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An analysis of the nonlinearizable classes of Emden-Liénard equations

M. Ahmed[1] **· A. H. Kara**[1,2]

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Abstract
We present a general method to construct first integrals for some classes of the well known second-order ordinary differential equations, viz., the Emden and Liénard classes of equations. The approach does not require a knowledge of a Lagrangian but, rather, uses the 'multiplier approach' [1, 2]. It is then shown how a study of the invariance properties and conservation laws are used to 'twice' reduce the equations to solutions. In this paper, we restrict our work to the nonlinearizable cases.

Keywords First integrals · Reduction · Nonlinearizable Emden and Liénard equations

Mathematics Subject Classification 70H33 · 37K06 · 70S10

1 Introduction

In [4], Tiwari et al. carry out a complete classification of the Lie point symmetry groups associated with the quadratic, ordinary differential (ode) Liénard type equation, $y'' + f(y)y'^2 + g(y) = 0$, for which the Lie analysis split into the maximal (eight parameter) symmetry group equivalent to $sl(3, R)$ which is linearizable and the non-maximal (three, two, and one parameter) symmetry groups. For the non-maximal cases, the identified equations are integrable and contain some examples such as the 'Mathews-Lakshmanan oscillator particle on a rotating parabolic well'.

In [10], we computed the first integrals of the various classes of the linearizable cases without recourse to constructing the respective Lagrangians and Noether's Theorem [3, 5]. In essence, the method generally assumes that there exists a relationship between symmetries and conservation laws [6, 7] and the computation adopts the method expounded in the 'multiplier approach' [1, 2]. In each case, a maximal five first integrals (conserved quantities) were obtained corresponding to the five independent multipliers. This matches with the equivalent five that are obtainable from the standard Lagrangian and variational symmetries of the linear free particle or oscillator equations via Noether's Theorem. In this paper, we focus on the first integrals of the classes of nonlinearizable quadratic Liénard type equations. The method

✉ A. H. Kara
Abdul.Kara@wits.ac.za

1 School of Mathematics, University of the Witwatersrand, Johannesburg, Wits 2050, South Africa

2 Centre of Excellence, Mathematical and Statistical Sciences (CoE MASS), Johannesburg, South Africa

of multipliers is, vastly, more useful as there is no need to construct a Lagrangian for the underlying ode. To demonstrate a possible relationship between Lie point symmetries and first integrals, we need only check if the first integral is invariant under the symmetry. The primary application of this relationship lies in the double reduction of the ode (see [6] and, more recently [8, 9] - in [10], the method for odes is presented in detail.

2 The Emden equations

I. The three paramenter case

It has been shown in [4] that a class of Emden equations generating a three dimensional algebra of Lie point symmetries is

$$y'' - \frac{2}{y}y'^2 + \lambda y^5 = 0$$

with a basis of symmetries being

$$X_1 = \tfrac{\partial}{\partial x}, \qquad X_2 = x^2\tfrac{\partial}{\partial x} - xy\tfrac{\partial}{\partial y} \qquad X_3 = 2x\tfrac{\partial}{\partial x} - y\tfrac{\partial}{\partial y} \tag{2.1}$$

so that

$$[X_1, X_2] = \tfrac{1}{2}X_3, \qquad [X_1, X_3] = 2X_1, \qquad [X_2, X_3] = -2X_2. \tag{2.2}$$

The multipliers $P(x, y, y')$ that lead lead to a 'total divergence' satisfies ([1, 2])

$$\tfrac{\delta}{\delta y}[P(x, y, y')(y'' - \tfrac{2}{y}y'^2 + \lambda y^5)]) = 0, \tag{2.3}$$

where $\frac{\delta}{\delta y}$ is the Euler operator. After some lengthy calculations we get that the multiplier functions take the form

$$P = \tfrac{f(x,y)-y'g(x,y)}{4y^7} \tag{2.4}$$

which we enumerate below to obtain

$$\begin{aligned} P_1 &= \tfrac{-x}{2y^3} - \tfrac{x^2}{2y^4}y', \\ P_2 &= \tfrac{-1}{2y^3} - \tfrac{x}{y^4}y,' \\ P_3 &= \tfrac{-y'}{y^4}. \end{aligned} \tag{2.5}$$

Each multiplier P_i leads to a first integral, I, that satisfy Eq. (2.3).
(a) Regarding P_1, we get

$$D_x I = \left(\frac{-x}{2y^3} - \frac{x^2}{2y^4}y'\right)\left(y'' - \frac{2}{y}y'^2 + \lambda y^5\right), \tag{2.6}$$

where $I = I(x, y, y')$ is the first integral and $D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots$ is the total derivative with respect to x. After expanding and applying the total derivative in Eq. (2.6), we obtain

$$I_x + y'I_y + y''I_{y'} = \frac{y'y'' + 3y'^2y + y'y^3}{y^6 + 3y^4y' + 3y^2y'^2 + y'^3}. \tag{2.7}$$

Splitting on y'' yields

$$\begin{aligned} y'' : I_{y'} &= -\tfrac{x^2 y'}{2y^4} - \tfrac{x}{2y^3}, \\ : I_x + y' I_y &= -\tfrac{x(xy+y)\left(\lambda y^6 - 2y'^2\right)}{2y^5}. \end{aligned} \tag{2.8}$$

Integrating Eq. (2.8) with respect to y' gives

$$I(x, y, y') = -\frac{y^2 \lambda x^2}{4} - \frac{1}{4y^2} + A(x, y), \tag{2.9}$$

where $A(x, y)$ is an arbitrary function of x and y. Substituting the value of $I(x, y, y')$ into Eq. (2.8) leads into the following

$$-A_x - \frac{xy^2}{2y^4} - \frac{y'}{2y^3} - \left(\frac{xy'\left(2xy' + 3y\right)}{2y^5} + A_y\right) y' = \frac{-x(xy' + y)(\lambda y^6 - 2y'^2)}{2y^5}. \tag{2.10}$$

After simplifying Eq. (2.10), we get that

$$A_x + y' A_y = 0. \tag{2.11}$$

Splitting Eq. (2.11) on y' gives

$$\begin{aligned} y' : A_x &= -\tfrac{y^2 \lambda x}{2}, \\ : A_y &= -\tfrac{y \lambda x^2}{2} + \tfrac{1}{2y^3}. \end{aligned} \tag{2.12}$$

Integrating Eq. (2.12) with respect to y yields

$$A(x, y) = -\frac{y^2 \lambda x^2}{4} - \frac{1}{4y^2} + B(x), \tag{2.13}$$

where $B(x)$ is an arbitrary function of x. After replacing A into Eq. (2.12) and integrating with respect to x, we obtain $B = C_1$, where C_1 is an arbitrary constant of integration. Therefore, the first integral is given by

$$I(x, y, y') = -\frac{x^2 y'^2}{4y^4} - \frac{xy'}{2y^3} - \frac{y^2 \lambda x^2}{4} - \frac{1}{4y^2} + C_1. \tag{2.14}$$

Thus, we may let the first integral corresponding to P_1 be

$$I_1(x, y, y') = -\frac{x^2 y'^2}{4y^4} - \frac{xy'}{2y^3} - \frac{y^2 \lambda x^2}{4} - \frac{1}{4y^2}.$$

It is easy to check that $DI_1 = \left(\frac{-x}{2y^3} - \frac{-x^2}{2y^4} y'\right)\left(y'' - \frac{2}{y} y'^2 + \lambda y^5\right)$.
(b) The first integral for the case P_2,viz.,

$$P_2 = \tfrac{-1}{2y^3} - \tfrac{-x}{y^4} y' \tag{2.15}$$

satisfy the divergent form

$$D_x I = -\frac{\left(2xy' + y\right)\left(\lambda y^6 + y'' y - 2y'^2\right)}{2y^5}. \tag{2.16}$$

On Expanding and applying the total derivative in Eq. (2.16), we obtain

$$I_x + y' I_y + y'' I_{y'} = -\frac{\left(2xy' + y\right)\left(\lambda y^6 + y'' y - 2y'^2\right)}{2y^5}. \tag{2.17}$$

Splitting on y'' yields

$$\begin{aligned} y'' : I_{y'} &= \tfrac{-xy'}{y^4} - \tfrac{1}{2y^3}, \\ : I_x + y' I_y &= -\tfrac{(2xy'+y)(\lambda y^6 - 2y'^2)}{2y^5}. \end{aligned} \tag{2.18}$$

Integrating Eq. (2.18) with respect to y' gives

$$I(x, y, y') = -\frac{xy'^2}{2y^4} - \frac{y'}{2y^3} + A(x, y), \tag{2.19}$$

where $A(x, y)$ is an arbitrary function of x and y. Substituting the value of $I(x, y, y')$ into Eq. (2.18) leads into the following

$$-A_x + \frac{y'^2}{2y^4} - \left(\frac{\left(4xy'^2 + 3y'y\right)}{2y^5} + A_y\right) y' = -\frac{\left(2xy' + y\right)\left(\lambda y^6 - 2y'^2\right)}{2y^5}. \tag{2.20}$$

After simplifying Eq. (2.20), we get that

$$A_x + y' A_y = 0. \tag{2.21}$$

Splitting Eq. (2.20) on y' gives

$$\begin{aligned} y' : A_x &= -\tfrac{y^2\lambda}{2}, \\ : A_y &= -xy\lambda. \end{aligned} \tag{2.22}$$

Integrating Eq. (2.22) with respect to y yields

$$A(x, y) = -\frac{xy^2\lambda}{2} + B(x), \tag{2.23}$$

where $B(x)$ is an arbitrary function of x. After replacing the value of A into Eq. (2.23) and integrating with respect to x, we obtain $B = C_1$, where C_1 is an arbitrary constant of integration. Therefore, the first integral is given by

$$I(x, y, y') = -\frac{xy'^2}{2y^4} - \frac{y'}{2y^3} - \frac{y^2\lambda x}{2} + C_1 \tag{2.24}$$

from which we can choose the first integral corresponding to P_2 to be

$$I_2(x, y, y') = -\frac{xy'^2}{2y^4} - \frac{y'}{2y^3} - \frac{y^2\lambda x}{2}$$

so that that $DI_2 = \left(\frac{-1}{2y^3} - \frac{-x}{y^4} y'\right)\left(y'' - \frac{2}{y} y'^2 + \lambda y^5\right)$.

(c) For

$$P_3 = -\tfrac{y'}{y^4} \tag{2.25}$$

we get

$$D_x I = -\frac{y'\left(\lambda y^6 + y'' y - 2y^2\right)}{y^5}. \tag{2.26}$$

and

$$I_x + y'I_y + y''I_{y'} = -\frac{y'\left(\lambda y^6 + y''y - 2y^2\right)}{y^5}. \tag{2.27}$$

Splitting on y'' yields

$$\begin{aligned} y'' : I_{y'} &= \tfrac{-y'}{y^4}, \\ : I_x + y'I_y &= -\tfrac{-y'\left(\lambda y^6 - 2y'^2\right)}{y^5}. \end{aligned} \tag{2.28}$$

Integrating Eq. (2.28) with respect to y' gives

$$I(x, y, y') = -\frac{y'^2}{y^4} + A(x, y), \tag{2.29}$$

where $A(x, y)$ is an arbitrary function of x and y. Substituting the value of $I(x, y, y')$ into Eq. (2.28) leads into the following

$$-\left(\frac{2y'^2}{y^5} + A_y\right)y' = -\frac{-y'\left(\lambda y^6 - 2y'^2\right)}{y^5}. \tag{2.30}$$

After simplifying Eq. (2.30), we get that

$$A_x + y'A_y = 0. \tag{2.31}$$

Splitting Eq. (2.31) on y' gives

$$\begin{aligned} y' : A_x &= 0, \\ : A_y &= -\lambda y. \end{aligned} \tag{2.32}$$

Integrating Eq. (2.32) with respect to y yields

$$A(x, y) = -\frac{\lambda y^2}{2} + B(x), \tag{2.33}$$

where $B(x)$ is an arbitrary function of x. After replacing the value of A into Eq. (2.32) and integrating with respect to x, we obtain $B = C_1$, where C_1 is an arbitrary constant of integration. Therefore, the first integral is given by

$$I(x, y, y') = -\frac{y'^2}{2y^4} - \frac{y^2\lambda}{2} + C_1. \tag{2.34}$$

Finally, a first integral corresponding to P_3 is

$$I_3(x, y, y') = -\frac{y'^2}{2y^4} - \frac{y^2\lambda}{2}$$

and $DI_3 = \left(\frac{-y'}{y^4}\right)\left(y'' - \frac{2}{y}y'^2 + \lambda y^5\right)$.

Note. The only direct 'association' between symmetry and first integral exists X_3 & I_2 and between X_1 & I_3. In these respective cases, a single symmetry would lead to 'double reduction' of $y'' - \frac{2}{y}y'^2 + \lambda y^5 = 0$ (for details, see [8, 9]). The second case is obvious and the association is not required for a general solution.

II. Two dimensional algebra case

The form of the second-order Emden ode generating a Lie point symmetry algebra of two dimensions is

$$y'' + \frac{1}{y}y'^2 + \frac{1}{4}\lambda y^3 = 0$$

with algebra

$$X_1 = \tfrac{\partial}{\partial x}, \qquad X_2 = x\tfrac{\partial}{\partial x} - y\tfrac{\partial}{\partial y}. \tag{2.35}$$

The multiplier condition is

$$\tfrac{\delta}{\delta y}[P(x, y, y')(y'' + \tfrac{1}{y}y'^2 + \tfrac{1}{4}\lambda y^3)]) = 0 \tag{2.36}$$

and after some lengthy calculations, the multiplier function takes the form

$$P = -y^2 y' \tag{2.37}$$

so that

$$D_x I = -y^2 y' \left(y'' + \frac{1}{y}y'^2 + \frac{1}{4}\lambda y^3 \right), \tag{2.38}$$

where $I = I(x, y, y')$ is the first integral and $D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots$ is the total derivative with respect to x. After Expanding and applying the total derivative in Eq. (2.38), we obtain

$$I_x + y' I_y + y'' I_{y'} = -\frac{yy'\left(y^4 + 4y''y + 4y'^2\right)}{4}. \tag{2.39}$$

Splitting on y'' yields

$$\begin{aligned} y'' &: I_{y'} = -y^2 y', \\ &: I_x + y' I_y = \tfrac{y'^2}{y} + \tfrac{\lambda y^3}{4}. \end{aligned} \tag{2.40}$$

Integrating Eq. (2.40) with respect to y' gives

$$I(x, y, y') = -\frac{y^2 y'^2}{2} + A(x, y), \tag{2.41}$$

where $A(x, y)$ is an arbitrary function of x and y. Substituting the value of $I(x, y, y')$ into Eq. (2.41) leads into the following

$$-A_x - \left(-yy'^2 + A_y\right) y' = -\frac{-yy'\left(\lambda y^4 + 4y'^2\right)}{4}. \tag{2.42}$$

After simplifying Eq. (2.42), we get that

$$A_x + y' A_y = 0. \tag{2.43}$$

Splitting Eq. (2.43) on y' gives

$$\begin{aligned} y' &: A_x = 0, \\ &: A_y = -\tfrac{\lambda y^5}{4}. \end{aligned} \tag{2.44}$$

Integrating Eq. (2.44) with respect to y yields

$$A(x, y) = -\frac{\lambda y^6}{24} + B(x), \tag{2.45}$$

where $B(x)$ is an arbitrary function of x. After replacing the value of A into Eq. (2.44) and integrating with respect to x, we obtain $B = C_1$, where C_1 is an arbitrary constant of integration. Therefore, the first integral is given by

$$I(x, y, y') = -\frac{1}{2}y^2y'^2 - \frac{1}{24}\lambda y^6 + C_1. \tag{2.46}$$

Thus, we may let the first integral corresponding to P_1 be

$$I(x, y, y') = -\frac{1}{2}y^2y'^2 - \frac{1}{24}\lambda y^6.$$

It is easy to check that $DI = \left(-y^2y'\right)\left(y'' + \frac{1}{y}y'^2 + \frac{1}{4}\lambda y^3\right)$.

Note. Even though the ode $y'' + \frac{1}{y}y'^2 + \frac{1}{4}\lambda y^3 = 0$ admits a two-dimensional algebra of Lie point symmetries, we obtain a single first integral. Nevertheless, the ode is completely integrable and, also, P is invariant under X_1 and conformally invariant under X_2 since $X_2 P = -6P$.

III. One dimensional algebra case

(a) The first form of the second-order ode generating a Lie point symmetry algebra of dimension one is the **Mathews-Lakshmanan Oscillator** equation

$$y'' - \frac{\lambda y y'^2}{\lambda y^2 + 1} + \frac{w_0^2 y}{\lambda y^2 + 1} = 0$$

with symmetry $X = \partial/\partial t$. The multiplier condition is

$$\frac{\delta}{\delta y}\left[P(x, y, y')\left(y'' - \frac{\lambda y y'^2}{\lambda y^2+1} + \frac{w_0^2 y}{\lambda y^2+1}\right)\right] = 0, \tag{2.47}$$

so that

$$P = -\frac{y'}{\lambda y^2+1} \tag{2.48}$$

and

$$D_x I = \left(-\frac{y'}{\lambda y^2 + 1}\right)\left(y'' - \frac{\lambda y y'^2}{\lambda y^2 + 1} + \frac{w_0^2 y}{\lambda y^2 + 1}\right), \tag{2.49}$$

where $I = I(x, y, y')$ is the first integral. After expanding and applying the total derivative in Eq. (2.49), we obtain

$$I_x + y'I_y + y''I_{y'} = -\frac{y'\left(\lambda y^2 y'' + \left(-\lambda y'^2 + w_0^2\right)y + y''\right)}{\left(\lambda y^2 + 1\right)^2}. \tag{2.50}$$

Splitting on y'' yields

$$\begin{aligned} y'' &: I_{y'} = -\frac{y'}{\lambda y^2+1}, \\ &: I_x + y'I_y = -\frac{\lambda y y'^2}{\lambda y^2+1} + \frac{w_0^2 y}{\lambda y^2+1}. \end{aligned} \tag{2.51}$$

Integrating Eq. (2.51) with respect to y' gives

$$I(x, y, y') = -\frac{y'^2}{2\lambda y^2 + 2} + A(x, y), \tag{2.52}$$

where $A(x, y)$ is an arbitrary function of x and y. Substituting the value of $I(x, y, y')$ into Eq. (2.51) leads into the following

$$-A_x - \left(\frac{y'^2 \lambda y}{\left(\lambda y^2 + 1\right)^2} + A_y\right) y' = \frac{y' y \left(-\lambda y'^2 + w_0^2\right)}{\left(\lambda y^2 + 1\right)^2}. \tag{2.53}$$

After simplifying Eq. (2.53), we get that

$$A_x + y' A_y = 0. \tag{2.54}$$

Splitting Eq. (2.54) on y' gives

$$\begin{aligned} y' &: A_x = 0, \\ &: A_y = -\tfrac{w_0^2 y}{\left(\lambda y^2+1\right)^2}. \end{aligned} \tag{2.55}$$

Integrating Eq. (2.55) with respect to y yields

$$A(x, y) = \frac{w_0^2}{2\lambda\left(\lambda y^2 + 1\right)} + B(x), \tag{2.56}$$

where $B(x)$ is an arbitrary function of x. After replacing the value of A into Eq. (2.55) and integrating with respect to x, we obtain $B = C_1$, where C_1 is an arbitrary constant of integration. Therefore, the first integral is, finally, given by

$$I(x, y, y') = -\frac{y'^2}{2(\lambda y^2 + 2)} + \frac{w_0^2}{2\lambda\left(\lambda y^2 + 1\right)}$$

and $DI = \left(-\frac{y'}{\lambda y^2+1}\right)\left(y'' - \frac{\lambda y y'^2}{\lambda y^2+1} + \frac{w_0^2 y}{\lambda y^2+1}\right)$.

(b) For **a particle in a rotating parabolic well**, the ode is

$$y'' + \frac{\lambda y y'^2}{\lambda y^2 + 1} + \frac{w_0^2 y}{\lambda y^2 + 1} = 0.$$

The multiplier functions take the form

$$P = f(x, y) - y' g(x, y) \tag{2.57}$$

which we showed to be

$$P = -(\lambda y^2 + 1) y' \tag{2.58}$$

leading to a first integral that satisfy Eq. (2.58)

$$D_x I = \left(-(\lambda y^2 + 1) y'\right)\left(y'' - \frac{\lambda y y'^2}{\lambda y^2 + 1} + \frac{w_0^2 y}{\lambda y^2 + 1}\right), \tag{2.59}$$

where $I = I(x, y, y')$ is the first integral and $D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots$ is the total derivative with respect to x. After Expanding and applying the total derivative in Eq. (2.59), we obtain

$$I_x + y' I_y + y'' I_{y'} = -\left(\lambda y^2 y'' + \left(\lambda y'^2 + w_0^2\right) y + y''\right) u_1. \tag{2.60}$$

Splitting on y'' yields

$$\begin{aligned} y'' : I_{y'} &= -\left(\lambda y^2 + 1\right) y', \\ : I_x + y' I_y &= \tfrac{\lambda y y'^2}{\lambda y^2+1} + \tfrac{w_0^2 y}{\lambda y^2+1}. \end{aligned} \tag{2.61}$$

Integrating Eq. (2.61) with respect to y' gives

$$I(x, y, y') = -\frac{1}{2}\lambda y^2 y'^2 - \frac{1}{2} y^2 + A(x, y), \tag{2.62}$$

here $A(x, y)$ is an arbitrary function of x and y. Substituting the value of $I(x, y, y')$ into Eq. (2.62). leads into the following

$$-A_x - \left(-\lambda y y'^2 + A_y\right) y' = -y\left(\lambda y'^2 + w_0^2\right) y'. \tag{2.63}$$

After simplifying Eq. (2.63), we get that

$$A_x + y' A_y = 0. \tag{2.64}$$

Splitting Eq. (2.64) on y' gives

$$\begin{aligned} y' : A_x &= 0, \\ : A_y &= -w_0^2 y. \end{aligned} \tag{2.65}$$

Integrating Eq. (2.65) with respect to y yields

$$A(x, y) = -\frac{w_0^2 y^2}{2} + B(x), \tag{2.66}$$

where $B(x)$ is an arbitrary function of x. After replacing the value of A into Eq. (2.65) and integrating with respect to x, we obtain $B = C_1$, where C_1 is an arbitrary constant of integration. Therefore, the first integral is given by

$$I(x, y, y') = -\frac{1}{2}\lambda y^2 y'^2 - \frac{1}{2} y'^2 - \frac{1}{2} w_0^2 y^2 + C_1. \tag{2.67}$$

Thus, we may let the first integral corresponding to P be

$$I(x, y, y') = -\frac{1}{2}\lambda y^2 y'^2 - \frac{1}{2} y'^2 - \frac{1}{2} w_0^2 y^2.$$

It is easy to check that $DI = \left(-(\lambda y^2 + 1)y'\right)\left(y'' - \frac{\lambda y y'^2}{\lambda y^2+1} + \frac{w_0^2 y}{\lambda y^2+1}\right)$.

Conclusions

We presented a general method to construct first integrals for classes of some well known second-order ordinary differential equations, viz., the nonlinearizable Emden and Liénard classes of equations. The approach does not require a knowledge of a Lagrangian and, instead, utilises the ‘multiplier approach’.

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