position relatively to others allowances in the budget set, it will be so for a positive linear functional defined on $\mathcal{Z}'_\Omega$. That allowance happens to be an Edgeworth equilibrium. This is what is confirmed by the following lemma.

Lemma 5.44. *Consider a Riesz economy in which the consumer's consumption set are cone positive and an allowance $(v_1, ..., v_q)$ produced by that economy. If a linear functional q defined on the subspace $\mathcal{Z}_\Omega$ is such that the market value of a dominant allocation $h \succ_i v_i$ is greater than the individual endowment ω_i (or income) of each economic actor i.e., $q.h > q.\omega_i$ then the given allowance is termed an equilibrium of Edgeworth.*

As said earlier, the notion of Edgeworth equilibrium is a price free notion just like the notion of Pareto efficient allocation. The above lemma asserts the existence of a supporting prices for an Edgeworth equilibrium. This fact illustrate the second welfare's theorem which states that in a lattice economy every weakly efficient allocation can be supported by prices.

Here we provide an illustration of the above lemma by considering a two consumers Riesz economy in which economic actor's taste are identified by the following numerical functions: $U_1(x) = \int_0^1 sx(s)\, dt$ and $U_2(x) = \int_0^1 (1-s)x(s)\, ds$. The space of the commodity of the economy is specified by $\mathcal{Z} = L_1[0,1]$ while economic actor's endowments are given by $\omega_1 = \omega_2 = 1/2.\chi_{\{}[0,1]\}$.

An observation of the above assumptions show that both utility functions are strictly monotone and linear, and thus convex. Here linearity means that these utilities functions are of the form $U(x) = q.x$. That is $U_1 = sx$ and $U_2 = (1-s)x$ where s and $1-s$ are linear functional defined on the space of the commodity $\mathcal{Z}$. From there , we deduce that $q_1(s) = s$ and $q_2(s) = 1-s$; and, as can be seen they are continuous. Since both utilities functions are defined on the subspace of $\mathcal{Z} = L_1[0,1]$ which is an order interval, they are weakly continuous. Now if both linear functionals belong to the space of price $\mathcal{H}'$ then the functional $q(s) = max\{1, 1-s\}$ shall behave in a way that the market value of any commodity good that would be judged better or let us say preferred by economic actor as being better than the dominant allowance of the budget set *i.e.*, $h \succ_i v_i$, the market value of that good might be greater than economic actor's income ω. This is just saying that $q.h > q.\omega_i$. This means that vector price q help the allowance (v_1, v_2) to keep its dominant position; it is thus an allocation of Walras and therefore by the above lemma, it is also an allowance of Edgeworth. If in contrary the linear functional q does not belong to the space of price (which could be the case when $\mathcal{H}'$ is inside $\mathcal{L}_\infty[0,1]$). In this case then the allowance (u_1, u_2) will only be an equilibrium of Edgeworth and not an equilibrium of Walras.

5.9.2 Core Equivalence Theorem

Here we present topological conditions under which an equilibrium of the Edgeworth, an equilibrium of Walras and the quasi-equilibrium are related. We first introduce the notion of an approximate quasi-equilibrium, which in fact as we shall see, serves as a bridge between the three others notion of equilibrium.

Definition 5.45. Let $(v_1, ..., v_q)$ be an allowance produced by an economic barter and ζ a positive real greater than zero such that a linear functional defined on the space of the commodity (independently of the real ζ) which assign to the endowment Ω, a market value of one unit and for any commodity good h that would be preferred to the produced allowance $(v_1, ..., v_q)$, one have that the market value of that commodity good $q.h$ shall exceed $q.\omega_i - \zeta$. In this situation, the allowance $(v_1, ..., v_q)$ is termed an approximate quasi-equilibrium.

Theorem 5.46. *A Riesz economy in which the consumption set of economic actors with convex and weakly upper semi-continuous preferences is modeled as a positive cone and the order interval* $[0, \Omega]$ *(the Edgeworth box) of the commodity space* $\mathcal{Z}$ *is weakly compact and, each economic actor desire the total endowment* Ω *of the economy, then each allowance that is an equilibrium of Edgeworth is also an approximate quasi-equilibrium.*

Proof. Let $(s_1, s_2, ..., s_m)$ be an allowance of Edgeworth produced by the economic barter under consideration and consider the set H_i of commodities good that are preferred than the one produced by the current economic barter.

$H_i = \{v \in \mathbb{R}^l_+ : v \succ_i s_i\}$ and $J_i = H_i - \Omega_i = \{v \in \mathbb{R}^l_+ : v + \Omega_i \succeq_i s_i\}$ where Ω_i is the individual endowment and $\mathbb{R}^l_+$ the commodity set. H_i and J_i are clearly nonempty and convex sets. We now consider the following convex hull: $H = co(\bigcup_{i=1}^q H_i) = \{ \sum_{i=1}^m \eta t_i : t_i \in T_i, \eta_i \geq 0 \ \forall iand \sum_{i=1}^m \eta_i = 1\}$. In light of the preceding theorem, we have that for a real $\zeta > 0$, zero does not belong to $\zeta.\Omega + \bar{H}$. In this condition, the separating hyperplane theorem guarantee the existence of a linear functional (a vector price)q for which $q.(h + \zeta.\Omega) > 0$ for all $h \in H$. If now we rewrite $q \cdot \Omega$ as $q \cdot \Omega = \frac{1}{\zeta}.(0 + \zeta \cdot \Omega)$ and for $q = \frac{q}{q.\omega}$ we shall have $q \cdot \Omega = 1$. Suppose now that there exist a commodity good v that is preferred to the commodity s_i for each economic actor i. Looking back at the definition of the set H, we shall realize that the commodity $v - \omega_i$ belong to the set H and thus the market value of the commodity $q \cdot (\zeta \cdot \omega + v - \omega_i)$ is greater or equal to zero. We thus deduce that $q \cdot v$ is greater or equal to $q \cdot \omega_i - \zeta$. We have then showed that the allowance $(s_1, ..., s_m)$ is an approximative quasi-equilibrium. □

Notice that the approximate quasi-equilibria are Walrasian equilibria for an appropriate sub-economy of the original economy.

Let us now embark in the process of relating various equilibria notions. We need first to observe that an allowance that is an equilibrium of Walras relatively to a linear functional (a price function) defined on $\mathcal{Z}'_\Omega$, is also an equilibrium of Edgeworth. So to relate an equilibrium of Edgeworth to an equilibrium of Walras in infinite dimensional commodity space, one need first to move to the finite dimension, the Riesz pair $< \mathcal{Z}_\Omega, \mathcal{Z}'_\Omega >$, and then characterize the Edgeworth equilibrium in term of supporting price. Finally we shall extend the price p in $\mathcal{Z}'_\Omega$ to a price q in $\mathcal{Z}'$. This is precisely the approach used by C.D Aliprantis, D.J.Brown and O.Burkinshaw [2].

5.9.3 Extending Supporting Price Process.

In the process of extending the supporting price p of an optimal allowance from the principal ideal $\mathcal{Z}'_\Omega$ to the price space $\mathcal{Z}'$, difficulties may arise, often due to the size of the ideal engender by the endowment Ω relative to the size of the space of the commodity $\mathcal{Z}$ of the original economy. Among these cases, one can point out the following situations:

- For a small Riesz commodity space, the extension may in general not pose a problem. If for instance the space of commodity is of dimension three $\mathbb{R}^3$ whilst the potential Ω is an element of the cone positive of $\mathbb{R}^2$, in this case we have that $\mathcal{Z}_\Omega \equiv \mathbb{R}^2$ and the price supporting any optimal allowance will lie in $\mathbb{R}^3$.

- If the total potential Ω belong to the interior of $\mathcal{Z}^+$ (or if it is an order unit) then $\mathcal{Z}_\Omega \equiv \mathcal{Z}$. This is for instance the case when Ω is an element of l_∞ and is uniformly bounded away from zero, we have $\mathcal{Z}_\Omega \equiv l_\infty$.

- For large Riesz spaces where $\mathcal{Z}_\Omega \neq \mathcal{Z}$,things work differently. One need to make use of the topological properties of the weak closure $\bar{\mathcal{Z}}_\Omega$ of $\mathcal{Z}_\Omega$.

For $\Omega \gg 0$, the ideal engendered by $\Omega, \mathcal{Z}_\Omega$, is closely related to the band engendered by $\Omega : \mathcal{B}_\Omega = \{x \in L : |x| \wedge nw \uparrow |x|\}$. In this case $\mathcal{Z}_\Omega \subseteq \mathcal{B}_\Omega$ and the fact that every band is ξ-closed imply that $\bar{\mathcal{Z}}_\Omega$ is equal to $\mathcal{B}_\Omega$. Hence, the ideal $\bar{\mathcal{Z}}_\Omega = \{x \in \mathcal{Z} : |x| \wedge nw \xrightarrow{\xi} |x|\}$ is a ξ-closure of $\mathcal{Z}_\Omega$. If in addition ξ is order continuous then $\bar{\mathcal{Z}}_\Omega = \mathcal{B}_\Omega$ and $\bar{\mathcal{Z}}_\Omega$ is the natural Riesz subspace for the domains of the prolonged Walrasian price system $\bar{p} \in \bar{Z}_\Omega$. To see it , consider the case where one need to relate an equilibrium of Edgeworth that belong to a Riesz commodity space $\mathcal{Z}$ to an equilibrium of Walras. In this case, one have to account with the fact that an equilibrium of Edgeworth in the Riesz space of commodity $\mathcal{Z}$ is automatically an equilibrium of the Edgeworth in $\mathcal{Z}_\Omega$ and thus an equilibrium of Walras relatively to the Riesz pair $\langle \mathcal{Z}_\Omega, \mathcal{Z}'_\Omega \rangle$. That is with respect to a price $p \in \mathcal{Z}'_\Omega$. Our problem now is to bring(extend) that price $p \in \mathcal{Z}'_\Omega$ to belong in $\mathcal{Z}'$. The first step in this procedure shall consist in prolonging p to a price $\bar{p} \in \bar{Z}'_w$ and for this reason, we say that $\bar{\mathcal{Z}}_\Omega$ is the natural Riesz subspace for the domains of the prolonged Walrasian price system $\bar{p} \in \bar{L}'_\Omega$.

5.9.4 Extended Walrasian Equilibrium

When $\mathcal{Z}_\Omega \neq \mathcal{Z}$ but $\bar{\mathcal{Z}}_\Omega \equiv \mathcal{Z}$ then we turn to S.F.Richard (1985)who has showed the possibility of prolonging a strongly monotone and continuous consumer's tastes to a consumer's tastes that are defined on a consumption set that is convex and closed and having at least one interior point. So for large spaces that possess such consumer's tastes, Walrasian prices in $\mathcal{Z}'_\Omega$ can be prolonged to Walrasian prices in $\bar{Z}'_\Omega$ where $\bar{Z}_\Omega$ is the closure of $\mathcal{Z}_\Omega$ relatively to the original topology on $\mathcal{Z}$. $\bar{Z}_\Omega$ thus happen to be the natural Riesz subspace for the domain of the prolonged Walrasian price system.

Examples of such spaces are:

- The Riesz commodity space $\mathcal{L}_1$ with strictly positive potential Ω and the topology ξ being an order continuous (norm topology on $\mathcal{L}_1$); in this case we have $\bar{\mathcal{Z}}_\Omega \equiv \mathcal{B}_\Omega \equiv \mathcal{Z}_1$.

- The Riesz commodity space $\mathcal{L}_2(\mu)$ with strictly positive potential Ω and norm continuous. Then $\bar{\mathcal{Z}}_\Omega \equiv \mathcal{B}_\Omega \equiv \mathcal{L}_2$.

- The space of sequences having real values, $\mathcal{R}_\infty$, with strictly positive potential. That space is known to be a Riesz and also as a complete Dedekind space, once endowed with the product topology that is locally convex-solid and an order continuous. In this condition, $\bar{\mathcal{Z}}_w \equiv \mathcal{R}_\infty$.

Definition 5.47 (*Extended Walrasian equilibrium*)**.** Whenever it can be possible to define a linear functional ψ from the closure $\bar{Z}_w^+$ to $\bar{R}^+$ in a way that the market value of the potential Ω be of one unit *i.e.*, $\psi(\Omega) = 1$ and if further the defined linear functional ψ is additive *i.e.*, $\psi(u + v) = \psi(u) + \psi(v)$ hold for all commodity good u, v in $\bar{Z}_w$; and, the linear functional support any dominant commodity good h of the $\bar{Z}_w$ for $h \succeq_i v_i$ one have that $\psi(h) \geq \psi(\omega)_i$. Then the allowance $(v_1, ..., v_q)$ produced by an economic barter will be termed an prolonged equilibrium of Walras.

In general all commodities good in $\bar{Z}_\Omega$ may not have finite market value from the defined price ψ. In this event the price system ψ may not belong to $\bar{Z}'_\Omega$; therefore if we provide finite value to the total potential Ω by setting its market value equal to one $i.e.\psi(\Omega) = 1$ then economically we can expect an allocation defined by an extended Walrasian price system to be Pareto optimal.

To sum up, if an allowance $(v_1, ..., v_q)$ is an equilibrium of Walras on the subspace $\mathcal{Z}_\Omega$ then it is fortiori a extended equilibrium of Walras. Under the same hypothesis one can obtain a supporting price ψ for a dominant allowance by expressing the market value of a commodity good belonging to $(\bar{Z}_\Omega^+)$ in the following way: $\psi(u) = \sup\{p(u \wedge n.\Omega)n = 1, 2, ...\}$, for a commodity good u in the subspace $(\bar{Z}_\Omega^+)$. One can thus assert that an equilibrium of Walras can be prolonged on $\mathcal{Z}_\Omega$ to an approximate quasi-equilibrium whenever consumer's tastes are uniformly proper and when $\bar{Z}_\Omega \neq \mathcal{Z}$. This fact then state the relationship between an equilibrium of Walras on $\mathcal{Z}_\Omega$ and the extended Walrasian equilibrium.

5.9.5 Approximate quasi-equilibrium

By considering the space, $C_a(H)$, the spaces of all countably additive measures on H; with H a metric space that is compact, one observe that the Riesz space $C_a(H)$ has a very large commodity space and therefore one possibly has $\bar{\mathcal{Z}}_\Omega \neq \mathcal{Z}$. In this case, uniformly properness is not capable of helping prolong a Walrasian prices system from $\mathcal{Z}'_\Omega$ to $\bar{Z}'_\Omega$. To help solve the problem, one need to consider the correspondence between the prices system in $\mathcal{Z}'_\Omega$ associated with the equilibrium of Walras in $\mathcal{Z}_\Omega$ and the sequence of priceq_n in $\mathcal{Z}'$ that support that allowance of Walras in the original economy. We then index each price system in $\mathcal{Z}'$. For each index we normalize by imposing the value of one to the total potential Ω, meaning that each price q_n restricted to $(\mathcal{Z}'_\Omega, ||.||_\infty)$ has norm one. That is the original definition of price is weaken so that consumer can consider all commodity vector in his budget set. By then, we economically expect any allocation defined by a prolonged Walrasian price system to be Pareto optimal. For a commodity good that would be (or pretend to be) preferred to the optimal allowance, $h \geq v_i \in \mathcal{Z}^+$ implies $q_n * h \geq q_n * \omega_i - 1/n$ We thus restrict each $q_n \in \mathcal{Z}'$ to $p_n \in \mathcal{Z}'_\Omega$. Now that the sequence p_n is in an order continuous ideal $\mathcal{Z}'_\Omega$ and the weakly compact order interval $[0, \Omega] \subset \mathcal{Z}'_\Omega$, we have that any subsets extracted from q_n is $\delta(< \mathcal{Z}'_\Omega, \mathcal{Z}_\Omega >)$-convergent to an accumulation point (vector) $p \in \mathcal{Z}'_\Omega$ and p is such that $p * \Omega = 1$. It follow from the above correspondence that there might exist a vector q in $\mathcal{Z}'$ for which p is the restriction on $\mathcal{Z}'_\Omega$. This process lead to an obtention of a family of prices in $\mathcal{Z}'$ which is called an approximate quasi-equilibrium.

Definition 5.48 (Approximate quasi-equilibrium)**.** If an allowance $(v_1, ..., v_q)$ produced by an economic barter is such that one can defined a positive linear functional q, depending on a real ζ, which assign a market value of one unit to the total potential Ω of the economy and for a commodity good h that can preferred to the optimal allowance $(v_1, ..., v_q)$, $h \geq_i v_i$ for some i, then one have that the market value of that commodity good might exceed consumer's income *i.e.*, $q.h \geq q.\omega_i$. In this case, one assert that the optimal allowance in question $(v_1, ..., v_q)$ is an approximate quasi-equilibrium.

Theorem 5.49. *If $(v_1, ..., v_q)$ is an allowance engender by the Riesz economic barter $\langle \mathcal{Z}, \mathcal{Z}' \rangle$ then there might exists a strictly positive linear functional q defined on $\mathcal{Z}_\Omega$ behaving in the way that*

- $q.\Omega = 1$*; and*
- *Any commodity good $s \in \mathcal{Z}_\Omega$ that is preferred than any others in the sub-commodity space $\mathcal{Z}_\Omega$ we have $q.s \geq q.\Omega_i$*

This is to say that every approximate quasiequilibrium is an allowance of Walras with respect to the economy having same primitive(endowment and economic actor's tastes) and with a commodity-price duality defined by $\langle \mathcal{Z}_\Omega, \mathcal{Z}'_\Omega \rangle$.

Proof. consider an allowance $(v_1, ..., v_q)$ that we assume being an approximate quasiequilibrium. To see that it is also a Walras equilibrium for an economy sharing same characteristic, we pick for each integer n a linear functional $0 \leq q_n \in \mathcal{L}^{|}prime$ behaving in a way that $q_n.\Omega = 1$ and for a commodity good $s \succeq v_i \in \mathcal{L}^+$ we have $q_n.s \geq q.\Omega_i - \frac{1}{n}$. That is each of the q_n restricted to the sub-commodity space $\mathcal{Z}_\Omega, ||.||_\infty$ has been normalized to 1. Now consider a $\sigma(\langle \mathcal{Z}_\Omega, \mathcal{Z}'_\Omega \rangle)$ accumulation point q of these linear functional(prices) q_n. We must then have $q \in \mathcal{Z}'_\Omega$ as each q_n has been restricted to $\mathcal{Z}'_\Omega$ and in addition the linear functional(price) q behave in a way that for a commodity good $s \in \mathcal{Z}_\Omega$ that is is preferred than any others commodity in $\mathcal{Z}_\Omega$ we have $q.s \geq q.\Omega_i$. It thus follows that the allowance $(v_1, ..., v_q)$ is an equilibrium of Walras for the economy having same characteristic than the original ones and whose Riesz dual system is $\sigma(\langle \mathcal{Z}_\Omega, \mathcal{Z}'_\Omega \rangle)$. □

5.9.6 Examples

A.Mass-Colell(1986) provided an example of an economy with a single consumer and two commodity goods. The taste of the single economic actor is identified by the following numerical function $U(u, v) = \sqrt{u} + \sqrt{v}$. Since the economy is about two goods, the commodity space $\mathcal{L} = \mathbb{R}^2$ and the total potential of the economy automatically form the initial endowment of that single consumer $\Omega = (1, 0)$. The defined utility function reveals a strongly monotone, continuous and quasi-concave consumer's preferences on $\mathbb{R}^2_+$. Hence providing the single consumer with the possibility of maximizing his utility. It follows that the initial endowment $W = (1, 0)$ is the Edgeworth equilibrium and from the topological properties of the preferences(strongly monotone and continuous) that there are uniformly proper and so, the Edgeworth equilibrium is also a quasi-equilibrium. The ideal generated by the total potential of the economy $\Omega = (1, 0)$ is $\mathcal{Z}_\Omega \equiv \mathcal{B}_\Omega = \{(x, 0) : x \in \mathbb{R}\}$. Since the the allocation $\Omega = (1, 0)$ is a quasi-equilibrium, it might exist a price system $p = (0, 1) \in \mathcal{Z}'_\Omega = \{(p_1, 0) : p_1 \in \mathbb{R}\}$ that supports the allocation $\Omega = (1, 0)$. The second example is taken from J.Ostroy(1984) who considered the Riesz dual system $\langle \mathcal{L}_1, \mathcal{L}_\infty \rangle$ over a σ-finite measure space. The topological norm on the commodity space $\mathcal{L}_1$ is order continuous, making the Riesz dual system symmetric. Assuming $\Omega \gg 0$ and strongly monotone, quasi-concave, uniformly proper and norm continuous preferences, we have that $\mathcal{Z}_\Omega \equiv \mathcal{B}_\Omega \equiv \mathcal{L}_1$ leading therefore to a Walrasian equilibrium for this economy. The preceding analysis holds true for the following symmetric Riesz dual system $\langle \mathcal{L}_2(\mu), \mathcal{L}_2(\mu) \rangle$.

Chapter 6

General Conclusion

Our focus in this Master Dissertation has been on infinite dimensional exchange economies, particularly on the properties of its equilibrium. From Adam Smith, Leon Walras, exchange economy is known to secure maximum prosperity to the society. Two mathematical models helped to conduct the equilibrium analysis of this infinite dimensional exchange economies: the continuum (or atomless) economy and the Riesz pair $\langle \mathcal{L}_w, \mathcal{L}'_w \rangle$ model.

Continuum economic has the merit of showing that indeed an infinite dimensional exchange economy reaches an equilibrium which is a core allocation and a Walras equilibrium. We learn from this framework that the size of a consumption sector induce convexity on the mean demand of the economy and therefore an approximate equilibrium price proportionally to the size of the economy. The larger the economy, the better the approximation to the equilibrium can be expected. Atomless economic's framework showed that, under strongly convex consumer's preferences, Walras allocations is uniquely determined by an equilibrium prices. We learned that, in a large economy, the dynamic of the redistribution of the total endowment in the commodity space provide useful information about allocations. By then, one can thus measure the extent to which the distribution of agents' characteristic determine the set of the distribution of Walras allocations of the economy. Furthermore, one can define a Walras equilibrium distribution in terms of distributions of agents' characteristics.

Further, we saw that an atomless economy can be considered as resulting from a convergence of a series of finite economies. This characterization of continuum economy permit a generalization of the Edgeworth's Limit Theorem. The revised version of the continuum economy show how the increase of economic actors in an economic barter, sequentially transform each allowance in the core to an allowance of Walras; moving therefore the commodity vector of economic agents to their demand set.

Mas-Colell (1995) pointed out that the Edgeworth box is remarkably a powerful tool in term of economic analysis. Riesz theory allowed us to properly explore the Edgeworth box and thus helped to confirm Mas-Colell statement. This Master Dissertation has used tool provided by Riesz theory to conduct topological analysis of the Edgeworth box, particularly the Riesz pair $\langle \mathcal{L}_w, \mathcal{L}'_w \rangle$. Riesz pair enabled us to move equilibrium analysis from the infinite dimensional economy, the original economy $\langle \mathcal{L}, \mathcal{L}' \rangle$ to the finite dimensional economy $\langle \mathcal{L}_w, \mathcal{L}'_w \rangle$; where it was possible to define three more equilibrium concepts: the quasiequilibrium, the approximate quasiequilibrium and the extended Walras equilibrium. Finally, the Riesz pair enabled us to relate these equilibrium concepts.

In addition the Riesz pair appeared to be an appropriate mathematical tool susceptible of paving the way of equilibrium for a dual situation. Globalization is actually subject of heated debate, it is praised by some for generating economic growth while it is accused by others of creating social inequality. The prosperity of any human organisation pass through its equilibrium. Year 2015 alone has recorded three negotiations: Iran's nuclear deal, Asia pacific Trade and COP21 agreement for climate change at Paris(France). We have all witness the way the Worldwide has followed and wished positive outcome of these negotiations. Positive outcome is nothing more than what is called equilibrium. Agreement among 195 countries involves bargain. That is a situation where all participants are equally satisfied and are all in the contract curve. Riesz theory has allowed us not only to consider an infinite dimensional organization but most importantly to investigate its equilibrium's existence and properties relative to that equilibrium. The Lausanne School has been instrumental from year 1874 to the entire civilisation in conceiving and mathematically formalising the notion of equilibrium. Leon Walras has been the predecessor, Vilfredo Pareto and Francis Edgeworth made it concise. The actual negotiation on climate change in Paris (France) is all about Pareto optimality who pointed out that equilibrium in any human society is reached when it is good to make someone better off without making anyone else worse off. Equilibrium require each participants to be willing to give up a portion of what he cherish the most in order to get what is good for him and the entire community. Mas-Colell expressed this equilibrium condition in Riesz language in term of uniform properness. Uniform properness has allowed the infinite dimensional analogue of the classical welfare's theorem of Debreu and Scarf (1963) as well as the second welfare theorem. It is shown that for large spaces with uniformly proper preferences, Walrasian prices in $\mathcal{L}'_w$ can be extended to Walrasian prices in $\bar{L}'_w$.

Bibliography

[1] C.D. Aliprantis and D.J. Brown (1983) *Equilibrium in Markets with a Riesz Spaces of commodities*,Journal of Mathematical Economics,11,189-207.

[2] C.D. Aliprantis, D.J. Brown O. and Burkinshaw (1987) *Edgeworth Equilibrium*, Econometrica, Vol 55,No 5,1109-1137.

[3] C.D. Aliprantis and O. Burkinshaw (2003) *Locally Solid Riesz Spaces with Application to Economics.* Newyork and London

[4] C.D. Aliprantis K.C.Border(2005)*Infinite Dimensional Analysis*, A Hitch Hikers'Guide Third Edition.

[5] Arrow,K.J. and F.H. Hahn, (1971)*General competitive analysis.*Holden-Day, Sanfrancisco,CA

[6] R.J.Aumann(1966)*Existence of a Competitive Equilibrium Markets with a Continuum of Traders*, Econometrica 34, 1-17.

[7] Bewley, T.F. (1972) *Existence of Equilibrium in Economics with infinity many commodities*, Journal of Economic Theory, 4, 514-540.

[8] Debreu, G. (1954) *Valuation Equilibrium and Pareto Optimum*, Proceeding of the National Academic of Sciences, 40, 588-592.

[9] Debreu, G. and Scarf H. (1963) *Limit Theorem on the core of an Economy*, International Economic Review, 4, 235-246.

[10] Debreu, G. (1975) *The rate of Convergence of the core of an Economy*, Journal of Mathematical Economics, 2, 1-7.

[11] Debreu, G. (1974) *Four Aspects of the Mathematical Theory of Economic Equilibrium*, Proceeding of the International Congress of Mathematicians, Vancouver, 65-77.

[12] Debreu, G. (1981) *Existence of competitive equilibrium*,Handbook of mathematical economics (North-Holland,Amsterdam)

[13] E. De Jonge A.C.M Van Rooij (1977)*Introduction to Riesz spaces*, Mathematical Centre Tracts 78 Amsterdam 77

[14] Walker, Donald A. (1983) *William Jaffe's Essay on Walras*, Cambridge, New York Cambridge University Press.

[15] Filipe Martins DA-Rocha (2002)*Equilibre general avec une double infinite d'agents et de biens*, Cermsem, CNRS,Universite Paris1.

[16] Landreth, Harry (1976) *History of Economic Theory*, Houghton Mifflin Company, Boston.

[17] Hildenbrand, W. (1974) *Core and equilibrium of a large economy*, Princeton University Press

[18] Hildenbrand, W. (1970) *On Economies with Many Agents*, Journal of Economic Theory

[19] Hong,L.*:On order-bounded subsets of Locally Solid Riesz Spaces*, Quaestiones Mathematicae 39(3) 2016:381-389 Department of Mathematics, Robert Morris University

[20] Jean-Jacques Laffont (1992)*Advances in Economic Theory: Volume2*,The Economic Journal World Congress of the Econometric Society in Barcelona(August 1990)

[21] Mass-Colell, A. (1974) *An equilibrium existence theorem without complete or transitive preferences*,Journal of Mathematical Economics,237-246.

[22] Mass-Colell, A. (1975) *A model of equilibrium with differentiated commodities*,Journal of Mathematical Economics,2,263-295.

[23] Mass-Colell, A. (1986) *The price equilibrium existence problem in topological vector lattices*,Econometrica,54,1039-1059

[24] A.Mas-Colell and S.F.Richard (1991)*A new approach to the existence of equilibria in vector lattices*,J.Econom. Theory 53,1-11.

[25] Nikaido, H. (1968) *Convex Structure and economic theory*, Academic Press, Newyork.

[26] OmidZabeti*Topological algebras of locally solid vector subspaces of order bounded operators*,Department of Mathematics, Robert Morris University of Sistan and Baluchestan (December 2016)

[27] Peleg,B. and M.E. Yaari (1970)*Markets with Countably Many Commodities*,International Economic Review, 11,369-377

[28] Konrad Podczeck (2001) *On Core - Walras equivalence for economies with a large commodity space*,Working paper, University of Vienna.

[29] Konrad Podczeck (2006) *A note on the convexity and compactness of the integral of Banach space valued correspondence*, University of Vienna.

[30] R.Radner (1967)*Efficienccy prices for infinite horizon production programes*,Review of Economic studies 34, 51-66.

[31] Samuelson, Paul A. and Nordhaus, William (1963) *Economics (Twelfth Edition)*, McGraw - Hill International Editions

[32] R.Tyrrell Rockafellar (1997)*Convex Analysis*,Princeton University Press

[33] A.W. WICKSTEAD*Vector and Banach Lattices*,Math.Ann(212)(1975)

[34] M.A.Khan And N.C.Yannelis (1991) *Equilibrium in Markets with a Continuum of agents and commodities, Equilibrium Theory in Infinite Dimensional spaces*, Springer-Verlag, Newyork.

[35] Yannelis, N.C., And W.R. Zame (1984) *Equilibria in Banach Lattices Without Ordered Preferences*,Institute for Mathematics and Its Applications, University of Minnesota.

Economic Equilibrium Analysis with Riesz Geometry and Measure Theory

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Abstract

Equilibrium is a concern of each individual in the society, of an organisation, and of the entire world. This Masters dissertation consider the analysis of economic equilibrium for an infinite dimensional economy. Two mathematic models are used to conduct the analysis: the continuum economics model and the Riesz dual system $\langle \mathcal{L}_w, \mathcal{L}'_w \rangle$ model. Continuum economics show how an economy with many economic actors reach both an equilibrium of Walras and a core allocation. The second model, based on the paper by Aliprantis, Burkinshaw, Brown, explore the equilibrium properties embodied in the Edgeworth box. Three more equilibrium's concepts(extended Walras equilibrium, the quasiequilibrium and the approximate quasiequilibrium) as well as the connection between them are presented in the last chapter of this dissertation.

0.1 DECLARRATION

I declare that this dissertation is my own, unaided work.
It is being submitted in partial fulfilment of the requirements for the degree of Master of Science by dissertation in the University of the Witwatersrand, Johannesburg.
It has not been submitted before for any degree or examination in any other university.

Francis Kalonji Mudibu

0.2 PREFACE(Acknowledge)

There is no economy where is social inequality. It has been a nice moment to deal with such a subject as an economic equilibrium, particularly when it has to be assessed with the tools of Riesz Geometry and theoretic measure. We cannot succeed alone, we need available and good peoples. Destiny led me to Professor B.A. Watson who not only suggested the subject, red the whole dissertation, corrected my English and made technical and helpful suggestions to improve the presentation of this dissertation. Professor Watson receive my deep thanks. To you God Almighty, without you many things would not be possible, let your invisible hand continue to lead us.