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Two-dimensional turbulent thermal classical far wake

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ABSTRACT

The two-dimensional turbulent thermal classical far wake is investigated. The turbulence is described by the Boussinesq hypothesis for the Reynolds stresses with Prandtl's mixing length model for the eddy viscosity and eddy thermal conductivity. Two conservation laws are derived for the thermal boundary layer equations for the far wake using the multiplier method and two conserved quantities are obtained from the conserved vectors and boundary conditions. The Lie point symmetry associated with the momentum and thermal conserved vectors is derived. It is found that the momentum and thermal mixing lengths are proportional. In obtaining the invariant solution the cases $\nu \neq 0$, $\kappa \neq 0$ and $\nu = 0$, $\kappa = 0$ needed to be treated separately, where ν is the kinematic viscosity and κ is the thermal conductivity of the fluid. When $\nu \neq 0$, $\kappa \neq 0$ the associated Lie point symmetry is determined in full but an analytical solution of the reduced ordinary differential equations could not be derived. The invariant solution is obtained numerically using a shooting method with the two conserved quantities as targets. The width of the wake is infinite. When $\nu = 0$, $\kappa = 0$ the width of the wake is finite. In order to obtain the associated Lie point symmetry in full, Prandtl's hypothesis that the mixing length is proportional to the width of the wake was imposed. The invariant solution is obtained analytically. The lines of constant mean temperature difference and the streamlines for the mean velocity deficit are plotted. The numerical and analytical solutions agree in the limit $\nu \to 0$, $\kappa \to 0$.

1. Introduction

In this paper the properties of the two-dimensional turbulent thermal far wake are investigated. The Reynolds decomposition is performed in which the velocity and temperature fields are expressed as the sum of a time average mean and a fluctuation. The Boussinesq hypothesis [1,2] for the Reynolds stresses in terms of the eddy viscosity ν_T and an analogue of Fourier's law of heat conduction for the mean temperature in terms of the eddy thermal conductivity κ_T are made [3]. The eddy viscosity and eddy thermal conductivity are formulated in terms of Prandtl's mixing length model and a momentum and thermal mixing length are introduced. The mixing length is the distance travelled across the mean flow by a fluid lump before mixing with the mean flow. The boundary layer approximation is applied to the far wake and the partial differential equations are analysed by deriving their conservation laws and associated Lie point symmetries.

The boundary layer equations for a turbulent thermal free jet derived by Mubai and Mason [4] are applicable to the turbulent thermal classical far wake. The partial differential equations for the mean flow are reduced to ordinary differential equations using the Lie point symmetry associated with the conserved vectors of the mean flow. The ordinary differential equations can be integrated at least once by the double reduction theorem of Sjöberg [5,6]. Two conserved vectors are derived. Lie point symmetry analysis has been applied to wake problems without thermal effects [7,8] but not to thermal wakes. Lie point symmetry methods have been applied to liquid jet problems [9,10] and to the turbulent thermal jet [4].

There are several methods for obtaining conservation laws and conserved vectors. These methods were compared in [11]. In this paper the conservation laws and conserved vectors are derived using the multiplier method [12,13]. Noether [14] was the first to relate symmetries to conservation laws. Kara and Mahomed [15,16] derived the condition for a conserved vector to be associated with a Lie point symmetry of the partial differential equations. We use this condition to derive the Lie point symmetry associated with the two conserved vectors of the mean flow of a turbulent thermal wake.

In the early work on the turbulent thermal classical wake by Prandtl [17] and Goldstein [18] the kinematic viscosity ν and the thermal conductivity κ of the fluid were approximated to zero because they are very small compared with the eddy viscosity ν_T and eddy thermal conductivity κ_T. Although ν and κ are small they are never

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zero in a real fluid. They play an important role in the mathematical analysis. In this paper we will consider two cases, when $\nu = 0$, $\kappa = 0$ and when $\nu \neq 0$, $\kappa \neq 0$, and we will investigate the role played by ν and κ in the derivation of the invariant solution and in the analysis of the results.

The problem will be formulated in terms of the stream function for the mean velocity deficit. We find that the invariant solution for the mean velocity deficit and the temperature difference for the case $\nu = 0$, $\kappa = 0$, is in a form convenient for plotting the streamlines of the mean velocity deficit and the lines of constant mean temperature difference. These lines are plotted and used to analyse the results.

The partial differential equations for the mean flow are reduced to a system of two ordinary differential equations which cannot in general be solved analytically when ν and κ are non-zero. They are solved numerically. We will use the shooting method applied by Mubai and Mason [4] to the turbulent thermal jet in which the two conserved quantities are treated as the targets.

This paper is structured as follows. In Section 2 we apply the boundary layer equations derived in [4] to the turbulent thermal far wake. The eddy viscosity and eddy thermal conductivity are modelled using Prandtl mixing lengths. The boundary conditions and governing partial differential equations are written in terms of a stream function and temperature difference. In Section 3 the conservation laws and conserved quantities for the classical thermal wake are derived. In Section 4 the full Lie point symmetry of the partial differential equations is derived. In Section 5 we use the conserved vectors obtained in Section 3 to derive the associated Lie point symmetry. In Section 6 we reduce the partial differential equations to ordinary differential equations using the associated Lie point symmetry. We obtain an analytical solution when $\nu = 0$ and $\kappa = 0$ and the numerical solution when $\nu \neq 0$ and $\kappa \neq 0$. In Section 7 we solve for the lines of constant mean temperature difference and the streamlines of the mean velocity deficit when $\nu = 0$ and $\kappa = 0$. In Section 8 the numerical solution when $\nu \neq 0$ and $\kappa \neq 0$ is compared with the analytical solution when $\nu = 0$ and $\kappa = 0$ and the streamlines of the velocity deficit and lines of constant temperature difference of the mean flow are drawn. In Section 9 we provide conclusions.

2. Mathematical formulation

Consider a two-dimensional turbulent thermal far-wake. The fluid is assumed to be incompressible and flows past a heated symmetrical fixed obstacle aligned with the flow with a constant mainstream velocity U. The obstacle supplies heat to the wake so that the maximum temperature is at the centreline of the wake. The viscous and thermal diffusion effects are only significant within the boundary of the wake and the mainstream flow can be approximated as being inviscid. The x-axis is parallel to the mainstream velocity and passes through the obstacle. The origin of the coordinate system is at the trailing edge of the obstacle. The y-axis is perpendicular to the x-axis as shown in Fig. 1.

Mubai and Mason [4] derived the boundary layer equations for the mean flow for a turbulent thermal jet. The equations remain valid for a turbulent thermal wake. The boundary layer equations are

$$\overline{v}_x \frac{\partial \overline{v}_x}{\partial x} + \overline{v}_y \frac{\partial \overline{v}_x}{\partial y} = \frac{\partial}{\partial y}\left[\left(\nu + l^2(x)\left|\frac{\partial \overline{v}_x}{\partial y}\right|\right)\frac{\partial \overline{v}_x}{\partial y}\right], \tag{2.1}$$

$$\overline{v}_x \frac{\partial \overline{T}}{\partial x} + \overline{v}_y \frac{\partial \overline{T}}{\partial y} = \frac{\partial}{\partial y}\left[\left(\frac{\kappa}{\rho c_v} + l(x)l_\theta(x)\left|\frac{\partial \overline{v}_x}{\partial y}\right|\right)\frac{\partial \overline{T}}{\partial y}\right], \tag{2.2}$$

where $\overline{v}_x(x,y)$ and $\overline{v}_y(x,y)$ are the $x-$ and $y-$components of the mean velocity, $\overline{T}(x,y)$ is the mean temperature, ν is the kinematic viscosity of the fluid, ρ and κ are the density and thermal conductivity of the fluid, c_v is the specific heat capacity of the fluid at constant volume, $l(x)$ and $l_\theta(x)$ are the momentum and thermal mixing lengths for the turbulent flow. In Eq. (2.2) viscous effects are neglected. The terms

$$\nu_T = l^2(x)\left|\frac{\partial \overline{v}_x}{\partial y}\right| \qquad \text{and} \qquad \kappa_T = \rho c_v l(x) l_\theta(x)\left|\frac{\partial \overline{v}_x}{\partial y}\right| \tag{2.3}$$

are the turbulent eddy kinematic viscosity and eddy thermal conductivity, respectively. The conservation of mass equation is

$$\frac{\partial \overline{v}_x}{\partial x} + \frac{\partial \overline{v}_y}{\partial y} = 0. \tag{2.4}$$

We write the governing partial differential equations (2.1), (2.2) and (2.4) in terms of the mean velocity deficit, $\overline{w}(x,y)$, and the mean temperature difference, $\overline{S}(x,y)$, defined by

$$\overline{v}_x(x,y) = U - \overline{w}(x,y) \qquad \text{and} \qquad \overline{S}(x,y) = \overline{T}(x,y) - T_B, \tag{2.5}$$

where T_B is the ambient temperature. Unlike $\overline{v}_x(x,y)$ and $\overline{T}(x,y)$, both $\overline{w}(x,y)$ and $\overline{S}(x,y)$ vanish at $y = \pm\infty$ or on a finite boundary. For a far-wake, we assume that $\overline{w} \ll U$, $\overline{v}_y \ll U$ and $\overline{S} \ll T_B$. We neglect terms containing second-order products of $\overline{w}$, $\overline{v}_y$ and $\overline{S}$ but we keep the turbulent terms because the eddy viscosity and eddy thermal conductivity are much greater than ν and κ. Substituting (2.5) into (2.1) and (2.2) and neglecting second-order terms in $\overline{w}$, $\overline{v}_y$ and $\overline{S}$ except the turbulent terms gives

$$U\frac{\partial \overline{w}}{\partial x} = \frac{\partial}{\partial y}\left[\left(\nu + l^2(x)\left|\frac{\partial \overline{w}}{\partial y}\right|\right)\frac{\partial \overline{w}}{\partial y}\right], \tag{2.6}$$

$$U\frac{\partial \overline{S}}{\partial x} = \frac{\partial}{\partial y}\left[\left(\frac{\kappa}{\rho c_v} + l(x)l_\theta(x)\left|\frac{\partial \overline{w}}{\partial y}\right|\right)\frac{\partial \overline{S}}{\partial y}\right]. \tag{2.7}$$

Substituting (2.5) into Eq. (2.4) gives

$$-\frac{\partial \overline{w}}{\partial x} + \frac{\partial \overline{v}_y}{\partial y} = 0. \tag{2.8}$$

Since the wake is symmetrical about the centre line and there is no cavity formation in the flow, $\overline{v}_y$ vanishes at $y = 0$ and $\overline{w}$ and $\overline{S}$ are maximum at $y = 0$. Hence

$$y = 0: \qquad \overline{v}_y(x,0) = 0, \qquad \frac{\partial \overline{w}}{\partial y}(x,0) = 0, \qquad \frac{\partial \overline{S}}{\partial y}(x,0) = 0. \tag{2.9}$$

Both ν_T and κ_T vanish on the boundary curve $y = y_B(x)$ and therefore from (2.3)

$$\frac{\partial \overline{w}}{\partial y}(x, y_B(x)) = 0. \tag{2.10}$$

Outside the turbulent wake, $y_B(x) < y < \infty$ and on the boundary $y = y_B(x)$, the fluid is at ambient temperature and the mean velocity deficit vanishes. Thus

$$\overline{S}(x, y_B(x)) = 0, \qquad \overline{w}(x, y_B(x)) = 0. \tag{2.11}$$

The normal component of the turbulent heat flux vector

$$\overline{h}_i = -\left(\kappa + \kappa_T\right)\frac{\partial \overline{S}}{\partial x_i} \tag{2.12}$$

vanishes at the boundary $y = y_B(x)$. Hence

$$\kappa\frac{\partial \overline{S}}{\partial y}(x, y_B(x)) = 0. \tag{2.13}$$

When $\kappa = 0$ the boundary condition (2.13) is identically satisfied. The boundary of the turbulent thermal wake in the upper half of the xy-plane may be a finite curve $y = y_B(x)$ or infinite for all positive values of x. The boundary curve is at $y = \pm\infty$ when $\nu \neq 0$ and $\kappa \neq 0$, as for the laminar thermal wake, but from the analytical solution we find that the boundary curve is finite when $\nu = 0$ and $\kappa = 0$.

Since the mean flow is two-dimensional and incompressible a stream function for the velocity deficit, $\overline{\psi}(x,y)$, can be introduced defined by

$$\overline{w} = \frac{\partial \overline{\psi}}{\partial y}, \qquad \overline{v}_y = \frac{\partial \overline{\psi}}{\partial x}. \tag{2.14}$$

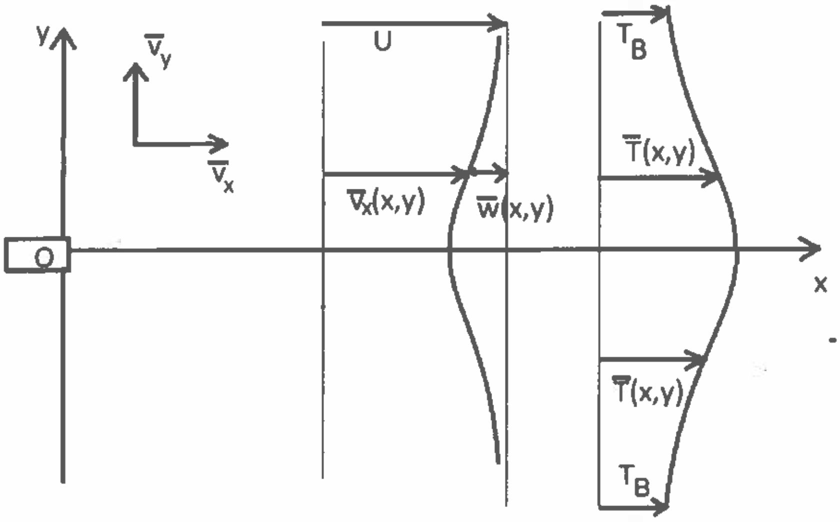

Fig. 1. The mean velocity and the mean temperature profiles of a two-dimensional turbulent thermal classical wake.

The conservation of mass equation for the mean flow, (2.4), is identically satisfied. Since the wake is symmetrical about the centreline we consider the upper half of the wake $0 < y < y_B(x)$. In this region $\frac{\partial \overline{w}}{\partial y} < 0$. Eqs. (2.6) and (2.7) become

$$U\frac{\partial^2\overline{\psi}}{\partial x\partial y} = \frac{\partial}{\partial y}\left[\left(\nu - l^2(x)\frac{\partial^2\overline{\psi}}{\partial y^2}\right)\frac{\partial^2\overline{\psi}}{\partial y^2}\right], \tag{2.15}$$

$$U\frac{\partial\overline{S}}{\partial x} = \frac{\partial}{\partial y}\left[\left(\frac{\kappa}{\rho c_v} - l(x)l_\theta(x)\frac{\partial^2\overline{\psi}}{\partial y^2}\right)\frac{\partial\overline{S}}{\partial y}\right]. \tag{2.16}$$

The boundary conditions are

$$y=0: \quad \frac{\partial\overline{\psi}}{\partial x}(x,0)=0, \quad \frac{\partial^2\overline{\psi}}{\partial y^2}(x,0)=0, \quad \frac{\partial\overline{S}}{\partial y}(x,0)=0, \tag{2.17}$$

$$y=y_B(x): \quad \frac{\partial\overline{\psi}}{\partial y}(x,y_B(x))=0, \quad \frac{\partial^2\overline{\psi}}{\partial y^2}(x,y_B(x))=0,$$
$$\overline{S}(x,y_B(x))=0, \quad \kappa\frac{\partial\overline{S}}{\partial y}(x,y_B(x))=0. \tag{2.18}$$

3. Conservation laws, conserved vectors and conserved quantities

In this section, the conservation laws, conserved vectors and conserved quantities for the partial differential equations (2.15) and (2.16) are derived.

3.1. Conservation laws and conserved vectors

When calculating conservation laws and Lie point symmetries we treat x, y, $\overline{\psi}$, $\overline{S}$ and all partial derivatives of $\overline{\psi}$ and $\overline{S}$ with respect to x and y as independent variables. We use the subscript notation to indicate that the partial derivatives are treated as independent variables. Hence, when calculating conservation laws, x, y, $\overline{\psi}$, $\overline{S}$, $\overline{\psi}_x$, $\overline{\psi}_y$, $\overline{S}_x$, $\overline{S}_y$, and all higher derivatives of $\overline{\psi}$ and $\overline{S}$ are treated as independent variables. We use the notation $\frac{\partial\overline{\psi}}{\partial x}$, $\frac{\partial\overline{\psi}}{\partial y}$, …, $\frac{\partial\overline{S}}{\partial x}$, $\frac{\partial\overline{S}}{\partial y}$, … when we treat $\overline{\psi}(x,y)$ and $\overline{S}(x,y)$ as functions of the independent variables x and y.

Writing the governing equations (2.15) and (2.16) in terms of the subscript notation we have

$$F_1(x,\overline{\psi}_{xy},\overline{\psi}_{yy},\overline{\psi}_{yyy}) = \nu\overline{\psi}_{yyy} - 2l^2(x)\overline{\psi}_{yy}\overline{\psi}_{yyy} - U\overline{\psi}_{xy} = 0, \tag{3.1}$$

$$F_2(x,\overline{\psi}_{yy},\overline{\psi}_{yyy},\overline{S}_x,\overline{S}_y,\overline{S}_{yy}) = \frac{\kappa}{\rho c_v}\overline{S}_{yy} - l(x)l_\theta(x)\overline{\psi}_{yyy}\overline{S}_y - l(x)l_\theta(x)\overline{\psi}_{yy}\overline{S}_{yy} - U\overline{S}_x = 0. \tag{3.2}$$

A conservation law of the system of partial differential equations (3.1) and (3.2) is defined by

$$D_xT^1 + D_yT^2\Big|_{F_1=0,F_2=0} = 0, \tag{3.3}$$

where

$$D_x = \frac{\partial}{\partial x} + \overline{\psi}_x\frac{\partial}{\partial\overline{\psi}} + \overline{S}_x\frac{\partial}{\partial\overline{S}} + \overline{\psi}_{xx}\frac{\partial}{\partial\overline{\psi}_x} + \overline{S}_{xx}\frac{\partial}{\partial\overline{S}_x} + \overline{\psi}_{xy}\frac{\partial}{\partial\overline{\psi}_y} + \overline{S}_{xy}\frac{\partial}{\partial\overline{S}_y} + \cdots, \tag{3.4}$$

$$D_y = \frac{\partial}{\partial y} + \overline{\psi}_y\frac{\partial}{\partial\overline{\psi}} + \overline{S}_y\frac{\partial}{\partial\overline{S}} + \overline{\psi}_{yx}\frac{\partial}{\partial\overline{\psi}_x} + \overline{S}_{yx}\frac{\partial}{\partial\overline{S}_x} + \overline{\psi}_{yy}\frac{\partial}{\partial\overline{\psi}_y} + \overline{S}_{yy}\frac{\partial}{\partial\overline{S}_y} + \cdots, \tag{3.5}$$

and T^1 and T^2 are the components of the conserved vector $T=(T^1,T^2)$. We use the multiplier method [11–13] to obtain conservation laws for the partial differential equations (3.1) and (3.2), which has been applied to fluid flow problems before [4,9,10]. The method requires that we obtain Λ_1 and Λ_2 that satisfy

$$\Lambda_1F_1 + \Lambda_2F_2 = D_xT^1 + D_yT^2 \tag{3.6}$$

for arbitrary functions $\overline{\psi}(x,y)$ and $\overline{S}(x,y)$ and F_1 and F_2 are defined in (3.1) and (3.1). The determining equations for Λ_1 and Λ_2 are obtained by applying Euler operators E_ψ and $E_{\overline{S}}$ to (3.6) and using the fact that these operators annihilate the divergence of the conserved vector. The Euler operator is defined by

$$E_\phi = \frac{\partial}{\partial\phi} - D_x\frac{\partial}{\partial\phi} - D_y\frac{\partial}{\partial\phi_y} + D_x^2\frac{\partial}{\partial\phi_{xx}} + D_xD_y\frac{\partial}{\partial\phi_{xy}} + D_y^2\frac{\partial}{\partial\phi_{yy}} - D_y^3\frac{\partial}{\partial\phi_{yyy}} - \cdots. \tag{3.7}$$

The multipliers can be assumed to depend on x, y, $\overline{\psi}$, $\overline{S}$ and higher derivatives of ψ and $\overline{S}$. Here we limit the dependence to x, y, $\overline{\psi}$ and $\overline{S}$ so that Λ_1 and Λ_2 can be computed readily by hand. Including derivatives of $\overline{\psi}$ and $\overline{S}$ in the dependence of Λ_1 and Λ_2 would require software packages such as GeM in Maple [19]. Solving for Λ_1 and Λ_2 from the determining equations obtained by applying the Euler operators $E_{\overline{\psi}}$ and $E_{\overline{S}}$ to (3.6) we find that Λ_1 is an arbitrary function of x and Λ_2 is a constant. To find the conserved vectors, it sufficient to assume that Λ_1 is a constant. With Λ_1 and Λ_2 as constants, the left hand side of (3.6) can be rewritten in a conservation form and we can extract the conserved vector (T^1,T^2). When $\Lambda_1=1$ and $\Lambda_2=0$ we find the momentum flux conserved vector

$$T^1 = U\overline{\psi}_y, \qquad T^2 = -\nu\overline{\psi}_{yy} + l^2(x)\overline{\psi}_{yy}^2 \tag{3.8}$$

and when $A_1 = 0$ and $A_2 = 1$ we find the thermal flux conserved vector

$$T^1 = U\overline{S}, \qquad T^2 = -\frac{\kappa}{\rho c_v}\overline{S}_y + l(x)l_\theta(x)\overline{\psi}_{yy}\overline{S}_y. \tag{3.9}$$

We denote the momentum flux conserved vector by $T_M = (T^1, T^2)$ and the thermal flux conserved vector $T_\theta = (T^1, T^2)$.

3.2. Conserved quantities

We now treat $\overline{\psi}(x,y)$ and $\overline{S}(x,y)$ as dependent variables that satisfy the system of governing partial differential equations (2.15), (2.16) and boundary conditions (2.17) and (2.18) in order to obtain conserved quantities. Hence the total derivatives in (3.3) become partial derivatives and

$$\frac{\partial T^1}{\partial x} + \frac{\partial T^2}{\partial y} = 0. \tag{3.10}$$

The momentum flux conserved vector T_M has components

$$T^1 = U\frac{\partial\overline{\psi}}{\partial y}, \qquad T^2 = -\nu\frac{\partial^2\overline{\psi}}{\partial y^2} + l^2(x)\left(\frac{\partial^2\overline{\psi}}{\partial y^2}\right)^2. \tag{3.11}$$

The thermal flux conserved vector T_θ has components

$$T^1 = U\overline{S}, \qquad T^2 = -\frac{\kappa}{\rho c_v}\frac{\partial\overline{S}}{\partial y} + l(x)l_\theta(x)\frac{\partial^2\overline{\psi}}{\partial y^2}\frac{\partial\overline{S}}{\partial y}. \tag{3.12}$$

We integrate equation (3.10) with respect to y from the centreline of the wake, $y = 0$, to the boundary of the wake, $y_B(x)$, for the momentum and thermal flux conserved vectors (3.11) and (3.12). In both cases the term $\frac{\partial T^2}{\partial y}$ in (3.10) vanishes after integration due to the boundary conditions (2.17) and (2.18). The equation therefore reduces to

$$\int_0^{y_B(x)} \frac{\partial T^1}{\partial x}dy = 0. \tag{3.13}$$

We use the theorem for differentiation under the integral sign [20] to obtain

$$\frac{d}{dx}\int_0^{y_B(x)} T^1 dy - \frac{dy_B}{dx}T^1(x, y_B(x)) = 0. \tag{3.14}$$

For both the momentum and thermal flux conserved vectors, the second term in (3.14) vanishes after applying boundary conditions (2.17) and (2.18). Hence, substituting (3.8) and (3.9) into (3.14) we find that

$$D = 2\rho U\int_0^{y_B(x)} \frac{\partial\overline{\psi}}{\partial y}(x,y)dy, \qquad K = 2\rho c_v U\int_0^{y_B(x)} \overline{S}(x,y)dy, \tag{3.15}$$

are constants independent of x.

4. Derivation of the Lie point symmetry

The Lie point symmetry generator

$$X = \xi^1(x,y,\overline{\psi},\overline{S})\frac{\partial}{\partial x} + \xi^2(x,y,\overline{\psi},\overline{S})\frac{\partial}{\partial y} + \eta^1(x,y,\overline{\psi},\overline{S})\frac{\partial}{\partial\overline{\psi}} + \eta^2(x,y,\overline{\psi},\overline{S})\frac{\partial}{\partial\overline{S}} \tag{4.1}$$

of the partial differential equations (2.15) and (2.16) satisfies the conditions

$$X^{[3]}F_1\Big|_{F_1=0,F_2=0} = 0, \qquad X^{[3]}F_2\Big|_{F_1=0,F_2=0} = 0, \tag{4.2}$$

where

$$X^{[3]} = X + \zeta_x^2\frac{\partial}{\partial\overline{S}_x} + \zeta_y^2\frac{\partial}{\partial\overline{S}_y} + \zeta_{xy}^1\frac{\partial}{\partial\overline{\psi}_{xy}} + \zeta_{yy}^1\frac{\partial}{\partial\overline{\psi}_{yy}} + \zeta_{yy}^2\frac{\partial}{\partial\overline{S}_{yy}} + \zeta_{yyy}^1\frac{\partial}{\partial\overline{\psi}_{yyy}}, \tag{4.3}$$

F_1 and F_2 are defined in (3.1) and (3.2) and

$$\zeta_i^\alpha = D_i\eta^\alpha - u_k^\alpha D_i\xi^k, \tag{4.4}$$

$$\zeta_{i_1\cdots i_{s-1}i_s}^\alpha = D_{i_s}\zeta_{i_1\cdots i_{s-1}}^\alpha - u_{ki_1\cdots i_{s-1}}^\alpha D_{i_s}\xi^k, \tag{4.5}$$

$$u^1 = \overline{\psi}, \qquad u^2 = \overline{S}. \tag{4.6}$$

The two equations in (4.2) are expanded and split into coefficients of the powers and products of the independent derivatives of $\overline{\psi}$ and $\overline{S}$ to form a system of linear partial differential equations for ξ^1, ξ^2, η^1, η^2, $l(x)$ and $l_\theta(x)$. Solving the system of partial differential equations, it is found that

$$X = \xi^1(x)\frac{\partial}{\partial x} + \left(a_1(x)y + a_2(x)\right)\frac{\partial}{\partial y} + \left(\left(a_1(x) + a_3\right)\overline{\psi} + a_4(x,y)\right)\frac{\partial}{\partial\overline{\psi}} + \left(a_5\overline{S} + a_6\right)\frac{\partial}{\partial\overline{S}} \tag{4.7}$$

provided that

$$\frac{d}{dx}\left(l^2(x)\right)\xi^1 - (3a_1(x) - a_3)l^2(x) + l^2(x)\frac{d\xi^1}{dx} = 0. \tag{4.8}$$

$$\frac{d}{dx}\left(l(x)l_\theta(x)\right)\xi^1 - (3a_1(x) - a_3)l(x)l_\theta(x) + l(x)l_\theta(x)\frac{d\xi^1}{dx} = 0, \tag{4.9}$$

$$2l^2(x)\frac{\partial^3 a_4}{\partial y^3} - U\left[\frac{da_1}{dx}y + \frac{da_2}{dx}\right] = 0, \qquad \nu\left[2\frac{\partial\xi^2}{\partial y} - \frac{d\xi^1}{dx}\right] + 2l^2(x)\frac{\partial^2 a_4}{\partial y^2} = 0, \tag{4.10}$$

$$\nu\frac{\partial^3 a_4}{\partial y^3} - U\frac{\partial^2 a_4}{\partial x\partial y} = 0, \qquad \frac{\kappa}{\rho c_v}\left[2a_1(x) - \frac{d\xi^1}{dx}\right] + l(x)l_\theta(x)\frac{\partial^2 a_4}{\partial y^2} = 0, \tag{4.11}$$

$$U\left[\frac{da_1}{dx}y + \frac{da_2}{dx}\right] - l(x)l_\theta(x)\frac{\partial^3 a_4}{\partial y^3} = 0, \tag{4.12}$$

where a_3, a_5 and a_6 are arbitrary constants.

5. Derivation of the associated Lie point symmetry

The Lie point symmetry (4.7) of the system of partial differential equations (2.15) and (2.16) satisfying (4.7) to (4.12) is associated with the conserved vector (T^1, T^2) provided [15,16]

$$X^{[\infty]}(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i) = 0, \qquad i = 1, 2, \tag{5.1}$$

where $X^{[\infty]}$ is the prolongation of X up to all orders required to evaluate $X^{[\infty]}(T^1)$ and $X^{[\infty]}(T^2)$. When Eq. (5.1) is expanded for $i = 1, 2$ it gives

$$X^{[\infty]}(T^1) + T^1 D_y(\xi^2) - T^2 D_y(\xi^1) = 0, \tag{5.2}$$

$$X^{[\infty]}(T^2) + T^2 D_x(\xi^1) - T^1 D_x(\xi^2) = 0. \tag{5.3}$$

The conserved vectors (3.8) and (3.9) contain derivatives up to second order in $\overline{\psi}$ and first order in $\overline{S}$. Hence we only need to prolongate the Lie point symmetry X up to second-order in $\overline{\psi}$ and first-order in $\overline{S}$. The prolongated Lie point symmetry is therefore

$$X^{[\infty]} = X + \zeta_y^1\frac{\partial}{\partial\overline{\psi}_y} + \zeta_{yy}^1\frac{\partial}{\partial\overline{\psi}_{yy}} + \zeta_y^2\frac{\partial}{\partial\overline{S}_y}, \tag{5.4}$$

where the prolongation terms are defined by (4.4) and (4.5).

5.1. Lie point symmetry associated with the momentum flux conserved vector

Expanding Eq. (5.2) with the conserved vector (3.8), using (4.4) and (4.5) for the prolongation terms and splitting the equation by powers of $\overline{\psi}_y$ it is found that a_4 is an arbitrary function of x, a_1 is a constant and $a_3 = -a_1$. By (4.10), a_2 is also a constant and

$$\xi^2(y) = a_1 y + a_2. \tag{5.5}$$

Eqs. (4.8) to (4.12) reduce to

$$\frac{d}{dx}\left(l^2(x)\right)\xi^1(x) - 4a_1 l^2(x) + l^2(x)\frac{d\xi^1}{dx} = 0, \tag{5.6}$$

$$\frac{d}{dx}\left(l(x)l_\theta(x)\right)\xi^1(x) - 4a_1 l(x)l_\theta(x) + l(x)l_\theta(x)\frac{d\xi^1}{dx} = 0, \tag{5.7}$$

$$\nu\left[2a_1 - \frac{d\xi^1}{dx}\right] = 0, \qquad \frac{\kappa}{\rho c_v}\left[2a_1 - \frac{d\xi^1}{dx}\right] = 0. \tag{5.8}$$

Next expanding (5.3) with the conserved vector (3.8) using (4.4) and (4.5) for the prolongation terms and splitting the equation by powers of $\overline{\psi}_{yy}$ we obtain (5.6) and the first equation in (5.8).

The Lie point symmetry associated with the conserved vector (3.8) becomes, for $\nu = 0$ and $\nu \neq 0$,

$$X = \xi^1(x)\frac{\partial}{\partial x} + \left(a_1 y + a_2\right)\frac{\partial}{\partial y} + a_4(x)\frac{\partial}{\partial \overline{\psi}} + \left(a_5\overline{S} + a_6\right)\frac{\partial}{\partial \overline{S}} \tag{5.9}$$

provided that Eqs. (5.6), (5.7) and (5.8) are satisfied.

5.2. Lie point symmetry associated with the momentum and thermal flux conserved vectors

Expanding Eq. (5.2) using the Lie point symmetry generator (5.9) and the conserved vector (3.9) and splitting the equation by powers of $\overline{S}$, we find that $a_5 = -a_1$ and $a_6 = 0$. Now expanding Eq. (5.3) using the Lie point symmetry generator (5.9), the conserved vector (3.9) using (4.4) and (4.5) for the prolongation terms and splitting the equation into coefficients of $\overline{S}_y$ and $\overline{\psi}_{yy}\overline{S}_y$ we obtain (5.7) and the second equation of (5.8).

The Lie point symmetry associated with the conserved vectors (3.8) and (3.9) is

$$X = \xi^1(x)\frac{\partial}{\partial x} + \left(a_1 y + a_2\right)\frac{\partial}{\partial y} + a_4(x)\frac{\partial}{\partial \overline{\psi}} - a_1\overline{S}\frac{\partial}{\partial \overline{S}} \tag{5.10}$$

provided that Eqs. (5.6) to (5.8) are satisfied.

5.3. Mixing lengths

In this subsection, we show using Eqs. (5.6) and (5.7) that the momentum and the thermal mixing lengths are proportional. From (5.6) we can make $\frac{d\xi^1}{dx}$ the subject and substitute it into (5.7) to obtain

$$\xi^1(x)\frac{d}{dx}\left(l(x)l_\theta(x)\right) = \frac{l_\theta(x)}{l(x)}\xi^1(x)\frac{d}{dx}\left(l^2(x)\right). \tag{5.11}$$

We consider the general case in which $\xi^1(x) \neq 0$. Expanding the derivatives in (5.11) and integrating the resulting equation gives

$$l_\theta(x) = Bl(x), \tag{5.12}$$

where B is a constant.

Eqs. (5.6) and (5.12) are equivalent to Eqs. (5.6) and (5.7). In the following subsections we work with Eqs. (5.6) and (5.12). Eq. (5.7) does not need to be considered further. The cases $\nu \neq 0$, $\kappa \neq 0$ and $\nu = 0$, $\kappa = 0$ are treated separately.

5.4. Case $\nu \neq 0$ and $\kappa \neq 0$

When $\nu \neq 0$ and $\kappa \neq 0$ the equations in (5.8) give

$$\xi^1(x) = 2a_1 x + a_7, \tag{5.13}$$

where a_7 is a constant. We are considering the general case when $\xi^1(x) \neq 0$ and $a_1 \neq 0$. Substituting (5.13) into (5.6) we obtain a variables separable differential equation, which, after integration, yields

$$l(x) = l_1\left(x + \frac{a_7}{2a_1}\right)^{\frac{1}{2}}, \tag{5.14}$$

where l_1 is a positive constant. From (5.12)

$$l_\theta(x) = l_2\left(x + \frac{a_7}{2a_1}\right)^{\frac{1}{2}}, \tag{5.15}$$

where $l_2 = Bl_1$.

Without loss of generality, we divide the Lie point symmetry (5.10) by a_1. Therefore, when $\nu \neq 0$ and $\kappa \neq 0$, the Lie point symmetry associated with the conserved vectors (3.8) and (3.9) is

$$X = \left(2x + a_7\right)\frac{\partial}{\partial x} + \left(y + a_2\right)\frac{\partial}{\partial y} + a_4(x)\frac{\partial}{\partial \overline{\psi}} - \overline{S}\frac{\partial}{\partial \overline{S}}. \tag{5.16}$$

The mixing lengths are

$$l(x) = l_1\left(x + \frac{1}{2}a_7\right)^{\frac{1}{2}}, \qquad l_\theta(x) = l_2\left(x + \frac{1}{2}a_7\right)^{\frac{1}{2}}. \tag{5.17}$$

5.5. Case $\nu = 0$ and $\kappa = 0$

When $\nu = 0$ and $\kappa = 0$, without loss of generality, we divide the associated Lie point symmetry generator (5.10) by a_1. Hence, the Lie point symmetry associated with the conserved vectors (3.8) and (3.9) is

$$X = \xi^{1*}(x)\frac{\partial}{\partial x} + (y + a_2^*)\frac{\partial}{\partial y} + a_4^*(x)\frac{\partial}{\partial \overline{\psi}} - \overline{S}\frac{\partial}{\partial \overline{S}} \tag{5.18}$$

provided from (5.6) and (5.12)

$$\frac{d}{dx}\left(l^2(x)\right)\xi^{1*} - 4l^2(x) + l^2(x)\frac{d\xi^{1*}}{dx} = 0 \tag{5.19}$$

and

$$l_\theta(x) = Bl(x), \tag{5.20}$$

where the asterisks denote division by a_1. We omit the asterisks going forward to simplify notation.

In order to obtain $l(x)$ and $\xi^1(x)$ a second equation in addition to (5.19) is required relating $l(x)$ and $\xi^1(x)$. This equation is derived using Prandtl's hypothesis [21].

6. Invariant solutions

In this section, the general form of the invariant solution will be derived for the two cases, $\nu \neq 0$, $\kappa \neq 0$ and $\nu = 0$, $\kappa = 0$. An analytical solution will be given for the case, $\nu = 0$ and $\kappa = 0$.

6.1. Case $\nu \neq 0$ and $\kappa \neq 0$.

The solution $\overline{\psi} = f(x, y)$ and $\overline{S} = g(x, y)$ is an invariant of the governing partial differential equations (2.15) and (2.16) provided

$$X(\overline{\psi} - f)\Big|_{\overline{\psi}=f(x,y)} = 0, \qquad X(\overline{S} - g)\Big|_{\overline{S}=g(x,y)} = 0. \tag{6.1}$$

Expanding the first equation in (6.1) using the Lie point symmetry generator (5.16) we obtain the partial differential equation

$$\left(2x + a_7\right)\frac{\partial f}{\partial x} + \left(y + a_2\right)\frac{\partial f}{\partial y} = a_4(x). \tag{6.2}$$

Using the method of characteristics to solve the partial differential equation (6.2) it is found that up to an arbitrary additive constant

$$\overline{\psi}(x, y) = F(\xi) + \frac{1}{2}\int^x \frac{a_4(\tau)d\tau}{\left(\tau + \frac{1}{2}a_7\right)}, \tag{6.3}$$

where F is an arbitrary function and

$$\xi = \frac{y + a_2}{\left(x + \frac{1}{2}a_7\right)^{\frac{1}{2}}}. \tag{6.4}$$

Expanding the second equation in (6.1) we obtain the partial differential equation

$$\left(2x + a_7\right)\frac{\partial g}{\partial x} + \left(y + a_2\right)\frac{\partial g}{\partial y} = -g. \tag{6.5}$$

Similarly, using the method of characteristics to solve the partial differential equation (6.5) we obtain

$$\overline{S}(x,y) = \left(x + \frac{1}{2}a_7\right)^{-\frac{1}{2}} G(\xi), \tag{6.6}$$

where G is an arbitrary function.

Writing the conserved quantity (3.15) in terms of the invariant solution (6.3) and (6.4) we obtain

$$D = 2\rho U \int_{L(x)}^{\xi_B(x)} \frac{dF}{d\xi} d\xi, \tag{6.7}$$

where

$$L(x) = \frac{a_2}{\left(x + \frac{1}{2}a_7\right)^{1/2}}, \qquad \xi_B(x) = \frac{y_B(x) + a_2}{\left(x + \frac{1}{2}a_7\right)^{1/2}}. \tag{6.8}$$

Since D is a constant independent of x it follows that $\frac{dD}{dx} = 0$ and using the theorem for differentiating under an integral sign [20] we obtain

$$\frac{d\xi_B}{dx}\frac{dF}{d\xi}(\xi_B(x)) - \frac{dL}{dx}\frac{dF}{d\xi}(L(x)) = 0. \tag{6.9}$$

The mean velocity deficit vanishes on the boundary of the wake and is nonzero on the centreline. Hence, by (2.14) and (6.3), $\frac{dF}{d\xi}$ also vanishes on the boundary of the wake, $\xi = \xi_B(x)$, and is nonzero at the centreline, $\xi = L(x)$. The first term of (6.9) therefore vanishes and Eq. (6.9) reduces to

$$\frac{dL}{dx} = 0. \tag{6.10}$$

Hence, $L(x)$ is a constant and therefore $a_2 = 0$. The upper bound of the integral in (6.7) becomes

$$\xi_B(x) = \frac{y_B(x)}{\left(x + \frac{1}{2}a_7\right)^{1/2}}. \tag{6.11}$$

Since the mean velocity gradient $\frac{\partial \overline{w}}{\partial y}$ vanishes at the boundary $y = y_B(x)$ it follows, by (2.14) and (6.3), that $\frac{d^2F}{d\xi^2}$ also vanishes at $\xi = \xi_B(x)$. Assuming that F is analytic at $\xi = \xi_B(x)$ and not a constant, the derivatives of F at $\xi_B(x)$ cannot be all zero. Therefore there exists an integer $n \geq 3$ such that

$$\frac{d^{n-1}F}{d\xi^{n-1}}(\xi_B(x)) = 0 \qquad \text{but} \qquad \frac{d^nF}{d\xi^n}(\xi_B(x)) \neq 0. \tag{6.12}$$

Differentiating the first equation in (6.12) with respect to x, we have

$$\frac{d^nF}{d\xi^n}(\xi_B(x))\frac{d\xi_B}{dx} = 0. \tag{6.13}$$

Therefore $\xi_B(x)$ is a constant and from (6.11)

$$y_B(x) = \xi_B \left(x + \frac{1}{2}a_7\right)^{1/2}. \tag{6.14}$$

The conserved quantity (6.7) becomes

$$D = 2\rho U \int_0^{\xi_B} \frac{dF}{d\xi} d\xi. \tag{6.15}$$

The half-width of the wake $y_B(x)$, given by (6.14), is proportional to the mixing lengths, (5.14) and (5.15). When the boundary of the wake is infinite, $\xi_B = \infty$. Writing the second conserved quantity (3.15) in terms of the invariant solution (6.6) and (6.4) we obtain

$$K = 2\rho c_v U \int_0^{\xi_B} G(\xi) d\xi. \tag{6.16}$$

The mean velocity deficit

$$\overline{w}(x,y) = \frac{\partial \overline{\psi}}{\partial y} = \frac{1}{\left(x + \frac{1}{2}a_7\right)^{\frac{1}{2}}} \frac{dF}{d\xi} \tag{6.17}$$

should be finite at all nonnegative values of x except at the trailing edge of the obstacle, $x = 0$. Hence the only possible value for a_7 is zero. This is consistent with the fact that the mixing lengths in (5.17) must vanish as $x \to 0$ because the flow is approximately laminar close to the obstacle.

The first boundary condition in (2.17) gives

$$a_4(x) - \xi \frac{dF}{d\xi}\bigg|_{\xi=0} = 0. \tag{6.18}$$

Since $\frac{dF}{d\xi}(0)$ is finite, because $\overline{w}(x,0)$ is given by (6.17), it follows that $a_4(x) = 0$. Writing boundary conditions (2.17) and (2.18) in terms of the invariant solution (6.3), (6.6) and (6.4), all with $a_2 = a_4 = a_7 = 0$, we obtain

$$\frac{d^2F}{d\xi^2}(0) = 0, \qquad \frac{dF}{d\xi}(\xi_B) = 0, \qquad \frac{d^2F}{d\xi^2}(\xi_B) = 0, \tag{6.19}$$

$$\frac{dG}{d\xi}(0) = 0, \qquad G(\xi_B) = 0, \qquad \frac{dG}{d\xi}(\xi_B) = 0. \tag{6.20}$$

Substituting the invariant solution (6.3), (6.6), (6.4) and the mixing lengths (5.14) and (5.15) into the governing partial differential equations (2.15) and (2.16) we obtain

$$\frac{d}{d\xi}\left[\left(\nu - l_1^2 \frac{d^2F}{d\xi^2}\right)\frac{d^2F}{d\xi^2}\right] + \frac{U}{2}\frac{d}{d\xi}\left(\xi\frac{dF}{d\xi}\right) = 0, \tag{6.21}$$

$$\frac{d}{d\xi}\left[\left(\frac{\kappa}{\rho c_v} - l_1 l_2 \frac{d^2F}{d\xi^2}\right)\frac{dG}{d\xi}\right] + \frac{U}{2}\frac{d}{d\xi}(\xi G) = 0. \tag{6.22}$$

The differential equations (6.21) and (6.22) can be integrated at least once. This demonstrates the double reduction theorem by Sjöberg [5, 6]. Eq. (6.21) agrees with the ordinary differential equation for the invariant solution of the classical wake derived by Hutchinson and Mason [8].

In summary, for $\nu \neq 0$ and $\kappa \neq 0$, the Lie point symmetry is

$$X = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \overline{S}\frac{\partial}{\partial \overline{S}}, \tag{6.23}$$

the invariant solution is of the form

$$\overline{\psi}(x,y) = F(\xi), \qquad \overline{S}(x,y) = x^{-1/2}G(\xi), \qquad \xi = \frac{y}{x^{1/2}}, \tag{6.24}$$

the mixing lengths and boundary of the wake are

$$l(x) = l_1 x^{1/2}, \qquad l_\theta(x) = l_2 x^{1/2}, \qquad y_B(x) = \xi_B x^{1/2} \tag{6.25}$$

and the conserved quantities are

$$D = 2\rho U \int_0^{\xi_B} \frac{dF}{d\xi} d\xi, \qquad K = 2\rho c_v U \int_0^{\xi_B} G(\xi) d\xi, \tag{6.26}$$

where $F(\xi)$ and $G(\xi)$ satisfy the system of ordinary differential equations (6.21) and (6.22) subject to the boundary conditions (6.19) and (6.20).

6.2. Case $\nu = 0$ and $\kappa = 0$

When $\nu = 0$ and $\kappa = 0$ there are two determining equations, (5.19) and (5.20), for the three unknowns $\xi^1(x)$, $l(x)$ and $l_\theta(x)$. We obtain the third equation by making use of Prandtl's hypothesis which states that for large values of the Reynolds number the motions in sections perpendicular to the direction of flow are mechanically and geometrically similar. With this hypothesis Prandtl showed that in the wake behind a solid body the mixing length is approximately proportional to the breadth of the wake [21].

An invariant solution, $\overline{\psi} = f(x,y)$ and $\overline{S} = g(x,y)$, of the system of partial differential equations (2.15) and (2.16) satisfies conditions (6.1). Expanding the first equation in (6.1) using the Lie point symmetry generator (5.18) we obtain the partial differential equation

$$\xi^1(x)\frac{\partial f}{\partial x} + (y + a_2)\frac{\partial f}{\partial y} = a_4(x). \tag{6.27}$$

Using the method of characteristics the solution of the partial differential equation (6.27) is obtained as

$$\overline{\psi}(x,y) = F(\xi) + \int^x \frac{a_4(\tau)d\tau}{\xi^1(\tau)}. \tag{6.28}$$

up to an arbitrary additive constant, where F is an arbitrary function and

$$\xi = (y + a_2)\exp\left(-\int^x \frac{d\tau}{\xi^1(\tau)}\right). \tag{6.29}$$

Similarly to the case $\nu \neq 0$, $\kappa \neq 0$, it can be shown that $a_2 = 0$ and the half-width of the wake, $y_B(x)$, is

$$y_B(x) = \xi_B \exp\left(\int^x \frac{d\tau}{\xi^1(\tau)}\right). \tag{6.30}$$

where ξ_B is a constant. By Prandtl's hypothesis [21], the mixing length $l(x)$ is proportional to the breadth of the wake. Therefore

$$l(x) = l_1 \exp\left(\int^x \frac{d\tau}{\xi^1(\tau)}\right), \tag{6.31}$$

where l_1 is a constant. Substituting (6.31) into (5.19) and solving for $\xi^1(x)$ gives

$$\xi^1(x) = 2x + a_7, \tag{6.32}$$

where a_7 is a constant. Substituting (6.32) into (6.31) we have

$$l(x) = l_1\left(x + \frac{1}{2}a_7\right)^{\frac{1}{2}}. \tag{6.33}$$

By (5.20) the thermal mixing length is

$$l_\theta(x) = l_2\left(x + \frac{1}{2}a_7\right)^{\frac{1}{2}}, \tag{6.34}$$

where $l_2 = Bl_1$ is a constant. Hence the invariant solution (6.28) and (6.29), by (6.32), becomes

$$\overline{\psi}(x, y) = F(\xi) + \int^x \frac{a_4(\tau)d\tau}{2\tau + a_7}, \tag{6.35}$$

where, since $a_2 = 0$,

$$\xi = y\exp\left(-\int^x \frac{d\tau}{\xi^1(\tau)}\right) = \frac{y}{\left(x + \frac{1}{2}a_7\right)^{1/2}}. \tag{6.36}$$

The mean velocity deficit is

$$\overline{w}(x, y) = \frac{\partial\overline{\psi}}{\partial y} = \frac{1}{\left(x + \frac{1}{2}a_7\right)^{\frac{1}{2}}}\frac{dF}{d\xi}. \tag{6.37}$$

Again, since $\overline{w}(x, y)$ must be finite at all nonnegative values of x except at $x = 0$ it follows that $a_7 = 0$. Therefore, the associated Lie point symmetry (5.18) becomes

$$X = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + a_4(x)\frac{\partial}{\partial\overline{\psi}} - \overline{S}\frac{\partial}{\partial\overline{S}}. \tag{6.38}$$

We now determine the invariant solution $\overline{S} = g(x, y)$ which satisfies the second condition in (6.1). Expanding this condition using the Lie point symmetry (6.38) we obtain the partial differential equation

$$2x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = -g. \tag{6.39}$$

Using again the method of characteristics to solve (6.39) we obtain

$$\overline{S}(x, y) = x^{-\frac{1}{2}}G(\xi), \tag{6.40}$$

where G is an arbitrary function and

$$\xi = \frac{y}{x^{\frac{1}{2}}}. \tag{6.41}$$

Substituting the invariant solution (6.28), (6.40) and (6.41) into the conserved quantities in (3.15) we obtain

$$D = 2\rho U\int_0^{\xi_B}\frac{dF}{d\xi}d\xi, \qquad K = 2\rho c_v U\int_0^{\xi_B} G(\xi)d\xi. \tag{6.42}$$

Again, the boundary condition

$$\frac{\partial\overline{\psi}}{\partial x}(x, 0) = 0 \tag{6.43}$$

gives $a_4(x) = 0$. Writing the remaining boundary conditions (2.17) and (2.18) in terms of the invariant solution (6.28) with $a_4(x) = 0$ and (6.40) we obtain

$$\frac{d^2F}{d\xi^2}(0) = 0, \qquad \frac{dF}{d\xi}(\xi_B) = 0, \qquad \frac{d^2F}{d\xi^2}(\xi_B) = 0, \tag{6.44}$$

$$\frac{dG}{d\xi}(0) = 0, \qquad G(\xi_B) = 0. \tag{6.45}$$

Substituting the invariant solution (6.35) with $a_4(x) = 0$, (6.40), (6.41) and the mixing lengths (6.33), (6.34) with $a_7 = 0$ into the governing partial differential equations (2.15) and (2.16), with $\nu = 0$ and $\kappa = 0$, we obtain

$$l_1^2\frac{d}{d\xi}\left[\left(\frac{d^2F}{d\xi^2}\right)^2\right] - \frac{U}{2}\frac{d}{d\xi}\left(\xi\frac{dF}{d\xi}\right) = 0, \tag{6.46}$$

$$l_1 l_2\frac{d}{d\xi}\left[\frac{d^2F}{d\xi^2}\frac{dG}{d\xi}\right] - \frac{U}{2}\frac{d}{d\xi}(\xi G) = 0. \tag{6.47}$$

The differential equations (6.46) and (6.47) can be integrated at least once. This demonstrates again the double reduction theorem of Sjöberg [5,6].

In summary, for $\nu = 0$ and $\kappa = 0$, the Lie point symmetry, the general form of the invariant solution, the two mixing lengths, the boundary of the wake and the two conserved quantities are again given by (6.23) to (6.26) and are the same as for $\nu \neq 0$ and $\kappa \neq 0$. The ordinary differential equations, (6.46) and (6.47), satisfied by $F(\xi)$ and $G(\xi)$ are obtained from (6.21) and (6.22) by setting $\nu = 0$ and $\kappa = 0$. The boundary conditions, (6.44) and (6.45), are the same as (6.19) and (6.20) for $\nu \neq 0$ and $\kappa \neq 0$ except that the condition $\frac{dG}{d\xi}(\xi_B) = 0$ does not apply when $\kappa = 0$. The derivation of the results for $\nu = 0$ and $\kappa = 0$ was at times different from that for $\nu \neq 0$ and $\kappa \neq 0$. For example, the Prandtl hypothesis was required for $\nu = 0$ and $\kappa = 0$ to determine the Lie point symmetry and the boundary of the wake.

Both Prandtl [17] and Goldstein [18] in their solution for the two-dimensional turbulent thermal wake assumed that the mixing lengths, $l(x)$ and $l_\theta(x)$, are proportional and also that they are proportional to the width of the wake. For both cases, $\nu \neq 0$, $\kappa \neq 0$ and $\nu = 0$, $\kappa = 0$ we were able to prove that $l(x)$ and $l_\theta(x)$ are proportional. For the case $\nu \neq 0$, $\kappa \neq 0$ we found that the mixing lengths are proportional to the width of the wake but we will find that the numerical results indicate that the width of the wake is infinite. For the case $\nu = 0$, $\kappa = 0$, the width of the wake is finite but Prandtl's hypothesis had to be invoked.

6.3. *Analytical solution for $\nu = 0$ and $\kappa = 0$*

We will consider the upper half of the wake. The differential equations for the invariant solution when $\nu = 0$ and $\kappa = 0$ are (6.46) and (6.47). Integrating (6.46), imposing the first boundary condition in (6.44) and taking the square root of the equation gives

$$l_1\frac{d^2F}{d\xi^2} = \pm\sqrt{\frac{U}{2}}\xi^{\frac{1}{2}}\left(\frac{dF}{d\xi}\right)^{\frac{1}{2}}. \tag{6.48}$$

The second derivative of F is real since

$$\overline{w}(x, y) = \frac{1}{x^{1/2}}\frac{dF}{d\xi} > 0. \tag{6.49}$$

For a classical wake, the spatial gradient in the y-direction of the mean velocity deficit is negative in the upper half of the wake. Hence we take the minus sign in (6.48). Eq. (6.48) is a first order differential equation in $\frac{dF}{d\xi}$ and it is variables separable. Separating the variables $H = \frac{dF}{d\xi}$ and ξ in (6.48) and integrating the resulting equation yields

$$\frac{dF}{d\xi} = \frac{U}{18l_1^2}\left(\xi_B^{3/2} - \xi^{3/2}\right)^2. \tag{6.50}$$

To calculate the conserved quantity D and the fluid variables $\overline{w}(x, y)$ and $\overline{v}_y(x, y)$ it is sufficient to derive $\frac{dF}{d\xi}$ and $F(\xi)$ does not need to be obtained. The second derivative $\frac{d^2F}{d\xi^2}$ is important for physical variables such as the shear stress τ_{xy}. Substituting (6.50) into (6.48) with the minus sign we obtain

$$\frac{d^2F}{d\xi^2} = -\frac{U}{6l_1^2}\xi^{1/2}\left(\xi_B^{3/2} - \xi^{3/2}\right). \tag{6.51}$$

We integrate (6.47) and apply the first boundary condition in (6.44). We then substitute (6.51) for the second derivative of F and integrate the resulting equation by making the change of variable $\eta = \xi^{3/2}$ to obtain

$$G(\xi) = G(0)\xi_B^{-3\frac{l_1}{l_2}}\left(\xi_B^{3/2} - \xi^{3/2}\right)^{2\frac{l_1}{l_2}}. \tag{6.52}$$

The boundary condition $G(\xi_B) = 0$ is identically satisfied. The constant $G(0)$ cannot be obtained from the boundary conditions. It will be determined using the second conserved quantity in (6.42).

Substituting (6.50) into the first conserved quantity in (6.42) and solving for ξ_B gives

$$\xi_B = \left(20\frac{l_1^2}{\rho U^2}D\right)^{\frac{1}{4}}. \tag{6.53}$$

Substituting (6.52) into the second conserved quantity in (6.42) and using the properties of Beta and Gamma functions [20] we find that

$$G(0) = \frac{3K}{4\rho c_v U\xi_B}\frac{\left(\frac{1}{3}+\frac{l_1}{l_2}\right)}{\frac{l_1}{l_2}}\frac{\Gamma\left(2\left(\frac{1}{3}+\frac{l_1}{l_2}\right)\right)}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(2\frac{l_1}{l_2}\right)}. \tag{6.54}$$

The physical variables become

$$\overline{w}(x,y) = \frac{1}{x^{1/2}}\frac{dF}{d\xi} = \frac{U}{18l_1^2x^{1/2}}\left(\xi_B^{3/2} - \xi^{3/2}\right), \tag{6.55}$$

$$\overline{v}_y(x,y) = -\frac{1}{2x}\frac{dF}{d\xi} = -\frac{U}{36l_1^2x}\left(\xi_B^{3/2} - \xi^{3/2}\right), \tag{6.56}$$

$$\overline{S}(x,y) = x^{-1/2}G(\xi) = x^{-1/2}G(0)\xi_B^{-3\frac{l_1}{l_2}}\left(\xi_B^{3/2} - \xi^{3/2}\right)^{2\frac{l_1}{l_2}}, \tag{6.57}$$

where ξ_B is determined in terms of D by (6.53).

6.4. Numerical solution for $\nu \neq 0$ and $\kappa \neq 0$

We solve the differential equations, (6.46) and (6.47), by using a shooting method described in Mubai and Mason [4] to convert the problem to an initial value problem. The targets of the shooting method are the conserved quantities given in (6.26). We solve the resulting initial value problem using the Python function scipy.integrate.odeint.

7. Lines of constant mean temperature difference and streamlines of the mean velocity deficit

The analytical solution for $\nu = 0$ and $\kappa = 0$ for the mean temperature difference and stream function lead directly to the analytical solution for the lines of constant mean temperature difference and the streamlines of the mean velocity deficit.

7.1. Lines of constant mean temperature difference for $\nu = 0$ and $\kappa = 0$

The lines of constant mean temperature difference are defined by

$$\overline{S}(x,y) = C, \tag{7.1}$$

where the constant C takes a range of discrete values. Hence, by (6.57) and (6.41), on a line of constant mean temperature difference

$$y = \xi_B\left[1 - \left(\frac{C}{G(0)}\right)^{\frac{l_2}{2l_1}}x^{\frac{l_2}{4l_1}}\right]^{2/3}x^{1/2}. \tag{7.2}$$

For y to be real it is necessary that

$$0 \leq x \leq \left(\frac{G(0)}{C}\right)^2. \tag{7.3}$$

When $C = 0$ the mean temperature difference $\overline{S}(x,y) = 0$ and (7.2) reduces to the boundary of the wake

$$y = \xi_B x^{1/2}. \tag{7.4}$$

For a given value of $C > 0$ the lines of constant mean temperature difference for $y > 0$ start at $x = 0$ with gradient ∞ and extend to $x = (G(0)/C)^2$ where the gradient is $-\infty$. When $C = 0$ the line reduces to the boundary of the wake. Graphs of the lines of constant mean temperature difference are presented in Fig. 6 with the other results in Section 8.

7.2. Streamlines of the mean velocity deficit for $\nu = 0$ and $\kappa = 0$

The stream function $\overline{\psi}(x,y)$ for the mean velocity deficit $\overline{w}(x,y)$ and $\overline{v}_y(x,y)$ satisfies (2.14). Hence if vector $\mathbf{A} = (-\overline{w}, \overline{v}_y)$ then

$$(\boldsymbol{A}\cdot\nabla)\overline{\psi} = 0 \tag{7.5}$$

and ψ is constant along curves with tangent vector $\boldsymbol{A}$ at each point. They are the streamlines, as seen by an observer moving with the mainstream velocity U, of the mean velocity deficit directed towards the obstacle. The curves are defined by

$$\overline{\psi}(x,y) = k_1, \tag{7.6}$$

where the constant k_1 is given a range of discrete values. Eq. (7.6) is written in terms of ξ by using (6.28), with $a_4(x) = 0$. Eq. (6.50) is integrated to obtain $F(\xi)$ and the constant of integration is incorporated in k_1. Hence

$$\frac{U}{72l_1^2}\xi\left[\xi^3 - \frac{16}{5}\xi_B^{3/2}\xi^{3/2} + 4\xi_B^3\right] = k_1. \tag{7.7}$$

Let $u = \frac{\xi}{\xi_B}$. Then

$$u^4 - \frac{16}{5}u^{5/2} + 4u = k_2, \tag{7.8}$$

where

$$k_2 = \frac{72l_1^2}{U\xi_B^4}k_1 \tag{7.9}$$

and the equation of a streamline of the mean velocity deficit is

$$y = u\xi_B x^{1/2}, \qquad 0 \leq x \leq \infty. \tag{7.10}$$

When $u = 1$, then $k_2 = \frac{9}{5}$ and (7.10) reduces to the boundary of the wake $y = \xi_B x^{\frac{1}{2}}$ while when $u = 0$, then $k_2 = 0$ and (7.10) reduces to the centre line $y = 0$. The boundary of the wake and the centre line are limiting streamlines of the mean velocity deficit in the upper half of the wake. The function

$$f(u) = u^4 - \frac{16}{5}u^{5/2} + 4u \tag{7.11}$$

is an increasing function of u for $0 \leq u \leq 1$ and therefore for each value of k_2 in the range $0 \leq k_2 \leq \frac{9}{5}$ there is only one value of u in the range $0 \leq u \leq 1$.

The streamlines of the mean velocity deficit are obtained as follows for a given value of ξ_B. Choose k_2 in the range $0 \leq k_2 \leq \frac{9}{5}$. Solve Eq. (7.8) for u. Then the equation of the streamline for a given value of k_2 is (7.10). By giving k_2 discrete values in the range $0 \leq k_2 \leq \frac{9}{5}$ the streamlines can be plotted. They are presented in Fig. 7 with the other results in Section 8.

8. Results and discussion

In this section we present the analytical and numerical results for the classical turbulent thermal wake. We show the role played by the kinematic viscosity and thermal conductivity in the analysis of the wake. For all plots in this section, $l_1 = 1$, $l_2 = 1$, $D = 1$, $K = 1$, $\rho = 1$,

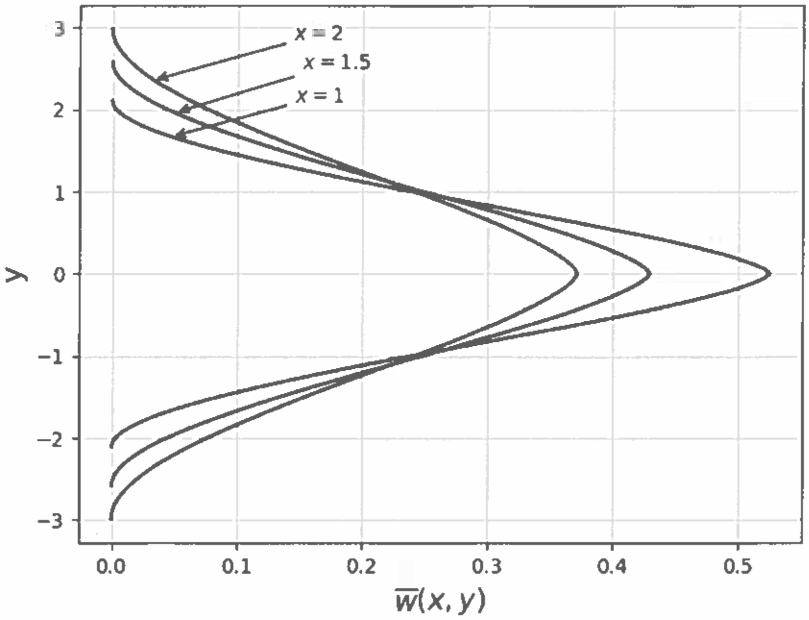

Fig. 2. The mean velocity deficit $\overline{w}(x, y)$ for $\nu = 0$ plotted against y at $x = 1$, $x = 1.5$ and $x = 2$ using the exact analytical solution (6.55).

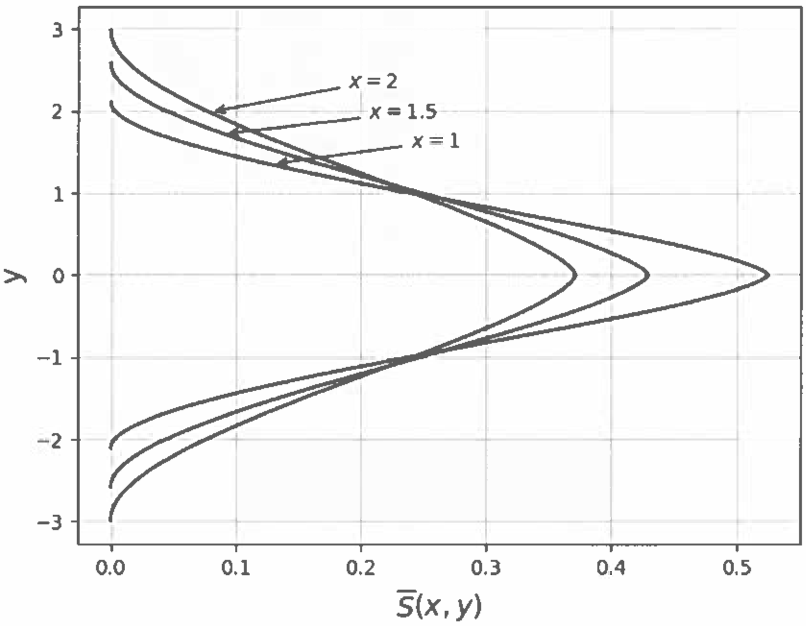

Fig. 3. The mean temperature difference $\overline{S}(x, y)$ for $\nu = 0$ and $\kappa = 0$ plotted against y at $x = 1$, $x = 1.5$ and $x = 2$ using the exact analytical solution (6.57).

$c_\nu = 1$ and $U = 1$. For these values, when $\nu = 0$ and $\kappa = 0$, Eqs. (6.53) and (6.54) give $\xi_B = 0.5254$ and $G(0) = 2.1147$.

Fig. 2 shows the plots for the mean velocity deficit profiles when $\nu = 0$ at $x = 1$, $x = 1.5$ and $x = 2$. The graphs are plotted using the analytical solution (6.55). We see that as the distance from the obstacle, x, increases the mean velocity deficit at the centreline, $\overline{w}(x, 0)$, decreases and the boundary of the wake increases. This shows that the turbulent diffusion increases with the distance from the obstacle, x. The flow becomes more turbulent as the distance from the obstacle increases. Turbulence increases the momentum diffusion in the mean flow. We also see that the boundary of the wake is finite.

Fig. 3 shows the plots for the mean temperature difference profiles when $\nu = 0$ and $\kappa = 0$ at $x = 1$, $x = 1.5$ and $x = 2$. The graphs are plotted using the analytical solution (6.57). We see that as the distance from the obstacle, x, increases the mean temperature difference at the centreline, $\overline{S}(x, 0)$, decreases and the boundary of the wake increases. This shows that turbulent thermal diffusion increases with the distance from the obstacle, x. The flow becomes more turbulent as the distance from the obstacle increases and the turbulence increases the thermal diffusion in the mean flow. We see that the thermal boundary is equal to the momentum boundary in Fig. 2.

Fig. 4 shows the plots for the mean velocity deficit profile at $x = 1$ for various values of kinematic viscosity. The graph for $\nu = 0$ is plotted using the analytical solution (6.55) and the graphs for $\nu \neq 0$ are plotted using

$$\overline{w}(x, y) = \frac{1}{x^{1/2}} \frac{dF}{d\xi}, \tag{8.1}$$

where $\frac{dF}{d\xi}$ is obtained by solving differential equations (6.46) and (6.47) using the shooting method numerical scheme described in Mubai and

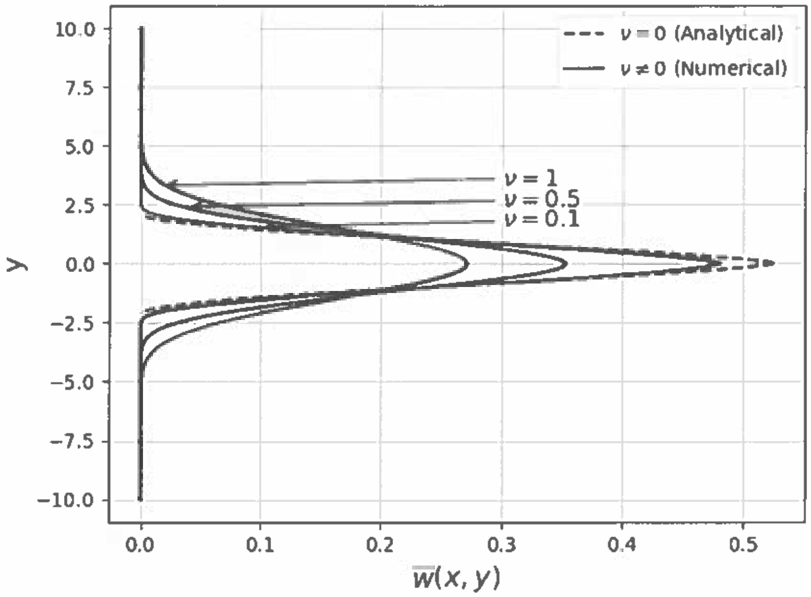

Fig. 4. The mean velocity deficit $\overline{u}(x, y)$ plotted at $x = 1$ against y for $\nu = 0$, $\nu = 0.1$, $\nu = 0.5$ and $\nu = 1$.

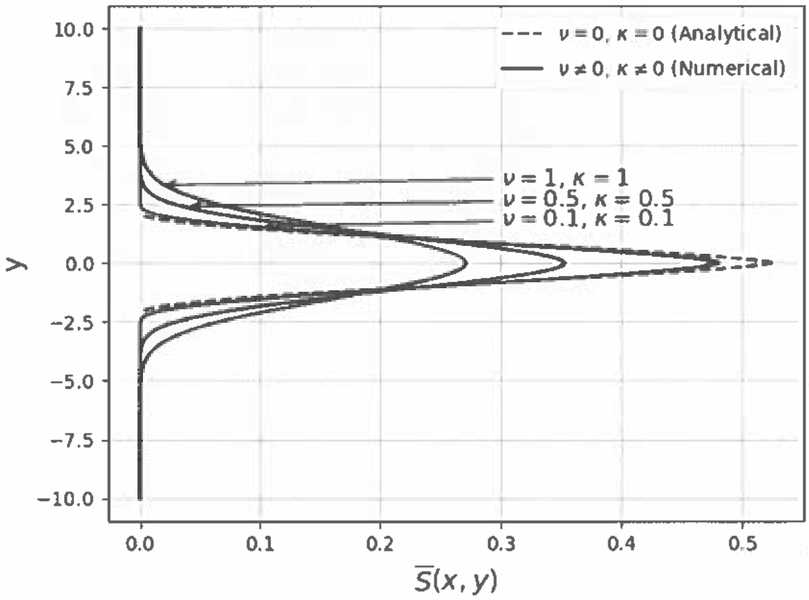

Fig. 5. The mean temperature difference $\overline{S}(x, y)$ plotted at $x = 1$ against y for $\nu = 0$, $\kappa = 0$; $\nu = 0.1$, $\kappa = 0.1$; $\nu = 0.5$, $\kappa = 0.5$ and $\nu = 1$, $\kappa = 1$.

Mason [4]. We see that, as the kinematic viscosity increases, the mean velocity deficit at the centreline, $\overline{u}(1, 0)$, decreases and the profile becomes broader along the y-axis. We also see that as the kinematic viscosity tends to zero in the numerical solution the mean velocity deficit becomes closer to the analytical solution for the mean velocity deficit when the kinematic viscosity vanishes. Hence, the numerical solution for the mean velocity deficit is in agreement with the analytical solution (6.55) when $\nu = 0$. Since any numerical scheme is limited by its tolerance we cannot definitively discern whether the width of the wake is finite or infinite using the numerical results when $\nu \neq 0$. We believe for physical reasons that the solution extends to $y = \pm\infty$ whenever $\nu \neq 0$ because the fluid remains viscous.

Fig. 5 shows the plots for the mean temperature difference profile at $x = 1$ for a range of values of kinematic viscosity and thermal conductivity. The graph for $\nu = 0$ and $\kappa = 0$ is plotted using the analytical solution (6.57) and the graphs for $\nu \neq 0$ and $\kappa \neq 0$ are plotted using

$$\overline{S}(x, y) = \frac{1}{x^{1/2}} G(\xi), \tag{8.2}$$

where $G(\xi)$ is obtained by solving the differential equations (6.46) and (6.47) using the shooting method numerical scheme described by Mubai and Mason [4]. We see that, as the kinematic viscosity and thermal conductivity increase, the mean temperature difference at the centreline, $\overline{S}(1, 0)$, decreases and the profile becomes broader along the y-axis. We also see that as the kinematic viscosity and the thermal conductivity tend to zero in the numerical solution the mean temperature difference tends closer to that obtained using the analytical solution for $\nu = 0$ and $\kappa = 0$. Hence, the numerical solution for the mean temperature difference is in agreement with the analytical solution (6.57). Since any numerical scheme is limited by its tolerance we cannot definitively discern whether the thermal boundary is finite using the numerical results when $\nu \neq 0$ and $\kappa \neq 0$. However, we believe

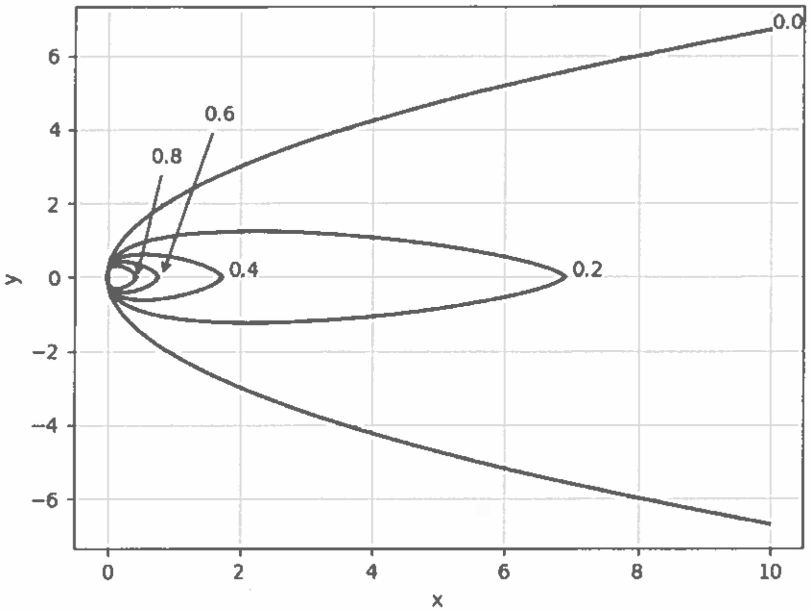

Fig. 6. The lines of constant mean temperature difference for $\nu = 0$ and $\kappa = 0$ plotted using (7.2). On the boundary of the wake the mean temperature difference is zero. The curves are labelled by values of the constant mean temperature difference C.

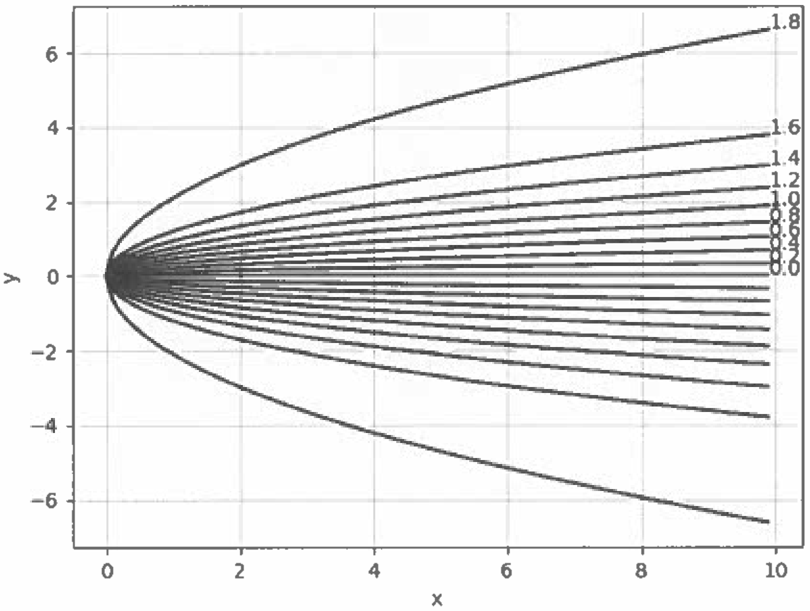

Fig. 7. Streamlines for the mean velocity deficit of the turbulent thermal classical wake when $\nu = 0$ plotted using (7.8) and (7.10) with $\xi_B = 0.5254$ and values of k_2 in the range $0 \le k_2 \le 1.8$ in the upper half of the wake. The boundary of the wake is $k_2 = 1.8$ and the curves in the upper half are reflected about the x-axis to obtain the streamlines in the lower half. The direction of flow as seen by an observer moving with the mainstream velocity U is towards the origin.

for physical reasons that when $\nu \neq 0$ and $\kappa \neq 0$ the solution for the mean temperature difference extends to $y = \pm\infty$ because the fluid remains thermally conducting.

Fig. 6 shows a plot of lines of constant mean temperature difference when $\nu = 0$ and $\kappa = 0$. The lines in the lower half of the wake are obtained by reflecting the curves in the upper half about the x-axis. The lines start at $x = 0$ with a gradient of infinite magnitude and end at $x = (G(0)/C)^2$ also with gradient of infinite magnitude. The quantities ξ_B and $G(0)$ are given by (6.53) and (6.54). Fig. 6 also shows the relation between the lines of constant mean temperature difference and the boundary of the wake. We see that the mean temperature difference is greater closer to the heated obstacle and decreases as the distance along the centreline increases. The mean temperature difference decreases as the lines of constant mean temperature difference become closer to the boundary of the thermal wake. The lines of constant mean temperature difference are closer to each other near the obstacle and more spread out as the lines approach the boundary of the thermal wake because the mean heat flux is greater near the heated obstacle and vanishes at the boundary of the thermal wake.

In Fig. 7 the streamlines for the mean velocity deficit are plotted for the turbulent thermal wake when $\nu = 0$ using Eqs. (7.8) and (7.10) for values of k_2 in the range $0 \leq k_2 \leq \frac{9}{5}$ in the upper half of the wake. The streamlines in the lower half of the wake are obtained by reflecting the streamlines in the upper half about the x-axis. The mean velocity deficit vector $(-\overline{w}, \overline{v}_y)$ as seen by an observer moving with the mainstream velocity U in the $+x$ direction is tangential to the curves at each point. For the same interval of k_2 the streamlines are closer together near the centre line than near the boundary of the wake because the mean velocity deficit is greater near the centre line and vanishes on the boundary. From (6.56)

$$\overline{v}_y = -\frac{U}{36 l_1^2 x}\left(\xi_B^{3/2} - \xi^{3/2}\right) < 0 \tag{8.3}$$

and the distance of the streamline from the centre line decreases as x decreases towards the obstacle.

9. Conclusions

It was sufficient to consider multipliers Λ_1 and Λ_2 that are functions only of x, y, $\overline{\psi}$ and $\overline{S}$. The calculations could therefore be performed manually. It was found that Λ_2 is a constant and that Λ_1 is an arbitrary function of x. It was also sufficient to take Λ_1 as a constant to derive two conserved vectors for the system of two partial differential equations. The two conserved quantities were derived in a systematic way from the conserved vectors and boundary conditions.

We found that for all values of the kinematic viscosity and thermal conductivity the thermal mixing length is proportional to the momentum mixing length. Otherwise the cases $\nu \neq 0$, $\kappa \neq 0$ and $\nu = 0$, $\kappa = 0$ needed to be treated separately. For $\nu \neq 0$, $\kappa \neq 0$ the associated Lie point symmetry could be determined completely. However an analytical solution of the reduced ordinary differential equations could not be obtained but a numerical solution was derived using a shooting method with the two conserved quantities as targets. The width of the wake is infinite in the $\pm y$-directions because for sufficiently large values of $|y|$, ν and κ will become larger than the eddy viscosity and thermal eddy conductivity and the turbulent wake behaves increasingly like a laminar wake which extends to $y = \pm\infty$. In comparison, for $\nu = 0$ and $\kappa = 0$, the associated Lie point symmetry could not be completely determined and required Prandtl's hypothesis that the mixing length is proportional to the width of the wake to complete the solution. An analytical solution could be derived and the width of the wake is finite. The analytical results lead to the derivation of the lines of constant mean temperature difference which were found to be closed curves emanating from the trailing end of the obstacle and the streamlines of the mean velocity deficit. Unlike the two-dimensional turbulent thermal jet [4] the numerical solution for the wake as $\nu \to 0$, $\kappa \to 0$ matched the analytical solution (with Prandtl's hypothesis) for $\nu = 0$, $\kappa = 0$. This provided a check on the numerical shooting method.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Erick Mubai reports was provided by University of the Witwatersrand Johannesburg. Erick Mubai reports a relationship with University of the Witwatersrand Johannesburg that includes: employment.

Data availability

Data will be made available on request.

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