Stuart F. Cullender*

Under the supervision of
Prof. C.C.A. Labuschagne
School of Mathematics
University of the Witwatersrand, Johannesburg
Private Bag 3, P O WITS 2050, South Africa

Generalized Martingale and Stopping Time Techniques in Banach Spaces†

October 15, 2007

* E-mail address: scullender@gmail.com

† A thesis submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, in fulfillment of the requirements for the degree of Doctor of Philosophy.

Abstract

Probability theory plays a crucial role in the study of the geometry of Banach spaces. In the literature, notions from probability theory have been formulated and studied in the measure free setting of vector lattices. However, there is little evidence of these vector lattice techniques being used in the study of geometry of Banach spaces. In this thesis, we fill this niche. Using the l-tensor product of Chaney-Shaefer, we are able to extend the available vector lattice techniques and apply them to the Lebesgue-Bochner spaces. As a consequence, we obtain new characterizations of the Radon Nikodým property and the UMD property.

Declaration

I declare that this thesis is my own, unaided work. It is being submitted for the Degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

Stuart F. Cullender

Date

Acknowledgments

I am indebted to the following people, who made it possible for me to write this thesis.

- My advisor, Coenraad Labuschagne, for his boundless patience, knowledge, ideas and support.
- My parents, Stephen and Hava Cullender, for their unconditional support and encouragement.
- My partner, Carey Caldecott, for bearing my first born son. Thank you for your support and affording me the space to write this thesis.
- Nico van Niekerk of Andisa Treasury Solutions, for employing the unemployable. Thank you for providing me with the means to support my family during the writing of this thesis.
- Colin Thornton of Dial-a-Nerd, whose company afforded me the opportunity of resuming my undergraduate studies, after my initial failures.
- All the authors of mathematical surveys, for bringing good mathematics to the masses.

Contents

Introduction

Martingales and stopping times have their origins in probability theory and have proven invaluable in the study of geometry of Banach spaces (cf. [11, 37, 44, 45, 14, 18, 88]). The medium in which probability theory and the geometry of Banach spaces mingle are the Lebesgue-Bochner spaces.

Let (Ω, Σ, μ) denote a finite measure space, Y a Banach space and $1 \leq p < \infty$. We denote the Banach space of p-integrable functions on Ω, taking values in Y, by $L^p(\mu, Y)$. These are the functions $f : \Omega \to Y$ for which the Bochner norm $\|f\|_p := \left(\int_\Omega \|f\|^p \,\mathrm{d}\mu\right)^{1/p}$ is finite.

There are two fundamental questions that arise when considering the Lebesgue-Bochner spaces:

1. If a Banach space Y has a certain property, does $L^p(\mu, Y)$ inherit this same property?
2. Given a property for $L^p(\mu, Y)$, what are implications on the geometry of Y?

The second question has lead to the identification of important classes of Banach spaces. The characterizations of these classes often exploit the interplay between the geometry of Banach spaces and probability theory. A classical example is the class of Banach spaces Y for which the Radon-Nikodým Theorem holds in $L^p(\mu, Y)$. Such Banach spaces are said to have the Radon-Nikodým property (RNP). The Radon-Nikodým property has been characterized with the use of martingales (cf. [37, 45, 20, 53]). Another, more recent example are the Banach spaces Y for which all martingale difference sequences are unconditional in $L^p(\mu, Y)$ for $1 < p < \infty$. Such Banach spaces are said to be unconditional for martingale difference sequences (UMD). Both the properties RNP and UMD have intrinsic geometrical characterizations which rely heavily on martingale and stopping time techniques. These geometric characterizations are independent of the Lebesue-Bochner spaces from which their respective definitions originated (cf. [14, 18, 37, 45]).

Many of the classical martingale results in $L^p(\mu)$ rely on the lattice properties of $L^p(\mu)$ rather than the underlying measure space. This has been observed by several authors who were able to transport some of the theory to the more general, measure free setting of vector lattices [101, 25, 95, 99, 96, 97, 98, 32, 108, 107, 61, 62, 63, 64, 65, 66, 67].

Our primary aim in this thesis is to study Troitsky's generalized notion of a martingale in a Banach lattice in [101] and some of its consequences in the Lebesgue-Bochner spaces. We use the generalization of the Lebesgue-Bochner spaces $L^p(\mu, Y)$ as provided by Chaney and Schaefer (cf. [19, 92]). Chaney and Schaefer generalized the Lebesgue-Bochner space $L^p(\mu, Y)$ to a completed tensor product with respect to the l-norm, which we denote $E\tilde{\otimes}_l Y$. Here, E denotes a Banach lattice, Y a Banach space and

$$\|u\|_l = \inf\left\{\left\|\sum_{i=1}^{n} |x_i|\|y_i\|\right\| : u = \sum_{i=1}^{n} x_i \otimes y_i\right\}$$

for all $u = \sum_{i=1}^{n} x_i \otimes y_i \in E \otimes Y$. In the case where $E = L^p(\mu)$ we have $E\tilde{\otimes}_l Y$ isometrically isomorphic to $L^p(\mu, Y)$.

We will extend Troitsky's theory to the l-tensor product of Chaney-Schaefer and develop techniques to provide new results concerning the above mentioned geometric properties.

In Chapter 1, we provide a preliminary understanding of the Lebesgue-Bochner spaces and their ties to the geometry of Banach spaces. Most of the material in this chapter is well known, but we still present proofs of important results that are used later on. Although this may bore those familiar with the subject, it is still interesting to note the contrast in technique as one passes from the Lebesgue-Bochner spaces to the measure-free setting of the l-tensor product.

The information on the l-tensor product in the literature is sparse. We therefore, in Chapter 2, provide an extensive exposition of the l-norm, which contains some new proofs of known results (cf. [28]). We outline three important formulas for calculating the l-norm. The lattice properties of the l-norm are also discussed. In particular, we show that order theoretic versions of injectivity and uniformity hold for the l-norm. We also present a new proof of a known result concerning the inheritance of a lattice ordering in $E\tilde{\otimes}_l Y$, in the case where Y is a Banach lattice (cf. [19, 92]). The chapter concludes with a direct proof of a result concerning the order continuity of the l-norm (cf. [68]).

We wish to demonstrate that the abstraction of a martingale in [101] is more than just a change in notation, but a vehicle for obtaining new results that cannot be formulated in the classical setting.

For instance, when considering the Lebesgue-Bochner space $L^p(\mu, Y)$ more generally as the l-tensor product $E\widetilde{\otimes}_l Y$ of a Banach lattice E and a Banach space Y, another natural question may be asked:

3. *Given a Banach space Y that endows $L^p(\mu, Y)$ with a certain property, for which Banach lattices E does this property hold in $E\widetilde{\otimes}_l Y$?*

We show in Chapter 3, by studying spaces of norm bounded martingales and their behavior under the l-tensor product, that this question has a non-trivial answer for the Radon Nikodým property. Indeed, our abstractions afford us a generalization of a well known characterization of the Radon Nikodým property: Y has the Radon Nikodým property if and only if every $L^p(\mu, Y)$-bounded martingale, $1 < p < \infty$, is norm convergent (cf. [45, Theorem II.2.2.2]). We show that this characterization holds in $E^*\widetilde{\otimes}_l Y$ for all order continuous dual spaces E^* of separable Banach lattices E. As a consequence of the techniques used in proving this result, we are able to show that Y^* has the Radon Nikodým property if and only if $(E\widetilde{\otimes}_l Y)^* = E^*\widetilde{\otimes}_l Y^*$ holds for every separable Banach lattice E with order continuous dual. This generalizes a well known duality result for the Lebesgue-Bochner spaces; i.e. [37, Chapter IV, §1, Theorem 1].

The techniques used in Chapter 3 also have other applications. In Chapter 4, we use these techniques to prove a Grothendieck style characterization of the Banach spaces $\ell^p(Y) := \{(y_i) \subset Y : (\|y_i\|) \in \ell^p\}$, $1 \leq p < \infty$ (cf. [36]). We show that $\ell^1(Y)$ is isometrically isomorphic $\mathcal{L}^{\mathrm{cas}}(c_0, Y)$, where $\mathcal{L}^{\mathrm{cas}}(c_0, Y)$ denotes the space of 'cone absolutely summing' operators from c_0 into Y. These are operators that map positive summable sequences to absolutely summable sequences. Similarly, we show $\ell^p(Y)$ is isometrically isomorphic to $\mathcal{L}^{\mathrm{cas}}(\ell^q, Y)$, where $\frac{1}{p} + \frac{1}{q} = 1$, $p \neq 1$.

In Chapter 5, we harness the theory in Chapter 3 to provide a complete description of convergent martingales in the l-tensor product. This description specializes to the following result for Lebesgue-Bochner spaces (cf. [26]).

Theorem *Let (Ω, Σ, μ) be a finite measure space, Y a Banach space and (Σ_n) an increasing sequence of sub σ-algebras of Σ. In order for $(f_n, \Sigma_n)_{n=1}^\infty$ to be a convergent martingale in $L^p(\mu, Y)$ $(1 \leq p < \infty)$ it is necessary and sufficient that, for each $i \in \mathbb{N}$, there exist a convergent martingale $(x_i^{(n)}, \Sigma_n)_{n=1}^\infty$ in $L^p(\mu)$ and $y_i \in Y$ such that, for each $n \in \mathbb{N}$, we have*

$$f_n(\omega) = \sum_{i=1}^{\infty} x_i^{(n)}(\omega) y_i \quad \textit{for all } \omega \in \Omega,$$

where

$$\left\| \sum_{i=1}^{\infty} \left| \lim_{n\to\infty} x_i^{(n)} \right| \right\|_{L^p(\mu)} < \infty \quad \textit{and} \quad \lim_{i\to\infty} \|y_i\| = 0.$$

The above theorem reveals the inner workings of norm convergent martingales, which yield further characterizations of the properties RNP and UMD. These results depend on the ability to represent elements of the l-tensor product as an infinite series. We present two methods for doing this. The first method, proved in [69], is similar in spirit to the well known representation of elements in the projective tensor product (see for example [90, Proposition 2.7]). This method yields the above stated theorem. The second method is to use 'bases with vector-valued coefficients'. This concept was studied by Figiel and Wojtaszczyk in [46]. We exhibit a simple condition for a basis in a Banach lattice E to be a 'Y-basis' of $E\widetilde{\otimes}_l Y$, for any Banach space Y. The advantage to Y-bases is the uniqueness of the series representation of elements of $E\widetilde{\otimes}_l Y$. The use of Y-bases provides analogous description results for convergent martingales.

In Chapter 6, we consider the notion of a martingale difference sequence (m.d.s.) in a Banach space Y. We study the space of 'm.d.s. multipliers', associated with a m.d.s. $(d_i) \subset Y$. This is the sequence space

$$A^{(d_i)} := \left\{ (\alpha_i) \subset \mathbb{R} : \sum_{i=1}^{\infty} \alpha_i d_i \text{ converges in } Y \right\},$$

endowed with the norm $\|\cdot\|_{A^{(d_i)}}$, defined by $\|(\alpha_i)\|_{A^{(d_i)}} = \sup_{n\in\mathbb{N}} \|\sum_{i=1}^n \alpha_i d_i\|$ for each $(\alpha_i) \in A^{(d_i)}$. The unit vectors (e_i) form a basis of $A^{(d_i)}$ that is equivalent to (d_i). Using the martingale techniques developed in Chapter 3, we are able to show that (d_i) is an unconditional m.d.s. if and only if $A^{(d_i)}$ can be renormed so that it becomes an order continuous Banach lattice under the ordering $(\alpha_i) \geq 0 \Leftrightarrow \alpha_i \geq 0$, for each $i \in \mathbb{N}$. We denote this Banach lattice again by $A^{(d_i)}$ and call it 'the Banach lattice of unconditional m.d.s. multipliers'.

We consider the l-tensor product of two martingale difference sequences. Using the technique of Gelbaum and Gil de Lamadrid in [49], we are able to show that if (ξ_i) is a m.d.s. in the Banach lattice E, relative to a positive filtration, and (η_j) is a m.d.s. in the Banach space Y, then $(\xi_i \otimes \eta_j)$ is a m.d.s. in $E\widetilde{\otimes}_l Y$, provided the sequence $(\xi_i \otimes \eta_j)$ is ordered in an appropriate manner. Consequently, if $1 \leq p < \infty$,

$(d_i) \subset L^p(\mu)$ is a classical m.d.s. basis and $(y_j) \subset Y$ a basis, then $(d_i \otimes y_j)$ is a basis of $L^p(\mu, Y)$.

It is a well known fact that the spaces $L^p(\mu)$ possess an unconditional basis for $1 < p < \infty$ (cf. [13, 72]). However, Aldous showed in [1] that this is not always true for $L^p(\mu, Y)$, even when $1 < p < \infty$ and Y has unconditional basis. This dashes any hope of showing that $(\xi_i \otimes \eta_j)$ is an unconditional m.d.s. in $E\widetilde{\otimes}_l Y$ when (ξ_i) and (η_j) are unconditional. Thus, in the pursuit of results on unconditionality, we consider the space $A^{(\xi_i)}\widetilde{\otimes}_l A^{(\eta_j)}$. Using a result of Popa (cf. [85]), we are able to show that $(e_i \otimes e_j)$ is an unconditional basis of $A^{(\xi_i)}\widetilde{\otimes}_l A^{(\eta_j)}$ when both $(\xi_i) \subset E$ and $(\eta_j) \subset Y$ are unconditional. Consequently, $(e_i \otimes e_j) \subset A^{(\xi_i)}\widetilde{\otimes}_l A^{(\eta_j)}$ is not necessarily equivalent to the m.d.s. $(\xi_i \otimes \eta_j) \subset E\widetilde{\otimes}_l Y$. However, if (ξ_i) is also a boundedly complete basis of E, and E has type p and cotype q, then we have the continuous embeddings

$$\ell^p(E)\widetilde{\otimes}_l Y \supset [(e_i \otimes \xi_i) \otimes \eta_j] \to A^{(\xi_i)}\widetilde{\otimes}_l A^{(\eta_j)} \to [(e_i \otimes \xi_i) \otimes \eta_j] \subset \ell^q(E)\widetilde{\otimes}_l Y,$$

defined by $(e_i \otimes \xi_i) \otimes \eta_j \mapsto e_i \otimes e_j \mapsto (e_i \otimes \xi_i) \otimes \eta_j$ for each $i, j \in \mathbb{N}$. The sequence $((e_i \otimes \xi_i) \otimes \eta_j)$ is an unconditional m.d.s. in both $\ell^p(E)\widetilde{\otimes}_l Y$ and $\ell^q(E)\widetilde{\otimes}_l Y$, but does not span either of these spaces. In the case when E has type and cotype 2, the m.d.s. $((e_i \otimes \xi_i) \otimes \eta_j) \subset \ell^2(E)\widetilde{\otimes}_l Y$ is equivalent to the unconditional basis $(e_i \otimes e_j) \subset A^{(\xi_i)}\widetilde{\otimes}_l A^{(\eta_j)}$.

In Chapter 7, we add the notion of a stopping time to Troitsky's theory of martingales in Banach lattices in [101]. Bounded stopping times in Riesz spaces have been studied in [62]. Our aim is to extend this theory to unbounded stopping times in the Banach lattice setting. A generalized definition of a stopping time in a Banach lattice, adapted to a filtration, is formulated. Our definition of a stopping time differs slightly from the definition in [62].

We show that a stopping time in an order continuous Banach lattice E can be considered as an unconditional Schauder decomposition. This allows access to the theory of Schauder decomposition and multipliers in the thesis of Witvliet (cf. [103]). Consequently, we are able to define a stopped martingale in an order continuous Banach lattice, with respect to an unbounded stopping time, adapted to an R-bounded filtration. This gives rise to an optional stopping theorem for unbounded stopping times. For R-bounded filtrations, this theorem states that a net of stopped martingales in an order continuous Banach lattice, indexed by a directed set of unbounded stopping times, is again a martingale.

We extend this optional stopping theorem to the l-tensor product $E\widetilde{\otimes}_l Y$, where Y is a Banach space. Consequently, we are able to characterize convergent nets of stopped martingales in $E\widetilde{\otimes}_l Y$. In particular, we get the following result.

Theorem *Let (Ω, Σ, μ) be a finite measure space, $1 < p < \infty$, and Y a Banach space. Suppose that $(f_i, \Sigma_i) \subset L^p(\mu, Y)$ is a convergent martingale and $\mathbb{D}$ a directed set of (not necessarily bounded) stopping times adapted to (Σ_i). Then the following statements are equivalent:*

(a) The net of stopped processes $\{f_\tau\}_{\tau\in\mathbb{D}}$ is convergent in $L^p(\mu, Y)$.

(b) For each $i \in \mathbb{N}$, there exist a $y_i \in Y$ and a convergent martingale $(x_j^{(i)}, \Sigma_j)_{j=1}^\infty \subset L^p(\mu)$ with $\{x_\tau^{(i)}\}_{\tau\in\mathbb{D}}$ convergent, so that $f_\tau = \sum_{i=1}^\infty x_\tau^{(i)}(\omega) y_i$ for each $\omega \in \Omega$ and $\tau \in \mathbb{D}$, where

$$\left\| \sum_{i=1}^{\infty} \left| \lim_{\tau\in\mathbb{D}} x_\tau^{(i)} \right| \right\| < \infty \qquad \textit{and} \qquad \lim_{i\to\infty} \|y_i\| = 0.$$

The above equivalence holds for $1 \le p < \infty$ with $(f_i, \Sigma_i) \subset L^p(\mu, Y)$ not necessarily convergent, provided $\mathbb{D}$ is a directed set of bounded stopping times.

The geometric notion of R-boundedness, first studied by Berkson and Gillespie in [8], plays a pivotal role here, as well as the property (α), introduced by Pisier in [83]. Also see a weaker form of property (α) studied by Kalton and Weis in [59].

Lastly, we apply the above techniques to unconditional Schauder decompositions in the Lebesgue-Bochner spaces. Bourgain noted in [9] that if Y is a UMD space with an unconditional basis, then $L^p(\mu, Y)$ has an unconditional basis for $1 < p < \infty$. We generalize this result to stopping times. We show that if Y is a UMD lattice possessing a stopping time, then the Schauder decomposition of $L^p(\mu, Y)$, formed from the product of any martingale decomposition of $L^p(\mu, Y)$ with this stopping time, is unconditional.

This study has produced several publications that have either appeared or are in press, namely [26, 29, 27, 25, 30, 28]. This thesis is arranged, more or less, in the order these papers developed. Due to the difficulty in obtaining and collating some of the literature, we have interspersed extensive preliminaries (some with new proofs) throughout the text, to make this work more accessible. We hope that someone with a modest background in functional analysis and probability theory will be able to understand the concepts presented here. Arguably, one of the motivations for studying 'measure free' stochastic processes in vector lattices is to avoid grubby set theoretic manipulations, intrinsic to probability theory. This motivation would amount to nil if the 'measure free' theory were difficult to access. In this spirit, we have included an appendix at the end, where the definitions and results used implicitly throughout the text may be found, together with suitable references.

Each chapter starts with an introduction where we provide an overview of the results contained therein, as well as their origin. Also, each chapter concludes with a ‘notes and remarks’ section where we provide insights not directly related to the material, directions for further reading and the occasional conjecture or problem.

1

Martingales and the geometry of Banach spaces

1.1 Introduction

We begin by providing the necessities to understand the Bochner integral and the Lebesgue-Bochner spaces as a completed tensor product. We extend all the necessary notions from classical probability theory to the Lebesgue-Bochner spaces and present a classical characterization of the Radon Nikodým property, namely: A Banach space Y has the Radon Nikodým property if and only if every $L^p(\mu, Y)$-bounded martingale, $1 < p < \infty$, is norm convergent. We provide a full proof for this result, as it plays an important role in our characterization of the Radon Nikodým property in Chapter 3. Our exposition of the Bochner integral and the Radon Nikodým property in Sections 1.2 and 1.3 is primarily a blend of Egghe [45] and Diestel and Uhl [37]. We refer the reader to these influential texts for the full story, as we only provide a small glimpse of the theory here. A reader who is familiar with the Lebesgue-Bochner spaces may forego the material in these sections.

Section 1.4 is concerned with unconditional convergence and the so called 'UMD property'. Here, we recall some essential results on unconditional convergence of series in Banach spaces. We then turn our attention to bases and basic sequences. We consider the Haar system and the sequence of Rademacher functions as important examples of basic sequences. Important results concerning the Rademacher functions, such as the Khinchin Inequality and Kahane's Inequality are recalled. This leads to the notions of type and cotype. Finally, after a discussion of the unconditionality of the Haar system, the section concludes with a discussion of UMD property and its relationship with the existence of unconditional bases in $L^p(\mu, Y)$, $1 < p < \infty$. The material in this section is based on Diestel, Jarchow and Tonge [36], Lindenstrauss and Tzafriri [71], Burkholder [18] and Rubio de Francia [88].

Sections 1.5 and 1.6 are closely related. Section 1.5 deals with unconditional Schauder decompositions and their interaction with collections of R-bounded operators. We present the elegant multiplier theorem of Clément, de Pagter, Sukochev and Witvliet in [22, 103]. This result plays a decisive role in Chapter 7, when viewed in the context of stopping times. Section 1.6 considers products of commuting pairs of Schauder decompositions. An important geometrical property of Pisier [84] is recalled, namely property (α). This property guarantees that the product of two unconditional Schauder decompositions is again unconditional. A generalization of the Stein Inequality is also presented here (cf. [94]). These results are important for the work done in Chapter 7. The material in Sections 1.5 and 1.6 is almost exclusively drawn from the comprehensive thesis of Witvliet [103]. We present the proofs of results that we will use in later Chapters. We highly recommend the thesis of Witvliet, and the references contained therein, for more background on unconditional Schauder decomposition and multiplier theorems.

1.2 The Lebesgue-Bochner spaces

Throughout, let Y denote a Banach space and (Ω, Σ, μ) denote a σ-finite measure space. We say a function $s : \Omega \to Y$ is *simple* if there exist $y_1, y_2, \ldots, y_n \in Y$ and sets $A_1, A_2, \ldots, A_n \in \Sigma$ such that $s = \sum_{i=1}^{n} y_i \chi_{A_i}$. Here, χ_{A_i} denotes the characteristic function of A_i, given by $\chi_{A_i}(\omega) = 1$ when $\omega \in A_i$ and $\chi_{A_i}(\omega) = 0$ when $\omega \notin A_i$. A function $f : \Omega \to Y$ is called *μ-measurable* if there exists a sequence of simple functions (s_n) with $\lim_{n\to\infty} \|s_n - f\| = 0$ μ-almost everywhere. We now recall the definition of the Bochner integral (cf. [37, 45]).

Definition 1.2.1 The *Bochner integral* of a simple function $s = \sum_{i=1}^{n} y_i \chi_{A_i}$, over a set $E \in \Sigma$, is defined as

$$\int_E s \,\mathrm{d}\mu = \sum_{i=1}^{n} \mu(A_i \cap E) y_i.$$

A μ-measurable function $f : \Omega \to Y$ is said to be *Bochner integrable* if there exists a sequence of simple functions (s_n) with $\lim_{n\to\infty} \int_\Omega \|s_n - f\| \,\mathrm{d}\mu = 0$. In this case, the Bochner integral of f, over a set $E \in \Sigma$, is

$$\int_E f \,\mathrm{d}\mu = \lim_{n\to\infty} \int_E s_n \,\mathrm{d}\mu.$$

The fact that this limit exists and is independent of the sequence of approximating simple functions is proved in the same manner as for the scalar case. There is a simple, yet important, test for Bochner integrability.

Theorem 1.2.2 (Bochner) *If $f : \Omega \to Y$ is a μ-measurable function, then f is Bochner integrable if and only if the scalar-valued function $\|f\| : \Omega \to \mathbb{R}$ is integrable.*

Proof. Suppose that f is Bochner integrable, then there exists a sequence of simple functions (s_n) for which $\lim_{n\to\infty}\int_\Omega \|s_n - f\|\,\mathrm{d}\mu = 0$ holds. Thus,

$$\int_\Omega \|f\|\,\mathrm{d}\mu = \int_\Omega \|(f-s_n)+s_n\|\,\mathrm{d}\mu \le \int_\Omega \|(f-s_n)\|\,\mathrm{d}\mu + \int_\Omega \|s_n\|\,\mathrm{d}\mu$$

for each $n \in \mathbb{N}$, whence $\int_\Omega \|f\|\,\mathrm{d}\mu < \infty$.

Conversely, let (s_n) be a sequence of simple functions that converge μ-almost everywhere to f. Fix a $\delta > 0$ and define

$$g_n(\omega) = \begin{cases} s_n(\omega), & \text{if } \|s_n(\omega)\| \le (1+\delta)\|f(\omega)\|; \\ 0, & \text{otherwise.} \end{cases}$$

Then (g_n) is a sequence of μ-measurable simple functions that converges to f μ-almost everywhere. Moreover, $(\|f - g_n\|)$ is dominated by the integrable function $(2+\delta)\|f\|$. The Lebesgue Dominated Convergence Theorem implies

$$\lim_{n\to\infty}\int_\Omega \|f-g_n\|\,\mathrm{d}\mu = \int_\Omega \lim_{n\to\infty}\|f-g_n\|\,\mathrm{d}\mu = 0$$

and the proof is complete. $\square$

The Dominated Convergence Theorem easily extends to the Bochner Integral by virtue of the above theorem. Also, if $f : \Omega \to Y$ is integrable, there exists a sequence of approximating simple functions (s_n) for which $\lim_{n\to\infty}\int_\Omega \|s_n\|\,\mathrm{d}\mu = \int_\Omega \|f\|\,\mathrm{d}\mu$, whence the inequality

$$\begin{aligned} &\left\|\int_E f\,\mathrm{d}\mu\right\| = \left\|\lim_{n\to\infty}\int_E s_n\,\mathrm{d}\mu\right\| = \lim_{n\to\infty}\left\|\int_E s_n\,\mathrm{d}\mu\right\| \\ &\le \lim_{n\to\infty}\int_E \|s_n\|\,\mathrm{d}\mu = \int_E \|f\|\,\mathrm{d}\mu \end{aligned} \tag{1.1}$$

holds for each $E \in \Sigma$. Another simple but important result, achieved by approximating by simple functions, is the following.

Proposition 1.2.3 *Let X and Y be Banach spaces with $f : \Omega \to X$ Bochner integrable. If $T : X \to Y$ is a bounded linear operator, then $Tf : \Omega \to Y$ is Bochner integrable and $\int_\Omega Tf\,\mathrm{d}\mu = T\left(\int_\Omega f\,\mathrm{d}\mu\right)$.*

Proof. Choose a sequence of X-valued simple functions (s_n) for which $\lim_{n\to\infty}\int_\Omega \|s_n - f\|\,\mathrm{d}\mu = 0$. Then (Ts_n) is a sequence of Y-valued step functions that converges μ-almost everywhere to Tf and $\int_\Omega Ts_n\,\mathrm{d}\mu = T\left(\int_\Omega s_n\,\mathrm{d}\mu\right)$ for each $n \in \mathbb{N}$. Moreover, $\lim_{n\to\infty}\int_\Omega \|Ts_n - Tf\|\,\mathrm{d}\mu \le \|T\|\lim_{n\to\infty}\int_\Omega \|s_n - f\|\,\mathrm{d}\mu = 0$ so that Tf is Bochner integrable. Consequently,

$$\begin{aligned}\int_\Omega Tf \,\mathrm{d}\mu &= \lim_{n\to\infty}\int_\Omega Ts_n \,\mathrm{d}\mu = \lim_{n\to\infty} T\left(\int_\Omega s_n \,\mathrm{d}\mu\right)\\ &= T\left(\lim_{n\to\infty}\int_\Omega s_n \,\mathrm{d}\mu\right) = T\left(\int_\Omega f \,\mathrm{d}\mu\right)\end{aligned}$$

completes the proof. □

Definition 1.2.4 For $1 \leq p < \infty$ and a Banach space Y, the *Bochner space* of (classes of μ-almost everywhere equal) p-integrable functions is the vector space

$$L^p(\mu, Y) = \left\{ f : \Omega \to Y \ \mu\text{-measurable} \ : \int_\Omega \|f\|^p \,\mathrm{d}\mu < \infty \right\}$$

together with the *Bochner norm*, defined by $\|f\|_p = \left(\int_\Omega \|f\|^p \,\mathrm{d}\mu\right)^{1/p}$ for all $f \in L^p(\mu, Y)$. In the case $p = \infty$,

$$L^\infty(\mu, Y) = \{ f : \Omega \to Y \ \mu\text{-measurable} \ : \ \text{ess-sup}\,\|f\| < \infty \}$$

and $\|f\|_\infty = \text{ess-sup}\,\|f\|$ for all $f \in L^\infty(\mu, Y)$. When μ is the Lebesgue measure, the associated Bochner spaces are referred to as the *Lebesgue-Bochner* spaces.

In the case $Y = \mathbb{R}$, the Bochner spaces reduce to the scalar-valued L^p-spaces. The fact that the Bochner spaces are Banach spaces follows in the same manner as for the scalar case. It is clear from the definition of Bochner integrability that the simple functions are dense in the Bochner spaces for $1 \leq p < \infty$. This simple fact is vital for the extension of many results from the scalar-valued case to the vector-valued case. It is also vital in the following decomposition of the Bochner spaces into a completed tensor product (cf. [19, 31, 34, 35, 37]).

For $1 \leq p < \infty$ define a bilinear mapping $\psi : L^p(\mu) \times Y \to L^p(\mu, Y)$ by

$$\psi(f, y)(\omega) = f(\omega)y \quad \text{ for all } \omega \in \Omega.$$

Then the induced linear map $\psi^\ell : L^p(\mu) \otimes Y \to L^p(\mu, Y)$ is described by

$$\psi^\ell(f \otimes y)(\omega) = f(\omega)y \quad \text{ for all } \omega \in \Omega$$

and is injective. Thus, $L^p(\mu, Y)$ contains a copy of $L^p(\mu) \otimes Y$ and we may induce the Bochner norm. The normed space $(L^p(\mu) \otimes Y, \|\cdot\|_p)$ is denoted $L^p(\mu) \otimes_{\Delta_p} Y$.

Let $S_p(\mu)$ denote the real-valued simple functions in $L^p(\mu)$, we write

$$S_p(\mu) \otimes Y := \left\{ \sum_{k=1}^n \chi_{A_k} \otimes y_k : n \in \mathbb{N},\ \chi_{A_k} \text{ integrable},\ \ y_k \in Y \right\}$$

and let

$$S_p(Y) := \left\{ \sum_{k=1}^{n} y_k \chi_{A_k} : n \in \mathbb{N},\ \chi_{A_k} \text{ integrable},\ \ y_k \in Y \right\}$$

denote the Y-valued simple functions in $L^p(\mu, Y)$. Then, the restricted map

$$\psi^\ell : S_p(\mu) \otimes Y \to S_p(Y)$$

clearly defines a one-to-one, norm preserving correspondence between $S_p(\mu) \otimes Y$ and $S_p(Y)$. Hence

$$\psi^\ell(S_p(\mu) \otimes Y) = S_p(Y) \subset \psi^\ell(L^p(\mu) \otimes_{\Delta_p} Y) \subset L^p(\mu, Y).$$

Since $S_p(\mu)$ is dense in $L^p(\mu)$ and $S_p(Y)$ is dense in $L^p(\mu, Y)$ it follows, by taking completions, that $L^p(\mu)\widetilde{\otimes}_{\Delta_p} Y$ is isometrically isomorphic to $L^p(\mu, Y)$.

We recall some classical definitions and results from probability theory. Throughout, let (Ω, Σ, μ) denote a finite measure space.

Definition 1.2.5 Let $1 \le p < \infty$ and Y be a Banach space.

(a) If Σ_1 is a sub σ-algebra of Σ, the *conditional expectation* of $f \in L^p(\mu)$ relative to Σ_1, denoted by $\mathbb{E}(f \mid \Sigma_1)$, is a Σ_1-measurable element of $L^p(\mu)$ which is given by

$$\int_A \mathbb{E}(f \mid \Sigma_1)\, \mathrm{d}\mu = \int_A f\, \mathrm{d}\mu \quad \text{for all } A \in \Sigma_1. \tag{1.2}$$

(b) A monotone increasing sequence (Σ_i) of sub σ-algebras of Σ is called a *filtration.*

(c) If (Σ_i) is a filtration, a sequence $(f_i) \subset L^p(\mu)$ is said to be *adapted* to (Σ_i) if each f_i is Σ_i-measurable.

(d) A sequence $(f_i) \subset L^p(\mu)$ adapted to (Σ_i) is called a *martingale* relative to (Σ_i) if $\mathbb{E}(f_m \mid \Sigma_n) = f_n$ for all $n \le m$. We use the notation (f_n, Σ_n) when there is a need to indicate the filtration involved.

(e) A sequence $(f_i) \subset L^p(\mu)$ adapted to (Σ_i) is called a *submartingale (supermartingale)* relative to (Σ_i) if $\mathbb{E}(f_m \mid \Sigma_n) \ge (\le) f_n$ for all $n \le m$.

(f) Let $(d_i) \subset L^p(\mu)$ be a sequence and $\sigma(d_1, \ldots, d_i)$ denote the smallest σ-algebra allowing $d_1, \ldots, d_i$ to be measurable. Then (d_i) is called a *martingale difference sequence (m.d.s.)* if

$$\mathbb{E}(d_{i+1} \mid \sigma(d_1, \ldots, d_i)) = 0$$

for each $i \in \mathbb{N}$.

(g) A sequence $(f_i) \subset L^p(\mu)$ is called *uniformly integrable* if for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\mu(E) < \delta, E \in \Sigma \Rightarrow \int_E |f_n| \,\mathrm{d}\mu < \varepsilon$$

for all $n \in \mathbb{N}$.

The above definition for (sub) (super) martingales extends to filtrations indexed by uncountable directed sets in the obvious manner. For a filtration (Σ_i), it follows from (1.2) that $\mathbb{E}(\,\cdot \mid \Sigma_i)$ is a positive contractive projection for each $i \in \mathbb{N}$ and, for $i \leq j$, we have

$$\mathbb{E}(\,\cdot \mid \Sigma_i) = \mathbb{E}(\mathbb{E}(\,\cdot \mid \Sigma_j) \mid \Sigma_i) = \mathbb{E}(\mathbb{E}(\,\cdot \mid \Sigma_i) \mid \Sigma_j),$$

which implies $\mathcal{R}\big(\mathbb{E}(\,\cdot \mid \Sigma_i)\big) \subset \mathcal{R}\big(\mathbb{E}(\,\cdot \mid \Sigma_j)\big)$.

If $1 < p < \infty$ and $(f_n) \subset L^p(\mu)$ with $\sup_{n\in\mathbb{N}} \|f_n\|_p < \infty$, it follows from Hölders inequality that (f_n) is uniformly integrable with $\sup_{n\in\mathbb{N}} \|f_n\|_1 < \infty$. In probability theory, a stronger definition for uniform integrability is often used: for each $\varepsilon > 0$, there exists $K > 0$ such that

$$\int_{\{\omega\in\Omega:|f_n(\omega)|>K\}} |f_n| \,\mathrm{d}\mu < \varepsilon$$

for all $n \in \mathbb{N}$. This is equivalent to Definition 1.2.5(g) when $\sup_{n\in\mathbb{N}} \|f_n\|_1 < \infty$ (cf. [21, pp. 96–97]). The Doob Convergence Theorem (cf. [41]) asserts that a $L^1(\mu)$-bounded martingale converges in $L^1(\mu)$-norm if and only if it uniformly integrable.

We will also make use of distributions; suppose that (Ω, Σ, μ) is a finite measure space and $f : \Omega \to \mathbb{R}$ is measurable. Then, the *(cumulative) distribution function* N_f of f is defined by $\mu(f^{-1}(B))$ for every Borel set $B \subset \mathbb{R}$. If $N_f(\{\omega \in \Omega : f(\omega) < -a\}) = N_f(\{\omega \in \Omega : f(\omega) > a\})$ for each $a \in \mathbb{R}$, then f is said to be *symmetric*. Observe that if f is a symmetrically distributed measurable function, then $N_f = N_{-f}$. The measurable functions $f_1, \dots, f_n$ are said to be *independent* whenever $\mu(\bigcap_{i=1}^n \{\omega \in \Omega : f_i(\omega) \in B_i\}) = \prod_{i=1}^n \mu(\{\omega \in \Omega : f_i(\omega) \in B_i\})$ for any Borel sets $B_1, \dots, B_n$. An infinite set of measurable functions is said to be independent if every finite subset is independent.

The notion of a conditional expectation can be extended to the vector-valued L^p-spaces with the aid of simple functions. Let Σ_1 be a sub σ-algebra of Σ. Define

$$\mathbb{E}(\,\cdot \mid \Sigma_1) : S_p(Y) \to S_p(Y)$$

by

$$\mathbb{E}\left(\sum_{i=1}^n \chi_{A_i} \otimes x_i \,\middle|\, \Sigma_1\right) = (\mathbb{E}(\,\cdot \mid \Sigma_1) \otimes \mathrm{id}_Y)\left(\sum_{i=1}^n \chi_{A_i} \otimes x_i\right) = \sum_{i=1}^n \mathbb{E}(\chi_{A_i} \mid \Sigma_1) \otimes x_i,$$

where $\mathbb{E}(\chi_{A_i} \mid \Sigma_1)$ denotes the conditional expectation of $\chi_{A_i} \in L^p(\mu)$. By Jensen's inequality, it follows that

$$\Delta_p\left(\mathbb{E}\left(\sum_{i=1}^n \chi_{A_i} \otimes x_i \,\middle|\, \Sigma_1\right)\right) \leq \Delta_p\left(\sum_{i=1}^n \chi_{A_i} \otimes x_i\right).$$

The *conditional expectation* $\mathbb{E}(\,\cdot \mid \Sigma_1) : L^p(\mu, Y) \to L^p(\mu, Y)$, of $f \in L^p(\mu, Y)$ relative to Σ_1, is the continuous extension of the operator $\mathbb{E}(\,\cdot \mid \Sigma_1) \otimes \mathrm{id}_Y$ on $S_p(Y)$ to $L^p(\mu, Y)$; it satisfies (1.2) and is a contractive projection. The notion of an adapted sequence, martingales and martingale difference sequences now extend to $L^p(\mu, Y)$ in the obvious manner.

Uniform integrability extends to $L^p(\mu, Y)$ as follows: A sequence $(f_n) \subset L^p(\mu, Y)$ is said to be *uniformly integrable* if $(\|f_n\|) \subset L^p(\mu)$ is uniformly integrable. Again, it follows easily from the Hölder inequality that if $\sup_{n\in\mathbb{N}} \|f_n\|_p < \infty$ for some $1 < p < \infty$, then (f_n) is uniformly integrable. It follows from the Maximal Lemma (cf. [45, Lemma II.1.5] or [37, Chapter V, §2, Lemma 7]) that every norm convergent martingale $(f_i) \subset L^p(\mu, Y)$ also converges μ-almost everywhere (cf. [37, Chapter V, §2, Theorem 8] or [45, Theorem II.1.6]). If $(f_i) \in L^p(\mu, Y)$ is just a sequence that converges in norm, then we can only find a subsequence of (f_i) that converges μ-almost everywhere (cf. [45, pp. 11–12]).

In general, the Doob Convergence Theorem does not hold in $L^p(\mu, Y)$ for any Banach space Y. However, there is a class of Banach spaces for which this theorem does hold. This is the theme of the next section.

1.3 The Radon-Nikodým property

In this section we define the Radon-Nikodým property and illustrate its intimate relationship with martingales. We start with the definition of a vector measure. Throughout, let (Ω, Σ, μ) be a finite measure space and Y be a Banach space.

Definition 1.3.1 (a) A function $F : \Sigma \to Y$ is called a *(countable additive) vector measure* if $F(\bigcup_{n=1}^\infty E_n) = \sum_{n=1}^\infty F(E_n)$ where $(E_n) \subset \Sigma$ is pairwise disjoint.

(b) A vector measure $F : \Sigma \to Y$ is called *μ-continuous* if $\lim_{\mu(E)\to 0, E\in\Sigma} F(E) = 0$.

(c) The *variation* of a vector measure $F : \Omega \to Y$ if defined as $|F|(E) = \sup_\pi \sum_{A\in\pi} \|F(A)\|$, where the supremum is taken over all finite partitions π of E.

(d) A vector measure $F : \Omega \to Y$ is said to be of *bounded variation* if $|F|(\Omega) < \infty$.

It can be shown that a vector measure of bounded variation is countably additive if and only if its variation is countably additive. Since we are only interested in countably additive vector measures, we will drop the reference to countable additivity.

One way vector measures of bounded variation arise is the Bochner integral. Indeed, If $f : \Omega \to Y$ is a Bochner integrable function, define $F(E) = \int_E f \,\mathrm{d}\mu$ for all $E \in \Sigma$. Then, the following result guarantees that F is a vector measure.

Proposition 1.3.2 *If $f : \Omega \to Y$ is a Bochner integrable function, then the following statements hold:*

(a) $\lim_{\mu(E)\to 0, E\in\Sigma} \int_E f \,\mathrm{d}\mu = 0$.
(b) If $(E_n) \subset \Sigma$ is pairwise disjoint, with $E := \bigcup_{n=1}^\infty E_n$, then $\int_E f \,\mathrm{d}\mu = \sum_{n=1}^\infty \int_{E_n} f \,\mathrm{d}\mu$. Moreover, $\sum_{n=1}^\infty \int_{E_n} f \,\mathrm{d}\mu$ converges absolutely.
(c) If $F(E) := \int_E f \,\mathrm{d}\mu$, then F is a vector measure of bounded variation and $|F|(E) = \int_E \|f\| \,\mathrm{d}\mu$ for all $E \in \Sigma$.

Proof. (a) Since $\lim_{\mu(E)\to 0, E\in\Sigma} \int_E \|f\| \,\mathrm{d}\mu = 0$, the result follows by (1.1).

(b) Notice that $\sum_{n=1}^\infty \int_{E_n} f \,\mathrm{d}\mu$ is absolutely summable by (1.1) and so is also summable by the completeness of Y. Lastly, observe

$$\left\| \int_{\bigcup_{n=1}^\infty E_n} f \,\mathrm{d}\mu - \sum_{n=1}^m \int_{E_n} f \,\mathrm{d}\mu \right\| = \left\| \int_{\bigcup_{n=m+1}^\infty E_n} f \,\mathrm{d}\mu \right\|$$

which goes to zero by part (a).

(c) The fact that F is a vector measure follows from parts (a) and (b). Let π be a finite partition of $E \in \Sigma$. Then, by (1.1),

$$\sum_{A\in\pi} \|F(A)\| = \sum_{A\in\pi} \left\| \int_A f \,\mathrm{d}\mu \right\| \le \sum_{A\in\pi} \int_A \|f\| \,\mathrm{d}\mu = \int_E \|f\| \,\mathrm{d}\mu.$$

Taking the supremum over all finite partitions of E on the right shows $|F|(E) \le \int_E \|f\| \,\mathrm{d}\mu$. An application of Bochner's theorem shows that F is of bounded variation.

For the reverse inequality, let $\varepsilon > 0$ and choose a sequence of simple functions (s_n) with $\lim_{n\to\infty} \int_\Omega \|s_n - f\| \,\mathrm{d}\mu = 0$. There exists an $N \in \mathbb{N}$ such that $n \ge N$ implies $\int_\Omega \|s_n - f\| \,\mathrm{d}\mu < \varepsilon/2$. Choose a finite partition π' of E such that $\sum_{A\in\pi'} \left\| \int_A s_n \,\mathrm{d}\mu \right\| = \int_E \|s_n\| \,\mathrm{d}\mu$. Select a finite refinement π of π' such that $|F|(E) - \sum_{B\in\pi} \left\| \int_B f \,\mathrm{d}\mu \right\| < \varepsilon/2$. Moreover, we still have $\sum_{B\in\pi} \left\| \int_B s_n \,\mathrm{d}\mu \right\| = \int_E \|s_n\| \,\mathrm{d}\mu$ and

$$\sum_{B\in\pi} \left| \left\| \int_B s_n \,\mathrm{d}\mu \right\| - \left\| \int_B f \,\mathrm{d}\mu \right\| \right| \le \int_E \|s_n - f\| \,\mathrm{d}\mu < \varepsilon/2$$

by (1.1). Consequently,

$$\begin{aligned}
&\left| |F|(E) - \int_E \|s_n\| \,\mathrm{d}\mu \right| = \left| |F|(E) - \sum_{B\in\pi} \left\| \int_B s_n \,\mathrm{d}\mu \right\| \right| \\
&\leq \left| |F|(E) - \sum_{B\in\pi} \left\| \int_B f \,\mathrm{d}\mu \right\| \right| + \sum_{B\in\pi} \left| \left\| \int_B s_n \,\mathrm{d}\mu \right\| - \left\| \int_B f \,\mathrm{d}\mu \right\| \right| \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Thus, $\lim_{n\to\infty} \int_E \|s_n\| \,\mathrm{d}\mu = \int_E \|f\| \,\mathrm{d}\mu = |F|(E)$. □

Corollary 1.3.3 *If f and g are Bochner integrable and $\int_E f \,\mathrm{d}\mu = \int_E g \,\mathrm{d}\mu$ for all $E \in \Sigma$, then $f = g$ μ-almost everywhere.*

Proof. The vector measure $F(E) := \int_E (f-g) \,\mathrm{d}\mu = 0$ for all $E \in \Sigma$. Hence, $0 = |F|(E) = \int_E \|f-g\| \,\mathrm{d}\mu$ by the above theorem. Consequently, $\|f-g\| = 0$ μ-almost everywhere, which completes the proof. □

It follows that every Bochner integrable function corresponds to a vector measure of bounded variation that is μ-continuous. In the case where $Y = \mathbb{R}$, the classical Radon Nikodým Theorem provides a converse; we recall the statement of the Radon-Nikodým Theorem for the vector-valued case:

Theorem 1.3.4 (Radon-Nikodým) *Let $G : \Sigma \to Y$ be a μ-continuous vector measure of bounded variation, then there exists a Bochner integrable function $g \in L^1(\mu, Y)$ such that $G(E) = \int_E g \,\mathrm{d}\mu$ for all $E \in \Sigma$.*

Unfortunately (or perhaps, fortunately) the above theorem is not always valid. For example, the Radon Nikodým Theorem fails when $Y = c_0$ or $Y = L^1(\mu)$ (cf. [37, Chapter III, §1, Examples 1 and 2]). The failure of this theorem lead to the identification of a class of Banach spaces for which the Radon-Nikodým theorem *does* hold.

Definition 1.3.5 A Banach space Y is said to have the *Radon Nikodým property with respect to the measure space* (Ω, Σ, μ) if, for every μ-continuous vector measure $G : \Sigma \to Y$ of bounded variation, there exists a Bochner integrable function $g \in L^1(\mu, Y)$ such that $G(E) = \int_E g \,\mathrm{d}\mu$ for all $E \in \Sigma$. Y is said to have the *Radon Nikodým property (RNP)* if Y has the Radon Nikodým property with respect to every finite measure space.

There is a long list of characterizations of the Radon Nikodým property, to which we will add a few more, later on. One of the more famous characterizations is the Lewis-Stegall Theorem (cf. [37, Chapter III, §1, Theorem 8]).

Theorem 1.3.6 (Lewis-Stegall) *A Banach space Y has the Radon-Nikodým property with respect to (Ω, Σ, μ) if and only if every bounded linear operator $T : L^1(\mu) \to Y$ admits a factorization $T = LS$*

$$\begin{tikzcd} L^1(\mu) \arrow[rr, "T"] \arrow[dr, "S"'] & & Y \\ & \ell_1 \arrow[ur, "L"'] & \end{tikzcd}$$

where $L : \ell_1 \to Y$ and $0 \leq S : L^1(\mu) \to \ell_1$ are continuous linear operators. In this case, for each $\varepsilon > 0$, L, S may be chosen such that $\|S\| \leq \|T\| + \varepsilon$ and $\|L\| \leq 1$.

We point out that the proof of the Lewis-Stegall Theorem presented in [37] shows that the operator $S : L^1(\mu) \to \ell_1$ in the above theorem is positive. This observation is crucial for our work in Chapter 3.

A classical characterization of when a dual space has the Radon Nikodým property is:

Theorem 1.3.7 *Let (Ω, Σ, μ) be a finite measure space, $1 \leq p < \infty$, and Y a Banach space. Then $L^p(\mu, Y)^* = L^q(\mu, Y^*)$ where $\frac{1}{p} + \frac{1}{q} = 1$, if and only if Y^* has the Radon-Nikodým property with respect to (Ω, Σ, μ).*

We now collect some sufficient conditions for a Banach space to have the Radon Nikodým property.

Proposition 1.3.8 *Let Y be a Banach space. Then, Y has RNP if one of the following statements hold:*

(a) Y is a Hilbert space.
(b) Y has a separable dual.
(c) Y is reflexive.
(d) Y has a boundedly complete basis.
(e) Y is a subspace of a Banach space with RNP.

The existence of a conditional expectation on $L^p(\mu)$ depends on the Radon Nikodým Theorem. However, conditional expectations always exist on $L^p(\mu, Y)$, even when the Radon Nikodým Theorem fails, as is easily seen by the above construction. This curious phenomenon gives us a clue that probability theory will play a decisive role in characterizing the Radon Nikodým property. We shall present a martingale characterization.

Let $(f_n, \Sigma_n) \subset L^p(\mu, Y)$ be a martingale, then $F(E) := \lim_{n\to\infty} \int_E f_n \,\mathrm{d}\mu$ exists for each $E \in \bigcup_{n=1}^{\infty} \Sigma_n$. Indeed, there is an $N \in \mathbb{N}$ such that $E \in \Sigma_n$ for all $n \geq N$. Hence, by (1.2), $\int_E f_N \,\mathrm{d}\mu = \int_E \mathbb{E}(f_n \,|\, \Sigma_N) \,\mathrm{d}\mu = \int_E f_n \,\mathrm{d}\mu = \lim_{n\to\infty} \int_E f_n \,\mathrm{d}\mu$. We use this in the following lemma.

Lemma 1.3.9 *Let $1 \leq p < \infty$ and $(f_n, \Sigma_n) \subset L^p(\mu, Y)$ be a martingale. Then (f_n, Σ_n) converges in $L^p(\mu, Y)$ if and only if there exists $f \in L^p(\mu, Y)$ such that, for each $E \in \bigcup_{n=1}^{\infty} \Sigma_n$, we have $F(E) = \lim_{n\to\infty} \int_E f_n \,\mathrm{d}\mu = \int_E f \,\mathrm{d}\mu$.*

Proof. Suppose $\lim_{n\to\infty} f_n = f$. It follows from (1.1) and Bochner's Theorem that the linear operator $L^p(\mu, Y) \to Y$, defined by $g \mapsto \int_E g \,\mathrm{d}\mu$, is bounded with norm less than or equal to one for all $E \in \Sigma$. Hence, $\lim_{n\to\infty} \int_E f_n \,\mathrm{d}\mu = \int_E f \,\mathrm{d}\mu$ for all $E \in \Sigma \supset \bigcup_{n=1}^{\infty} \Sigma_n$.

Conversely, suppose there exists $f \in L^p(\mu, Y)$ such that, for each $E \in \bigcup_{n=1}^{\infty} \Sigma_n$, we have $F(E) = \lim_{n\to\infty} \int_E f_n \,\mathrm{d}\mu = \int_E f \,\mathrm{d}\mu$. Let Σ_∞ denote the σ-algebra generated by $\bigcup_{n=1}^{\infty} \Sigma_n$ and set $f_\infty = \mathbb{E}(f \,|\, \Sigma_\infty)$. Then, by (1.2), $F(E) = \int_E f_\infty \,\mathrm{d}\mu$ for all $E \in \bigcup_{n=1}^{\infty} \Sigma_n$ and $\mathbb{E}(f_\infty | \Sigma_n) = f_n$ for all $n \in \mathbb{N}$. Observe that, since $1 \leq p < \infty$, simple functions that are measurable with respect to the algebra $\bigcup_{n=1}^{\infty} \Sigma_n$ are dense in $L^p_\infty := L^p(\Omega, \Sigma_\infty, \mu|_{\Sigma_\infty}, Y) \ni f_\infty$. Thus, for $\varepsilon > 0$, there exists a simple function $s_\varepsilon \in L^p_\infty$ that is $\bigcup_{n=1}^{\infty} \Sigma_n$-measurable with $\|s_\varepsilon - f_\infty\|_p < \varepsilon/2$. Consequently, there exists an $N \in \mathbb{N}$ such that $\mathbb{E}(s_\varepsilon \,|\, \Sigma_n) = s_\varepsilon$ for all $n \geq N$. Hence,

$$\begin{aligned} \|f_n - f_\infty\|_p &\leq \|f_n - s_\varepsilon\|_p + \|s_\varepsilon - f_\infty\|_p \\ &= \|\mathbb{E}(f_\infty - s_\varepsilon \,|\, \Sigma_n)\|_p + \|s_\varepsilon - f_\infty\|_p \\ &\leq 2\|s_\varepsilon - f_\infty\|_p \\ &< \varepsilon. \end{aligned}$$

Since L^p_∞ is a closed subspace of $L^p(\mu, Y)$, the proof is complete. □

We are now in a position to reproduce the following well known characterization of the Radon Nikodým property.

Theorem 1.3.10 *The following statements are equivalent for a Banach space Y:*

(a) Y has the Radon Nikodým property.
(b) Every uniformly integrable martingale $(f_n) \subset L^1(\mu, Y)$ with $\sup_{n\in\mathbb{N}} \|f_n\|_1 < \infty$ converges in $L^1(\mu, Y)$.
(c) Every martingale $(f_n) \subset L^1(\mu, Y)$ with $\sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$ converges in $L^1(\mu, Y)$.
(d) Every martingale $(f_n) \subset L^p(\mu, Y)$ $(1 < p < \infty)$ with $\sup_{n\in\mathbb{N}} \|f_n\|_p < \infty$ converges in $L^p(\mu, Y)$.

Proof. We show the implications (a)⇒(b)⇒(c)⇒(a), (d)⇒(c) and (b)⇒(d).

(a)⇒(b) Suppose that Y has RNP and let $(f_n, \Sigma_n) \subset L^1(\mu, Y)$ be a uniformly integrable martingale with $\sup_{n\in\mathbb{N}} \|f_n\|_1 < \infty$. For $E \in \bigcup_{n=1}^{\infty} \Sigma_n$, the limit $F(E) := \lim_{n\to\infty} \int_E f_n \,\mathrm{d}\mu$ exists. Let Σ_∞ denote the σ-algebra generated by $\bigcup_{n=1}^{\infty} \Sigma_n$, we will show that F is a μ-continuous vector measure on Σ_∞, of bounded variation.

For a sequence $(E_i) \subset \bigcup_{n=1}^{\infty} \Sigma_n$ we may consider the union $A := \bigcup_{i=1}^{\infty} E_i$ to be disjoint. Let $\varepsilon > 0$, then by the uniform integrability of (f_n), there exists $N \in \mathbb{N}$ such that $l > k \geq N$ implies $\left\|\sum_{i=k}^{l} \int_{E_i} f_n \,\mathrm{d}\mu\right\| \leq \int_{\bigcup_{i=k}^{l} E_i} \|f_n\| \,\mathrm{d}\mu < \varepsilon$ for all $n \in \mathbb{N}$. Consequently,

$$\sum_{i=1}^{\infty} \lim_{n\to\infty} \int_{E_i} f_n \,\mathrm{d}\mu = \lim_{n\to\infty} \sum_{i=1}^{\infty} \int_{E_i} f_n \,\mathrm{d}\mu = \lim_{n\to\infty} \int_A f_n \,\mathrm{d}\mu = F(A)$$

exists in Y. Hence, $F(A^c) = F(\Omega \setminus A) = \lim_{n\to\infty} \int_\Omega f_n \,\mathrm{d}\mu - \lim_{n\to\infty} \int_A f_n \,\mathrm{d}\mu$ also exists. It follows that $F(A)$ exists for all $A \in \Sigma_\infty$.

A similar argument shows that F is countably additive on Σ_∞. Indeed, let $(A_i) \subset \Sigma_\infty$ be pairwise disjoint. Then, using uniform integrability as above and the existence of $F(A_i)$ for each $i \in \mathbb{N}$, we have $\sum_{i=1}^{\infty} F(A_i) = \sum_{i=1}^{\infty} \lim_{n\to\infty} \int_{A_i} f_n \,\mathrm{d}\mu = \lim_{n\to\infty} \sum_{i=1}^{\infty} \int_{A_i} f_n \,\mathrm{d}\mu = \lim_{n\to\infty} \int_{\bigcup_{i=1}^{\infty} A_i} f_n \,\mathrm{d}\mu = F(\bigcup_{i=1}^{\infty} A_i)$.

To show that F is of bounded variation on Σ_∞, let $\pi \subset \Sigma_\infty$ be a finite partition of Ω. Then,

$$\begin{aligned} \sum_{A\in\pi} \|F(A)\| &= \sum_{A\in\pi} \left\| \lim_{n\to\infty} \int_A f_n \,\mathrm{d}\mu \right\| = \lim_{n\to\infty} \sum_{A\in\pi} \left\| \int_A f_n \,\mathrm{d}\mu \right\| \\ &\leq \sup_{n\in\mathbb{N}} \sum_{A\in\pi} \int_A \|f_n\| \,\mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int_\Omega \|f_n\| \,\mathrm{d}\mu < \infty. \end{aligned}$$

Lastly, we show that F is μ-continuous. Let $\mu(A_i) \subset \Sigma_\infty$ with $\lim_{i\to\infty} \mu(A_i) = 0$ and $\varepsilon > 0$. By the uniform integrability of (f_n), there exists $N \in \mathbb{N}$ such that $i \geq N$ implies $\left\|\int_{A_i} f_n \,\mathrm{d}\mu\right\| \leq \int_{A_i} \|f_n\| \,\mathrm{d}\mu < \varepsilon$ for all $n \in \mathbb{N}$. Consequently, $\|F(A_i)\| < \varepsilon$.

Since Y has RNP, there exists $f \in L^1_\infty := L^1(\Omega, \Sigma_\infty, \mu|_{\Sigma_\infty}, Y)$ such that

$$F(E) = \lim_{n\to\infty} \int_E f_n \,\mathrm{d}\mu = \int_E f \,\mathrm{d}\mu$$

for all $E \in \Sigma_\infty$. Since L^1_∞ is a closed subspace of $L^p(\mu, Y)$, an application of Lemma 1.3.9 shows that (b) is true.

(b)⇒(c) Let $(f_n) \subset L^1(\mu, Y)$ be a martingale with $\sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$. Then $\sup_{n\in\mathbb{N}} \|f_n\|_1 \leq \sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$. Moreover, since

$$\mu(\{\omega \in \Omega : \|f_n(\omega)\| > \sup_{n\in\mathbb{N}} \|f_n\|_\infty\}) = 0$$

for each $n \in \mathbb{N}$, we have

$$\int_{\{\omega\in\Omega:\|f_n(\omega)\|>\sup_{n\in\mathbb{N}}\|f_n\|_\infty\}} \|f_n\|\,\mathrm{d}\mu = 0$$

for each $n \in \mathbb{N}$, showing that (f_n) in uniformly integrable. An application of (b) confirms (c).

(c)⇒(a) Assume (c) to be true and let (Ω, Σ, μ) denote any finite (complete) measure space. First, assume that $F : \Sigma \to Y$ is a vector measure with the following property:

$$\exists\, K > 0 \text{ such that } \|F(E)\| \le K\mu(E) \text{ for all } E \in \Sigma. \tag{1.3}$$

Let $\mathcal{P} = \{\pi \subset \Sigma : \pi \text{ a finite partition of } \Omega\}$, directed by refinement. Define

$$f_\pi = \sum_{A\in\pi} \frac{F(A)}{\mu(A)}\chi_A$$

for each $\pi \in \mathcal{P}$, using the convention $0/0 = 0$. Denote by Σ_π the σ-algebra generated by π. It is readily verified that $(f_\pi, \Sigma_\pi)_{\pi\in\mathcal{P}} \subset L^1(\mu, Y)$ is a martingale. Also,

$$\|f_\pi\|_\infty = \sup_{A\in\pi} \frac{\|F(A)\|}{\mu(A)} \le K < \infty$$

for all $\pi \in \mathcal{P}$. Consequently, $\sup_{\pi\in\mathcal{P}} \|f_\pi\|_\infty < \infty$.

Suppose, for a moment, that (c) implies that $(f_\pi, \Sigma_\pi)_{\pi\in\mathcal{P}}$ is convergent in $L^1(\mu, Y)$; i.e. there exists $f \in L^1(\mu, Y)$ such that $\lim_{\pi\in\mathcal{P}} \|f_\pi - f\|_1 = 0$. Because

$$\mathcal{P}_E := \{\pi \in \mathcal{P} : \pi \ge \{E, \Omega \setminus E\}\}$$

is cofinal in $\mathcal{P}$ for each $E \in \Sigma$, we have

$$\int_E f\,\mathrm{d}\mu = \lim_{\pi\in\mathcal{P}} \int_E f_\pi\,\mathrm{d}\mu = \lim_{\pi\in\mathcal{P}_E} \int_E f_\pi\,\mathrm{d}\mu = \lim_{\pi\in\mathcal{P}_E} \sum_{A\in\pi, A\subset E} \frac{F(A)}{\mu(A)}\mu(A\cap E) = F(E)$$

for all $E \in \Sigma$, as required.

To show that (c) implies that $(f_\pi, \Sigma_\pi)_{\pi\in\mathcal{P}}$ is convergent, assume $(f_\pi, \Sigma_\pi)_{\pi\in\mathcal{P}}$ is not convergent. Then, there is $\varepsilon > 0$ such that, for each $\pi \in \mathcal{P}$, there exist $\pi \le \pi', \pi'' \in \mathcal{P}$ with $\|f_{\pi'} - f_{\pi''}\|_1 \ge 2\varepsilon$. Hence, either $\|f_\pi - f_{\pi'}\|_1 \ge \varepsilon$ or $\|f_\pi - f_{\pi''}\|_1 \ge \varepsilon$. Choose $\pi \le \gamma \in \{\pi', \pi''\}$ such that $\|f_\pi - f_\gamma\|_1 \ge \varepsilon$. Continuing this process, we can inductively extract a countable martingale $(f_{\pi_n}, \Sigma_{\pi_n})_{n\in\mathbb{N}} \subset (f_\pi, \Sigma_\pi)_{\pi\in\mathcal{P}}$ that contradicts (c).

To prove (a), let $G : \Sigma \to Y$ be a μ-continuous measure of bounded variation. It follows from the completeness of (Ω, Σ, μ) that $|G|$ is also μ-continuous. By the classical Radon-Nikodým Theorem, there exists $0 \le \varphi \in L^1(\mu)$ such that

$$|G|(E) = \int_E \varphi \, \mathrm{d}\mu \tag{1.4}$$

for all $E \in \Sigma$. Observe $\|G(E)\| \le |G|(E)$ for all $E \in \Sigma$ so that G satisfies property (1.3) with respect to the finite measure space $(\Omega, \Sigma, |G|)$. By the above, there exists $g \in L^1(|G|, Y)$ such that

$$G(E) = \int_E g \, \mathrm{d}|G| \tag{1.5}$$

for all $E \in \Sigma$. Combining (1.4) and (1.5), we obtain $G(E) = \int_E g\varphi \, \mathrm{d}\mu$ for all $E \in \Sigma$, with $g\varphi \in L^1(\mu, Y)$.

(d)⇒(c) Let $(f_n) \subset L^1(\mu, Y)$ be a martingale with $\sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$. Then $\sup_{n\in\mathbb{N}} \|f_n\|_p \le \sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$. By (d), (f_n) is convergent in $L^p(\mu, Y)$, which implies convergence in $L^1(\mu, Y)$.

(b)⇒(d) Let $(f_n, \Sigma_n) \subset L^p(\mu, Y)$ be a martingale with $\sup_{n\in\mathbb{N}} \|f_n\|_p < \infty$. By Hölders inequality, (f_n) is uniformly integrable with $\sup_{n\in\mathbb{N}} \|f_n\|_1 < \infty$. Consequently, (f_n) converges to a function $f \in L^1(\mu, Y)$. Thus, by Lemma 1.3.9,

$$F(E) = \lim_{n\to\infty} \int_E f_n \, \mathrm{d}\mu = \int_E f \, \mathrm{d}\mu$$

for all $E \in \bigcup_{n=1}^\infty \Sigma_n$. Since (f_n) is a martingale, (f_n) also converges to f μ-almost everywhere. By Fatou's Lemma, we have

$$\int_\Omega \|f\|^p \, \mathrm{d}\mu \le \lim_{n\to\infty} \int_\Omega \|f_n\|^p \, \mathrm{d}\mu \le \sup_{n\in\mathbb{N}} \|f_n\|_p^p < \infty,$$

whence $f \in L^p(\mu, Y)$. Another application of Lemma 1.3.9 proves (d). □

1.4 Unconditional convergence and the UMD property

Before we delve into the intricacies of Banach spaces that are unconditional for martingale difference sequences (UMD), we review some results on unconditional convergence.

Definition 1.4.1 Let X be a normed space and $(x_n) \subset X$ be a sequence.

(a) The series $\sum_{n=1}^\infty x_n$ is said to *converge to* $x \in X$ if $\lim_{n\to\infty} \|\sum_{i=1}^n x_i - x\| = 0$. In this case, we say that (x_i) is *summable*.

(b) The series $\sum_{n=1}^\infty x_n$ is said to be *unconditionally convergent* if, for every permutation π of $\mathbb{N}$, the series $\sum_{n=1}^\infty x_{\pi(n)}$ is convergent. In this case, the sequence (x_n) is said to be *unconditionally summable.*

(c) The series $\sum_{n=1}^{\infty} x_n$ is said to be *unordered summable* to $x \in X$ if, for every $\varepsilon > 0$, there exists a finite set $N_0 \subset \mathbb{N}$ so that for every finite set N with $N_0 \subset N \subset \mathbb{N}$, we have $\| \sum_{n \in N} x_n - x\| < \varepsilon$.

(d) The series $\sum_{n=1}^{\infty} x_n$ is said to be *absolutely convergent* if the series $\sum_{n=1}^{\infty} \|x_n\|$ is convergent. In this case, the sequence (x_n) is said to be *absolutely* summable.

In finite dimensional normed spaces, a series is absolutely summable if and only if it is unconditionally summable. It is well known that a normed space is a Banach space if and only if every absolutely summable sequence is unconditionally summable (cf. [36, Proposition 1.1]). However, as a consequence of the Dvoretzky-Rogers Theorem (cf. [36, Theorem 1.2]), every infinite dimensional Banach space contains a sequence that is unconditionally summable but not absolutely summable. Unconditional summability has many useful equivalent forms, which are collected in [36, Theorem 1.9]. We state this theorem, which is referred to often throughout the text.

Theorem 1.4.2 *For a sequence (x_n) in a Banach space Y, the following statements are equivalent:*

(a) (x_n) is unconditionally summable.

(b) (x_n) is unordered summable.

(c) (x_n) is subseries summable, that is, for every strictly increasing sequence (k_n) of natural numbers, the series $\sum_{n=1}^{\infty} x_{k_n}$ is convergent.

(d) (x_n) is sign summable, that is, $\sum_{n=1}^{\infty} \theta_n x_n$ converges for every choice of signs $\theta_n = \pm 1$.

(e) $(\lambda_n x_n)$ is summable for every $(\lambda_n) \in \ell_\infty$.

(f) (x_n) is weakly subseries summable.

(g) (x_n) is weakly sign summable.

(h) $(\lambda_n x_n)$ is weakly summable for every $(\lambda_n) \in \ell_\infty$.

(i) The operator $T : Y^* \to \ell_1$, defined by $Tx^* = (\langle x_n, x^*\rangle)_{n=1}^{\infty}$, is compact.

(j) $(b_n) \mapsto \sum_{n=1}^{\infty} b_n x_n$ defines a compact operator from ℓ_∞ into Y.

(k) $(b_n) \mapsto \sum_{n=1}^{\infty} b_n x_n$ defines a compact operator from c_0 into Y.

(l) $(b_n) \mapsto \sum_{n=1}^{\infty} b_n x_n$ defines a bounded operator from ℓ_∞ into Y.

It ought to be mentioned that the equivalence of weak subseries summability and strong subseries summability is a famous result due to Orlicz and Pettis (cf. [36, Theorem 1.8]). This elegant result allows access to the rest of the weak characterizations of unconditional summability.

Given a Banach space Y, define the vector space of unconditionally summable sequences in Y, denoted by $\ell_1[Y]$. The above theorem suggests a norm for $\ell_1[Y]$, namely:

$$\|(x_n)\|_\varepsilon := \sup\left\{\sum_{n=1}^{\infty}|\langle x_n, x^*\rangle| : x^* \in Y^*, \|x^*\| \leq 1\right\} \tag{1.6}$$

$$= \sup\left\{\left\|\sum_{n=1}^{\infty} b_n x_n\right\| : \|(b_n)\|_\infty \leq 1\right\}. \tag{1.7}$$

Formula (1.6) is the norm of the operator T defined in Theorem 1.4.2(i) and formula (1.7) is the norm of the adjoint of T; i.e. the operator defined in Theorem 1.4.2(l). It is a simple exercise to show that $\ell_1[Y]$ is a Banach space using the second of these formulae. Moreover, using the compactness of the operator T, it can be shown that $\ell_1[Y]$ is isometric to $\ell_1\widetilde{\otimes}_\varepsilon Y$ (cf. [90, Example 3.4]), hence our choice of the notation $\|\cdot\|_\varepsilon$.

We focus our attention on a special kind of sequence, namely:

Definition 1.4.3 Let Y denote a Banach space.

(a) A sequence (y_i) in a Banach space Y is called a *basis* if each element $y \in Y$ has a unique series expansion $y = \sum_{i=1}^{\infty} \alpha_i y_i$, where (α_i) is a sequence of scalars.

(b) A sequence $(u_j) \subset Y$ is called a *block basis of the basis* (y_i) if $u_j = \sum_{i=n_j+1}^{n_{j+1}} \alpha_i y_i$ for each $j \in \mathbb{N}$, with (α_i) scalars and $n_1 < n_2 < \ldots$ an increasing sequence of natural numbers.

(c) A basis $(y_i) \subset Y$ is called *unconditional* if the unique series expansion $\sum_{i=1}^{\infty} \alpha_i y_i$ of each element in Y converges unconditionally.

(d) A sequence $(y_i) \subset Y$ is called a *basic sequence* if (y_i) is a basis of its closed linear span, which we denote by $[y_i]$.

(e) If (y_i) is an unconditional basis of $[y_i]$, then (y_i) is referred to as an *unconditional basic sequence*.

(f) Two basic sequences (x_i) and (y_i) are called *equivalent* if, given a sequence of scalars (α_i), we have $\sum_{i=1}^{\infty} \alpha_i x_i$ convergent if and only if $\sum_{i=1}^{\infty} \alpha_i y_i$ convergent. Equivalently, (x_i) and (y_i) are equivalent if there exists an isomorphism T from $[x_i]$ onto $[y_i]$ such that $Tx_i = y_i$ for each $i \in \mathbb{N}$.

If Y is a Banach space with basis (y_i), define the norm $\|\cdot\|_0$ on Y by $\|x\|_0 = \sup_{n\in\mathbb{N}} \|\sum_{i=1}^{n} \alpha_i y_i\|$ for all $x = \sum_{i=1}^{\infty} \alpha_i y_i \in Y$. It is clear that $\|x\| \leq \|x\|_0$ for each $x \in Y$, and an easy argument shows that $(Y, \|\cdot\|_0)$ is complete. The Open Mapping Theorem implies that the norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent. Define the *natural projections* associated with (y_i) as the sequence of projections (P_i) on Y given by

$P_n\left(\sum_{i=1}^{\infty}\alpha_i y_i\right)=\sum_{i=1}^{n}\alpha_i y_i$ for each $\sum_{i=1}^{\infty}\alpha_i y_i\in Y$ and $n\in\mathbb{N}$. It follows, by the above remarks and the Principle of Uniform Boundedness, that (P_i) is a uniformly bounded collection of linear projections. The quantity $K=\sup_{i\in\mathbb{N}}\|P_i\|<\infty$ is known as the *basis constant*. Thus, a basis can be characterized as follows (cf. [71, Proposition 1.a.3.]):

Proposition 1.4.4 *Let (y_i) be a sequence in a Banach space Y. Then (y_i) is a basis of Y if and only if the following three conditions hold:*

(a) $y_i\neq 0$ for all $i\in\mathbb{N}$.

(b) There is a constant K so that, for every choice of scalars (α_i) and positive integers $n<m$, we have

$$\left\|\sum_{i=1}^{n}\alpha_i y_i\right\|\leq K\left\|\sum_{i=1}^{m}\alpha_i y_i\right\|.$$

(c) The closed linear span of (y_i) is all of Y.

Parts (a) and (b) of the above proposition characterize basic sequences. If $K=1$ for the basis (y_i) in the above proposition, then (y_i) is referred to as a *monotone basis.* If $\|y_i\|=1$ for each $i\in\mathbb{N}$, then (y_i) is said to be *normalized.*

The unit vectors $e_i=(\delta_{ik})_{k=1}^{\infty}$ $(i=1,2,\ldots)$ form a monotone and normalized basis in each of the spaces c_0 and ℓ_p, $1\leq p<\infty$.

A more interesting example of a monotone basis is the *Haar system* in $L^p(\mu)$, $1\leq p<\infty$. Here, μ is the Lebesgue measure on the unit interval. The Haar system is defined by $\chi_1=\mathbf{1}$ and

$$\chi_{2^k+l}(\omega)=\begin{cases}1, & \text{if } \omega\in[(2l-2)2^{-(k+1)},(2l-1)2^{-(k+1)}];\\ -1, & \text{if } \omega\in((2l-1)2^{-(k+1)},(2l)2^{-(k+1)}];\\ 0, & \text{otherwise}\end{cases}$$

for $k=0,1,2,\ldots$, $l=1,2,\ldots,2^k$. We will write (χ_i) to denote the Haar system in its given order. Let Σ_n be the σ-algebra generated by $\sum_{i=1}^{n}\chi_i$ for each $n\in\mathbb{N}$. Then, it is readily verified that $(\sum_{i=1}^{n}\alpha_i\chi_i,\Sigma_n)_{n\in\mathbb{N}}$ is a martingale for every choice of scalars (α_i). Consequently, for $n\leq m$, we have $\|\sum_{i=1}^{n}\alpha_i\chi_i\|=\|\mathbb{E}(\sum_{i=1}^{m}\alpha_i\chi_i\,|\,\Sigma_n)\|\leq\|\sum_{i=1}^{m}\alpha_i\chi_i\|$. Lastly, since $[\chi_i]$ contains all the diadic step functions, Proposition 1.4.4 shows that (χ_i) is a basis of $L^p(\mu)$ for $1\leq p<\infty$. It is easy to see that the Haar system is a sequence of symmetrically distributed, independent functions. Consequently, the Haar system is a martingale difference sequence. A basis of $L^p(\mu)$ that is also a martingale difference sequence will be referred to as a *m.d.s. basis.*

Definition 1.4.5 Let (Ω, Σ, μ) be the unit interval equipped with the Lebesgue measure. The sequence of *Rademacher functions* (r_i) is defined to be the block basic sequence of the Haar system, given by

$$r_1(\omega) = \chi_1(\omega) \quad \text{ and } \quad r_i(\omega) = \sum_{k=2^{i-2}+1}^{2^{i-1}} \chi_k(\omega) = \operatorname{sign}(\sin(2^i \pi \omega)),$$

for all $\omega \in \Omega$ and $i \geq 2$. We mention here that whenever the Rademacher functions are used, it will be implicitly assumed that the underlying measure space is the Lebesgue interval.

The Rademacher sequence is also symmetrically distributed and independent. Another useful feature of the Rademacher functions is orthogonality; if $0 < n_1 < n_2 < \ldots < n_k$ and $p_1, \ldots, p_k \geq 0$ are natural numbers, then it is easily verified that

$$\int_0^1 r_{n_1}^{p_1} \cdot r_{n_2}^{p_2} \cdot \ldots \cdot r_{n_k}^{p_k} \, \mathrm{d}t = \begin{cases} 1, & \text{if each } p_j \text{ is even;} \\ 0, & \text{otherwise.} \end{cases}$$

This feature is used in the proof of the classical inequality of Khinchin (cf. [36, Theorem 1.10]):

Theorem 1.4.6 (Khinchin's Inequality) *For any $0 < p < \infty$, there are positive constants A_p and B_p such that*

$$A_p \left(\sum_{i=1}^n |\alpha_i|^2 \right)^{1/2} \leq \left(\int_0^1 \left| \sum_{i=1}^n \alpha_i r_i(t) \right|^p \mathrm{d}\mu \right)^{1/p} \leq B_p \left(\sum_{i=1}^n |\alpha_i|^2 \right)^{1/2}$$

for every choice of scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$.

This inequality has many far reaching consequences. An obvious consequence is that the Rademacher functions are equivalent to the unit vector basis in ℓ^2 for all $0 \leq p < \infty$. Hence, (r_i) is an unconditional basic sequence in $L^p(\mu)$ for $1 \leq p < \infty$, but not an unconditional basis. It can also be shown that $[r_i]$ is complemented in $L^p(\mu)$ for $1 < p < \infty$ and not complemented when $p = 1$. A generalization of the Khinchin's inequality is the classical result of Kahane (cf. [36, Theorem 11.1]):

Theorem 1.4.7 (Kahane's Inequality) *If $0 < p, q < \infty$ and Y is a Banach space, then there is a constant $K_{p,q} > 0$ for which*

$$\left\| \sum_{i=1}^n r_i \otimes y_i \right\|_{L^q(\mu, Y)} \leq K_{p,q} \left\| \sum_{i=1}^n r_i \otimes y_i \right\|_{L^p(\mu, Y)}$$

holds for every choice of vectors $y_1, y_2, \ldots, y_n \in Y$.

Unfortunately, a Khinchin style inequality of the form

$$A_p \left(\sum_{i=1}^{n} \|y_i\|^2 \right)^{1/2} \leq \left\| \sum_{i=1}^{n} r_i \otimes y_i \right\|_{L^p(\mu,Y)} \leq B_p \left(\sum_{i=1}^{n} \|y_i\|^2 \right)^{1/2}$$

does not hold in general; e.g. when $Y = c_0$ or $Y = \ell^1$. The notions of type and cotype are crafted to determine the deviation of a Banach space from this inequality.

Definition 1.4.8 Let Y be a Banach space.

(a) Y is said to have *type p* if there is a constant $M \geq 0$ such that

$$\left\| \sum_{i=1}^{n} r_i \otimes y_i \right\|_{L^2(\mu,Y)} \leq M \left\| \sum_{i=1}^{n} e_i \otimes y_i \right\|_{\ell^p(Y)}$$

holds for every choice of vectors $y_1, y_2, \ldots, y_n \in Y$. The smallest constant for which the above inequality holds is called *type p constant* of Y and is denoted $T_p(Y)$.

(b) Y is said to have *cotype q* if there is a constant $M \geq 0$ such that

$$\left\| \sum_{i=1}^{n} e_i \otimes y_i \right\|_{\ell^q(Y)} \leq M \left\| \sum_{i=1}^{n} r_i \otimes y_i \right\|_{L^2(\mu,Y)}$$

holds for every choice of vectors $y_1, y_2, \ldots, y_n \in Y$. The smallest constant for which the above inequality holds is called the *cotype q constant* of Y and is denoted $C_q(Y)$.

In the above definition, Kahane's inequality tells us that we may replace $\|\cdot\|_{L^2(\mu,Y)}$ with $\|\cdot\|_{L^r(\mu,Y)}$, $0 < r < \infty$, provided there is a suitable adjustment of constants. Every Banach space has type ≤ 1 and cotype ∞. It is not possible to have a non-trivial Banach space with type > 2 or cotype < 2 (cf. [72, 36]). Hilbert spaces have both type and cotype equal to 2 (cf. [36, Corollary 11.8]). A Banach space has the same type or cotype as its bidual (cf. [36, Corollary 11.9]). If a Banach Y space has type p, then its dual has cotype q and $C_q(Y^*) \leq T_p(Y)$ where $\frac{1}{p} + \frac{1}{q} = 1$ (cf. [36, Proposition 11.10]). Thus, a Banach space must have finite cotype if its dual has non trivial type. The converse fails when considering ℓ^1, which has cotype 2, and its dual ℓ^∞, which only has trivial type.

We pass to unconditional bases. Suppose (y_i) is an unconditional basis of a Banach space Y. By the Closed Graph Theorem, for any $\sigma \subset \mathbb{N}$, the map $P_\sigma : Y \to Y$ defined by $P_\sigma \left(\sum_{i=1}^{\infty} \alpha_i y_i \right) = \sum_{i \in \sigma} \alpha_i y_i$ is a bounded linear projection. Similarly, for every choice of signs $\theta = (\theta_i)$, we have a bounded linear operator $M_\theta : Y \to Y$ given by $M_\theta\left(\sum_{i=1}^\infty \alpha_i y_i\right) = \sum_{i=1}^\infty \theta_i \alpha_i y_i$. Moreover, by the Principle of Uniform Boundedness, we have that $\sup_\sigma \|P_\sigma\|$ and $\sup_\theta \|M_\theta\|$ are finite and these quantities are related by the inequality

$$\sup_\sigma \|P_\sigma\| \leq \sup_\theta \|M_\theta\| \leq 2 \sup_\sigma \|P_\sigma\|.$$

The quantity $\sup_\theta \|M_\theta\|$ is known as the *unconditional constant* of the unconditional basis (y_i) and is always larger than or equal to the basis constant. Consequently, a basis (y_i) is unconditional if and only if there is a constant $M > 0$ such that

$$\left\| \sum_{i=1}^n \theta_i \alpha_i y_i \right\| \leq M \left\| \sum_{i=1}^n \alpha_i y_i \right\|$$

holds for every choice of scalars (α_i), every choice of signs (θ_i) and $n \in \mathbb{N}$.

An easy, but useful, consequence of the Hahn Banach Theorem is the following result concerning unconditional constants (cf. [71, Proposition 1.c.7]).

Proposition 1.4.9 *Let (y_i) be an unconditional basic sequence with unconditional constant M. Then for every choice of scalars (α_i) such that $\sum_{i=1}^n \alpha_i y_i$ is convergent and every sequence $(\lambda_i) \in \ell^\infty$, we have*

$$\left\| \sum_{i=1}^\infty \lambda_i \alpha_i y_i \right\| \leq 2M \|(\lambda_i)\|_\infty \left\| \sum_{i=1}^\infty \alpha_i y_i \right\|.$$

(In the real case we may take M instead of $2M$).

The Haar system is an unconditional basis of $L^p(\mu)$ for $1 < p < \infty$ (cf. [72, Theorem 2.c.5] and [13, Theorem 9]). The unconditionality of the Haar system was first proved by Paley in 1931 (cf. [80]). The space $L^1(\mu)$ is not isomorphic to a subspace of a space with unconditional basis (cf. [71, Proposition 1.d.1]). Consequently, the Haar system is not an unconditional basis of $L^1(\mu)$.

Later, in 1966, Burkholder and Gundy discovered that every martingale difference sequence in $L^p(\mu)$, $1 < p < \infty$, is unconditional (cf. [13]). Dor and Odell [42] and, independently, Pełczyński and Rosenthal [81] proved that every monotone basis of $L^p(\mu)$ is unconditional. Their proofs use a characterization by Andô; every contractive projection on $L^p(\mu)$ $(1 \leq p < \infty)$ that preserves the constant functions is, in fact, a conditional expectation (cf. [5] and [87]).

Maurey [75] showed the unconditional constant of the Haar system is bigger than the unconditional constant of any other martingale difference sequence. On the other hand, Olevskiĭ [77, 78] showed the unconditional constant of the Haar system is smaller than the unconditional constant of any unconditional basis of

$L^p(\mu)$. Observing this, Burkholder proved the unconditional constant of the Haar system in $L^p(\mu)$ is $p \vee (\frac{p}{p-1}) - 1$ where $1 < p < \infty$ (cf. [15, 17, 18]). Moreover, if (P_n) is a sequence of contractive projections on $L^p(\mu)$ with $P_0 = 0$ and $P_i P_j = P_j P_i = P_j$ for all $j \le i$, then the inequality

$$\left\| \sum_{n=1}^{\infty} \alpha_n (P_n - P_{n-1}) f \right\|_p \le \left[p \vee \left(\frac{p}{p-1} \right) - 1 \right] \|f\|_p$$

holds for all $1 < p < \infty$, $f \in L^p(\mu)$ and $|\alpha_n| \le 1$ (cf. [42, 16]).

In view of the fact that $L^p(\mu)$ always possesses an unconditional basis when $1 < p < \infty$, a natural question was raised in [37, p. 116]: Does $L^p(\mu, Y)$ possess an unconditional basis if $1 < p < \infty$ and Y is a Banach space with unconditional basis? This question remained open for a number of years before being answered negatively in 1978 by Aldous in [1]. He showed, using a delicate probabilistic argument, that if $L^p(\mu, Y)$ has an unconditional basis, then Y is necessarily a UMD space. Since all UMD spaces are reflexive, it follows that $L^p(\mu, Y)$ cannot have an unconditional basis if $Y = \ell^1$ or $Y = c_0$, even though the unit vectors form an unconditional basis of Y. Herewith, the definition of a UMD space:

Definition 1.4.10 A Banach space Y is said to be *unconditional for martingale difference sequences (UMD)* if $U_p(Y)$ is finite for all $1 < p < \infty$. Here, $U_p(Y)$ denotes the least $1 \le M \le \infty$ such that

$$\left\| \sum_{i=1}^{n} \theta_i d_i \right\|_p \le M \left\| \sum_{i=i}^{n} d_i \right\|_p$$

holds for every martingale difference sequence $(d_i) \subset L^p(\mu, Y)$, every choice of signs (θ_i) and every $n \in \mathbb{N}$. In this definition, the measure space must be purely non-atomic.

Thus, if $\sum_{i=1}^{\infty} \alpha_i d_i$ converges in $L^p(\mu, Y)$ where $1 < p < \infty$ and Y is UMD, then $\sum_{i=1}^{\infty} \alpha_i d_i$ converges unconditionally.

Pisier showed that, for a Banach space Y, the finiteness of $U_p(Y)$ for all $1 < p < \infty$ is equivalent to the finiteness of $U_p(Y)$ for some $1 < p < \infty$ (cf [75]). This fact also follows naturally from an intrinsic geometric characterization of UMD spaces, proved by Burkholder in [14];

Theorem 1.4.11 (Burkholder) *A Banach space is UMD if and only if it is ζ-convex. Here, a Banach space Y is said to be ζ-convex if there exists a biconvex function $\zeta : Y \times Y \to \mathbb{R}$ such that $\zeta(0,0) > 0$ and $\zeta(x,y) \le \|x+y\|$ if $\|x\| = \|y\| = 1$.*

It was noted by Maurey in [75], and proved by Aldous in [1, Proposition 2], that all UMD spaces are necessarily super-reflexive (also see [9]). However, there are super-reflexive spaces that are not UMD (cf. [9, 83]).

The class of UMD spaces appears to be quite small, with most of the examples arising from classical analysis. Verifying even the simplest example of a UMD space, namely $\mathbb{R}$, involves the deep inequalities of Burkholder. It also follows from Burkholder's geometrical characterization that every Hilbert space is a UMD space. Indeed, the required biconvex function is given by $\zeta(\cdot,\cdot) = 1 + (\cdot,\cdot)$ where $(\cdot,\cdot)$ denotes the inner product. Other examples of UMD spaces include $L^p(\mu)$ and ℓ^p for $1 < p < \infty$. It is immediate that if Y is a UMD space then $\ell^p(Y)$ and $L^p(\mu, Y)$ are also UMD spaces, for $1 < p < \infty$. The Schatten p-classes for $1 < p < \infty$ are another example.

The property of UMD is invariant under isomorphism. Moreover, UMD spaces have non-trivial type and finite cotype. The dual of a UMD space is also a UMD space. We may also consider a smaller class of martingale difference sequences when testing for UMD spaces (cf. [88]).

Proposition 1.4.12 *Let $1 < p < \infty$. For a Banach space Y, the following are equivalent:*

(a) Y is a UMD space.

(b) There exists a constant M such that, for any choice of signs (θ_i), we have
$$\left\| \theta_1 \mathbb{E}(f \,|\, \mathcal{W}_1) + \sum_{i=2}^{n} \theta_i \left(\mathbb{E}(f \,|\, \mathcal{W}_i) - \mathbb{E}(f \,|\, \mathcal{W}_{i-1}) \right) \right\|_p \leq M \left\| \mathbb{E}(f \,|\, \mathcal{W}_1) + \sum_{i=2}^{n} \left(\mathbb{E}(f \,|\, \mathcal{W}_i) - \mathbb{E}(f \,|\, \mathcal{W}_{i-1}) \right) \right\|_p$$
for all $f \in L^p(\mu, Y)$ and $n \in \mathbb{N}$. Here, $\mathcal{W}_i$ denotes the σ-algebra generated by the i-th Rademacher function.

Martingales adapted to the filtration $(\mathcal{W}_n)$ in the above proposition are referred to as *Walsh-Paley martingales*. If $(f_n, \mathcal{W}_n)$ is a Walsh-Paley martingale, then $f_n = \sum_{i=1}^{2^n-1} \chi_i \otimes y_i$, where $y_1, y_2, \ldots, y_{2^n-1} \in Y$ and $n = 1, 2, \ldots$.

For more background and elementary properties of UMD spaces, we refer the reader to [88, 18].

1.5 Schauder decompositions and R-boundedness

We pass to a more general notion than that of a basis.

Definition 1.5.1 Let Y be a Banach space. A sequence (D_i) of bounded linear projections on Y is called a *Schauder decomposition* of Y if

(a) $D_iD_j = 0$ whenever $i \neq j$ and
(b) $y = \sum_{i=1}^{\infty} D_i y$ for all $y \in Y$.

The corresponding *partial sum projections* (P_n) are defined by $P_n = \sum_{i=1}^{n} D_i$. Note that $\bigcup_{i=1}^{\infty} \mathcal{R}(P_i)$ is dense in Y. In an analogous fashion to the natural projections associated with a basis, it follows from the Principle of Uniform Boundedness that (P_n) is uniformly bounded. Hence, (D_i) is also uniformly bounded.

Notice that the sequence $(D_i y)$ is a basic sequence in Y for each $y \in Y$. Indeed, for $n < m$ and any scalars (α_i) we have

$$\left\| \sum_{i=1}^{n} \alpha_i D_i y \right\| = \left\| P_n \left(\sum_{i=1}^{m} \alpha_i D_i y \right) \right\| \leq \sup_{n \in \mathbb{N}} \|P_n\| \left\| \sum_{i=1}^{m} \alpha_i D_i y \right\| .$$

Now use Proposition 1.4.4.

If $\dim(D_i) = 1$ for each $i \in \mathbb{N}$, any sequence (y_i) with $y_i \in \mathcal{R}(D_i)$ for each $i \in \mathbb{N}$ forms a basis for Y. In this case, each $y \in Y$ has a unique basis expansion $y = \sum_{i=1}^{\infty} \alpha_i y_i$ where the α_i's are scalars.

Definition 1.5.2 let Y be a Banach space and (D_i) be a Schauder decomposition of Y. The decomposition (D_i) will be called *unconditional* if there exists a constant M for which

$$\left\| \sum_{i=1}^{n} \theta_i D_i y \right\| \leq M \left\| \sum_{i=1}^{n} D_i y \right\|$$

holds for all choices of signs (θ_i), all $n \in \mathbb{N}$ and all $y \in Y$. The smallest M for which this inequality holds is called the *unconditional constant* of (D_i).

As a consequence of the Principle of Uniform Boundedness, the above definition is equivalent to the unconditional convergence of $\sum_{i=1}^{\infty} D_i y$ for every $y \in Y$ (cf. [103, Lemma 1.2.5]).

An important class of a Schauder decompositions arise from filtrations. If (Ω, Σ, μ) is a finite measure space, Y a Banach space and (Σ_i) a filtration with $\Sigma_i \uparrow \Sigma$, then the Schauder decomposition of $L^p(\mu, Y)$, $1 \leq p < \infty$, formed by the differences $(\mathbb{E}(\,\cdot \mid \Sigma_i) - \mathbb{E}(\,\cdot \mid \Sigma_{i-1}))$ is called a *martingale decomposition.* Note that if $1 < p < \infty$ and Y is a UMD space, then all martingale decompositions are unconditional. Conversely, by Proposition 1.4.12, if the filtration generated by the Rademacher functions determines an unconditional Schauder decomposition of $L^p(\mu, Y)$, $1 < p < \infty$, then Y is a UMD space.

Using this observation, we can derive a characterization of UMD spaces that is similar to Theorem 1.3.10. The following definition will be useful in the formulation of this result.

Definition 1.5.3 Let $1 \leq p < \infty$ and Y be a Banach space. If $(f_n) \subset L^p(\mu, Y)$ is a martingale, we define the ± 1-*transform* of (f_n) to be the martingale (g_n) defined by $g_n = \sum_{i=1}^n \theta_i (f_i - f_{i-1})$ for each $n \in \mathbb{N}$, where (θ_i) denotes a choice of signs. Here, we observe the convention $f_0 = 0$.

Theorem 1.5.4 *Let $1 < p < \infty$ and Y be a Banach space. Then the following statements are equivalent:*

(a) Y is a UMD space.

(b) Every ± 1 transform of every martingale $(f_n) \subset L^p(\mu, Y)$ with $\sup_{n\in\mathbb{N}} \|f_n\|_p < \infty$ converges in $L^p(\mu, Y)$.

Proof. (a)$\Rightarrow$(b) Suppose Y is a UMD space and $(f_n) \subset L^p(\mu, Y)$ is a martingale with $\sup_{n\in\mathbb{N}} \|f_n\|_p < \infty$. Define $d_i = f_i - f_{i-1}$ for each $i \in \mathbb{N}$ and $g_n = \sum_{i=1}^n \theta_i d_i$ for each $n \in \mathbb{N}$, where (θ_i) is any choice of signs. By the definition of a UMD space, we have

$$\sup_{n\in\mathbb{N}} \|g_n\|_p = \sup_{n\in\mathbb{N}} \left\| \sum_{i=1}^n \theta_i d_i \right\|_p \leq U_p(Y) \sup_{n\in\mathbb{N}} \left\| \sum_{i=1}^n d_i \right\| = U_p(Y) \sup_{n\in\mathbb{N}} \|f_n\|_p < \infty.$$

Since Y is a UMD space, Y is reflexive. Thus, Y also has the Radon Nikodým property. By Theorem 1.3.10, (g_n) converges in $L^p(\mu, Y)$.

(b)$\Rightarrow$(a) Let $f \in L^p(\mu, Y)$ and define $d_1 = \mathbb{E}(f \mid \Sigma_1)$ and $d_i = \mathbb{E}(f \mid \Sigma_i) - \mathbb{E}(f \mid \Sigma_{i-1})$ for $i \geq 2$, where (Σ_i) denotes a filtration with $\Sigma_i \uparrow \Sigma$. Consider the martingale (f_n) defined by $f_n = \sum_{i=1}^n d_i$ for each $n \in \mathbb{N}$. Then $\sup_{n\in\mathbb{N}} \|f_n\|_p \leq \|f\|_p < \infty$. By (b), every ± 1 transform of (f_n) is convergent in $L^p(\mu, Y)$. It follows that (Σ_i) determines an unconditional Schauder decomposition of $L^p(\mu, Y)$. Consequently, Y is a UMD space by Proposition 1.4.12. $\square$

The Rademacher functions play a role in determining whether a Schauder decomposition is unconditional. The results that follow are taken from the thesis of Witvliet (cf. [103]).

Proposition 1.5.5 *Let (D_i) be a Schauder decomposition of the Banach space Y. The following statements are equivalent:*

(a) The decomposition (D_i) of Y is unconditional.

(b) There exists a $1 \le p < \infty$ *such that*

$$M_p^{-1}\left\|\sum_{i=1}^n D_i y\right\| \le \left\|\sum_{i=1}^n r_i \otimes D_i y\right\|_{\Delta_p} \le M_p \left\|\sum_{i=1}^n D_i y\right\|$$

holds for some constant $M_p > 0$*, for all* $y \in Y$ *and* $n \in \mathbb{N}$*.*

(c) For all $1 \le p < \infty$ *there exists* $M_p > 0$ *such that*

$$M_p^{-1}\left\|\sum_{i=1}^n D_i y\right\| \le \left\|\sum_{i=1}^n r_i \otimes D_i y\right\|_{\Delta_p} \le M_p \left\|\sum_{i=1}^n D_i y\right\|$$

holds, for all $y \in Y$ *and* $n \in \mathbb{N}$*.*

Proof. (c)⇒(b) Obvious.

(a)⇒(c) First note that if the decomposition (D_i) is unconditional, then there exists a constant $M > 0$ so that

$$M^{-1}\left\|\sum_{i=1}^n D_i y\right\| \le \left\|\sum_{i=1}^n \theta_i D_i y\right\| \le M \left\|\sum_{i=1}^n D_i y\right\|$$

holds for any choice of signs (θ_i) and $y \in Y$. The right inequality holds by definition. For the left inequality, let $x = \sum_{i=1}^n \theta_i D_i y$ for some $y \in Y$. By definition, $\|\sum_{i=1}^n \theta_i D_i x\| = \|\sum_{i=1}^n \theta_i^2 D_i y\| \le M \|\sum_{i=1}^n D_i x\| = M \|\sum_{i=1}^n \theta_i D_i y\|$. Hence, if (Ω, Σ, μ) is the Lebesgue interval and (r_i) the sequence of Rademacher functions, then

$$M^{-1}\left\|\sum_{i=1}^n D_i y\right\| \le \left\|\sum_{i=1}^n r_i(\omega) D_i y\right\| \le M \left\|\sum_{i=1}^n D_i y\right\|$$

holds for each $\omega \in \Omega$. Consequently,

$$M_p^{-1}\left\|\sum_{i=1}^n D_i y\right\| \le \left\|\sum_{i=1}^n r_i \otimes D_i y\right\|_{\Delta_p} \le M_p \left\|\sum_{i=1}^n D_i y\right\|$$

holds for all $1 \le p < \infty$.

(b)⇒(a) Let (θ_i) be any choice of signs, then r_i and $\theta_i r_i$ have the same distribution. Consequently,

$$\left\|\sum_{i=1}^n \theta_i D_i y\right\| \le M_p \left\|\sum_{i=1}^n \theta_i r_i \otimes D_i y\right\|_{\Delta_p} = M_p \left\|\sum_{i=1}^n r_i \otimes D_i y\right\|_{\Delta_p} \le M_p^2 \left\|\sum_{i=1}^n D_i y\right\|.$$

□

If (D_i) is an unconditional Schauder decomposition of a Banach space Y, Theorem 1.4.2 implies that any sequence $\lambda = (\lambda_i) \in \ell^\infty$ induces a bounded operator $T_\lambda : Y \to Y$ defined by $T_\lambda y = \sum_{i=1}^\infty \lambda_i D_i y$ for each $y \in Y$. Conversely, if $\lambda = (\lambda_i)$ is a sequence such that T_λ is bounded, then $\lambda \in \ell^\infty$.

More generally, one could consider a sequence of operators $L = (L_i) \subset \mathcal{L}(Y)$ with the property $L_i(\mathcal{R}(D_i)) \subset \mathcal{R}(D_i)$ for each $i \in \mathbb{N}$, or equivalently, $D_i L_i D_i = L_i D_i$ for each $i \in \mathbb{N}$. For such a sequence, consider the induced map $T_L : Y \to Y$ defined by

$$T_L y = \sum_{i=1}^\infty L_i D_i y \tag{1.8}$$

for each $y \in Y$. When is this map bounded? The answer to this question leads to the definition of R-boundedness.

Indeed, if $k \in \mathbb{N}$, then $D_k T_L = \sum_{i=1}^\infty D_k D_i L_i D_i = D_k L_k D_k = L_k D_k$ on the dense set $Y_0 := \bigcup_{i=1}^\infty \mathcal{R}(P_i)$, where (P_i) are the corresponding partial sum projections of (D_i). Proposition 1.5.5 implies

$$M_2^{-1}\|y\| \leq \left\| \sum_{i=1}^\infty r_i \otimes D_i y \right\|_{\Delta_2} \leq M_2 \|y\|$$

for all $y \in Y_0$. Since $T_L(Y_0) \subset Y_0$, we obtain the inequality

$$M_2^{-1}\|T_L y\| \leq \left\| \sum_{k=1}^\infty r_k \otimes L_k D_k y \right\|_{\Delta_2} \leq M_2 \|T_L y\|$$

from above. It follows that $\|T_L\| < \infty$ if and only if there exists a constant $M > 0$ such that

$$\left\| \sum_{i=1}^\infty r_i \otimes L_i y_i \right\|_{\Delta_2} \leq M \left\| \sum_{i=1}^\infty r_i \otimes y_i \right\|_{\Delta_2}$$

holds for all sequences $(y_i) \subset Y_0$ with $y_i \in \mathcal{R}(D_i)$ for each $i \in \mathbb{N}$. The definition of 'randomized boundedness' of a collection $(L_i) \subset \mathcal{L}(Y)$ is crafted to guarantee the boundedness of the induced operator T_L.

Definition 1.5.6 Let Y be a Banach space. A collection $\mathcal{T} \subset \mathcal{L}(X)$ is said to be *R-bounded* if there exists a constant $M > 0$ such that

$$\left\| \sum_{i=1}^n r_i \otimes T_i x_i \right\|_{\Delta_2} \leq M \left\| \sum_{i=1}^n r_i \otimes x_i \right\|_{\Delta_2}$$

holds for all $(T_i)_{i=1}^n \subset \mathcal{L}(X)$, $(x_i)_{i=1}^n \subset X$ and $n \in \mathbb{N}$. Here, the sequence (r_i) denotes the Rademacher functions.

The property of R-boundedness is stable under the operation of taking closures.

Proposition 1.5.7 *Let Y be a Banach space. If $\mathcal{T} \subset \mathcal{L}(Y)$ is an R-bounded collection, then its closure (in the operator norm topology) is also R-bounded with the same R-bound as $\mathcal{T}$.*

Proof. Let $\overline{\mathcal{T}}$ denote the closure of $\mathcal{T}$. Choose $T_1, T_2 \ldots T_n \in \overline{\mathcal{T}}$ and vectors $y_1, y_2, \ldots, y_n \in Y$. For each $1 \leq i \leq n$ there exists a sequence $(T_k^{(i)}) \subset \mathcal{T}$ such that $\lim_{k\to\infty} \|(T - T_k^{(i)})y_i\| = 0$. Consequently,

$$\left\|\sum_{i=1}^n r_i \otimes T_i y_i\right\|_{\Delta_2} \leq \left\|\sum_{i=1}^n r_i \otimes (T_i - T_k^{(i)})y_i\right\|_{\Delta_2} + \left\|\sum_{i=1}^n r_i \otimes T_k^{(i)} y_i\right\|_{\Delta_2}$$
$$\leq \sum_{i=1}^n \left\|(T_i - T_k^{(i)})y_i\right\| + M \left\|\sum_{i=1}^n r_i \otimes y_i\right\|_{\Delta_2}$$

Taking the limit as $k \to \infty$ gives the result. □

Observe that R-boundedness implies uniform boundedness when $n = 1$. On a Hilbert space, R-boundedness and uniform boundedness are equivalent (cf. [103, Lemma 2.2.3]).

In [103, Lemma 2.2.10] it is shown that if $\mathcal{T}$ is countable in the above definition, then $\mathcal{T} = \{T_i : i \in \mathbb{N}\}$ is R-bounded with R-bound M if and only if

$$\left\|\sum_{i=1}^n r_i \otimes T_i y_i\right\|_{\Delta_2} \leq M \left\|\sum_{i=1}^n r_i \otimes y_i\right\|_{\Delta_2}$$

holds for all vectors $y_1, y_2, \ldots, y_n \in Y$ and $n \in \mathbb{N}$. This result is used to show that if (D_i) is an unconditional decomposition of a Banach space Y, then (D_i) is R-bounded (cf. [103, Lemma 2.2.12]).

Let (Σ_i) denote a filtration, then it follows from the work of Stein in [94] that the sequence of operators $(\mathbb{E}(\,\cdot\,|\Sigma_i))$ on $L^p(\mu)$ is R-bounded for $1 < p < \infty$. More generally, it can be shown that if (D_i) is an unconditional Schauder decomposition of a Banach space possessing property (α) (cf. [84]), or even the weaker property (Δ) (cf. [59]), then the sequence of partial sum projections corresponding to (D_i) is R-bounded. This result is shown in [10] and [103, Theorem 2.4.3]. We shall study property (α) and present the proof of [103, Theorem 2.4.3] in the next section.

We conclude with the following important multiplier theorem (cf. [103, Theorem 2.2.4] or [22]).

Theorem 1.5.8 (Clément-de Pagter-Sukochev-Witvliet) *Let (D_i) be an unconditional Schauder decomposition of the Banach space Y, with unconditional* *constant K. Suppose that $\mathcal{T} \subset \mathcal{L}(Y)$ is R-bounded with R-bound M. If $L = (L_i) \subset \mathcal{T}$ is such that $L_i D_i = D_i L_i D_i$ for all $i \in \mathbb{N}$, then the series*

$$T_L y := \sum_{i=1}^{\infty} L_i D_i y$$

is convergent in Y for all $y \in Y$, and defines a bounded linear operator $T_L : Y \to Y$ with $\|T_L\| \leq K^2 M$.

Proof. Let $y \in Y$ and $n \leq m$ be natural numbers. Using Proposition 1.5.5 and the definition of R-boundedness, we have

$$\begin{aligned}
\left\|\sum_{i=n}^{m} L_i D_i y\right\| &= \left\|\sum_{i=n}^{m} D_i L_i D_i y\right\| = \left\|\sum_{k=1}^{m} D_k \left(\sum_{i=n}^{m} D_i L_i D_i y\right)\right\| \\
&\leq K \left\|\sum_{k=1}^{m} r_k \otimes D_k \left(\sum_{i=n}^{m} D_i L_i D_i y\right)\right\|_{\Delta_2} = K \left\|\sum_{k=1}^{m}\sum_{i=n}^{m} r_k \otimes D_k \left(D_i L_i D_i y\right)\right\|_{\Delta_2} \\
&= K \left\|\sum_{i=n}^{m} r_i \otimes L_i D_i y\right\|_{\Delta_2} \leq KM \left\|\sum_{i=n}^{m} r_i \otimes D_i y\right\|_{\Delta_2} \leq K^2 M \left\|\sum_{i=n}^{m} D_i y\right\| \to 0
\end{aligned}$$

as $n, m \to \infty$. Consequently, $\|T_L y\| \leq K^2 M \|y\|$ for all $y \in Y$. $\square$

For more on R-bounded collections, we refer the reader to [22, 103, 59].

1.6 Property (α) and product Schauder decompositions

In this section, we consider products of unconditional Schauder decompositions and a special property that guarantees this product decomposition is unconditional. This section is also an exposition of results contained in the thesis of Witvliet (cf. [103]).

Definition 1.6.1 Let Y be a Banach space. A pair of Schauder decompositions (D_i) and (D'_j) of Y are said to *commute* if $D_i D'_j = D'_j D_i$ for all $i, j \in \mathbb{N}$.

In what follows, let (D_i) and (D'_j) denote commuting Schauder decompositions of a Banach space Y. Using the idea of Gelbaum and Gil de Lamadrid in [49] for constructing the tensor product basis with respect to a uniform crossnorm, one can define an order for the collection $(D_i D'_j)$.

Definition 1.6.2 Let (D_i) and (D'_j) be commuting decompositions of the Banach space Y. We define the *square ordering* on the collection $(D_i D'_j)$ to be the ordering of the indices (i, j) along the squares; i.e., $(i_1, j_1) \leq (i_2, j_2)$ when one of the following conditions hold:

(a) $\max\{i_1, j_1\} < \max\{i_2, j_2\}$,

(b) $\max\{i_1, j_1\} = \max\{i_2, j_2\}$ and $i_1 < i_2$ or

(c) $\max\{i_1, j_1\} = \max\{i_2, j_2\} = i_1 = i_2$ and $j_1 \geq j_2$.

We shall use the notation S_k for the set consisting of the first k ordered pairs of indices (i, j) in the square ordering.

The following diagram illustrates the above definition.

$$
\begin{array}{ccccccc}
(1,1) & \rightarrow & (1,2) & & (1,3) & & (1,4) \\
 & & \downarrow & & \downarrow & & \downarrow \\
(2,1) & \leftarrow & (2,2) & & (2,3) & & (2,4) \\
 & & & & \downarrow & & \downarrow \\
(3,1) & \leftarrow & (3,2) & \leftarrow & (3,3) & & (3,4) \\
 & & & & & & \downarrow \\
 & & \cdots & \longleftarrow & (4,3) & \leftarrow & (4,4)
\end{array}
$$

It is evident that $\bigcup_{i,j\in\mathbb{N}} \mathcal{R}(D_i D'_j)$ is dense in Y. It also follows from the definition of the square ordering that corresponding partial sum projections $P_n = \sum_{(i,j)\in S_n} D_i D'_j$ are uniformly bounded. In fact,

$$
\sup_{n\in\mathbb{N}} \|P_n\| \leq 3 \left(\sup_{n\in\mathbb{N}} \left\| \sum_{i=1}^{n} D_i \right\| \right) \left(\sup_{n\in\mathbb{N}} \left\| \sum_{j=1}^{n} D'_j \right\| \right).
$$

Indeed, Let (S_i) denote the corresponding partial sum projections of (D_i) and (T_j) denote the corresponding partial sum projections of (D'_j). Then

$$
P_k = \begin{cases} S_i T_i & ; \ k = i^2 \\ S_i T_i + S_{k-i^2} D_{i+1} & ; \ i^2 < k \leq i^2 + i + 1 \\ S_{i+1} T_{i+1} - D'_{i+1} T_{(i+1)^2 - k} & ; \ i^2 + i + 1 < k < (i+1)^2 \end{cases}
$$

for each $k \in \mathbb{N}$. Thus,

$$
\|S_i T_i\| \leq \|S_i\| \|T_i\| \leq \left(\sup_{n\in\mathbb{N}} \|S_n\| \right) \left(\sup_{n\in\mathbb{N}} \|T_n\| \right),
$$

$$
\|S_{k-i^2} D_{i+1}\| \leq \|S_{k-i^2}\| \|D_{i+1}\| \leq 2 \left(\sup_{n\in\mathbb{N}} \|S_n\| \right) \left(\sup_{n\in\mathbb{N}} \|T_n\| \right)
$$

and

$$
\|D'_{i+1} T_{(i+1)^2-k}\| \leq \|D'_{i+1}\| \|T_{(i+1)^2-k}\| \leq 2 \left(\sup_{n\in\mathbb{N}} \|S_n\| \right) \left(\sup_{n\in\mathbb{N}} \|T_n\| \right),
$$

from which the assertion follows. Consequently, the collection $(D_i D'_j)$ arranged in the square ordering is a Schauder decomposition of Y. We will revisit this technique later on.

A natural question to ask is whether $(D_i D'_j)$ is unconditional if (D_i) and (D'_j) are both unconditional. In general, the answer is negative, as is shown in [103, Proposition 5.3.5]. There is, however, a class of Banach spaces where this property does hold. Taylor made to suit our purpose is:

Definition 1.6.3 A Banach space Y is said to have *property* (α) if there exists a constant $\alpha > 0$ such that

$$\int_0^1\int_0^1 \left\| \sum_{1\le i,j\le n} \theta_{ij} r_i(s) r_j(t) y_{ij} \right\|^2 \mathrm{d}s\,\mathrm{d}t \le \alpha^2 \int_0^1\int_0^1 \left\| \sum_{1\le i,j\le n} r_i(s) r_j(t) y_{ij} \right\|^2 \mathrm{d}s\,\mathrm{d}t$$

for all vectors $(y_{ij}) \subset Y$, choices of signs (θ_{ij}), and $n \in \mathbb{N}$. Here, (r_i) denotes the sequence of Rademacher functions.

Property (α) was introduced by Pisier in [84] and is independent of the UMD property. Like the UMD property, $L^p(\mu, Y)$ inherits property (α) from Banach spaces Y with property (α). Property (α) has a special interaction with R-bounded collections of operators.

Lemma 1.6.4 *Let Y be a Banach space with property (α) and $\mathcal{T} \subset \mathcal{L}(Y)$ be R-bounded with R-bound M. Then there exists a constant $K > 0$ such that*

$$\int_0^1\int_0^1 \left\| \sum_{1\le i,j\le n} r_i(s) r_j(t) T_{ij} y_{ij} \right\|^2 \mathrm{d}s\,\mathrm{d}t \le K^2 \int_0^1\int_0^1 \left\| \sum_{1\le i,j\le n} r_i(s) r_j(t) y_{ij} \right\|^2 \mathrm{d}s\,\mathrm{d}t$$

for all $T_{ij} \in \mathcal{T}$, all $y_{ij} \in Y$ ($1 \le i, j \le n$) and all $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$ and (r'_{ij}) denote a $n \times n$ matrix of the first n^2 Rademacher functions. Since Y has property (α), we have

$$\begin{aligned}
&\int_0^1\int_0^1 \left\| \sum_{1\le i,j\le n} r_i(s) r_j(t) T_{ij} y_{ij} \right\|^2 \mathrm{d}s\,\mathrm{d}t \\
&= \int_0^1\int_0^1 \left\| \sum_{1\le i,j\le n} \left(r'_{ij}(u)\right)^2 r_i(s) r_j(t) T_{ij} y_{ij} \right\|^2 \mathrm{d}s\,\mathrm{d}t \\
&\le \alpha^2 \int_0^1\int_0^1 \left\| \sum_{1\le i,j\le n} r'_{ij}(u) r_i(s) r_j(t) T_{ij} y_{ij} \right\|^2 \mathrm{d}s\,\mathrm{d}t
\end{aligned}$$

for all $u \in [0,1]$. Integrating and using the R-boundedness of $\mathcal{T}$ yields

$$
\begin{aligned}
&\int_0^1\!\!\int_0^1 \Big\| \sum_{1\le i,j\le n} r_i(s)r_j(t)T_{ij}y_{ij} \Big\|^2 \,\mathrm{d}s\,\mathrm{d}t \\
&\le \alpha^2 \int_0^1 \left(\int_0^1\!\!\int_0^1 \Big\| \sum_{1\le i,j\le n} r'_{ij}(u)r_i(s)r_j(t)T_{ij}y_{ij} \Big\|^2 \,\mathrm{d}s\,\mathrm{d}t \right) \mathrm{d}u \\
&= \alpha^2 \int_0^1\!\!\int_0^1 \Big\| \sum_{1\le i,j\le n} r'_{ij} \otimes T_{ij}\big(r_i(s)r_j(t)y_{ij}\big) \Big\|_{\Delta_2}^2 \,\mathrm{d}s\,\mathrm{d}t \\
&\le \alpha^2 M^2 \int_0^1\!\!\int_0^1 \Big\| \sum_{1\le i,j\le n} r'_{ij} \otimes r_i(s)r_j(t)y_{ij} \Big\|_{\Delta_2}^2 \,\mathrm{d}s\,\mathrm{d}t \\
&= \alpha^2 M^2 \int_0^1 \left(\int_0^1\!\!\int_0^1 \Big\| \sum_{1\le i,j\le n} r'_{ij}(u)r_i(s)r_j(t)y_{ij} \Big\|^2 \,\mathrm{d}s\,\mathrm{d}t \right) \mathrm{d}u \\
&\le \alpha^4 M^2 \int_0^1\!\!\int_0^1\!\!\int_0^1 \Big\| \sum_{1\le i,j\le n} r_i(s)r_j(t)y_{ij} \Big\|^2 \,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}u \\
&= \alpha^4 M^2 \int_0^1\!\!\int_0^1 \Big\| \sum_{1\le i,j\le n} r_i(s)r_j(t)y_{ij} \Big\|^2 \,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}
$$

Setting $K = \alpha^2 M$ completes the proof. □

Theorem 1.6.5 *Let Y be a Banach space with property (α) and let (D_i) be an unconditional Schauder decomposition of Y. If $\mathcal{T} \subset \mathcal{L}(Y)$ is an R-bounded collection of operators, then*

$$\mathcal{S} := \{T_L : L = (L_i) \subset \mathcal{T} \text{ with } L_iD_i = D_iL_iD_i \text{ for all } i \in \mathbb{N}\}$$

is R-bounded. Here, T_L is defined as in (1.8) and well defined by Theorem 1.5.8.

Proof. We start with a special case. Select $S_1, S_2, \ldots, S_n \in \mathcal{S}_0$ where

$$\mathcal{S}_0 := \left\{ \sum_{i=1}^n D_iL_i : (L_i)_{i=1}^n \subset \mathcal{T} \text{ with } L_iD_i = D_iL_iD_i \text{ for all } 1 \le i \le n, n \in \mathbb{N} \right\}.$$

Without loss of generality, we may assume that $0 \in \mathcal{T}$. Consequently, there is $N \in \mathbb{N}$ such that $S_i = \sum_{j=1}^m L_j^{(i)} D_j$ for all $1 \le i \le n$ and $m \ge N$.

Using the R-boundedness of $\mathcal{T}$, Lemma 1.6.4 and Proposition 1.5.5 twice, we have

$$
\begin{aligned}
\left\| \sum_{i=1}^{n} r_i \otimes S_i y_i \right\|_{\Delta_2}^2 &= \int_0^1 \left\| \sum_{j=1}^{m} D_j \left(\sum_{i=1}^{n} r_i(s) L_j^{(i)} D_j y_i \right) \right\|^2 \mathrm{d}s \\
&\le M_2^2 \int_0^1 \left\| \sum_{j=1}^{m} r_j \otimes D_j \left(\sum_{i=1}^{n} r_i(s) L_j^{(i)} D_j y_i \right) \right\|_{\Delta_2}^2 \mathrm{d}s \\
&= M_2^2 \int_0^1 \int_0^1 \left\| \sum_{i=1}^{n} \sum_{j=1}^{m} r_i(s) r_j(t) L_j^{(i)} D_j y_i \right\|^2 \mathrm{d}s\, \mathrm{d}t \\
&\le M_2^2 K^2 \int_0^1 \int_0^1 \left\| \sum_{i=1}^{n} \sum_{j=1}^{m} r_i(s) r_j(t) D_j y_i \right\|^2 \mathrm{d}s\, \mathrm{d}t \\
&= M_2^2 K^2 \int_0^1 \left\| \sum_{j=1}^{m} r_j \otimes D_j \left(\sum_{i=1}^{n} r_i(s) y_i \right) \right\|_{\Delta_2}^2 \mathrm{d}s \\
&= M_2^4 K^2 \int_0^1 \left\| \sum_{j=1}^{m} D_j \left(\sum_{i=1}^{n} r_i(s) y_i \right) \right\|^2 \mathrm{d}s \\
&= M_2^4 K^2 \left\| \sum_{i=1}^{n} r_i \otimes \left(\sum_{j=1}^{m} D_j y_i \right) \right\|_{\Delta_2}^2 .
\end{aligned}
$$

Taking the limit as $m \to \infty$ shows that the set $\mathcal{S}_0$ is R-bounded. Proposition 1.5.7 completes the proof as $\mathcal{S}$ is the closure of $\mathcal{S}_0$. $\square$

Corollary 1.6.6 *If (D_i) is an unconditional Schauder decomposition of a Banach space Y with property (α), then the set $\mathcal{S} = \{\sum_{i\in\sigma} D_i : \sigma \subset \mathbb{N}\}$ is R-bounded.*

Proof. Apply the above theorem with $\mathcal{T} = \{0, \mathrm{id}_Y\}$ $\square$

Corollary 1.6.7 (Stein's Inequality) *Let (Ω, Σ, μ) be a finite measure space, $1 < p < \infty$ and (Σ_i) be a filtration. Then the sequence of conditional expectations $(\mathbb{E}(\,\cdot \mid \Sigma_i))$ on $L^p(\mu)$ is R-bounded.*

Proof. We may assume $\Sigma_i \uparrow \Sigma$. Since $L^p(\mu)$ has property (α) and $(\mathbb{E}(\,\cdot \mid \Sigma_i))$ are the corresponding partial sum projections of an unconditional decomposition of $L^p(\mu)$, the result follows easily from the above corollary. $\square$

Theorem 1.6.8 *Let (D_i) and (D'_j) be a pair of commuting unconditional Schauder decompositions of a Banach space Y. If Y has property (α) then the product decomposition $(D_i D'_j)$ is unconditional.*

Proof. We have already ascertained that $(D_iD'_j)$ is a Schauder decomposition. To prove unconditionality, let (η_{ij}) be an $m \times m$ matrix with entries in $\{0,1\}$. For each $1 \le i \le m$, define the finite sets $\sigma_i := \{j \in \mathbb{N} : \eta_{ij} = 1\}$. Then,

$$\sum_{i=1}^{m}\sum_{j=1}^{m} \eta_{ij} D_i D'_j y = \sum_{i=1}^{m} \left(\sum_{j\in\sigma_i} D'_j \right) D_i y$$

for each $y \in Y$. Since Y has property (α), it follows from Corollary 1.6.6 and Theorem 1.5.8 that there is a constant $M > 0$ so that

$$\left\| \sum_{i=1}^{m}\sum_{j=1}^{m} \eta_{ij} D_i D'_j y \right\| \le M \left\| \sum_{i=1}^{m}\sum_{j=1}^{m} D_i D'_j y \right\|$$

holds for each $y \in Y$. Consequently, each expansion $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} D_i D'_j y$ is subseries summable and thus, by Theorem 1.4.2, unconditionally summable. An application of the Principle of Uniform Boundedness completes the proof. □

1.7 Notes and remarks

In Section 1.3, we have only focused on norm convergence of martingales. There are is also a classical characterization of the Radon Nikodým property in terms of almost everywhere convergence.

Theorem 1.7.1 *A Banach space Y has the Radon Nikodým property if and only if every $L^1(\mu, Y)$-bounded martingale converges almost everywhere.*

This result is due to A. Ionescu Tulcea, C. Ionescu Tulcea (cf. [53]) and Chatterji (cf. [20]). Notice how we can drop the requirement of uniform integrability in Theorem 1.3.10(b), provided we trade in norm convergence for almost everywhere convergence.

In the Introduction, we mentioned that the Radon Nikodým property has an intrinsic geometrical characterization. This geometrical property is known as ‘dentability’.

Definition 1.7.2 Let Y be a Banach space.

(a) A bounded set $D \subset Y$ is called *dentable* if for each $\varepsilon > 0$, there exists $x \in D$ such that $x \notin \overline{\mathrm{co}}\,(D \setminus B_\varepsilon(x))$. Here, $\overline{\mathrm{co}}$ denotes the closed convex hull and $B_\varepsilon(x) = \{y \in Y : \|x - y\| \le \varepsilon\}$.
(b) If every bounded set in Y is dentable, then Y is called *dentable.*

Contributions by Davis, Huff, Maynard, Phelps and Rieffel yielded the following deep characterization of the Radon-Nikodym property.

Theorem 1.7.3 (Davis-Huff-Maynard-Phelps-Rieffel) *A Banach space Y has the Radon Nikodým property if and only if it is dentable.*

We refer the reader to [45, Theorem II.2.3.3] and [37, Chapter V, §3, Theorem 7] for more details, as well as for references to the original papers. The above result has a similar quality to Burkholder's geometrical characterization of the UMD property in [14] (Theorem 1.4.11). That is, both of these characterizations are independent of the Lebesgue-Bochner spaces. Burkholder also proved the following martingale characterization of the UMD property in [14].

Theorem 1.7.4 (Burkholder) *A Banach space Y has the UMD property if and only if every ± 1-transform of every $L^1(\mu, Y)$-bounded martingale converges almost everywhere.*

This result bears a marked resemblance to Theorem 1.7.1. By the same token, Theorem 1.3.10 and Theorem 1.5.4 also resemble each other.

The UMD property has an important interaction with the R-boundedness of partial sum projections associated with unconditional Schauder decompositions. In Corollary 1.6.6, we exhibited Witvliet's generalization of the Stein inequality (cf. [103, Corollary 2.3.5]). It was shown that if (D_i) is an unconditional decomposition of a Banach space Y with property (α), then the collection of operators $\{\sum_{i\in\sigma} D_i : \sigma \subset \mathbb{N}\}$ is R-bounded on Y. We ought to mention that this result was generalized further by Witvliet.

Definition 1.7.5 A Banach space Y is said to have *property $(\varDelta)$* if there exists a constant $\alpha > 0$ such that

$$\int_0^1\int_0^1 \left\| \sum_{1\le i\le j\le n} \theta_{ij} r_i(s) r_j(t) y_{ij} \right\|^2 \mathrm{d}s\,\mathrm{d}t \le \alpha^2 \int_0^1\int_0^1 \left\| \sum_{1\le i,j\le n} r_i(s) r_j(t) y_{ij} \right\|^2 \mathrm{d}s\,\mathrm{d}t$$

for all vectors $(y_{ij}) \subset Y$, choices of signs (θ_{ij}), and $n \in \mathbb{N}$. Here, (r_i) denotes the sequence of Rademacher functions.

It is evident that if a Banach space has property (α), then it also has property $(\varDelta)$. Property $(\varDelta)$ was introduced by Kalton and Weis in [59], where it was also shown that every UMD space possesses this property. Witvliet generalized the Stein inequality to Banach spaces with property $(\varDelta)$ (cf. [103, Theorem 2.4.3]).

Theorem 1.7.6 (Stein's Inequality) *If (D_i) is an unconditional decomposition of a Banach space Y with property $(\varDelta)$, then the collection of operators $\{\sum_{i=1}^n D_i : n \in \mathbb{N}\}$ is R-bounded on Y.*

Consequently, the above result holds for any unconditional Schauder decomposition of a UMD space. This implies that all martingale decompositions of $L^p(\mu, Y)$ ($1 < p < \infty$) are R-bounded, provided Y has the UMD property. The case $Y = \mathbb{R}$ is crucial for the theory of stopping times in Chapter 7.

2

Generalized vector-valued L^p-spaces

2.1 Introduction

Having sampled some of the classical theory of the Lebesgue-Bochner spaces in the previous chapter, it is time to free ourselves from the confines of a measure space. We have seen that the Lebesgue-Bochner space $L^p(\mu, Y)$ may be decomposed as the completed tensor product $L^p(\mu)\widetilde{\otimes}_{\Delta_p}Y$ for $1 \leq p < \infty$, where Δ_p is the induced Bochner norm. Our purpose is to exhibit a reasonable crossnorm $\|\cdot\|_l$ on the tensor product of a general Banach lattice E and a Banach space Y that 'extends' the Bochner norm. That is to say, when $E = L^p(\mu)$, the norms $\|\cdot\|_l$ and Δ_p coincide on $E \otimes Y$.

In Section 2.2, we consider the class of cone absolutely summing operators that map from a Banach lattice to a Banach space. These are the operators that map positive summable sequences to absolutely summable sequences. The cone absolutely summing norm on this class of operators is used to induce the l-norm on the tensor product $E \otimes Y$. Consequently, the theory of cone absolutely summing operators plays a central role. We recall some basic properties, as well as important duality and extension results. Our work is taken primarily from the comprehensive survey of Banach lattices by Schaefer [92].

Section 2.3 shows how the class of cone absolutely summing operators can be used to induce the l-norm on the tensor product $E \otimes Y$, as well as how the l-norm extends the Bochner norm. The l-norm was studied by Schaefer in [92]. Independently, Jacobs studied the Δ-norm on $E \otimes Y$ and Chaney studied the M-norm on the tensor product $X \otimes F$, of a Banach space X and a Banach lattice F. We show that the transpose of the M-norm, the l-norm and the Δ-norm all coincide on $E \otimes Y$. This fact is known, but not clearly seen from the literature. Since it provides us with multiple formulae for calculating the l-norm, and forms the foundation for most of our work, we present a complete, elementary proof for this result. Some parts of the proof are original and can be found in [28].

It is well known that the Bochner norm is not an injective, uniform crossnorm. In Section 2.4, we show that weaker order theoretic versions of these properties hold for the l-norm. We make extensive use of these properties throughout this thesis. We also show that if E and F are both Banach lattices, then the completed l-tensor product $E\widetilde{\otimes}_l F$ is a Banach lattice. This result was shown by Chaney for the M-norm and by Schaefer for the l-norm, using operator theoretic arguments. We provide a new proof for this result which relies on Fremlin's fundamental construction of a Riesz tensor product of Archimedean vector lattices [47]. Our proof for this result can also be found in [28].

This chapter concludes with Section 2.5, which shows that the l-tensor product of order continuous Banach lattices is again order continuous. This result was first shown by Popa in [85]. Later, a direct proof was found, which appeared in [68]. We present this proof.

2.2 Cone absolutely summing operators

The origin of cone absolutely summing operators lies in the following definition of Dinculeanu found in [38, 39, 40].

Definition 2.2.1 (Dinculeanu) Suppose Y is a Banach space and $1 \leq p < \infty$. Let $T : L^p(\mu) \to Y$ be a bounded linear operator. We define the *triple bar norm* of T by

$$|\!|\!|T|\!|\!|_p = \sup\left\{\sum_{i=1}^{n} \|\alpha_i T(\chi_{A_i})\|_Y : s = \sum_{i=1}^{n} \alpha_i \chi_{A_i} \text{ a simple function}, \|s\|_p \leq 1\right\}.$$

The finiteness of $|\!|\!|T|\!|\!|_p$ is characterized by the following theorem, which can be found in [37, Chapter IV, §4, Theorem 8].

Theorem 2.2.2 *Suppose Y is a Banach space and $1 \leq p < \infty$. For $T \in \mathcal{L}(L^p(\mu), Y)$ we have $|\!|\!|T|\!|\!|_p < \infty$ if and only if T maps positive convergent series in $L^p(\mu)$ to absolutely convergent series in Y.*

Proof. Note that T maps positive convergent series in $L^p(\mu)$ to absolutely convergent series in Y if and only if

$$\|T\|_L := \sup\left\{\sum_{i=1}^{n} \|Tf_i\| : (f_i) \subset L^p(\mu)_+, \left\|\sum_{i=1}^{n} f_i\right\|_p \leq 1\right\} < \infty. \tag{2.1}$$

It is easy to see that $\|\!|T|\!\|_p \leq \|T\|_L$ which proves the 'if' part of the theorem. For the converse, show $\|\!|T|\!\|_p \geq \|T\|_L$ by approximating by simple functions. □

Dinculeanu showed in [38, 39, 40] that a bounded operator $T : L^p(\mu) \to Y$ has $\|\!|T|\!\|_p < \infty$ if and only if there exists a Y-valued, μ-continuous vector measure F of q-bounded variation ($\frac{1}{p} + \frac{1}{q} = 1$) such that $Tf = \int_\Omega f \, \mathrm{d}F$ for all $f \in L^p(\mu)$. Given this result, a number of characterizations of the Radon Nikodým property in terms of operators mapping from $L^p(\mu)$ $(1 < p < \infty)$ into a Banach space Y emerged (cf. [104, 102, 19]).

Schaefer studied the class of cone absolutely summing operators in [92, 91], as well its dual counterpart, the class of majorizing operators.

Definition 2.2.3 Let E, F be Banach lattices and X, Y Banach spaces. If $T \in \mathcal{L}(E, Y)$ and $S \in \mathcal{L}(X, F)$ then,

(a) T is called *cone absolutely summing* if for every unconditionally summable sequence $(x_i) \subset E_+$, (Tx_i) is absolutely summable in Y. The space of cone absolutely summing operators from E into Y is denoted by $\mathcal{L}^{\mathrm{cas}}(E, Y)$.
(b) S is called *majorizing* if for every null sequence $(x_i) \subset X$, (Tx_i) is contained in an order interval in F. The space of majorizing operators to from X into F is denoted by $\mathcal{L}^{\mathrm{maj}}(X, F)$.

Cone absolutely summing operators serve as an order theoretic counter part to the absolutely summing operators studied in [36]. In the terminology of Krivine (cf. [60]), cone absolutely summing operators are known as 1-*concave* operators (see also [72, p. 45]).

A candidate for the norm of a cone absolutely summing operator is suggested by (2.1) and the following lemma.

Lemma 2.2.4 *Let E be a Banach lattice and $(x_n) \subset E_+$. If (x_n) is unconditionally summable, then $\|(x_n)\|_\varepsilon = \|\sum_{n=1}^\infty x_n\|$. Consequently, (x_n) is unconditionally summable if and only if it is summable.*

Proof. For each $x^* \in E^*$ and $x \in E_+$ we have $|\langle x, x^* \rangle| \leq \langle x, |x^*| \rangle$. Since $\|\,|x^*|\,\| = \|x^*\|$, it follows that

$$\begin{aligned}
\left\|\sum_{n=1}^{\infty} x_n\right\| &= \sup\left\{\left|\left\langle\sum_{n=1}^{\infty} x_n, x^*\right\rangle\right| : x^* \in E^*, \|x^*\| \le 1\right\} \\
&= \sup\left\{\left\langle\sum_{n=1}^{\infty} x_n, x^*\right\rangle : x^* \in E^*_+, \|x^*\| \le 1\right\} \\
&= \sup\left\{\sum_{n=1}^{\infty} \langle x_n, x^*\rangle : x^* \in E^*_+, \|x^*\| \le 1\right\} \\
&= \sup\left\{\sum_{n=1}^{\infty} |\langle x_n, x^*\rangle| : x^* \in E^*, \|x^*\| \le 1\right\} \\
&= \|(x_n)\|_\varepsilon < \infty.
\end{aligned}$$

An application of Theorem 1.4.2 completes the proof. □

Since convergence and unconditional convergence of positive series in a Banach lattice are equivalent, we are justified in the following definition.

Definition 2.2.5 Let E, F be Banach lattices and X, Y Banach spaces. If $T \in \mathcal{L}^{\mathrm{cas}}(E, Y)$ and $S \in \mathcal{L}^{\mathrm{maj}}(X, F)$ then,

(a) the cone absolutely summing norm of T is defined as

$$\|T\|_{\mathrm{cas}} = \sup\left\{\sum_{i=1}^{n} \|Tx_i\| : (x_i)_{i=1}^n \subset E_+, \left\|\sum_{i=1}^{n} x_i\right\| \le 1\right\},$$

(b) the majorizing norm of S is defined as

$$\|S\|_{\mathrm{maj}} = \sup\left\{\left\|\sup_{1\le i\le n} |Tx_i|\right\| : (x_i)_{i=1}^n \subset X, \sup_{1\le i\le n} \|x_i\| \le 1\right\}.$$

It follows that $\mathcal{L}^{\mathrm{cas}}(E, Y)$ and $\mathcal{L}^{\mathrm{maj}}(X, F)$ are Banach spaces under their respective norms and the inclusion maps $\mathcal{L}^{\mathrm{cas}}(E, Y) \hookrightarrow \mathcal{L}(E, Y)$ and $\mathcal{L}^{\mathrm{maj}}(X, F) \hookrightarrow \mathcal{L}(X, F)$ are continuous with norm less than or equal to one (cf. [92, Chapter IV, §3, Proposition 3.6]). Moreover, we have the following theorem (cf. [92, Chapter IV, §3, Theorem 3.8]):

Theorem 2.2.6 *Let E be a Banach lattice and Y be a Banach space, then the canonical map $T \mapsto T^*$ from $\mathcal{L}(E, Y)$ into $\mathcal{L}(Y^*, E^*)$ maps $\mathcal{L}^{\mathrm{cas}}(E, Y)$ isometrically into $\mathcal{L}^{\mathrm{maj}}(Y^*, E^*)$. A corresponding assertion is valid for $\mathcal{L}^{\mathrm{maj}}(X, F)$, where X is a Banach space and F is a Banach lattice.*

For a comprehensive exposition of cone absolutely summing and majorizing maps, we refer the interested reader to [92, Chapter IV, §3]. We conclude this section with a useful extension result (cf. [92, Chapter IV, §3, Proposition 3.9]).

Theorem 2.2.7 *Let E_0 be a Banach sublattice of the Banach lattice E, and let Y be any Banach space. If $T_0 \in \mathcal{L}^{\mathrm{cas}}(E_0, Y)$, then T_0 possesses an extension $T \in \mathcal{L}^{\mathrm{cas}}(E, Y)$ such that $\|T\|_{\mathrm{cas}} = \|T_0\|_{\mathrm{cas}}$.*

2.3 The l-tensor product of a Banach lattice and a Banach space

In the early 1970's, two important contributions to the theory of tensor products, which involve the tensor product of a Banach lattice and a Banach space, were independently made by Chaney (cf. [19]) and by Schaefer (cf. [92]).

If E is a Banach lattice and Y is a Banach space, Chaney introduced the *M-norm* on $Y \otimes E$ as

$$\|u\|_M := \left\| \sup\left\{ \left| \sum_{i=1}^n x^*(x_i) y_i \right| : \|x^*\| \leq 1 \right\} \right\|$$

for all $u = \sum_{i=1}^n x_i \otimes y_i \in Y \otimes E$. He also showed that the M-norm is equal to the $|\mu|$-norm on $Y \otimes E$, defined by

$$\|u\|_{|\mu|} = \inf\left\{ \left\| \sum_{i=1}^n \|x_i\| \, |y_i| \right\| : u = \sum_{i=1}^n x_i \otimes y_i \right\}$$

for all $u \in Y \otimes E$ (cf. [19, Theorem 1.4]). The $|\mu|$-norm is the transpose of the Δ-norm on $E \otimes Y$, given by

$$\|u\|_\Delta = \inf\left\{ \left\| \sum_{i=1}^n \|y_i\| \, |x_i| \right\| : u = \sum_{i=1}^n x_i \otimes y_i \right\}$$

for all $u \in E \otimes Y$. The Δ-norm was introduced earlier by Jacobs in his thesis, see [54].

Schaefer considered the *l-norm* on $E \otimes Y$, where the l-norm is the norm induced on $E \otimes Y$ by the cone absolutely summing norm on the Banach space of cone absolutely summing operators $T\colon E^* \to Y$. Denote by $\|\cdot\|_l$ the norm on $E \otimes Y$ induced by $\|\cdot\|_{\mathrm{cas}}$ under the canonical embedding of $E \otimes Y$ into $\mathcal{L}^{\mathrm{cas}}(E^*, Y)$, defined by

$$\sum_{i=1}^n x_i \otimes y_i = u \mapsto L_u$$

where $L_u x^* = \sum_{i=1}^n \langle x_i, x^* \rangle y_i$ for all $x^* \in E^*$; i.e.,

$$\|\cdot\|_l := \|\cdot\|_{\mathrm{cas}}\big|_{E \otimes Y}.$$

Under the canonical identification $u \mapsto L_u$, of $E \otimes Y$ with a subspace of $\mathcal{L}^{\mathrm{cas}}(E^*, Y)$ and $Y \otimes E$ with a subspace of $\mathcal{L}^{\mathrm{maj}}(Y^*, E^{**})$, the transpose map $u \mapsto {}^t u$

from $E \otimes Y$ onto $Y \otimes E$ corresponds with the formation of adjoints $L_u \mapsto L_u^*$ from $\mathcal{L}^{\mathrm{cas}}(E^*, Y)$ into $\mathcal{L}^{\mathrm{maj}}(Y^*, E^{**})$. Consequently, the norm induced on $Y \otimes E$ by $\|\cdot\|_{\mathrm{maj}}$ under the canonical embedding of $Y \otimes E$ into $\mathcal{L}^{\mathrm{maj}}(Y^*, E)$ is the transpose of the norm $\|\cdot\|_l$. We denote this norm by $\|\cdot\|_m$; i.e.,

$$\|\cdot\|_{{}^t l} = \|\cdot\|_m := \|\cdot\|_{\mathrm{maj}}\big|_{Y \otimes E}.$$

An intriguing result, proved by Schaefer, is that the dual of $E \otimes_l Y$ is isometrically isomorpic to $\mathcal{L}^{\mathrm{cas}}(E, Y^*)$ under the canonical map $f \mapsto T_f$, defined by $\langle x, T_f y\rangle = \langle x \otimes y, f\rangle$ (cf. [92, Chapter IV, §7, Theorem 7.4]).

A similar result was proved earlier by Jacobs in his thesis. He proved that the dual of $E \otimes_\Delta Y$ is canonically isometric to $\mathcal{L}^{\mathrm{cas}}(E, Y^*)$. A proof of the theorem of Jacobs can also be found in the thesis of Jeurninck, see [56, p. 104].

Since $\|z\| = \sup\{|\langle z, z^*\rangle| : z^* \in Z^*, \|z^*\| \leq 1\}$ for all $z \in Z$, in any Banach space Z, it follows from the duality results of Jacobs and Schaefer that $\|\cdot\|_\Delta = \|\cdot\|_l$ on $E \otimes Y$. Together with Chaney's result that $\|\cdot\|_M = \|\cdot\|_{|\mu|}$ on $Y \otimes E$, we have

$$\|\cdot\|_{{}^t M} = \|\cdot\|_\Delta = \|\cdot\|_l \text{ on } E \otimes Y,$$

where $\|\cdot\|_{{}^t M}$ denotes the transpose of $\|\cdot\|_M$.

An important property of the l-tensor product is its connection with the Lebesgue-Bochner spaces. Let Y denote a Banach space, (Ω, Σ, μ) denote a σ-finite measure space and $1 \leq p < \infty$. If $E = L^p(\mu)$, then the completed l-tensor product $E \widetilde{\otimes}_l Y$ is isometric to $L^p(\mu, Y)$ under the canonical mapping $f \otimes x \mapsto f(\,\cdot\,)x$ (cf. [19, 92]). We summarize with the following theorem:

Theorem 2.3.1 (Chaney-Jacobs-Jeurnink-Schaefer) *Let E be a Banach lattice and Y a Banach space. The following norms of $u = \sum_{i=1}^n x_i \otimes y_i \in E \otimes Y$ are equal:*

(a) $\|u\|_l := \sup\left\{\sum_{j=1}^m \left\|\sum_{i=1}^n x_j^*(x_i) y_i\right\| : (x_j^*)_{j=1}^m \subset E_+^*, \left\|\sum_{j=1}^m x_j^*\right\| \leq 1\right\},$

(b) $\|u\|_{{}^t M} := \left\|\sup\left\{\left|\sum_{i=1}^n y^*(y_i) x_i\right| : \|y^*\| \leq 1\right\}\right\|,$

(c) $\|u\|_\Delta := \inf\left\{\left\|\sum_{i=1}^n \|y_i\|\,|x_i|\right\| : u = \sum_{i=1}^n x_i \otimes y_i\right\}.$

In particular, when $E = L^p(\mu)$ with $1 \leq p < \infty$ and (Ω, Σ, μ) a σ-finite measure space, we have all three of the above norms equal to the Bochner norm Δ_p on $E \otimes Y$, induced by $L^p(\mu, Y)$.

The techniques used in the literature to prove the above result are not easily accessible. Also, some of the literature itself is difficult to obtain. Due to the importance of the above result to the rest of this thesis, we present a complete, elementary proof.

Our strategy will be to show that $E \otimes_{t_M} Y$ isometrically embeds into $E^{**} \otimes_{t_M} Y$. We then show that similar embeddings hold for $\|\cdot\|_l$ and $\|\cdot\|_\Delta$. Thus, the result will follow after proving that these three norms co-incide on $E^{**} \otimes Y$. We conclude by showing that $\|\cdot\|_{t_M}$ and Δ_p coincide on $E \otimes Y$ when $E = L^p(\mu)$ and $1 \leq p < \infty$. Before we start, the definition of $\|\cdot\|_{t_M}$ needs to be checked:

Lemma 2.3.2 *Let E be a Banach lattice and Y a Banach space. For finite sequences $(x_i)_{i=1}^n \subset E$ and $(y_i)_{i=1}^n \subset Y$ we have $\sup\{|\sum_{i=1}^n y^*(y_i)x_i| : \|y^*\| \leq 1\} \in E$. Moreover, this supremum can be approximated in norm by finite suprema of elements from $\{|\sum_{i=1}^n y^*(y_i)x_i| : \|y^*\| \leq 1\}$.*

Proof. Let $A := \{|\sum_{i=1}^n y^*(y_i)x_i| : \|y^*\| \leq 1\}$ and $e := \sum_{i=1}^n \|y_i\|\,|x_i|$, then it is clear that $A \subset [0,e]$. Consider the ideal in E generated by e, denoted E_e, together with the gauge norm defined by $\|x\|_e := \inf\{\lambda > 0 : x \in [-\lambda e, \lambda e]\}$. Then $(E_e, \|\cdot\|_e)$ is an AM-space with order unit e. We show that A is precompact in $(E_e, \|\cdot\|_e)$. Let $(w_j) \subset A$, then, for each $j \in \mathbb{N}$, we have $w_j = |\sum_{i=1}^n y_j^*(y_i)x_i|$, where $\|y_j^*\| \leq 1$. By Banach-Alaoglu, the dual unit ball of Y is w^*-compact. Thus, there exists a subsequence $(y_{j_k}^*)$ that converges in the w^*-topology to a functional y^* in the unit ball of Y^*; i.e. $y_{j_k}^*(y) \to y^*(y)$ as $k \to \infty$ for all $y \in Y$. Thus, by letting $w = |\sum_{i=1}^n y^*(y_i)x_i| \in A$, we have that

$$|w_{j_k} - w| \leq \sup_{1 \leq i \leq n} |y_{j_k}^*(y_i) - y^*(y_i)| \sum_{i=1}^n |x_i| \to 0 \quad \text{as } k \to \infty.$$

Hence, $(w_j) \subset A$ has a subsequence that converges relatively uniformly and thus in norm to $w \in A$ in $(E_e, \|\cdot\|_e)$. This shows that A is precompact in $(E_e, \|\cdot\|_e)$. Since every precompact subset in an AM-space has a supremum (cf. [76, Theorem 2.1.12]), we have $\sup A \in E_e \subset E$. The last part of the assertion follows from [76, Corollary 2.1.13] and the fact that the inclusion map $E_e \hookrightarrow E$ is continuous. $\square$

The above lemma shows that $\|\cdot\|_{t_M}$ is well defined and allows us to prove:

Theorem 2.3.3 *Let E be a Banach lattice and Y be a Banach space. Then $E \otimes_{t_M} Y$ is isometrically embedded into $E^{**} \otimes_{t_M} Y$.*

Proof. Let $i_E : E \to E^{**}$ denote the canonical Riesz isometry. We have that

$$\begin{aligned}&\left\| i_E\left(\sup\left\{\left|\sum_{i=1}^n y^*(y_i)x_i\right| : \|y^*\|\le 1\right\}\right)\right\|_{E^{**}}\\ &\ge \left\|\sup\left\{ i_E\left(\left|\sum_{i=1}^n y^*(y_i)x_i\right|\right) : \|y^*\|\le 1\right\}\right\|_{E^{**}}\end{aligned}$$

holds for all $u=\sum_{i=1}^n x_i\otimes y_i\in E\otimes Y$. For the reverse inequality, let $\varepsilon>0$. Then, by Lemma 2.3.2, there exist functionals $y_1^*,\dots,y_m^*$ in the dual unit ball of Y such that

$$\left\|\sup\left\{\left|\sum_{i=1}^n y^*(y_i)x_i\right| : \|y^*\|\le 1\right\} - \sup_{1\le j\le m}\left|\sum_{i=1}^n y_j^*(y_i)x_i\right|\right\|_E<\varepsilon.$$

Since i_E is a Riesz isometry, this implies that

$$\left\| i_E\left(\sup\left\{\left|\sum_{i=1}^n y^*(y_i)x_i\right| : \|y^*\|\le 1\right\}\right) - \sup_{1\le j\le m} i_E\left(\left|\sum_{i=1}^n y_j^*(y_i)x_i\right|\right)\right\|_{E^{**}}<\varepsilon.$$

Hence,

$$\begin{aligned}&\left\| i_E\left(\sup\left\{\left|\sum_{i=1}^n y^*(y_i)x_i\right| : \|y^*\|\le 1\right\}\right)\right\|_{E^{**}}\\ &< \left\|\sup_{1\le j\le m} i_E\left(\left|\sum_{i=1}^n y_j^*(y_i)x_i\right|\right)\right\|_{E^{**}}+\varepsilon\\ &\le \left\|\sup\left\{ i_E\left(\left|\sum_{i=1}^n y^*(y_i)x_i\right|\right) : \|y^*\|\le 1\right\}\right\|_{E^{**}}+\varepsilon.\end{aligned}$$

The fact that ε is arbitrary yields

$$\begin{aligned}&\left\| i_E\left(\sup\left\{\left|\sum_{i=1}^n y^*(y_i)x_i\right| : \|y^*\|\le 1\right\}\right)\right\|_{E^{**}}\\ &\le \left\|\sup\left\{ i_E\left(\left|\sum_{i=1}^n y^*(y_i)x_i\right|\right) : \|y^*\|\le 1\right\}\right\|_{E^{**}}.\end{aligned}$$

By the above reasoning and the fact that i_E is a Riesz isometry, it follows that

$$\begin{aligned}\|(i_E\otimes \mathrm{id}_Y)(u)\|_{E^{**}\otimes_{t_M}Y} &= \left\|\sup\left\{ i_E\left(\left|\sum_{i=1}^n y^*(y_i)x_i\right|\right) : \|y^*\|\le 1\right\}\right\|_{E^{**}}\\ &= \left\| i_E\left(\sup\left\{\left|\sum_{i=1}^n y^*(y_i)x_i\right| : y^*\in \|y^*\|\le 1\right\}\right)\right\|_{E^{**}}\\ &= \left\|\sup\left\{\left|\sum_{i=1}^n y^*(y_i)x_i\right| : \|y^*\|\le 1\right\}\right\|_E\\ &= \|u\|_{E\otimes_{t_M}Y}\end{aligned}$$

for all $u=\sum_{i=1}^n x_i\otimes y_i\in E\otimes Y$, showing that $E\otimes_{t_M}Y$ is isometrically embedded into $E^{**}\otimes_{t_M}Y$. $\square$

The next theorem is more general than the above result. It is a result that is key to many of the main results throughout this thesis. We will revisit this result in the next section.

Theorem 2.3.4 *(a) Let E be a Banach lattice and Y a Banach space. If E_0 is a Banach lattice with $i_0 : E_0 \to E$ a Riesz isometry and Y_0 a Banach space with $j_0 : Y_0 \to Y$ an isometry, then $i_0 \otimes j_0 : E_0 \otimes_l Y_0 \to E \otimes_l Y$ is an isometry.*

(b) Let X be a Banach space and F a Banach lattice. If X_0 is a Banach space with $i_0 : X_0 \to X$ an isometry and F_0 a Banach lattice with $j_0 : F_0 \to F$ a Riesz isometry, then $i_0 \otimes j_0 : X_0 \otimes_m F_0 \to X \otimes_m F$ is an isometry.

Proof. Let $T \in \mathcal{L}^{\text{cas}}(E^*, Y_0)$ with $j_0 : Y_0 \to Y$ an isometry, it follows that

$$\begin{aligned} \|j_0 \circ T\|_{\text{cas}} &= \sup \left\{ \sum_{i=1}^n \|(j_0 \circ T)(x_i^*)\| : (x_i^*)_{i=1}^n \subset E_+^*, \left\| \sum_{i=1}^n x_i^* \right\| = 1 \right\} \\ &= \sup \left\{ \sum_{i=1}^n \|T(x_i^*)\| : (x_i^*)_{i=1}^n \subset E_+^*, \left\| \sum_{i=1}^n x_i^* \right\| = 1 \right\} \\ &= \|T\|_{\text{cas}}. \end{aligned}$$

Consequently, if $u \in E \otimes Y_0$, then $\|u\|_l = \|(\text{id}_E \otimes j_0)(u)\|_l$ so that

$$(\text{id}_E \otimes j_0) : E \otimes_l Y_0 \to E \otimes_l Y$$

is an isometry.

Next, consider $T \in \mathcal{L}^{\text{maj}}(X^*, F_0)$ with $j_0 : F_0 \to F$ now a Riesz isometry. Then we have

$$\begin{aligned} \|j_0 \circ T\|_{\text{maj}} &= \sup \left\{ \left\| \sup_{1 \le i \le n} |(j_0 \circ T)(x_i^*)| \right\| : (x_i^*)_{i=1}^n \subset X^*, \sup_{1 \le i \le n} \|x_i^*\| \le 1 \right\} \\ &= \sup \left\{ \left\| j_0 \left(\sup_{1 \le i \le n} |T(x_i^*)| \right) \right\| : (x_i^*)_{i=1}^n \subset X^*, \sup_{1 \le i \le n} \|x_i^*\| \le 1 \right\} \\ &= \sup \left\{ \left\| \sup_{1 \le i \le n} |T(x_i^*)| \right\| : (x_i^*)_{i=1}^n \subset X^*, \sup_{1 \le i \le n} \|x_i^*\| \le 1 \right\} \\ &= \|T\|_{\text{maj}}. \end{aligned}$$

Consequently, if $u \in X \otimes F_0$, then $\|u\|_m = \|(\text{id}_X \otimes j_0)(u)\|_m$ so that

$$(\text{id}_X \otimes j_0) : X \otimes_m F_0 \to X \otimes_m F$$

is an isometry.

Since $\|\cdot\|_m = \|\cdot\|_{{}^tl}$, it follows that if $i_0 : X_0 \to X$ is an isometry, then $i_0 \otimes \text{id}_F : X_0 \otimes_m F \to X \otimes_m F$ is an isometry, from which (b) is readily deduced.

Similarly, since $\|\cdot\|_l = \|\cdot\|_{{}^tm}$, it follows that if $i_0 : E_0 \to E$ is a Riesz isometry, then $i_0 \otimes \text{id}_Y : E_0 \otimes_l Y \to E \otimes_l Y$ is an isometry, from which (a) is now evident. $\square$

In particular, it follows from the above result that $E \otimes_l Y$ embeds into $E^{**} \otimes_l Y$ isometrically. To prove a similar result for the Δ-norm we need a duality result from the thesis of Jacobs (cf. [54]). The proof presented here is due to the author and appears in [28].

By noting that for $u \in \sum_{i=1}^n x_i \otimes y_i \in E \otimes Y$, we have the decomposition

$$u = \sum_{i=1}^n (\|y_i\|x_i)^+ \otimes (y_i/\|y_i\|) + \sum_{i=1}^n (\|y_i\|x_i)^- \otimes (-y_i/\|y_i\|),$$

it follows that

$$\|u\|_\Delta = \inf\left\{ \left\| \sum_{i=1}^n x_i \right\| : u = \sum_{i=1}^n x_i \otimes y_i,\ x_i \geq 0,\ \|y_i\| = 1 \right\}.$$

Thus, for $f \in (E \otimes_\Delta Y)^*$, we have

$$\begin{aligned} \|f\| &= \sup\left\{ |f(u)| : u = \sum_{i=1}^n x_i \otimes y_i,\ \|u\|_\Delta \leq 1 \right\} \\ &= \sup\left\{ |f(u)| : u = \sum_{i=1}^n x_i \otimes y_i,\ x_i \geq 0,\ \|y_i\| = 1,\ \left\| \sum_{i=1}^n x_i \right\| \leq 1 \right\}, \end{aligned} \tag{2.2}$$

which will be used in the proof of:

Theorem 2.3.5 (Jacobs) *Let E be a Banach lattice and Y a Banach space. Then $(E \otimes_\Delta Y)^*$ is isometric to $\mathcal{L}^{\text{cas}}(E, Y^*)$.*

Proof. Consider the map from $(E \otimes_\Delta Y)^*$ into $\mathcal{L}(E, Y^*)$ given by $f \mapsto T_f$ where $\langle y, T_f x\rangle = f(x \otimes y)$ for all $x \in E$ and $y \in Y$. By (2.2), we have

$$\begin{aligned} \|f\| &= \sup\left\{ |f(u)| : u = \sum_{i=1}^n x_i \otimes y_i,\ x_i \geq 0,\ \|y_i\| = 1,\ \left\| \sum_{i=1}^n x_i \right\| \leq 1 \right\} \\ &= \sup\left\{ \left| \sum_{i=1}^n \langle y_i, T_f x_i\rangle \right| : (x_i)_{i=1}^n \subset E_+,\ (y_i)_{i=1}^n \subset Y,\ \|y_i\| = 1,\ \left\| \sum_{i=1}^n x_i \right\| \leq 1 \right\} \\ &= \sup\left\{ \sup\left\{ \left| \sum_{i=1}^n \langle y_i, T_f x_i\rangle \right| : \right.\right. \\ &\qquad\qquad \left.\left. (y_i)_{i=1}^n \subset Y,\ \|y_i\| = 1 \right\} : (x_i)_{i=1}^n \subset E_+,\ \left\| \sum_{i=1}^n x_i \right\| \leq 1 \right\}. \end{aligned}$$

Note that for any sequence $(y_i)_{i=1}^n \subset Y$ with $\|y_i\| = 1$, there exists a sequence of scalars $\alpha_1, \ldots, \alpha_n$ with $|\alpha_i| = 1$ for $i = 1, \ldots, n$ such that

$$\left| \sum_{i=1}^n \langle y_i, T_f x_i\rangle \right| \leq \sum_{i=1}^n |\langle y_i, T_f x_i\rangle| = \sum_{i=1}^n |\langle \alpha_i y_i, T_f x_i\rangle| = \left| \sum_{i=1}^n \langle \alpha_i y_i, T_f x_i\rangle \right|.$$

Consequently, we have

$$\|f\| = \sup\left\{ \sup\left\{ \sum_{i=1}^n |\langle y_i, T_f x_i\rangle| : \right.\right.$$
$$\left.\left. (y_i)_{i=1}^n \subset Y,\ \|y_i\| = 1 \right\} : (x_i)_{i=1}^n \subset E_+,\ \left\|\sum_{i=1}^n x_i\right\| \le 1 \right\}.$$

Notice for each $\varepsilon > 0$ there exists y_i with $\|y_i\| = 1$, for $i = 1, \dots, n$, so that

$$\sup\{\langle y_i, T_f x_i\rangle : \|y_i\| = 1\} - \varepsilon/n < \langle y_i, T_f x_i\rangle \le \sup\{\langle y_i, T_f x_i\rangle : \|y_i\| = 1\}.$$

Summing over i gives

$$\sum_{i=1}^n \sup\{\langle y_i, T_f x_i\rangle : \|y_i\| = 1\} - \varepsilon$$
$$< \sum_{i=1}^n \langle y_i, T_f x_i\rangle \le \sum_{i=1}^n \sup\{\langle y_i, T_f x_i\rangle : \|y_i\| = 1\}.$$

This implies

$$\sum_{i=1}^n \sup\{\langle y_i, T_f x_i\rangle : \|y_i\| = 1\} = \sup\left\{ \sum_{i=1}^n \langle y_i, T_f x_i\rangle : (y_i)_{i=1}^n \subset Y,\ \|y_i\| = 1 \right\},$$

and so

$$\begin{aligned}\|f\| &= \sup\left\{ \sum_{i=1}^n \sup\{|\langle y_i, T_f x_i\rangle| : \|y_i\| = 1\} : x_i \ge 0,\ \left\|\sum_{i=1}^n x_i\right\| \le 1 \right\} \\ &= \sup\left\{ \sum_{i=1}^n \|T_f x_i\| : x_i \ge 0,\ \left\|\sum_{i=1}^n x_i\right\| \le 1 \right\} \\ &= \|T_f\|_{\mathrm{cas}}.\end{aligned}$$

Thus, the map $f \mapsto T_f$ is a linear isometry of $(E \otimes_\Delta Y)^*$ into $\mathcal{L}^{\mathrm{cas}}(E, Y^*)$ which is easily seen to be surjective. □

We can now prove the following result. The proof is adapted from [19].

Theorem 2.3.6 *Let E be a Banach lattice and Y be a Banach space. Then $E \otimes_\Delta Y$ is isometrically embedded into $E^{**} \otimes_\Delta Y$.*

Proof. Let $i_E : E \to E^{**}$ denote the canonical Riesz isometry and $i_Y^* : Y^{***} \to Y^*$ denote the adjoint of the canonical isometry from Y into Y^{**}. Consider the map from $\mathcal{L}^{\mathrm{cas}}(E^{**}, Y^*)$ into $\mathcal{L}^{\mathrm{cas}}(E, Y^*)$ defined by $T \mapsto T \circ i_E$. We shall show this map is a metric surjection. Note already for any $T \in \mathcal{L}^{\mathrm{cas}}(E^{**}, Y^*)$, the map $T \circ i_E$ is just the restriction of T to E; thus, it follows by the definition of $\|\cdot\|_{\mathrm{cas}}$ that $\|T \circ i_E\|_{\mathrm{cas}} \le \|T\|_{\mathrm{cas}}$.

Now let $R \in \mathcal{L}^{\mathrm{cas}}(E, Y^*)$ and let $S = i_Y^* \circ R^{**}$. Then, for all $x \in E$ and $y \in Y$, we have

$$\begin{aligned}\langle y, (S \circ i_E)x\rangle &= \langle y, (i_Y^* \circ R^{**} \circ i_E)x\rangle = \langle y, (R^* \circ i_Y)^* i_E(x)\rangle \\ &= \langle (R^* \circ i_Y)y, i_E(x)\rangle = \langle x, R^* i_Y(y)\rangle = \langle Rx, i_Y(y)\rangle \\ &= \langle y, Rx\rangle,\end{aligned}$$

which shows that $S \circ i_E = R$. Thus, the following diagram commutes.

$$\begin{tikzcd} E^{**} \arrow[r, "R^{**}"] \arrow[dr, "S"] & Y^{***} \arrow[d, "i_Y^*"] \\ E \arrow[u, "i_E"] \arrow[r, "R"] & Y^* \end{tikzcd}$$

As a consequence of the isometric embedding $\mathcal{L}^{\mathrm{cas}}(E, Y^*) \hookrightarrow \mathcal{L}^{\mathrm{cas}}(E^{**}, Y^{***})$ given by Theorem 2.2.6, we obtain

$$\begin{aligned}\|S\|_{\mathrm{cas}} &= \|i_Y^* \circ R^{**}\|_{\mathrm{cas}} \\ &= \sup\left\{\sum_{i=1}^n \|(i_Y^* \circ R^{**})x_i\| : (x_i)_{i=1}^n \subset E_+, \left\|\sum_{i=1}^n x_i\right\| \leq 1\right\} \\ &\leq \sup\left\{\sum_{i=1}^n \|i_Y^*\|\|R^{**}x_i\| : (x_i)_{i=1}^n \subset E_+, \left\|\sum_{i=1}^n x_i\right\| \leq 1\right\} \\ &= \sup\left\{\sum_{i=1}^n \|R^{**}x_i\| : (x_i)_{i=1}^n \subset E_+, \left\|\sum_{i=1}^n x_i\right\| \leq 1\right\} \\ &= \|R^{**}\|_{\mathrm{cas}} = \|R\|_{\mathrm{cas}} = \|S \circ i_E\|_{\mathrm{cas}},\end{aligned}$$

which shows that $S \in \mathcal{L}^{\mathrm{cas}}(E^{**}, Y^*)$ and the map $T \mapsto T \circ i_E$ is surjective. So for each $R \in \mathcal{L}^{\mathrm{cas}}(E, Y^*)$, there exists $S \in \mathcal{L}^{\mathrm{cas}}(E^{**}, Y^*)$ such that

$$S \circ i_E = R \quad \text{and} \quad \|S\|_{\mathrm{cas}} = \|R\|_{\mathrm{cas}}.$$

Hence, $\|R\|_{\mathrm{cas}} = \inf\{\|T\|_{\mathrm{cas}} : T \in \mathcal{L}^{\mathrm{cas}}(E^{**}, Y^*),\, T \circ i_E = R\}$, showing that the map $T \mapsto T \circ i_E$ is indeed a metric surjection from $\mathcal{L}^{\mathrm{cas}}(E^{**}, Y^*)$ onto $\mathcal{L}^{\mathrm{cas}}(E, Y^*)$.

But $(E \otimes_\Delta Y)^* = \mathcal{L}^{\mathrm{cas}}(E, Y^*)$ and $(E^{**} \otimes_\Delta Y)^* = \mathcal{L}^{\mathrm{cas}}(E^{**}, Y^*)$ by Theorem 2.3.5. Thus, it follows that $E \otimes_\Delta Y$ is isometrically embedded into $E^{**} \otimes_\Delta Y$ and the proof is complete. $\square$

To complete the first part of the proof of Theorem 2.3.1, we now show that the norms $\|\cdot\|_l$, $\|\cdot\|_{\mathfrak{t}M}$ and $\|\cdot\|_\Delta$ coincide on $E^{**} \otimes Y$.

Consider the norms $\|\cdot\|_l$ and $\|\cdot\|_{\mathfrak{t}M}$ on $E^* \otimes Y$. Let $u = \sum_{i=1}^n x_i^* \otimes y_i \in E^* \otimes Y$, then the induced map $L_u \in \mathcal{L}_{\mathrm{cas}}(E, Y)$ is defined by $L_u(x) = \sum_{i=1}^n x_i^*(x) y_i$ for all $x \in E$. Consequently, by Theorem 2.2.6, we have $\|u\|_l := \|L_u\|_{\mathrm{cas}} = \|L_u^*\|_{\mathrm{maj}}$, where $L_u^* \in \mathcal{L}^{\mathrm{maj}}(Y^*, E^*)$ is given by $\langle x, L_u^*(y^*)\rangle = \sum_{i=1}^n y^*(y_i) x_i^*(x)$, yielding

$$\|u\|_l = \sup\left\{\left\|\sup_{1\leq j\leq m}|L_u^*(y_j^*)|\right\| : (y_j^*)_{j=1}^m \subset Y^*,\ \|y_j^*\|\leq 1\right\}$$
$$= \sup\left\{\left\|\sup_{1\leq j\leq m}\left|\sum_{i=1}^n y_j^*(y_i)x_i^*\right|\right\| : (y_j^*)_{j=1}^m \subset Y^*,\ \|y_j^*\|\leq 1\right\}. \qquad (2.3)$$

Note that E^* is a Dedekind complete Banach lattice and the set

$$\left\{\sup_{1\leq j\leq m}\left|\sum_{i=1}^n y_j^*(y_i)x_i^*\right| : (y_j^*)_{j=1}^m \subset Y^*,\ \|y_j^*\|\leq 1\right\}\subset E_+^*$$

is upwards directed and bounded above by $\sum_{i=1}^n \|y_i\|\,|x_i^*|$. Thus, we may interchange the norm and supremum in (2.3), which gives

$$\|u\|_l = \left\|\sup\left\{\sup_{1\leq j\leq m}\left|\sum_{i=1}^n y_j^*(y_i)x_i^*\right| : (y_j^*)_{j=1}^m \subset Y^*,\ \|y_j^*\|\leq 1\right\}\right\|$$
$$= \left\|\sup\left\{\left|\sum_{i=1}^n y^*(y_i)x_i^*\right| : \|y^*\|\leq 1\right\}\right\|$$
$$= \|u\|_{{}^tM}.$$

Now consider the norms $\|\cdot\|_\Delta$ and $\|\cdot\|_{{}^tM}$ on $E^*\otimes Y$. Suppose that $u=\sum_{i=1}^\infty x_i^*\otimes y_i \in E^*\otimes Y$ with $\{x_i^*\}_{i=1}^n$ a mutually disjoint set. Then

$$\|u\|_{{}^tM} = \left\|\sup\left\{\left|\sum_{i=1}^n y^*(y_i)x_i^*\right| : \|y^*\|\leq 1\right\}\right\| = \left\|\sup\left\{\bigvee_{i=1}^n |y^*(y_i)|\,|x_i^*| : \|y^*\|\leq 1\right\}\right\|$$
$$= \left\|\bigvee_{i=1}^n \left(|x_i^*|\,\sup\{|y^*(y_i)| : \|y^*\|\leq 1\}\right)\right\| = \left\|\sum_{i=1}^n \|y_i\|\,|x_i^*|\right\|$$
$$\geq \inf\left\{\left\|\sum_{i=1}^n \|y_i\|\,|x_i^*|\right\| : u=\sum_{i=1}^n x_i^*\otimes y_i\right\}$$
$$= \|u\|_\Delta.$$

An easy application of the triangle inequality shows that we in fact have $\|u\|_{{}^tM} = \|u\|_\Delta$. Hence, if the set

$$\left\{\sum_{i=1}^n x_i^*\otimes y_i \in E^*\otimes Y : \{x_i^*\}_{i=1}^n \text{ mutually disjoint}\right\}$$

is dense in $E^*\widetilde{\otimes}_\Delta Y$, we have the norms $\|\cdot\|_\Delta$ and $\|\cdot\|_{{}^tM}$ equal on $E^*\otimes Y$. We will show that this is indeed the case, after we complete the proof of Theorem 2.3.1.

To this end we observe that, from the above results, we have the diagram

$$\begin{tikzcd}
E \otimes_l Y \arrow[r, "i_E \otimes \mathrm{id}_Y"] \arrow[d, leftrightarrow, "I_1"'] & E^{**} \otimes_l Y \arrow[d, leftrightarrow, "H_1"] \\
E \otimes_{t_M} Y \arrow[r, "i_E \otimes \mathrm{id}_Y"] \arrow[d, leftrightarrow, "I_2"'] & E^{**} \otimes_{t_M} Y \arrow[d, leftrightarrow, "H_2"] \\
E \otimes_\Delta Y \arrow[r, "i_E \otimes \mathrm{id}_Y"] & E^{**} \otimes_\Delta Y
\end{tikzcd}$$

where H_1 and H_2 are surjective isometries and $i_E : E \to E^{**}$ denotes the canonical Riesz isometry. It follows that I_1 and I_2 are also surjective isometries induced by H_1 and H_2 respectively, and so the norms $\|\cdot\|_l$, $\|\cdot\|_{t_M}$ and $\|\cdot\|_\Delta$ are equal on $E \otimes Y$.

For the last part of the proof, let $E = L^p(\mu)$ with $1 \le p < \infty$. For any $u = \sum_{i=1}^n \chi_{A_i} \otimes y_i \in S(\mu) \otimes Y$, we have

$$\|u\|_{t_M} = \left\| \sup \left\{ \left| \sum_{i=1}^n y^*(y_i)\chi_{A_i} \right| : \|y^*\| \le 1 \right\} \right\|_p = \left\| \sum_{i=1}^n \|y_i\|\chi_{A_i} \right\|_p = \Delta_p(u)$$

as above. Since $S(\mu) \otimes Y$ is dense in $E \widetilde{\otimes}_{\Delta_p} Y$, the result follows.

The density of $S(\mu) \otimes Y$ in $L^p(\mu)\widetilde{\otimes}_{\Delta_p} Y$ is generalized to the Δ-tensor product by the following result, which is still required in the proof of Theorem 2.3.1. This result was proved by Chaney in [19], using representation theory. We provide an elementary proof, via Freudenthal's Spectral Theorem.

Theorem 2.3.7 *Let E be a Banach lattice with principle projection property and let Y be a Banach space. Then the set of step-functions*

$$\left\{ \sum_{i=1}^n x_i \otimes y_i \in E \otimes Y : \{x_i\}_{i=1}^n \text{ mutually disjoint} \right\}$$

is dense in $E \widetilde{\otimes}_\Delta Y$.

Proof. Let $f \in E \widetilde{\otimes}_\Delta Y$ and $\varepsilon > 0$. There exists $u = \sum_{i=1}^n x_i \otimes y_i \in E \otimes Y$ with

$$\left\| f - \sum_{i=1}^n x_i \otimes y_i \right\|_\Delta < \frac{\varepsilon}{2}. \tag{2.4}$$

Now let $g_\varepsilon = (\sum_{i=1}^n |x_i|)/\| \sum_{i=1}^n |x_i| \|$, then for each i $(1 \le i \le n)$, the fact that $|x_i| \le \sum_{i=1}^n |x_i|$ implies that $|x_i|$ is an element of the ideal generated by g_ε. Thus, by Freudenthal's Spectral Theorem (cf. [3, 6.9], [76, Theorem 1.2.18] or [106, Theorem 33.2]), for arbitrary $\delta > 0$ and i $(1 \le i \le n)$, there exist linear combinations of disjoint components of g_ε, denoted $s_\delta^{(i)}$, with

$$0 \le |x_i| - s_\delta^{(i)} < \frac{\delta g_\varepsilon}{n \left(\sup_{1 \le i \le n} \|y_i\| \right)}.$$

Thus, for each i $(1 \leq i \leq n)$, we have

$$0 \leq \|y_i\|(|x_i| - s_\delta^{(i)}) \leq \frac{\delta g_\varepsilon \|y_i\|}{n\left(\sup_{1\leq i\leq n}\|y_i\|\right)} \leq \frac{\delta g_\varepsilon\left(\sup_{1\leq i\leq n}\|y_i\|\right)}{n\left(\sup_{1\leq i\leq n}\|y_i\|\right)} = \frac{\delta g_\varepsilon}{n},$$

and summing over i gives

$$0 \leq \sum_{i=1}^n \|y_i\||x_i| - \sum_{i=1}^n \|y_i\| s_\delta^{(i)} \leq \frac{\delta g_\varepsilon n}{n} = \delta g_\varepsilon.$$

By taking the norm and using the fact that $\|g_\varepsilon\| = 1$, we obtain the inequality

$$\left\| \sum_{i=1}^n \|y_i\||x_i| - \sum_{i=1}^n \|y_i\| s_\delta^{(i)} \right\| \leq \delta \|g_\varepsilon\| = \delta.$$

Setting $\delta = \varepsilon/2$ gives

$$\left\| \sum_{i=1}^n x_i \otimes y_i - \sum_{i=1}^n s_\varepsilon^{(i)} \otimes y_i \right\|_\Delta \leq \frac{\varepsilon}{2}. \tag{2.5}$$

Choose a disjoint collection of components of g_ε, denoted $\{q_l : 1 \leq l \leq r\}$, so that each $s_\varepsilon^{(i)}$ can be written as $s_\varepsilon^{(i)} = \sum_{j=1}^r \gamma_j^{(i)} q_j$ for $i = 1, \ldots, n$. Hence, $s_\varepsilon^{(i)} \otimes y_i = \left(\sum_{j=1}^r \gamma_j^{(i)} q_j\right) \otimes y_i = \sum_{j=1}^r (q_j \otimes \gamma_j^{(i)} y_i)$, and summing over i gives

$$\sum_{i=1}^n s_\varepsilon^{(i)} \otimes y_i = \sum_{i=1}^n \sum_{j=1}^r (q_j \otimes \gamma_j^{(i)} y_i) = \sum_{j=1}^r \left(q_j \otimes \sum_{i=1}^n \gamma_j^{(i)} y_i \right). \tag{2.6}$$

Finally, by (2.4), (2.5) and (2.6), we get

$$\begin{aligned}
&\left\| f - \sum_{j=1}^r \left(q_j \otimes \sum_{i=1}^n \gamma_j^{(i)} y_i \right) \right\|_\Delta \\
&\leq \left\| f - \sum_{i=1}^n x_i \otimes y_i \right\|_\Delta + \left\| \sum_{i=1}^n x_i \otimes y_i - \sum_{i=1}^n s_\varepsilon^{(i)} \otimes y_i \right\|_\Delta \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

where the q_j's are mutually disjoint and the proof is complete. □

2.4 Lattice properties of the *l*-norm

If X and Y are Banach spaces and α is a norm on $X \otimes Y$, we denote the normed space $(X \otimes Y, \alpha)$ by $X \otimes_\alpha Y$, its norm completion by $X \widetilde{\otimes}_\alpha Y$ and its continuous dual by $(X \otimes_\alpha Y)^*$. The norm of an element $u \in X \widetilde{\otimes}_\alpha Y$ will be denoted $\alpha_{X,Y}(u)$ when there is a need to distinguish the Banach spaces involved or simply $\alpha(u)$ if there is no risk of ambiguity. A norm α on $X \otimes Y$ is called a *reasonable crossnorm* (cf. [31, 52, 34, 35, 37]) if α satisfies the conditions:

(a) For $x \in X$ and $y \in Y$, $\alpha(x \otimes y) \leq \|x\| \, \|y\|$.

(b) For $x^* \in X^*$ and $y^* \in Y^*$, $x^* \otimes y^* \in (X \otimes_\alpha Y)^*$ and $\|x^* \otimes y^*\| \leq \|x^*\| \, \|y^*\|$.

It is well known that the inequalities in (a) and (b) may be replaced by equality. It is also well known that α is a reasonable crossnorm on $X \otimes Y$ if and only if $\pi(u) \leq \alpha(u) \leq \varepsilon(u)$ for every $u \in X \otimes Y$ ([31, 90]). Hence, it is readily verified from Theorem 2.3.1 and the inequalities

$$\sup\left\{\left\|\sum_{i=1}^n y^*(y_i)x_i\right\| : \|y^*\| \leq 1\right\} \leq \left\|\sum_{i=1}^n \|y_i\| \, |x_i|\right\| \leq \sum_{i=1}^n \|y_i\|\|x_i\|$$

that $\|\cdot\|_l$ is indeed a reasonable crossnorm.

Let X, X_0, Y and Y_0 be Banach spaces. If $S : X_0 \to X$ and $T : Y_0 \to Y$ are bounded linear maps, then a reasonable crossnorm α is called a *uniform crossnorm* if $S \otimes T : X_0 \otimes_\alpha Y_0 \to X \otimes_\alpha Y$ satisfies

$$\|S \otimes T\| \leq \|S\|\|T\|.$$

Since the inequality $\|S \otimes T\| \geq \|S\|\|T\|$ holds for all reasonable crossnorms α, equality holds in the definition of uniform crossnorms. In the case where X_0 is a closed subspace of X, Y_0 is a closed subspace of Y and α is a uniform crossnorm, we have that $\alpha_{X,Y}(u) \leq \alpha_{X_0,Y_0}(u)$. This inequality can be strict and thus $E_0\widetilde{\otimes}_\alpha Y_0$ need not be a subspace of $E\widetilde{\otimes}_\alpha Y$. A uniform crossnorm for which $\alpha_{X_0,Y_0}(u) = \alpha_{X,Y}(u)$ holds for each closed subspace X_0 of X and Y_0 of Y is called *injective*.

Pisier noted that the Bochner norm Δ_p is not an injective uniform crossnorm for $1 < p < \infty$ (see [31, p. 147]). However, for $1 \leq p < \infty$, it is known that the Bochner norm Δ_p, induced by $L^p(\mu, X)$, has the property that if $0 \leq S : L^p(\mu) \to L^p(\mu)$ (note that any positive operator between Banach lattices is bounded, thus S is also bounded) and $T : Y \to Y$ is a bounded map, then $S \otimes T : L^p(\mu, Y) \to L^p(\mu, Y)$ has the property that

$$\|S \otimes T\| = \|S\|\|T\| \tag{2.7}$$

(see [37, 69]). Property (2.7) extends to the l-tensor and the m-tensor products as stated below (cf. [69]).

Theorem 2.4.1 *(a) Let E_1 and E_2 be Banach lattices and let Y_1 and Y_2 be Banach spaces. Let $T_1 : E_1 \to E_2$ be a positive linear operator and $T_2 : Y_1 \to Y_2$ be a bounded linear operator. Then $\left\|(T_1 \otimes T_2)u\right\|_l \leq \|T_1\| \, \|T_2\| \, \|u\|_l$ for all $u \in E_1 \otimes Y_1$.*

(b) Let X_1 and X_2 be Banach spaces and let F_1 and F_2 be Banach lattices. Let $T_1 : X_1 \to X_2$ be a bounded linear operator and $T_2 : F_1 \to F_2$ be a positive linear operator. Then $\left\|(T_1 \otimes T_2)u\right\|_m \leq \|T_1\| \, \|T_2\| \, \|u\|_m$ for all $u \in X_1 \otimes F_1$.

Proof. Let $u \in E_1 \otimes_\Delta Y_1$. Then $u = \sum_{i=1}^n x_i \otimes y_i$ and

$$(T_1 \otimes T_2)(u) = \sum_{i=1}^n T_1 x_i \otimes T_2 y_i.$$

Thus, by Theorem 2.3.1,

$$\begin{aligned}\|(T_1 \otimes T_2)u\|_l &= \left\| \sum_{i=1}^n T_1 x_i \otimes T_2 y_i \right\|_l \leq \left\| \sum_{i=1}^n \|T_2 y_i\| \, |T_1 x_i| \right\| \\ &\leq \left\| \sum_{i=1}^n \|T_2 y_i\| \, T_1(|x_i|) \right\| \leq \|T_1\| \, \|T_2\| \, \left\| \sum_{i=1}^n \|y_i\| \, |x_i| \right\|.\end{aligned}$$

Consequently,

$$\|(T_1 \otimes T_2)u\|_l \leq \|T_1\| \, \|T_2\| \, \|u\|_l.$$

The proof for the m-norm is similar. $\square$

In Theorem 2.3.4 we showed that the l-norm exhibits a weaker form of injectivity: if E_0 is a closed Riesz subspace of E and Y_0 is a closed subspace of Y, then $E_0 \widetilde{\otimes}_l Y_0$ is a closed subspace of $E \widetilde{\otimes}_l Y$.

These properties motivate the following definition:

Definition 2.4.2 If E, E_0 are Banach lattices and Y, Y_0 Banach spaces with $0 \leq S : E_0 \to E$, $T : Y_0 \to Y$ bounded linear maps, then a reasonable crossnorm α is called

(a) *left order uniform* (or in short, *left uniform*) if $\|S \otimes T\| \leq \|S\| \|T\|$,
(b) *left order injective* (or in short, *left injective*) if $S \otimes T : E_0 \widetilde{\otimes}_\alpha Y_0 \to E \widetilde{\otimes}_\alpha Y$ is an isometry, provided that S is a Riesz isometry and T is an isometry.

The notions of a *right order uniform crossnorm* and a *right order injective crossnorm* are defined in a symmetrical manner.

We now pass to the l-tensor product of two Banach lattices. We rely on a fundamental construction of the Riesz tensor product of Archimedean Riesz spaces.

Definition 2.4.3 Let E and F be Archimedean Riesz spaces. We denote the *projective cone* of $E \otimes F$ by

$$E_+ \otimes F_+ := \left\{ \sum_{i=1}^n x_i \otimes y_i \, : \, (x_i, y_i) \in E_+ \times F_+, \, n \in \mathbb{N} \right\}.$$

D.H. Fremlin (in [47]) constructed an Archimedean Riesz space $E \overline{\otimes} F$ with the following properties:

(RBi) If $(x, y) \in E \times F$, then $|x| \otimes |y| = |x \otimes y|$ in $E\overline{\otimes}F$.

(F) If G is any Archimedean Riesz space such that $E \otimes F$ is a vector subspace of G and $|x| \otimes |y| = |x \otimes y|$ in G for all $(x, y) \in E \times F$, then $E\overline{\otimes}F$ is the Riesz subspace of G generated by $E \otimes F$.

(SS) If E_0 and F_0 are Riesz subspaces of E and F respectively, then $E_0\overline{\otimes}F_0$ is a Riesz subspace of $E\overline{\otimes}F$.

(ru-D)$_+$ If $z \in (E\overline{\otimes}F)_+$, then there exists $(x, y) \in E_+ \times F_+$ with the property that for each $\varepsilon > 0$ there exists $v_\varepsilon \in E_+ \otimes F_+$ such that $|z - v_\varepsilon| \leq \varepsilon x \otimes y$; moreover, $v_\varepsilon \in E_+ \otimes F_+$ may be chosen such that $v_\varepsilon \leq z$ (see [51]).

Let E and F be Banach lattices. We are interested in those reasonable crossnorms α on $E \otimes F$ which have extensions to $E\overline{\otimes}F$ in such a way that (the extension of) α is a Riesz norm on $E\overline{\otimes}F$. Such reasonable crossnorms are called *order reasonable crossnorms*. The following theorem was proved in [70].

Theorem 2.4.4 *Let E and F be Banach lattices. If α is a reasonable crossnorm on $E \otimes F$, then $|\alpha|$, defined by*

$$|\alpha|(u) = \inf\Big\{\alpha(v) : v \in E_+ \otimes F_+ \text{ and } |u| \leq v\Big\} \qquad \text{for all } u \in E\overline{\otimes}F,$$

is a Riesz norm on $E\overline{\otimes}F$ and a reasonable crossnorm on $E \otimes F$ with the property that $E\widetilde{\otimes}_{|\alpha|}F$ is a Banach lattice, with positive cone the $|\alpha|$-closure of the projective cone in $E \otimes F$. Moreover, α can be extended to a Riesz norm on $E\overline{\otimes}F$ if and only if $\alpha = |\alpha|$ on $E \otimes F$.

The above theorem motivates the following definition.

Definition 2.4.5 (a) A left (right) order uniform crossnorm that is also an order reasonable crossnorm will be referred to as a *left (right) uniform Riesz crossnorm.*

(b) A left (right) order injective crossnorm that is also an order reasonable crossnorm will be referred to as a *left (right) injective Riesz crossnorm.*

The l-norm (m-norm) is an example of a left (right) uniform, left (right) injective Riesz crossnorm, as the following results show.

Theorem 2.4.6 (Chaney-Schaefer) *If E and F are Banach lattices, then*

(a) $E\widetilde{\otimes}_M F$ is a Banach lattice, with positive cone the M-closure of the projective cone of $E \otimes F$.

(b) $E\widetilde{\otimes}_l F$ is a Banach lattice, with positive cone the l-closure of the projective cone of $E \otimes F$.

The above result was proved by Chaney in [19] and Schaefer in [92] using operator techniques. We present a new proof that uses the above mentioned construction of Fremlin. Our proof appears in [28]. We start with a trivial lemma pertaining to the monotonicity of the Δ-norm.

Lemma 2.4.7 *Let E and F be Banach lattices and let $u \in E \otimes F$. Then $\|u\|_\Delta \leq \|u+v\|_\Delta$ for all $v \in E \otimes F$.*

Proof. Let $u = \sum_{i=1}^n x_i \otimes y_i \in E \otimes F$ and $v = \sum_{i=1}^n a_i \otimes b_i \in E \otimes F$. From the definition of the Δ-norm and the fact that F is Riesz normed, we have

$$\|u\|_\Delta \leq \left\| \sum_{i=1}^n \|y_i\| \, |x_i| \right\| \leq \left\| \sum_{i=1}^n (\|y_i\| \, |x_i| + \|b_i\| \, |a_i|) \right\|,$$

which holds for all representations of u and v. But then $\|u\|_\Delta \leq \|u+v\|_\Delta$. $\square$

Definition 2.4.8 Let E and F be Archimedean Riesz spaces. The cone on $E \otimes F$ induced by $(E\overline{\otimes}F)_+$ is denoted by $[E_+ \otimes F_+]$.

Using property (Ru-D)$_+$, the cone $[E_+ \otimes F_+]$ can be characterized as follows: $z \in [E_+ \otimes F_+]$ if and only if there exists $(x,y) \in E_+ \times F_+$ such that $z + \varepsilon(x \otimes y) \in E_+ \otimes F_+$ for all $\varepsilon > 0$ (cf. [48]).

Theorem 2.4.9 *Let E and F be Banach lattices. If $u \in E \otimes F$, then*

$$\|u\|_\Delta = \inf \{ \|v\|_\Delta : |u| \leq v \in E_+ \otimes F_+ \}$$

where $\leq$ denotes the order with respect to the cone $(E\overline{\otimes}F)_+$. Consequently, $\|\cdot\|_\Delta$ is a left uniform, left injective Riesz crossnorm.

Proof. Let $u = \sum_{i=1}^n x_i \otimes y_i \in E \otimes F$, then $|u| \in E\overline{\otimes}F$ and $|u| = |\sum_{i=1}^n x_i \otimes y_i| \leq \sum_{i=1}^n |x_i| \otimes |y_i|$. Thus,

$$\left\{ \left\| \sum_{i=1}^n \|y_i\| \, |x_i| \right\| : u = \sum_{i=1}^n x_i \otimes y_i \right\} \subset \left\{ \left\| \sum_{i=1}^n \|y_i\| \, |x_i| \right\| : |u| \leq \sum_{i=1}^n |x_i| \otimes |y_i| \right\}.$$

Hence,

$$\inf \left\{ \left\| \sum_{i=1}^n \|y_i\| \, |x_i| \right\| : |u| \leq \sum_{i=1}^n |x_i| \otimes |y_i| \right\} \leq \|u\|_\Delta.$$

Thus, if $v \in E_+ \otimes F_+$ has the representation $\sum_{i=1}^n a_i \otimes b_i$, it follows that

$$\begin{aligned} &\inf \{ \|v\|_\Delta : |u| \leq v \in E_+ \otimes F_+ \} \\ &\leq \inf \left\{ \left\| \sum_{i=1}^n \|b_i\| \, a_i \right\| : |u| \leq \sum_{i=1}^n a_i \otimes b_i \in E_+ \otimes F_+ \right\} \\ &\leq \|u\|_\Delta. \end{aligned}$$

For the reverse inequality, consider $u \in E \otimes F$ and all $\sum_{i=1}^n x_i \otimes y_i \in E \otimes F$ for which the inequality $|u| \leq \sum_{i=1}^n |x_i| \otimes |y_i|$ holds with respect to the ordering induced by $(E\overline{\otimes}F)_+$. Represent $\sum_{i=1}^n |x_i| \otimes |y_i| \in E \otimes F$ by

$$\sum_{i=1}^n |x_i| \otimes |y_i| = \sum_{i=1}^n a_i \otimes b_i,$$

where $a_i \in E$ and $b_i \in F$. Then $u \leq \sum_{i=1}^n a_i \otimes b_i$ in $(E \otimes F, [E_+ \otimes F_+])$. Thus, there exists $(x, y) \in E_+ \times F_+$ such that for any $\varepsilon > 0$ there exists $v_\varepsilon \in E_+ \otimes F_+$ such that

$$\left(\sum_{i=1}^n a_i \otimes b_i - u\right) + \varepsilon(x \otimes y) = v_\varepsilon.$$

Hence,

$$u + v_\varepsilon = \sum_{i=1}^n a_i \otimes b_i + \varepsilon x \otimes y$$

and by Lemma 2.4.7, we have

$$\|u\|_\Delta \leq \|u + v_\varepsilon\|_\Delta \leq \left\|\sum_{i=1}^n \|b_i\|\, |a_i| + \varepsilon\|y\|\, x\right\| \leq \left\|\sum_{i=1}^n \|b_i\|\, |a_i|\right\| + \varepsilon\|y\|\|x\|.$$

But $\varepsilon > 0$ is arbitrary and so

$$\|u\|_\Delta \leq \left\|\sum_{i=1}^n \|b_i\|\, |a_i|\right\|,$$

which gives

$$\begin{aligned}\|u\|_\Delta &\leq \inf\left\{\left\|\sum_{i=1}^n \|b_i\|\, |a_i|\right\| : \sum_{i=1}^n |x_i| \otimes |y_i| = \sum_{i=1}^n a_i \otimes b_i\right\} \\ &= \left\|\sum_{i=1}^n |x_i| \otimes |y_i|\right\|_\Delta .\end{aligned}$$

Consequently,

$$\begin{aligned}\|u\|_\Delta &\leq \inf\left\{\left\|\sum_{i=1}^n |x_i| \otimes |y_i|\right\|_\Delta : |u| \leq \sum_{i=1}^n |x_i| \otimes |y_i|\right\} \\ &= \inf\left\{\|v\|_\Delta : |u| \leq v \in E_+ \otimes F_+\right\}.\end{aligned}$$

□

Proof of Theorem 2.4.6 By Theorem 2.4.9, we have $\|u\|_\Delta = \|u\|_{|\Delta|}$ for all $u \in E \otimes F$. Thus, Δ is a norm on $E \otimes F$ with the desired properties as described in (b), by Theorem 2.4.4. But, as shown in Theorem 2.3.1, we have $\|\cdot\|_\Delta = \|\cdot\|_l$ on $E \otimes F$. Hence, Theorem 2.4.6 part (b) is proved. Part (a) now follows from Theorem 2.3.1 and the fact that the transposition map is a Riesz isometry. □

2.5 Order continuity of the l-norm

In the previous section we saw that the l-tensor product of two Banach lattices is again a Banach lattice. Now, we shall prove that the l-tensor product of two order continuous Banach lattices is again order continuous. We recall some facts about regular operators.

Let E and F be Riesz spaces. An operator $T : E \to F$ is called *regular* if $T = T_1 - T_2$ where $T_1, T_2 : E \to F$ are both positive operators. We denote the space of regular operators by $L^r(E,F)$. If F is Dedekind complete, then $L^r(E,F)$ becomes a Dedekind complete Riesz space with a modulus given by $|T|(x) = \sup\{|Ty| : |y| \leq x\}$ for all $T \in L^r(E,F)$ and $x \in E_+$. In this case, $L^r(E,F)$ contains all the order bounded operators. Recall that an operator $T : E \to F$ is *order bounded* if the image of any order bounded set in E, under T, is order bounded in F. It is clear that every regular operator is already order bounded.

In the case where E and F are Banach lattices, we may equip $L^r(E,F)$ with the norm $\|\cdot\|_r$ defined by

$$\|T\|_r = \inf\{\|S\| : 0 \leq S \in L^r(E,F), |Tx| \leq S|x| \,\forall\, x \in E_+\}$$

for all $T \in L^r(E,F)$, whence $\|T\|_r \leq \|T\|$. Then, $(L^r(E,F), \|\cdot\|_r)$ is a Banach space, which we denote by $\mathcal{L}^r(E,F)$. If F is Dedekind complete, then $\mathcal{L}^r(E,F)$ becomes a Banach lattice with $\|T\|_r = \|\,|T|\,\|$ for each $T \in \mathcal{L}^r(E,F)$ (cf. [76, Proposition 1.3.6] or [92, Chapter IV, §1, Proposition 1.4]).

In [85], Popa showed that if E and F are order continuous Banach lattices, then $E\widetilde{\otimes}_m F$ is an order continuous Banach lattice. In [68], a direct proof for Popa's result is given, which yields as a bonus, that if F has property (P), then $E\widetilde{\otimes}_m F$ is an order ideal of $\mathcal{L}^r(E^*, F)$.

Definition 2.5.1 A Banach lattice F is said to have *property (P)* if there exists a positive, contractive projection $F^{**} \to F$, where F (under evaluation) is identified with a vector sublattice of its bidual F^{**}.

It is readily verified that all dual Banach lattices have property (P) and that all Banach lattices with property (P) are Dedekind complete. If E and F are Banach lattices, with F possessing property (P). Then, by [92, Chapter IV, §4, Theorem 4.3], it follows that $\mathcal{L}^{\mathrm{maj}}(E,F)$ is a Banach sublattice of $\mathcal{L}^r(E,F)$. In fact, $\mathcal{L}^{\mathrm{maj}}(E,F)$ is an ideal of $\mathcal{L}^r(E,F)$. This result is used in the proof of the following:

Theorem 2.5.2 *If E and F are Banach lattices, each with order continuous norm, then $E\widetilde{\otimes}_m F$ has order continuous norm. Moreover, if F has property (P), then $E\widetilde{\otimes}_m F$ is an ideal in $\mathcal{L}^r(E^*, F)$.*

Proof. Since F is a Banach lattice with order continuous norm, F is Dedekind complete. Thus, the space of regular maps $\mathcal{L}^r(E^*, F)$ is a Dedekind complete Banach lattice.

Since continuous linear operators of finite rank are regular, it follows that $E \otimes F \subset \mathcal{L}^r(E^*, F)$. The ideal $A_{E\otimes F}$ generated by $E \otimes F$ in $\mathcal{L}^r(E^*, F)$ is contained in $\mathcal{L}^{\mathrm{maj}}(E^*, F)$; observe, by the definition of a majorizing operator, $0 \le S \le T$, $T \in \mathcal{L}^{\mathrm{maj}}(E^*, F)$ and $S \in \mathcal{L}^r(E^*, F)$ imply $S \in \mathcal{L}^{\mathrm{maj}}(E^*, F)$. Now, for $0 \le S \in A_{E\otimes F} \subset \mathcal{L}^r(E^*, F)$ we can find an operator of finite rank (in fact, of rank one) that majorizes S, implying that S itself is majorizing. Consequently, $A_{E\otimes F} \subset \mathcal{L}^{\mathrm{maj}}(E^*, F)$.

We claim that the $\|\cdot\|_{\mathrm{maj}}$-closure of $A_{E\otimes F}$ has order continuous norm. Since $A_{E\otimes F}$ is Dedekind complete, it suffices to show that

$$T_n \downarrow 0 \text{ in } A_{E\otimes F} \Rightarrow \|T_n\|_{\mathrm{maj}} \downarrow 0.$$

Let (T_n) be a sequence in $A_{E\otimes F}$ such that $T_n \downarrow 0$ and select $x \in E_+$ and $y \in F_+$ such that $0 \le T_n \le x \otimes y$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$ be given. Since E has order continuous norm, it follows from [3, 12.17] that there exists $\phi \in E^*_+$ such that

$$\langle x, (|\psi| - \phi)_+\rangle < \varepsilon \quad \text{for all } \|\psi\| \le 1.$$

Thus, if $(\psi_i)_{i=1}^k$ are functionals in the unit ball of E^*, then

$$\begin{aligned} |T_n\psi_i| &\le T_n|\psi_i| \\ &\le T_n(|\psi_i| - \phi)_+ + T_n\phi \\ &\le (|\psi_i| - \phi)_+(x)y + T_n\phi \\ &\le \varepsilon y + T_n\phi \end{aligned}$$

for each i $(1 \le i \le k)$ so that

$$\left\| \sup_{1\le i\le k} |T_n\psi_i| \right\| \le \varepsilon\|y\| + \|T_n\phi\|.$$

By the order continuity of the norm on F, we have that $\|T_n\| \downarrow 0$. Consequently, $\|T_n\|_{\mathrm{maj}} \downarrow 0$, which completes the proof of the claim.

To complete the proof of the theorem, it suffices to show that $A_{E\otimes F} \subset E\widetilde{\otimes}_m F$. To this end, let $0 \le T \in A_{E\otimes F}$ and choose $x \in E_+$ and $y \in F_+$ such that $T \le x \otimes y$.

We denote the boolean algebra of components of $x \otimes y$ by $C_{x\otimes y}$. Since $A_{E\otimes F}$ is an ideal in $\mathcal{L}^r(E^*, F)$ and $x \otimes y \in A_{E\otimes F}$, we have

$$C_{x\otimes y} \subset A_{E\otimes F}.$$

Let $\varepsilon > 0$ be given. By Freudenthal's Spectral Theorem (cf. [3, 6.9], [76, Theorem 1.2.18] or [106, Theorem 33.2]), there exist scalars $\alpha_1, \alpha_2, \dots, \alpha_k$ and disjoint $C_1, C_2, \dots, C_k \in C_{x\otimes y}$ such that

$$0 \le T - \sum_{i=1}^{k} \alpha_i C_i < \varepsilon(x \otimes y).$$

Consequently, T can be approximated in the $\|\cdot\|_{\text{maj}}$-norm by an element of the form $\sum_{i=1}^{k} \alpha_i C_i$ where $\alpha_1, \alpha_2, \dots, \alpha_k$ are scalars and $C_1, \dots, C_k \in C_{x\otimes y}$.

Before continuing, we fix some notation. For any set $D \subset E$, we let

$$\begin{aligned}
D^\sigma &:= \{x \in E : x_n \uparrow x \text{ for some sequence } (x_n) \subset D\} \\
D_\sigma &:= \{x \in E : x_n \downarrow x \text{ for some sequence } (x_n) \subset D\} \\
D^\eta &:= \{x \in E : x_\lambda \uparrow x \text{ for some net } \{x_\lambda\} \subset D\} \\
D_\eta &:= \{x \in E : x_\lambda \downarrow x \text{ for some net } \{x_\lambda\} \subset D\}.
\end{aligned}$$

Using this notation, it follows by de Pagter's Component Theorem (cf. [79] or [2, Theorem 2.6]) that we have

$$C_{x\otimes y} = \Big(((S_{x\otimes y})^\sigma)_\eta \Big)^\eta \tag{2.8}$$

where $S_{x\otimes y}$ consists exactly of elements of the form

$$\sum_{i=1}^{m} Q_i(x \otimes y)P_i^* \tag{2.9}$$

with the $Q_i(x \otimes y)P_i^*$ mutually disjoint and $Q_i : F \to F$, $P_i^* : E^* \to E^*$ band projections. Since the norm $\|\cdot\|_{\text{maj}}$ is order continuous on $A_{E\otimes F}$ and $C_{x\otimes y} \subset A_{E\otimes F}$, it follows from (2.8) that every C_i can be approximated in the norm $\|\cdot\|_{\text{maj}}$ by an element of the form (2.9). However, if $Q : F \to F$ and $P^* : E^* \to E^*$ are band projections, then

$$Q(x \otimes y)P^* = (P^{**}x) \otimes (Qy)$$

where $P^{**}x \in E$, since E is an ideal of E^{**} by the order continuity of the norm on E (cf. [76, Theorem 2.4.2]) and $P^{**}x \le x$. Thus, T can be approximated in the norm $\|\cdot\|_{\text{maj}}$ by an element of the projective cone of $E \otimes F$. Consequently, T is an element of the $\|\cdot\|_{\mathrm{maj}}$-closure of the projective cone. It now follows from Theorem 2.4.6 that $A_{E\otimes F} \subset E\widetilde{\otimes}_m F$ so that $E\widetilde{\otimes}_m F$ has order continuous norm.

Now suppose that F has property (P), then by the remark preceding the theorem, $\mathcal{L}^{\mathrm{maj}}(E^*, F)$ is a Banach lattice and an ideal in $\mathcal{L}^r(E^*, F)$. Since $E\widetilde{\otimes}_m F$ is the $\|\cdot\|_{\mathrm{maj}}$-closure of the ideal $A_{E\otimes F}$, it follows that $E\widetilde{\otimes}_m F$ is an ideal of $\mathcal{L}^{\mathrm{maj}}(E^*, F)$ and thus of $\mathcal{L}^r(E^*, F)$. $\square$

Corollary 2.5.3 *Let E and F be Banach lattices, each with order continuous norm, then $E\widetilde{\otimes}_l F$ has order continuous norm. Moreover, if F has property (P), then $E\widetilde{\otimes}_l F$ is an ideal in $\mathcal{L}^r(E^*, F)$.*

Proof. This follows from the fact that the transposition map is a Riesz isometry. $\square$

Observe that the above result also holds for the completed tensor products $E\widetilde{\otimes}_M F$, $E\widetilde{\otimes}_{{}^tM} F$, $E\widetilde{\otimes}_{|\mu|} F$ and $E\widetilde{\otimes}_{\Delta} F$ by Theorem 2.3.1.

2.6 Notes and remarks

In Section 2.4, we identified the notions of left order uniform and left order injective. We showed that the l-norm is an order reasonable crossnorm, possessing both of these properties. This is only part of the picture.

Let X and Y be Banach spaces. We recall that a reasonable crossnorm α on $X\otimes Y$ is called *projective* if, whenever X_1 and Y_1 are quotients of X and Y respectively, we have that $X_1 \otimes_\alpha Y_1$ is a quotient of $X \otimes_\alpha Y$.

It is well known that the Bochner norm Δ_p is not a projective norm (cf. [31]). However, it does possess an order theoretic version of projectivity, studied in [69]. We recall some terminology.

Let E and F be Banach lattices and let $T : E \to F$ be a positive linear operator; if $[0, Tx] = T[0, x]$ for all $x \in E_+$, then T is called *interval preserving*; and if $T[0, x]$ is dense in $[0, Tx]$ for all $x \in E_+$, then T is called *almost interval preserving.*

It is shown in [76, p. 42] that if E and F are Banach lattices and $T : E \to F$ is a positive linear mapping; then T is a Riesz homomorphism if and only if T^* is (almost) interval preserving; and T is almost interval preserving if and only if T^* is a Riesz homomorphism. The following result was proved in [69].

Theorem 2.6.1 *(a) Let E and E_1 be Banach lattices and let Y and Y_1 be Banach spaces. If $q_0 : E \to E_1$ is an almost interval preserving metric surjection and $q_1 : Y \to Y_1$ is a metric surjection, then $q_0 \otimes q_1 : E\widetilde{\otimes}_l Y \to E_1\widetilde{\otimes}_l Y_1$ is a metric surjection.*

(b) *Let X and X_1 be Banach spaces and let F and F_1 be Banach lattices. If $q_1 : F \to F_1$ is an almost interval preserving metric surjection and $q_0 : X \to X_1$ is a metric surjection, then $q_0 \otimes q_1 : X\widetilde{\otimes}_m F \to X_1\widetilde{\otimes}_m F_1$ is a metric surjection.*

In view of the above result, it is sensible to make the following definition.

Definition 2.6.2 If E and E_1 are Banach lattices, Y and Y_1 are Banach spaces, $q_0 : E \to E_1$ an almost interval preserving metric surjection and $q_1 : Y \to Y_1$ a metric surjection, then a reasonable crossnorm α is called *left order projective* (or in short, *left projective*) if $q_0 \otimes q_1 : E\widetilde{\otimes}_\alpha Y \to E_1\widetilde{\otimes}_\alpha Y_1$ is a metric surjection. The notion of a *right order projective* crossnorm is defined in a symmetrical manner.

It is intriguing that the l-norm enjoys both left injectivity and left projectivity, as well as left uniformity. For this reason, we have the formulae in Theorem 2.3.1 for calculating the l-norm. In particular, we may take the infimum over all representations; i.e.

$$\|u\|_l = \inf\left\{\left\|\sum_{i=1}^n \|y_i\|\,|x_i|\right\| : u = \sum_{i=1}^n x_i \otimes y_i\right\} \qquad \text{for all } u \in E\widetilde{\otimes}_l Y,$$

or, we may take the supremum over functionals in the dual unit ball; i.e.

$$\|u\|_l = \left\|\sup\left\{\left|\sum_{i=1}^n y^*(y_i)x_i\right| : \|y^*\| \le 1\right\}\right\| \qquad \text{for all } u \in E\widetilde{\otimes}_l Y.$$

The latter formula has the advantage of being independent of the representation $u = \sum_{i=1}^n x_i \otimes y_i$. On the other hand, as we shall see in Chapter 5, every tensor $u \in E\widetilde{\otimes}_l Y$ is characterized by a (non unique) series representation $u = \sum_{i=1}^\infty x_i \otimes y_i$ where $\|\sum_{i=1}^\infty |x_i|\| < \infty$ and $\lim_{i\to\infty} \|y_i\| = 0$. Moreover,

$$\|u\|_l = \inf\left\{\left\|\sum_{i=1}^\infty |x_i|\right\| \sup_{i\in\mathbb{N}} \|y_i\| : u = \sum_{i=1}^\infty x_i \otimes y_i, \left\|\sum_{i=1}^\infty |x_i|\right\| < \infty, \lim_{i\to\infty} \|y_i\| = 0\right\}.$$

In Section 2.4, we saw that the l-tensor product of two Banach lattices is again a Banach lattice. It is interesting to note that this property is not shared by any of Grothendieck's 'natural tensor norms', introduced in [52] (also see [31, 90]). This was shown by D. Pérez-García and I. Villanueva in [82], where they exhibit a Banach lattice E such that $E\widetilde{\otimes}_\alpha E$ does not have the so called 'Gordon-Lewis property' for any natural tensor norm α. Consequently, $E\widetilde{\otimes}_\alpha E$ cannot be isomorphic to a Banach lattice. Thus, a natural tensor norm is not an order reasonable crossnorm. The question of whether this is true for every tensor norm remains open. For background reading on the Gordon-Lewis property (and local unconditional structure), the reader would do well to consult [36].

In general, when considering the l-tensor product $E\widetilde{\otimes}_l Y$ of a Banach lattice E with a Banach space Y, a slight change in the category of operators on E allows for the use of numerous tensor norm techniques. We often exploit this fact throughout this thesis.

3

Martingales and the Radon Nikodým property

3.1 Introduction

There have been a number of authors that have considered stochastic processes in a general vector lattice. In particular, we are interested in the work of Troitsky, who formulated a generalized notion for a martingale in a Banach lattice [101]. One drawback to this approach is that we can only apply this theory to the Lebesgue-Bochner space $L^p(\mu, Y)$ when it is a Banach lattice. Even in this case, one can say little about the geometry of Y using these methods. Our purpose in this chapter is to remedy this problem, using the l-tensor product. Because the l-norm can be used to extend a scalar valued L^p-space to its vector-valued counterpart, it can also be used to extend the theory of Troitsky to the vector-valued setting, without the requirement of an ordering. With this goal in mind, we start by exhibiting the results of Troitsky in [101].

In Section 3.2, we formulate the notions of a filtration and a martingale in a general Banach space. We then study the Banach space of norm bounded martingales in this setting, as well as the subspaces of norm convergent, weakly convergent and weak* convergent martingales. A generalization of Doob's convergence is presented. Unless otherwise mentioned, the results in this section are due to Troitsky [101]. Although the work of Troitsky is formulated for Banach lattices, there are many results that do not require an order structure.

Having studied the convergence of martingales in Banach spaces, we now look at the impact of adding a lattice structure in Section 3.3. We modify the Banach space definitions of a filtration and a martingale appropriately for the Banach lattice setting. This allows for the formulation of submartingales and supermartingales. Our focus is the inheritance of a lattice structure in the space of norm bounded martingales from the underlying Banach lattice. Again, unless otherwise mentioned, the results in this section are due to Troitsky [101].

Section 3.4 is pivotal to our study of these abstract martingales in the vector-valued setting. We consider the l-tensor product of a filtration on a Banach lattice E with a filtration on a Banach space Y. Our considerations reveal an important distributive property of the space of norm convergent martingales on $E\widetilde{\otimes}_l Y$. This property allows for an explicit description of norm convergent martingales in the Lebesgue-Bochner spaces, which is studied in Chapter 5. The material in this section is original an can be found in [26].

The abstract martingale techniques developed in previous sections can now be used to provide some answers to the following question, mentioned in the Introduction:

- *Given a Banach space Y that endows $L^p(\mu, Y)$ with a certain property, for which Banach lattices E does this property hold in $E\widetilde{\otimes}_l Y$?*

In Chapter 1, we saw that a Banach space Y has the Radon Nikodým property if and only if every $L^p(\mu, Y)$-bounded martingale, $1 < p < \infty$, is norm convergent. We show, in Section 3.5, that this result holds in the l-tensor product $E^*\widetilde{\otimes}_l Y$, for all order continuous duals E^* of separable Banach lattices E. In proving this result, we consider martingales in the space of cone absolutely summing operators. As a consequence, we are able to generalize another classical result, namely Theorem 1.3.7. We prove that if Y is a Banach space, then Y^* has the Radon Nikodým property if and only if $E^*\widetilde{\otimes}_l Y^* = (E\widetilde{\otimes}_l Y)^*$ for all separable Banach lattices E with order continuous dual. To our knowledge, the results in this section are new.

3.2 Martingales in Banach spaces

Throughout, let Y denote a Banach space and (Ω, Σ, μ) denote a finite measure space. If (Σ_i) is a filtration, then the sequence of corresponding conditional expectations $(\mathbb{E}(\cdot \,|\, \Sigma_i))$ constitute a sequence of contractive commuting projections on $L^p(\mu)$, with increasing range. This observation suggests an abstract definition for a filtration on a Banach space.

Definition 3.2.1 Let Y be a Banach space.

(a) If $T_i : Y \to Y$ is a contractive projection and $T_{i\wedge j} = T_i T_j$ for each $i, j \in \mathbb{N}$, then the sequence of projections (T_i) is called a *BS-filtration* on Y.

(b) If (T_i) is a BS-filtration on Y, then a sequence $(f_i) \subset Y$ is said to be *adapted* to (T_i) if $f_i \in \mathcal{R}(T_i)$ for each $i \in \mathbb{N}$.

(c) If (T_i) is a BS-filtration on Y, then (f_i, T_i) is called a *martingale* on Y if $T_i f_j = f_i$ for all $i \leq j$.

We use the prefix 'BS' to indicate that the filtration acts on a Banach space. Later, we will introduce an additional definition for a filtration on a Banach lattice, for which we use the prefix 'BL'.

It follows from the above definition that if (f_i, T_i) is a martingale, then $(f_i) \subset Y$ is adapted to the BS-filtration (T_i). It also follows that $\mathcal{R}(T_i) \subset \mathcal{R}(T_j)$ for $i \leq j$.

Our primary example of a BS-filtration, other than the classical L^p-space setting, are the partial sum projections corresponding to a Schauder decomposition. The notion of a BS-filtration (T_i) on a Banach space Y is weaker than that of a Schauder decomposition of Y. Indeed, for a BS-filtration, we do not require $x = \lim_{i\to\infty} T_i x$ for all $x \in Y$.

Now consider

$$M(Y, T_i) = \{(f_i, T_i) : (f_i, T_i) \text{ is a martingale in } Y\}.$$

Then $M(Y, T_i)$ is a vector space if we define $(f_i, T_i) + (g_i, T_i) = (f_i + g_i, T_i)$ and $\lambda(f_i, T_i) = (\lambda f_i, T_i)$ for all $\lambda \in \mathbb{R}$. The map $\Theta : M(Y, T_i) \to Y^{\mathbb{N}}$, defined by $\Theta((f_i, T_i)) = (f_i)$, is a linear injection.

It is well known that the space of all norm bounded sequences on Y, denoted by $\ell_\infty(Y) := \left\{(y_i) \in Y^{\mathbb{N}} : \sup_{i\in\mathbb{N}} \|y_i\| < \infty\right\}$, is a Banach space with respect to the norm $\|(y_i)\|_\infty := \sup_{i\in\mathbb{N}} \|y_i\|$. We induce this norm on a subspace of $M(Y, T_i)$.

Definition 3.2.2 Let Y be a Banach space and (T_i) a BS-filtration on Y. We define the *space of norm bounded martingales* as

$$\mathcal{M}(Y, T_i) = \left\{(f_i, T_i) \in M(Y, T_i) : \sup_{i\in\mathbb{N}} \|f_i\| < \infty\right\},$$

together with the norm defined by $\|(f_i, T_i)\| = \sup_{i\in\mathbb{N}} \|f_i\|$ for all $(f_i, T_i) \in \mathcal{M}(Y, T_i)$.

It is evident that $\mathcal{M}(Y, T_i)$ is a normed space with respect to $\|\cdot\|$ and that Θ is an isometry from $\mathcal{M}(Y, T_i)$ into $\ell_\infty(Y)$. Notice that if $(f_i, T_i) \in \mathcal{M}(Y, T_i)$, then

$$\|(f_i, T_i)\| = \sup_{i\in\mathbb{N}} \|f_i\| = \lim_{i\to\infty} \|f_i\|.$$

This fact follows easily from $\|f_i\| = \|T_i f_{i+1}\| \leq \|f_{i+1}\|$.

Theorem 3.2.3 *Let Y be a Banach space and (T_i) a BS-filtration on Y. Then $\mathcal{M}(Y, T_i)$ is a Banach space.*

Proof. Select a Cauchy sequence $(f_i^{(n)}, T_i)$ from $\mathcal{M}(Y, T_i)$. Since $\ell^\infty(Y)$ is a Banach space, the sequence $(f_i^{(n)})$ converges to a limit $(f_i) \in \ell^\infty(Y)$. It suffices to check that this limit is a martingale. Indeed, for $i \leq j$, we have $f_i^{(n)} = T_i f_j^{(n)}$ for each $n \in \mathbb{N}$. Since each T_i is bounded, we have $\lim_{n\to\infty} T_i f_j^{(n)} = T_i f_j = f_i$ as required. $\square$

Let $I_i = \mathrm{id}_Y$ for all $i \in \mathbb{N}$, where id_Y denotes the identity map on Y. Then (I_i) is a (trivial) BS-filtration on Y and

$$(f_i, I_i) \in \mathcal{M}(Y, I_i) \iff (f_i) \text{ is a constant sequence in } Y.$$

If we define $\Psi\colon Y \to \mathcal{M}(Y, I_i)$ by $\Psi(f) = (f_i, I_i)$, where $f = f_i$ for all $i \in \mathbb{N}$, then Y is isometrically isomorphic to $\mathcal{M}(Y, I_i)$. Plainly, every martingale in $\mathcal{M}(Y, I_i)$ is convergent.

Definition 3.2.4 Let Y be a Banach space and (T_i) a BS-filtration on Y. We define the *space of norm convergent martingales* by

$$\mathcal{M}_{\mathrm{nc}}(Y, T_i) = \{(f_i, T_i) \in \mathcal{M}(Y, T_i) : (f_i) \text{ is norm convergent in } Y\}.$$

Since convergent sequences are norm bounded, we have $\mathcal{M}_{\mathrm{nc}}(Y, T_i) \subset \mathcal{M}(Y, T_i)$. We verify that $\mathcal{M}_{\mathrm{nc}}(Y, T_i)$ is a complete with respect to the norm induced by $\mathcal{M}(Y, T_i)$.

Theorem 3.2.5 *Let Y be a Banach space and (T_i) a BS-filtration on Y. Then $\mathcal{M}_{\mathrm{nc}}(Y, T_i)$ is a Banach space.*

Proof. Let $(f_i^{(n)}, T_i)$ be a Cauchy sequence in $\mathcal{M}_{\mathrm{nc}}(Y, T_i)$. Since $\mathcal{M}(Y, T_i)$ is a Banach space by Theorem 3.2.3, the sequence $(f_i^{(n)}, T_i)$ converges to a limit $(f_i, T_i) \in \mathcal{M}(Y, T_i)$. We check that this limit is a convergent martingale. Let $\varepsilon > 0$ and select $n \in \mathbb{N}$ such that $\sup_{i\in\mathbb{N}} \|f_i^{(n)} - f_i\| < \varepsilon/3$. Since $(f_i^{(n)})_{i=1}^\infty$ is a Cauchy sequence, there exists $N > 0$ so that $i, j \geq N$ implies $\|f_i^{(n)} - f_j^{(n)}\| \leq \varepsilon/3$. Consequently,

$$\|f_i - f_j\| \leq \|f_i - f_i^{(n)}\| + \|f_j - f_j^{(n)}\| + \|f_i^{(n)} - f_j^{(n)}\| \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,$$

whence (f_i) is a Cauchy sequence in Y. This completes the proof. $\square$

To describe $\mathcal{M}_{\mathrm{nc}}(Y, T_i)$, we use the following analogue of a Lemma 1.3.9 (cf. [37, Chapter 5, §2, Corollary 2]). This result and its corollary can be found in [26].

Proposition 3.2.6 *Let Y be a Banach space and (T_i) a BS-filtration on Y. Then $f \in \overline{\bigcup_{i=1}^\infty \mathcal{R}(T_i)}$ if and only if $\lim_{i\to\infty} \|T_i f - f\| = 0$.*

Proof. Suppose that $\lim_{i\to\infty} T_i f = f$. It is evident that $T_i f \in \mathcal{R}(T_i)$ for each $i \in \mathbb{N}$ so that $f \in \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$. Conversely, suppose that $f \in \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$. Then there exists a sequence $(f_n) \subset \bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$ such that $\lim_{n\to\infty} f_n = f$. Thus, for each $\varepsilon > 0$, there exists $n \in \mathbb{N}$ so that $\|f_n - f\| < \varepsilon/2$. Since (T_i) is a filtration on Y, there exists an $N_\varepsilon \in \mathbb{N}$ such that $i \geq N_\varepsilon$ implies $f_n \in \mathcal{R}(T_i)$. Hence,

$$\begin{aligned}
\|T_i f - f\| &\leq \|T_i f - f_n\| + \|f_n - f\| \\
&= \|T_i(f - f_n)\| + \|f_n - f\| \\
&\leq \|f - f_n\| + \|f_n - f\| \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon
\end{aligned}$$

completes the proof. □

Corollary 3.2.7 *Let Y be a Banach space and (f_i, T_i) a martingale in Y. Then (f_i, T_i) converges to f if and only if $f \in \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$ and $f_i = T_i f$ for all $i \in \mathbb{N}$.*

Proof. Suppose (f_i, T_i) converges to f, then it is clear that $f \in \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$. Also, for $i \leq j$, we have $T_i f_j = f_i$ so that $\lim_{j\to\infty} T_i f_j = T_i f = f_i$. Conversely, by the above proposition, we have $\|T_i f - f\| = \|f_i - f\| \to 0$ as $i \to \infty$, which completes the proof. □

We recall the following definition from [101].

Definition 3.2.8 Let Y be a Banach space and (T_i) a BS-filtration on Y.

(a) If $\lim_{i\to\infty} T_i y = y$ for all $y \in Y$, then the BS-filtration (T_i) is said to be *dense* in Y.
(b) If (f_i, T_i) is a martingale, then (f_i, T_i) is called *fixed* if there exists $f \in Y$ such that $f_i = T_i f$ for all $i \in \mathbb{N}$. In this case, (f_i, T_i) is said to be *fixed on f*.

By Proposition 3.2.6, it is obvious that a BS-filtration (T_i) on a Banach space Y is dense in Y if and only if $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)} = Y$. Also, Corollary 3.2.7 implies that all convergent martingales are fixed, but not all fixed martingales are convergent, unless the corresponding BS-filtration is dense. This gives us an intermediate space of martingales.

Definition 3.2.9 Let Y be a Banach space and (T_i) a BS-filtration on Y. Define the *space of fixed martingales* by

$$\mathcal{M}_{\mathrm{f}}(Y, T_i) = \{(f_i, T_i) \in \mathcal{M}(Y, T_i) : \exists\ f \in Y \text{ so that } T_i f = f_i\ \forall\ i \in \mathbb{N}\}.$$

Observe that if $(f_i, T_i) \in \mathcal{M}_{\mathrm{f}}(Y, T_i)$ is fixed on $f \in Y$, then $\sup_{i\in\mathbb{N}} \|f_i\| = \sup_{i\in\mathbb{N}} \|T_i f\| \leq \|f\| < \infty$. So plainly, we have $\mathcal{M}_{\mathrm{nc}}(Y, T_i) \subset \mathcal{M}_{\mathrm{f}}(Y, T_i) \subset \mathcal{M}(Y, T_i)$. In general, $\mathcal{M}_{\mathrm{f}}(Y, T_i)$ need not be complete unless $\mathcal{M}_{\mathrm{nc}}(Y, T_i) = \mathcal{M}_{\mathrm{f}}(Y, T_i)$ or $\mathcal{M}_{\mathrm{f}}(Y, T_i) = \mathcal{M}(Y, T_i)$. We extend the notion of a dense BS-filtration in a Banach space with the following definition.

Definition 3.2.10 Let Y be a Banach space and (T_i) a BS-filtration on Y. We say that (T_i) is *complemented* in Y if there exists a contractive projection $T_\infty : Y \to Y$ with $\mathcal{R}(T_\infty) = \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$ and $T_i T_\infty = T_\infty T_i = T_i$ for all $i \in \mathbb{N}$.

Note that any BS-filtration that is dense in a Banach space Y is complemented by the identity operator. Consequently, we may replace the word ‘dense’ with the word ‘complemented’.

Proposition 3.2.11 *Let Y be a Banach space and (T_i) a BS-filtration on Y. If (T_i) is complemented in Y, we have $\mathcal{M}_{\mathrm{nc}}(Y, T_i) = \mathcal{M}_{\mathrm{f}}(Y, T_i)$. In this case, $\mathcal{M}_{\mathrm{f}}(Y, T_i)$ is complete.*

Proof. If (T_i) is complemented in Y by $T_\infty : Y \to Y$ and $(f_i, T_i) \in \mathcal{M}_{\mathrm{f}}(Y, T_i)$ is fixed on some $f \in Y$, then (f_i, T_i) is also fixed on $T_\infty f \in \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$. Consequently, (f_i, T_i) is convergent by Corollary 3.2.7. The completeness is taken care of by Theorem 3.2.5 $\square$.

If (Σ_i) is a classical filtration, then $(\mathbb{E}(\,\cdot\,|\Sigma_i))$ is complemented by $\mathbb{E}(\,\cdot\,|\bigvee_{i=1}^{\infty} \Sigma_i)$, where $\bigvee_{i=1}^{\infty} \Sigma_i$ denotes the σ-algebra generated by $\bigcup_{i=1}^{\infty} \Sigma_i$. Consequently, a classical martingale is convergent if and only if it is fixed.

A well known result of Dunford and Pettis asserts that a sequence $(f_i) \subset L^1(\mu)$ is norm bounded (i.e. $\sup_{i\in\mathbb{N}} \|f_i\|_1 < \infty$) and uniformly integrable if and only if (f_i) is relatively weakly compact in $L^1(\mu)$ (cf. [43]). Note that we are using the weaker form of uniform integrability in Definition 1.2.5. Replacing uniform integrability with relative weak compactness allows us to generalize Doob's Convergence Theorem to the Banach space setting.

Theorem 3.2.12 *Let Y be a Banach space and (T_i) a BS-filtration on Y. If $(f_i, T_i) \in M(Y, T_i)$ and (f_i) is relatively weakly compact in Y, then $(f_i, T_i) \in \mathcal{M}_{\mathrm{f}}(Y, T_i)$. If, in addition, (T_i) is complemented in Y, then $(f_i, T_i) \in \mathcal{M}_{\mathrm{nc}}(Y, T_i)$.*

Proof. Let $(f_i, T_i) \in M(Y, T_i)$ with (f_i) relatively weakly compact in Y. By the Eberlein-Šmulian Theorem (cf. [33]), there exists a subsequence (f_{i_k}) that converges weakly to some $f \in Y$. Since bounded maps are weakly continuous, we have $T_j f_{i_k} \to$

$T_j f$ weakly as $k \to \infty$ for all $j \in \mathbb{N}$. Since $T_j f_{i_k} = f_j$ for large k, we have $f_j = T_j f$ for each $j \in \mathbb{N}$. If (T_i) is complemented, we have $\mathcal{M}_{\mathrm{nc}}(Y, T_i) = \mathcal{M}_{\mathrm{f}}(Y, T_i)$ by Proposition 3.2.11. This completes the proof. □

For a reflexive Banach space Y, the weak and weak* topologies coincide. By the Banach Alaoglu Theorem, every norm bounded martingale is relatively weakly compact. Thus, we have an immediate corollary.

Corollary 3.2.13 *Let Y be a reflexive Banach space and (T_i) a BS-filtration on Y. Then, $\mathcal{M}_{\mathrm{f}}(Y, T_i) = \mathcal{M}(Y, T_i)$. In the case where (T_i) is complemented in Y, we even have $\mathcal{M}_{\mathrm{nc}}(Y, T_i) = \mathcal{M}_{\mathrm{f}}(Y, T_i) = \mathcal{M}(Y, T_i)$.*

We pass to weakly convergent martingales. Motivated by the results on weak convergence of martingales in [101], we make the following definition.

Definition 3.2.14 Let Y be a Banach space and (T_i) a BS-filtration on Y. Define the *space of weakly convergent martingales* by

$$\mathcal{M}_{\mathrm{wc}}(Y, T_i) = \{(f_i, T_i) \in \mathcal{M}(Y, T_i) : (f_i) \text{ is weakly convergent in } Y\}.$$

Proposition 3.2.15 *Let Y be a Banach space and (T_i) a BS-filtration on Y, then $\mathcal{M}_{\mathrm{wc}}(Y, T_i) \subset \mathcal{M}_{\mathrm{f}}(Y, T_i)$.*

Proof. Suppose $(f_i, T_i) \subset \mathcal{M}_{\mathrm{wc}}(Y, T_i)$ converges weakly to $f \in Y$. Since bounded operators are also weakly continuous, it follows that $T_j f_i \to T_j f$ weakly as $i \to \infty$ for all $j \in \mathbb{N}$. Consequently, for $j \leq i$, we ascertain from $f_j = T_j f_i \to T_j f$ weakly as $i \to \infty$, that $f_j = T_j f$ for each $j \in \mathbb{N}$. □

Again, $\mathcal{M}_{\mathrm{wc}}(Y, T_i)$ need not be complete unless $\mathcal{M}_{\mathrm{wc}}(Y, T_i) = \mathcal{M}_{\mathrm{nc}}(Y, T_i)$ or $\mathcal{M}_{\mathrm{wc}}(Y, T_i) = \mathcal{M}(Y, T_i)$.

Corollary 3.2.16 *Let Y be a Banach space and (T_i) a BS-filtration on Y. If (T_i) is complemented in Y, a martingale $(f_i, T_i) \in M(Y, T_i)$ is norm convergent if and only if it is weakly convergent. In this case, $\mathcal{M}_{\mathrm{wc}}(Y, T_i)$ is complete.*

Proof. By the above proposition, we have the inclusions $\mathcal{M}_{\mathrm{nc}}(Y, T_i) \subset \mathcal{M}_{\mathrm{wc}}(Y, T_i) \subset \mathcal{M}_{\mathrm{f}}(Y, T_i)$. By Proposition 3.2.11 we have $\mathcal{M}_{\mathrm{nc}}(Y, T_i) = \mathcal{M}_{\mathrm{f}}(Y, T_i)$ and the assertion follows immediately. An application of Theorem 3.2.5 completes the proof. □

The following result can be found in [26].

Proposition 3.2.17 *Let Y be a Banach space and (T_i) a BS-filtration on Y. Then $L : \mathcal{M}_{\mathrm{nc}}(Y,T_i) \to \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$, defined by $L((f_i,T_i)) = \lim_{i\to\infty} f_i$, is a surjective isometry.*

Proof. It follows easily that L is well defined, linear and

$$\|L\| = \sup\{\|L(f_i,T_i)\| : \|(f_i,T_i)\| \leq 1\} \leq 1.$$

To see that L is a surjection, let $f \in \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$. Then $T_i f \to f$ in norm and $(T_i f, T_i)$ is a martingale on Y such that $L((T_i f, T_i)) = f$. Also, $L((f_i,T_i)) = 0$ implies that $\lim_{i\to\infty} f_i = 0$ and Corollary 3.2.7 assures us that $f_i = T_i 0 = 0$ for each $i \in \mathbb{N}$. Thus, it follows that L is injective. Furthermore,

$$\|L^{-1}\| = \sup\left\{\|L^{-1}f\| : \|f\| \leq 1\right\} = \sup\left\{\sup_{i\in\mathbb{N}} \|T_i f\| : \|f\| \leq 1\right\} \leq 1,$$

which completes the proof that L is a surjective isometry. □

Corollary 3.2.18 *Let (T_i) be a BS-filtration on a Banach space Y. Then $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)} = Y$ if and only if $\mathcal{M}_{\mathrm{nc}}(Y,T_i)$ is isometrically isomorphic to $\mathcal{M}(Y,I_i)$, where $I_i = \mathrm{id}_Y$ for all $i \in \mathbb{N}$.*

Proof. This can easily been seen from the fact that $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)} = \mathcal{M}_{\mathrm{nc}}(Y,T_i)$ and $Y = \mathcal{M}(Y,I_i)$. □

To summarize, we have established the following inclusions for a BS-filtration (T_i) on a Banach space Y:

$$\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)} = \mathcal{M}_{\mathrm{nc}}(Y,T_i) \subset \mathcal{M}_{\mathrm{wc}}(Y,T_i) \subset \mathcal{M}_{\mathrm{f}}(Y,T_i) \subset \mathcal{M}(Y,T_i),$$

with equality holding throughout in the case where Y is reflexive and (T_i) is complemented in Y.

We now consider dual spaces, which gives access to weak* convergent martingales. Again, the following definition is motivated by the results on weak* convergence of martingales, found in [101].

Definition 3.2.19 Let Y^* be the dual of a Banach space and (T_i) a BS-filtration on Y^*. Define the *space of weak* convergent martingales* by

$$\mathcal{M}_{\mathrm{w*c}}(Y^*,T_i) = \{(f_i,T_i) \in \mathcal{M}(Y^*,T_i) : (f_i) \text{ is weak* convergent in } Y^*\}.$$

Again, $\mathcal{M}_{\mathrm{w*c}}(Y^*, T_i)$ need not be complete. If (T_i) is a BS-filtration on a Banach space Y, it is easily verified that the sequence of adjoint operators (T_i^*) on Y^* is also a BS-filtration. Indeed, each T_i^* is a contractive projection and $T_i^* T_j^* = (T_j T_i)^* = T_{i\wedge j}^*$ for all $i, j \in \mathbb{N}$.

Definition 3.2.20 Let Y be a Banach space and (T_i) a BS-filtration on Y. We refer to the BS-filtration of adjoint operators (T_i^*) on Y^* as the *dual filtration.*

Proposition 3.2.21 *Let Y be a Banach space and (T_i) a BS-filtration on Y. Then we have the inclusion $\mathcal{M}_{\mathrm{w*c}}(Y^*, T_i^*) \subset \mathcal{M}_{\mathrm{f}}(Y^*, T_i^*)$. Moreover, if (T_i) is complemented in Y, then $\mathcal{M}_{\mathrm{w*c}}(Y^*, T_i^*) = \mathcal{M}_{\mathrm{f}}(Y^*, T_i^*)$.*

Proof. Let $(f_i^*, T_i^*) \in \mathcal{M}_{\mathrm{w*c}}(Y^*, T_i^*)$ with weak* limit $f^* \in Y^*$. Since T_i^* is adjoint, and thus weak*-continuous for each $i \in \mathbb{N}$, we have $T_j^* f_i^* \to T_j^* f^*$ weak* as $i \to \infty$. On the other hand, for $j \leq i$, we have $T_j^* f_i^* = f_j^*$. Consequently, $f_j^* = T_j^* f^*$ for each $j \in \mathbb{N}$, so that $(f_i^*, T_i^*) \in \mathcal{M}_{\mathrm{f}}(Y^*, T_i^*)$.

Now suppose that (T_i) is complemented in Y by the contractive projection $T_\infty : Y \to Y$ and $(f_i^*, T_i^*) \in \mathcal{M}_{\mathrm{f}}(Y^*, T_i^*)$ is fixed on $f^* \in Y^*$. By Proposition 3.2.6, we have $\|T_i T_\infty f - T_\infty f\| \to 0$ as $i \to \infty$ for each $f \in Y$. Consequently,

$$\langle f, f_i^* \rangle = \langle f, T_i^* f^* \rangle = \langle T_i T_\infty f, f^* \rangle \to \langle T_\infty f, f^* \rangle = \langle f, T_\infty^* f^* \rangle$$

as $i \to \infty$ for all $f \in Y$. Thus, $(f_i^*, T_i^*) \in \mathcal{M}_{\mathrm{w*c}}(Y^*, T_i^*)$. $\square$

We conclude with:

Theorem 3.2.22 *Let Y be a Banach space and (T_i) a BS-filtration on Y, then $\mathcal{M}(Y, T_i) \subset \mathcal{M}_{\mathrm{f}}(Y^{**}, T_i^{**})$. In the case (T_i^*) is complemented in Y^*, we have*

$$\mathcal{M}(Y, T_i) \subset \mathcal{M}_{\mathrm{w*c}}(Y^{**}, T_i^{**}) = \mathcal{M}_{\mathrm{f}}(Y^{**}, T_i^{**}).$$

Proof. Let $Z = \bigcup_{i=1}^\infty \mathcal{R}(T_i^*)$. We start by showing that $\mathcal{M}(Y, T_i) \hookrightarrow Z^*$ isometrically. Consider the map $\mathcal{M}(Y, T_i) \to Z^*$, defined by $(f_i, T_i) := F \mapsto f_F^{**}$, where

$$\langle f^*, f_F^{**} \rangle = \lim_{i\to\infty} \langle f_i, f^* \rangle$$

for all $f^* \in Z$. We first check that the map $F \mapsto f_F^{**}$ is well defined. Indeed, if $f^* \in Z$, then $f^* \in \mathcal{R}(T_i^*)$ for some $i \in \mathbb{N}$. Consequently, $T_i^* f^* = f^*$ and, for every $j \geq i$, we get

$$\langle f_j, f^* \rangle = \langle f_j, T_i^* f^* \rangle = \langle T_i f_j, f^* \rangle = \langle f_i, f^* \rangle.$$

Thus, the sequence $(\langle f_i, f^*\rangle)$ is eventually constant; i.e. f_F^{**} exists. It is now readily verified that the map $F \mapsto f_F^{**}$ is linear. Moreover,

$$|\langle f^*, f_F^{**}\rangle| = \lim_{i\to\infty} |\langle f_i, f^*\rangle| \le \lim_{i\to\infty} \|f_i\|\|f^*\| \le \sup_{i\in\mathbb{N}} \|f_i\|\|f^*\|,$$

which shows $\|f_F^{**}\| \le \|(f_i, T_i)\|$. For the reverse inequality, let $j \in \mathbb{N}$ and, using the Hahn-Banach Theorem, choose a norm one functional $y^* \in Y^*$ with $\langle f_j, y^*\rangle = \|f_j\|$. Then

$$\|f_F^{**}\| \ge |\langle T_j^* y^*, f_F^{**}\rangle| = \lim_{i\to\infty} |\langle f_i, T_j^* y^*\rangle| = \lim_{i\to\infty} |\langle T_j f_i, y^*\rangle| = |\langle f_j, y^*\rangle| = \|f_j\|.$$

Consequently, $\|f_F^{**}\| \ge \sup_{j\in\mathbb{N}} \|f_j\| = \|(f_i, T_i)\|$ and the map $F \mapsto f_F^{**}$ is an isometry.

To complete the first part of the proof let $F = (f_i, T_i) \in \mathcal{M}(Y, T_i)$ and, using the Hahn-Banach Theorem, extend the corresponding functional f_F^{**} uniquely to a functional $\widehat{f}_F^{**}$ on Y^*. Then, for every $y^* \in Y^*$ and $j \in \mathbb{N}$,

$$\begin{aligned}
&\langle y^*, T_j^{**}\widehat{f}_F^{**}\rangle = \langle T_j^* y^*, \widehat{f}_F^{**}\rangle = \langle T_j^* y^*, f_F^{**}\rangle \\
&= \lim_{i\to\infty} \langle f_i, T_j^* y^*\rangle = \lim_{i\to\infty} \langle T_j f_i, y^*\rangle = \langle f_j, y^*\rangle.
\end{aligned}$$

Consequently, $f_i = T_i^{**}\widehat{f}_F^{**}$ for all $i \in \mathbb{N}$ and $(f_i, T_i) \in \mathcal{M}_{\mathrm{f}}(Y^{**}, T_i^{**})$. For the last part of the proof, assume (T_i^*) is complemented in Y^* and apply Proposition 3.2.21.
□

It follows easily from the above proof that, in the case (T_i^*) is complemented in Y^*, we also have the isometric embedding $\mathcal{M}(Y, T_i) \hookrightarrow Y^{**}$ because $\overline{Z}$ is complemented in Y^* by a norm one projection. Consequently, we have $\mathcal{M}(Y, T_i) \hookrightarrow Z^* = \overline{Z}^* \hookrightarrow Y^{**}$.

If we add the above results to our previous summary, we get the following inclusions for a BS-filtration (T_i), with complemented dual filtration in Y^*, on a Banach space Y:

$$\begin{aligned}
\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)} = \mathcal{M}_{\mathrm{nc}}(Y, T_i) &\subset \mathcal{M}_{\mathrm{wc}}(Y, T_i) \\
&\subset \mathcal{M}_{\mathrm{w*c}}(Y, T_i) \subset \mathcal{M}_{\mathrm{f}}(Y, T_i) \subset \mathcal{M}(Y, T_i) \\
&\subset Y^{**} = \mathcal{M}_{\mathrm{nc}}(Y^{**}, T_i^{**}) \subset \mathcal{M}_{\mathrm{wc}}(Y^{**}, T_i^{**}) \\
&\subset \mathcal{M}_{\mathrm{w*c}}(Y^{**}, T_i^{**}) = \mathcal{M}_{\mathrm{f}}(Y^{**}, T_i^{**}) \\
&\subset \mathcal{M}(Y^{**}, T_i^{**}),
\end{aligned}$$

where all the above inclusions are isometries.

3.3 Martingales in Banach lattices

Let (Ω, Σ, μ) denote a finite measure space and $1 \leq p < \infty$. It is well known that the spaces $L^p(\mu)$ are Banach lattices. Moreover, if $\Sigma_1 \subset \Sigma$, then $\mathbb{E}(\,\cdot \mid \Sigma_1) : L^p(\mu) \to L^p(\mu)$ is strictly positive and $\mathcal{R}(\mathbb{E}(\,\cdot \mid \Sigma_1))$ is a closed Riesz subspace of $L^p(\mu)$. In fact, $\mathcal{R}(\mathbb{E}(\,\cdot \mid \Sigma_1)) = L^p(\Omega, \Sigma_1, \mu|_{\Sigma_1})$. It turns out that these properties are key to accessing vector-valued results concerning filtrations.

We consider martingales in a Banach lattice. With the added lattice structure, it is essential to add to the definition of a BS-filtration.

Definition 3.3.1 Let E be a Banach lattice.

(a) A BS-filtration (T_i) on E for which $T_i \geq 0$ for each $i \in \mathbb{N}$ is called a *positive BS-filtration* on E. If T_i is strictly positive (i.e. T_i is positive and $T_i|x| = 0 \Rightarrow x = 0 \;\forall\, x \in E$) for each $i \in \mathbb{N}$, then (T_i) is called a *strictly positive BS-filtration.*
(b) A positive BS-filtration (T_i) on E for which $\mathcal{R}(T_i)$ is a (closed) Riesz subspace of E, for each $i \in \mathbb{N}$, is called a *BL-filtration* on E.
(c) If (T_i) is a positive BS-filtration on E, then (f_i, T_i) is called a *submartingale (supermartingale)* on E if $T_i f_j \geq (\leq) f_i$ for all $i \leq j$.

Observe in the above definition that a sequence is a martingale if and only if it is both a submartingale and a supermartingale. In the case where a submartingale (supermartingale) (f_i, T_i) is norm bounded, we will still use the notation $\|(f_i, T_i)\| = \sup_{i \in \mathbb{N}} \|f_i\|$.

The notion of a positive BS-filtration coincides with Troitsky's notion of a filtration on a Banach lattice in [101]. Also, the notion of a submartingale (supermartingale) in the above definition is consistent with [101]. However, there is a subtle difference between this definition and Definition 1.2.5(e). Indeed, the classical definition requires that a submartingale (supermartingale) be adapted to the corresponding filtration. In the definition above, there is no guarantee that $f_i \in \mathcal{R}(T_i)$ for all $i \in \mathbb{N}$ if (f_i, T_i) is a submartingale (supermartingale), unless (f_i, T_i) is also a martingale.

In the case where a positive BS-filtration (T_i) is complemented in the Banach lattice E by a contractive projection $T_\infty : E \to E$, it follows that T_∞ is positive. Indeed, if $f \in E_+$, then $T_i f \in E_+$ for each $i \in \mathbb{N}$. Thus, Proposition 3.2.6 implies $\lim_{i \to \infty} T_i f = T_i T_\infty f = T_\infty f \geq 0$. Consequently, T_∞ is positive.

In [92, Chapter III, §11, Proposition 11.5] it is shown that if $T : E \to E$ is a strictly positive projection on a Banach lattice E, then $\mathcal{R}(T)$ is a Banach sublattice of E. Consequently, every strictly positive BS-filtration is a BL-filtration.

In [101], it is remarked that if (T_i) is a positive BS-filtration on a Banach lattice E with $\overline{\bigcup_{i=1}^{\infty}\mathcal{R}(T_i)}$ a Riesz subspace of E, then one may as well assume $\overline{\bigcup_{i=1}^{\infty}\mathcal{R}(T_i)} = E$. When only considering positive BS-filtrations on a Banach lattice E, this is indeed the case. However, if one plans to extend the theory to the l-tensor product, it is important to maintain the distinction between $\overline{\bigcup_{i=1}^{\infty}\mathcal{R}(T_i)}$ and E. We, therefore, avoid this assumption.

We define a partial ordering on the space of martingales defined on a Banach lattice relative to a positive BS-filtration.

Definition 3.3.2 Let E be a Banach lattice and (T_i) a positive BS-filtration on E. We define a partial ordering on $\mathcal{M}(E,T_i)$ by

$$(f_i, T_i) \geq 0 \Longleftrightarrow f_i \geq 0 \text{ for all } i \in \mathbb{N}.$$

The following result can be found in [26].

Proposition 3.3.3 *Let E be a Banach lattice and (T_i) a BL-filtration on E. If $L : \mathcal{M}_{\mathrm{nc}}(E,T_i) \to \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(T_i)}$ is defined by $L((f_i,T_i)) = \lim_{i\to\infty} f_i$, then $\mathcal{M}_{\mathrm{nc}}(E,T_i)$ is a Banach lattice and $L : \mathcal{M}_{\mathrm{nc}}(E,T_i) \to \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(T_i)}$ is a surjective Riesz isometry.*

Proof. It was shown in Proposition 3.2.17 that L is a surjective isometry. To see that L is positive is trivial, because if $(f_i,T_i) \geq 0$, then $f_i \geq 0$ for each $i \in \mathbb{N}$ and $\lim_{i\to\infty} f_i \geq 0$. Similarly, L^{-1} is positive, because if $0 \leq f \in \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(T_i)}$ then $T_i f \geq 0$ for each $i \in \mathbb{N}$; hence, $L^{-1}(f) = (T_i f, T_i) \geq 0$.

Since $\overline{\bigcup_{i=1}^{\infty}\mathcal{R}(T_i)}$ is a Riesz space, it follows that $\mathcal{M}_{\mathrm{nc}}(E,T_i)$ is also a Riesz space. Indeed, for $F, G \in \mathcal{M}_{\mathrm{nc}}(E,T_i)$, it is readily verified that $L^{-1}(L(F) \vee L(G))$ is the least upper bound in $\mathcal{M}_{\mathrm{nc}}(E,T_i)$ of $\{F, G\}$.

Thus, by the preceding part, L is a surjective Riesz isometry. Since $\|\cdot\|_E$ is a Riesz norm, the martingale norm is also a Riesz norm. Furthermore, since $\overline{\bigcup_{i=1}^{\infty}\mathcal{R}(T_i)}$ is a Banach lattice, $\mathcal{M}_{\mathrm{nc}}(E,T_i)$ is a Banach lattice. $\square$

For a Banach lattice E and positive BS-filtration (T_i) on E, the space $\mathcal{M}(E,T_i)$ is a partially ordered vector space with respect to the order defined in Definition 3.3.2. In general, $\mathcal{M}(E,T_i)$ need not be a lattice with respect to this ordering. To achieve a lattice ordering, we assume more about the order structure of E.

Definition 3.3.4 A Banach lattice E is said to be a *Kantorovič-Banach space (KB-space)* if every increasing norm bounded sequence is also norm convergent.

The spaces $L^p(\mu)$ $(1 \leq p < \infty)$ serve as classical examples of KB-spaces. With this additional structure, Troitsky [101] obtained the following result.

Theorem 3.3.5 *If E is a KB-space and (T_i) a positive BS-filtration on E, then $\mathcal{M}(E,T_i)$ is a Banach lattice with lattice operations given by*

$$(f_i,T_i)\vee(g_i,T_i)=\left(\lim_{i\to\infty}T_j(f_i\vee g_i),T_j\right)_{j=1}^{\infty},$$

$$(f_i,T_i)\wedge(g_i,T_i)=\left(\lim_{i\to\infty}T_j(f_i\wedge g_i),T_j\right)_{j=1}^{\infty}\quad \textit{and}$$

$$|(f_i,T_i)|=\left(\lim_{i\to\infty}T_j(|f_i|),T_j\right)_{j=1}^{\infty}$$

for all $(f_i,T_i),(g_i,T_i)\in\mathcal{M}(E,T_i)$.

Notice that the intuitive guess of $|(f_i,T_i)|=(|f_i|,T_i)$ is incorrect. One does not have to look far for a counter example: Consider the Banach lattice $L^1(\mu)$ with respect to the Lebesgue interval. The sequence $f_1=0$ and $f_i=\chi_{[0,1/2)}-\chi_{[1/2,1]}$ for all $i\geq 2$ is a martingale with respect to the filtration (Σ_i), defined by $\Sigma_1=\{\emptyset,[0,1]\}$ and $\Sigma_i=\{\emptyset,[0,\frac{1}{2}),[\frac{1}{2},1],[0,1]\}$ for all $i\geq 2$. However, $|f_i|=1$ for all $i\in\mathbb{N}$, which is not a martingale relative to (Σ_i). To prove the above theorem, we need a lemma from [101].

Lemma 3.3.6 *Let E be a Banach lattice and (T_i) a positive BS-filtration on E. If (f_i,T_i) and (g_i,T_i) are two norm bounded submartingales in E, then the following statements hold:*

(a) For each $j\in\mathbb{N}$, the sequence $(T_j(f_i\vee g_i))_{i=j}^{\infty}$ is increasing, norm bounded by $\|(f_i,T_i)\|+\|(g_i,T_i)\|$, and bounded below by $f_j\vee g_j$.

(b) If $(T_j(f_i\vee g_i))_{i=j}^{\infty}$ converges to h_j for each $j\in\mathbb{N}$, then $(h_i,T_i)\in\mathcal{M}(E,T_i)$ and (h_i,T_i) is the least martingale satisfying $(f_i,T_i)\leq(h_i,T_i)$ and $(g_i,T_i)\leq(h_i,T_i)$. In the case where $(f_i,T_i),(g_i,T_i)\in\mathcal{M}(E,T_i)$ with $(g_i,T_i)=(-f_i,T_i)$, we have $(f_i,T_i)\vee(-f_i,T_i)=(h_i,T_i)=|(f_i,T_i)|\in\mathcal{M}(E,T_i)$ and $\|\,|(f_i,T_i)|\,\|=\|(f_i,T_i)\|$.

Proof. (a) Since each T_j is positive, we have $T_j(f_i\vee g_i)\geq(T_jf_i)\vee(T_jg_i)\geq f_j\vee g_j$, for $j\leq i$. Moreover,

$$T_j(f_{i+1}\vee g_{i+1})=T_jT_i(f_{i+1}\vee g_{i+1})\geq T_j((T_if_{i+1})\vee T_i(g_{i+1}))\geq T_j(f_i\vee g_i).$$

Lastly,

$$\|T_j(f_i\vee g_i)\|\leq\|f_i\vee g_i\|\leq\||f_i|+|g_i|\|\leq\|(f_i,T_i)\|+\|(g_i,T_i)\|.$$

This proves part (a).

(b) To see $(h_i,T_i)\in\mathcal{M}(E,T_i)$, observe for $i\leq j$ that

$$T_i h_j = T_i(\lim_{k\to\infty} T_j(f_k \vee g_k)) = \lim_{k\to\infty} T_i T_j(f_k \vee g_k)) = \lim_{k\to\infty} T_i(f_k \vee g_k) = h_i,$$

and $\|h_i\| \leq \|(f_i, T_i)\| + \|(g_i, T_i)\|$ for all $i \in \mathbb{N}$. Thus, (h_i, T_i) is a martingale and $\|(h_i, T_i)\| < \infty$.

Since, by part (a), $T_j(f_i \vee g_i) \geq f_j \vee g_j$ for all $i \geq j$, it follows that $h_j \geq f_j \vee g_j$ for all $j \in \mathbb{N}$. Consequently, $(h_i, T_i) \geq (f_i, T_i)$ and $(h_i, T_i) \geq (g_i, T_i)$. Now suppose $(z_i, T_i) \in \mathcal{M}(E, T_i)$ such that $(z_i, T_i) \geq (f_i, T_i)$ and $(z_i, T_i) \geq (g_i, T_i)$. Then $z_i \geq f_i \vee g_i$ for all $i \in \mathbb{N}$, whence $z_j = T_j z_i \geq T_j(f_i \vee g_i)$ for all $j \leq i$. Taking the limit as $i \to \infty$ yields $z_j \geq h_j$. Thus, $(z_i, T_i) \geq (h_i, T_i)$.

For the case where $(f_i, T_i), (g_i, T_i) \in \mathcal{M}(E, T_i)$ with $(g_i, T_i) = (-f_i, T_i)$, we obtain $(f_i, T_i) \vee (-f_i, T_i) = (h_i, T_i) = |(f_i, T_i)| \in \mathcal{M}(E, T_i)$. Moreover,

$$\|h_j\| = \lim_{i\to\infty} \|T_j |f_i|\| \leq \lim_{i\to\infty} \|f_i\| = \|(f_i, T_i)\|$$

and, for $j \leq i$, we have $|f_j| = |T_j f_i| \leq T_j |f_i|$. Consequently,

$$\|h_j\| = \lim_{i\to\infty} \|T_j |f_i|\| \geq \|f_j\|,$$

so that $\|(f_i, T_i)\| = \| \, |(f_i, T_i)| \, \|$. □

Combining the above lemma with the KB property yields:

Proof of Theorem 3.3.5: Let $(f_i, T_i), (g_i, T_i) \in \mathcal{M}(E, T_i)$ where E is a KB-space. Then Lemma 3.3.6(a) asserts that $(T_j(f_i \vee g_i))_{i=j}^{\infty}$ is an increasing sequence that is norm bounded for each $j \in \mathbb{N}$. Since E is a KB-space, $h_j := \lim_{i\to\infty} T_j(f_i \vee g_i)$ exists for each $j \in \mathbb{N}$. Lemma 3.3.6(b) implies that $(h_i, T_i) \in \mathcal{M}(E, T_i)$ and $(h_i, T_i) = (f_i, T_i) \vee (g_i, T_i)$. In particular, we have $|(f_i, T_i)| \in \mathcal{M}(E, T_i)$ with $\|(f_i, T_i)\| = \| \, |(f_i, T_i)| \, \|$ for each $(f_i, T_i) \in \mathcal{M}(E, T_i)$. This completes the proof. □

Corollary 3.3.7 *If E is an AL-space and (T_i) a positive BS-filtration on E, then $\mathcal{M}(E, T_i)$ is also an AL-space.*

Proof. Since E is an AL-space, E is a KB-space. Theorem 3.3.5 implies that $\mathcal{M}(E, T_i)$ is a Banach lattice. Let $(f_i, T_i), (g_i, T_i) \in \mathcal{M}(E, T_i)_+$, then $f_i, g_i \in E_+$ for each $i \in \mathbb{N}$ whence $\|f_i + g_i\| = \|f_i\| + \|g_i\|$. Consequently, we have

$$\|(f_i, T_i) + (g_i, T_i)\| = \lim_{i\to\infty} \|f_i + g_i\| = \lim_{i\to\infty} (\|f_i\| + \|g_i\|) = \|(f_i, T_i)\| + \|(g_i, T_i)\|.$$

□

Corollary 3.3.8 *For every $1 \leq p < \infty$, we have that $\mathcal{M}(L^p(\mu), \Sigma_i)$ is a Banach lattice for every filtration (Σ_i).*

Proposition 3.3.9 *Let E be a Banach lattice with the KB-property and (T_i) a positive BS-filtration on E. If (s_i, T_i) is a norm bounded submartingale in E, then there exists a unique least martingale $(f_i, T_i) \in \mathcal{M}(E, T_i)$ such that $s_i \leq f_i$ for each $i \in \mathbb{N}$. Moreover, $\|(f_i, T_i)\| \leq \|(s_i, T_i)\|$.*

Proof. Suppose that (s_i) is a norm bounded submartingale relative to (T_i). By Lemma 3.3.6(a), $(T_j(s_i))_{i=j}^{\infty}$ is an increasing sequence that is norm bounded by $\sup_{i\in\mathbb{N}} \|s_i\|$ for each $j \in \mathbb{N}$. Since E is a KB-space, $(T_j(s_i))_{i=j}^{\infty}$ converges in norm to $h_j \in E$ for each $j \in \mathbb{N}$. Moreover, Lemma 3.3.6(b) implies that $(h_i, T_i) \in \mathcal{M}(E, T_i)$. Since Lemma 3.3.6(b) also implies that (h_i, T_i) is the least martingale which dominates (s_i), it follows that (h_i, T_i) is unique. Lastly,

$$\|h_i\| = \lim_{j\to\infty} \|T_i s_j\| \leq \lim_{j\to\infty} \|s_j\| = \sup_{i\in\mathbb{N}} \|s_i\|.$$

Thus, $\|(h_i, T_i)\| \leq \|(s_i, T_i)\|$. $\square$

If a Banach lattice E is not necessarily a KB-space, there are also conditions we can put on a positive BS-filtration (T_i) so that $\mathcal{M}(E,T_i)$ has a lattice ordering.

Lemma 3.3.10 *An increasing norm bounded sequence contained in a finite-dimensional subspace of a Banach lattice has a supremum and converges to it in norm.*

Proof. Suppose that (x_i) is an increasing sequence contained in the unit ball of a finite dimensional subspace of a Banach lattice. By continuity of the lattice operations, $x_n \leq \lim_{i\to\infty} x_{n_i}$ for every $n \geq 1$ and every convergent subsequence (x_{n_i}). Therefore, if (x_{m_i}) is another convergent subsequence, then $\lim_{i\to\infty} x_{m_i} \leq \lim_{i\to\infty} x_{n_i}$. It follows that all convergent subsequences of (x_i) have the same limit. Since the unit ball of a finite dimensional space is compact, the proof is complete. $\square$

Proposition 3.3.11 *Let E be a Banach lattice and (T_i) a positive BS-filtration on E with each T_i of finite rank, then $\mathcal{M}(E, T_i)$ is a Banach lattice with the same lattice operations as in Theorem 3.3.5.*

Proof. Let $(f_i, T_i), (g_i, T_i) \in \mathcal{M}(E, T_i)$, then $(T_j(f_i \vee g_i))_{i=j}^{\infty} \subset \mathcal{R}(T_j)$ for each $j \in \mathbb{N}$. Since $\mathcal{R}(T_j)$ is finite-dimensional, $(T_j(f_i \vee g_i))_{i=j}^{\infty}$ is convergent for each $j \in \mathbb{N}$ by Lemma 3.3.10. The result now follows from Lemma 3.3.6 in a similar manner to the proof of Theorem 3.3.5. $\square$

Definition 3.3.12 Let E be a Banach lattice and (T_i) a positive BS-filtration on E. Define the *space of regular martingales* to be

$$\mathcal{M}_{\mathrm{r}}(E,T_i) = \{(f_i,T_i) \in \mathcal{M}(E,T_i) : \exists\,(g_i,T_i) \in \mathcal{M}(E,T_i)_+ \\ \text{such that } (f_i,T_i) \le (g_i,T_i)\}.$$

It is evident that a martingale is regular if and only if it is the difference of two positive martingales.

Proposition 3.3.13 *Let E be a Banach lattice and (T_i) a positive BS-filtration on E. If E has order continuous norm, then $\mathcal{M}_{\mathrm{r}}(E,T_i)$ is a Banach lattice with the same lattice operations as in Theorem 3.3.5. Moreover, if E is a KB-space, then $\mathcal{M}_{\mathrm{r}}(E,T_i) = \mathcal{M}(E,T_i)$*

Proof. Let $(f_i,T_i),(g_i,T_i) \in \mathcal{M}_{\mathrm{r}}(E,T_i)$. There are two positive martingales $(\widehat{f}_i,T_i)$, $(\widehat{g}_i,T_i)$ such that $(f_i,T_i) \le (\widehat{f}_i,T_i)$ and $(g_i,T_i) \le (\widehat{g}_i,T_i)$. Lemma 3.3.6(a) implies that $(T_j(f_i \vee g_i))_{i=j}^{\infty}$ is increasing for each $j \in \mathbb{N}$. Also,

$$T_j(f_i \vee g_i) \le T_j(\widehat{f}_i \vee \widehat{g}_i) \le T_j(\widehat{f}_i + \widehat{g}_i) = \widehat{f}_j + \widehat{g}_j$$

for each $j \in \mathbb{N}$. Thus, $(T_j(f_i \vee g_i))_{i=j}^{\infty}$ is order bounded for each $j \in \mathbb{N}$. The order continuity of E implies that E is Dedekind complete. Consequently, $h_j := \sup_{j \le i < \infty} T_j(f_i \vee g_i) \in E$, and the order continuity of E gives $\lim_{i\to\infty} \|T_j(f_i \vee g_i) - h_j\| = 0$ for each $j \in \mathbb{N}$. From Lemma 3.3.6(b) we see that $(h_i,T_i) \in \mathcal{M}(E,T_i)$. Moreover, $(h_i,T_i) \in \mathcal{M}_{\mathrm{r}}(E,T_i)$ since $(h_i,T_i) \le (\widehat{f}_j + \widehat{g}_j,T_i)$, whence $\mathcal{M}_{\mathrm{r}}(E,T_i)$ is a Riesz space.

Lemma 3.3.6(b) implies that $\|\,|(f_i,T_i)|\,\| = \|(f_i,T_i)\|$ for all $(f_i,T_i) \in \mathcal{M}_{\mathrm{r}}(E,T_i)$. Thus, to show that $\mathcal{M}_{\mathrm{r}}(E,T_i)$ is a Banach lattice, we need only show that $\mathcal{M}_{\mathrm{r}}(E,T_i)$ is complete. Select a Cauchy sequence $(f_i^{(n)},T_i) \subset \mathcal{M}_{\mathrm{r}}(E,T_i)$. Since $\mathcal{M}(E,T_i)$ is complete, there exists $(f_i,T_i) \in \mathcal{M}(E,T_i)$ such that $\lim_{n\to\infty} \|(f_i^{(n)},T_i) - (f_i,T_i)\| = 0$. By selecting a suitable subsequence, we may assume $\|(f_i^{(n+1)},T_i) - (f_i^{(n)},T_i)\| < 2^{-n}$ for all $n \in \mathbb{N}$. By the first part of the proof $|(f_i^{(n+1)},T_i) - (f_i^{(n)},T_i)|$ exists for each $n \in \mathbb{N}$. Consequently, the series

$$\sum_{n=1}^{\infty} \left|(f_i^{(n+1)},T_i) - (f_i^{(n)},T_i)\right|$$

converges in $\mathcal{M}(E,T_i)$. Set $(\widehat{f}_i,T_i) = \sum_{n=1}^{\infty} |(f_i^{(n+1)},T_i) - (f_i^{(n)},T_i)|$. Plainly, $(\widehat{f}_i,T_i) \ge 0$. Now observe

$$\begin{aligned}
(f_i^{(m)},T_i) &= (f_i^{(1)},T_i) + \sum_{n=1}^{m-1} \Big((f_i^{(n+1)},T_i) - (f_i^{(n)},T_i)\Big) \\
&\le \left|(f_i^{(1)},T_i)\right| + \sum_{n=1}^{m-1} \left|(f_i^{(n+1)},T_i) - (f_i^{(n)},T_i)\right| \le \left|(f_i^{(1)},T_i)\right| + (\widehat{f}_i,T_i).
\end{aligned}$$

Thus, $\lim_{n\to\infty}(f_i^{(n)}, T_i) \le |(f_i^{(1)}, T_i)| + (\widehat{f}_i, T_i)$ and so $(f_i, T_i) \in \mathcal{M}_{\mathrm{r}}(E, T_i)$.

To complete the proof, assume E is a KB-space. Theorem 3.3.5 implies that $\mathcal{M}(E,T_i)$ is a Banach lattice. But then, $(f_i,T_i) \le |(f_i,T_i)| \in \mathcal{M}(E,T_i)$ for all $(f_i,T_i) \in \mathcal{M}(E,T_i)$. Consequently, $\mathcal{M}(E,T_i) = \mathcal{M}_{\mathrm{r}}(E,T_i)$. □

We have seen that in order for $\mathcal{M}(E,T_i)$ (respectively, $\mathcal{M}_{\mathrm{r}}(E,T_i)$) to be a Banach lattice, we need to assume that E is a KB-space (order continuous). In this case, it is natural to inquire whether the space $\mathcal{M}(E,T_i)$ ($\mathcal{M}_{\mathrm{r}}(E,T_i)$) is also a KB-space (order continuous). In order to prove this, we need an extra assumption on the underlying filtration. We recall the following definition from [101].

Definition 3.3.14 Let E be a Banach lattice and (T_i) a positive BS-filtration on E. We say (T_i) is *bounded below on* E_+ if there exist $i \in \mathbb{N}$ and a constant $\delta > 0$ such that $\|T_i f\| \ge \delta \|f\|$ for every $f \in E_+$.

If $\Sigma_1 \subset \Sigma$ and $0 \le f \in L^1(\mu)$, then

$$\|\mathbb{E}(f \mid \Sigma_1)\|_1 = \int_\Omega |\mathbb{E}(f \mid \Sigma_1)| \,\mathrm{d}\mu = \int_\Omega \mathbb{E}(f \mid \Sigma_1) \,\mathrm{d}\mu = \int_\Omega f \,\mathrm{d}\mu = \int_\Omega |f| \,\mathrm{d}\mu = \|f\|_1.$$

Consequently, any classical filtration on $L^1(\mu)$ is bounded below.

Theorem 3.3.15 *Let E be a Banach lattice and (T_i) a positive BS-filtration on E that is bounded below on E_+. Then the following statements hold.*

(a) If E has order continuous norm, then $\mathcal{M}_{\mathrm{r}}(E,T_i)$ is an order continuous Banach lattice.

(b) If E is a KB-space, then $\mathcal{M}(E,T_i)$ is a KB-space.

Proof. (a) It follows from Proposition 3.3.13 that $\mathcal{M}_{\mathrm{r}}(E,T_i)$ is a Banach lattice. Let $((f_i^{(\alpha)}, T_i))_\alpha \subset \mathcal{M}_{\mathrm{r}}(E,T_i)_+$ be a downwards directed net with infimum 0. For a fixed $i \in \mathbb{N}$, the sequence $(f_i^{(\alpha)}) \subset E_+$ is decreasing. Since E is order continuous (and thus, Dedekind complete), $f_i := \inf_\alpha f_i^{(\alpha)} \in E_+$ and $\lim_\alpha \|f_i^{(\alpha)} - f_i\| = 0$. For $i \le j$, we have

$$T_i f_j = T_i(\lim_\alpha f_j^{(\alpha)}) = \lim_\alpha T_i(f_j^{(\alpha)}) = \lim_\alpha f_i^{(\alpha)} = f_i$$

and $\sup_{i\in\mathbb{N}} \|f_i\| < \infty$. Thus, $(f_i,T_i) \in \mathcal{M}_{\mathrm{r}}(E,T_i)_+$. Moreover, it follows from $0 \le (f_i,T_i) \le (f_i^{(\alpha)},T_i)$ for all α and $((f_i^{(\alpha)},T_i))_\alpha \downarrow 0$ that $(f_i,T_i) = 0$. Consequently, $\lim_\alpha f_i^{(\alpha)} = 0$ for all $i \in \mathbb{N}$. Since (T_i) is bounded below on E_+, there exist $i \in \mathbb{N}$ and a constant $\delta > 0$ such that $\|T_i f\| \ge \delta\|f\|$ for all $f \in E_+$. Hence, for every $j \in \mathbb{N}$ with $i \le j$, we have

$$\|f_i^{(\alpha)}\| = \|T_i f_j^{(\alpha)}\| \geq \delta \|f_j^{(\alpha)}\|.$$

Taking the limit as $j \to \infty$, we obtain $\|(f_i^{(\alpha)}, T_i)\| \leq \frac{1}{\delta}\|f_i^{(\alpha)}\|$ for each α. Now, taking the limit as $\alpha \to \infty$, we observe $\|(f_i^{(\alpha)}, T_i)\| \leq \frac{1}{\delta}\|f_i^{(\alpha)}\| \to 0$. This completes the proof of (a).

(b) Since E is supposedly a KB-space, E is also order continuous. Theorem 3.3.5 implies that $\mathcal{M}(E, T_i)$ is a Banach lattice. Moreover, Proposition 3.3.13 implies that $\mathcal{M}(E, T_i) = \mathcal{M}_{\mathrm{r}}(E, T_i)$. Hence, by part (a), $\mathcal{M}(E, T_i)$ is an order continuous Banach lattice. Let $((f_i^{(n)}, T_i))_{n\in\mathbb{N}} \subset \mathcal{M}(E, T_i)$ be an increasing sequence with $\sup_{n\in\mathbb{N}} \|(f_i^{(n)}, T_i)\| \leq K < \infty$. For each $i \in \mathbb{N}$, the sequence $(f_i^{(n)})_{n=1}^{\infty} \subset E$ is increasing and norm bounded by K. Thus, there exists f_i such that $\lim_{n\to\infty} \|f_i^{(n)} - f_i\| = 0$ for each $i \in \mathbb{N}$. Moreover, $\sup_{i\in\mathbb{N}} \|f_i\| \leq K < \infty$. Also, for $i \leq j$, we have

$$T_i f_j = T_i(\lim_{\alpha} f_j^{(\alpha)}) = \lim_{\alpha} T_i(f_j^{(\alpha)}) = \lim_{\alpha} f_i^{(\alpha)} = f_i.$$

Thus, $(f_i, T_i) \in \mathcal{M}(E, T_i)$ and $((f_i^{(n)}, T_i))_{n\in\mathbb{N}} \uparrow (f_i, T_i)$. Since $\mathcal{M}(E, T_i)$ is order continuous, we have $\lim_{n\to\infty} \|(f_i^{(n)}, T_i) - (f_i, T_i)\| = 0$ and the proof is complete. $\square$

In general, a classical filtration (Σ_i) is not bounded below on $L^p(\mu)$ for $1 < p < \infty$, so the above proof does not apply to $\mathcal{M}(L^p(\mu), \Sigma_i)$ and $\mathcal{M}_{\mathrm{r}}(L^p(\mu), \Sigma_i)$. However, Corollary 3.2.13 comes to our aid. Since $L^p(\mu)$ is reflexive and $\mathbb{E}(\,\cdot \mid \Sigma_i)$ is complemented, it follows that $\mathcal{M}(L^p(\mu), \Sigma_i) = \mathcal{M}_{\mathrm{r}}(L^p(\mu), \Sigma_i) = \mathcal{M}_{\mathrm{nc}}(L^p(\mu), \Sigma_i)$. But $\mathcal{M}_{\mathrm{nc}}(L^p(\mu), \Sigma_i)$ is Riesz isometric to $L^p(\Omega, \vee_{i=1}^{\infty}\Sigma_i, \mu|_{\vee_{i=1}^{\infty}\Sigma_i})$, by Theorem 3.3.3. Consequently, $\mathcal{M}(L^p(\mu), \Sigma_i)$ and $\mathcal{M}_{\mathrm{r}}(L^p(\mu), \Sigma_i)$ are both KB-spaces.

3.4 Filtrations on the l-tensor product

Let (Ω, Σ, μ) denote a finite measure space, Y a Banach space and $1 \leq p < \infty$. Our aim in this section is to make the necessary preparations for characterizing the Radon Nikodým property, using the abstract martingale theory developed in the preceding sections.

In general, we cannot use the definition of a BL-filtration on $L^p(\mu, Y)$, unless $L^p(\mu, Y)$ is a Banach lattice. Also, the class of BS-filtrations on $L^p(\mu, Y)$ is too large to characterize RNP. To overcome this, we consider the l-tensor product of a BL-filtration on a Banach lattice E with a BS-filtration on a Banach space Y. The following results are stated in terms of left uniform, left injective crossnorms, of which the l-norm is a special case.

Lemma 3.4.1 *(a) Let E be a Banach lattice and Y a Banach space. If α is a left uniform, left injective crossnorm on $E \otimes Y$, $0 \leq S : E \to E$ and $T : Y \to Y$ bounded projections and $\mathcal{R}(S)$ a (closed) Riesz subspace of E, then $S \otimes_\alpha T \colon E\widetilde{\otimes}_\alpha Y \to E\widetilde{\otimes}_\alpha Y$ is a bounded projection with norm $\|S\|\|T\|$ and range $S(E)\widetilde{\otimes}_\alpha T(Y)$, which is a closed subspace of $E\widetilde{\otimes}_\alpha Y$.*

(b) A symmetrical result holds if α is a right uniform, right injective crossnorm on $Y \otimes E$.

Proof. Since α is a left order uniform crossnorm, it follows that $\|S \otimes T\| = \|S\|\|T\|$; consequently, the continuous extension $S \otimes_\alpha T : E\widetilde{\otimes}_\alpha Y \to E\widetilde{\otimes}_\alpha Y$ is bounded. To see that $S \otimes_\alpha T$ is a projection, let $u \in E\widetilde{\otimes}_\alpha Y$. Then there exists a sequence $(u_j) \subset E \otimes Y$ such that $u_j \to u$ in norm. Representing each u_j as $\sum_{i=1}^{n_j} x_i^{(j)} \otimes y_i^{(j)}$, we conclude that

$$(S \otimes_\alpha T)^2(u_j) = \sum_{i=1}^{n_j} S^2(x_i^{(j)}) \otimes T^2(y_i^{(j)}) = \sum_{i=1}^{n_j} S(x_i^{(j)}) \otimes T(y_i^{(j)}) = (S \otimes_\alpha T)(u_j).$$

By the continuity of $S \otimes_\alpha T$, it follows that $(S \otimes_\alpha T)^2(u) = (S \otimes_\alpha T)(u)$. As $S(E)$ is a closed Riesz subspace of E and $T(Y)$ a closed subspace of Y, the left order injectivity of the α-norm gives

$$(S \otimes_\alpha T)(E \otimes Y) = S(E) \otimes T(Y) \subset S(E)\widetilde{\otimes}_\alpha T(Y) \hookrightarrow E\widetilde{\otimes}_\alpha Y \quad \text{(isometrically)}.$$

Thus, $S(E) \otimes T(Y) \subset (S \otimes_\alpha T)(E\widetilde{\otimes}_\alpha Y) \subset S(E)\widetilde{\otimes}_\alpha T(Y)$. As $S \otimes_\alpha T$ is a bounded projection and thus has closed range, it follows that $(S \otimes_\alpha T)(E\widetilde{\otimes}_\alpha Y) = S(E)\widetilde{\otimes}_\alpha T(Y)$.
□

Theorem 3.4.2 *(a) Let E be a Banach lattice and Y a Banach space. If α is a left uniform, left injective crossnorm on $E \otimes Y$, (S_i) a positive BS-filtration on E and (T_i) a BS-filtration on Y, then $(S_i \otimes_\alpha T_i)$ is a BS-filtration on $E\widetilde{\otimes}_\alpha Y$. Moreover, if (S_i) is a BL-filtration, then $\mathcal{R}(S \otimes_\alpha T_i) = \mathcal{R}(S_i)\widetilde{\otimes}_\alpha \mathcal{R}(T_i)$ for each $i \in \mathbb{N}$.*

(b) A symmetrical result holds if α is a right uniform, right injective crossnorm on $Y \otimes E$.

Proof. The proof follows easily from Lemma 3.4.1 and its proof. □

In particular, if (T_i) is a BL-filtration on a Banach lattice E, it can be naturally extended to the vector valued setting by considering $(T_i \otimes_l \mathrm{id}_Y)$ on $E\widetilde{\otimes}_l Y$. This extension is consistent with the sequence of operators $(\mathbb{E}(\,\cdot \mid \Sigma_i))$ on $L^p(\mu, Y)$, corresponding to the classical filtration (Σ_i). We now consider the l-tensor product of BL-filtrations.

Lemma 3.4.3 *(a) Let E and F be Banach lattices. If α is a left uniform, left injective Riesz crossnorm on $E \otimes F$, $S : E \to E$ and $T : F \to F$ positive contractive projections with ranges (closed) Riesz subspaces of E and F respectively, then $(S \otimes_\alpha T) : E\widetilde{\otimes}_\alpha F \to E\widetilde{\otimes}_\alpha F$ is a positive contractive projection with range $S(E)\widetilde{\otimes}_\alpha T(F)$, which is a closed Riesz subspace of $E\widetilde{\otimes}_\alpha F$.*

(b) A symmetrical result holds if α is a right uniform, right injective Riesz crossnorm on $E \otimes F$.

Proof. By Lemma 3.4.1, it suffices to show that $S \otimes_\alpha T \geq 0$ with range $S(E)\widetilde{\otimes}_\alpha T(F)$ a Riesz subspace of $E\widetilde{\otimes}_\alpha F$.

Since α is an order reasonable crossnorm, it follows that $E\widetilde{\otimes}_\alpha F$ is a Banach lattice with $E_+ \otimes F_+$ α-dense in $(E\widetilde{\otimes}_\alpha F)_+$. Since $(S \otimes T)(E_+ \otimes F_+) \subset E_+ \otimes F_+$ and $S \otimes T : E \otimes_\alpha F \to E \otimes_\alpha F$ is continuous, in fact $\|S \otimes T\| = \|S\|\|T\|$, we get that $0 \leq S \otimes_\alpha T : E\widetilde{\otimes}_\alpha F \to E\widetilde{\otimes}_\alpha F$.

By the left order injectivity of α, we have that $S(E)\widetilde{\otimes}_\alpha T(F)$ is a closed subspace of $E\widetilde{\otimes}_\alpha F$. Also, by property (SS) in Section 2.4, we get that $S(E)\overline{\otimes}T(F)$ is a Riesz subspace of $E\overline{\otimes}F$ and is thus also a Riesz subspace of $E\widetilde{\otimes}_\alpha F$. Since $S(E)\overline{\otimes}T(F)$ is dense in $S(E)\widetilde{\otimes}_\alpha T(F)$, it follows that $S(E)\widetilde{\otimes}_\alpha T(F)$ is a closed Riesz subspace of $E\widetilde{\otimes}_\alpha F$. □

Theorem 3.4.4 *(a) Let E and F be Banach lattices. If α is a left uniform, left injective Riesz crossnorm on $E \otimes F$, (S_i) and (T_i) BL-filtrations (positive BS-filtrations) on E and F respectively, then $(S_i \otimes_\alpha T_i)$ is a BL-filtration (positive BS-filtration) on $E\widetilde{\otimes}_\alpha F$.*

(b) A symmetrical result holds if α is a right uniform, right injective Riesz crossnorm on $E \otimes F$.

Proof. The proof follows easily from Lemma 3.4.3 and its proof. □

The following result is the corner stone of many of the main results in this thesis. Note that the proof relies on the definition of the l-norm (m-norm).

Theorem 3.4.5 *(a) If (S_i) is a BL-filtration on the Banach lattice E and (T_i) is a BS-filtration on the Banach space Y, then*

$$\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i)} \ \widetilde{\otimes}_l \ \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)} \ = \ \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i \otimes_l T_i)}.$$

(b) If (J_i) is a BS-filtration on the Banach space X and (K_i) is a BL-filtration on the Banach lattice F, then

$$\overline{\bigcup_{i=1}^{\infty}\mathcal{R}(J_i)}\ \widetilde{\otimes}_m\ \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(K_i)}\ =\ \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(J_i\otimes_m K_i)}.$$

Proof. We will prove the first equality, the second is derived similarly.

($\supset$): Let $y \in \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(S_i\otimes_l T_i)}$ and $\varepsilon > 0$ be given. Select $y_0 \in \mathcal{R}(S_i \otimes_l T_i)$ for some $i \in \mathbb{N}$ such that $\|y - y_0\|_l < \varepsilon$. Since $\mathcal{R}(S_i \otimes_l T_i) = S_i(E)\,\widetilde{\otimes}_l\, T_i(Y)$ and $S_i(E)\,\widetilde{\otimes}_l\, T_i(Y) \subset \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(S_i)}\ \widetilde{\otimes}_l\ \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(T_i)}$ by the left order injectivity of the l-norm, it follows that $y \in \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(S_i)}\ \widetilde{\otimes}_l\ \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(T_i)}$.

($\subset$): Let $y \in \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(S_i)}\ \widetilde{\otimes}_l\ \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(T_i)}$ and $\varepsilon > 0$ be given. Select $y_0 \in \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(S_i)} \otimes \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(T_i)}$ such that $\|y - y_0\|_l < \varepsilon/2$. Let $y_0 = \sum\limits_{i=1}^{n_0} a_i \otimes y_i$, where $a_i \in \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(S_i)}$ and $y_i \in \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(T_i)}$. Select $v_i \in \bigcup_{i=1}^{\infty}\mathcal{R}(T_i)$ such that

$$\|y_i - v_i\|_Y < \frac{\varepsilon}{4\sum_{i=1}^{n_0}\|a_i\|}$$

and select $b_i \in \bigcup_{i=1}^{\infty}\mathcal{R}(S_i)$ such that

$$\|a_i - b_i\|_E < \frac{\varepsilon}{4\sum_{i=1}^{n_0}\|v_i\|}.$$

Let $z_1 = \sum\limits_{i=1}^{n_0} b_i \otimes v_i$. Then $z_1 \in \bigcup_{i=1}^{\infty}\mathcal{R}(S_i\otimes_l T_i)$,

$$y_0 - z_1 = \sum_{i=1}^{n_0}\Big(a_i \otimes (y_i - v_i) + (a_i - b_i)\otimes v_i\Big),$$

$$\|y_0 - z_1\|_l \le \left\|\sum_{i=1}^{n_0}\Big(\|y_i - v_i\|\,|a_i| + \|v_i\|\,|a_i - b_i|\Big)\right\|_E < \varepsilon/4 + \varepsilon/4 = \varepsilon/2,$$

and

$$\|y - z_1\|_l \le \|y - y_0\|_l + \|y_0 - z_1\|_l \le \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus, $y \in \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(S_i\otimes_l T_i)}$. $\square$

Thus, one has the following distributive property:

Corollary 3.4.6 *(a) If (S_i) is a BL-filtration on the Banach lattice E and (T_i) is a BS-filtration on the Banach space Y, then*

$$\mathcal{M}_{\mathrm{nc}}\left(E\,\widetilde{\otimes}_l\, Y, S_i\otimes_l T_i\right) = \mathcal{M}_{\mathrm{nc}}(E, S_i)\,\widetilde{\otimes}_l\,\mathcal{M}_{\mathrm{nc}}(Y, T_i).$$

(b) If (J_i) is a BS-filtration on the Banach space X and (K_i) is a BL-filtration on the Banach lattice F, then

$$\mathcal{M}_{\mathrm{nc}}\left(X\,\widetilde{\otimes}_m\, F, J_i\otimes_m K_i\right) = \mathcal{M}_{\mathrm{nc}}(X, J_i)\,\widetilde{\otimes}_m\,\mathcal{M}_{\mathrm{nc}}(F, K_i).$$

Proof. We only prove (a), since the proof for (b) is similar. By Propositions 3.2.17, Proposition 3.3.3 and Theorem 3.4.2, we have that $\overline{\bigcup_{i=1}^{\infty}\mathcal{R}(S_i)}$ is Riesz isometric to $\mathcal{M}_{\rm nc}(E,S_i)$, $\overline{\bigcup_{i=1}^{\infty}\mathcal{R}(T_i)}$ is isometric to $\mathcal{M}_{\rm nc}(Y,T_i)$ and $\mathcal{M}_{\rm nc}\left(E\,\widetilde{\otimes}_l\,Y, S_i\otimes_l T_i\right)$ is isometric to $\overline{\bigcup_{i=1}^{\infty}\mathcal{R}(S_i\otimes_l T_i)}$. By Theorem 3.4.5, we have

$$\overline{\bigcup_{i=1}^{\infty}\mathcal{R}(S_i\otimes_l T_i)} \;=\; \overline{\bigcup_{i=1}^{\infty}\mathcal{R}(S_i)}\;\widetilde{\otimes}_l\;\overline{\bigcup_{i=1}^{\infty}\mathcal{R}(T_i)},$$

from which the above assertion is now clear. $\square$

Corollary 3.4.7 *Let E be a Banach lattice and Y a Banach space. If (S_i) is a complemented BL-filtration on E and (T_i) is a complemented BS-filtration on Y, then $(S_i\otimes T_i)$ is a complemented BS-filtration on $E\,\widetilde{\otimes}_l\,Y$.*

Proof. Assume (S_i) and (T_i) are complemented by the contractive projections $0\le S_\infty: E\to E$ and $T_\infty : Y\to Y$ respectively. By Theorem 3.4.5 and Lemma 3.4.1, we have

$$\mathcal{R}(S_\infty\otimes_l T_\infty)=\mathcal{R}(S_\infty)\widetilde{\otimes}_l\mathcal{R}(T_\infty)=\overline{\bigcup_{i=1}^{\infty}\mathcal{R}(S_i)}\;\widetilde{\otimes}_l\;\overline{\bigcup_{i=1}^{\infty}\mathcal{R}(T_i)}=\overline{\bigcup_{i=1}^{\infty}\mathcal{R}(S_i\otimes_l T_i)}.$$

Consequently, by Theorem 3.4.2, $(S_i\otimes_l T_i)$ is a BS-filtration complemented by $S_\infty\otimes_l T_\infty$. $\square$

3.5 A characterization of the Radon Nikodým property

To characterize the Radon Nikodým property, we still require a fair amount of preparation. We first consider BS-filtrations on the space of cone absolutely summing operators from a Banach lattice E to a Banach space Y.

Proposition 3.5.1 *Let E be a Banach lattice and Y a Banach space. Suppose that (T_i) is a BL-filtration on E. Then the sequence $(\widehat{T_i})$ of maps $\widehat{T_i}:\mathcal{L}^{\rm cas}(E,Y)\to\mathcal{L}^{\rm cas}(E,Y)$, defined by $\widehat{T_i}F=F\circ T_i$ for each $F\in\mathcal{L}^{\rm cas}(E,Y)$ and $i\in\mathbb{N}$, is a BS-filtration on $\mathcal{L}^{\rm cas}(E,Y)$.*

Proof. Since (T_i) is a BL-filtration, $F\circ T_i\in\mathcal{L}^{\rm cas}(E,Y)$ and $\widehat{T_i}$ is a well defined, linear projection for each $i\in\mathbb{N}$. It also follows from

$$
\begin{aligned}
\|\widehat{T}_i F\|_{\mathrm{cas}} &= \sup\left\{\sum_{j=1}^{n}\|FT_i x_j\| : (x_j)_{i=1}^{n} \subset E_+, \left\|\sum_{j=1}^{n} x_j\right\| \le 1\right\} \\
&= \sup\left\{\sum_{j=1}^{n}\|F x_j\| : (x_j)_{i=1}^{n} \subset \mathcal{R}(T_i)_+, \left\|\sum_{j=1}^{n} x_j\right\| \le 1\right\} \\
&\le \sup\left\{\sum_{j=1}^{n}\|F x_j\| : (x_j)_{i=1}^{n} \subset E_+, \left\|\sum_{j=1}^{n} x_j\right\| \le 1\right\} \\
&= \|F\|_{\mathrm{cas}}
\end{aligned}
$$

that each $\widehat{T}_i$ is bounded and $\sup_{i\in\mathbb{N}}\|\widehat{T}_i\| = 1$. Moreover,

$$
\widehat{T}_i\widehat{T}_j F = F \circ T_j \circ T_i = F \circ T_{i\wedge j} = \widehat{T}_{i\wedge j} F
$$

for each $F \in \mathcal{L}^{\mathrm{cas}}(E, Y)$ and $i, j \in \mathbb{N}$. Consequently, $(\widehat{T}_i)$ is a BS-filtration on $\mathcal{L}^{\mathrm{cas}}(E, Y)$. $\square$

In view of the above proposition, we are justified in making the following definition.

Definition 3.5.2 Let E be a Banach lattice and Y a Banach space. Suppose that (T_i) is a BL-filtration on E. Then $(\widehat{T}_i)$, as defined in Proposition 3.5.1, is called the *BS-filtration on $\mathcal{L}^{\mathrm{cas}}(E, Y)$ induced by (T_i).*

We exhibit a known characterization of cone absolutely summing operators (cf. [92, Chapter IV, §3, Proposition 3.3]).

Lemma 3.5.3 *Let E be a Banach lattice, Y a Banach space and $l > 0$. For any bounded operator $T : E \to Y$ the following statements are equivalent:*

(a) T is cone absolutely summing with $\|T\|_{\mathrm{cas}} \le l$.
(b) There exists $x_T^* \in E_+^*$ so that $\|x_T^*\| \le l$ and $\|Tx\| \le \langle |x|, x_T^*\rangle$ for all $x \in E$.
(c) There exist an AL-space L, $0 \le T_1 \in \mathcal{L}(E, L)$ and $T_2 \in \mathcal{L}(L, Y)$ such that $T = T_2 \circ T_1$ where $\|T_1\| \le l$ and $\|T_2\| \le 1$.

In the case where E is separable, we may take $L = L^1(\mu)$ in (c), where (Ω, Σ, μ) is a finite measure space.

Proof. (a)⇒(b) Define the map $\rho_T : E_+ \to \mathbb{R}$ by

$$
\rho_T(x) = \sup\left\{\sum_{i=1}^{\infty}\|Tx_i\| : (x_i) \in (\ell^1\widetilde{\otimes}_\varepsilon E)_+, \sum_{i=1}^{\infty} x_i = x\right\} \tag{3.1}
$$

for each $x \in E_+$. Since $\|T\|_{\mathrm{cas}} \le l$, we have $\rho_T(x) \le l\|x\|$ for all $x \in E_+$. Clearly, ρ_T is homogenius. Moreover, ρ_T is additive on E_+. To see this, let $x, y \in E_+$ and

$(x_i),(y_i) \in (\ell^1\widetilde{\otimes}_\varepsilon E)_+$ such that $\sum_{i=1}^\infty x_i = x$ and $\sum_{i=1}^\infty y_i = y$. Define the sequences $(\widehat{x}_i),(\widehat{y}_i) \in (\ell^1\widetilde{\otimes}_\varepsilon E)_+$ by

$$(\widehat{x}_1,\widehat{x}_2,\widehat{x}_3,\dots) = (x_1,0,x_2,0,x_3,0,\dots)$$

and

$$(\widehat{y}_1,\widehat{y}_2,\widehat{x}_3,\dots) = (0,y_1,0,y_2,0,y_3,\dots)$$

respectively. Then $\sum_{i=1}^\infty \widehat{x}_i + \widehat{y}_i = x+y$ and

$$\rho_T(x+y) \geq \sum_{i=1}^\infty \|T\widehat{x}_i + T\widehat{y}_i\| = \sum_{i=1}^\infty \|T\widehat{x}_i\| + \sum_{i=1}^\infty \|T\widehat{y}_i\| = \sum_{i=1}^\infty \|Tx_i\| + \sum_{i=1}^\infty \|Ty_i\|.$$

Taking appropriate suprema on the right hand side yields $\rho_T(x+y) \geq \rho_T(x)+\rho_T(y)$.

For the reverse inequality, let $z = x+y$ and choose any sequence $(z_n) \in (\ell^1\widetilde{\otimes}_\varepsilon E)_+$ satisfying $\sum_{i=1}^\infty z_i = z$. Define $w_n = \sum_{i=1}^{n-1} z_i$, $u_n = w_n \wedge x$, $v_n = (w_n - x)^+$, $x_n = u_{n+1} - u_n$ and $y_n = v_{n+1} - v_n$ for each $n \in \mathbb{N}$. Observe that $\lim_{n\to\infty} u_n = x$ and $\lim_{n\to\infty} v_n = y$. Consequently, $u_n \uparrow x$ and $v_n \uparrow y$ and $x_n, y_n \in E_+$ for each $n \in \mathbb{N}$. Moreover,

$$\begin{aligned}
x_n + y_n &= w_{n+1}\wedge x - w_n \wedge x + (w_{n+1}-x)^+ - (w_n - x)^+ \\
&= (w_{n+1}\wedge x - w_n\wedge x) + (w_{n+1}\vee x - x - (w_n \vee x - x)) \\
&= |w_{n+1}\wedge x - w_n \wedge x| + |w_{n+1}\vee x - w_n \vee x| \\
&= |w_{n+1} - w_n| \\
&= z_n
\end{aligned}$$

for each $n \in \mathbb{N}$. Thus, $(x_i),(y_i) \in (\ell^1\widetilde{\otimes}_\varepsilon E)_+$ with $\sum_{i=1}^\infty x_i = x$, $\sum_{i=1}^\infty y_i = y$ and

$$\sum_{i=1}^\infty \|z_i\| \leq \sum_{i=1}^\infty \|x_i\| + \sum_{i=1}^\infty \|y_i\| \leq \rho_T(x)+\rho_T(y).$$

Since $(z_n) \in (\ell^1\widetilde{\otimes}_\varepsilon E)_+$ was an arbitrary sequence satisfying $\sum_{i=1}^\infty z_i = z$, taking the appropriate supremum on the left hand side yields $\rho_T(x+y) \leq \rho_T(x)+\rho_T(y)$.

Hence, ρ_T may be uniquely extended to a linear functional x_T^* on E (cf. [106, Lemma 20.1]) of norm $\|x_T^*\| \leq l$. Thus, by the construction of x_T^*, we have

$$\|Tx\| \leq \|Tx^+\| + \|Tx^-\| \leq \langle x^+, x_T^*\rangle + \langle x^-, x_T^*\rangle = \langle |x|, x_T^*\rangle$$

for all $x \in E$.

(b)⇒(c) Assume (b) to be true. Then there exists a functional $x_T^* \in E_+^*$ so that $\|x_T^*\| \leq l$ and $\|Tx\| \leq \langle |x|, x_T^*\rangle$ for all $x \in E$. The map $E \to \mathbb{R}_+$, defined by

$x \mapsto \langle |x|, x_T^* \rangle$, defines a L-norm on E. Let $N := \{x \in E : \langle |x|, x_T^* \rangle = 0\}$, then N is a closed ideal in E, E/N is a Riesz space and the quotient map $q : E \to E/N$ is a Riesz homomorphism. It is not difficult to see that the quotient norm, defined by

$$\|q(x)\| = \inf\{\langle |x - z|, x_T^* \rangle : z \in N\}$$

for all $x \in E$, is an L-norm on E/N. Let L denote the norm completion of E/N with respect to this L-norm. Then L is an AL-space and

$$q : E \to L \tag{3.2}$$

is a Riesz homomorphism with dense range and norm $\|q\| \leq l$. Let $T_1 = q$ and define $\tau : \mathcal{R}(T_1) \to Y$ by $\tau(T_1 x) = Tx$ for each $x \in E$. Then τ is bounded with norm $\|\tau\| \leq 1$. Since $\mathcal{R}(T_1)$ is dense in L, we have $T_2 : L \to Y$ where T_2 denotes the unique continuous extension of τ. Consequently, $\|T_1\| \leq l$, $\|T_2\| \leq 1$ and $T = T_2 \circ T_1$, as required.

(c)⇒(a) Suppose (c) holds and let $(x_i) \subset E_+$ be an unconditionally summable sequence. Since $T_1 \geq 0$ we have $(T_1 x_i) \subset L_+$ and, since L is an AL-space, $\sum_{i=1}^{\infty} \|T_1 x_i\| = \| \sum_{i=1}^{\infty} T_1 x_i\|$. By the Hahn Banach Theorem, there exists $x^* \in L^*$, with $\|x^*\| \leq 1$, such that

$$\left\| \sum_{i=1}^{\infty} T_1 x_i \right\| = \left\langle \sum_{i=1}^{\infty} T_1 x_i, x^* \right\rangle = \left\langle \sum_{i=1}^{\infty} x_i, T_1^* x^* \right\rangle \leq l \left\| \sum_{i=1}^{\infty} x_i \right\|.$$

Using the fact that $\|T_2\| \leq 1$, we obtain

$$\sum_{i=1}^{\infty} \|T x_i\| = \sum_{i=1}^{\infty} \|T_2 T_1 x_i\| \leq \sum_{i=1}^{\infty} \|T_1 x_i\| = \left\| \sum_{i=1}^{\infty} T_1 x_i \right\| \leq l \left\| \sum_{i=1}^{\infty} x_i \right\|.$$

Thus, T is cone absolutely summing with $\|T\|_{\text{cas}} \leq l$.

To complete the proof, assume E is separable. Then, by [92, Chapter II, §6, Proposition 6.2], there exists a quasi-interior point $0 \leq e \in E$. Since the map given by (3.2) is a Riesz homomorphism with dense range, it follows by [92, Chapter II, §6, Proposition 6.4] that $T_1 e$ is a quasi-interior point of L. But L is an AL-space and is thus order continuous. Hence, $T_1 e$ is also a weak order unit of L by [92, Chapter II, §6, Proposition 6.5] and [76, Theorem 2.4.2]. It follows by Kakutani's representation theorem for AL-spaces (cf. [57] or [72, Theorem 1.b.2]) that L is Riesz and isometrically isomorphic to $L^1(\mu)$, where (Ω, Σ, μ) may be chosen to be finite. □

Proposition 3.5.4 *Let E be a Banach lattice with order continuous dual and Y a Banach space. Suppose that (T_i) is a BL-filtration on E and $(\widehat{T_i})$ is the BS-filtration* *on $\mathcal{L}^{\mathrm{cas}}(E,Y)$ induced by (T_i). If $(F_i, \widehat{T}_i) \in \mathcal{M}(\mathcal{L}^{\mathrm{cas}}(E,Y), \widehat{T}_i)$, then there exists $0 \leq (f_i^*, T_i^*) \in \mathcal{M}(E^*, T_i^*)$ such that $\|F_i x\| \leq \langle |x|, f_i^*\rangle$ for each $x \in E$ and $i \in \mathbb{N}$.*

Proof. By Lemma 3.5.3 there exists, for each F_i, a positive functional $x^*_{F_i} \in E^*$ with $\|x^*_{F_i}\| \leq \sup_{i\in\mathbb{N}} \|F_i\|_{\mathrm{cas}} := l$ and $\|F_i x\| \leq \langle |x|, x^*_{F_i}\rangle$ for each $x \in E$ and $i \in \mathbb{N}$. Define $s_i^* \in E^*_+$ by

$$\langle x, s_i^*\rangle = \langle T_i x, x^*_{F_i}\rangle$$

for each $x \in E$ and $i \in \mathbb{N}$. Then, $\sup_{i\in\mathbb{N}} \|s_i^*\| < l$ and, since $x^*_{F_i} \geq 0$, we get

$$\|F_i x\| = \|\widehat{T}_i F_i x\| = \|F_i T_i x\| \leq \langle |T_i x|, x^*_{F_i}\rangle \leq \langle T_i |x|, x^*_{F_i}\rangle = \langle |x|, s_i^*\rangle \tag{3.3}$$

for all $x \in E$ and $i \in \mathbb{N}$. We now show that (s_i^*, T_i^*) is a submartingale. Let $i \leq j$ and $x \in E_+$. Then, by (3.1),

$$\begin{aligned}
\langle x, T_i^* s_j^*\rangle &= \langle T_i x, s_j^*\rangle = \langle T_j T_i x, x^*_{F_j}\rangle = \langle T_i x, x^*_{F_j}\rangle \\
&= \sup\left\{\sum_{n=1}^{\infty} \|F_j x_n\| : (x_n) \in (\ell^1 \widetilde{\otimes}_\varepsilon E)_+, \sum_{n=1}^{\infty} x_n = T_i x\right\} \\
&\geq \sup\left\{\sum_{n=1}^{\infty} \|F_j T_i x_n\| : (x_n) \in (\ell^1 \widetilde{\otimes}_\varepsilon E)_+, \sum_{n=1}^{\infty} x_n = T_i x\right\} \\
&= \sup\left\{\sum_{n=1}^{\infty} \|\widehat{T}_i F_j x_n\| : (x_n) \in (\ell^1 \widetilde{\otimes}_\varepsilon E)_+, \sum_{n=1}^{\infty} x_n = T_i x\right\} \\
&= \sup\left\{\sum_{n=1}^{\infty} \|F_i x_n\| : (x_n) \in (\ell^1 \widetilde{\otimes}_\varepsilon E)_+, \sum_{n=1}^{\infty} x_n = T_i x\right\} \\
&= \langle T_i x, x^*_{F_i}\rangle \\
&= \langle x, s_i^*\rangle.
\end{aligned}$$

Since $s_i^*(x) \leq T_i^* s_j^*(x)$ for all $x \in E_+$, it follows that $s_i^* \leq T_i^* s_j^*$. Consequently, (s_i^*, T_i^*) is a submartingale. Since E^* is order continuous, it follows that E^* has the KB-property (cf. [76, Theorem 2.4.14]). Consequently, by Proposition 3.3.9, there exists a unique least martingale $0 \leq (f_i^*, T_i^*) \in \mathcal{M}(E^*, T_i^*)$ that dominates the submartingale (s_i^*, T_i^*), with $\sup_{i\in\mathbb{N}} \|f_i^*\| \leq \sup_{i\in\mathbb{N}} \|s_i^*\| \leq l$. By (3.3),

$$\|F_i x\| \leq \langle |x|, s_i^*\rangle \leq \langle |x|, f_i^*\rangle$$

for all $x \in E$, and the proof is complete. □

Theorem 3.5.5 *Let E be a Banach lattice with order continuous dual and Y a Banach space. Suppose that (T_i) is a BL-filtration on E and $(\widehat{T}_i)$ is the BS-filtration on $\mathcal{L}^{\mathrm{cas}}(E,Y)$ induced by (T_i). Then $\mathcal{M}_{\mathrm{f}}(\mathcal{L}^{\mathrm{cas}}(E,Y), \widehat{T}_i) = \mathcal{M}(\mathcal{L}^{\mathrm{cas}}(E,Y), \widehat{T}_i)$.*

Proof. The inclusion $\mathcal{M}_{\mathrm{f}}(\mathcal{L}^{\mathrm{cas}}(E,Y),\widehat{T}_i) \subset \mathcal{M}(\mathcal{L}^{\mathrm{cas}}(E,Y),\widehat{T}_i)$ is obvious. For the reverse inclusion, let $(F_i,\widehat{T}_i) \in \mathcal{M}(\mathcal{L}^{\mathrm{cas}}(E,Y),\widehat{T}_i)$. By Proposition 3.5.4, there exists $0 \le (f_i^*,T_i^*) \in \mathcal{M}(E^*,T_i^*)$ such that $\|F_i x\| \le \langle |x|, f_i^*\rangle$ for each $x \in E$ and $i \in \mathbb{N}$. Let $\sup_{i\in\mathbb{N}} \|f_i^*\| := l$ and define $f^* : \bigcup_{i=1}^{\infty} \mathcal{R}(T_i) \to \mathbb{R}$ by

$$\langle x, f^*\rangle = \lim_{i\to\infty} \langle x, f_i^*\rangle$$

for each $x \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$. Observe that f^* is well defined. Indeed, for $x \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$ there exists $i \in \mathbb{N}$ such that $x \in \mathcal{R}(T_i)$. Consequently, $i \le j$ implies

$$\langle x, f_i^*\rangle = \langle x, T_i^* f_j^*\rangle = \langle T_i x, f_j^*\rangle = \langle x, f_j^*\rangle.$$

Thus, $\langle x, f^*\rangle = \lim_{i\to\infty}\langle x, f_i^*\rangle$ exists for each $x \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$. Evidently, f is positive, linear and the inequality

$$|\langle x, f^*\rangle| = \lim_{i\to\infty} |\langle x, f_i^*\rangle| \le \lim_{i\to\infty} \|f_i^*\|\|x\| = l\|x\|$$

shows that f^* is also bounded with norm $\|f^*\| \le l$.

Now define a map $F : \bigcup_{i=1}^{\infty} \mathcal{R}(T_i) \to Y$ by

$$Fx = \lim_{i\to\infty} F_i x$$

for each $x \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$. The map F is well defined because, for each $x \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$, there is some $i \in \mathbb{N}$ for which $x \in \mathcal{R}(T_i)$. Thus, $i \le j$ implies

$$F_i x = \widehat{T}_i F_j x = F_j T_i x = F_j x$$

so that $Fx = \lim_{i\to\infty} F_i x$ exists for each $x \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$. It is now evident that F is linear. Moreover, since $\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$ is a Riesz subspace of E, we have

$$\|Fx\| = \lim_{i\to\infty} \|F_i x\| \le \lim_{i\to\infty} \langle |x|, f_i^*\rangle = \langle |x|, f^*\rangle \le l\|x\|$$

for all $x \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$. Thus, F is bounded. Let $\overline{f}^*$ and $\overline{F}$ denote the unique continuous extensions of f^* and F respectively to the Banach sublattice $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$ of E. Then we have $\|\overline{F}x\| \le \langle |x|, \overline{f}^*\rangle$ for all $x \in \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$. Consequently, Lemma 3.5.3 implies $\overline{F} \in \mathcal{L}^{\mathrm{cas}}(\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}, Y)$. By Theorem 2.2.7, $\overline{F}$ possesses an extension $\overline{F}_\infty \in \mathcal{L}^{\mathrm{cas}}(E,Y)$ with $\|\overline{F}\|_{\mathrm{cas}} = \|\overline{F}_\infty\|_{\mathrm{cas}}$. Finally,

$$\widehat{T}_i \overline{F}_\infty x = F T_i x = \lim_{j\to\infty} F_j T_i x = \lim_{j\to\infty} \widehat{T}_i F_j x = F_i x$$

for all $x \in E$ and $i \in \mathbb{N}$. Thus, $(F_i,\widehat{T}_i) \in \mathcal{M}_{\mathrm{f}}(\mathcal{L}^{\mathrm{cas}}(E,Y),\widehat{T}_i)$. $\square$

We continue our preparations with the next lemma, which is a simple restatement of well known facts about order continuity of the norm in dual Banach lattices. We recall that a Banach space Y is said to have the *Schur property* if every weakly convergent sequence also converges in norm. The space ℓ^1 has the Schur property, by Schur's Theorem (cf. [36, Theorem 1.7]).

Lemma 3.5.6 *Let E be a Banach lattice such that E^* has order continuous norm. If $T : E \to \ell^1$ is a positive linear operator, then T is compact.*

Proof. Let $T : E \to \ell^1$ be a positive operator. Denote the restriction of T^* to c_0 by $T^*|_{c_0}$. Then $T^*|_{c_0} : c_0 \to E^*$ is positive. But E^* is a KB-space by [76, Theorem 2.4.14]; thus, $T^*|_{c_0}$ is weakly compact (cf. [92, Chapter II, §5, Proposition 5.15]). Consequently, $(T^*|_{c_0})^* : E^{**} \to \ell^1$ is compact because ℓ^1 has the Schur property. Hence, $T = (T^* \,|_{c_0})^* \,|_E$ is compact. □

Lastly, we need the following characterization of the l tensor product, which is shown in [69, Theorem 5.2].

Theorem 3.5.7 *Let E be a Banach lattice, Y a Banach space and $T \in \mathcal{L}(E, Y)$. Then $T \in E^*\widetilde{\otimes}_l Y$ if and only if there exist $0 \le S \in \mathcal{L}(E, \ell^1)$ and $R \in \mathcal{L}(\ell^1, Y)$ such that S is compact and $T = R \circ S$; further, $\|T\|_{\text{cas}} = \inf \|R\| \, \|S\|$ where the infimum is taken over all such factorizations of T.*

We are now prepared to characterize the Radon Nikodým property:

Theorem 3.5.8 *Let Y be a Banach space. Then the following statements are equivalent:*

(a) *Y has the Radon-Nikodým property.*

(b) *$E^*\widetilde{\otimes}_l Y = \mathcal{L}^{\text{cas}}(E, Y)$ for all separable Banach lattices E with order continuous dual.*

(c) *$\mathcal{M}(E^*\widetilde{\otimes}_l Y, T_i^* \otimes_l \text{id}_Y) = \mathcal{M}_{\text{f}}(E^*\widetilde{\otimes}_l Y, T_i^* \otimes_l \text{id}_Y)$ for all separable Banach lattices E with order continuous dual and all BL-filtrations (T_i) on E.*

(d) *$\mathcal{M}(E^*\widetilde{\otimes}_l Y, T_i^* \otimes_l \text{id}_Y) = \mathcal{M}_{\text{nc}}(E^*\widetilde{\otimes}_l Y, T_i^* \otimes_l \text{id}_Y)$ for all separable Banach lattices E with order continuous dual and all complemented, strictly positive BS-filtrations (T_i) on E.*

(e) *$\mathcal{M}(E^*\widetilde{\otimes}_l Y, T_i^* \otimes_l \text{id}_Y) = \mathcal{M}_{\text{nc}}(E^*, T_i^*)\widetilde{\otimes}_l Y$ for all separable Banach lattices E with order continuous dual and all complemented, strictly positive BS-filtrations (T_i) on E.*

(f) $\mathcal{M}(E\widetilde{\otimes}_l Y, T_i \otimes_l \mathrm{id}_Y) = \mathcal{M}(E, T_i)\widetilde{\otimes}_l Y$ *for all separable reflexive Banach lattices* E *and all complemented, strictly positive BS-filtrations* (T_i) *on* E.

(g) $\mathcal{M}(E\widetilde{\otimes}_l Y, T_i \otimes_l \mathrm{id}_Y) = \mathcal{M}_{\mathrm{nc}}(E\widetilde{\otimes}_l Y, T_i \otimes_l \mathrm{id}_Y)$ *for all separable reflexive Banach lattices* E *and all complemented, strictly positive BS-filtrations* (T_i) *on* E.

Proof. We prove the implications (a)⇒(b)⇒(c)⇒(d)⇒(e)⇒(f)⇒(g)⇒(a).

(a)⇒(b) Let E be a separable Banach lattice with order continuous dual. Let $T \in \mathcal{L}^{\mathrm{cas}}(E, Y)$. By Lemma 3.5.3, there exist a finite measure space (Ω, Σ, μ) and operators $0 \le T_1 \in \mathcal{L}(E, L^1(\mu))$ and $T_2 \in \mathcal{L}(L^1(\mu), Y)$ such that $T = T_2 \circ T_1$ where $\|T_1\| \le \|T\|_{\mathrm{cas}}$ and $\|T_2\| \le 1$. Since Y has the Radon-Nikodým property, the Lewis-Stegall Theorem (Theorem 1.3.6) guarantees the existence of operators $0 \le S_1 \in \mathcal{L}(L^1(\mu), \ell^1)$ and $S_2 \in \mathcal{L}(\ell^1, Y)$ such that $T_2 = S_2 \circ S_1$. Since E^* is order continuous, the positive operator $S_1 \circ T_1 : E \to \ell^1$ is compact by Lemma 3.5.6. Hence, $T \in E^*\widetilde{\otimes}_l X$ by Theorem 3.5.7.

(b)⇒(c) Suppose E is a separable Banach lattice with order continuous dual and (T_i) is a BL-filtration on E. Let $(f_i, T_i^* \otimes_l \mathrm{id}_Y) \in \mathcal{M}(E^*\widetilde{\otimes}_l Y, T_i^* \otimes \mathrm{id}_Y)$. By (b), $E^*\widetilde{\otimes}_l Y$ is isometric to $\mathcal{L}^{\mathrm{cas}}(E, Y)$ under the continuous extension of the canonical isometry $E^* \otimes_l Y \to \mathcal{L}^{\mathrm{cas}}(E, Y)$, given by $u \mapsto L_u$, where $L_u = \sum_{i=1}^n \langle \cdot, x_i^* \rangle y_i$ for $u = \sum_{i=1}^n x_i^* \otimes y_i$. Let $(F_i) \subset \mathcal{L}^{\mathrm{cas}}(E, Y)$ for which $f_i \mapsto F_i$ for each $i \in \mathbb{N}$. Suppose $i \le j$ and select a sequence $(u_k) \subset E^* \otimes_l Y$ such that $\lim_{k\to\infty} u_k = f_j$. Then, we also have $\lim_{k\to\infty} L_{u_k} = F_j$. For each $k \in \mathbb{N}$, choose a representation $u_k = \sum_{i=1}^{n_k} x_i^{*(k)} \otimes y_i^{(k)}$ and observe that

$$(T_i^* \otimes \mathrm{id}_Y)u_k \mapsto \sum_{i=1}^{n_k} \langle \cdot, T_i^* x_i^{*(k)} \rangle y_i^k = L_{u_k} \circ T_i = \widehat{T}_i L_{u_k}$$

for all $k \in \mathbb{N}$. Taking the limit as $k \to \infty$ yields $(T_i^* \otimes_l \mathrm{id}_Y)f_j \mapsto \widehat{T}_i F_j$. On the other hand, $(T_i^* \otimes_l \mathrm{id}_Y)f_j = f_i \mapsto F_i$. Thus, $\widehat{T}_i F_j = F_i$ from which $(F_i, \widehat{T}_i) \in \mathcal{M}(\mathcal{L}^{\mathrm{cas}}(E, Y), \widehat{T}_i)$ follows. By Theorem 3.5.5, there exists $F_\infty \in \mathcal{L}^{\mathrm{cas}}(E, Y)$ such that $\widehat{T}_i F_\infty = F_i$ for each $i \in \mathbb{N}$. Let $f_\infty \in E^* \otimes_l Y$ such that $f_\infty \mapsto F_\infty$. A similar argument to the above shows that $(T_i^* \otimes_l \mathrm{id}_Y)f_\infty \mapsto \widehat{T}_i F_\infty = F_i$ for each $i \in \mathbb{N}$. Consequently, $(T_i^* \otimes_l \mathrm{id}_Y)f_\infty = f_i$ for each $i \in \mathbb{N}$, implying that $(f_i, T_i^* \otimes_l \mathrm{id}_Y) \in \mathcal{M}_{\mathrm{f}}(E^*\widetilde{\otimes}_l Y, T_i^* \otimes_l \mathrm{id}_Y)$.

(c)⇒(d) If (T_i) is a complemented, strictly positive BS-filtration on a Banach lattice E, then the dual filtration (T_i^*) is also a complemented, strictly positive BS-filtration on E^*. Hence, (T_i^*) is a complemented BL-filtration on E^*. Consequently, $(T_i^* \otimes_l \mathrm{id}_Y)$ is a complemented BS-filtration on $E^*\widetilde{\otimes}_l Y$, by Corollary 3.4.7. Thus, the implication follows immediately from Proposition 3.2.11.

(d)⇒(e) Since $\mathcal{M}(E^*\widetilde{\otimes}_l Y, T_i^* \otimes_l \mathrm{id}_Y) = \mathcal{M}_{\mathrm{nc}}(E^*\widetilde{\otimes}_l Y, T_i^* \otimes_l \mathrm{id}_Y)$ holds and (T_i^*) a BL-filtration on E^*, we have

$$\mathcal{M}_{\mathrm{nc}}(E^*\widetilde{\otimes}_l Y, T_i^* \otimes_l \mathrm{id}_Y) = \mathcal{M}_{\mathrm{nc}}(E^*, T_i^*)\widetilde{\otimes}_l \mathcal{M}_{\mathrm{nc}}(Y, \mathrm{id}_Y) = \mathcal{M}_{\mathrm{nc}}(E^*, T_i^*)\widetilde{\otimes}_l Y$$

by Corollary 3.4.6. Thus, (e) follows immediately.

(e)⇒(f) Since E is a separable reflexive Banach lattice, E^{**} is order continuous and E^* is separable (cf. [92, Chapter II, §5, Theorem 5.16]). By (e) and the reflexivity of E, it follows that

$$\begin{aligned}\mathcal{M}(E\widetilde{\otimes}_l Y, T_i \otimes_l \mathrm{id}_Y) &= \mathcal{M}(E^{**}\widetilde{\otimes}_l Y, T_i^{**} \otimes_l \mathrm{id}_Y)\\ &= \mathcal{M}_{\mathrm{nc}}(E^{**}, T_i^{**})\widetilde{\otimes}_l Y\\ &= \mathcal{M}_{\mathrm{nc}}(E, T_i)\widetilde{\otimes}_l Y.\end{aligned}$$

By the reflexivity of E and Corollary 3.2.13, $\mathcal{M}_{\mathrm{nc}}(E, T_i)$ is Riesz and isometrically isomorphic to $\mathcal{M}(E, T_i)$. Consequently, $\mathcal{M}(E\widetilde{\otimes}_l Y, T_i \otimes_l \mathrm{id}_Y) = \mathcal{M}(E, T_i)\widetilde{\otimes}_l Y$, as required.

(f)⇒(g) As above, $\mathcal{M}_{\mathrm{nc}}(E, T_i)$ is Riesz and isometrically isomorphic to $\mathcal{M}(E, T_i)$. Another application of Corollary 3.4.6 reveals that

$$\mathcal{M}(E, T_i)\widetilde{\otimes}_l Y = \mathcal{M}_{\mathrm{nc}}(E, T_i)\widetilde{\otimes}_l \mathcal{M}_{\mathrm{nc}}(Y, \mathrm{id}_Y) = \mathcal{M}_{\mathrm{nc}}(E\widetilde{\otimes}_l Y, T_i \otimes_l \mathrm{id}_Y).$$

Thus, $\mathcal{M}(E\widetilde{\otimes}_l Y, T_i \otimes_l \mathrm{id}_Y) = \mathcal{M}_{\mathrm{nc}}(E\widetilde{\otimes}_l Y, T_i \otimes_l \mathrm{id}_Y)$ by (f).

(g)⇒(a) For all finite measure spaces (Ω, Σ, μ) and $1 < p < \infty$, the Banach lattice $L^p(\mu)$ is separable and reflexive. By (g), it follows that $\mathcal{M}(L^p(\mu, Y), \Sigma_i) = \mathcal{M}_{\mathrm{nc}}(L^p(\mu, Y), \Sigma_i)$ for every filtration (Σ_i). Thus, Y has the Radon Nikodým property by Theorem 1.3.10. □

The above theorem allows us to generalize Theorem 1.3.7, which characterizes the 'Asplund spaces'. A Banach space Y is called an *Asplund space* if Y^* has the Radon Nikodým property.

Theorem 3.5.9 *Let Y be a Banach space. Then Y is an Asplund space if and only if $E^*\widetilde{\otimes}_l Y^* = (E\widetilde{\otimes}_l Y)^*$ for all separable Banach lattices E with order continuous dual.*

Proof. By Theorem 3.5.8, Y^* has the Radon Nikodým property if and only if $E^*\widetilde{\otimes}_l Y^* = \mathcal{L}^{\mathrm{cas}}(E, Y^*)$ for all separable Banach lattices E with order continuous dual. But Theorem 2.3.5 implies that $E^*\widetilde{\otimes}_l Y^* = \mathcal{L}^{\mathrm{cas}}(E, Y^*) = (E\widetilde{\otimes}_l Y)^*$, which completes the proof. □

It is important to note that the above theorem does not include the case $E = L^1(\mu)$. However, by Theorem 1.3.7, Y is an Asplund space if and only if $L^1(\mu, Y)^* = L^\infty(\mu, Y^*)$ for all finite measure spaces (Ω, Σ, μ).

3.6 Notes and remarks

At the beginning of the chapter, we bifurcated the results of Troitsky in [101]; Section 3.2 contained results that relied on norm structure only (i.e. the theory of BS-filtrations) and Section 3.3 contained results that required an additional order structure (i.e. the theory of BL-filtrations). In Section 3.4, we exploited the properties of left order uniformity and left order injectivity to study the tensor product of BL-filtrations with BS-filtrations. In Theorem 3.4.5 and Corollary 3.4.6, we showed that the space of norm convergent martingales distributes over the l-tensor product. This result features heavily in Chapter 5 and Chapter 7. As mentioned, the proof of this result relies on the definition of the l-norm. However, a second glance at the proof reveals that these results also hold for any left order uniform, left order injective crossnorm that is smaller than the l-norm.

Let $1 < p < \infty$ and Y be a Banach space. As we have seen, we have the isometric embedding $L^p(\mu, Y) \hookrightarrow \mathcal{L}^{\text{cas}}(L^q(\mu), Y)$ where $\frac{1}{p} + \frac{1}{q} = 1$. By Theorem 3.5.5, every $L^p(\mu, Y)$-bounded martingale, $1 < p < \infty$, is fixed on some element in $\mathcal{L}^{\text{cas}}(L^q(\mu), Y)$. This fact is somewhat surprising and gives new insight into Theorem 1.3.10. Indeed, when Y has the Radon Nikodým property, the space $\mathcal{L}^{\text{cas}}(L^q(\mu), Y)$ collapses down to $L^p(\mu, Y)$ (see Theorem 3.5.8), forcing every $L^p(\mu, Y)$-bounded martingale to be fixed on an element in $L^p(\mu, Y)$. Consequently, every $L^p(\mu, Y)$-bounded martingale must converge.

It is natural to ask whether an analogue of Theorem 3.5.8 exists for the UMD property. This is tantamount to asking:

- If Y is a UMD space, for which Banach lattices E does every ± 1-transform of every bounded martingale in $E^* \widetilde{\otimes}_l Y$ converge?

Unfortunately, we have not made much progress in answering this question. We suspect that the only Banach lattices E that satisfy the above question are separable abstract L^p-spaces, $1 < p < \infty$. Another related question is:

- If Y is a UMD space, for which Banach lattices E do we have $E^* \widetilde{\otimes}_l Y = \mathcal{L}^{\text{cas}}(E, Y)$?

Since every UMD space already has the Radon Nikodḿ property, the class of separable Banach lattices with order continuous dual will satisfy the above question, by Theorem 3.5.8. Given that the UMD property is substantially stronger than the Radon Nikodým property, is it possible to extend this class of Banach lattices?

4

Intermission

4.1 A description of $\ell^p_{\mathrm{strong}}(Y)$

The technique used in the previous chapter of factorizing operators in the characterization of the Radon Nikodým property also has other applications. In this brief interlude, we use these techniques to produce a Grothendieck style characterization of the space of p-summable sequences in a Banach space.

Let Y be a Banach space, $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Denote the space of weakly p-summable sequences by

$$\ell^p_{\mathrm{weak}}(Y) = \Big\{ (y_n) \ : \ y_n \in Y \text{ and } (\langle y_n, y^* \rangle) \in \ell^p \ \forall \ y^* \in Y^* \Big\},$$

endowed with the norm ϵ_p, given by

$$\epsilon_p\big((y_n)\big) = \begin{cases} \sup \Big\{ \big(\sum_n |\langle y_n, y^* \rangle|^p\big)^{1/p} : \|y^*\| \leq 1 \Big\}, & 1 \leq p < \infty; \\ \sup \left\{ \sup_{n \in \mathbb{N}} |\langle y_n, y^* \rangle| : \|y^*\| \leq 1 \right\}, & p = \infty. \end{cases}$$

Then $\ell^p_{\mathrm{weak}}(Y)$ is a Banach space (cf. [36] and [55, §19.4]). The space

$$(c_0)_{\mathrm{weak}}(Y) = \Big\{ (y_n) \ : \ y_n \in Y \text{ and } \big(\langle y_n, y^* \rangle\big) \in c_0 \ \forall \ y^* \in Y^* \Big\}$$

is a closed subspace of $\ell^\infty_{\mathrm{weak}}(Y)$ (cf. [36] and [55, §19.4]). The following description of $\ell^p_{\mathrm{weak}}(Y)$ was formulated by Grothendieck (cf. [36]):

Theorem 4.1.1 *Let Y be a Banach space and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

(a) $\ell^p_{\mathrm{weak}}(Y)$ is isometrically isomorphic to $\mathcal{L}(\ell^q, Y)$ for $1 < p \leq \infty$ and $\ell^1_{\mathrm{weak}}(Y)$ is isometrically isomorphic to $\mathcal{L}(c_0, Y)$.

(b) $\ell^p_{\mathrm{weak}}(Y^*)$ is isometrically isomorphic to $\mathcal{L}(Y, \ell^p)$ for $1 \leq p \leq \infty$ and $(c_0)_{\mathrm{weak}}(Y^*)$ is isometrically isomorphic to $\mathcal{L}(Y, c_0)$.

In (a) the isomorphism is given by Γ, defined by

$$\Gamma : (y_n) \mapsto T_{(y_n)}, \quad T_{(y_n)}\big((\lambda_n)_{n=1}^\infty\big) = \sum_{n=1}^\infty \lambda_n y_n, \quad (\lambda_n) \in \ell^q.$$

In (b) the isomorphism is given by Θ,

$$\Theta : (y_n^*) \mapsto T_{(y_n^*)}, \quad T_{(y_n^*)}(x) = (\langle y, y_n^*\rangle)_{n=1}^\infty, \quad y \in Y.$$

The reader is referred to [12, 70] for an order analogue of the above, for the case where Y is a Banach lattice.

Motivated by the operator description of $\ell^p_{\text{weak}}(Y)$ as stated in Theorem 4.1.1, we proceed to describe $\ell^p_{\text{strong}}(Y)$ in terms of the cone absolutely summing operators. Here,

$$\ell^p_{\text{strong}}(Y) = \left\{ (y_i) \,:\, y_i \in Y, \sum_{i=1}^\infty \|y_i\|^p < \infty \right\}$$

is the Banach space of all absolutely p-summable sequences (y_n) in Y with respect to the norm

$$\Delta_p((y_i)) = \left(\sum_{i=1}^\infty \|y_i\|^p \right)^{1/p} \qquad \text{(cf. [34, 35, 36]).}$$

Since the finite sequences in $\ell^p_{\text{strong}}(Y)$ form a dense subset of $\ell^p_{\text{strong}}(Y)$ we have that $\ell^p_{\text{strong}}(Y)$ isometrically isomorphic to $l^p \widetilde{\otimes}_{\Delta_p} Y$. Moreover, by Theorem 2.3.1, we have that $\Delta_p = \|\cdot\|_l$ on $\ell^p \otimes Y$. The isometric isomorphism is obtained as the continuous extension of the linearization of the bilinear map $\psi\colon \ell^p \times Y \to \ell^p_{\text{strong}}(Y)$, defined by $\psi((\lambda_n)_{n=1}^\infty, y) = \sum_{n=1}^\infty \lambda_n y$.

We want to show that the canonical embeddings $\ell^1 \widetilde{\otimes}_l Y \hookrightarrow \mathcal{L}^{\text{cas}}(c_0, Y)$ and $\ell^p \widetilde{\otimes}_l Y \hookrightarrow \mathcal{L}^{\text{cas}}(\ell^q, Y)$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $1 \le p < \infty$, are surjective isometries. To achieve this objective, we will need to re-examine Lemma 3.5.3. In an effort to minimize the turning of pages, we restate this lemma.

Lemma 4.1.2 *Let E be a Banach lattice, Y a Banach space and $l \in \mathbb{R}_+$. Then the following statements are equivalent for $T \in \mathcal{L}(E, Y)$.*

(a) $T \in \mathcal{L}^{\text{cas}}(E, Y)$ with $\|T\|_{\text{cas}} \le l$.
(b) There exists $x^* \in E^*_+$ such that $\|x^*\| \le l$ and $\|Tx\| \le x^*(|x|)$ for all $x \in E$.
(c) There exist an AL-space L, $T_1 \in \mathcal{L}_+(E, L)$ and $T_2 \in \mathcal{L}(L, Y)$ such that $T = T_2 \circ T_1$ where $\|T_1\| \le l$ and $\|T_2\| \le 1$.

By the L-space theorem of Kakutani, every AL-space is Riesz and isometrically isomorphic to some $L^1(\mu)$-space, where (Ω, Σ, μ) is some measure space (cf. [57, 58, 72, 92]). Let (Ω, Σ, μ) be a σ-finite measure space, $1 \le p < \infty$ and $T \in \mathcal{L}^{\mathrm{cas}}(L^p(\mu), Y)$. We claim that the AL-space in the factorization of T in Lemma 4.1.2(c) can be taken as $L^1(\mu)$; i.e., a different measure space does not have to be introduced:

To substantiate our claim, it follows from Lemma 4.1.2(b) that there exists $f \in L^q(\mu)_+$, where $\frac{1}{p} + \frac{1}{q} = 1$, such that $\|f\|_q \le l$ and for all $g \in L^p(\mu)$,

$$\|Tg\| \le \int_\Omega |g| f \,\mathrm{d}\mu = \int_\Omega |g| \,\mathrm{d}\mu_f,$$

with $d\mu_f = f\,\mathrm{d}\mu$. Let $\Omega_f := \{\omega \in \Omega : f(\omega) \neq 0)\}$ and let P_B be the band projection of $L^0(\mu) := \{f\colon \Omega \to \mathbb{R} : f \text{ is } \mu\text{-measurable}\}$ onto the band consisting of all functions in $L^0(\mu)$ which vanish on $\Omega \backslash \Omega_f$. Since

$$\int_\Omega |P_B g| \,\mathrm{d}\mu_f = \int_\Omega |g| f \,\mathrm{d}\mu \le \|f\|_q \,\|g\|_p \quad \text{for all } g \in L^p(\mu),$$

the map P_B maps $L^p(\mu)$ into $L^1(\mu_f)$ and has norm $\|f\|_q \le l$. Using the fact that $P_B g = g$ μ_f-almost everywhere, one sees that the image of $L^p(\mu)$ is an ideal in $L^1(\mu_f)$. Its closure is a band in $L^1(\mu_f)$ and is in fact equal to $L^1(\mu_f)$, because the image of a weak order unit in $L^p(\mu)$ (which exists by the σ-finiteness of the measure space) is a weak order unit in $L^1(\mu_f)$. Define $T_2^*\colon P_B(L^p(\mu)) \to Y$ by $T_2^* h = Th$ for all $h \in P_B(L^p(\mu))$. Then

$$\|T_2^* h\| = \|Th\| \le \int_\Omega |h| f \,\mathrm{d}\mu = \int_\Omega |h| \,\mathrm{d}\mu_f = \|h\|_{L^1(\Omega,\mu_f)}$$

and so we can extend T_2^* by continuity to $L^1(\mu_f)$ and the extended operator is of norm less than or equal to one.

We have thus shown that the AL-space L in Theorem 4.1.2(b) can be taken to be the space $L^1(\mu_f)$.

Using Ando's theorem (cf. [6], [71, Lemma 1.b.9] or [92, Chapter III, §11, Theorem 11.4]), one can do better and show that L can be taken to be the space $L^1(\mu)$. To see this, consider the multiplication operator $M_f : L^1(\mu_f) \to L^1(\mu)$, defined by $M_f g = fg$ for all $g \in L^1(\mu)$. Using the fact that $f \ge 0$, it readily follows that M_f is an isometric Riesz isomorphism (into). By Ando's theorem, there exists a contractive projection P_A of $L^1(\mu)$ onto the closed Riesz subspace $M_f(L^1(\mu_f))$ and so the identity operator I on $L^1(\mu_f)$ has the factorization $I = M_f^{-1} \circ P_A \circ M_f$. This shows that T has the factorization claimed in Lemma 4.1.2(c) with $L = L^1(\mu)$, $T_1 = M_f \circ P_B|_{L^p(\mu)}$ and $T_2 = T_2^* \circ M_f^{-1} \circ P_A$, where $P_B|_{L^p(\mu)}$ denotes the restriction of P_B to $L^p(\mu)$. This completes the proof of our claim. We are now able to prove our main result.

Lemma 4.1.3 *Let Y be a Banach space. Then the canonical injections $\ell^1\widetilde{\otimes}_l Y \hookrightarrow \mathcal{L}^{\text{cas}}(c_0, Y)$ and $\ell^p\widetilde{\otimes}_l Y \hookrightarrow \mathcal{L}^{\text{cas}}(\ell^q, Y)$, for $\frac{1}{p}+\frac{1}{q}=1$ and $p \neq 1$, are surjections and therefore surjective isometric isomorhpisms.*

Proof. Consider the case $p=1$. Let $T \in \mathcal{L}^{\text{cas}}(c_0, Y)$. Since c_0 is a Banach sublattice of ℓ^∞, T has an extension $S \in \mathcal{L}^{\text{cas}}(\ell^\infty, Y)$ such that $\|T\|_{\text{cas}} = \|S\|_{\text{cas}}$, by Theorem 2.2.7. But then S has a factorization $S = S_2 \circ S_1$, where $S_1 \in \mathcal{L}_+(\ell^\infty, \ell^1)$ and $S_2 \in \mathcal{L}(\ell^1, Y)$, by the remarks following Lemma 4.1.2. Since $(\ell^\infty)^*$ is an AL-space, it has order continuous norm so that S_1 is compact, by Lemma 3.5.6. But then T has a factorization $T = S_2 \circ (S_1|_{c_0})$, where $S_1|_{c_0}$ is compact. So, by Theorem 3.5.7, $T \in \ell^1\widetilde{\otimes}_l Y$.

The case $1 < p < \infty$ is similar, but without the complication of first making an extension as in the case $p=1$. □

Theorem 4.1.4 *Let Y be a Banach space and $\frac{1}{p}+\frac{1}{q}=1$. Then*

(a) $\ell^1_{\text{strong}}(Y)$ is isometrically isomorphic to $\mathcal{L}^{\text{cas}}(c_0, Y)$.
(b) $\ell^p_{\text{strong}}(Y)$ is isometrically isomorphic to $\mathcal{L}^{\text{cas}}(\ell^q, Y)$ for $\frac{1}{p}+\frac{1}{q}=1$ and $p \neq 1$.

Proof. Since $\ell^p_{\text{strong}}(Y)$ is isometrically isomorphic to $\ell^p\widetilde{\otimes}_l Y$, as mentioned above, the proof of the result is completed by applying Lemma 4.1.3. □

The reader is referred to [23] for more on the interplay between sequence spaces and cone absolutely summing operators, as well as strongly majorizing operators. Our techniques differ from those used in [23].

5

Convergent martingales

5.1 Introduction

In this chapter, we provide some explicit characterizations of convergent martingales in the l-tensor product. The main ingredient in these characterizations are representation theorems for elements of the completed l-tensor product $E\widetilde{\otimes}_l Y$, of a Banach lattice E and a Banach space Y.

In Section 5.2, we present a representation theorem for elements in the l-tensor product that is analogous to the well known representation theorem for elements in the projective tensor product (see for instance [90, Proposition 2.7]). The l-tensor product version of this result is considerably more difficult and appears in [69]. Our presentation of this result is adapted from this paper.

In Section 5.3, the representation theorem in Section 5.2 is used, in conjunction with the distributive result for the space of norm convergent martingales on the l-tensor product (Theorem 3.4.5), to prove a complete description of convergent martingales in the l-tensor product. Combining this description with the martingale characterizations of the Radon Nikodým property, studied in Chapter 3, yields another form of the Radon Nikodým property. This description can be also be applied to martingale difference sequences. As a consequence, when we specialize to the Lebesgue-Bochner spaces, we obtain a description of the UMD property. The results is this section are original and some of them appear in [26].

One drawback to the representation theorem, studied in Section 5.2, is that there are uncountably many representations for an element in the l-tensor product. Consequently, the description theorem for convergent martingales in the l-tensor product, offered in Section 5.3, has uncountably many representations. To improve this situation, we examine the notion of a basis with 'vector-valued coefficients' in Section 5.4. This concept is considered by Figiel and Wojtaszczyk in [46] and is the origin of our presentation. We identify an easy criterion for a basis in a Banach lattice E

to be a so called 'Y-basis' for the l-tensor product $E\widetilde{\otimes}_l Y$. With mild assumptions on E, this provides a unique way to represent elements of $E\widetilde{\otimes}_l Y$.

In Section 5.5, the theory of bases with vector-valued coefficients is used to provide another description of norm convergent martingales in the l-tensor product. In contrast to the results in Section 5.3, the description shown here is unique. Analogues of the descriptions of the Radon Nikoým property and the UMD property, given in Section 5.3, are deduced.

5.2 Representing the elements in the *l*-tensor product

We present a characterization of elements of $E\widetilde{\otimes}_l Y$, as can be found in [69], which will allow us to describe norm convergent martingales in the l-tensor product. We start with a lemma from [69, Theorem 3.2].

Lemma 5.2.1 *Let E be Banach a lattice and Y a Banach space. Then,*

$$\|u\|_\Delta = \inf \left\{ \left\| \sum_{i=1}^n |x_i| \right\| \sup_{1\le i\le n} \|y_i\| : u = \sum_{i=1}^n x_i \otimes y_i \right\}$$

for all $u = \sum_{i=1}^n x_i \otimes y_i \in E \otimes Y$.

Proof. Let $u = \sum_{i=1}^n x_i \otimes y_i$. Then

$$\left\| \sum_{i=1}^n \|y_i\| \, |x_i| \right\| \le \left\| \sum_{i=1}^n |x_i| \right\| \sup_{1\le i\le n} \|y_i\|.$$

Thus,

$$\|u\|_\Delta \le \inf \left\{ \left\| \sum_{i=1}^n |x_i| \right\| \sup_{1\le i\le n} \|y_i\| : u = \sum_{i=1}^n x_i \otimes y_i \right\}.$$

On the other hand,

$$u = \sum_{i=1}^n \|y_i\| x_i \otimes \frac{y_i}{\|y_i\|} ;$$

and clearly

$$\left\| \sum_{i=1}^n \|y_i\| \, |x_i| \right\| \sup_{1\le i\le n} \left\| \frac{y_i}{\|y_i\|} \right\| \le \left\| \sum_{i=1}^n \|y_i\| \, |x_i| \right\|.$$

Consequently,

$$\inf\left\{\left\|\sum_{i=1}^{n}|x_i|\right\|\sup_{1\le i\le n}\|y_i\| : u=\sum_{i=1}^{n}x_i\otimes y_i\right\}$$
$$\le \left\|\sum_{i=1}^{n}\|y_i\|\,|x_i|\right\|\sup_{1\le i\le n}\left\|\frac{y_i}{\|y_i\|}\right\| \le \left\|\sum_{i=1}^{n}\|y_i\|\,|x_i|\right\|,$$

so that

$$\inf\left\{\left\|\sum_{i=1}^{n}|x_i|\right\|\sup_{1\le i\le n}\|y_i\| : u=\sum_{i=1}^{n}x_i\otimes y_i\right\}\le \|u\|_{\Delta}.$$

Thus,

$$\|u\|_{\Delta}=\inf\left\{\left\|\sum_{i=1}^{n}|x_i|\right\|\sup_{1\le i\le n}\|y_i\| : u=\sum_{i=1}^{n}x_i\otimes y_i\right\}.$$

□

The following result is adapted from [69, Theorem 4.1]

Theorem 5.2.2 *Let E be a Banach lattice and Y a Banach space. Then, $u\in E\widetilde{\otimes}_l Y$ if and only if $u=\sum_{i=1}^{\infty}x_i\otimes y_i$, where $(|x_i|)\subset E$ is an unconditionally summable sequence and $(y_i)\subset Y$ is a null sequence. Moreover,*

$$\|u\|_l=\inf\left\{\left\|\sum_{i=1}^{\infty}|x_i|\right\|\sup_{i\in\mathbb{N}}\|y_i\| : u=\sum_{i=1}^{\infty}x_i\otimes y_i,\ \left\|\sum_{i=1}^{\infty}|x_i|\right\|<\infty,\ \lim_{i\to\infty}\|y_i\|=0\right\}.$$

Proof. Let $(|a_i|)\subset E$ be an unconditionally summable sequence and $(y_i)\subset Y$ be a null sequence. By Proposition 2.2.4, we have $\|(|a_i|)\|_{\varepsilon}=\|\sum_{i=1}^{\infty}|a_i|\|<\infty$. Let $u_r=\sum\limits_{i=1}^{r}a_i\otimes y_i$. Then, for each $q,r\in\mathbb{N}$ with $q\le r$ we have, by Lemma 5.2.1,

$$\|u_r-u_q\|_{\Delta}\le\left\|\sum_{i=q+1}^{r}|a_i|\right\|\sup_{q+1\le i\le r}\|(y_i)\|\to 0$$

as $q,r\to\infty$. Thus, (u_r) is a Cauchy sequence that converges to $u:=\sum\limits_{i=1}^{\infty}a_i\otimes y_i$ in $E\widetilde{\otimes}_{\Delta}Y$. Moreover, $\|u_r\|_{\Delta}\le\|\sum_{i=1}^{\infty}|a_i|\|\sup_{i\in\mathbb{N}}\|y_i\|$ for each $r\in\mathbb{N}$. Thus,

$$\|u\|_{\Delta}\le\left\|\sum_{i=1}^{\infty}|a_i|\right\|\sup_{i\in\mathbb{N}}\|y_i\|. \tag{5.1}$$

Conversely, let $u\in E\widetilde{\otimes}_{\Delta}Y$ and $\varepsilon>0$ be given. Then, there exists a sequence (u_n) in $E\otimes Y$ such that

$$\|u-u_i\|_{\Delta}<(1/2)^{2i+1}\varepsilon$$

for $i = 0, 1, 2, \ldots$. Then

$$\|u_{i+1} - u_i\|_\Delta \leq \|u_{i+1} - u\|_\Delta + \|u - u_i\|_\Delta < (1/4)^i \varepsilon$$

for $i = 0, 1, 2, \ldots$. Hence, by Lemma 5.2.1, $u_{i+1} - u_i$ has a representation

$$u_{i+1} - u_i = \sum_{n=1}^{n_{i+1}} a_n^{(i+1)} \otimes y_n^{(i+1)},$$

with $a_n^{(i+1)} \in E$, $y_n^{(i+1)} \in Y$,

$$\left\| \sum_{n=1}^{n_{i+1}} \left| a_n^{(i+1)} \right| \right\| \sup_{1 \leq n \leq n_{i+1}} \left\| y_n^{(i+1)} \right\| < (1/4)^i \varepsilon,$$

$$\left\| \sum_{n=1}^{n_{i+1}} \left| a_n^{(i+1)} \right| \right\| \leq \left[(1/4)^i \varepsilon \right]^{1/2}$$

and

$$\sup_{1 \leq n \leq n_{i+1}} \left\| y_n^{(i+1)} \right\| \leq \left[(1/4)^i \varepsilon \right]^{1/2}.$$

Since $\|u_0\|_\Delta \leq \|u\|_\Delta + \|u - u_0\|_\Delta < \|u\|_\Delta + \varepsilon/2$, by Lemma 5.2.1, there is a representation

$$u_0 = \sum_{n=1}^{n_0} a_n^{(0)} \otimes y_n^{(0)}$$

with $a_n^{(0)} \in E$, $y_n^{(0)} \in Y$,

$$\left\| \sum_{n=1}^{n_0} \left| a_n^{(0)} \right| \right\| \sup_{1 \leq n \leq n_0} \left\| y_n^{(0)} \right\| < \|u\|_\Delta + \varepsilon/2,$$

$$\left\| \sum_{n=1}^{n_0} \left| a_n^{(0)} \right| \right\| \leq (\|u\|_\Delta + \varepsilon/2)^{1/2}$$

and

$$\sup_{1 \leq n \leq n_0} \left\| y_n^{(0)} \right\| \leq (\|u\|_\Delta + \varepsilon/2)^{1/2}.$$

From the choices of the representations of $u_{i+1} - u_i$, it follows that, for any $k \in \mathbb{N}$,

$$\left\| u - \sum_{i=0}^{k} \sum_{n=1}^{n_i} a_n^{(i)} \otimes y_n^{(i)} \right\|_\Delta = \|u - u_k\|_\Delta < (1/2)^{2k+1} \varepsilon;$$

i.e. the series $\sum_{i=0}^{\infty} \sum_{n=1}^{n_i} a_n^{(i)} \otimes y_n^{(i)}$ converges to u in $E \widetilde{\otimes}_\Delta Y$. At this point, it is less cumbersome to relabel some of the sequences. Consider the composed sequences given by

$$(a_i) := \left(a_1^{(0)}, \ldots, a_{n_0}^{(0)}, a_1^{(1)}, \ldots, a_{n_1}^{(1)}, a_1^{(2)}, \ldots\right)$$

and

$$(y_i) := \left(y_1^{(0)}, \ldots, y_{n_0}^{(0)}, y_1^{(1)}, \ldots, y_{n_1}^{(1)}, y_1^{(2)}, \ldots\right).$$

Let $n'_k = n_0 + n_1 + \cdots + n_k$ for each $k \in \mathbb{N} \cup \{0\}$. Then, for each $k \in \mathbb{N} \cup \{0\}$,

$$\left\|\sum_{i=1}^{n'_k} |a_i|\right\| \leq \sum_{i=0}^{k} \left\|\sum_{n=1}^{n_i} \left|a_n^{(i)}\right|\right\| \leq (\|u\|_\Delta + \varepsilon/2)^{1/2} + \varepsilon^{1/2}$$

and, similarly,

$$\sup_{1 \leq i \leq n'_k} \|y_i\| \leq \sum_{i=0}^{k} \sup_{1 \leq n \leq n_i} \|y_n^{(i)}\| \leq (\|u\|_\Delta + \varepsilon/2)^{1/2} + \varepsilon^{1/2}.$$

Also, for each $k \in \mathbb{N} \cup \{0\}$,

$$\sum_{i=0}^{k} \sum_{n=1}^{n_i} a_n^{(i)} \otimes y_n^{(i)} = \sum_{i=1}^{n'_k} a_i \otimes y_i.$$

Thus, the series $\sum_{i=1}^{\infty} a_i \otimes y_i$ converges to u in $E\widetilde{\otimes}_\Delta Y$.

To show that $(|a_i|)$ is unconditionally summable, let

$$\xi_i := \begin{cases} (1/2^k)^{1/2}, & \text{for } n'_k + 1 \leq i \leq n'_{k+1},\ k \in \mathbb{N} \cup \{0\} \\ 1, & \text{for } 1 \leq i \leq n'_0 \end{cases}$$

and let $\bar{a}_i := a_i/\xi_i$ for all $i \in \mathbb{N}$. Clearly, $(\xi_i) \in c_0$. For large $m, n \in \mathbb{N}$ with $m < n$, select $j, l \in \mathbb{N}$ such that $n'_j < m < n \leq n'_{j+l}$. Then,

$$\|(|\bar{a}_i|)_{i=1}^{n} - (|\bar{a}_i|)_{i=1}^{m-1}\|_\varepsilon = \sup\left\{\sum_{i=m}^{n} |\langle |\bar{a}_i|, x^*\rangle| : x^* \in E^*, \|x^*\| \leq 1\right\}$$
$$\leq \sum_{k=j}^{j+l} \sup\left\{\sum_{i=n'_k+1}^{n'_{k+1}} |\langle |\bar{a}_i|, x^*\rangle| : x^* \in E^*, \|x^*\| \leq 1\right\}.$$

By the definition of (ξ_i) and Proposition 2.2.4 it follows that

$$
\begin{aligned}
&\sum_{k=j}^{j+l} \sup \left\{ \sum_{i=n'_k+1}^{n'_{k+1}} |\langle |\bar{a}_i|, x^* \rangle| : x^* \in E^*, \|x^*\| \le 1 \right\} \\
&= \sum_{k=j}^{j+l} 2^{k/2} \left\| \sum_{i=n'_k+1}^{n'_{k+1}} |a_i| \right\| \\
&= \sum_{k=j}^{j+l} 2^{k/2} \left\| \sum_{n=1}^{n_{k+1}} \left| a_n^{(k+1)} \right| \right\| \\
&\le \sum_{k=j}^{j+l} 2^{k/2} \left[(1/4)^k \varepsilon \right]^{1/2} = \varepsilon^{1/2} \sum_{k=j}^{j+l} (1/2)^{k/2} \\
&\le \varepsilon^{1/2} \left(1 + \frac{1}{\sqrt{2}} \right).
\end{aligned}
$$

Thus, $(|\bar{a}_i|)$ is unconditionally summable. Consequently, by Theorem 1.4.2(e), $(|a_i|) = (\xi_i |\bar{a}_i|)$ is also unconditionally summable.

We now show $\lim_{i\to\infty} \|y_i\| = 0$. As before, let $\bar{y}_i := y_i/\xi_i$ for all $i \in \mathbb{N}$. For large $m, n \in \mathbb{N}$ with $m < n$, select $j, l \in \mathbb{N}$ such that $n'_j < m < n \le n'_{j+l}$. Then,

$$
\|(\bar{y}_i)_{i=1}^{n} - (\bar{y}_i)_{i=1}^{m-1}\|_{\ell_\infty(Y)} = \sup_{m \le i \le n} \|\bar{y}_i\| \le \sum_{k=j}^{j+l} \left(\sup_{n'_k+1 \le i \le n'_{k+1}} \|\bar{y}_i\| \right).
$$

By the definition of (ξ_n), it follows that

$$
\begin{aligned}
&\sum_{k=j}^{j+l} \left(\sup_{n'_k+1 \le i \le n'_{k+1}} \|\bar{y}_i\| \right) = \sum_{k=j}^{j+l} 2^{k/2} \left(\sup_{n'_k+1 \le i \le n'_{k+1}} \|y_i\| \right) \\
&= \sum_{k=j}^{j+l} 2^{k/2} \left(\sup_{1 \le n \le n_{k+1}} \left\| y_n^{(k+1)} \right\| \right) \\
&\le \sum_{k=j}^{j+l} 2^{k/2} \left[(1/4)^k \varepsilon \right]^{1/2} = \varepsilon^{1/2} \sum_{k=j}^{j+l} (1/2)^{k/2} \\
&\le \varepsilon^{1/2} \left(1 + \frac{1}{\sqrt{2}} \right).
\end{aligned}
$$

Thus, $(\bar{y}_i)$ is a bounded sequence. Consequently, $\lim_{i\to\infty} \|y_i\| = \lim_{n\to\infty} \|\xi_i \bar{y}_i\| = 0$ because $(\xi_i) \in c_0$. Furthermore,

$$
\left\| \sum_{i=1}^{\infty} |a_i| \right\| \le (\|u\|_\Delta + \varepsilon/2)^{1/2} + \varepsilon^{1/2}
$$

and

$$
\sup_{i \in \mathbb{N}} \|(y_i)\| \le (\|u\|_\Delta + \varepsilon/2)^{1/2} + \varepsilon^{1/2}.
$$

Hence,

$$\left\|\sum_{i=1}^{\infty}|a_i|\right\|\sup_{i\in\mathbb{N}}\|(y_i)\| \leq \|u\|_{\Delta}+f(\varepsilon), \tag{5.2}$$

where f is a positive real valued function with $f(\varepsilon) \to 0$ as $\varepsilon \to 0$. It follows from (5.1) and (5.2) that the norm equality holds. Theorem 2.3.1 completes the proof. □

We also have the following corollary (cf. [69, Corollary 4.2])

Corollary 5.2.3 *The sequence (x_i) in Theorem 5.2.2 can be chosen such that $(x_i) \subset E_+$.*

Proof. If $u = \sum_{i=1}^{\infty} x_i \otimes y_i \in E\widetilde{\otimes}_l Y$ with $(|x_i|) \subset E$ unconditionally summable and $(y_i) \subset Y$ a null sequence, then

$$u = \sum_{i=1}^{\infty}(x_i^+ - x_i^-)\otimes y_i = \sum_{i=1}^{\infty}(x_i^+\otimes y_i + x_i^-\otimes(-y_i)),$$

with $(x_1^+, x_1^-, x_2^+, x_2^-, \dots) \subset E_+$ unconditionally summable by Lemma 2.2.4 and $(y_1, -y_1, y_2, -y_2, \dots) \subset Y$ a null sequence. □

5.3 A description of convergent martingales in the l-tensor product

In this section, we are concerned with representation theorems for convergent martingales in the Lebesgue-Bochner spaces. For added convenience in proving our first result, we restate Theorem 5.2.2: If E is a Banach lattice and Y a Banach space, then $u \in E\widetilde{\otimes}_l Y$ if and only if $u = \sum_{i=1}^{\infty} x_i \otimes y_i$, where

$$\left\|\sum_{i=1}^{\infty}|x_i|\right\|_E < \infty \quad \text{and} \quad \lim_{i\to\infty}\|y_i\|_Y = 0. \tag{5.3}$$

We now prove:

Theorem 5.3.1 *Let (S_n) be a BL-filtration on a Banach lattice E and (T_n) a BS-filtration on a Banach space Y. For a martingale $M = (f_n, S_n \otimes_l T_n)_{n=1}^{\infty}$ in $E\widetilde{\otimes}_l Y$, the following statements are equivalent:*

(a) M is convergent in $E\widetilde{\otimes}_l Y$.

(b) For each $i \in \mathbb{N}$, there exist convergent martingales $(x_i^{(n)}, S_n)_{n=1}^{\infty}$ and $(y_i^{(n)}, T_n)_{n=1}^{\infty}$ in E and Y respectively such that, for each $n \in \mathbb{N}$, we have

$$f_n = \sum_{i=1}^{\infty} x_i^{(n)} \otimes y_i^{(n)},$$

where

$$\left\| \sum_{i=1}^{\infty} \left| \lim_{n\to\infty} x_i^{(n)} \right| \right\| < \infty \quad \textit{and} \quad \lim_{i\to\infty} \left\| \lim_{n\to\infty} y_i^{(n)} \right\| = 0.$$

Proof. (a)⇒(b) Let $M = (f_n, S_n \otimes_l T_n)_{n=1}^{\infty}$ be a convergent martingale in $E\widetilde{\otimes}_l Y$. Then, by Theorem 3.4.5, M corresponds to an element

$$f \in \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i \otimes_l T_i)} = \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i)} \ \widetilde{\otimes}_l \ \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$$

and thus, by the remark preceding this theorem, we have $f = \sum_{i=1}^{\infty} x_i \otimes y_i$ where (5.3) holds. Then, for each $n \in \mathbb{N}$, we have $f_n = (S_n \otimes_l T_n)(\sum_{i=1}^{\infty} x_i \otimes y_i)$. Now let $x_i^{(n)} := S_n(x_i)$ and $y_i^{(n)} := T_n(y_i)$ for each $i \in \mathbb{N}$. Then, by Corollary 3.2.7,

$$f_n = \sum_{i=1}^{\infty} S_n(x_i) \otimes T_n(y_i) = \sum_{i=1}^{\infty} x_i^{(n)} \otimes y_i^{(n)}$$

where $(x_i^{(n)}, S_n)_{n=1}^{\infty}$ and $(y_i^{(n)}, T_n)_{n=1}^{\infty}$ are convergent martingales in E and Y, with limits x_i and y_i respectively, so that

$$\left\| \sum_{i=1}^{\infty} \left| \lim_{n\to\infty} x_i^{(n)} \right| \right\| < \infty \quad \text{and} \quad \lim_{i\to\infty} \left\| \lim_{n\to\infty} y_i^{(n)} \right\| \to 0$$

hold.

(b)⇒(a) For each $i \in \mathbb{N}$, let $x_i = \lim_{n\to\infty} x_i^{(n)}$ and $y_i = \lim_{n\to\infty} y_i^{(n)}$. Then the sequences (x_i) and (y_i) satisfy (5.3) so that Theorem 3.4.5 implies

$$f := \sum_{i=1}^{\infty} x_i \otimes y_i \in \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i)} \ \widetilde{\otimes}_l \ \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)} \ = \ \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(S_i \otimes_l T_i)}.$$

Then, for each $n \in \mathbb{N}$, we have

$$f_n = \sum_{i=1}^{\infty} x_i^{(n)} \otimes y_i^{(n)} = \sum_{i=1}^{\infty} S_n(x_i) \otimes T_n(y_i) = (S_n \otimes_l T_n) f.$$

It now follows that $M := (f_n, S_n \otimes_l T_n)_{n=1}^{\infty}$ is a convergent martingale, by Corollary 3.2.7. □

Note that a symmetrical result holds for the m-norm. This result can be specialized to the Lebesgue-Bochner spaces as follows.

Corollary 5.3.2 *Let (Ω, Σ, μ) denote a finite measure space, $(\Sigma_n)_{n=1}^{\infty}$ a filtration, Y a Banach space and $1 \leq p < \infty$. For a martingale $(f_n, \Sigma_n)_{n=1}^{\infty}$ in $L^p(\mu, Y)$, the following statements are equivalent:*

(a) $(f_n, \Sigma_n)_{n=1}^{\infty}$ *is convergent in* $L^p(\mu, Y)$.

(b) *For each* $i \in \mathbb{N}$, *there exist a convergent martingale* $(x_i^{(n)}, \Sigma_n)_{n=1}^{\infty}$ *in* $L^p(\mu)$ *and* $y_i \in Y$ *such that, for each* $n \in \mathbb{N}$, *we have*

$$f_n = \sum_{i=1}^{\infty} x_i^{(n)} \otimes y_i,$$

where

$$\left\| \sum_{i=1}^{\infty} \left| \lim_{n\to\infty} x_i^{(n)} \right| \right\|_{L^p(\mu)} < \infty \qquad \text{and} \qquad \lim_{i\to\infty} \|y_i\| = 0.$$

Proof. In the case where $E = L^p(\mu)$ $(1 \le p < \infty)$, $S_n = \mathbb{E}(\,\cdot\,|\Sigma_n)$ (where (Σ_n) is a filtration in the classical sense) and $T_n = \mathrm{id}_Y$ for each $n \in \mathbb{N}$, the proof now follows as a simple consequence of Theorem 5.3.1. $\square$

Combining Theorem 5.3.1 with Theorem 3.5.8, we obtain the following characterization of the Radon Nikodým property.

Theorem 5.3.3 *Let* Y *be a Banach space. Then the following conditions are equivalent:*

(a) Y *has the Radon Nikodým property.*

(b) *For every separable reflexive Banach lattice* E *and every complemented, strictly positive BS-filtration* (S_i) *on* E, *we have* $(f_n) \in \mathcal{M}(E\widetilde{\otimes}_l Y, S_n \otimes_l \mathrm{id}_Y)$ *if and only if for each* $i \in \mathbb{N}$, *there exist* $(x_i^{(n)}, S_n)_{n=1}^{\infty} \in \mathcal{M}_{\mathrm{nc}}(E, S_i)$ *and* $y_i \in Y$ *such that, for each* $n \in \mathbb{N}$, *we have*

$$f_n = \sum_{i=1}^{\infty} x_i^{(n)} \otimes y_i,$$

where

$$\left\| \sum_{i=1}^{\infty} \left| \lim_{n\to\infty} x_i^{(n)} \right| \right\| < \infty \qquad \text{and} \qquad \lim_{i\to\infty} \|y_i\| = 0.$$

Corollary 5.3.2 can be restated to characterize convergence of martingale difference sequences:

Theorem 5.3.4 *Let* (Ω, Σ, μ) *denote a finite measure space,* (Σ_n) *a filtration,* Y *a Banach space and* $1 \le p < \infty$. *For a martingale difference sequence* $(d_k) \subset L^p(\mu, Y)$ *relative to* (Σ_n), *the following statements are equivalent:*

(a) *The series* $\sum_{k=1}^{\infty} d_k$ *converges in* $L^p(\mu, Y)$.

(b) *For each $i \in \mathbb{N}$, there exist a martingale difference sequence $(d_k^{(i)})_{k=1}^{\infty} \subset L^p(\mu)$ relative to (Σ_n), with $\sum_{k=1}^{\infty} d_i^{(k)}$ convergent, and $y_i \in Y$ such that, for each $k \in \mathbb{N}$, we have*

$$d_k = \sum_{i=1}^{\infty} d_i^{(k)} \otimes y_i,$$

where

$$\left\| \sum_{i=1}^{\infty} \left| \sum_{k=1}^{\infty} d_i^{(k)} \right| \right\|_{L^p(\mu)} < \infty \quad \text{and} \quad \lim_{i\to\infty} \|y_i\| = 0.$$

Proof. (a)$\Rightarrow$(b) Suppose that $\sum_{k=1}^{\infty} d_k$ converges in $L^p(\mu, Y)$. Define $f_n = \sum_{k=1}^{n} d_k$, then (f_n, Σ_n) is a convergent martingale. By Theorem 5.3.2, for each $i \in \mathbb{N}$, there exist a convergent martingale $(x_i^{(n)}, \Sigma_n)_{n=1}^{\infty}$ in $L^p(\mu)$ and $y_i \in Y$ such that, for each $n \in \mathbb{N}$, we have

$$f_n = \sum_{k=1}^{n} d_k = \sum_{i=1}^{\infty} x_i^{(n)} \otimes y_i = \sum_{i=1}^{\infty} \sum_{k=1}^{n} \left(x_i^{(k)} - x_i^{(k-1)} \right) \otimes y_i,$$

where

$$\left\| \sum_{i=1}^{\infty} \left| \sum_{k=1}^{\infty} \left(x_i^{(k)} - x_i^{(k-1)} \right) \right| \right\|_{L^p(\mu)} < \infty \quad \text{and} \quad \lim_{i\to\infty} \|y_i\| = 0.$$

Here, we use the convention of $x_i^{(0)} = 0$ for each $i \in \mathbb{N}$. Define $d_i^{(k)} = (x_i^{(k)} - x_i^{(k-1)})$ for each $i, k \in \mathbb{N}$. Then $(d_i^{(k)})_{k=1}^{\infty} \subset L^p(\mu)$ is a martingale difference sequence relative to (Σ_n) for each $i \in \mathbb{N}$. Consequently,

$$\begin{aligned} d_k &= (\mathbb{E}(\,\cdot \mid \Sigma_k) - \mathbb{E}(\,\cdot \mid \Sigma_{k-1}))(f_n) \\ &= (\mathbb{E}(\,\cdot \mid \Sigma_k) - \mathbb{E}(\,\cdot \mid \Sigma_{k-1})) \left(\sum_{i=1}^{\infty} \sum_{k=1}^{n} d_i^{(k)} \otimes y_i \right) \\ &= \sum_{i=1}^{\infty} d_i^{(k)} \otimes y_i, \end{aligned}$$

for all $k \leq n$ and $n \in \mathbb{N}$, where

$$\left\| \sum_{i=1}^{\infty} \left| \sum_{k=1}^{\infty} d_i^{(k)} \right| \right\|_{L^p(\mu)} < \infty \quad \text{and} \quad \lim_{i\to\infty} \|y_i\| = 0.$$

(b)$\Rightarrow$(a) Define $x_n^{(i)} = \sum_{k=1}^{n} d_k^{(i)}$ for each $i \in \mathbb{N}$. Then $(x_n^{(i)}, \Sigma_n)_{n=1}^{\infty}$ is a martingale in $L^p(\mu)$ for each $i \in \mathbb{N}$. Thus,

$$f_n := \sum_{k=1}^{n} d_k = \sum_{k=1}^{n} \sum_{i=1}^{\infty} d_i^{(k)} \otimes y_i = \sum_{i=1}^{\infty} \left(\sum_{k=1}^{n} d_i^{(k)} \right) \otimes y_i = \sum_{i=1}^{\infty} x_n^{(i)} \otimes y_i,$$

where

$$\left\| \sum_{i=1}^{\infty} \left| \lim_{n\to\infty} x_i^{(n)} \right| \right\|_{L^p(\mu)} < \infty \quad \text{and} \quad \lim_{i\to\infty} \|y_i\| = 0$$

hold. By Theorem 5.3.2, $\sum_{k=1}^{n} d_k = f_n$ converges in $L^p(\mu, Y)$. $\square$

Corollary 5.3.5 *Let (Ω, Σ, μ) denote a finite measure space, (Σ_n) a filtration, Y a Banach space and $1 \le p < \infty$. For a martingale difference sequence $(d_k) \subset L^p(\mu, Y)$ relative to (Σ_n), the following statements are equivalent:*

(a) The series $\sum_{k=1}^{\infty} d_k$ is unconditionally convergent.

(b) For any choice of signs (θ_k) there exist, for each $i \in \mathbb{N}$, a martingale difference sequence $(d_k^{(i)})_{k=1}^{\infty} \subset L^p(\mu)$ relative to (Σ_n), with $\sum_{k=1}^{\infty} d_i^{(k)}$ convergent, and $y_i \in Y$ such that, for each $k \in \mathbb{N}$, we have

$$d_k = \sum_{i=1}^{\infty} \theta_k d_i^{(k)} \otimes y_i,$$

where

$$\left\| \sum_{i=1}^{\infty} \left| \sum_{k=1}^{\infty} d_i^{(k)} \right| \right\|_{L^p(\mu)} < \infty \quad \text{and} \quad \lim_{i\to\infty} \|y_i\| = 0.$$

Proof. Suppose (a) holds, then $\sum_{k=1}^{\infty} \theta_k d_k$ converges for every choice of signs (θ_k). Thus, (b) follows directly from Theorem 5.3.4. Conversely, suppose (b) holds. Another application of Theorem 5.3.4 shows that $\sum_{k=1}^{\infty} \theta_k d_k$ must converge for every choice of signs (θ_k). Thus, (a) holds by Theorem 1.4.2. $\square$

The above corollary can now be used to characterize the UMD property:

Theorem 5.3.6 *Let $1 < p < \infty$ and Y be a Banach space. Then the following statements are equivalent:*

(a) Y is a UMD space.

(b) For every choice of signs (θ_k) and every martingale difference sequence $(d_k) \subset L^p(\mu, Y)$ with $\sup_{n\in\mathbb{N}} \| \sum_{k=1}^{n} d_k\|_p < \infty$, there exist for each $i \in \mathbb{N}$, a martingale difference sequence $(d_i^{(k)})_{k=1}^{\infty} \subset L^p(\mu)$, with $\sum_{k=1}^{\infty} d_i^{(k)}$ convergent, and $y_i \in Y$ such that, for each $k \in \mathbb{N}$, we have

$$d_k = \sum_{i=1}^{\infty} \theta_k d_i^{(k)} \otimes y_i,$$

where

$$\left\| \sum_{i=1}^{\infty} \left| \sum_{k=1}^{\infty} d_i^{(k)} \right| \right\|_{L^p(\mu)} < \infty \quad \text{and} \quad \lim_{i\to\infty} \|y_i\| = 0.$$

Proof. (a)⇒(b) Suppose that Y is a UMD-space and that $(d_k) \subset L^p(\mu, Y)$ is a martingale difference sequence with $\sup_{n\in\mathbb{N}} \|\sum_{i=1}^n d_i\| < \infty$. It follows from Theorem 1.5.4 that $\sum_{i=1}^\infty d_i$ converges unconditionally in $L^p(\mu, Y)$. Thus, by Corollary 5.3.5, (b) must hold.

(b)⇒(a) By Corollary 5.3.5, every martingale difference sequence $(d_k) \subset L^p(\mu, Y)$ with $\sup_{n\in\mathbb{N}} \|\sum_{k=1}^n d_k\|_p < \infty$ converges unconditionally. Thus, Y is a UMD space by Theorem 1.5.4. □

5.4 Bases with vector-valued coefficients

The disadvantage of Theorem 5.3.2 is that the representation of a convergent martingale in $L^p(\mu, Y)$ is not unique. A glance at Theorem 5.2.2 reveals that there are uncountably many representations. To remedy this problem, we need to be able to represent elements of $E\widetilde{\otimes}_l Y$ uniquely. To achieve this, we use the notion of a basis with 'vector valued coefficients' outlined in [46, pp. 588-590]. We specialize the definition to suit our purposes.

Definition 5.4.1 Let E be a Banach lattice and Y a Banach space.

(a) A sequence $(f_i) \subset E$ is said to be a *basis with vector coefficients* for $E\widetilde{\otimes}_l Y$, or simply a Y*-basis* of $E\widetilde{\otimes}_l Y$, provided that for each $u \in E\widetilde{\otimes}_l Y$ there exists a unique sequence $(y_i) \subset Y$ such that $u = \sum_{i=1}^\infty f_i \otimes y_i$.
(b) The Y-basis $(f_i) \subset E$ is said to be *unconditional* if $u = \sum_{i=1}^\infty f_i \otimes y_i$ converges unconditionally for each $u \in E\widetilde{\otimes}_l Y$.

In the same manner as for bases, we define the *natural projections* $(\widehat{P}_n)$ on $E\widetilde{\otimes}_l Y$, associated to the Y-basis (f_i), by

$$\widehat{P}_n\left(\sum_{i=1}^\infty f_i \otimes y_i\right) = \sum_{i=1}^n f_i \otimes y_i$$

for all $u = \sum_{i=1}^\infty f_i \otimes y_i \in E\widetilde{\otimes}_l Y$ and $n \in \mathbb{N}$. By the Principle of Uniform Boundedness, we have that

$$\mathrm{bc}((f_i), E, Y) := \sup_{n\in\mathbb{N}} \|\widehat{P}_n\| < \infty.$$

The quantity $\mathrm{bc}((f_i), E, Y)$ is referred to as the *basis constant* for the Y-basis $(f_i) \subset E$ with respect to Y.

If (f_i) is an unconditional Y-basis, (θ_i) any choice of signs and $\sigma \subset \mathbb{N}$, the Closed Graph Theorem assures us that the operators $\widehat{M_\theta} : E\widetilde{\otimes}_l Y \to E\widetilde{\otimes}_l Y$ and $\widehat{P}_\sigma : E\widetilde{\otimes}_l Y \to E\widetilde{\otimes}_l Y$, defined respectively by

$$\widehat{M}_\theta\left(\sum_{i=1}^{\infty} f_i \otimes y_i\right) = \sum_{i=1}^{\infty} \theta_i f_i \otimes y_i$$

and

$$\widehat{P}_\sigma\left(\sum_{i=1}^{\infty} f_i \otimes y_i\right) = \sum_{i\in\sigma} f_i \otimes y_i,$$

for each $u = \sum_{i=1}^{\infty} f_i \otimes y_i \in E\widetilde{\otimes}_l Y$, are bounded. The Principle of Uniform Boundedness implies $\sup_\theta \|\widehat{M}_\theta\| < \infty$ and $\sup_\sigma \|\widehat{P}_\sigma\| < \infty$. We define the *unconditional constant* of the Y-basis $(f_i) \subset E$, with respect to Y, to be

$$\operatorname{ubc}((f_i), E, Y) = \sup_\theta \|\widehat{M}_\theta\|.$$

Notice that we have $\operatorname{bc}((f_i), E, Y) \leq \sup_\sigma \|\widehat{P}_\sigma\| \leq \operatorname{ubc}((f_i), E, Y) \leq 2\sup_\sigma \|\widehat{P}_\sigma\|$. It follows from the same argument as for bases that (f_i) is a Y-basis (or unconditional Y-basis) for $E\widetilde{\otimes}_l Y$ if and only if $\operatorname{bc}((f_i), E, Y)$ (or $\operatorname{ubc}((f_i), E, Y)$) is finite.

Since the l-norm is a left uniform, left injective crossnorm, it follows that the space $E\widetilde{\otimes}_l Y$ is 'regular' in the sense of [46, p. 588]. Thus, if Y_0 is a closed subspace of Y, we have $\operatorname{bc}((f_i), E, Y_0) \leq \operatorname{bc}((f_i), E, Y)$ and $\operatorname{ubc}((f_i), E, Y_0) \leq \operatorname{ubc}((f_i), E, Y)$. In particular, we have that $(f_i) \subset E$ is a basis (or unconditional basis) of E if $(f_i) \subset E$ is a Y-basis (or unconditional Y-basis) of $E\widetilde{\otimes}_l Y$.

It is readily verified that the natural projections $(\widehat{P}_n)$ on $E\widetilde{\otimes}_l Y$, associated with the Y-basis $(f_i) \subset E$, are given by maps $P_n \otimes_l \operatorname{id}_Y : E\widetilde{\otimes}_l Y \to E\widetilde{\otimes}_l Y$, where (P_n) are the natural projections associated to the basis (f_i) of E. Consequently, we have the formula

$$\operatorname{bc}((f_i), E, Y) = \sup_{n\in\mathbb{N}} \|P_n \otimes_l \operatorname{id}_Y\|.$$

Similarly, if (f_i) is an unconditional Y-basis, then we also have

$$\operatorname{ubc}((f_i), E, Y) = \sup_\theta \|M_\theta \otimes_l \operatorname{id}_Y\|,$$

where $M_\theta : E \to E$ is defined by $M_\theta(\sum_{i=1}^{\infty} \alpha_i f_i) = \sum_{i=1}^{\infty} \theta_i \alpha_i f_i$ for all $\sum_{i=1}^{\infty} \alpha_i f_i \in E$ and every choice of signs $\theta = (\theta_i)$.

In view of the above discussion, the following question arises: Given a basis (f_i) of a Banach lattice E, for which Banach spaces Y is (f_i) a Y-basis of $E\widetilde{\otimes}_l Y$? The following lemma provides a partial answer (cf. [46, Lemma 32]):

Proposition 5.4.2 *Let $1 \leq p < \infty$ and $E = L^p(\mu)$. Let Y be a Banach space isometric to a subspace of a quotient space of an L^p-space. If $T : E \to E$ is a bounded linear operator, then $T \otimes_l \operatorname{id}_Y : L^p(\mu)\widetilde{\otimes}_l Y \to L^p(\mu)\widetilde{\otimes}_l Y$ is bounded with norm less than or equal to $\|T\|$.*

The above proposition implies that any basis of $L^p(\mu)$ is also a Y-basis of $L^p(\mu, Y)$, where Y is a subspace of quotient space of a L^p-space. By placing a restriction on the basis $(f_i) \subset E$, we can substantially increase the class of Banach spaces Y for which (f_i) is a Y-basis of $E\widetilde{\otimes}_l Y$.

Theorem 5.4.3 *Let E be a Banach lattice and $(f_i) \subset E$ a basis. If the natural projections (P_i) associated to (f_i) are positive then, for any Banach space Y, (f_i) is a Y-basis for $E\widetilde{\otimes}_l Y$.*

Proof. Let Y be any Banach space, then $(f_i) \subset E$ is a Y basis if and only if $\mathrm{bc}((f_i), E, Y) < \infty$. Since the l-norm is a left uniform crossnorm, it follows that

$$\mathrm{bc}((f_i), E, Y) = \sup_{n\in\mathbb{N}} \|P_n \otimes_l \mathrm{id}_Y\| = \sup_{n\in\mathbb{N}} \|P_n\| < \infty,$$

and the proof is complete. □

Corollary 5.4.4 *Let $1 \leq p < \infty$, then any basis of $L^p(\mu)$, which is also a martingale difference sequence (e.g. the Haar system), is a Y-basis for $L^p(\mu, Y)$ for every Banach space Y.*

Proof. The result follows from Theorem 5.4.3 and the fact that the associated natural projections are conditional expectations, which are positive. □

If $1 < p < \infty$, it follows from Proposition 1.4.12 that a Banach space Y has the UMD property if and only if the Haar system is an unconditional Y-basis of $L^p(\mu, Y)$.

5.5 A unique representation for convergent martingales in the l-tensor product

We can now prove a unique representation result for convergent martingales in the l-tensor product.

Theorem 5.5.1 *Let E be a Banach lattice and $(h_i) \subset E$ be a basis so that the associated natural projections are positive. Let (S_n) be a complemented BL-filtration on E and (T_n) a BS-filtration on any Banach space Y. Then, for a martingale $M = (f_n, S_n \otimes_l T_n)_{n=1}^{\infty}$ in $E\widetilde{\otimes}_l Y$, the following statements are equivalent:*

(a) M is convergent in $E\widetilde{\otimes}_l Y$.

(b) For each $i \in \mathbb{N}$, there exist unique convergent martingales $(x_i^{(n)}, S_n)_{n=1}^\infty$ and $(y_i^{(n)}, T_n)_{n=1}^\infty$ in E and Y respectively such that, for each $n \in \mathbb{N}$, we have

$$f_n = \sum_{i=1}^\infty x_i^{(n)} \otimes y_i^{(n)},$$

where $x_i^{(n)} = S_n(h_i)$ for each $i, n \in \mathbb{N}$ and the series $\sum_{i=1}^\infty h_i \otimes \left(\lim_{n\to\infty} y_i^{(n)}\right)$ converges in $E\widetilde{\otimes}_l Y$.

In the case where (h_i) is normalized, the sequence $(\lim_{n\to\infty} y_i^{(n)})_{i=1}^\infty \subset Y$ is weakly null.

Proof. (a)⇒(b) Let $M = (f_n, S_n \otimes_l T_n)_{n=1}^\infty$ converge to $f \in E\widetilde{\otimes}_l Y$. Theorem 5.4.3 implies that f has a unique representation $f = \sum_{i=1}^\infty h_i \otimes y_i$. We claim that $(y_i) \subset \overline{\bigcup_{i=1}^\infty \mathcal{R}(T_i)}$. Indeed, Corollary 3.2.7, Theorem 3.4.5 and the left injectivity of the l-norm imply

$$f \in \overline{\bigcup_{i=1}^\infty \mathcal{R}(S_i \otimes_l T_i)} = \overline{\bigcup_{i=1}^\infty \mathcal{R}(S_i)} \,\widetilde{\otimes}_l\, \overline{\bigcup_{i=1}^\infty \mathcal{R}(T_i)} \hookrightarrow E \,\widetilde{\otimes}_l\, \overline{\bigcup_{i=1}^\infty \mathcal{R}(T_i)}.$$

Another application of Theorem 5.4.3 shows that f also has a unique representation $f = \sum_{i=1}^\infty h_i \otimes \widehat{y}_i \in E \,\widetilde{\otimes}_l\, \overline{\bigcup_{i=1}^\infty \mathcal{R}(T_i)}$. Since $E \,\widetilde{\otimes}_l\, \overline{\bigcup_{i=1}^\infty \mathcal{R}(T_i)}$ is a closed subspace of $E \,\widetilde{\otimes}_l\, Y$, it follows that $\widehat{y}_i = y_i$ for each $i \in \mathbb{N}$ and $(y_i) \subset \overline{\bigcup_{i=1}^\infty \mathcal{R}(T_i)}$.

For each $n \in \mathbb{N}$, we have $f_n = (S_n \otimes_l T_n)(\sum_{i=1}^\infty h_i \otimes y_i)$. Now let $x_i^{(n)} := S_n(h_i)$ and $y_i^{(n)} := T_n(y_i)$ for each $i, n \in \mathbb{N}$. Then

$$f_n = \sum_{i=1}^\infty S_n(h_i) \otimes T_n(y_i) = \sum_{i=1}^\infty x_i^{(n)} \otimes y_i^{(n)}$$

where $(x_i^{(n)}, S_n)_{n=1}^\infty \in \mathcal{M}_{\mathrm{nc}}(E, T_i)$ by Proposition 3.2.11 and, by Corollary 3.2.7, $(y_i^{(n)}, T_n)_{n=1}^\infty \in \mathcal{M}_{\mathrm{nc}}(Y, S_i)$ for each $i \in \mathbb{N}$. Since $\lim_{n\to\infty} y_i^{(n)} = y_i$ for each $i \in \mathbb{N}$, the series $\sum_{i=1}^\infty h_i \otimes (\lim_{n\to\infty} y_i^{(n)})$ converges in $E\widetilde{\otimes}_l Y$.

(b)⇒(a) For each $i \in \mathbb{N}$, let $y_i = \lim_{n\to\infty} y_i^{(n)}$. Then $\sum_{i=1}^\infty h_i \otimes y_i$ converges in $E\widetilde{\otimes}_l Y$. Corollary 3.2.7 implies that $(y_i) \subset \overline{\bigcup_{i=1}^\infty \mathcal{R}(T_i)}$. Assume (S_i) is complemented in E by the contractive projection $S_\infty : E \to E$. By the left injectivity of the l-norm, $\overline{\bigcup_{i=1}^\infty \mathcal{R}(S_i)} \,\widetilde{\otimes}_l\, \overline{\bigcup_{i=1}^\infty \mathcal{R}(T_i)}$ is a closed subspace of $E\widetilde{\otimes}_l Y$. Thus, Lemma 3.4.1 and Theorem 3.4.5 imply

$$\begin{aligned} f &:= \sum_{i=1}^\infty (S_\infty h_i) \otimes y_i \\ &= (S_\infty \otimes_l \mathrm{id}_Y)\left(\sum_{i=1}^\infty h_i \otimes y_i\right) \in \overline{\bigcup_{i=1}^\infty \mathcal{R}(S_i)} \,\widetilde{\otimes}_l\, \overline{\bigcup_{i=1}^\infty \mathcal{R}(T_i)} = \overline{\bigcup_{i=1}^\infty \mathcal{R}(S_i \otimes_l T_i)}. \end{aligned}$$

Also, for each $n \in \mathbb{N}$, we have

$$f_n = \sum_{i=1}^{\infty} x_i^{(n)} \otimes y_i^{(n)} = \sum_{i=1}^{\infty} S_n(S_\infty h_i) \otimes T_n(y_i) = (S_n \otimes_l T_n)f.$$

It now follows from Corollary 3.2.7 that $M := (f_n, S_n \otimes_l T_n)_{n=1}^{\infty}$ is a convergent martingale.

To complete the proof, let $\varepsilon > 0$. Because the series $\sum_{i=1}^{\infty} h_i \otimes (\lim_{n\to\infty} y_i^{(n)})$ converges in $E\widetilde{\otimes}_l Y$, there exists $N > 0$ so that $n \geq m \geq N$ implies

$$\left\| \sum_{i=m}^{n} \left\langle \left(\lim_{n\to\infty} y_i^{(n)} \right), y^* \right\rangle h_i \right\| \leq \left\| \sup \left\{ \left| \sum_{i=m}^{n} \left\langle \left(\lim_{n\to\infty} y_i^{(n)} \right), y^* \right\rangle h_i \right| : \|y^*\| \leq 1 \right\} \right\| < \varepsilon$$

for each $y^* \in Y$, $\|y^*\| \leq 1$. Since $\inf_{i\in\mathbb{N}} \|h_i\| > 0$, it follows that $(\lim_{n\to\infty} y_i^{(n)})_{i=1}^{\infty}$ is a weakly null sequence (cf. [93, Chapter I, §3, Lemma 3.1]). $\square$

The above result can be specialized to:

Theorem 5.5.2 *Let (Ω, Σ, μ) denote a finite measure space, $(\Sigma_n)_{n=1}^{\infty}$ a filtration, Y a Banach space, $1 \leq p < \infty$ and (h_i) any m.d.s. basis of $L^p(\mu)$. For a martingale $(f_n, \Sigma_n) \subset L^p(\mu, Y)$, the following statements are equivalent:*

(a) (f_n, Σ_n) is convergent in $L^p(\mu, Y)$.

(b) For each $i \in \mathbb{N}$, there exists a unique $y_i \in Y$ such that, for each $n \in \mathbb{N}$, we have

$$f_n = \sum_{i=1}^{\infty} \mathbb{E}(h_i \,|\, \Sigma_n) \otimes y_i,$$

where the series $\sum_{i=1}^{\infty} h_i \otimes y_i$ converges in $L^p(\mu, Y)$.

In the case where (h_i) is normalized, the sequence $(y_i) \subset Y$ is weakly null.

Proof. In the case where $E = L^p(\mu)$ $(1 \leq p < \infty)$, $S_n = \mathbb{E}(\,\cdot\,|\Sigma_n)$ (where (Σ_n) is a filtration in the classical sense) and $T_n = \mathrm{id}_Y$ for each $n \in \mathbb{N}$, the proof follows as a simple consequence of Theorem 5.5.1. $\square$

The above result bears a strong resemblance to Corollary 3.2.7, but with the added structure of a Y-basis. With the above characterization of convergent martingales at hand, we can produce an analogue of Theorem 5.3.3, which characterizes the Radon Nikodým property.

Theorem 5.5.3 *Let Y be a Banach space. Then the following statements are equivalent.*

(a) Y has the Radon Nikodým property.

(b) *For every reflexive Banach lattice E possessing a basis $(h_i) \subset E$ with positive natural projections, and every complemented strictly positive BS-filtration (S_i) on E, we have $(f_n) \in \mathcal{M}(E\widetilde{\otimes}_l Y, S_n \otimes_l \mathrm{id}_Y)$ if and only if, for each $i \in \mathbb{N}$, there exist unique $(x_i^{(n)}, S_n)_{n=1}^{\infty} \in \mathcal{M}_{\mathrm{nc}}(E, S_i)$ and $y_i \in Y$ such that, for each $n \in \mathbb{N}$, we have*

$$f_n = \sum_{i=1}^{\infty} x_i^{(n)} \otimes y_i^{(n)},$$

where $x_i^{(n)} = S_n(h_i)$ for each $i, n \in \mathbb{N}$ and the series $\sum_{i=1}^{\infty} h_i \otimes y_i$ converges in $E\widetilde{\otimes}_l Y$.

Proof. (a)⇒(b) Suppose that Y has the Radon Nikodým property. By Theorem 3.5.8, it follows that $\mathcal{M}(E\widetilde{\otimes}_l Y, S_i \otimes_l \mathrm{id}_Y) = \mathcal{M}_{\mathrm{nc}}(E\widetilde{\otimes}_l Y, S_i \otimes_l \mathrm{id}_Y)$ for every separable reflexive Banach lattice E and every complemented, strictly positive BS-filtration (S_i) on E. Certainly, if E possesses a basis, then it is separable. Thus, (b) holds by Theorem 5.5.1.

(b)⇒(a) By Theorem 5.5.1, $\mathcal{M}(E\widetilde{\otimes}_l Y, S_i \otimes_l \mathrm{id}_Y) = \mathcal{M}_{\mathrm{nc}}(E\widetilde{\otimes}_l Y, S_i \otimes_l \mathrm{id}_Y)$ for every reflexive Banach lattice E possessing a basis $(h_i) \subset E$ with positive natural projections, and every complemented, strictly positive BS-filtration (S_i) on E. Since $L^p(\mu)$ is a reflexive Banach lattice for $1 < p < \infty$, possessing such a basis, (a) follows immediately from Theorem 1.3.10. □

Using similar reasoning as in the proof of Theorem 5.3.6, one can deduce an analogous characterization of the UMD property.

Theorem 5.5.4 *Let $1 < p < \infty$ and Y be a Banach space. Then the following statements are equivalent:*

(a) *Y is a UMD space.*

(b) *For every choice of signs (θ_k) and every martingale difference sequence $(d_k) \subset L^p(\mu, Y)$, with $\sup_{n\in\mathbb{N}} \|\sum_{k=1}^{n} d_k\|_p < \infty$, there exist for each $i \in \mathbb{N}$, a martingale difference sequence $(d_i^{(k)})_{k=1}^{\infty} \subset L^p(\mu)$, with $\sum_{k=1}^{\infty} d_i^{(k)}$ convergent, and unique $y_i \in Y$ such that, for each $k \in \mathbb{N}$, we have*

$$d_k = \sum_{i=1}^{\infty} \theta_k d_i^{(k)} \otimes y_i,$$

where $d_i^{(k)} = \mathbb{E}(h_i \,|\, \Sigma_k) - \mathbb{E}(h_i \,|\, \Sigma_{k-1})$ for each $i, k \in \mathbb{N}$ and the series $\sum_{i=1}^{\infty} h_i \otimes y_i$ converges in $L^p(\mu, Y)$.

5.6 Notes and remarks

Let E denote a Banach lattice, Y a Banach space and suppose that $(h_i) \subset E$ is a Y-basis of $E\widetilde{\otimes}_l Y$.

Although Theorem 5.5.1 provides a unique description of convergent martingales, it is not as descriptive as its non-unique counterpart, Theorem 5.3.1. To make Theorem 5.5.1 more informative, we need a characterization of the sequences $(y_i) \subset Y$ for which the series $\sum_{i=1}^{\infty} h_i \otimes y_i$ converges in $E\widetilde{\otimes}_l Y$.

In the case where (h_i) is normalized (or semi-normalized), it is necessary that (y_i) be weakly null, by Theorem 5.5.1. This, however, is far from a sufficient condition for the convergence of $\sum_{i=1}^{\infty} h_i \otimes y_i$ in $E\widetilde{\otimes}_l Y$. Indeed, if (y_i) is weakly null, then $(\theta_i y_i)$ is weakly null for every choice of signs (θ_i). Consequently, if '(y_i) weakly null' were to be a sufficient condition for the convergence of $\sum_{i=1}^{\infty} h_i \otimes y_i$ in $E\widetilde{\otimes}_l Y$, then every $\sum_{i=1}^{\infty} h_i \otimes y_i \in E\widetilde{\otimes}_l Y$ would converge unconditionally. Thus, any normalized Y-basis of $E\widetilde{\otimes}_l Y$ would be unconditional. In the case of the normalized Haar system, Corollary 5.4.4 would imply that every Banach space is a UMD space, which is false.

On the other hand, define $\widehat{h}_i = (2^{-i}/\|h_i\|)h_i$ for each $i \in \mathbb{N}$. Then $(\widehat{h}_i) \subset E$ is a Y-basis of $E\widetilde{\otimes}_l Y$ and $\sum_{i=1}^{\infty} |\widehat{h}_i|$ converges in E. By Theorem 5.2.2, it follows that if $(y_i) \subset Y$ is a null sequence, then $\sum_{i=1}^{\infty} \widehat{h}_i \otimes y_i$ converges in $E\widetilde{\otimes}_l Y$. In fact, there exists an injective linear operator $\Phi : c_0(Y) \to E\widetilde{\otimes}_l Y$, of norm less than one, defined by $\Phi(y_i) = \sum_{i=1}^{\infty} \widehat{h}_i \otimes y_i$.

Now suppose that E is a (separable) reflexive Banach lattice and Y is a reflexive Banach space. Since E is a separable Banach lattice with order continuous dual and Y has the Radon Nikodým property, we have $(E\widetilde{\otimes}_l Y)^* = E^*\widetilde{\otimes}_l Y^*$, by Theorem 3.5.9. Consider the adjoint $\Phi^* : E^*\widetilde{\otimes}_l Y^* \to \ell^1(Y)$, defined by $f \mapsto (T_f h_i)_{i=1}^{\infty}$, where $f \mapsto T_f$ denotes the canonical isometry $(E\widetilde{\otimes}_l Y)^* \to \mathcal{L}^{\mathrm{cas}}(E, Y^*)$. The map Φ^* bears a resemblance to the operator $T : X^* \to \ell^1$ in Theorem 1.4.2(i). However, unlike T, Φ^* does not always enjoy the property of compactness. The compactness of T arises from the Schur property of ℓ^1. In constrast, it is necessary for Y to have the Schur property in order for $\ell^1(Y)$ to have the Schur property.

By taking the double adjoint of Φ and applying Theorem 3.5.9 again, we may extend Φ to $\Phi^{**} : \ell^\infty(Y) \to E\widetilde{\otimes}_l Y$. Consequently, if $(y_i) \in \ell^\infty(Y)$, then the series $\sum_{i=1}^{\infty} \widehat{h}_i \otimes y_i$ converges unconditionally in $E\widetilde{\otimes}_l Y$. Thus, the condition '(y_i) a bounded sequence' is not necessary for the convergence of $\sum_{i=1}^{\infty} \widehat{h}_i \otimes y_i$ in $E\widetilde{\otimes}_l Y$.

6

Martingale difference sequences

6.1 Introduction

We turn our attention to generalized martingale difference sequences. In Section 6.2, we study the notion of a martingale difference sequence (m.d.s.) in a Banach space Y. This notion is slightly stronger than that of a basic sequence, since every martingale difference sequence is a basic sequence, but a basic sequence is only a martingale difference sequence in its closed linear span. Our aim is to study unconditional martingale difference sequences.

In section 6.3, we introduce the space of 'm.d.s. multipliers', associated with a m.d.s. $(d_i) \subset Y$. This is the sequence space

$$A^{(d_i)} := \left\{ (\alpha_i) \subset \mathbb{R} : \sum_{i=1}^{\infty} \alpha_i d_i \text{ converges in } Y \right\}$$

endowed with the norm $\|\cdot\|_{A^{(d_i)}}$, defined by $\|(\alpha_i)\|_{A^{(d_i)}} = \sup_{n\in\mathbb{N}} \|\sum_{i=1}^{n} \alpha_i d_i\|$ for each $(\alpha_i) \in A^{(d_i)}$. The unit vectors (e_i) form a basis of $A^{(d_i)}$ that is equivalent to (d_i). Using the martingale techniques developed in Chapter 3, we are able to show that (d_i) is an unconditional m.d.s. if and only if $A^{(d_i)}$ can be renormed so that it becomes an order continuous Banach lattice under the ordering $(\alpha_i) \geq 0 \Leftrightarrow \alpha_i \geq 0$, for each $i \in \mathbb{N}$. We denote this Banach lattice again by $A^{(d_i)}$ and call it 'the Banach lattice of unconditional m.d.s. multipliers'.

Next, in Section 6.4, we consider the l-tensor product of two martingale difference sequences. Using the technique of Gelbaum and Gil de Lamadrid in [49], we are able to show that if (ξ_i) is a m.d.s. in the Banach lattice E, relative to a positive BS-filtration, and (η_j) is a m.d.s. in the Banach space Y, then $(\xi_i \otimes \eta_j)$ is a m.d.s. in $E\widetilde{\otimes}_l Y$, provided the sequence $(\xi_i \otimes \eta_j)$ is ordered in an appropriate manner. Consequently, if $1 \leq p < \infty$, $(d_i) \subset L^p(\mu)$ is a classical m.d.s. basis and $(y_j) \subset Y$ a basis, then $(d_i \otimes y_j)$ is a basis of $L^p(\mu, Y)$.

If (d_i) is an unconditional m.d.s. in a Banach lattice E, then $A^{(d_i)}$ is a Banach lattice whose order structure may differ from that of E. In Section 6.5, we study the relationship between the ordering in $A^{(d_i)}$ and the ordering in E. To this end, we identify a property possessed by a large class of martingale difference sequences, called 'random positive equivalence'. We show, using the Maurey-Khinchin inequality (cf. [74, 36, 72]), that every unconditional m.d.s. in a Banach lattice with cotype $q < \infty$ has random positive equivalence. In this case, there exists a regular, order continuous mapping $R : A^{(d_i)} \to \ell^q(E)$, defined by $Re_i = e_i \otimes d_i$ for each $i \in \mathbb{N}$.

The regular map $R : A^{(d_i)} \to \ell^q(E)$ is useful when we consider the l-tensor product of unconditional martingale difference sequences in Section 6.6. As already mentioned, Aldous showed in [1] that, if $L^p(\mu, Y)$ $(1 < p < \infty)$ possesses an unconditional basis, then Y must be a UMD (and thus, reflexive) space. This result suggests that if (ξ_i) is an unconditional m.d.s. in the Banach lattice E, relative to a positive BS-filtration, and (η_j) is an unconditional m.d.s. in the Banach space Y, then $(\xi_i \otimes \eta_j)$ need not be an unconditional m.d.s. in $E\widetilde{\otimes}_l Y$. In view of this, we consider the the space $A^{(\xi_i)}\widetilde{\otimes}_l A^{(\eta_j)}$.

Using the result of Popa (Theorem 2.5.2), we are able to show that $(e_i \otimes e_j)$ is an unconditional basis of $A^{(\xi_i)}\widetilde{\otimes}_l A^{(\eta_j)}$, provided $(\xi_i) \subset E$ and $(\eta_j) \subset Y$ are unconditional. Consequently, the basis $(e_i \otimes e_j) \subset A^{(\xi_i)}\widetilde{\otimes}_l A^{(\eta_j)}$ need not be equivalent to the m.d.s. $(\xi_i \otimes \eta_j) \subset E\widetilde{\otimes}_l Y$. However, if (ξ_i) is also a boundedly complete basis of E, and E has type p and cotype q, then we have continuous embeddings

$$\ell^p(E)\widetilde{\otimes}_l Y \supset [(e_i \otimes \xi_i) \otimes \eta_j] \to A^{(\xi_i)}\widetilde{\otimes}_l A^{(\eta_j)} \to [(e_i \otimes \xi_i) \otimes \eta_j] \subset \ell^q(E)\widetilde{\otimes}_l Y,$$

defined by $(e_i \otimes \xi_i) \otimes \eta_j \mapsto e_i \otimes e_j \mapsto (e_i \otimes \xi_i) \otimes \eta_j$ for each $i, j \in \mathbb{N}$. The sequence $((e_i \otimes \xi_i) \otimes \eta_j)$ is an unconditional m.d.s. in both $\ell^p(E)\widetilde{\otimes}_l Y$ and $\ell^q(E)\widetilde{\otimes}_l Y$, but does not span either of these spaces. In the case when E has type and cotype 2, the m.d.s. $((e_i \otimes \xi_i) \otimes \eta_j) \subset \ell^2(E)\widetilde{\otimes}_l Y$ is equivalent to the unconditional basis $(e_i \otimes e_j) \subset A^{(\xi_i)}\widetilde{\otimes}_l A^{(\eta_j)}$.

The results in this chapter are original and have appeared in [29]. We warn the reader that [29] contains a mathematical error. Consequently, a corrigendum [27] has been published. See the Notes and remarks section at the end of this chapter for more information on where we went wrong.

6.2 Martingale difference sequences in a Banach space

The filtrations that we encounter in this chapter are not necessarily uniformly bounded by 1, but by some constant K. We, therefore, need generalized notions of a BS-filtration and a BL-filtration, as introduced in Chapter 3.

Definition 6.2.1 Let E be a Banach lattice and Y a Banach space.

(a) A sequence (T_i) of projections on Y with the property that $T_{i\wedge j} = T_iT_j$, for each $i, j \in \mathbb{N}$, is called a *K-BS-filtration* on Y if $\sup_{i\in\mathbb{N}} \|T_i\| = K < \infty$. Note that 1-BS-filtration is simply a BS-filtration.

(b) A sequence (T_i) of positive projections on E, with $\mathcal{R}(T_i)$ a closed Riesz subspace of E and $T_{i\wedge j} = T_iT_j$ for each $i, j \in \mathbb{N}$, is called a *M-BL-filtration* on E if $\sup_{i\in\mathbb{N}} \|T_i\| = M < \infty$. Note that 1-BL-filtration is simply a BL-filtration.

(c) If (T_i) is a K-BS-filtration (M-BL-filtration) on Y (on E), then (f_i, T_i) is called a *K-martingale (M-martingale)* on Y (on E) if $T_if_j = f_i$ for all $i \le j$. A 1-martingale will simply be referred to as a martingale.

The notions of a 'positive' and 'strictly positive' K-BS-filtration on a Banach lattice are defined in a similar manner to Definition 3.3.1. The above definition extends to collections of projections indexed by any directed set, in an obvious manner. In Chapter 7, we will encounter M-BL-filtrations indexed by directed sets of stopping times.

A notational change in the proofs of Proposition 3.2.6 and Corollary 3.2.7 shows that the same results hold for K-BS-filtrations and K-martingales, namely:

Proposition 6.2.2 *Let Y be a Banach space and (T_i) a K-BS-filtration on Y. Then $f \in \overline{\bigcup_{i=1}^\infty \mathcal{R}(T_i)}$ if and only if $\lim_{i\to\infty} \|T_if - f\| \to 0$.*

Corollary 6.2.3 *Let Y be a Banach space and (f_i, T_i) a K-martingale in Y. Then (f_i, T_i) converges to f if and only if $f \in \overline{\bigcup_{i=1}^\infty \mathcal{R}(T_i)}$ and $f_i = T_if$ for all $i \in \mathbb{N}$.*

We now recall the classical definition of a martingale difference sequence. Let $(d_i) \subset L^p(\mu)$ be a sequence and $\sigma(d_1, \dots, d_i)$ denote the smallest σ-algebra allowing $d_1, \dots, d_i$ to be measurable. Then, as mentioned earlier, (d_i) is called a martingale difference sequence (m.d.s.) if

$$\mathbb{E}(d_{i+1} \mid \sigma(d_1, \dots, d_i)) = 0 \tag{6.1}$$

for each $i \in \mathbb{N}$. Notice that $(\sigma(d_1, \dots, d_i))_{i=1}^\infty$ is a filtration and

$$d_{i+1} \in \mathcal{R}\Big(\mathbb{E}\Big(\cdot \mid \sigma(d_1, \dots, d_{i+1})\Big) - \mathbb{E}\Big(\cdot \mid \sigma(d_1, \dots, d_i)\Big)\Big)$$

for each $i \in \mathbb{N}$. Conversely, if (Σ_i) is a filtration and $(g_i) \subset L^p(\mu)$ is a sequence such that $g_{i+1} \in \mathcal{R}(\mathbb{E}(\,\cdot \mid \Sigma_{i+1}) - \mathbb{E}(\,\cdot \mid \Sigma_i))$ for each $i \in \mathbb{N}$, then it follows from (1.2) that

$$0 = \int_A \mathbb{E}(g_{i+1} \mid \Sigma_i)\mathrm{d}\mu = \int_A \mathbb{E}(g_{i+1} \mid \sigma(g_1, \dots, g_i))\mathrm{d}\mu \quad \text{for all } A \in \sigma(g_1, \dots, g_i),$$

which implies (g_i) satisfies (6.1) and is, therefore, an m.d.s. . Using this characterization, we introduce an abstract notion for a m.d.s. in a Banach space.

Let (T_i) be a K-BS-filtration and $i < j$, then $T_j - T_i$ is a projection due to the fact that the T_i's commute. This justifies the following definition.

Definition 6.2.4 Let (T_i) be a K-BS-filtration on a Banach space Y. Then the *difference projections* (D_i) *relative to* (T_i) are given by $D_1 = T_1$ and $D_i = T_i - T_{i-1}$ for $i \geq 2$.

It is then clear that $T_i = \sum_{k=1}^{i} D_k$ for each $i \in \mathbb{N}$ and that $D_i D_j = 0$ whenever $i \neq j$.

Definition 6.2.5 Let (D_i) be the difference projections relative to a K-BS-filtration (T_i) on a Banach space Y. Then a sequence (d_i) is called a *K-martingale difference sequence (K-m.d.s.) relative to* (T_i) if $d_i \in \mathcal{R}(D_i)$ for each $i \in \mathbb{N}$. A 1-m.d.s. will simply be referred to as a m.d.s. .

A sequence $(d_i) \subset L^p(\mu)$ obeying (6.1) will be called a *classical m.d.s.* and is clearly a special case of a m.d.s. in the above definition.

Notice that $D_i d_j = d_j$ whenever $i = j$ and $D_i d_j = 0$ whenever $i \neq j$. The sequence of partial sums $f_i = \sum_{k=1}^{i} d_k$, for each $i \in \mathbb{N}$, form a K-martingale with respect to (T_i). Conversely, if (f_i, T_i) is a K-martingale then the sequence of differences, defined by $d_1 = f_1$ and $d_i = f_i - f_{i-1}$ for $i \geq 2$, forms a K-m.d.s. relative to (T_i).

If (D_i) is the sequence of difference projections relative to a K-BS-filtration (T_i) on a Banach space Y with $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)} = Y$ then, for each $x \in Y$, Proposition 6.2.2 asserts that $(\sum_{k=1}^{i} D_k)x = T_i x \to x$ as $i \to \infty$. Thus $\sum_{k=1}^{\infty} D_k x = x$, which gives $Y = \oplus_{i=1}^{\infty} \mathcal{R}(D_i)$.

Suppose that (d_i) is a K-m.d.s. in a Banach space Y relative to (T_i) and that (α_i) is a sequence of scalars. Then the partial sums $f_i = \sum_{k=1}^{i} \alpha_k d_k$ form a K-martingale with respect to (T_i). If $i < j$, then

$$\|f_i\| = \left\| \sum_{k=1}^{i} \alpha_k d_k \right\| = \left\| T_i \left(\sum_{k=1}^{j} \alpha_k d_k \right) \right\| \leq K \left\| \sum_{k=1}^{j} \alpha_k d_k \right\| = K \|f_j\|.$$

Hence, (d_i) is a basic sequence with basis constant K. On the other hand, if (x_i) is a basic sequence in a Banach space Y with basis constant K, then (x_i) is a K-m.d.s. in

$[x_i]$, relative to the associated natural projections (P_i) on $[x_i]$. In short, we have the following result:

Proposition 6.2.6 *Every K-m.d.s. in a Banach space is a basic sequence with basis constant K. Also, every basic sequence in a Banach space with basis constant K is a K-m.d.s. in its closed linear span.*

If (d_i) is a K-m.d.s. relative to (T_i), it is easily observed that $[d_i] \subset \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$ which is, in general, necessarily strict. Indeed, consider the m.d.s. of Rademacher functions (r_i) in $L^p(\mu)$ for $1 \leq p < \infty$. Since (r_i) is a block basis of the Haar system, it follows that $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(\mathbb{E}(\,\cdot\,|\,\sigma(r_1,\ldots,r_i)))} = L^p(\mu)$. On the other hand, it follows from Khinchin's inequality (Theorem 1.4.6), that $[r_i]$ is isomorphic to ℓ^2, in which case, the inclusion $[r_i] \subset \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(\mathbb{E}(\,\cdot\,|\,\sigma(r_1,\ldots,r_i)))}$ is certainly strict. The next result characterizes the situation.

Proposition 6.2.7 *Let (d_i) be a K-m.d.s. in a Banach space Y relative to (T_i). Then $[d_i] = \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$ if and only if* $\operatorname{rank}(T_i) = i$ *for each* $i \in \mathbb{N}$.

Proof. Suppose $[d_i] = \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$ and let (D_i) be the difference projections relative to (T_i), then it follows from the above discussion that $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)} = \oplus_{i=1}^{\infty}\mathcal{R}(D_i)$. On the other hand, for $f \in [d_i]$, we have a unique basis expansion so that $f = \sum_{i=1}^{\infty} \alpha_i d_i = \sum_{i=1}^{\infty} D_i f$. The uniqueness of both these expansions implies $\alpha_i d_i = D_i f$ for each $i \in \mathbb{N}$ so that $T_i f = \sum_{k=1}^{i} \alpha_k d_k$. Thus, (T_i) are just the natural projections associated to the basic sequence (d_i). But then $\operatorname{rank}(T_i) = i$ for each $i \in \mathbb{N}$.

Conversely, it is sufficient to show $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)} \subset [d_i]$, since the reverse inclusion is always true. Since the T_i's have increasing ranges and $\operatorname{rank}(T_i) = i$ for each $i \in \mathbb{N}$, the difference projections (D_i) relative to (T_i) are all of rank one. As before, $f \in \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$ has a unique expansion $f = \sum_{i=1}^{\infty} D_i f$. The fact that $\dim(\mathcal{R}(D_i)) = 1$ implies there exists a scalar α_i such that $D_i f = \alpha_i d_i$ for each $i \in \mathbb{N}$. It follows that $\sum_{i=1}^{n} D_i f = \sum_{i=1}^{n} \alpha_i d_i \in \operatorname{span}(d_i)$ for each $n \in \mathbb{N}$, which completes the proof. $\square$

It is apparent from the above proof that, in the case where $[d_i] \subset \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$, the restriction $T_i|_{[d_i]}$ of T_i to $[d_i]$ is just the i-th natural projection on $[d_i]$, associated to the basic sequence (d_i), for each $i \in \mathbb{N}$.

In view of the fact that every K-m.d.s. is a basic sequence, we formulate the analogous notion of an unconditional K-m.d.s. .

Definition 6.2.8 Let (d_i) be a K-m.d.s. in a Banach space Y. Then (d_i) is said to be *unconditional* if there exists a constant $M > 0$ such that for every choice of scalars (α_i), signs (θ_i) and natural numbers n, we have

$$\left\| \sum_{i=1}^{n} \theta_i \alpha_i d_i \right\| \leq M \left\| \sum_{i=1}^{n} \alpha_i d_i \right\|.$$

The smallest constant M for which the above inequality holds is called the *unconditional constant* of (d_i).

Note that M in the above definition is never smaller than K. It is evident that if (d_i) is an unconditional K-m.d.s. , then it forms an unconditional basis of $[d_i]$ with unconditional constant M.

6.3 The Banach lattice of unconditional m.d.s. multipliers

Our goal in this section is to characterize the unconditionality of a K-m.d.s. in terms of a sequence space. We first recall some basic definitions from [93].

Definition 6.3.1 Let X and Y be Banach spaces.

(a) A sequence $(x_i) \subset X$ is said to *dominate* a sequence $(y_i) \subset Y$ provided for all sequences of scalars (α_i) we have

$$\sum_{i=1}^{\infty} \alpha_i x_i \text{ converges} \Rightarrow \sum_{i=1}^{\infty} \alpha_i y_i \text{ converges.}$$

In this case we shall use the notation $(x_i) \succcurlyeq (y_i)$.

(b) We shall say that (x_i) *strictly dominates* (y_i) if there exists a bounded linear mapping $T : [x_i] \to [y_i]$ such that $Tx_i = y_i$ for each $i \in \mathbb{N}$. In this case we shall write $(x_i) \succ (y_i)$.

(c) The sequences (x_i) and (y_i) are said to be *equivalent* if $(x_i) \succcurlyeq (y_i) \succcurlyeq (x_i)$ and *strictly equivalent* if $(x_i) \succ (y_i) \succ (x_i)$. In these cases, we shall use the notations $(x_i) \sim (y_i)$ and $(x_i) \approx (y_i)$ respectively.

It is immediate that if $(x_i) \approx (y_i)$ then $[x_i]$ is isomorphic to $[y_i]$ under the bounded linear map that takes x_i to y_i for each $i \in \mathbb{N}$. Strict domination clearly implies domination and strict equivalence clearly implies equivalence, but the reverse implications need not be true (cf. [93, pp. 69–74]). The following result is taken from [93, Chapter I, §8, Theorem 8.1].

Proposition 6.3.2 *Let X and Y be Banach spaces and $(x_i) \subset X$, $(y_i) \subset Y$ be sequences. Then the following statements hold:*

(a) We have $(x_i) \succ (y_i)$ if and only if there exists a constant $K > 0$ such that $\| \sum_{i=1}^{n} \alpha_i y_i \| \leq K \| \sum_{i=1}^{n} \alpha_i x_i \|$ holds for every choice of scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$.

(b) If (x_i) is a basic sequence, then $(x_i) \succcurlyeq (y_i)$ if and only if $(x_i) \succ (y_i)$.

(c) If (x_i) and (y_i) are both basic sequences, then $(x_i) \sim (y_i)$ if and only if $(x_i) \approx (y_i)$.

Proof. (a) Suppose $(x_i) \succ (y_i)$, then there exists a bounded linear map $u : [x_i] \to [y_i]$ such that $u(x_i) = y_i$ for each $i \in \mathbb{N}$. Thus, $\| \sum_{i=1}^n \alpha_i y_i\| \leq K \| \sum_{i=1}^n \alpha_i x_i\|$ holds for every choice of scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ with $K = \|u\|$.

Conversely, define a linear map $u : \operatorname{span}(x_i) \to [y_i]$ by $u(\sum_{i=1}^n \alpha_i x_i) = \sum_{i=1}^n \alpha_i y_i$. Then u is well defined since $\sum_{i=1}^n \alpha_i x_i = 0$ implies $\|u(\sum_{i=1}^n \alpha_i x_i)\| = \| \sum_{i=1}^n \alpha_i y_i\| \leq K \| \sum_{i=1}^n \alpha_i x_i\| = 0$. Moreover, for arbitrary $\sum_{i=1}^n \alpha_i x_i \in \operatorname{span}(x_i)$, we have $\|u(\sum_{i=1}^n \alpha_i x_i)\| \leq K \| \sum_{i=1}^n \alpha_i x_i\|$ so that u is bounded. Taking the unique continuous extension of u to $[x_i]$ completes the proof of (a).

(b) Only the implication $(x_i) \succcurlyeq (y_i) \Rightarrow (x_i) \succ (y_i)$ requires a proof. Since (x_i) is a basis of $[x_i]$, we have the convergence of $u(x) := \sum_{i=1}^\infty \alpha_i y_i$ for each $x = \sum_{i=1}^\infty \alpha_i x_i \in [x_i]$. The map $u : [x_i] \to [y_i]$ defined in this manner is linear and maps x_i to y_i for each $i \in \mathbb{N}$. Moreover, we have $u(x) = \lim_{n\to\infty} u_n(x)$, where $u_n(x) := \sum_{i=1}^n \alpha_i y_i$ for all $x = \sum_{i=1}^\infty \alpha_i x_i \in [x_i]$. Each u_n is continuous because

$$\|u_n(x)\| = \left\| \sum_{i=1}^n x_i^*(x) y_i \right\| \leq \sum_{i=1}^n |x_i^*(x)| \|y_i\| \leq \left(\sum_{i=1}^n \|x_i^*\| \|y_i\| \right) \|x\|.$$

Here, $(x_i^*) \subset [x_i]^*$ denotes the functionals biorthogonal to the basis (x_i), i.e. $\langle x_i, x_j^* \rangle = \delta_{ij}$ for all $i, j \in \mathbb{N}$. Consequently, an application of the Principle of Uniform Boundedness shows that u is continuous.

(c) Apply (b) to $(x_i) \succcurlyeq (y_i) \succcurlyeq (x_i)$. $\square$

Definition 6.3.3 Let (x_i) be a sequence in a Banach space Y such that $x_i \neq 0$ for each $i \in \mathbb{N}$. We define the normed linear space of sequences of coefficients of (x_i) to be

$$A^{(x_i)} = \left\{ (\alpha_i) \subset \mathbb{R} : \sum_{i=1}^\infty \alpha_i x_i \text{ converges in } Y \right\},$$

endowed with the norm $\| \cdot \|_{A^{(x_i)}}$, defined by $\|(\alpha_i)\|_{A^{(x_i)}} = \sup_{n \in \mathbb{N}} \|\sum_{i=1}^n \alpha_i x_i\|$ for each $(\alpha_i) \in A^{(x_i)}$.

The following result is taken from [93, Chapter I, §3, Proposition 3.1] and [93, Chapter I, §8, Proposition 8.1].

Theorem 6.3.4 *Let (x_i) be a sequence in a Banach space Y such that $x_i \neq 0$ for each $i \in \mathbb{N}$. Then the following statements hold:*

(a) $A^{(x_i)}$ *is a Banach space.*

(b) The unit vectors $e_i = (\delta_{ik})_{k=1}^{\infty}$ $(i = 1, 2, \ldots)$ *constitute a basis of* $A^{(x_i)}$ *such that* $(e_i) \sim (x_i)$ *and* $(e_i) \succ (x_i)$.

Proof. (a) Observe that, for $(\alpha_i) \in A^{(x_i)}$, the sequence $(\|\sum_{i=1}^{n} \alpha_i x_i\|)_{n=1}^{\infty}$ converges. Therefore, $\|(\alpha_i)\|_{A^{(x_i)}}$ is finite. Moreover, since $x_i \neq 0$ for each $i \in \mathbb{N}$, it follows that $\|(\alpha_i)\|_{A^{(x_i)}} = 0 \Leftrightarrow (\alpha_i) = 0$ so that $\|\cdot\|_{A^{(x_i)}}$ is indeed a norm.

Select a Cauchy sequence $((\alpha_i^{(k)}))_{k=1}^{\infty}$ from $A^{(x_i)}$. For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $m \geq k \geq N$ implies

$$\left\|(\alpha_i^{(m)}) - (\alpha_i^{(k)})\right\|_{A^{(x_i)}} = \sup_{n\in\mathbb{N}} \left\|\sum_{i=1}^{n} (\alpha_i^{(m)} - \alpha_i^{(k)}) x_i\right\| < \varepsilon.$$

Consequently,

$$\left\|(\alpha_n^{(m)} - \alpha_n^{(k)}) x_n\right\| \leq \left\|\sum_{i=1}^{n} (\alpha_i^{(m)} - \alpha_i^{(k)}) x_i\right\| + \left\|\sum_{i=1}^{n-1} (\alpha_i^{(m)} - \alpha_i^{(k)}) x_i\right\| < 2\varepsilon.$$

Since $x_i \neq 0$ for each $i \in \mathbb{N}$, we may write

$$\left|(\alpha_n^{(m)} - \alpha_n^{(k)})\right| < 2\varepsilon/\|x_n\|$$

for each $n \in \mathbb{N}$. Thus, for each $n \in \mathbb{N}$, the sequence $(\alpha_n^{(k)})_{k=1}^{\infty}$ is convergent to a scalar, α_n say. Since $\|\sum_{i=1}^{n} (\alpha_i^{(m)} - \alpha_i^{(k)}) x_i\| < \varepsilon$ for each $n \in \mathbb{N}$ it follows, by taking the limit as $m \to \infty$, that

$$\left\|\sum_{i=1}^{n} (\alpha_i - \alpha_i^{(k)}) x_i\right\| < \varepsilon$$

for each $n \in \mathbb{N}$. Then, for $k \geq N$ and $l \in \mathbb{N}$, we obtain

$$\begin{aligned}
\left\|\sum_{i=n+1}^{n+l} \alpha_i x_i\right\| &= \left\|\sum_{i=n+1}^{n+l} (\alpha_i - \alpha_i^{(k)}) x_i + \sum_{i=n+1}^{n+l} \alpha_i^{(k)} x_i\right\| \\
&\leq \left\|\sum_{i=1}^{n} (\alpha_i - \alpha_i^{(k)}) x_i\right\| + \left\|\sum_{i=1}^{n+l} (\alpha_i - \alpha_i^{(k)}) x_i\right\| + \left\|\sum_{i=n+1}^{n+l} \alpha_i^{(k)} x_i\right\| \\
&< 2\varepsilon + \left\|\sum_{i=n+1}^{n+l} \alpha_i^{(k)} x_i\right\|.
\end{aligned}$$

Since Y is complete and the series $\sum_{i=1}^{\infty} \alpha_i^{(k)} x_i$ converges, it follows that the series $\sum_{i=1}^{\infty} \alpha_i x_i$ also converges, so that $(\alpha_i) \in A^{(x_i)}$. Lastly, by the above,

$$\left\|(\alpha_i) - (\alpha_i^{(k)})\right\|_{A^{(x_i)}} = \sup_{n\in\mathbb{N}} \left\|\sum_{i=1}^{n} (\alpha_i^{(k)} - \alpha_i) x_i\right\| \leq \varepsilon$$

and the proof of (a) is complete.

(b) If $(\alpha_i) \in A^{(x_i)}$, then $\sum_{i=1}^{\infty} \alpha_i x_i$ converges and

$$\left\| (\alpha_i) - \sum_{i=1}^{m} \alpha_i e_i \right\| = \sup_{m+1 \leq n < \infty} \left\| \sum_{i=m+1}^{n} \alpha_i x_i \right\| \to 0$$

as $m \to \infty$. Thus, the series $\sum_{i=1}^{\infty} \alpha_i e_i$ converges to (α_i). On the other hand, if $\sum_{i=1}^{\infty} \alpha_i e_i = 0$, then

$$\begin{aligned} \|(\alpha_i)\|_{A^{(x_i)}} &= \sup_{n \in \mathbb{N}} \left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \leq \lim_{n \to \infty} \sup_{1 \leq k \leq n} \left\| \sum_{i=1}^{k} \alpha_i x_i \right\| \\ &= \lim_{n \to \infty} \left\| \sum_{i=1}^{n} \alpha_i e_i \right\|_{A^{(x_i)}} = \left\| \sum_{i=1}^{\infty} \alpha_i e_i \right\|_{A^{(x_i)}} = 0, \end{aligned}$$

so that $\alpha_i = 0$ for each $i \in \mathbb{N}$. Thus, every $(\alpha_i) \in A^{(x_i)}$ has a unique expansion $\sum_{i=1}^{\infty} \alpha_i e_i$. Consequently, (e_i) is a basis of $A^{(x_i)}$.

We have already seen that the convergence of $\sum_{i=1}^{\infty} \alpha_i x_i$ implies the convergence of $\sum_{i=1}^{\infty} \alpha_i e_i$. For the reverse implication, observe the inequality

$$\left\| \sum_{i=n+1}^{n+m} \alpha_i x_i \right\| \leq \sup_{n+1 \leq k \leq n+m} \left\| \sum_{i=n+1}^{k} \alpha_i x_i \right\| = \left\| \sum_{i=n+1}^{n+m} \alpha_i e_i \right\|_{A^{(x_i)}}$$

for each $n, m \in \mathbb{N}$. Thus, the convergence of $\sum_{i=1}^{\infty} \alpha_i e_i$ implies the convergence of $\sum_{i=1}^{\infty} \alpha_i x_i$ by the completeness of Y. Thus, $(e_i) \sim (x_i)$. An application of Proposition 6.3.2(b) completes the proof. □

Definition 6.3.5 If Y is a Banach space and $(x_i) \subset Y$ is a sequence, then the map from $A^{(x_i)}$ into Y, given by $(\alpha_i) \mapsto \sum_{i=1}^{\infty} \alpha_i x_i$, will be referred to as the *co-ordinate map.*

It is evident that the co-ordinate map for any sequence is linear and of norm one. If (x_i) is a basic sequence, Theorem 6.3.4 and Proposition 6.3.2(c) imply that $(e_i) \approx (x_i)$. Thus, $A^{(x_i)}$ is isomorphic to $[x_i]$ under the co-ordinate map. We shall mainly consider $A^{(d_i)}$ where (d_i) is a K-m.d.s. (and thus a basic sequence).

Definition 6.3.6 Let (d_i) be a K-m.d.s. in a Banach space. The order on $A^{(d_i)}$, defined by

$$(\alpha_i) \geq 0 \Longleftrightarrow \alpha_i \geq 0 \quad \text{for each } i \in \mathbb{N},$$

is called the *sequential ordering* induced by (d_i). The set

$$A_+^{(d_i)} := \{(\alpha_i) \in A^{(d_i)} : (\alpha_i) \geq 0\}$$

is called the *positive cone* induced by (d_i).

Evidently, $\lambda A^{(d_i)} \subset A^{(d_i)}$ where $\lambda \in \mathbb{R}_+$, $A^{(d_i)} + A^{(d_i)} \subset A^{(d_i)}$ and $A_+^{(d_i)} \cap (-A_+^{(d_i)}) = \{0\}$. Thus, $(A^{(d_i)}, A_+^{(d_i)})$ is a partially ordered vector space.

Lemma 6.3.7 *Suppose that (d_i) is an unconditional K-m.d.s. , then $A^{(d_i)}$ is a Dedekind complete Riesz space under the sequential ordering. Furthermore, $A^{(d_i)}$ can be renormed so that it becomes a Dedekind complete Banach lattice with the unit vectors (e_i) as an unconditional basis.*

Proof. Let $(\alpha_i) \in A^{(d_i)}$. By Theorem 1.4.2, the unconditional convergence of $\sum_{i=1}^{\infty} \alpha_i d_i \in [d_i]$ implies that the series $\sum_{i=1}^{\infty} |\alpha_i| d_i$, $\sum_{i=1}^{\infty} \alpha_i^+ d_i$ and $\sum_{i=1}^{\infty} \alpha_i^- d_i$ also converge in $[d_i]$, where $\alpha_i^+ := \max\{0, \alpha_i\}$ and $\alpha_i^- := \max\{0, -\alpha_i\}$. Hence, $(|\alpha_i|), (\alpha_i^+), (\alpha_i^-) \in A^{(d_i)}$. It is clear from the definition of the sequential ordering that

$$|(\alpha_i)| = (\alpha_i) \vee (-(\alpha_i)) = (\alpha_i \vee (-\alpha_i)) = (|\alpha_i|) \in A^{(d_i)}.$$

Consequently, $A^{(d_i)}$ is a Riesz space. In addition, we have $(\alpha_i) = (\alpha_i^+) - (\alpha_i^-)$ and $(\alpha_i^+) \wedge (\alpha_i^-) = 0$, thus $(\alpha_i)^+ = (\alpha_i^+)$ and $(\alpha_i)^- = (\alpha_i^-)$. Moreover, the unconditional convergence of $\sum_{i=1}^{\infty} \alpha_i d_i \in [d_i]$ also implies the convergence of $\sum_{i=1}^{\infty} \gamma_i d_i$ whenever $|\gamma_i| \le |\alpha_i|$ for each $i \in \mathbb{N}$ (cf. [71, Proposition 1.c.6.]). Thus, $(\gamma_i) \in A^{(d_i)}$ provided that $|\gamma_i| \le |\alpha_i|$ for each $i \in \mathbb{N}$. It is now evident from the Dedekind completeness of $\mathbb{R}$ that $(A^{(d_i)}, A_+^{(d_i)})$ is a Dedekind complete Riesz space.

Since $(e_i) \approx (d_i)$, with (d_i) an unconditional basic sequence, it follows that (e_i) is an unconditional basis of $A^{(d_i)}$. It follows from Proposition 1.4.9 that, for all $(\alpha_i) \in A^{(d_i)}$ and $(\lambda_i) \in \ell^\infty$, we have $\|\sum_{i=1}^{\infty} \lambda_i \alpha_i e_i\| \le M \|(\lambda_i)\|_\infty \|\sum_{i=1}^{\infty} \alpha_i e_i\|$ where M is the unconditional constant of (e_i). Thus, if $(\alpha_i), (\beta_i) \in A^{(d_i)}$ with $|(\alpha_i)| \le |(\beta_i)|$, it follows that $\|(\alpha_i/\beta_i)\|_\infty \le 1$. Consequently,

$$\left\| \sum_{i=1}^{\infty} \alpha_i e_i \right\| = \left\| \sum_{i=1}^{\infty} (\alpha_i/\beta_i) \beta_i e_i \right\| \le M \left\| \sum_{i=1}^{\infty} \beta_i e_i \right\|.$$

Thus, $|(\alpha_i)| \le |(\beta_i)|$ implies $\|(\alpha_i)\| \le M \|(\beta_i)\|$, i.e. $A^{(d_i)}$ is a partially ordered Banach space. Define $\|\cdot\|_0$ on $A^{(d_i)}$ by

$$\|(\alpha_i)\|_0 = \sup\{\|(\beta_i)\| : |(\beta_i)| \le |(\alpha_i)|\},$$

for each $(\alpha_i) \in A^{(d_i)}$. Then it is readily verified that $\|\cdot\|_0$ is an equivalent norm on $A^{(d_i)}$ and $(A^{(d_i)}, A_+^{(d_i)}, \|\cdot\|_0)$ is a Dedekind complete Banach lattice. $\square$

If (d_i) is an unconditional K-m.d.s. in a Banach lattice E, we shall denote the Dedekind complete Banach lattice obtained by renorming $A^{(d_i)}$ again by $A^{(d_i)}$. Note that, after renorming, the unit vectors (e_i) are now an unconditional basis of $A^{(d_i)}$ with unconditional constant one and the co-ordinate map from $A^{(d_i)}$ onto $[d_i]$ is still of norm one. We refer to $A^{(d_i)}$ as *the Banach lattice of m.d.s. multipliers*. It should be clear that $A^{(d_i)}$ is isomorphic to $[d_i]$ but not necessarily Riesz isomorphic; in fact, $[d_i]$ need not even be a Riesz subspace of E.

Theorem 6.3.8 *Let (d_i) be a K-m.d.s. in a Banach space Y, then (d_i) is unconditional if and only if $A^{(d_i)}$ is an order continuous Banach lattice.*

Proof. Suppose that (d_i) is an unconditional K-m.d.s. . Then Lemma 6.3.7 asserts that $A^{(d_i)}$ is a Dedekind complete Banach lattice. It is now sufficient to show that every positive, order bounded, disjoint sequence in $A^{(d_i)}$ converges in norm to zero (cf. [106, Theorem 17.14]).

To this end, let $(x_k) \subset A^{(d_i)}$ be a positive disjoint sequence which is order bounded. By the Dedekind completeness of $A^{(d_i)}$, it follows that $\sup_k x_k \in A^{(d_i)}$. Let $s = \sup_k x_k$ and define the sequence of partial sums (s_j) by $s_j = \sum_{k=1}^{j} x_k$. Since (x_k) is disjoint, we have that $s_j = \bigvee_{k=1}^{j} x_k$ for each $j \in \mathbb{N}$ with $s_j \uparrow s$. We claim that $s_j \to s$ in norm. To see this, for each $j \in \mathbb{N}$ let

$$\sigma_j = \bigcup_{1 \le k \le j} \left\{ i \in \mathbb{N} : x_k = (\alpha_i^{(k)}),\, \alpha_i^{(k)} \neq 0 \right\}$$

and define the family of projections (P_{σ_j}) on $A^{(d_i)}$ by $P_{\sigma_j}((\gamma_i)) = \sum_{i \in \sigma_j} \gamma_i e_i$ for each $(\gamma_i) \in A^{(d_i)}$. By Lemma 6.3.7, we have that (e_i) is an unconditional basis of $A^{(d_i)}$. Thus, (P_{σ_j}) is a BS-filtration. Now observe

$$x_j \wedge x_k = (\alpha_i^{(j)}) \wedge (\alpha_i^{(k)}) = (\alpha_i^{(j)} \wedge \alpha_i^{(k)}) = 0$$

for $j \neq k$. Hence $\alpha_i^{(j)} \wedge \alpha_i^{(k)} = 0$, giving either $\alpha_i^{(j)} = 0$ or $\alpha_i^{(k)} = 0$ for each $i \in \mathbb{N}$. Thus, the sequence $(\alpha_i^{(k)})_{k=1}^{\infty}$ has at most one non-zero element for each $i \in \mathbb{N}$. As a consequence, (s_j, P_{σ_j}) is a martingale and $P_{\sigma_j}(s) = s_j$ for each $j \in \mathbb{N}$, with $s \in \overline{\bigcup_{j=1}^{\infty} \mathcal{R}(P_{\sigma_j})}$. An appeal to Corollary 3.2.7 gives $s_j \to s$ in norm, which proves the claim. It is now evident that $\|x_{k+1}\| = \|s_{k+1} - s_k\| \to 0$, since (s_k) is a Cauchy sequence.

Conversely, suppose that $A^{(d_i)}$ is an order continuous Banach lattice under the sequential ordering. Note that the order continuity of the norm on $A^{(d_i)}$ implies that $A^{(d_i)}$ is Dedekind complete. Since $A^{(d_i)}$ is a Riesz space, we may decompose any element uniquely as the difference of two disjoint positive elements and so we need only consider positive elements.

To this end, let $f = \sum_{i=1}^{\infty} \alpha_i e_i \in A_+^{(d_i)}$ and let $(n_r)_{r=1}^{\infty}$ be a strictly increasing sequence of natural numbers. For each $k \in \mathbb{N}$, define $x_k = \sum_{r=1}^{k} \alpha_{n_r} e_{n_r}$. Then (x_k) is an increasing sequence which is bounded above by f. The Dedekind completeness of $A^{(d_i)}$ implies that $x_k = \sum_{r=1}^{k} \alpha_{n_r} e_{n_r} \uparrow \sum_{r=1}^{\infty} \alpha_{n_r} e_{n_r} := x \in A_+^{(d_i)}$. Now $(x - x_k) \downarrow 0$ and the order continuity of the norm implies $\|x - x_k\| \to 0$. Hence, the series $\sum_{r=1}^{\infty} \alpha_{n_r} e_{n_r}$ is summable, from which we deduce the unconditional summability of $f = \sum_{i=1}^{\infty} \alpha_i e_i$ by Theorem 1.4.2. Thus, (e_i) is an unconditional basis of $A^{(d_i)}$ with $(e_i) \approx (d_i)$, which completes the proof. $\square$

As an immediate consequence of this result, we obtain a familiar characterization of an unconditional basis that can be found in [93, Chapter II, §16, Proposition 16.2.].

Corollary 6.3.9 *Let Y be a Banach space and $(x_i) \subset Y$ be a basic sequence. Then (x_i) is an unconditional basic sequence if and only if $[x_i]$ can be renormed so that it is an order continuous Banach lattice with order induced by the cone*

$$C_+^{(x_i)} := \left\{ \sum_{i=1}^{\infty} \alpha_i x_i \in [x_i] : \alpha_i \geq 0 \text{ for each } i \in \mathbb{N} \right\}.$$

Proof. We have that (x_i) is a K-m.d.s. in $[x_i]$, relative the associated natural projections. Since (e_i) is a basis of $A^{(x_i)}$ with $(e_i) \approx (x_i)$, it follows that $([x_i], C_+^{(x_i)})$ is Riesz isomorphic to $(A^{(x_i)}, A_+^{(x_i)})$, provided that (x_i) is unconditional. The result now follows by inducing the (equivalent) norm $\|\cdot\|_{A^{(x_i)}}$ on $[x_i]$. $\square$

6.4 The l-tensor product of martingale difference sequences

Using the idea of Gelbaum and Gil de Lamadrid in [49] for constructing the tensor product basis with respect to a uniform crossnorm, we construct the l-tensor product of two martingale difference sequences. This construction resembles the earlier work on product Schauder decompositions in Section 1.6.

Definition 6.4.1 Let (ξ_i) and (η_j) be sequences in the Banach spaces X and Y respectively. Define the *square ordering* on the sequence of tensors $(\xi_i \otimes \eta_j)$ to be the ordering of the indices (i, j) along along the squares, i.e., $(i_1, j_1) \leq (i_2, j_2)$ when one of the following conditions hold:

(a) $\max\{i_1, j_1\} < \max\{i_2, j_2\}$,
(b) $\max\{i_1, j_1\} = \max\{i_2, j_2\}$ and $i_1 < i_2$ or
(c) $\max\{i_1, j_1\} = \max\{i_2, j_2\} = i_1 = i_2$ and $j_1 \geq j_2$.

The square ordering on $(\xi_i \otimes \eta_j)$ is illustrated by the diagram after Definition 1.6.2. Again, we shall use the notation S_k for the set consisting of the first k ordered pairs of indices (i, j) in the square ordering.

Let E be a Banach lattice and Y a Banach space. Suppose that $(\xi_i) \subset E$ and $(\eta_j) \subset Y$ are basic sequences with $[\xi_i]$ a Riesz subspace of E. Since the l-norm is a reasonable crossnorm, it follows that $[\xi_i]\widetilde{\otimes}_l[\eta_j] = [\xi_i \otimes \eta_j]$. Moreover, the left order injectivity of the l-norm implies that $[\xi_i]\widetilde{\otimes}_l[\eta_j]$ is a closed subspace of $E\widetilde{\otimes}_l Y$.

Proposition 6.4.2 *Let (S_i) be a positive K_1-BS-filtration on the Banach lattice E, (T_j) be a K_2-BS-filtration on the Banach space Y and define the sequence (P_k) by*

$$P_k = \begin{cases} S_i \otimes_l T_i & ;\ k = i^2 \\ S_i \otimes_l T_i + S_{k-i^2} \otimes_l (T_{i+1} - T_i) & ;\ i^2 < k \leq i^2 + i + 1 \\ S_{i+1} \otimes_l T_{i+1} - (S_{i+1} - S_i) \otimes_l T_{(i+1)^2-k} & ;\ i^2 + i + 1 < k < (i+1)^2 \end{cases}$$

for each $k \in \mathbb{N}$. Then (P_k) is a K-BS-filtration on $E\widetilde{\otimes}_l Y$ where $K \leq 3K_1K_2$. Moreover, if (ξ_i) and (η_j) are martingale difference sequences relative to (S_i) and (T_j) respectively, then the sequence $(\xi_i \otimes \eta_j)$ with the square ordering is a K-m.d.s. in $E\widetilde{\otimes}_l Y$ relative to (P_k).

Proof. Since (S_i) is a positive K-BS-filtration and (T_j) is a K-BS-filtration we have, for each $i \in \mathbb{N}$, that

$$\|S_i \otimes_l T_i\| = \|S_i\|\|T_i\| \leq K_1K_2,$$

$$\|S_{k-i^2} \otimes_l (T_{i+1} - T_i)\| = \|S_{k-i^2}\|\|(T_{i+1} - T_i)\| \leq 2K_1K_2$$

and

$$\begin{aligned} \|(S_{i+1} - S_i) \otimes_l T_{(i+1)^2-k}\| &\leq \|S_{i+1} \otimes_l T_{(i+1)^2-k}\| + \|S_i \otimes_l T_{(i+1)^2-k}\| \\ &= \|T_{(i+1)^2-k}\|(\|S_{i+1}\| + \|S_i\|) \\ &\leq 2K_1K_2, \end{aligned}$$

from which we deduce $\sup_{k\in\mathbb{N}} \|P_k\| \leq 3K_1K_2$. Hence, (P_k) is uniformly bounded on $E\widetilde{\otimes}_l Y$.

Using the fact that (S_i) and (T_j) are K-BS-filtrations, we first show that P_k is a projection for each $k \in \mathbb{N}$. The case where k is a perfect square is trivial. For the case $i^2 < k \leq i^2 + i + 1$, for some $i \in \mathbb{N}$, we have

$$\begin{aligned}
P_k^2 &= (S_i \otimes_l T_i + S_{k-i^2} \otimes_l (T_{i+1} - T_i))^2 \\
&= (S_i \otimes_l T_i)^2 + (S_i \otimes_l T_i)(S_{k-i^2} \otimes_l (T_{i+1} - T_i)) + \\
&\qquad (S_{k-i^2} \otimes_l (T_{i+1} - T_i))(S_i \otimes_l T_i) + (S_{k-i^2} \otimes_l (T_{i+1} - T_i))^2 \\
&= S_i \otimes_l T_i + (S_i S_{k-i^2}) \otimes_l (T_i T_{i+1} - T_i^2) + \\
&\qquad (S_{k-i^2} S_i) \otimes_l (T_{i+1} T_i - T_i^2) + S_{k-i^2} \otimes_l (T_{i+1} - T_i) \\
&= S_i \otimes_l T_i + 0 + 0 + S_{k-i^2} \otimes_l (T_{i+1} - T_i) \\
&= P_k.
\end{aligned}$$

For the case $i^2 + i + 1 < k < (i+1)^2$, for some $i \in \mathbb{N}$, we have

$$\begin{aligned}
P_k^2 &= (S_{i+1} \otimes_l T_{i+1} - (S_{i+1} - S_i) \otimes_l T_{(i+1)^2-k})^2 \\
&= (S_{i+1} \otimes_l T_{i+1})^2 - (S_{i+1} \otimes_l T_{i+1})((S_{i+1} - S_i) \otimes_l T_{(i+1)^2-k}) - \\
&\qquad ((S_{i+1} - S_i) \otimes_l T_{(i+1)^2-k})(S_{i+1} \otimes_l T_{i+1}) + ((S_{i+1} - S_i) \otimes_l T_{(i+1)^2-k})^2 \\
&= S_{i+1} \otimes_l T_{i+1} - (S_{i+1}^2 - S_{i+1} S_i) \otimes_l T_{i+1} T_{(i+1)^2-k} - \\
&\qquad (S_{i+1}^2 - S_i S_{i+1}) \otimes_l T_{(i+1)^2-k} T_{i+1} + (S_{i+1} - S_i) \otimes_l T_{(i+1)^2-k} \\
&= S_{i+1} \otimes_l T_{i+1} - 2(S_{i+1} - S_i) \otimes_l T_{(i+1)^2-k} + (S_{i+1} - S_i) \otimes_l T_{(i+1)^2-k} \\
&= S_{i+1} \otimes_l T_{i+1} - (S_{i+1} - S_i) \otimes_l T_{(i+1)^2-k} \\
&= P_k.
\end{aligned}$$

To prove that (P_k) is a K-BS-filtration, we need only show that $P_k = P_k P_{k+1} = P_{k+1} P_k$ for each $k \in \mathbb{N}$. This presents us with five cases for each $i \in \mathbb{N}$: $k = i^2$, $i^2 < k < i^2 + i + 1$, $k = i^2 + i + 1$, $i^2 + i + 1 < k < (i+1)^2 - 1$ and $k = (i+1)^2 - 1$. The verification of these cases is a tedious but trivial exercise and will be omitted.

For the last part of the proof, it follows from the definition of the square ordering that

$$P_n \left(\sum_{(i,j) \in S_m} \xi_i \otimes \eta_j \right) = \sum_{(i,j) \in S_n} \xi_i \otimes \eta_j$$

for $n \leq m$. This gives $\xi_i \otimes \eta_j \in \mathcal{R}(P_k - P_{k-1})$ for each $k \in \mathbb{N}$, where $\{(i,j)\} = S_k \backslash S_{k-1}$. Here, P_0 is defined to be zero and S_0 to be the empty set. $\square$

In the case where (S_i) and (T_j) are the natural projections associated with the bases (ξ_i) and (η_j) respectively, it is evident that (P_k) are the natural projections associated with the basic sequence $(\xi_i \otimes \eta_j)$ with the square ordering, and so we obtain the following corollary.

Corollary 6.4.3 *Let be E be a Banach lattice with a basis (ξ_i) possessing the property that the natural projections associated with (ξ_i) are positive. If Y is a Banach* *space with basis (basic sequence) (η_j), then the sequence $(\xi_i \otimes \eta_j)$ with the square ordering is a basis (basic sequence) of $E\widetilde{\otimes}_l Y$.*

Proof. The case where (η_j) is a basis for Y follows from Proposition 6.4.2 and the fact that $[\xi_i \otimes \eta_j] = E\widetilde{\otimes}_l Y$. For the case where (η_j) is a basic sequence in Y, it follows from the first part of the proof that $(\xi_i \otimes \eta_j)$ is a basis of $E\widetilde{\otimes}_l[\eta_j]$. The left order injectivity of the l-norm now implies that $E\widetilde{\otimes}_l[\eta_j]$ is a closed subspace of $E\widetilde{\otimes}_l Y$. Thus, $(\xi_i \otimes \eta_j)$ is a basic sequence in $E\widetilde{\otimes}_l Y$. $\square$

In particular, when $1 \leq p < \infty$ and $E = L^p(\mu)$, we obtain the following result:

Corollary 6.4.4 *Let $1 \leq p < \infty$ and (y_j) be a basic sequence in a Banach space Y. If (d_i) is a classical m.d.s. in $L^p(\mu)$, then the sequence $(d_i \otimes y_j)$ with the square ordering is a basic sequence in $L^p(\mu, Y)$. If, in addition, we have $[d_i] = L^p(\mu)$ then the sequence $(d_i \otimes y_j)$ with the square ordering is a basis of $L^p(\mu, Y)$ provided (y_j) is a basis of Y.*

Proof. Since (d_i) is a m.d.s. relative to a positive K-BS-filtration, the result follows from Proposition 6.4.2, Corollary 6.4.3 and the fact that $L^p(\mu, Y)$ is isometric to $L^p(\mu)\widetilde{\otimes}_l Y$. $\square$

Note that the Haar system is an example of a classical m.d.s. for which the linear span is dense in $L^p(\mu)$, thus $L^p(\mu, Y)$ has a basis if Y has a basis and $1 \leq p < \infty$. Aldous showed in [1, Proposition 3] that if a classical m.d.s. formed a basis of $L^p(\mu, Y)$ for a Banach space Y, then Y is necessarily one dimensional. This result can be generalized to the l-tensor product as follows:

Proposition 6.4.5 *Let E be a Banach lattice and Y a Banach space. If there exists a BL-filtration on E so that $(T_i \otimes_l \mathrm{id}_Y - T_{i-1} \otimes_l \mathrm{id}_Y)$ is a Schauder decomposition of $E\widetilde{\otimes}_l Y$ with $\mathrm{rank}\,((T_i \otimes_l \mathrm{id}_Y) - (T_{i-1} \otimes_l \mathrm{id}_Y)) = 1$ for each $i \in \mathbb{N}$, then $\dim(Y) = 1$.*

Proof. Note that for any BL-filtration (T_i) on E, $(T_i \otimes_l \mathrm{id}_Y)$ is a BS-filtration on $E\widetilde{\otimes}_l Y$ by Theorem 3.4.2. Moreover, if (T_i) is a BL-filtration that is dense in E, then $(T_i \otimes_l \mathrm{id}_Y)$ is a BS-filtration that is dense in $E\widetilde{\otimes}_l Y$. Thus, $(T_i \otimes_l \mathrm{id}_Y - T_{i-1} \otimes_l \mathrm{id}_Y)$ is a Schauder decomposition of $E\widetilde{\otimes}_l Y$. Now suppose that $\mathrm{rank}\,((T_i \otimes_l \mathrm{id}_Y) - (T_{i-1} \otimes_l \mathrm{id}_Y)) = 1$ for each $i \in \mathbb{N}$. By Lemma 3.4.1,

$$\dim(\mathcal{R}(T_i \otimes_l \mathrm{id}_Y)) = \dim(\mathcal{R}(T_i)\widetilde{\otimes}_l Y) = i$$

for each $i \in \mathbb{N}$. Thus,

$$\dim(\mathcal{R}(T_i))\dim(Y) = \dim(\mathcal{R}(T_i) \otimes Y) \leq \dim(\mathcal{R}(T_i)\widetilde{\otimes}_l Y) = i,$$

so that $\dim(Y) \leq i/\dim(\mathcal{R}(T_i)) \leq i$ for all $i \in \mathbb{N}$. Consequently, we must have $\dim(Y) = 1$. $\square$

It is important to note that the definition of a K-m.d.s. determines a larger class of sequences than the definition of a classical m.d.s. . In the latter case, it is therefore possible to have a K-m.d.s. as a basis of $L^p(\mu, Y)$ for which Y could be infinite dimensional.

6.5 Martingale difference sequences with random positive equivalence

For an unconditional K-m.d.s. (d_i) in a Banach lattice E, we are faced with the problem of relating the sequential ordering on $A^{(d_i)}$ to the ordering on E. We introduce the following property for a K-m.d.s. in a Banach lattice.

Definition 6.5.1 Let E be a Banach lattice and (r_i) denote the sequence of Rademacher functions. A K-m.d.s. (d_i) in E is said to have *random positive equivalence* if $L^2(\mu, E) \supset (r_i \otimes d_i) \sim (d_i) \sim (r_i \otimes |d_i|) \subset L^2(\mu, E)$.

There is a special class of Banach lattice for which every unconditional K-m.d.s. has random positive equivalence. Before we can define this class, we need to give meaning to the expression $(\sum_{i=1}^n |x_i|^p)^{1/p}$ for $1 \leq p < \infty$ and $x_1, x_2, \ldots, x_n \in E$, where E is an arbitrary Banach lattice. To achieve this, we use the functional calculus of Krivine (cf. [60, 72]):

Proposition 6.5.2 *Let $x_1, x_2, \ldots, x_n$ be elements of a Banach lattice E.*

(a) *For $1 \leq p < \infty$, $(\sum_{i=1}^n |x_i|^p)^{1/p} = \sup\left\{\sum_{i=1}^n \alpha_i x_i : (\sum_{i=1}^n |\alpha_i|^q)^{1/q} \leq 1\right\}$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

(b) $\sup_{1\leq i\leq n} |x_i| = \sup\{\sum_{i=1}^n \alpha_i x_i : \sum_{i=1}^n |\alpha_i| \leq 1\}$.

The above proposition is consistent with the meaning of $(\sum_{i=1}^n |x_i|^p)^{1/p}$ and $\sup_{1\leq i\leq n} |x_i|$ when E is a scalar-valued function space. In view of the above functional calculus, the following definition is justified:

Definition 6.5.3 Let E be a Banach lattice and $1 \leq p, q \leq \infty$.

(a) E is called *p-convex* if there is a constant $M > 0$ so that

$$\left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \right\| \leq M \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{1/p}, \quad \text{if } 1 \leq p < \infty$$

or

$$\left\| \sup_{1 \leq i \leq n} |x_i| \right\| \leq M \sup_{1 \leq i \leq n} \|x_i\|, \quad \text{if } p = \infty$$

for every choice of vectors $x_1, x_2, \ldots, x_n \in E$. The smallest value of M for which the above holds is denoted by $M^{(p)}(E)$.

(b) E is called *q-concave* if there is a constant $M > 0$ so that

$$\left(\sum_{i=1}^{n} \|x_i\|^q \right)^{1/q} \leq M \left\| \left(\sum_{i=1}^{n} |x_i|^q \right)^{1/q} \right\|, \quad \text{if } 1 \leq q < \infty$$

or

$$\sup_{1 \leq i \leq n} \|x_i\| \leq M \left\| \sup_{1 \leq i \leq n} |x_i| \right\|, \quad \text{if } q = \infty$$

for every choice of vectors $x_1, x_2, \ldots, x_n \in E$. The smallest value of M for which the above holds is denoted by $M^{(q)}(E)$.

In [36, Theorem 16.17], it is shown that a Banach lattice E has finite concavity if and only if it has finite cotype. It is important to note that, although a Banach lattice may have cotype $q < \infty$ and concavity $q' < \infty$, the constants q and q' are not necessarily equal (cf. [36, p. 332]). We are interested in Banach lattices with finite concavity, the reason being, the following generalization of the Khinchin inequality (cf. [36, Theorem 16.11] and [74, 72]):

Theorem 6.5.4 (The Maurey-Khinchin Inequality) *Let $1 \leq q < \infty$. If E is a q-concave Banach lattice, then for each $0 < p < \infty$, there are constants $C_1, C_{p,q} > 0$ so that, for every choice of $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in E$, we have*

$$C_1 \left\| \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \right\| \leq \left\| \sum_{i=1}^{n} r_i \otimes x_i \right\|_{\Delta_p} \leq C_{p,q} \left\| \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \right\|.$$

Here, (r_i) denotes the sequence of Rademacher functions.

It can be shown that the above inequality holds in a Banach lattice E if and only if E has finite concavity (or cotype). We can now show that any unconditional K-m.d.s. in a Banach lattice with finite concavity has random positive equivalence.

Proposition 6.5.5 *Let E be a Banach lattice with concavity $q < \infty$. Then every unconditional K-m.d.s. in E has random positive equivalence. In fact, we have $L^2(\mu, Y) \supset (r_i \otimes d_i) \approx (d_i) \approx (r_i \otimes |d_i|) \subset L^2(\mu, Y)$.*

Proof. Let (r_i) denote the Rademacher functions and suppose that (d_i) is an unconditional K-m.d.s. in E with unconditional constant M. The Maurey-Khinchin inequality, for $p = 2$, yields

$$C_1 \left\| \left(\sum_{i=1}^n |\alpha_i d_i|^2 \right)^{1/2} \right\| \leq \left\| \sum_{i=1}^n \alpha_i (r_i \otimes d_i) \right\|_{\Delta_2} \leq C_{2,q} \left\| \left(\sum_{i=1}^n |\alpha_i d_i|^2 \right)^{1/2} \right\|$$

and

$$C_1 \left\| \left(\sum_{i=1}^n |\alpha_i d_i|^2 \right)^{1/2} \right\| \leq \left\| \sum_{i=1}^n \alpha_i (r_i \otimes |d_i|) \right\|_{\Delta_2} \leq C_{2,q} \left\| \left(\sum_{i=1}^n |\alpha_i d_i|^2 \right)^{1/2} \right\|$$

for all scalars $\alpha_1, \ldots, \alpha_n$. Consequently, by Proposition 6.3.2(a), we have $(r_i \otimes d_i) \approx (r_i \otimes |d_i|)$. Moreover, using the unconditionality of (d_i), we obtain

$$C_1 M^{-1} \left\| \left(\sum_{i=1}^n |\alpha_i d_i|^2 \right)^{1/2} \right\| \leq \left\| \sum_{i=1}^n \alpha_i d_i \right\| \leq M C_{2,q} \left\| \left(\sum_{i=1}^n |\alpha_i d_i|^2 \right)^{1/2} \right\|$$

for all scalars $\alpha_1, \ldots, \alpha_n$. Thus, $L^2(\mu, Y) \supset (r_i \otimes d_i) \approx (d_i) \approx (r_i \otimes |d_i|) \subset L^2(\mu, Y)$ by Proposition 6.3.2(a), and the proof is complete. □

The spaces $L^p(\mu)$ have type $\min\{2, p\}$ and cotype $\max\{2, p\}$ for each $1 \leq p < \infty$ (cf. [4, pp. 126–127] or [72]). Moreover, Burkholder (and Gundy) showed in [13, Theorem 9] that every classical m.d.s. in $L^p(\mu)$, for $1 < p < \infty$, is unconditional. Thus, every classical m.d.s. in $L^p(\mu)$ has random positive equivalence provided $1 < p < \infty$.

Theorem 6.5.6 *Let E be a Banach lattice with cotype $q < \infty$ and $(d_i) \subset E$ an unconditional K-m.d.s. . Then, the map $R : A^{(d_i)} \to \ell^q(E)$, defined by $R(e_i) = e_i \otimes d_i$ for each $i \in \mathbb{N}$, is regular and order continuous.*

Proof. Since E has finite cotype, it also has finite concavity. Consequently, Proposition 6.5.5 implies that (d_i) has random positive equivalence so that $A^{(d_i)} = A^{(r_i \otimes d_i)} = A^{(r_i \otimes |d_i|)}$ (as sets). Moreover, Proposition 6.5.5 implies

$$L^2(\mu, E) \supset (r_i \otimes d_i) \approx (d_i) \approx (r_i \otimes |d_i|) \subset L^2(\mu, E).$$

Consequently, $A^{(d_i)}$, $A^{(r_i \otimes d_i)}$, and $A^{(r_i \otimes |d_i|)}$ are all isomorphic to each other. Moreover, since E is of cotype $q < \infty$, it follows from Proposition 6.3.2(a) that

$$L^2(\mu, E) \supset (r_i \otimes d_i) \succ (e_i \otimes d_i) \subset \ell^q(E)$$

and

$$L^2(\mu, E) \supset (r_i \otimes |d_i|) \succ (e_i \otimes |d_i|) \subset \ell^q(E).$$

Consequently, the maps $R_1, R_2 : A^{(d_i)} \to \ell^q(E)$, defined by

$$R_1((\alpha_i)) = \frac{1}{2}\left(\sum_{i=1}^{\infty} \alpha_i e_i \otimes |d_i| + \sum_{i=1}^{\infty} \alpha_i e_i \otimes d_i\right)$$

and

$$R_2((\alpha_i)) = \frac{1}{2}\left(\sum_{i=1}^{\infty} \alpha_i e_i \otimes |d_i| - \sum_{i=1}^{\infty} \alpha_i e_i \otimes d_i\right)$$

for all $(\alpha_i) \in A^{(d_i)}$, are well defined, linear and bounded.

It follows from $R_1((\alpha_i)) = \sum_{i=1}^{\infty} \alpha_i e_i \otimes (d_i^+)$, $R_2((\alpha_i)) = \sum_{i=1}^{\infty} \alpha_i e_i \otimes (d_i^-)$ and

$$(R_1 - R_2)((\alpha_i)) = \sum_{i=1}^{\infty} \alpha_i e_i \otimes (d_i^+) - \sum_{i=1}^{\infty} \alpha_i e_i \otimes (d_i^-) = \sum_{i=1}^{\infty} \alpha_i e_i \otimes d_i = R((\alpha_i))$$

for each $(\alpha_i) \in A^{(d_i)}$, that $R = R_1 - R_2$, where R_1 and R_2 are positive; i.e. R is regular.

Since Theorem 6.3.8 implies that $A^{(d_i)}$ has order continuous norm, the regularity of R implies that R is order continuous. $\square$

To apply the above result, we consider the l-tensor product of unconditional K-m.d.s. ' s.

6.6 The *l*-tensor product of unconditional martingale difference sequences

If (ξ_i) is an unconditional K_1-m.d.s. in a Banach lattice E, relative to a positive K-BS-filtration, and (η_j) is an unconditional K_2-m.d.s. in a Banach space Y, then it does not follow that $(\xi_i \otimes \eta_j)$ is an unconditional K-m.d.s. in $E\widetilde{\otimes}_l Y$. Indeed, let $E = L^p(\mu)$, (χ_i) denote the Haar system in E, $Y = \ell_1$ and (e_j) denote the unit vector basis in Y. Note that (χ_i) is an unconditional m.d.s. in E relative to a positive BS-filtration and (e_j) is an unconditional m.d.s. in Y. By the result of Aldous in [1, Theorem 1], Y is super reflexive if $L^p(\mu, Y)$ possesses an unconditional basis. This fact and Corollary 6.4.4 imply that if $(\chi_i \otimes e_j)$ were an unconditional K-m.d.s. , then ℓ_1 would be reflexive, which is certainly not the case. In sight of this, we pursue a weaker result.

Recall that if E and F are Banach lattices. We denote the projective cone of $E \otimes F$ by

$$E_+ \otimes F_+ := \left\{ \sum_{i=1}^{n} x_i \otimes y_i \ : \ (x_i, y_i) \in E_+ \times F_+ \right\}.$$

Moreover, Theorem 2.4.6 asserts that $E\widetilde{\otimes}_l F$ is a Banach lattice and that the positive cone of $E\widetilde{\otimes}_l F$ is the l-closure of the projective cone $E_+ \otimes F_+$. Also, the theorem of Popa (Theorem 2.5.2) implies that $E\widetilde{\otimes}_l F$ is an order continuous Banach lattice if E and F are order continuous Banach lattices. We use these results in the following proposition.

Proposition 6.6.1 *Let X and Y be Banach spaces. If $(\xi_i) \subset X$ and $(\eta_j) \subset Y$ are both unconditional K-m.d.s. 's, then $(e_i \otimes e_j)$ is an unconditional basis of $A^{(\xi_i)}\widetilde{\otimes}_l A^{(\eta_j)}$ with unconditional constant one.*

Proof. By Lemma 6.3.7, we have that (e_i) and (e_j) are unconditional bases of the respective Banach lattices $A^{(\xi_i)}$ and $A^{(\eta_j)}$. It follows from the above remarks that $A^{(\xi_i)}\widetilde{\otimes}_l A^{(\eta_j)}$ is a Banach lattice with positive cone the l-closure of the projective cone $A_+^{(\xi_i)} \otimes A_+^{(\eta_j)}$. Corollary 6.4.3 implies that the sequence $(e_i \otimes e_j)$ with the square ordering is a basis of $A^{(\xi_i)}\widetilde{\otimes}_l A^{(\eta_j)}$. We claim that

$$(A^{(\xi_i)}\widetilde{\otimes}_l A^{(\eta_j)})_+ = \left\{ \sum_{i,j \in \mathbb{N}} \alpha_{ij}(e_i \otimes e_j) : \alpha_{ij} \geq 0 \text{ for each } i, j \in \mathbb{N} \right\}. \tag{6.2}$$

Indeed, it is clear that $\alpha_{ij} \geq 0$ for each $i, j \in \mathbb{N}$ implies that $\sum_{i,j\in\mathbb{N}} \alpha_{ij}(e_i \otimes e_j) \geq 0$. Conversely, suppose $\sum_{i,j\in\mathbb{N}} \alpha_{ij}(e_i \otimes e_j) \geq 0$, we wish to show that $\alpha_{ij} \geq 0$ for each $i, j \in \mathbb{N}$. Let $i, j, r, s \in \mathbb{N}$ and assume $j \neq s$. Using the fact that $\otimes$ is a Riesz bimorphism, i.e. $|x \otimes y| = |x| \otimes |y|$ for all $(x, y) \in E \times F$ (cf. [47] and [69]), we deduce $(e_i \otimes e_j) \wedge (e_r \otimes e_s) = 0$ from the mutual disjointness of $(e_j) \subset A^{(\eta_j)}$. Similarly, if $i \neq r$, then $(e_i \otimes e_j) \wedge (e_r \otimes e_s) = 0$ follows from the mutual disjointness of $(e_i) \subset A^{(\xi_i)}$. Thus, $(e_i \otimes e_j)$ is a mutually disjoint set so that

$$\sum_{i,j\in\mathbb{N}} \alpha_{ij}(e_i \otimes e_j) = \left| \sum_{i,j\in\mathbb{N}} \alpha_{ij}(e_i \otimes e_j) \right| = \sum_{i,j\in\mathbb{N}} |\alpha_{ij}|(e_i \otimes e_j),$$

giving $\alpha_{ij} \geq 0$ for each $i, j \in \mathbb{N}$. This proves (6.2).

By Theorem 6.3.8 we have that $A^{(\xi_i)}$ and $A^{(\eta_j)}$ are order continuous Banach lattices. Thus, by the theorem of Popa, $A^{(\xi_i)}\widetilde{\otimes}_l A^{(\eta_j)}$ is also an order continuous Banach lattice. It now follows from (6.2) and Corollary 6.3.9 that $(e_i \otimes e_j)$ is an unconditional basis of $A^{(\xi_i)}\widetilde{\otimes}_l A^{(\eta_j)}$.

To see that the unconditional constant of $(e_i \otimes e_j)$ is one, let $\theta = (\theta_{ij})$ be any choice of signs and $\sum_{i,j\in\mathbb{N}} \alpha_{ij}(e_i \otimes e_j) \in A^{(\xi_i)}\widetilde{\otimes}_l A^{(\eta_j)}$. Then, using the fact that $\|\cdot\|_l$ is a Riesz norm on $A^{(\xi_i)}\widetilde{\otimes}_l A^{(\eta_j)}$, we obtain

$$\left\| M_\theta \left(\sum_{i,j\in\mathbb{N}} \alpha_{ij}(e_i \otimes e_j) \right) \right\|_l = \left\| \sum_{i,j\in\mathbb{N}} \theta_{ij}\alpha_{ij}(e_i \otimes e_j) \right\|_l$$
$$= \left\| \sum_{i,j\in\mathbb{N}} |\alpha_{ij}|(e_i \otimes e_j) \right\|_l = \left\| \sum_{i,j\in\mathbb{N}} \alpha_{ij}(e_i \otimes e_j) \right\|_l .$$

Hence, $\{M_\theta\}$ is uniformly bounded by one. This completes the proof. □

For the unit vector basis $(e_i) \subset \ell^p$, we have $A^{(e_i)} = \ell^p$ for all $1 \le p < \infty$. Thus, we have the following corollary.

Corollary 6.6.2 *Let Y be a Banach space and $(y_j) \subset Y$ be an unconditional basis (basic sequence). Then $(e_i \otimes y_j)$ is an unconditional basis (basic sequence) of $\ell^p(Y)$ for all $1 \le p < \infty$.*

Since the spaces $\ell^2 \widetilde{\otimes}_\pi \ell^2$ and $\ell^2 \widetilde{\otimes}_\varepsilon \ell^2$ are not Banach lattices, we obtain the following negative result, found in [49]:

Corollary 6.6.3 *The sequence $(e_i \otimes e_j)$ is not an unconditional basis of $\ell^2 \widetilde{\otimes}_\pi \ell^2$ nor $\ell^2 \widetilde{\otimes}_\varepsilon \ell^2$.*

From the above discussion, it follows that the basis $(e_i \otimes e_j) \subset A^{(\chi_i)} \widetilde{\otimes}_l \ell^1$ is not equivalent to the basis $(\chi_i \otimes e_j) \subset L^p(\mu, \ell^1)$. The reason is that the order structure of $L^p(\mu)$ differs from the order structure of $A^{(\chi_i)}$, and plays a large role in calculating the l-norm. This leaves us with a question: What is the relationship between $A^{(\chi_i)} \widetilde{\otimes}_l \ell^1$ and $L^p(\mu, \ell^1)$? In the results that follow, we will provide some partial answers.

Theorem 6.6.4 *Let E be a Banach lattice with cotype $q < \infty$ and Y a Banach space. Assume $(\xi_i) \subset E$ is an unconditional K_1-m.d.s. , then for any sequence $(\eta_j) \subset Y$ we have*

$$A^{(\xi_i)} \widetilde{\otimes}_l A^{(\eta_j)} \supset (e_i \otimes e_j) \succ ((e_i \otimes \xi_i) \otimes \eta_j) \subset \ell^q(E) \widetilde{\otimes}_l Y.$$

Moreover, if (ξ_i) is an unconditional K_1-m.d.s. relative to a positive K_1-BS-filtration and $(\eta_j) \subset Y$ is an unconditional K_2-m.d.s. , then $(e_i \otimes e_j)$ is an unconditional basis of $A^{(\xi_i)} \widetilde{\otimes}_l A^{(\eta_j)}$ and $((e_i \otimes \xi_i) \otimes \eta_j) \subset \ell^q(E) \widetilde{\otimes}_l Y$ is an unconditional K-m.d.s. .

Proof. Let $(\eta_j) \subset Y$ be a sequence. Let S denote the linear map from $A^{(\xi_i)}$ into $\ell^q(E)$ with $S(e_i) = e_i \otimes \xi_i$ for each $i \in \mathbb{N}$ and T denote the co-ordinate map from $A^{(\eta_j)}$ into Y. Note that T has norm one. By Theorem 6.5.6, we have that S is bounded and regular. Thus $S = S_1 - S_2$ where S_1 and S_2 are positive maps. Since the l-norm is a left order uniform crossnorm, the map

$$S \otimes T : A^{(\xi_i)} \otimes_l A^{(\eta_j)} \to \ell^q(E) \otimes_l Y$$

is bounded because $\|S \otimes T\| = \|S_1 \otimes T - S_2 \otimes T\| \leq \|S_1\|\|T\| + \|S_2\|\|T\| = \|S_1\| + \|S_2\|$. Thus, the unique continuous extension

$$S \otimes_l T : A^{(\xi_i)} \widetilde{\otimes}_l A^{(\eta_j)} \to \ell^q(E) \widetilde{\otimes}_l Y \tag{6.3}$$

has the properties $(S \otimes_l T)(e_i \otimes e_j) = (e_i \otimes \xi_i) \otimes \eta_j$ for each $i, j \in \mathbb{N}$ and

$$\|S \otimes_l T\| \leq \|S_1\| + \|S_2\| := K_S < \infty. \tag{6.4}$$

This shows that $(e_i \otimes e_j) \succ ((e_i \otimes \xi_i) \otimes \eta_j)$.

To complete the proof, assume $(\eta_j) \subset Y$ is an unconditional K_2-m.d.s. , $(\xi_i) \subset E$ is an unconditional K_1-m.d.s. relative to a positive K_1-BS-filtration (T_i) and let (P_n) denote the natural projections associated with the basis $(e_i) \subset \ell^q$. Then, Theorem 3.4.4 implies that $(P_i \otimes T_i)$ is a positive K_1-BS-filtration on $\ell^q(E)$. It is evident that $(e_i \otimes \xi_i) \subset \ell^q(E)$ is a K_1-m.d.s. relative $(P_i \otimes_l T_i)$. Hence, Proposition 6.4.2 implies that $((e_i \otimes \xi_i) \otimes \eta_j)$, with the square ordering, is a K-m.d.s. in $\ell^q(E)\widetilde{\otimes}_l Y$. To prove unconditionality, observe that

$$\ell^q(E)\widetilde{\otimes}_l Y \supset ((e_i \otimes \xi_i) \otimes \eta_j) \approx ((e_i \otimes \xi_i) \otimes e_j) \subset \ell^q(E)\widetilde{\otimes}_l A^{(\eta_j)},$$

and $(e_i \otimes \xi_i) \otimes e_j)$ is a mutually disjoint sequence in $\ell^q(E)\widetilde{\otimes}_l A^{(\eta_j)}$.

Lastly, Proposition 6.6.1 implies that $(e_i \otimes e_j)$ is an unconditional basis of $A^{(d_i)}\widetilde{\otimes}_l A^{(y_j)}$. $\square$

Corollary 6.6.5 *Let $1 < p < \infty$ and (d_i) be a classical m.d.s. in $L^p(\mu)$ and $(y_j) \subset Y$ be a basic sequence. Then there exists a constant $K > 0$ for which*

$$\left(\sum_{i \in \mathbb{N}} \left(\|d_i\|_p \left\| \sum_{j \in \mathbb{N}} \alpha_{ij} y_j \right\| \right)^{\max\{2,p\}} \right)^{1/\max\{2,p\}} \leq K \left\| \sum_{i \in \mathbb{N}} \left\| \sum_{j \in \mathbb{N}} \alpha_{ij} y_j \right\| d_i \right\|_p$$

holds for any choice of finitely supported scalars (α_{ij}).

Proof. Since (d_i) is a classical m.d.s. in $L^p(\mu)$, it follows that (d_i) is an unconditional m.d.s. relative to a positive BS-filtration. Also, recall that $L^p(\mu)$ is a Banach lattice with cotype $\max\{2, p\}$. Hence, the conditions of the above theorem are satisfied and we have

$$A^{(d_i)}\widetilde{\otimes}_l A^{(y_j)} \supset (e_i \otimes e_j) \succ ((e_i \otimes d_i) \otimes y_j) \subset \ell^{\max\{2,p\}}(L^p(\mu))\widetilde{\otimes}_l Y.$$

More precisely, let S denote the map from $A^{(d_i)}$ into $\ell^{\max\{2,p\}}(L^p(\mu))$ defined by $e_i \mapsto e_i \otimes d_i$ for each $i \in \mathbb{N}$, T denote the co-ordinate map from $A^{(y_j)}$ into Y and U denote the co-ordinate map from $A^{(d_i)}$ into E. By the above theorem, the map $S \otimes_l T$ given by (6.3) is bounded. Note that $\|S \otimes_l T\|$ is less than the constant $K_S > 0$ defined in (6.4). Thus,

$$\left\| \sum_{i,j \in \mathbb{N}} \alpha_{ij}((e_i \otimes d_i) \otimes y_j) \right\|_l \leq K_S \left\| \sum_{i,j \in \mathbb{N}} \alpha_{ij}(e_i \otimes e_j) \right\|_l \tag{6.5}$$

for every choice of finitely supported scalars (α_{ij}). The result now follows from calculating the l-norm in the above inequality. Indeed, let $n = \min\{k \in \mathbb{N} : (i,j) \in S_k \ \forall\ \alpha_{ij} \neq 0\}$, then we may write

$$\sum_{(i,j) \in S_n} \alpha_{ij}((e_i \otimes d_i) \otimes y_j) = \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_{ij}((e_i \otimes d_i) \otimes y_j),$$

where $m = \max(S_n \setminus S_{n-1})$ and $\alpha_{ij} = 0$ for each $(i,j) \in S_{m^2} \setminus S_n$. On the left hand side of inequality (6.5) we have, by Theorem 2.3.1(b) and the mutual disjointness of $(e_i \otimes d_i)$, that

$$\begin{aligned}
&\left\| \sum_{(i,j) \in S_n} \alpha_{ij}((e_i \otimes d_i) \otimes y_j) \right\|_l = \left\| \sum_{i=1}^{m} (e_i \otimes d_i) \otimes \left(\sum_{j=1}^{m} \alpha_{ij} y_j \right) \right\|_l \\
&= \left\| \sup \left\{ \left| \sum_{i=1}^{m} \left\langle \sum_{j=1}^{m} \alpha_{ij} y_j, y^* \right\rangle (e_i \otimes d_i) \right| : \|y^*\| \leq 1 \right\} \right\|_{\ell^{\max\{2,p\}}(L^p(\mu))} \\
&= \left\| \sum_{i=1}^{m} \left\| \sum_{j=1}^{m} \alpha_{ij} y_j \right\| (e_i \otimes |d_i|) \right\|_{\ell^{\max\{2,p\}}(L^p(\mu))} \\
&= \left(\sum_{i=1}^{m} \left(\|d_i\|_p \left\| \sum_{j=1}^{m} \alpha_{ij} y_j \right\| \right)^{\max\{2,p\}} \right)^{1/\max\{2,p\}} .
\end{aligned}$$

Similarly, by Theorem 2.3.1(b) and the positive mutual disjointness of (e_i) in $A^{(d_i)}$, we obtain from the right hand side of inequality (6.5)

$$\begin{aligned}
&K_S \left\| \sum_{(i,j) \in S_n} \alpha_{ij}(e_i \otimes e_j) \right\|_l = K_S \left\| \sum_{i=1}^{m} e_i \otimes \left(\sum_{j=1}^{m} \alpha_{ij} e_j \right) \right\|_l \\
&= K_S \left\| \sup \left\{ \left| \sum_{i=1}^{m} \left\langle \sum_{j=1}^{m} \alpha_{ij} e_j, y^* \right\rangle e_i \right| : \|y^*\| \leq 1 \right\} \right\|_{A^{(d_i)}} \\
&= K_S \left\| \sum_{i=1}^{m} \left\| \sum_{j=1}^{m} \alpha_{ij} e_j \right\|_{A^{(y_j)}} e_i \right\|_{A^{(d_i)}} \leq K_S \|U^{-1}\| \|T^{-1}\| \left\| \sum_{i=1}^{m} \left\| \sum_{j=1}^{m} \alpha_{ij} y_j \right\| d_i \right\|_p .
\end{aligned}$$

Setting $K = K_S\|U^{-1}\|\|T^{-1}\|$ completes the proof. $\square$

Corollary 6.6.6 *Let $1 < p < \infty$ and (d_i) be a classical m.d.s. in $L^p(\mu)$. Then there exists a constant $K > 0$ for which*

$$\left(\sum_{i\in\mathbb{N}}\left(\|d_i\|_p\left\|\sum_{j\in\mathbb{N}}\alpha_{ij}y_j\right\|\right)^{\max\{2,p\}}\right)^{1/\max\{2,p\}} \leq K\left\|\sum_{i,j\in\mathbb{N}}\|\alpha_{ij}y_j\|\, d_i\right\|_p$$

holds for any sequence $(y_i) \subset Y$ and choice of finitely supported scalars (α_{ij}).

Proof. Let $(y_i) \subset Y$ denote any sequence. Using the same notation as in the proof of Theorem 6.6.5, there is a constant $K_S > 0$ given by (6.4) such that (6.5) holds for every choice of finitely supported scalars (α_{ij}). Since K_S depends only on the map S and not on the map T, a moment's reflection reveals that K_S is independent of the sequence $(y_i) \subset Y$. Thus, (6.5) holds for any sequence $(y_i) \subset Y$ and choice of finitely supported scalars (α_{ij}). As in the proof of the above theorem, the result now follows from calculating the l-norm in inequality (6.5), but with one small difference. Indeed, the left hand side of (6.5) is calculated in the same manner as in Theorem 6.6.5. For the right hand side of (6.5), use the positive mutual disjointness of (e_i) in $A^{(d_i)}$ to obtain

$$\begin{aligned}
&K_S\left\|\sum_{i,j\in\mathbb{N}}\alpha_{ij}(e_i\otimes e_j)\right\|_l = K_S\left\|\sum_{j=1}^m\left(\sum_{i=1}^m\alpha_{ij}e_i\right)\otimes e_j\right\|_l \\
&\leq K_S\left\|\sum_{j=1}^m\left|\sum_{i=1}^m\alpha_{ij}e_i\right|\|e_j\|_{A^{(y_j)}}\right\|_{A^{(d_i)}} = K_S\left\|\sum_{j=1}^m\left(\sum_{i=1}^m|\alpha_{ij}|e_i\right)\|y_j\|\right\|_{A^{(d_i)}} \\
&= K_S\left\|\sum_{i=1}^m\left(\sum_{j=1}^m\|\alpha_{ij}y_j\|\right)e_i\right\|_{A^{(d_i)}} \leq K_S\|U^{-1}\|\left\|\sum_{i=1}^m\left(\sum_{j=1}^m\|\alpha_{ij}y_j\|\right)d_i\right\|_p \\
&= K_S\|U^{-1}\|\left\|\sum_{i,j\in\mathbb{N}}\|\alpha_{ij}y_j\|d_i\right\|_p.
\end{aligned}$$

Since the constant $K_S\|U^{-1}\|$ is independent of the choice of sequence (y_i) and scalars (α_{ij}), setting $K = K_S\|U^{-1}\|$ completes the proof. $\square$

In particular, suppose $1 < p < \infty$ and $(d_i) \subset L^p(\mu)$ is a classical m.d.s. . Then there is a constant $K > 0$ such that

$$\left(\sum_{i=1}^n(\|y_i\|\|d_i\|_p)^{\max\{2,p\}}\right)^{1/\max\{2,p\}} \leq K\left\|\sum_{i=1}^n\|y_i\|d_i\right\|_p$$

holds for every choice of finite sequence $y_1, y_2, \ldots, y_n$. To access a similar, reverse inequality we resort to a duality argument.

Definition 6.6.7 Let Y be a Banach space and $(x_i) \subset Y$ be a basis.

(a) The functionals $(x_i^*) \subset Y^*$ defined by $\langle x_i, x_j^* \rangle = \delta_{ij}$ for each $i, j \in \mathbb{N}$ are said to be *biorthogonal* to (x_i).
(b) The basis (x_i) is said to be *shrinking* if (x_i^*) is a basis of Y^*.
(c) The basis (x_i) is said to be *boundedly complete* if for every sequence of scalars (α_i) such that $\sup_{n\in\mathbb{N}} \|\sum_{i=1}^n \alpha_i x_i\| < \infty$, the series $\sum_{n=1}^\infty \alpha_i x_i$ converges.

Observe that if (x_i) is a basis of a Banach space Y, then the functionals (x_i^*) biorthogonal to (x_i) form a K-m.d.s. relative to the dual filtration (T_i^*), where K is the basis constant of (x_i). Although (x_i^*) is a basic sequence, (x_i^*) need not be a basis of Y^*. Indeed, in order for (x_i^*) to be a basis of Y^*, it is necessary for Y^* to be separable. So, if $Y = \ell^1$, then the functionals biorthogonal to (e_i) are not a basis of ℓ^∞.

It can be shown that if (x_i) is a shrinking basis of the Banach space Y, then the functionals (x_i^*), biorthogonal to (x_i), form a boundedly complete basis of Y^*. Moreover, if (x_i) is a boundedly complete basis of Y, then Y is the dual of a Banach space with a shrinking basis and (x_i) is biorthogonal to this basis (cf. [71, Proposition 1.b.4]). Combining the notions of shrinking and boundedly complete, one can show: A Banach space Y with basis (x_i) is reflexive if and only if (x_i) is both shrinking and boundedly complete (cf. [71, Theorem 1.b.5]).

Proposition 6.6.8 *Let Y be a Banach space, $(x_i) \subset Y$ a shrinking unconditional basis and $(x_i^*) \subset Y^*$ denote the functionals biorthogonal to (x_i). Then*

$$A^{(x_i^*)} \supset (e_i) \approx (e_i^*) \subset (A^{(x_i)})^*,$$

where $(e_i^*) \subset (A^{(x_i)})^*$ denotes the functionals biorthogonal to the unconditional basis $(e_i) \subset A^{(x_i)}$. Moreover, the isomorphism $A^{(x_i^*)} \to (A^{(x_i)})^*$, defined by $e_i \mapsto e_i^*$ for each $i \in \mathbb{N}$, is a surjective Riesz isomorphism between Banach lattices.

Proof. Let S denote the co-ordinate map from $A^{(x_i)}$ onto Y and consider its adjoint $S^* : Y^* \to (A^{(x_i)})^*$. Then

$$\langle e_j, S^* x_i^* \rangle = \langle S e_j, x_i^* \rangle = \langle x_j, x_i^* \rangle = \delta_{ij} = \langle e_j, e_i^* \rangle$$

for each $i, j \in \mathbb{N}$ shows that $S^* x_i^* = e_i^*$ for each $i \in \mathbb{N}$. Thus, $Y^* \supset (x_i^*) \succ (e_i^*) \subset (A^{(x_i)})^*$. Similarly, the bounded linear map $(S^{-1})^* : (A^{(x_i)})^* \to Y^*$ has the property $(S^{-1})^* e_i^* = x_i^*$ for each $i \in \mathbb{N}$. Consequently, $Y^* \supset (x_i^*) \approx (e_i^*) \subset (A^{(x_i)})^*$. Thus, $A^{(x_i^*)} \supset (e_i) \approx (x_i^*) \approx (e_i^*) \subset (A^{(x_i)})^*$.

Since (x_i) is unconditional, (x_i^*) is an unconditional K-m.d.s. in Y^*. Consequently, $A^{(x_i^*)}$ and $(A^{(x_i)})^*$ are Banach lattices by Theorem 6.3.8. Let $T : A^{(x_i^*)} \to (A^{(x_i)})^*$ be the isomorphism onto $[e_i^*]$ defined by $Te_i = e_i^*$ for each $i \in \mathbb{N}$.

We now show that T is a Riesz isomorphism between Banach lattices. It is evident that $0 \leq \sum_{i=1}^{\infty} \alpha_i e_i^* \in (A^{(x_i)})^*$ when $\alpha_i \geq 0$ for each $i \in \mathbb{N}$, since $e_i^* \geq 0$ for each $i \in \mathbb{N}$. Conversely, let $0 \leq f^* \in (A^{(x_i)})^*$. Then

$$\langle x, f^* \rangle = \left\langle \sum_{i=1}^{\infty} \alpha_i e_i, f^* \right\rangle = \left\langle x, \sum_{i=1}^{\infty} f^*(e_i) e_i^* \right\rangle$$

for each $x = \sum_{i=1}^{\infty} \alpha_i e_i \in A^{(x_i)}$. Thus, $f^* = \sum_{i=1}^{\infty} f^*(e_i) e_i^*$ converges in the weak* topology, with $f^*(e_i) \geq 0$ for each $i \in \mathbb{N}$. Since (x_i) is a shrinking basis, so is $(e_i) \subset A^{(x_i)}$. Consequently, $[e_i^*] = (A^{(x_i)})^*$ and $f^* = \sum_{i=1}^{\infty} f^*(e_i) e_i^*$ converges in the norm topology. It follows that $0 \leq \sum_{i=1}^{\infty} \alpha_i e_i^* \in (A^{(x_i)})^*$ if and only if $\alpha_i \geq 0$ for each $i \in \mathbb{N}$. Thus, T is a Riesz isomorphism from $A^{(x_i^*)}$ onto $(A^{(x_i)})^*$. $\square$

Theorem 6.6.9 *Assume E is a Banach lattice with type p and cotype q where $1 < p \leq 2 \leq q < \infty$. Let $(\xi_i) \subset E$ be a boundedly complete unconditional basis and Y a Banach space. Then, for any basic sequence $(\eta_j) \subset Y$, we have*

$$\ell^p(E)\widetilde{\otimes}_l Y \supset ((e_i \otimes \xi_i) \otimes \eta_j) \succ (e_i \otimes e_j) \succ ((e_i \otimes \xi_i) \otimes \eta_j) \subset \ell^q(E)\widetilde{\otimes}_l Y,$$

where $(e_i \otimes e_j)$ is the basis of $A^{(\xi_i)}\widetilde{\otimes}_l A^{(\eta_j)}$. In particular, if $p = q = 2$, then

$$A^{(\xi_i)}\widetilde{\otimes}_l A^{(y_j)} \supset (e_i \otimes e_j) \approx ((e_i \otimes \xi_i) \otimes \eta_j) \subset \ell^2(E)\widetilde{\otimes}_l Y.$$

Proof. Since $(\xi_i) \subset E$ is a boundedly complete basis, E has a predual F with a shrinking basis (b_i) so that (ξ_i) is biorthogonal to (b_i). Since E has type p, it follows that F^{**} and F have cotype p^* where $\frac{1}{p} + \frac{1}{p^*} = 1$.

Let S denote the linear map from $A^{(b_i)}$ into $\ell^{p^*}(F^{**})$ with $S(e_i) = e_i \otimes b_i$ for each $i \in \mathbb{N}$. By Theorem 6.5.6, we have that S is bounded and regular. Thus, $S = S_1 - S_2$ where S_1 and S_2 are positive maps. Consequently, $S^*|_{\ell^p(E)} : \ell^p(E) \to (A^{(b_i)})^*$ is regular, because $S^* = (S_1 - S_2)^* = S_1^* - S_2^*$ where S_1^* and S_2^* are positive. Since

$$\langle e_i, S^*(e_j \otimes \xi_j) \rangle = \langle Se_i, e_j \otimes \xi_j \rangle = \langle e_i \otimes b_i, e_j \otimes \xi_j \rangle = \delta_{ij} = \langle e_i, e_j^* \rangle$$

for all $i, j \in \mathbb{N}$, we have $S^*|_{\ell^p(E)}(e_i \otimes \xi_i) = e_i^*$ for each $i \in \mathbb{N}$, where $(e_i^*) \subset (A^{(b_i)})^*$ are the functionals biorthogonal to the basis $(e_i) \subset A^{(b_i)}$. Proposition 6.6.8 implies the existence of a positive bounded linear map $U : (A^{(b_i)})^* \to A^{(\xi_i)}$ with $Ue_i^* = e_i$ for each $i \in \mathbb{N}$. Consequently, the map $U \circ S^*|_{\ell^p(E)} : \ell^p(E) \to A^{(\xi_i)}$ is regular with $(U \circ S^*|_{\ell^p(E)})(e_i \otimes \xi_i) = e_i$ for each $i \in \mathbb{N}$.

Now choose any basic sequence $(\eta_j) \subset Y$ and let T denote the co-ordinate map from $A^{(\eta_j)}$ onto $[\eta_j]$. Since the l-norm is a left order uniform crossnorm, the map

$$(U \circ S^*|_{\ell^p(E)}) \otimes T^{-1} : \ell^p(E) \otimes_l [\eta_j] \to A^{(\xi_i)} \otimes_l A^{(\eta_j)}$$

is bounded because

$$\begin{aligned}\|(U \circ S^*|_{\ell^p(E)}) \otimes T^{-1}\| &= \|(U \circ S_1^*|_{\ell^p(E)}) \otimes T^{-1} - (U \circ S_2^*|_{\ell^p(E)}) \otimes T^{-1}\| \\ &\le \|U\|\|T^{-1}\|(\|S_1\| + \|S_2\|).\end{aligned}$$

Thus, the unique continuous extension

$$(U \circ S^*|_{\ell^p(E)}) \otimes_l T^{-1} : \ell^p(E)\widetilde{\otimes}_l[\eta_j] \to A^{(\xi_i)}\widetilde{\otimes}_l A^{(\eta_j)}$$

has the properties $((U \circ S^*|_{\ell^p(E)}) \otimes T^{-1})((e_i \otimes \xi_i) \otimes \eta_j) = (e_i \otimes e_j)$ for each $i, j \in \mathbb{N}$. Since $\ell^p(E)\widetilde{\otimes}_l[\eta_j]$ is a closed subspace of $\ell^p(E)\widetilde{\otimes}_l Y$, this shows that

$$\ell^p(E)\widetilde{\otimes}_l Y \supset ((e_i \otimes \xi_i) \otimes \eta_j) \succ (e_i \otimes e_j) \subset A^{(\xi_i)}\widetilde{\otimes}_l A^{(\eta_i)}.$$

Lastly, since E has cotype q, it follows from Theorem 6.6.4 that $A^{(\xi_i)}\widetilde{\otimes}_l A^{(\eta_j)} \supset (e_i \otimes e_j) \succ ((e_i \otimes \xi_i) \otimes \eta_j) \subset \ell^q(E)\widetilde{\otimes}_l Y$. $\square$

Using the proof of Corollary 6.6.5 and, bearing in mind that $L^p(\mu)$ has type $\min\{2,p\}$ and cotype $\max\{2,p\}$, Theorem 6.6.9 specializes easily to the following corollary.

Corollary 6.6.10 *Let (d_i) be a m.d.s. basis of $L^p(\mu)$ where $1 < p < \infty$. If $(y_j) \subset Y$ is a basic sequence, then there exists a constant $K > 0$ for which*

$$\begin{aligned} K^{-1}\left(\sum_{i\in\mathbb{N}}\left(\|d_i\|_p\left\|\sum_{j\in\mathbb{N}}\alpha_{ij}y_j\right\|\right)^{\max\{2,p\}}\right)^{1/\max\{2,p\}} &\le \left\|\sum_{i\in\mathbb{N}}\left\|\sum_{j\in\mathbb{N}}\alpha_{ij}y_j\right\| d_i\right\|_p \\ \le K\left(\sum_{i\in\mathbb{N}}\left(\|d_i\|_p\left\|\sum_{j\in\mathbb{N}}\alpha_{ij}y_j\right\|\right)^{\min\{2,p\}}\right)^{1/\min\{2,p\}} & \end{aligned}$$

holds for any choice of finitely supported scalars (α_{ij}).

If (d_i) is a normalized m.d.s. basis of $L^p(\mu)$, then the above inequality becomes

$$\begin{aligned} K^{-1}\left(\sum_{i\in\mathbb{N}}\left\|\sum_{j\in\mathbb{N}}\alpha_{ij}y_j\right\|^{\max\{2,p\}}\right)^{1/\max\{2,p\}} &\le \left\|\sum_{i\in\mathbb{N}}\left\|\sum_{j\in\mathbb{N}}\alpha_{ij}y_j\right\| d_i\right\|_p \\ \le K\left(\sum_{i\in\mathbb{N}}\left\|\sum_{j\in\mathbb{N}}\alpha_{ij}y_j\right\|^{\min\{2,p\}}\right)^{1/\min\{2,p\}} &, \end{aligned}$$

which is consistent with inequalities (1.9) and (1.10) in [4, p. 128].

6.7 Notes and remarks

We regret to announce that there is a mathematical error in [29, p. 1300], which adversely effects the Abstract, Theorems 5.2 and 7.2, and Corollaries 7.3 and 7.4 of that paper.

We recall that if X and Y are Banach spaces, a sequence $(x_i) \subset X$ is said to be *equivalent* to a sequence $(y_i) \subset Y$ provided, for all scalar sequences (α_i), we have $\sum_{i=1}^{\infty} \alpha_i x_i$ convergent $\Leftrightarrow \sum_{i=1}^{\infty} \alpha_i y_i$ convergent. If $(x_i) \subset X$ is a basic sequence that is equivalent to a sequence $(y_i) \subset Y$, then there exists a constant K such that $\|\sum_{i=1}^{n} \alpha_i y_i\| \leq K\|\sum_{i=1}^{n} \alpha_i x_i\|$ for any choice of scalars $\alpha_1, \ldots, \alpha_n$ and $n \in \mathbb{N}$ (see Theorem 6.3.2).

We noted in [29, p. 1300] that the following inequality holds: For any unconditional basic sequence $(x_n) \subset L^p(\mu)$ with unconditional constant M, where $1 \leq p < \infty$, there exist constants K_1 and K_p (K_p dependent on p) for which

$$K_1(MK_p)^{-1} \left\| \sum_{i=1}^{n} \alpha_i x_i \right\|_{L^p(\mu)} \leq \int_0^1 \left\| \sum_{i=1}^{n} r_i(t) \alpha_i |x_i| \right\|_{L^p(\mu)} \mathrm{d}t$$
$$\leq K_1^{-1}(MK_p) \left\| \sum_{i=1}^{n} \alpha_i x_i \right\|_{L^p(\mu)}$$

holds for all scalars $\alpha_1, \ldots, \alpha_n$ (where (r_i) denotes the sequence of Rademacher functions). Using the unconditionality of (x_i) and the above inequality, we concluded (incorrectly) that (x_i) is equivalent to $(|x_i|)$. This conclusion arose from our erroneous interpretation of the word ‘equivalent’ used in [4, p. 128]. Our conclusion is easily seen to be true when (x_i) is a mutually disjoint sequence. But, in general, this is hardly the case, as the following simple counterexample shows.

Let $(r_i) \subset L^2(\mu)$ denote the sequence of Rademacher functions. Since (r_i) is equivalent to the unit vector basis of ℓ^2, it follows that (r_i) is an unconditional basic sequence. If our claim were true, then there exists a constant $K > 0$ such that

$$\left\| \sum_{i=1}^{n} \alpha_i |r_i| \right\|_2 \leq K \left\| \sum_{i=1}^{n} \alpha_i r_i \right\|_2$$

for any choice of scalars $\alpha_1, \ldots, \alpha_n$ and $n \in \mathbb{N}$. But then

$$n = \left\| \sum_{i=1}^{n} \mathbf{1} \right\|_2 = \left\| \sum_{i=1}^{n} |r_i| \right\|_2 \leq K \left\| \sum_{i=1}^{n} r_i \right\|_2 = K \left(\sum_{i=1}^{n} \int_0^1 r_i^2(t)\, \mathrm{d}t \right)^{1/2} = Kn^{1/2}$$

for all $n \in \mathbb{N}$, which is absurd.

The above note appears in the corrigendum [27]. The results in Section 6.5 and Section 6.6 serve as a correction to [29].

7

Stopping times

7.1 Introduction

An important notion in probability theory is that of a stopping time (cf. [45, 44]). In this chapter, we add the notion of a stopping time to Troitsky's theory of martingales in Banach lattices (cf. [101]). Bounded stopping times in Riesz spaces have been studied in [62]. Our aim is to extend this theory to unbounded stopping times in the Banach lattice setting. To achieve this, we use the results of Witvliet, studied in Sections 1.5 and 1.6.

In Section 7.2, we recall the classical definitions of a stopping time (adapted to a filtration) and a stopped process. After reviewing some basic properties of these definitions, a generalized definition of a stopping time in a Banach lattice, adapted to a positive BS-filtration, is deduced. Our definition of a stopping time differs slightly from the definition in [62]. We also consider a stopped process in a Banach lattice, with respect to a stopping time adapted to a positive BS-filtration. A natural ordering on the set of all stopping times, adapted to a positive BS-filtration, is discussed. This allows us to consider nets of stopped processes in a Banach lattice.

Section 7.3 contains the connection between stopping times in a Banach lattice and the theory of Schauder decomposition and multipliers. It is shown that a stopping time in an order continuous Banach lattice is an unconditional Schauder decomposition. Thus, using the multiplier theorem of Witvliet (Theorem 1.5.8), we are able to define a stopped martingale in an order continuous Banach lattice, with respect to an unbounded stopping time, adapted to a positive R-bounded BS-filtration. This gives rise to an optional stopping theorem for unbounded stopping times. For positive R-bounded BS-filtrations, this theorem states that a net of stopped martingales in an order continuous Banach lattice, indexed by a directed set of unbounded stopping times, is again a martingale. The section concludes with a characterization of convergent nets of stopped martingales in an order continuous Banach lattice.

The theory of unbounded stopping times is restricted to positive R-bounded filtrations. By the Stein inequality (Corollary 1.6.7), classical filtrations on $L^p(\mu)$ are R-bounded when $1 < p < \infty$.

The theory in Section 7.3 is formulated on an order continuous Banach lattice E. In Section 7.4, we extend this theory to the l-tensor product $E\widetilde{\otimes}_l Y$, where Y is a Banach space. With an optional stopping theorem for $E\widetilde{\otimes}_l Y$ at hand, we are able to characterize convergent nets of stopped martingales in $E\widetilde{\otimes}_l Y$ in a similar fashion to the results in Section 5.3.

Lastly, in Section 7.4, we apply the above techniques to unconditional Schauder decompositions in the Lebesgue-Bochner spaces. Bourgain noted in [9] that if Y is a UMD space with an unconditional basis, then $L^p(\mu, Y)$ has an unconditional basis for $1 < p < \infty$. We generalize this result to stopping times. We show that if Y is a UMD lattice possessing a stopping time, then the Schauder decomposition of $L^p(\mu, Y)$, formed from the product of any martingale decomposition of $L^p(\mu, Y)$ with this stopping time, is unconditional. The results in this chapter are original and have appeared in [30].

7.2 Stopping times in Banach lattices

We first recall some basic definitions and results on stopping times from [44, 45]. Throughout, let (Ω, Σ, μ) denote a finite measure space.

Definition 7.2.1 Let (Σ_i) denote a filtration and $1 \le p < \infty$.

(a) A *stopping time* adapted to (Σ_i) is a map $\tau : \Omega \to \mathbb{N} \cup \{\infty\}$ such that $\tau^{-1}(\{1, ..., i\}) \in \Sigma_i$ for each $i \in \mathbb{N}$. A stopping time τ is said to be *bounded* if there exists $n \in \mathbb{N}$ such that $\tau(\omega) \le n$ almost everywhere on Ω, i.e. up to sets of measure zero, $\tau^{-1}(\{1, ..., n\}) = \Omega$.

(b) If τ is a bounded stopping time and $(f_i) \subset L^p(\mu)$ is a sequence, both adapted to a filtration (Σ_i), then the *stopped process* is the pair (f_τ, Σ_τ) where

$$f_\tau = \sum_i \chi_{\tau^{-1}(\{i\})} f_i \quad \text{and} \quad \Sigma_\tau = \left\{A \subset \Omega : A \cap \tau^{-1}(\{i\}) \in \Sigma_i \ \forall\, i \in \mathbb{N}\right\}.$$

(c) The set of all stopping times adapted to the filtration (Σ_i) is denoted by $\mathbb{T}^*$. The subset of $\mathbb{T}^*$ consisting of all bounded stopping times is denoted $\mathbb{T}$.

We define a partial ordering on $\mathbb{T}^*$ as follows: If $\sigma, \tau \in \mathbb{T}^*$, define $\sigma \le \tau$ if and only if $\sigma(\omega) \le \tau(\omega)$ for almost all $\omega \in \Omega$. It is easily verified that if $\sigma, \tau \in \mathbb{T}$ are (bounded) stopping times, then $\sigma \vee \tau$ and $\sigma \wedge \tau$ are again (bounded) stopping times.

Doob's optional stopping theorem (cf. [86, 100]) asserts that for a martingale (f_i, Σ_i) in $L^p(\mu)$ and $\sigma, \tau \in \mathbb{T}$ with $\sigma \leq \tau$, we have $f_\sigma = \mathbb{E}(f_\tau \mid \Sigma_\sigma)$. Consequently, if $\mathbb{D} \subset \mathbb{T}$ is a directed set, the sequence $(f_\tau, \Sigma_\tau)_{\tau \in \mathbb{D}}$ is a martingale.

There is a correspondence between a bounded stopping time τ adapted to a filtration (Σ_i) and a commuting sequence (P_i) of linear band projections on $L^p(\mu)$, as was established in [62].

Indeed, for $f \in L^p(\mu)$, define the projection $P_i f = f \cdot \chi_{\tau^{-1}(\{1,\ldots,i\})}$ for each $i \in \mathbb{N}$. Then, for $i \leq j$ and $0 \leq f \in L^p(\mu)$, the inequality $0 \leq P_i f \leq P_j f \leq f$ gives $0 \leq P_i \leq P_j \leq \mathrm{id}_{L^p(\mu)}$, which implies (P_i) is an increasing sequence of (contractive) band projections on $L^p(\mu)$.

Also, notice that $\tau^{-1}(\mathbb{N}) = \Omega$. Hence, $\sup_i P_i f = \sup_i \chi_{\tau^{-1}(\{1,\ldots,i\})} f = \chi_\Omega f = f$ for all $0 \leq f \in L^p(\mu)$. Moreover, (P_i) satisfies $P_i P_j = P_{i \wedge j}$. This follows directly from identity

$$\chi_{\tau^{-1}(\{1,\ldots,i\})} \cdot \chi_{\tau^{-1}(\{1,\ldots,j\})} = \chi_{\tau^{-1}(\{1,\ldots,i\}) \cap \tau^{-1}(\{1,\ldots,j\})} = \chi_{\tau^{-1}(\{1,\ldots,i \wedge j\})}.$$

Lastly, since $\tau^{-1}(\{1, \ldots, i\}) \in \Sigma_i$ for each $i \in \mathbb{N}$, it follows from (1.2) that

$$P_i \mathbb{E}(f \mid \Sigma_j) = \chi_{\tau^{-1}(\{1,\ldots,i\})} \mathbb{E}(f \mid \Sigma_j) = \mathbb{E}(\chi_{\tau^{-1}(\{1,\ldots,i\})} f \mid \Sigma_j) = \mathbb{E}(P_i f \mid \Sigma_j)$$

for all $i \leq j$ and $f \in L^p(\mu)$.

Thus, a stopping time τ adapted to a filtration (Σ_i) is an increasing sequence (P_i) of commuting band projections on $L^p(\mu)$ for which $T_j P_i = P_i T_j$ for all $i \leq j$, where each $T_j := \mathbb{E}(\,\cdot \mid \Sigma_j)$. Furthermore, if τ is bounded, then there exists n_0 such that $P_i = \mathrm{id}_{L^p(\mu)}$ for all $i \geq n_0$.

Motivated by the above observations, we formulate a definition for a stopping time on a Banach lattice:

Definition 7.2.2 Let E be a Banach lattice with the projection property and let (T_i) be a positive BS-filtration on E. A *stopping time* on E adapted to (T_i) is a sequence of band projections $\mathcal{P} = (P_i)$ satisfying the following properties:

(a) $P_i P_j = P_{i \wedge j}$ for all $i, j \in \mathbb{N}$,
(b) $T_j P_i = P_i T_j$ for all $i \leq j$, and
(c) $P_i f \uparrow f$ for all $f \in E_+$.

We say that $\mathcal{P}$ is a *bounded stopping time* if there exists an $n_0 \in \mathbb{N}$ such that $P_i = \mathrm{id}_E$ for all $i \geq n_0$. We say $\mathcal{P}$ *stops* at $m \in \mathbb{N}$ if m is the least natural number for which $P_m = \mathrm{id}_E$. For ease of notation, we will always assume $P_0 = 0$.

Note that a stopping time is nothing more than a specialized BL-filtration. Let (f_i) be a sequence adapted to (Σ_i) and consider the stopped process (f_τ, Σ_τ) where τ is a bounded stopping time. For each $i \in \mathbb{N}$ and $A \in \Sigma_\tau$, it follows from (1.2) that

$$\begin{aligned}
&\int_A \chi_{\tau^{-1}(\{i\})}\mathbb{E}(f \,|\, \Sigma_i)\,\mathrm{d}\mu = \int_{A\cap\tau^{-1}(\{i\})} \mathbb{E}(f \,|\, \Sigma_i)\,\mathrm{d}\mu = \int_{A\cap\tau^{-1}(\{i\})} f\,\mathrm{d}\mu \\
&= \int_A \chi_{\tau^{-1}(\{i\})} f\,\mathrm{d}\mu = \int_A \mathbb{E}(\chi_{\tau^{-1}(\{i\})} f \,|\, \Sigma_\tau)\,\mathrm{d}\mu.
\end{aligned}$$

Hence, $\mathbb{E}(\chi_{\tau^{-1}(\{i\})} f \,|\, \Sigma_\tau) = \chi_{\tau^{-1}(\{i\})}\mathbb{E}(f \,|\, \Sigma_i) = \mathbb{E}(\chi_{\tau^{-1}(\{i\})} f \,|\, \Sigma_i)$ almost everywhere. Thus, for each $f \in L^p(\mu)$, we obtain

$$\begin{aligned}
&\mathbb{E}(f \,|\, \Sigma_\tau) = \mathbb{E}\left(\sum_{i=1}^{\infty} \chi_{\tau^{-1}(\{i\})} f \,\middle|\, \Sigma_\tau\right) = \sum_{i=1}^{\infty} \mathbb{E}(\chi_{\tau^{-1}(\{i\})} f \,|\, \Sigma_\tau) \\
&= \sum_{i=1}^{\infty} \chi_{\tau^{-1}(\{i\})}\mathbb{E}(f \,|\, \Sigma_i) = \sum_{i=1}^{\infty} (P_i - P_{i-1})T_i f \quad \text{(see also [62]).}
\end{aligned}$$

This observation leads to a natural definition for a stopped process in a Banach lattice.

Definition 7.2.3 Let E be a Banach lattice with the projection property and let $(f_i) \subset E$ be a sequence adapted to a positive BS-filtration (T_i) on E.

(a) We denote the set of all stopping times adapted to (T_i) by $\mathbb{T}^*$ and the set of all bounded stopping times adapted to (T_i) by $\mathbb{T} \subset \mathbb{T}^*$.

(b) Let $\mathcal{P} \in \mathbb{T}$. We define the *stopped process* to be the pair $(f_\mathcal{P}, T_\mathcal{P})$, where

$$f_\mathcal{P} = \sum_{i=1}^{\infty} (P_i - P_{i-1}) f_i$$

and $T_\mathcal{P}$ is the bounded linear operator defined by

$$T_\mathcal{P} f = \sum_{i=1}^{\infty} (P_i - P_{i-1}) T_i f \quad \text{for all } f \in E.$$

This definition also applies to $\mathcal{P} \in \mathbb{T}^*$, if $(f_\mathcal{P}, T_\mathcal{P})$ exists.

The partial ordering on the set of all stopping times $\mathbb{T}^*$ adapted to some positive BS-filtration translates to the following: If $\mathcal{P} = (P_i)$, $\mathcal{Q} = (Q_i) \in \mathbb{T}^*$, then

$$\mathcal{P} \le \mathcal{Q} \Longleftrightarrow Q_i \le P_i \quad \text{for each } i \in \mathbb{N}.$$

Indeed, if $\tau, \sigma \in \mathbb{T}^*$ with $\tau(\omega) \le \sigma(\omega)$ almost everywhere, then $\{\omega \in \Omega : \sigma(\omega) \le i\} \subset \{\omega \in \Omega : \tau(\omega) \le i\}$, showing that $\chi_{\sigma^{-1}(\{1,\dots,i\})} \le \chi_{\tau^{-1}(\{1,\dots,i\})}$ for all $i \in \mathbb{N}$. Thus, $\mathcal{P} \vee \mathcal{Q} = (P_i Q_i)$ and $\mathcal{P} \wedge \mathcal{Q} = (P_i + Q_i - P_i Q_i)$ (cf. [73, Theorem 30.1]). It is easy to verify that $\mathbb{T}^*$ is a lattice with respect to this partial ordering, and that $\mathbb{T}$ is a sublattice of $\mathbb{T}^*$.

It is obvious that $T_{\mathcal{P}}$ is a positive projection. In the case where $E = L^p(\mu)$, $T_{\mathcal{P}} = \mathbb{E}(\,\cdot \mid \Sigma_\tau)$ is also a contractive projection. This implies that $\sup_{\mathcal{P}\in\mathbb{T}} \|T_{\mathcal{P}}\| = 1$. It is not clear that this property holds in a general Banach lattice. We, therefore, make additional assumptions.

7.3 R-bounded filtrations on Banach lattices

If (T_i) is a positive BS-filtration on a Banach lattice E, we at least require the collection $\{T_{\mathcal{P}}\}_{\mathcal{P}\in\mathbb{T}}$ be uniformly bounded. To achieve this, we pass to the theory of unconditional Schauder decompositions.

Proposition 7.3.1 *Let E be a Banach lattice with order continuous norm and (P_i) a (not necessarily bounded) stopping time adapted to some positive BS-filtration on E. Define the sequence of operators (D_i) on E by $D_i = P_i - P_{i-1}$ for each $i \in \mathbb{N}$. Then (D_i) is an unconditional Schauder decomposition with unconditional constant one.*

Proof. Due to the fact that $P_iP_j = P_{i\wedge j}$ for all $i, j \in \mathbb{N}$, the D_i's are bounded linear projections with $D_iD_j = 0$ whenever $i \neq j$. Since (P_i) is a stopping time, $P_if \uparrow f$ for all $f \in E_+$. Thus, by the order continuity of E, we have $\|f - P_if\| \to 0$ for all $f \in E_+$. For $f \in E$, note that $\sum_{k=1}^i D_kf = P_if = P_if^+ - P_if^- \to f^+ - f^- = f$ as $i \to \infty$. Thus, (D_i) is a Schauder decomposition.

For unconditionality, let (k_j) denote a strictly increasing sequence of natural numbers. For $f \in E_+$, we have the increasing sequence $(\sum_{j=1}^i D_{k_j}f)_{i=1}^\infty$ bounded above by f. Since order continuity implies Dedekind completeness, it follows that $\sup_i \sum_{j=1}^i D_{k_j}f \in E$. Using the order continuity of E again yields $\|\sup_n \sum_{j=1}^n D_{k_j}f - \sum_{j=1}^i D_{k_j}f\| \to 0$ as $i \to \infty$. From this, we infer that $\sum_{i=1}^\infty D_if$ is sub-series summable for every $f \in E$. The unconditionality of (D_i) follows from Theorem 1.4.2.

Lastly, let (θ_k) denote any choice of signs. Notice that $D_i = P_i - P_{i-1} \le P_i \le \mathrm{id}_E$ for each $i \in \mathbb{N}$. Hence, (D_i) is also a sequence of band projections. It follows from [106, Corollary 32.4] that, for any $f \in E$, $|D_if| \wedge |D_jf| = D_i|f| \wedge D_j|f| = 0$ for $i \neq j$. Thus, $|\sum_{k=1}^i \theta_kD_kf| = \sum_{k=1}^i |\theta_kD_kf| = \sum_{k=1}^i |D_kf| = |\sum_{k=1}^i D_kf|$. Since the norm on E is a Riesz norm, $\|\sum_{k=1}^i \theta_kD_kf\| = \|\sum_{k=1}^i D_kf\|$, which completes the proof. $\square$

Classical examples of Banach lattices with order continuous norm are the spaces $L^p(\mu)$ for $1 \leq p < \infty$.

It is impossible to have $\dim \mathcal{R}(D_i) = 1$ for all $i \in \mathbb{N}$ when $1 \leq p < \infty$, unless the underlying measure space is purely atomic.

Indeed, for $1 < p < \infty$ ($p \neq 2$), it would imply that $L^p(\mu)$ has an unconditional basis with unconditional constant one. This contradicts the fact that the Haar system is the unconditional basis of $L^p(\mu)$ with smallest unconditional constant $p \vee (\frac{p}{p-1}) - 1 > 1$ for $p \neq 2$ (cf. [15, 17, 18]). When $p = 2$, the divergence of $\sum_{i=1}^{\infty} \theta_i D_i f$ in $L^r(\mu)$ for some choice of signs (θ_i), $1 < r < p$ and $f \in L^r(\mu)$ implies divergence in $L^2(\mu)$. The case $p = 1$ is covered by [71, Proposition 1.d.1], which asserts that $L^1(\mu)$ is not isomorphic to any subspace of a Banach space with unconditional basis.

In what follows, we will consider stopping times adapted to positive R-bounded BS-filtrations on order continuous Banach lattices. Recall that if (Σ_i) is a classical filtration, then Corollary 1.6.7 implies that the sequence of operators $(\mathbb{E}(\,\cdot \mid \Sigma_i))$ on $L^p(\mu)$ is R-bounded when $1 < p < \infty$. This is sufficient for the convergence of a stopped process with respect to an unbounded stopping time. For convenience, we recall Theorem 1.5.8.

Theorem 7.3.2 (Clément-de Pagter-Sukochev-Witvliet) *Let (D_i) be an unconditional Schauder decomposition of the Banach space Y, with unconditional constant K. Suppose that $\mathcal{T} \subset \mathcal{L}(Y)$ is R-bounded with R-bound M. If $(T_i) \subset \mathcal{T}$ is such that $T_i D_i = D_i T_i D_i$ for all $i \in \mathbb{N}$, then the series*

$$Sx := \sum_{i=1}^{\infty} T_i D_i x$$

is convergent in Y for all $x \in Y$, and defines a bounded linear operator $S : Y \to Y$ with $\|S\| \leq K^2 M$.

In the context of stopping times on order continuous Banach lattices, the following theorem is now immediate:

Theorem 7.3.3 *Let E be an order continuous Banach lattice and (T_i) a positive R-bounded BS-filtration on E with R-bound M. If $\mathcal{P} = (P_i)$ is a (not necessarily bounded) stopping time adapted to (T_i), then*

$$T_{\mathcal{P}} f := \sum_{i=1}^{\infty} (P_i - P_{i-1}) T_i f$$

for all $f \in E$ defines a bounded linear projection with $\|T_{\mathcal{P}}\| \leq M$. Consequently, $\sup_{\mathcal{P} \in \mathbb{T}^*} \|T_{\mathcal{P}}\| \leq M$.

Proof. The proof is a direct consequence of Proposition 7.3.1 and the fact that T_i is a projection that commutes with P_j for all $i \in \mathbb{N}$ and $j \leq i$. $\square$

This yields an optional stopping theorem for unbounded stopping times:

Theorem 7.3.4 *Let E be a Banach lattice and (T_i) a positive BS-filtration on E. Then the following statements hold:*

(a) If E is order continuous and (T_i) is R-bounded, then $T_\mathcal{P} = T_\mathcal{P} T_\mathcal{Q} = T_\mathcal{Q} T_\mathcal{P}$ for $\mathcal{P}, \mathcal{Q} \in \mathbb{T}^*$ with $\mathcal{P} \leq \mathcal{Q}$.
(b) If E has the projection property and (f_i) is a martingale relative to (T_i), then $f_\mathcal{P} = T_\mathcal{P} f_\mathcal{Q}$ for $\mathcal{P}, \mathcal{Q} \in \mathbb{T}$ with $\mathcal{P} \leq \mathcal{Q}$.
(c) If E is order continuous, (T_i) R-bounded and (f_i) a fixed martingale relative to (T_i), then $f_\mathcal{P}, f_\mathcal{Q} \in E$ and $f_\mathcal{P} = T_\mathcal{P} f_\mathcal{Q}$ for $\mathcal{P}, \mathcal{Q} \in \mathbb{T}^*$ with $\mathcal{P} \leq \mathcal{Q}$. Moreover, $\sup_{\mathcal{P} \in \mathbb{T}^*} \|f_\mathcal{P}\| < \infty$.

Proof. (a) Let $\mathcal{P} = (P_i), \mathcal{Q} = (Q_i) \in \mathbb{T}^*$ with $\mathcal{P} \leq \mathcal{Q}$. Since E has the projection property, it follows that

$$(P_j - P_{j-1})(Q_i - Q_{i-1}) = (Q_i - Q_{i-1})(P_j - P_{j-1})$$

for all $i, j \in \mathbb{N}$ (cf. [106, Theorem 32.1]). Moreover, since $\mathcal{P} \leq \mathcal{Q}$ and $m \leq n$ imply $Q_m \leq P_n$, it follows that $P_n Q_m = Q_m P_n = Q_m$ for all $m, n \in \mathbb{N}$ with $m \leq n$. Consequently, for $i + 1 \leq j$, we have

$$\begin{aligned}(Q_i - Q_{i-1})(P_j - P_{j-1}) &= Q_i P_j - Q_{i-1} P_j - Q_i P_{j-1} + Q_{i-1} P_{j-1} \\ &= Q_i - Q_{i-1} - Q_i + Q_{i-1} \\ &= 0.\end{aligned}$$

These observations, together with the identities

- $T_i P_j = P_j T_i$,
- $T_i(Q_i - Q_{i-1}) = (Q_i - Q_{i-1}) T_i$,
- $T_j(P_j - P_{j-1}) = (P_j - P_{j-1}) T_j$ and
- $T_i T_j = T_j T_i = T_j$,

for all $i, j \in \mathbb{N}$ with $j \leq i$, yield

$$\begin{aligned}
&\left(\sum_{i=1}^{n}(Q_i - Q_{i-1})T_i\right)\left(\sum_{j=1}^{n}(P_j - P_{j-1})T_j\right)f \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}(Q_i - Q_{i-1})T_i(P_j - P_{j-1})T_j f = \sum_{i=1}^{n}\sum_{j=1}^{n} T_i(Q_i - Q_{i-1})(P_j - P_{j-1})T_j f \\
&= \sum_{i=1}^{n}\sum_{j=1}^{i} T_i(Q_i - Q_{i-1})(P_j - P_{j-1})T_j f = \sum_{i=1}^{n}\sum_{j=1}^{i}(Q_i - Q_{i-1})(P_j - P_{j-1})T_i T_j f \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}(Q_i - Q_{i-1})(P_j - P_{j-1})T_j f = \left(\sum_{i=1}^{n}(Q_i - Q_{i-1})\right)\left(\sum_{j=1}^{n}(P_j - P_{j-1})T_j\right)f \\
&= Q_n\left(\sum_{j=1}^{n}(P_j - P_{j-1})T_j\right)f,
\end{aligned}$$

for all $n \in \mathbb{N}$ and $f \in E$. Taking the limit as $n \to \infty$, Theorem 7.3.3 and Proposition 7.3.1 imply $T_{\mathcal{Q}}T_{\mathcal{P}}f = T_{\mathcal{P}}f$. Similar reasoning yields

$$\begin{aligned}
&\left(\sum_{j=1}^{n}(P_j - P_{j-1})T_j\right)\left(\sum_{i=1}^{n}(Q_i - Q_{i-1})T_i\right)f \\
&= \sum_{i=1}^{n}\sum_{j=1}^{i} T_j(P_j - P_{j-1})(Q_i - Q_{i-1})T_i f = \sum_{i=1}^{n}\sum_{j=1}^{i} T_j T_i(P_j - P_{j-1})(Q_i - Q_{i-1})f \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} T_j(P_j - P_{j-1})(Q_i - Q_{i-1})f = \left(\sum_{j=1}^{n}(P_j - P_{j-1})T_j\right)\left(\sum_{i=1}^{n}(Q_i - Q_{i-1})\right)f \\
&= \left(\sum_{j=1}^{n}(P_j - P_{j-1})T_j\right)Q_n f,
\end{aligned}$$

for all $n \in \mathbb{N}$ and $f \in E$. Taking the limit as $n \to \infty$, another application of Theorem 7.3.3 and Proposition 7.3.1 shows $T_{\mathcal{P}}T_{\mathcal{Q}}f = T_{\mathcal{P}}f$. Consequently,

$$T_{\mathcal{Q}}T_{\mathcal{P}}f = T_{\mathcal{P}}f = T_{\mathcal{P}}T_{\mathcal{Q}}f$$

for all $f \in E$. This completes the proof of (a).

(b) If $\mathcal{Q}$ stops at $n \in \mathbb{N}$, then $\mathcal{P}$ stops at $m \in \mathbb{N}$ for some $m \leq n$. Thus, for a martingale (f_i) relative to (T_i), it follows from the proof of (a) that

$$\begin{aligned}
&T_{\mathcal{P}}f_{\mathcal{Q}} = T_{\mathcal{P}}\sum_{i=1}^{\infty}(Q_i - Q_{i-1})f_i = T_{\mathcal{P}}\sum_{i=1}^{\infty}(Q_i - Q_{i-1})T_i f_n = T_{\mathcal{P}}T_{\mathcal{Q}}f_n = T_{\mathcal{P}}f_n \\
&= \sum_{i=1}^{\infty}(P_i - P_{i-1})T_i f_n = \sum_{i=1}^{\infty}(P_i - P_{i-1})f_i = f_{\mathcal{P}}.
\end{aligned}$$

(c) Assume for some $f \in E$, $f_i = T_i f$ for each $i \in \mathbb{N}$. Then,

$$f_{\mathcal{Q}} = \sum_{i=1}^{\infty}(Q_i - Q_{i-1})f_i = \sum_{i=1}^{\infty}(Q_i - Q_{i-1})T_i f = T_{\mathcal{Q}} f \in E,$$

by Theorem 7.3.3. Similarly, $f_{\mathcal{P}} = T_{\mathcal{P}} f \in E$. By (a) we have $f_{\mathcal{P}} = T_{\mathcal{P}} f = T_{\mathcal{P}} T_{\mathcal{Q}} f = T_{\mathcal{P}} f_{\mathcal{Q}}$. Lastly, observe that $\sup_{\mathcal{P}\in\mathbb{T}^*} \|f_{\mathcal{P}}\| = \sup_{\mathcal{P}\in\mathbb{T}^*} \|T_{\mathcal{P}} f\| \leq \sup_{\mathcal{P}\in\mathbb{T}^*} \|T_{\mathcal{P}}\|\|f\| < \infty$, by Theorem 7.3.3. $\square$

Corollary 7.3.5 *Let E be an order continuous Banach lattice and (T_i) a positive R-bounded BS-filtration on E with R-bound M. Then for any directed set $\mathbb{D} \subset \mathbb{T}^*$ we have that $\{T_{\mathcal{P}}\}_{\mathbb{P}\in\mathbb{D}}$ is a M-BS-filtration on E.*

The uniform boundedness of $\{T_{\mathcal{P}}\}_{\mathcal{P}\in\mathbb{T}}$ is essential for the next convergence result, which resembles Corollary 3.2.7.

Theorem 7.3.6 *Let E be an order continuous Banach lattice and (T_i) a positive R-bounded BS-filtration on E with R-bound M. Then the following statements hold:*

(a) If $\mathbb{D} \subset \mathbb{T}^*$ is a directed set, then $\lim_{\mathcal{P}\in\mathbb{D}} \|T_{\mathcal{P}} f - f\| = 0$ if and only if $f \in \overline{\bigcup_{\mathcal{P}\in\mathbb{D}} \mathcal{R}(T_{\mathcal{P}})}$.

(b) If (f_i) is a martingale relative to (T_i) and $\mathbb{D} \subset \mathbb{T}$ is a directed set, then $\lim_{\mathcal{P}\in\mathbb{D}} \|f_{\mathcal{P}} - g\| = 0$ if and only if $g \in \overline{\bigcup_{\mathcal{P}\in\mathbb{D}} \mathcal{R}(T_{\mathcal{P}})}$ and $f_{\mathcal{P}} = T_{\mathcal{P}} g$ for each $\mathcal{P} \in \mathbb{D}$. Moreover, if (f_i) is fixed, we may take $\mathbb{D} \subset \mathbb{T}^*$.

Proof. (a) Suppose that $\lim_{\mathcal{P}} T_{\mathcal{P}} f = f$. It readily follows from $T_{\mathcal{P}} f \in \mathcal{R}(T_{\mathcal{P}})$ for each $\mathcal{P} \in \mathbb{D}$ that $f \in \overline{\bigcup_{\mathcal{P}\in\mathbb{D}} \mathcal{R}(T_{\mathcal{P}})}$. Conversely, suppose that $f \in \overline{\bigcup_{\mathcal{P}\in\mathbb{D}} \mathcal{R}(T_{\mathcal{P}})}$. Then, for each $\varepsilon > 0$, there exists $f_\varepsilon \in \bigcup_{\mathcal{P}\in\mathbb{D}} \mathcal{R}(T_{\mathcal{P}})$ such that $\|f_\varepsilon - f\| < \varepsilon/(M+1)$. From Theorem 7.3.4(a) we may infer the existence of a $\mathcal{P}_0 \in \mathbb{D}$ with the property $f_\varepsilon \in \mathcal{R}(T_{\mathcal{P}})$ for all $\mathcal{P} \geq \mathcal{P}_0$. Hence, for all $\mathcal{P} \geq \mathcal{P}_0$, it follows from Theorem 7.3.3 that

$$\begin{aligned}
\|T_{\mathcal{P}} f - f\| &\leq \|T_{\mathcal{P}} f - f_\varepsilon\| + \|f_\varepsilon - f\| \\
&= \|T_{\mathcal{P}}(f - f_\varepsilon)\| + \|f_\varepsilon - f\| \\
&\leq M\|f - f_\varepsilon\| + \|f_\varepsilon - f\| \\
&< M\varepsilon/(M+1) + \varepsilon/(M+1) \\
&= \varepsilon,
\end{aligned}$$

which completes the proof of (a).

(b) Suppose $(f_{\mathcal{P}})$ converges to g, then it is clear that $g \in \overline{\bigcup_{\mathcal{P}\in\mathbb{D}} \mathcal{R}(T_{\mathcal{P}})}$. Also, by Theorem 7.3.4(b), we have $T_{\mathcal{P}} f_{\mathcal{Q}} = f_{\mathcal{P}}$ for all $\mathcal{P}, \mathcal{Q} \in \mathbb{D}$ with $\mathcal{P} \leq \mathcal{Q}$. Thus, $\lim_{\mathcal{Q}\in\mathbb{D}} T_{\mathcal{P}} f_{\mathcal{Q}} = T_{\mathcal{P}} g = f_{\mathcal{P}}$. Conversely, by part (a), we have $\|T_{\mathcal{P}} g - g\| = \|f_{\mathcal{P}} - g\| \to$

0, as required. For the case where (f_i) is fixed and $\mathbb{D} \subset \mathbb{T}^*$, it is enough to observe Theorem 7.3.4(c). $\square$

7.4 A Characterization of convergent nets of stopped processes in vector-valued L^p-spaces

We start with a lemma:

Lemma 7.4.1 *Let E be an order continuous Banach lattice and Y a Banach space. Suppose (T_i) is a R-bounded BL-filtration on E, with R-bound M. Then the following statements hold for $E\widetilde{\otimes}_l Y$:*

(a) If $\mathcal{P} \in \mathbb{T}^*$, then $T_\mathcal{P} : E \to E$ is positive and $\mathcal{R}(T_\mathcal{P})$ is a closed Riesz subspace of E. Consequently, $T_\mathcal{P} \otimes_l \mathrm{id}_Y : E\widetilde{\otimes}_l Y \to E\widetilde{\otimes}_l Y$ is a bounded linear projection with range $\mathcal{R}(T_\mathcal{P})\widetilde{\otimes}_l Y$; moreover, $\sup_{\mathcal{P}\in\mathbb{T}^*} \|T_\mathcal{P} \otimes_l \mathrm{id}_Y\| \le M$.

(b) $\overline{\bigcup_{\mathcal{P}\in\mathbb{D}} \mathcal{R}(T_\mathcal{P})}$ a closed Riesz subspace of E, for any directed set $\mathbb{D} \subset \mathbb{T}^*$.

(c) $\overline{\bigcup_{\mathcal{P}\in\mathbb{D}} \mathcal{R}(T_\mathcal{P})}\widetilde{\otimes}_l Y = \overline{\bigcup_{\mathcal{P}\in\mathbb{D}} \mathcal{R}(T_\mathcal{P} \otimes_l \mathrm{id}_Y)}$, for any directed set $\mathbb{D} \subset \mathbb{T}^*$.

Proof. (a) Since $(P_i - P_{i-1})$ is a band projection and T_i is positive for each $i \in \mathbb{N}$, the positivity of $T_\mathcal{P}$ follows easily. Now let $f \in \mathcal{R}(T_\mathcal{P})$. Observing that $|T_i f| = T_i|T_i f|$ for each $i \in \mathbb{N}$ and $|\sum_{i=1}^n (P_i - P_{i-1})T_i f| = \sum_{i=1}^n (P_i - P_{i-1})|T_i f|$ for each $n \in \mathbb{N}$, it follows that

$$\begin{aligned} T_\mathcal{P}|f| &= T_\mathcal{P}|T_\mathcal{P} f| = T_\mathcal{P}\left|\sum_{i=1}^\infty (P_i - P_{i-1})T_i f\right| = T_\mathcal{P}\sum_{i=1}^\infty (P_i - P_{i-1})|T_i f| \\ &= \sum_{i=1}^\infty (P_i - P_{i-1})T_i|T_i f| = \sum_{i=1}^\infty (P_i - P_{i-1})|T_i f| = \left|\sum_{i=1}^\infty (P_i - P_{i-1})T_i f\right| \\ &= |T_\mathcal{P} f| = |f|. \end{aligned}$$

Hence, $|f| \in \mathcal{R}(T_\mathcal{P})$, proving that $\mathcal{R}(T_\mathcal{P})$ is a Riesz space. Since $T_\mathcal{P}$ is a projection, it follows $\mathcal{R}(T_\mathcal{P})$ is also closed in E. The fact that $T_\mathcal{P} \otimes_l \mathrm{id}_Y : E\widetilde{\otimes}_l Y \to E\widetilde{\otimes}_l Y$ is a bounded linear projection with range $\mathcal{R}(T_\mathcal{P})\widetilde{\otimes}_l Y$ and norm $\|T_\mathcal{P}\|$ now follows from Lemma 3.4.1. Consequently, $\sup_{\mathcal{P}\in\mathbb{T}^*} \|T_\mathcal{P} \otimes_l \mathrm{id}_Y\| \le M$, by Theorem 7.3.3.

(b) The proof of (b) is a trivial consequence of (a).

(c) Let $y \in \overline{\bigcup_{\mathcal{P}\in\mathbb{D}} \mathcal{R}(T_\mathcal{P} \otimes_l \mathrm{id}_Y)}$ and $\varepsilon > 0$ be given. Select $y_0 \in \mathcal{R}(T_\mathcal{P} \otimes_l \mathrm{id}_Y)$ for some $\mathcal{P} \in \mathbb{D}$, such that $\|y - y_0\|_l < \varepsilon$. Part (a) and the left order injectivity of the l-norm imply that $y_0 \in \mathcal{R}(T_\mathcal{P} \otimes_l \mathrm{id}_Y) = \mathcal{R}(T_\mathcal{P})\widetilde{\otimes}_l Y \hookrightarrow \overline{\bigcup_{\mathcal{P}\in\mathbb{D}} \mathcal{R}(T_\mathcal{P})}\widetilde{\otimes}_l Y$. Thus, $y \in \overline{\bigcup_{\mathcal{P}\in\mathbb{D}} \mathcal{R}(T_\mathcal{P})}\widetilde{\otimes}_l Y$.

For the reverse inclusion, let $y \in \overline{\bigcup_{\mathcal{P}\in\mathbb{D}} \mathcal{R}(T_{\mathcal{P}})}\widetilde{\otimes}_l Y$ and $\varepsilon > 0$ be given. Select $y_0 \in \overline{\bigcup_{\mathcal{P}\in\mathbb{D}} \mathcal{R}(T_{\mathcal{P}})} \otimes Y$ such that

$$\|y - y_0\|_l < \varepsilon/2.$$

Let $y_0 = \sum_{i=1}^{n_0} a_i \otimes y_i$, where $a_i \in \overline{\bigcup_{\mathcal{P}\in\mathbb{D}} \mathcal{R}(T_{\mathcal{P}})}$ and $y_i \in Y$. Select $b_i \in \bigcup_{\mathcal{P}\in\mathbb{D}} \mathcal{R}(T_{\mathcal{P}})$ such that

$$\|a_i - b_i\| < \varepsilon/\left(2\sum_{i=1}^{n_0} \|y_i\|\right).$$

Let $z = \sum_{i=1}^{n_0} b_i \otimes y_i$. Then, by part (a), $z \in \bigcup_{\mathcal{P}\in\mathbb{D}} \mathcal{R}(T_{\mathcal{P}} \otimes_l \mathrm{id}_Y)$. Also,

$$y_0 - z = \sum_{i=1}^{n_0} (a_i - b_i) \otimes y_i,$$

$$\|y_0 - z\|_l \leq \| \sum_{i=1}^{n_0} \|y_i\| \, |a_i - b_i| \| < \varepsilon/2$$

and

$$\|y - z\|_l \leq \|y - y_0\|_l + \|y_0 - z\|_l < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus, $y \in \overline{\bigcup_{\mathcal{P}\in\mathbb{D}} \mathcal{R}(T_{\mathcal{P}} \otimes_l \mathrm{id}_Y)}$. $\square$

Corollary 7.4.2 *Let E be an order continuous Banach lattice and (T_i) a R-bounded BL-filtration on E with R-bound M. Then for any directed set $\mathbb{D} \subset \mathbb{T}^*$ we have that $\{T_{\mathcal{P}}\}_{\mathbb{P}\in\mathbb{D}}$ is a M-BL-filtration on E.*

As already mentioned, if (T_i) is a BL-filtration on E, then it may be extended to $E\widetilde{\otimes}_l Y$ by considering $(T \otimes_l \mathrm{id}_Y)$. Similarly, if $\mathcal{P} = (P_i) \in \mathbb{T}^*$, it is easy to check that $\mathcal{P}$ extends to $E\widetilde{\otimes}_l Y$ by considering $(P_i \otimes_l \mathrm{id}_Y)$. The next lemma shows that $T_{\mathcal{P}}$ can be extended to $E\widetilde{\otimes}_l Y$ in the same manner, under some additional assumptions.

Lemma 7.4.3 *Let E be an order continuous Banach lattice and Y a Banach space. Suppose (T_i) is a positive R-bounded BS-filtration on E, with R-bound M. If $\mathcal{P} = (P_i) \in \mathbb{T}^*$, then*

$$\widehat{T}_{\mathcal{P}} f := \sum_{i=1}^{\infty} \left[(P_i \otimes_l \mathrm{id}_Y) - (P_{i-1} \otimes_l \mathrm{id}_Y)\right](T_i \otimes_l \mathrm{id}_Y) f$$

converges for each $f \in E\widetilde{\otimes}_l Y$ and defines a bounded linear projection on $E\widetilde{\otimes}_l Y$ with $\|\widehat{T}_{\mathcal{P}}\| \leq M$.

Proof. Let $u = \sum_{j=1}^{n} x_j \otimes y_j \in E \otimes Y$ and observe that

$$\begin{aligned}
(T_{\mathcal{P}} \otimes \mathrm{id}_Y)u &= \sum_{j=1}^{n} T_{\mathcal{P}} x_j \otimes y_j = \sum_{j=1}^{n}\sum_{i=1}^{\infty}(P_i - P_{i-1})T_i x_j \otimes y_j \\
&= \sum_{j=1}^{n}\sum_{i=1}^{\infty}\big[(P_i \otimes_l \mathrm{id}_Y) - (P_{i-1} \otimes_l \mathrm{id}_Y)\big](T_i \otimes_l \mathrm{id}_Y)(x_j \otimes y_j) \\
&= \sum_{i=1}^{\infty}\big[(P_i \otimes_l \mathrm{id}_Y) - (P_{i-1} \otimes_l \mathrm{id}_Y)\big](T_i \otimes_l \mathrm{id}_Y)u \\
&= \widehat{T}_{\mathcal{P}} u.
\end{aligned}$$

Hence, by taking the unique continuous extension of the bounded operator $T_{\mathcal{P}} \otimes \mathrm{id}_Y$, we obtain $T_{\mathcal{P}} \otimes_l \mathrm{id}_Y = \widehat{T}_{\mathcal{P}}$ on $E\widetilde{\otimes}_l Y$. An application of Lemma 7.4.1(a) completes the proof. $\square$

As a consequence of the above lemma, the assertions of Theorem 7.3.4 and Theorem 7.3.6 trivially extend to $E\widetilde{\otimes}_l Y$. In what follows, the distinction between these theorems and their extended counterparts will be implicit.

We recall Theorem 5.2.2 again for convenience: if E is a Banach lattice and Y a Banach space, then $u \in E\widetilde{\otimes}_l Y$ if and only if $u = \sum_{i=1}^{\infty} x_i \otimes y_i$, where

$$\Big\| \sum_{i=1}^{\infty} |x_i| \Big\|_E < \infty \quad \text{and} \quad \lim_{i\to\infty} \|y_i\|_Y = 0. \tag{7.1}$$

Using this representation theorem and the above results, we can now give a description of convergent nets of stopped processes in the l-tensor product, namely:

Theorem 7.4.4 *Let E be an order continuous Banach lattice and Y a Banach space. Suppose (T_i) is a R-bounded BL-filtration on E. In addition, suppose that $(f_i) \subset E\widetilde{\otimes}_l Y$ is a martingale relative to the BS-filtration $(T_i \otimes_l \mathrm{id}_Y)$, and $\mathbb{D} \subset \mathbb{T}$ is a directed set. Then the following statements are equivalent:*

(a) The net of stopped processes $\{f_{\mathcal{P}}\}_{\mathcal{P}\in\mathbb{D}}$ is convergent in $E\widetilde{\otimes}_l Y$.

(b) For each $i \in \mathbb{N}$, there exist a $y_i \in Y$ and a fixed martingale $(x_j^{(i)}, T_j)_{j=1}^{\infty} \subset E$ with $\{x_{\mathcal{P}}^{(i)}\}_{\mathcal{P}\in\mathbb{D}} \subset E$ convergent, so that $f_{\mathcal{P}} = \sum_{i=1}^{\infty} x_{\mathcal{P}}^{(i)} \otimes y_i$ for each $\mathcal{P} \in \mathbb{D}$, where

$$\left\| \sum_{i=1}^{\infty} \left| \lim_{\mathcal{P}\in\mathbb{D}} x_{\mathcal{P}}^{(i)} \right| \right\| < \infty \qquad \textit{and} \qquad \lim_{i\to\infty} \|y_i\| = 0.$$

Moreover, if $(f_i) \subset E\widetilde{\otimes}_l Y$ is fixed, we may take $\mathbb{D} \subset \mathbb{T}^*$.

Proof. Suppose $\{f_{\mathcal{P}}\}_{\mathcal{P}\in\mathbb{D}}$ is convergent in $E\widetilde{\otimes}_l Y$. Then, by Theorem 7.3.6(b) and Lemma 7.4.1(c), it follows that

$$\lim_{\mathcal{P}\in\mathbb{D}} f_{\mathcal{P}} \in \overline{\bigcup_{\mathcal{P}\in\mathbb{D}} \mathcal{R}(T_{\mathcal{P}} \otimes_l \mathrm{id}_Y)} = \overline{\bigcup_{\mathcal{P}\in\mathbb{D}} \mathcal{R}(T_{\mathcal{P}})}\widetilde{\otimes}_l Y.$$

By the remark preceding this theorem, $\lim_{\mathcal{P}\in\mathbb{D}} f_{\mathcal{P}} = \sum_{i=1}^{\infty} x_i \otimes y_i$ where (7.1) holds. By Theorem 7.3.6(b), $f_{\mathcal{P}} = (T_{\mathcal{P}} \otimes_l \mathrm{id}_Y)(\sum_{i=1}^{\infty} x_i \otimes y_i)$ for each $\mathcal{P} \in \mathbb{D}$. For each $i \in \mathbb{N}$, define $x_j^{(i)} = T_j x_i$, for each $j \in \mathbb{N}$. Then $(x_j^{(i)})_{j=1}^{\infty}$ is a fixed martingale relative to (T_j), so that $x_{\mathcal{P}}^{(i)} = T_{\mathcal{P}} x_i \in E$ for each $i \in \mathbb{N}$. Hence,

$$f_{\mathcal{P}} = \sum_{i=1}^{\infty} T_{\mathcal{P}} x_i \otimes y_i = \sum_{i=1}^{\infty} x_{\mathcal{P}}^{(i)} \otimes y_i,$$

where $\{x_{\mathcal{P}}^{(i)}\}_{\mathcal{P}\in\mathbb{D}}$ is convergent in E, by Theorem 7.3.6(b). Thus, $\left\| \sum_{i=1}^{\infty} \left| \lim_{\mathcal{P}\in\mathbb{D}} x_{\mathcal{P}}^{(i)} \right| \right\| < \infty$ and $\lim_{i\to\infty} \|y_i\| = 0$ hold.

Conversely, for each $i \in \mathbb{N}$, let $x_i = \lim_{\mathcal{P}\in\mathbb{D}} x_{\mathcal{P}}^{(i)}$. Theorem 7.3.6(b) implies $x_{\mathcal{P}}^{(i)} = T_{\mathcal{P}} x_i$ for each $i \in \mathbb{N}$ and $(x_i) \subset \overline{\bigcup_{\mathcal{P}\in\mathbb{D}} \mathcal{R}(T_{\mathcal{P}})}$. Also, (x_i) and (y_i) satisfy (7.1) so that Lemma 7.4.1(c) implies

$$f := \sum_{i=1}^{\infty} x_i \otimes y_i \in \overline{\bigcup_{\mathcal{P}\in\mathbb{D}} \mathcal{R}(T_{\mathcal{P}})}\widetilde{\otimes}_l Y = \overline{\bigcup_{\mathcal{P}\in\mathbb{D}} \mathcal{R}(T_{\mathcal{P}} \otimes_l \mathrm{id}_Y)}.$$

Then, for each $\mathcal{P} \in \mathbb{D}$,

$$f_{\mathcal{P}} = \sum_{i=1}^{\infty} x_{\mathcal{P}}^{(i)} \otimes y_i = \sum_{i=1}^{\infty} T_{\mathcal{P}}(x_i) \otimes y_i = (T_{\mathcal{P}} \otimes_l \mathrm{id}_Y) f.$$

It now follows from Theorem 7.3.6(b) that $\{f_{\mathcal{P}}\}_{\mathcal{P}\in\mathbb{D}}$ is convergent.

Lastly, if (f_i) is fixed, then $\{f_{\mathcal{P}}\}_{\mathcal{P}\in\mathbb{D}} \subset E\widetilde{\otimes}_l Y$ for $\mathbb{D} \subset \mathbb{T}^*$, by Theorem 7.3.4(c). Thus, the result also holds in this case. $\square$

The above result specializes the following:

Theorem 7.4.5 *Let (Ω, Σ, μ) be a finite measure space, $1 < p < \infty$, and Y a Banach space. Suppose that $(f_i, \Sigma_i) \subset L^p(\mu, Y)$ is a convergent martingale and $\mathbb{D}$ a directed set of (not necessarily bounded) stopping times adapted to (Σ_i). Then the following statements are equivalent:*

(a) The net of stopped processes $\{f_\tau\}_{\tau\in\mathbb{D}}$ is convergent in $L^p(\mu, Y)$.

(b) For each $i \in \mathbb{N}$, there exist a $y_i \in Y$ and a convergent martingale $(x_j^{(i)}, \Sigma_j)_{j=1}^{\infty} \subset L^p(\mu)$ with $\{x_\tau^{(i)}\}_{\tau\in\mathbb{D}}$ convergent, so that $f_\tau = \sum_{i=1}^{\infty} x_\tau^{(i)}(\omega) y_i$ for each $\omega \in \Omega$ and $\tau \in \mathbb{D}$, where

$$\left\| \sum_{i=1}^{\infty} \left| \lim_{\tau\in\mathbb{D}} x_\tau^{(i)} \right| \right\| < \infty \qquad \textit{and} \qquad \lim_{i\to\infty} \|y_i\| = 0.$$

The above equivalence holds for $1 \le p < \infty$ *with* $(f_i, \Sigma_i) \subset L^p(\mu, Y)$ *not necessarily convergent, provided* $\mathbb{D}$ *is a directed set of bounded stopping times.*

Proof. In the case $E = L^p(\mu)$, where $1 < p < \infty$ and $\mathbb{D} \subset \mathbb{T}^*$, it follows that E is order continuous and the BL-filtration (T_i) on E given by $(\mathbb{E}(\,\cdot \mid \Sigma_i))$ (where (Σ_i) is a classical filtration) is R-bounded by Corollary 1.6.7. The result now follows by noting that any martingale in $L^p(\mu)$ or $L^p(\mu, Y)$, relative to (Σ_i), is fixed if and only if it is convergent.

For the case $p = 1$ and $\mathbb{D} \subset \mathbb{T}$, we have $\sup_{\tau \in \mathbb{D}} \|\mathbb{E}(\,\cdot \mid \Sigma_\tau)\| \le 1$ even though $(\mathbb{E}(\,\cdot \mid \Sigma_i))$ is not necessarily R-bounded. Consequently, the result holds for $1 \le p < \infty$ and $(f_i, \Sigma_i) \subset L^p(\mu, Y)$ not necessarily convergent, provided that $\mathbb{D}$ is a directed set of bounded stopping times. □

7.5 Unconditional Schauder decompositions in vector-valued L^p-spaces

In this section, we give an application of the above techniques to a result concerning the existence of unconditional Schauder decompositions in vector-valued L^p-spaces.

Recall that Aldous showed in [1, Proposition 4] that if $L^p(\mu, Y)$ possesses an unconditional basis, then Y is a UMD space. Conversely, if Y is a UMD space with unconditional basis, then $L^p(\mu, Y)$ has unconditional basis, as was noted by Bourgain in [9]. We now generalize this converse to unconditional Schauder decompositions in $L^p(\mu, Y)$.

Theorem 7.5.1 *Let* $1 < p < \infty$ *and* Y *be a Banach lattice. If* $(\mathbb{E}_i)$ *is a martingale decomposition of* $L^p(\mu)$ *and* (P_j) *any stopping time on* Y*, then*

$$\big\{(\mathbb{E}_i - \mathbb{E}_{i-1}) \otimes_{\Delta_p} (P_j - P_{j-1})\big\}_{i,j \in \mathbb{N}}$$

is an unconditional Schauder decomposition of $L^p(\mu, Y)$*, provided that* Y *possesses the UMD property. In particular, if* Y *is a Banach space with unconditional basis, then* $L^p(\mu, Y)$ *has an unconditional basis, provided* Y *possesses the UMD property.*

Proof. Define the bounded operators $D_i = (\mathbb{E}_i - \mathbb{E}_{i-1}) \otimes_{\Delta_p} \mathrm{id}_Y$ and $D'_j = \mathrm{id}_{L^p(\mu)} \otimes_{\Delta_p} (P_j - P_{j-1})$ on $L^p(\mu, Y)$ for all $i, j \in \mathbb{N}$. Then it is easily verified that $D_i D'_j = D'_j D_i$ for all $i, j \in \mathbb{N}$.

Assume Y has the UMD property. Then Y is reflexive, and thus has the Radon-Nikodým property (cf. [37]). This implies Y is an order continuous Banach lattice (cf. [45, p. 74]). Hence, $L^p(\mu, Y)$ is also an order continuous Banach lattice.

Moreover, $(\mathrm{id}_{L^p(\mu)} \otimes_{\Delta_p} P_j)f \uparrow f$ for all $f \in L^p(\mu, Y)_+$ so that $(\mathrm{id}_{L^p(\mu)} \otimes_{\Delta_p} P_j)$ is a stopping time adapted to the BS-filtration $(\mathbb{E}_i \otimes_{\Delta_p} \mathrm{id}_Y)$ on $L^p(\mu, Y)$. By Proposition 7.3.1, (D'_j) is an unconditional Schauder decomposition of $L^p(\mu, Y)$. On the other hand, the UMD property of Y implies that (D_i) is also an unconditional Schauder decomposition of $L^p(\mu, Y)$.

By Theorem 1.6.8, it remains to show that $L^p(\mu, Y)$ has property (α). It is remarked in [103, Remark 2.3.2] that any Banach lattice with finite cotype has property (α) (cf. [84, Proposition 2.1] and [36, Theorem 14.1]). In our case, $L^p(\mu, Y)$ is a UMD Banach lattice. However, Banach spaces that possess the UMD property already have finite cotype, as was noted by Aldous in the proof of [1, Proposition 2].

In particular, if Y is a Banach space possessing an unconditional basis, it can be renormed so that it becomes an order continuous Banach lattice Y_0 (see Theorem 6.3.8 and its corollary). The natural coordinate projections (P_j) on Y_0 constitute a stopping time. It follows from the above that $\{D_i D'_j\}$ is an unconditional decomposition on $L^p(\mu, Y_0)$ with $\dim(D_i D'_j) = 1$, whenever $\dim(D_i) = 1$. Since $L^p(\mu, Y)$ is isomorphic to $L^p(\mu, Y_0)$ and the latter has an unconditional basis, the proof is complete. $\square$

7.6 Notes and remarks

Theorem 7.4.4 may be considered as an extension of Theorem 5.3.1 when the Banach lattice E is order continuous and the BL-filtration (T_i) on E is R-bounded. Indeed, consider the sequence of stopping times on E, adapted to (T_i), defined by $\mathcal{P}_1 = (\mathrm{id}_E, \mathrm{id}_E, \mathrm{id}_E, \ldots)$, $\mathcal{P}_2 = (0, \mathrm{id}_E, \mathrm{id}_E, \ldots)$, $\mathcal{P}_3 = (0, 0, \mathrm{id}_E, \ldots)$, ... Since $T_{\mathcal{P}_i} = T_i$ for each $i \in \mathbb{N}$, it is evident that Theorem 7.4.4 reduces to Theorem 5.3.1. Of course, we have assumed that the filtration on Y is the trivial filtration. However, Theorem 7.4.4 can easily be improved to accommodate any BS-filtration on Y.

A

Appendix

A.1 Riesz spaces

The material in this section can be found in [73, 105, 106, 76, 92]. We focus on a special class of vector space endowed with partial ordering.

Definition A.1.1 Let X be a partially ordered set.

(a) If every subset of X consisting of two elements has a supremum and an infimum then X is called a *lattice*.
(b) We denote $\sup\{x, y\}$ by $x \wedge y$ and $\inf\{x, y\}$ by $x \vee y$ for all $x, y \in X$.
(c) If X is a lattice, then X is called a *distributive lattice* if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in X$.

Definition A.1.2 Let E be a real vector space.

(a) If E has a partial ordering so that
  (i) $f \leq g \Rightarrow f + h \leq g + h$ for every $f, g, h \in E$ and
  (ii) $f \geq 0 \Rightarrow \alpha f \geq 0$ for every non-negative $\alpha \in \mathbb{R}$,

  then E is called an *ordered vector space*.
(b) Let E be an ordered vector space, then the subset $C_E = \{f \in E : f \geq 0\}$ is called the *positive cone* of E. An ordered vector space E with its positive cone C_E is denoted (E, C_E). The cone C_E is said to be *generating* if $E = C_E - C_E$, *proper* if $C_E \cap (-C_E) = \{0\}$ and *Archimedean* if it follows from $y - nx \in C_E$ for all $n \in \mathbb{N}$, with $y \in C_E$ and $x \in E$, that $-x \in C_E$.
(c) Let E be an ordered vector space, then for $f, g \in E$ with $f \leq g$ we define an *order interval* $[f, g]$ by $[f, g] := \{h \in E : f \leq h \leq g\}$.
(d) If E is an ordered vector space and a lattice, then E is called a *Riesz space*. We use the notation E_+ for the positive cone of a Riesz space E. If E_+ is Archimedean, then E is called an *Archimedean Riesz space*.

(e) Let E be a Riesz space, then for all $f \in E$ we have the notations $f^+ = f \vee 0$, $f^- = (-f) \vee 0 = -(f \wedge 0)$ and $|f| = f \vee (-f)$. We call f^+ and f^- the *positive* and *negative* parts of f respectively.

We collect some elementary consequences of the above definitions, the proofs of which can be found in [106, Theorems 5.1, 5.2, 5.5 and 6.1].

Proposition A.1.3 *Let E be a Riesz space and $f, g \in E$. Then the following statements hold:*

(a) $f^+, f^- \in E_+$ *and* $|-f| = |f|$.
(b) $f = f^+ - f^-$, $f^+ \wedge f^- = 0$ *and* $|f| = f^+ + f^-$, *moreover* E_+ *is proper and generating.*
(c) $f \vee g = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$ *and* $f \wedge g = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|$.
(d) $||f| - |g|| \leq |f+g| \leq |f| + |g|$.
(e) E *is an infinitely distributive lattice.*

The decomposition $f = f^+ - f^-$ is unique in the sense that $f = u - v$ with $u \wedge v = 0$, $u \geq 0$ and $v \geq 0$ if and only if $u = f^+$ and $v = f^-$ (cf. [106, Theorem 5.6]). This is known as the *minimal decomposition*, for if $f = u - v$ with $u \geq 0$ and $v \geq 0$, then $f^+ \leq u$ and $f^- \leq v$ (cf. [106, Theorem 5.6]).

It is evident from (c) in the above proposition that E is a Riesz space if and only if $f \in E$ implies $|f| \in E$. We look at some algebraic structures found in Riesz spaces.

Definition A.1.4 Let E be a Riesz space.

(a) $R \subset E$ is called a *Riesz subspace* if R is a linear subspace of E and for all $x, y \in R$ we have $x \wedge y \in R$ and $x \vee y \in R$.
(b) $S \subset E$ is called *solid* if $f \in S \Rightarrow \left[-|f|, |f|\right] \subset S$.
(c) $A \subset E$ is called an *ideal* if A is a solid linear subspace.
(d) An ideal $B \subset E$ is called a *band* when the supremum (if it exists) of every subset of B that is bounded above is an element of B.

From the above definition we can deduce that $A \subset E$ is an ideal if and only if A is a linear subspace, $f \in A \Leftrightarrow |f| \in A$ and $0 \leq g \leq f \in A \Rightarrow g \in A$. It is also worth noting that any ideal $A \subset E$ is a Riesz subspace of E and that the intersection or algebraic sum of any two ideals is again an ideal.

The statements in the next definition are justified by [76, Propositions 1.2.5 and 1.2.6].

Definition A.1.5 Let E be a Riesz space and $D \subset E$.

(a) The ideal A_D *generated* by D is the smallest ideal containing D and can be expressed as

$$A_D = \bigcup \Big\{ n\big[\text{-}y, y\big] : n \in \mathbb{N},\ y = |x_1| \vee \ldots \vee |x_r|,\ x_1, \ldots, x_r \in D \Big\}.$$

In the case where $D = \{f\}$ we denote A_D by A_f. We call A_f the *principle ideal generated* by f and it can be expressed as

$$A_f = \bigcup \Big\{ n\big[\text{-}|f|, |f|\big] : n \in \mathbb{N} \Big\}.$$

We also use the notation E_f to denote A_f, as is customary in some of the literature.

(b) The band B_A *generated by an ideal* A is the smallest band containing A and can be expressed as

$$B_A = \Big\{ g \in E : |g| = \sup \big(\big[0, |g|\big] \cap A \big) \Big\}.$$

The band generated by the principle ideal A_f is called the *principle band* generated by f and is denoted by B_f, which can be expressed as

$$B_f = \Big\{ g \in E : |g| = \sup \big\{ |g| \wedge n|f| : n \in \mathbb{N} \big\} \Big\}.$$

(c) An element $0 < e \in E_+$ is called a *strong order unit* if $A_e = E$.

(d) An element $0 < e \in E_+$ is called a *weak order unit* if $B_e = E$. Note that $0 < e \in E_+$ is a weak order unit of E if and only if for every $f \in E_+$ we have that $f = \sup\{f \wedge ne : n \in \mathbb{N}\}$.

It is shown in [76, Corollary 1.2.14] that a positive element of a Banach lattice is a strong order unit if and only if it is an interior point of E_+.

Definition A.1.6 Let E be a Riesz space. We say that $f, g \in E$ are *disjoint* if $|f| \wedge |g| = 0$ and we write $f \perp g$. If D is a non empty subset of E, then the set $D^d = \{f \in E : f \perp g \ \ \forall\, g \in D\}$ is called the *disjoint complement* of D. If D_1 and D_2 are both non empty subsets of E such that $d_1 \perp d_2$ for all $d_1 \in D_1$ and $d_2 \in D_2$, then D_1 and D_2 are said to be *disjoint* and is denoted $D_1 \perp D_2$.

Mutually disjoint elements exhibit some important properties which are listed below (cf. [106, Theorems 8.1, 8.2 and 8.4]).

Proposition A.1.7 *Let E be a Riesz space and D be non empty subset of E. Then the following statements hold:*

(a) If $f_0 = \sup D$ *and for* $f \in E$ *we have* $f \perp g$ *for all* $g \in D$*, then* $f \perp f_0$*.*

(b) If $\{f_1, \ldots, f_n\}$ *is a mutually disjoint set of non-zero elements, then this set is linearly independent.*

(c) For $f, g \in E$ *with* $f \perp g$ *we have* $|f+g| = |f-g| = |f|+|g| = ||f|-|g|| = |f| \vee |g|$*.*

(d) D^d *is a band.*

(e) $D \subset D^{dd}$*,* $D^d = D^{ddd}$ *and* $D^d \cap D^{dd} = \{0\}$*.*

We turn our attention to sequences and nets.

Definition A.1.8 Let E be a Riesz space and (f_n) be a sequence in E.

(a) If (f_n) is an increasing (decreasing) sequence, we shall write $f_n \uparrow$ $(f_n \downarrow)$, moreover if $f = \sup_n f_n$ $(f = \inf_n f_n)$ exists in E, then we shall write $f_n \downarrow f$ $(f_n \uparrow f)$.

(b) We say (f_n) converges *in order* to f if there exists a sequence (p_n) in E such that $p_n \downarrow 0$ and $|f - f_n| \leq p_n$ for all $n \in \mathbb{N}$. We denote this by $f_n \to f$ (ord).

(c) Let $0 < u \in E$. Then (f_n) is said to converge *u-uniformly* to f if given $\varepsilon > 0$, there exists N_ε such that $n \geq N_\varepsilon \Rightarrow |f - f_n| < \varepsilon u$. We denote this $f_n \to f$ $(u\text{-}un)$.

(d) If E is Archimedean, E is said to be *uniformly complete* if for every $u > 0$ in E, every u-uniform Cauchy sequence has a limit in E.

In general, we are not guaranteed unique u-uniform limits in a Riesz space unless it is Archimedean, in this case u-uniform convergence implies order convergence.

Definition A.1.9 A non-empty subset D in a Riesz space E is said to be upwards (downwards) directed if for any two elements f and g in D there exists an element h in D such that $h \geq f \vee g$ $(h \leq f \wedge g)$. We denote this as $D \uparrow$ $(D \downarrow)$ and if $f_0 = \sup D$ $(f_0 = \inf D)$ exists in E we shall write $D \uparrow f_0$ $(D \downarrow f_0)$.

If D is an arbitrary set which is bounded above (below), then by adjoining all finite suprema (infima) to D, we can turn D into an upwards (downwards) directed set without altering the set of upper (lower) bounds of D.

Definition A.1.10 Let E be a Riesz space.

(a) A Riesz space E is said to be *Dedekind complete* if the supremum of every subset of E that is bounded above is an element of E and *Dedekind σ-complete* if the supremum of every countable subset of E that is bounded above is an element of E.

(b) Any band B in E satisfying $B \oplus B^d = E$ is called a *projection band.*

(c) If every band in E is a projection band, then E is said to have the *projection property.*

(d) If every principle band in E is a projection band, then E is said to have the *principle projection property.*

The next important structural result is called *the main inclusion theorem* and can be found in [106, Theorem 12.3].

Theorem A.1.11 *Let E be a Riesz space, then*

(a) E Dedekind complete $\Rightarrow$ E has the projection property $\Rightarrow$ E has the principle projection property $\Rightarrow$ E is Archimedean.

(b) E Dedekind complete $\Rightarrow$ E Dedekind σ-complete $\Rightarrow$ E has the principle projection property $\Rightarrow$ E is Archimedean.

We conclude this section with Freudenthal's Spectral Theorem. We first state some auxiliary definitions and results, which can be found in [106, 76, 92]

Definition A.1.12 Let E be a Riesz space with $0 < e \in E$. We call $p \in E_+$ a *component* of e if $p \wedge (e - p) = 0$.

Note in the above definition that p is a component of e if and only if $(e - p)$ is a component. The proof of the next result can be found in [106, Theorem 3.7].

Proposition A.1.13 *Let E be a Riesz space with $0 < e \in E$. The set $C_e := \{p \in E_+ : e \wedge (p - e) = 0\}$ of components of e is a lattice with respect to the ordering inherited from E.*

Definition A.1.14 Let E be a Riesz space with $0 < e \in E$ and suppose $K = \{k_1, \ldots, k_p\}$ is a set of non-zero, mutually disjoint components of e such that $e = \sum_{i=1}^{p} k_i$. By a *disjoint refinement* of K we mean a set $M = \{m_1, \ldots, m_r\}$ (where $p \leq r$) of non-zero, mutually disjoint components of e such that

$$e = \sum_{i=1}^{r} m_i \quad \text{and} \quad k_i = \sum_{\substack{m_s \leq k_i \\ 1 \leq s \leq r}} m_s \quad \text{for each } 1 \leq i \leq p.$$

Proposition A.1.15 *Let E be a Riesz space with $0 < e \in E$. Suppose $K = \{k_1, \ldots, k_p\}$ and $L = \{l_1, \ldots, l_q\}$ are sets each consisting of non-zero, mutually disjoint components of e such that $e = \sum_{i=1}^{p} k_i = \sum_{j=1}^{q} l_j$. Then there exists a disjoint refinement M of both K and L.*

Definition A.1.16 Let E be a Riesz space with $0 < e \in E$. Any $s \in E$ for which there exist pairwise disjoint components $p_1, p_2, \ldots, p_n$ of e and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ such that $s = \sum_{i=1}^n \alpha_i p_i$, is called an *e-step-function.* We may assume without loss of generality that $\sum_{i=1}^n p_i = e$, in this case $s = \sum_{i=1}^n \alpha_i p_i$ is known as the *standard representation* of s and is not unique.

We note that although standard representations are not unique, the α_i's are uniquely determined for a particular standard representation. It follows from Proposition A.1.15 that any two e-step functions may be written as a linear combination of the same components.

Proposition A.1.17 *Let E be a Riesz space with $0 < e \in E$. The set of all e-step-functions*

$$S(E) := \left\{ \sum_{i=1}^{n} \alpha_i p_i : p_i \text{ a component of } e,\ \alpha_i \in \mathbb{R},\ n \in \mathbb{N} \right\}$$

is a Riesz subspace of E.

The proof of the next result can be found in [106, Theorem 33.2].

Theorem A.1.18 (Freudenthal's Spectral Theorem) *Let E be a Riesz space with the principle projection property and let $0 < e \in E$. Then for any $f \geq 0$ in the principle ideal A_e, there exist sequences (s_n) and (t_n) of positive e-step-functions such that $s_n \uparrow f$ and $t_n \downarrow f$ hold e-uniformly.*

A.2 Normed Riesz spaces and Banach lattices

In this section we introduce the notion of a norm on a Riesz space in a compatible manner and look at some related results and properties. The material in this section can be found in [7, 106, 76, 92].

Definition A.2.1 Let X be a real vector space.

(a) A map $\|\cdot\| : X \to \mathbb{R}$ is called a *norm* if
  (i) $\|f\| > 0$ for all $f \in X$ and $\|f\| = 0$ if and only if $f = 0$,
  (ii) $\|\alpha f\| = |\alpha| \|f\|$ for all $f \in X$ and $\alpha \in \mathbb{R}$ and
  (iii) $\|f+g\| \leq \|f\| + \|g\|$ for all $f, g \in X$ (this is known as the triangle inequality).
(b) The pair $(X, \|\cdot\|)$ is called a *normed space.*
(c) If $(X, \|\cdot\|)$ is complete with respect to the norm, i.e. every norm Cauchy sequence has a limit in X, then $(X, \|\cdot\|)$ is called a *Banach space.*

(d) The set $\mathrm{ball}(X) := \{x \in X : \|x\| \leq 1\}$ is called the *closed unit ball* in X.

An order structure can be added to the normed structure in a compatible way.

Definition A.2.2 Let E be a Riesz space

(a) If E is equipped with a norm $\|\cdot\|$, then $\|\cdot\|$ is called a *Riesz norm* if for all $f, g \in E$ with $|f| \leq |g|$, we have that $\|f\| \leq \|g\|$.
(b) A Riesz space E equipped with a Riesz norm is called a *normed Riesz space.*
(c) If a normed Riesz space E is complete with respect to the norm, then E is called a *Banach lattice.*

Note that every normed Riesz space is Archimedean. In general, the normed topology does not coincide with the order topology. We establish some relationships between the different modes of convergence (cf. [106, Theorems 10.3, 15.3, 15.4 and 15.7]).

Proposition A.2.3 *Let E be a normed Riesz space and (f_n) a sequence in E. Then the following statements hold:*

(a) $f_n \to f$ (u-un) implies $f_n \to f$ (ord).
(b) $f_n \to f$ (u-un) implies $f_n \to f$ (norm).
(c) $f_n \uparrow$ and $f_n \to f$ (norm) implies $f_n \uparrow f$.
(d) $f_n \to g$ (ord) and $f_n \to f$ (norm) implies $f = g$.
(e) If D is an upwards directed set in E such that D converges in norm to f_0, then $f_0 = \sup D$.

Definition A.2.4 Let E be a Riesz space.

(a) E is said to be *order separable* if every subset of E possessing a supremum in E contains a finite or countable subset having the same supremum.
(b) E is said to be *super Dedekind complete* if E is order separable and Dedekind complete.
(c) The normed Riesz space E is said to have *order continuous norm* if for any subset $D \downarrow 0$ in E, we have $\inf\{\|f\| : f \in D\} = 0$. The norm is said to be *σ-order continuous* if for any sequence $f_n \downarrow 0$ in E we have $\|f_n\| \downarrow 0$.

It is evident that sequences that converge in the order topology of a Banach lattice with order continuous norm, also converge in the norm topology. The next result can be found in [106, Theorem 17.8].

Theorem A.2.5 *Any Banach lattice having order continuous norm is super Dedekind complete.*

We list some important characterizations of Banach lattices with order continuous norm (cf. [106, Theorems 17.9 and 17.14]).

Theorem A.2.6 *For a Banach lattice E the following conditions are equivalent:*

(a) E has order continuous norm.
(b) E has σ-order continuous norm and E is Dedekind σ-complete.
(c) Every sequence in E which is increasing and bounded above converges in norm.
(d) Every order bounded disjoint sequence in E converges in norm to zero.

We present some special types of Banach lattice which are commonly found in mathematical analysis.

Definition A.2.7 Let $(E, \|\cdot\|)$ denote a normed Riesz space.

(a) $(E, \|\cdot\|)$ is called an *L-normed* space if $\|\cdot\|$ satisfies $\|x+y\| = \|x\| + \|y\|$ for all $x, y \in E_+$. An L-normed Banach lattice is called an AL-space.
(b) $(E, \|\cdot\|)$ is called an *M-normed* space if $\|\cdot\|$ satisfies $\|x \vee y\| = \|x\| \vee \|y\|$ for all $x, y \in E_+$. An M-normed Banach lattice is called an AM-space.

Every non-zero positive element in a Banach lattice can generate an AM-space, as the next proposition shows (cf. [92, Chapter II, §7, Proposition 7.2]).

Proposition A.2.8 *Let E be a Banach lattice. For each $e \in E_+$ the gauge function of $[-e, e]$, given by*

$$p_e(x) := \inf\{\lambda \in \mathbb{R} : \lambda e \leq x \leq \lambda e\} \quad \textit{for all } x \in E,$$

is an M-norm on the principle ideal E_e so that (E_e, p_e) is an AM-space with order unit e and unit ball $[-e, e]$. Moreover, the canonical inclusion $E_e \to E$ is continuous.

A.3 Operator theory on normed spaces

The reader is referred to [7, 24, 89] for a comprehensive presentation of the material in this section.

Definition A.3.1 Let X and Y be vector spaces.

(a) We shall call a map $T : X \to Y$ a *linear operator* if we have $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for each $\alpha, \beta \in \mathbb{R}$, $x, y \in X$. Note that we sometimes denote $T(x)$ by Tx.

(b) A linear operator $P : X \to X$ is called a *projection* if $P^2x = P(Px) = Px$ for all $x \in X$.
(c) We shall denote by $\mathrm{id}_Z : Z \to Z$ the *identity* operator on a vector space Z, which is defined by $\mathrm{id}_Z(x) = x$ for all $x \in Z$.

Definition A.3.2 Let X and Y be vector spaces and $T : X \to Y$ be a linear operator.

(a) We denote the *range* of T by $\mathcal{R}(T) = \{y \in Y : \exists\, x \in X$ so that $Tx = y\}$. We note that $\mathcal{R}(T)$ is a vector subspace of Y.
(b) We denote the *kernel* or *null space* of T by $\mathcal{N}(T) = \{x \in X : Tx = 0\}$. We note that $\mathcal{N}(T)$ is a vector subspace of X.
(c) We define the *rank* of a linear operator to be the dimension of $\mathcal{R}(T)$ as a vector space.
(d) We define the *nullity* of a linear operator to be the dimension of $\mathcal{N}(T)$ as a vector space.

Definition A.3.3 Let X and Y vector spaces.

(a) We shall denote by $L(X, Y)$ the vector space of all linear operators from X into Y. If $X = Y$ then we shall write $L(X, X)$ as $L(X)$.
(b) In the case where $Y = \mathbb{R}$, we shall write $L(X, Y)$ as $X^\#$. The elements of $X^\#$ are called *linear functions* and $X^\#$ is called the *algebraic dual* of X.
(c) $X^{\#\#} = (X^\#)^\#$ is called the *algebraic bidual* of X.

We note that a vector space X can be canonically embedded as a subspace of its bidual under the injective linear mapping $i_X : X \to X^{\#\#}$ defined by $\langle x^\#, i_X(x)\rangle = \langle x, x^\#\rangle$ for all $x \in X$ and $x^\# \in X^\#$. We may view this as an abstract containment and denote this as $X \subset X^{\#\#}$.

Definition A.3.4 Let X and Y denote normed spaces, and $T : X \to Y$ denote a linear operator.

(a) $T : X \to Y$ is called *bounded* if there exists a constant $C > 0$ such that $\|Tx\| \leq C\|x\|$ for all $x \in X$.
(b) $T : X \to Y$ is called *open* if $T(O)$ is open in Y for every open set $O \subset X$.
(c) $T : X \to Y$ is called an *isometry* if $\|Tx\| = \|x\|$ for all $x \in X$.
(d) $T : X \to Y$ is called an *isomorphism* if the exists a $K > 0$ such that $K^{-1}\|x\| \leq \|Tx\| \leq K\|x\|$ for all $x \in X$.

(e) $T : X \to Y$ is called a *metric surjection* if T is surjective and

$$\|y\| = \inf\{\|x\| : x \in X,\, Tx = y\}$$

for every $y \in Y$. Metric surjections are sometimes referred to as *quotient operators.*

It is easily shown that a linear operator is bounded if and only if it is continuous, therefore we will use these terms interchangeably.

Note that part (e) in the above definition is equivalent to $T : X \to Y$ mapping the open unit ball of X onto the open unit ball of Y. This implies that Y is isometrically isomorphic to the quotient space $X/\mathcal{N}(T)$.

Definition A.3.5 Let X and Y be normed spaces.

(a) We define the normed space $\mathcal{L}(X,Y)$ by $\mathcal{L}(X,Y) := \{T \in L(X,Y) : \text{T is bounded}\}$ together with the *operator norm* $\|\cdot\|$ defined by $\|T\| = \sup\{\,\|Tx\| : \|x\| \leq 1\}$ for all $T \in \mathcal{L}(X,Y)$. If $X = Y$ then we shall write $\mathcal{L}(X,X)$ as $\mathcal{L}(X)$.
(b) In the case where $Y = \mathbb{R}$, we shall write $\mathcal{L}(X,Y)$ as X^*. The elements of X^* are called *linear functionals* and X^* is called the *continuous dual* of X.
(c) We call $X^{**} = (X^*)^*$ the *continuous bidual* of X.

If X is a normed space and Y is a Banach space, then $\mathcal{L}(X,Y)$ is also a Banach space with respect to the operator norm. In particular, we have that X^* is a Banach space.

We note that a normed space X can be canonically embedded as a subspace of its bidual under the isometry $i_X : X \to X^{**}$ defined by $\langle x^*, i_X(x)\rangle = \langle x, x^*\rangle$ for all $x \in X$ and $x^* \in X^*$. Again, we see this as an abstract containment where the normed structure is preserved and we denote this as $X \hookrightarrow X^{**}$. The elements of $X \hookrightarrow X^{**}$ are sometimes referred to as *induced linear functionals* on X^*. Since X^{**} is always a Banach space, the closure of X in X^{**} is complete, which shows every normed space has a completion.

We now state some fundamental results from functional analysis.

Theorem A.3.6 *(a)* (Open Mapping Theorem) *A bounded linear surjection acting between Banach spaces is open.*
(b) (Closed Graph Theorem) *A linear operator between acting Banach spaces is bounded if and only if its graph is closed.*
(c) (Principle of Uniform Boundedness) *Let X and Y be Banach spaces and $S \subset \mathcal{L}(X,Y)$. If* $\sup\{\|Tx\| : T \in S\} < \infty$ *for all $x \in X$, then* $\sup\{\|T\| : T \in S\} < \infty$.

(d) (Hahn-Banach) *If f is a bounded linear functional on a subspace of a normed space, then f extends to the whole space with preservation of norm.*

Corollary A.3.7 (Hahn-Banach)

(a) If X is a normed linear space and $x \in X$, then there exists $x^* \in X^*$ of norm 1 such that $x^*(x) = \|x\|$.
(b) If X is a normed space, then for all $x \in X$ we have $\|x\| = \sup\{|x^*(x)| : \|x^*\| \leq 1, x^* \in X^*\}$.
(c) If X is a normed space and $x^*(x) = 0$ for all $x^* \in \mathrm{ball}(X^*)$, then $x = 0$; i.e. $\mathrm{ball}(X^*)$ separates the points in X.

Definition A.3.8 Let X and Y be normed spaces.

(a) Let $T \in \mathcal{L}(X, Y)$. We define the *adjoint* $T^* : Y^* \to X^*$ by

$$\langle x, T^*y^* \rangle = \langle Tx, y^* \rangle$$

for all $y^* \in Y^*$ and $x \in X$.
(b) For $T \in \mathcal{L}(X, Y)$, we call $T^{**} : X^{**} \to Y^{**}$ the *second adjoint* of T.

We collect some useful results involving adjoints.

Proposition A.3.9 *Let X and Y be normed spaces, then the following statements hold:*

(a) The mapping $T \mapsto T^*$ is an isometry of $\mathcal{L}(X, Y)$ into $\mathcal{L}(Y^*, X^*)$.
(b) The second adjoint $T^{**} : X^{**} \to Y^{**}$ is a unique continuous extension of $T : X \to Y$; if X is reflexive, then $T^{**} = T$.
(c) $T : X \to Y$ is an isometry if and only if $T^* : Y^* \to X^*$ is a metric surjection.
(d) $T : X \to Y$ is a metric surjection if and only if $T^* : Y^* \to X^*$ is an isometry.
(e) If X and Y are Banach spaces, then a bounded linear operator $T : X \to Y$ has closed range if and only if $T^* : Y^* \to X^*$ has closed range.

We define other weaker-than-norm topologies that exist on Banach spaces and their duals.

Definition A.3.10 Let X be a normed space.

(a) The topology on X generated by the norm is called the *strong topology* on X.
(b) The weakest topology on X allowing all the linear functionals in X^* to be continuous is called the *weak topology* on X, and is denoted by $\sigma(X, X^*)$.
(c) The weak topology on X^* is denoted by $\sigma(X^*, X^{**})$.

(d) The weakest topology on X^* allowing all the induced linear functionals in X to be continuous is called the *weak* topology* on X^*, and is denoted by $\sigma(X^*, X)$. Evidently $\sigma(X^*, X)$ is weaker than $\sigma(X^*, X^{**})$.

(e) A normed space X is called *reflexive* if $X = X^{**}$, in this case the weak and the weak* topologies on X^* coincide.

Theorem A.3.11 (BANACH-ALAOGLU) *If X is a normed space, then the closed unit ball of X^* is $\sigma(X^*, X)$ compact.*

The above theorem implies that every sequence in the closed unit ball of X^* has a weak* convergent subsequence.

A.4 Operator theory on Riesz spaces

In this section, we identify various classes of operators between (normed) Riesz spaces. The material in this section is taken from [106, 76, 92].

Definition A.4.1 Let E and F denote Riesz spaces and $T \in L(E, F)$.

(a) T is called *positive* if $T(E_+) \subset F_+$. If T is positive we write $T \geq 0$ and we denote the space of positive operators by $L_+(E, F)$. It is clear that $L(E, F)$ becomes an ordered vector space under the ordering defined by $T_1 \geq T_2 \Leftrightarrow T_1 - T_2 \geq 0$ for all $T_1, T_2 \in L(E, F)$.

(b) T is called *regular* if $T = T_1 - T_2$ with $T_1, T_2 \in L_+(E, F)$. We denote the space of regular operators by $L^r(E, F)$, which is a vector subspace of $L(E, F)$. Note that $L_+(E, F)$ is a proper and generating positive cone of $L^r(E, F)$.

(c) T is called *order bounded* if T maps any order interval in E into an order interval in F. We denote the space of order bounded operators by $L^b(E, F)$, which is a vector subspace of $L(E, F)$.

(d) T is called *order continuous* if T is regular and for any downwards directed set $D \subset E$ with $D \downarrow 0$ we have $\inf\{|Tf| : f \in D\} = 0$ in F. The vector space of all order continuous operators in $L^r(E, F)$ is denoted by $L^n(E, F)$.

(e) T is called *σ-order continuous* if T is regular and for any sequence (f_n) in E with $f_n \downarrow 0$ we have $\inf\{|Tf_n| : n \in \mathbb{N}\} = 0$ in F. The vector space of all σ-order continuous operators in $L^r(E, F)$ is denoted by $L^c(E, F)$.

(f) T is called a *Riesz homomorphism* if for all $f, g \in E$ we have $T(f \vee g) = T(f) \vee T(g)$. A *Riesz isomorphism* is an injective Riesz homomorphism.

Note that ψ is a Riesz homomorphism if and only if $|\psi(f)| = \psi(|f|)$ for all $f \in E$. Riesz homomorphisms are necessarily positive.

The next theorem collects some fundamental structural results involving operators defined above. These results can be found in [106, Theorems 18.3, 18.4 and 20.2] and [76, Proposition 1.3.9] respectively.

Theorem A.4.2 *(a) For the Riesz spaces E and F, we have the inclusion*

$$L^n(E, F) \subset L^c(E, F) \subset L^r(E, F) \subset L^b(E, F) \subset L(E, F).$$

(b) Let E be a Banach lattice and F be a normed Riesz space, then we have the inclusion $L^r(E, F) \subset L^b(E, F) \subset \mathcal{L}(E, F)$.

(c) For E and F Riesz spaces with F Dedekind complete, we have

$$L^r(E, F) = L^b(E, F).$$

Moreover, we have that $L^r(E, F)$ is a Dedekind complete Riesz space with $L_+(E, F)$ as positive cone.

(d) For E and F Riesz spaces with F Dedekind complete, we have that $L^n(E, F)$ and $L^c(E, F)$ are bands in the Dedekind complete Riesz space $L^r(E, F)$.

In view of (b) in the above theorem, we shall denote $L_+(E, F)$, $L^r(E, F)$ and $L^b(E, F)$ by $\mathcal{L}_+(E, F)$, $\mathcal{L}^r(E, F)$ and $\mathcal{L}^b(E, F)$ respectively whenever E is a Banach lattice and F is a normed Riesz space. The next proposition can be found in [76, Proposition 1.3.6].

Proposition A.4.3 *Let E and F be Banach lattices. For every $T \in \mathcal{L}^r(E, F)$ we define the r-norm of T by*

$$\|T\|_r = \inf\{\|S\| : S \in \mathcal{L}_+(E, F),\ |Tx| \le S|x|\ \forall\ x \in E_+\}.$$

$(\mathcal{L}^r(E, F), \|\cdot\|_r)$ is a Banach space. Moreover, $\|T\| \le \|T\|_r$ for every regular operator $T : E \to F$. If F, in addition, is Dedekind complete, then $(\mathcal{L}^r(E, F), \|\cdot\|_r)$ is a Banach lattice such that $\|T\|_r = \|\,|T|\,\|$ for every regular operator $T : E \to F$.

We now turn our attention to dual spaces.

Definition A.4.4 Let E be a Riesz space.

(a) We define the *order dual* of E to be $E^\sim = L^b(E, \mathbb{R})$.

(b) We define the *order continuous dual* of E to be $E_n^\sim = L^n(E, \mathbb{R})$.

(c) We define the *σ-order continuous dual* of E to be $E_c^\sim = L^c(E, \mathbb{R})$.

Note that $E^\sim$ is a Dedekind complete Riesz space and that $E_n^\sim$ and $E_c^\sim$ are bands in $E^\sim$. It is clear that $E_n^\sim \subset E_c^\sim \subset E^\sim$. The next result explains the relationships between the normed dual and order dual on a normed Riesz space. The proofs can be found in [106, Theorems 25.8 and 25.10].

Proposition A.4.5 *Let E be a normed Riesz space, then the following statements hold:*

(a) E^* is an ideal in $E^\sim$.

(b) E^* is a Dedekind complete Banach lattice with respect to the ordering inherited from $E^\sim$. Moreover, if G is an upwards directed set of positive elements in E^* such that $G \uparrow \varphi_0$, then $\{\|\varphi\| : \varphi \in G\} \uparrow \|\varphi_0\|$.

(c) If E is a Banach lattice, then $E^\sim = E^*$.

(d) If E is a Banach lattice with order continuous norm then $E^\sim = E_n^\sim = E^*$.

The canonical embedding $i_E : E \to E^{**}$ is in fact a Riesz homomorphism, as well as an isometry, as the next result indicates. The following theorem is easily derived from the above proposition and [106, pp. 204–205].

Theorem A.4.6 *Let E be a normed Riesz space and $i_E : E \to E^{**}$ be the canonical embedding defined by $\langle x^*, i_E(x)\rangle = \langle x, x^*\rangle$ for all $x^* \in E^*$. Then the following statements are true.*

(a) $i_E : E \to E^{**}$ is a Riesz homomorphism and an isometry whose range is contained in $(E^*)_n^\sim$.

(b) $i_E : E \to E^{**}$ preserves arbitrary suprema and infima if and only if $E^* \subset E_n^\sim$. In particular, this is true when E is an order continuous Banach lattice.

AM-spaces and AL-spaces share a duality. The proof of the next proposition can be found in [92, Chapter II, §9, Proposition 9.1].

Proposition A.4.7 *The dual of each M-normed space is an AL-space, and the dual of each L-normed space is an AM-space.*

A.5 Tensor products of vector spaces

The material in this section is taken from [50, 90]. Throughout this section let X, Y and Z denote real vector spaces.

Definition A.5.1 A map $\varphi : X \times Y \to Z$ is called *bilinear* if we have

(a) $\varphi(\alpha x + \beta y, z) = \alpha\varphi(x, z) + \beta\varphi(y, z)$ for each $\alpha, \beta \in \mathbb{R}$, $x, y \in X$ and $z \in Y$ and
(b) $\varphi(x, \gamma y + \eta z) = \gamma\varphi(x, y) + \eta\varphi(x, z)$ for each $\gamma, \eta \in \mathbb{R}$, $x \in X$ and $y, z \in Y$.

Definition A.5.2 We write $B(X\times Y, Z)$ for the vector space of all bilinear mappings from $X\times Y$ into Z. If Z is $\mathbb{R}$, then we just write $B(X\times Y)$. The elements of $B(X\times Y)$ are called *bilinear forms*.

Definition A.5.3 The *tensor product* $X\otimes Y$ of the vector spaces X and Y is defined to be the vector subspace of $B(X \times Y)^{\#}$ generated by the set

$$\Big\{x \otimes y \in B(X \times Y)^{\#} : \exists\ (x, y) \in X \times Y \quad \text{such that } \langle\varphi, x \otimes y\rangle = \varphi(x, y)\ \forall\ \varphi \in B(X \times Y)\Big\}.$$

An element $u \in X \otimes Y$ is called a *tensor* and is of the form $u = \sum_{i=1}^{n} x_i \otimes y_i$ where $x_i \in X$, $y_i \in Y$ and $i = 1, \dots n$.

It is easy to see that the map $\otimes : X \times Y \to X \otimes Y$, defined by $(x, y) \mapsto x \otimes y$, is bilinear and thus exhibits properties of a multiplication. For any $u \in X \otimes Y$ with $u \neq 0$ there exists a smallest number $n \in \mathbb{N}$ such that $u = \sum_{i=1}^{n} x_i \otimes y_i$ where the x_i's *and* the y_i's are linearly independent. Note that this representation is not unique.

Definition A.5.4 If $u \in X\otimes Y$ with $u \neq 0$ and $u = \sum_{i=1}^{n} x_i\otimes y_i$ where n is minimal, then n is called the *rank* of u. A tensor of rank one (i.e. $u = x \otimes y$) is called an *elementary tensor*. Note that $0 \otimes y = x \otimes 0 = 0$ and has rank zero.

We provide a means of identifying the zero tensor. This result can be found in [90, Proposition 1.2].

Proposition A.5.5 *The following statements are equivalent for* $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$*:*

(a) $u = 0$.
(b) $\sum_{i=1}^{n} x^{\#}(x_i)y^{\#}(y_i) = 0$ *for all* $x^{\#} \in X^{\#}$ *and* $y^{\#} \in Y^{\#}$.
(c) $\sum_{i=1}^{n} y^{\#}(y_i)x_i = 0$ *for all* $y^{\#} \in Y^{\#}$.
(d) $\sum_{i=1}^{n} x^{\#}(x_i)y_i = 0$ *for all* $x^{\#} \in X^{\#}$.

The main purpose of tensor products is *linearize* bilinear maps. The following result can be found in [90, Proposition 1.4].

Proposition A.5.6 *Let* X*,* Y *and* Z *be arbitrary vector spaces, then* $L(X \otimes Y, Z)$ *is isomorphic to* $B(X \times Y, Z)$ *under the mapping* $f \mapsto f \circ \otimes$ *for all* $f \in L(X \otimes Y, Z)$*. In particular, if* Z *is* $\mathbb{R}$*, then we have* $B(X \times Y) = (X \otimes Y)^{\#}$.

It follows that every bilinear map φ from $X \times Y$ into Z induces a unique linear map φ^ℓ from $X \otimes Y$ into Z so that the following diagram commutes:

$$\begin{tikzcd} X \times Y \arrow[r, "\varphi"] \arrow[d, "\otimes"'] & Z \\ X \otimes Y \arrow[ur, "\varphi^\ell"'] & \end{tikzcd}$$

An important consequence of the this result is that the tensor product $X \otimes Y$ of vector spaces X and Y always exists and is unique up to isomorphism. This result also enables us to characterize a tensor by means of embedding $X \otimes Y$ into some familiar vector spaces.

Consider the bilinear map $\Phi : X \times Y \to B(X^\# \times Y^\#)$ defined by

$$\langle (x^\#, y^\#), \Phi(x, y)\rangle = x^\#(x)y^\#(y)$$

for all $(x, y) \in X \times Y$ and $(x^\#, y^\#) \in X^\# \times Y^\#$. Linearizing, we obtain an injection $\Phi^\ell : X \otimes Y \to B(X^\# \times Y^\#)$ where $\langle (x^\#, y^\#), \Phi^\ell(x \otimes y)\rangle = x^\#(x)y^\#(y)$ for all $x \otimes y \in X \otimes Y$ and $(x^\#, y^\#) \in X^\# \times Y^\#$. This enables us to view a tensor in $X \otimes Y$ as a bilinear form on $X^\# \times Y^\#$ whose action is defined by

$$\langle (x^\#, y^\#), u\rangle = \left\langle (x^\#, y^\#), \sum_{i=1}^{n} x_i \otimes y_i \right\rangle = \sum_{i=1}^{n} x^\#(x_i)y^\#(y_i)$$

for all $u \in X \otimes Y$ and $(x^\#, y^\#) \in X^\# \times Y^\#$. Thus, we have a canonical embedding $X \otimes Y \subset B(X^\# \times Y^\#)$.

Analogous to the above, the bilinear map $\Phi : X^\# \times Y^\# \to B(X \times Y)$ defined by $\langle (x, y), \Phi(x^\#, y^\#)\rangle = x^\#(x)y^\#(y)$ for all $(x^\#, y^\#) \in X^\# \times Y^\#$ and $(x, y) \in X \times Y$ yields a canonical embedding $X^\# \otimes Y^\# \subset B(X \times Y) = (X \otimes Y)^\#$.

We can induce two more useful embeddings in the following way. Define the bilinear map $\Phi : X \times Y \to L(X^\#, Y)$ by

$$\langle x^\#, \Phi(x, y)\rangle = x^\#(x)y$$

for all $(x, y) \in X \times Y$ and $x^\# \in X^\#$. Linearizing, we obtain an injection $\Phi^\ell : X \otimes Y \to L(X^\#, Y)$ where $\langle x^\#, \Phi^\ell(x \otimes y)\rangle = x^\#(x)y$ for all $x \otimes y \in X \otimes Y$ and $x^\# \in X^\#$. This enables us to view a tensor in $X \otimes Y$ as a linear operator from $X^\#$ into Y whose action is defined by

$$\langle x^{\#}, u\rangle = \left\langle x^{\#}, \sum_{i=1}^{n} x_i \otimes y_i \right\rangle = \sum_{i=1}^{n} x^{\#}(x_i) y_i$$

for all $u \in X \otimes Y$ and $x^{\#} \in X^{\#}$. Thus, we have a canonical embedding $X \otimes Y \subset L(X^{\#}, Y)$. By a symmetrical argument we obtain another embedding $X \otimes Y \subset L(Y^{\#}, X)$. We summarize these embeddings in the following proposition.

Proposition A.5.7 *For the vector spaces X and Y we have the following embeddings:*

(a) $X \otimes Y \subset B(X^{\#} \times Y^{\#})$,
(b) $X^{\#} \otimes Y^{\#} \subset B(X \times Y) = (X \otimes Y)^{\#}$,
(c) $X \otimes Y \subset L(X^{\#}, Y)$ *and*
(d) $X \otimes Y \subset L(Y^{\#}, X)$.

This justifies the following definition.

Definition A.5.8 Let X and Y be vector space and $u \in X \otimes Y$.

(a) We denote the bilinear form in $B(X^{\#} \times Y^{\#})$ induced by u by B_u.
(b) We denote the two linear operators induced by u by L_u and R_u respectively where $L_u \in L(X^{\#}, Y)$ and $R_u \in L(Y^{\#}, X)$.

It is worth noting that the rank of a tensor $u \in X \otimes Y$ and the rank of the induced operators L_u and R_u coincide.

A.6 Tensor products of Banach spaces

In this section we look at some ways of equipping the tensor product of two Banach spaces with a norm, in particular we look at the projective and injective norms. The material in this section is taken almost exclusively from [90].

Definition A.6.1 Let X, Y and Z be normed spaces.

(a) We call a bilinear mapping $\varphi : X \times Y \to Z$ *bounded* if there exists a constant $C > 0$ such that $\|\varphi(x, y)\| \leq C\|x\|\|y\|$ for all $x \in X$ and $y \in Y$.
(b) We define the normed vector space

$$\mathcal{B}(X \times Y, Z) = \{\varphi \in B(X \times Y, Z) : \varphi \text{ is bounded}\}$$

with the norm $\|\cdot\|$ defined by $\|\varphi\| = \sup\{\|\varphi(x, y)\| : \|x\| \leq 1, \|y\| \leq 1\}$ for all $\varphi \in \mathcal{B}(X \times Y, Z)$. If Z is $\mathbb{R}$, then we just write $\mathcal{B}(X \times Y)$. The elements of $\mathcal{B}(X \times Y)$ are called *bounded bilinear forms*.

If X, Y are normed spaces and Z is a Banach space, then $\mathcal{B}(X \times Y, Z)$ is a Banach space with respect to $\|\cdot\|$. In particular, we have that $\mathcal{B}(X \times Y)$ is a Banach space. The following proposition is clear.

Proposition A.6.2 *For the Banach spaces X and Y, we have the following embeddings:*

(a) $X \otimes Y \hookrightarrow \mathcal{B}(X^* \times Y^*)$.
(b) $X^* \otimes Y^* \hookrightarrow \mathcal{B}(X \times Y)$.
(c) $X \otimes Y \hookrightarrow \mathcal{L}(X^*, Y)$.
(d) $X \otimes Y \hookrightarrow \mathcal{L}(Y^*, X)$.

Definition A.6.3 Let X and Y be Banach spaces.

(a) If we equip $X \otimes Y$ with a norm α, we shall denote the normed space $(X \otimes Y, \alpha)$ by $X \otimes_\alpha Y$ and its norm completion by $X \widetilde{\otimes}_\alpha Y$. We shall sometimes use the notation $\alpha_{X,Y}(u)$ to denote the norm of a tensor u in the tensor product $X \otimes Y$ if there is a chance of ambiguity.
(b) We say a norm α on $X \otimes Y$ is a *reasonable crossnorm* if it has the following properties
   (i) $\alpha(x \otimes y) \leq \|x\|\|y\|$ for all $x \in X$ and $y \in Y$ and
   (ii) for all $x^* \in X^*$ and $y^* \in Y^*$, the linear functional $x^* \otimes y^*$ on $X \otimes Y$ is bounded and $\|x^* \otimes y^*\| \leq \|x^*\|\|y^*\|$.
(c) We define the *projective norm* on $X \otimes Y$ to be
$$\pi(u) = \inf \left\{ \sum_{i=1}^n \|x_i\|\|y_i\| : u = \sum_{i=1}^n x_i \otimes y_i \right\}$$
for all $u \in X \otimes Y$.
(d) We define the *injective norm* on $X \otimes Y$ to be $\varepsilon(u) = \|L_u\|$ for all $u \in X \otimes Y$ where L_u is the induced linear operator in $\mathcal{L}(X^*, Y)$.

In view of the fact that the norm of L_u and the norms of the induced maps $R_u \in \mathcal{L}(Y^*, X)$ and $B_u \in \mathcal{B}(X^* \times Y^*)$ coincide, we have that

$$\begin{aligned}
\varepsilon(u) &= \sup \left\{ \left\| \sum_{i=1}^n x^*(x_i) y_i \right\| : \|x^*\| \leq 1 \right\} \\
&= \sup \left\{ \left\| \sum_{i=1}^n y^*(y_i) x_i \right\| : \|y^*\| \leq 1 \right\} \\
&= \sup \left\{ \left| \sum_{i=1}^n y^*(y_i) x^*(x_i) \right| : \|x^*\| \leq 1, \|y^*\| \leq 1 \right\}
\end{aligned}$$

for all $u = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$.

Note that $X \widetilde{\otimes}_\varepsilon Y$ is just the closure of $X \otimes Y$ in $\mathcal{L}(X^*, Y)$ with respect to the operator norm. The next result can be found in [90, Propositions 2.1, 2.3 and 3.1].

Proposition A.6.4 *Let X and Y be Banach spaces. Then π and ε are reasonable crossnorms on $X \otimes Y$ and $\varepsilon(u) \leq \pi(u)$ for all $u \in X \otimes Y$.*

Reasonable crossnorms are characterized by the following result which can be found in [90, Proposition 6.1].

Proposition A.6.5 *Let X and Y be Banach spaces.*

(a) A norm α on $X \otimes Y$ is a reasonable crossnorm if and only if $\varepsilon(u) \leq \alpha(u) \leq \pi(u)$ for all $u \in X \otimes Y$.

(b) If α is a reasonable crossnorm on $X \otimes Y$, then $\alpha(x \otimes y) = \|x\|\|y\|$ for all $x \in X$ and $y \in Y$.

(c) If α is a reasonable crossnorm on $X \otimes Y$, then for all $x^* \in X^*$ and $y^* \in Y^*$ we have that the norm of the linear functional $x^* \otimes y^*$ on $(X \otimes Y, \alpha)$ satisfies $\|x^* \otimes y^*\| = \|x^*\|\|y^*\|$.

Definition A.6.6 Let W, X, Y, Z be Banach spaces.

(a) Let $S \in \mathcal{L}(X, W)$ and $T \in \mathcal{L}(Y, Z)$. Then the tensor product of S and T is the unique linear mapping $S \otimes T : X \otimes Y \to W \otimes Z$, defined by $S \otimes T(x \otimes y) = (Sx) \otimes (Ty)$ for all $x \in X$ and $y \in Y$.

(b) We call a reasonable crossnorm α on $X \otimes Y$ a *uniform crossnorm* if for any bounded linear operators $S : X \to W$, $T : Y \to Z$ we have that the operator $S \otimes T : X \otimes_\alpha Y \to W \otimes_\alpha Z$ is bounded and $\|S \otimes T\| \leq \|S\|\|T\|$. We denote the unique continuous (norm preserving) extension of $S \otimes T$ by $S \otimes_\alpha T : X \widetilde{\otimes}_\alpha Y \to W \widetilde{\otimes}_\alpha Z$.

Note that the norms ε and π are uniform crossnorms.

Definition A.6.7 Let X and Y be Banach spaces.

(a) We call a reasonable crossnorm α on $X \otimes Y$ *injective* if whenever E and F are subspaces of X and Y respectively, the norm induced on $E \otimes F$ by the norm on $X \otimes_\alpha Y$ coincides with the norm on $E \otimes_\alpha F$.

(b) We call a reasonable crossnorm α on $X \otimes Y$ *projective* if whenever W and Z are quotients of X and Y respectively with quotient operators $Q : X \to W$ and $R : Y \to Z$, we have that $Q \otimes_\alpha R : X \otimes_\alpha Y \to W \otimes_\alpha Z$ is also a quotient operator. That is to say that $W \otimes_\alpha Z$ is a quotient of $X \otimes_\alpha Y$.

As is evident from their names, the uniform crossnorms ε and π are respectively injective and projective.

Definition A.6.8 Let X and Y be Banach spaces and α be a reasonable crossnorm on $X \otimes Y$. The *transpose map* $u \mapsto {}^t u$ from $X \otimes Y$ onto $Y \otimes X$ is given by $u = \sum_{i=1}^n x_i \otimes y_i \mapsto {}^t u := \sum_{i=1}^n y_i \otimes x_i$. The *transpose* of α, denoted by ${}^t\alpha$, is the reasonable crossnorm on $X \otimes Y$ defined by ${}^t\alpha(u) = \alpha({}^t u)$ for all $u \in Y \otimes X$. The norm α is called *symmetric* if $\alpha(u) = {}^t\alpha(u)$ for all $u \in X \otimes Y$ and *asymmetric* if it is not symmetric.

In the above definition, the transpose map defines a canonical isometric isomorphism between $X \otimes Y$ and $Y \otimes X$.

Definition A.6.9 A uniform crossnorm α is called *finitely generated* if for pair of Banach spaces X and Y and every $u \in X \otimes Y$, we have

$$\alpha_{X,Y}(u) = \inf\{\alpha(u; M \otimes N) : u \in M \otimes N,\, \dim M < \infty,\, \dim N < \infty\}.$$

A *tensor norm* is defined to be a finitely generated uniform crossnorm.

Thus, the behavior of a tensor norm is completely determined by its values on tensor products of finite dimensional spaces. The norms ε and π are both symmetric tensor norms.

References

1. D.J. Aldous, *Unconditional bases and martingales in $L^p(F)$*, Math. Proc. Camb. Phil. Soc. **85** (1979), 117–123.
2. C.D. Aliprantis and O. Burkinshaw, *The components of a positive operator*, Math. Z. **184** (1983), 245–257.
3. ______, *Positive operators*, Academic Press, Orlando, New York, San Diego, London, 1985.
4. D. Alspach and E. Odell, *L_p spaces*, Handbook of the Geometry of Banach spaces, vol. 1, North-Holland, Amsterdam, London, New York, Oxford, Paris, Shannon, Tokyo, 2001, pp. 123–159.
5. T. Andô, *Contractive projections in L^p spaces*, Pacific J. Math. **17** (1966), 391–405.
6. ______, *Banachverbände und positive projectionen*, Math. Z. **109** (1969), 121–130.
7. B. Beauzamy, *Introduction to Banach spaces and their geometry*, Mathematics Studies, no. 68, North-Holland, Amsterdam, New York, Oxford, 1982.
8. E. Berkson and T.A. Gillespie, *Spectral decompositions and harmonic analysis on UMD spaces*, Studia Math. **112** (1994), no. 1, 13–49.
9. J. Bourgain, *Some remarks on Banach spaces in which martingale difference sequences are unconditional*, Ark. Mat. **21** (1983), 163–168.
10. ______, *Vector valued singular integrals and the H^1-BMO duality*, Probability Theory and Harmonic Analysis, Marcel Dekker, New York, 1986, pp. 1–19.
11. R.D. Bourgin, *Geometric aspects of convex sets with the Radon-Nikodým property*, Lecture Notes in Math., no. 993, Springer-Verlag, Berlin, 1983.
12. Q. Bu and G. Buskes, *The Radon-Nikodým Property for tensor products of Banach lattices*, Positivity **10** (2006), no. 2, 365–390.
13. D.L. Burkholder, *Martingale transforms*, Ann. Math. Stat. **37** (1966), 1494–1504.
14. ______, *A geometrical characterization of Banach spaces in which martingale difference sequences are unconditional*, Ann. Probab. **9** (1981), 997–1011.
15. ______, *A nonlinear partial differential equation and the unconditional constant of the Haar system in L^p*, Bull. Amer. Math. Soc. **7** (1982), 591–595.
16. ______, *Boundary value problems and sharp inequalities for martingale transforms*, Ann. Probab. **12** (1984), 647–702.
17. ______, *An elementary proof of an inequality of R.E.A.C. Paley*, Bull. London. Math. Soc. **17** (1985), 474–478.
18. ______, *Martingales and singular integrals in Banach spaces*, Handbook of the Geometry of Banach spaces, vol. 1, North-Holland, Amsterdam, London, New York, Oxford, Paris, Shannon, Tokyo, 2001, pp. 233–269.

19. J. Chaney, *Banach lattices of compact maps*, Math. Z. **129** (1972), 1–19.
20. S.D. Chatterji, *Martingale convergence and the Radon-Nikodým theorem in Banach spaces*, Math. Scand. **22** (1968), 21–41.
21. K.L. Chung, *A course in probability theory*, second ed., Probability and Mathematical Statistics, no. 21, Academic Press, New York, 1974.
22. P. Clément, B. de Pagter, F.A. Sukochev, and H. Witvliet, *Schauder decomposition and multiplier theorems*, Studia Math. **138** (2000), 135–163.
23. J.J. Conradie and J. Swart, *On Schwartz operator ideals determined by the order-bounded sets in a Banach lattice*, Bull. Inst. Math. Acad. Sinica **11** (1983), no. 4, 525–548.
24. J. B. Conway, *A course in functional analysis*, Springer-Verlag, 1990.
25. S.F. Cullender, W.-C. Kuo, C.C.A. Labuschagne, and B.A. Watson, *Measure-free martingales with application to classical martingales*, Soft Methods for Integrated Uncertainty Modelling (Bristol, England), Advances in Soft Computing, vol. 37, Springer-Verlag, 2006, pp. 121–128.
26. S.F. Cullender and C.C.A. Labuschagne, *A description of norm-convergent martingales on vector-valued L^p-spaces*, J. Math. Anal. Appl. **323** (2006), no. 1, 119–130.
27. ———, *Corrigendum to "Unconditional martingale difference sequences in Banach spaces" [J. Math. Anal. Appl. 326 (2007) 1291–1309]*, J. Math. Anal. Appl. **338** (2007), no. 1, 751–752.
28. ———, *A note on the M-norm of Chaney-Schaefer*, Quaestiones Math. **30** (2007), no. 2, 151–158.
29. ———, *Unconditional martingale difference sequences in Banach spaces*, J. Math. Anal. Appl. **326** (2007), no. 2, 1291–1309.
30. ———, *Unconditional Schauder decompositions and stopping times in the Lebesgue-Bochner spaces*, J. Math. Anal. Appl. **336** (2007), no. 2, 849–864.
31. A. Defant and K. Floret, *Tensor norms and operator ideals*, Mathematics Studies, no. 176, North-Holland, Amsterdam, London, New York, Tokyo, 1993.
32. R. DeMarr, *A martingale convergence theorem in vector lattices*, Canad. J. Math. **18** (1966), 424–432.
33. J. Diestel, *Sequences and Series in Banach spaces*, Springer-Verlag, New York, 1984.
34. J. Diestel, J.H. Fourie, and J. Swart, *The metric theory of tensor products, part 1*, Quaestiones Math. **25** (2002), 37–72.
35. ———, *The metric theory of tensor products, part 2*, Quaestiones Math. **25** (2002), 73–94.
36. J. Diestel, H. Jarchow, and A. Tonge, *Absolutely summing operators*, Cambridge studies in advanced mathematics, no. 43, Cambridge University Press, 1995.
37. J. Diestel and J.J. Uhl, *Vector measures*, Mathematical Surveys, no. 15, A.M.S., Providence, Rhode Island, 1977.
38. N. Dinculeanu, *Integral representation of linear operators. I*, Stud. Cerc. Mat. **18** (1966), 349–385.
39. ———, *Integral representation of linear operators II*, Stud. Cerc. Mat. **18** (1966), 483–536.
40. ———, *Linear operations on L^p-spaces*, Vector and Operator Valued Measures and Applications, Academic Press, New York, 1973, pp. 109–124.
41. J.L. Doob, *Stochastic processes*, John Wiley and Sons Inc., New York, 1990, A reprint of the 1953 original.
42. L.E. Dor and E. Odell, *Monotone bases in L_p*, Pacific J. Math. **60** (1975), 51–61.
43. N. Dunford and B.J. Pettis, *Linear operators on summable functions*, Trans. Amer. Math. Soc. **47** (1940), 323–392.

44. G.A. Edgar and L. Sucheston, *Stopping times and directed processes*, Encyclopedia Math. Appl., vol. 47, Cambridge University Press, 1992.
45. L. Egghe, *Stopping time techniques for analysts and probabilists*, London Mathematical Society Lecture Note Series, no. 100, Cambridge University Press, London, New York, New Rochelle, Melbourne, Sydney, 1984.
46. T. Figiel and P. Wojtaszczyk, *Special bases in function spaces*, Handbook of the Geometry of Banach spaces, vol. 1, North-Holland, Amsterdam, London, New York, Oxford, Paris, Shannon, Tokyo, 2001, pp. 561–597.
47. D.H. Fremlin, *Tensor products of Archimedean vector lattices*, Amer. J. Math. **94** (1972), 778–798.
48. D.H. Fremlin and M. Talagrand, *A decomposition theorem in tensor products of Archimedean Riesz spaces*, Mathematika **26** (1974), 302–305.
49. B. Gelbaum and J. Gil de Lamadrid, *Bases of tensor products of Banach spaces*, Pacific J. Math. **11** (1961), 1281–1286.
50. W.H. Greub, *Multilinear algebra*, Springer-Verlag, Berlin, Heidelberg, New York, 1967.
51. J.J. Grobler and C.C.A. Labuschagne, *An f-algebra approach to the Riesz tensor product of Archimedean Riesz spaces*, Quaestiones Math. **12** (1989), 425–438.
52. A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques*, Bol. Soc. Mat. São Paulo **8** (1956), 1–79.
53. A. Ionescu Tulcea and C. Ionescu Tulcea, *Abstract ergodic theorems*, Trans. Amer. Math. Soc. **107** (1963), 107–124.
54. H. Jacobs, *Ordered topological tensor products*, Ph.D. thesis, University of Illinois, 1969.
55. H. Jarchow, *Locally convex spaces*, B.G. Teubner, Stuttgart, 1981.
56. G.A.M. Jeurnink, *Integration of functions with values in a Banach lattice*, Ph.D. thesis, University of Nijmegen, The Netherlands, 1982.
57. S. Kakutani, *Concrete representation of abstract L-spaces and the mean ergodic theorem*, Ann. of Math. **42** (1941), 523–537.
58. ______, *Concrete representation of abstract M-spaces*, Ann. of Math. **42** (1941), 994–1024.
59. N.J. Kalton and L. Weis, *The H^∞-calculus and sums of closed operators*, Math. Ann. **321** (2001), 319–345.
60. J.L. Krivine, *Thèoremés de factorisation dans les espaces réticulés*, Séminaire Maurey-Schwartz 1973–1974 (École Polytechnique, Paris), Exposes 22–23, 1974.
61. W.-C. Kuo, *Stochastic processes on vector lattices*, Ph.D. thesis, University of the Witwatersrand, Johannesburg, 2006.
62. W.-C. Kuo, C.C.A. Labuschagne, and B.A. Watson, *Discrete-time stochastic processes on Riesz spaces*, Indag. Math. (New Series) **15** (2004), 435–451.
63. ______, *An upcrossing theorem for martingales on Riesz spaces*, Soft methodology and random information systems, Springer-Verlag, 2004, pp. 101–108.
64. ______, *Conditional expectations on Riesz spaces*, J. Math. Anal. Appl. **303** (2005), 509–521.
65. ______, *Zero-one laws for Riesz space and fuzzy random variables*, Fuzzy Logic, Soft Computing and Computational Intelligence (Beijing), Springer-Verlag and Tsinghua University Press, 2005, pp. 393–397.
66. ______, *Convergence of Riesz space martingales*, Indag. Math. **17** (2006), 453–461.
67. ______, *Ergodic theory and the strong law of large numbers on Riesz spaces*, J. Math. Anal. Appl. **325** (2007), 422–437.

68. C.C.A. Labuschagne, *A note on the order continuity of the norm of $E\widetilde{\otimes}_m F$*, Arch. Math. **62** (1994), 335–337.
69. ———, *Characterizing the one-sided tensor norms Δ_p and ${}^t\Delta_p$*, Quaestiones Math. **27** (2004), 339–363.
70. ———, *Riesz reasonable cross norms on tensor products of Banach lattices*, Quaestiones Math. **27** (2004), 243–266.
71. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I*, A Series of Modern Surveys in Mathematics, no. 92, Springer-Verlag, Berlin, Heidelberg, New York, 1977.
72. ———, *Classical Banach spaces II*, A Series of Modern Surveys in Mathematics, no. 97, Springer-Verlag, Berlin, Heidelberg, New York, 1979.
73. W.A.J. Luxemburg and A.C. Zaanen, *Riesz spaces volume I*, North-Holland Mathematical Library, North-Holland, Amsterdam, London, 1971.
74. B. Maurey, *Type et cotype dans les espaces munis de structures locales inconditionnelles*, Séminaire Maurey-Schwartz 1973–1974 (École Polytechnique, Paris), Exposes 24–25, 1974.
75. ———, *Système de Haar*, Séminaire Maurey-Schwartz 1974–1975 (École Polytechnique, Paris), 1975.
76. P. Meyer-Nieberg, *Bannach lattices*, Springer-Verlag, Heidelberg, Berlin, New York, 1991.
77. A.M. Olevskiĭ, *Fourier series and Lebesgue functions*, Uspehi Mat. Nauk **22** (1967), 237–239, (Russian).
78. ———, *Fourier Series with Respect to General Orthogonal Systems*, Springer, New York, 1975.
79. B. de Pagter, *The components of a positive operator*, Indag. Math. **45** (1983), 229–241.
80. R.E.A.C. Paley, *A remarkable series of orthogonal functions*, I. Proc. London Math. Soc. **34** (1931), 241–264.
81. A. Pełczyński and H. Rosenthal, *Localization techniques in L^p spaces*, Studia Math. **52** (1975), 263–289.
82. D. Pérez-García and I. Villanueva, *There is no lattice preserving natural tensor norm*, Quaestiones Math. **27** (2004), no. 3, 267–273.
83. G. Pisier, *Un exemple concerntant la super-réflexivité*, Séminaire Maurey-Schwartz 1974–1975 (École Polytechnique, Paris), 1975.
84. ———, *Some results on Banach spaces without local unconditional structure*, Comp. Math. **37** (1978), 3–19.
85. N. Popa, *Die Permanenzeigenschaften der Tensorprodukte von Banachverbänden*, Romanian-Finnish Seminar on Complex Analysis 1976 (Bucharest), LNM 743, Springer-Verlag, 1979, pp. 627–647.
86. P.E. Protter, *Stochastic integration and differential equations*, second ed., Springer-Verlag, 2005, Version 2.1.
87. M.M. Rao, *Two characterizations of conditional probability*, American Mathematical Society **59** (1976), no. 1, 75–80.
88. J.L. Rubio de Francia, *Martingale and integral transforms of Banach space valued functions*, Probability and Banach spaces 1985 (Zaragoza), Lecture Notes in Math., no. 1221, Springer-Verlag, 1986, pp. 195–222.
89. W. Rudin, *Functional analysis*, McGraw Hill, 1991.
90. R.A. Ryan, *Introduction to tensor products of Banach spaces*, Springer-Verlag, London, 2002.
91. H.H. Schaefer, *Normed tensor products of Banach lattices*, Israel J. Math. **13** (1973), 400–415.
92. ———, *Banach lattices and positive operators*, Die Grundlehren der mathematischen Wissenschaften, no. 215, Springer-Verlag, Berlin, Heidelberg, New York, 1974.

93. I. Singer, *Bases in Banach spaces I*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, no. 154, Springer-Verlag, 1970.

94. E. M. Stein, *Topics in harmonic analysis related to the Littlewood-Paley theory*, Annals of Mathematics Studies, no. 63, Princeton University Press, Princeton N.J., 1970.

95. G. Stoica, *Martingales in vector lattices*, Bull. Math. de la Soc. Sci. Math. de Roumanie **34** (1990), no. 82, 357–362.

96. ______, *On some stochastic-type operators*, Analele Universitătii Bucuresti, Mathematică **39** (1990), 58–62.

97. ______, *Vector valued quasi-martingales*, Stud. Cerc. Math. **42** (1990), 73–79.

98. ______, *The structure of stochastic processes in normed vector lattices*, Stud. Cerc. Math. **46** (1994), 477–486.

99. ______, *Weak laws of large numbers in some non-commutative spaces*, Bull. London Math. Soc. **32** (2000), 471–476.

100. D. W. Stroock, *Lectures on stochastic analysis: Diffusion theory*, Cambridge University Press, 1987.

101. V. Troitsky, *Martingales in Banach lattices*, Positivity **9** (2005), 437–456.

102. J.J. Uhl, *On a class of operators on Orlicz spaces*, Studia Math. **40** (1971), 17–22.

103. H. Witvliet, *Unconditional Schauder decompositions and multiplier theorems*, Ph.D. thesis, Delft University of Technology, 2000.

104. T.K. Wong, *On a class of absolutely p-summing operators*, Studia Math. **39** (1971), 181–189.

105. A.C. Zaanen, *Riesz spaces II*, North-Holland Mathematical Library, North-Holland, Amsterdam, New York, Oxford, 1983.

106. ______, *Introduction to operator theory in Riesz spaces*, Springer-Verlag, Berlin, Heidelberg, New York, 1997.

107. R. Zaharopol, *On the ergodic theorem of E. Hopf*, J. Math. Anal. Appl. **178** (1993), no. 1, 70–86.

108. ______, *The filling scheme in Archimedean Riesz spaces*, Rocky Mountain J. Math. **24** (1994), no. 3, 1163–1190.