The Geometry and Topology of the Gauge Theory/Gravity Correspondence.

by

Warren Anthony Carlson

Thesis presented in fulfilment of the requirements for the degree of

Master of Science in Physics

at the University of the Witwatersrand

School of Physics,
University of the Witwatersrand,
PO Box Wits, Johannesburg, 2050, South Africa

Supervisor: Prof. R.S. de Mello Koch

September 2010

Abstract

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W.A. Carlson

School of Physics,

University of the Witwatersrand,

PO Box Wits, Johannesburg, 2050, South Africa

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The well studied AdS/CFT correspondence describes a duality between gauge theories and string theories. What is currently lacking is a dictionary translating gauge theory correlators in space-time to string theory correlators on a world-sheet. We investigate aspects of world-sheet correlators of closed string theories dual to large N $SU(N)$ gauge theories following a prescription by Gopakumar. The prescription presents a concrete mapping of gauge theory correlators to string theory correlators. Gauge theory correlators are translated to string world-sheet correlators via a reparametrization by inverse Schwinger parameters and a mapping to the moduli space of Riemann surfaces (world-sheets) via a special class of quadratic differentials know as Strebel differentials.

Acknowledgements

I offer my sincerest gratitude to my supervisor Prof. Robert de Mello Koch for his patience and enthusiasm. I have learnt a great deal under his guidance. I am thankful for his honesty and his willingness to help, and am grateful for the time and effort that he invested in me. I would also like to thank Nicholas J. Park, Grant Mashile, Mthokozisi Masuku and Alia Zaidi for comments and discussion.

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Declaration

I, the undersigned, hereby declare that the work contained in this thesis is my own original work and that I have not previously in its entirety or in part submitted it at any university for a degree.

Signature: ....

W.A. Carlson

Date: 16 September 2010

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Chapter 1

Introduction

The current theories describing the fundamental processes of nature are Quantum Field Theories (QFT's). In these theories, all fundamental processes in nature are described by excitations of some fields. These field excitations manifest as point-like particles which interact locally with other point-like particles. These theories have proven accurate descriptions of nature under current tests. There are, however, indications that new effects and processes are involved at shorter distances and higher energies than we are currently able to probe. The scales at which such effects come into play are on the order of the Planck scale. In such regimes the effects of quantum gravity become important. Although quantum field theories have accurately unified the electromagnetic, the weak and the strong interactions, they have thus far been unsuccessful in incorporating gravity.

One method to incorporate gravity into a consistent quantum theory is to replace the notion that natural processes at a microscopic level involve fundamental point-like objects, as excitations of some fields, by a notion of fundamental objects as extended 1-dimensional objects [1]. These objects, called strings, oscillate and the low energy oscillatory modes of these strings appear as point-like particles. The type of particle that such a string oscillation may look like is determined by the mode of the oscillation. This idea was originally developed in an attempt to explain the strong interaction in Quantum Chromodynamics (QCD). QCD is a gauge theory based on the group $SU(N)$ with $N = 3$. 't Hooft suggested that the theory might simplify in the large N limit [2].

It is believed that large N $SU(N)$ gauge theories are dual to closed string theory with a string coupling constant $g_s \sim 1/N$, in the limit of large N, with fixed 't Hooft coupling $\lambda = g_{YM}^2 N$. In this way gauge theories are connected to string theories in the large N limit [3]. Our goal is to translate a gauge theory into a string theory where the correlators of the theory sum over string world-sheets. By this translation, Feynman diagrams of the gauge theory are re-interpreted as closed string diagrams. Following the 't Hooft double line notation [2], gauge theory Feynman diagrams are re-interpreted as closed Riemann surfaces with boundaries.

We would like to construct the string theory dual to a specific large N gauge theory. It is not known how to derive the closed string theory dual to standard gauge theories [4]. One prescription for mapping gauge theory correlators to world-sheet correlators, in the 3+1-dimensional case, is presented by Gopakumar [5; 6; 7]. This prescription involves Strebel differentials to relate gauge theory Feynman diagrams to the moduli space of Riemann Surfaces.

In Chapter 2, we show how 0-dimensional gauge theory Feynman diagrams are interpreted as Riemann surfaces with boundaries, following the 't Hooft double line notation. It then becomes evident that gauge theory correlators may be interpreted as summations over Riemann surfaces. In Chapter 3, a brief overview of the string world-sheet action is presented. The object of interest in this scrip is the Strebel differential and its relation to the study of Riemann surfaces. An overview of Strebel differentials and their relation to the Moduli space of Riemann surfaces may be found in Chapter 4. Using the Strebel differential, gauge theory correlators are mapped to integrals over the moduli space of Riemann surfaces. In Chapter 5 we give an example of Gopakumar's prescription of mapping $3+1$-dimensional gauge theory correlators to string world-sheet correlators.

Chapter 2

Matrix Models

2.1 Gaussian Matrices

The expectation values of operators play a central role in quantum theory. Computing such expectation values amounts to evaluating integrals of the form

$$\langle \mathcal{O} \rangle = \frac{\int [\mathrm{d}M] \mathcal{O} e^{-\frac{1}{2}\mathrm{tr}\left(M^2\right) - \frac{g}{N}\mathrm{tr}\left(M^4\right)}}{\int [\mathrm{d}M] e^{-\frac{1}{2}\mathrm{tr}(M^2) - \frac{g}{N}\mathrm{tr}(M^4)}}, \tag{2.1.1}$$

where $\mathcal{O}$ is the operator whose expectation value we wish to compute and M is an $N \times N$ matrix satisfying a particular set of conditions. We now develop a formulation to evaluate these integrals.

Let M be an $N \times N$ Hermitian matrix. Given this,

$$M = M^\dagger,$$

which we may rewrite as

$$M_{ij} = {M^*}_{ji}.$$

It is evident that

$$\begin{aligned} {M_{ij}}^{\Re} =& {M_{ji}}^{\Re}, \\ {M_{ij}}^{\Im} =& -{M_{ji}}^{\Im}, \end{aligned}$$

where the quantities ${M_{ij}}^{\Re}$ and ${M_{ij}}^{\Im}$ are the real and imaginary components of M_{ij}, respectively. Define the measure, $[\mathrm{d}M]$, in Equation (2.1.1) as,

$$[\mathrm{d}M] = \prod_{i=1}^{N} \prod_{i<j}^{N} \mathrm{d}M_{ii} \mathrm{d}{M_{ij}}^{\Re} \mathrm{d}{M_{ij}}^{\Im}.$$

Let J be any hermitian $N \times N$ matrix and $\alpha, \beta \in \mathbb{R}$ and $\alpha > 0$. Define the generating function $Z[J]$ by

$$Z[J] = \frac{\int [\mathrm{d}M]\, e^{-\alpha \mathrm{tr}\left(M^2\right) + \beta \mathrm{tr}(JM)}}{\int [\mathrm{d}M]\, e^{-\alpha \mathrm{tr}(M^2)}}. \tag{2.1.2}$$

The function $Z[J]$ is a generating function of Gaussian expectation values. All correlators in the free theory may be constructed from $Z[J]$. Therefore, $Z[J]$ gives a complete description of the free theory.

Evaluating $Z[J]$ yields,

$$\begin{aligned} Z[J] &= \frac{\int [\mathrm{d}M]\, e^{-\alpha \mathrm{tr}\left(M^2\right) + \beta \mathrm{tr}(JM)}}{\int [\mathrm{d}M]\, e^{-\alpha \mathrm{tr}(M^2)}} \\ &= e^{\mathrm{tr}\left(\alpha\left(\frac{\beta}{2\alpha} J\right)^2\right)} \frac{\int [\mathrm{d}M] e^{-\mathrm{tr}\left(\alpha\left(M - \frac{\beta}{2\alpha} J\right)^2\right)}}{\int [\mathrm{d}M] e^{-\alpha \mathrm{tr}(M^2)}} \\ &= e^{\mathrm{tr}\left(\frac{\beta^2}{4\alpha} J^2\right)} \frac{\int [d\tilde{M}] e^{-\alpha \mathrm{tr}\left(\tilde{M}^2\right)}}{\int [\mathrm{d}M] e^{-\alpha \mathrm{tr}(M^2)}} \\ &= e^{\mathrm{tr}\left(\frac{\beta^2}{4\alpha} J^2\right)}. \end{aligned} \tag{2.1.3}$$

When $\frac{\mathrm{d}}{\mathrm{d}J_{kl}}\left[Z[J]\right]$ is evaluated at $J = 0$, it follows that

$$\left.\frac{\mathrm{d}}{\mathrm{d}J_{kl}}\left[Z[J]\right]\right|_{J=0} = \beta \frac{\int [\mathrm{d}M] M_{lk} e^{-\alpha \mathrm{tr}\left(M^2\right)}}{\int [\mathrm{d}M] e^{-\alpha \mathrm{tr}(M^2)}}.$$

Define the expectation value $\langle \mathcal{O} \rangle_0$ by

$$\langle \mathcal{O} \rangle_0 = \frac{\int [\mathrm{d}M] \mathcal{O} e^{-\alpha \mathrm{tr}\left(M^2\right)}}{\int [\mathrm{d}M] e^{-\alpha \mathrm{tr}(M^2)}}, \tag{2.1.4}$$

where the zero subscript denotes expectation value in the free theory. Since each differentiation of $Z[J]$ with respect to the matrix element J_{kl} extracts

M_{lk} from the integrand, it follows that

$$\left\langle \prod_{i=1}^{n} M_{k_i l_i} \right\rangle_0 = \frac{1}{\beta^n} \left(\prod_{i=1}^{n} \frac{\mathrm{d}}{\mathrm{d}J_{l_i k_i}} \right) \left[Z[J] \right] \Bigg|_{J=0} . \tag{2.1.5}$$

Note that the factors of β^n in the expectation value and the generating functional cancel.

Using Equation (2.1.5),

$$\begin{aligned} \langle M_{kl} \rangle_0 &= 0, \\ \langle M_{k_1 l_1} M_{k_2 l_2} \rangle_0 &= \frac{1}{2\alpha} \delta_{k_1 l_2} \delta_{l_1 k_2}, \end{aligned} \tag{2.1.6}$$

and,

$$\begin{aligned} \langle M_{k_1 l_1} M_{k_2 l_2} M_{k_3 l_3} M_{k_4 l_4} \rangle_0 = &\frac{1}{4\alpha^2} \left(\delta_{k_3 l_4} \delta_{k_4 l_3} \delta_{k_1 l_2} \delta_{k_2 l_1} + \delta_{k_2 l_3} \delta_{k_3 l_2} \delta_{k_4 l_1} \delta_{k_1 l_4} + \right. \\ &\left. \delta_{k_2 l_4} \delta_{k_4 l_2} \delta_{k_1 l_3} \delta_{k_3 l_1} \right) . \end{aligned} \tag{2.1.7}$$

The assignment of $J = 0$ eliminates all J-dependent terms, so

$$\left\langle \prod_{i=1}^{2n+1} M_{k_i l_i} \right\rangle_0 = 0. \tag{2.1.8}$$

The pairing of Kronecker delta functions for a matrix element pair is called a Wick contraction of that pair of matrix elements. Equation (2.1.7) may be rewritten as

$$\begin{aligned} \langle M_{k_1 l_1} M_{k_2 l_2} M_{k_3 l_3} M_{k_4 l_4} \rangle_0 = &\langle M_{k_1 l_1} M_{k_2 l_2} \rangle_0 \langle M_{k_3 l_3} M_{k_4 l_4} \rangle_0 + \\ &\langle M_{k_1 l_1} M_{k_3 l_3} \rangle_0 \langle M_{k_2 l_2} M_{k_4 l_4} \rangle_0 + \\ &\langle M_{k_1 l_1} M_{k_4 l_4} \rangle_0 \langle M_{k_2 l_2} M_{k_3 l_3} \rangle_0 , \end{aligned} \tag{2.1.9}$$

which may also be derived using Wick's Theorem [8]. We now set $\alpha = \frac{1}{2}$.

Let U be any unitary $N \times N$ matrix, then

$$U^\dagger U = U U^\dagger = I.$$

Under

$$M \mapsto U^\dagger M U, \tag{2.1.10}$$

we find,

$$M^\dagger \mapsto \left(U^\dagger M U\right)^\dagger = U^\dagger M^\dagger \left(U^\dagger\right)^\dagger = U^\dagger M^\dagger U = U^\dagger M U.$$

This transformation preserves hermiticity. M^k transforms to

$$\begin{aligned} M^k &\mapsto \left(U^\dagger M U\right)^k \\ &= \overbrace{\left(U^\dagger M U\right)\left(U^\dagger M U\right)\ldots\left(U^\dagger M U\right)}^{k\text{times}} \\ &= U^\dagger M^k U, \end{aligned}$$

and so,

$$\operatorname{tr}\left(M^k\right) \mapsto \operatorname{tr}\left(M^k\right).$$

We can define an equivalence relation, $\sim$, on the set of $N \times N$ Hermitian matrices by

$$M \sim M' \Leftrightarrow M' = U^\dagger M U. \tag{2.1.11}$$

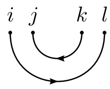
The Matrix model is a model for Yang-Mills theory. Yang-Mills theory has local symmetry. Physical observables are invariant under symmetry transformations. This ensures that value of an observable is independant of the co-ordinate system used to compute it. Electromagnetic theory makes this clear where the observed electric and magnetic fields are defined in terms of a scalar potential and a vector potential. The potentials uniquely define the observed fields, while the observed fields do not uniquely define the potentials. These potentials are not observable. A modification of the potentials which leaves the observables invariant is an symmetry of the theory and is called a *gauge transformation*. Treating Equation (2.1.10) as a local symmetry in the matrix model implies that physical observables are invariant under this symmetry. From this, $\mathrm{tr}\left(M^k\right)$ is physical for any k.

2.2 Ribbon Graphs

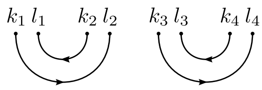
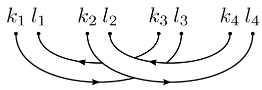
We now develop a diagrammatic notation to evaluate expectation values in the matrix model. From Equation (2.1.6), the value of the Wick contraction of matrix elements M_{ij} and M_{kl} is,

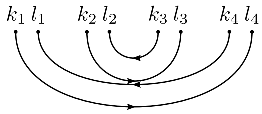
$$\langle M_{ij} M_{kl} \rangle_0 = \delta_{il}\delta_{jk}.$$

In the new notation each matrix is represented by a pair of labelled dots. In this notation, Equation (2.1.6) becomes

$\langle M_{ij} M_{kl} \rangle_0 =$  .

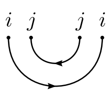
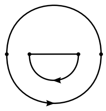
We say that the two matrices are connected by a ribbon. Similarly, Equation (2.1.7) becomes

$\langle M_{k_1 l_1} M_{k_2 l_2} M_{k_3 l_3} M_{k_4 l_4} \rangle_0 =$  $+$  $+$

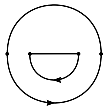
 .

The line segment connecting dots is directed towards the second matrix index that it connects. The orientation of line segments ensures that ribbons do not twist.

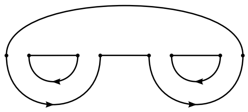
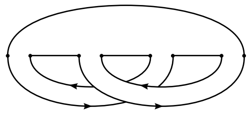
In operators involving traces of matrices, all matrix indices are summed over. In these cases, matrix indices are dummy variables and we need not keep track of them. This allows us to simplify the notation. Dots carrying the same label in a single matrix product are connected by a line segment and the dummy indices are omitted. In the new notation,

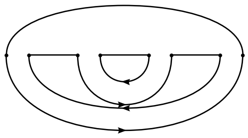
 $=$  $= \delta_{ii}\delta_{jj}.$

Each closed loop represents a contracted Kronecker delta. For an $N \times N$ matrix M, summing over repeated indices gives $\delta_{ii} = N$. Therefore, in the new notation,

$$\left\langle \mathrm{tr}\left(M^2\right)\right\rangle_0 = \langle M_{ij} M_{ji} \rangle_0 = \delta_{ii}\delta_{jj} =$$  $$= N^2.$$

Similarly,

$$\left\langle \mathrm{tr}\left(M^4\right)\right\rangle_0 =$$  $+$  $+$



$$=2N^3 + N.$$

While keeping in mind that ribbons may not twist, the orientation arrows on the ribbons are omitted. The diagram representing a product of Wick contractions in an expectation value is called a *ribbon graph.* Every operator $\mathcal{O}$ is a product of traces of an arbitrary number of matrices. Each such trace of matrices in $\mathcal{O}$ is drawn as a circular bulge in the ribbon graphs in the sequel.

Suppose that the sum of the powers of matrices in $\tilde{\mathcal{O}}$ were odd. Since ribbons graphs are diagrammatic representations of index contractions, there exists one ribbon in each diagram which does not connect two pairs of indices on the matrix M. This corresponds to an uncontracted pair of indices on the source matrix J, in the generating functional, which leaves a multiplicative factor of J in the correlator. Under the assignment of $J = 0$, the correlator vanishes. This is shown by Equation (2.1.8). Therefore, any ribbon graph which contains a pair of indices which cannot be contracted necessarilty takes the value zero. Therefore, there exists no non-zero ribbon graph representation of $\left\langle \tilde{\mathcal{O}} \right\rangle$.

For an operator $\mathcal{O}$ where the sum of matrix powers is $2n$, there are $2n!$ ways to pair matrices in Wick contractions. Products of Wick contractions are symmetric over the ordering of the Wick contractions, so the number of distinct products of Wick contractions is reduced by a factor of $n!$. Since each Wick contraction is also symmetric under the swapping of two matrices in a single contraction, the number of distinct Wick contractions is further reduced by a factor of two for each pair of matrices in $\mathcal{O}$. Therefore, the number of Wick contractions in $\langle\mathcal{O}\rangle_0$ is

$$\mathcal{N} = \frac{2n!}{2^n n!} = (2n-1)!!. \tag{2.2.1}$$

For each combination of Wick contractions in $\langle\mathcal{O}\rangle_0$ there corresponds a ribbon graph. For operators of increasing sum of matrix powers, the number of diagrams grows semi-factorially.

The Schwinger-Dyson Equation [9; 10] is a calculational tool which simplifies the computation of correlators. It is also useful for computing the weighting and form of ribbon graphs in the perturbation expansion of correlators. A brief overview of the Schwinger-Dyson equation is given in Appendix A.1 and an illustration of its usefullness for constructing ribbon graphs is presented in Appendix A.2.

Ribbon graphs may be thought of as triangulations of two dimensional oriented surfaces or manifolds. In a triangulation of a smooth manifold, $\mathcal{M}_h$, of genus h, the connected region bounded by the edges of a graph is called a *face*. In such a comparison traces correspond to vertices in the triangulation, ribbons correspond to edges and loops correspond to faces. For any triangulation on $\mathcal{M}_h$ consisting of v vertices, e edges and f faces, the Euler characteristic equation for $\mathcal{M}_h$ is

$$v - e + f = 2 - 2h. \tag{2.2.2}$$

The *Euler character*, χ, of $\mathcal{M}_h$ can be given in terms of the genus, h, of $\mathcal{M}_h$ as

$$\chi = 2 - 2h. \tag{2.2.3}$$

We now scale the theory such that the value of each ribbon graph reflects the Euler character of the surface which it triangulates. To do this, consider the rescaling of M such that

$$M \mapsto \frac{1}{\sqrt{N}} M = M' \tag{2.2.4}$$

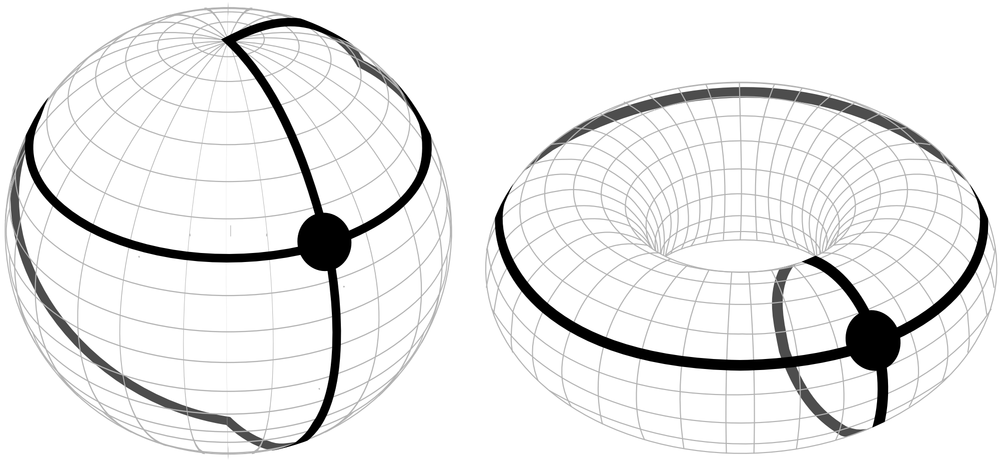
where N is the rank of M. Then,

$$M_{ij} = \sqrt{N} M'_{ij}.$$

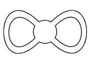
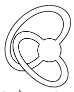
The vacuum diagrams of the interacting theory of Equation (2.1.1) are scaled to give

$$\int [\mathrm{d}M'] e^{-\frac{N}{2}\mathrm{tr}\left(M'^2\right) - gN\mathrm{tr}\left(M'^4\right)} = \sum_{k=0}^{\infty} \frac{(-gN)^k}{k!} \int [\mathrm{d}M'] \left(\mathrm{tr}\left(M'^4\right)\right)^k e^{-\frac{N}{2}\mathrm{tr}\left(M'^2\right)}.$$

Following Equation (2.1.6), this rescaling sets the value of each ribbon to $\frac{1}{N}\delta_{ij}\delta_{kl}$ and assigns a value of N to each $\mathrm{tr}\left(M'^4\right)$. We introduce a factor of N for each trace in the operator whose expectation value we wish to compute. The value of each ribbon graph carries a factor of N raised to the power $V - R + L$ where V is the number of vertices, R is the number of ribbons connecting vertices and L is the number of closed loops in the graph. Following Equation (2.2.2), $V - R + L = 2 - 2h$. Thus, under this recalling each vacuum graph is proportional to N^{χ}. It follows that the value of a correlator is dominated by those contributions corresponding to surfaces of maximal χ, or minamal genus [3]. These are surfaces with the topology of a sphere or plane and are refered to as *planar* diagrams. Two examples of Riemann surfaces triangulated by ribbon graphs are presented in Figure 2.1.



(a) (b)

Figure 2.1: Triangulations of Riemann surfaces by ribbon graphs: after a rescaling, the corresponding ribbon graph formulation gives, $N\left\langle \mathrm{tr}\left(M^4\right)\right\rangle_0 =$ 2  $+$  . (a) The first term is identified with a triangulation of a sphere and (b) the second term is identified with a triangulation of a torus.

Following the above discussion, vertices are equipped with a factor of N. To equip each vertex in a correlator with a factor of N, each trace in $\mathcal{O}$ is multiplied by N. For correlators in the interacting theory described by Equation (2.1.1), the expectation value of $\mathcal{O}$ is scaled giving,

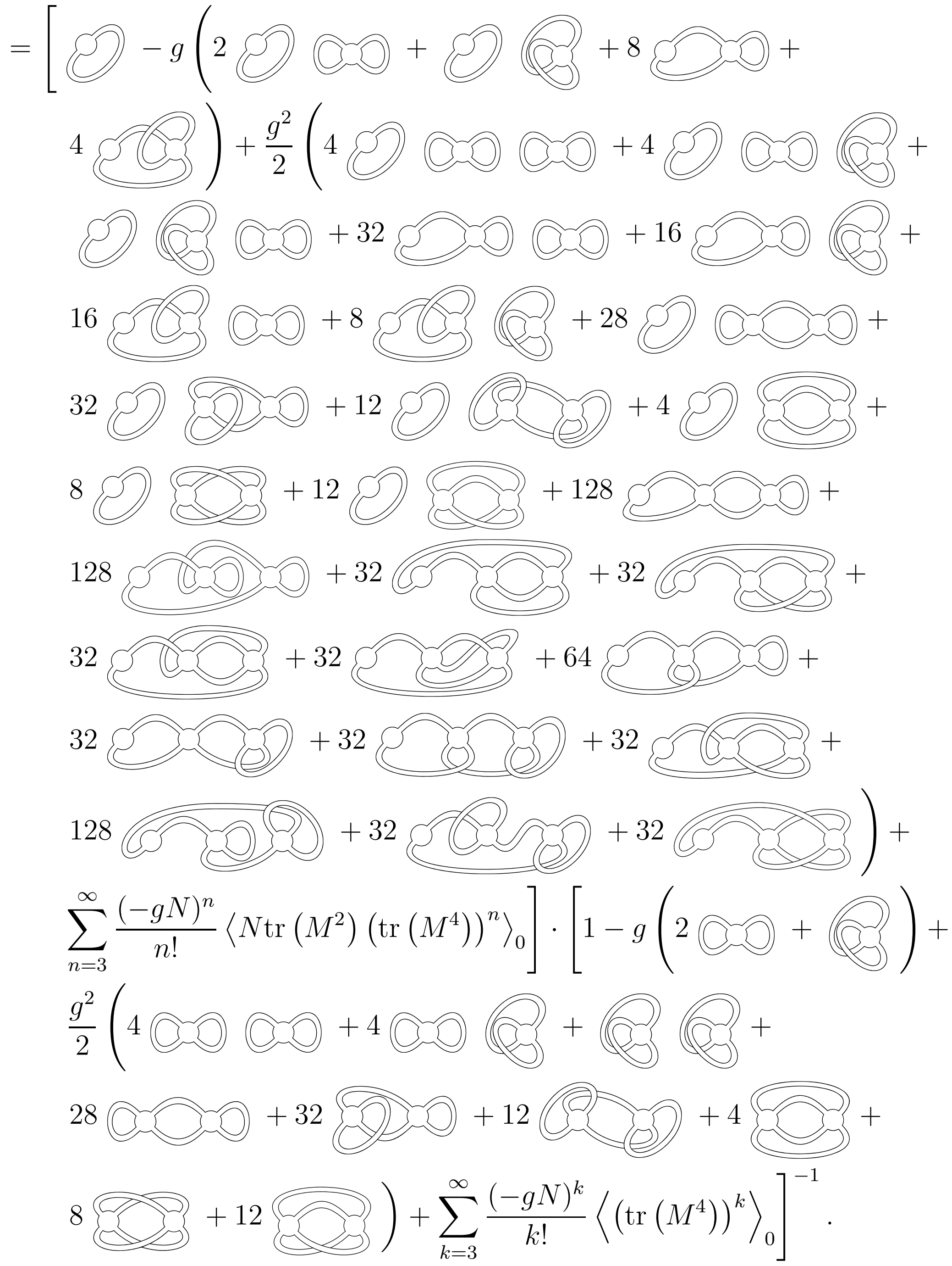
$$\begin{aligned} N^p\left\langle \mathcal{O}\right\rangle &= \frac{\int [\mathrm{d}M] N^p \mathcal{O}\; e^{-\frac{N}{2}\mathrm{tr}\left(M^2\right)-gN\mathrm{tr}\left(M^4\right)}}{\int [\mathrm{d}M] e^{-\frac{N}{2}\mathrm{tr}(M^2)-gN\mathrm{tr}(M^4)}} \\ &= \frac{\sum_{n=0}^{\infty}\frac{(-gN)^n}{n!}\int [\mathrm{d}M] N^p\mathcal{O}\left(\mathrm{tr}\left(M^4\right)\right)^n e^{-\frac{N}{2}\mathrm{tr}\left(M^2\right)}}{\sum_{k=0}^{\infty}\frac{(-gN)^k}{k!}\int [\mathrm{d}M]\left(\mathrm{tr}\left(M^4\right)\right)^k e^{-\frac{N}{2}\mathrm{tr}(M^2)}}, \end{aligned} \tag{2.2.5}$$

where $\mathcal{O}$ is the scaled operator and N^p is the correction factor such that each trace in $\mathcal{O}$ carries a factor of N.

From Equation (2.2.5), the expectation value of any operator $\mathcal{O}$, is calculable in terms of a series of free theory expectation values. To illustrate this, suppose we were to evaluate $\left\langle \mathrm{tr}\left(M^2\right)\right\rangle$. Following Equation (2.2.4) and (2.2.5),

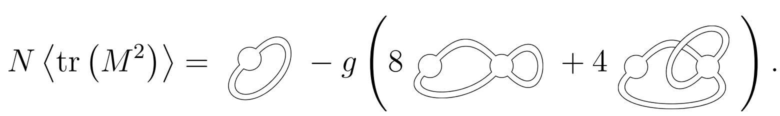
$$N\left\langle \mathrm{tr}\left(M^2\right)\right\rangle = \frac{\sum_{n=0}^{\infty}\frac{(-gN)^n}{n!}\int [\mathrm{d}M] N\mathrm{tr}\left(M^2\right)\left(\mathrm{tr}\left(M^4\right)\right)^n e^{-\frac{N}{2}\mathrm{tr}\left(M^2\right)}}{\sum_{k=0}^{\infty}\frac{(-gN)^k}{k!}\int [\mathrm{d}M]\left(\mathrm{tr}\left(M^4\right)\right)^k e^{-\frac{N}{2}\mathrm{tr}(M^2)}}$$

$$
= \frac{\sum_{n=0}^{\infty} \frac{(-gN)^n}{n!} \left\langle N \mathrm{tr}\left(M^2\right) \left(\mathrm{tr}\left(M^4\right)\right)^n \right\rangle_0}{\sum_{k=0}^{\infty} \frac{(-gN)^k}{k!} \left\langle \left(\mathrm{tr}\left(M^4\right)\right)^n \right\rangle_0}
$$

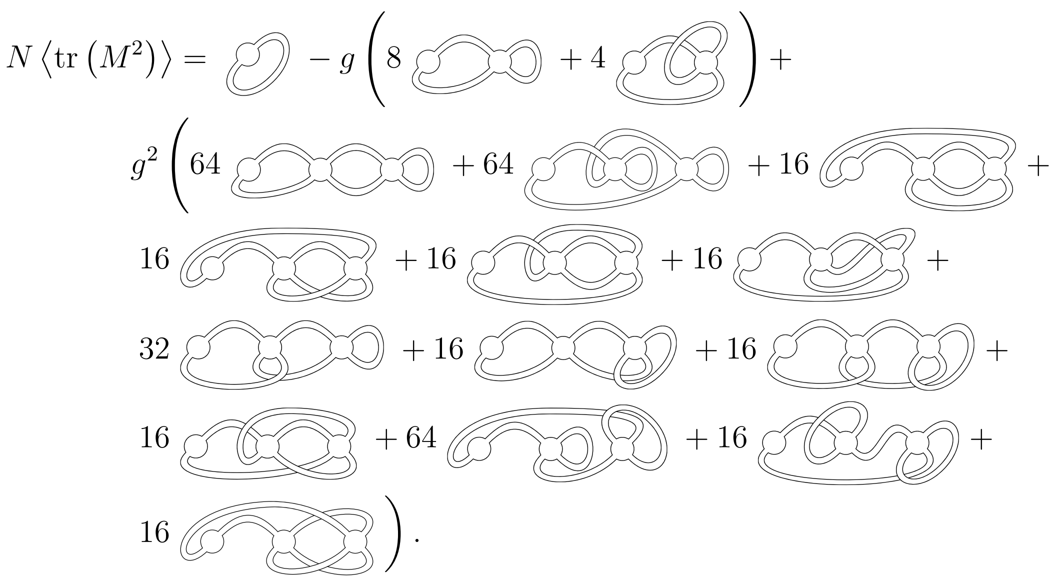


The above expression passes some simple validity tests. The above correlator is a weighted sum of expectation values of operators, $\langle \mathcal{O}_{2m} \rangle_0$. Each operator is a product of traces over matrices, where the sum of matrix powers is some even integer, $2m$. Since the value of each expectation value is equal to the sum of Wick contractions, there is a total of $(2m-1)!!$ terms in each expectation value. The terms in the correlator are organized in powers of the interaction parameter g. For the g^0 term in $\langle \mathrm{tr}\,(M^2)\rangle$ there is one term. There are $5!! = 15$ terms for g^2. For the g^n term, there are $(2+4^n-1)!!$ terms. Also, for each order in g, the sum of ribbon graphs includes every permutation of ribbon connections with respect to the number and order of the vertices associated with the operators in the expectation value.

We may compute the expectation value of any such operator by expanding the series in the Equation (2.2.5) to any order in g. To first order in g,



To second order in g,



The effect of normalization of the integral in Equation (2.2.5) is to eliminate vacuum ribbon graphs from the expectation value. There is a one-to-one correspondence between ribbon graphs of genus h and smooth surfaces of genus h. Since expectation values are a series of triangulations, an operator expectation value may now be interpreted as of as sum over oriented surfaces.

2.3 Classical Limits of the Matrix Model

Until now we have studied a matrix model in 0-dimensional Euclidean space, where $\langle \mathcal{O} \rangle = \int [\mathrm{d}X] \mathcal{O} e^{-S}$ with S as the action of the system. We could also study quantum mechanics in Minkowski space, where $\langle \mathcal{O} \rangle = \int [\mathrm{d}X] \mathcal{O} e^{iS'/\hbar}$ with the action S'. We now investigate some details of expectation values in general. Our observations are true in both cases.

For any operator $\mathcal{O} = \prod_{k=0}^{l} \mathcal{O}_k$, the expectation value $\langle \mathcal{O} \rangle$ is

$$\langle \mathcal{O} \rangle = \int [\mathrm{d}X] \ \mathcal{O} e^{i\frac{S'}{\hbar}} = \int [\mathrm{d}X] e^{i\frac{S'}{\hbar}} \prod_{k=0}^{l} \mathcal{O}_k.$$

This is a summation over configurations of the system, normalized such that

$$\int [\mathrm{d}X] e^{iS'/\hbar} = 1$$

In this sense, $e^{iS'/\hbar}$ can be interpreted as the statistical weight μ_j of the configuration $\mathcal{C}_j$. The expectation value $\langle \mathcal{O} \rangle$ is a weighted sum,

$$\langle \mathcal{O} \rangle = \sum_{i=0}^{n} \mu_i \prod_{k=0}^{l} \mathcal{O}_{ki},$$

where n is the total number of configurations of the system and $\mathcal{O}_{kj}$ is the value of the k-th operator on the configuration $\mathcal{C}_j$.

A classical system has a single configuration, $\mathcal{C}_c$, therefore,

$$\mu_j = \begin{cases} 1 & \text{for } i = c \\ 0 & \text{for } i \neq c. \end{cases}$$

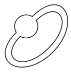
So, in the classical regime,

$$\langle \mathcal{O} \rangle = \sum_{j=0}^{n} \mu_j \prod_{k=0}^{l} \mathcal{O}_{kj} = \mu_c \prod_{k=0}^{l} \mathcal{O}_{kc} = \prod_{k=0}^{l} \mathcal{O}_{kc}.$$

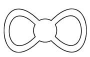
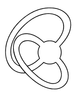
Thus, in the classical limit of a theory,

$$\left\langle \prod_{j=0}^{k} \mathcal{O}_j \right\rangle = \prod_{j=0}^{k} \langle \mathcal{O}_j \rangle. \tag{2.3.1}$$

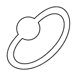
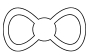
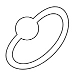
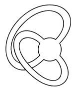
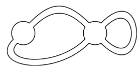
Following the free theory of Equation (2.1.4),

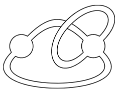
$$\left\langle \operatorname{tr}\left(M^2\right)\right\rangle_0 =$$  $$= N^2,$$

and

$$\left\langle \operatorname{tr}\left(M^4\right)\right\rangle_0 = 2$$  $+$  $$= 2N^3 + N = 2N^3\left(1 + \frac{1}{2}\frac{1}{N^2}\right).$$

Now,

$$\left\langle \operatorname{tr}\left(M^2\right)\operatorname{tr}\left(M^4\right)\right\rangle_0 = 2$$   $+$   $+ 8$  $+$

4 

$$= 2N^5 + 9N^3 + 4N$$
$$= 2N^5\left(1 + \frac{9}{2}\frac{1}{N^2} + 2\frac{1}{N^4}\right)$$

For large N, $\frac{1}{N} \to 0$, so

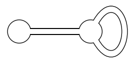
$$\left\langle \operatorname{tr}\left(M^2\right)\operatorname{tr}\left(M^4\right)\right\rangle_0 = \left\langle \operatorname{tr}\left(M^2\right)\right\rangle_0 \left\langle \operatorname{tr}\left(M^4\right)\right\rangle_0 = 2N^5\left(1 + O\left(\frac{1}{N^2}\right)\right).$$

Extending this to the interacting theory of Equation (2.1.1), we introduce the 't Hooft coupling $\lambda = gN$, and find similarly to first order in λ,

$$\begin{aligned}\left\langle \operatorname{tr}\left(M^2\right)\operatorname{tr}\left(M^2\right)\right\rangle &= \left\langle \operatorname{tr}\left(M^2\right)\right\rangle \left\langle \operatorname{tr}\left(M^2\right)\right\rangle \\ &= \left(N^4 - 16\lambda N^4 + O\left(\lambda^2 N^4\right)\right)\left(1 + O\left(\frac{1}{N^2}\right)\right).\end{aligned}$$

This may be extended to any order in λ for any operator $\mathcal{O}$. In the limit $N \to \infty$, expectation values factorize giving a classical limit of the theory.

In the classical limit, the expectation values of operators involving products of traces of matrices raised to odd powers are zero. This follows from following Equation (2.1.8). For example,

$$\left\langle \operatorname{tr}\left(M\right)\operatorname{tr}\left(M^3\right)\right\rangle_0 =$$  $$= 3N^2,$$

whereas,

$$\langle \mathrm{tr}\,(M) \rangle_0 \left\langle \mathrm{tr}\,\left(M^3\right) \right\rangle_0 = 0 \cdot \left(3N^2 + O\left(\frac{1}{N}\right)\right) = 0.$$

For any such operator, the classical limit is zero.

Generally, expectation values of $\mathcal{O}$ may be written as

$$\left\langle \prod_{j=0}^{k} \mathcal{O}_j \right\rangle = \left(\prod_{j=0}^{k} \langle \mathcal{O}_j \rangle\right)\left(1 + O(\hbar)\right).$$

Terms of the form $\frac{1}{N}$ in the matrix model control the correction to the classical limit. In this way the limit $\frac{1}{N} \to 0$ is the limit of $\hbar \to 0$, which leads to classical physics.

2.4 AdS/CFT Correspondence

It was demonstrated that expectation values in the matrix model factorize in the large N limit. The terms which dominate the factorization are those with the largest number of closed loops in the ribbon graph representations. These are the terms with disconnected ribbon graphs. Since the space-time dimension of the theory does not effect the N-dependence of the expectation values, this is true for any number of space-time dimensions. Therefore, the quantum field theory of matrices in the large N limit is equivalent to a classical theory. The AdS/CFT correspondence is a conjecture of what the classical theory describing the large N limit of a specific matrix model is. In this correspondence, the matrix model is $\mathcal{N} = 4$ super Yang-Mills theory in $3+1$-dimensions. Below we give a brief overview of the AdS/CFT correspondence, for a more detailed treatment of this conjecture see [3] and [11].

We may study open and closed strings in string theory. The objects on which open strings may end are called *Dirichlet membranes* or *Dp-branes*, and are described by their spacial dimension, p. These Dp-branes are dynamical objects having tension and charge, and preserve certain symmetries. Since open strings may interact to form closed strings, any theory of open strings is also a theory of closed strings. The action of an open string theory S, may be partitioned to give,

$$S = S_{\text{open}} + S_{\text{closed}} + S_{int}. \tag{2.4.1}$$

where S_{open} describes open strings, a $3+1$-dimensional theory, on a Dp-brane, S_{closed} describes closed strings in a $9+1$-Dimensional theory, and S_{int} describes the interaction of open strings with closed strings.

The energy of a string is determined by the the modes of oscillation of the *center-of-mass* wave-function and of the string itself. Open string theory in $3+1$-dimensions is described by $\mathcal{N} = 4$ super Yang-Mills theory, $S_{3+1}^{N=4\ \text{SYM}}$, which is a gauge theory with gauge group $SU(N)$. The closed string theory in $9+1$-dimensions is described by massless type IIB string theory in the super gravity limit, $S_{\mathcal{M}_{10}\ \text{SUGRA}}^{\text{IIB}}$. In the low energy limit, the string center-of-mass wavelengths are so large that they do not "see" the Dp-branes, so they do not interact with them and the interaction term of the open strings with closed strings vanishes. In this effective theory, the action of Equation (2.4.1) becomes

$$S \mapsto S_{3+1}^{N=4\ \text{SYM}} + S_{\mathcal{M}_{10}\ \text{SUGRA}}^{\text{IIB}}. \tag{2.4.2}$$

Black Dp-brane solutions in type IIB string theory are generalizations of Black Holes. For $p = 3$, this $D3$-brane solution describes a black hole extended in three spacial dimensions. There are three spacial dimensions and one temporal dimension parallel to the $D3$-brane. The associated physics of this system may be partitioned, one part describing closed strings far from the horizon, $S_{\mathcal{M}_{10}}^{\text{closed}}$, the second describing closed strings in the near horizon limit, $S_{\text{Hor}}^{\text{closed}}$, and a third describing the interaction of strings far from the horizon with strings close to the horizon or Hawking radiation, S_{Hawking},

$$S = S^{\text{closed}}_{\mathcal{M}_{10}} + S^{\text{closed}}_{\text{Hor}} + S_{\text{Hawking}}. \tag{2.4.3}$$

In the low energy limit, only the long wavelength modes of $S^{\text{closed}}_{\mathcal{M}_{10}}$, $S^{\text{IIB}}_{\mathcal{M}_{10}\ \text{SUGRA}}$, contribute. The shorter wavelength oscillatory modes of the string itself are red-shifted in the super gravity limit and also contribute to the this low energy limit. The near horizon limit portion of the action becomes type IIB closed string theory on an $AdS_5 \times S^5$ background, $S^{\text{IIB}}_{AdS_5 \times S^5}$. Hawking radiation does not contribute in this limit. The effective low energy limit of Equation (2.4.3) is

$$S = S^{\text{IIB}}_{\mathcal{M}_{10}\ \text{SUGRA}} + S^{\text{IIB}}_{AdS_5 \times S^5}. \tag{2.4.4}$$

Term for term comparison of Equation (2.4.2) and Equation (2.4.4) reveals that $\mathcal{N} = 4$ super Yang-Mills theory in $3+1$-dimensions is dual to type IIB closed string theory on an $AdS_5 \times S^5$ background in the super gravity limit.

Following the conjecture of the AdS/CFT correspondence, type IIB closed string theory is the classical theory dual to $\mathcal{N} = 4$ super Yang-Mills theory in $3+1$-dimensions in the large N limit. Some of the questions one may ask about this duality include:

- Does this correspondence extend to finite N, and if so, what does the finite N regime describe in the string theory?
- How does one derive the dual string theory from the matrix model?

To answer these questions, we will try to invent a string theory such that the sum over surfaces equals the sum of ribbon graphs for any expectation value that we evaluate. This string theory will not be unique. Finally, we determine whether such a string theory is sensible. This program was initiated for $3+1$-dimensions by Gopakumar [7] and later, in 0-dimensions, by Razamat [12].

Chapter 3

String Action

3.1 The Action Principle

We would like to build a model for a string theory dual to a large N gauge theory. To do this, we look at the dynamics of such systems. We may formulate the equations of motion for each system in terms of an *action* [1; 13; 14; 15]. The action of a system is that quantity which specifies the evolution of the system. By Hamilton's *Principle of Least Action*, the evolution of the system acts to extremize the action [15]. The action, S, of a system is formulated according to

$$S = \int_{\mathcal{C}} \mathrm{d}t \; L, \tag{3.1.1}$$

where L is the *Lagrangian* of the system traversing a path C. In field theory, the action of a system is an integral over an N-dimensional space-time,

$$S = \int_{\mathcal{V}} \mathrm{d}^N x \; \mathcal{L}, \tag{3.1.2}$$

where $\mathcal{L}$ is the *Lagrangian Density* and $\mathcal{V}$ is now the volume of the space-time.

To extract the equations of motion of system from the action, S, consider the evolution of a system described by the field ϕ,

$$S\left(\mathcal{V};\phi\right)=\int_{\mathcal{V}}\mathrm{d}^{N}x\;\mathcal{L}\left(\phi;\partial_{j}\phi\right). \tag{3.1.3}$$

By Hamilton's *Variational Principle* the equations of motion of the system, may be determined from a variation of S induced by varying ϕ by $\delta\phi$. That is

$$\begin{aligned}\phi &\mapsto \phi+\delta\phi,\\ \partial_j\phi &\mapsto \partial_j\phi+\partial_j\delta\phi.\end{aligned}$$

By Taylor series expansion,

$$\begin{aligned}\mathcal{L}\left(\phi;\partial_j\phi\right)&\mapsto\mathcal{L}\left(\phi+\delta\phi;\partial_j\left(\phi+\delta\phi\right)\right)\\ &=\mathcal{L}\left(\phi;\partial_j\phi\right)+\left[\delta\phi\left(\frac{\partial\mathcal{L}}{\partial\phi}\right)+\partial_j\delta\phi\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_j\phi\right)}\right)\right]+\left[\frac{\left(\delta\phi\right)^2}{2!}\left(\frac{\partial^2\mathcal{L}}{\partial\phi^2}\right)+\right.\\ &\left.\frac{\left(\partial_j\delta\phi\right)^2}{2!}\left(\frac{\partial^2\mathcal{L}}{\partial\left(\partial_j\phi\right)^2}\right)+\left(\delta\phi\right)\left(\partial_j\delta\phi\right)\frac{\partial^2\mathcal{L}}{\partial\phi\partial\left(\partial_j\phi\right)}\right]+\ldots\\ &=\mathcal{L}+\delta\mathcal{L}.\end{aligned}$$

To first order in ϕ and $\partial_j\phi$,

$$\delta\mathcal{L}=\delta\phi\left(\frac{\partial\mathcal{L}}{\partial\phi}\right)+\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_j\phi\right)}\right)\partial_j\delta\phi.$$

The first order correction to the action is,

$$\begin{aligned}\delta S&=\int_{\mathcal{V}}\mathrm{d}^Nx\;\delta\mathcal{L}\\ &=\int_{\mathcal{V}}\mathrm{d}^Nx\left[\delta\phi\left(\frac{\partial\mathcal{L}}{\partial\phi}\right)+\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_j\phi\right)}\right)\partial_j\delta\phi\right].\end{aligned}$$

By chain rule,

$$\partial_j\left[\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_j\phi\right)}\right)\delta\phi\right]=\delta\phi\partial_j\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_j\phi\right)}\right)+\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_j\phi\right)}\right)\partial_j\delta\phi,$$

or,

$$\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_j\phi\right)}\right)\partial_j\delta\phi=\partial_j\left[\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_j\phi\right)}\right)\delta\phi\right]-\delta\phi\partial_j\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_j\phi\right)}\right).$$

Thus,

$$\begin{aligned}\delta S &= \int_{\mathcal{V}} \mathrm{d}^N x \left(\delta\phi \left(\frac{\partial \mathcal{L}}{\partial \phi} \right) + \partial_j \left[\left(\frac{\partial \mathcal{L}}{\partial (\partial_j \phi)} \right) \delta\phi \right] - \delta\phi \partial_j \left(\frac{\partial \mathcal{L}}{\partial (\partial_j \phi)} \right) \right) \\ &= \int_{\mathcal{V}} \mathrm{d}^N x \left(\left(\frac{\partial \mathcal{L}}{\partial \phi} \right) - \partial_j \left(\frac{\partial \mathcal{L}}{\partial (\partial_j \phi)} \right) \right) \delta\phi + \int_{\mathcal{V}} \mathrm{d}^N x \ \partial_j \left[\left(\frac{\partial \mathcal{L}}{\partial (\partial_j \phi)} \right) \delta\phi \right].\end{aligned}$$

Notice that, by the Divergence Theorem, the last term is an integral over the surface $\mathcal{S}$, bounding $\mathcal{V}$, having the surface element $\mathrm{d}a$,

$$\int_{\mathcal{V}} \mathrm{d}^N x \ \partial_j \left[\left(\frac{\partial \mathcal{L}}{\partial (\partial_j \phi)} \right) \delta\phi \right] = \oint_{\mathcal{S}} \mathrm{d}a_j \left(\frac{\partial \mathcal{L}}{\partial (\partial_j \phi)} \right) \delta\phi.$$

Since there must be zero variation at the boundary of the space-time in which the system evolves, $\delta\phi$ is zero over $\mathcal{S}$. Therefore, the surface term is zero. By requiring that S is stationary under an arbitrary change $\delta\phi$,

$$\delta S = 0 \leftrightarrow \left(\frac{\partial \mathcal{L}}{\partial \phi} \right) - \partial_j \left(\frac{\partial \mathcal{L}}{\partial (\partial_j \phi)} \right) = 0.$$

The equations

$$\partial_j \left(\frac{\partial \mathcal{L}}{\partial (\partial_j \phi)} \right) = \left(\frac{\partial \mathcal{L}}{\partial \phi} \right), \tag{3.1.4}$$

are the Euler-Lagrange field equations. The Euler-Lagrange equations specify the equations of motion for a given system.

Since the physics of a system must be independent of the co-ordinate system used to describe that system, we require that the action of a system is invariant under co-ordinate transformation and reparametrization. We now develop this notion of an action describing a system in natural units, $c = \hbar = 1$, first for a relativistic point particle and then for a relativistic string.

3.2 World-Line Action

From Differential Geometry, the *fundamental quadratic form* of space-time is,

$$\mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}X^\mu \mathrm{d}X^\nu, \tag{3.2.1}$$

where $g_{\mu\nu}$ is the *fundamental bilinear form* of space-time, describes the separation of adjacent space-time events. Equation (3.2.1) is referred to as the *space-time interval.*

The locus of points traced by a particle on a space-time trajectory, $X^\mu(\lambda)$, is called the *world-line* of the particle. The *proper-time*, τ, as measured by a particle on its world-line is the time interval as measured by an observer in the instantaneous rest frame of the particle. The proper-time of a particle defines the arc-length parameter on the world-line of the particle, see Figure 3.1. In differential form the proper-time relation is

$$\mathrm{d}\tau^2 = g_{\mu\nu}\mathrm{d}X^\mu\mathrm{d}X^\nu. \tag{3.2.2}$$

When measured along the world-line of a particle, $\mathcal{C}$, the proper-time is

$$\tau = \int_{\mathcal{C}} \sqrt{g_{\mu\nu}\mathrm{d}X^\mu\mathrm{d}X^\nu}. \tag{3.2.3}$$

The world-line of a particle is that path in space-time along which the proper-time of the particle is an extremum of the proper-time relation. The world-line parametrization of Equation (3.2.3) determines an action S_τ, for a free point particle,

$$S_\tau = -m\int_{\mathcal{C}} \sqrt{g_{\mu\nu}\mathrm{d}X^\mu\mathrm{d}X^\nu}. \tag{3.2.4}$$

Equation (3.2.4) is a summation over all particle space-time trajectories connecting the terminal points of motion and has dimensions $[S_\tau] = 1$ in natural units.

Now we investigate whether the action of Equation (3.2.4) is natural. Let Λ be an invertible transformation. Under the co-ordinate transformation

$$X^\mu \mapsto X'^\mu = \Lambda^\mu{}_\sigma X^\sigma, \tag{3.2.5}$$

the covariant metric tensor $g_{\mu\nu}$ transforms by

$$g_{\mu\nu} \mapsto g'_{\mu\nu} = \left(\Lambda^{-1}\right)^\sigma{}_\mu \left(\Lambda^{-1}\right)^\theta{}_\nu \, g_{\sigma\theta}.$$

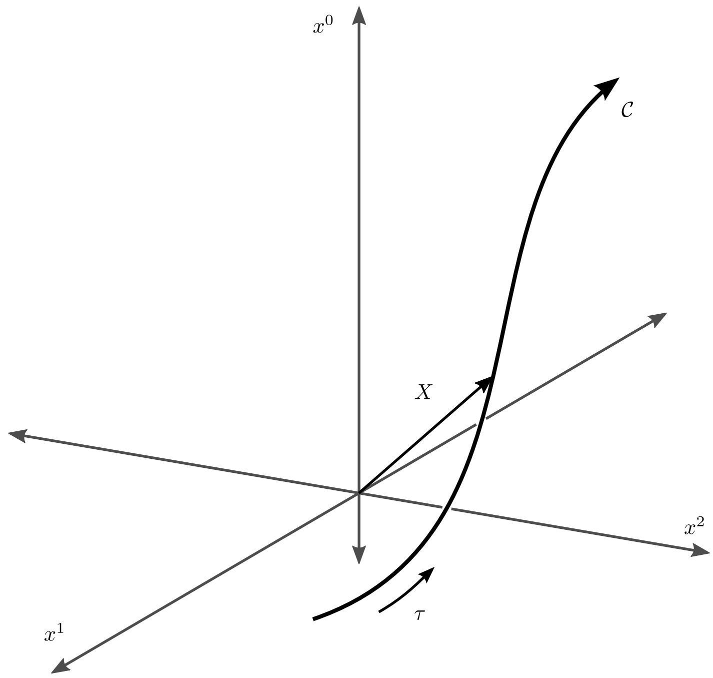


Figure 3.1: The world-line of a particle may be parametrized by the world-line arc-length. This arc-length is the proper-time of the particle measured along the world-line.

The action transforms by

$$\begin{aligned} S_\tau \mapsto & S'_\tau \\ &= -m \int_{\mathcal{C}} \sqrt{g'_{\mu\nu} \mathrm{d}X'^\mu \mathrm{d}X'^\nu} \\ &= -m \int_{\mathcal{C}} \sqrt{\left(\Lambda^{-1}\right)^\sigma{}_\mu \left(\Lambda^{-1}\right)^\theta{}_\nu g_{\sigma\theta} \Lambda^\mu{}_\alpha \mathrm{d}X^\alpha \Lambda^\nu{}_\beta \mathrm{d}X^\beta} \\ &= -m \int_{\mathcal{C}} \sqrt{\left(\Lambda^{-1}\right)^\sigma{}_\mu \Lambda^\mu{}_\alpha \left(\Lambda^{-1}\right)^\theta{}_\nu \Lambda^\nu{}_\beta g_{\sigma\theta} \mathrm{d}X^\alpha \mathrm{d}X^\beta} \\ &= -m \int_{\mathcal{C}} \sqrt{g_{\alpha\beta} \mathrm{d}X^\alpha \mathrm{d}X^\beta}. \end{aligned}$$

The action S_τ is invariant under co-ordinate transformation.

Now consider the reparametrization of Equation (3.2.4) by,

$$\lambda \mapsto \lambda' = \lambda'(\lambda), \tag{3.2.6}$$

where $\lambda'(\lambda)$ is a monotone function of λ. Since λ' is monotone, $X'^\mu\left(\lambda'(\lambda)\right) =$

$X^\mu(\lambda)$ and the world-lines plotted by X'^μ and X^μ are identical. Now,

$$\mathrm{d}\lambda' = \left(\frac{\partial \lambda'}{\partial \lambda}\right)\mathrm{d}\lambda = \left(\frac{\mathrm{d}\lambda'}{\mathrm{d}\lambda}\right)\mathrm{d}\lambda.$$

Note that $\mathrm{d}X^\mu = \left(\frac{\mathrm{d}X^\mu}{\mathrm{d}\lambda}\right)\mathrm{d}\lambda$. Under this reparametrization,

$$\begin{aligned}
S_\tau \mapsto & S'_\tau \\
= & -m\int_{\mathcal{C}}\sqrt{g_{\mu\nu}\left(\frac{\mathrm{d}X'^\mu}{\mathrm{d}\lambda'}\right)\mathrm{d}\lambda'\left(\frac{\mathrm{d}X'^\nu}{\mathrm{d}\lambda'}\right)\mathrm{d}\lambda'} \\
= & -m\int_{\mathcal{C}}\sqrt{g_{\mu\nu}\left(\frac{\mathrm{d}X^\mu}{\mathrm{d}\lambda}\right)\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\lambda'}\right)\left(\frac{\mathrm{d}\lambda'}{\mathrm{d}\lambda}\right)\mathrm{d}\lambda\left(\frac{\mathrm{d}X^\nu}{\mathrm{d}\lambda}\right)\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\lambda'}\right)\left(\frac{\mathrm{d}\lambda'}{\mathrm{d}\lambda}\right)\mathrm{d}\lambda} \\
= & -m\int_{\mathcal{C}}\sqrt{g_{\mu\nu}\left(\frac{\mathrm{d}X^\mu}{\mathrm{d}\lambda}\right)\mathrm{d}\lambda\left(\frac{\mathrm{d}X^\nu}{\mathrm{d}\lambda}\right)\mathrm{d}\lambda}\,.
\end{aligned}$$

This could also have been shown by noting that, since $X'^\mu\left(\lambda'(\lambda)\right) = X^\mu(\lambda)$,

$$-m\int_{\mathcal{C}}\sqrt{g_{\mu\nu}\mathrm{d}X'^\mu\mathrm{d}X'^\nu} = -m\int_{\mathcal{C}}\sqrt{g_{\mu\nu}\mathrm{d}X^\mu\mathrm{d}X^\nu}.$$

Therefore, S_τ is also invariant under reparametrization. The action of Equation (3.2.4) is both co-ordinate and parametrization invariant, hence it is natural. Rewriting S_τ, taking proper-time as the parameter, yields,

$$S_\tau = \int_{\mathcal{C}}\mathrm{d}\tau\left(-m\sqrt{g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu}\right), \tag{3.2.7}$$

where the dot notation signifies differentiation with respect to proper-time.

Extremizing the Equation (3.2.7) with respect to X^μ produces the *geodesic* equations for the point particle. Assuming that $g_{\mu\nu} = g_{\mu\nu}(X)$, $X^\mu = X^\mu(\tau)$ and τ is chosen as an affine parameter satisfying

$$g_{\mu\nu}\left(\frac{\mathrm{d}X^\mu}{\mathrm{d}\tau}\right)\left(\frac{\mathrm{d}X^\nu}{\mathrm{d}\tau}\right) = 1, \tag{3.2.8}$$

the Euler-Lagrange equations give

$$\left(\frac{\partial L}{\partial X^\alpha}\right) = \frac{-m}{2}\frac{\left(\frac{\partial g_{\mu\nu}}{\partial X^\alpha}\right)\dot{X}^\mu\dot{X}^\nu}{\sqrt{g_{\sigma\theta}\dot{X}^\sigma\dot{X}^\theta}}, \tag{3.2.9}$$

and

$$\left(\frac{\partial L}{\partial \dot{X}^\alpha}\right) = \frac{-m g_{\alpha\nu}\dot{X}^\nu}{\sqrt{g_{\sigma\theta}\dot{X}^\sigma\dot{X}^\theta}}. \tag{3.2.10}$$

By Equation (3.2.8),

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{1}{\sqrt{g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu}}\right] = 0,$$

so,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left(\frac{\partial L}{\partial X^\alpha}\right)\right] = -m\frac{\left(\frac{\mathrm{d}g_{\alpha\nu}}{\mathrm{d}X^\mu}\right)\dot{X}^\mu\dot{X}^\nu + g_{\alpha\nu}\ddot{X}^\nu}{\sqrt{g_{\sigma\theta}\dot{X}^\sigma\dot{X}^\theta}}.$$

Following Equations (3.2.9) and (3.2.10), solving for $\ddot{X}^\alpha$ gives

$$\ddot{X}^\alpha = -\Gamma^\alpha{}_{\mu\nu}(g)\dot{X}^\mu\dot{X}^\nu, \tag{3.2.11}$$

where the *affine connection* $\Gamma^\alpha{}_{\mu\nu}(g)$ defined by

$$\Gamma^\alpha{}_{\mu\nu}(g) = \frac{1}{2}g^{\alpha\beta}\left[\left(\frac{\mathrm{d}g_{\beta\nu}}{\mathrm{d}X^\mu}\right) + \left(\frac{\mathrm{d}g_{\beta\mu}}{\mathrm{d}X^\nu}\right) - \left(\frac{\partial g_{\mu\nu}}{\partial X^\beta}\right)\right], \tag{3.2.12}$$

is the associated *Christoffel Symbol* of the second kind.

In the flat space-time of Minkowski Space,

$$g = \eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3.2.13}$$

In this space-time $\left(\frac{\mathrm{d}g_{\mu\nu}}{\mathrm{d}X^\alpha}\right) = 0$ and substituting this into Equation (3.2.11) gives $\ddot{X}^\alpha = 0$. By this account, the acceleration of the particle in this flat space-time is zero.

Under the assignment of $\tau = X^0 = t$, we may write

$$g_{\mu\nu}\left(\frac{\partial X^\mu}{\partial \tau}\right)\left(\frac{\partial X^\nu}{\partial \tau}\right) = 1 - \vec{v}\cdot\vec{v},$$

where $\vec{v}$ is the 3-space velocity of the particle. In the non-relativistic (low velocity) limit, $\vec{v} \cdot \vec{v} \ll 1$, so,

$$\begin{aligned} S_\tau &= \int_{\mathcal{C}} \mathrm{d}\tau \left(-m\sqrt{g_{\mu\nu}\mathrm{d}\dot{X}^\mu \mathrm{d}\dot{X}^\nu} \right) \\ &= \int_{\mathcal{C}} \mathrm{d}\tau \left(-m\sqrt{1 - \vec{v} \cdot \vec{v}} \right) \\ &\approx \int_{\mathcal{C}} \mathrm{d}\tau \left(\frac{1}{2}mv^2 - m^2 \right). \end{aligned}$$

In this low velocity limit for the free particle, the kinetic energy of the particle is $\frac{1}{2}mv^2$. The relativistic rest mass energy of the particle is m. Classical physics is recovered in the low velocity limit. Equation (3.2.11) specifies the acceleration, $\ddot{X}^\alpha$, of the particle of mass, m, in the space-time background.

An electrically charged point particle couples to the Maxwell field A^μ, consisting of the scalar potential ϕ and the vector potential $\vec{A}$,

$$A^\mu = \left(\phi; \vec{A} \right)^\mu . \tag{3.2.14}$$

Since the Maxwell field couples to the point particle along the world-line of the particle, there is a contribution of the Maxwell field to the action. One candidate for this is,

$$S_M = \int \mathrm{d}X^\mu q A_\mu, \tag{3.2.15}$$

where A_μ is the field and q is the charge coupling of the particle to the field. Under the invertible co-ordinate transformation of Equation (3.2.5), the field A_μ transforms by

$$A'_\mu = \left(\Lambda^{-1}\right)^\eta{}_\mu A_\eta .$$

The action transforms by,

$$\begin{aligned} S_M \mapsto & S'_M \\ &= \int_{\mathcal{C}} \mathrm{d}X'^\mu q A'_\mu \\ &= \int_{\mathcal{C}} \mathrm{d}X^\eta \Lambda^\mu{}_\eta q \left(\Lambda^{-1}\right)^\sigma{}_\mu A_\sigma \end{aligned}$$

$$= \int_{\mathcal{C}} \mathrm{d}X^{\mu} q A_{\mu}.$$

So, this action is invariant under co-ordinate transformations.

Under the reparametrization of Equation (3.2.6) the action will transform by

$$\begin{aligned} S_M \mapsto & S'_M \\ &= \int_{\mathcal{C}} \mathrm{d}\lambda' \left(\frac{\partial X'^{\mu}}{\partial \lambda'}\right) q A_{\mu} \\ &= \int_{\mathcal{C}} \mathrm{d}\lambda \left(\frac{\partial \lambda'}{\partial \lambda}\right)\left(\frac{\partial \lambda}{\partial \lambda'}\right)\left(\frac{\partial X'^{\mu}}{\partial \lambda}\right) q A_{\mu} \\ &= \int_{\mathcal{C}} \mathrm{d}\lambda \left(\frac{\partial X'^{\mu}}{\partial \lambda}\right) q A_{\mu}. \end{aligned}$$

Again, could also have been shown by recalling that $X'^{\mu}(\lambda'(\lambda)) = X^{\mu}(\lambda)$ and so,

$$\int_{\mathcal{C}} \mathrm{d}X'^{\mu} q A_{\mu} = \int_{\mathcal{C}} \mathrm{d}X^{\mu} q A_{\mu}.$$

This action is also invariant under reparametrization and may therefore, be rewritten in terms of the proper-time τ,

$$S_M = \int_{\mathcal{C}} \mathrm{d}\tau q A_{\mu} \dot{X}^{\mu}. \tag{3.2.16}$$

Equations (3.2.7) and (3.2.16) are now combined to define the natural action for an electrically charged particle,

$$S_{pp} = \int_{\mathcal{C}} \mathrm{d}\tau \left(-m\sqrt{g_{\mu\nu}\dot{X}^{\mu}\dot{X}^{\nu}} + q A_{\mu}\dot{X}^{\mu}\right). \tag{3.2.17}$$

Since this action is parametrization invariant, we choose τ as an affine parameter in the sequel.

We now construct the equations of motion of an electrically charged particle in flat space-time. The Lagrangian is

$$L_{pp} = -m\sqrt{g_{\mu\nu}\dot{X}^{\mu}\dot{X}^{\nu}} + q A_{\mu}\dot{X}^{\mu}. \tag{3.2.18}$$

By the Euler-Lagrange equations,

$$\begin{aligned}\left(\frac{\partial L_{pp}}{\partial X^\alpha}\right) =& \frac{\partial}{\partial X^\alpha}\left[-m\sqrt{g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu} + qA_\mu\dot{X}^\mu\right] \\ =& \frac{-m}{2}\frac{\left(\frac{\partial g_{\mu\nu}}{\partial X^\alpha}\right)\dot{X}^\mu\dot{X}^\nu}{\sqrt{g_{\sigma\theta}\dot{X}^\sigma\dot{X}^\theta}} + q\left(\frac{\partial A_\mu}{\partial X^\alpha}\right)\dot{X}^\mu\end{aligned} \tag{3.2.19}$$

and

$$\begin{aligned}\left(\frac{\partial L_{pp}}{\partial \dot{X}^\alpha}\right) =& \frac{\partial}{\partial \dot{X}^\alpha}\left[-m\sqrt{g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu} + qA_\mu\dot{X}^\mu\right] \\ =& -m\frac{g_{\alpha\nu}\dot{X}^\nu}{\sqrt{g_{\sigma\theta}\dot{X}^\sigma\dot{X}^\theta}} + qA_\alpha.\end{aligned}$$

For any affine world-line parameter, $g_{\mu\nu}\mathrm{d}\dot{X}^\mu\dot{X}^\nu = 1$, which we will freely use.. Also,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left(\frac{\partial L_{pp}}{\partial \dot{X}^\alpha}\right)\right] =& \frac{\mathrm{d}}{\mathrm{d}\tau}\left[-m\sqrt{g_{\mu\nu}\dot{X}^\mu\dot{X}^\nu} + qA_\mu\dot{X}^\mu\right] \\ =& -m\frac{\left(\frac{\mathrm{d}g_{\alpha\nu}}{\mathrm{d}X^\mu}\right)\dot{X}^\mu\dot{X}^\nu + g_{\alpha\nu}\ddot{X}^\nu}{\sqrt{g_{\sigma\theta}\dot{X}^\sigma\dot{X}^\theta}} + q\left(\frac{\mathrm{d}A_\alpha}{\mathrm{d}X^\mu}\right)\dot{X}^\mu.\end{aligned} \tag{3.2.20}$$

After substituting Equations (3.2.19) and (3.2.20) into the Euler-Lagrange equations for L_{pp}, solving for $\ddot{X}^\alpha$ gives,

$$\ddot{X}^\alpha = -\Gamma^\alpha{}_{\mu\nu}\dot{X}^\mu\dot{X}^\nu + \frac{q}{m^2}g^{\alpha\beta}F_{\mu\beta}\dot{X}^\mu, \tag{3.2.21}$$

where the totally anti-symmetric *Maxwell field strength tensor*, $F_{\mu\nu}$, is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3.2.22}$$

The Maxwell field gives rise to the electric field, $\vec{E}$, and magnetic field, $\vec{B}$, which are functions of the vector potential $\vec{A}$ and the scaler field ϕ, with $\vec{E} = -\nabla\phi - \left(\frac{\partial\vec{A}}{\partial t}\right)$ and $\vec{B} = \nabla\times\vec{A}$.

A component wise study of Equation (3.2.21) reveals that for $\mu = 0$ and $\alpha = i$,

$$\left[\left(\frac{\partial A_i}{\partial X^0}\right) - \left(\frac{\partial A_0}{\partial X^i}\right)\right]\dot{X}^0 = -\left(\frac{\mathrm{d}A^0}{\mathrm{d}X^i}\right) - \left(\frac{\mathrm{d}A^i}{\mathrm{d}X^0}\right).$$

By assignment of $A^0 = \phi$, this becomes

$$\left[\left(\frac{\partial A_i}{\partial X^0}\right) - \left(\frac{\partial A_0}{\partial X^i}\right)\right] \dot{X}^0 = -\nabla\phi - \left(\frac{\partial A^i}{\partial \tau}\right) = \vec{E}.$$

For $\mu = j$ and $\alpha = i$,

$$\left[\left(\frac{\partial A_i}{\partial X^j}\right) - \left(\frac{\partial A_j}{\partial X^i}\right)\right] \dot{X}^j = \vec{v} \times \left(\nabla \times \vec{A}\right) = \vec{v} \times \vec{B}.$$

Therefore, the particle experiences the electromagnetic force,

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right), \tag{3.2.23}$$

which is the well known *Lorentz force.*

3.3 World-Sheet Action

In the case of the point particle, the relativistic action was determined from the one dimensional path traced by the particle in space-time. For a one dimensional extended object, like a string, a two dimensional surface, known as a *world-sheet*, is traced in space-time. The point particle action was formulated in terms of the arc-length parameter of the world-line. The world-sheet action for a string is now analogously formulated.

Let $\mathcal{S}$ be a surface parametrized by the position vector $\vec{X}$, where $X^i = X^i(\sigma^1;\sigma^2)$ and σ^1 and σ^2 are co-ordinates in the parameter space of $\mathcal{S}$. The *tangent vector* $\vec{u}_{(p_0;\vec{p})}$ is the vector tangent to $\mathcal{S}$ at the point p_0 in the direction of $\vec{p}$. The area, A, of $\mathcal{S}$ is the sum of all area elements a_{ij}, determined from the area of the parallelogram between the tangent vectors $\vec{u}$ and $\vec{v}$, of length $\Delta\sigma_1$ and $\Delta\sigma_2$, see Figure 3.1. Then,

$$\begin{aligned}
A &= \lim_{N\to\infty} \sum_{i=0}^{N} \sum_{j=0}^{N} a_{ij} \\
&= \lim_{N\to\infty} \sum_{i=0}^{N} \sum_{=0}^{N} \left\| \vec{u}\big|_{\left((\sigma_i^1;\sigma_j^2);\vec{\sigma^1}\right)} \times \vec{v}\big|_{\left((\sigma_i^1;\sigma_j^2);\vec{\sigma^2}\right)} \right\|
\end{aligned}$$

$$= \lim_{N\to\infty} \sum_{i=0}^{N} \sum_{j=0}^{N} \left\| \Delta\sigma^1 \partial_{\sigma^1} \vec{X} \times \Delta\sigma^2 \partial_{\sigma^2} \vec{X} \right\| \Big|_{\left(\sigma_i^1;\sigma_j^2\right)}$$

$$= \lim_{N\to\infty} \sum_{i=0}^{N} \sum_{j=0}^{N} \Delta\sigma^1 \Delta\sigma^2 \left\| \partial_{\sigma^1} \vec{X} \times \partial_{\sigma^2} \vec{X} \right\| \Big|_{\left(\sigma_i^1;\sigma_j^2\right)}$$

or,

$$A = \int_S \mathrm{d}\sigma^1 \mathrm{d}\sigma^2 \left\| \partial_{\sigma^1} \vec{X} \times \partial_{\sigma^2} \vec{X} \right\|.$$

Note that,

$$\left(\partial_{\sigma^1} \vec{X} \times \partial_{\sigma^2} \vec{X} \right)^i = \epsilon^i{}_{jk} \partial_{\sigma^1} X^j \partial_{\sigma^2} X^k,$$

so,

$$\left\| \partial_{\sigma^1} \vec{X} \times \partial_{\sigma^2} \vec{X} \right\|^2 = \partial_{\sigma^1} X_j \partial_{\sigma^1} X^j \partial_{\sigma^2} X_k \partial_{\sigma^2} X^k - \partial_{\sigma^1} X_n \partial_{\sigma^2} X^n \partial_{\sigma^2} X_m \partial_{\sigma^1} X^m.$$

The area integral now becomes

$$A = \int_S \mathrm{d}\sigma^1 \mathrm{d}\sigma^2 \sqrt{\partial_{\sigma^1} X_j \partial_{\sigma^1} X^j \partial_{\sigma^2} X_k \partial_{\sigma^2} X^k - \partial_{\sigma^1} X_n \partial_{\sigma^2} X^n \partial_{\sigma^2} X_m \partial_{\sigma^1} X^m}. \tag{3.3.1}$$

We require that distances measured on the surface agree with distances measured in the space in which the surface is embedded. For a given tangent vector $\mathrm{d}\vec{u} = \mathrm{d}\vec{u}\,(\sigma^1;\sigma^2)$, the length of $\mathrm{d}\vec{u}$ is given by $\mathrm{d}s^2 = \mathrm{d}\vec{u} \cdot \mathrm{d}\vec{u}$. From the position vector $\vec{X}$ construct the tangent vector $\mathrm{d}u^i = \left(\frac{\partial X^i}{\partial \sigma^a}\right) \mathrm{d}\sigma^a$. The quantity $\mathrm{d}\vec{u} \cdot \mathrm{d}\vec{u}$ is now given by,

$$\mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}X^\mu \mathrm{d}X^\nu = g_{\mu\nu} \left(\frac{\partial X^\mu}{\partial \sigma^a}\right) \left(\frac{\partial X^\nu}{\partial \sigma^b}\right) \mathrm{d}\sigma^a \mathrm{d}\sigma^b = h_{ab} \mathrm{d}\sigma^a \mathrm{d}\sigma^b, \tag{3.3.2}$$

where h_{ab} is the induced metric on the world-sheet which takes the matrix form,

$$h_{ab} = g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu. \tag{3.3.3}$$

Equation (3.3.1) can be re-written in the form,

$$A = \int_S \mathrm{d}\sigma^1 \mathrm{d}\sigma^2 \sqrt{\det(h)}. \tag{3.3.4}$$

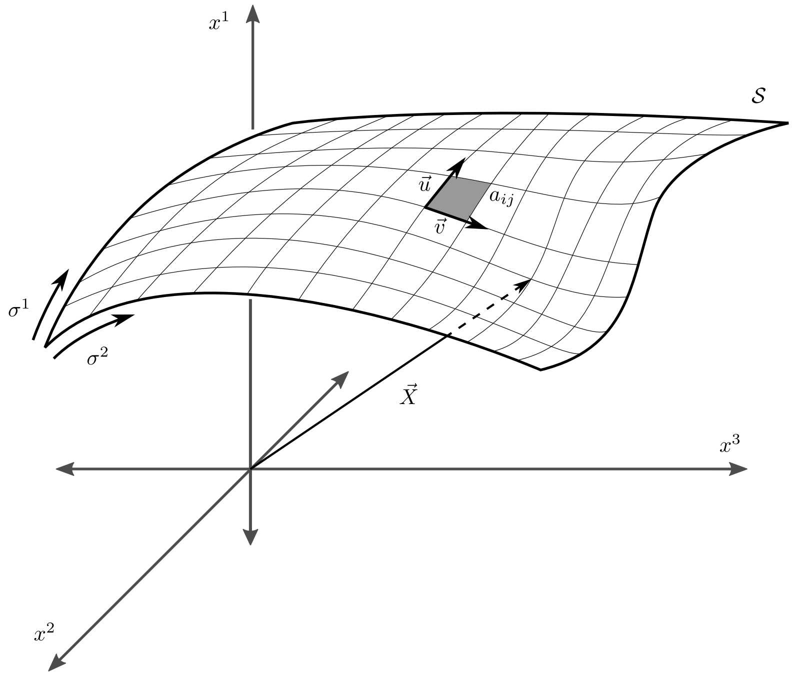


Figure 3.1: The area element of a_{ij} of a surface parametrized by σ^1 and σ^2.

Consider the following change of variables for Equation (3.3.4),

$$\sigma^i \mapsto \sigma'^i = \sigma'^i \left(\sigma^1; \sigma^2\right). \tag{3.3.5}$$

The corresponding change in measure is

$$\mathrm{d}\sigma^1 \mathrm{d}\sigma^2 = J\left(\frac{\partial(\sigma^1;\sigma^2)}{\partial(\sigma'^1;\sigma'^2)}\right)\mathrm{d}\sigma'^1\mathrm{d}\sigma'^2 = \left|\det\left(M\right)\right| \mathrm{d}\sigma'^1\mathrm{d}\sigma'^2,$$

where M is a matrix describing the coordinate transformation, having elements $M^i{}_j = \left(\frac{\partial\sigma^i}{\partial\sigma'^j}\right)$. Similarly, the inverse co-ordinate change provides

$$\mathrm{d}\sigma'^1\mathrm{d}\sigma'^2 = J\left(\frac{\partial(\sigma'^1;\sigma'^2)}{\partial(\sigma^1;\sigma^2)}\right)\mathrm{d}\sigma^1\mathrm{d}\sigma^2 = \left|\det\left(M'\right)\right| \mathrm{d}\sigma^1\mathrm{d}\sigma^2,$$

where M' is the matrix describing the inverse coordinate transformation, having elements $M'^i{}_j = \left(\frac{\partial\sigma'^i}{\partial\sigma^j}\right)$. By the forms of M and M' is is clear that,

$$\left|\det\left(M\right)\right|\left|\det\left(M'\right)\right| = \left|\det\left(MM'\right)\right| = \left|\det\left(M'M\right)\right| = 1.$$

The induced metric also changes,

$$h_{ij} = h'_{ab} \left(\frac{\partial \sigma'^a}{\partial \sigma^i} \right) \left(\frac{\partial \sigma'^b}{\partial \sigma^j} \right).$$

In matrix notation this gives

$$h_{ij} = h'_{ab} M'^a{}_i M'^b_j = \left(M'^T \right)_i^{\ a} h'_{ab} M'^b{}_j,$$

and taking the determinant on each side of the equality sign gives,

$$\det(h) = \det\left(M'^T h' M' \right) = \det\left(M'^T \right) \det(h') \det(M') = \det(h') \det(M')^2.$$

Therefore,

$$\sqrt{\det(h)} = \sqrt{\det(h')} \left| \det(M') \right|.$$

Under this reparametrization,

$$\begin{aligned}
A &\mapsto A' \\
&= \int_{\mathcal{S}} \mathrm{d}\sigma'^1 \mathrm{d}\sigma'^2 \sqrt{\det(h')} \\
&= \int_{\mathcal{S}} \mathrm{d}\sigma^1 \mathrm{d}\sigma^2 \left| \det(M) \right| \sqrt{\det(h)} \left| \det(M') \right| \\
&= \int_{\mathcal{S}} \mathrm{d}\sigma^1 \mathrm{d}\sigma^2 \left| \det(M) \right| \left| \det(M') \right| \sqrt{\det(h)} \\
&= \int_{\mathcal{S}} \mathrm{d}\sigma^1 \mathrm{d}\sigma^2 \sqrt{\det(h)}.
\end{aligned}$$

Therefore, the area functional is invariant under an arbitrary change of variables.

To construct an appropriate string action, we note that in that same way that the action of the world-line is proportional to the length of the world-line, the string action is proportional to the area of the world-sheet. The action constructed from the area functional is a natural action. The parameters σ^1 and σ^2 are relabelled τ and σ, respectively, where τ is related to the proper-time on the string, and σ is related to the positions along the string. In parameter space, τ ranges from negative infinity to positive infinity, while σ may be chosen to range over a finite interval $[0, \ell]$. To ensure that the area functional is well defined, the object under the square root of Equation (3.3.4) must be positive. When analytically continuing from Euclidean-space metric to a Minkowski metric (with τ and σ as time-like and space-like co-ordinates, respectively), the time component is Wick rotated and the entire area element is multiplied by a negative sign. The dimension of the area functional in space-time is LT. Since the action must have dimension 1 in natural units, we introduce the quantity $-T_0$ having dimension L^{-2}. The quantity T_0 is the string *tension* and has units of force. This gives rise to the world-sheet action,

$$S_{NG} = -T_0 \int_{\mathcal{S}} \mathrm{d}\tau \mathrm{d}\sigma \sqrt{-\det(h)}. \tag{3.3.6}$$

This is the *Nambu-Goto* string action. Equation (3.3.6) is a summation over all string world-sheets interpolating between the initial and final string configurations.

As in the case of the world-line action, we now determine the corresponding equations of motion for the world-sheet from the Nambu-Goto action of Equation (3.3.6). The Lagrangian for the Nambu-Goto action is

$$\mathcal{L}_{NG} = -T_0 \sqrt{-\det(h)}. \tag{3.3.7}$$

Extremizing Equation (3.3.7) using Equation (3.1.4), we find $\left(\frac{\partial \mathcal{L}_{NG}}{\partial X^\mu}\right) = 0$ and

$$\begin{aligned}\partial_\nu \left(\frac{\partial \mathcal{L}_{NG}}{\partial(\partial_\nu X^\mu)}\right) &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left(\frac{\partial \mathcal{L}_{NG}}{\partial\left(\frac{\partial X^\mu}{\partial \tau}\right)}\right)\right] + \frac{\mathrm{d}}{\mathrm{d}\sigma}\left[\left(\frac{\partial \mathcal{L}_{NG}}{\partial\left(\frac{\partial X^\mu}{\partial \sigma}\right)}\right)\right] \\ &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left(\frac{\partial \mathcal{L}_{NG}}{\partial(\partial_\tau X^\mu)}\right)\right] + \frac{\mathrm{d}}{\mathrm{d}\sigma}\left[\left(\frac{\partial \mathcal{L}_{NG}}{\partial(\partial_\tau X^\mu)}\right)\right],\end{aligned}$$

where the dot and prime notation signify differentiation with respect to τ and σ respectively. Define the conjugate momenta P^τ_μ and P^σ_μ by

$$P^\tau_\mu = \left(\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}\right) = -T_0 \frac{\left(\dot{X}_\nu X'^\nu\right) X'_\mu - \left(X'_\nu X'^\nu\right) \dot{X}_\mu}{\sqrt{-\det(h)}}, \tag{3.3.8}$$

$$P^{\sigma}_{\mu} = \left(\frac{\partial \mathcal{L}}{\partial X'^{\mu}}\right) = -T_0 \frac{\left(\dot{X}_{\nu} X'^{\nu}\right) \dot{X}_{\mu} - \left(\dot{X}_{\nu} \dot{X}^{\nu}\right) X'_{\mu}}{\sqrt{-\det\left(h\right)}}. \tag{3.3.9}$$

Now,

$$\partial_{\nu} \left(\frac{\partial \mathcal{L}_{NG}}{\partial \left(\partial_{\nu} X^{\mu}\right)}\right) = \left(\frac{\mathrm{d}P^{\tau}_{\mu}}{\mathrm{d}\tau}\right) + \left(\frac{\mathrm{d}P^{\sigma}_{\mu}}{\mathrm{d}\sigma}\right) \tag{3.3.10}$$

and the equations of motion are

$$\left(\frac{\mathrm{d}P^{\tau}_{\mu}}{\mathrm{d}\tau}\right) + \left(\frac{\mathrm{d}P^{\sigma}_{\mu}}{\mathrm{d}\sigma}\right) = 0. \tag{3.3.11}$$

The equations of motion of the string must be solved subject to boundary conditions. The first condition is that the specific endpoint $\sigma_* \in \{0; \ell\}$ of the string is free,

$$P^{\sigma}_{\mu}(\tau; \sigma_*) = 0.$$

This is the *free endpoint condition.* This condition places no restriction on the position of string co-ordinate endpoint σ_*. The second is that the end points of the string are fixed throughout the motion,

$$\left(\frac{\partial X^{\mu}}{\partial \tau}\right) = 0,$$

with $\mu \neq 0$. This is the *Dirichlet boundary condition.* This condition fixes the μ-co-ordinate of a string endpoint in time. These boundary conditions may be imposed in a combination of ways. Either the free endpoint condition or the Dirichlet boundary condition my be imposed at each end point and for each spacial direction. The Dirichlet boundary conditions arise when the endpoints of a string are attached to a physical object. Closed strings have no end points and so must satisfy a periodic boundary condition,

$$X^{\mu}\left(\tau; \sigma\right) = X^{\mu}\left(\tau; \sigma + \ell\right).$$

In a similar manner to the world-line action, we now introduce a charge contribution to the world-sheet action. Since Maxwell charge is carried by points, as in the case of the point particle, strings also carry charge. Since strings are different from point particles, we expect that there exists some new charge carried by strings and some new field to which these charges couple. We therefore introduce the totally antisymmetric gauge field tensor analogue of the Maxwell gauge field, the Kalb-Ramond field, $B_{\mu\nu}$. From this object we construct the string analogue to the point particle electromagnetic action.

Similar to the world-line action, we note that there exists a coupling of the Kalb-Ramond gauge field to the world-sheet tangent vectors,

$$S_{KR} = -\int_{\mathcal{S}} \mathrm{d}\tau \mathrm{d}\sigma\ B_{\mu\nu}\left(X\left(\tau;\sigma\right)\right)\left(\frac{\partial X^{\mu}}{\partial \tau}\right)\left(\frac{\partial X^{\nu}}{\partial \sigma}\right).$$

This coupling is dimensionless in natural units, so $B_{\mu\nu}$ has dimensions of L^{-2}. Since $B_{\mu\nu}$ is totally anti-symmetric only the antisymmetric components on the world-sheet contribute, so

$$S_{KR} = -\frac{1}{2}\int_{\mathcal{S}} \mathrm{d}\tau \mathrm{d}\sigma\ B_{\mu\nu}\left(\frac{\partial X^{\mu}}{\partial \tau}\right)\wedge\left(\frac{\partial X^{\nu}}{\partial \sigma}\right) \tag{3.3.12}$$

where $\wedge$ is the totally antisymmetric product satisfying $a\wedge b = -b\wedge a$, known as the *wedge product*.

Consider again the co-ordinate transformation of Equation (3.2.5). Under this co-ordinate transformation the Kalb-Ramond field, $B_{\mu\nu}$, transforms by

$$B_{\mu\nu} \mapsto B'_{\mu\nu} = \left(\Lambda^{-1}\right)^{\sigma}{}_{\mu}\left(\Lambda^{-1}\right)^{\theta}{}_{\nu} B_{\sigma\theta}.$$

The action transforms by

$$\begin{aligned}
S_{KB} \mapsto & S'_{KB} \\
&= -\frac{1}{2}\int_{\mathcal{S}} \mathrm{d}\tau \mathrm{d}\sigma\ B'_{\mu\nu}\left(\frac{\partial X'^{\mu}}{\partial \tau}\right)\wedge\left(\frac{\partial X'^{\nu}}{\partial \sigma}\right) \\
&= -\frac{1}{2}\int_{\mathcal{S}} \mathrm{d}\tau \mathrm{d}\sigma\ \left(\Lambda^{-1}\right)^{\sigma}{}_{\mu}\left(\Lambda^{-1}\right)^{\theta}{}_{\nu} B_{\sigma\theta}\Lambda^{\mu}{}_{\zeta}\left(\frac{\partial X^{\zeta}}{\partial \tau}\right)\wedge\Lambda^{\nu}{}_{\xi}\left(\frac{\partial X^{\xi}}{\partial \sigma}\right)
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\int_{\mathcal{S}} \mathrm{d}\tau \mathrm{d}\sigma \ \left(\Lambda^{-1}\right)^{\sigma}{}_{\mu} \left(\Lambda^{-1}\right)^{\theta}{}_{\nu} \Lambda^{\mu}{}_{\zeta}\Lambda^{\nu}{}_{\xi} B_{\sigma\theta} \left(\frac{\partial X^{\zeta}}{\partial \tau}\right) \wedge \left(\frac{\partial X^{\xi}}{\partial \sigma}\right) \\
&= -\frac{1}{2}\int_{\mathcal{S}} \mathrm{d}\tau \mathrm{d}\sigma \ B_{\mu\nu} \left(\frac{\partial X^{\mu}}{\partial \tau}\right) \wedge \left(\frac{\partial X^{\nu}}{\partial \sigma}\right).
\end{aligned}
$$

So, S_{τ} is invariant under co-ordinate transformation.

Now, consider again the reparametrization of Equation (3.3.5), where $\sigma^1 = \tau$ and $\sigma^2 = \sigma$

$$
\begin{aligned}
S_{KR} \mapsto & S'_{KR} \\
&= -\frac{1}{2}\int_{\mathcal{S}} \mathrm{d}\tau' \mathrm{d}\sigma' \ B_{\mu\nu} \left(\frac{\partial X'^{\mu}}{\partial \tau'}\right) \wedge \left(\frac{\partial X'^{\nu}}{\partial \sigma'}\right) \\
&= -\frac{1}{2}\int_{\mathcal{S}} \mathrm{d}\tau \mathrm{d}\sigma \ J\left(\frac{\partial(\tau';\sigma')}{\partial(\tau;\sigma)}\right) B_{\mu\nu} \left[\left(\frac{\partial X^{\mu}}{\partial \tau}\right)\left(\frac{\partial \tau}{\partial \tau'}\right) + \left(\frac{\partial X^{\mu}}{\partial \sigma}\right)\left(\frac{\partial \sigma}{\partial \tau'}\right)\right] \\
&\quad \wedge \left[\left(\frac{\partial X^{\nu}}{\partial \sigma}\right)\left(\frac{\partial \sigma}{\partial \sigma'}\right) + \left(\frac{\partial X^{\nu}}{\partial \tau}\right)\left(\frac{\partial \tau}{\partial \sigma'}\right)\right] \\
&= -\frac{1}{2}\int_{\mathcal{S}} \mathrm{d}\tau \mathrm{d}\sigma \ J\left(\frac{\partial(\tau';\sigma')}{\partial(\tau;\sigma)}\right) \left[\left(\frac{\partial \tau}{\partial \tau'}\right)\left(\frac{\partial \sigma}{\partial \sigma'}\right) - \left(\frac{\partial \tau}{\partial \sigma'}\right)\left(\frac{\partial \sigma}{\partial \tau'}\right)\right] \times \\
&\quad B_{\mu\nu} \left(\frac{\partial X^{\mu}}{\partial \tau}\right) \wedge \left(\frac{\partial X^{\nu}}{\partial \sigma}\right),
\end{aligned}
$$

where $J\left(\frac{\partial(\tau';\sigma')}{\partial(\tau;\sigma)}\right)$ is the Jacobian for the co-ordinate transformation $(\tau;\sigma) \mapsto (\tau';\sigma')$. Define

$$
\gamma\left(\frac{\partial(\tau;\sigma)}{\partial(\tau';\sigma')}\right) = \left[\left(\frac{\partial \tau}{\partial \tau'}\right)\left(\frac{\partial \sigma}{\partial \sigma'}\right) - \left(\frac{\partial \tau}{\partial \sigma'}\right)\left(\frac{\partial \tau}{\partial \sigma'}\right)\right].
$$

Now,

$$
\begin{aligned}
S_{KR} \mapsto & S'_{KR} \\
&= -\frac{1}{2}\int_{\mathcal{S}} \mathrm{d}\tau \mathrm{d}\sigma \ J\left(\frac{\partial(\tau';\sigma')}{\partial(\tau;\sigma)}\right) \gamma\left(\frac{\partial(\tau';\sigma')}{\partial(\tau;\sigma)}\right) B_{\mu\nu} \left(\frac{\partial X^{\mu}}{\partial \tau}\right) \wedge \left(\frac{\partial X^{\nu}}{\partial \sigma}\right) \\
&= -\frac{\operatorname{sgn}(\gamma)}{2}\int_{\mathcal{S}} \mathrm{d}\tau \mathrm{d}\sigma \ B_{\mu\nu} \left(\frac{\partial X^{\mu}}{\partial \tau'}\right)\left(\frac{\partial X^{\nu}}{\partial \sigma'}\right),
\end{aligned}
$$

where $\operatorname{sgn}(\gamma)$ is the sign of $\gamma\left(\frac{\partial(\tau;\sigma)}{\partial(\tau';\sigma')}\right)$. Since $\gamma\left(\frac{\partial(\tau;\sigma)}{\partial(\tau';\sigma')}\right)$ is the inverse of $J\left(\frac{\partial(\tau';\sigma')}{\partial(\tau;\sigma)}\right)$, S_{KR} maps to $\pm S_{KR}$ depending on the sign of $\gamma\left(\frac{\partial(\tau;\sigma)}{\partial(\tau';\sigma')}\right)$. The negative sign comes when the co-ordinate transformation include a world-sheet parity, $z \mapsto -z$ or $z \mapsto -z^*$.

Combining Equation (3.3.6) and Equation (3.3.12) we find the world-sheet action of the relativistic string,

$$S_{WS} = -\int_{\mathcal{S}} \mathrm{d}\tau \mathrm{d}\sigma \left(T_0 \sqrt{-\det(h)} + \frac{1}{2} B_{\mu\nu} \left(\frac{\partial X^{\mu}}{\partial \tau} \right) \wedge \left(\frac{\partial X^{\nu}}{\partial \sigma} \right) \right). \tag{3.3.13}$$

Again, the equations of motions are solved subject to boundary conditions. For further treatment of this see [1], [13] and [14].

To understand a theory of strings, we must understand the surfaces traced out by their motion. A technology used to describe these surfaces is described in the next Chapter.

Chapter 4

Strebel Differentials

4.1 Quadratic Differential

This study of the Riemann surfaces traced out by strings in space-time relies on a special kind of quadratic differential, called a *Strebel Differential*, in particular the theorem by K. Strebel [16]. We begin our analysis with a review of quadratic differentials, see [17]. Let $\Sigma_{g,n}$ be a compact Riemann surface of genus g with n marked points, and let ϕ be a meromorphic function on $\Sigma_{g,n}$. A quadratic differential q is an expression of the form

$$q(z) = \phi(z)(\mathrm{d}z)^2. \tag{4.1.1}$$

A curve $\gamma(t)$ in $\Sigma_{g,n}$ such that $q(\gamma(t))$ is real, is called a *vertical curve*, or simply *vertical*, if it satisfies the relation $\phi(\gamma(t))\,(\gamma'(t))^2 < 0$, and is called a *horizontal curve*, or *horizontal*, if $\phi(\gamma(t))\,(\gamma'(t))^2 > 0$, where the prime notation identifies differentiation with respect to curve the parameter, t. This terminology is motivated by the simplest such differential on $\mathbb{C}$, being $q_0 = (\mathrm{d}z)^2$. The horizontal lines $\gamma_h(t) = at + ic$ with $t \in \mathbb{R}$ are the horizontal curves of q and the vertical lines $\gamma_v(t) = b + it$ with $t \in \mathbb{R}$ are the vertical curves of q.

To understand the behaviour of quadratic differentials, let $z \in \mathbb{C}$ and define

$\phi(z)$ by,

$$\phi(z) = \sum_{j=1}^{\infty} a_j (z - b_j)^{-j} + \sum_{k=0}^{\infty} c_k (z - d_k)^k. \tag{4.1.2}$$

Shift $z \mapsto z + z' = z_0$, by some fixed $z' \in \mathbb{C}$, such that ϕ has a zero of order m at $z_0 = 0$. Then, near $z_0 = 0$, $q(z_0)$ takes the form,

$$q(z_0) \approx \alpha z_0^m (\mathrm{d}z)^2.$$

In polar coordinates,

$$\phi(r;\theta) \approx \rho e^{i\beta} \left(re^{i\theta}\right)^m,$$

where $r, \rho \in \mathbb{R}_+$ and $\theta, \beta \in [0; 2\pi)$. Now,

$$\mathrm{d}z = e^{i\theta}\mathrm{d}r + ire^{i\theta}\mathrm{d}\theta,$$

and

$$\mathrm{d}z^2 = e^{2i\theta}\left(\mathrm{d}r^2 + 2ir\mathrm{d}r\ \mathrm{d}\theta - r^2\mathrm{d}\theta^2\right).$$

Therefore,

$$q(z_0) \approx \rho e^{i\beta} \left(re^{i\theta}\right)^m e^{2i\theta} \left((\mathrm{d}r)^2 + 2ir\mathrm{d}r\ \mathrm{d}\theta - r^2(\mathrm{d}\theta)^2\right).$$

To determine the behaviour of the differential in the vicinity of a zero, suppose that θ is constant and that r is allowed to vary. Then, $\mathrm{d}\theta = 0$ and so,

$$q(z_0) \approx \rho r^m e^{i\theta(m+2)+i\beta} \left(\mathrm{d}r^2\right).$$

Supposing that q is horizontal imposes a restriction on the value of θ in the exponential. That is,

$$e^{i(m+2)\theta+i\beta} > 0.$$

The restriction that this places on the value of θ is

$$(m+2)\theta + \beta = 2\pi n,$$

or,

$$\theta = \frac{2\pi n - \beta}{(m+2)}, \tag{4.1.3}$$

where $n \in \mathbb{Z}$. By this restriction, there exists $m + 2$ unique values for θ, each separated by $\frac{2\pi n - \beta}{(m+2)}$. Therefore, there are $m + 2$ intersecting horizontal curves at a zero of order m.

Near a simple pole, a quadratic differential takes the approximate form

$$q(z) \approx \frac{a}{z} \mathrm{d}z^2, \tag{4.1.4}$$

where $m = -1$, so by Equation (4.1.3), there can be no locally intersecting horizontal curves. For a second order pole, with residue $p \in \mathbb{R}_+$, the quadratic differential is locally of the form

$$q(z) \approx \frac{-p^2}{(2\pi)^2 z^2} \mathrm{d}z^2, \tag{4.1.5}$$

for which $m = 2$, so there are no horizontal curves emanating from second order poles. For circular curves, parametrized on the interval $t \in [0; 2\pi)$,

$$\gamma(t) = r_0 e^{it},$$

we find,

$$\begin{aligned} \phi\left(\gamma(t)\right)\left(\gamma'(t)\right)^2 &= \frac{-p^2}{(2\pi)^2 \left(r_0 e^{it}\right)^2} \left(r_0 i e^{it}\right)^2 \\ &= \frac{p^2}{(2\pi)^2} > 0. \end{aligned}$$

Therefore, circular curves centred on the second order pole, are horizontal in the region of a second order pole. Straight lines emanating from the second order pole, parametrized on the interval $t \in [0; \infty)$, have the form

$$\gamma(t) = t e^{i\theta}.$$

We find

$$\phi\left(\gamma(t)\right)\left(\gamma'(t)\right)^2 = \frac{-p^2}{(2\pi)^2 \left(t e^{i\theta}\right)^2} \left(e^{it}\right)^2$$

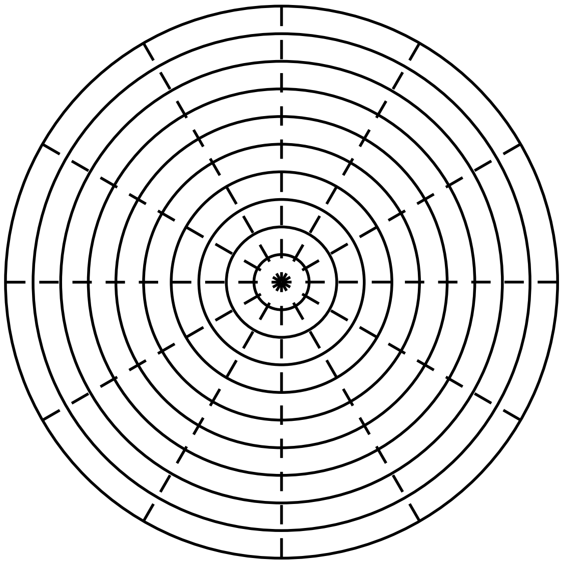


Figure 4.1: The vertical and horizontal curves of $\frac{-1}{z^2}(\mathrm{d}z)^2$.

$$= \frac{-p^2}{(2\pi)^2\, t^2} < 0.$$

Hence, straight lines emanating from second order poles are vertical, see Figure 4.1.

It is useful to distinguish the set of simple closed horizontal curves from other horizontal curves of a quadratic differential. The set of horizontal curves which are complementary to the set of simple closed horizontal curves is called the set of *critical curves* of the differential. In particular, these curves intersect at the zeros of the differential. The set of critical curves of a quadratic differential is complicated. Therefore, we consider a special type of quadratic differential called a *Strebel* differential.

4.2 Strebel Differentials

We are now ready to state the theorem of Strebel [16]. Let $g, n \in \mathbb{Z}_+$ and $2g + n > 2$. For every Riemann surface $\Sigma_{g,n}$ of genus g with n marked points $\{x_1; \ldots; x_n\}$ and n positive numbers $\{p_1; \ldots; p_n\}$, there exists a unique quadratic differential, q, that is holomorphic on $\Sigma_{g,n} \setminus \{x_1; \ldots; x_n\}$ and has second order poles at each marked point, x_i. Every compact horizontal curve $\mathcal{C}$ is a simple loop circling around one of the poles, say x_i, and it satisfies

$$p_i = \oint_{\mathcal{C}} \sqrt{q}, \tag{4.2.1}$$

where the branch of the square root is chosen such that the integral is positive with respect to the positive orientation of $\mathcal{C}$ that is determined by the complex structure of $\Sigma_{g,n}$. The set of simple closed horizontal curves about each pole foliates a punctured disc, called a *ring-domain*, and the boundary of these discs is the union of the critical curves whose endpoints are the zeros of the differential. The union of all non-compact horizontal curves forms a closed subset $\Sigma_{g,n}$ of measure zero. A quadratic differential satisfying these requirements is called a *Strebel differential.*

The space of conformally equivalent classes of compact Riemann surfaces of genus g with n punctures is called *Moduli Space*, $\mathcal{M}_{g,n}$. Each point in moduli space defines a set of conformally equivalent surfaces. The points in moduli space are numerical parameters which specify the properties of a Riemann surface. The space, $\mathcal{M}_{g,n} \times \mathbb{R}^n_+$, constructed by assigning a sequence of positive real numbers $\{p_1; \ldots; p_n\}$ to each point in $\mathcal{M}_{g,n}$, is called the *Decorated Moduli Space of Riemann Surfaces.* By Strebel's theorem, there exists a unique Strebel differential for each point in the decorated moduli space.

4.3 Cell Decomposition of Moduli Space.

The critical curves of the Strebel differential q, on $\Sigma_{g,n}$ forms a graph which is embedded into $\Sigma_{g,n}$ where each zero of the critical curve is a vertex and each curve between vertices is an edge, this graph is called the *critical graph* of the Strebel differential q. Define the set of 0-cells as the set of vertices of the critical graph and the set of edges of the critical graph as the set of 1-cells. The ring-domains bounded by the edges of the critical graph constitute the set of 2-cells. The collection of 0-cells, 1-cells and 2-cells defines a *cell-decomposition* of $\Sigma_{g,n}$. The unique Strebel differential on a specific marked Riemann surface $\Sigma_{g,n}$ is said to induce a cell decomposition of $\Sigma_{g,n}$.

Given a Strebel differential, q, on $\Sigma_{g,n}$, we assign a length $l_{z_iz_j}$ to each edge z_iz_j of the critical graph,

$$l_{z_iz_j} = \int_{z_i}^{z_j} \sqrt{q} = \int_{t_i}^{t_j} \mathrm{d}t \left(\frac{\partial\gamma(t)}{\partial t}\right) \sqrt{\phi(\gamma(t))}, \tag{4.3.1}$$

where z_i and z_j are end vertices of the edge z_iz_j and the integration is performed in the orientation for which the integral is positive. Since the curve is horizontal, this integral is real. By this assignment of lengths, the graph defined by the horizontal curve $\gamma_h(t)$ is a metric graph. Note that under the reparametrization $t \mapsto t' = f(t)$, the edge length $l_{z_iz_j}$ becomes,

$$\begin{aligned}
l_{z_iz_j} &\mapsto l'_{z_iz_j} \\
&= \int_{t'_j}^{t'_j} \mathrm{d}t' \left(\frac{\partial\gamma(t')}{\partial t'}\right) \sqrt{\phi(\gamma(t'))} \\
&= \int_{f^{-1}(f(t_i))}^{f^{-1}(f(t_j))} \mathrm{d}t \left(\frac{\partial f(t)}{\partial t}\right) \left(\frac{\partial\gamma(f(t))}{\partial t}\right) \left(\frac{\partial t}{\partial f(t)}\right) \sqrt{\phi(\gamma(t))} \\
&= \int_{t_i}^{t_j} \mathrm{d}t \left(\frac{\partial\gamma(t)}{\partial t}\right) \sqrt{\phi(\gamma(t))}.
\end{aligned}$$

Therefore, each length assignment is independent of parametrization. Following Equations (4.2.1) and (4.3.1), every horizontal curve has an edge length uniquely determined by the Strebel differential, q. Each ring-domain about a marked point, x_i, is foliated by horizontal curves of equal length. The "size", but not the length, of successive horizontal curve increases approaching the critical curve of the Strebel differential, see Figure 4.1 for an illustration of what is meant by "size". At the critical curve the circumference of the marked

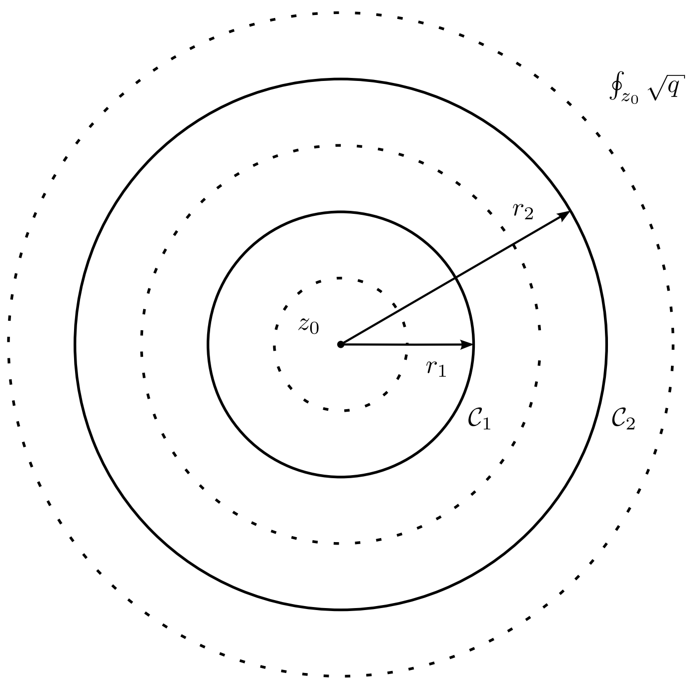


Figure 4.1: The concentric simple closed horizontal curves $\mathcal{C}_1$ and $\mathcal{C}_2$, centred at z_0, at distanced r_1 and r_2 from z_0, respectively, are assigned the same length of $\oint_{z_0} \sqrt{q}$.

point x_i is the sum of the critical curve edge lengths on the boundary of the ring-domain centred at x_i.

Given these length assignments, we define the space of all metric graphs of genus g with n faces and with all vertices at least trivalent, $RG_{g,n}$. A generic Riemann surface is mapped to a Strebel differential which has at most simple zeros. The trivalent graphs of the corresponding Strebel differential sweep out top dimensional cells of $RG_{g,n}$. Riemann surfaces which map to graphs with higher order vertices are part of a lower dimensional cell in the complex. Cells corresponding to trivalent graphs in this decomposition are called *top-cells*, the cells corresponding to graphs with vertices having valence greater than three are ordered by their dimension relative to the top-cells as *next-to top-cells* and *next-to next-to top-cells* and so on.

Different cells connect to each other where the length of an edge is collapsed

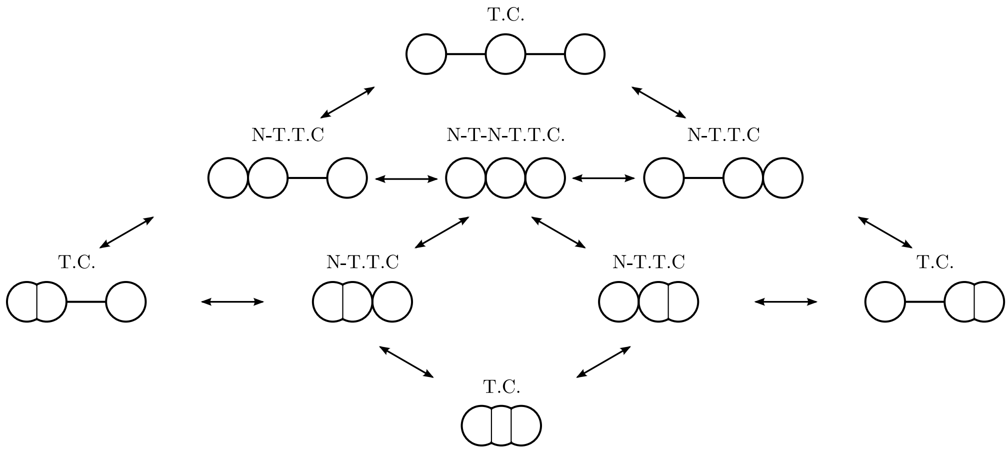


Figure 4.2: An example of the Strebel Differential critical graph cell-decomposition of the $\mathcal{M}_{0,4} \times \mathbb{R}_+^4$ showing cell relations between Top Cells (T.C), Next-To Top Cells (N-T.T.C.) and Next-To-Next-To Top Cells (N-T-N-T.T.C.).

to zero. One can move from one cell to another by crossing an edge of zero length. This may be done in various channels whereby an edge is collapsed, contracting together two vertices in one channel and then expanding the contracted vertex in another channel giving rise to a new edge of non-zero length, see Figure 4.2.

For each point in $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$ there corresponds, via the result of Strebel, a unique point in $RG_{g,n}$. This point is obtained by constructing the unique Strebel differential on the corresponding surface and computing the sequence of edge lengths. The reverse mapping is more complicated. The Riemann surface $\Sigma_{g,n}$, with $g, n \in \mathbb{Z}_+$ and $2g + n > 2$, may be viewed as a collection of semi-infinite cylinders glued together. The circumference of a cylinder is the sum of the lengths of the edges along which that cylinder is glued to the surrounding cylinders. Let $\Gamma \in RG_{g,n}$ and let q be a quadratic differential. Let e_{ij} be an edge of Γ bounded by vertices at v_i and v_j. For each edge e_{ij} of length l_{ij} construct an infinite strip of constant width l_{ij}, in $\mathbb{C}$, parallel to the imaginary axis. On this strip we have the differential

$$q(z) = (\mathrm{d}z)^2, \tag{4.3.2}$$

in terms of the co-ordinates of the plane. By Equation (4.1.3) there are m intersecting horizontal curves at each zero of q of order $m-2$. Then every quadratic differential has a form

$$q(\omega) = \frac{m^2}{4}\omega^{m-2}\left(\mathrm{d}\omega\right)^2, \tag{4.3.3}$$

locally about a zero of degree $m-2$. To patch each strip for adjacent edges together we note that from Equations (4.3.3) and (4.3.2) locally about v_i,

$$(\mathrm{d}z)^2 = \frac{m^2}{4}\omega^{m-2}\left(\mathrm{d}\omega\right)^2. \tag{4.3.4}$$

Solving Equation (4.3.4) subject to the initial conditions, $\omega = z = 0$ corresponding to point $z_i = v_i$, for the co-ordinate transform, we find about v_i,

$$\omega(z) = (1)^{\frac{1}{m}} z^{\frac{2}{m}}. \tag{4.3.5}$$

This maps the portion of each strip about the vertex v_i into a wedge of angle $\frac{2\pi}{m}$. The differentials $(\mathrm{d}z)^2$ for each strip are transformed into a differential of the form in Equation (4.3.3). We may choose a local co-ordinate u about each pole of residue p_i of the differential q,

$$q(u) = \frac{-p_i^2}{4\pi^2}\frac{(\mathrm{d}u)^2}{u^2}. \tag{4.3.6}$$

This defines a local co-ordinate disc about each pole. When several edges bound a face, halves of the corresponding strips are glued to form a cylinder. Let z be the co-ordinate on the strip in $\mathbb{C}$,

$$(\mathrm{d}z)^2 = \frac{-p_i^2}{4\pi^2}\frac{(\mathrm{d}u)^2}{u^2}. \tag{4.3.7}$$

Solving the Equation 4.3.7 for the local co-ordinate u gives

$$u_i(z) = Ae^{-2\pi i z/p_i}, \tag{4.3.8}$$

where A is a constant of integration. This patches together co-ordinate discs. The residue p_i is the sum of all the edge lengths of the edges bounding the ring-domain centred at the pole u_i. The Riemann surface formed by the collection of co-ordinate patches can be shown to be unique [18]. These mappings define an isomorphism on $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$ and $RG_{g,n}$. By this isomorphism $RG_{g,n}$ induces a cell decomposition on $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$. The Strebel differential is said to induce a cell decomposition on the decorated moduli space.

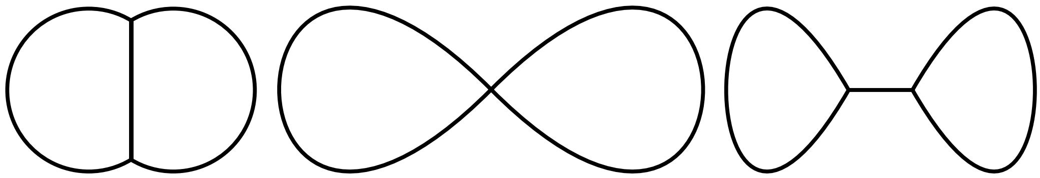
4.4 Localization on Moduli Space

The dimension of each cell in the cell decomposition of $RG_{g,n}$ is specified by the number of edge length parameters of the corresponding critical graph of the associated Strebel differential. Euler's theorem states that the number of edges, e, of a trivalent graph having n faces and embedded in a surface of genus g, is given by,

$$e = 6g - 6 + 3n. \tag{4.4.1}$$

The metric graphs associated with the Strebel differential are restricted to a subset of $RG_{g,n}$ corresponding to the cells where the number of edges of the critical graph corresponds to the number of edge length assignments of $RG_{g,n}$. Since the cell decomposition of $RG_{g,n}$ is isomorphic to the cell decomposition of $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$, the set of metric graphs corresponding to a Strebel differential are restricted to a subset of $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$. The circumference of each pole of the Strebel differential is sum of the lengths of the edges of the critical graph which border the ring-domain about each pole. By the positivity of the circumferences, for each edge assignment there corresponds an equation of constraint which is satisfied by the circumferences.

To demonstrate this localization, consider the sphere with three ordered



(a) (b) (c)

Figure 4.1: Critical graphs of the three-point function: (a) $\Delta > 0$, (b) $\Delta = 0$, and (c) $\Delta < 0$.

marked points, $(z_1; z_2; z_3)$, as presented in [17]. By the $SL(2;\mathbb{C})$ symmetry, z_1, z_2 and z_3 are placed at 0, 1 and ∞, respectively. Choose a triple $(p_0; p_1; p_\infty)$ of positive real numbers. The residue of the pole at $z = i$ is p_i. The most general Strebel differential for the sphere with three marked points is

$$q = \frac{-1}{4\pi^2}\left(a\left(\frac{\mathrm{d}z}{z}\right)^2 + b\left(\frac{\mathrm{d}z}{1-z}\right)^2 + c\left(\frac{\mathrm{d}z}{z(1-z)}\right)^2\right). \tag{4.4.2}$$

The residues p_i may now be computed explicitly in terms of a, b and c following Equation (4.2.1). Solving for a, b and c in terms of the p_i's gives,

$$\begin{aligned} a &= \frac{1}{2}\left(p_0^2 + p_\infty^2 - p_1^2\right) \\ b &= \frac{1}{2}\left(p_1^2 + p_\infty^2 - p_0^2\right) \\ c &= \frac{1}{2}\left(p_0^2 + p_1^2 - p_\infty^2\right). \end{aligned} \tag{4.4.3}$$

These equations give an explicit formulation of the Strebel differential as a function of co-ordinates of the decorated moduli space. Note that these equations also enforce that a, b and c are real valued.

Solving for the zero of the differential in Equation (4.4.2) yields,

$$z = \frac{a \pm i\sqrt{ab + bc + ca}}{a + b}. \tag{4.4.4}$$

The foliation of the sphere by the Strebel differential in Equation (4.4.2) depends on the discriminant of Equation (4.4.4). Define,

$$\Delta = ab + bc + ca. \tag{4.4.5}$$

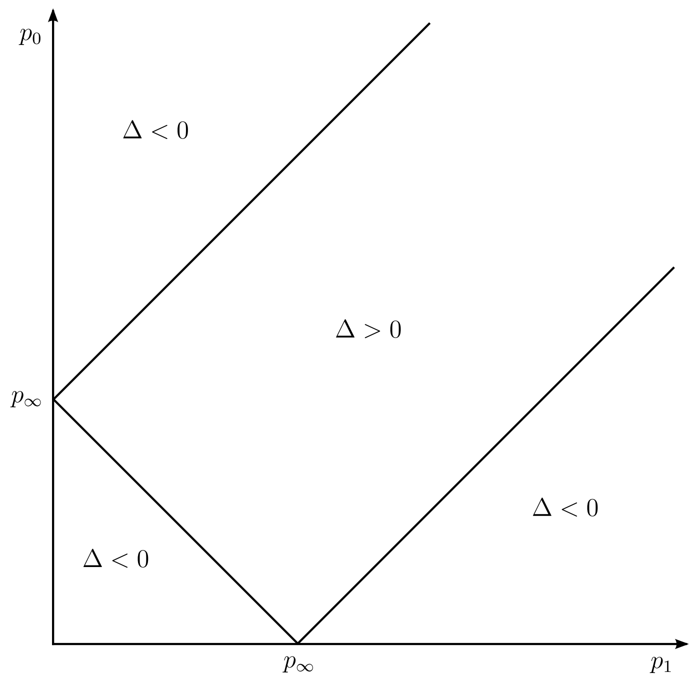


Figure 4.2: The partitioning of Moduli Space into seven domains by the three-point function at a slice of constant p_∞.

Note that Equation (4.4.5) may be rewritten as,

$$\Delta = \frac{1}{4}\left(p_0 + p_1 + p_\infty\right)\left(p_0 + p_\infty - p_1\right)\left(p_1 + p_\infty - p_0\right)\left(p_0 + p_1 - p_\infty\right). \quad (4.4.6)$$

Therefore, Δ is real valued and depends only on the residues p_i and not on the positions of the poles. The choice of the values of the poles fixes the foliation.

There are three possibilities for the value of Δ giving rise to three different foliations of the sphere, see Figure 4.1. These possibilities are,

- $\Delta > 0$: The Strebel differential has two complex zeros. The critical graph is trivalent with two vertices and three edges. At each vertex, horizontal curves intersect with an angular separation of $\frac{2\pi}{3}$, see Figure 4.1a.

- $\Delta = 0$: The Strebel differential a single real zero at $z = \frac{a}{a+b}$. The critical graph has one vertex with a valence of four. At this vertex, horizontal curves intersect with an angular separation of $\frac{\pi}{2}$, see Figure 4.1b.

- $\Delta < 0$: The Strebel differential has two real zeros. The critical graph is trivalent with two vertices and three edges. At each vertex, horizontal curves intersect with an angular separation of $\frac{2\pi}{3}$. The topological type of this case is different to that of the case where $\Delta > 0$, see Figure 4.1c.

The discriminant of Equation (4.4.4) partitions $\mathbb{R}^3_+$ into seven domains, three $\mathbb{R}^3_+$ subspaces where $\Delta < 0$, three $\mathbb{R}^2_+$ subspaces there $\Delta = 0$, and one $\mathbb{R}^3_+$ subspace where $\Delta > 0$. This partitioning of moduli space is depicted in Figure 4.2. It is now clear that for the special case of $\Delta = 0$, the Strebel differential of Equation (4.4.2) localizes on a 2-dimensional subspace of the 3-dimensional decorated moduli space. In general, the Strebel differential will localize on a subspace of moduli space whenever one of its zeros is of order greater than one. This corresponds to a critical graph where one vertex has valence greater than three.

Chapter 5

Gopakumar's Prescription

5.1 Schwinger Parametrization

The mapping from gauge theory correlators to correlators on the world-sheet using the Strebel differential, as presented by Gopakumar [7], goes via a reparametrization of the gauge theory Feynman diagrams using Schwinger parameters. All operators in the gauge theory are assumed to be normal ordered. This eliminates divergences due to self contractions. For position-space correlators, propagators are recast as integrals, using the mathematical identity

$$\frac{1}{(x_i - x_j)^2} = \int_0^\infty \mathrm{d}\sigma \ e^{-\sigma (x_i - x_j)^2}, \tag{5.1.1}$$

where σ is the *Schwinger parameter*. General gauge theory correlators are products of such factors. There are restrictions on the functional relation between the propagators and the Schwinger parameters. Any monotone function f satisfying $f(0) = 0$ and $f(\infty) = \infty$ would be suitable. It would be interesting to determine how a different choice of parametrization of the gauge theory propagators would effect the translation of gauge theory correlators to world-sheet correlators.

For a Feynman graph constructed on a Riemann surface, the connectedness

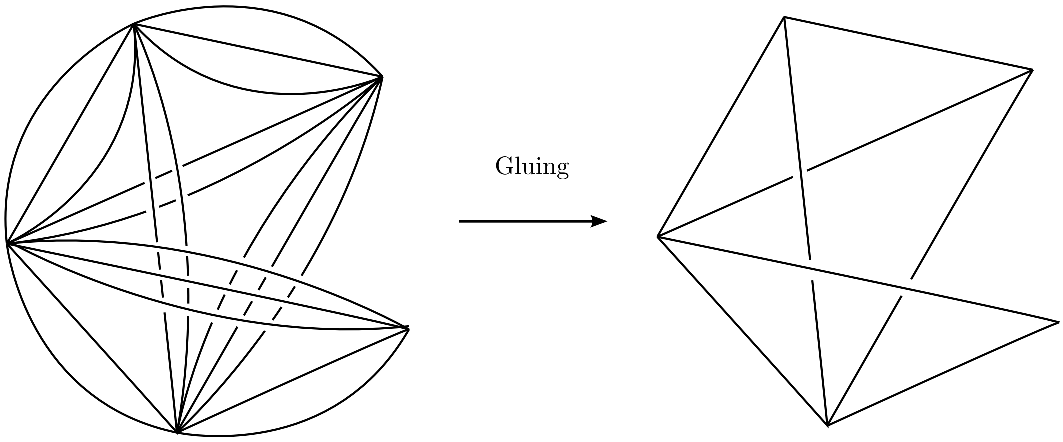


Figure 5.1: Homotopically equivalent propagators of a planar Feynman graph are glued together. The double-line notation is suppressed in this representation.

of the space in which it is embedded provides a well defined homotopy operation on the graph. Homotopically equivalent edges in the Feynman graph may be combined to form a single edge, giving rise to a propagator over an effective Schwinger parameter,

$$\tilde{\sigma}_k = \sum_{l=1}^{m_k} \sigma_{k_l}, \tag{5.1.2}$$

where m_k is the number of homotopically equivalent propagators and σ_{k_i} is the Schwinger parameter for the i-th homotopic propagator. In this way all homotopic edges in the Feynman graph may be collapsed into a single edge carrying a compensated dependence on the Schwinger parameter. These homotopic edges are said to be *reduced*, or *glued*. The resulting graph is called a *skeleton graph*, and the process of reducing edges is called a *skeleton reduction*, see Figure 5.1.

Consider the expectation value

$$\left\langle \prod_{i=1}^{n} \mathcal{O}_i \right\rangle.$$

Let J_{ij} denote the number of contractions between the i-th and j-th operators, following the identity

$$\frac{1}{n!}\int_0^\infty \mathrm{d}x\; x^{n-1}e^{-ax} = \frac{1}{a^n},$$

general Feynman propagators, with E_r reduced edges where the ij-th reduced edge is formed by gluing J_{ij} lines, take the form,

$$\prod_{\substack{1\leq i\leq j\\ J_{ij}>0}}^{Er} (x_i - x_j)^{-2J_{ij}} = \int_0^\infty \prod_{\substack{1\leq i\leq j\\ J_{ij}>0}}^{E_r} \mathrm{d}\tilde{\sigma}_{ij}\; \frac{\tilde{\sigma}_{ij}^{J_{ij}-1}}{J_{ij}!} e^{-\tilde{\sigma}_{ij}(x_i-x_j)^2}. \tag{5.1.3}$$

5.2 Schwinger Parameters and Moduli Space

The skeleton graphs of gauge theory correlators have vertices of valence greater than or equal to one. Since critical graphs of Strebel differentials are generally trivalent, the conversion of gauge theory graphs to critical graphs of the Strebel differential incorporates a graph duality. By this duality, field theory insertions, specifying the vertices of the Feynman graphs, map to the poles of the quadratic differential, which correspond to the bounded ring-domains of the critical graph. Also, the faces of the Feynman graphs map to vertices, corresponding to the zeros of the critical curves of the critical graphs. Each edge of the Feynman graph is mapped, by the graph duality, to an edge of the critical dual graph. In so doing, an edge of a Feynman graph having an associated effective Schwinger parameter $\tilde{\sigma_{ij}}$, is mapped to an edge of the critical dual graph with length l_{ij}. Integrating over $\tilde{\sigma_{ij}}$ in Equation (5.1.3) corresponds to integrating over the set of all lengths for the corresponding edge, l_{ij}. Equation (5.1.3) contains an integral over the space of metric graphs.

For a trivalent graph, Γ, Euler's theorem states that the number of edges, e, is

$$e = 6g - 6 + 3v = 6g - 6 + 2n + n \tag{5.2.1}$$

where n is the number of faces and g is the genus of the surface triangulated by Γ. It was shown by Kontsevich, Harer and Zagier [18; 19] and others that the number of real moduli for a Riemann surface of genus g with n marked points is $6g-6+2n$. The additional n in Equation (5.2.1) corresponds to the moduli p_i of the decoration $\mathbb{R}_+^n$, on $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$. Hence, integrating over all edges of Γ corresponds to integrating over the full set of real moduli of Riemann surfaces in the decorated moduli space. Integration over the space of metric graphs corresponds to an integration over the decorated moduli space of Riemann surfaces. Therefore, integration over the Schwinger parameters corresponds to integration over the conformal equivalence classes of Riemann surfaces. The gauge theory correlator is now an integral over the string world-sheets. There is now a concrete method mapping gauge theory correlators to world-sheet correlators.

5.3 Operator Product Expansions

The basic object of interest in perturbation theory is the path integral expectation of a product of local operators. It is important to understand the behaviour of these expectation values in the limit that two operators are taken to approach each other. The tool that gives a systematic description of this limit is the *Operator Product Expansion* (OPE). It states that the product of two local operators may be approximated to arbitrary accuracy by a sum of local operators,

$$\mathcal{O}_i(x_i)\mathcal{O}_j(x_j) = \sum_k C_{i,j,k}(x_i - x_j)\mathcal{O}_k(x_j). \tag{5.3.1}$$

This approximation is valid inside an arbitrary expectation value only when the separation distance $|x_i - x_j|$ of the operators $\mathcal{O}_i(x_i)$ and $\mathcal{O}_j(x_j)$ is small compared to the separation distance to the next nearest operator. The coefficient functions $C_{i,j,k}(x_i - x_j)$ are dependent only on the separation distance and indices i, j, k and not on any other operators. By convention the terms of the OPE arranged in order of decreasing size in the limit $x_1 \to x_2$.

An OPE is like a Laurent series expansion in complex analysis, where the validity of the expansion holds on an open disk centred at the point of one operator insertion, with the radius being the smallest distance from the centre of the expansion to the next nearest operator insertion. In the same sense that the Laurent series is of central importance to analysis, the operator product expansion is of central importance to in Quantum Field Theory. As in the case of the Laurent series, we write an OPE as the sum of singular terms plus a sum of non-singular terms. OPE's are useful for asymptotic expansions, where the dominant behaviour is described the singular terms.

Observables are invariants of a theory so, under symmetry transformation group $SL(2;\mathbb{C})$,

$$z \mapsto \frac{az+b}{cz+d}, \tag{5.3.2}$$

where $ad - bc = 1$, physical operators transform as scalars. We will study the world-sheet theory in the conformal gauge. In this gauge the physical world-sheet vertex operators should have the scaling dimension $(h;\bar{h}) = (1;1)$. When this is true, integrated vertex operators transform as scalars under conformal transformation. In particular, we expect each single trace gauge-invariant operator $\mathcal{O}(x) = \mathrm{tr}\,(\cdots(x))$ of the space-time theory to correspond to a world-sheet vertex operator $\mathcal{V}_{\mathcal{O}}(x;\sigma)$ of dimension $(1;1)$. For the scalar field Φ in the adjoint representation of $SU(N)$, the gauge invariants of the theory include $\mathrm{tr}\,(\Phi^n)$ for $n \geq 1$.

A natural object to study on the world-sheet theory is the world-sheet OPE. In the world-sheet theory dual to the gauge theory, vertex operators $\mathcal{O}_i(x)$ in space-time theory are replaced by vertex operators $\mathcal{V}_{\mathcal{O}_i}(\sigma;x)$ on the world-sheet, where x is the space-time co-ordinate and σ is the world-sheet co-ordinate. The product of two physical vertex operators, in the limit that $\sigma_i \to \sigma_j$, should have an expansion in local world-sheet operators of the form

$$\mathcal{V}_{\mathcal{O}_i}(x_i;\sigma_i)\mathcal{V}_{\mathcal{O}_j}(x_j;\sigma_j) = \sum_{k=0} C_{i,j,k}(x_i - x_j) \left|\sigma_i - \sigma j\right|^{h_k - 2} \left|\bar{\sigma_i} - \bar{\sigma j}\right|^{\bar{h}_k - 2} \mathcal{V}_{\mathcal{O}_k}(x_j), \tag{5.3.3}$$

where the operators $\mathcal{V}_{\mathcal{O}_k}$ which appear have world-sheet scaling dimensions $(h_k; \bar{h}_k)$. The world-sheet theory dual to a gauge theory will have a combination of both physical and non-physical operators. Physical operators on the world-sheet correspond to physical operators in space-time. The space-time interpretation of non-physical operators on the world-sheet is unclear.

The OPE of a correlator is a smooth function of the position of the field theory insertions of the gauge theory. Since some world-sheet OPE's are localized on some subspace of the decorated moduli space they must be proportional to some delta function distribution on the subspace where the world-sheet correlator is non-zero. In these cases, the world-sheet OPE reduces to an integral over delta function distributions, making the OPE of such a correlator ill defined. Also, not all possible pole positions for a world-sheet correlator lead to valid operator expansions since there are some choices of pole positions which lead to contradictions on the world-sheet theory side, as will be shown.

5.4 Localization of World-Sheet Correlators

Gopakumar's prescription translates the gauge theory correlator to a world-sheet correlator by means of the reduced Schwinger parameters. Integration over the Schwinger parameter is translated to an integral over the decorated moduli space. The expected number of integration parameters corresponds to the number of edges of the critical graph of the Strebel differential, which is the dimension of the parameter space bounded by Equation (5.2.1).

The number of integration parameters must match the number of edges of the associated metric graph. In some cases the restrictions on the integration parameters are replaced by an integration over a delta function distribution of those parameter. In these cases there is non-zero contribution to the correlator on some restricted domain, so the integration over moduli space is over a set of parameters which is smaller than the set of edge length parameters of the associated metric graph. The world-sheet correlator is said to *localize*.

Recall from Section 4.4 the localization of the three-point Strebel differential on the sphere. There are three different foliation of the sphere by the Strebel differential of Equation (4.4.2), as depicted in Figure 4.1. For the case where the discriminant of Equation (4.4.4) is zero, the Strebel differential is localized on a 2-dimensional subspace of the moduli space. Figure 4.1b contributes to the correlation function of the form

$$\left\langle \mathrm{tr}\left(\Phi^{m_1}\left(x_1\right)\right) \mathrm{tr}\left(\Phi^{m_1+m_2}\left(x_2\right)\right) \mathrm{tr}\left(\Phi^{m_2}\left(x_3\right)\right)\right\rangle_{S^2}.$$

For the case where Figure 4.1b is the only contributing graph the the world-sheet amplitude, there is a non-zero contribution only from the 2-dimensional subspace of the moduli space where the discriminant of Equation (4.4.4) is equal to zero.

Upon evaluating the OPE, a limit is taken of the world-sheet correlator in which two poles of the Strebel differential approach each other. This amounts to rewriting the Strebel differential in terms of poles placed at points, say, t and $-t$. The separation of the poles in $\Sigma_{g,n}$ is simply $2|t|$. The world-sheet correlator is then rewritten in terms of the separation parameter t and the limit that $t \to 0$ is computed. This corresponds to

$$\sum_k C_{i,j,k}(x_i - x_j)\, |t|^{h_k+\bar{h}_k-4}\, \mathcal{V}_{\mathcal{O}_k}(x_j),$$

in the operator product expansion. In some cases the equations of constraint on the values of the poles and the positions of the poles place restrictions on the values of t in the Strebel differential, see the Π four-point function in [4]. In this OPE of the world-sheet correlator, the number of integration parameters is less than that expected by Equation (5.2.1) and the world-sheet OPE is said to *localize*.

5.5 The Five-Point Function Diagram

We now proceed to compute the operator product expansion of a single gauge theory correlator. Consider the following five-point gauge theory correlator in $3+1$-dimensions, where each trace factor is normal ordered.

$$\left\langle \mathrm{tr}\left(\Phi^{m_1}\left(x_1\right)\right) \mathrm{tr}\left(\Phi^{m_1+m_2}\left(x_2\right)\right) \mathrm{tr}\left(\Phi^{m_2+m_3}\left(x_3\right)\right) \mathrm{tr}\left(\Phi^{m_3+m_4}\left(x_4\right)\right) \mathrm{tr}\left(\Phi^{m_4}\left(x_5\right)\right)\right\rangle, \tag{5.5.1}$$

where $m_i \in \mathbb{Z}_+$. One possible planar Feynman diagram contributing to the correlator of Equation (5.5.1) is presented in Figure 5.1. From Figure 5.2b the corresponding Strebel differential must have five second order poles to match the five vertices of the associated Feynman graph. It is also clear from Figure 5.2b that the dual graph has a single vertex of valence eight, so the corresponding Strebel differential must have a zero of order six, since at each zero of order m there are $m+2$ intersecting horizontal curves in the critical graph of the Strebel differential. The positions of these poles may be fixed under an $SL(2;\mathbb{C})$ transformation to $z = \{-t; t; 1; s; \infty\}$. The vertex is chosen to be at $z = c_0$. The decorated moduli space for this example is $\mathcal{M}_{0,5} \times \mathbb{R}^5_+$ for which the corresponding Strebel differential is localized on a 4-dimensional subspace of the 9-dimensional space $RG_{0,5}$. The most general Strebel differential for the

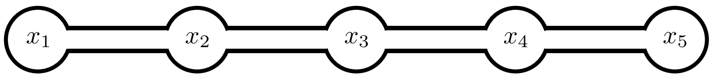


Figure 5.1: The gauge theory Feynman graph for the five point chain correlator in $3+1$ dimensions.

five point amplitude is

$$q = \frac{-p_\infty^2}{4\pi^2} \frac{(c_0 - z)^6}{(z-1)^2(z-s)^2(z-t)^2(z+t)^2} \mathrm{d}z^2. \tag{5.5.2}$$

This Strebel differential has two complex moduli, t and s.

There are ten choices for the positions of the poles at t and $-t$ on the diagram, up to a swapping of $-t$ and $+t$, not counting the permutations on the positions for the poles at 1, s and ∞. These choices are

$$(-t, t) = \{(a;b), (a;c), (a;d), (a;e), (b;c), (b;d), (b;e), (c;d), (c;e), (d;e)\}.$$

We label the residue of a pole, p_i, by the position of the pole, i, on the world-sheet. Each p_i, corresponds to an element of decoration $\mathbb{R}_+^5$. Denote p_{+t} and p_{-t} by p_+ and p_-, respectively. Equations of constraint relating the circumferences of the poles may be derived by rewriting the equations specifying the circumferences of each pole in terms of the lengths of the edges of the critical curves. These equations specified in terms of p_- and $p_+$are listed in Table 5.1.

The square root used to determine the circumferences of the poles introduces an ambiguity in the sign of each pole. This ambiguity is uniquely fixed by the equation of constraint for each labelling scheme. The signs are listed in Table 5.2. There exist equivalent labellings under swapping of $+t$ and $-t$. Under swapping $+t$ and $-t$, with appropriate relabelling of a, b and c, we find that the equivalent diagram labelling schemes are,

$$(a;b) \sim (d;e); \quad (a;c) \sim (c;e); \quad (a;d) \sim (b;e); \quad (b;c) \sim (c;d); \quad (a;e); \quad (b;d).$$

Therefore, there are six distinct labellings for $(-t;+t)$, up to a swapping of $-t$ and t and relabelling of a, b and c.

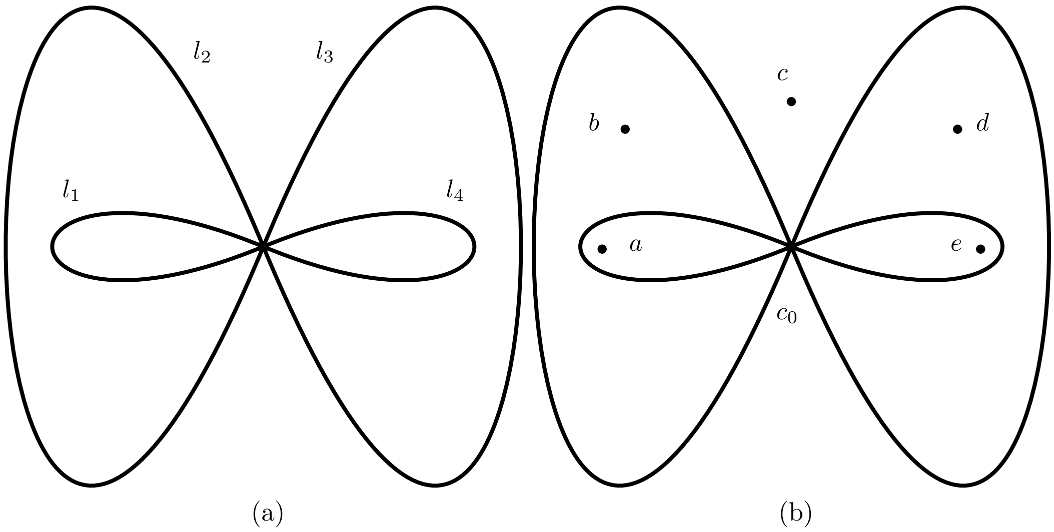


Figure 5.2: The Dual graphs of the five point chain correlator showing (a) the edge assignments and (b) pole and zero positions of the five point correlator.

(-t;+t)	Constraint
$(a;b)$	$p_- = p_+ - p_a + p_b - p_c$
$(a;c)$	$p_- = -p_+ + p_a + p_b - p_b$
$(a;d)$	$p_- = p_+ + p_a - p_b - p_c$
$(a;e)$	$p_- = -p_+ + p_a - p_b + p_c$
$(b;c)$	$p_- = p_+ + p_a - p_b + p_c$
$(b;d)$	$p_- = -p_+ + p_a + p_b + p_c$
$(b;e)$	$p_- = p_+ + p_a + p_b - p_c$
$(c;d)$	$p_- = p_+ - p_a + p_b - p_c$
$(c;e)$	$p_- = -p_+ - p_a + p_b + p_c$
$(d;e)$	$p_- = p_+ + p_a - p_b + p_c$

Table 5.1: The positivity constraint equations for the Strebel circumferences.

The ratios of the circumferences are given by

$$\begin{aligned} \frac{p_1}{p_\infty} &= \frac{\gamma_1(c-1)^3}{(1-s)(1-t)(1+t)} \\ \frac{p_s}{p_\infty} &= \frac{\gamma_s(c-s)^3}{(1-s)(s-t)(s+t)} \\ \frac{p_\pm}{p_\infty} &= \frac{\gamma_\pm(c\mp t)^3}{2t(1\mp t)(s\mp t)}. \end{aligned} \tag{5.5.3}$$

Now we define,

$$\begin{aligned} F_1 &= [\gamma_1(1-s)(1-t)(1+t)]^{\frac{1}{3}} \\ F_s &= [\gamma_s(1-s)(s-t)(s+t)]^{\frac{1}{3}} \\ F_\pm &= [\gamma_\pm 2t(1\mp t)(s\mp t)]^{\frac{1}{3}}\,, \end{aligned} \tag{5.5.4}$$

giving us the convenient forms for each circumference ratio,

$$\begin{aligned} \frac{p_1}{p_\infty} &= \alpha_1^3 = \frac{(c_0-1)^3}{F_1^3} \\ \frac{p_s}{p_\infty} &= \alpha_s^3 = \frac{(c_0-s)^3}{F_1^3} \\ \frac{p_\pm}{p_\infty} &= \alpha_\pm^3 = \frac{(c_0\mp t)^3}{F_\pm^3}. \end{aligned} \tag{5.5.5}$$

Equation (5.5.5) implies

$$c_0 = 1-\alpha_1 F_1 = \alpha_+ F_+ + t = \alpha_- F_- - t = s - \alpha_s F_s, \tag{5.5.6}$$

which solving for t gives,

$$2t = \alpha_- F_- + \alpha_+ F_+. \tag{5.5.7}$$

Since the α_i's are ratios of positive real numbers, we find

$$\begin{aligned} 2\mathrm{Re}\,(t) &= \alpha_+ \mathrm{Re}\,(F_+) + \alpha_- \mathrm{Re}\,(F_-) \\ 2\mathrm{Im}\,(t) &= \alpha_+ \mathrm{Re}\,(F_+) + \alpha_- \mathrm{Re}\,(F_-)\,. \end{aligned} \tag{5.5.8}$$

Rewriting the above equations gives

$$\alpha_\pm = 2\frac{\mathrm{Im}\,(t)\,\mathrm{Re}\,(F_\mp) - \mathrm{Re}\,(t)\,\mathrm{Im}\,(F_\mp)}{\mathrm{Re}\,(F_+)\,\mathrm{Im}\,(F_-) - \mathrm{Re}\,(F_-)\,\mathrm{Im}\,(F_+)}. \tag{5.5.9}$$

Defining $N_\pm$ and Δ by

$$N_\pm = \mathrm{Im}\,(t)\,\mathrm{Re}\,(F_\mp) - \mathrm{Re}\,(t)\,\mathrm{Im}\,(F_\mp)\,, \tag{5.5.10}$$

and

$$\Delta = \mathrm{Re}\,(F_+)\,\mathrm{Im}\,(F_-) - \mathrm{Re}\,(F_-)\,\mathrm{Im}\,(F_+)\,, \tag{5.5.11}$$

we have

$$\alpha_\pm = 2\frac{N_\pm}{\Delta}. \tag{5.5.12}$$

Equation (5.5.6) with the requirement that the α_i's are real gives rise to an equation of constraint on moduli space,

$$0 = \mathrm{Im}\left(\frac{t + \alpha_+ F_+ - 1}{F_1}\right). \tag{5.5.13}$$

This constraint will cause the correlator to localize on a 3-dimensional subspace of the moduli space.

To evaluate the OPE of this correlator on the world-sheet, we must consider the limit that $t \to -t$, which is equivalent to considering the limit that $t \to 0$. Let $t = |t|\, e^{i\phi}$ and $s = |s|\, e^{i\theta_s}$. We may make t and $-t$ distinct by introducing a phase shift of $\pi k_\pm$ such that

$$\pm t = |t|\, e^{i(\phi + \pi k_\pm)}, \tag{5.5.14}$$

with $k_\pm \in \mathbb{Z}$ and k_+ is even and k_- is odd. Now for small t,

$$\begin{aligned} F_\pm =& \left[\gamma_\pm 2t(1 \mp t)(s \mp t)\right]^{\frac{1}{3}} \\ \approx& \Big[2\gamma_\pm st\Big]^{\frac{1}{3}} \left[1 \mp \frac{1}{3}\left(1 + s^{-1}\right) t\right] \\ \approx& \gamma_\pm^{\frac{1}{3}} \left(2|s||t|\right)^{\frac{1}{3}} e^{i\frac{\phi+\theta_s}{3}} \mp \gamma_\pm^{\frac{1}{3}} \frac{1}{3} \left(2|s||t|^4\right)^{\frac{1}{3}} e^{i\frac{4\phi+\theta_s+3\pi k_\mp}{3}} \\ & \mp \gamma_\pm^{\frac{1}{3}} \frac{1}{3} \left(2|s|^{-2}|t|^4\right)^{\frac{1}{3}} e^{i\frac{4\phi+\theta_s+\pi k_\mp}{3}}. \end{aligned}$$

Since $\gamma_\pm$ determines the sign of poles, introduce $\tilde{k}_\pm \in \mathbb{Z}$ such that

$$\gamma_\pm = e^{i\pi \tilde{k}_\pm} \in \{-1; 1\}. \tag{5.5.15}$$

Now,

$$\begin{aligned} F_\pm =& \left(2|s||t|\right)^{\frac{1}{3}} e^{i\frac{\phi+\theta_s+\tilde{k}_\pm}{3}} + \frac{1}{3}\left(2|s||t|^4\right)^{\frac{1}{3}} e^{i\frac{4\phi+\theta_s+\pi(\tilde{k}_\pm+3k_\mp)}{3}} \\ & + \frac{1}{3}\left(2|s|^{-2}|t|^4\right)^{\frac{1}{3}} e^{i\frac{4\phi-2\theta_s+\pi(\tilde{k}_\pm+3k_\mp)}{3}}, \end{aligned}$$

which gives

$$\begin{aligned}\mathrm{Re}\,(F_\pm) = &(2|s||t|)^{\frac{1}{3}}\cos\left(\frac{\phi+\theta_s+\tilde{k}_\pm}{3}\right)\\ &+\frac{1}{3}\left(2|s||t|^4\right)^{\frac{1}{3}}\cos\left(\frac{4\phi+\theta_s+\pi(\tilde{k}_\pm+3k_\mp)}{3}\right)\\ &+\frac{1}{3}\left(2|s|^{-2}|t|^4\right)^{\frac{1}{3}}\cos\left(\frac{4\phi-2\theta_s+\pi(\tilde{k}_\pm+3k_\mp)}{3}\right),\end{aligned} \tag{5.5.16}$$

and

$$\begin{aligned}\mathrm{Im}\,(F_\pm) = &(2|s||t|)^{\frac{1}{3}}\sin\left(\frac{\phi+\theta_s+\tilde{k}_\pm}{3}\right)\\ &+\frac{1}{3}\left(2|s||t|^4\right)^{\frac{1}{3}}\sin\left(\frac{4\phi+\theta_s+\pi(\tilde{k}_\pm+3k_\mp)}{3}\right)\\ &+\frac{1}{3}\left(2|s|^{-2}|t|^4\right)^{\frac{1}{3}}\sin\left(\frac{4\phi-2\theta_s+\pi(\tilde{k}_\pm+3k_\mp)}{3}\right).\end{aligned} \tag{5.5.17}$$

By defining

$$\beta = \frac{\pi(\tilde{k}_- - \tilde{k}_+)}{3}, \tag{5.5.18}$$

and

$$H = \sin(\phi+\beta)+\sin(\phi-\beta)+\frac{1}{|s|}\left(\sin(\phi-\theta+\beta)\sin(\phi-\theta-\beta)\right), \tag{5.5.19}$$

we have

$$\begin{aligned}\alpha_\pm^3 = &\frac{2^2|t|^2\sin\left(\frac{2\phi-\theta_s+\pi\tilde{k}_\mp}{3}\right)^3}{|s|\sin(\beta)^2\left(\sin(\beta)+|t|H\right)} - \frac{2^2|t|^3\sin\left(\frac{\phi+\theta_s+\pi\tilde{k}_\mp}{3}\right)\sin\left(\frac{2\phi-\theta_s-\pi\tilde{k}_\mp}{3}\right)^2}{|s|\sin(\beta)^2\left(\sin(\beta)+|t|H\right)}\\ &\mp\frac{2^2|t|^3\sin\left(\frac{\phi-2\theta_s+\pi\tilde{k}_\mp}{3}\right)\sin\left(\frac{2\phi-\theta_s-\pi\tilde{k}_\mp}{3}\right)^2}{|s|\sin(\beta)^2\left(\sin(\beta)+|t|H\right)}.\end{aligned} \tag{5.5.20}$$

At the leading order in t for $t \to 0$, we find

$$\frac{p_\pm}{p_\infty} = \alpha_\pm^3 = \frac{2^2|t|^2}{|s|}\left[\frac{\sin\left(\frac{2\phi-\theta_s+\pi\tilde{k}_\mp}{3}\right)}{\sin\left(\frac{\pi(\tilde{k}_- - \tilde{k}_+)}{3}\right)}\right]^3. \tag{5.5.21}$$

We may fix the phases $\tilde{k}_+$ and $\tilde{k}_-$ by enforcing the positivity of the residues. Note that $\phi \in [0; 2\pi)$ and $\theta_s \in [0; 2\pi)$, therefore $2\phi - \theta_s \in [-2\pi; 4\pi)$. The phase choices are

$$2\phi - \theta_s \in (-2\pi; -\pi) \Rightarrow (k_-; k_+) = (3; 2)$$
$$2\phi - \theta_s \in (-\pi; 0) \Rightarrow (k_-; k_+) = (5; 4)$$
$$2\phi - \theta_s \in (0; \pi) \Rightarrow (k_-; k_+) = (5; 4), (1; 2)$$
$$2\phi - \theta_s \in (\pi; 2\pi) \Rightarrow (k_-; k_+) = (5; 0), (3; 4)$$
$$2\phi - \theta_s \in (2\pi; 3\pi) \Rightarrow (k_-; k_+) = (1; 0), (3; 4)$$
$$2\phi - \theta_s \in (3\pi; 4\pi) \Rightarrow (k_-; k_+) = (0; 5), (3; 2).$$

Also, as $t \to 0$, $c_0 \to 0$. This implies $\frac{\gamma_s}{\gamma_1} s = \frac{p_s}{p_1} \geq 0$. Thus, s is real valued. The sign of s, given by $\frac{\gamma_s}{\gamma_1}$, sets the phase to $\theta_s = 2\pi N$ for positive s, or to $\theta_s = \pi + 2\pi N$ when s is negative, where $N \in \mathbb{Z}$. Also, Equation (5.5.13) imposes a delta function constraint on s and t causes the OPE to localize on some 3-dimensional subspace of the moduli space. At leading order in $|t|$, this delta function constraint is a functional relation among the moduli $|s|$, ϕ and θ_b. We may therefore substitute any function which integrates to one on the constraint space into the OPE. As a result, θ dependence is not well defined. The OPE of the corresponding world-sheet correlator is no longer a smooth function of the moduli $|s|$, ϕ and θ_b. However, the OPE does have a well defined $|t|$ dependence at leading order since the delta function of Equation (5.5.13) is independent of $|t|$ for small $|t|$.

We now recast the gauge theory correlator as a correlator on the world-sheet. Reparametrization of the gauge theory correlator of Equation (5.5.1) via the effective Schwinger parameters produces

$$I = \prod_{i=1}^{4} (x_i - x_{i+1})^{-2m_i} = \int \prod_{j=1}^{4} \mathrm{d}\sigma_j \frac{\sigma_j^{m_j - 1}}{m_j!} e^{-\sigma_j (x_j - x_{j+1})^2}. \tag{5.5.22}$$

The form of the OPE, to leading order is dependant only on the position of t. This means that the choices of locations for the other field theory insertions are redundant. To see this, consider first the case where we have chosen the field theory insertions as follows $(-t; +t; \infty; 1; s) = (a; b; c; d; e)$, then,

$$I = \int \frac{\mathrm{d}p_-\mathrm{d}p_+\mathrm{d}p_\infty\mathrm{d}p_1}{m_1!m_2!m_3!m_4!} (p_-)^{m_1-1} (p_+ - p_-)^{m_2-1} (p_\infty - p_+ + p_-)^{m_3-1} (p_1)^{m_4-1} \times$$
$$\exp\left[-p_-(x_1-x_2)^2\right] \exp\left[-(p_+-p_-)(x_2-x_3)^2\right] \exp\left[-p_1(x_4-x_5)^2\right] \times$$
$$\exp\left[-(p_\infty - p_+ + p_-)(x_3-x_4)^2\right]$$
$$= \int \frac{\mathrm{d}\frac{p_-}{p_\infty}\mathrm{d}\frac{p_+}{p_\infty}\mathrm{d}\frac{p_1}{p_\infty}\mathrm{d}p_\infty}{m_1!m_2!m_3!m_4!} \left(\frac{p_-}{p_\infty}\right)^{m_1-1} \left(\frac{p_+}{p_\infty} - \frac{p_-}{p_\infty}\right)^{m_2-1} \left(1 - \frac{p_+}{p_\infty} + \frac{p_-}{p_\infty}\right)^{m_3-1} \times$$
$$\left(\frac{p_1}{p_\infty}\right)^{m_4-1} p_\infty^{m_1+m_2+m_3+m_4-1} \exp\left[E\right],$$

where,

$$E = -\,p_\infty \left(\frac{p_-}{p_\infty}\left\{(x_1-x_2)^2 - (x_2-x_3)^2 + (x_3-x_4)^2\right\}\right.$$
$$\left. + \frac{p_+}{p_\infty}\left\{(x_2-x_3)^2 - (x_3-x_3)^2\right\} + \frac{p_1}{p_\infty}\left\{(x_4-x_4)^2\right\}\right).$$

After integrating over p_∞ and performing a co-ordinate change from $\frac{p_i}{p_\infty}$ to $|t|, |s|, \phi$ using the Jacobian

$$J\left(\frac{\partial\left(\frac{p_+}{p_\infty}; \frac{p_-}{p_\infty}; \frac{p_1}{p_\infty}\right)}{\partial(|t|; |s|; \phi)}\right) = \frac{-64\gamma_1 |t|^3 e^{i\theta_s} \sin\left(\frac{2\phi - \theta_s - \pi\tilde{k}_-}{3}\right)^2 \sin\left(\frac{2\phi - \theta_s - \pi\tilde{k}_+}{3}\right)^2}{|s|^2\left(|s|e^{i\theta_s} - 1\right)^2 \sin\left(\frac{\pi(\tilde{k}_- - \tilde{k}_+)}{3}\right)^5},$$

at leading order where $t \to 0$ yields

$$I_{1a} = \int \mathrm{d}|t|\ |t|^{2m_1+2m_2-1} \tilde{F}_{1a}(|s|; \phi; \mathrm{x}), \tag{5.5.23}$$

where $\tilde{F}_{1a}(|s|; \phi; \mathrm{x})$ is some function of world-sheet co-ordinates $|s|$ and ϕ and the space-time co-ordinates $\mathrm{x} = (x_i)$. Similarly, choosing the field theory insertions as follows $(-t; +t; \infty; 1; s) = (a; b; c; e; d)$ yields

$$I_{1b} = \int \mathrm{d}|t|\ |t|^{2m_1+2m_2-1} \tilde{F}_{1b}(|s|; \phi; \mathrm{x}) \tag{5.5.24}$$

and $\tilde{F}_{1b}(|s|;\phi;\mathrm{x})$ is some new function of co-ordinates $|s|, \phi$ and $\mathrm{x} = (x_i)$. It is not difficult to show that choosing $(-t;+t;\infty;1;s) = (a;b;x;y;z)$ where $x,y,z = \{c,d,e\}$ always yields an operator product expansion of the form

$$I_1 = \int \mathrm{d}|t|\ |t|^{2m_1+2m_2-1}\tilde{F}_1(|s|;\phi;\mathrm{x}). \tag{5.5.25}$$

To leading order, the associated Jacobians are

$$J\left(\frac{\partial\left(\frac{p_+}{p_\infty};\frac{p_-}{p_\infty};\frac{p_1}{p_\infty}\right)}{\partial(|t|;|s|;\phi)}\right) = \frac{-64\gamma_1|t|^3e^{i\theta_s}\sin\left(\frac{2\phi-\theta_s-\pi\tilde{k}_-}{3}\right)^2\sin\left(\frac{2\phi-\theta_s-\pi\tilde{k}_+}{3}\right)^2}{|s|^2\left(|s|e^{i\theta_s}-1\right)^2\sin\left(\frac{\pi(\tilde{k}_--\tilde{k}_+)}{3}\right)^5},$$

$$J\left(\frac{\partial\left(\frac{p_+}{p_\infty};\frac{p_-}{p_\infty};\frac{p_s}{p_\infty}\right)}{\partial(|t|;|s|;\phi)}\right) = \frac{-64\gamma_s|t|^3e^{i\theta_s}\sin\left(\frac{2\phi-\theta_s-\pi\tilde{k}_-}{3}\right)^2\sin\left(\frac{2\phi-\theta_s-\pi\tilde{k}_+}{3}\right)^2}{|s|^2\left(|s|e^{i\theta_s}-1\right)^2\sin\left(\frac{\pi(\tilde{k}_--\tilde{k}_+)}{3}\right)^5},$$

$$J\left(\frac{\partial\left(\frac{p_+}{p_\infty};\frac{p_s}{p_\infty};\frac{p_1}{p_\infty}\right)}{\partial(|t|;|s|;\phi)}\right) = \frac{8\gamma_1\gamma_s|t|^3e^{i\phi}\sin\left(2\frac{2\phi-\theta_s-\pi\tilde{k}_-}{3}\right)\sin\left(\frac{2\phi-\theta_s-\pi\tilde{k}_-}{3}\right)}{|s|^2\left(|s|e^{i\theta_s}-1\right)^2\sin\left(\frac{\pi(\tilde{k}_--\tilde{k}_+)}{3}\right)^3},$$

$$J\left(\frac{\partial\left(\frac{p_-}{p_\infty};\frac{p_s}{p_\infty};\frac{p_1}{p_\infty}\right)}{\partial(|t|;|s|;\phi)}\right) = \frac{8\gamma_1\gamma_s|t|^3e^{i\phi}\sin\left(2\frac{2\phi-\theta_s-\pi\tilde{k}_+}{3}\right)\sin\left(\frac{2\phi-\theta_s-\pi\tilde{k}_+}{3}\right)}{|s|^2\left(|s|e^{i\theta_s}-1\right)^2\sin\left(\frac{\pi(\tilde{k}_--\tilde{k}_+)}{3}\right)^3}.$$

There are five distinct OPE's at leading order,

$$\begin{array}{lllcl}
(-t;+t) & = & (a;b) & : & I_A = \int \mathrm{d}|t|\ |t|^{2m_1+2m_2-1}\tilde{F}_A(|s|,\phi,\mathrm{x}) \\
(-t;+t) & = & (a;c);(a;d) & : & I_B = \int \mathrm{d}|t|\ |t|^{2m_1+1}\tilde{F}_B(|s|,\phi,\mathrm{x}) \\
(-t;+t) & = & (a;e) & : & I_C = \int \mathrm{d}|t|\ |t|^{2m_1+2m_4-1}\tilde{F}_C(|s|,\phi,\mathrm{x}) \\
(-t;+t) & = & (b;c);(c;d) & : & I_D = \int \mathrm{d}|t|\ |t|^{3}\tilde{F}_D(|s|,\phi,\mathrm{x}) \\
(-t;+t) & = & (b;e);(c;e);(d;e) & : & I_E = \int \mathrm{d}|t|\ |t|^{2m_4+1}\tilde{F}_E(|s|,\phi,\mathrm{x}),
\end{array}$$

where $\tilde{F}_X(|s|,\phi,\mathrm{x})$ are functions of the world-sheet co-ordinates $|s|$ and ϕ, and the space-time co-ordinates x, which are dependant on the placement choice of poles at 1, s and ∞ relative to pole positions a, b, c, d and e.

We may now read off the conformal dimensions from the leading order terms in each operator product expansion,

$$\begin{aligned}
I_A : &\quad h + \bar{h} - 4 = 2m_1 + 2m_2 - 1 &\Rightarrow&\quad h + \bar{h} = 2m_1 + 2m_2 + 3 \\
I_B : &\quad h + \bar{h} - 4 = 2m_1 + 1 &\Rightarrow&\quad h + \bar{h} = 2m_1 + 5 \\
I_C : &\quad h + \bar{h} - 4 = 2m_1 + 2m_4 - 1 &\Rightarrow&\quad h + \bar{h} = 2m_1 + 2m_4 + 3 \\
I_D : &\quad h + \bar{h} - 4 = 3 &\Rightarrow&\quad h + \bar{h} = 7 \\
I_E : &\quad h + \bar{h} - 4 = 2m_4 + 1 &\Rightarrow&\quad h + \bar{h} = 2m_4 + 5.
\end{aligned}$$

Clearly there is no choice of m_1, m_2, m_3 or m_4 which would result in $h+\bar{h} = 2$. The operator product expansions of these the five point correlator on the world-sheet have no physical operators. Furthermore, is is interesting to note that the leading order terms in these expansions are non-singular, which contradicts our expectation that the OPE has singular terms at leading order.

Case	-t	+t	∞	1	s	γ_-	γ_+	γ_1	γ_s
1	a	b	c	d	e	-1	-1	-1	-1
				e	d	-1	-1	+1	+1
			d	c	e	+1	+1	-1	+1
				e	c	+1	+1	-1	+1
			e	c	d	-1	-1	+1	+1
				d	c	-1	-1	-1	-1
2	a	c	b	d	e	+1	-1	+1	+1
				e	d	+1	-1	-1	-1
			d	b	e	+1	-1	+1	+1
				e	b	+1	-1	-1	-1
			e	b	d	-1	+1	-1	+1
				d	b	-1	+1	-1	+1
3	a	d	b	c	e	+1	+1	-1	+1
				e	c	+1	+1	+1	-1
			c	b	e	-1	-1	-1	-1
				e	b	-1	-1	+1	+1
			e	b	c	-1	-1	-1	-1
				c	b	-1	-1	+1	+1
4	a	e	b	c	d	+1	-1	-1	-1
				c	d	+1	-1	+1	+1
			c	b	d	-1	+1	-1	+1
				d	b	-1	+1	-1	+1
			d	b	c	+1	-1	+1	+1
				c	b	+1	-1	-1	-1
5	b	c	a	d	e	+1	+1	-1	-1
				e	d	+1	+1	+1	+1
			d	a	e	-1	-1	-1	+1
				e	a	-1	-1	-1	+1
			e	a	d	+1	+1	+1	+1
				d	a	+1	+1	-1	-1
6	b	d	a	c	e	+1	-1	-1	+1
				e	c	+1	-1	-1	+1
			c	a	e	+1	-1	+1	-1
				e	a	+1	-1	+1	-1
			e	a	c	+1	-1	+1	-1
				c	a	+1	-1	+1	-1

Table 5.2: The signs of circumference for each inequivalent five point diagram labelling scheme are shown in the above table. There are six distinct labelling schemes specifying the relative placements of $-t$ and $+t$.

Chapter 6

Comments on Gopakumar's Prescription

As was seen in previous sections, Gopakumar's prescription does suffer some short comings. Firstly, Strebel's theorem only works for Riemann surface $\Sigma_{g,n}$ where $2 - 2g > n$. This means that not all gauge theory correlators are mappable to correlators on a world-sheet. Furthermore, any gauge theory correlator having a reduced Feynman graph whose dual has at least one vertex with valence less than three does not have a Strebel graph assiciated with it. This follows from the fact that every Strebel critical graph has vertices of valence at least three. Also, disconnected Feynman graphs cannot be mapped by this prescription and neither can Feynman graphs with topologically simple closed loops. This puts a major restriction on the types correlators which may be computed.

Also, the Strebel differential induces a cell decomposition of moduli space. The values of the residues of the poles of the Strebel differential must satisfy equations of constraint determined by the permutations on the labelings of the poles. Then for each equation of constraint, the Strebel differential is defined on a cell of the decorated moduli space. Furthermore, the Strebel differential may localize on a subspace of that cell of moduli space. When mapping gauge theory correlators via the Strebel differential only a subset of all possible surfaces are summed over as determined by the domain on which the Strebel differential localizes. So the translation from a gauge theory, where the path integral sums over all possible paths interpolating between the initial and final configurations of the system, to a the string theory, where the path integral sums over all possible world-sheets interpolating between the initial and final configurations of the system, covers only a subset of all the possible surfaces.

Another short coming of the prescribed mapping is that gauge theory correlators which have valid Operator Product Expansions in Minkowski may not have smooth (analytic) operator product expansions on the world-sheet. This is due to the world-sheet correlator localizing on a subspace of moduli space. Also, since physical operators in an operator product expansion have scaling dimension (1;1), gauge theory correlators may not have physical operators on the world-sheet side, as was shown.

On a more positive note, this mapping does present a concrete method for mapping a gauge theory correlators to correlators on a world-sheet. We must now ask a few questions of this prescription,

1. Since the prescription appears to have so many problems, is it simply wrong?

2. If the prescription is wrong, can it be corrected by an appropriate modification?

3. Supposing that the prescription is correct, what does this tell us about string theory?

As a point of ongoing research, we are investigating the implications of using gauge theory correlators composed of Schur polynomials, and localizing at single points in moduli space, to determine whether is it possible use mass localization of gauge theory correlators to cover the whole of moduli space. It is suspected that such a modification of Gopakumar's prescription is valid when the moduli space of such correlators coincide. It is hoped that this limiting case will shed some light on the how to construct a viable mapping from a gauge theory to a string theory. Also, it is hoped that this investigation will shed some light on the nature of string theory correlators and the associated moduli spaces.

Appendices

Appendix A

The Schwinger-Dyson Equation and Ribbon Graphs

A.1 The Schwinger-Dyson Equation

We want to compute expectation values of operators of the form $\prod_{i=1}^{k} \mathrm{tr}\,(M^{n_i})$. To do so we must evaluate equations of the form Equation (2.1.5) which is non-trivial. Suppose $\mathcal{O} = \prod_{i=1}^{k} \mathrm{tr}\,(M^{n_i})$. Note that,

$$\int [\mathrm{d}M] \frac{\mathrm{d}}{\mathrm{d}M_{ij}} \left[\left(M^{n_k-1}\right)_{ij} \prod_{l=1}^{k-1} \mathrm{tr}\,(M^{n_l}) e^{-\alpha \mathrm{tr}\left(M^2\right)} \right]$$

is a surface term and by, $\left[e^{-x^2}\right]_{-\infty}^{+\infty} = 0$, evaluates to zero. So,

$$\int [\mathrm{d}M] \frac{\mathrm{d}}{\mathrm{d}M_{ij}} \left[\left(M^{n_k-1}\right)_{ij} \prod_{l=1}^{k-1} \mathrm{tr}\,(M^{n_l}) e^{-\alpha \mathrm{tr}\left(M^2\right)} \right] = 0. \tag{A.1.1}$$

After differentiating and rewriting the above expression, we find

$$\begin{aligned} \langle \mathcal{O} \rangle_0 =& \frac{1}{2\alpha} \left\langle \frac{\mathrm{d}}{\mathrm{d}M_{ij}} \left[\left(M^{n_k-1}\right)_{ij} \right] \prod_{l=1}^{k-1} \mathrm{tr}\,(M^{n_l}) \right\rangle_0 + \\ & \frac{1}{2\alpha} \left\langle \left(M^{n_k-1}\right)_{ij} \frac{\mathrm{d}}{\mathrm{d}M_{ij}} \left[\prod_{l=1}^{k-1} \mathrm{tr}\,(M^{n_l}) \right] \right\rangle_0 . \end{aligned}$$

This new expression for $\langle \mathcal{O} \rangle_0$ is a sum of expectation values of operators having a sum of matrix powers two less that that of $\mathcal{O}$. Equation (A.1.1) is known as the *Schwinger-Dyson* equation.

The Schwinger-Dyson equation expresses $\langle \mathcal{O} \rangle_0$ as a series of combinations of expectation values of simpler operators. To see this, suppose $\mathcal{O} = \mathrm{tr}\,(M^4)$, then by Equation (A.1.1),

$$\left\langle \mathrm{tr}\left(M^4\right)\right\rangle_0 = 2N\left(\frac{1}{2\alpha}\left\langle \mathrm{tr}\left(M^2\right)\right\rangle_0\right) + \frac{1}{2\alpha}\left\langle \mathrm{tr}\,(M)\,\mathrm{tr}\,(M)\right\rangle_0,$$

where M is an $N \times N$ matrix. The sum of powers in each term of the expansion of $\langle \mathrm{tr}\,(M^4)\rangle_0$ is two less that that of $\langle \mathrm{tr}\,(M^4)\rangle_0$.

A.2 Constructing Ribbon Graphs

Differentiation of the exponential term in Equation (A.1.1) serves to reconstruct the desired correlator, which does not appear explicitly, while differentiation of the other factors in Equation (A.1.1) constructs a series of correlators whose sum is equal to the desired correlator once all permutations of a single Wick contraction has taken place. The differentiation operation in Equation (A.1.1) simulates the Wick contraction of matrix elements by replacing Wick contracted elements with an appropriate factor. The product rule of differentiation ensures that every combination of single contractions is computed. Choosing the matrix element M_{ij} as the parameter by which to differentiate is analogous to identifying an index pair with which to Wick contract other index pairs. Diagrammatically, this is analogous to choosing one index dot pair in a trace and finding all combinations of joining that pair to another index pair with a ribbons.

Connecting two pairs of index dots by a ribbon produces a new vertex having valence two less than the sum of the valencies of the vertices that were connected to construct it. This new vertex carries a co-efficient determined by the Wick contraction which formed this vertex. Since the Schwinger-Dyson equation expresses the expectation value of an operator as a series of combinations of operator product expectation values, the combinations of unconnected index pairs are those given in the solution to the Schwinger-Dyson Equation. From this, the connections of the associated ribbon graphs may be read from the output of the Schwinger-Dyson equation.

Suppose one wanted to identify all the associated diagrams of

$$\left\langle \mathrm{tr}\left(M^2\right)\mathrm{tr}\left(M^4\right)\right\rangle_0 .$$

By the Schwinger-Dyson equation,

$$\begin{aligned}\int [\mathrm{d}M]\mathrm{tr}\left(M^2\right)\mathrm{tr}\left(M^4\right)e^{-\alpha\mathrm{tr}\left(M^2\right)} =& \frac{1}{2\alpha}N^2\int [\mathrm{d}M]\mathrm{tr}\left(M^4\right)e^{-\alpha\mathrm{tr}\left(M^2\right)}+\\ & 4\left[\frac{1}{2\alpha}\int [\mathrm{d}M]\mathrm{tr}\left(M^4\right)e^{-\alpha\mathrm{tr}\left(M^2\right)}\right].\end{aligned}$$

After the first contraction there is one graph having two propagator loops and four equivalent graphs connected by a single ribbon. The graph containing the loops is formed by contracting indices in $\mathrm{tr}\,(M^2)$, whereas the graphs without loops are those formed by contracting a index pair in $\mathrm{tr}\,(M^2)$ with a pair in $\mathrm{tr}\,(M^4)$, see Figure A.1.

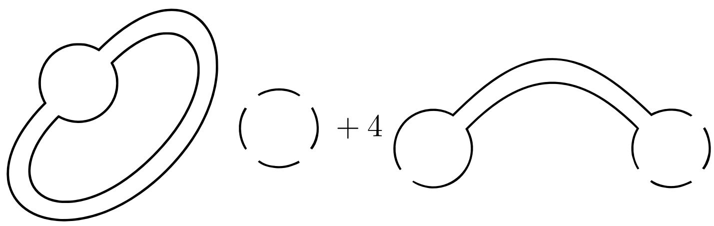
$\langle \mathrm{tr}\,(M^2)\,\mathrm{tr}\,(M^4)\rangle_0 \rightarrow$ 

Figure A.1: The partially connected vertices of $\langle \mathrm{tr}\,(M^2)\,\mathrm{tr}\,(M^4)\rangle_0$ after one contraction.

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