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Research for Doctorate Degree

Resourcing Learners' Misconceptions on Hyperbolic Functions through Mediation of ICTs at a Rural Secondary School

By

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A thesis submitted to the School of Education, Faculty of Humanities, University of the Witwatersrand, Johannesburg, in fulfillment of the requirements for the degree of Doctor of Philosophy.

Declaration

I declare that this thesis is my original work. It is submitted to the faculty of humanities, University of Witwatersrand for the degree of Philosophy. It has not been submitted before for any other degree or examination in any other university. All the sources that I have used or quoted have been indicated and acknowledged by means of comprehensive references.

Signature **Date:** 04/11/2021

Fanuel Matindike

Dedication

I dedicate this thesis to my late father (Saimon Gorivoto Matindike) and my late mother (Livinia Masona) who sacrificed everything they had to provide education for me.

Acknowledgements

This study could not have been possible without the support and collaborative efforts received from other people.

First and foremost, I thank my supervisor, Prof J.P. Makonye for his guidance and unsurpassable support. He allowed me to explore, make mistakes and through his concise and expert comments, propelled me in the right direction. That way, I felt I was growing academically at every stage of my study.

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I also extend my acknowledgements to the PhD research weekend organizers for consistently organizing enriching and inspiring programmes. Each time I made presentations, I received insightful feedback. When I listened to testimonies presented by others who had completed their PhDs, my spirit was always inspired and uplifted.

To the participating school, the SGB, teachers and the participating learners, I extend my gratitude for their willingness to avail themselves. This study could not have been completed without their collegial and convivial support.

Above all, I give glory and honor to God the Almighty. I can do all things through the strength that the Lord gives me.

Abstract

This study was motivated by the persistent challenge of poor learner performance in mathematics in South Africa, particularly in marginalized communities. A closer analysis of examination and diagnostic reports from both local exit examinations and international comparative tests pointed to some concerning issues in the state of school mathematics in South Africa. Firstly, there is a widening gap in performance in mathematics between more affluent schools and schools in marginalized communities. Secondly, recurring learner errors and misconceptions under functions expose pedagogical shortcomings in this topic. Thirdly, in this technological era, the integration of ICT tools in the teaching of mathematics is lagging far behind and mostly non-existent in marginalized schools. Based on this background, this study sought to explore how ICT-mediated instruction could enhance learner conceptual understanding of functions and minimize their misconceptions in this topic in a rural context.

The study was guided by three research questions. First, it sought to ascertain the learners' errors and misconceptions under hyperbolic functions. Second, it aimed to establish the impact of ICT-mediated instruction in deepening conceptual understanding of concepts under functions, thereby minimizing leaners' misconceptions and third, the study proposed an ICT-mediated instructional model that can provide a context-based framework to teachers on the effective use of ICT tools in the teaching of functions.

The constructivist theories of Social constructivism, Activity theory and APOS theory were merged to provide the theoretical lens for this study. The theoretical argument informing the study was that ICT tools, if properly used, enriches the social (pedagogical) context of the learner and facilitates the reduction of the cognitive gap in the zone of proximal development (ZPD). The nature of issues under investigation in this study naturally called for a qualitative research methodology. Understanding errors and misconceptions involve in-depth study of learners' internal thinking patterns and reasoning. Similarly, ICT-mediated instruction is highly contextual and situational. Thus, the qualitative case study methodology was most appropriate as it allowed for in-depth understanding of the scenario under study.

The study found that learners possess a wide range of misconceptions that makes them susceptible to errors in dealing with hyperbolic functions. Algebraic misconceptions came out as the major constraining factor impeding learners' ability to execute procedural calculations. This was exasperated by the finding that learners' conception of hyperbolic functions was restricted to the symbolic (or algebraic) form. Their lack of exposure to multiple representations was reflected in their graphical misconceptions and several representational errors displayed. Calculations were mechanistic and procedural with no understanding of concepts involved. Misconceptions emanating from the asymptotic nature of hyperbolic functions were evident when learners struggled to connect different representations

of the same function and to deal with graphical transformations. Failure to understand the language and notational symbols used under functions was another source of errors.

The main findings from this study confirmed that proper ICT integration has potential to deepen conceptualization of functions. The deepening of understanding was explained in terms of Dubinsky's *action – process – object –* and *schema* (APOS) conception levels. Before the ICT-mediated intervention lessons, learners were mainly operating at *pre-function* or *action*-conception level of hyperbolic functions. This conception level is characterized by procedures *without* understanding and memorization of rules. After the lessons, there was glaring evidence that learners had progressed to higher levels of conception. Mental processes of interiorisation, encapsulation and thematisation were evident in learners' responses after the intervention.

Although the study was context based with limited generalizability, it proposed an ICT-mediated instructional model which highlights affordances and constraints provided by ICTs in the teaching of functions. The model acknowledges that ICT-mediated instruction is situational and underscores the need for the teacher to take cognizance of contextual factors. Its transferability to other contexts should be done with the full knowledge of the context in which this study was undertaken.

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Abbreviations and Acronyms

ANA	:	Annual National Assessment
APOS	:	Actions, Processes, Objects and Schema
CAPS	:	Curriculum and Assessment Policy Statement
DBE	:	Department of Basic Education
FET	:	Further Education and Training
ICT	:	Information and Communication Technology
MKOs	:	More Knowledgeable Others
MLMMS	:	Mathematical Literacy, Mathematics and Mathematical Sciences
NSC	:	National Senior Certificate
PCK	:	Pedagogical Content Knowledge
RNCS	:	Revised National Curriculum
SSA	:	Sub-Saharan Africa
TMSS	:	Trends in Mathematics and Science Study
TPACK	:	Technological Pedagogical Content Knowledge
UNESCO	:	United Nations Scientific Organization
ZPD	:	Zone of Proximal Development

CHAPTER 1: BACKGROUND TO THE STUDY

1.1 Introduction

The South African school curriculum in general and in particular, the mathematics one has undergone various transformations, innovations and refinements in a bid to make it more relevant to the South African context. However, despite these curriculum transformations, the quality of learner performance in mathematics is persistently below expected standard (Graven, 2014; Spaul & Kotze, 2015; Taylor & Taylor, 2013; Visser, Juan & Feza, 2015). The Department of Basic Education (DBE) Diagnostic Reports (2011-2016) reveal glaring evidence that learners struggle to deal with questions involving functions. This is reflected by a long list of errors and misconceptions that students display in answering questions on functions. The misconceptions signify that learners lack a deep and conceptual understanding of the notion of function. This study problematizes the teaching and learning of the function concept in the South African school context. The function concept is widely viewed as a 'unifying concept' (Carlson, Oehrtman & Thompson 2005; Ponte, 1992; Sheely, 1996), as it connects several other concepts in mathematics.

The study is premised on the view that learners' misconceptions can be taken advantage of and used as resources to advance learner understanding of concepts (Borasi, 1994; Magen-Nagar, 2016; Sarwadi & Shahrill, 2014; Lourens, Molefe & Brodie, 2011). Sarwadi and Shahrill noted that the immediate concerns of the classroom teacher is to determine if conceptual difficulties are present in their students and help them realise the need for changing their mental representations. In an endeavour to deal with learners' misconceptions and advance conceptual understanding of concepts, one innovation that has not been taken full advantage of is the integration of ICT tools in mathematics teaching and learning. The study provides research evidence that integration of ICT has potential to radically transform mathematics teaching and learning. Unfortunately, schools and teachers are failing to unlock this potential (Ertmer, 1999; Howie & Blignant, 2009; Nkula & Krauss, 2014; Wilson-Strydom & Thomson, 2015) due to various reasons, some of which are discussed in this chapter. This chapter presents the statement of the problem, which articulates the existing knowledge gaps that this study intends to fill.

To guide and give direction to this study, the aims and research questions of the study are given. The significance of the study is included to illustrate how the study contributes to the existing body of knowledge in mathematics education. The key terms in this study were also explained.

1.2. Context of the Study

The quality of school learners' general performance in mathematics remains a persistent challenge in South Africa. This assertion is evidenced by the low performance levels in mathematics both in local exit examinations (DBE, 2015; Makgato & Mji, 2006) and in international comparison tests (Howie, 2001; Reddy, 2006). The National Senior Certificate (NSC) mathematics diagnostic reports, from 2011 to 2015 (DBE, 2011-2015), reveal that the number of candidates who achieved mathematics at 30% or above ranged from 49% to 59%. Those who achieved at 40% or better ranged from 32% to 41%. Evidently, the percentage pass rate and quality of passes remain significant causes for concern. If this trend of poor performance continues, learners are more likely to eventually get frustrated, lose confidence and begin to shun the subject again (Makgato & Mji, 2006; Reddy, Prinsloo, Visser, Winnaar, Arends & Rogers , 2012). As an educator in one of the most disadvantaged communities in Limpopo province, I have observed that most learners opt for the relatively simpler subject, Mathematical Literacy with the hope of achieving better results.

The challenge of learner poor performance in mathematics is worse in the more economically disadvantaged communities in South Africa. Given South Africa's political history, the widening economic disparity between the rich and the poor continue to influence the education system in very complex ways (Graven, 2001). A case in point is when the minister of basic education, as cited in the DBE report (2012), noted that Grade nine learners in Limpopo province scored the worst results in the mathematics Annual National Assessment (ANA) with marks ranging from 9% to 17%. This suggests that as learners enter the FET phase, they have generally mastered less than 20% of the mathematics concepts covered. Inevitably, chances are high that such learners will not pursue mathematics at FET phase. This scenario has far reaching consequences as the nation must produce matric graduates who

are eligible to study key professions in the fields of accounting, engineering, commerce and science. It also raises the question of equity in mathematics education.

As Makonye (2009) notes, equity in mathematics education can be looked at in terms of both formal access and substantial access. On one hand, Makonye elucidates that formal access is concerned with the provision of legislative and regulatory frameworks enabling all children getting access to schools and being in class to study mathematics. On the other hand, Makonye (2009) further explains that substantive access is when learners are fruitfully engaged in mathematical tasks resulting in them attaining mathematical proficiency. One would argue that curriculum reforms in South Africa have afforded learners formal access to mathematics but there exists a gap between formal access and substantive access in some groups of learners. I subscribe to Brodie's (2002) argument that while the education system cannot be blamed for the exogenous inequalities (human, social, cultural and material resources at learners' disposal), it can contribute to overcoming such exogenous inequalities through specific interventions. One such intervention is the main aim of this investigation, that is, equitable integration of ICTs in teaching and learning.

International ratings show that South Africa's level of performance in mathematics does not compare well with other countries. For example in international comparison tests like the RAND Mathematics Study Panel (2002) and the Trends in Mathematics and Science Study (TMSS)(2011), South African learners' achievement in mathematics was found at the bottom of the pile (Reddy et al, 2012). As one of the biggest economies in Africa, the South African education system is relatively well-resourced as compared to other African nations especially in mathematics and science, as these are the key drivers in the economic and technical advancement of the country. One then wonders why South Africa has such a comparatively low performance in mathematics.

Researchers attribute poor performance in mathematics in South Africa to a wide range of factors which include widening economic and social inequalities (Graven, 2014), linguistic factors (Valero, Graven,

Jurdack, Meaney & Pentedo 2012), teachers' PCK (Pournara, hodgen, Adler & Pillay, 2015), availability of teaching time, teacher absenteeism and poor school management (Organisation for Economic Co-operation and Development (OECD), 2008). While these factors do influence performance in mathematics, I believe that the key determinant factor is the teachers' instructional practices in the mathematics classroom. Some researchers have noted that mathematics teaching in South Africa is largely traditional, mechanistic and out-dated (Grussendorf, Booyse, & Burroughs, 2014; Ndlovu & Moll, 2016; Nkula & Krauss, 2014; OECD, 2008; Valero et al, 2012). Grussendorf et al (2014), in particular, noted that students learn mathematical concepts without understanding them because they are taught procedures without conceptual understanding. A conceptual approach to mathematics teaching empowers students to solve problems from their own understanding (Alagic, 2003). Informed by these observations, this study advocated for the establishment of a mathematics classroom atmosphere that aids conceptual understanding in a technology-mediated environment.

Agreeably, several intervention programmes have been set up by the DBE and other stakeholders in an attempt to resolve the challenge of low performance in mathematics but very little has been achieved (Ndlovu & Moll, 2016 & Grusserndorf et al, 2014). Teachers have constantly been called upon to increase their contact time with learners through extra lessons. In-service training of mathematics teachers has been undertaken and institutions of higher learning have introduced bridging courses targeted at improving and aligning the content and pedagogical knowledge of mathematics and science teachers to meet the curriculum expectations (Makgato & Mji, 2006). Another notable intervention strategy was the setting up of *Dinaledi* schools, whose focus was to increase the number of learners studying mathematics and physical sciences and increase the pass rate and achievement in these subjects (DBE, 2005). These intervention strategies might be working in some isolated cases but overly there has not been significant improvement in learner performance and achievement in mathematics.

One intervention that has not been taken full advantage of is the integration of ICTs in the teaching and learning of mathematics. This study argues that efforts should be directed towards radically and

massively transforming pedagogical practices in ICT-mediated mathematics classrooms. There is need to explore new learning opportunities afforded through the integration of new ICT tools in the teaching and learning of mathematics. In the next section I put integration of ICTs in mathematics education into perspective and expose the digital gaps in the South African context.

1.3. Integration of ICTs in Mathematics Education

In this digital age, computers and other technological gadgets are used in every sphere of our daily lives. Elagic (2003) remarked that we are teaching a new generation; born and educated surrounded by emerging technologies. This has transformed the way they interact with the world around them and has exposed them to much information. Thus, failure to harness and integrate ICT in the teaching and learning process is tantamount to making education insensitive and ultimately irrelevant to the present-day society.

Various studies have been carried out to establish and evaluate the benefits of using ICTs in mathematics education (Nkula & Krauss, 2014; Howie & Blignant, 2009; Mdlongwa, 2012; Becta, 2003). Becta (2003), for example, identified enhancement of collaboration among students and promotion of communication and sharing of knowledge as the key benefits of ICTs in mathematics teaching and learning. Mdlongwa (2012) indicates that using ICT makes learning more exciting, authentic and relevant, affording learners more time for analysis and discussion. Alagic (2003) further notes that the use of ICT in mathematics teaching supports and facilitates conceptual development, exploration and reasoning. He further clarifies that this is achieved by removing computational constraints, making the study of mathematics topics that were impractical without technology more accessible. The benefits of ICT integration, as argued by the above scholars, facilitate the achievement of the key demands in the mathematics curriculum, such as development of reasoning capacity, problem-solving and critical thinking skills.

Furthermore, ICT opens wider opportunities for learners to share ideas and learn from each other, which is a vital component in building mathematical knowledge (Hardman, 2005). In tandem with

constructivist pedagogy, ICT tools afford students the opportunity to explore, solve problems and identify patterns and relationships on their own (Keong et al. 2005). As noted above, technology reduces the burden of long and time-consuming computational working which have no real learning in them, and channel the learners' effort on identifying patterns, trends and relationships.

In the same vein, Muijs and Reynolds (2005) categorised four benefits of ICT that I found of particular relevance to the teaching and learning of functions. These are: presentation of information, modelling, interactivity and quick and automatic completion of routine tasks. By presentation of information, Muijs and Reynolds (2005) refer to the visual presentation of mathematical information. This includes computer simulation of real situations (Lagrange, 2014) that enables learners to gain better understanding of a concept. Similarly, modelling of different scenarios allows learners to identify emerging patterns and trends with easy. Interactivity is when learners get immediate feedback while solving a problem and share their findings or observations with fellow learners. As alluded to above, quick and automatic completion of routine tasks enables learners to concentrate on real learning. In drawing graphs, for example, learners' efforts may be spend on algebraic calculation of x and y values and misses out on identification of key features of the graph.

In developing economies, research suggests that ICT policies are available on paper but there are no deliberate and pragmatic strategic plans to ensure successful implementation of these policies (Hennessy, Hawisson & Wamakote, 2010). With specific reference to South Africa, the White Paper on e-education was introduced in 2003 with a three-phased implementation plan that was expected to be fully-fledged by 2013 (Howie & Blignant, 2009). At its maturity stage, the White Paper envisaged that all educators should integrate ICT in teaching and learning. Needless to say, the targeted implementation period has since elapsed but in reality, the integration of ICT tools in teaching and learning is at its embryotic stage (Ndlovu & Mji, 2011), particularly in rural schools. Even in provinces like Gauteng, where significant strides have been made in terms of provision of ICT resources, accessibility of ICT tools to learners is still an issue (Ndlovu, 2009). In her study on ICT accessibility to learners, Ndlovu

found out that learners were only taught instrumental ICT skills, most learners had contact with a computer 30 minutes once a week; and internet was rarely connected. Such findings confirm the view that availability of ICT physical resources does not guarantee accessibility and subsequent integration in teaching and learning. Clearly there is still a huge gap between what government aims to achieve through ICT integration in education and what is happening on the ground.

With regard to schools in disadvantaged communities, Anderson (2005) observed that the disparity in school-based uses of new technologies exacerbates the plight of already disadvantaged social groups. Shedding more light on this, Anderson found that minority groups and other poor students are less likely to be exposed to computers for effective learning as compared to their economically and socially advantaged peers and are less likely to have teachers professionally trained on technology use. This inevitably creates a digital divide defined on the basis of socio-economic inequalities.

In the context of South Africa, despite substantial efforts by the government and other stakeholders to ensure accessibility of ICTs to learners and teachers in public schools, digital equity has not been attained (Gudmundsdottir, 2010; Tustin, Goetz and Heydenrych, 2012). Tustin et al (2012) broadly viewed digital divide as the gap that exists in society between those who are knowledgeable on how to and can use ICTs to their advantage and those who cannot. This definition was broadened from the narrow view that refers to digital divide as inequality of access to ICTs to include the ability to use available ICT resources properly. Following this argument, it is clear that even if a school is provided with all the latest technology but does not know how and when to use them, it still leads to digital divides. I personally have observed that some schools have computers and other ICT tools but the challenge is that these ICT tools are either underutilised or just kept in their closets. It is against this background that this study hoped to contribute towards bridging this digital divide in schools by empowering mathematics teachers in disadvantaged contexts and ensure that the little ICT resources available are gainfully utilised to upgrade quality of teaching and learning.

I believe effective ICT integration in mathematics education at school level is contextual and can only be achieved with a clear understanding of the dictates and confinements of the mathematics national curriculum. The current mathematics school curriculum in South Africa can best be understood by tracing back its historical developments, particularly post democratic South Africa. The next section highlights the mathematics school curriculum changes since the inception of democracy in South Africa.

1.4. A Short Synopsis of the Mathematics School Curriculum changes in South Africa post-democracy

Since 1994, the new democratic South African government embarked on radical educational reforms to completely overhaul the education system (OECD, 2008). A new curriculum, Curriculum 2005 (C2005) was developed for implementation (Graven, 2001). In C2005, the mathematics subject was replaced with a broader learning area known as Mathematical Literacy, Mathematics and Mathematical Sciences (MLMMS) (Graven, 2001). In this learning area, there was more emphasis on social constructivist, learner-centred and more integrated approaches to teaching (Graven, 2001). Teachers had to re-align their pedagogical practices if the expectations of this new curriculum had to be realised. Secondly, there was a shift away from the 'absolutist paradigm' which viewed Mathematics as an existing body of infallible objective truths (Pudi, 2006). It portrayed mathematics as a human creation, a result of human's attempt to understand and manipulate his/her environment.

The major criticism levelled against C2005 was its failure to provide clarity of curriculum and to adequately empower educators to confidently fulfil their expected roles. Critiques further argued that C2005 was an example of an international policy borrowing with its roots in New Zealand, Australia, Canada and United States (OCED, 2008; Chisholm, 2005). These arguments led to the Revised National Curriculum Statement (RNCS) in 2002. The ushering in of the new RNCS presented no major ideological shifts in the view and value of Mathematics. Its main purpose was identified as to ensure that all learners passing through the FET phase acquire a functional mathematical knowledge that empowers them to make sense of society (DBE, 2003). RNCS was designed on the basis that

competence in mathematical processes such as investigating, generalising and proving were more important than the acquisition of content knowledge for its own sake (DBE, 2003).

The distinguishing feature of the revised NCS was its emphasis on explicitly prescribed learning outcomes. The Mathematics content was structured into four learning outcomes, with specific assessment standards under each outcome. These however became the major weak points of this curriculum. The report of the Task Team for the Review of the Implementation of the NCS (OECD, 2009) noted that focussing on outcomes failed to give clarity on content, concepts and skills. Agreeably, teachers may view and interpret a learning outcome differently and thus the content and processes adopted to achieve that outcome will vary. These shortcomings led to the introduction and subsequent rolling out of CAPS from 2012.

The mathematics CAPS document defines mathematics as a human activity that involves observing, representing and investigating patterns and qualitative relationships in physical and social phenomenon (DBE, 2011). Such a view of mathematics implies a shift in the underpinning educational approach which has significant pedagogical implications. There was a glaring shift from the objective-based NCS, described as 'a learner-centred and activity-based approach' (DBE, 2005) to CAPS which is described as 'an active, critical approach to learning, rather than rote and uncritical learning of given mathematics' (DBE, 2011, p.4). The focus has shifted towards conceptual mathematical knowledge and development of cognitive skills in individual learners (Grussendorf et al., 2014). These cognitive skills are identified in the mathematics CAPS document as 'logical and critical thinking, problem solving and decision making' (DBE, 2011, p.8).

The question that arises is: 'To what extent does mathematics teachers understand the demands of this new curriculum and to what extent they have aligned their pedagogical practices to meet the curriculum demands?' Valero et al (2012) pointed out that mathematics teaching approaches are not in tandem with changing philosophies in mathematics education. The mismatch between curriculum demands and pedagogical practices in the classroom disadvantages the learners because they will not be able to

develop the knowledge and skills expected out of them. Orton (2004) noted that teachers may theoretically claim to be constructivist in their approach, but what happens in the classroom may still be behaviourist in nature. This results in learners generally failing to perform mathematical tasks to the expected levels.

One topic in the mathematics school curriculum that has proven challenging both to teachers and learners is that of functions. This study problematizes the teaching and learning of functions in the South African high school mathematics and develops alternative intervention strategies and resources to enhance understanding of concepts under functions.

1.5. Functions in the South African School Mathematics Curriculum

The main mathematical idea in this thesis is the concept of function. The mathematical definition of a function evolved over centuries and can be traced back to Descartes (1596 – 1650) as shall be discussed later in greater detail (see Section 4.2). It was not until the work of Leibniz (1646 – 1716) that the term 'function' was used and later developed by mathematicians like Dirichlet (1805 – 1859) and Bourbaki (Kleiner, 1989). Without delving into a detailed discussion on the historical development of the notion of function at this point, I note that the definitions of function presented in learners' textbooks are derived from the contributions made by these great mathematicians. In high school mathematics, the most prevalent textbook definition is that a function is a correspondence between two sets (a domain and a range) such that every element in the domain has exactly one element in the range that corresponds to it (Vinner, 1983). Of course, there are differences in the descriptions of functional relationships between dependent and independent variable given in textbooks. The verbs used include 'depends', 'relates', 'corresponds' and 'connects'. This explains why the formal definition of a function presents a pedagogical challenge to learners.

The notion of function developed into undoubtedly one of the big ideas in mathematics which justifies its inclusion into the school mathematics curriculum. In the early twentieth century, mathematicians such as Klein pursued the view that 'functional thinking' should be made the unifying notion of school

mathematics (Ponte, 1992; Kleiner, 1989). Klein considered functions as the 'soul of mathematics'. Similarly, Makonye (2014) argues that the function concept builds the discipline of mathematics as it is the substance that binds mathematical concepts and processes. This argument cannot be disputed because several fields of mathematics including mathematical analysis, the theories of differential and integral calculus and functional analysis deal with functions either directly or indirectly. The notion of function also features predominantly in other learning disciplines such as Physics and Life Sciences (Ponte, 1992; Kleiner, 1989).

While the inclusion of functions in the school curriculum is well justified, the problem is that much of the research in mathematics education has revealed difficulties associated with the teaching and learning of the function concept (Makonye, 2014; Even, 1990; Kalchman & Koedinger, 2005). Surprisingly the challenge seems to stretch across the board from school, university and pre-service teacher training. Kalchman & Koedinger (2005), for example, argued that lack of awareness and appreciation of the fundamental laws governing functions by teachers presents a pedagogical challenge. Makonye (2014) made a similar observation that the notion of function is often not understood by both teachers and learners. Therefore, a deeper conceptual understanding of functions plays a vital role as a spring board propelling a learner to the deeper and more abstract realm of Mathematics.

In the South African context, the notion of function constitutes a fundamental concept in secondary school Mathematics. According to the CAPS document for Mathematics in the FET phase (see Appendix A), the prescribed content under functions expect learners to: have an understanding of the concept of function and inverse function; be able to identify the features of a function, including domain, range, intercepts, asymptotes, symmetry, intervals of increase and decrease; be able to switch comfortably between representations of function as tables, graphs, words and formulae; have the ability to use different representations of functions to model and/or solve contextual and mathematical problems (FET mathematics CAPS document). Under the topic of 'Functions and Graphs', the

document specifies that the following functions should be covered: linear functions, quadratic functions, hyperbolic functions, exponential, trigonometric and cubic functions.

Clearly, the content prescription contains some abstract concepts and higher-order cognitive skills which may not be attained via the traditional expository methodologies. One conspicuous component of this content prescription is the emphasis it places on multiple representations of functions. As Even (1998) argues, a rich and deep conceptual understanding can easily be achieved through the ability to identify and represent a concept in different representations, and to flexibly move from one representation to another. Such different representations of functions may be afforded through the use of modern technologies.

The study focussed on hyperbolic functions for two main reasons. First, one would appreciate that it would not be possible to sufficiently investigate the issues under study if all types of functions were included. Thus, I had to focus on one type of function. Secondly, preliminary literature reviewed revealed that studies on errors and misconceptions under functions seem to focus on quadratic functions and cubic functions under differential calculus. Very little research has been done on hyperbolic functions despite the fact that they do possess some features not present in other types of functions. The asymptotic nature of hyperbolic graphs and the fractional nature of their algebraic equations make them slightly unique from other functions. As one would expect, these features are likely to present pedagogical challenges to teachers and learners.

1.6. Errors and misconceptions under functions

Mathematical errors refer to the mistakes which learners commit during calculation or during the solution of mathematical problems (Godden, Mbekwa & Julie, 2013; Luneta & Makonye, 2010). Luneta and Makonye (2010) viewed an error as a mistake, slip, blunder or inaccuracy visible in learners' written work resulting in learners not obtaining a perfect score in routine assessments or in examinations. Errors are broadly categorised into unsystematic and systematic errors (Godden et al, 2013; Olivier, 1989; Sarwadi & Shahrill, 2014). Unsystematic errors are described as 'unintended, non-recurring wrong

answers' which learners can easily correct by themselves (Luneta & Makonye, 2010). Such errors are mainly due to carelessness and therefore easy to correct. On the other hand, systematic errors are more structural and deeply ingrained in the sense of being caused by internal mental constructions referred to as misconceptions (Godden et al, 2013). This makes systematic errors or misconceptions more resistant to change and hence difficult to eliminate than the unsystematic ones.

Borasi (1994) views errors as 'springboards for enquiry' implying that errors point to some important issues for further exploration in the learning of mathematics. It is important to note that misconceptions make sense to the learners as a result of conceptual links that they make in line with knowledge they acquired previously (Nesher, 1987). It is therefore critical that errors and misconceptions be embraced in teaching and learning as they inform us of mental processes in each individual learner. Louren, Molefe and Brodie (2011) underscored the importance of errors and misconceptions in informing teachers' instructional practices. Errors present evidence to the teacher on the extent to which the expected learning result has not been achieved and suggest the course of action to be taken. It is also clear that teaching in ways that do not embrace errors results in misconceptions developing or being perpetuated. It is for this reason that teachers need to understand the actual type of errors that learners make and dig down to understand why they make those errors. This study rests on the view that understanding learners' errors can help teachers develop teaching strategies that give learners an opportunity to engage with and reflect upon their misconceptions.

There is ubiquitous research evidence that learners exude a variety of errors as they engage with problems related to functions (Godden et al, 2013; Lourens et al, 2011; Sarwadi & Shahrill, 2014; Luneta & Makonye, 2010) . Luneta and Makonye (2010), for example, concluded that the errors and misconceptions on functions are manifestations of the learners' lack of a deep and structural understanding of concepts under functions. Likewise, Sarwadi and Shahril (2014) remarked that such errors and misconceptions confirm the claims that when teaching functions, teachers tend to concentrate on the manipulation of algebraic and symbolic expressions which have no real meaning to the learners.

Abdullah (2010) observed that even in textbooks, the concept of function is mainly restricted to symbolic manipulation, with little or no reference on how the learning context and multiple-representations of functions can be enriched in the classroom activities. He further argues that the premature introduction of the formal definition and symbolism of the function concept confuses learners, given the abstract nature of the concepts involved. Teachers who want to promote the learning of functions for conceptual understanding are faced with a huge challenge of designing instructional strategies that address learners' prior knowledge and minimise misconceptions to facilitate better construction of knowledge. Unfortunately there seem to be a dearth of research on classroom based intervention strategies that can assist teachers to deal with learners' misconceptions. Fortunately, the challenge is huge but not insurmountable. The study demonstrates that proper integration of ICT tools in the teaching and learning of functions can broaden learning opportunities and affords learners an opportunity to critically reflect on their conceptions of concepts.

1.7. Autograph Technology

The integration of ICTs in mathematics education may be achieved through the use of commercially made software specifically designed for the teaching and learning of mathematics. Examples of such software include Goemeter Sketchpad (GSP), Geogebra, Graphic Calculators and Autograph (Tarmizi, Yunus, Ayub & Bakar, 2009). If used strategically, teaching mathematics with such dynamic software has potential to make the learning environment more effective and enjoyable (Bakar, Ayub & Tarmizi, 2010).

However, it must be taken into account that pedagogical practices take place in different and complex contextual conditions which include students, institutional environment, epistemology, time issues and material. Teachers must balance the different elements in their working environment, especially when using technology, by looking at their practices within specific contexts. This study seeks to explore the impact of using Autograph software in promoting the conceptual understanding of concepts under functions. I conveniently chose to use the Autograph software mainly because of its affordability and

availability at the institution where this study was carried out and its appropriateness in the teaching and learning of functions. Other technological tools such as geogebra, geometer sketchpad and graphic calculators are also handy in the teaching of functions.

Autograph is a dynamic, interactive software package that can be used in teaching a wide range of topics, spanning from pre-Algebra through Calculus (Bakar et al, 2010). The software can be used in the teaching of graphs, transformations and vectors (in 2D and 3D) and is appropriate for enhancing understanding of concepts in calculus, trigonometry, polar and parametric plotting and differential equations. The software can also be used for drawing statistical graphs, functions and for transforming shapes. As Baker et al (2010) explain, Autograph affords learners an opportunity to change and animate graphs or shapes already plotted to encourage understanding of concepts. As a new technology, Tarmizi et al (2009) indicate that many studies are on-going that are examining the effectiveness of using Autograph in teaching and learning mathematics. This study aims to explore the impact of using the Autograph software in enhancing conceptual understanding of hyperbolic functions, thereby minimising learner misconceptions around this topic.

1.8. Statement of the Problem

The main problem that was driving this study was that the notion of function, which is arguably the central and unifying concept (Ponte, 1992; Kleiner, 1989) in the mathematics school curriculum in South Africa, is presenting pedagogical challenges both to teachers and learners. Consequently, studies show that learners hold a variety of misconceptions on this concept which impede on their knowledge construction in other related topics and their application of mathematical knowledge in other disciplines. In this technological era, integration of ICTs is one intervention that is yet to be fully harnessed to address this challenge. Researchers like Nkula and Krauss (2014) and Mdlongwa (2012) report that ICT integration in South Africa, in general, is at its infant stage and is almost non-existent in marginalised schools. A number of constraints have been noted (Ertmer, 1999; Wilson-Strydom and Thomson, 2015) and chief among them include shortage of ICT resources, time constraints and lack of technical and

pedagogical guidance on how to use ICTs. The little ICT resources available in marginalised schools can be used to bridge the widening digital gap.

Dreyfus and Eisenberg (1982) argues that the pedagogical challenges of the function concept arise from three aspects that include a large number of connected sub-concepts, the way functions can glue together seemingly unrelated subjects and its multiple representations in a number of different settings. Exemplifying these complexities, Sajka (2003) highlighted that $f(x)$ can be used to represent both the name of a function or the value of function f. In her study, exploring the understanding of symbols used in functional notion, Sajka (2003) identified how some intrinsic intricacies of the function concept and the limited contexts in which some symbols occur in teaching impede on learner's understanding. Sajka (2003) concluded that the connection that exists between the concept of function and the concept of the formula of a function is indistinguishable. Furthermore, Abdullah (2005) pointed out that functional symbols were obstacles to most of the students. For example, Abdullah observed that students were not aware that x is a variable in the notation $f(x)$. They retained $f(x)$ even though x was already given a value. As a result, students equated $f(x-4) = f(x) - 4$. In dealing with such a complex and elusive concept, teachers often ask learners to manipulate algebraic equations and expressions, and compute answers with no understanding of the concepts they are dealing with. Clearly, this strong procedural bias has not been effective in promoting understanding of fundamental function concepts.

In South Africa, examination and diagnostic reports expose some common errors made by learners under functions that seem to be recurring. Examples of such errors include failure to correctly identify x- and y-intercepts and wrongly applying concepts of domain, range and asymptotic behaviour of functions. The reports also report that learners fail to understand the effects of parameters a, p and q on: $f(x) = a(x-p)^2 + q$; $g(x) = \frac{a}{x-p} + q$ and $h(x) = a^{x+p} + q$. Such errors are clear manifestation of the lack of conceptual understanding of functional concepts. The errors expose the deficiencies of the traditional pedagogical approach used in the teaching and learning of functions. The obvious weakness with emphasis on procedures *without* understanding is that it exposes learners to misconceptions. For

example learners struggle to manipulate algebraic expressions correctly, such as wrongly equating $2.2^x = 4^x$ and $2\log_2 x = \log_2 2x$. The central argument in this study is that promoting knowledge of functional concepts gives meaning to procedures, reduces chances of errors and misconceptions and empowers learners to apply mathematical knowledge functions in various contexts.

Ndlovu and Moll (2016) are of the opinion that new ICT tools need to be explored as they can further deepen learners' understanding by making semi abstract concepts more visible. Unfortunately, the quality use of ICTs in teaching and learning mathematics in South Africa has proven to be problematic to schools and teachers (see for example Howie & Blignant, 2009; Nkula & Krauss, 2014). More specifically, studies reveal the complexities of the interplay of technology and teaching and learning (Elagic, 2003; Horkanson & Hooper, 2000; Wilson-Strydom & Thomson, 2014; Hardman, 2005). Wilson-Strydom and Thomson (2005), for example, observed that many schools that have access to ICT resources tend to concentrate on learning *about* ICTs, rather than learning *with* or *through* the use of ICTs. Similarly, Horkanson and Hooper (2000) observed that ICT integration in mathematics classrooms is largely representational in nature and not generative.

On the one hand, representational use is where ICT tools are simply used to represent information in another form, such as the graph of $f(x)$ being drawn on the Euclidean plane. This robs learners' opportunity to grapple with the question and link the graph to the phenomenon it represents. A graph drawn by computer is a sterile representation if a learner does not understand in the first place how that graph links with its formula form. The representation is not generative. Representational level of ICT usage seems to reinforce traditional teaching approaches and does not promote knowledge construction. On the other hand, generative use views ICT tools as 'cognitive tools' or 'mediational tools' (Lim, 2003) that engage learners in knowledge re-construction (Alagic, 2003). The development of top end cognitive skills such as problem-solving and critical thinking is helped by generative use of ICT tools.

As mentioned previously, studies have shown that the teaching and learning of functions at high school level is problematic and challenging. Learners are struggling in understanding the underlying concepts

in this topic and teachers seem to be running short of pedagogical ideas on how best to present this topic. Traditional teaching approaches appear to have failed to help learners. This study pursues the argument that the integration of ICTs in the classroom has the potential to revolutionise pedagogical practices. In particular, the study focussed on exploring ICT-mediated teaching of concepts under hyperbolic functions with the aim of deepening conceptual knowledge and minimise learners' misconceptions. It further aimed to come up with a contextualised and classroom-based ICT-mediated instructional model that teachers can use for their teaching of functions.

1.9. Objectives and Purpose of the Study

This study was premised on the need to improve technology-based pedagogical practices in the teaching and learning of functions. It aims to unveil the actual errors and misconceptions that learners hold in dealing with functions tasks. The research investigated how the integration of ICT tools as an intervention strategy has the potential to deepen conceptualisation of the concepts under functions and minimise learner misconceptions around this topic. Using a grade 11 class sample at a rural school, the study hoped to achieve the following objectives:

1. To identify the common errors and misconceptions that learners hold in dealing with concepts under hyperbolic functions.
2. To explore the impact of the use of ICT as mediational tools in enhancing conceptual understanding of concepts under functions.
3. To build a pedagogical model for ICT-mediated instruction in deepening conceptual understanding of functions.

1.10. Research Questions

To have a deeper and focused perspective of the problem under study, the following specific questions were addressed:

1. What errors and misconceptions do learners hold on hyperbolic functions?

2. What impact does ICT-mediated instruction have in enhancing conceptual understanding of concepts under hyperbolic functions?
3. What pedagogical model for ICT-mediated instruction could this study recommends in deepening learner conceptual understanding and minimise misconceptions on hyperbolic functions?

1.10. Significance of the Study

The significance of this study rests on the need to ensure effective teaching and learning of functions. The study aims to examine the teaching of mathematical concepts under functions and graphs in an ICT-mediated environment, by exploring conceptual understanding through the use of such cognitive tools in a classroom set up. Research has revealed that mathematics curriculum development in an ICT-based environment is a complex task requiring on-going inquiry, feedback and refinement (Dreyfas, 2002; Elagic 2003). Mathematics teachers find themselves at the centre of this complex innovation, and in most cases they respond by either avoiding the innovation or by re-aligning it to reinforce their traditional teaching practices (Ertmer, 1999). However, the reality that the teachers are facing is that they are teaching a modern generation that is immersed in technology in every sphere of their lives. Thus, the failure to integrate this technology in the classroom renders teaching and learning irrelevant and detached from social reality. Teachers cannot navigate through the complexities of ICT integration on their own. They need guidance and visible support structures to ensure meaningful and quality ICT integration. It is in this spirit that I believe this study will be a professional guide on ICT integration for conceptual learning. The study will hopefully challenge mathematics teachers to introspectively re-examine their pedagogical practices in light of the ubiquitous availability of ICT tools.

The defining feature of this study is its conceptual orientation. A conceptual approach to learning aims for students to solve problems by working from their own understanding (Glaserfeld, 1988). While this cannot be easily achieved, once created, it is easily maintained (Thompson, 1994). The characteristic features of conceptual learning include knowledge deepening and knowledge creation. UNESCO (2011)

elaborated on this by saying that knowledge deepening is, ' … learning more relevant to the work place and to social life … knowledge of school subjects to understand and solve complex problems encountered in real world situations of work and life …' (p.25). Thus, teaching for conceptual understanding empowers learners and its benefits go beyond examinations and school life. Instead of superficial coverage of topics and reinforcing procedures through drill and practice, this study focuses on deeper understanding of concepts, principles and procedures, and how these are organised and interconnected with and across other concepts (Bransford, Brown & Cocking, 2001) to form a complex schema of hyperbolic functions.

According to the mathematics CAPS document (DBE, 2011), one of the key aims of the mathematics curriculum is to develop problem solving and cognitive skills. The document further elaborates that teaching should not be limited to 'how' but should rather feature the 'when' and the 'why' problem types. The 'how' problems refer to procedural and computational skills, while the 'when' and 'why' type of problems require learners to have a deeper understanding of concepts and meanings behind mathematical procedures. Clearly, the curriculum advocates a conceptually oriented type of curriculum delivery. As already highlighted in the earlier sections of this chapter, the integration of technology can play an important role in enhancing teaching and learning of mathematics. ICTs afford students opportunities to use visualisations and simulations to explore, understand and apply complex knowledge.

It is understood that most schools in developing countries encounter barriers to the use of ICTs ranging from lack of computer resources and infrastructure, technical support, administrative support and sufficient teacher training (Ertmer et al, 2012). However, even where resources are available, the use of ICTs is not a central feature in many classrooms. Even in developed countries, research findings convey the limited extent to which the ICT tools have been incorporated into educational systems, particularly in teaching and learning (UNESCO, 2011). This study hopes to contribute towards achieving the envisaged goal, where everyday classroom pedagogical practice is ICT-driven.

It has already been argued that one of the key determinants for the successful integration of ICT is the availability of a strong, clear and practicable ICT policy framework. The challenge in this regard is that policies may be there on paper, but there are no deliberate efforts or clear strategic plans to enforce implementation of these policies. As Edmore (2005) argues, some policies are merely symbolic acts available for political reasons or simply to please donors or special interest groups, without effecting the change they are intended for. In response to such symbolic policies, school administrators and teachers make surface changes that have no real impact on teaching and learning. Teachers may reinforce their old practices with the new innovation by selecting parts of the reform that suits them, without making fundamental changes.

This study has its basis on the belief that research plays a pivotal role in operationalizing policies by providing information to policy makers and support and guidance to policy implementers. With regard to ICT integration in education, policy makers need information to test the authenticity of the hype and bold claims of technology advocates (UNESCO, 2011). In other words, policy makers need evidence that justifies the huge financial resources needed for ICT integration. They want to know if the use of computers really enhances teaching and learning, and whether students' access to and use of ICT will promote their learning of school subjects. The findings of this study provide valuable insights to policy makers on the practicability of ICT integration in enhancing mathematical learning in disadvantaged schools. Mathematics teachers will be challenged to re-examine their use if ICTs in terms of whether they foster conceptual understanding or re-enforce procedural learning. The next section explains the key concepts in this study.

1.12. Operational Definitions

Function

The definition of function as used in this study was derived from Dirichlet and Bourbaki's contributions towards the evolution of the notion of function. Vinner and Dreyfus (1989) called it the Dirichlet-Bourbaki concept whereby a function is viewed as 'a correspondence between two nonempty sets that

assigns to every element in the first set (the domain) exactly one element in the second set (the codomain)' p.357. This view of function is more acceptable in modern mathematics because it brings out the *one-valuedness* and the *arbitrariness* nature of functions. It is, however, important to note that in the high school curriculum and textbooks, the concept of function is used in its narrow sense, only referring to functions whose rule of correspondence is given in terms of an algebraic formula.

Mathematical Conception

Mathematical conception refers to a combination of internal representations and associations evoked by a mathematical concept, the subjective universe of human knowledge (Sfard, 1991). It is a cognitive process of knowledge construction whereby students learn ways in which the concepts they are learning about are connected (Cope & Kalantzis, 2015). It involves the students' level of understanding of a concept developing from the particular to the general; from the concrete to the abstract; making deductions and hypothesising or looking for trends and relationships. In the context of this study, the term mathematical conception was used in relation to Dubinsky's four-stage process of mathematics concept development (APOS), whereby learners deal with mathematical situations by constructing mental actions, processes, objects and organising these into schemas (Makonye, 2017) to make sense of the problem situation.

Misconception

Misconception is a conception that is not in agreement with the opinion commonly agreed on by experts on a particular discipline (Kucuk & Isleyen, 2011). It is the inconsistency between the concept that we want students to learn and the internal representation they build in their minds (Michael, 2002). It can be viewed as knowledge that contradicts with currently accepted body of mathematical knowledge, and must be overcome, minimised or eliminated for students to achieve better understanding.

ICT Tools

Information and communication technology (ICT) is a blanket term that includes a collection of communication devices or applications such as radio, television, cellular phones, computers and network hardware and software and satellite systems, as well as the various services associated with them(UNESCO, 2002). Alkamel and Chouthaiwale (2018) defines it as a broad set of technological tools used to communicate, create, disseminate, store and manage information. In this study, ICT tools refer to any technological device used, applied or integrated in the teaching and learning activity in an endeavour to facilitate learner understanding.

Interiorization

According to the APOS theory, interiorization is a transitional change from a procedural activity (*action*-conception) to be able to do the same activity again imagining some understanding of the processes involved (*process*-conception) (Breidenback, Dubinsky, Hawks & Nicholas, 1992). Syaiful, Kamid and Marssal (2014) further clarify that an individual begins to reflect on an action thereby forming a an internal cognitive structure that performs the same operation as the action, but the action is wholly performed in the mind of the individual. In the context of this study, interiorization was considered evident if, for example, a learner was able to explain why we substitute $x = 0$ when calculating y-intercept and vice-versa and move away from being able to mechanically calculate x-and y-intercepts with no understanding of the procedures.

Encapsulation

As explained by Breidenback et al (1992), encapsulation occurs when one becomes aware of a *process* as a totality and realises that transformations can act on that totality. In APOS terms, we say the process has been encapsulated into a cognitive object. Encapsulation condenses the processes into a cognitive object that one can change and be manipulated (Chimhande, Naidos & Stols, 2017). In this study, learners were considered to have encapsulated hyperbolic function if they could easily and flexibly switch from symbolic to graphical representation or vice-versa and be able to apply transformations on a given hyperbolic function.

Thematization

A mathematical topic comprises many *actions*, *processes* and *objects* that need to be organised and linked into a coherent structure called a *schema* (Maharaj, 2010). Thus a schema is viewed as a dynamic collection of action-conceptions, processes, objects and other schema and the relationship between them (Cooley, Baker & Triguerous, 2000). A mathematical idea is then said to have been thematized when an individual imagine of it as a total entity made up of several components that can be decomposed to recover its components. In the case of hyperbolic functions, I expected a learner who had thematized hyperbolic functions to schema level to be able to link and connect the concepts of intercepts, asymptotes, domain, range and axes of symmetry to deal with an unfamiliar problem situation.

1.13. Conclusion of the chapter

It has been shown that the teaching and learning of mathematics in South African schools is characterised by a number of shortcomings. More specifically, the chapter has problematized the teaching and learning of functions. The function concept is a central mathematical idea that 'unifies' or connects with several other mathematical concepts. Available evidence suggests that learners struggle to understand the concepts under functions. This is evident from the errors and misconceptions that learners persistently display as they attempt to solve problems involving functions. Researchers have so far agreed that the use of ICT tools has enormous potential to improve the quality of teaching and learning of mathematics. However, not much has been done to ascertain the impact of using ICT tools on addressing learners' misconceptions and deepening conceptual understanding. This study, therefore, sought to explore the extent to which use of ICT tools as an intervention strategy, may assist learners in reflecting on their misconceptions and address them in the context of a rural high school. The next chapter discusses the theoretical framework for this study.

CHAPTER 2: THEORETICAL FRAMEWORK

2.1 Introduction

In the preceding chapter, I described the research context and problematized the teaching and learning of functions in the school mathematics curriculum in South Africa. It was also shown that though the use of ICTs for teaching and learning has the potential to radically transform pedagogical practices in mathematics classrooms, their use is either at an embryotic stage or non-existent. In this chapter, I discuss the theories that formed the theoretical framework of this study.

As Nyamupangedengu (2015) puts it, a theoretical framework is a form of representation of phenomena to be studied. It is used to see where there are overlaps, contradictions, refinements or qualifications (Miles & Huberman, 1994). In this study, I used a theoretical framework to provide a base on which my study was anchored and it provided the lens for the literature review, the research methodology, and most importantly the analysis and interpretation of the data. What one sees and what one does not see is influenced by the theoretical perspective (Nyamupanedengu, 2015) and that perspective provides meaning for the phenomena being studied and place findings within a larger context.

Since my study was centred on learner acquisition of mathematical knowledge, the chapter begins by briefly discussing two dichotomous views on mathematical knowledge, namely absolutism and fallibilism. I found this critical because the way we view learners' errors and misconceptions, to a large extent, hinges on some ontological and epistemological assumptions that we hold. The fallibist-flavoured learning theories of social-constructivism, APOS theory and the Activity theory are discussed in more detail as they form the theoretical base of the study. Though the theories are discussed separately, they have fundamental linkages as demonstrated under the conceptual framework of the study.

2.2. The nature of mathematical knowledge

A study on pedagogical aspects of mathematics would inevitably require a sufficient understanding of how we view the nature of mathematical knowledge. In this section, I briefly discuss two contrasting

philosophical perspectives, that is, *absolutism* and *fallibism*. They differ on their ontological and epistemological assumptions.

2.2.1. Absolutism

Absolutism views mathematical knowledge as a body of infallible and objective truths, independent of the affairs and values of humanity (Ernest, 2002). In other words, mathematical knowledge is viewed as existing *'out-there',* consisting of certain, absolute and unchallengeable truths. That external body of knowledge has to be 'transmitted' or 'poured' into the learners' minds. Despite its inadequacies and its present disreputable status in academia, I included this traditional philosophy here because its features are still evident in the present day mathematics education systems (Dummett, 1977; Ernest, 1998). Ernest (1998) pointed out that the government-driven curriculum attempts to breakdown absolute knowledge into hierarchical components or structures which teachers are then expected to disseminate to students, who are viewed as passive absorbers, at various levels. This observation resonates well with the education system in South Africa. The curriculum is centrally designed at national level and cascaded down to schools to be 'transmitted' to the learners. This administrative reality seems to affirm that mathematics is given, absolute and unchallengeable.

The absolutist paradigm gave birth to behaviourist learning theories, which are not part of the theoretical framework of this study. Their belief that knowledge exist independent of human beings and is absolutely true was later proven to be flawed (Ben-Ari, 2001). In line with their philosophical underpinnings, the learners' mind is viewed as 'clean slate' that can be filled with knowledge (Orton, 2004). Drill and practice and focus on procedural knowledge epitomises behaviourism. In theory we condemn these teaching practices but research has shown that teachers are still practicing them. It has been argued in the preceding chapter that teaching of mathematics in South Africa is characterised by teacher-centred approaches and the promotion of procedural knowledge. In this study, I refer to those teaching practices as traditional pedagogy.

2.2.2. Fallibism

As a result of the underpinning flaws in absolutism, *fallibism* has become the more accepted view in the philosophy of mathematics. The original seminal figure to this philosophy, as reported by Tymoczk (1986), is Imre Lakatos (1976; 1978). Lakatos (1978) developed a philosophy of mathematics premised on the view that mathematics is a human activity that encompasses what mathematicians do and have done, with all their imperfections that characterise any human activity or creation (Dengate & lerman, 1995; Ernest, 2002). According to this philosophical perspective, mathematics is a human creation and as such, fallible and open to correction. This position has formed the basis of constructivism and post-modern thought in mathematics education. Modern researches and current learning theories in mathematics education are built around the principles of constructivism and its epistemological assumptions (Dengate& Lerman, 1995; Steep, 1988). I discuss constructivism in more detail below and how its principles formed the theoretical basis for this study.

2.3. Constructivism

This study was largely informed by the constructivist learning theories. As noted in the previous section, constructivism emerged from the philosophical assumptions of fallibilism and its major proponent is Ernest Van Glasersfeld (1983, 1984, and 1986). According to Glasersfeld (1995), constructivism;

> '… is an unconventional approach to the problems of knowledge and knowing. It starts from the assumption that knowledge … is in the hands of persons and that the thinking subject has no alternative but to construct what he or she knows on the basis of his or her experience' (p.1).

Firstly, it is critical to note from the onset that constructivism aligns knowledge to human construction on the basis of personal experience. The ontological knowledge is rejected. As Bern-Ari (2001) argues, we can never truly 'know' anything; hence the theory of existence cannot influence educational practices. Secondly, Glasersfeld implied that epistemology is non-foundationalist and fallible. In other words, knowledge is a human creation and never perfect. It is thus very critical in my study because though we expect learners to acquire the acceptable body of knowledge, it is fine for them to make errors

and have misconceptions in their process of knowledge construction. The errors and misconceptions are then refined and aligned to the acceptable body of knowledge. In other words, mistakes are part of the learning process. We all learnt to ride a bicycle by falling down, developing some permanent scars, but there after riding becomes a second nature.

Thirdly, as Bern-Ari (2001) puts it, constructivists believe that knowledge creation is achieved by cognitively reflecting on existing knowledge. This principle directly emanates from Jean Piaget's theory of knowledge acquisition by children (Glasersfeld, 1995). Constructivist principles tend to reflect democratic and egalitarian attributes that shift focus of learning, both from the teacher and the content, to the learner (Dengate & Lerman, 1995). This shift is evident in the adoption of child-centred learning, mixed-ability classes, individualised learning schemes and the emphasis on problem solving and investigational activities in the educational policy documents (Walkerdine, 1984).

These constructivist ideas resonate well with the major focus of this study. From my discussion under the research context in chapter 1, it came out that teaching of functions in South Africa is constructivist on paper but largely behaviourist in practice (see Makonye, 2017; Chimhande, Naidoo & Stols, 2017). The White Paper on e-education envisages teaching of mathematics to advance constructivist constructs like deepening of knowledge, conceptual understanding and knowledge creation (DBE, 2013). Clearly there is a gap between expected pedagogical practices and the reality of what happens in the mathematics classroom.

2.4. Constructivist principles that influenced teaching and learning

In mathematics education, as Kilpatrick (1995) explains, constructivism is a theory of knowledge acquisition and not a theory of teaching or instruction. However, constructivists seek to derive implications for practice from the theory. Glasersfeld (1983) extracted some key constructs for educational practice from the constructivist principles which I found pertinent to this study and are discussed below.

2.4.1. Teaching versus training

Constructivists view teaching as practices that focus at generating understanding. In the context of the teaching and learning of mathematics, Glasersfeld (1995) indicated that the focus is on conceptual form of knowledge. On the other hand, training is a practice that aims to reinforce repetitive behaviour. Training is associated with traditional teaching approaches where teachers give instructions and engenders mastery of skill through drill-and-practice (Glasersfeld, 1995). In support of this view, Sarwadi and Shahrill (2014) observed that teaching of mathematics in schools is more focussed on the mastery of rules, procedures and formulae used to arrive at correct answers without understanding of basic concepts. Such teaching approaches provide favourable breeding environments for errors and misconceptions.

2.4.2. 'Inside' versus 'outside'

Another essential contribution made by constructivists is the particular focus on processes inferred as *inside* the student rather than on overt *external* responses (Glasersfeld, 1983). A constructivist teacher should strive to make inferences about what a student is thinking and make efforts to gain access into the learner's mind. Thus in line with constructivist thinking, external responses or behaviour displayed by a learner should be viewed as manifestations of internal mental processes, and should be made use of to make inferences about the learner's thinking pattern. In doing calculations or solving mathematical problems, learners commit errors which are a result of existing conceptual gaps entrenched in the learners' conceptual schemes (Luneta & Makonye, 2010; Godden et al, 2013; Olivier, 1989). Teachers rarely diagnose learners' individual errors to identify the root cause of these errors and how best they can be addressed.

2.4.3. Constructed versus transferred

Constructivists also make a useful distinction between knowledge being constructed by the learner and knowledge being transferred to the learner during teaching. According to Davis and Mason (1986), teachers who subscribe to transfer of knowledge make the following statements, " ... 'I got the ideas across....' ; 'some students pick things up quickly.....'; '... the teacher is a medium for delivering

curriculum to students.....' ."(pp. 8-9). Constructivists reject these views of instruction and believe their task is to change the teachers' knowledge transfer model. Glasersfeld (1983) pointed out that teachers with a constructivist view of learning should not see themselves as delivery agents of an education system, but more as gardeners, tour guides or learning councillors. Constructivists envisage a radically transformed role of the teacher from a transfer agent of knowledge to more of a facilitator and creator of a conducive environment for knowledge creation.

2.5. Constructivism and Misconceptions

One of the most attractive and promising contribution of constructivist work is the attention given to learners' misconceptions. The way constructivists view misconceptions hinges on Piaget's theory of cognitive change. As Muijs and Reynolds (2000) explain, cognitive processes can be classified as one of the following: adaption and organisation. Organisation is largely an internal process involving re-arranging and 'linking up items of previous learning to form a strongly interconnected cognitive system' (Muijs & Reynolds, 2000: 275).

Muijs and Reynolds (2000) further clarify that adaption occurs in two ways: assimilation and accommodation. In assimilation, the learner attempts to link new concepts, skills and information into her previously acquired cognitive structure. The existing cognitive structure must then be modified in order to create room for new knowledge. This, according to Piaget, is known as accommodation. In view of these mental processes, constructivists argue that during assimilation and accommodation, chances are that a learner may develop 'concepts' which do not fit well into the existing cognitive structures or which are incongruent to socially accepted body of mathematical knowledge (Kilpatrick, 1987) and we call these incongruent cognitive structures misconceptions.

My conception of mathematical misconception in this study drew heavily from the above theoretical background. The learners' deviations from teacher's expectations are regarded by constructivists as clues to students' thinking (Dengate & Lerman, 1995). Another way of looking at it is that by diagnosing and understanding learners' misconceptions, the teacher is able to assist the learner in correcting his/her

own misconceptions. Constructivists further argue that learners' misconceptions are better understood when seen as arising from alternative constructions of meaning rather than as failures in communication. In light of this argument, this study viewed misconceptions as integral components of knowledge creation and concept formation which should be harnessed to facilitate conceptual understanding in learners.

2.6. Constructivist theories that informed this study

As discussed in the preceding section, constructivism is a broad paradigm that subscribes to a set of basic beliefs that represent a worldview and define the nature of the world (Guba & Lincolin, 1994). From an educational perspective, it is a theoretical perspective on which most modern learning theories are built. This study was informed by three constructivist learning theories namely: social-constructivism, APOS theory and the Activity theory. Below, I discuss these learning theories and how they informed my study.

2.6.1. Social Constructivism

The social constructivism theory is grounded on the contribution of the soviet psychologist Lev Vygotsky (1896 – 1934). Vygotsky believed that what constitutes the discipline of mathematics is the work and experience of mathematicians (Thomson, 2002). Vygotsky's perspective partly agrees with radical constructivists, such as Glasersfeld and Piaget, on the view that knowledge is constructed through internal mental processes but adds another dimension that these constructions cannot be complete without the systematic intervention of the 'more knowledgeable others' (MKOs). His critical argument is that children, based on their previous experiences, have their own understanding and beliefs, and the responsibility of adults is to guide the children's thinking in order to develop that thinking to the level of a more scientific, conceptual understanding (Dengate & Lerman, 1995). It is this theoretical claim that formed the back-borne of this study. The school or teachers in particular, may not be blamed for the learners' individual differences or social backgrounds but they definitely can

facilitate the learners' academic achievement by manipulating and enriching the learner's pedagogical context. One way of enriching the pedagogical set up is through the integration of ICTs.

The other reason for adopting social constructivism for this study is its base line view on learning as a social process (Gredler, 1997). The theory argues that learning does not take place only within an individual (as viewed by radical constructivists) nor is it a passive absorption of knowledge as dictated and shaped by external forces (as viewed by behaviourist). Rather, learning occurs when individuals are actively engaged in social activities. The social context in which learning takes place is regarded by social constructivists as very critical. Now ICTs, by their very nature, are social gadgets. In this digital era, our social interactions and relationships in every sphere of our lives are driven and influenced by ICTs to a larger extent. The learners we teach in schools are immersed in technological devises in their everyday life. If the school fail to take cognisance of this social reality and harness it to enhance learning, it is bound to fail.

Gredler (1997) identified two aspects of social learning contexts. Firstly, it is the historical developments acquired by the learner as a member of a socio-cultural context. These refer to symbol systems such as language, mathematical systems and logic. Effective learning should take cognisance of these cultural symbolic systems already existing in the learners' cognitive structures. Secondly, the learners' social interaction with the MKOs of the society is critical. Gredler (1997) asserted that social meaning of important symbol systems and how to use them cannot be realised without the social interaction with MKOs. To illustrate the importance of the MKOs, Vygotsky (1978), introduced the concept of the 'zone of proximal development' (ZPD), which has proved quite relevant to teaching and learning. The key social constructivist principles that provided important insights into this study are discussed separately below.

- **The Zone of Proximal Development (ZPD)**

The ZPD is the cognitive gap that exists between what children can accomplish on their own and what they can accomplish in collaboration with another more competent person (Thomson, 2002). Ernest

(1999) explains it as the difference between a learner's actual level of development (what a learner has already mastered) and what the same learner can achieve when provided with educational support (potential development). Ernest (1999) further elaborates that by closing this cognitive gap, the learner is taken to a higher level of cognitive development. If properly integrated, ICT tools have the potential to enrich the interactions within the learning context and facilitate the reduction of the gap between actual and potential levels of cognitive development. The teacher's role would be to provide 'conflict-generating' problems and design instructional activities that provide opportunities for problem solving, thereby propelling learners to higher levels of thinking.

- **Scaffolding**

The term scaffolding was not originally used by Vygotsky. Rather, it was introduced by Wood, Bruner and Ross (1976) in an attempt to operationalize Vygotsky's idea of teaching in the ZPD (Wells, 1999). The term is commonly used to describe the different kinds of educational support which learners receive as they interact with parents, teachers and other mentors targeted towards the acquisition of news skills, concepts and deeper levels of understanding (Maybin, Mercer & Stierer, 1992). It involves structuring instructional activities in small steps based on tasks that learners have already mastered in familiar contexts (Thomson, 2002). It is important to note that within the classroom, the more knowledgeable one is not always the teacher. Confrey (1986) noted that other learners who have demonstrated mastery of tasks and concepts can be placed in collaborative groups to assist each other.

Wells and Chang-Wells (2002) provide three defining features of educational scaffolding. Firstly, scaffolding is characterised by the dialogical nature of the interactions in which knowledge is co-constructed. This underscores the need for dialogue between the teacher and the learners. Secondly, it is the significance of the kind of learning in which knowing is imbedded. In other words, the quality of any learning activity should be evaluated based on its ability to enable learners to carry out a task which they would not be able to manage on their own. The third feature is the role of artefacts that mediate

learning. This view re-affirms the role of ICTs, as modern artefacts, in supporting learning and improves the quality of social interaction in the classroom.

Wells and Chang-Wells (2002) provided very critical insights on ICT integration. In the first place, they acknowledge the role of mediating artefacts (ICTs in this case) in facilitating knowledge construction. However, the mere presence of ICT tools in the classroom does not guarantee effective learning. They underscored the need for dialogue or interaction within the learning context. This study operationalized this view by exploring how ICT-mediated instruction could promote learner-learner interaction, learner-teacher interaction and learner-tool interaction. In line with the principle of scaffolding, every pedagogical activity involving ICTs should include a challenging task that engages learners in problem-solving and critical thinking. This automatically dispels 'representational' use of ICTs

- **ICTs as 'Mediational Tools'**

Another essential component of concept formation, as Identified by Vygotsky (1978), is semiotic mediation. Vygotsky's central argument is that all human mental processes or functions are regarded as consequences of mediated activity. The mediatory role may be played by a psychological tool or sign (such as words, graphs or algebra) or a physical tool (like a computer), (Vygotsky, 1978). Thus, in a sense, Vygotsky saw action mediated by signs and symbols as linking the external social world to the internal human mental process.

In the area of mathematics education, Radford (2000) noted that mathematical signs or tools mediate two processes. The first one is the mental representation of a mathematical concept in the individual (internal). Secondly, it is the individual's interaction with the already accepted and socially sanctioned mathematical world (external). This view resonates with Vygotsky's fundamental principle that the individual's knowledge construction is a product of cognitive processes facilitated by the individual's social context.

Thus, mediating tools are cognitive tools that bridge the gap between external social reality and the human internal (mental) reality. Functions are abstract representations of some real-world phenomena which are external to the learner's mind. Unfortunately, we present them to learners in their abstract nature, in the form of definitions, symbols and algebraic equations. Consequently, learners find it difficult to assimilate and accommodate this abstract knowledge because there is nothing familiar or concrete, they can link them to. My argument is that ICTs have the potential to provide visual or concrete images that learners can associate functions with.

In view of the above observations, I found Vygotsky's ideas particularly useful to this study because they provide the essential linkages between the learner, mediating tools and mediation activities and concept formation, which form the focal objective of this study. The integration of ICT tools in a classroom set up involves a complex system of interacting components involving learners, concepts, ICT tools and other contextual factors.

The weakness of social constructivism, in my view, is that it provides broad constructs or ideas for consideration in teaching and learning, but it does not provide specific classroom-based guidelines on the implementation of those ideas. This study involved classroom-based intervention and as a result, I needed another theory or model that could guide me in designing the intervention programme. To augment Vygotsky's theory, the Activity theory provided a useful framework on how to effectively connect interacting components in an ICT-mediated classroom.

2.6.2. The Activity Theory

Basically, the Activity Theory was developed in response to the failure by Vygotsky's learning theory to provide an analytical framework that situates learning within a wider and broader context (Lim & Chai, 2004), and to account for the complexity and the fluid nature of the learning activities (Hardman, 2005). The first generation of Activity Theory, as Hardman (2005) explains it, is grounded on Vygotsky's notion of mediation but extends to demonstrate how cognitive processes happen within a collective context of interacting components. The second generation of Activity Theory was based on

the work of Alexei Leontiev (1981) who further developed the theory to include the distinction between individual action, operation and activity (Engestrom, 1987). It is Engestrom who modified the Activity theory to include interacting components of subject, object tools division of labour, community, rules and outcomes (Murphy & Rodinguez-Manzanares, 2008). Figure 2.1 below illustrates the basic relationships between the dimensions in Engestrom's activity system.

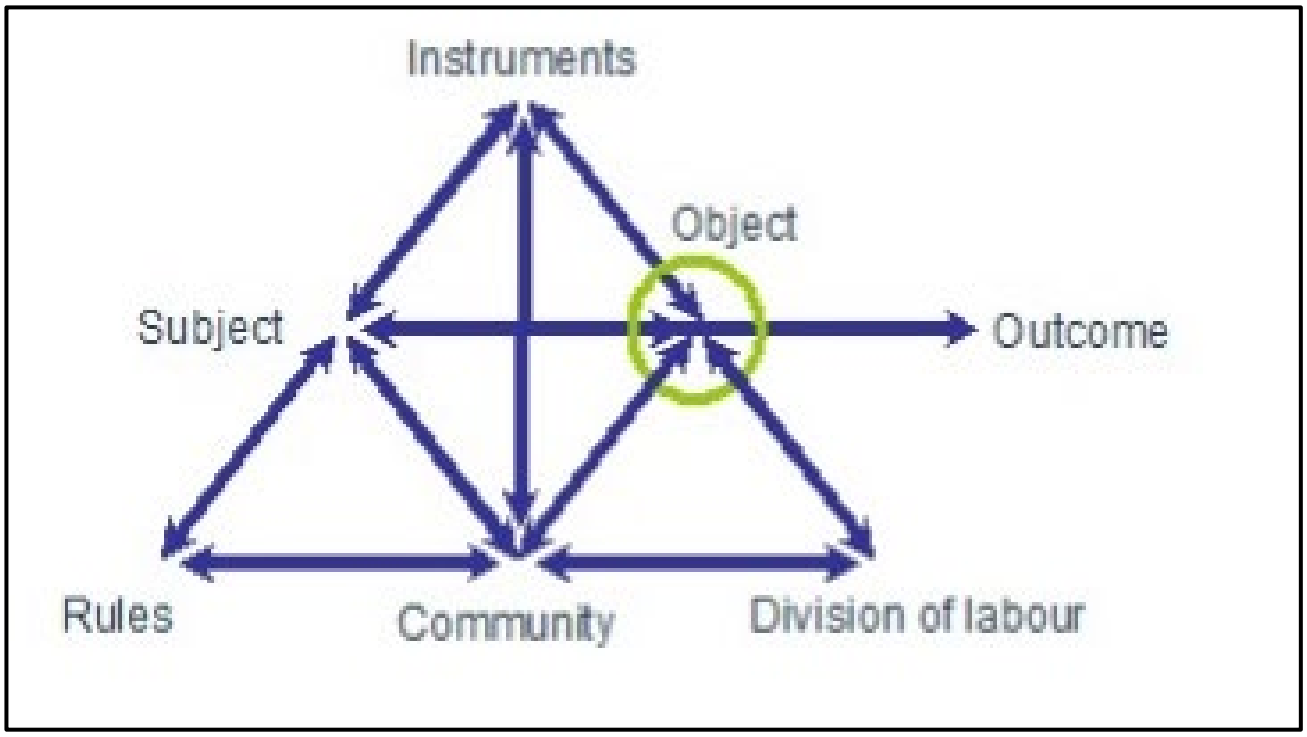


Figure 2. 1 An Activity system (Adopted from Hardman, 2008)

The triangular diagram illustrates the multiple relationships possible between the components within the activity system. Goncalves, Sousa and Zacarius (2013) note that the central focus is to understand the activity system in its entirety, rather than the individual components' separate connections. They further highlight that Activity Theory is a useful framework in understanding the collective nature of human work and its praxis. Human work can only be fully understood and analysed within the context in which it occurs. It is, therefore, important to consider the actions of everyone in the activity system, in terms of: who is involved? what are their motives and goals? which rules apply in the context? and what tools are used in the community where the activity occurs (Goncalves, et al. 2013).

I found the second-generation Activity Theory quite useful to this study because it identifies all key components involved in a learning context and can be modified to suit the prevailing circumstances. The interacting components are connected together into an activity system, viewed by Engestrom (1987) as a unit of analysis. The integration of ICT in the learning context brings in other dimensions that make the context more inclusive and more complex. The Activity theory was useful in developing the instructional activities for each of the intervention lessons and allowed me to observe the actual processes by which activities shaped or were shaped by the context. Further details on how the components of the activity system were used are unpacked under the conceptual framework in the next chapter.

Both social constructivism and the Activity theory could not adequately provide the theoretical backing needed in this study. The second research question of the study sought to establish the impact of the ICT-mediated instructional intervention on learners' conceptual understanding of hyperbolic functions. I therefore needed a framework that would assist me in qualitatively establishing the change in learners' understanding after the intervention lessons. To achieve this, I incorporated Dubinsky's APOS theory.

2.6.3. Dubinsky's APOS Theory

According to Dubinsky (2000), the understanding of a mathematical concept is the result of mental construction and re-construction towards the development of mathematical objects. The construction and reconstruction is achieved through mental activities of *actions*, *processes* and *objects* of mathematics which are eventually organised into a *schema* to solve mathematical problems. In reference to these key mental constructions, Dubinsky referred to this theory as APOS Theory (Dubinsky & McDonald, 2002).

The APOS Theory is a constructivist theory that seeks to explain how the achievement of concept learning occurs (Syaiful, Kamid & Marshal, 2014) and is an extension of Piaget's idea of reflective abstraction in explaining the development of logical thinking in children (Dubinsky, 2000). Dubinsky sought to explain the development of higher order mathematical thinking, particularly in college

students. However, as Syaiful et al. (2014) point out, the theory has been relevant in understanding students' learning of several topics such as calculus, abstract algebra, statistics and others.

Several other researchers have used APOS theory to explore the learners' conception of the function concept (Breidenbach et al., 1992; Brijlall & Ndlovu, 2013; Chimhande, 2013) and to evaluate learners' understanding of the function concept (Weller, Arnon & Dubinsky, 2011). Dubinsky and Wilson's (2013) study only focussed on high school learners' action and process conception levels and deliberately ignored the object and schema levels. Brijlall and Ndlovu (2013) used APOS theory to evaluate grade 12 learners' understanding of linear programming. They only considered up to the object conception level.

Similarly, Chimhande (2013) used the theory to examine the likely mental structures required to learn the function concept and to evaluate their level of understanding after a Realistic Mathematics Education (RME) intervention approach. In this study, the APOS theory was used to determine the learners' conception levels of hyperbolic functions before and after the ICT-mediated intervention lessons to establish if the lessons facilitated any growth in conception level.

As Dubinsky (2000) puts it, an *action* is a transformation of objects perceived by the individual as an external process, and as requiring step-by-step procedures on how to execute the operation. Watson and Moritz (2000) explained it as the transformation in an individual's thinking as a result of stimuli from outside. In relation to functions, the Action-conception of function may include familiar objects of numbers and repeatedly carrying out manipulations on numbers. At this level, the student is limited to procedural operations involving formulae, substitution and manipulation of algebra. For example, given: $\boldsymbol{f(x) = 2x^2 - 4}$, students at the *action*-conception level focus on the individual components of the process, that is substituting an input value $\boldsymbol{x}$ to get an output value $\boldsymbol{y}$.

The APOS theory is quite handy in the analysis of learner performance in mathematical tasks. It enables the researcher to determine the level of learner understanding by pointing to mental constructions of

actions, *processes*, *objects* and/or *schemas*. However, the APOS Theory does not provide a clear guidance on how computer tools can be integrated in a classroom situation to facilitate learner transition from one mental construction to another. The theory is confined to explaining internal mental processes without elaborating on external factors that facilitate those mental processes. Further discussion on how APOS theory was applied in this study is under conceptual framework in the next chapter.

2.7. Linking the theories

No single learning theory could be found to provide sufficient theoretical backing that I needed for this study. I therefore drew from Pegg and Tall's (2005) idea of theoretical triangulation. They defined theoretical triangulation as 'the process of compiling relevant theoretical perspectives and practitioner explanations, assessing their strengths, weaknesses and appropriateness, and using some subset of these as the focus of empirical investigation' (p.43). The implication of this description is that different theories of learning can be merged or blended to complement each other, whilst they still maintain their separate identity. Drawing from this idea, the theories of social-constructivism, Activity theory and APOS theory were merged in this study within the broad constructivist paradigm. I therefore drew analytical insights from the interconnectedness of the theories and their complementary constructs to build an integrated theoretical framework, as illustrated in figure 2.2 below. A more detailed discussion on how the identified constructs were used in this study is contained in the next chapter.

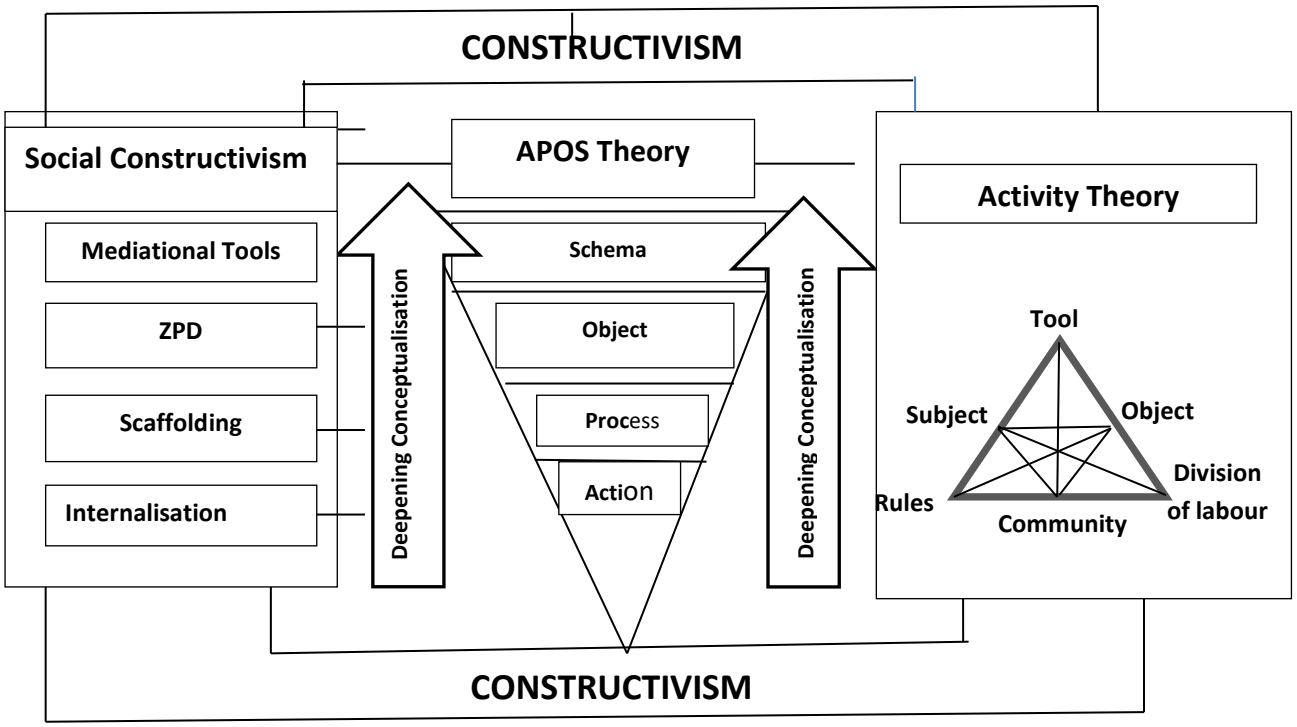


Figure 2. 2. Diagrammatic representation of the theoretical framework of the study

2.8. Conclusion of the chapter

This chapter presented the learning theories that provided the theoretical backing for this study. The study was rooted in the broad constructivism paradigm, which emerged from the fallibist philosophical perspective of the world. Its key epistemological assumption is that it places humans at the centre of knowledge. Constructivism gave birth to several learning theories, but this study merged three of these to form its theoretical framework, namely, social constructivism, activity theory and APOS theory. Social constructivism was chosen because it acknowledges that learning cannot be complete without the systematic intervention of the learner's social context. In other words, by deliberately manipulating and shaping the learning context we can facilitate and enhance internal knowledge construction. This view resonated well with the central focus of this study, which was, to enrich the pedagogical context through integration of ICTs and facilitate knowledge construction. To augment social constructivism theory, I incorporated the Activity theory to provide specific guidelines and insights on designing and implementation of an ICT-mediated intervention programme. To assess and evaluate the impact of the

intervention programme on learner conceptual understanding, Dubinsky's APOS theory was found most appropriate. These theories were discussed separately but they were linked and connected to achieve one purpose, which is, deepening learner conceptual understanding. In the next chapter, I zoom into the theoretical constructs, extracted from the above theories, and how they shaped the course of the study.

CHAPTER 3: CONCEPTUAL FRAMEWORK

3.1. Introduction

In Chapter 2, I discussed the three learning theories that were merged to provide the theoretical base for the study. This chapter unpacks the theoretical constructs, extracted from the learning theories and other relevant literature, which influenced this study. According to Miles and Huberman (2004), the conceptual framework of a study articulates, either diagrammatically, graphically or in a descriptive format, the main theoretical constructs or variables, and their interconnectedness. They further clarify that the conceptual framework helps in placing data in convenient categories for analysis. In the present study, the conceptual framework assisted me in the categorisation of learners' errors and misconceptions, development of the ICT-mediated intervention lessons and analysis of learners' understanding of hyperbolic functions.

In line with the three research questions of the study, the conceptual framework was structured in three overlapping sections. The first section of the conceptual framework provided the theoretical lens on the analysis and categorisation of learners' errors and misconceptions. The second section focuses on theoretical constructs that informed the development and implementation of the ICT-mediated intervention lessons. The third section dissects the APOS theory showing how the constructs of *action*, *process*, *object* and *scheme* were used to assess and evaluate the impact of the intervention lessons. These sections of the conceptual framework were discussed separately and applied at different phases of the study but they are closely linked and they do inform each other. It is this linkage that ultimately informed a holistic analysis of data in this study.

3.2. Conceptual Framework for Categorization of Errors and Misconceptions

The first research question of this study sought to find out the errors learners made in dealing with hyperbolic function problems and infer misconceptions underlying the errors. The errors were extracted from the learners' written responses in the pre-test and their misconceptions were inferred from their oral responses to the task-based interviews. To achieve this, I needed a protocol which provided codes

to categorise the errors displayed by the learners and the theoretical lens to infer on the misconceptions underpinning the errors.

In developing the protocol, a wide range of literature on different classifications of mathematical errors was consulted. I eventually settled on classifications developed by Donaldson (1963), Hirst (2003) and more recently, Makonye (2011). The error categories by Donaldson (1963) were developed for different subject disciplines. However, they have been utilised by other researchers to analyse learners' errors in other mathematics topics (Orton, 1983; Makonye, 2010; Luneta, 2015 ; Brodie & Berger, 2010). Given the interrelatedness of mathematical knowledge, I found these error clarifications quite relevant and appropriate in analysing errors under hyperbolic functions. The error categories are illustrated in Fig 3.1 below.

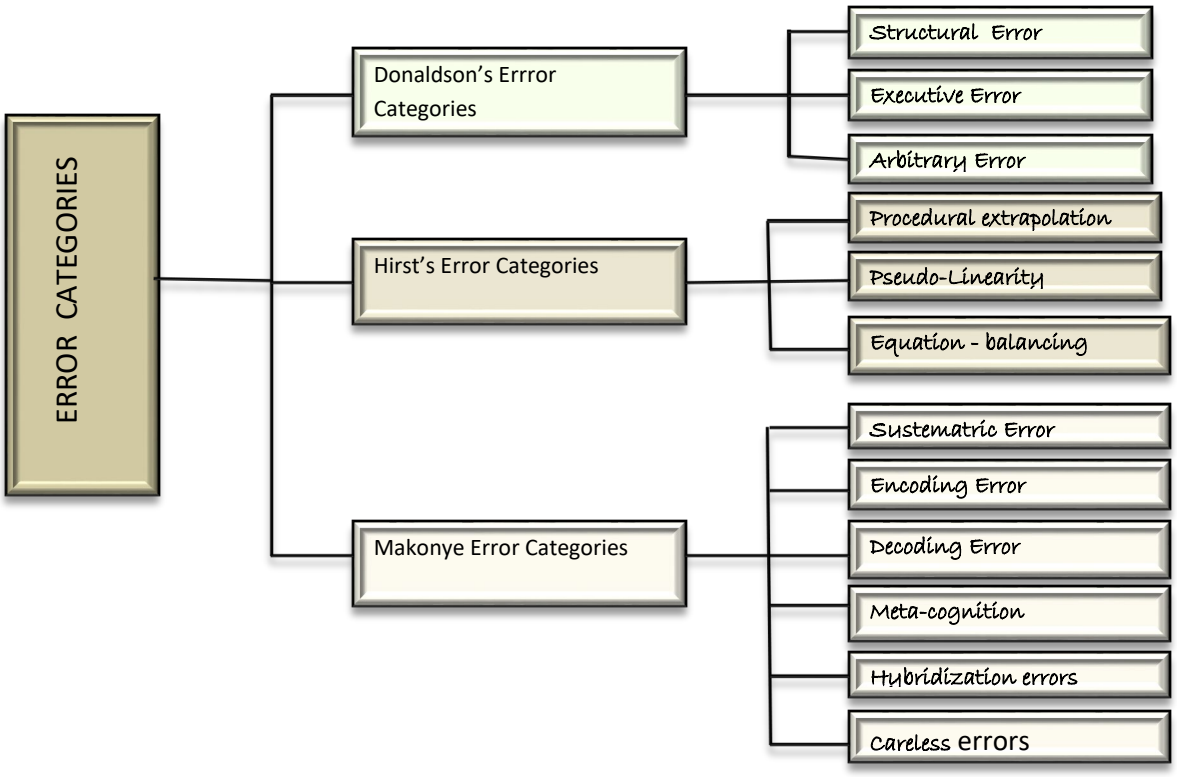


Figure 3. 1.Protocol for categorisation of learners' errors and misconceptions

The protocol provided a general error framework that assisted in the identification and the categorisation of the errors displayed by the learners. It is noteworthy that given the diverse and dynamic nature of

errors that learners exude in different mathematical contexts, the list of error categories can never be exhaustive. For purposes of this study, the focus was on developing a protocol that was broad enough to accommodate and account for the wide range of errors that learners portrayed as they responded to the test items.

Donaldson (1963) suggested three error categories that occur while learners learn mathematics. These are *arbitrary*, *structural* and *executive* errors. Executive errors involve failure to carry out procedural calculations or manipulations even though the required concept has been understood. Structural errors arise from failure to identify the relationships in the problem or to understand the fundamental principle leading to the solution of the problem (Donaldson, 1963). The author further characterised arbitrary errors as failure by learners to recognise part of the available information and act on the rest. Such errors occur mostly with learners who want questions to fit what they are familiar with.

Hirst (2003) proposed three generic error categories namely: *procedural extrapolation*, *pseudo-linearity* and *equation balancing* errors. In *procedural extrapolation*, the learners attempt to extend previously learnt procedure to new situations similar to one learnt in the past. In other words, it is an over-generalisation of a valid procedure in new situations where it causes errors (Hirst, 2003). In algebra, *Pseudo-linearity* errors include errors like: $(a+b)^2 = a^2 + b^2$; $sin\,(\alpha+\beta) = sin\alpha + sin\beta$ and $log\,(x-y) = log\,x - logy$. Norman and Pritchard (1994) described such errors as misgeneralised distributivity. Equation balancing errors, as Hirst (2003) explains, emanate from learners' previous experiences. In balancing equations and creating equivalent fractions, learners are taught to '*do the same thing to both sides of the equation'* and '*to multiply the numerator and denominator by the same number'.* Learners may then want to extend this to: $\frac{a}{b} = \frac{a+2}{b+2}$ or $\frac{f(x)}{h(x)} = \frac{f\prime(x)}{h\prime(x)}$.

In a study on errors made by learners on differential calculus in answering NCS Grade 12 mathematics examinations items, Makonye (2010) identified 22 types of errors. I however discarded some of the identified error categories because they either fell under the categories discussed above or were neither

applicable nor relevant in the context of this study. I eventually extracted six error categories from Makonye's (2010) study which are briefly explained below.

- **Systematic errors** - this category of errors occur as a result of some underlying thinking or previously acquired schema in a learner's mind, which the learner is trying to apply in a new phenomenon. They emanate from an invisible misconception and are therefore intentional and consistent (Makonye, 2011). Often, the learner is not aware that the thinking is faulty and may even defend their thinking against any teaching or instruction that may counter their 'conception'. Thus, systematic errors are inevitably more difficult to reverse or resolve.
- **Decoding Errors** – These errors occur when learners convert word problems in common language to mathematical symbols or vice-versa. When students fail to correctly decode or interpret mathematical language or symbols, they end up answering unintended questions. Makonye (2011) exemplified that given the statement: *There are three times as many girls as there are boys* and a learner decode this as $3g = b$, where g stands for girls and b for boys. That becomes an incorrect representation of the given statement.
- **Meta-cognition errors** – These are errors that arise as a result of the learner failing to reflect on the sensibility of what they have done or are doing. As Makonye (2011) explains it, such errors occur when learners fail to monitor their own thinking on a mathematical solution. The third principle, which I find very critical, involves the importance of students' ability to engage in metacognitive processes and cross-checking their own understanding. Kalchman and Koedinger (2005) reiterate the importance of students engaging in their metacognitive skills and monitoring their own solutions as they go. Functions are often generalized in algebra and require students to operate at a higher level of abstraction when confronted with a problem situation. Failure to come to terms with these generalisations results in students having little or no clue even when a solution has gone wrong. Thus, this demands the learner's ability to interpret and make sense of the numerical answers they got.

- **Hybridization errors** – These errors do occur when a learner thinks that some mathematical concepts are related when in fact they are not (Makonye, 2011). Such errors are likely to arise if learners have a weak or muddled conceptual understanding of a concept and fail to see how the concept is connected to other related concepts. In the context of functions in general, hybridization errors are more likely to occur because we deal with different types of functions possessing some commonalities but each type having some unique features that separates it from the other types.
- **Careless errors** – these errors are sometimes referred to as unsystematic or random errors. They could be caused by the lack of concentration, tiredness or anxiety. The errors are non-recurring, and learners can easily self-correct if they develop the habit of checking and verifying their answers where possible. Careless errors should not be overlooked because when a problem involves several procedural steps, a careless mistake may further complicate subsequent steps or result in unreasonable solutions. The learner may possess the mathematical competence to solve a problem, but a careless error may disrupt the flow and sensibility of a solution resulting in an incorrect solution.

Admittedly, the error categories identified above are neither exhaustive nor mutually exclusive. From a constructive perspective, errors occur as learners uniquely interpret new situations with the lens of their prior knowledge (Makonye, 2010). Thus, errors arise out of learners' attempt to generalise what they have learnt to new situations by way of assimilation and accommodation (Piaget, 1968). The central argument underlying this study is that misconceptions can be analysed as indicators of cognitive processes that can be re-channelled to bring about complete concept development. As noted earlier on, one defining feature of constructivism is its emphasis on the process of construction and re-construction of individual knowledge as enhanced by external, socially shared experiences (Vygotsky, 1978). In line with this perspective, no one deliberately teaches learners the errors and misconceptions that they have. Rather, the errors and misconceptions are an expression of their internal re-construction processes as they make sense of new mathematical concepts (Smith, diSessa & Roschelle, 1993). The identification

and categorisation of learners' errors is simply inadequate. Learners should be given an opportunity to clarify and explain the thinking behind their working. This places the researcher (or the teacher) in a better position to correctly infer on the learners' underlying misconceptions. This study did not end at identification and categorisation of learners' errors but went further to dig deeper and diagnose the underlying misconceptions. This was necessary because the next phase of the study had to develop and administer an ICT-mediated instructional intervention programme targeted at deepening conceptual understanding and minimising the misconceptions.

3.3 Conceptual Framework for the ICT- mediated Instructional Intervention

Having identified the learners' errors and misconceptions, the next research question of the study sought to explore the impact of ICT-mediated instructional intervention on learners' conceptual understanding. I then had to develop and deliver an ICT-based teaching programme for the learners on hyperbolic functions.

The underlying argument for the introduction of computers into the classroom is that their use can have a positive impact on learner performance (Hardman, 2005). Literature reviewed for purposes of this study showed that there is a current dearth of in-depth classroom-based studies exploring on how teachers can effectively integrate ICT tools to enhance students' understanding of mathematical concepts. To ensure that the fullest potential of the ICT-mediated instructional intervention was realised, I drew heavily from the blended theoretical constructs of social constructivism and the third generation Activity theory as proposed by Engestrom (1987). Figure 3.2 below is a diagrammatic representation of the pedagogical framework for the ICT-mediated intervention lessons.

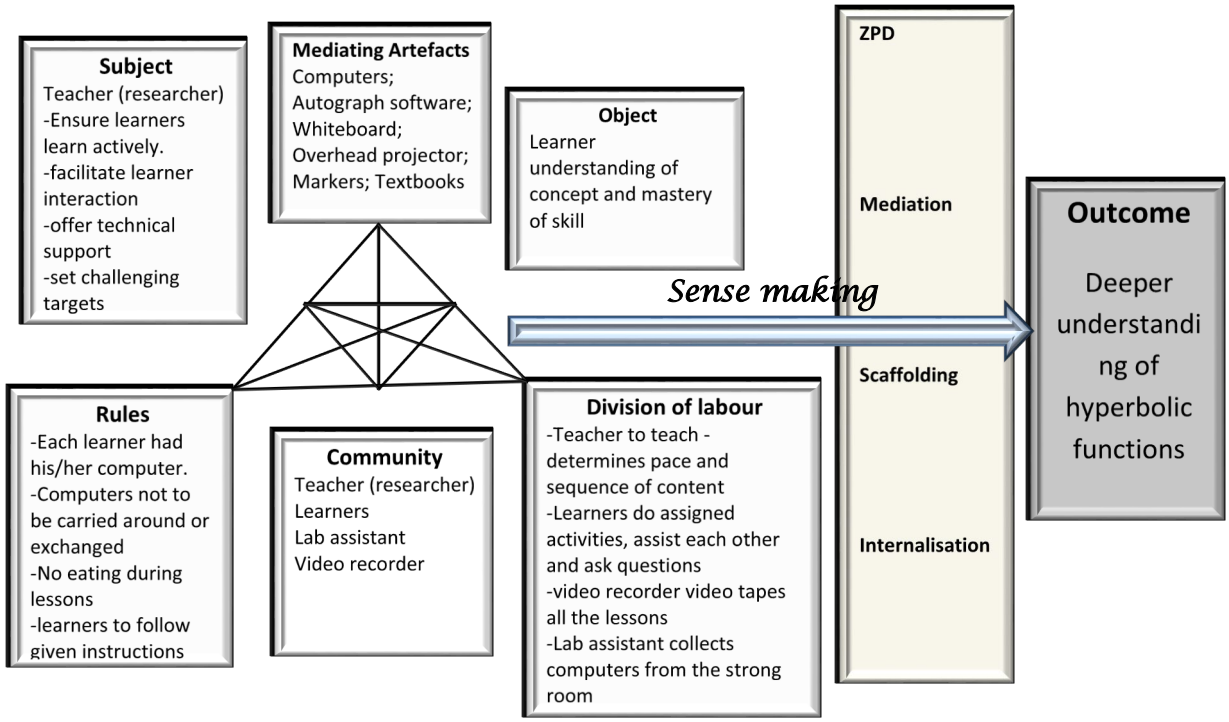


Figure 3. 2. The conceptual framework for the ICT-mediated instructional intervention

I concur with Hardman's (2008) observation that a pedagogical activity is complex and multifaceted. The definition of pedagogy itself is dynamic because there is no one prescribed pedagogical approach for any given pedagogical context. Hardman (2008) went on to define pedagogy as:

> a structured process whereby a culturally more experienced peer or teacher uses cultural tools to mediate or guide a novice into established, relatively stable ways of knowing and being within a particular, institutionalised context, in such a way that the knowledge and skills the novice acquires lead to relatively lasting changes in the novice's behaviour (p.66).

This conceptualisation of pedagogy resonates well with social constructivist constructs, which informed this study. It brings out the idea of MKOs (teacher or fellow learner) using cultural tools to mediate (ICTs as meditational tools) a novice into more stable ways of knowing (scaffolding in ZPD). It acknowledges that a pedagogical activity is a 'structured process', implying that it is not haphazard but

carefully and strategically planned. The Activity theory was developed to operationalize these constructs and make them applicable in a variety of contexts.

The Activity theory's central theme, initiated by Vygotsky (1978) and further developed by Leont'ev (1978) and Luria (1979), is that human psychological processes develop through socially and culturally mediated practical experiences (Cole, 1996). An *activity* in this case, is viewed as a form of shared human experience that is connected to and associated with various actions targeted towards an object (purpose or motive) in a socially and culturally structured environment (Yamazumi, 2006). The current version of the Activity learning theory, as developed by Engestrom (1987), is premised on the viewpoint of an activity system as a basic unit of analysis. As illustrated in figure 3.2 above, the *subject* acts on the *object* in order to transform it using *mediating artefacts* to arrive at specific *outcomes*. The subject's position and actions are governed by the *rules* of the system, his/her *community* and *division of labour*.

For the instructional intervention in this study, Engestrom's activity system dimensions were applied as explained below:

- **Subject** - The subject in the activity system represents the individuals and social nature of human activity, which is guided by a purpose (Goncalves Sousa & Zacarias, 2013). In this case the subjects were the thirteen learners who participated in this study and the teacher, who was also the researcher. Drawing from the Social Constructivist perspective, learners were viewed as unique individuals with unique mathematical understandings based on their social and cultural activities. Instructional activities were designed to allow learner to learner and teacher to learner interaction, thereby promoting scaffolding within the learners' ZBD.
- **Object** – The object is what the human activity in the system is intended to achieve (Goncalves at al, 2013). Engestrom (1987) identified the object as the 'raw material' or 'problem space' at which the activity is directed and which is moulded and transformed into outcomes with the help of physical and symbolic, external and internal mediating instruments. In this case, the object is the targeted mathematical concept or skill that learners should understand or master by engaging in the

pedagogical activity. That concept(s) or skill should ultimately contribute towards the achievement of the intended outcome at the end of the intervention programme.

- **Mediating Artefacts and Signs –** The dimension of mediating artefacts and signs was again drawn from Vygotsky, who notes that in real life, human beings use artefacts (or tools) to accomplish tasks that would otherwise have remained out of reach (Bussi & Mariotti, 2014). Vygotsky further argued that mental activities are supported and ehnanced by means of signs that are a product of the internalization process. His work focussed on functioning of natural language and hence interpreted signs as linguistic signs or psychological tools. He however went on to give a list of examples of psychological tools which included non-linguistic signs, for example: works of art, diagrams, maps and drawings (Vygotsky, 1981). This study was centred on technological tools as mediating artefacts and more specifically, these included computers, Autograph software, overhead projector, whiteboard and other supporting material.

The importance of mediation in learning is brought out by Vygotsky when he claimed that in the ZPD, cognitive development is modelled by the process of internalization. He went on to define internalization as 'the internal reconstruction of an external operation' (Vygotsky, 1978, p.56). In other words, the internal process of knowledge construction by an individual is generated by external socially shared experiences. ICTs mediate the relationship between external processes (in the learning context) and internal processes (knowledge construction).

- **Community** – Activity theorists refer to a community as the participants of an activity system (Engestrom, 1987). They constitute members of a community who participate in acting on the shared object (Cole & Engestrom, 1993). In a wider sense, a community includes the school, teacher unions, mathematics associations, curriculum advisors and the department of education (DBE). In the study, the community was restricted to the classroom context which included only the teacher and the learners. Other participants, who might not have been directly involved in the teaching and learning, were the computer lab assistant and the video recorder.

- **Rules** – The human actions or social interactions in an activity system are governed and driven by certain rules, norms and conventions (Hardman, 2005). The rules specify and regulate the appropriate procedures and acceptable interactions between participants within the activity system (Goncalves, Sousa & Zacarius , 2013). Over and above the rules provided by the national, departmental and school policies, I agreed with the learners a set of social and instructional rules which were to be adhered to during the intervention programme. Social rules included how learners conducted themselves as they interacted with fellow learners and with the teacher and other elderly participants. The key was mutual respect and taking the programme seriously. Instructional rules included respecting other participant's opinion, following instructions, handling computers with care and focus (not digressing). The agreed rules were pasted on the wall of the classroom.
- **Division of labour -** Division of labour specifies how tasks, responsibilities and roles are allocated to members, including the power relations among members in the activity system (Murphy & Rodriguez-Manzanares, 2008). In a classroom context, the traditional role of the teacher is to teach, and the role of learners is to learn. However, as Hardman (2005) notes, the introduction of computers in the classroom can drastically transform the traditional roles of the teacher and learners. In this case my responsibility as the teacher-cum-researcher was to develop the pedagogical activities, decide on learning material and ensure their availability. I also delivered all the lessons, deciding on the pace and the sequence of the learning activities. Learners were expected to follow instructions, actively participate in the learning activities, respond to questions and ask their own questions. The computer lab assistant ensured that all computers were battery-charged before the lessons. In line with school policy, he was the only person allowed to take out the computers from the strong room and take them back after use. I also had a video recorder who recorded all the lessons, without interfering with the lessons.

Researchers who subscribe to the Activity theory (Goncalves, et al.,2013) point out that the complexity in the theory is that the interacting dimensions in the activity system must be understood in totality and not the components' separate connections. The integration of ICTs in the activity system inevitably

brings in other dimensions that make the context more complex and more inclusive. The manner in which the teacher orchestrates the complex connections in the activity system is largely influenced by the epistemic assumptions he holds. My approach basically was to infuse the social-constructivist pedagogical constructs to inform my actions and decisions and my responses to scenarios as they unfolded during the pedagogical activities.

3.4. Application of APOS Theory to evaluate concept development

Both social-constructivism and Activity theory were not useful in evaluating how the intervention lessons impacted on the learners' conception of hyperbolic functions. As explained in the previous chapter (see Section 2.6.3 above), I incorporated Dubinsky's APOS theory to develop an analytical framework that I used to determine the learners' conception levels. There are two ways in which the theory was used. Firstly, based on APOS mental constructs, I developed a genetic decomposition on hyperbolic functions which enabled me to do a comparative analysis of learners' conception levels before and after the intervention lessons. Secondly, it enabled me to identify any indications of transition from one conception level to the other as a result of the intervention lessons. The two ways are explained in more detail below:

3.4.1 A Genetic Decomposition

In APOS literature, a genetic decomposition of a concept is a hierarchically organised set of mental constructs which might illustrate how the concept can develop in the mind of a learner (Asiala, Brown, De Vries, Dubinsky, Matthews & Thomas, 1996; Syaiful, Kamid & Marssal, 2010 ; Breidenbach, Hawks & Nicholas, 1992 ; Maharaj, 2010 ; De Morois & Tall, 1996). In the data analysis of this study, the genetic decomposition specifies the particular *actions*, *processes*, and *objects* that are organised into a hyperbolic function *schema*. This procedure was important as it provided an essential theoretical lens upon which the learners' levels of conception were determined and explained. It subsequently allowed the researcher to provide meaningful possible explanations of the errors displayed by learners. This analytical framework was particularly useful in comparing what learners could do or could not do before

and after the instructional intervention. I referred to this genetic decomposition as the *APOS-Hyperbolic function.*

Table 3.1 below presents summary of the indicators or criteria for each level of conception of hyperbolic functions. These indicators were adapted from several other researchers who applied the APOS theoretical framework in other related mathematical topics (Chimhande, Naidoo & Stols, 2017; Dubinsky & MacDonald, 2012; Syaiful et al, 2014; Maharaj, 2010; Breidenback et al, 1992).

Criteria of Learners at Action-Conception Level	
What learners can do:	**What learners can not do:**
See a function as a relationship between two sets Can substitute numbers into a function expressed algebraically. Calculate x-and y-intercepts of hyperbola expressed algebraically. Determine the equation of asymptotes of a hyperbolic function given algebraically. Give the domain and range of a hyperbolic function expressed algebraically. Exhibit strong tendencies to recall definitions of concepts verbatim as they are given in class. Use rules without reason.	Cannot see the input-output relationship as one thing ie ***f(x)*** is what you get out of ***f*** when you put ***x*** into ***f***. Cannot explain how and why the procedure they use works. Cannot explain why at the x-intercept $y = 0$ and at the y-intercept $x = 0$ Cannot explain why $x = p$ and $y = q$ are asymptotes Cannot represent a hyperbolic function graphically.
Criteria of Learners at Process-Conception Level	
What learners can do:	**What learners can not do:**
Explain the meaning of critical points, that is, asymptotes and intercepts. Attempt to give explanations of the procedures. Sketch the graph of a given hyperbolic function by first calculating critical points. Think of curve sketching as an entire activity and internalise the procedure.	Adequately explain why they take the steps of the procedures that they use Cannot attempt questions which require interpretation of sketched graphs. Struggle to derive the equation of a function from a sketched graph. Can hardly see the connection between the graph and its equation
Criteria of learners at Object-Conception Level	
What learners can do:	**What learners can not do:**
Connect symbolic and graphical representations of the function. Extract key features of a hyperbolic function from a drawn graph. Describe how a hyperbolic function has been transformed by looking at the graph of a transformed function. Relate a transformed symbolic function to its graphical representation.	Cannot easily switch from graphical to symbolic representation. Cannot see that the critical points connect the graph and its equation. Cannot solve non-routine problems involving hyperbolic functions.
Criteria of Learners at Scheme-Conception	
What learners can do:	**What learners can not do:**
Connect all concepts involved into a total entity and soive unfamiliar problems. Switch from graph to equation and from equation back to graph with easy. Assert that the critical points connect the graph and its function Link the critical points located on a drawn graph and the ones calculated algebraically.	

Table 3. 1. Indicators for each APOS conception level of hyperbolic functions (APOS-Hyperbola)

3.4.2. Transition from one APOS level to another

The second analytical strategy employed to further evaluate and establish the impact of the instructional intervention was drawn from APOS theorists' claim that concept development occurs as learners' progress from one conception level to another. Thus, the central focus here was to determine if the ICT-mediated intervention facilitated the development of the internal mental processes needed to propel learners from a lower level of conception to a higher one. Figure 3.2 below illustrates the mental process involved in the transition from one conception level to the other.

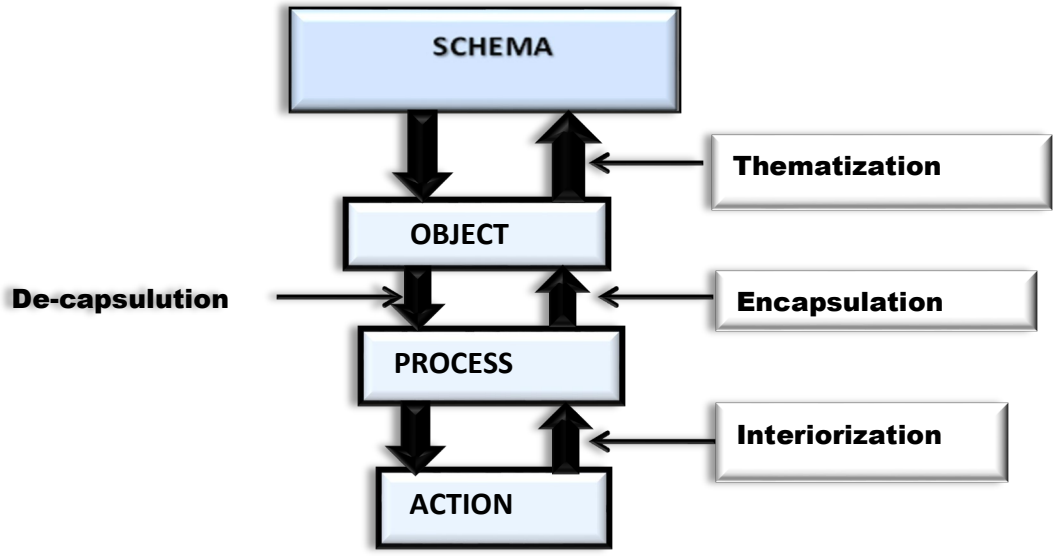


Figure 3. 3. Mental processes involved in APOS theory

According to Dubinsky (2000), when an action is repeated by an individual, the individual reflects upon it and make an internal mental construction called a *process*. At this stage, the individual begins to imagine the process of performing an operation wholly in the mind, without actually doing it. The process can then be reversed or combined with other processes.

Sfard (1992) described this transition from *action* to *process* as interiorization. It is a development from a procedural operation to some understanding of the processes involved (Syaiful et al., 2014). Generally, under functions, interiorization is the process that propels a student to move from viewing a 'function' as a meaningless algebraic expression and begin to condense the processes involved in functions. According to the APOS-Hyperbola (see Table 3.1 above), interiorization would be evident if a learner

could move away from being able to use rules to determine intercepts and asymptotes to being able to explain the meaning behind the rules. In other words, learners that had attained a *process*-conception were expected to give meaningful justifications for their procedures or working and demonstrate understanding of concepts like intercepts, asymptotes and axis of symmetry.

An *object* develops from a *process* whereby an individual begins to view the process as a whole and appreciates that transformations can act on it (Dubinsky & McDonald, 2002). Piaget used the word encapsulation to describe the transition from *process* to *object* (Syaiful et al., 2014). In reference to the genetic decomposition explained above, a learner who had attained the *object*-conception level was expected to flexibly switch between symbolic and graphical representation of hyperbolic functions. The learner was expected to be able to transform a function according to a given rule and describe the transformation(s) applied to the function. The transformations were restricted to vertical and horizontal translations, that is $f(x) \pm q$ and $f(x \pm p)$ respectively, and reflection in the x- and y-axis, that is $f(x) \rightarrow -f(x)$ and $f(x) \rightarrow f(-x)$.

Finally, a *schema* is a complex cognitive structure comprising of an individual's collection of *actions*, *processes* and *objects* and connected to other schemas by general principles to form a framework in the individual's mind (Dubinsky & McDonald, 2002). Dubinsky and McDonald further indicate that the framework so formed must be coherent as it forms the basis for which one can determine which phenomenon are in the scope of the schema and which are not. Likewise, Tall and Vinner (1981) used the term 'concept image' to refer to the cognitive framework associated with a concept. According to the APOS-Hyperbola, a learner who had developed a hyperbolic schema was expected to connect and link all the concepts under hyperbolic functions into a coherent total entity. For example, when confronted with an unfamiliar problem, the learner was expected to switch from a graph to an equation and from equation back to a graph with relative easy. The learner was expected to demonstrate the ability to link the critical points located on a drawn graph to the ones calculated algebraically and assert that the critical points connect the graph and its function.

3.5. Conclusion of the chapter

This chapter discussed the specific theoretical constructs that influenced and shaped this study. Vygotsky's Social constructivism theory provided the main analytical insights on creating a favourable learning context that facilitated knowledge creation. His notions of mediation, ZPD, scaffolding and internalization were useful in the delivery of the lessons.

To operationalize Vygotsky's learning constructs, the Activity theory provided a pedagogical framework that situated learning in an ICT-mediated context (Lim & Chai, 2004). The ICT-mediated learning context was, thus, viewed as an activity system connecting subject, object, mediational tools, community, division of labour and rules (Hardman, 2005). The teacher, as the driver of the activity system, orchestrated the inter-connected components from a social-constructivist perspective. To assess and evaluate the impact of the ICT-mediated intervention, Dubinsky's APOS constructs of action, process, object and schema provided an analytical framework to determine the level of learners' conception of hyperbolic functions before and after the intervention lessons. The next chapter presents a review of related literature.

CHAPTER 4: LITERATURE REVIEW

4.1. Introduction

This chapter presents a review of research findings on learners' conceptions and misconceptions around the concept of function. The review also focuses on studies on the integration of ICTs in the teaching and learning of mathematics. Generally, research studies show that teaching and learning of function concepts is very complex, with even high performing students possessing weak understanding of the concepts (Thompson, 1994; Carlson & Oehrtman, 2005). The difficulty in the conceptualisation of the function concept arises from its complexity and the abstract nature of the concept itself (Ponte, 1992; Sheely, 1996; Kleiner, 1989) and is perpetuated by teaching approaches that largely favour procedural knowledge at the expense of conceptual understanding (Makonye, 2014; Carlson & Oehrtman, 2005; Kalchman & Koedinger, 2005).

Scholars generally agree on the claim that the introduction of technological tools in mathematics classrooms has the potential to enhance the effectiveness of pedagogical practices (Mdlongwa, 2014; Strydom-Wilson &Thomson, 2005, Becta, 2003). Studies on the integration of ICTs in mathematics teaching and learning in underdeveloped and developed countries are reviewed in this chapter. The envisaged contribution of this study to the existing knowledge gaps has been established and demonstrated in this chapter too.

4.2. Overview on the historical development of the notion of function

The notion of function is the central mathematical notion in this thesis. It is therefore necessary to have some understanding of the historical development of the concept from a mathematical perspective. Kleiner (1989) notes that the evolution of the function concept can be traced back to over 4000 years ago. In those pre-calculus ancient times, the idea of function was regarded as 'a result of the human endeavour to come to terms with changes observed and experienced in the surrounding world' (Sierpinska, 1992, p.3). However, at that time mathematicians lacked the necessary algebraic pre-

requisites needed to develop a concept of function (Kleiner, 1989). Thus, as Kleiner (1989) notes, their ideas were not regarded as mathematics.

Later, the developments in astronomy subsequently led to the establishment of the study of motion as a major problem of science (Kleiner, 1989). This led to the development of symbolic algebra, linked to geometry through the work of Descartes (1596 – 1650). A function was then presented as an equation between two variables geometrically represented by a curve indicating dependence between two variables (Sears, Tran, Lee & Thomas, 2016).

Newton (1642 – 1727), who was the first to demonstrate how functions could be developed in infinite power series, used the term 'fluent' to designate independent variables and 'relata quantities' to refer to dependent variables (Ponte, 1992). However, it was Leibniz (1646 – 1716) who was the first to use the tern function in 1673. Leibniz viewed a function as the dependence of geometrical quantities. He introduced the terms 'constant', 'variable' and 'parameter' (Sears et al. 2016). With developments in study of curves by algebraic methods, there arose the need for a term to represent quantities that were dependent on one another by means of an analytical expression. Finally, the term function was adopted in the work of Leibniz and Bernoilli (1667 – 1748). However, during that time, the idea of function was confined to the notion of analytical expression which resulted in several incoherencies and imposed limitations on the types of functions that could be considered.

The limitation above was later dealt with by Dirichlet (1805 -1859), who separated the function concept from its analytical representation. He presented the notion of function in terms of an *arbitrary* correspondence between variables representing numerical sets such that any value of the independent variable is associated with *a unique* value of the dependent variable (Ponte, 1992). Dirichlet's definition was the first to mention the *one-valuedness* and *arbitrariness* of a function. One-valuedness means for each element of the independent variable, there is one and only element in the dependent variable. This was necessary as it made the analysis of functions manageable (Even, 1990). Arbitrariness is the quality

whereby the value of a function at any point is independent of the value at the other points but also that the domain and range can be random sets.

Developments in different areas of mathematics in the end of the 19th century and the beginning of the 20th century, such as analysis, set theory and algebra, led to a more generalised view of the function concept. For example, in 1939 Bourbaki gave a more abstract and set theoretical definition of a function as:

> Let E and F be two sets, which may or may not be distinct. A relation between a variable x of E and a variable element y of F is called a functional relation in y if for all x in E, there exists a unique y in F which is in the given relation with x (Weiner, 1989, p.299).

The definition made specific reference to domain and range which was not included in earlier definitions. It also did not make reference at all to number sets. In Sfard's terminology, the definition is structural (Sfard, 1991). Sfard (1991) meant that a function was now viewed as an entity in its own right, and not just a process acting on objects. To sum up the historical development of the function concept, Sfard (1991) noted that the development of the concept could be seen as a transition from an operational to a structural perspective of the concept.

The significance of this historical background is that it highlights how the function concept evolved from being a non-mathematical notion to one of the fundamental concepts in modern mathematics. Realising the significance of the function concept, mathematicians successfully advocated for its inclusion in the school curriculum in the 20th century. Klein, for example, argued that 'functional thinking should pervade all of mathematics and at school, and students should be brought up to functional thinking' (Sierpinska, 1992, p.32). However, as noted in Chapter 1 of this study, research on the function concept from an educational perspective reveals difficulties related to the learning of the concept. I now look at what research has shown about learner conception of the function concept.

4.3. Research on the teaching and learning of the function concept

Given the role that the function concept plays as a 'unifying concept' in mathematics, researchers in mathematics education have shown a lot of interest in investigating the teaching and learning of the concept. Vinner and Dreyfus (1989) concede that numerous studies on students' conceptions of the function concept reveal inconsistencies both within conceptions and between conceptions and definition of a function.

The inconsistencies cut across high school, tertiary and teacher education schooling. Scholars seem to agree on the observation that learning the function concept is so complex that even high performing students possess a shallow understanding of the concept (Thompson, 1994 ; Carlson & Oehrtman, 2005). Thompson (1994), for example, argues that the reasoning capacity and the levels of conception needed for a deep and flexible understanding of the concept of function are more complex and surpasses what the designers of curriculum and instruction assume.

Vinner and Dreyfus (1989) used the theoretical constructs of concept definition and concept image to examine the conception of a function possessed by university students and junior high school teachers. One of their key finding was that the participants defined and viewed the function concept differently. They concluded that the concept images of a function invoked in different situations were mutually inconsistent. This observation was supported by Dubinsky and Harel (1992) who found restrictions and incoherence in students' function conceptions. They cited examples of misconceptions whereby students believed that the input and output values of all functions need to have numbers and that all graphs of functions need to be continuous Thus, though the function concept appear to be straightforward, it is a complex one even for tertiary students.

Kalchman and Koedinger (2005) came up with three key principles in the teaching and learning of functions. Firstly, they suggest that new knowledge should be built on the foundation of students' existing knowledge and understanding. Emphasis is made on the need for students and teachers to continually connect and link students' experiences outside the mathematics classroom and their learning

experiences in the classroom. In this regard, Kalchman and Koedinger (2005) stressed that instruction must be built around a 'bridging context' which is familiar to the learners.

The second principle in teaching functions, as enunciated by Kalchman and Koedinger (2005), is that learners need to acquire both a deep relational (conceptual) understanding of function and procedural (operational) fluency. The scholars argue that the fundamental principle in functions is that of a dependent relationship, that is, 'the value of one thing depends on, is determined by or is a function of another' p.32. This notion must be well conceptualized by the learners. Contrarily, the tendency by teachers is to focus on algebraic expressions and leading learners to substitute input values to determine the output. Kalchman and Koedinger (2005) further argue that presenting the formal definition of a function prematurely does not necessarily bring out to students the idea that the value of one variable is determined by the value of another variable, and that the rule tells us how they are related.

The third principle, according to Kalchman and Koedinger (2005), entails the need for students to engage in metacognitive processes and their ability to monitor their understanding as they go. They argued that mathematical relationships are abstracted and generalized in algebra and hence, students must operate at a higher level of abstraction to come to terms with these generalisations. In their own words, they said:

> ... instruction should assist students not only in thinking *with* mathematical procedures and concepts, but also in thinking *about* procedures and concepts and in reflecting on and articulating their own thinking and learning' (p 371).

Therefore, there is need to empower students to think about problems in multiple ways (strategies) and with multiple tools (representations). This enables them to draw both quantitative and qualitative conclusions based on the scenario presented to them. The ideas relating to multiple strategies, multiple representations and learner's qualitative interpretation of their solution were quite pertinent to this study. My point of departure is that proper integration of ICTs can help to foster these practices and engage learners in developing a coherent mental construction of functions.

Makonye (2014) explored the teaching of functions using a Realistic Mathematics Education approach. He notes that the teaching and learning of the function concept is often divorced from the learners' everyday contexts. He further argues that formal mathematical symbolism such as $f(x)$ are sometimes prematurely introduced to learners, resulting in learners developing misconceptions on the concept. As an example, Makonye (2014) proposed the 'Matchstick approach' as the basis for teaching different functions representation. What captured my interest to this illustration is that it epitomizes how different representations can be used to develop the function concept. To start with, matchsticks are arranged to form successive squares by adding matchsticks to the previous one, as shown in Figure 3.1 below.

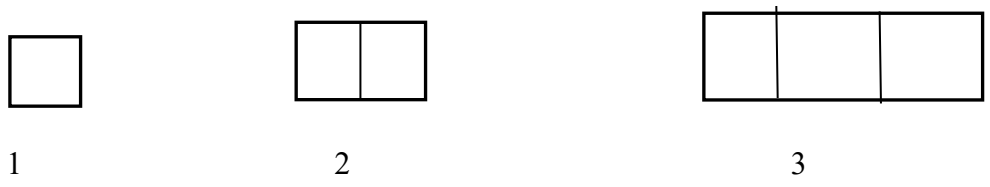


Figure 4. 1. Matchstick Pattern (Adopted from Makonye 2014, p.657)

Learners can then engage with this pattern guided by the following questions: (i) How many matchsticks are required to make 55 squares? (ii) Determine the number of squares which can be made by 226 matchsticks and, (iii) draw a table showing the number of squares and the corresponding number of matchsticks. The Matchsticks Problem can then be represented in tabular form, using an algebraic equation or a straight-line graph. This invokes in students the understanding that a functional relationship can be represented in a variety of ways. While such an approach may be effective in introducing the concept of function, it may not be adequate to address other complex concepts and representations under functions.

In a similar study, Chimhande (2013) used design research to explore how Realistic Mathematics Education (RME) activities can improve the teaching and learning of functions at grade 11 level. He collected data through content analysis of learners' written responses and task-based interviews which were then analysed using APOS theory. The major finding of his study was that incorporating learners'

conceptions and RME activities in designing instructional activities facilitates learners' transitional progression through APOS conception levels. While I concur that RME is one approach which can help learners to connect external representation (socio-cultural) of functions to their internal representations (concept images), it is not always easy for teachers to design real activities that can effectively model more abstract functions. The next section focuses on previous research findings on students' conceptions and misconceptions of the function concept

4.4. Studies on learners' errors and misconceptions under functions.

Hazzan and Goldernberg (1997) highlighted the importance of knowing and understanding how students conceptualize the notion of a function and the need to identify their challenges and misunderstandings as they try to conceptualize the concept. Conception, as viewed by Smith, Disessa and Roschelle (1993), is the result of a mental construction process, compatible with other aspects of a learner's knowledge system and is stable and robust when confronted with other conceptions.

Research evidence reveal that students have specific conceptions of a function that are at variance with expert ideas (Carlson & Oehrtman , 2005; Clement, 1985; Abdullah, 2005; Blomhoj, 1997). In that case, they become misconceptions. It is through understanding these conceptions or misconceptions that we can develop instructional strategies to ensure conceptual understanding. The key question that guided my search for relevant literature was to find out why the function concept is difficult for learners to understand.

Carlson and Oehrtman (2005) note that one persistent challenge students have is that of distinguishing between visible attributes of a physical situation and similar attributes on the graph of a function that models the situation. They explain that when dealing with structures modelling concrete situations, there are topographical structures within the real setting itself. These may include features like a winding racetrack, the meanders of a road up a hilly terrain or the shape of a low-lying basin being filled with water. Such features may be confused with subtle features of a graphical representation of functions such as turning points, concavity and points of inflection. Thus, Carlson and Oehrtman (2005) argue

that several graphical misconceptions can be traced back to confusing visual attributes of the real situation and the shape of the graph.

Similarly, Clement (1985) identified two common misconceptions in graphing. The first one he named *slope-height confusion,* whereby learners mistakenly use the graphical feature of height instead of slope to represent speed. This applies, for example in distance-time graphs where the graph may be viewed as an object going up and not as change in speed with time. The second misconception was described as *treating the graph as a picture.* In this case, the learners view the graph as a lateral picture on the problem situation. They erroneously link the shape of the graph to some visual characteristics of the real scene it represents. These misconceptions further confirm that the graphical representation of a function is an abstract notion from the learners' perspective. They struggle to link the graph to something they already know in real life.

Some of the challenges that students face in comprehending functions emerge from the formal definition of a function itself. As Abdullah (2005) noted, we often introduce the concept function as a special mapping of two sets, where each object is mapped onto one image. Abdullah (2005) argued that this definition made it difficult for learners to deal with calculating *fg(x)* and illustrating the mapping of this function. Some students could obtain *fg(x)* algebraically but could not illustrate diagrammatically the direction of the mapping of *fg(x).* This illustration epitomises how learners may grasp algebraic procedures without attaching any meaning to those procedures.

Similarly, Blomhoj (1997) studied Danish students in the ninth grade, and asked them the question, 'What do you say about how x relate to y, given the expression $y = x + 5$?' The responses obtained included: 'x is five bigger than y'or 'x is five times bigger than y'. Excerpts from interviews with the students show some contradictions between verbally expressing the relationship and providing tables which clearly illustrate that y is five more than x. The interpretation provided by Blomhoj (1997) was that one misconception arising from a restricted conception of 'function as machines' in school could be the misleading conclusion that the independent variable changes its value and is transformed into the

dependent variable. This observation is very critical because most textbooks generally introduce functions through 'function as machines'. This further stress the need to expose learners to multiple representations of functions which affords learners a more comprehensive and coordinated understanding of the concept. This study believes that this can be effectively achieved through the use of ICT tools.

Studies reveal other errors and misconceptions which are not directly under functions but are connected or related to functions (Seyepu, 2015; Baidoo, 2019). Seyepu (2015) analysed errors in derivatives of trigonometric functions on chemical engineering university students. One of the startling findings from this research was that a considerable number of the students struggled to understand the subtraction of fractions. He gave a specific example that students could not calculate 1 minus $\frac{3}{4}$. It also came out in the same study that the students could also not manipulate multiplication by the number 0. These are unexpected errors at university level, but they simply reflect the cumulative nature of misconceptions and how challenges faced by learners under functions may be caused by misconceptions from other topics.

Finally, Baidoo (2019) analysed selected grade 10 learners' performance in algebraic fractions and the errors they make in Eastern Cape, South Africa. One of Baidoo's (2019) findings was that learners have conceptual shortcomings in their simplification of algebraic fractions. He concluded that the learners' conceptual gap, their failure to understand mathematical language and procedural errors are the key hinderances to their manipulation of algebraic fractions. If these findings are anything to go by, one would expect learners to inevitably experience challenges in dealing with hyperbolic functions because, by their nature, they involve manipulation and simplification of algebraic fractions.

The studies discussed above were carried out in different contexts and at different levels of mathematics education, focussing on different aspects of the function concept. Some were done with high school learners, others with university students and students in pre-service teacher training. The studies that involved tertiary students focussed on aspects of functions beyond the high school prescribed content.

On the other hand, most of the studies that involved high school learners looked at learner conception of functions in general, and none zoomed on hyperbolic functions. The following section is focussed on studies that looked at the multiple representations of functions.

4.5. Studies on the Role of Multiple Representation of Function.

One common challenge that learners encounter in dealing with functions, emerging from the literature reviewed, is the learners' inability to connect the different representations of the function concept. In simple terms, a representation is something that stands for something else (Chimhande, 2013). With regard to mathematical concepts, representations can be internal or external (Goldin, 2001).

Goldin (2001) notes that external representations are visual symbolic features that can be written or spoken. In the context of functions, these include algebraic expressions and notations, graphs, flow diagrams, table of values, ordered pairs and everyday real-life situations. Goldin (2001) refers to internal representations as mental images that learners develop as they interact with and reflect upon the external representations. In line with the constructivist argument of internal knowledge construction (Vygotsky, 1978), each learner's internal representation of the function concept is unique.

Even (1998) argues that when students can identify and represent a given scenario in different representations, and flexibly switch from one representation to another, it empowers them to see rich relationships and develop a deeper and broader conception of that scenario. Thus, exposure to multiple representations enriches and strengthens one's ability to and flexibility in dealing with an unfamiliar problem situation. The linkages between multiple representations provide insights into the understanding of all the dimensions, as well as the many facets of the concept. Ainsworth (1999) reiterated that using multiple representation is likely to capture a learner's interest and, consequently play an important role in creating conditions for effective learning. Ainsworth (1999) further argued that multiple external representations (MERs) confer not only motivational benefit, but also deepen the learners' understanding of the subject being taught.

Computer-based learning, through the many multi-representational environments associated with them, can support a wide range of learning activities in the classroom. As Confrey (1992) posits, computer-based learning ranges from the use of general tools such as spreadsheets and powerpoint to more specialised mathematical software such as Function Probe, GeoGebra and Autograph. However, there is no consensus among researchers on the effectiveness of multi-representational environments in the learning of mathematics. Ainsworth (1999), for instance, cautions that research designed to investigate the effectiveness of multi-representational environments in enhancing learning has produced mixed results. Other studies have revealed that learners struggle to work with MERs and thus fail to benefit from the claimed learning affordances (Even, 1998; Thompson, 1992).

Ponte (1992) observed that most high school students have challenges in abstract thinking. He elaborates that many students find it difficult to deal with cartesian graphs and algebraic expressions. To counter this obstacle, Ponte (1992) suggests that the teaching of functions should expose learners, in a balanced way, to the three most essential forms of representation, that is numerical, graphical and algebraic forms. The author, however, reiterates that the analytical and geometrical representations of function should not be allowed to de-emphasize the role played by numerical aspects on learning functions, which are tables and computations. Ponte's (1992) argument seems to be based on the view that learners find it difficult to learn concepts under functions because they are expected to deal with predominantly more abstract aspects of the concept without regard of their natural groundings. This is evident in that when required to interpret functional relationships given through cartesian graphs, students are more confident with and resort to numerical reasoning strategies. From the learner's perspective, numbers are considered as essential, familiar and concrete entities onto which more abstract mathematical concepts can be built.

Multiple representations of a concept, as exemplified above, help learners to see links, emerging patterns and relationships between concepts (Kilpatrick et al., 2001). While such real-life situation based activities are very powerful in the introduction of function concepts, some more complex and abstract

concepts may not be easily modelled without the use of technology. Ainsworth (1999) concedes that several recent technological developments may have a central role in the study of functions. Ainsworth (1999) further notes that these technological instruments, wisely used, facilitate the process of conjecturing, testing and generalizing. Against this background, this study is aimed at examining the use of computer technology to propel learner understanding of functions, especially the more abstract realm of the concept. The main assumption of this study is that effective introduction of ICT tools into the mathematics classroom exposes learners to multiple representation of the function concept and demystify the connection between the various forms of representation.

4.6. Surface and deep mathematical knowledge

The dichotomous notions of mathematical knowledge or mathematical learning have attracted considerable attention in mathematics education research (Skemp, 1976; Fanskanger & Bjuland, 2018; Sfard, 1997; Hiebert & Lefevre, 1986). Different terminology has been used by different researchers to highlight the valuable distinction between two notions of mathematical knowledge. In this section I refer to the notions of 'surface' and 'deep' learning (Fanskanger & Bjuland, 2018), 'operational' and 'structural' knowledge (Sfard, 1997), 'instrumental' and relational' knowledge (Skemp, 1976) and 'conceptual' and 'procedural' knowledge (Hiebert & Lefevre, 1986)

Research on deep and surface learning originated from Sweden in the mid-1970s (Fanskanger & Bjuland, 2018). Franskanger and Bjuland described surface learning as involving superficial engagement with a task with a focus on the memorisation or the application of ideas or facts without any reflection or understanding. From the literal sense of surface, it means the knowledge only scratches the basic facts or skills needed to perform a specific task, without worrying about the underlying principles behind those facts. On the other hand, deep learning involves a focus on how various aspects of the content relate to each other, with an intention to gain better understanding (Fanskanger & Bjuland, 2018; Rellero, 2016). Despite this distinction, deep learning requires a strong foundation in factual

knowledge. However, deep knowledge does not stop at factual information but, as Rellero (2016) puts it, transforms factual knowledge to practically usable knowledge in a problem-solving context.

In a similar distinction, Sfard (1997) used the terms operational and structural mathematical knowledge. Mathematics is operational when it is used for calculations like addition, subtraction, multiplication and division. Clearly, imparting operational mathematics is similar to surface learning because you only equip learners with operational skills or facts to perform a task at hand. Learners have no choice but to memorise the facts or procedures because they cannot store or link them to their existing cognitive structures. Sfard (1997) views structural mathematics as involving the fundamental notions that connect superficially disjoint mathematical ideas. Just like in deep learning, acquiring the structural notion of mathematics is when one begins to see the connections and relationships between mathematical concepts.

Referring to the same two notions of mathematics identified above, Skemp (1976) used the words instrumental and relational knowledge. Instrumental understanding involves the skill to carry out mathematical computations and the use of formulae correctly. According to Skemp (1976), instrumental knowledge involves rote memorisation of mathematical facts, rules and formulae, without necessarily knowing why and how the rules are derived. When instrumental knowledge is extrapolated in other mathematical contexts without a deeper structural understanding, errors and misconceptions are inevitable (Hirst, 2003). Relational understanding stresses the understanding of a relationship and connections between mathematical ideas and concepts (Skemp, 1976). This level of understanding attempts to tie together bits of mathematical ideas to form a logical and coherent mental structure. It results in building a flexible, meaningful and deep mathematical knowledge that is general in nature (Suh, 2007). Once a learner has developed such a solid mathematical structure, it becomes a foundation upon which new mathematical knowledge can easily be built in the mind.

Hiebert and Lefevre (1986) used the terms procedural and conceptual. Just like surface, operational and instrumental knowledge, they described procedural knowledge as knowledge of carrying out a

mathematical task, the *know-how* of mathematics, but not the *know-why*. Eisenburg, Borko, Underhill, Brown, Jones and Agard (1993) added that procedural knowledge is the acquisition of computational skills, procedures, rules and algorithms for solving problems. Hiebert and Lefevre (1986) elaborate that conceptual knowledge is that knowledge which is part of a network, comprising of individual components of information and the relationship between these pieces of information. They underscore the need to see mathematics as a connected body of knowledge.

The question one may ask is: why is it important to talk about these two notions of mathematical knowledge? In Chapter 1, as I elaborated on the problem statement of this study, it came out that the teaching of school mathematics in South Africa largely favours surface or procedural knowledge (see, for example, Makonye, 2014). Teachers who subscribe to surface knowledge regard mathematics as a toolkit consisting of rules and formulae, which when correctly recalled can help to get answers to mathematical problems and leads to success in examinations. Subsequently, learners blindly perceive mathematics as composed of mysterious procedures which when correctly followed leads to a pass i examinations (Hiebert & Carpenter, 1992). This type of knowledge may sound quite favourable to mathematics teachers because it provides speedy and accurate shortcuts to obtain answers to specific mathematics questions. With pressure mounting on teachers to make learners pass, surface teaching and learning becomes the order of the day. Unfortunately, if learners have a shallow, weak or distorted knowledge of concepts, they are prone to misconceptions and experience problems in assimilating and accommodating incoming knowledge. Function, as a unifying mathematical concept (Ponte, 1998), requires learners to have a deep and relational understanding of the concept that enables them to link and connect with other mathematical ideas.

This study aimed to explore the use of ICTs as cognitive tools to facilitate deep concept formation. The study had a conceptual orientation in the teaching and learning of hyperbolic functions in particular. The following section is devoted to research studies done on ICT integration in teaching and learning.

4.7. Integration of ICTs in teaching and learning.

In this section, I reviewed previous studies that have investigated the integration of ICT in teaching and learning. Admittedly, the type and level of ICT integration in the education system vary greatly between different countries depending on their level of economic development. In more economically developed countries such as the United Kingdom (UK), the United States of America (USA) and Singapore, the level of ICT integration is at a more advanced stage than in less developed economies (Collie & Lewis, 2011; Polly & Ausband, 2009; Lim & Hang, 2003).

For the purposes of this study, I looked at studies on ICT integration in advanced economies and in developing countries before zooming into research studies on ICT integration in South Africa. The focus was to draw some lessons from other countries that may be relevant and applicable in South Africa.

4.7.1 Studies on ICT Integration in Developed Countries.

A study by Lim and Hang (2003) used the Activity Theory framework to study the ICT integration processes in Singapore schools. The study sought to explore the question of how ICTs have been integrated in Singapore schools such that they promote higher order cognitive skills. Their study revealed that ICT tools had been introduced in all subjects by developing software in line with curriculum objectives. At every stage, from primary to secondary school, learners were expected to acquire specific ICT skills and necessary competencies. However, the study exposed some internal contradictions within the ICT integration system. In some cases, ICT was 'merely bolted on to existing classroom activities' (Lim & Hang, 2003, p.60). They discovered that teachers conducted lessons in the same way they could with handwritten overhead transparencies and overhead projectors. Teachers raised concerns of inflexible school timetables and the pressure of year-end examinations as negating the potentials of computers.

In Malaysia, Keong, Horani and Daniel (2005) carried out a survey on the use of ICT in mathematics teaching in secondary schools. The main objective of the study was to help teachers in their use of ICTs in teaching. To achieve this, the study sought to investigate the ICT applications used by teachers and

how the applications were used in the classroom. They reported some useful findings from their survey. Firstly, the study revealed that the most popular application packages used by mathematics teachers were: word processing (71%), spread sheets (51%), search engines (44%), presentation software (37%) and drill and practice software (24%).

These applications are representational in nature and add very little value to cognitive development (Hokanson & Hooper, 2000). Keong et al. (2005) suggested that mathematics teachers needed more time to learn to use applications like graphical applications and simulations programmes that have the potential to promote and encourage explorations and higher order thinking. Secondly, the survey revealed that the most common barriers to integration included lack of time in school schedules, lack of adequate technical support and lack of knowledge on the ways to use ICT to enhance the curriculum. These barriers are mainly teacher-centred and underscore the need for teacher development and support for effective integration. If support structures are persistently missing, it may impact negatively on their attitudes towards ICTs.

Ainley (2011) examined the use of ICTs by Australian mathematics and science teachers. The study was based on SITES2006 which confirmed that Australian mathematics teachers were among the leading users of ICT. The study found that mathematics teachers in Australia had high levels of confidence in their capacity to use ICTs. Despite the high level of ICT usage, the study reports that a considerable number of Australian teachers had not received ICT-related professional development training in terms of the technical and pedagogical use of ICTs. To some extent, this stifled effective integration of ICT in teaching. The study also cited time constraints as a common barrier to ICT integration on the part of teachers (Ainley, 2011). The issue of time constraints as a barrier to ICT integration also emerged in Keong et al.'s (2005) study in Malaysia. The study was not explicit on its findings about how teachers used ICTs in their lessons. While we may gain some insights on ICT use in teaching and learning, Malaysian and Australian educational contexts are more technologically advanced than South Africa and the research findings in these contexts may not be applicable to the South African context.

Tarmizi et al (2009) explored the instructional efficiency of using the Autograph Software in the teaching and learning of quadratic functions on Form 4 secondary students in Malaysia. The study investigated the effects of integrating technology on various performance levels. Surprisingly, what came out of the study was that the performance outcome of the experimental group was significantly lower than that of the conventional group (Tarmizi et al, 2009). The researchers cited several factors that might have led to these findings, which included time constraints, lack of focus and teacher factor. Such findings corroborate the view that mere use of ICTs in the classroom does not necessarily result in effective learning. Several other contextual factors must be considered and expertly managed for effective learning.

Ayub et al., (2008) compared students' perceptions on mathematical software utilised in mathematics teaching and learning in Malaysian secondary schools namely: Geometer Sketchpad, Graphic Calculator and Autograph. The study was based on the dimensions of perceived ease of use, perceived usefulness and behaviour intention related to use of the technology. The study concluded that the perception of students towards the use of graphic calculator, Geometer Sketchpad and Autograph was good (Ayub et al, 2008). The study revealed that the use of the three mathematical software conformed to the students' expectation in term of usefulness, ease of use and behaviour intention. Such findings may not be applicable in the South African context, especially in marginalised areas, where the level of technological competency of learners is very low.

I drew some very pertinent insights from the above studies. Firstly, the availability of ICTs does not necessarily translate to effective integration. Studies in Malaysia and Singapore have clearly shown that ICT use in the classroom is still representational. Teachers use ICTs to reinforce their traditional approach to teaching. Secondly, despite the presence of ICT tools, teachers still cite other barriers like time constraints, pressure to cover content and lack of technical support. Technical support in this case refers to the technological pedagogical content knowledge (PTACK) (Koehler & Mishra, 2009;

Shulman, 1986). Teachers may have the technical knowledge on how to operate and use ICT tools but linking them to pedagogy and content is a huge challenge to teachers.

4.7.2. Studies on ICT Integration in Developing countries.

The National Survey of ICT in South Africa schools (2003) reports on the general consensus among researchers that ICT use in developing countries is hampered by lack of clear policy and strategic framework. A strong and concise legislative and policy framework is inevitably essential to unleash the development potential imbedded in ICTs (Crede & Mansell, 1998). Crede and Mansell (1998) present a critical review of ICT policies in developing countries and argue that technology investment in developing countries is often premised on models borrowed from the developed world, which are unaffordable and inappropriate. They made a contestable claim that ICT policies in developing countries are 'copied and pasted' from other countries and are based on models which are irrelevant, unaffordable and unsustainable. In my view, policies are there on paper, but the challenge is that there is no coordination on their implementation.

Hennessy, Hawisson and Wamakote (2010) synthesized research literature on teachers' use of ICT in primary and secondary schools in Sub-Saharan African (SSA). As mentioned earlier, they noted that formulation and implementation of national policies on ICT use is an essential pre-requisite, alongside with supportive local policies and school leadership. The researchers admit that significant strides have been made towards learners gaining access to ICTs, but 'infrastructures necessary for deploying technological resources are lacking in low income countries' (Hennesy et al, 2010, p.41). These include, among others, the availability of electricity and internet connectivity.

Hennessy et al. (2010) report that a survey in African countries such as Mauritius, Ghana and Zimbabwe has shown that the most prevalent barriers to the use of computers were lack of time available in classes, lack of time for planning and lack of school policies on computers usage. While unavailability of technological tools is inevitably a major barrier to ICT integration, their availability does not necessarily

guarantee effective usage. Focussing on Nigeria, Tella, Toyobo, Adika and Adeyinike (2007) report that the lack of expertise in using ICT hinders teacher's readiness and confidence in using ICT.

A follow up question to the above findings would be: 'What ICT skills do teachers need to integrate ICT in their teaching?' UNESCO (2005) identified three ICT competency levels: technological literacy, knowledge deepening and knowledge creation. Lack of technological literacy on the part of the teacher inevitably makes him/her reluctant and resistant to the use of ICTs. Of particular importance to this study is knowledge deepening and knowledge creation. UNESCO (2009) clarifies that knowledge deepening and creation is achieved when students can apply the knowledge of school subjects that they have learnt to solve problems encountered in real life situations. This expectation demands teachers to adopt learner-centred teaching. The teacher's role would be to design carefully structured problem tasks and provide guidance and support in students' collaborative and goal-oriented projects. In guiding learner's understanding of concepts, UNESCO (2009) proposes that teacher need to employ open-ended technology tools that are specific to subject areas.

This review has so far revealed that ICT integration in more developed countries has reached a more advanced stage in terms of availability ICT infrastructure, teachers' technological literacy levels and their willingness to use ICTs in their classrooms. However, in research in Singapore and Malaysia has shown that use of ICT in teaching and learning is still inadequate and marred with internal and external contradictions. In developing countries ICT integration is at its very infant stage. The gap that exists is that while technological resources are not adequate, the little ICT resources available are not being properly utilized. Since this study was carried out in South Africa, I now focus on the state of ICT integration in South Africa.

4.7.3. ICT Integration in South Africa

This section is divided into two sub-sections. The first sub-section is a critical review of the national policy and legislative framework that promote the role of ICTs in the education system. The second sub-section is devoted to studies on ICT integration in South Africa.

4.7.3.1. Policy and Legislative Framework

The ushering in of the new political dispensation in South Africa in 1994 saw new educational policies being developed and implemented which substantially transformed the educational landscape in the country. The separate, racially defined education system was abolished and replaced by the establishment of a single education system (Lundall & Howell, 2000). The underlying philosophy, principles and goals for the new education system are articulated in the White Paper 1 on education and training (DOE, 1995). The South African Schools Act (SASA)(1996) provides the regulatory framework on how schools are governed, organized and funded. These two documents form the basis upon which other educational policies are crafted and shaped. Thus, any strategy or policy developed around ICTs in South African schools has to be aligned to the principles outlined in these policy frameworks.

A very comprehensive policy statement dealing with ICTs in schools is the Technology Enhanced Learning Investigation (TELI) commissioned by the DBE in 1995 (Lundall & Howell, 2000). TELI contains several potentially useful recommendations for the promotion of technology enhanced learning in schools. The concept of e-education in the South African context is explicitly outlined in the White Paper on e-education (DBE, 2005). According to the White Paper, e-education is about connecting learners to other learners, teachers to professional support services and provision of platforms for learning online. The policy document clarifies that e-education is not restricted to the development of computer literacy and technical skills necessary to operate various types of ICTs, but it also aims to enhance teaching and learning through communication and collaboration by using ICTs. The policy document envisages ICT use that promotes the acquisition of cognitive skills such as comprehension, reasoning, problem-solving and creative thinking (DBE, 2013).

Metaphorically speaking, the ICT policy documents have been criticised for providing a destination without giving directions on how to reach that destination. Mooketsi and Chagona (2014) observed that there is a disparity between the government's expectations and the practices of teachers. In trying to explain the causes for this disparity, separate studies by Du Plessis and Webb (2012) and Padayachee (2017) made a similar observation that the current ICT policies provide very limited support and guidance to schools and teachers on how they can effectively integrate ICTs within the South African context. The lack of clarity in the policy documents at national level is evident in the uncoordinated and differential approach in integrating ICTs in schools. Schools that are better resourced have made significant strides in ICT integration while those in marginalised communities are lagging far behind.

4.7.3.2. Studies on ICT integration in mathematics teaching in South Africa

Hardman (2005) undertook an exploratory case study of computer use in a primary mathematics classroom. Her research was driven by the argument that computer tools introduce conflict within the context in which they are used, resulting in transformation of the practices in that context. The study adopted an exploratory case study design and was centred on a teacher using computers to teach mathematics in a rural farm school. The researcher reports that computers were just sufficient for children to work on in pairs and the learner profile is one of disadvantaged background in terms of socio-economic status.

One important finding from this study is that the introduction of computers into a mathematics lesson force contradiction within the lesson and such contradictions are suggestive of change. For example, one such contradiction relates to the use of computers either as instruments for recall, recording and practice or as instruments for exploration. Teachers are also confronted with the conflict of ensuring that the curriculum is covered by well-performing students and the need to foster creative inquirers, capable of constructing meaning from their environment.

Howie and Blignaut (2009) worked on a study to establish South Africa's readiness to integrate ICTs in mathematics teaching. Data was drawn from the national dataset for SITES2006.The study raised a

wide range of issues around ICT integration, which include: access to computers and the internet, availability of technology, obstacles to realising pedagogical goals and teacher development (Howie & Blignaut, 2009). What captured my interest most were the findings with regard to ICT use in mathematics and science classrooms. The study reports that less than 20% of mathematics teachers used ICT in their teaching. Only 5% reported that they were using ICT at least once a week. What comes out clearly is that while considerable progress is being made on provision of and access to ICT facilities, capacity building on the part of teachers is lagging behind.

Ogbonnaya (2010) explored on improving the teaching and learning of quadratic functions using ICTs in South Africa. The study exposed several challenges students have in understanding the concept of function. The study was based on the Activity Theory framework and employed a practical action research paradigm (Ogbonnaya, 2010). Graph-sketching software was used to teach parabolic functions and compared to the conventional approach. The key finding of this study was that learners performed better in tests and assignments after the use of the software compared to when they learnt in the traditional approach. It also came out in the study that students who were taught with the software demonstrated better conceptual understanding of the concept of parabolic function. Another important benefit that came out was that the use of the software raised the level of motivation of the learners. Such findings endorse what several previous studies have given as one of the key benefits of ICT use (Becta, 2003; Hardman, 2005; Berger & Cretchley, 2005). However, the type of questions set in the tests and assignments do not really test for conceptual understanding. For example, ability to calculate the x- and y-intercepts and turning points of a given function may not adequately demonstrate that the learners have gained conceptual understanding. Also, the manner in which the graph plotting software was made use of was representational and not thought provoking.

4.8. Quality use of ICT and Enhancement of Learning.

Wilson-Strydom and Thomson (2014) argue that the adoption and integration of computers is a challenging and complex process for schools and teachers. This is expected considering that there is

little or no previous experience in the use of computers to support teaching and learning. Several studies have revealed that many schools that have access to ICTs tend to focus on learning *about* ICTs, rather than learning *with* or *through* the use of ICTs (Jonassen, Peck & Wilson, 1999 ; Wilson-Strydom & Thomson, 2014; Bialobrzaska & Cohen, 2005). Quality use of ICTs for teaching and learning must be looked at from the point of view of the extent to which it advances deep learning.

4.8.1. Levels of ICT Integration

UNESCO (2002) proposed a four-stage model of ICT integration. The first stage was described as *Emerging.* Schools at this stage begin to purchase some computing equipment, software and facilities. In this initial stage, administrators and teachers begin to explore the possibilities and consequences of using ICT for school administration and school curriculum (UNESCO 2002). Linking this to Ertmer's (1999) model of barriers to ICT integration, one would say schools at this emerging stage are beginning to break first order barriers.

The second stage of ICT integration is referred to as *Applying*. At this level, new insights and understanding of the contribution of ICT to learning has developed. As Wilson-Strydom and Thomson (2015) describe it, the use of ICTs at this stage by school administrators and teachers is extended to involve the use of ICT for tasks already carried out in school management and in the curriculum.

In a separate three-level integration model, Bialobrzeska and Cohen (2005) identified this level as 'functional practice'. Teachers use the computer in basic and functional ways such as word processing, document presentation, spread-sheets, producing graphs and searching for information on the internet. The important aspect at this stage is that a computer is used as an aid to assist in a task that previously would have been done in some other way, but which can be done more effectively using a computer. While this level of integration is valuable, one may argue that it has very little contribution to the development of cognitive skills in learners.

The third stage, according to the UNESCO (2002) report, is '*Infusing'.* This level involves imbedding ICT across the curriculum. Schools that have reached this level employ a wide range of computer-based technologies in laboratories, classrooms and administrative offices (UNESCO, 2002). Bialobrazeska andCohen (2005) described this level of integration as integrative practice. They view it as that level when teachers begin to fuse their purposes or job with activities of the computer and new ways of teaching and learning begin to emerge.

The final stage of integration is that of '*transformational practice'*. The UNESCO report explains this stage as that when schools and teachers use ICTs to rethink and renew school organisation and curriculum in more creative ways. This level of integration is about new ideas, new thinking and innovative use of ICTs to improve quality of learning. According to Bialobrazeska and Cohen (2005), learning occurs as a result of activities and opportunities which exist in computer-supported environments. This view forms the foundation upon which this study is premised. The study argues that ICTs open new learning avenues and expose learners to expanded learning opportunities which previously did not exist. Unfortunately, as discussed in the previous sections of this review, this level of integration has generally not been attained in the South African school classrooms. It is therefore my hope that this study will provide impetus and a functional guideline to mathematics educators that facilitates and propels their progression from functional ICTs users to more transformative users of ICT in the classroom.

4.8.2. 'Representational' versus 'Generative' Use of ICTs

Closely linked to the stages or levels of ICT integration, it is equally important to consider the types of integration. Wilson-Strydom and Thomson (2015) differentiated between two types of integration, namely *representational* and *generative* use of ICTs**.** *Representational* use describes how computers are used to merely re-present information in another medium (Wilson-Strydom & Thomson, 2015). In this case, computers are made use of within a task, but their usage is merely to represent information in another form and not to facilitate the reconstruction new information.

The epistemological assumption underpinning *representational* use of ICTs is that knowledge is absolute, definable and re-presentable (Hokanson & Hooper, 2000). Following this argument, if a teacher views mathematical knowledge from an absolutist perspective, she is likely to limit the use of computers to representational uses. Hokanson and Hooper (2000) further stress that using ICTs as representational tools is partially integrative. The benefits of computers in teaching and learning can only be realised when their use goes beyond the representational usage. As envisaged in the DoE White Paper on e-education, ICTs should be incorporated to advance higher order cognitive skills and apparently that cannot be realised through representational type of integration.

The concept of *generative* use of ICTs, as Horkanson and Hooper (2000) put it, has its roots in the constructivist view of knowledge and learning. The assumption here is that knowledge is fallible and not a finished product that can be transmitted from one individual to another. In this model learning is viewed as a process of individually constructing knowledge (Wilson-Strydom & Thomson, 2015). Effective incorporation of ICTs should facilitate and promote knowledge construction within each individual learner. Teachers who subscribe to this theoretical perspective are more likely to incorporate ICTs in their teaching in more *generative* ways.

The generative use of computers is supported by the socio-constructivist view of knowledge and learning. As alluded to under the theoretical framework of this thesis, social-constructivists argue that knowledge construction is not confined to an individual's internal view of it, but rather is a process of negotiating and extracting meaning out of a given social context (Hardman, 2005). In that sense, computers open opportunities for learners to continuously re-construct and refine knowledge on the basis of their experience and inter-subjective exchange.

Some researchers refer to ICTs as 'cognitive tools' or 'mediational tools' (Lim, 2003; Jonassen & Reeves, 1996), expressing both the cognitive and socio-cognitive power inherent in technology, if properly integrated. The *generative* integration of ICTs resonates perfectly well with the theoretical framework on which this study is premised. As mentioned previously, the focus of this study is on

deepening conceptual understanding of functions through the use of ICTs. The concept of *generative* integration is adopted in this investigation.

4.9. Conclusion of the chapter

Research reveals that the teaching and learning of functions in schools presents challenges both to teachers and learners. Teachers seem to place more emphasis on teaching of procedures and manipulation of algebraic expressions at the expense of conceptual understanding. Subsequently, learners have a 'surface understanding' of concepts under functions. This is evidenced by the common errors and misconceptions that learners exude as they attempt to solve problems involving functions.

In this technological age, numerous studies have shown that the integration of ICT in teaching and learning has potential to advance conceptual understanding. Unfortunately, the available technology in schools is mainly being used for administrative or representational purposes with very little contribution to the cognitive development of the learners. There seems to be a dearth of research on the impact of using technological tools in addressing learners' misconceptions. Not much has been done also to ascertain the effectiveness of ICT tools in enhancing conceptual knowledge in mathematics particularly in rural contexts.

In the context of South Africa, considerable strides have been made to improve the availability of and access to ICT resources, but of major concern is the fact that the little ICT resources available in schools are not being fully utilised to enhance the teaching and learning of mathematics. Reluctance to the use of ICTs in the teaching and learning could be attributed to inadequate teacher professional development. This study aims to establish the impact of ICT integration in addressing identified learners' misconceptions on functions. It explores how the use of ICTs can enhance conceptual knowledge in mathematics. The next chapter discusses the research methodology adopted for this study.

CHAPTER 5: RESEARCH METHODOLOGY

5.1. Introduction

This chapter is devoted to discussing the research methodology adopted for this study. The chapter begins by reminding the reader on the research questions that the study sought to address. It then describes the research paradigm underpinning the methodology used and the justification of the paradigm that was adopted. The research instruments and data collection procedure are unpacked and issues relating to research trustworthiness and ethics are explained.

All research is premised on some underlying ontological, epistemological and methodological assumptions (Guba & Lincolin, 1994). These assumptions define what constitutes a 'valid' research problem and which research methods are most appropriate to tackle the problem. Krauss (2005) notes that two broad conceptions of social reality exist and their philosophical assumptions influence research methodology. On the one hand, as Kuhn (1962) puts it, the positivist paradigm forms the basis for quantitative research methods. On the other hand, we have the anti-positivist or interpretivist paradigm which has given rise to qualitative research designs (Guba & Lincolin, 1994; Krauss, 2005). As is expounded in the coming sections, this study subscribes to the interpretivism paradigm and consequently, it adopts the qualitative research methodology

5.2. Research questions

Before delving into the details of my research methodology, I outline the research questions of this study to put the reader in better perspective of what the research aimed to achieve. As highlighted in Section 1.9 of this thesis, the study first sought to establish the errors learners made in dealing with hyperbolic functions and uncover the misconceptions underlying those errors.

The next research question sought to assess the impact of an ICT-mediated instructional intervention in enhancing conceptual understanding of concepts under hyperbolic functions, thereby minimising the errors and misconceptions. Based on the participants' and the researcher's experiences, the study

ultimately aimed to develop an ICT-mediated instructional model that could be used by mathematics teachers in their teaching of functions.

Linking the research questions to the theoretical framework (see Chapter 2) of the study influenced the methodological route I adopted. From a constructivist perspective, learners' errors and misconceptions arise as a result of their internal knowledge construction processes. The understanding of learner misconceptions requires an in-depth study of each learner's unique interpretation of the experiences presented to him or her.

The merged constructivist theories of social constructivism, Activity theory and APOS theory provided the theoretical backing in tackling the second research question. It is noteworthy that ICT-mediated instruction is highly contextual, and its impact can only be adequately assessed if located within a particular context. A case study approach was most appropriate to achieve this.

The third research question was grounded on insights that could be drawn from the learners' own perspectives and interpretation of their experiences during an ICT-mediated instruction, which could result in a proposed ICT-mediated pedagogical model. In the next section, I explain how the interpretivist research paradigm was best suited for this study, resulting in the adoption of the qualitative research approach.

5.3. Research Paradigm underpinning this study.

Research in the social sciences, and educational research in particular, is paradigmatic in nature. Krauss (2005) noted that the word paradigm originates from the Greek word for exhibiting side by side. Kuhn (1962) explained that the concept of paradigm refers to different sets of assumptions, commitments, procedures and theories of social affairs that provides a perspective on how we look at science. Research processes and procedures are influenced by certain philosophical assumptions that the researcher subscribe to. Considering this viewpoint, as Jick (1979) puts it, paradigm provides the theoretical base upon which problems for inquiry are selected and defined, and the methodological approach to use is decided. Guba and Lincolin (1994) are of the opinion that research paradigm is the

belief system or 'worldview' that guides the investigator, not only in choices of methods but also in ontological and epistemological fundamental ways. In sum, a research paradigm is the philosophical framework that guides or directs the whole research process. In other words, the way one views social reality has a significant influence on how one views the research phenomenon and the approach to investigate it.

Considering the above conception of research paradigm, two main philosophical views exist in educational research, namely, the traditional view (or positivism) and the interpretive view (or anti-positivism) (Cohen & Manion, 2007). These contending perspectives and their corresponding reflections in educational research, stem from their different conceptions of social reality and social behaviour. As discussed under the theoretical framework of this study (see Chapter 2), this study was premised on constructivist principles which naturally resonates with the interpretivist research paradigm. I highlight below the assumptions that aligned this study to interpretivism.

The first assumption that separates the two views is of an ontological kind. Ritche, Lewis, Nicholas and Ormston (2014) view ontology as the nature of reality and what is to know about the world. The interpretivist paradigm views reality as fundamentally mind-dependent (Creswell, 2012; Krauss, 2005) and is only knowable through the human mind and socially constructed meanings (Ritche et al, 2014). No reality exists independent of these.

One of the key contributors to anti-positivism, Wilhelm Dilthey (1833-1911) pointed out that natural sciences differ from humanities in the sense that natural sciences seek to explain natural laws and patterns whereas humanities strive to understand the unique individual in his entire concrete setting (Winchester, 1988). Considering Dilthey's viewpoint, there is no natural law or pattern that can adequately define how learning occurs. The way a learner responds to or interprets his or her experiences within a learning context is unique. Gaining access to these unique and internal interpretations may only be achieved through in-depth interrogation of the subjects and trying to understand the phenomenon from their perspective.

The second assumption is epistemological in nature. Epistemology addresses issues on how we come to know social reality. From a positivist perspective, knowledge is hard, objective and tangible (Cohen & Manion, 2007), and as such the object of study is independent of the researcher (Krauss, 2005).

Interpretivism argue that knowledge is acquired through the meanings attached to the phenomenon studied (Krauss, 2005). From this perspective, knowledge is seen as personal, subjective and unique (Cohen & Manion, 2007). A researcher who subscribes to this conception of knowledge would interact with the subjects of study to obtain data and is driven by the assumption that the best way to investigate any phenomenon is to view it in its context (Guba & Lincolin, 1994). In other words, the best way to know what is going on in any given phenomenon is to become deeply involved in it, move into the context being studied and experience what it is like to be part of it.

Thirdly, the other assumption has to do with the relationship between human beings and their environment. From the positivist perspective, human beings are portrayed as responding mechanically and deterministically to their environment (Cohen & Manion, 2007), and therefore are products of their environment, controlled like puppets (Burrell & Morgain, 1979). This view completely negates the constructivist perspective on learning.

The interpretivist paradigm argues that human beings are initiators of their own actions with free will and creativity, producing their own environments (Krauss, 2005; Cohen & Manion, 2007). It therefore follows that the principal concern of inquiry is to gain an understanding of the way in which the individuals create, modify and interpret the world in which they find themselves in. In undertaking this study from an interpretivist perspective, I subscribe to the view that learners are active knowledge creators and initiators. Any instructional intervention should actively involve them, and its impact should be assessed based on learners' perspectives about it.

The set of assumptions discussed above had direct implications for the methodological approach adopted in this study. As Pandey and Pandey (2015) highlight, one's ontological and epistemological viewpoint will in turn influence the appropriate research methods. On one hand, when one subscribes to the positivist paradigm, the scientific investigation will be directed at analysing relationships between identified variables of the phenomenon (Cohen & Manion, 2007). In that case, the design of inquiry will be predominantly quantitative. On the other hand, if one favours interpretivism, emphasis is on an in-depth understanding of an individual case rather than the general and the universal (Creswell, 2012). In this study, as indicated earlier, I took the qualitative route. In the next section, I unpack the defining features of qualitative research and the justification for taking the qualitative route.

5.4. Qualitative Research

Qualitative research stems from the interpretivist paradigm of social reality, which stresses the importance of individuals' experiences in the creation of their social world (Burrell & Morgain, 1979). Denzin and Lincolin (2000, p.3) posit that qualitative research is concerned with exploring the phenomenon from the 'interior' and 'taking the perspectives and accounts of research participants as a starting point'. The focus is on the 'how', 'what' and 'why' questions rather than 'how many' (Ritche et al, 2014). It follows that research questions cannot be phrased in a structured or closed way. Rather, qualitative questions are stated in a way that one can best learn from participants.

Strategies or methodologies associated with the qualitative approach include *Ethnography* (Lester, 1999), *Grounded theory* (Strauss & Corbin, 1998), *Case studies* (Stake, 1995) and *Phenomenology* (Lester, 1999; Stanley & Wise, 1993). Below, I explain why I adopted the qualitative research design for this study.

5.4.1. Rationale for adopting qualitative research

In selecting a research methodology, Guba (1981) hints that 'it is proper to select that paradigm whose assumptions are best met by the phenomenon being studied' p.78. One distinguishing feature of

qualitative research is that it is naturalistic and attempts to study the subjects in their natural settings. Denzin and Lincoln (2002) explain that qualitative researchers adopt interpretive and naturalistic approaches to their subject matter and attempt to make sense of a phenomenon in terms of the interpretations people bring to them. In the context of learning, complex processes are involved. For the learning environment to be understood, one has to understand the complex interactions of the several variables involved. Worse still, the introduction of ICT tools adds another complex dimension in the learning process. Qualitative research allows the researcher to explore and to discover issues about the problem out of the complex phenomenon.

Another defining feature of qualitative research is that it is mainly interested with processes rather than the product of an inquiry (Lincoln & Guba, 1985). I found this view very critical and pertinent to this study. For example, the first research question of this study did not end at identifying what errors learners make but went further to unearth the underlying misconceptions and infer on possible reasons why learners make those errors.

This study was not only interested in whether the integration of ICTs enhances learner understanding but also on exploring and understanding the processes that lead to enhanced understanding. Domegan and Fleming (2007) argued that the assessment of learning is better achieved during the learning process and not after the learning has taken place. The description of the learning processes is more valuable that dwelling on the outcomes. If I had chosen quantitative research approach, it was likely going to obscure some of the insights and experiences of the participants that I needed to understand and expose.

Qualitative researchers depend on a variety of data sources. These include observation and participant observation, interviews, documents and texts and the researcher's impressions and reactions (Lincoln & Guba, 1985). The collected data is then collated and presented as descriptive narration with words in an attempt to understand the phenomenon in its unperturbed natural setting. Obtaining data about

the same participants in the same setting from several sources helps to ensure the credibility of the data.

Thick descriptions of data are meant to capture the diverse perspectives of the research participants in their interpretation of reality and the researcher's own interpretation of the outcomes of the study. It allows any other reader who goes through the data to come up with his own outcomes of findings from the data. The data sources in this study were participant observation, task-based and open-ended interviews and document analysis.

5.4.2. Weaknesses of qualitative research

Qualitative research is not without its weaknesses and it is important for the researcher to be aware of these shortcomings so as to minimise their impact on the credibility of the research findings. Qualitative research acknowledges researcher's subjectivity. This is why the 'biases, motivational interest or perspectives of the inquirer' must be identified and made explicit throughout the study (Lincoln & Guba, 1985, p.29). Researcher bias may influence the design of the study, the selection of participants and the data collection procedures. To counter bias in this study, I strived to be non-manipulative of the research process and took into account the unperturbed and unfiltered views of the participants.

Critics of qualitative research argue that sources of data or the subjects may not be equally credible (Guba, 1981; Merriam, 1992). In the first place the sample may not be representative of the larger population. However, qualitative researchers do not normally claim that their findings are generalizable. Instead, their focus is on having an in-depth understanding of the phenomenon under study.

An additional argument that has been raised against qualitative research is that any group that is studied is altered to a certain extent by the presence of the researcher, and consequently data collected is somewhat skewed (Guba, 1981). Closely linked to this criticism is the accusation that it takes time

to build trust with participants that warrants full and self-representation. This is particularly a challenge in short-term observational studies.

However, as pointed out earlier, qualitative researchers rely on multiple data sources from the same subjects. By triangulating multiple data sources, it helps to check and cross-check the authenticity and credibility of the collected data. In defence of qualitative research, Merriam (1992) posits that its main thrust is on ensuring credibility and confirmability of findings on the phenomenon under study, rather than imposing quantitative ideas of generalizability on qualitative research. In the next section, I zoom on the specific research design used for the study.

5.5. The research design

A research design is generally viewed as a plan for a study used in collecting and analysing the data. Pandey and Pandey (2015) described it as the plan, structure and strategy of investigation conceived so as to obtain answers to research questions. A more penetrating definition was provided by Plano Clark and Ivankova (2016) when they viewed a research design as 'a formal and defined set of procedures for collecting, analysing and interpreting data to address a specific type of research purpose' (p.138). These definitions demonstrate that a research design is like a blueprint that explains how an investigation will be carried out to achieve its intended purpose. As discussed in the previous section, the research design has to be aligned with the research paradigm and has to be most appropriate to address the issues under investigation.

This study sought to investigate learners' conception of hyperbolic functions and to explore how the introduction of ICTs in their learning could enhance their conception of concepts under functions. Since learners vary in their learning patterns and their conception of what they learn, exploring their understanding and interpretation of their learning experiences require an in-depth study of a few learners within a particular context. To achieve this, I adopted the qualitative case study research design. The key features of a case study research design and its appropriateness to this study is discussed in the following sections.

5.6. Qualitative case study design

A comprehensive definition of a case study is given by Harling (2002) who says:

> … a holistic inquiry that investigates a contemporary phenomenon within its natural setting. It involves a collection of in-depth and detailed data that was rich in content from multiple sources of information including direct observation, interviews and documents. The multiple sources of information provide the wide array of information needed to provide an in-depth picture (p.29).

The definition brings out the naturalistic nature of a case study. This study explored and investigated learning processes within an ICT mediated instructional context. The study was classroom based and the participants were studied within their usual classroom setting.

The only deviation from their usual setting was the introduction of ICTs in their learning. Another defining feature of case study as defined in Harlington's definition is the multiplicity of data sources. This emanates from the quest by qualitative researchers to have a deeper and detailed understanding of the phenomenon under study. The focus of a case study is to answer the 'how' and 'why' questions (Yin, 2003). To achieve this, case study uses a multiple of data sources as strategy to capture the same data from different perspectives. As shall be explicated in the next sections, this study employed document analysis, task-based and open-ended interviews, direct and participant observation to collect data.

This study, more specifically, adopted the descriptive and exploratory case study approach. As Yin (2003) explains, descriptive case study aims to give a detailed description of a phenomenon and the real-life context in which it occurred. Exploratory case study is most appropriate in exploring those interventions or phenomenon in which the outcomes are not clear and multi-dimensional. This study was descriptive in the sense that it involved the designing and implementation of an ICT based instructional intervention. It was explorative because its main focus was to explore how ICTs can best be integrated in the teaching and learning of functions. As I indicated earlier, learning is a very

complex process and any attempt to predict how and why learners respond to any instructional intervention can never be done with high degree of certainty. From a constructivist perspective, learning is about knowledge construction and constructed knowledge is not truth that remains stable and generalizable across all contexts. Rather, it is dynamic and exists within a specific context. Given the constructivist theoretical perspective and the interpretive position this study adopted, I considered the case study methodology most appropriate to address the nature of research questions in this study.

To sum up the defining features of case study, Merriam (1998) identifies four essential features of a case study namely: *particularistic, descriptive, heuristic and inductive*. Particularistic case study refers to a situation whereby the study is directed towards a single event, process or situation. In this case, the study focused on a grade 11 mathematics class at a high school in Limpopo province. The content coverage in the study was restricted to hyperbolic functions only. The descriptive quality emerges from the rich-descriptions and extensive set of details provided by the researcher regarding the phenomenon under study.

As Merriam (1998) explains, case study is heuristic in the sense that it advances deeper and detailed understanding of the phenomenon. It presents rich data from a multiple of sources that allows the researcher to make a deep sense of the scenario from a variety of perspectives.

Lastly, a case study is inductive because the basis used to determine generalisations emerges from the data obtained from a case or a few cases studied (Merriam, 1998). However, Tellis (1997) cautions that case studies do not claim to be representative, but their emphasis is on what can be learnt from a single or from a multiple of cases studied. Their underlying philosophy is not to prove but to improve. This study sought to improve the quality of ICT integration in the teaching and learning of functions, through the development of a model which might be applicable in other similar contexts.

No research methodology is perfect, and a case study is non-exceptional. The weaknesses of a case study emerge from the criticisms levelled against qualitative research in general. It lacks representativeness and statistical generalizability (Miles & Huberman, 1994). It has, however, been

noted already the focus is not to generalise findings but to understand the phenomenon in depth (Merriam, 1998). Generalizability may be achieved through multiple case studies in multiple settings. The richness and complexity of data collected means the data is open to different representations and subject to researcher bias. The question on how researcher bias can be minimised in qualitative research was dealt with in Section 5.4.2 above.

Merriam (1998) concedes that qualitative case studies in education are often premised on theoretical principles, models and concepts. An inductive approach is then used to support or challenge theoretical assumptions. Drawing from this argument, Figure 5.1 below is a diagrammatic representation of the framework for the research design of this study.

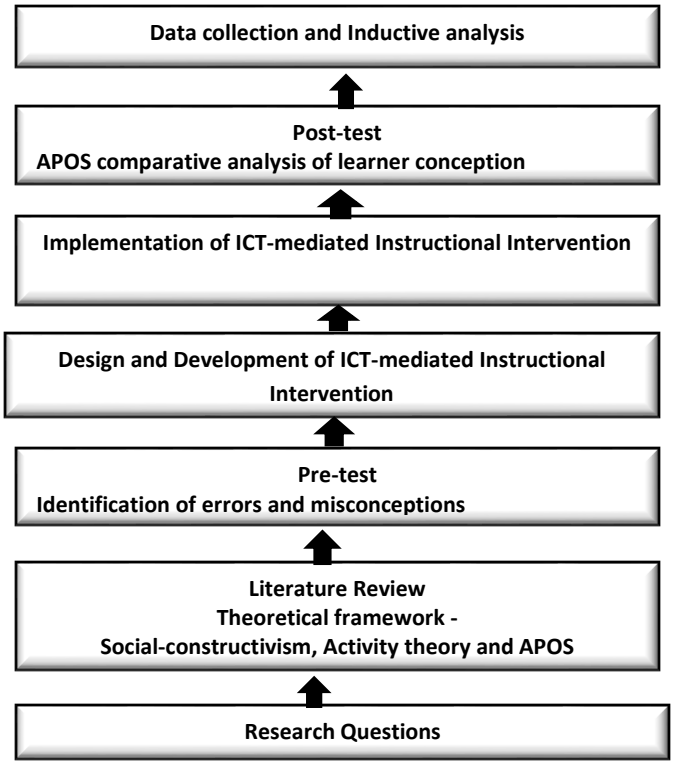


Figure 5. 1. Diagrammatic representation of the research design adopted for the study

5.7. The Research Site

The study was carried out at a rural secondary school in the Limpopo province. The school serves three villages in an economically disadvantaged community. Most of the learners come from low-

income families that depend on government social grants. Some of the learners travel an average of five kilometres to school. The Limpopo DoE has recently provided a bus to ferry the learners to and fro school, thanks to the *Scholar Transport* initiative. Due to the low economic status of the community, the school is classified under *Quintile 1* and hence the parents do not pay anything for the education of their children.

When data was collected the school had a total enrolment of 250 learners. Of these 130 were girls and 120 boys. Generally, the learners were not well exposed to ICTs, especially computers. The school had recently established a computer lab, but only Grades 11 and 12 had access to these computers. The total staff establishment for the school was nine teachers, one lab assistant, one administration clerk and two cleaners. All educators at the school are professionally qualified with teaching experience ranging from ten years and above. Only five of the educators are computer literate. These educators attended a course on computers skills and a workshop on ICT integration. The others have had very little exposure to computers.

The school offered the following subjects at FET level: Sepedi Home Language (HL); English First Additional Language (FAL); Mathematics; Mathematical Literacy; Geography; Life Sciences; Physical Sciences; Agricultural Sciences and Life Orientation. Once learners enter FET level, they have to choose either Mathematics or Mathematical Literacy and either Agricultural sciences or Physical Science. The academic performance of the school has remarkably improved in the past three years to an average pass rate of 70%. Previously, the school had been in the underperforming category for several years. In the case of Mathematics, the school performance in the matric examinations has oscillated around 55% in the past five years.

The school was conveniently chosen for two main reasons. Firstly, I once taught at this school which made it easier for me to gain access to the participants and other resources needed for the purposes of this study. Secondly, the school had recently received a donation of fifty laptops, two overhead projectors among other ICT tools specifically for teaching and learning purposes. Though the ICT

resources and infrastructure were not adequate for the whole school, they were sufficient to overcome the challenge of lack of ICT resources which stands out as the major barrier in most rural schools.

5.8. Sampling

Qualitative researchers use non-probability sampling techniques whereby the investigator selects research participants or sites that can best help one to fully understand the phenomenon under study (Creswell, 2012). As Patton (1990) puts it, the other term for qualitative sampling is purposeful sampling. The defining feature in purposive sampling is that the researcher intentionally selects individuals or sites for a specific purpose. Patton (1999) notes that the qualitative standard used in choosing participants and sites is whether they are 'information rich' or not (p.16). Thus, in qualitative research one may decide to study a site, individuals, groups or some combination (Creswell, 2012). Purposive sampling was most appropriate for this study.

The participants in this study were thirteen (13) Grade 11 learners at the school who were studying pure mathematics. The school had a total of 41 learners in Grade 11 and out of these, only 13 participants were selected from the school because they were the only ones doing pure mathematics in Grade 11. The Grade 11 mathematics class was conveniently chosen because according to the CAPS document, the content on quadratic, hyperbolic and exponential functions is sufficiently covered in Grade 11. The 13 learners were doing Mathematics since Grade 10 and were taught by a qualified and experienced educator.

5.9. Data Sources

5.9.1. Document analysis

Document analysis in this study was based on learners' written responses in the pre- and post-diagnostic tests. Tests provide an impressive and flexible way of collecting data and they vary extraordinarily in terms of type, scope and level of sophistication (Cohen & Manion, 2007). The tests used in this study were diagnostic in nature. A diagnostic test is an in-depth test to discover particular strengths, weaknesses and difficulties that a learner is experiencing (Cohen & Manion, 2007). It is

particularly designed to unearth specific areas of weakness and strength. Diagnostic tests include several items about the same concept. This allows the investigation to expose difficulties in a learners' understanding.

The pre-test provided data on common errors learners made around hyperbolic functions. Triangulating the learners' written responses and their oral responses during task-based interviews enabled the learners' misconceptions to be inferred. The question items were derived from a variety of sources including past examination papers and learners' textbooks. The structure of the question items and their cognitive demand was informed by the APOS-conception levels, as guided by the genetic decomposition I developed for the study (see item 3.3.1) .The post test was similar to the pre-test but not exactly the same. The items in the post-test were developed by slightly altering each item in the pre-test but ensuring that the items tested the same concept and/or skill and maintained the same cognitive demand. This was done to minimise the effect of memorising and duplicating answers from the pre-test but at the same time allowing for comparative analysis of responses in the pre- and post-test. The table 5.1 below shows the APOS item analysis of the test instruments.

Question Number	Item Number	Concept or skill covered	APOS level
1	1.1	Determining x-and y-intercepts from a given hyperbolic equation.	A & P
	1.2	Extracting the equations of the vertical and horizontal asymptotes from a given hyperbolic equation.	A
	1.3	Use the critical points calculated in 1.1 and 1.2 above to sketch the graph of the hyperbolic function.	P
2	2.1	Given the graph of $f(x) = \frac{a}{x-p} + q$, learners were asked to write down the equations of the vertical and horizontal asymptote.	P
	2.2	Using the graph in 2.1 above, learners were asked to apply the horizontal translation $f(x \pm p)$ and find the equations of asymptotes of the transformed graph.	O

	2.3	Using the graph in 2.1 above, learners were asked to apply the vertical translation $f(x) \pm q$ and determine the domain and range of the transformed graph.	O
	2.4	Using the graph of $f(x)$ in 2.1 above, learners were asked to determine values x for which $f(x) \leq q$ or $f(x) \geq q$	P & O
3	3.1	Given a hyperbolic function with equation $y = g(x)$ and having the following properties: ▪ Domain: $x \in R, x \neq p$ ▪ Range: $y \in R, y \neq q$ ▪ Passes through the point $(a; b)$ Learners were then asked to determine the equation of $g(x)$.	S
	3.2	Given a hyperbola of the form: $k(x) = \frac{a}{x+p} + q$, passing through the point $(a; b)$ and its axes of symmetry intersecting at point $(t; s)$, learners were asked to find the equation of $k(x)$.	S

A - Action; P- Process; O – Object; S - Schema

Table 5. 1. APOS item analysis of the pre- and post-test instruments

5.9.2. Interviews

In qualitative interviews, the researcher asks one or more participants general and open-ended questions and records their responses (Creswell, 2012). The use of open-ended questions allows participants to express their opinions without being influenced by any perspectives of the researcher or past research findings (Oppenheim, 1992). Open-ended questions attract open-ended responses (Creswell, 2012), which open room for further probing and extraction of further information. Two types of interviews were carried out in this study, namely *task-based* and *open-ended interviews.*

5.9.2.1. Task-based Interviews

Task-based interview is a particular form of clinical interview (Piaget, 1975) designed so that research participants interact with the interviewer in the context of a task environment that is carefully crafted

for purposes of the interview (Goldin, 2000). In the interviews, the researcher has an active role, involving students to explain their task solutions. The diagnostic power of this approach is drawn from the opportunity the researcher had to question and probe student thinking. In this study, task-based interviews allowed learners to explain, justify and clarify their written responses both in the pre- and post-test. The interviews allowed the researcher to gain deeper understanding of the learners' misunderstandings and the resultant errors, thereby providing better access to the misconceptions underlying the errors. It was also through such interviews that the learners' APOS level of conception could be established. The task-based interviews were done either on one-to-one basis or focus group interviews in cases where learners exuded similar error patterns.

5.9.2.2. Open-ended Interviews

Open-ended interviews were used to elicit learners' experiences during the intervention programme and how they interpreted their experiences. The central focus of the interviews was to identify the affordances and constraints provided by the ICT tools, based on learners' perspectives. In a friendly and relaxed environment, learners were given an opportunity to share their experiences focussing on whether ICTs assisted them in any way to understand better the content taught better and what challenges they encountered. The interviews were done on a one-to-one basis and were all audio-taped for analysis.

5.9.3. Observations

Observation, as defined by Mills (2011), is the process of gathering first-hand information by observing research participants' actions and interactions within a research site. From a qualitative perspective, the defining feature of observation is that it provides the researcher an opportunity to collect 'live' data from naturally occurring situations (Cohen & Manion, 2007). The investigator can observe directly at what is occurring 'in situ' rather than relying on second-hand information. One of the major advantages of observation is that it enables researchers to investigate individuals who have

challenges in verbalising their ideas (Creswell, 2012) and provides a reality check (Robson, 2002) against individuals who act differently from what they say they do.

In this study, the researcher assumed the role of a participant observer. Spradely (1980) notes that participant observation is an investigatory role adopted by researchers when they actively participate in the setting they observe. Spradely (1980) further argues that the researcher assumes the role of an 'inside' observer. During clinical interviews, observation played a very important role because it enabled me to see things that participants were not even aware of or what they were uncomfortable to discuss (Patton, 2002). These included the level of confidence displayed as they explained their responses and their facial expression if they were sure of what they were saying. I recorded the learners' non-verbal expressions in my journal during the interviews.

During the intervention lessons, learners were observed in different instructional settings like learner-learner interaction, learner- teacher interaction, learner-tool interaction and learner-content interaction. Fortunately, I had the privilege of being allowed to video-tape all the intervention lessons. Observations were done after each of the lessons together with a colleague (who is an experienced mathematics educator). I solicited the help of a colleague in a bid to minimise researcher bias. Since I was the designer and implementer of the instructional programme, and also playing the role of teacher-cum-researcher, chances were that my reflection of the lessons would unconsciously overlook certain aspects.

To ensure that our observations focussed on pertinent issues, I developed an observation tool that my colleague and I used independently before comparing notes. The observation tool was adapted from Hardman (2008) and is given in Table 5.2 below:

Activity Theory Dimensions	Questions to ask when analysing evaluative episodes	Observation notes
Object	▪ What was the focus of the activity / episode? ▪ Was the focus on transmitting practical (or technical) skills? ▪ Was there any attempt to develop and reinforce learners' content knowledge? ▪ Were there any probing questions to develop metacognitive skills and higher mental functions?	
Mediating ICT tools	▪ Was the use of ICTs representational (to represent work, notes or content covered)? ▪ Was the use of ICTs generative (to promote mathematical thinking and problem solving)? ▪ Was the use of ICTs linked to the targeted content or concept? ▪ Did the use of ICT tools facilitate learner-content interaction? ▪ Did the use of ICT tool promote learner-learner interaction and learner-teacher interaction?	
Division of labour	▪ What were the major roles played by the teacher? Eg controlling learners, offering technical skills, elaborating maths concepts, probing and asking questions, ▪ What were the major roles played by the learners? Eg listening to the teacher, engaged in group and individual activities, answering question orally, asking questions.	
Rules	▪ What kind of instructional rules were observable? ▪ How did learners respond to teacher's questions? ▪ How did the teacher respond to learners' contributions? ▪ How did learners interact with each other? ▪ How were learners with technical challenges assisted? ▪ Was the order / discipline during the episode?	
Community	▪ Was there productive and orderly interaction between the learners? ▪ Was the teacher working together with learners towards the object? ▪ Was the video recorder always well positioned – not obstructing or drawing learners' attention?	

Table 5. 2. Observation guide for the intervention lessons (Adapted from Hardman, 2008, p.76)

Drawing from the Activity theory dimensions of object, tools, rules, division of labour and community (see Section 3.3), the observation guide provided guiding questions that helped to give a broader picture of what was happening during the pedagogical activities. Critical reflections were done on the lessons focussing on what went well and areas for further improvement. The reflections and other comments were recorded on the observation guide for further analysis and were useful in preparation for the next lessons and in the subsequent analysis.

5.10. Data Collection Procedure

In line with the research design for this study, as shown in Figure 5.3, data collection was done in three phases. The first phase involved administering the pre-test on hyperbolic functions, followed

by the task-based interviews. The central focus of phase one was to identify learners' errors and misconceptions.

The second phase involved the development and implementation of the ICT-mediated instructional intervention programme. The post-test was administered after the intervention lessons and it was followed by another session of task-based interviews. Phase two sought to evaluate the impact of the intervention lessons on learner conceptualisation of concepts under hyperbolic functions.

The third phase involved the open-ended interview questions on learners' experiences throughout the intervention programme. Data collected from open-ended interviews was triangulated with data from document analysis and the researcher's observations to develop the proposed ICT-mediated instructional model. Details of each of the phases are described separately below.

5.10.1. Phase 1

This phase began by administering of the pre-test (see Appendix J) on hyperbolic functions to the learners. This was done in August 2017. According to the mathematics Annual Teaching Plan (ATP) for that year, learners had been taught the topic in term two. In my preliminary situation analysis of the research site, I had already verified with the subject teacher that no ICT tool was used in teaching the topic. The teaching approach where no ICTs were used is described in this study as the traditional approach. The learners were tested on their understanding of concepts under hyperbolic functions after they were initially taught without the use of ICTs. The purpose of the pre-test was two-fold. First, it allowed me the opportunity to identify the common errors and misconceptions learners displayed in dealing with problems under hyperbolic functions. Secondly, the pre-test helped me in establishing the learners' conception level (in APOS terms) of hyperbolic functions before the intervention lessons.

The pre-test was followed by the task-based interviews based on the learners' written responses in the pre-test. The interviews were centred on the items in the pre-test and afforded learners an

opportunity to explain and clarify their written responses in the test. It was through the interview data that I could dig deep into why learners responded the way they did. It also allowed me to place the learners in the appropriate conception level depending on the level of understanding they displayed. The task-based interviews were mainly one-on-one, but focus-group interviews were done on three occasions where learners displayed similar error patterns or where some learners indicated that they were uncomfortable to be interviewed alone. All the interviews were audio-taped, with approval from the learners. The interview data was then transcribed for subsequent analysis.

5.10.2. Phase 2

Informed and guided by the emerging errors and misconceptions, the next phase was the development of the ICT-mediated intervention programme on hyperbolic functions. In developing the intervention programme, I solicited expert guidance from one of the district Curriculum Advisors (subject specialist). The Curriculum Advisor had facilitated at one of the district workshops on ICT integration and I was confident he would offer valuable insights in this regard. He assisted by installing Autograph software on all the computers that were to be used for this study and he made valuable input on learning activities based on the software. All the lessons were video-taped for analysis. The ICT-mediated instructional intervention programme was structured as outlined below.

5.10.2.1. Introduction to Autography

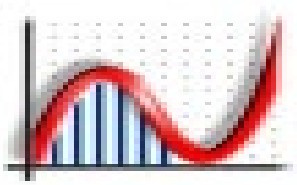
Autograph is dynamic software for teaching calculus, algebra and coordinate geometry. Its environment has 2D and 3D graphing capabilities for transformations, conic sections, vectors, slope and derivatives (Tarmazi, Mohd, Bakar & Yunus, 2008). In real time, users can observe and visualize functions, equations and calculations. Tarmazi et al. (2008) further note that Autograph can be used for drawing statistical graphs, functions and vectors for transforming shapes. It allows users to change and animate graphs, shapes or vectors already plotted to facilitate the understanding of concepts. In this study, Autography assisted learners in plotting hyperbolic graphs more quickly and accurately.

It allowed learners to identify the key features of hyperbolic functions and visualise the effect of parameters in the general equation of hyperbolic functions.

Data collected from the learners' profiles revealed that the majority of the learners had never used a computer before. It was therefore imperative that before the commencement of the intervention lessons, learners had to be familiarized with the computer keyboard and the Autograph software itself. The first two sessions of the intervention lessons were devoted to basic computer literacy, learning how to use the computer keyboard and plotting graphs using Autography. Initially the sessions were scheduled to take one hour each, but out of excitement for using computers for the first time, most of the learners ended up voluntarily extending the time in the computer lab. During these introductory sessions, the learners were basically taken through the following steps, which were adapted from the Autograph manual:

Step 1: How to launch Autography

To launch Autograph, double-click on the desktop Autograph icon:



Or you can launch Autograph from:

Start → *Program* → *Autograph 3.20*

Step 2: The level selector

Autograph can be used at two levels, that is, the *Standard* and *Advanced* levels. The user would first be required to select the level from the two options as indicated below



The Standard Level – designed for ages 11 – 16 years. The interface is simpler, with larger icons and you cannot move out degrees into radians.

Advanced Level – the interface includes many more options for the discerning user, including calculus, probability distributions and equations in 3D.

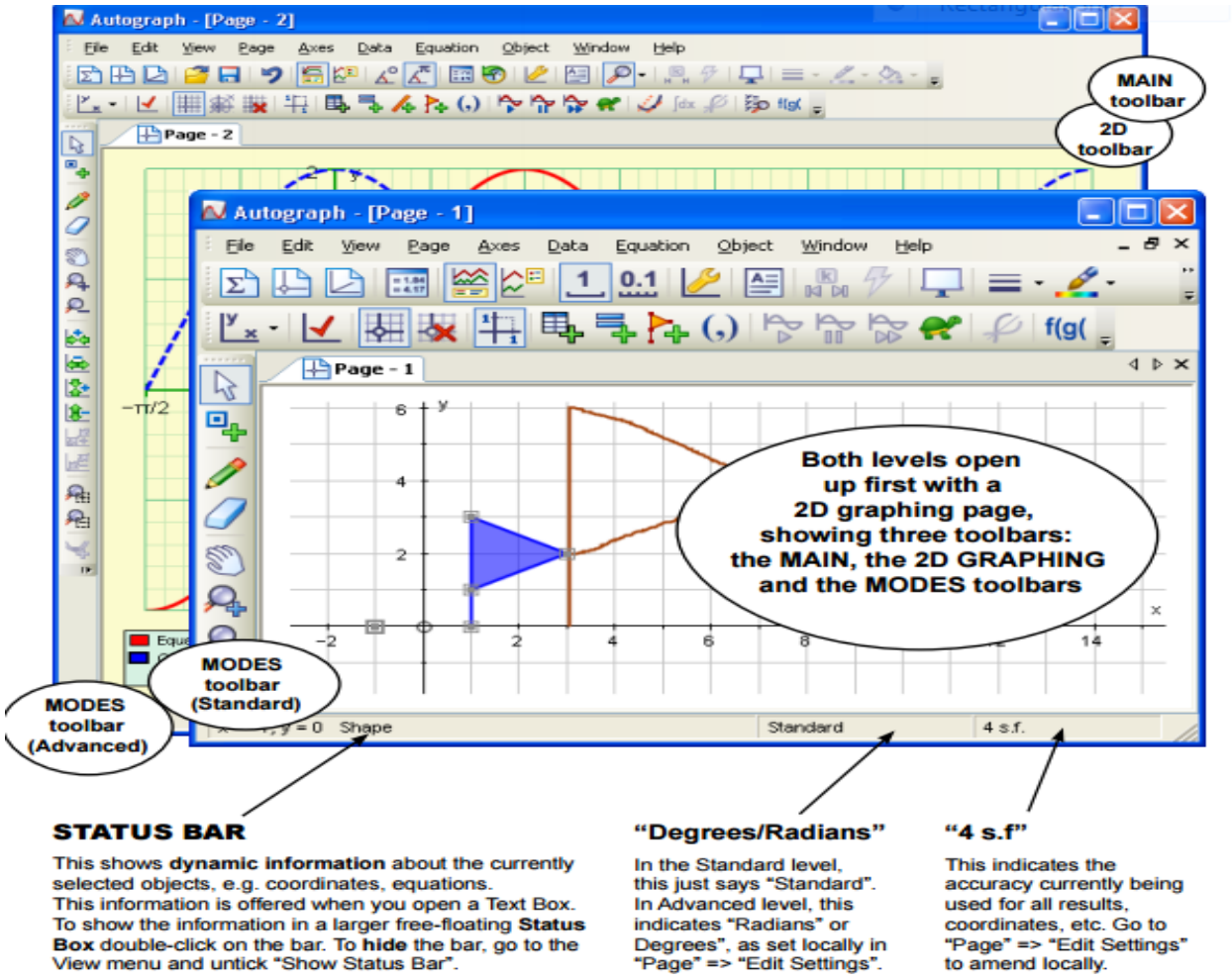
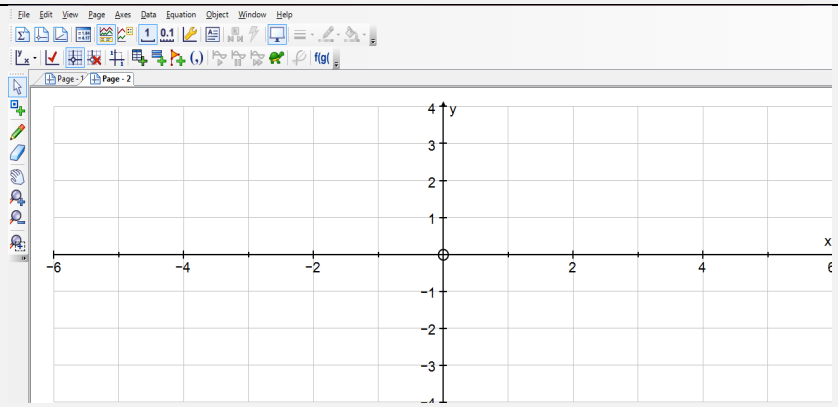
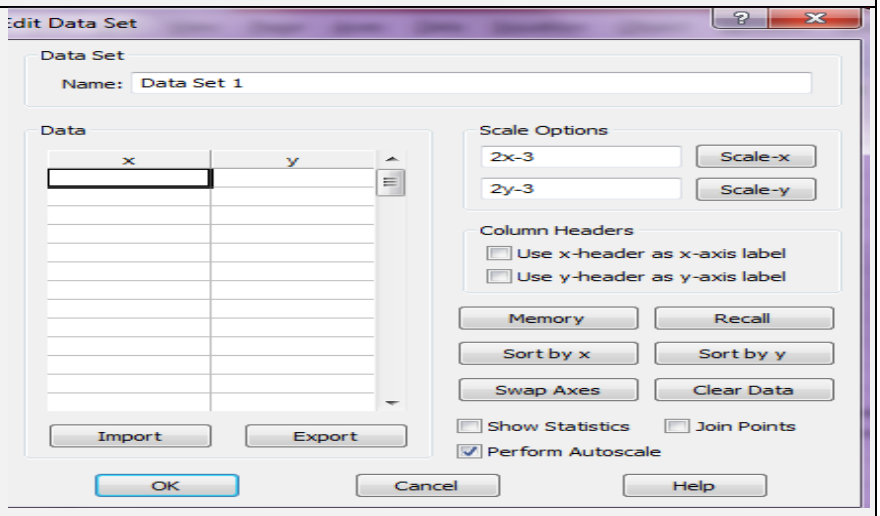


Figure 5. 2. The main toolbar for both standard and advanced levels

For purposes of the intervention lessons, learners were selecting the *standard* level option. This option was chosen because it is user-friendly and its toolbar is less confusing, especially for beginners.

Step 3: Plotting points and drawing graphs

Autograph allows one to plot points on the Cartesian plane and draw graphs of the form $y = f(x)$. This was an important skill, which learners had to have because it formed an integral part of the lessons. Table 5.3 below shows the instructions followed by the learners in plotting points and drawing graphs.

INSTRUCTION	AUTOGRAPH DISPLAY
a) Open the 2D page with 2D toolbar and the standard x – y axes	
b) Click the icon  ENTER XY DATA SET to a data set for the function $f(x) = 2x + 1$. Then press OK	

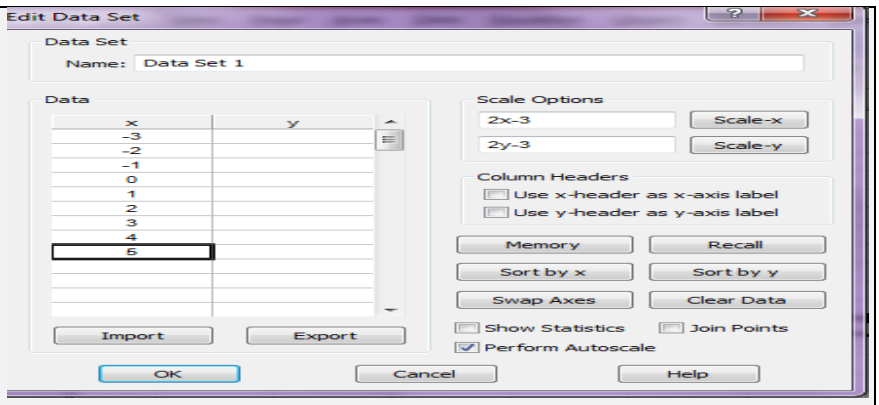
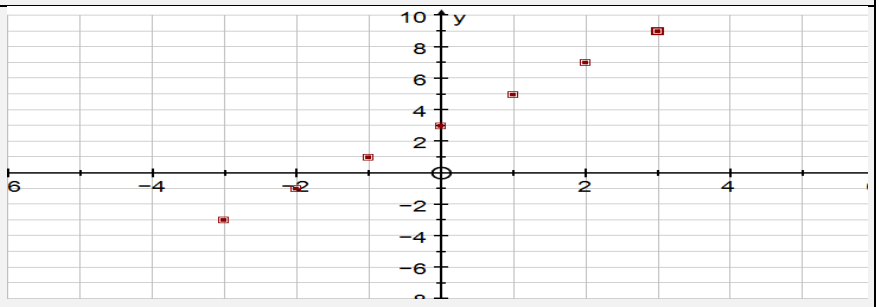
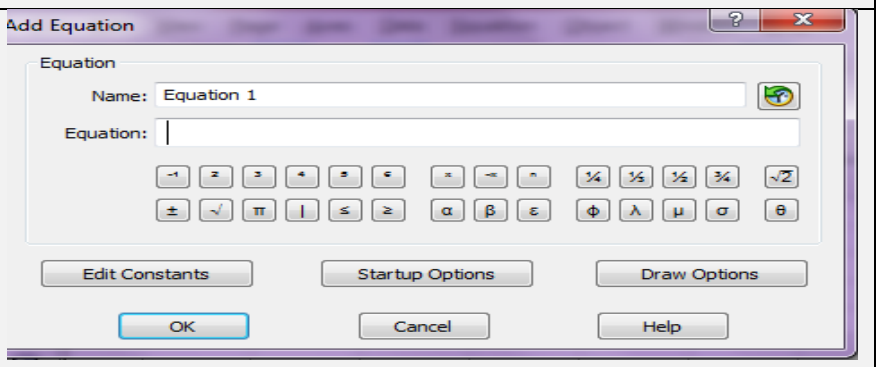
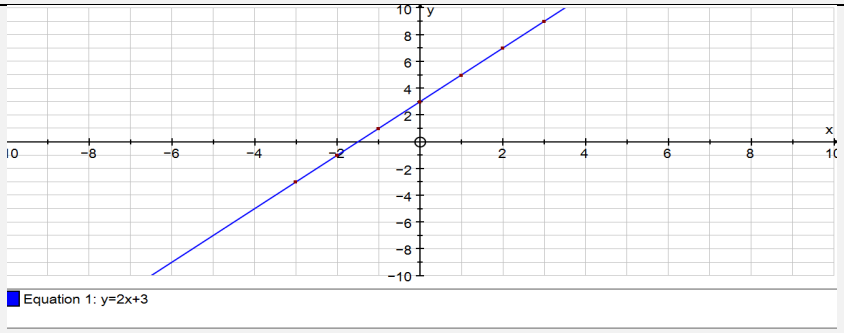
c) ENTER the point manually. Click on the first cell, enter the first item, and then press ENTER, and so on.	
d) The plotted points will be displayed as shown.	
e) Entering equation – Click the icon  ENTER EQUATION to create an equation. Then click OK	
f) From the equation entered, you will get this graph displayed	

Table 5. 3. Instructions on plotting points and drawing graphs using Autograph

Step 4: Changing the scale for axes.

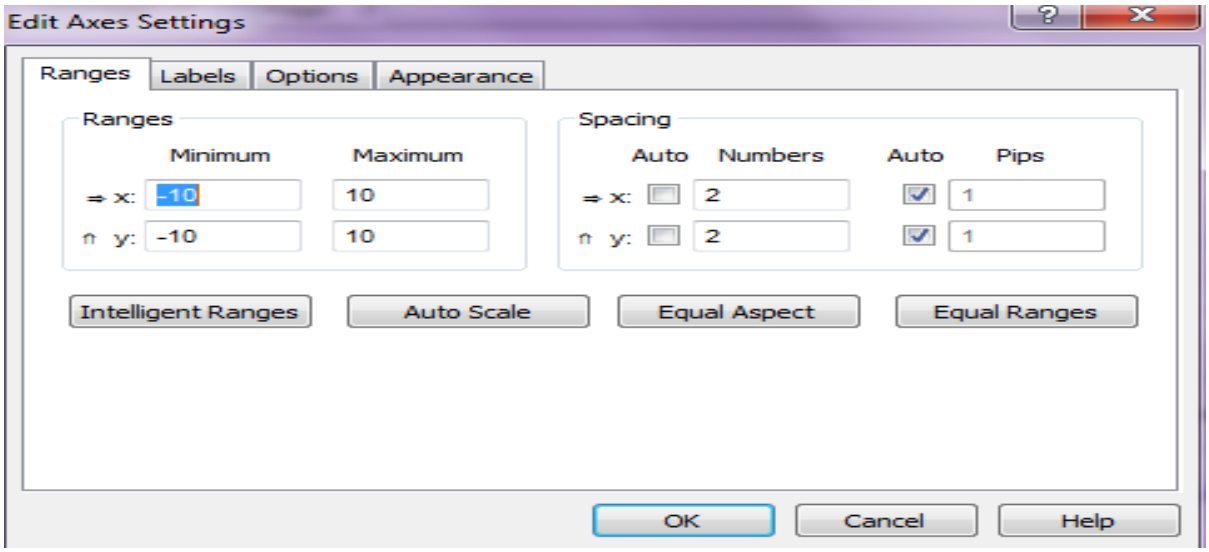


Figure 5. 3. The Autography dialogue box for changing scale for the axes

Step 5: Saving your work

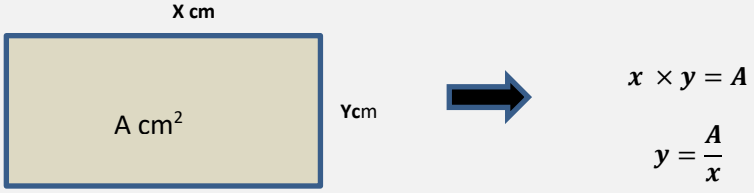
These introductory lessons provided step-by-step activities that the learners needed to undergo with intermittent guidance from the teacher. The learners were given a number of activities and exercises involving the plotting of points and drawing a variety of graphs using the Autograph software. Examples of graphs were drawn mainly from linear and quadratic functions which the learners had already covered in class. As one would expect, the learners struggled a lot at first but with more and more practice, they improved noticeably. At the end of each session, some of the learners requested to remain in the computer lab for further practice on their own. By the end of Day 2, the researcher

was left with no doubt that all the learners had acquired sufficient competency levels in using Autograph to embark on the actual intervention lessons.

5.10.2.2. The Intervention lessons

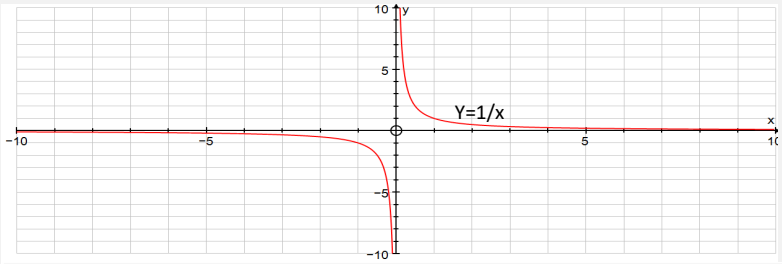
- ***Lesson 1***

To give learners an idea of how hyperbolic equations come about, inversely proportional relations were used. Learners were asked to consider a rectangle with length $x\ cm$, breadth $y\ cm$ and area $A\ cm^2$. The learners could then form the equation: $xy = A$, which could then be written as $y = \frac{A}{x}$. The outline of activities in lesson 1 is given in the Table 5.4 below.

1. TOPIC : Hyperbolic Functions **DURATION :** 1 hr	
2. OBJECT: • Key features of a hyperbolic function • The effect of a on $f(x) = \frac{a}{x}$	
3. MEDIATING ARTIFACTS (TOOLS)	Laptops; Overhead projector; Textbooks; Notebooks
4. RULES	• Each learner to use the computer allocated to him or her. • Learners to follow instructions and should not use computers for other purposes besides what they are asked to. • If a learner is stuck or needs assistance during the lesson, he/she should raise up his/her hand.
5. COMMUNITY	• Learners • Teacher • Lab assistance
6. DIVISION OF LABOUR:	
TEACHER AND LEARNER ACTIVITIES	
Introduction Learners asked to consider a rectangle with length *x cm,* breadth *y cm* and area *A* cm^2 . They form an equation in *x*, *y* and *A*, and make y the subject.  $x \times y = A$ $y = \frac{A}{x}$	

Drawing the graph of : $y = \frac{1}{x}$

Teacher demonstrates on the overhead projector how to draw the graph of $y = \frac{1}{x}$ using Autograph. Learners use Autography to draw the graph on their computers. Learners discuss in pairs and write down key features of the graph in their notebooks. Learners also asked to describe what happens to the graph as *x* nears 0 and as *y* nears 0.



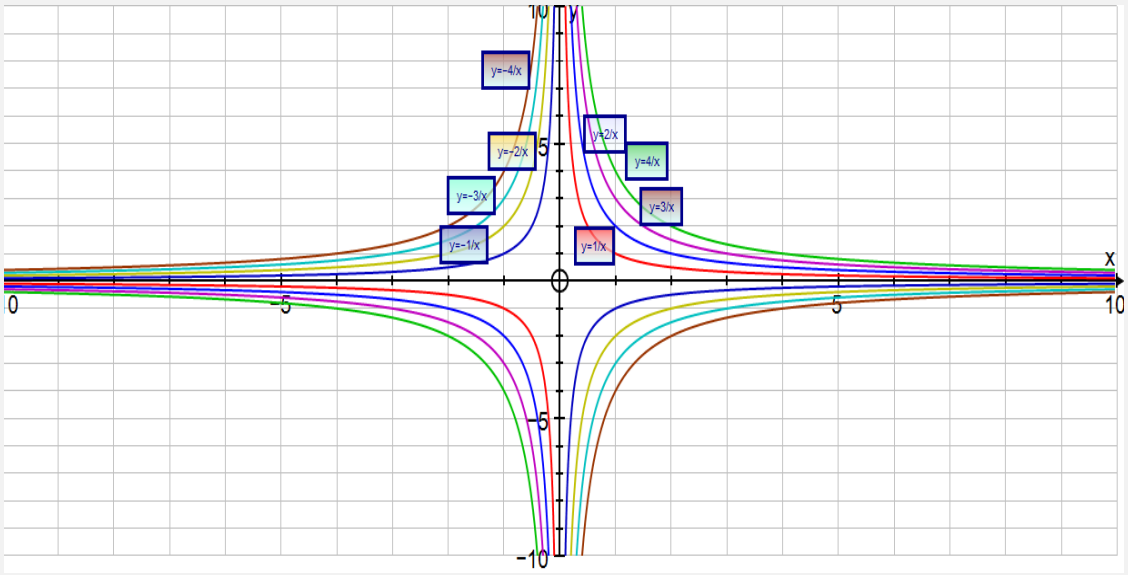
Key features:
-Has a horizontal and vertical asymptote.
-Is made up two symmetric curves in two opposite quadrants.
-has two axes of symmetry.
- Doxmain $x \in R, x \neq 0$

- Range: $y \in R, y \neq 0$

The effect of *a* on $y = \frac{a}{x}$

- Teacher instructs learners to use Autograph and draw graphs for the following hyperbolic functions on the same set of axes:

$y_1 = \frac{1}{x}$; $y_2 = \frac{2}{x}$; $y_3 = \frac{3}{x}$; $y_4 = \frac{-1}{x}$; $y_5 = \frac{-2}{x}$; $y_6 = \frac{-3}{x}$



- In groups of 3, learners use their graphs to explain the effect of *a* on the shape and quadrants of the graph. They answer the following questions:

i) For a ≥ 0, f(x) lies in …… and …… quadrants.

ii) For a < 0, f(x) lies in ….. and ….. quadrants.

iii) As a increases, f(x) tends to ……..

iv) *Discuss the effect of a on the graph of* $y = \frac{a}{x}$

The graph animation

- Using the constant controller and the animation controller, teacher demonstrates graph animation on the projector.
- Learners use graph animation on their computers to visualise the effect of a on the graph of $y = \frac{a}{x}$

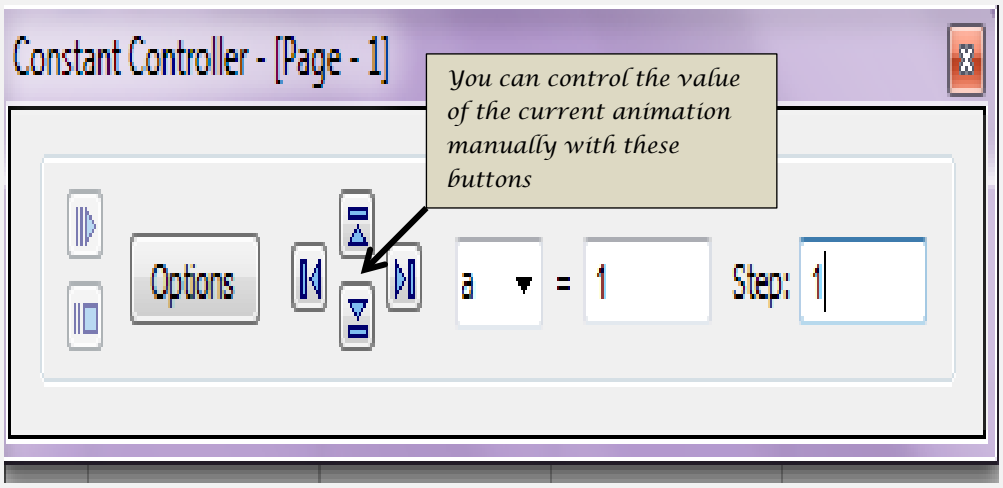


Table 5. 4. Outline of learning activities for lesson 1

- ***Lesson 2***

The central object of lesson 2 was for learners to understand the effect of the parameter $\boldsymbol{q}$ on $f(x) = \frac{a}{x} + q$ and be able to sketch the graph of the function from its key features. An Analysis of the learners' written responses in the pre-test exposed several shortcomings and misconceptions learners have in dealing with translation of functions. Translation is a mapping that changes the position of points by sliding them to other positions. What seemed challenging to learners was to visualise each point of the original function moving the same distance parallel to the y-axis and why the vertical asymptote remained unchanged. The outline of learning activities for the lesson is shown in Table 5.5 below.

TOPIC : Hyperbolic Functions **DURATION:** 1 hr

OBJECT:

- The effect of q on f(x) $= \frac{a}{x} + q$
- Features of f(x) $= \frac{a}{x} + q$ and sketching the graph.

MEDIATIONAL TOOLS	Computers; Overhead projector; Textbooks; Notebooks Whiteboard; Whiteboard markers.
RULES:	• Each learner to use the computer allocated to him or her. • Learners to follow instructions and should not use computers for other purposes besides what they are asked to. • If a learner is stuck or needs assistance during the lesson, he/she should raise up his/her hand.

DIVISION OF LABOUR:

Teacher and Learner Activities

Introduction:

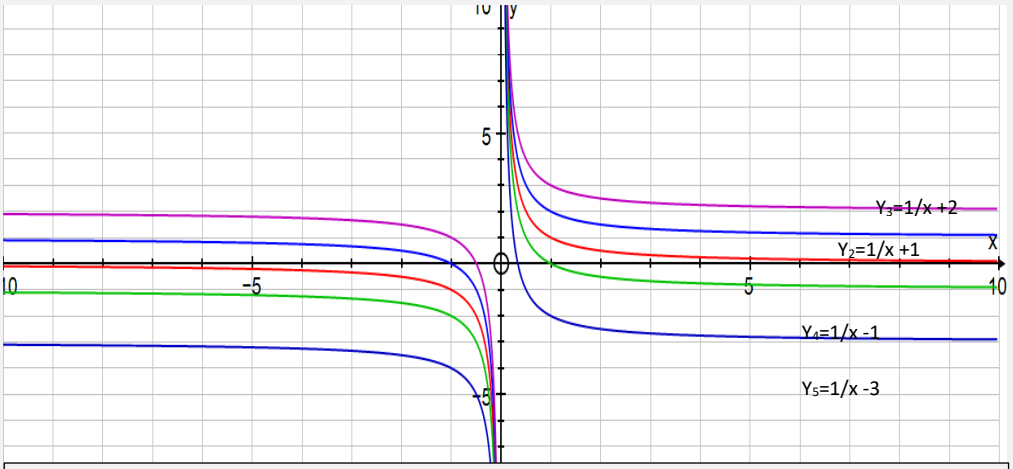
Recap questions on the key features of hyperbolic functions and the effect of a on the graph of f(x) $= \frac{a}{x}$

Drawing graphs of the form : $y = \frac{a}{x} + q$

Learners are instructed to use Autograph to draw the following graphs on the same set of axes:

$y_1 = \frac{1}{x}$; $y_2 = \frac{1}{x} + 1$; $y_3 = \frac{1}{x} + 2$; $y_4 = \frac{1}{x} - 1$; $y_5 = \frac{1}{x} - 3$

Autograph display:- The graphs would appear as shown below on Autograph

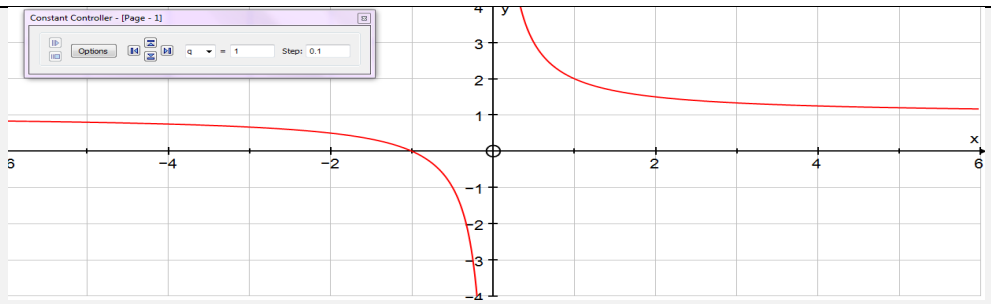


Learners use the graphs above to complete the table below:

Equation	*Vertical asymptote*	*Horizontal asymptote*	*Domain*	*Range*
$y_1 = \frac{1}{x}$				
$y_2 = \frac{1}{x} + 1$				
; $y_3 = \frac{1}{x} + 2$				
; $y_4 = \frac{1}{x} - 1$				
$y_5 = \frac{1}{x} - 3$				

Graph animation

- Using the constant controller and the animation controller, teacher demonstrates graph animation on the projector.
- Learners use graph animation on their computers to visualise the effect of q on the graph of $y = \frac{a}{x}$ +q.



Effect of q on f(x) = $\frac{a}{x} + q$

Ask learners to use their graphs to discuss the effect of changing the value of q on the shape and features of the graph. They complete the following statements:

v) For $q > 0$ $f(x)$ .....

vi) For $q < 0$, $f(x)$ .....

vii) If f(x) is translated q units up or down, how does this transform the equation of $f(x)$?

Table 5. 5. Outline of lesson 2

- ***Lesson 3***

Activities for lesson 3 were similar to those of lesson 2, but the object of lesson 3 was on the effect of the parameter p on $f(x) = \frac{a}{x+p} + q$. In the pre-test, learners had some challenges in dealing with transformations of the form $f(x \pm p)$. Firstly, they could not decode the functional notation itself. Secondly, they struggled to visualise the horizontal translation of the function and its effect on key features like asymptotes, domain and range. Research has also confirmed that the effect of parameter p presents some cognitive obstacles to learners in the sense that adding p shifts the graph in the negative direction and when p is negative the graph shifts in the positive direction (Eisenburg & Dreifus, 1994; Zazkis, Liljedahl & Gadonsky, 2003). The teaching and learning activities for lesson 3 are provided in Table 5.6 below.

TOPIC : Hyperbolic Functions **DURATION**: 1 hr

OBJECT:

- Effect of p on f(x) = $\frac{a}{x-p} + q$
- Identify the characteristics of f(x) = $\frac{a}{x-p} + q$ and sketch the graph.

MEDIATION TOOLS	Laptops; Overhead projector; Textbooks; Notebooks Whiteboard.
RULES	• Each learner to use the computer allocated to him or her. • Learners to follow instructions and should not use computers for other purposes besides what they are asked to. • If a learner is stuck or needs assistance during the lesson, he/she should raise up his/her hand.

DIVISION OF LABOUR

TEACHER AND LEARNER ACTIVITIES

Introduction

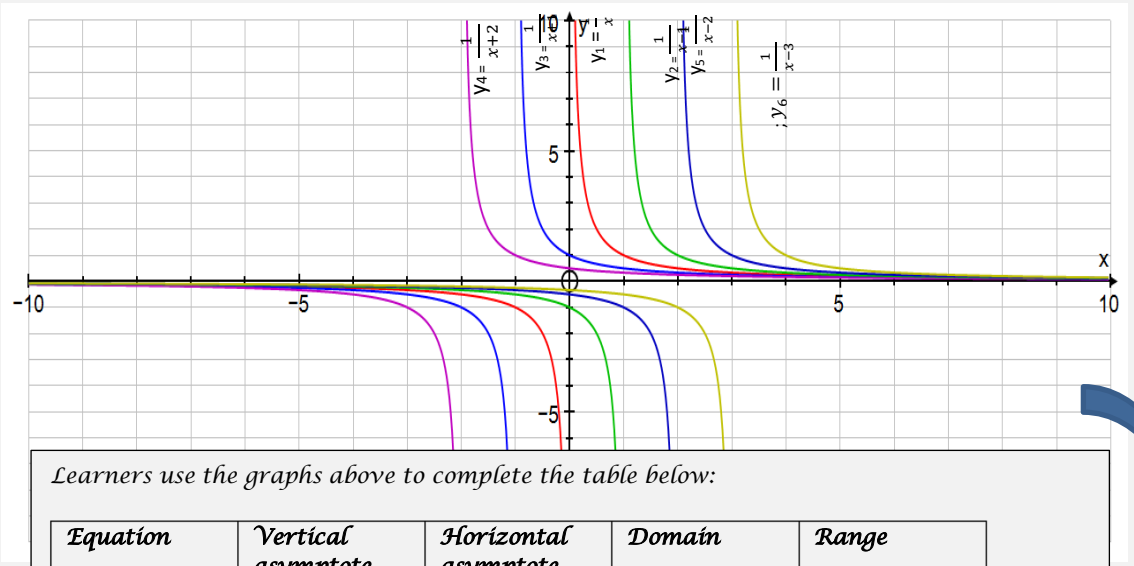
Teacher asks recap questions on the effects of a and q on the graph of f(x) = $\frac{a}{x} + q$

Drawing the graph of : y = $\frac{a}{x-p}$

Teacher instructs learners to use Autograph to draw the following graphs on the same set of axes:

- $y_1 = \frac{1}{x}$; $y_2 = \frac{1}{x-1}$; $y_3 = \frac{1}{x+1}$; $y_4 = \frac{1}{x+2}$; $y_5 = \frac{1}{x-2}$; $y_6 = \frac{1}{x-3}$

Autograph display:



Learners use the graphs above to complete the table below:

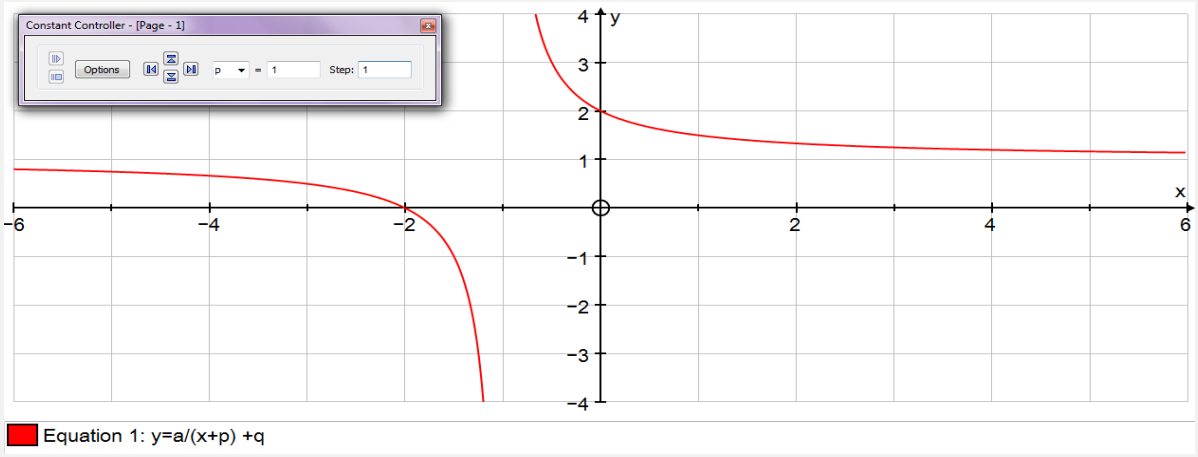
Equation	*Vertical asymptote*	*Horizontal asymptote*	*Domain*	*Range*
$y_1 = \frac{1}{x}$				
$y_2 = \frac{1}{x} + 1$				
; $y_3 = \frac{1}{x} + 2$				
; $y_4 = \frac{1}{x} - 1$				
$y_5 = \frac{1}{x} - 3$				

Effect of parameter p

Graph Animation

Using the constant controller and the animation controller, learners visualise the movement of the graph as the value of ***p*** changes.

Autograph display:



As learners view the animation on their computers, they complete the following statements:

i) A change in ***p*** causes a …… shift

ii) If the value of ***p*** increases, the vertical asymptote and the whole graph ……

iii) If the value of ***p*** changes, then the …… asymptote of the hyperbola will shift.

iv) If the value of ***p*** decreases, the graph …… and the vertical asymptote …….

v) In your owns words describe the effect of ***p*** on the graph of $f(x) = \frac{a}{x+p} + q$

Table 5. 6. Outline of lesson 3

- ***Lesson 4***

The activities for lesson 4 were designed to assist learners develop a better conceptual understanding of the intercepts of a function. In the analysis of learners' errors and misconceptions, it emerged that learners make several errors when calculating the x-and y-intercepts. The errors were basically procedural in nature, but further probing revealed that they emanated from lack of understanding of the procedures. A graphic illustration of this was when learners failed to justify why they substituted $y = 0$ for the x-intercept and $x = 0$ for the y-intercept. Given a hyperbolic graph, some learners even failed to finger-point at the x-and y-intercept on the graph. This simply demonstrated that learners

had memorised the procedure for calculating intercepts, but had not understood what an intercept is.

The lesson activities are shown in Table 5.7 below.

TOPIC : Hyperbolic Functions **DURATION:** 1 hr	
OBJECT:	• Identifying x- and y-intercepts on a graph • Determining coordinates of x-and y-intercepts algebraically given the function $f(x) = \frac{a}{x+p} + q$
MEDIATING TOOLS	Computers; Overhead projector; Notebooks; Markers Whiteboard.
PRIOR KNOWLEDGE	• Learners have done x- and y-intercepts on quadratic functions
RULES	• Each learner to use the computer allocated to him or her. • Learners to follow instructions and should not use computers for other purposes besides what they are asked to. • If a learner is stuck or needs assistance during the lesson, he/she should raise up his/her hand.
DIVISION OF LABOUR:	
Teacher and Learner Activities:	

Introduction

Question and answer on the meaning of x- and y-intercepts and how we calculate them.

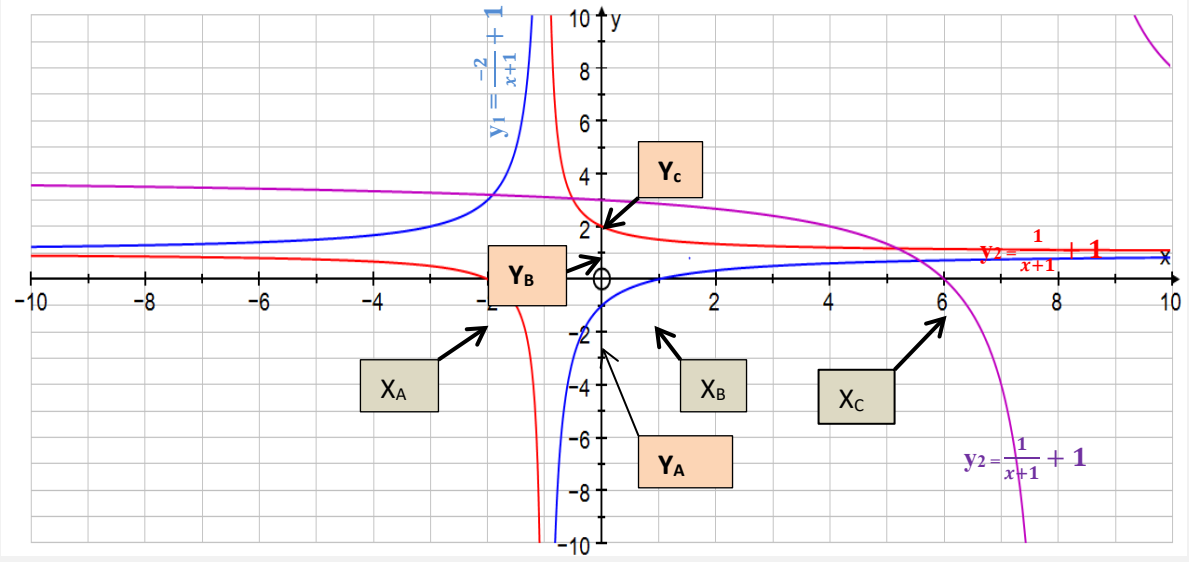
Identifying the x-and y-intercepts from the graph of : $\mathbf{y = \frac{a}{x-p} + q}$

- Teacher instructs learners to use Autograph to draw the following graphs on their computers on the same set of axes.

  $y_1 = \frac{-2}{x+1} + 1$; $y_2 = \frac{1}{x+1} + 1$; $y_3 = \frac{8}{x-8} + 4$.

- For each of the graphs drawn, learners use the Autograph **point mode** to label the x- and y- intercepts. X-intercepts were labelled X_A, X_B and X_C and similarly, y-intercepts were labelled Y_A, Y_B and Y_C respectively.

Autograph display:



Learners used the graphs above to complete the table below:

EQUATION	X-INTERCEPT	Y-INTERCEPT
$y_1 = \frac{-2}{x+1} + 1$	X_A (.... ; ....)	Y_A (... ; ...)
$y_2 = \frac{1}{x+1} + 1$	X_B (..... ; ....)	Y_B (... ; ...)
$y_3 = \frac{8}{x-8} + 4$.	X_C (... ; ...)	Y_C (... ; ...)

What do you notice?

Determining x-and y-intercept through calculation

- From the preceding activity, teacher stressed the observation that for x-intercept the value of y is always 0 and for y-intercept, x is always 0. Teacher then used the example $g(x) = \frac{2}{x+1} + 2$ to demonstrate how x- and y-intercepts can be determined by calculation.
- Learners were then asked to determine by calculation the x- and y-intercepts for the three functions used in the previous function. They compared their answers with those obtained from the graph.

Table 5. 7. Outline of lesson 4

- ***Lesson 5 and 6***

Activities for lesson 5 were designed to facilitate learners' understanding of the concepts asymptotes, domain and range. Learners' responses in the pre-test and data collected from interviews revealed that learners had a mechanistic understanding of these concepts. Given the symbolic equation of a hyperbolic function, learners could easily give the equations of asymptotes, the domain and the range. However, further probing showed that most of the learners had no understanding of what an asymptote is, and how it is connected to the domain and range.

Some learners defined an asymptote as 'a dotted line that divides the graph', while others interchanged asymptotes and the x- and y-axes. This conceptual shortcoming limited learners' ability in tacking higher order problems. The approach used during this lesson was mainly guided discovery. The Autograph software assisted learners in analysing the behaviour of the given function as x and y approach infinity. The key *object* of the lesson activities was to enable learners to identify the vertical and horizontal asymptotes of hyperbolic functions, first from the graph of the function and from its algebraic equation. The lesson activities are given in Table 5.8 below.

TOPIC : Hyperbolic Functions	**DURATION:** 1 hr
OBJECT:	• Defining horizontal and vertical asymptote • Identifying the vertical and horizontal asymptote from the graph of: $f(x) = \frac{a}{x+p} + q$. • Connect asymptotes to domain and range.
MEDIATING TOOLS	Computers; Overhead projector; Notebooks; Markers Whiteboard.
PRIOR KNOWLEDGE	• Learners can extract the equation of asymptotes from the algebraic equation: $y = \frac{a}{x+p} + q$ • Learners met vertical asymptotes under tangent functions.
RULES	• Each learner to use the computer allocated to him or her. • Learners to follow instructions and should not use computers for other purposes besides what they are asked to. • If a learner is stuck or needs assistance during the lesson, he/she should raise up his/her hand.
DIVISION OF LABOUR:	

Teacher and Learner Activities:

Introduction

Recap questions on functions with asymptotes and how we determine the asymptotes.

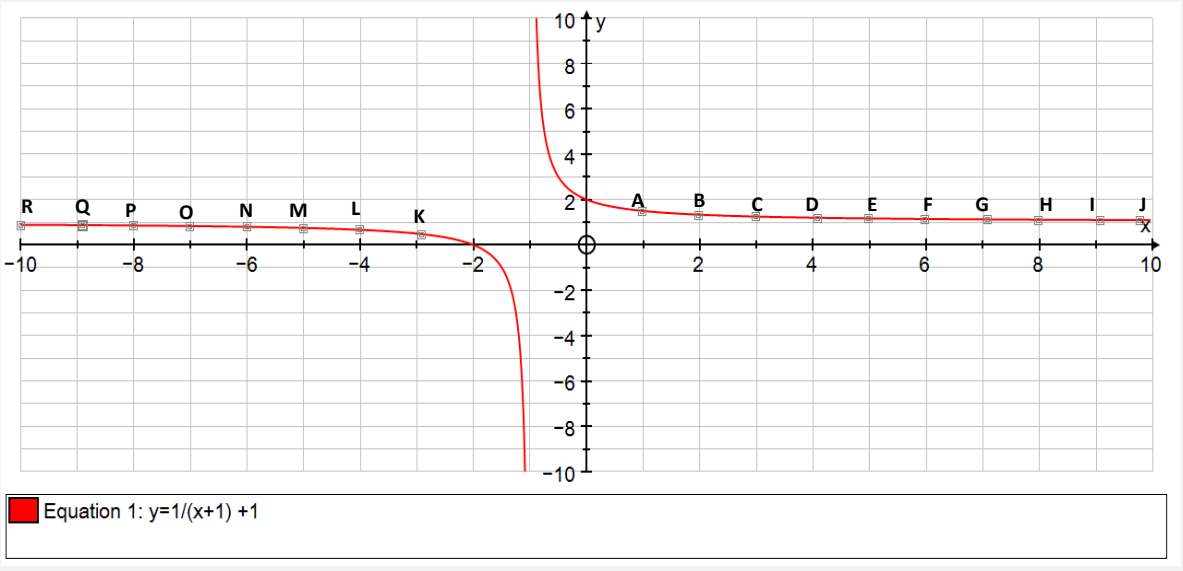
Identifying the horizontal asymptote from the graph of : $\mathbf{y = \frac{a}{x-p} + q}$

- Teacher instructed learners to use Autograph to draw the graph of the function given below:

  $f(x) = \frac{1}{x+1} + 1$

- Learners independently described the *end behaviour* of $f(x)$ as x approaches ∞ or -∞ and shared their responses.
- To assist learners in investigating this, learners were instructed to use the *point mode*  to mark points **A** to **J** and K to **R**, as illustrated in the Autograph display below.

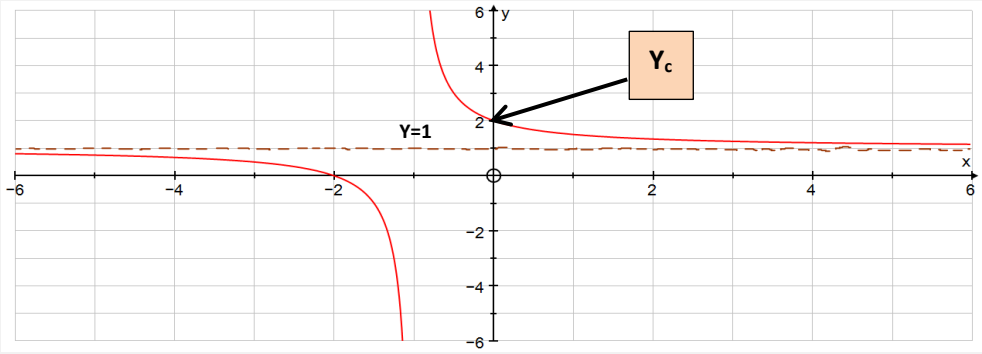
Autograph display:



- Using the *point mode*, learners completed the table below:

Behaviour of f as $x \to \infty$	**Behaviour of f as $x \to -\infty$**
A (… ; …)	K (… ; …)
B (…; …)	M (…; …)
C (… ;…)	N(… ;…)
D(… ; …)	O(… ; …)
E(… ; …)	P(… ; …)
F(… ; …)	Q(… ; …)
G(… ; …)	R(… ; …)
H(… ; …)	S(… ; …)
I(… ; …)	
J(… ; …)	

- Teacher then reiterated that when $f(x)$ approaches a particular number, L, as x approaches ∞ or $-\infty$, then the line $y = L$ is called *a horizontal asymptote* on the graph of f. In this example, the horizontal asymptote was $y = 1$.
- Using the Autograph *scribble* , learners marked and labelled the horizontal asymptote on their graphs



How is the range (output values) of f related to the graph of $y = f(x)$?

- Learners were given an opportunity to reflect on the following questions in pairs before reporting to the whole class.
  - i) Does the graph of $y = f(x)$ cross the line $y = 1$?
  - ii) Is there a value of x for which $f(x) = 1$? Show by calculation.
  - iii) How would you define a horizontal asymptote in your own words?

Table 5. 8. Outline of learning activities for lessons 5 and 6

All the lessons were video-taped and observations were done as outlined in Section 5.5.3. After the intervention lessons, the post-test was administered under the same controlled conditions as the pre-test. This was followed by another session of task-based interviews. The main focus was on trying to determine if there was any significant change in learners' level of conception of the concepts covered.

The task-based interviews were based on question items as learners explained and clarified their written responses. The interviews were audio-taped and later transcribed for analysis.

During the interviews, I also recorded any non-verbal communication made by the learners as they gave their oral responses. These were recorded in my reflexive journal. Data collected from the document analysis of learners' written responses in the test were triangulated with task-based interview and observation data.

5.6.3. Phase 3

The final phase of the data collection process was mainly targeted towards addressing the third research question of this study. The learners were subjected to open-ended interview questions to share their experiences and perceptions about the whole intervention programme. Firstly, the interviews sought to find out what learners found helpful to them in terms of their interaction with other learners, the teacher, the computer tool and the content taught.

Secondly, learners were expected to share some of the challenges they encountered which might have hindered their understanding of the content taught. Thirdly, they were given room to share their suggestions on what could have been done in another way to make the lessons more beneficial to them. Data collected from the open-ended interviews was triangulated and verified by the video-taped lessons and my own personal observations as a participant observer.

As already explained earlier, an atmosphere was created which allowed learners to talk about their experiences during the whole intervention programme. The learners were questioned and probed on what exactly they found helpful during the lessons and what did not work for them. It was critical to obtain honest opinion from the learners, hence the need to ensure that the interviews were done in a friendly and non-threatening environment. What worked in my favour was that the interviews were done after several weeks interacting with the learners. I believed that they had developed trust and faith in me as their teacher. The interview data was transcribed and inductively analysed to come up

with affordances and constraints that constituted components of the ICT-mediated instructional model that this study is proposing.

5.11. Data Analysis

Qualitative analysis, as Bogdan and Bikhen (2003) defines it, involves sorting the data, organising them, classifying them into manageable subgroups, coding them, synthesising them and searching for sub-themes, themes and patterns. They argue that the central focus of analysis of qualitative data is to identify patterns, themes and meanings. To achieve this, Bogdan and Bikhen (2003) suggest that the initial stage in data analysis would be the categorisation and organisation of the data. At this stage, data is sorted and organised into file folders according to their source, that is, interview data, observation data and document data.

Data collected in word form through interviewing learners will be transcribed for analysis. The next level in data analysis would be exploring the general sense of the data. As Creswell (2012) puts it, this process involves 'memoing ideas, thinking about organisation of data and considering whether you need more data' (p.243). Agar (1980) suggested that one needs to read the transcripts in their entirety several times and get immersed into the details of the data collected.

In the context of this study, data analysis had to respond to the three research questions that the research sought to answer. The analysis of data was done and presented in three phases. The initial phase involved the identification of errors and detection of misconceptions from the learners' written responses in the pre-test and their oral responses during task-based interviews.

The categorisation of errors was informed by the error categorisation protocol (see Section 3.2) adapted from error types identified by other researchers. During error analysis, other errors emerged which did not fit into the pre-determined categories. As a result new error categories were developed. Error diagnosis and detection of misconceptions was made possible through the analysis of task-based interview data. Learners were asked to explain their responses and probed until the root cause of the error was established. However, this could not be done with absolute degree of certainty because

misconceptions are internal cognitive structures. We can only infer from tracing the learners' thinking patterns based on what they say.

The second phase of data analysis sought to establish the impact of the ICT-mediated instructional intervention on learners' conceptual understanding. It involved a comparative analysis of learners' written and oral responses before and after the intervention programme. The central objective at this stage was to find out if there was any qualitative change in learners' level of conception of hyperbolic functions. The level of conception (or understanding) was explained in terms of Dubinsky's mental constructions of *Action*, *Process*, *Object* and *Schema* (APOS) conception (see Conceptual framework of this study). The deepening of learners' conception of hyperbolic functions was explained in terms of mental processes of *interiorization*, *encapsulation* and *thematization*.

The third and final phase of data analysis was the inductive analysis of the open-ended interview data. Grounded on learners' experiences and perspectives, the analysis sought to establish ICT affordances and constraints that constituted the ICT-mediated instructional model. To facilitate this process, the interview data was transcribed and coded.

Coding is the process of segmenting and labelling text to form descriptions and broad themes in the data (Strauss & Corbin, 1990). This stage allowed the researcher to identify and tentatively name the conceptual categories into which the data would be grouped. The identification of themes and meaning-making are the ultimate goals in qualitative inquiry. The data analysis process adopted in this study is presented in Figure 5.4 below.

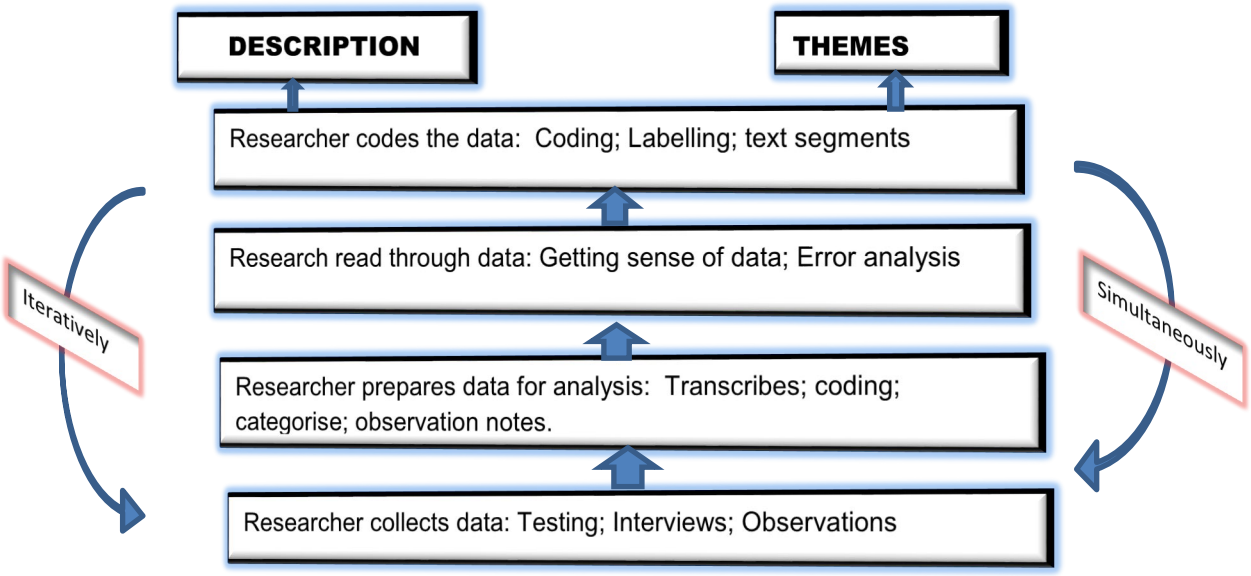


Figure 5. 4. Data analysis framework (Adapted from Creswell, 2012, p.237)

What stands out clearly from the framework above is that data collection and analysis processes were iterative and inductive in nature. Monly (1978) explains inductive reasoning as the study of a number of individual cases leading to a hypothesis and eventually to a generalisation. In the context of this investigation, it involved moving from analysis of the detailed data (written responses, transcribed interview data, notes from observation guides) to the general codes and themes which constituted the ICT-mediated pedagogical model that this study is proposing. It was iterative in the sense that the researcher had to 'cycle back and forth between data collection and analysis' (Creswell, 2012, p.238).The themes can be viewed as similar codes aggregated together to form a major idea (Lofland & Lofland, 1996). They form the core element in qualitative data analysis. The identification of emerging themes, featuring the actual words and live experiences of participants, is an important result of qualitative data analysis that provides richness to the findings and their meaning.

It is however critical to note that the themes and meanings emerging from this study are subjective as they depend on the researcher's own interpretation and perspective of the collected data. This point leads to issues of trustworthiness of the study discussed below.

5.12. Trustworthiness of the Study

One other distinguishing feature of qualitative research is the criteria used to ensure the credibility of research data. Several studies have demonstrated that qualitative researchers have attempted to respond to the issues of reliability and validity in their own qualitative manner (Morrow, 2005; Rolfe, 2004; Shenton, 2004; Guba, 1981). Shenton (2004), for example, noted that qualitative researchers prefer to use different terminology to distinguish themselves from the positivist paradigm.

Guba (1981) came up with four constructs that qualitative researchers need to consider in their pursuit of trustworthiness, namely, *credibility (*internal validity), *transferability* (external validity / generalizability), *dependability* (reliability) and *confirmability* (objectivity). I discuss below the trustworthiness of this study based on Guba's constructs.

5.12.1. Credibility

Credibility in qualitative research is viewed as the extent to which research data and findings are congruent with reality (Lincon & Guba, 1985; Shenton, 2004; Meriam, 1998). Apparently, this is problematic because according to the interpretivist paradigm, reality is fluid and relative to meaning that people construct within social contexts. However, some concrete ways were used to improve the credibility of the findings of this study. One such technique is triangulation of data collection methods (Shenton, 2004). The use of observation, task-based and open-ended interviews and document analysis helped to ensure the credibility of the data collected. These data sources gave room for follow-up and iterative questioning, whereby the researcher had to go back to the informants seeking clarity on previously raised matters through rephrased questions (Lincon & Guba, 1985). This was useful to expose contradictions and falsehoods.

This study also enhanced credibility by the use of thick descriptions of the phenomenon under study. Geertz (1983) defined thick descriptions as detailed, rich descriptions not only of participants' experiences of the phenomenon, but also of the contexts in which these experiences occur. These included the exact words spoken by the participants and a true representation of actions observed. It

also encompassed a detailed and graphic description of the context under which data is collected. This allows readers and other researchers to make informed assessment of the authenticity of the data collected.

5.12.2. Transferability

In positivist terms, transferability is analogous to external validity or generalizability which refers to the 'extent to which the reader is able to generalise the findings of a study to his or her context' (Morrow, 2005, p.252). Given the small sample sizes and absence of statistical analysis in qualitative research, its findings cannot be said to be generalizable in the conventional sense. However qualitative researchers argue that although each case may be unique, it is an example within a broader group (Morrow, 2005; Shenton, 2004). This implies that the prospect of transferability holds in qualitative research. It is the responsibility of the investigator to ensure that adequate contextual information about the phenomenon is provided to facilitate transferability. This allows the reader to determine how far he or she can transfer the findings in his or her own context.

5.12.3. Dependability

Dependability is analogous to reliability. From a positivist perspective, quantitative researchers strive to show that if the study were repeated in the same context, with the same participants, similar results would be obtained (Shenton, 2004). This approach is obviously not applicable in qualitative research. As discussed earlier on, qualitative research is premised on the view that human behaviour is dynamic, highly contextual and changes continually depending on how people respond to surrounding factors. The complexity of social reality is compounded by the fact that the quality of inferences depends on the personal constructions of meanings based on the researcher's skill and expertise in gathering and interpreting information (Meriam, 1998).

Qualitative researchers believe that dependability may be addressed by ensuring that processes within the study are reported in detail (Meriam, 1998; Shenton, 2004; Morrow, 2005). This enables future researchers to repeat the work, though not necessarily to obtain the same results. This was achieved in this study by explaining the assumptions and theories backing the study, contextual factors, data sources and a detailed explanation of data collection and analysis.

5.12.4. Confirmability

In positivist terms, confirmability is described as objectivity of a research study. Quantitative researchers claim that objectivity is achieved using research instruments that are independent of human skill or perception (Patton, 1990). However, this argument is disputable because the tests, questionnaires and interview questions used are designed by humans, implying that the intrusion of human biases is inevitable. For qualitative researchers, the biggest worry is ensuring that the study's findings are a true reflection of the experiences and the views of the informants, rather than the views and preferences of the researcher (Shenton, 2004).

One key criterion to address confirmability is for the researcher to acknowledge and admit his or her predispositions (Meriam, 1998). The researcher has to be transparent about his personal beliefs, strengths and weaknesses. In addition, a detailed and critical methodological description will allow for an 'audit trail' (Shenton, 2004), whereby readers can trace the course of the study step-by-step. In order to facilitate auditing by other researchers, the researcher will archive all collected data in a well organised and retrievable form so that they can made available whenever necessary.

5.13. Ethical Considerations

This being largely a qualitative study, I had to interact deeply with the participants and thus entering their personal space in terms of their opinions, weaknesses and individual learning challenges in order to collect data. Creswell (2003) cautions that qualitative researchers should always realise that while they are doing their research, they are encroaching on the private spaces of their research participants. Thus, as Creswell (2003) suggests, the researcher has an obligation to respect the rights,

needs, values and desires of the informants. In light of this reality, this study adhered to the following ethical considerations: permission of access; informed consent; honesty and trust; privacy, confidentiality and anonymity.

5.13.1. Permission of access

I officially sought approval from the Limpopo Department of Education (LDoE) to carry out the study in this province (see appendix B) and was granted permission (see appendix C). This was done at provincial, circuit and school levels. At school level, the researcher also officially asked for permission from the school governing board (SGB) to use school facilities and other resources for purposes of this study (see appendix D).

5.13.2. Informed consent

The researcher informed and explicitly explained to the learners the purpose and nature of data to be collected and their role during data collection. I made it very clear to learners that the research was for academic purposes only and their participation was absolutely voluntary. Learners' informed consent was also sought to audiotape interviews and videotape intervention lessons (see appendix E). Since the participants were minors, informed consent was also sought from their parents and guardians (see appendix G).

5.13.3. Anonymity and confidentiality

The study involved diagnostic testing and analysis of learners' individual errors and misconceptions and interaction with participants at individual level. Total anonymity may not have been possible. However, learners were assured that strict confidentiality and anonymity will be exercised by removing any identifying characteristics in recording collected data. The researcher use pseudo names

throughout the subsequent analysis and reporting of the collected data. The real name of the school where the study was undertaken and the name of the principal were never disclosed.

5.13.4. Honest and Trust

As a doctorate thesis, I was committed to ensuring that the research findings are credible and will contribute towards the mathematics education body of knowledge. I strictly adhered to all ethical guidelines, ensuring honesty and trustworthiness of the data collected and the subsequent data analysis.

5.14. Pilot Study

A pilot study was carried out using ten grade 11 mathematics learners from a secondary school in one of the neighbouring villages. The learners were ferried to the research site on three Saturdays. To ensure safety and security of the learners, an Isuzu twin-cab was used to transport the learners to the research site. The twin-cab carried five learners per trip and were driven by a mature and experienced driver. The learners wrote the diagnostic test and were taught three of the intervention lessons. This gave the researcher an opportunity to test the dependability of the research instruments.

5.15. Conclusion of the chapter

The study adopted the qualitative case study research approach. This research methodological approach resonates well with the theoretical framework underpinning this study. Basically, knowledge is viewed as a human construction and a result of individual mental processes. In line with this theoretical standpoint, the learners' conception or misconception of concepts under hyperbolic functions could only be understood through a qualitative and in-depth analysis of learners' written and oral responses.

The designing, development and implementation of an instructional intervention programme is highly contextual and could best be explored through a case study approach. Typical of a qualitative case study and driven by the need to ensure credibility of research findings, this study employed multiple data sources, and these include document analysis, interviews and observations. The chapter described the sources of data, sampling of participants, data collection procedure and analysis. Issues relating to trustworthiness of the study and ethical considerations were also unpacked. The next three chapters focus on analysis of collected data.

CHAPTER 6: ANALYSIS OF LEARNERS' ERRORS AND MISCONCEPTIONS AND DISCUSSION OF FINDINGS

6.1. Introduction

This chapter is aimed at addressing the first research question of this study which sought to establish the errors learners made in dealing with hyperbolic functions and inferring on the misconceptions underlying those errors. Drawing from Borasi (1994), the constructive epistemology underlying error analysis as a springboard for inquiry has radical implications for the very notion of 'error'.

In this study, a mathematical error is interpreted in the most comprehensive way possible. The analysis presented in this study was based on learners' written responses to the question items in the pre-test and their oral responses during the task-based interviews held after the pre-test. The analysis approach adopted was blended, that is, it was both deductive and inductive. It was initially deductive in the sense that I approached the data with a protocol for error categorisation developed from similar studies done in other disciplines and other mathematics topics. However, during error analysis and categorisation, some errors emerged which could not be placed in any of the categories in the protocol. These were then inductively placed under new categories. Before delving into the details of the analysis, I briefly discuss deductive and inductive approaches to data analysis and how they were used in this part of the study.

6.2. Deductive and Inductive Analysis

Research data can be analysed either inductively or deductively or through a combination of both approaches. Several researchers seem to agree on the view that vigorous analysis can be achieved through forward and backward movement of deductive and inductive logic until patterns in the data emerge in a way that research questions could be answered (Miles & Huberman, 2004; Meriam, 1992). Furthermore, Miles, Huberman and Saldana (2014) advocate a combination of a deductive and inductive analysis concurrent with data collection. They base their argument on the view that interweaving data collection and data analysis helps the researcher to cycle back and forth between thinking about the existing data.

Deductive analysis is based on a theory and pre-determined assumptions in relation to that theory in the face of empirical data (Miles & Huberman, 2004). Thomas and James (2006) consider deductive analysis as aimed at testing whether data are consistent with prior assumptions or hypothesis identified or constructed by an investigator. What stands out clearly in deductive data analysis is that data is examined and analysed on the basis of a pre-determined or pre-constructed theory, in order to prove or disprove the theory.

This study is built on a conceptual framework (see chapter 3). The learners' errors and misconceptions were deductively analysed based on error categories developed by other researchers (Donaldson, 1963; Hirst, 2003; Makonye, 2010). This deductive coding approach was necessitated by the fact that several other researchers have looked at errors and misconceptions in other related mathematics topics. However, given the diversity of learners and the complexity of contextual factors in learning environments, error categories can never be exhaustive. Thus imbedded in this deductive analysis is inductive coding as other codes progressively emerged during data collection and analysis.

The categorisation of learners' errors and the subsequent analysis was done through the theoretical lens of Dubinsky's APOS theory. This theory enabled the researcher to develop predetermined indicators or criteria that allowed learners to be placed in different levels of conception of the concepts on the basis of their performance in the pre- and post-diagnostic test. A comparative analysis of the learners' conception levels before and after the ICT mediated instructional intervention did put the researcher in a position to make informed conclusions on the impact of the instructional mediation on conceptual understanding of the learners.

Inductive analysis, on the other hand, starts with the observation and examination of events or processes with the aim of reaching more general explanations induced by those events or processes (Miles, Huberman & Saldana, 1994). It refers to the approaches that primarily use in-depth analysis of the raw data to derive themes or a model through interpretation made from raw data (Strauss & Cobin, 1998). What these scholars are indicating is that under inductive analysis, the researcher

begins with a phenomenon or an area of study and develops a model or theory about the underlying structure of experiences and processes that are evident in the data. This approach resonates perfectly well with the third research question of this study.

6.3. Profile of Learner participants

The sample of this research study consisted of thirteen Grade 11 learners who were studying mathematics at the school. In terms of gender, seven of the learners were boys and six were girls. The ages of the learners ranged from sixteen to twenty. All the learners, except for two, came from the three villages surrounding the school and did their primary education at the three feeder schools located in each of the villages. The other two learners transferred to the school from other schools in the same district. Generally, all the learners were from low income households whereby the major source of income was social grants.

With reference to computer competence, all the learners indicated that they had no computers at home and had never had access to or used a computer before. At school, they were afforded a chance to use computers once a week, but the computers were very few and they could not get ample time to acquire any computer skills. Throughout the study, the learners were referred to using pseudonyms. Each participant was asked to create a pseudonym of their choice. Their identity was only known by the researcher. The profile of each of the learner participant, as drawn from the interview information, is described in the subsequent paragraphs.

6.3.1. Learner #1: Oscar

Oscar is a boy of eighteen years old. He lives with his mother and father in the nearby village and both of his parents are not working. Oscar is the third son in a family of six boys and five girls. They survive on a social grant which the father and mother are earning. On computer skills, Oscar indicated that he had never used a computer before. Asked why he never made use of school computers, Oscar said that he was never taught how to use the computers. He indicated that for research, he uses his cell phone.

6.3.2. Learner #2: Pebetsi

Pebetsi is a girl of sixteen years old. She is the last born in a single parent family of five boys and two girls. Her mother is not working, and their major source of income is a social grant. She did her primary education at one of the feeder primary schools. Pebetsi indicated that she never had access to computers at home. The only moment she attempted to use a computer was when they were asked to type a geography project which they had done in groups. Her computer skills were very minimal.

6.3.3. Learner #3: Mmathapelo

Mmathapelo is a girl aged seventeen years old. Her father is a teacher and her mother is a social worker. The family resides in the village where the school is located. She indicated that they have a computer at home, but she was never interested in learning how to use a computer. Even at school, she never visited the computer lab. To her, learning to use a computer was unnecessary and time wasting as she was not studying computers as a subject,

6.3.4. Learner #4: Joel

Joel transferred to the school in question in grade ten from another school in the same district. He is a boy of sixteen years old. His parent is a single mother who is not working, and he is getting social grant. Joel revealed that at the school where he did primary education, they had access to computers, and they did computer lessons once a week. Though he was not very competent in basic computer skills, he had some understanding on how to use a computer.

6.3.5. Learner #5: Mpho

Mpho is a boy aged eighteen years and comes from a family of three girls and five boys. His father is working and staying in Gauteng. One of the challenges he was quick to note was the long distance he travels to school. His resides in a village about six kilometres away from the school. On some days when the scholar transport was not available, he had to walk to school. Mpho repeated grade 10 after

failing to meet the minimum requirements for promotion. He also had no background on the use of computers, both at home and at school.

6.3.6. Learner #6: Gabriel

Gabriel is a boy of sixteen years old and comes from a single parent family. His mother and brother are working. Gabriel admitted that he had good computer skills, especially on Word processing. At home, his brother has a computer that he frequently uses to type documents in Word. Even at school, he is one of the few learners who frequent the computer lab for research and typing.

6.3.7. Learner #7: Precious

Precious is a girl of seventeen years old. Both of her parents are available, but they are not working. She expressed a lot of interest in learning computer skills, but she never had the opportunity to do so. She indicated that at home they have no computer and at school, the time allocated for using computers was very short, especially for beginners. Precious could not hide her excitement to take part in this study as she hoped this would give her an opportunity to learn about computers.

6.3.8. Learner #8: Sekgobela

Sekgobela is the only daughter to a single mother who is working in a nearby town. She is sixteen years old and is staying alone at home with her younger siblings. Her major challenge was that she stayed in a village about five kilometres away from the school, and hence could not attend any extra lessons arranged after normal school lessons or in the morning before lessons start. To accommodate her in this study, special transport arrangements were made for her as the intervention lessons were done late afternoon after normal school hours. When asked if she had any background on basic computer skills, she expressed nervousness as she confessed that her fingers had never touched a computer keyboard.

6.3.9. Learner #9: Kopano

Kopano is a girl of seventeen years old. Her father passed on and she is staying with her mother. The mother is not working, and they are surviving on a social grant. She repeated grade 10 after failing to meet the minimum requirements for promotion. At first, she was very hesitant to participate in the study for the reason that she had never used a computer before. She was very nervous and thought the project was going to add an extra burden to her.

6.3.10. Learner #10: Fogo

Fogo is a girl of sixteen years old. Both her parents passed on and she is staying with her grandmother. When asked if she had ever used a computer before, she simply indicated that she knew nothing about computers. She was, however, very excited about participating in the project as she hoped it would equip her with some computer skills.

6.3.11. Learner #11: Ledwaba

Ledwaba joined the school when he was doing grade 10. He was coming from another public school in another neighbouring district. He is a boy of eighteen years old. Ledwaba indicated that both his father and mother were not available, and he was staying with his aunt. The previous year he had repeated grade 10 after failing to meet promotion requirements. On his competency in basic computer skills, Ledwaba pointed out that he attended some computer lessons at his former school. Although he could not say that he was competent in using a computer, he had some knowledge of basic computer skills.

6.3.12. Learner #12: Ricky

Ricky is a boy of twenty years old. From the interview, one could easily note that Ricky has a complicated social background. According to him, both his parents are not available and the grandmother who was taking care of them passed on. This left him the head of the family, taking care of his younger siblings. As one would expect, this situation impacted on Rick's schooling in a number of ways. His school attendance was inconsistent and on several occasions, he dropped out of school.

On his competence in basic computer skills, Ricky indicated that he had never used a computer before. To accommodate Ricky in this study, special transport arrangements had to be made to ferry him home after the lessons.

6.3.13. Learner #13: Mashabela

Lastly, Mashabela is a boy of sixteen years old. He stayed in a village about six kilometres from school and used special transport to and fro school every day. Both his parents are available, and his father is working. On computer skills, Mashabela revealed that they have a family computer at home. He was able to use applications like Word and Excel. He could not hide his excitement to participate in the study as he had special interest in the use of computers.The profile of learner participants discussed above is summarized in Table 6.1 below.

Learner No	Pseudonym	Gender	Age	Parents	Family source of Income	No of years in phase	Competence on basic computer skills
#1	Oscar	Male	18	Both parents available	Social grant	2	None
#2	Pebetsi	Female	16	Only mother available	Social grant	2	None
#3	Mmathapelo	Female	17	Both parents available	Father is a government employee	2	None
#4	Joel	Male	16	Both parents not available	Social grant	2	None
#5	Mpho	Male	18	Both parents are available	Father is working	3	None
#6	Gabriel	Male	16	Only mother available	Mother is working	2	Understands the basics of Word and Excel
#7	Precious	Female	17	Both parents available	Social grant	2	None
#8	sekgobela	Female	16	Only mother available	Mother is working	2	None
#9	Kopano	Female	17	Only mother available	Social grant	3	None
#10	Fogo	Female	16	Only mother available	Social grant	2	None
#11	Ledwaba	Male	18	Both father and mother not available	Social grant	3	Understands the basics of Word and Excel
#12	Ricky	Male	20	Both parents not available	Social grant	3	None
#13	Mashabela	Male	16	Both parents available	Father is working	2	Understands the basics of Word and Excel

Table 6. 1. Summary of the profile of the learner participants

Table 6.1 shows that seven of the learner participants were boys and six were girls. This represented a fair gender balance though gender was not an important factor for consideration in this study. The ages of the learners ranged from sixteen to twenty years. Nine of the learners belong to the same age cohort of between sixteen and seventeen years and these learners never repeated a grade. The other four learners were eighteen years old and above. From the information gathered during interviews, these learners either repeated earlier grades after failing to meet promotion requirements or dropped out of school at one point. Information on the table reveals that the four learners already had three years in the FET phase. This also reflected the low to average cognitive ability of the learners. They were slow and struggled to grasp simple mathematical concepts.

Understanding the learners' social and family background was important as this assisted in understanding the learners better. Literature confirms that the socio-economic background of learners influence their learning in a number of ways (Spaull, 2006; Visser, Juan & Feza, 2015). Table 6.1 above shows that three of the learners had both parents available, seven learners were from single parent families and three had no parents. Most of those with single parents were not staying with the parent. The parents were either working or looking for work in the urban areas. This left the majority of the learners either staying alone at home or under the care of grandparents.

This social reality inevitably impacts on the learners' learning as they lack strong support structures at home and, in this study, it was evident in some of the learners lacking focus, concentration and motivation. They needed to be constantly reminded of the objective of the project and how they stood to benefit out of it. In some cases, a learner from a child-headed family would request to be excused from attending a lesson to attend to some urgent matter at home. This would mean the lesson had to be re-scheduled to ensure that all the leaners attend all the scheduled lessons.

What is also clear from the learners' profile is that the majority of them did not have any basic computer skills. Only three of the learners indicated that they had some knowledge of the basics of

Word and **Excel** computer applications. The other ten learners admitted that they had never used a computer before. Given that all the learners were from low to middle – income families, they had no computers at home.

At school, they had computers but did not have a properly structured programme designed to equip learners with computer skills. The majority of the learners never visited the computer lab because they did not know how to use the computers. The Autography software does not require complex computer skills, but it requires the user to be able to use a computer keyboard comfortably. To ensure that learners acquired these basic computer skills, the first two sessions of the intervention lessons were devoted to teaching learners on how to use a computer. The focus of these two sessions was on using the computer keyboard and sketching of graph using Autography. Though some learners would have needed more time, the majority of them could easily use the software after the two sessions.

6.4. Data Sources

The analysis of learners' errors and misconceptions was based on the learners' written responses to the question items in the pre-test and their responses to the task-based interview questions. The pre-test was diagnostic in nature as it aimed to extract and expose learners' knowledge gaps and their common errors and misconceptions in dealing with concepts on hyperbolic functions. Each question item was ear-marked to assess the learners' ability (or inability) to perform a particular mathematical skill or their understanding (or lack of it) of a particular concept under hyperbolic functions. The question items were drawn from a variety of sources ranging from past examination questions, and prescribed CAPS textbooks to other intervention manuals. To ensure that the question items catered for all cognitive levels, the compilation of the question items was informed by the APOS- levels of conception and the genetic decomposition (see Table 3.1) developed as the analytical framework for the study.

The task-based interviews were done after the pre-test as I analysed learners' written responses to the question items. The central focus of these interviews was to allow learners' an opportunity to clarify

and explain the thinking behind their working or answer. This was drawn from the constructivists' argument that during assimilation and accommodation, chances are that a learner may develop 'concepts' which do not fit well into the existing cognitive structures or which are incongruent to socially accepted body of mathematical knowledge (Kilpatrick, 1987).

The errors and misconceptions are results of internal mental processes as learners construct and re-construct their own knowledge. The interviews were either one-on-one or focus-group in cases where learners exuded similar error patterns. All interviews were transcribed and attached to the learner's pre-test script. The subsequent analyses and error categorisation was based on the error categorisation protocol briefly discussed in 6.5 below.

6.5. Error Categorization Protocol

As I indicated earlier, the analytical approach initially adopted for analysing learners' errors and misconceptions was deductive analysis. I approached the data with a set of predetermined error categories, which I called 'error categorisation protocol'. The error categories were expounded in greater detail under the conceptual framework of this study (see Figure 3.1) and I briefly outline them here as a reminder to the reader. The protocol consisted of a total of twelve error categories drawn from three different researchers. The first three categories were drawn from Donaldson's (1963) multi-discipline error categories which he described as *structural*, *executive* and *arbitrary* errors. The next set of three error categories were from Hirst (2003), who came up with general error categories in mathematics, which were not topic-specific. He described the error categories as *procedural extrapolation*, *pseudo-linearity* and *equation-balancing*. The last set of errors was drawn from Makonye's (2010) study on learners' errors under differential calculus. Makonye (2011) came up with a long list of error categories which I condensed to six categories which I found relevant to this study. These were *systematic*, *encoding*, *decoding*, *meta-cognition*, *hybridization* and *careless* errors. All the error categories noted above provided the analytical framework for the identification and

categorisation of learners' errors in this study. It is however critical to note that these error categories were neither mutually exclusive nor exhaustive.

As the analysis and categorisation of errors progressed, new error categories emerged and others were renamed to suit the context of the topic. The analysis of the learners' written responses were analysed in conjunction with transcribed data from the task-based interviews. The interview data provided insights into the thinking behind the learners' errors and allowed the researcher to make informed inferences on the possible misconceptions underlying these errors.

6.6. Item by Item Analysis of Learner Errors and Misconceptions

In this section, an item by item analysis was made on learners' written responses in the pre-test. Informed by the APOS theoretical framework, in conjunction with the specified cognitive levels in the CAPS document, each question item targeted a specific concept or skill at varying conception levels.

The question items were broad enough to cover all relevant concepts and skills as specified in the CAPS document. The learners' written responses were categorised as: 'No attempt', 'Incorrect', 'Partially correct' or 'Perfectly correct'. If a learner left a question item without attempting it, I was interested to find out why. Was it because the learner did not understand what the question demanded or was it that s/he did not know how to tackle the problem. In cases where a learner presented a 'perfectly correct' response, I would ask the learner to check if s/he followed procedures with or without understanding. The central focus was, however, on 'incorrect' and 'partially correct' responses. Even responses that did not seem to make any mathematical sense were considered as legitimate starting points and valuable sources of information about the learners' learning process. The learners' transcribed responses to interview questions provided valuable data that gave meaning and 'sense' to the written responses from the learners' perspective.

6.6.1 Analysis of learners' responses in Item 1.1

Question item 1.1 of the pre-test required the learners to calculate the x- and y-intercepts of the hyperbolic function: $g(x) = \frac{6}{x-2} + 3$. Nearly all the learners demonstrated some understanding of the correct procedure in determining the intercepts of the function. Only Precious seemed to be confusing the concept of intercepts to that of asymptotes as shall be shown later. Twelve of the learners picked the correct procedure, but seven of them could not arrive at the correct solution. The dominant error was that of *executive* error. The following vignette from Lerato's response epitomises the common error that the learners were making.

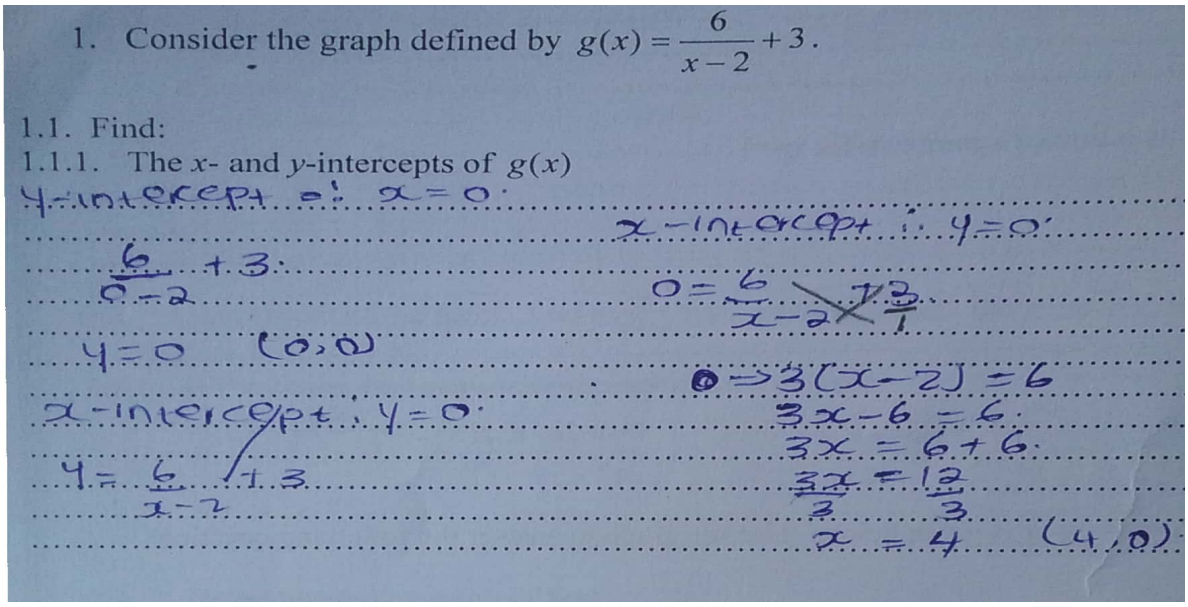


Figure 6. 1. Lerato's response to item 1.1

Lerato's response shows that she correctly picked the correct procedure to calculate the x- and y-intercepts. She indicated that when calculating the y-intercept, the value of x is 0 and for x-intercept$f(x) = 0$. She however experienced difficulties in the execution of algebraic procedures to arrive at the correct answer. As indicated by crossing lines on her working, she erroneously applied cross multiplication over addition. This can be classified as an executive error, which resulted from a *procedural extrapolation error*. The learner clearly assumed that if we can cross multiply, $\frac{6}{x-2} =$

3, then we can do the same for: $\frac{6}{x-2} + 3$. This error resulted in five of the learners wrongly getting the x-intercept as $x = 4$ instead of $x = 0$.

Another algebraic shortcoming was demonstrated by Sekgobela. His response is shown in Figure 6.2 in the vignette below.

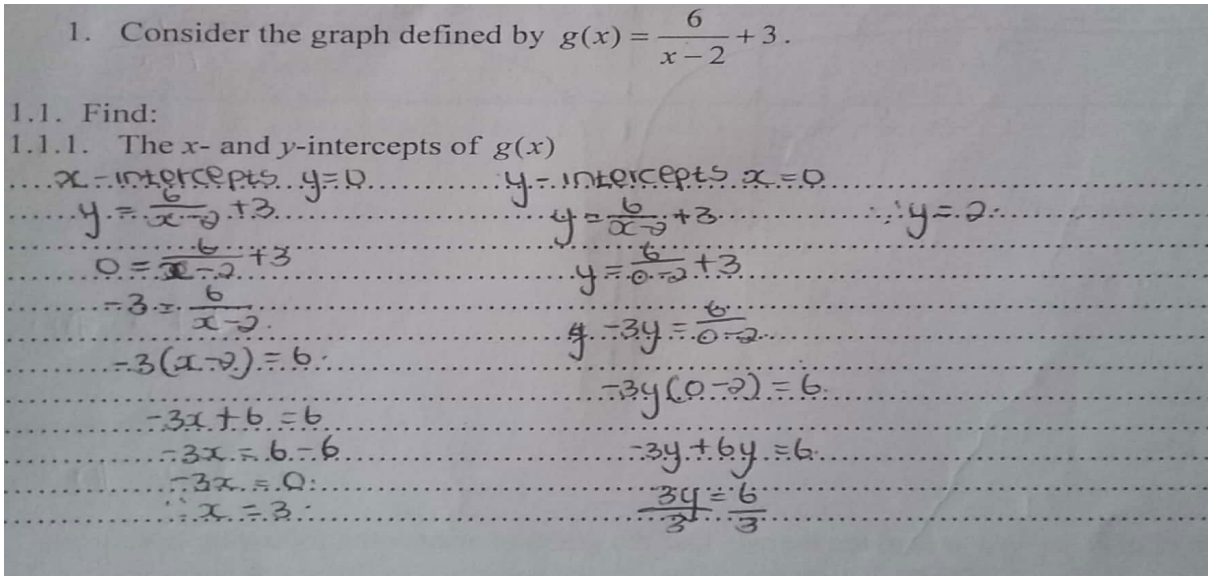


Figure 6. 2 Sekgobela's response to item 1.1

It is clear that the learner had a good understanding of the required procedure to solve the problem. He executed most of the steps correctly but seemed to have problems in dealing with the number 0. In the first instance, he correctly simplified up to the stage where he got: $-3x = 0$. Interestingly, the learner went on to write: $\therefore x = 3$. Most probably, the learner divided both sides by -3 but assumed that a number divided into 0 remains unchanged. In the second instance, he tried to expand , $-3y(0-2) = 6$ and got: $-3y + 6y = 6$. Again, in expanding the brackets, the learner multiplied $-3y$ by 0 and got $-3y$. When asked to explain this during interviews, the learner checked with his calculator and simply said, '… it was just a mistake'. The learner's verbal response suggested that this could have been a *careless* or *unsystematic* error. This was a case of *operational extrapolation* in relation to the number 0. Mathematically we know that adding any real number to or subtracting from 0, the number remains unchanged. The learner might have wrongly extrapolated this to multiplication by 0 and division into 0. In my view, this exposes learners' over dependence on

calculators which makes them loose sense and meaning of important operational and algebraic principles.

The only learner who showed no understanding on how to calculate intercepts of a function was Fogo. Her response to this question displayed a muddled conception of the concepts intercept and asymptote. She presented her answer as shown in Figure 6.3 below.

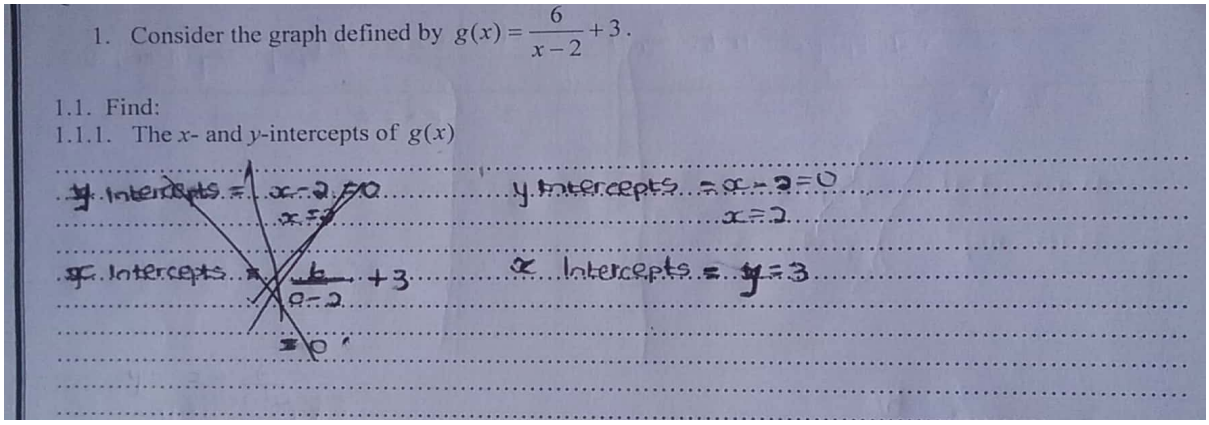


Figure 6. 3 Fogo's response to item 1.1

The learner demonstrated insufficient understanding of what an intercept is and how to calculate its coordinates. During interviews, it came out that the learner could not even identify intercepts and asymptotes on a drawn graph. She only reproduced the memorised equations which were not applicable to the question. This is a typical *structural* error.

To establish whether learners had a *process* conception of the concept of intercept, learners were asked during interviews to explain why they substituted 0 for x when calculating x-intercept and 0 for $f(x)$ when calculating y-intercept. Surprisingly, all the learners, including those who gave perfectly correct answers, could not accurately justify why they presented their responses that way. They could not give any meaningful explanation to the procedure they used. Below is a transcribed extract of the interview I had with Sekgobela.

> **R**: Let's look at question 1.1.1. The question required you to calculate the x- and y-intercepts. You started by saying, for x-intercept, $y = 0$ and for y-intercept, $x = 0$, can you explain why you say so?

Sekgobela: I think when we say $x = 0$ or $y = 0$ we make the equation quite simple and we can find the value of x and y easily and more quickly. Substitution by 0 is simple and I think that's the reason we choose 0.

R: So if we choose any other number which is not 0, do we still get a correct answer?

Sekgobela: If we substitute any other number which is not 0, the function will be undefined. So we choose 0 so that we can be able to find the x- and the y-intercepts.

R: What is your understanding of x- and y-intercepts?

Sekgobela: The x-and y-intercepts are the vertical and horizontal lines on the hyperbolic graph. The x-intercept is the vertical line and the y-intercept is the horizontal line.

What one can read from this conversation is that the learner could not give the appropriate justification for the steps and procedure she had taken to solve the problem. To Sekgobela, 0 is conveniently chosen to make calculations faster and easier. She contradicted herself when she said that that choosing any other number would make the function undefined, which is not true anyway. Her conception of an intercept was incorrect. She seemed to think that x- and y-intercepts are the x-and y-axes. Throughout the learner's explanation, she made no reference to the graph of the function. That failure to relate the symbolic equation of the function to its graph of the function seemed to be the major obstacle to the learner.

Lerato responded to the same question as follows:

R: Looking at question 1.1, the question required you to calculate the x- and y-intercepts. I can see you started by saying, for x-intercept, $y = 0$ and for y-intercept, $x = 0$, can you explain why you say so?

Lerato: I think it's just a given formula that we have to use. If we need the x- intercept, we substitute 0 for y and vice-versa. It's a given principle that we use.

R: Ok, fine. The next question I want to ask is, 'Why do you choose 0 and not any other number to find the x-and y-intercepts?'

Lerato: I think if we choose another number which is not 0, we will not get the correct answer.

R: What is your understanding of x- and y-intercepts?

Lerato: I think the x-intercept is the horizontal axis and the y-intercept is the vertical line.

What I find most interesting in Lerato's response is that she considered substituting x or y by 0 as just a given rule or principle with no mathematical justification. The response epitomises the general conception learners had in responding to this question item. What this implied was that learners might have just memorised how to determine the intercepts of a function without a graphical mental picture of what these points are.

Dhlamini and Luneta (2016) described this as procedural mathematical proficiency. It is all about the 'knowing how' of mathematical knowledge without understanding the 'why' part. Procedural proficiency is limited in that it limits the learners' capacity to tackle mathematical problems of higher order cognitive demand. In APOS terms, this confirms why learners' responses were confined to the action level and struggled with higher conception level questions.

6.6.2 Analysis of learners' responses under Item 1.2

This question tested the learners' ability to derive equations of asymptotes of a hyperbolic function from its algebraic equation. Understanding the concept of asymptote is central to understanding of hyperbolic functions. The CAPS syllabus requires learners to deal with hyperbolic functions of the form: $g(x) = \frac{a}{x+p} + q$ where $x = -p$ is the vertical asymptote and $y = q$ is the horizontal asymptote. The question item required learners to extract the equation of the vertical and horizontal asymptotes from the given equation of the function.

Eight of the learners did not have problems in answering this question as they presented the equations of the asymptotes correctly. The other five of the learners presented partially correct responses. The learners' lack of conceptual understanding of what an asymptote is evident in different ways. Some of the learners could link the vertical and horizontal asymptotes to the constants 2 and 3 in the equation but could not give their answers as equations. A good example of this error is illustrated in Mpho's response in Figure 6.4 below.

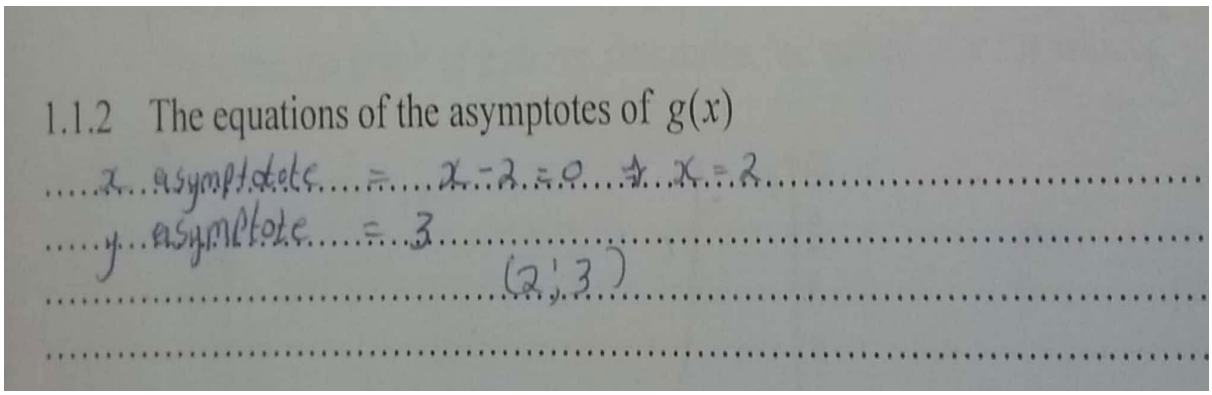


Figure 6. 4 Mpho's response to item 1.2

Apparently, Mpho's conception of an asymptote is a point and not a line. This could be classified as a *structural* error because the learner seemed to lack a correct concept image of an asymptote. Four other learners displayed similar conceptual limitations in their responses. Though they did not give their answers in coordinate form, they simply stated the vertical asymptote as a number 2 and the horizontal asymptote as 3. This could suggest that the learners might have viewed 2 as x-coordinate and 3 as the y-coordinate. Tall and Bakar's (1992) observed that students have problems with constant functions. Thy further note that given expressions such as y = c, where c is a constant, students were unaware that these represents function and could not identify their graphs as functions.

This initial observation prompted the investigation of the learners' conception of the concept of asymptote and their views on what it tells us about the function. Presented below is an extract of the transcribed interview I had with Precious and Mmathapelo.

> **R** : Let's move on to the next question which was asking for the equations of the horizontal and vertical asymptote. For horizontal asymptote, you said $x - 2 = 0 \Rightarrow x = 2$ and $y = 3$. Can you explain how you got this?
>
> **Precious:** $x - 2$ we get it from the function given, so to make it an equation, we say $x - 2 = 0$ and then we take 2 to the right-hand side so that we get $x = 2$ as the vertical asymptote.
>
> **R**: And how about the horizontal asymptote?
>
> **Precious**: If we look at the given equation, we see that 3 is a constant, therefore we just take $y = 3$ as our horizontal asymptote.
>
> **R:** Mmathapelo you responded to this question in the same manner, can you share your own explanation?

Mmathapelo : I also think that we take $x-2$ given in the equation and form an equation $x-2=0$ so that we are able to find the value of x. On horizontal asymptote, we take the constant 3 as it is and it becomes $y=3$.

R: Now let me ask a similar question to that of 1.1 above. When you say $x-2=0$, why are you choosing 0 and not any other number?

Precious: I have no idea why we are choosing 0.

Mmathapelo: I think we choose 0 because it's not going to change our asymptote. If choose another number and when we take 2 to the right hand side it is going to add that number and that is going to change our vertical asymptote.

R: All right, that interesting. What is your understanding of an asymptote?

Precious: Asymptote is an imaginary line on the graph.

R : What does that line tell us about the function?

Mmathaphelo : The lines of asymptote tell us the quadrants in which the graph lies. The vertical and horizontal asymptote will make four quadrants, so the graph will lie in the first and third quadrants or the second and fourth quadrants.

This extract reveals that though the learners were able to state the equations of the asymptotes correctly, they had an unclear understanding of what exactly an asymptote is. They might have acquired a memorised understanding that given a hyperbolic function: $(x)=\frac{a}{x-p}+p$, then $x-p=0$ is the vertical asymptote and $y=q$ is the horizontal asymptote. This was evident in their failure to explain why the denominator $x-2$ is equated to 0. This explained why the learners subsequently struggled to explain what an asymptote is. This again confirmed the emerging view that learners had a mechanical understanding of procedures without structural understanding of the rationale behind those procedures. Precious and four other learners described an asymptote as an 'imaginary line' but could not further elaborate what the line tells us about the function. Mmathaphelo conceived asymptotes as lines that divide the plane into four quadrants. Clearly, these asymptotic misconceptions are likely to breed contradictions when it comes to exponential functions and logarithmic functions with one asymptote and other functions with oblique asymptotes.

6.6.3 Analysis of learners' responses under Item 1.3

This question required learners to use their responses in items 1.1 and 1.2 to sketch the graph of $g(x)$. It tested the learners' ability to correctly transform the given function from its symbolic form to its graphical form. Four of the learners presented their sketch graphs perfectly well. They were able to clearly show the asymptotes and the intercepts. Even during interviews, these learners demonstrated a sound knowledge of the connection between the algebraic form of the function and its graphical representation.

The other four learners presented graphs that were partially correct. They displayed representational misconceptions, which subsequently led to errors in their graphical representation. The errors partly emanated from the errors which the learners had already made in items 1.1 and 1.2. A good illustration of this is Ricky's sketch graph shown in Figure 6.5 below.

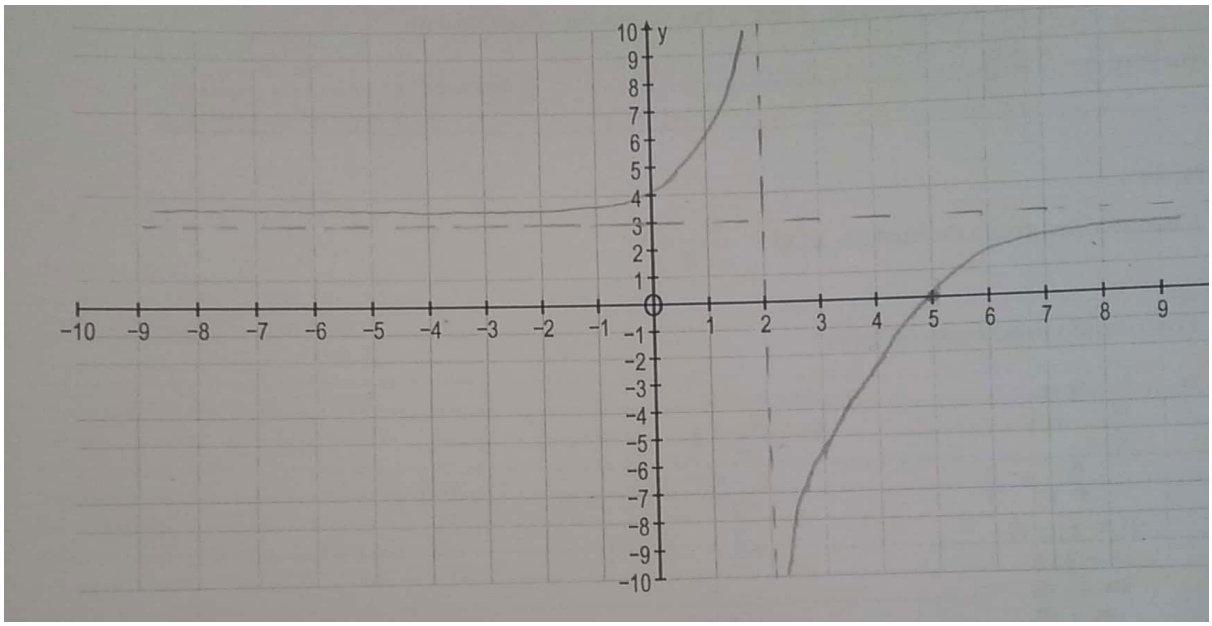


Figure 6. 5. Ricky's response to item 1.3

Ricky demonstrated reasonable understanding of how to sketch the graph of a hyperbolic graph and how its shape looks like. However, the learner had erroneously calculated the x-intercept as $(5; 0)$in item 1.1 and this might have led the learner to assume that the symmetric curves lie in quadrants 2 and 4. This could be a case of *meta-cognition error*. Meta-cognition error arises due to a learner's failure to reflect on the sensibility of what s/he has done (Makonye, 2011). The learner should have

realised that in general, if $g(x) = \frac{a}{x-p} + p$ and the value of $\boldsymbol{a}$ is positive, then the graph of $g(x)$ should lie in the 1st and 3rd quadrants. Also, the y-intercept shown on the graph, $(0\,;\,4)$ is not consistent to what the learner obtained in his calculations. If the learner had reflected on these inconsistencies, he would have realised the error in his calculations and probably correct it.

Another notable error in this category of learners was that of presenting graphs with only one curve instead of two symmetrical curves. Gabriel's response in Figure 6.6 below is a typical example of this kind of an error.

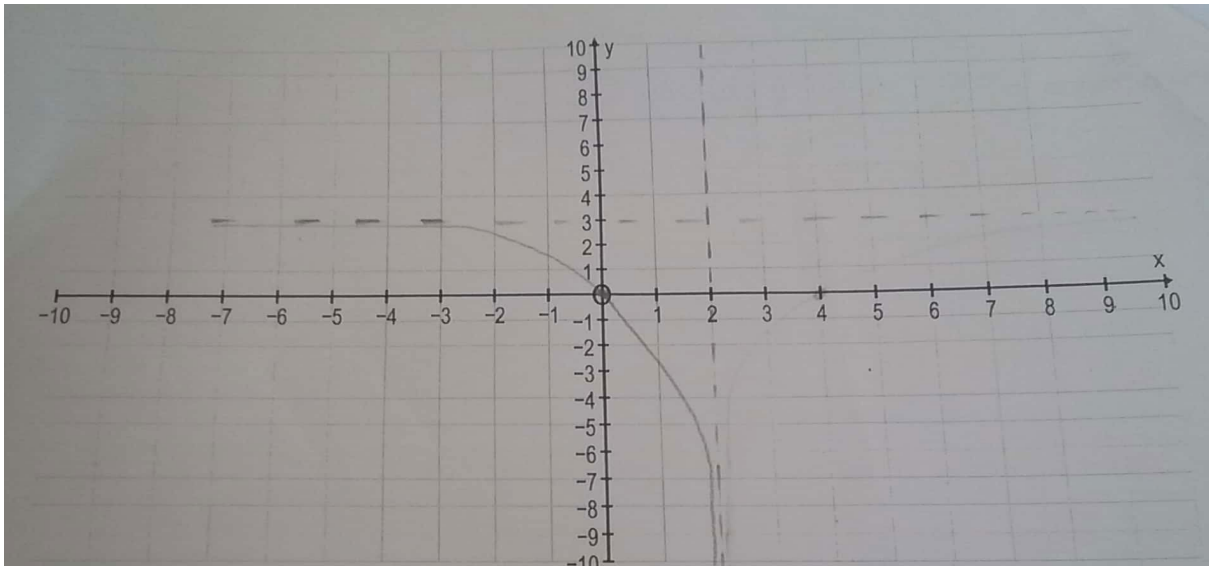


Figure 6. 6. Gabriel's response to item 1.3

What attracted my attention to the three learners who presented their graphs in this way was that their calculations in items 1.1 and 1.2 were perfectly correct. I then got interested in finding out what could have led to this error. All the three learners, interviewed separately, indicated that after realising that the x-intercept and the y-intercept of the function was the same point, $(0\,;\,0)$, they thought this could be a special case where the graph has only one curve. This could be classified as a *logically invalid inference* error because the learners made a wrong conclusion from the given scenario. Probably the learners were used to cases where the function has separate x- and y- intercepts which assisted them in sketching the graph.

The remaining five learners presented sketch graphs that depicted a fragmented understanding of the graphical representation of a hyperbolic function. The learners demonstrated that they could not connect the algebraic form of the function to its graph. Pebetsi and Ledwaba's graphs shown in Figures 6.7 and 6.8, respectively, below exemplify the graphs presented by this category of learners.

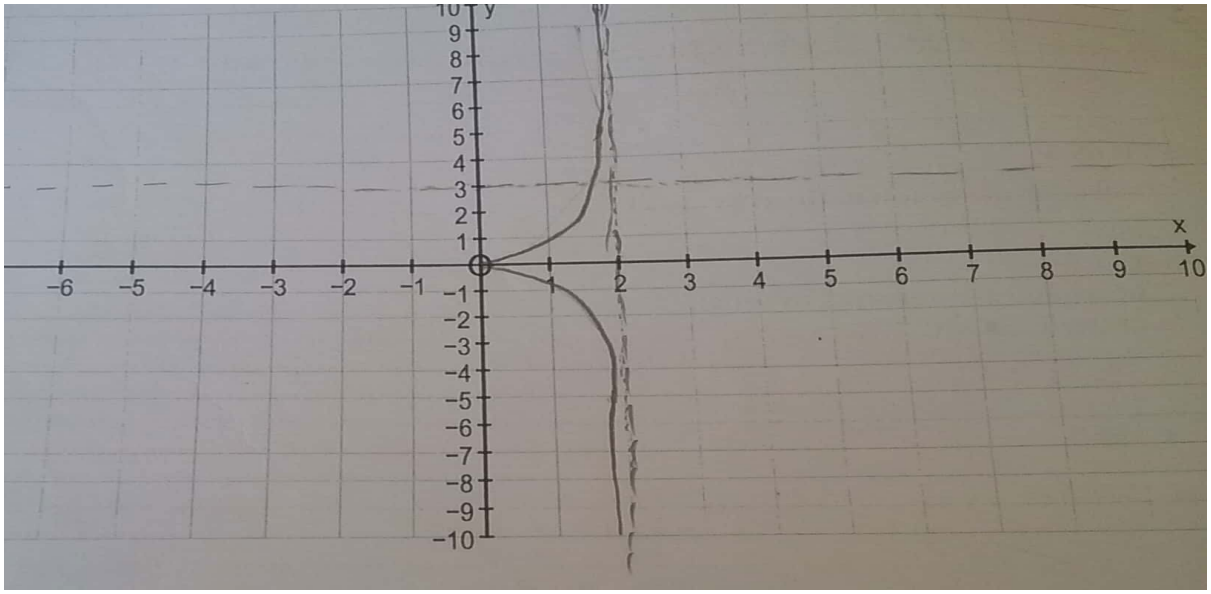


Figure 6. 7. Pebetsi's response to item 1.3

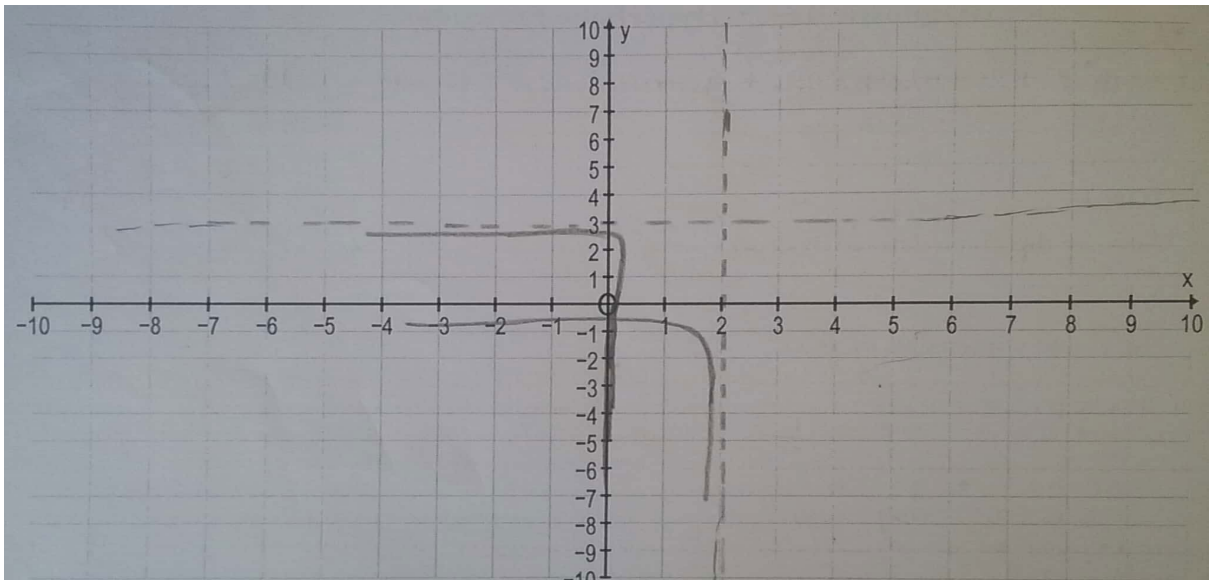


Figure 6. 8. Ledwaba's response to item 1.3

Both learners managed to show the horizontal and vertical asymptotes. What seemed to be challenging to these learners was to show the link between the graph and its asymptotes. In Pebetsi's

sketch graph, two curves meet at the origin and approaching the vertical asymptote in opposite directions. The other curve cuts the horizontal asymptote. Of course, the learner attempted to bring out the connection between the curve and the vertical asymptote but in a disjointed and inconsistent manner. The fact that one of the curves cuts the horizontal asymptote is an indication that the concept of asymptote is not yet fully developed in the learner's mind. It also appears like the learner was also confused by finding the x-and y-intercept as the same point. This may explain why the two curves were drawn meeting at the origin.

Similarly, Ledwaba's sketch shows that the curves of the graph are not correctly positioned. Though the learner displayed some understanding of how the graph of the function behaves as it approaches the asymptotes, she seems to think that the x- and y- axes are also asymptotes of g(x). Consequently, her graph is mixed up and not a true representation of the given function. What one could easily note was that the learners made little or no reference to the critical points calculated in the previous questions in sketching the graphs. This observation was, however, not surprising because in the previous analysis, it came out that learners calculated x-and y-intercepts and equations of asymptotes without an understanding of what they stand for.

Another error that was noted in this category of learners was that of confusing the graph of the hyperbola to that of a quadratic function. This error was categorised in this study as a *hybridization* error, where a learner thinks that some mathematical concepts are connected when in fact they are not. Fog's sketch graph in Figure 6.9 below epitomises this error.

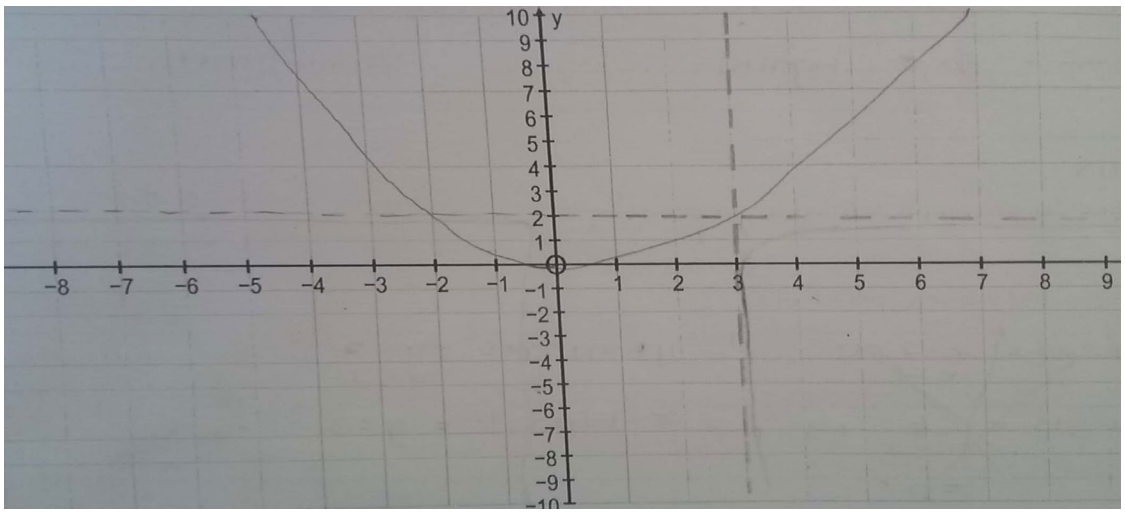


Figure 6. 9. Fog's response to item 1.3

Makonye (2011) explains that such errors are often due to a muddled conceptual understanding of the concepts. In this case, the learner seemed to have disjointed mental images of parabolic and hyperbolic graphs without an understanding of their key features. Attempting to connect lines of asymptotes to a parabolic curve is a clear demonstration of a deficiency in structural understanding of the concepts involved in graphing a function.

During interviews, it was not easy to infer the thinking behind these learners' errors. When asked to explain why they sketched their graphs the way they did, these learners gave very brief responses, which contained very little explanation in them. They gave responses such as:

- I wasn't sure how the graph looks like (Mmathapelo)
- I didn't know how to draw the graph (Fogs)
- Eish! These graphs are confusing to me (Lerato)

Explaining the challenges that students face in switching from one representation of a function to another, Even (1998) describes two approaches to the understanding of functions. Even (1998) draws a distinction between a *point-wise* approach and the *global* approach to the behaviour of functions. A point-wise approach involves point-by-point plotting, reading or dealing with discrete points of a function (Even, 1998). The point-wise approach focuses on specific points or values of the function

without a connected picture of the concepts involved. On the other hand, the global approach looks at the total picture of the function and its behaviour. Sketching the graph of a function given in algebraic form requires a global picture of a function in terms of its key features. A learner operating at point-wise level is more likely to find sketching the graph very challenging.

I found the theoretical distinction between the *point-wise* and the *global* approach to be relevant to this analysis. Even the mathematics CAPS document prescribes and expects Grade 10 to 12 learners to 'generate as many graphs as necessary, initially by means of point-by-point plotting, supported by available technology, to make and test conjectures …' (DBE, 2011:12). The document further requires learners to work on the relationships between variables in terms of numerical, graphical, verbal and symbolic representations of functions and convert flexibly between these representations. Learners are expected to develop from a point-wise understanding of functions to a global understanding which enhances their flexibility in switching from one representation to another.

6.6.4 Analysis of learners' responses under Item 2.1

Question 2 basically required learners to answer questions based on the given graph of a hyperbolic function. It tested the learners' ability to extract the key features of the function from the graph and interpret graphical information. In question item 2.1 learners were asked to identify and write down the equations of asymptotes from the graph of $g(x)$. The question was classified as a low order one, with a low cognitive demand. No working was required as the learners were simply expected to identify and write down the asymptotes from the graph.

Surprisingly, only three of the learners managed to give correct responses. Even during interviews, these learners demonstrated that they were able to identify the asymptotes based on the shape of the curves. The other three of the learners never attempted to answer this question. When asked why they left the question unanswered, they all indicated that they could not identify the asymptote because there was no dotted line on the graph. Interestingly, the learners were able to give correct equations

of asymptotes in item 1.2. The learners could give the equations of asymptotes if the function is given in its standard algebraic form but could not identify asymptotes from the graph.

The difficulty in navigating between the algebraic and graphical representations was also evident in the seven learners who attempted this question with partial or no success. The majority of these learners attempted to work out the equation of the graph before writing its asymptotes. Joel's response shown in Figure 6.10 below exemplifies this.

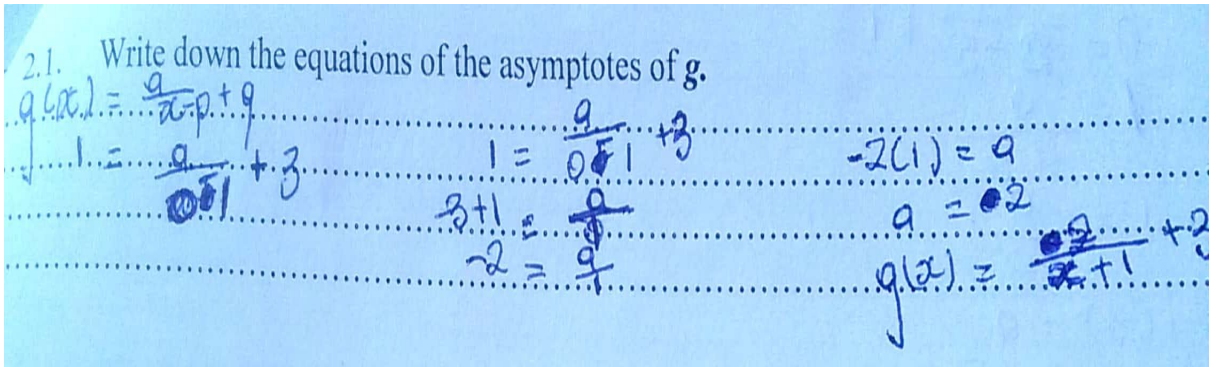


Figure 6. 10. Joel's response to item 2.1

Looking at the substitution that Joel did in the initial stage of his working, it is clear that he managed to correctly identify the asymptotes from the graph. Asked for the reason why he chose to determine the equation of $g(x)$ first, instead of simply writing the equations of asymptotes as seen from the graph, Joel had this to say:

> **Joel** : Eish! I thought this was the easier method for me because once I find the equation of g(x), then I can easily write the equations of asymptotes.
>
> **R** : But you clearly substituted p and q, why didn't you write the equation straight away?
>
> **Joel** : (*Shaking his head*) I did not know how to write the equation.

Asked the same question, Kgothatso responded as follows:

> **Kgothatso** : I did not quite understand this question. I thought the question was asking for the equation of g(x).
>
> **R** : If you had understood the question correctly, how would you answer it?
>
> **Kgothatso**: I think I would use the same method.

These responses further exposed the learners' rigidity in dealing with hyperbolic functions from an algebraic perspective. Their understanding of the function was limited to the algebraic form and this made it very difficult and uncomfortable for them to deal with problems involving the graph. Kgothatso might have committed an *arbitrary* error by deliberately changing the question to suit what she was comfortable with. The other challenge that might have exasperated learners' problems in this question was their failure to give the equation of vertical and horizontal lines. This was evident in about five of these learners who correctly identified the values of $\boldsymbol{p}$ and $\boldsymbol{q}$ but could not give the required equations.

DeMorois and Tall (1996) described the conceptual development of the function concept in terms of vertical and horizontal development. By horizontal development, they referred to the various forms of representations including geometric, numeric, verbal and symbolic. They further noted that most students have challenges in switching from one form of representation to another. They observed that if students lack breadth (different representations) in their understanding of function, it hampers their depth or vertical (structural) understanding. The above results seem to corroborate this assertion. The learners were confined to the algebraic form of hyperbolic function that they would attempt to approach every problem algebraically, even if the problem could easily be solved through other representations.

The other three learners gave wrong answers because they lacked an understanding of what an asymptote is. As what has already emerged earlier in this analysis, these learners seemed to have a muddled understanding of the concepts: asymptote, intercept and x-and y-axes. This was evident during interviews when these learners were asked to finger-point asymptotes on the graph and they either pointed to x-and-y intercepts or x-and y- axes. This constitutes a deep *structural* shortcoming. This underscores the inadequacy of a mechanistic understanding of functions limited to its standard

algebraic form. It constraints the learners' ability in seeing the interconnectedness of the concepts involved.

6.6.5 Analysis of learners' responses under Item 2.2

In this question item, learners were asked to give the equations of asymptotes of $g(x+3)$, transformed from $g(x)$. There were two possible approaches to tackle this problem. Learners with a good understanding of the process definition of a function could use the algebraic approach, whereby they would substitute x for $x+3$ to obtain $g(x+3)=\frac{a}{x+2}+3$. It would then be easier for the learner to identify the asymptotes. Alternatively, learners with a good graphical understanding of the function could use the graphical approach. This required the learners to realise that $g(x+3)$ means the graph of $g(x)$ has been shifted 3 units to the left, hence the vertical asymptotes shifts 2 units leftwards and the horizontal asymptotes remains unchanged.

None of the learners managed to answer this question item correctly. Three learners (Gabriel, Joel and Oscar) came closer to the correct solution by attempting to use the algebraic approach but made some *executive* errors and could not reach the final solution. A good illustration of this is Oscar's response given in Figure 6.11 below.

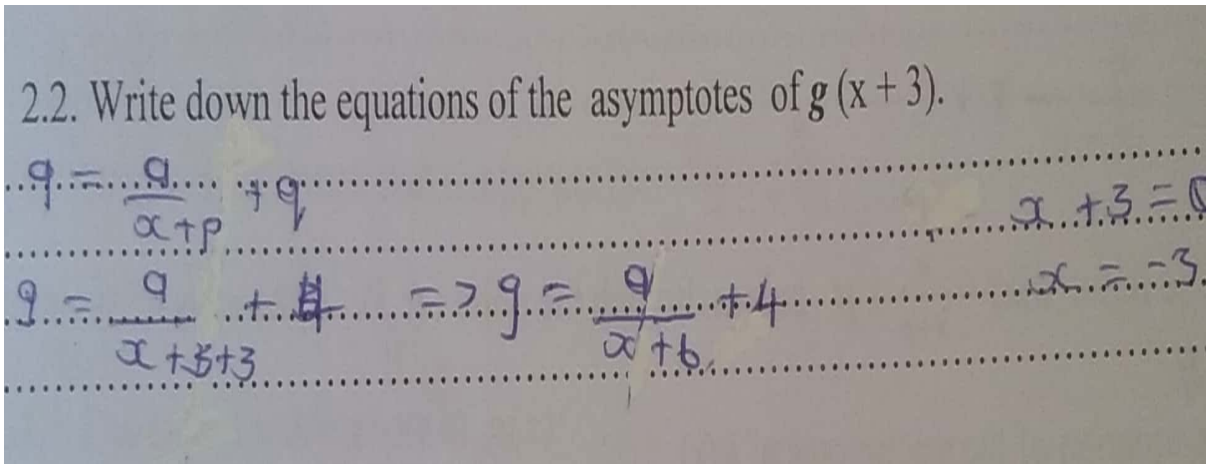


Figure 6. 11. Oscar's response to item 2

The encouraging thing to note in Oscar's response was that he had a correct interpretation of $g(x+3)$. He was able to substitute $x+3$ for x in the equation of $g(x)$. Though he substituted in the wrong equation of $g(x)$, which emanated from the error in item 2.1, he demonstrated some understanding of the input–output process definition of a function.

All the three learners who used this approach did not adequately answer the question as they left their answers as an equation of $g(x+3)$ instead of giving the equations of asymptotes. Such an error could be classified as an *unsystematic* or *careless* error, which arises from the learners' failure to pay attention to the demands of the question.

The most common error which the majority of the other learners made in this question item was that of *decoding* or *interpretation* error. This error occurs when a learner fails to understand mathematical symbols or terminology and subsequently ends up answering an unintended question. In this particular question, it was clear that learners failed to decode the functional notation $g(x+3)$. Fog's response given in Figure 6.12 below exemplifies how ten of the learners responded to this question.

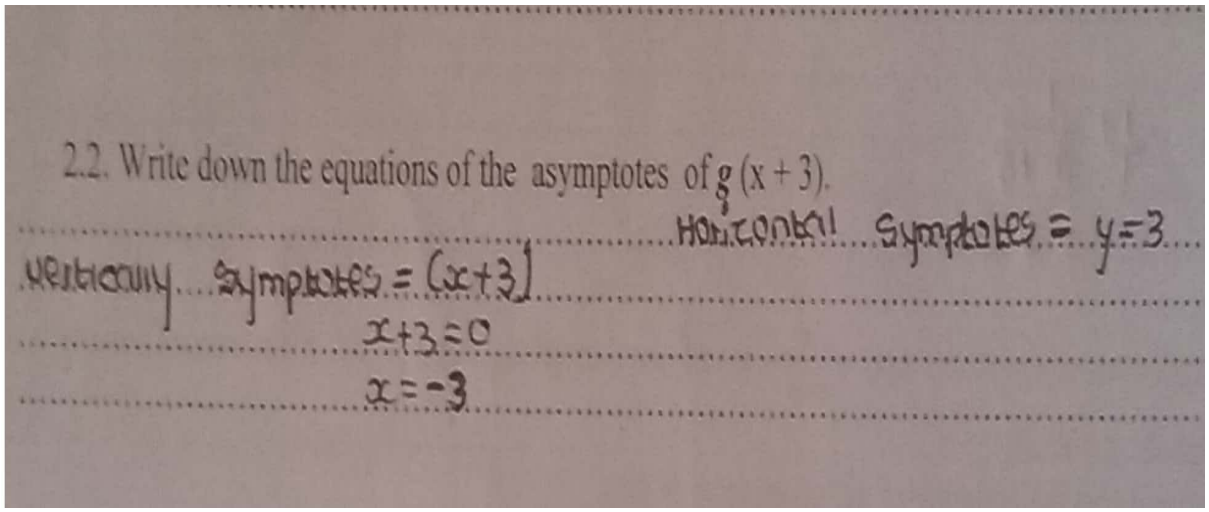


Figure 6. 12. Fog's response to item 2.2

What was clear was that the learner was challenged by the functional notation $g(x+3)$. He could not 'see' the connection between $g(x)$ and $g(x+3)$. As shown in the response above, most learners erroneously equated $x+3$ to 0 and obtained the vertical asymptote as $x=-3$. To get more insight

on the learners' thinking, I asked the learners to explain their own understanding of $g(x+3)$. The following is an extract of the transcribed conversation I had with Ricky:

> **R**: Question 2.2 required you to find the asymptotes of $g(x+3)$. Can you share with me your understanding of this notation: $g(x+3)$.
>
> **Ricky**: When we are given $g(x+3)$, I think it means our vertical asymptote is $x+3=0$ which $x=-3$
>
> **R**: So what then will you say about the horizontal asymptote?
>
> **Ricky**: I think it means the horizontal asymptote is equal to 0.
>
> **R**: Do you think there is any connection between the function of $g(x)$ and that of $g(x+3)$ and if so, what is the connection?
>
> **Ricky**: Hmmm I think there is no connection between the two because the asymptotes are different.

The above conversation does confirm the view that that the learners could not decode or interpret the meaning of $g(x+3)$. By equating$x+3=0$, Ricky most probably overgeneralised the rule that in determining the vertical asymptote, we equate the denominator $x \pm p$ to 0. All the ten learners made the same incorrect generalisation, and none could link $g(x)$ to $g(x+3)$. This observation corroborates Abdullah's (2010) finding that the symbol $f(x)$ presents one of the greatest obstacles to most students. Abdullah (2010) reports that students equate $f(x+h)$ to $f(x)+h$. The source of this challenge can be traced back to the inadequate understanding of the process definition of the function concept itself.

6.6.6 Analysis of learners' responses under Item 2.3

This question required learners to give the domain and range of $h(x)$, a new function obtained after translating $g(x)$ 3 units upwards. In their responses, learners displayed a memorised understanding of the domain and range of a hyperbolic function. Table 6.below presents the responses given by each of the learners to this question item.

Learner	Domain	Range
Kopano	$x \in R, x \neq 3$	$y \in R, y \neq -1$
Mpho	$x \in R, x \neq 3$	$y \in R, y \neq 3$
Gabriel	$x \in R, x \neq 6$	$y \in R, y \neq 4$
Mmathapelo	$x \in R, x \neq 0$	$y \in R, y \neq 4$
Oscar	$x \in R, x \neq -3$	$y \in R, y \neq 6$
Sekgobela	$x \in R, x \neq 1$	$x \in R, y \neq 3$
Ricky	$x \in R$	$no\ attempt$
Mashabela	$x \in R, x \neq 3$	$y \in R, y \neq -3$
Joel	$x \in R, x \neq 1$	$y \in R, y \neq 6$
Pebetsi	$x \in R, x \neq -3$	$y \in R, y \neq 3$
Fogs	$x \in R, x \neq -3$	$y \in R, y \neq 3$
Precious	$x \in R, x \neq 3$	$y \in R, y \neq 3$
Ledwaba	$x \in R, x \neq 6$	$y \in R, y \neq 1$

Table 6. 2. Learners' responses to item 2.3

Nearly all the learners demonstrated an understanding that given a hyperbolic function, $f(x) = \frac{a}{x-p} + q$, then the domain is written as $x \in R, x \neq p$ and the range $y \in R, y \neq q$. This was with the exception of Ricky who seemed unclear on how to write down the domain and range of the function. However, only Joel managed to present the correct response. Asked to explain his response, Joel indicated that he had used the algebraic approach. He first calculated the equation of $g(x)$ and added $g(x) + 3$ to obtain the equation of the transformed function. This further exposed the learner's shortcoming in

dealing with graphs. The problem could easily be tackled graphically without all the laborious working.

The major obstacle learners faced in this question was that the function was given in graphical form. As already discussed under item 2.1, learners could not identify the vertical and horizontal asymptotes from the given hyperbolic graph. In this case, learners were further required to translate the given graph three units up before giving its domain and range. It came out during the interviews that the majority of the learners could not figure out the effect of this vertical shift on the critical points of the graph. About six of the learners indicated that they shifted the x-axis three units up and the y-axis three units to the left or right. This confirmed the asymptotic misconception highlighted earlier in this analysis. These learners erroneously viewed the x- and y-axes as the horizontal and vertical asymptotes of the function.

Another category of learners correctly identified the vertical and horizontal asymptote of $g(x)$ but went on to shift the horizontal asymptote 3 units up and the vertical asymptote 3 units to the left or right. They argued that since $g(x)$ was shifted 3 units upwards, both asymptotes shift by 3 units. It was evidently apparent that the learners lacked a clear mental image of the effect of translating a graph. This further confirmed the absence of the *object*-conception of hyperbolic functions.

6.6.7 Analysis of learners' responses under Item 2.4

As shown in Table 6.2 above, two learners did not attempt this question and eleven provided incorrect responses. The question required learners to use the graph of $g(x)$ to determine the values of x for which $g(x) \leq 0$ and for which $g(x) < 2$.

The type of responses provided by all the learners simply displayed their inability to interpret graphical information. Ricky and Gabriel attempted to use the algebraic approach, but they got stuck because the equation of the function was not given in algebraic terms. During interviews, it emerged that most of the learners were confused on how the inequalities are linked to the graph of $g(x)$. They

could hardly conceptualise or *decode* the meaning of $g(x) \leq 0$. This *interpretation error* confirms Selden and Selden's (1992) assertion that visual information in the form of graphs is more difficult for students to learn, harder to teach and considered less mathematical by students. They further note that graphs pack in lots of information in a nonlinear fashion and thus students have difficulty in extracting visual information from a graph.

6.6.8 Analysis of learners' responses under Item 3.1

Question items under question 3 were *schema*-level questions. They were higher order questions which tested the learners' ability to see the linkages between the concepts under hyperbolic functions. In item 3.1, learners were expected to bring out the connection between domain, range and the asymptotes. Five of the learners managed to correctly substitute the values for the vertical and horizontal asymptote from the given information. However, they either could not substitute the given point to find the value of ***a*** or made *executive* errors in solving the algebraic equation with fractions. A typical example is shown in the vignette in Figure 6.13 below.

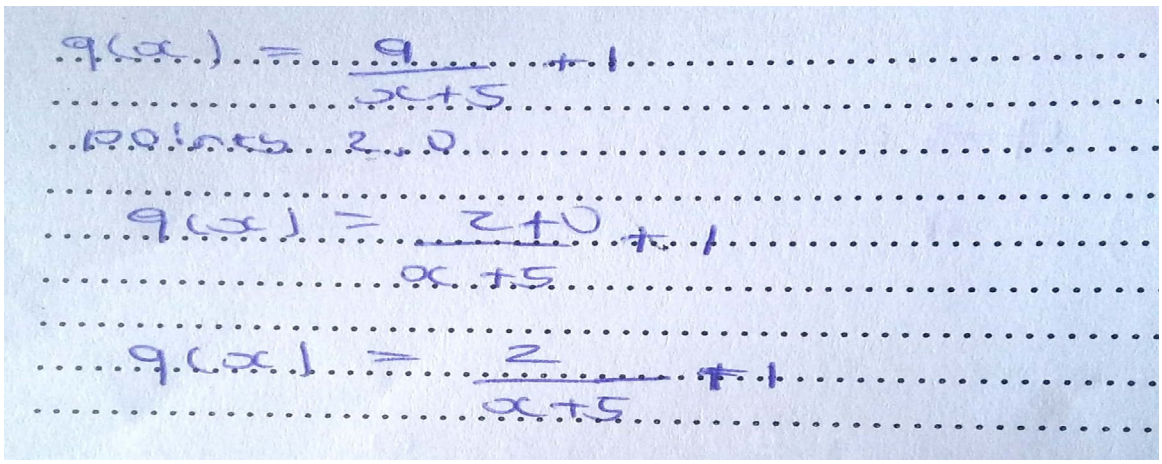


Figure 6. 13. Mmathapelo's response to Item 3.1

Initially, Mmathapelo was able to substitute the values of ***p*** and ***q*** in the standard formula, though she made an error of sign in substituting ***p***. This was a very important step because it implied that she was able to identify and extract the vertical and horizontal asymptote from the given domain and range respectively. However she went on to make a gross error by adding the x- and the y- coordinate of

the given point to find the value of ***a***. During the interview it was not easy to clearly figure out the thinking behind the learner's error. She could not give any explanation for adding the x-value and y-value of the given point. The adding of the input and output values exposed the learner's deficiency in her understanding of the process definition of a function.

A common shortcoming displayed by most of the learners was that they were only able to reproduce the general equation $g(x) = \frac{a}{x-p} + p$ as it was given in class, but their subsequent substitution simply exposed their lack of conceptual understanding. Pebetsi's response in Figure 6.14 below illustrates this muddled understanding of the connectedness of the critical points of a hyperbolic function.

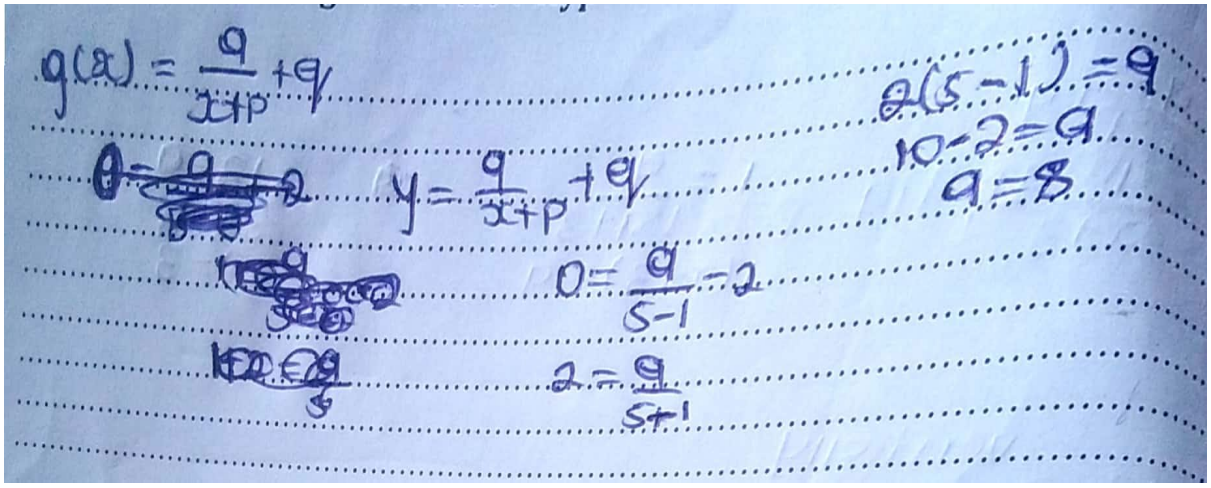


Figure 6. 14. Pebetsi's response to Item 3.1

Pebetsi's response epitomises a category of six learners who had a routinized understanding of hyperbolic functions. Their substitution into the formula was random without any reference or link to the critical points of the function given. During interviews, it emerged that nearly all of these learners could identify what the constants ***p*** and ***q*** stand for, but they could not extract their values from the given domain and range. I categorised this as a *structural* error emanating from lack of understanding of underlying concepts.

6.6.9 Analysis of learners' responses under Item 3.2

This question item required learners to use the interconnectedness of the symmetrical nature of a hyperbolic graph, its asymptotes and the graph itself. It demanded learners to have a well-developed concept image of the symbolic and graphical representation of hyperbolic functions. In APOS terms, learners were expected to have attained a *schema* conception of hyperbolic functions to successfully tackle this question.

The most critical step in tackling this question item was to identify the vertical and horizontal asymptote from the point of intersection of the axes of symmetry. Only Joel managed to do this and when asked to justify his response, he articulated the link between the point of intersection of the lines of symmetry to the point of intersection of the asymptotes. The flexibility of the learner in manoeuvring between graphical and symbolic representation made him to see how the critical points connected the graph and its equation.

A closer look at the other learners' responses revealed a blurred and mixed-up conception of the axes of symmetry and asymptotes. Most of the learners in this category identified the vertical and horizontal asymptotes as $x = 4$ and $y = 3$ respectively. Their thinking was that since it was given that point $(4; 3)$ lies on the graph, then both asymptotes also pass through that point. The point of intersection of the axes of symmetry was subsequently substituted to find the value of $\boldsymbol{a}$. Asked to justify this, Gabriel said that:

'*since the given hyperbola passes through point (4;3), then the asymptotes should pass through this point because they are part of the graph'*.

This response reflected a fragmented or disjoint conception of the critical points of the function. I classified this as *fragmentation* error.

Sub-question item 3.2.2 was a follow-up process question which required learners to correctly substitute their responses in 3.2.1 to determine the equation of the function. Despite some *executive* errors already discussed in previous question items, there were some unexpected errors identified in

this question item. Even if a learner obtained incorrect responses in sub-item 3.2.1, one would have expected the learner to consistently substitute in the given formula. This was not the case in Fog's response given in Figure 6.15 below.

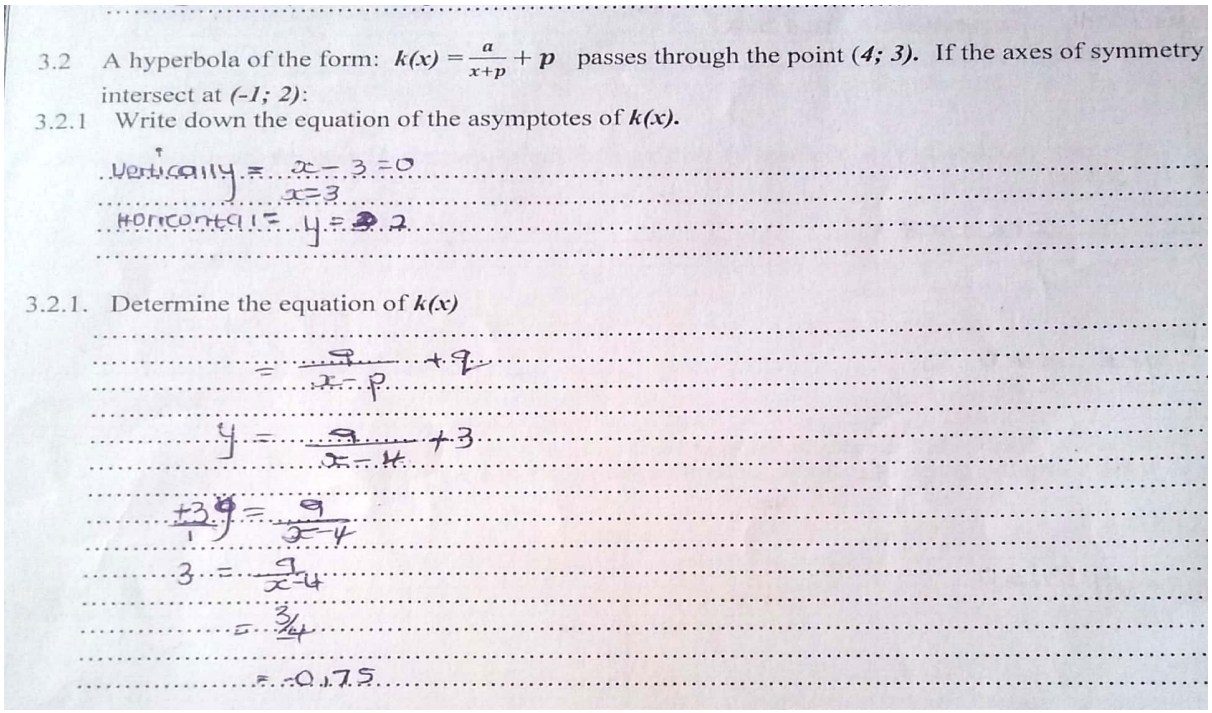


Figure 6. 15. Fog's response to Item 3.2

Fog erroneously stated the vertical asymptote as $x = 3$ and the horizontal asymptote as $y = 2$. When coming to substitution in the next sub-item, 4 and 3 were put in place of the vertical and horizontal asymptote respectively. Attempts to figure out the learner's line of thinking were fruitless as she did not attempt to give any justification to her working. She simply said, '… I did not understand what I was supposed to do'. However, this was not surprising because looking at the learner's responses in previous question items, it was evident that the learner had deep *conceptual* shortcomings. This exposes the struggle that learners go through if they have a conception of functions restricted to algebraic formulae with no understanding of concepts involved.

Another unexpected error was displayed by Pebetsi and Makobane. Both of these learners mixed up hyperbolic and linear functions as illustrated in Makobane's response in Figure 6.16 below.

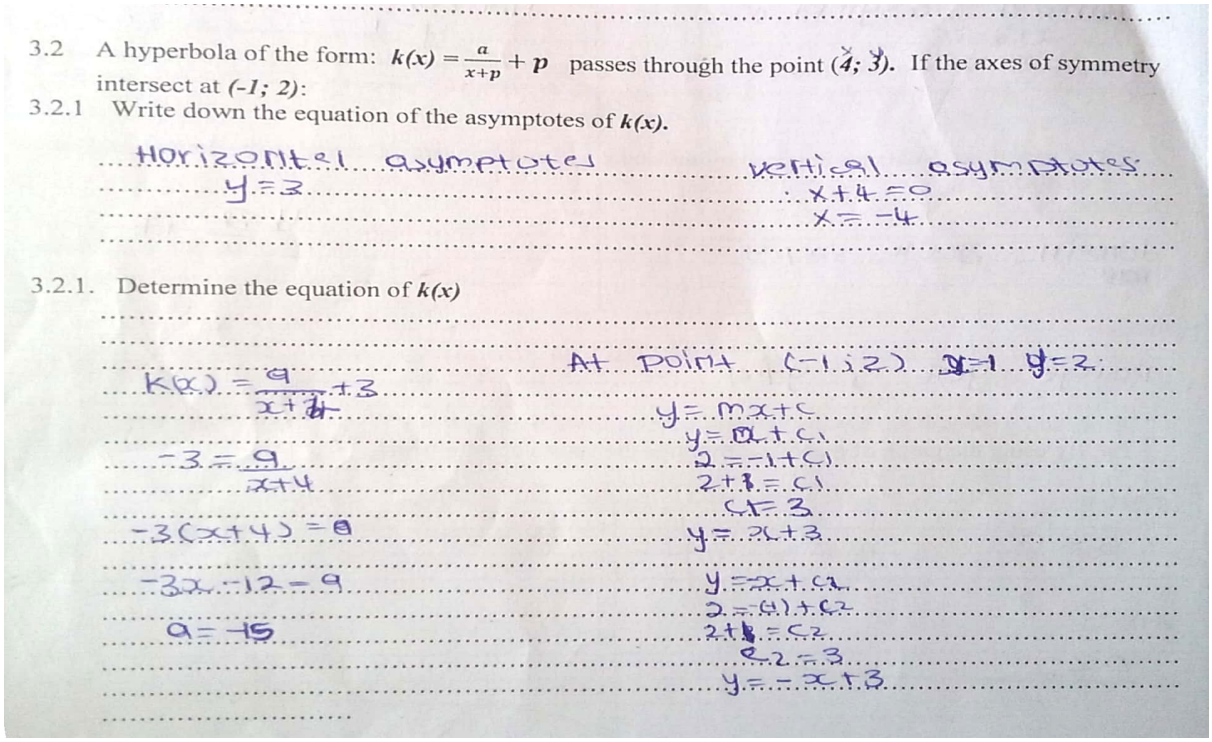


Figure 6. 16. Makobane's response to question item 3.2

In determining the equation of the hyperbolic function, $k(x)$, the learner makes a substitution (incorrectly though) into the general equation given. As she proceeds to the next step, she substitutes point $(-1; 2)$ into a linear equation. Apparently, there is no connection between the two steps. When asked to explain this, the learner indicated that she thought the question required her to calculate the equation of the axes of symmetry. She however could not justify why she used the hyperbolic equation in the first place. Going by her explanation, the error could be categorised as a *decoding* error due misunderstanding of the language and terminology used. However, given the fragmented way in which the learner mixed the two functions, one may view this as a case of *hybridization* error as the learner might have thought there is a connection between the two functions when in fact there is no connection.

6.7. Discussion of the findings on errors and misconceptions

In our everyday life, errors are viewed as simple mistakes people make that, once identified, may be corrected or rectified. Literature reviewed in Chapter 4 of this study showed that mathematical errors are not easy to identify and to correct, though teachers generally treat them as such. What makes mathematical errors more complex and not easy to deal with is that the errors we see displayed by learners are results of some cognitive structure deep seated in the learners' mind (Zembat, 2008; Michael, 2002; Clement, 1993). These underlying cognitive structures are called misconceptions.

A misconception is a perception or conception that is not compatible with the opinion commonly agreed on by experts on a particular subject (Zembat, 2008). Some scholars view it as students' conception that produces a systematic pattern of errors (Clement, 1993), while others such as Michael (2002) define it as the inconsistency between the concept that we want students to learn and the mental model that they build in their minds.

What is common in all these definitions is that misconceptions contradict or are incompatible with the accepted body of knowledge. They are a conception or perception that one has developed and reveals or manifest themselves through errors. In light of these theoretical insights, identifying learners' errors without diagnosing the misconceptions underlying them is not only inadequate, but futile.

Analysis of data in this study has revealed several errors that learners make under hyperbolic functions. In line with what I have highlighted in the preceding paragraph, an attempt was made during data analysis to infer on the misconceptions underlying the identified errors. This was made possible mainly from the data collected during task-based interviews when learners explained and justified their written responses. I should however admit that establishing learners' misconceptions may never be done with a high degree of certainty. We can only infer on learners' internal thinking processes based on how they respond to problem questions and what they say. I came up with five misconceptions which I discuss below and the common errors emanating from each of the

misconceptions. The misconceptions are: algebraic; conceptual; asymptotic; graphical and notational misconceptions.

6.7.1. Algebraic Misconceptions

Dealing with hyperbolic functions involves algebraic fractions. Algebraic skills are developed from much lower grades and one would expect that by the time learners get to grade 11, they would have mastered the art of simplifying algebraic fractions and solving equations with fractions. This assumption is not always true, as was confirmed by the findings from this study.

Learners displayed a number of shortcomings in their execution of algebraic manipulations. Though algebraic misconceptions may not be directly linked to the understanding of function concepts, they negatively influence the learners' ability to solve a given problem. In the context of hyperbolic functions, learners are confronted with algebraic fractions when determining x-and y-intercepts and points of intersection. Usually after determining these critical points, they may be required to sketch the graph of the function and once they made errors in their algebraic manipulations, their graphical representation is distorted. Examples of errors that emanated from algebraic misconceptions are highlighted in the responses given in Figure 6.17 below.

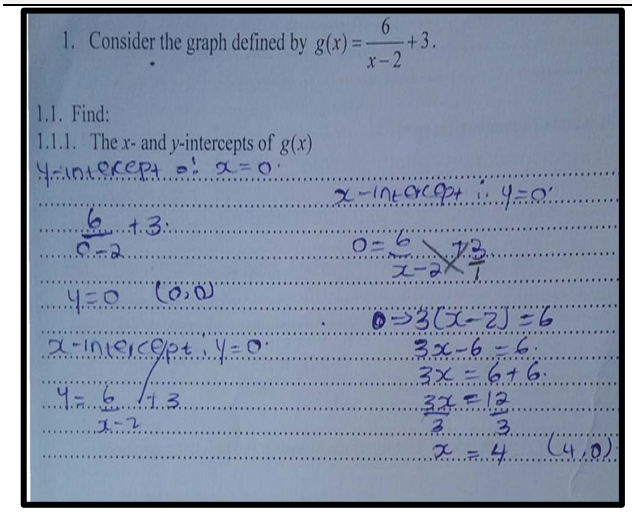
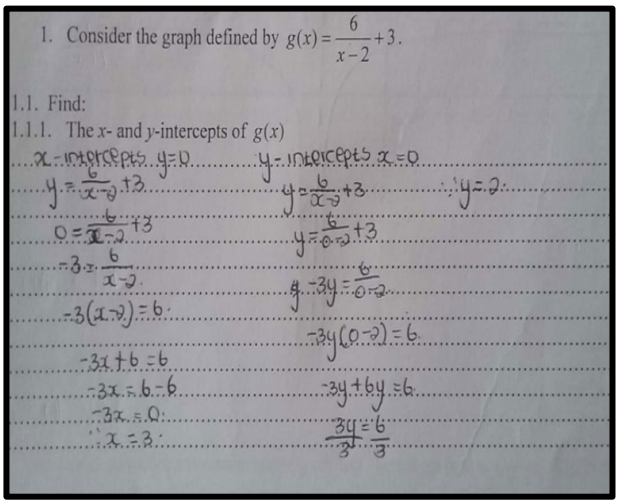
Response A	Response B
	

Figure 6. 17. Learner responses showing algebraic errors

The following error categories were identified under algebraic misconceptions:

- **Executive errors** – As Donaldson (1963) explains, executive errors involve the failure to carry out procedures even though the required concept has been understood. Both learner A and B picked the correct approach but failed to execute algebraic procedures to solve the equation.
- **Procedural-extrapolation errors** – From response A, the learner applied cross multiplication over addition. The learner erroneously extrapolated that if we can cross multiply: $\frac{\mathbf{6}}{\boldsymbol{x-2}} = \mathbf{3}$, we can do the same for $\frac{6}{x-2} + 3$. Usually such misconceptions arise when learners are taught procedures without an understanding of the mathematical justification behind the procedures. When asked to explain their responses, the learners who made this error did not see any problem with their working. They were even surprised why they could not obtain the correct answer. This underscores the need to mathematically empower learners by making them realize why some procedures or rules may work in specific contexts but not in other contexts.

  Learner B displayed a surprising error of operational extrapolation in relation to the number 0. In the first instance, the learner got the equation$-3x = 0$ and went on to say $\therefore x = -3$. Clearly she divided both sides by -3 and failed to divide -3 into 0. In the second instance, the learner failed to expand: $-3y\ (0 - 2) = 0$ and got$-3y + 6y = 6$**.** In both instances, the learner is wrongly and consistently applying the additive identity property of the number 0. This property, which is applied when carrying out addition operations, states that when a number is added to zero it gives the same number. The learner is extrapolating this property to apply on multiplication operation. Such an error at this level is evidence of the cumulative nature of misconceptions. Most likely, the learner picked-up this misconception in the early senior phase and carried it through to the FET phase. In my view, such confusion may arise

from making learners too reliant on calculators, which makes them loose the sense of numbers, particularly in the lower grades of their mathematics learning.

6.7.2. Conceptual Misconceptions

The findings of this study revealed that some of the errors learners made emanated from their lack of conceptual understanding of the concepts under hyperbolic functions. These concepts include intercepts, asymptotes, domain, range, x- and y-axes and the axis of symmetry. The learners' written and verbal responses demonstrated that they only had a mechanical or procedural understanding of how to determine the features of the function but lacked an understanding of what they represent. This conceptual shortcoming exposed them to a number of errors when confronted with unfamiliar problem situations.

Literature reviewed on conceptual knowledge (the *know-why*) and procedural knowledge (the *know-how*) of mathematics has shown that while these two notions of mathematical knowledge complement each, learners with a strong conceptual understanding stand a better chance of carrying out procedures with minimal errors and can assimilate and accommodate new knowledge more easily (Skemp, 1976; Suh, 2007; Hiebert & Carpenter,1992; Hiebert & Lefevre,1986; Eisenhart et al, 1993).

Functions, as a unifying mathematical concept (Ponte, 1998), requires learners to have a deep and relational understanding of the concept that enables them to link and connect with other mathematical ideas. The notion of function permeates through many other mathematical ideas. A shallow, rigid and instrumental understanding of the concept makes it difficult for learners to navigate through other mathematical ideas.

The findings of this study confirmed Hiebert and Carpenter's (1992) observation that, in most cases, mathematics is presented to learners as mysterious procedures which when correctly followed lead to success in examinations. The following findings from this study attest to this observation:

- **Intercepts given as equations not a point** – In answering questions that required the x- and y-intercepts, some learners attempted to derive the equation of a line. During task-based interviews, I tried to check if these learners understand what an intercept is. I presented a hyperbolic graph to them and asked them to finger-point the x- and y-intercept. As I had suspected, they could not identify the intercepts. Some even pointed the axes as intercepts.
- **Learners failed to justify why they substituted $x = 0$ for y-intercept and $y = 0$ for x-intercept** – In responding to the question, which required the coordinates of the x- and y-intercepts, nearly all the learners demonstrated some procedural understanding that for x-intercept, we substitute $y = 0$ and for y-intercept we substitute $x = 0$. When asked to explain or justify the approach individually, none of them gave a sensible mathematical explanation. Even those who gave perfectly correct responses had no idea as to why they substituted x and y with 0. To them, it was a given rule and because it worked for them, they were comfortable with it.
- **Failure to link domain and range to the asymptote** – The inability by the learners to connect the domain and range of a hyperbolic function to its asymptotes further exposed their lack of conceptual understanding. They only had a memorized understanding that when $f(x) = \frac{a}{x-p} + q$, then the domain is written as: $x \epsilon R, x \neq p$ and the range: $y \varepsilon R, y \neq q$. When the algebraic equation was slightly twisted or transformed, the majority of the learners failed to give the domain and range. On the other hand, when given the domain and range of a hyperbolic function, the learners failed to extract its asymptotes.

6.7.3. Asymptotic Misconceptions

The concept of asymptote is very central in understanding of hyperbolic functions, and yet it was the least understood by the learners. The learners' responses during interviews revealed that they had either a wrong or an incomplete conception of an asymptote. In the first place, asymptotes are usually shown on a graph using dotted lines. When they did not see the dotted line, they assumed the

asymptote was not there. On the question item which required learners to write down the equations of asymptotes given on a hyperbolic graph, the majority either did not attempt the question or gave wrong responses. Those who did not attempt indicated that they thought the asymptote was not there. They could not figure out the asymptotes from the shape of the graph. A few other learners presented the asymptotes as a point and not an equation. Their thinking was that if the vertical asymptote is $x = 2$ and the horizontal asymptote $y = 3$, then the asymptote is the point (2; 3).

When each of the learners were asked to describe their own understanding of an asymptote, the common response was that it is, ' … a dotted line which divides the graph'. Others described it as '… dotted lines that determine the quadrants in which the graph will lie'. The responses reflected that learners lacked a correct concept image of what an asymptote is and how it relates to other concepts like domain, range and axis of symmetry.

6.7.4. Graphical Misconceptions

One of the most prevalent weaknesses that emerged from the analysis of the learners' responses was that their conception of functions was confined to the algebraic representation. They struggled to represent a function given in symbolic terms as a graph or to identify the critical features of a function from a given graph. They could not appreciate the fact that the algebraic equation of a function and its graph are representations of one thing. This shortcoming resulted in the following errors:

- **Representational errors -** No link or incoherence between algebraic equation and graph. Sometimes the graph was drawn cutting the asymptotes.
- **Meta-cognition errors** – Learners failed to reflect on the sensibility of calculations done. For example, given $\boldsymbol{f(x) = \frac{a}{x+p} + q}$, the sign of $\boldsymbol{a}$ indicates the quadrants in which the graph should lie. If learners erroneously calculated the x- and y-intercepts and found them in the wrong quadrants, they could not reflect on their working to correct the error.

- **Logically - Invalid Inferences** – In the case where both the x- and y-intercept was the origin, most of the learners wrongly concluded that the graph would have one curve instead of two symmetrical curves. They could not give any reasonable justification for their inference, except that they were used to cases where the intercepts were two distinct points.
- **Hybridization error –** Some of the learners were confusing the graph of a hyperbolic function to that of the quadratic function. After showing the vertical and horizontal asymptotes, they still went further to draw a parabolic curve. This simply confirmed the view that the learners had no understanding of what the graphs were showing.

One major weakness of the traditional approach of teaching functions is that it is restricted to the algebraic representation of functions with limited exposure to other forms of representation. This deficiency makes it difficult for learners to grasp the link between different representations of a function or switch between them.

This finding corroborates Even's (1998) observation that the ability to identify and represent the same thing in different representations, and flexibility in moving from one representation to another, allow learners to see rich relationships and develop a better conceptual understanding. The integration of ICTs enriches the learning contexts and provides teachers and learners with unlimited opportunities for multiple representations of functions. However, the realisation of this potential depends on how well the technological tools are used to foster knowledge construction.

6.7.5. Notational Misconceptions

Learning obstacles emanate, largely, from failure by the learners to correctly read the meaning of functional notations and symbols. Mathematical language is largely symbolic and notational. The learners' misconceptions of the symbolic have a wrong conception of the symbolic representation of a mathematical idea can be a barrier to their understanding of concepts. The following errors resulted from notational misconceptions:

- **Interpretation Errors –** learners failed to correctly interpret questions such as: 'For what values of x is f(x)$\boldsymbol{f(x) \geq 0}$?
- **Decoding and Encoding Errors –** Learners failed to understand the meaning of symbols and terminology resulting in answering unintended questions. For example, learners struggled to see the link between $\boldsymbol{f(x+3)}$; $\boldsymbol{f(x)+h}$; $\boldsymbol{f(-x)}$ and the original function f(x).

6.8. Conclusion of the chapter

The findings presented in this chapter revealed that learners make several errors mainly as a result of their restricted conception of hyperbolic functions. Some errors were not directly linked to hyperbolic functions but emanated from algebraic misconceptions. It came out that learners' understanding of hyperbolic functions was largely mechanistic, procedural and restricted to the algebraic representation of the function. This exposed them to conceptual misconceptions.

The lack of exposure to other forms of representation of the function made learners very uncomfortable in dealing with graphical representations. Thus, graphical misconceptions constrained the learners' ability to flexibly switch from symbolic representation to graphical representation of a given function and vice-versa.

The asymptotic nature of hyperbolic functions presented conceptual and graphical challenges to the learners. The concept of asymptote and what it represents was not understood at all. This was evident in the representational errors that learners made. Some of the errors resulted from the failure to understand the language and notational symbols used under functions. This resulted in learners answering unintended questions or completely failing to understand the question.

So far, this study has identified the learners' errors and diagnosed the misconceptions underlying the errors. The next phase of the study is to develop an ICT-mediated instructional programme targeted at enhancing learner understanding of hyperbolic functions and bridge the knowledge gaps.

CHAPTER 7: AN ANALYSIS OF THE IMPACT OF THE ICT-MEDIATED INSTRUCTIONAL INTERVENTION AND DISCUSSION OF FINDINGS.

7.1. Introduction

The central focus of this chapter was to address the second research question of this study which sought to establish the impact of ICT-mediated instructional intervention on learner conceptualisation of concepts under hyperbolic functions. The preceding chapter unearthed errors and misconceptions displayed by the learners in dealing with problem situations in this topic.

The findings from this study revealed that learners' understanding of functions was mainly procedural and mechanical, with no understanding of concepts involved. This chapter presents a detailed qualitative comparative analysis of learners' responses in the pre- and post-test and the discussion of the findings.

As explicated under the operational definition of terms (see Chapter 1), ICT in education is understood as an umbrella term referring to any technological tool or device integrated in the teaching and learning process. In this study, the intervention lessons were centred on the use of Autograph software in conjunction with other supporting technological tools like projectors, whiteboard and whiteboard markers. I begin the chapter by briefly discussing the analytical framework adopted for data analysis.

7.2. The Analytical Framework

The APOS theory provided the theoretical lens to establish whether there was any change in the learners' conception of concepts under hyperbolic functions before and after the instructional intervention. As detailed under the conceptual framework (see chapter 3) of this study, APOS theory assumes that mathematical knowledge which is possessed by an individual is the interaction outcome with other people and the result of his/her mental constructions in comprehending mathematical ideas (Syaiful et al, 2014). The proponents of this theory identified four mental constructions namely: ***A****ction,* ***P****rocess,* ***O****bject* and ***S****cheme*.

Dubinsky et al. (2000) argue that comprehending a mathematical concept begins with the manipulation of existing mental constructions or manipulating physical objects to form *action.* When action is repeated or contemplated upon, it is interiorized to form the *process* which is then encapsulated (crystallised) to form the *object.* The object can be decomposed (or decapsulated) to become *process* again. The *action*, *process* and *object* constructions can be thematized or organised in order to generate a *scheme* (Syaiful et al, 2000; Maharaj, 2010; Dubinsky & McDonald, 2002). The diagram in figure 3.2 illustrates these mental constructions and the mental processes connecting them.

The *action-process-object* and *scheme* mental constructions were viewed in this study as developmental layers that depict the level of conception or depth of understanding of a concept. The genetic decomposition (also presented under the conceptual framework), specified indicators under each APOS conception level in relation to hyperbolic functions (APOS-Hyperbola). In analysing the learners' responses to the question items and their responses to interview questions, the indicators allowed the researcher to determine the learners' levels of conception before and after the instructional intervention. Of paramount importance in this whole process was an attempt to establish any evidence of whether the instructional intervention facilitated the mental processes of interiorization (from action to process), encapsulation (from process to object) and thematization (from object to scheme).

It is, however, difficult to determine with certainty an individual's conception level of a concept. This is mainly because we are dealing with abstract internal mental processes. Even Dubinsky et al. (1992) reported the difficulty they encountered in determining whether an individual's function concept was limited to action or that he/she had a process conception. Thus, any interpretation of individual responses may not be made with absolute reliability. In an attempt to counter this challenge, I triangulated the analysis of the learners' written responses with what they said during interviews as

they explained or justified their written responses. The following section presents the APOS analysis of the learners' responses in the pre-test

7.3. APOS Analysis of Learners' responses in the Pre- test

The purpose of the pre-test in this study was two-fold. Firstly, the pre-test helped the researcher to establish the learners' errors and misconceptions under hyperbolic functions. Secondly, the pre-test was used to establish the learners' APOS level of conception of hyperbolic functions before the instructional intervention. In accordance with the work schedule provided by the DoE in the province, the topic of functions was taught in term two around May. Initially, the learners were taught without the use of any ICT tool, except for the non-programmable calculators. This approach is described in this study as the traditional approach. The analysis of learners' responses in the pre-test aimed to establish the level of conception that the learners attained after having been taught without the use of ICT tools.

Each learner's response for each question item was classified under one of the four categories: '**No attempt'**, '**Incorrect**', '**Partially correct**' or '**Perfectly correct**'. 'No attempt' means the question item was left blank. 'Incorrect response' is when the approach used to answer the question was wrong and the working did not in any way lead to the solution of the problem. A 'partially correct' response is when some steps of the working were done correctly or some parts of the question were answered correctly but some were incorrectly done. A 'perfectly correct' response is when the most appropriate approach was used to solve the problem and the working was correctly done.

Table 7.1 below presents a summary of learners' responses in each of the question items in the pre-test in terms of APOS conception levels.

Question No	Question Item	APOS Level	No Attempt	Incorrect Response	Partially Correct	Perfectly Correct
1	1.1	A	0	1	3	9
	1.2	A	0	0	4	9
	1.3	P	0	4	6	3
2	2.1	P	1	7	3	2
	2.2	O	0	13	0	0
	2.3	P	1	5	6	1
	2.4	O	2	11	0	0
3	3.1	O	0	6	6	1
	3.2	S	0	11	1	1
	3.3	S	0	12	1	0

Key: A – *Action*; P – *Process*; O – *Object*; S - *Schema*

Table 7. 1. Summary of learners' responses to each question item

Table 7.1 shows the question items in the pre-test, the APOS level of each of the question item and the number of responses under each of the response categories described above. The APOS level signifies the cognitive effort required to respond to that problem and was informed by the criteria stipulated in the genetic decomposition. The response categories reflect the quality of responses given by the learners and inform us on what the learners were able to do and what they were not able to do. Of particular interest at this point is the comparative analysis of the quality of learner

responses under each APOS level. A graphical illustration of the responses under each level is given below,

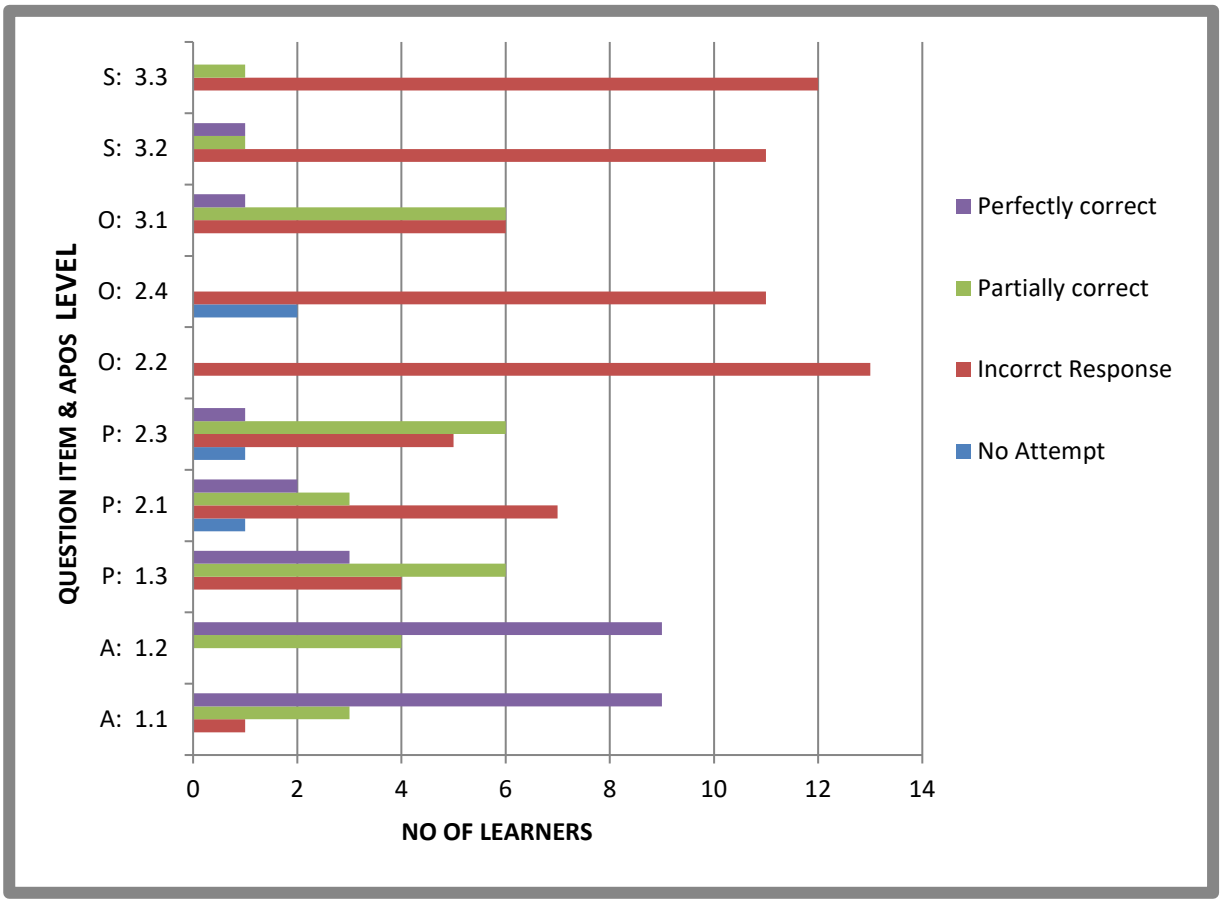


Figure 7. 1. A compound bar graph showing question items and number of responses under each response category

Figure 7.1 shows that questions falling under *action- level* have high number of 'perfectly correct' and 'partially correct responses' and much lower responses under 'no attempt' and 'incorrect responses'. At *process-level*, the number of 'perfectly correct' responses drop noticeably and adds on to the partially correct responses. There is an increase in incorrect responses at process level. As we move up to the question items under *object-level* and *schema-level*, we note that very few responses were perfectly correct. Most of them were either incorrect or partially correct. To further illustrate this point, the table below shows the average percentage of each response category under each APOS level.

Table 7.2 below shows the average percentage of each response category under each APOS level.

APOS LEVEL	NO ATTEMPT	INCORRECT RESPONSE	PARTIALLY CORRECT	PERFECTLY CORRECCT
A	0%	3.8%	26.9%	69.2%
P	5.1%	41.0%	38.5%	15.4%
O	5.1%	76.9%	15.4%	2.6%
S	0%	88.5%	7.7%	3.9%

Table 7. 2. The average percentage of responses for each APOS level

As highlighted above, the APOS level reflects the cognitive demand of the question items. Cognitive demand, as Doyle (1988) defines it, is the cognitive processes students are required to use in accomplishing a task. If we focus our attention at the lowest APOS level, that is *action* level, the questions required learners to reproduce previously learnt facts or procedures.

Stein, Smith, Hennington and Silver (1993) described such tasks as memorisation tasks. Specifically, in this case, the questions required learners to calculate the coordinates of the x and y-intercepts given hyperbolic algebraic equation. The learners were also expected to determine the equation of asymptotes from the equation. Such questions have a low cognitive demand as they are characterised by mere reproduction of what has been learnt with no connection to the concepts or meaning that underlie the facts. From Table 7.2, we note that 69.2% of the learners provided perfectly correct

responses to questions under action level, 26.9% provided partially correct responses and only 3.8% gave incorrect responses. None of the learners left a question not attempted at *action* level.

The results for the *action*-conception level are presented on the pie chart in Figure 7.2 below:

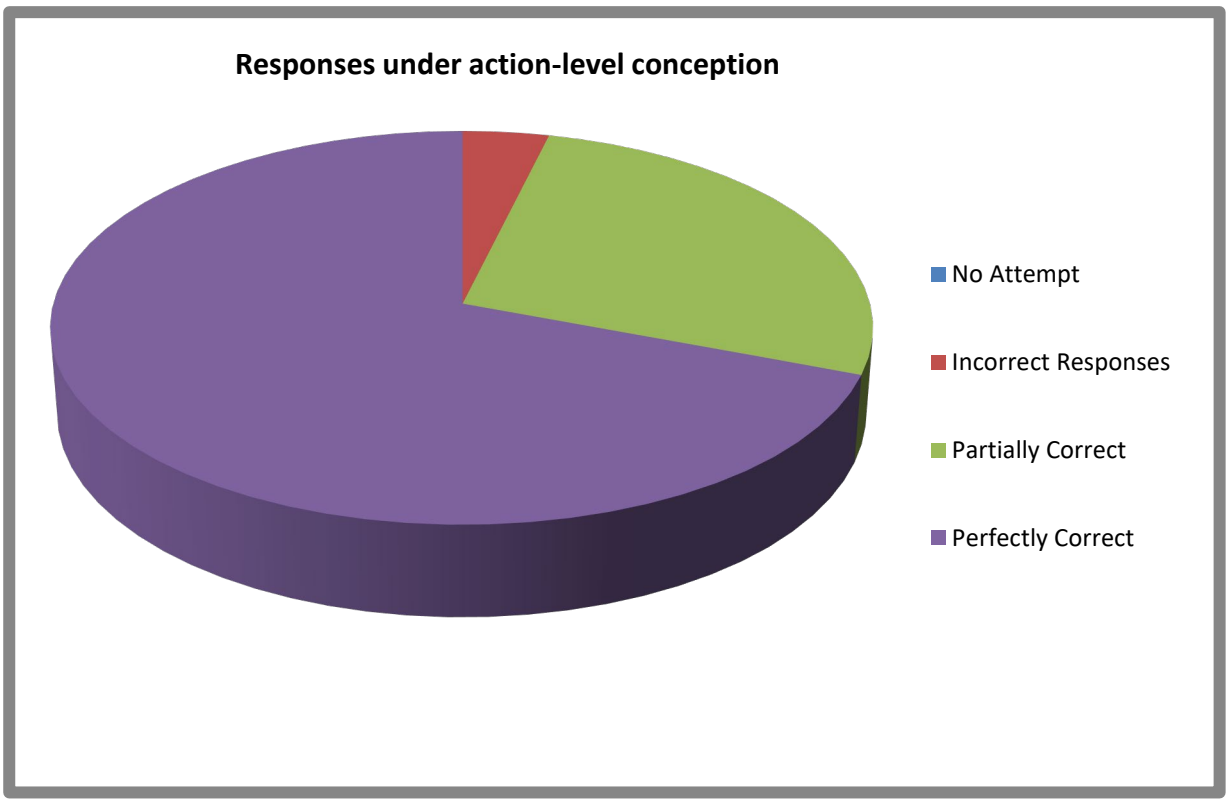


Figure 7. 2. Learner responses to Action-level question items

The responses revealed that eight of the learners had no problems in dealing with problems at *action* level. Their responses were perfectly correct without any errors. Responses from four learners were partially correct. These learners picked the correct procedure or approach, but they made some errors in executing their calculations, as shall be detailed in the next section. The incorrect responses came from one learner who demonstrated no knowledge of the concepts of intercepts and asymptotes. The learner seemed to be operating at the pre-function or pre-action conception level.

The next level of cognitive demand is the *process* level. Stein et al. (1993) described this level as 'procedure *without* connection'. The key focus at this conception stage is the correct execution of procedures to produce the correct answer, with no mathematical understanding. Table 7.2 reveals that only 15.4% managed to give perfectly correct responses. This represents a huge drop from the

previous conception level. 38.5% presented partially correct responses and 41% responded incorrectly. The pie chart in Figure 7.3 below illustrates the proportion of each response category at the *process*-level.

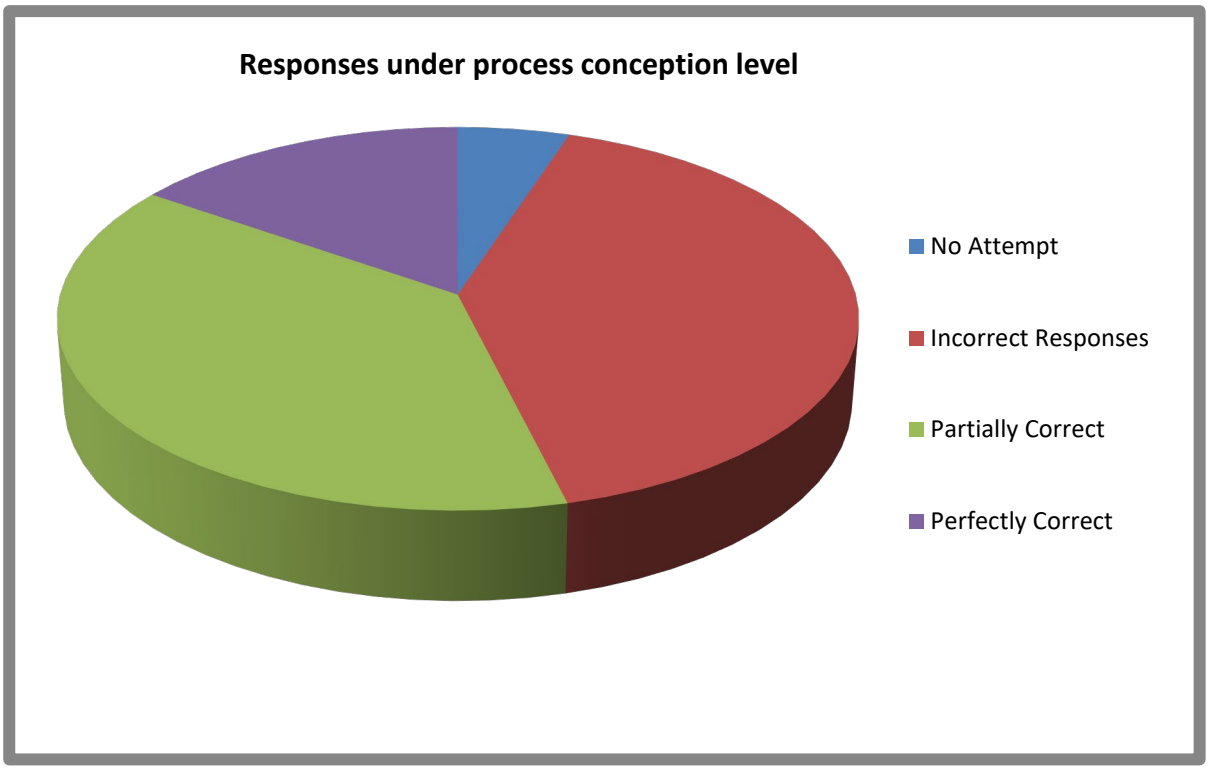


Figure 7. 3. Learner responses to Process-level question items

The question items under the process-level required learners to use the features of the hyperbolic function they had calculated to sketch the graph. They were also required to identify asymptotes from a given hyperbolic graph. About 5.1% of the responses were 'no attempt'. It came out later in the interviews that whenever learners were not sure of the procedure to use, they preferred to leave the question not attempted.

An interesting observation was that these learners never attempted to use the results they obtained in items 1.1 and 1.2. One possible explanation to this is that the learners did not recognise the connection between what they calculated and the graph. We also notice that the number of incorrect responses increased significantly. These learners attempted to sketch the graph but struggled to show the connection between the algebraic equation and its graph. The category of 'partially correct' and 'perfectly correct' responses also dropped significantly. Perfectly correct responses came from two

learners and partially correct responses came from three learners. These learners demonstrated that they had attained some *process*-conception, but, the prevalence of errors was an indication that the conception was not yet fully developed, as shall be discussed later.

According to the genetic decomposition (see Table 3.1), learners who had attained the *object* level of conception were expected to be able to describe how a given hyperbolic function has been transformed by looking at its graph or equation and relate the transformed function to its graph. The question items required learners to give the asymptotes, domain and range of a transformed hyperbolic function. The transformations were restricted to vertical and horizontal translations. The proportion of learners' responses is illustrated in the pie-chart below. The proportion of responses under the *object* conception is illustrated in the pie-chart in Figure 7.4 below.

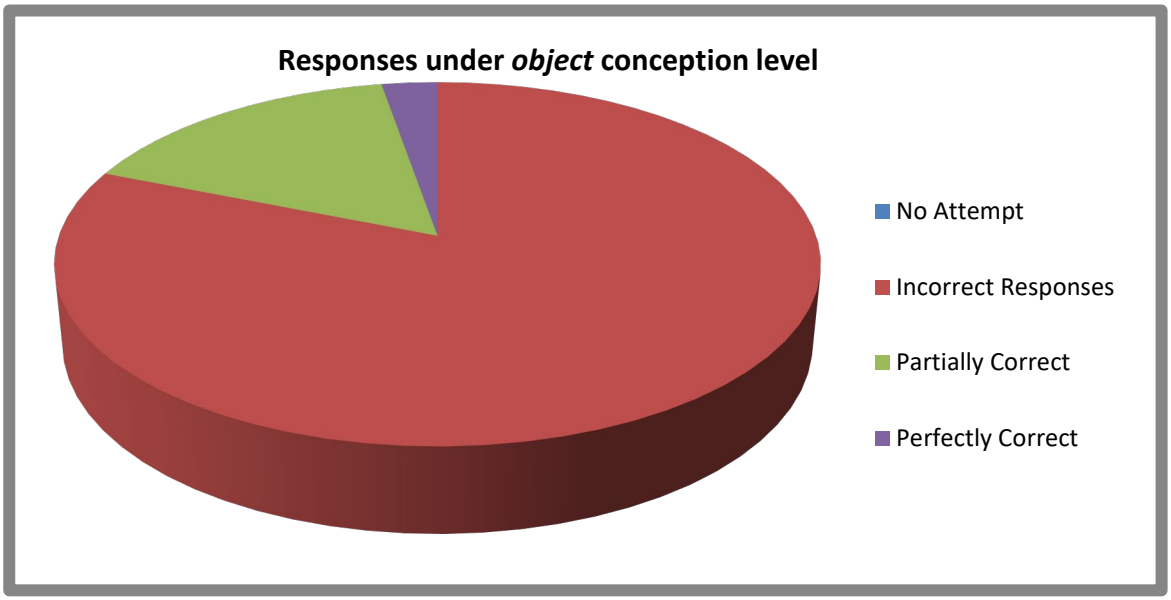


Figure 7. 4. Learner responses to *Object*-level question items

The results show that only 2.6% of the responses were perfectly correct. These responses came from one learner who demonstrated a strong *object*-conception. 5.4% of the written responses were partially correct and these came from two learners. 79.6% of the responses were incorrect and 5.1% of the responses were no attempts.

Focussing on the *schema* level, we notice that only 3.9% of the responses were perfectly correct, 7.7% were partially correct and 88.5% were incorrect. The results are illustrated in the form of a pie-chart in Figure 7.5 below.

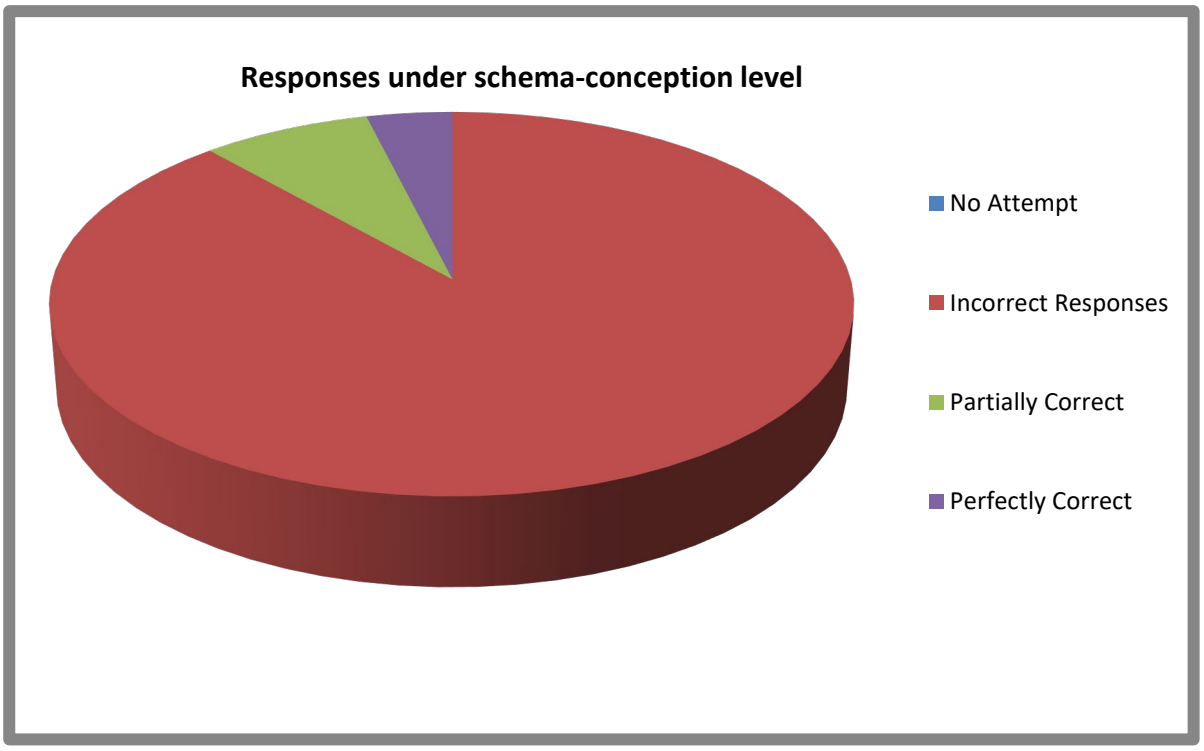


Figure 7. 5. Learner responses to *Schema*-level question items

The bulk of the responses were presented as 'incorrect responses'. This was not surprising because question items at this level demanded a relatively higher cognitive effort and deeper understanding of concepts. *Schema* level questions called for learners to explore the nature of concepts and their relationships.

The preceding analysis has already revealed that learners found it difficult to operate beyond the *action*-conception level. This dimension of understanding was missing in most of the learners. Failure to have a strong *process*- and *object*-conception of the concepts makes it very difficult for learners to apply what they have learnt in unfamiliar and non-routine problems.

From this preliminary overall analysis of learners' responses in the pre-test, I noted that the majority (about 70%) of the learners were operating at the *action*-conception level of understanding hyperbolic

functions. About 25% were at the process-conception level and only a handful (less than 5%) displayed some traits of the *object* and *schema* conception level.

The learners were then taken through ICT-mediated instructional intervention lessons, which were planned and administered as described in the next section.

7.4. Reflections on the Intervention lessons

Before presenting my analysis of the impact of the ICT-mediated intervention lessons, I found it prudent to take the reader through my personal reflections of what transpired during the intervention lessons. My reflections were informed by my experiences and observations during the lessons as a participant observer and by the post-lesson observations done based on the video clips after each of the lessons.

As I explained under Section 5.5.3, I did post-lesson observations with the assistance of a colleague, who is an experienced mathematics educator. The observations and reflections were recorded in my reflexive journal and were very valuable throughout my analysis of data.

The planning and delivery of the intervention lessons was informed by the conceptual framework (see Section 3.3) based on the third generation Activity theory as propounded by Engestrom (1987) and social-constructivist constructs of mediation, ZPD, scaffolding and internalisation. The key assumption underlying the activity system is that the introduction of computers as meditational artefacts inevitably forces a change in the conventional classroom activity system. It challenges the usual ways of interacting around the object of the system and consequently requires new ways of acting.

To sum up how the Activity theory informed the classroom computer-mediated activity system in the classroom, the teacher (s*ubjec*t) acted on the learners' understanding (*object*) in order to transform it using computer-based activities (*mediating artefact*) in order to arrive at deeper understanding of mathematical content (*outcome*).

The teacher and the learners' pedagogic positions were influenced by rules, which included everyday classroom rules, computer lab rules and the general school rules. These were communicated to the learners, agreed upon and pasted on the classroom wall. The teacher, the learners and the lab technician constituted the community of the activity system, all working together on the targeted *object*. There was division of labour whereby learners were expected to engage with the learning activities presented to them by the teacher and the lab-technician provided technical assistance regarding the functionality and operation of the computers.

- **Reflection on lesson 1**

The *object* of this lesson was to enable learners to identify the key features of a hyperbolic function and describe the effect of $\boldsymbol{a}$ on $(x) = \frac{a}{x}$. This content was covered from grade 10, but given the fact that the learners were initially taught without computers, it was necessary to ensure that learners have a good conceptual base by beginning with concepts covered in grade 10. The CAPS document does not require learners to solve problems involving real life application of hyperbolic functions. Even the prescribed learners' textbooks introduce hyperbolic functions by simply giving its algebraic equation.

The lesson went on as planned without any serious challenges. The learners did not show any difficulties in using the Autography software to draw the graphs. Although each learner was working on his/her computer, they required individual attention at various points during the lesson. Consequently, they completed individual tasks at different times which made it a challenge when switching from individual activity to group work activity or whole class discussion. Video evidence reveals that learners collaborated a lot, even during individual activities. When they got stuck, they preferred to check with their peers before calling the teacher or the lab assistant.

Another interesting observation was that the learners seemed to have a different *object* to that of the teacher. The focus and excitement of the learners was centred on the computer itself, and not the content or concepts to be learnt. Once they managed to correctly draw the graphs on the computer,

they would have achieved their objective without even attempting to answer questions based on the graphs. The teacher had to constantly redirect the learners' focus and emphasize the need to make conclusions based on the information gleaned from their graphs.

- **Reflection on lesson 2**

Activities planned for this lesson were done successfully. Learners managed to draw the graphs quickly as they demonstrated improved proficiency in using the Autograph software. This gave learners more time for discussion on the patterns emerging from the changing features of the graphs. In addition, learners were able to identify the vertical and horizontal asymptotes, the domain and range for each of the functions, but they still struggled to give a general description of the effect of the parameter $\boldsymbol{q}$. They observed that the vertical asymptote and the domain remained unchanged as we add or subtract $\boldsymbol{q}$, but they could not bring out a general verbal description of the effect of parameter $\boldsymbol{q}$ on $f(x)$. It was only after the graph animation, whereby learners could visualise the graph shifting up and down, that the learners noted that when $\boldsymbol{q}$ is positive the graph moves vertically upwards and when its negative it moves vertically downwards.

The learners' homework exercise showed that nearly all them were able to extract the asymptotes, the domain and range from the algebraic equation and subsequently managed to sketch its graph. However, in answering the questions, some learners still used the algebraic approach without using the knowledge of vertical translation of the function.

- **Reflections on lesson 3**

The activities for this lesson went on without any hiccups. The Autograph software enabled learners to draw the graphs accurately and much faster. Nearly all the learners did not encounter any challenges in using the graphs to complete the table. They could easily identify the asymptotes, the domain and range for each of the graphs. However, the persistent weakness displayed by the learners was that they had a 'point-wise' conception of the functions and their graphs. They could only

interpret the graphs in terms of their isolated key features but could not view the function as a total entity. This limited conception made it difficult for learners to give a general conjecture on the effect of $\boldsymbol{p}$.

Graph animation was useful in assisting learners to overcome this conceptual limitation. As learners changed the values of the parameter $\boldsymbol{p}$ on the constant controller, they could visualise the whole graph moving 'in real-time' to the left (if $\boldsymbol{p}$ is positive) and to the right (if $\boldsymbol{p}$ is negative). It was only after this activity that the learners were able to describe $f(x+p)$ as a horizontal translation to the left or right.

- **Reflections on Lesson 4**

Having drawn the graphs of the three functions given, learners worked in pairs and labelled the x- and y-intercepts. This was relatively easy for the learners as they assisted each other and compared their graphs. Nearly all the learners were able to correctly complete the table by writing the coordinates of the identified intercept points. Surprisingly, more than half of the learners struggled to bring out that for all x-intercepts, the value of y is 0 and vice-versa. This was the focus of the activity and once learners had acquired a graphical understanding of intercepts, they would better appreciate the algebraic procedure in calculating intercepts.

The algebraic calculations of the x-and y-intercepts were done by the learners individually. Some of the learners were still making several executive errors but they were able to correct themselves as they compared their working amongst themselves and their solutions to the graphs. Decoding the functional notation$f(0)$ and $f(x)=0$ was a challenge to most of the learners. They failed to note that $f(0)$ is the y-intercept and $f(x)=0$ gives the x-intercept.

- **Reflection on lesson 5 and 6:**

The activities for lesson 5 were similar to those in lesson 6. The central focus of lesson 6 was to guide learners to discover the *end behaviour* of the graph as x approaches a certain value, $\boldsymbol{a}$, from the left

and from the right. Eventually, learners managed to bring out that as x approaches $\boldsymbol{a}$ from one side, $f(x)$ approaches ∞ and as x approaches $\boldsymbol{a}$ from the other side, $f(x)$ approaches $-\infty$. I consolidated the learners' understanding of a vertical asymptote by giving the following definition:

Let $\boldsymbol{a}$ be a real number. The line given by $x = a$ is a vertical asymptote of the graph $y = f(x)$ if at least one of the following statements is true:

$$As\ x \to a, f(x) \to \infty$$

$$As\ x \to a, f(x) \to -\infty$$

The lesson activities for lessons 5 and 6 were initially planned to be covered in one lesson but after taking note of the time frame learners took to complete the activities, the lesson was split into two. The lessons were targeted towards enhancing the learners' understanding of the concepts asymptote, domain and range. An analysis of the learners' written and oral responses prior to the lessons revealed that learners wrongly conceived asymptotes as 'dotted lines that divide the graph of a hyperbolic function'. Others used the terms asymptotes and axes interchangeably.

Learners were taken through guided discovery activities which enabled them to identify vertical and horizontal asymptotes from the graph by looking at the behaviour of the graph as $x \to \pm\infty$ and as $y \to \pm\infty$. The use of the *point mode* on Autography enabled learners to quickly complete the given activity. Though learners could easily see that $f(x)$ was approaching a certain value, $y = L$, they could not easily link this to the horizontal asymptote. Using a similar approach, learners were to investigate on their own the behaviour of the graph as $y \to \pm\infty$.

It was enlightening to see learners engaging with a probing question posed to them: *If* $y = 1$ is the horizontal asymptote, is there a value of x for which $f(x) = 1$*?* They attempted to use their graphs and some argued that as x continue to increase, we will eventually get that value of x. It was only after proving by calculation that we cannot find a value of x for which $f(x) = 1$ that they got convinced. This activity allowed learners an opportunity to appreciate that a hyperbolic function is

undefined at a certain value of x and y. As a result, they saw the connection between the asymptotes, the domain and range of a given hyperbolic function.

7.5. A comparative analysis of learners' responses in the pre-and post-test

This section presents a qualitative comparative analysis of the learners' responses in the pre-and post-test. As explained in earlier sections, the learners' written responses to each question item were categorised as: *No attempt*, *Incorrect*, *Partially correct* or *perfectly correct*. Each question item targeted a particular level of conception in terms of the APOS theory. In Table 7.3 below, I compare the quality of responses under each APOS level in the pre- and post-test.

QUESTION	**QUESTION ITEM**	**APOS LEVEL**	**NO ATTEMPT**		**INCORRECT RESPONSE**		**PARTIALLY CORRECT**		**PERFRCTLY CORRECT**	
			PRE	POST	PRE	POST	PRE	POST	PRE	POST
	1.1	A	0	0	1	1	3	3	9	9
1	1.2	A	0	0	0	0	4	1	9	12
	1.3	P	0	0	4	0	6	7	3	6
2	2.1	P	1	0	7	4	3	0	2	9
	2.2	O	0	0	13	8	0	5	0	0
	2.3	P	1	0	5	4	6	4	1	5
	2.4	O	2	0	11	8	0	4	0	1
3	3.1	O	0	0	6	2	6	8	1	3
	3.2.1	S	0	0	11	3	1	4	1	6
	3.2.2	S	0	0	12	4	1	6	0	3

Key: A- Action ; P- *Process* ; O- *Object* ; S- *Scheme*

Table 7. 3. Comparative analysis of learners' responses in the pre- and post-test

Table 7.3 shows the number of responses under each response category for each question item and to get a clearer picture of the quality of responses under each APOS level, Table 7.4 below illustrates the average percentage of responses for each APOS level in the pre-and post-test.

APOS LEVEL	NO ATTEMPT		INCORRECT RESPONSE		PARTIALLY CORRECT		PERFECTLY CORRECT	
	PRE-TEST	POST-TEST	PRE-TEST	POST-TEST	PRE-TEST	POST-TEST	PRETEST	POST-TEST
A	0	0	4%	4%	27%	27%	69%	69%
P	5%	0%	41%	21%	36%	28%	15%	51%
O	5%	0%	77%	46%	15%	44%	3%	10%
S	5%	0%	79%	27%	8%	38%	4%	35%

Table 7. 4. The average percentage of responses under each APOS level in the Pre-and Post-test

The comparative analysis of learners' responses in the pre- and post-test, as illustrated in Tables 7.3 and 7.4 above, was done per each conception level as detailed in the following sections.

7.5.1. A comparison of Learners' responses under *action*-conception level in the pre- and post-test.

From Table 7.3, we notice items that in question falling under the *action* level, there was a small difference the quality of responses in the pre-and post-test. Table 7.4 confirms this observation as the

percentage of responses under each response category remained the same. Figure 7.6 below provides a graphical picture of the learners' responses under the action conception level.

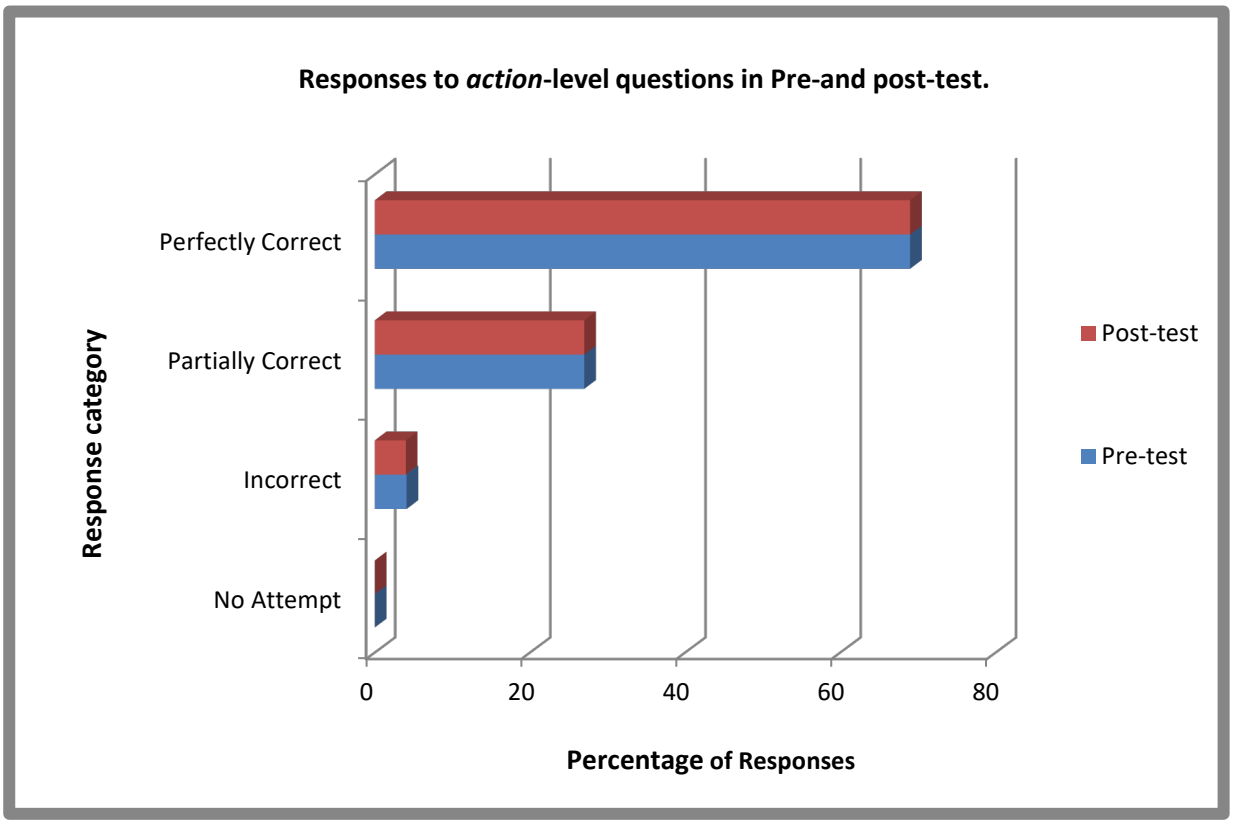


Figure 7. 6. A comparison of the Learners' responses to *action*-level questions in the pre-and post-test

From an item by item comparative analysis, we note that there was no change in the number of responses under each response category for question item 1.1. There was one learner who still could not answer this question even after going through the intervention lessons. Again, there was no change in the three learners who presented partially correct responses before and after the lessons. What these results seemed to suggest was that there were certain errors or misconceptions which could not be cleared during the intervention lessons. Considering that the question item required algebraic procedural fluency, this observation is not surprising. The central focus of the intervention lessons was more on deepening understanding of concepts than algebraic procedures. The intervention lessons might have failed to correct some algebraic misconceptions.

Despite the overall stagnancy in *action*-conception, there was a slight positive change in question item 1.2. The number of partially correct responses dropped from four before the lessons to only one after the lessons. Consequently, the number of perfectly correct responses rose from nine to twelve. This indicates some positive change because the ability to perfectly extract critical features of a function from its algebraic equation lays a good foundation for deeper understanding. This change may not be directly attributed to the use of the graphing software, but the repeated class activities in dealing with hyperbolic functions in algebraic form might have reinforced their *action* conception of asymptotes. While the intervention lessons might have strengthened the learners' *action* conception, they were not so successful in eradicating algebraic errors and could not develop one learner from a *pre-action* to *action* conception level.

7.5.2. A comparison of learners' responses under process level questions in the pre-and post-test

Questions items under the *process* conception showed a general improvement in the quality of the learners' responses. In question 1.3, for example, the number of perfectly correct responses increased from three to six, partially correct responses slightly increased from six to seven and incorrect responses dropped from four to zero. These results suggest that after the lessons, all the learners demonstrated some understanding on how to sketch the hyperbolic graph from its algebraic equation. However, seven partially correct responses indicate that errors and misconceptions were not completely wiped out.

A similar pattern emerged under item 2.1. Perfectly correct responses significantly increased from two to nine, partially correct responses dropped from three to zero and incorrect responses shrank from seven to four. A notable change in this question item was that after the lessons, the majority of the learners were able to correctly identify the lines of asymptote and perfectly give their equations. There was no change on the four of the learners who still struggled to identify lines of asymptote from a given graph of the function.

Question item 2.3 showed a positive shift, whereby the quality of responses had improved. The number of perfectly correct responses increased from one to five, partially correct responses dropped from six to four, incorrect responses marginally dropped from five to four and no attempts reduced from one to zero. What stands out in this question item is that after the lessons, the majority of the learners either gave a partially correct or a perfectly correct response. The four learners who presented incorrect responses had problems in translating the given graph three units upwards. The overall change in responses under process in the pre-and post-test is shown in Figure 7.7 below.

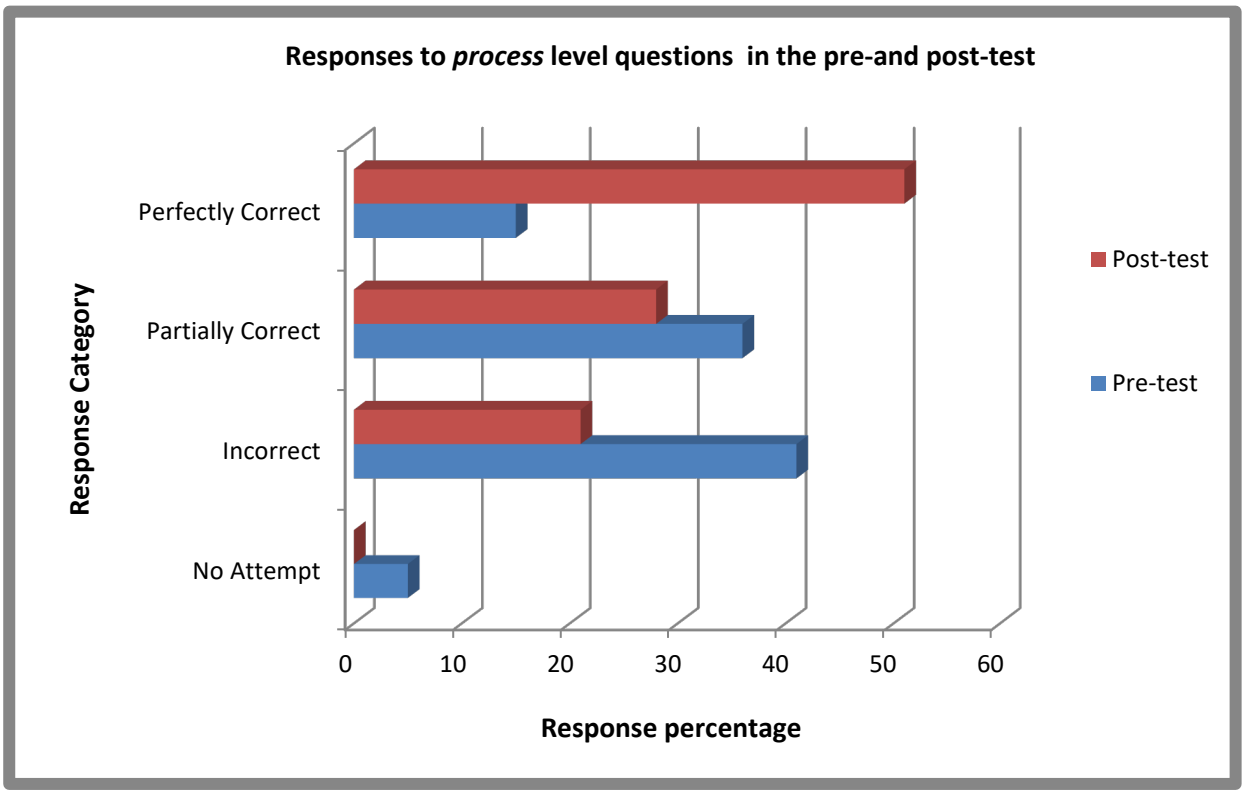


Figure 7. 7. A compound graph comparing learners' responses to *process*-level questions in the pre-and post-test

Generally the learners' responses on *process* level questions after the lessons seemed to reflect a significant rise in their *process* conception of the function. Possibly learners who initially presented partially correct responses might have corrected their errors to give perfectly correct responses. Similarly, learners who presented incorrect responses in the pre-test might have been better conceptually enlightened and managed to give partially correct or perfectly correct responses. This

suggests an improvement in the depth of learners' conception level from *action* to *process*. Further analysis in the next section will illustrate this emerging observation.

7.5.3. A comparison of the learners' responses to Object- level questions in the pre-and post-test

While question items under the *object* conception level generally reflected some improvement in the post-test, the majority of the learners demonstrated a very weak *object* conception. From Table 7.3, we note that in item 2.2, none of the learners provided a perfectly correct response both before and after the lessons. Partially correct responses increased from zero to five and incorrect responses dropped from thirteen to eight. Likewise, in item 2.4 perfectly correct responses slightly increased from zero to one, partially correct responses increased from zero to four and incorrect responses slightly dropped from five to four. In item 3.1, perfectly correct responses increased from one to three, partially correct responses increased from six to eight and incorrect responses dropped from six to two. To get a clearer picture of this comparative analysis, Figure 7.8 below illustrates learners' overall responses to object level question.

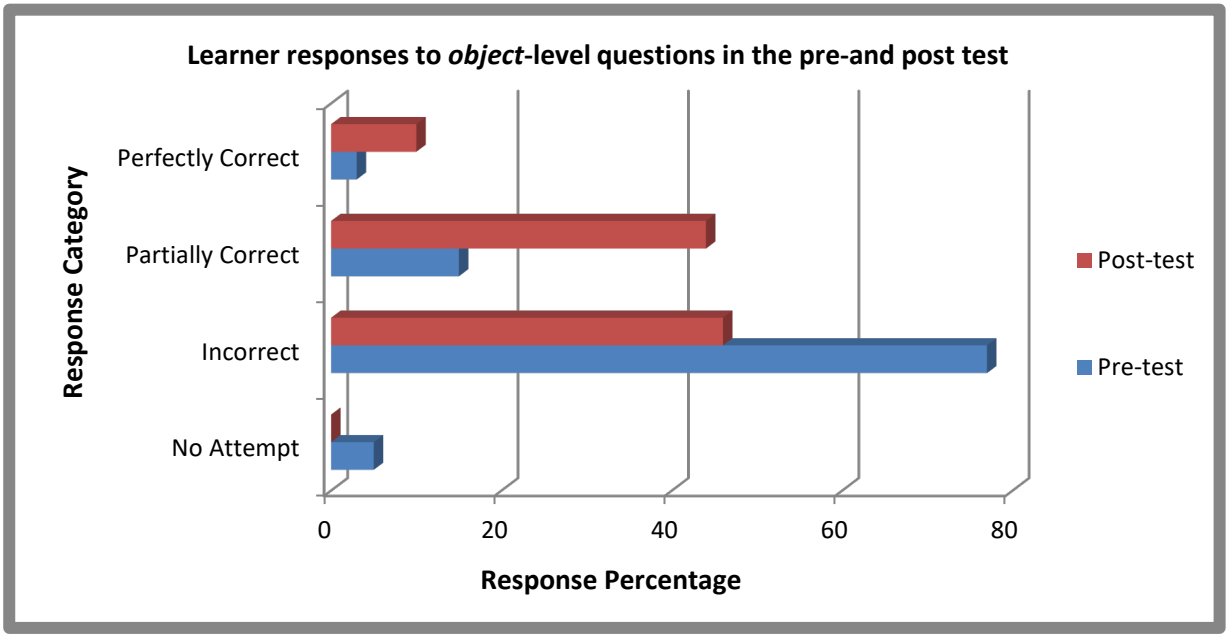


Figure 7. 8 A compound bar graph comparing learners' responses to *object* -level questions in the pre-and post-test.

There was a significant drop in incorrect responses under *object* level questions and a higher concentration of partially correct responses after the lessons. This suggests that the majority of the learners' object conception before the lessons was either very weak or totally absent. This improved during the lessons. A relatively high number of partially correct responses suggested that some errors and misconceptions had not been completely eradicated.

7.5.4. Comparison of Learners' responses under schema- level questions in the pre-and post-test

Focussing attention on *schema* level question items, we notice a significant drop in the number of incorrect responses. Item 3.2.1 had eleven incorrect responses in the pre-test and only three in the post-test. Partially correct responses increased from one to four and perfectly correct responses increased from one to six. A similar trend was witnessed in item 3.2.2. The number of incorrect responses reduced from twelve to four, partially correct responses increased from one to six and perfectly correct responses increased from zero to three. Figure 7.9 below provides a graphic illustration of the learners' responses in the pre-and post-test at *schema*-level.

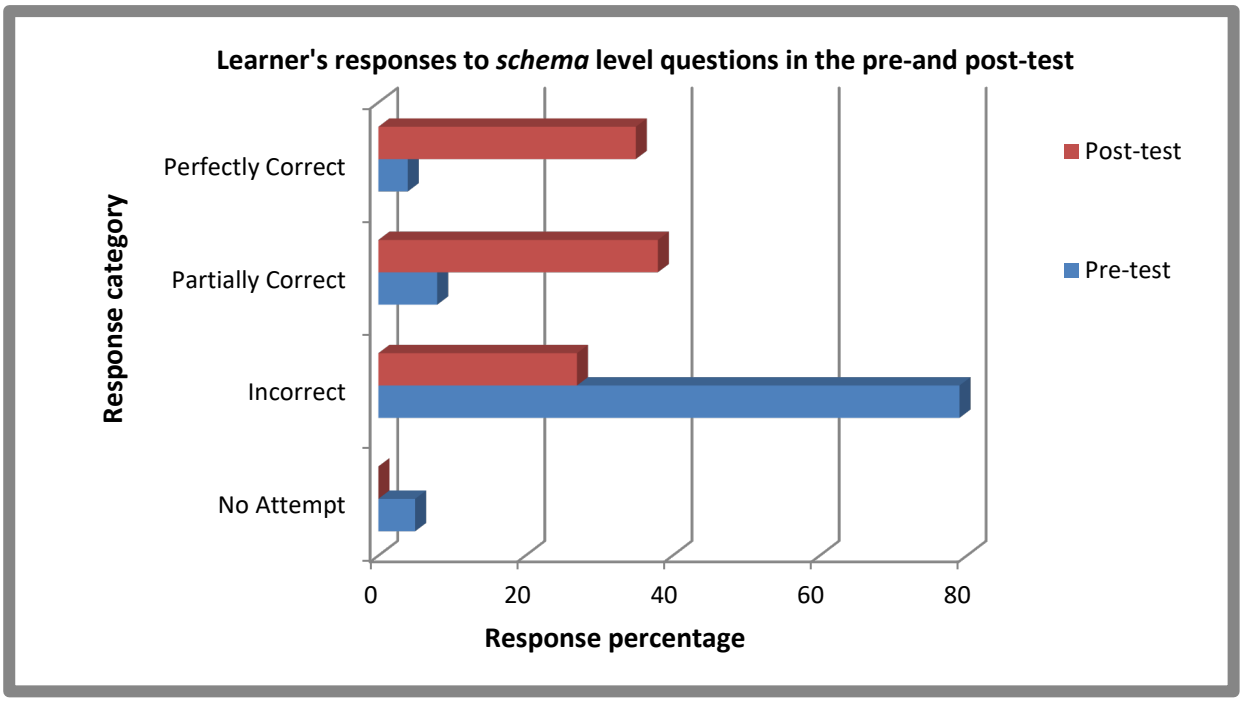


Figure 7. 9. A compound bar graph showing learner's responses under schema level questions in the pre-and post-test

There was a significant improvement in the quality of learners' responses after the intervention lessons. The percentage of 'no attempt' responses dropped from four to zero. This suggests that before the intervention lessons some learners had no idea on how to approach *schema* level questions and they never attempted to answer them. After the lessons they had courage to tackle the questions, though their responses were incorrect in most cases.

Learners found these question items very challenging in the pre-test because they could not identify the connection between the concepts of intercepts, asymptotes and axes of symmetry. In the post-test, the quality of responses improved, such that the learners were able to see the link between these critical points and this enabled them to interpret and respond to the questions meaningfully. While these results may not necessarily imply that these learners had attained *schema* conception level, they did reflect some growth in the learners' understanding of concepts in a more connected and coherent way.

7.6. Discussion of the findings

The above comparative analysis of learners' responses in the pre-test and post-test generally revealed some compelling evidence that mental processes of interiorization, encapsulation and thematization might have taken place as a result of the intervention lessons. According to the APOS theory, these mental processes are essential because they enable the learner to progress from one conception level to another, which is usually higher.

In response to the second research question of this study, I present some evidence to show that the instructional intervention might have facilitated these transitional mental processes to take place. The discussion on the findings was based on what the learners could not do before the intervention and what they could do after the intervention. The quality of the learners' written responses and their oral responses after the lessons was indicative of deeper conceptual understanding.

7.6.1. Indications of transition from action to process (Interiorization)

Interiorization is a change from a procedural activity, to be able to do the same activity again imagining some understanding of the processes involved (Syaiful et al. 2014). In other words, an individual begins to reflect on an action thereby forming a mental structure that performs the same operation as the action, but wholly in the mind of the individual. According to the genetic decomposition developed for this study, (see Table 5.1), interiorization would be evident if a learner could move away from being able to mechanically calculate x-and y-intercepts and determining equations of asymptotes to being able to explain why at x-intercept $y = 0$ and at y-intercept $x = 0$ and why $x = p$ and $y = q$ are asymptotes.

Table 7.5 below shows juxtaposed extracts of transcribed conversations I had with Mmathapelo before and after the intervention lessons.

Before Intervention lessons	**After intervention lessons**
R : Let's look at question 1.1.1. The question required you to calculate the x- and y-intercepts. You started by saying, for x-intercept, y=0 and for y-intercept, x=0, can you explain why you say so? **Mmathapelo:** When we say x=0, we are trying to remove the variable x so that we remain with one variable y. we can then be able to calculate the value of y. **R** : But why do we choose 0, why don't we put any other number? **Mmathapelo:** I think we choose 0 because it is a neutral number and will not affect the equation or the function itself. **R** : What is your understanding of the intercepts of a function? **Mmathaphelo**: I think intercepts are the points where we see the graph is turning.	**R** : Let's look at question 1.1.1. The question required you to calculate the x- and y-intercepts. You started by saying, for x-intercept, y=0 and for y-intercept, x=0, can you explain why you say so? **Mmathapelo:** Yes, for x-intercept we say f(x)=0 and for y-intercept x=0. **R** : Why do we choose 0, why don't we put any other number? **Mmathaphelo:** We know that any point that lies on the y-axis, the value of x is 0 and any point on the x-axis the value of y is 0. **R** : What is your understanding of the intercepts of a function? **Mmathaphelo**: The x-intercept is where the graph is cutting the x-axis and y-intercept is where the graph is cutting the y-axis

Table 7. 5 Extracts of transcribed interview with Mmathapelo before and after the lessons.

Before the lessons, Mmathapelo was able to follow the procedure in calculating the x- and $y-$intercepts but she could not give meaningful explanation as to why she followed that procedure. She confused the concept of intercept with that of turning point which does not apply to hyperbolic functions. After the lessons, Mmathapelo seemed to have acquired a better mental prototype of what an intercept is. She could bring out the fact that where the graph cuts the x-axis the value of y is always 0 and vice-versa. After the intervention, she could justify and better appreciate why we substitute $y = 0$ and $x = 0$ when calculating the coordinates of the x- and y-intercepts respectively.

A similar trend also emerged with regard to the concept of asymptote. In the pre-test, learners could only mechanically extract the equations of asymptotes from the standard equation of $g(x)$. When asked to explain what an asymptote is and to justify why these lines were asymptotes, they visibly became uncomfortable and could not give any sensible explanation. It was only after the lessons that most of them demonstrated a better conceptual image of what an asymptote stands for. Table 7.6 below shows juxtaposed extracts of a transcribed interview I had with Gabriel before and after the intervention lessons.

Before Intervention Lessons	**After intervention Lessons**
R: Let's move on to the next question which was asking for the equations of the horizontal and vertical asymptote. For horizontal asymptote, you said $x - 2 = 0 \Rightarrow x = 2$ and y = 3. Can you explain how you got this? **GABRIEL**: On vertical asymptote, when our denominator is $x - 2$, we say $x - 2 = 0$ then we take 2 to the right and we get $x = 2$ as the value of x. On horizontal asymptote, we just put it as it appears on the equation. The value of q becomes the value of y so it becomes $y = 3$. **R**: That's right, but why do you equate the denominator to 0? **GABRIEL** : I think by equating to 0 we make an equation that allows us to find the value of x.	**R**: Moving on to item 1.2, I can see you equated $x + 4 = 0$ to find vertical asymtote as $x = -4$ and the horizontal asymptote as $y = 3$. Can you explain why you did this? **GABRIEL**: First of all we know that an expression is invalid or undefined when its denominator is 0. So we equate the denominator to 0 so that we get the vertical asymptote. **R**: You are right. What about the horizontal asymptote? Why did you say $y = 3$? **GABRIEL**: Ummm I think for horizontal asymptote it's just given. We know from the general equation of the hyperbolic function that the constant, q, is the horizontal asymptote.

R: What is your understanding of an asymptote? **GABRIEL**: Asymptote is an imaginary line that cuts the graph into two parts. **R**: You are right, it's a line. But what does that line tell us about the hyperbolic function? **GABRIEL** : Asymptotes cuts the graph into two curves.	**R**: Can you share your own understanding of an asymptote. **GABRIEL**: Asymptotes are dotted lines that we see on a graph, whereby the graph gets closer and closer to the line but does not touch it or cross it. **R**: I like that. Why do you think the graph can not touch or cross this line? **GABRIEL**: I am not sure, but I think it's because the line divides or separates the graph.

Table 7. 6. Extracts of transcribed interview with Gabriel before and after the lessons.

Initially, Gabriel portrayed a good procedural fluency in determining the equations of asymptotes from the algebraic equation. However, when asked to justify why he followed that procedure, he could not give a meaningful explanation. He could not explain why he equated the denominator to zero to obtain the vertical asymptote. As one would expect, the learner struggled to explain what an asymptote is and what it tells us about the function. This reflected an *action*- conception of the concept of asymptote.

After the lessons, we notice an improvement in the learner's conceptualisation of asymptote though he is still struggling to accommodate it. He could explain that the function is undefined when the denominator is zero, which is very important in understanding asymptotes in rational functions. However, the learner could not use the same argument to justify the horizontal asymptote. In explaining what an asymptote is, Gabriel correctly describes the behaviour of the graph as it approaches an asymptote. This may imply that the learner had developed a visual mental image of the concept.

Gabriel's description of an asymptote had some limitations which, if not addressed, may result in misconceptions. He said, '… an asymptote is a 'dotted' line…', implying that if he does not see a dotted line, then there is no asymptote. It may also mean that whenever the learner sees a dotted line, he thinks that it is an asymptote. The learner also failed to link the notion of a function being undefined

when the denominator is zero to the question of why its graph cannot touch or cross its asymptotes. This, in my view, is a conceptual gap, which still needs to be bridged.

The above illustrations depict some conceptual growth as learners progressed from an *action* conception to a *process* conception where they could explain the meaning behind the rules and procedures. During the intervention lessons, learners were afforded an opportunity to use Autograph to draw several hyperbolic graphs on the computer. They were then engaged in group and individual activities to identify and extract the critical points for each of the graphs they had drawn. These activities facilitated the interiorization of the concepts of intercept and asymptote.

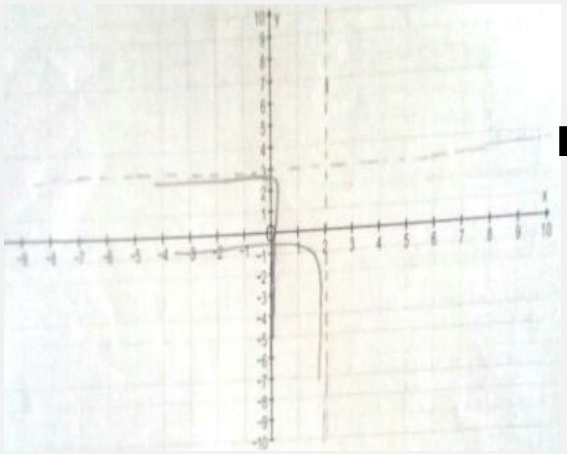
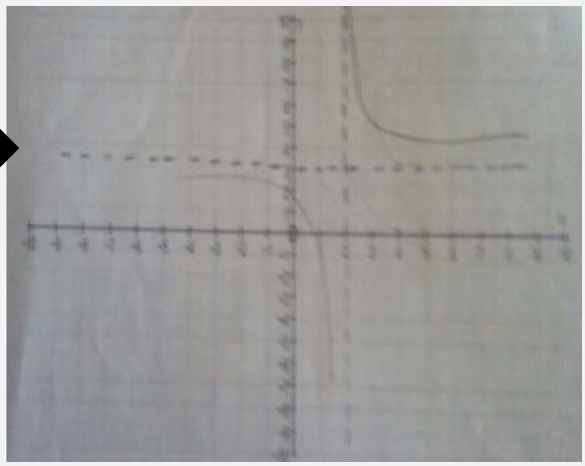
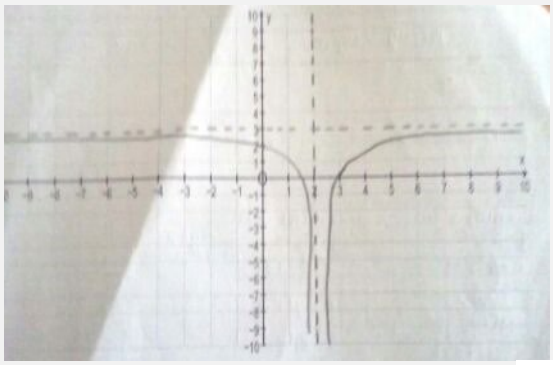
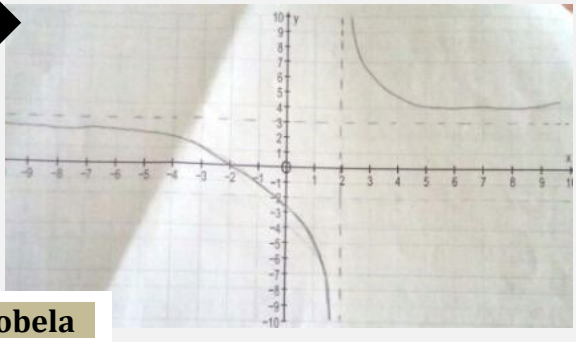
7.6.2. Indications of transition from *process* to *object* (Encapsulation)

In APOS literature, encapsulation is when an individual reflects on a particular set of processes until he or she can construct transformations on the mathematical object (Chimhande, Naidos & Stols, 2017). Thus, a process can be viewed as a totality, realising that transformations can act on the totality (Maharaj, 2010). Encapsulation condenses the processes into a cognitive object that one can change, manipulate or transform. In determining whether learners had encapsulated hyperbolic functions or not, I checked for the learners' ability and flexibility in switching from symbolic to graphical representation or vice-versa and their ability to understand transformations on hyperbolic functions.

7.6.2.1. Switching from symbolic to graphical representation

In APOS-Hyperbola (see Table 3.1), I noted that learners who had encapsulated hyperbolic functions can easily connect symbolic and graphical representations of the function. The flexibility in switching from one representation of a function to another allows one to see rich relationships, broaden and deepen conceptual understanding and strengthen one's ability to solve problems (Even, 1998). Question item 1.3 of the diagnostic test was developed to test the learners' ability to sketch the graph of a hyperbolic function given in algebraic form. They were expected to use the critical points calculated in item 1.1 and 1.2 to sketch the graph.

Comparing the sketches presented by the learners before and after the intervention lessons, I got a clear sense that leaners had vastly improved their ability to represent the function graphically. Figure 7.10 below juxtaposes responses of three learners before and after the lessons.

Before the lessons	**After the lessons**
	
Mmathapelo	
	
Sekgobela	

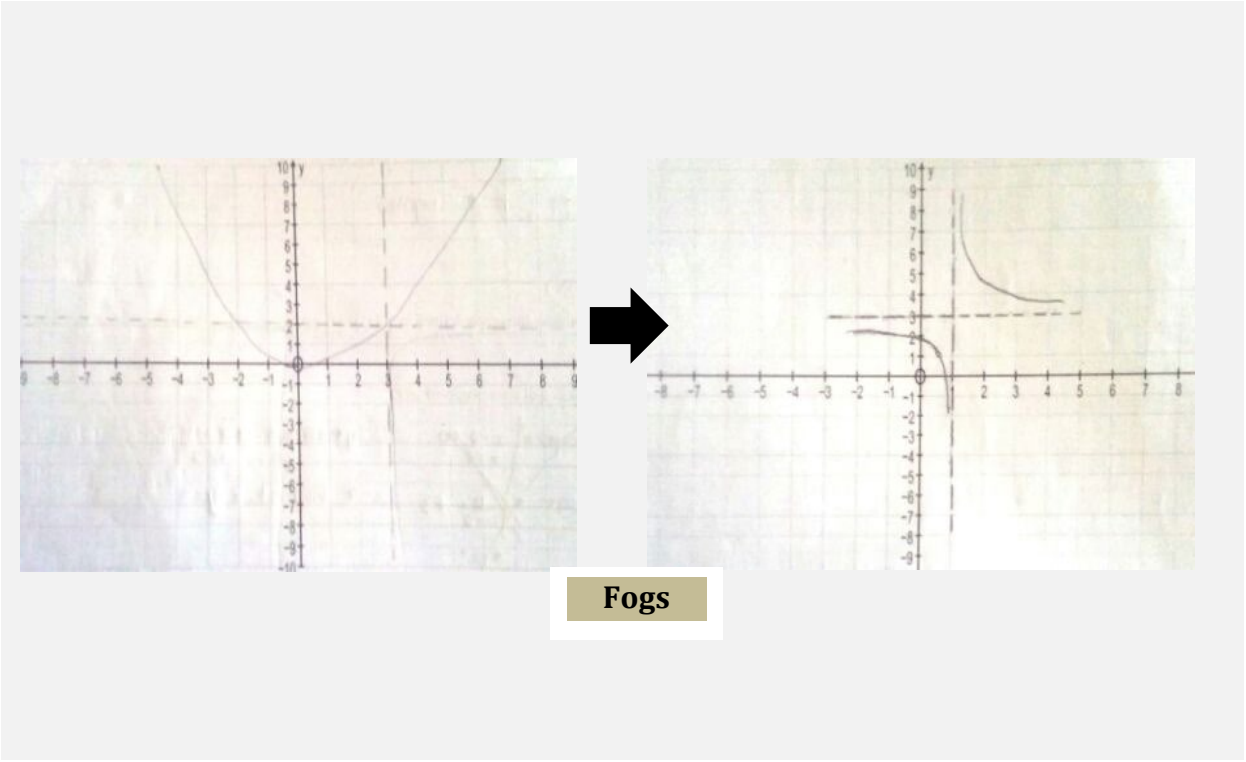


Figure 7. 10. Comparison of learners' graphical representations before and after the intervention lessons

Mmathapelo's sketch graph in the pre-test was mixed up. The graph had no connection between the critical points calculated and the graph itself. The vertical and horizontal asymptotes were correctly drawn but with no link to the graph. This lack of coordination simply reflected that the learner failed to view the symbolic and graphical representation as two different representations of one object. In the post-test, the sketch had improved drastically, though not yet perfect. The curves were placed in the correct quadrants and she demonstrated a better understanding of the behaviour of the graph as it approaches the asymptotes.

Similarly, Sekgobela initially sketched a graph that reflected a muddled understanding of the graphical representation of a hyperbolic function. The curves were drawn in wrong quadrants and what was mostly concerning was that there was no reference to the x- and y-intercepts calculated in item 1.2. This, again, portrayed a learner who had challenges in transforming a function from its algebraic form to its graphical representation. In the post-test, we notice an improved graphical representation of $g(x)$. The curves were placed in the correct quadrants and the learner demonstrated a clearer understanding of the connection between the graph and its asymptotes. There is evidence

that the learner was now able to link the critical points of a function to its graph. The only error that was evident was that the x-and y-intercepts were not correct and this emanated from the erroneous calculations done in item 1.2.

Fogs started off with a disjointed and fragmented understanding of the different types of functions. Apparently, he confused the graph of a quadratic function with that of a hyperbola. He drew a curve that cut across the lines of asymptotes, which showed that he had a limited understanding of what an asymptote is. The learner also interchanged the vertical and the horizontal asymptote. Though Fogs is a very weak leaner, we notice a remarkable improvement in his sketch in the post-test. He generally managed to bring out a correct graphical representation of a hyperbolic function, with asymptotes correctly placed and the symmetrical curves well linked to the asymptotes. Of course, just like the two learners discussed above, the learner had made some errors in calculating the x- and y-intercepts.

I chose to illustrate only these three learners here because they had initially shown serious deficiencies in their ability to sketch the graph of the given function. However, as shown in Table 7.3, about six learners who initially presented partially correct sketch graphs managed to self-correct their errors in the post-test. This indicates a conceptual shift from a 'process-orientated' conception, where critical points were calculated and viewed as isolated features of the function, to a more 'object-oriented' conception where the critical points were seen in totality, that is, as integrated parts of one *object*. The process-oriented conception is similar to Even's (1998) point-wise approach which is limited to plotting, reading and dealing with discreet points or values from a given graph. On the other hand, the object-oriented conception matches Even's global approach to a graph which focuses on the behaviour of the graph.

7.6.2.2. Transformations on hyperbolic functions

Transformation of functions posed enormous challenges to the learners in the pre-test. The functions $f(ax), f(-x), f(x) + k$ and $f(x + k)$ correspond to a dilation, reflection, vertical and horizontal translation on $f(x)$ respectively. In an extensive exploration on students' understanding of function

transformations, Eisenberg and Dreifus (1994) recognized the difficulty learners have in visualising translations of graphs. According to the APOS-Hyperbola developed for this analysis, one of the indicators of learners with an *object*-conception of hyperbola was that they should be able to describe the effect of transforming the function to its critical points and relate the transformed hyperbolic function to its graphical representation.

Question 2 of both the pre- and post-test was based on the given graph of the hyperbolic function $g(x) = \frac{a}{x-p} + q$. In the pre-test learners were asked to determine the asymptotes of $g(x+3)$. This question could easily be answered graphically if learners could realise that $g(x+3)$ represents a horizontal translation of three units to the left. As indicated earlier, all the learners provided wrong responses to this question in the pre-test. The obstacle was their failure to decode the meaning of $g(x+3)$. In the post-test, learners were required to determine the asymptotes of $f(x+1)$. Eight of the learners did not show any noticeable improvement and five of them demonstrated better understanding of what the question demanded. I got more interested in the five learners to find out if there was any change in their interpretation of the problem before and after the lessons.

Comparing what the learners said during interviews before and after the lessons, I got the sense that there was a remarkable change in the way they approached this question. I liked the way Joel described and justified his approaches. Table 7.7 below shows the conversation I had with Joel before and after the lessons.

Before Intervention Lessons	**After Intervention Lessons**
R : Lets move to question 2.2, which required you to find the asymptotes of $g(x+3)$. Can you share with me your understanding of this notation: $g(x+3)$. **JOEL** : When we are given $g(x+3)$, I think it means our vertical asymptote is $x+3=0$ which gives $x=-3$	**R**: Moving on to question 2.2, which required you to find the asymptotes of $f(x+1)$, can you explain how you arrived at your answer. **JOEL**: Firstly I had to find the equation of $g(x)$ using the graph given. I calculated and obtained: $g(x) = \frac{2}{x+2} - 1$. Then where I see x I put $x+1$, to get: $f(x+1) = \frac{2}{x+1+2} - 1$, which means: $f(x+1) =$

R : So what then do you say about the horizontal asymptote? **JOEL** : I think it means the horizontal asymptote is not there, because as you can see, the value of $\boldsymbol{q}$ is not given. **R** : Do you think there is any connection between the graph of $g(x)$ and that of $g(x+3)$? If so, explain why? **JOEL** : Hmmm I think there is no connection between the two because the asymptotes are different.	$\frac{2}{x+3} - 1$. Thus the vertical asymptote is $x = -3$ and horizontal asymptote $y = -1$ **R**: Waal, I like that. Now do you see any connection between $f(x)$ and $f(x+1)$? **JOEL**: I can say there is a connection because if we look at the two equations, they are almost the same. The values of $\boldsymbol{a}$ and $\boldsymbol{q}$ are the same, the only difference is the vertical asymptote which has chnged from 1 to -3. **R**: I like that observation. Now do you think there could be another way of determining the asymptotes of $g(x+1)$ without having to find the equation of $g(x)$ first? **JOEL**: Hmmm I never thought about that, …. I think there must be a way but I can't figure out how.

Table 7. 7. Joel's conception of $g(x+3)$ before and after the intervention lessons

A detailed analysis of Joel's responses before the lessons was presented in part 1 of this analysis. For purposes of this comparison, I note that the learner failed to decode the meaning of the functional notation $g(x+3)$. He wrongly associated $x+3$ with $x-p$ in the denominator of the general equation of the hyperbola. Consequently, the learner could not see any connection between $g(x)$ and $g(x+3)$.

After the lessons, Joel correctly applied the process definition (input-output definition) of a function to determine the equation of $f(x+1)$. Though his approach was long and winding, it allowed him to begin to see the vital connection between $g(x)$ and $g(x+3)$. It was during the interview that Joel discovered that the two functions have similar equations, with the only difference being on the value of $\boldsymbol{p}$**.** This discovery on the part of the learner, in my view, provided the first step towards the development of the notion of horizontal translation of a function.

In tackling the same question, it was only Gabriel who used the graphical approach and managed to arrive at his answer without much struggle. Below is the conversation I had with Gabriel after the lessons:

R: Moving on to question 2.2, which required you to find the asymptotes of $f(x+1)$, can you explain how you arrived at your answer.

GABRIEL: For this question, we know that $f(x+1)$ means the graph of $f(x)$ has been shifted 3 units to the right. So I looked at the graph of $f(x)$ and shifted the vertical asymptote from -2 to -3.

R: Waal that's wonderful! What about the horizontal asymptote?

GABRIE: The horizontal asymptote remains unchanged because it's a horizontal line

It was interesting to note that Gabriel was able to correctly interpret the meaning of $f(x+1)$ in transformation terms and managed to link this to the graph without any algebraic calculations. It was clear that the learner had fully encapsulated the *process*-oriented (operational) approach into a more *object*-oriented (structural) approach.

Despite this drastic conceptual growth, Gabriel made an error by shifting the graph to the right instead of the left. This could be a misconception, which originates from cognitive obstacles (Eisenburg & Dreifus, 1994) in the conceptual development of the learner. Learners' experiences with numbers suggest that adding 3 results in moving in the positive direction (right) and subtracting 3 results in moving in the negative direction (left). Unfortunately, horizontal translation of functions negates this notion and it takes some time and effort to correct this.

Question item 2.3 was also based on the graph of $g(x)$ and it tested the learners' ability to vertically translate the graph of $g(x)$ before giving its domain and range. One would have expected learners to find this question much simpler because the translation was stated in very simple and explicit terms. An analysis of the learners' initial responses revealed a number of challenges. To state them briefly, most of the learners could not identify the asymptotes on the drawn graph, translate the graph 3 units up and link the domain and range to the vertical and horizontal asymptote, respectively.

It was encouraging to note that in the post-test most of the learners had overcome these challenges. My attention was drawn to about six learners who initially provided either incorrect or partially correct responses but managed to correct their mistakes in the post-test. My concern was to ascertain if there was any change in their understanding or interpretation of the problem. One of the learners in

this category was Oscar. I present, in Table 7.8 below, the conversations I had with Oscar before and after the lessons.

Before Lessons	After Lessons
R : Here is the graph of g(x), can you show me the asymptotes of this function. **Oscar** : Asymptotes? (*Looking at the graph*) Hmmm, its this line and this one (*pointing correctly at the asymptotes*). **R** : That's fine. What are the equations of these lines? **Oscar** : This one is x=3 and this one is y=1. (*Interchanges the equations of the vertical and horizontal asymptote*) **R** : Now item 2.3 required you to determine the domain and range of a function obtained after shifting g(x) 3 units upwards. Can you share with me how you arrived at your answer? **Oscar** : Since g(x) is shifted 3 units, I moved the horizontal assymptote 3 units up and the vertical asymptote to the left. **R** : Ok! I get you. But why did you shift the vertical asymptote to the left? **Oscar** : (*shaking his head*) I was not sure in which direction to move it.	**R**: Looking at the graph of $g(x)$, can you explain how you obtained the equations of asymptotes? **Oscar**: If we look at the graph, we can see that this curve (*pointing on the graph*) is moving towards this vertical line but it does not touch it. So this line is the umm…. vertical asymptote. If we also look this side, we see that the graph is getting closer and closer to this horizontal line, so that is the horizontal asymptote. **R**: That's good. What are the equations of these lines? **Oscar**: The vertical one is $x = -2$ and the horizontal line is $y = -1$. (*This time the learner did not interchange the two equations*) **R**: That's wonderful. Now item 2.3 required you to give the domain and range of $h(x)$, obtained after shifting $f(x)$ 3 units to the right and 4 units down. How did you get the answers you have given? **Oscar**: Right, $f(x)$ is shifted 4 units up, it means the graph will move 4 steps down. So the horizontal asymptote will move from -1 to ummm … (*counting on the graph*), yah to -5. So the horizontal asymptote becomes $y = -5$. The vertical asymptote shifts 3 units to the right, which gives $x = 1$. So we write the domain as $x \in \boldsymbol{R},\ x \neq 1$and the range, $y \in \boldsymbol{R}, y \neq -5$. **R** : That's great!

Table 7. 8. A comparison of Oscar's conception of $g(x) + 3$ before and after the intervention lessons

Oscar, unlike the other learners (Mashabela, Precious, Fogs, Pebetsi and Mmathapelo), was able to identify the asymptotes from the drawn graph even before the intervention. As alluded to in the earlier sections of data analysis, the learners were used to seeing the asymptotes marked with a dotted line, and if they did not see the dotted line, they thought there were no asymptotes. Oscar struggled to give the equations of the vertical and horizontal asymptotes after a vertical translation. He struggled to explain the effect of shifting $g(x)$ upwards. His thinking was that the shifting of the whole graph up

or down changed the position of the vertical asymptote. This was a common misconception that featured in most of the learners. This confirmed the view that learners lacked a 'global' understanding of the function and its graph. Instead, they viewed the different parts of the graph as isolated components not linked to each other.

In the conversation after the lessons, it was noticeable that Oscar was more elaborate and articulate in the way he explained his procedures. During the interview, he spoke with more confidence and more excitement. His explanation of how he identified asymptotes displayed a clear understanding of the behaviour of the graph as it approaches its asymptotes. What was even more impressive was the way he explained the effect of shifting the function upwards on its domain and range. These observations suggest that the learner had managed to overcome the misconception of thinking that both asymptotes shift, though he struggled to justify why the vertical asymptote remained unchanged in this case. I got a sense that during the lessons, Oscar had picked a better visual image of a hyperbolic graph. He viewed the whole graph in totality as one unit, which could be transformed or manipulated according to any given rule. The same could be said for all the other learners in this category. Though they were not as elaborate as Oscar was in their explanations, they displayed a remarkable improvement in their approach in dealing with the graph.

I should however admit that even after the lessons, not all of the learners managed to attain the *object*-conception of hyperbolic functions. Eight of the learners still struggled with horizontal translation of the function and four learners could not vertically translate the graph. This finding corroborates Zazkis, Liljandahl and Gadowsky's (2003) findings that learners have difficulty in comprehending and visualising translations of graphs, particularly horizontal translation. They attributed this to the fact that the direction of horizontal translation of a function graph is inconsistent with intuitive expectations, thereby creating a dissonance for the learners. This dissonance is further aggravated by the fact that vertical translation operates as 'expected', that is, $f(x) + 3$ is a vertical upward translation and $f(x) - 3$ is a vertical downward translation. Learners are faced, not only with a

counter-intuitive behaviour of functions, but some behavioural inconsistency in the sense that some translations act according to their expectations while others do not.

Another possible source of the difficulty explained above is the instructional context in which the discussion of transformations of functions takes place. According to the CAPS document, transformation of functions is imbedded in the context of functions, rather than in the context of transformations. The topic of transformations was scrapped off from the new FET mathematics curriculum and learners only learn complex concepts such as translation, reflection and dilation in the already complex context of functions. In my view, this scenario presents a pedagogical obstacle, which reinforces the learning difficulty, rather than trying to overcome it.

7.6.3. Indications of transition from *object* to *schema* (Thematization)

It has already been highlighted that the *schema*-conception level is the highest level of understanding and is connected to other existing schemas. Once the development of a concept has matured into a *schema* in the learner's mind, he/she becomes more flexible in approaching a problem situation. Piaget (1983) and Garcia (1989) viewed schema development as a slow process in which the individual becomes aware of its components and their relations. In other words, the individual has reached the level where he/she is able to reflect on the meaning of the components and the relations that make up the schema. Arnon, Cottril, Dubinsky, Octac, Fuentes, Trigueros and Weller (2014) consider a schema to have been thermalized if an individual can think of it as a total entity and perform actions on it. Such actions include operating it, comparing it with another schema or decomposing it to recover its components (de-thematization).

A hyperbolic function *schema* would consist of components (or concepts) like intercepts, asymptotes, domain, range and axis of symmetry all connected together in form of an algebraic equation, a graph or any other form of representation. In line with the APOS-Hyperbola, I expected a learner who had thematized the hyperbolic function schema to be able to link these critical components located on the drawn graph and the ones calculated using the algebraic equation. I expected such a learner to be able

to comprehend the fact that critical points connect the graph and its function. When confronted with a problem and grappling to solve it, the learner must be able to flexibly switch from equation to graph and from graph to equation.

Two question items in the diagnostic test were aimed at testing whether learners had reached the schema level or not. In the post-test, the question was phrased as follows:

3.1. A hyperbolic function, $f(x)$, has the following features:

- The point $B(-1;0)$ is the y-intercept
- The domain of f is real numbers, but $x \neq 2$
- The range of f is real numbers, but $y \neq 3$
- f is a decreasing function

Determine the equation of $f(x)$.

An analysis of how the learners approached a similar question in the pre-test exposed their conceptual shortcomings. Generally learners had a memorised understanding that given $g(x) = \frac{a}{x+p} + q$, then $x = -p$ and $y = p$ are asymptotes but they could not link the asymptotes to the given domain and range. A few of the learners managed to identify the asymptotes from the domain and range but their substitution was flawed and they made several executive errors. When asked to explain their approach to this question, the following common responses were recorded:

Pebetsi: This question was a bit confusing to me. I did not understand how I could find the equation because we are not given the asymptotes. I just ended up substituting the numbers given into the equation, but I did not understand at all

Mmathapelo: I think if they say x ≠ 5, it means the value of p is 5 and when y ≠ 1, then q is 1. So I substituted but I was not sure how to proceed from there.

Mpho: This question was very difficult. I did not know how to get the values of p nand q. But I know if we can find the value of **p** and **q**, we can then substitute the point on the graph so that we find the value **a**.

Precious: Myself this question I did not understand at all. I did not even know how to start. I just substituted the numbers into the equation of the hyperbola but I cannot give the reason why.

> **Gabriel**: This question was very challenging to me. But I think since the domain is x ≠ 1, we can get the value of **p** from there and range says y ≠ 5, we also get the value of **q** from there. I was not sure on how to proceed to find the equation of the function.

Responses from Pebetsi, Mpho and Precious showed that they had no clue on how they could use the given information to find the equation of $g(x)$. To them, there was no link between the given domain and range and the asymptotes or the values of $\boldsymbol{p}$ and $\boldsymbol{q}$. Consequently, their substitution into the formula was just random without any sensible justification. Gabriel and Mmathaphelo's responses represent another set of learners who managed to extract the values of $\boldsymbol{p}$ and **q** from the domain and range but failed to substitute the given point to determine the value of $\boldsymbol{a}$. Their major limitation was that they had a disjointed conception of the critical points of the function. They failed to bring out why the domain and range were defined as $x \neq 5$ and $y \neq 1$ respectively, and link this to the vertical and horizontal asymptote. Again, their thinking seemed to be restricted to the algebraic representation, without a visual prototype of how the graph would look like. This limitation made it difficult for the learners to substitute the given point and calculate $\boldsymbol{a}$.

In the post-test, there was an improvement in the quality of the responses, but with little evidence of the schematization of the hyperbolic function schema. Only three learners (Gabriel, Joel and Ricky), demonstrated some evidence of thematization of the concepts. Not only did these learners present perfectly correct responses in the post-test, their explanations during interviews demonstrated that they had crystallized the concepts into a malleable concept image. When asked to explain his calculations after the lessons, Gabriel had this to say:

> From the given information, we are given the domain of $\boldsymbol{f}$ as $x \in R, x \neq 2$. This means $x = 2$ is a vertical asymptote and the value of $\boldsymbol{p}$ is -2. The range is given as $y \in R, y \neq 3$. What this also mean is that $y = 3$ is the horizontal asymptote and the value of $\boldsymbol{q}$ is 3. If we substitute in the general equation, we get: $\mathrm{f(x)} = \frac{\mathrm{a}}{\mathrm{x}-2} + 3$. Then to find the value of $\boldsymbol{a}$, we are given that point, $(-1; 0)$ lies on the graph. I therefore substituted the point into the equation to find the value of $\boldsymbol{a}$.

Gabriel's explanation illustrates that he had developed a clear understanding of the link between the defined domain and the vertical asymptote and the given range and the horizontal asymptote. Though

the learner's thinking was still confined to the algebraic approach, he was able to perfectly connect all concepts involved in a coherent manner. Joel, on the other hand, approached the question from a graphical perspective. By marking the given critical points on the Cartesian plane, the learner was able to create a visual image of the function. Though the question had not asked for the graph of the function, it was interesting to observe how the learner easily switched between the graphical and the symbolic representation as he grappled to find the equation of the function.

This study set out to test whether learners had attained the *schema* level of conception and the question item in relation to this was phrased as follows:

> 3.2. The axes of symmetry of a hyperbolic function,$k(x)$, intersect at the turning point of a quadratic function, $h(x) = \frac{1}{2}(x-2)^2 - 9$. If the y-intercept of $k(x)$ is $(0; -11)$, determine its equation.

An analysis of the learners' responses to a similar question before the intervention lessons revealed a number of errors and misconceptions. Learners displayed *fragmentation* errors, *hybridization* errors, *decoding* and *executive* errors (see Chapter 6). Their major challenge was that they failed to identify the connection between the axes of symmetry and the asymptotes. In APOS terms, it further confirmed that most of the learners were operating at the *action* level of conception, with only a few showing some signs of the *process* and *object*-conception level. None were at *schema* level of conception.

This particular question item in the post-test was complex because it required learners to see the connection between the graph of a hyperbolic function and that of a quadratic function. Nevertheless, there was generally a noticeable improvement in the quality of learners' responses to this question after the intervention.

A few of the learners did not show any meaningful change in their responses to this question. During the task-based interviews, I tried to figure out why these learners could not make significant progress in their response to this question. What came out was that their major challenge was their failure to

decode what exactly the question asked for. The wording used in the question seemed to be beyond their comprehension and a barrier to their attempts to solve it.

During conversations with this category of learners, it came out that when the question was re-phrased or clarified using simpler language, they were able to suggest meaningful approaches to solving the problem. This further confirmed the view that it is inadequate to judge learners' level of conception of a concept solely on the basis of their written responses. For one reason or the other, a learner may not attempt a question or presents an incorrect response but still possesses considerable level of understanding of the concepts in question.

My analysis of learners' responses to this question item was deliberately targeted towards those learners who demonstrated that they had thematized the concepts under hyperbolic functions. Both Gabriel and Joel had approached the problem from a graphical perspective. I was particularly attracted to Joel's explanation of his approach. Below is the conversation I had with him.

> **R**: Finally, let's look at the last question item 3.2. May you share with me how you arrived at this answer?
>
> **Joel**: We are told that the axes of symmetry of the hyperbolic function intersect at the turning point of the quadratic function. From the given equation of the quadratic function, its turning point is $(2; -9)$. It means the axes of symmetry of the hyperbola intersect at this point. We also know that the axes of symmetry cross each other like this (*sketching on a piece of paper as shown below*)

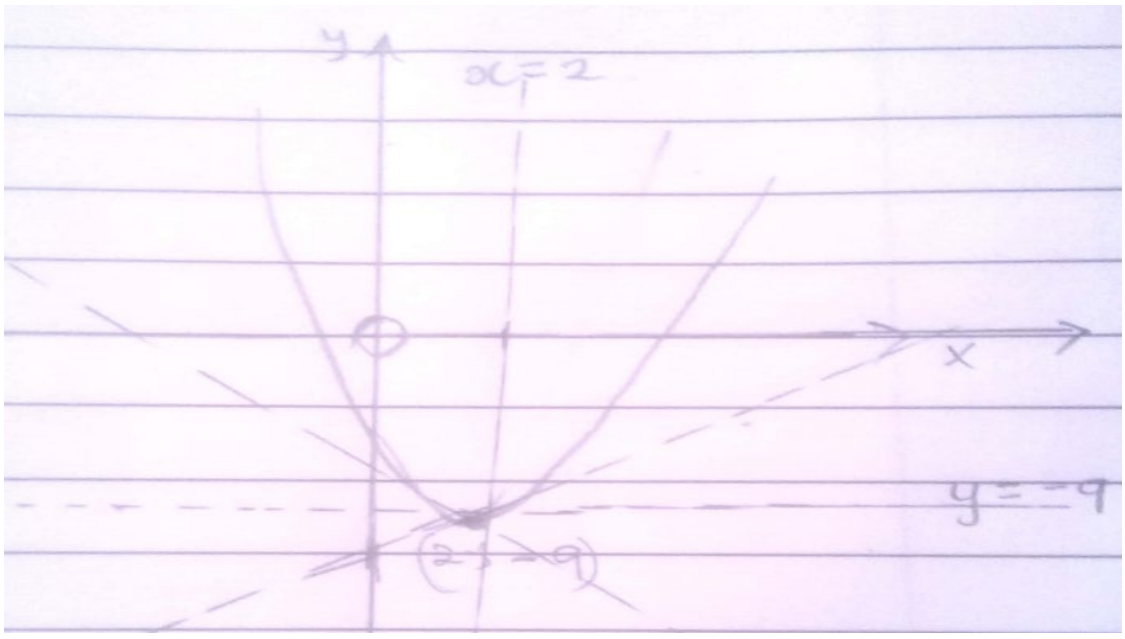


Figure 7. 11 Joel's partial graphical representation of his response to question item 3.2.

We know that where the lines of symmetry cross each other, the vertical and horizontal asymptotes also cross each other at the same point. So we can draw our asymptotes like this *(drawing the vertical and horizontal asymptote*). So we can see that the vertical asymptote is $x = 2$ and the horizontal asymptote is $y = -9$. ….. now wants we have the values of **p** and **q** we substitute into the general equation to get: $k(x) = \frac{a}{x+2} - 9$.

R: That's interesting. How then did you proceed?

Joel: Eeeeh, we are given a point on the graph $(0; -11)$, so we substitute $x = 0$ and $y = -11$ and find the value of **a**. The point also tells us that the graphs are in the 1st and 3rd quadrants like this (*drawing the curves*), and the value of **a** should be positive.

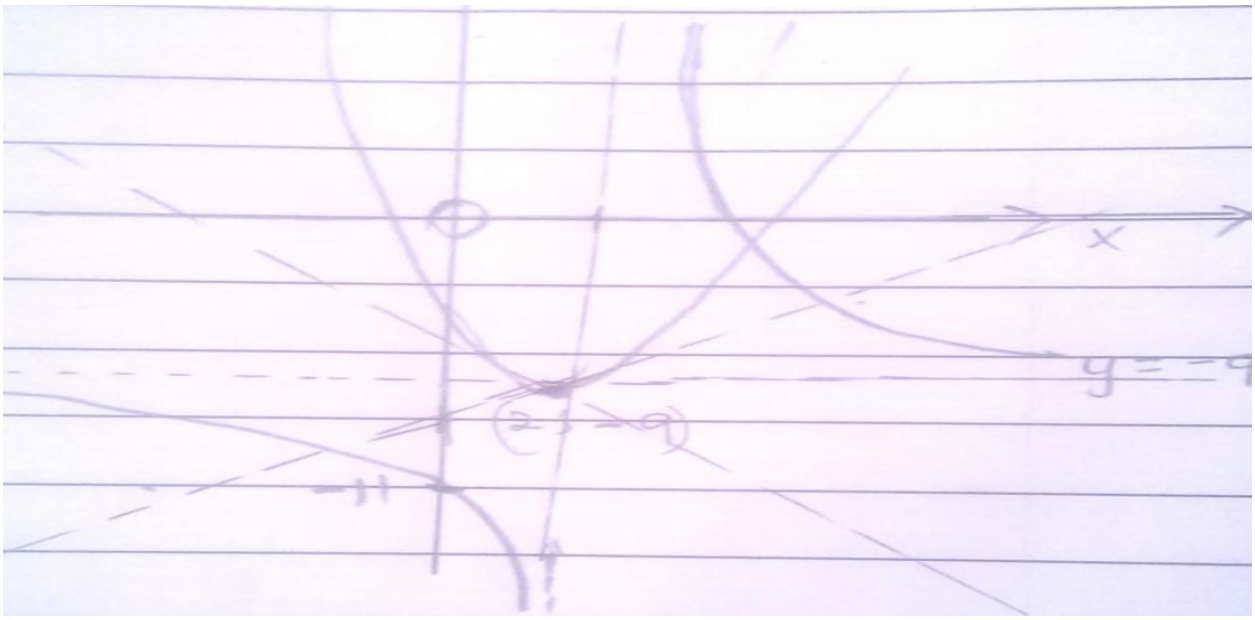


Figure 7. 12 Joel's complete graphical representation of his response to question item 3.2.

R: Wonderful! Now lets look at your substitution of **p** and **q** into the equation of k(x). If the denominator is $x + 2$ as you have put it, are we going to get the vertical asymptote as $x = 2$?

Joel: Ummm …. no, we get the vertical asymptote as $x = -2$. So I made a mistake, but ummm, $x = 2$ so the value of **p** is 2. I am not sure now, it's confusing me.

Joel demonstrated a strong ability to synthesize and link the graphical properties of a hyperbolic and a quadratic function and used them in totality to solve the problem. In APOS theory, in order to consider a learner to have thematized a schema, evidence must be found that he or she can act on a schema as an object. Given the point of intersection of the axes of symmetry, Joel was able to connect this to the vertical and horizontal asymptotes and used his graphical schema to visually show how the critical points are connected.

According to the genetic decomposition model developed for this study, the ability of a learner to flexibly switch from equation to graph and from graph back to equation was given as one of the indicators of a learner who has a fully developed hyperbolic schema. The learner was also expected to be able to link the critical points located on the drawn graph and the ones calculated using the algebraic equation. Joel was able to produce a graphical illustration of $k(x)$ and managed to switch on to the symbolic representation to derive the algebraic equation of the function. He, however, made an error in that if the vertical asymptote is $x = 2$, then the denominator would be $x - 2$ and not $x + 2$. This is an unsystematic error which occurred during the learner's explanation.

Gabriel approached the problem in a slightly different way but managed to bring out the connectedness of the concepts involved. Though he did not sketch the graph of $k(x)$, his explanation demonstrated a well-developed concept image of the hyperbolic function. Drawing from Piaget and Garcia (1889), thematization of a schema refers to mental construction of a schema so that it can be dissected, broken down, examined by its parts, reassembled and brought to bear in appropriate situations. In his oral response, Joel was able to demonstrate and bring out a coherent and complete coordination of the given properties of the function and used them as a totality to solve the problem. That kind of adaptive reasoning, according to the genetic decomposition model, can only be achieved by a learner who has constructed some level of a hyperbolic schema in her mind.

Ricky, on the other hand, had developed an algebraic hyperbolic schema. His thinking and approach to the problem seemed to be in algebraic terms. His written response to this question was perfectly correct with all calculations done correctly. However, when asked to explain and justify his working verbally, he struggled a lot. Inferring on his character, he is generally reserved and always struggles to explain himself in words. He was uncomfortable when asked to verbally explain himself. Consequently, further probing did not yield any result as the learner could not reveal adequate information to establish the thinking behind his approach to the question.

From an APOS perspective, probing and a conversation between the learner and the interviewer can help determining whether the responses are neither happenstance nor based on some memorised actions, but rather based upon the knowledge that had been constructed (Cooley, Trigueros & Baker, 2007). In my view, this clearly points to the limitation of the APOS theory in determining with certainty the level of conception a learner might have achieved with regard to a particular concept. Thematization of a schema is a mental and internal process, but the investigation of a schema is an external viewing process, with only the researcher's perceptions of available responses to assess understanding (Amon, Cottril and Dubinsky, 2013). In the case of Ricky, I suspected that the learner might have thematized the hyperbolic schema from an algebraic perspective but was still struggling to connect to the graphical representation. This might explain why the learner struggled to explain the connection between the point of intersection of the axes of symmetry and the asymptotes.

The above finding confirms Gagatsis and Shiakalli's (2004) observation that mathematics students have problems in passing from one representation to another and generally avoid the graphical representation while preferring the algebraic expression. Literature on conceptualisation of functions generally view a function schema confined to one form of representation as incomplete (Gagatsis & Shiakalli, 2004; Even, 1990; Cooley et al, 2007).

However, in the case of Ricky, following his responses to questions that required schema level of conception, there was compelling evidence that he had somehow thematized hyperbolic functions, as he was able to logically connect the critical points of the function and connect them algebraically. I suspected that his schema was tilted towards the symbolic representation and, in order to reach maturity, he need to be able to strike a balance between the algebraic and graphical representations. Even the APOS theory proponents attest to the view that evolution of a schema is not necessarily linear, but rather described as a progression within non-linear stages of concept development (Cooley et al.2007; Amon et al. 2013).

7.7. Conclusion of the chapter

This chapter has revealed that there was notable improvement in the learners' level of conception of hyperbolic functions after the intervention lessons. Before the lessons, the majority of the learners operated at the *action* conception level. They mechanically followed procedures in determining intercepts, asymptotes, axes of symmetry, domain and range with no understanding of the concepts involved. Consequently, their calculations were marred with structural and executive errors. After the lessons, the quality of learners' responses improved and there was evidence of improvement in level of conceptual understanding of the learners. Even the weaker learners showed some evidence of transition from an *action* conception to a *process* conception. Not only did they show improvement in their written responses, they were able to explain, justify and give reasons for their procedures.

The findings of this study showed that learners manged to develop some *object* level of conception. These learners demonstrated their ability to switch with easy from the symbolic representation to the graphical representation, which they struggled to do in the pre-test. This category of learners was able to deal with problems involving translations of hyperbolic functions. They could interpret the meaning of $f(x) + q$ and $f(x + p)$ both from an algebraic and graphical perspective.

It was evident that some of the learners, though few, managed to thematise the hyperbolic function schema. These learners were able to connect, coordinate and synthesize the critical features of a hyperbola and use them in totality to tackle higher order problem situations.

In this context, the question that one may ask is: what exactly might have facilitated or enhanced the deepening of the learners' conceptual understanding. Admittedly, classroom learning is a result of complex interacting contextual factors. The mere introduction and use of the graphical software and other supporting ICT tools may not automatically guarantee better understanding of concepts. The next chapter attempts to zoom into this question and find out what aspects of the intervention might have contributed to the growth in learners' conceptual understanding. Based on the learners' experiences during the intervention lessons and their perspectives on what worked for them, the next

chapter lays the foundation for an ICT-mediated pedagogical model that teachers may use for ICT integration in a similar contextual setting.

CHAPTER 8: ANALYSIS OF OPEN-ENDED INTERVIEW DATA AND DISCUSSION OF FINDINGS

8.1. Introduction

Chapter 7 presented the ICT-mediated instructional intervention and an APOS based comparative analysis of learners' responses before and after the instruction intervention lessons. The findings generally showed that the learners had an improved conceptual understanding of concepts under hyperbolic functions after the intervention.

This chapter was guided by the third research question of this study, which sought to build an ICT-mediated instructional model that teachers may use for ICT integration in a similar contextual setting. The model is informed by the learners' experiences during the intervention lessons and how they interpreted their experiences. As a participant observer, the researcher's experiences, throughout the data collection process, informed the triangulation of what the learners said and made sense of their experiences.

8.2. The theoretical basis

Any educational intervention is a complex intervention consisting of several components acting synergistically. What happens in a classroom situation involves multi-faceted interactions on one of the most complex subjects. Drawing from the role of cognitive artefacts, which has its historical basis in Vygotsky's approach of semiotic mediation (Bussi & Mariotti, 2014), the nature of human cognitive development is a result of the interweaving of the natural dimension (internal mental functions) and the social or cultural dimension (external social factors).

According to Vygotsky (1978) the connection between the external processes and the internal processes is facilitated through the process of *internalization*, which Vygotsky defined as '… the internal reconstruction of an external operation',(p.56). In other words, internalization is basically viewed as the process of construction of individual knowledge as generated by socially shared experiences. In other words, though knowledge acquisition is an internal process, systematic

educational intervention helps to enhance and facilitate knowledge construction. In light of this theoretical background, I argued that soliciting and analysing the participants' experiences during the intervention programme sheds light on affordances and constraints induced by ICT-mediated instruction.

8.3. Data sources

After the intervention lessons, learners were subjected to open-ended interview questions so that they could freely share their experiences, views and challenges with regard to the intervention lessons. Three main questions guided the interviews. The first question asked learners' views on ways in which the computers might have assisted them to better their understanding of the concepts taught. The second question allowed learners to share what sort of assistance they expected from the teacher and from their peers. The third question sought to find out the challenges learners encountered during the intervention programme that might have hindered or delayed their understanding of the concepts taught. All the interviews were audio-taped and transcribed for analysis.

Additional notes were collected from field notes on the lesson observation guide. All the intervention lessons were video-taped (with the learners' consent) for subsequent analysis. After each of the lessons, I would sit down and watch the lesson together with two of my colleagues who are also experienced educators. We would critically evaluate and reflect on the lesson focussing on what worked well and areas for further improvement. I noted down all the comments raised during this session and these constituted my field notes from the lesson observation. The field notes informed my preparation for the next lesson and in the subsequent data analysis.

8.4. How the data was analysed

Data analysis in this section was based on the grounded theory (Glaser & Strauss, 1967). I employed inductive data analysis of the transcribed interview data and the scribbled field notes on the lesson observation guides. I did not adopt any existing analytical framework because I wanted to be open-minded (Dey, 1999), without any predetermined or preconceived expectations of what was going to

come out of the data (Charmaz, 1990). The objective was to allow ICT affordances and constraints to emerge from the data.

The first phase of data analysis was initial coding (Strauss & Cobin, 1990). I read the transcribed interview data and texts on the lesson observation notes several times. I then identified and highlighted any word or phrase that I found relevant and at the same time employed memo writing to kick-start the analytical process (Charmaz, 1990). The highlighted words or phrases were used to inductively generate as many codes as possible from the data. The second phase of analysis was intermediate coding. At this stage, similarities and differences between the codes were identified leading to main categories. The categories were constantly reviewed and refined, identifying relationships between them. The final stage was the identification of themes that were represented as a set of interrelated concepts. This process of analysis was repeated for each of the main interview question. Eventually, the theme/s emerging from each of the main interview questions were connected and presented in form of an ICT-mediated pedagogical instruction model. An example of how the coding process was done for each of the interview questions is given in Table 8.1 below.

QUESTION: In your view, in what way would you say the use of computers in the lessons helped you to understand the concepts taught under hyperbolic functions?				
Learner	**Learner's Response (Verbatim)**	**Initial coding**	**Intermediate coding**	**Advanced Coding**
Pebetsi	I can say the use of computers helped me a lot because when I was drawing the graphs on the computer, I could easily see how the graph looks like and I could see how the graph changes as the equation changes. For example, I did not quite understand that when the value of **a** is positive the graph lies in the 1st and 3rd quadrants and when its positive, the graph lies in the 2rd and 4th quadrants. When the values of p and q changed, I could clearly see how the graph was shifting horizontally and vertically	**Positive ICT perception** **Drawing graphs** **Visualize** **Connecting graph and equation** **Effect of parameters** **Translation of graph** **Position of graph in quadrants**	**Aroused Interest** **Tool acceptance** **Efficient** **Representation**	**Motivation**
Ricky	Personally, I think the use of computers made it easier for me to see the things I were struggling to understand before. Now if I am given the values of the constants **a**, **p** and **q**, I can tell how the graph will look like. For example I always struggled to understand that when **p** is positive, the vertical asymptote would be negative. …… you see, when we	Visualise Effect of parameters Graphical representation Horizontal translation Comparison to	**Animation**	**Visual Representation**

	draw the graphs on our own, we sometimes make mistakes and we don't draw the graphs properly. This makes it difficult to make conclusions or see any connection between the graphs. With a computer you can see the graph moving left and right or up and down. That made it easier for me to understand how the graph changes as the equation changes.	traditional approach **Accuracy** **Making deductions /conclusions** **Graph animation** **Easier** **Graph transformation**		

Table 8. 1. An example of how the coding process was done for each of the interview questions

The coding process illustrated in Table 8.1 above was meticulously repeated for each of the three main interview questions. Though the interview data might have been analysed, question by question, the identified codes and the subsequently emerging categories and themes were inter-related. It is this interconnectedness of the emerging themes that I hope will bring new insights in the context of ICT-mediated instruction in a rural setting.

Table 8.2 below presents a summary of the initial codes, intermediate codes (main categories) and the advanced codes (themes) for each of the main interview question.

Interview Question	Initial Codes	Intermediate Codes	Advanced Codes
1) In your view, in what way would you say the use of computers in the lessons helped you to understand the concepts taught under hyperbolic functions?	-Positive ICT perception -Drawing graphs -Visualize -Connecting graph and equation -Effect of parameters -Translation of graph -Position of graph in quadrants -Graphical representation -Better than traditional approach -Accuracy -Making deductions / conclusions -Graph animation -Easier -Aroused interest - Fast -Confusing graphs -Inability to differentiate graphs -Eradicates mistakes -using the scribble pen	Aroused Interest Tool acceptance ICT Efficiency Representation Barriers / limitations Animation	Motivation Representational Competency Class Interactivity Constraints
2) Can you tell us what form of assistance you expected to get from the teacher and from your fellow learners during the lessons?	-sometimes got stuck --struggled with key board -colleague also busy -disturbing -frustrating -teacher busy assisting others -left behind -no adequate assistance -struggled on my own -answering activity questions	-Learner-learner interaction -Learner-teacher interaction -Learner-computer interaction -learner-content interaction	
3) Can you share some of the challenges you faced during the lessons which might have affected your learning.	-entering the equation-using computer keyboard -teacher too fast -struggled to adjust scale on the axes -graphs too many and confusing -computer switched off -slow in using keyboard -disturbed by other learners	-computer proficiency -pace of delivery -individual differences	

Table 8. 2. Summary of the coding process and emerging themes

Table 8.2 above shows that four themes emerged from the data, namely; motivation, visual representation, class interactivity and constraints. Admittedly, these are broad concepts and the discussion around them will be confined to ICT- affordances within the contextual setting of this study. As explicated earlier, the themes emerged from the learners' shared experiences, views and perspectives based on the ICT – mediated intervention lessons. The themes may be considered as key factors that might have acted as *enablers* or *hindrances* to learners' understanding of concepts. The next section is focused in presenting a detailed discussion of these themes.

8.5. Discussion of the findings

I now present the findings from the interview data, according to the four themes that emerged from the data analyses namely; motivation, visual representation, class interactivity and technological constraints. The discussion is based on the learners' views and perspectives after the intervention lessons, together with my own observations as I was an active participant during the data collection process. My interpretation of what the learners said or did was informed and guided by both the theoretical framework of this study and the literature reviewed. Each of the themes is discussed separately but an attempt is made to show their interconnectedness.

8.5.1. Theme 1: Motivation

Motivation is undeniably regarded as one of the major enablers of academic achievement (Linnenbrink & Pintrich, 2002; Mitchell, 1993). Motivation is simply defined as a strong personal interest in a particular subject or activity. This common understanding of motivation is not only narrow and simplistic but perceives motivation as a static and quantifiable trait (Mitchell, 1993). My view of motivation draws heavily from the social cognitivists who argue for a more comprehensive and multifaceted understanding of student motivation (Linnenbrink & Pintrich, 2002). Their central argument is that motivation is situational, contextual and domain-specific. This suggests that the level of motivation varies according to context. An understanding of how and why students are motivated for better achievement in various contexts is important.

When learners were asked to share their experiences from using computers in the classroom, nearly all of them expressed some excitement and interest. The excitement was evidently stimulated by the mere exposure to a computer. In response to this question, Mpho had this to say:

> … I also found the lessons very interesting because some of us, like myself, I had never lay my fingers on a computer, so i learnt quite a lot during the lessons

Mmathapelo expressed similar sentiments in response to the same question. Her first statement was:

> Personally I enjoyed the lessons very much. What I liked most was the drawing of graphs using the computer, it was amazing. … I thing it also helped us to gain some computer skills because computers are very important these days.

The learners expressed what Mitchell (1993) described as *situational interest*. Situational interest is stimulated by the features on the learning context, which may be short-term or long-term. Mitchell (1993) further explains that situational interest is initially aroused by 'catch' factors, for example, an exciting computer programme or innovative instructional practices. In the context of my study, the 'catch' factor was the computer and the graphing software. Considering the socio-economic background of the learner participants in my study, the majority of them admitted that they had no computers at home and had never used a computer before. The introduction of computers into the classroom was a strong catch factor that stimulated their interest in the lessons.

One of the weaknesses of the 'catch' factors is that they are more extrinsic and short-term. To ensure the interest is sustained, Mitchell (1993) proposed for 'hold' factors. These include empowering learner engagement in the task and content mastery. The 'hold' factors came out when learners expressed how the use of computers overcame the obstacles they encountered with the traditional approach. The following extract exemplifies this point:

> **Joel:**…. You see, before using the computer we used to draw the graphs free-hand. To me I found the use of computer much easier because it produces the graphs as they should be and very fast. When we draw with our free-hand, we sometimes make mistakes, making it difficult to see how the graphs look like and difficult to make conclusions.

According to Joel, and several others who raised this point, traditional approach of teaching graphs presents a number of learning obstacles to the learners. It is not only time-consuming but involves laborious calculations that expose them to errors and mistakes.

The introduction of graph sketching technological devices ameliorates and eliminates some of these obstacles, thereby affording learners more opportunities to engage in investigating effects, relationships and making deductions. It is not surprising that the majority of the learners welcomed the innovation, describing is it as fast, accurate and exciting. The net effect of all this is that it raises the learners' *self-efficacy* (Bandura, 1997). Bandura describes *self-efficacy* as the individual's beliefs about their performance in a particular context or a specific task. It ultimately results in higher levels of effort and increased persistence on difficult tasks. This was confirmed in this study because even after the sessions, learners would request to remain in the computer lab working on their tasks. The point here is that once learners begin to realise some improvement in their performance of a task or mastery of content, they definitely develop a sense of achievement, which motivates or propels them to achieve more.

To sum up this discussion on the impact of ICT-mediated instruction on motivation, I present the issues that have come up in figure 8.1 below.

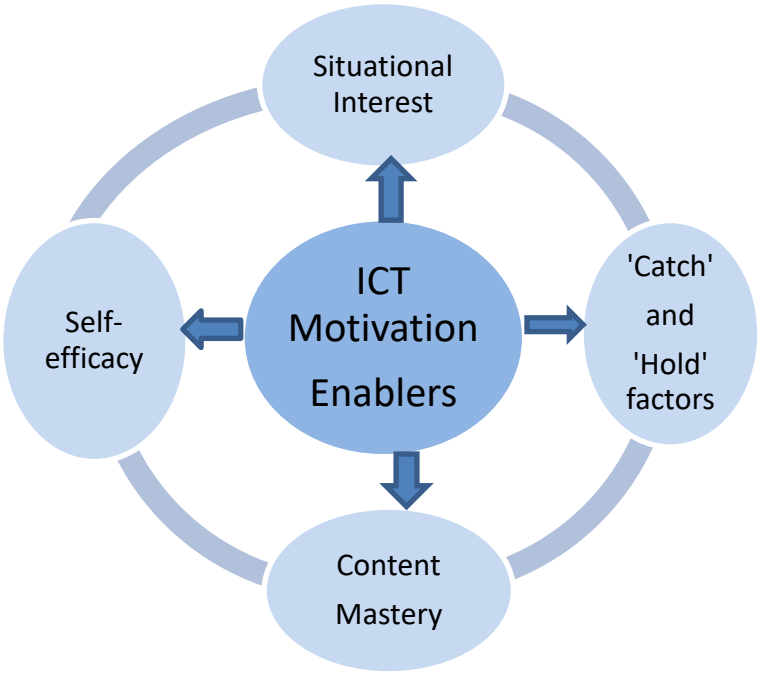


Figure 8. 1. ICT as motivation enabler

The findings of this study have confirmed that ICT tools are imbedded with motivational potential. However, the question of 'what' motivates a learner and 'why' is highly contextual. In this study, it has come out that the mere introduction of computers to the learners stimulated their interest. This was attributed to their socio-economic background, whereby the majority of them where using computers for the first time. This initial motivational 'catch' has to be sustained through 'hold' factors like content mastery and improved self-efficacy.

8.5.2. Theme 2: Representational Competency

Another computer affordance that emerged from the learners' narrations was that of representational competency. An analysis of the learners' errors and misconceptions that was presented in chapter 6 of this study revealed that learner understanding of hyperbolic functions was restricted to the symbolic or algebraic representation. They struggled to connect a function to its graphical representation. This probably explains why, after the intervention lessons, nearly all the learners stressed how they benefited from 'seeing' or visualising the graphical representation of the function in the computer. The following extracts help to clarify this argument:

> 'I could easily see how the graph looks like …' (Pebetsi)
>
> '…made it easier for me to see the things I struggled to understand before ...' (Ricky)
>
> '… I could see the graph moving left or right and up or down...' (Precious)
>
> 'I could see the graphs moving …' (Oscar)

These statements from the learners underscored the power of visual representation. What one can draw from the learners' statements is that by visualising the graphical representation of the function, they could identify how the graph looks like (maybe its key features), clear some misconceptions and visualise translations on graphs. Literature has confirmed that functions are complex and abstract

concepts, not easily accessible to the learners and no single representation can depict all aspects of the function concept (Arcavi, 2003).

Visual representations, especially graphs, may fail to enhance students' learning if not used properly. Matthews and Rau (2017) cautioned that multiple representations may confuse students rather than aid their learning. This came out in the data when two of the learners indicated that they found the graphs confusing to them. Precious had this to say:

> … drawing of graphs using the computer was very exciting to me. It was very fast and when seeing many graphs at one time, you could easily see how the graphs look like. But, hmmm at first those graphs were confusing to me, I could not differentiate the graphs

In the same vein, Pebetsi admitted that she struggled to answer activity questions using the graphs because they were too many and confusing to her. She expressed it as follows:

> … to me entering the equations and drawing the graphs on the computer was not a problem at all. My problem was on trying to answer the questions, because when I looked at the graphs, they were too many and could not understand them ….

Observations I made during the lessons confirmed the challenges raised by Pebetsi and Precious. I observed that learners were excited about drawing graphs on the computer but struggled to interpret them to respond to the given questions. The mere representation of a concept in another form may not mean anything to the learners if they cannot interpret it. There are representational competencies that learners should acquire in order to benefit from multiple representations.

Matthews and Rau (2017) identified two representational competencies that I found critical in unlocking the enabling potential in visual representations, namely, *visual fluency* and *connectional fluency*. Visual fluency entails the ability to connect the visual representation to concepts and seeing 'meaning' in them (Matthews & Rau, 2017). Learners begin to map concepts onto mental 'prototypes' or mental images. In the context of graphical representation of a function, visual fluency is when learners are able to identify critical features like asymptotes, turning points, intercepts and axis of symmetry. They should be able to compare and contrast two or more graphical representations,

identifying similarities and differences, patterns and relationships. The two learners identified above initially lacked visual fluency, hence they found the graphs confusing and could not make any sense out of them. What this tells us is that presenting multiple representations to learners without equipping them with skills to extract meaning out of them is counter-productive.

Connectional fluency entails the ability to connect multiple representations to one another and to flexibly switch between multiple representations (Matthews & Rau, 2017). In the case of functions, they can be represented algebraically (or symbolically), graphically, as ordered pairs or using a flow-diagram. Given the complexity and the abstract nature of the concepts under functions, no single representation can depict all aspects of the function concept. This study has revealed that learner understanding of hyperbolic functions was restricted to the algebraic form and this limited their capacity to solve higher-order questions.

The interconnectedness of these issues is given in figure 8.2 below.

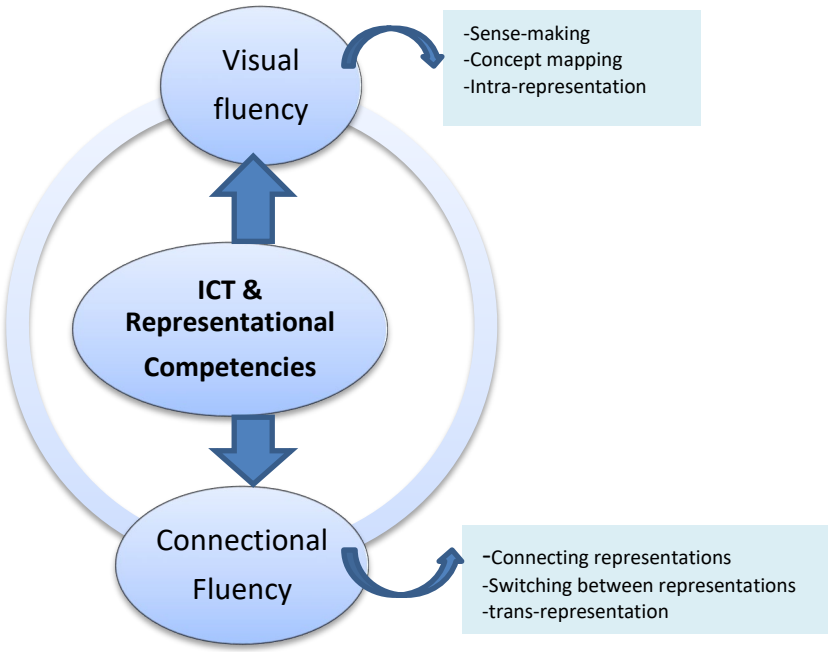


Figure 8. 2. A diagram showing ICT representational competencies

What has come out of this discussion is that ICT tools present efficient and effective opportunities for multiple representations of mathematical concepts in general and functions specifically. However,

learners need to be equipped with visual and connectional competencies for them to benefit out of these representational opportunities. Teachers integrating ICT in their classrooms have to make conscious and targeted efforts to develop visual and connectional fluencies in their learners.

8.5.3. Theme 3: Class Interactivity

Learning in a classroom context comprises a set of inter-related resource components, including people and objects, which are constantly in interaction. The people involved include teachers, learners and support staff, and objects commonly include the teaching and learning material, for example, books, pen and meditational artefacts. The introduction of ICT tools into the classroom adds another complicated component onto the already complex interactions going on in a classroom set up. This section of my discussion aims to bring out the role ICTs play and their influence on various classroom interactions. At the centre of all these complexities is the teacher, who is expected to orchestrate all the components in a classroom context to achieve desired outcomes.

Different scholars suggest different categories of interactions in an ICT-mediated classroom, but this discussion is guided by four sub-categories, which were evident from the data. Drawing insights from Beauchamp (2011), the sub-categories were identified as: *technical* interaction (learner-ICT tool interaction), *pedagogical* interaction (teacher – learner interaction), *conceptual* interaction (learner-concept interaction) and *social* interaction (learner-learner interaction). Each of these categories was considered as a sub-theme under class interactivity. The discussion under each sub-theme was based on what learners said about the interactions under that category. The intention was to find out whether ICT facilitated or hindered interaction from the learners' perspective. As mentioned earlier, this discussion is premised on the social constructivist learning theory. The effectiveness of any interaction was assessed on the basis of its ability to promote active knowledge construction by the learner.

8.5.3.1. Sub-theme 1: Technical Interaction

Technical interaction is the interaction between the learner and the technological tool (computer) (Beauchamp, 2006). During the intervention lessons, each learner had his/her own desktop computer installed with Autograph software. Learners interacted with computers individually when they were asked to draw graphs, make some annotations or adjustments on the graph. A number of learners reported several difficulties they encountered, especially in the first 2 to 3 lessons. Sekgobela had this to say:

> At first I had problems in entering the equation onto the computer. I was always behind because I had forgotten how to enter the equation.

Despite the fact that the first two lessons were wholly devoted to teaching the learners on how to use the software to draw graphs, Sekgobela still struggled to enter the equation. She was not the only one. Mpho had her own challenge which she expressed as follows:

> … I struggled a lot in the first lessons. I could not use the keyboard, I was very slow. I also struggled to use the eraser and the scribbler to draw asymptotes on the graph.

Ricky had his own challenge. He had this to say:

> … what gave me a challenge in using the software was on adjusting the scale and the axes before drawing the graph. It took me a long time to do it correctly … .

What is common about the problems highlighted above is that they emanated from the learners' lack of technological proficiency on the use of the computer and the software. When prompted to describe how they felt when they got stuck on the computer during the lessons, the learners used words such as, 'confused', 'frustrated', 'demotivated' and 'anxious'. What stands out clearly from this finding is that, for any productive technical interaction between the learner and the ICT tool, the learner must have sufficient technical competency to use the tool. If that is not the case, that technical interaction will definitely frustrate the learner and destroy his/her self-efficacy.

I also observed that during technical interactions some learners, especially those who were faster and more technically fluent, would digress from the focus of the lesson and begin to explore other things that had nothing to do with the lesson objectives. Some were taking photos of themselves, others were loading music and pictures onto the computers, while others were searching on the internet. Learner-computer interaction in the classroom requires close monitoring from the teacher and constantly directing the focus of the learners on the task or activity.

8.5.3.2. Sub-theme 2: Pedagogical Interaction

Pedagogical interaction refers to the interaction between the teacher and the learner. This took various forms depending on the nature of activity or task learners are engaged in. Teacher – learner interaction took place during whole-class teaching, individual and group work activities. In whole-class teaching, teacher made use of a projector to project diagrams, illustrations, demonstrations and explanations on the whiteboard.

When asked to comment on their interaction with the teacher, most of the learners expressed satisfaction with the clarity of the teacher's instructions, explanations and illustrations. About four learners particularly applauded the use of a projector saying that it was very helpful. Joel explained it as follows:

> … personally, what I found very helpful from the teacher's explanation was the use of a projector. It was very easy to follow what the teacher was saying because I could see it on the whiteboard …. It made it easy for me to use my computer as I could easily follow the teacher's demonstrations.

From a social constructivist perspective, whole-class teaching is usually discouraged because it is characterised with teacher-centeredness and learner passivity. However, when it is flavoured with technological tools like computers, projectors and the whiteboard, it captures the learners' interest, provides visual images and draws the learners' attention towards the targeted aspect or concept.

On teacher's interaction with individual learners or group of learners, several learners raised the concern that they could not get assistance from the teacher as promptly as they would have wanted. I liked the way Gabriel articulated it when he said:

> I can say yes, we got the assistance we needed from the teacher, but sometimes we could not get help as quickly as we wanted. You would find that the time you need help, the teacher would be busy helping other struggling learners. That left us stuck and frustrated as a group, not knowing how to proceed.

Gabriel's concern, together with several others who raised it, concurs with what came out in my reflection of the lessons particularly the first three lessons. Indisputably, the more learners are actively engaged in their learning, the more individual attention they need. With the introduction of ICTs, the pedagogical setting gets complex mainly because learner-tool (technical) interaction exposes learners' unique challenges that need to be addressed. Research confirms that ICT integration adds more classroom management burden on teachers' shoulders (Lundell & Howell, 2000), and this challenge is frequently cited by teachers as one of the reasons why they are reluctant to use ICTs in their classroom (Vandeyar, 2015).

8.5.3.3. Sub-theme 3: Conceptual Interaction

Conceptual interaction is the interactivity that obtains between the learners and ideas or concepts. It is the defining characteristic of the teaching and learning process (Beauchamp, 2006). In the absence of conceptual interaction, no learning would have taken place. Drawing from the social-constructivist perspective, learners construct new knowledge within the social context of the classroom. Construction of knowledge is influenced by a number of factors, and key among them includes the learners' past experiences, existing schema, available resources and the pedagogical setting. In the context of this study, the central focus was on establishing, from the learners' perspective, if the use of ICTs facilitates learners' internal construction of knowledge.

It has already been expounded under the theme of representational competency that learners benefited from the representational power of the technological tools. More specifically, learners' technical interaction with the computers as they drew graphs made it easier for them to identify features like asymptotes, intercepts and axes of symmetry on the graph.

Several of the learners admitted that they initially struggled to comprehend the effect of the parameters ***a***, ***p*** and ***q*** on the graph of the hyperbolic function. Even the analysis of learners' errors in this study revealed that the learners' errors mostly emanated from their failure to link the algebraic equation to the graph of the function. Graph animation, whereby learners could visualise in real-time the graph moving as they changed the parameter, was effective in fostering better understanding of the graphical representation. Ledwaba explained it as follows:

> … you see when you change values of **a** on the computer, you could see the graph changing quadrants. Even the values of **p** and **q**, you could see the graph moving to the left or right. It helped me because at first I did not understand it at all. Now once I see the equation of the hyperbolic function, I can tell the asymptotes, the quadrants and the axis of symmetry.

The transformation of functions and graph interpretation has been identified as the major challenges that learners encounter in dealing with functions in general. The use of computers provided learners with an opportunity to interact with function concepts, both from a graphical and algebraic point of view almost simultaneously. This enhanced their connectional competency.

One observation I made during the lessons was that during ICT-mediated pedagogic activities, there were some contradictions between what the teacher aimed to achieve and what the learners considered to be the objective of the activity. For example, when the learners were tasked to sketch graphs using Autograph, the teacher aimed to make learners identify patterns and make conjectures but, most learners viewed the drawing of graphs as an end in itself. Their excitement was on using the computer and they had to be constantly reminded to use their graphs to answer the given questions.

In some cases, once learners finished drawing their graphs, they easily digressed and began to explore other things in the computer which had nothing to do with the object of the lesson. This suggests that when learners are exposed to use of computers, they require strict supervision and monitoring by the teacher to ensure conceptual engagement by the learners. This extra classroom management burden has been noted by teachers as one of the barriers to ICT integration (Ertmer, 1999).

8.5.3.4. Sub-theme 4: Social Interaction

Drawing from the socio-constructivist conception of interactivity, social interaction in this study was viewed as any meaningful dialogue among the learners, which could either be between one learner and the other learner(s) or in group settings. In most cases, learner–learner interaction was initiated by the learners themselves at any point during the lesson. Even during technical interaction, when learners were expected to be working individually on their computers, social interaction continued to take place as learners compared their graphical displays and provided technical assistance to each other in the absence of the teacher. According to the way the lessons were generally structured, learner -learner interaction was expected to take place when learners worked in pairs or groups to discuss given questions based on their graphs. However, one observation I made during the lessons was that the use of computers by learners seemed to spark-off a lot of social interaction amongst the learners. During whole class teaching, learners would always want to compare what the teacher is illustrating, demonstrating or projecting with what they have on their computers.

Asked to comment on whether they benefited from their interaction with their fellow learners during the lessons, learners' responses expressed mixed reactions. Others felt that they benefited from their interaction with fellow learners while others indicated that they could not receive assistance from their colleagues. Mmathapelo expressed it as follows:

> … my colleague helped me a lot. You see, I was struggling a lot at first in using the computer. Each time I got stuck, my colleague would stop whatever she was doing to assist me. If both of us got stuck, then we would call the teacher to assist us.

Pebetsi expressed the same view when she indicated that her colleague assisted her in the interpretation of the graphs. She had this to say:

> … I had no problem in drawing the graphs. However, the graphs were very confusing to me at first. They were too many and could not see the difference between them. My colleagues helped me a lot because when we answered the questions during the activities, they explained to me how to differentiate between the graphs

The above narrations underscore the role of the 'more knowledgeable others' in facilitating cognitive growth within the learner's zone of proximal development (Bussi & Mariotti, 2014). In this case, the use of computers presented opportunities for social interaction, which in turn strengthened technical and conceptual interaction. It further confirmed the view that in a classroom setting, the 'more knowledgeable one' is not always the teacher. This is particularly true when new technological tools are involved. The challenge that this 'rich' pedagogic setting may bring to the teacher is to manage the social interactions so that they remain meaningful, gainful and task-oriented.

The learners who raised the concern of not getting any assistance from their colleagues were mainly those who had technical challenges in using the computers. One such learner was Fogs who had this to say:

> … I was always behind. I struggled with such things like drawing of graphs, using the eraser and the scribble pen in drawing asymptotes and axis of symmetry. When I tried to get assistance from my colleague, I would find him busy with his computer.

Precious also raised the same concern when she indicated that she felt very bad when she had to always ask her friend to assist her.

The theme and the sub-themes around the interplay between ICT and class interaction are presented in Figure 8.3 below.

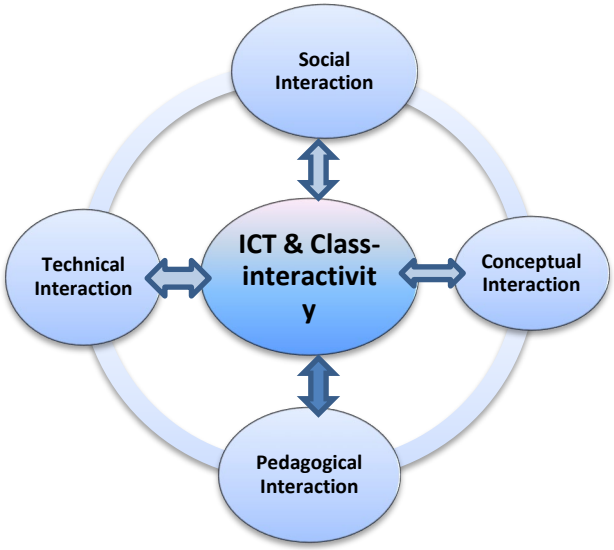


Figure 8. 3. Interacting components under ICT and Class-Interactivity

The findings of this study demonstrated that the introduction of ICTs in the classroom enriches class-interactivity. The ultimate goal of any pedagogical setting is to promote learner-concept interaction, resulting in deepened and broadened understanding of the concept. This study has found that the interaction between the other components in the didactical activity is essential in the achievement of the intended goal. For example, if the interaction between the learner and the ICT tool (technical interaction) is not effective, it becomes a barrier towards the achievement of the intended goal. Similarly, if the meditational tool does not promote or allow for learner-learner interaction, then learning is limited and restricted. Above all, the effectiveness of ICT activities depends on the pedagogical approach employed by the teacher. The teacher is at the centre of a pedagogical setting and the orchestrator of all the interacting components. Thus, the teacher's pedagogical belief or philosophy becomes critical in this case. This study has shown that learners get frustrated when they fail to get individual attention from teacher.

8.5.4 Theme 4: Constraints

A constraint in this study was viewed as any condition that forms an impediment or that makes it difficult to make progress or to achieve an objective. In the context of ICT mediated instruction, constraints are conditions that make it difficult to realise the potential embedded in ICTs. Constraints

are not divorced from affordances. Affordances provide potential for action (Kennewell, 2001) and if these affordances are not well-managed or not well-implemented, they become constraints. The three themes discussed above highlighted the affordances that the ICTs provided and interestingly, some constraints emerged out of these potentialities. I highlight below some of the constraints that emerged under each of the affordances.

Motivation was identified as one of the affordances provided by the ICTs. The motivation was more of 'situational interest' stimulated by the mere use of computers as a 'catch factor'. This emanated from the contextual background of the learners, whereby most of them were using computers for the first time. This condition became a constraint because it created a contradiction between what the teacher aimed to achieve and what learners considered to be the *object* of the didactical activity. For example, once learners had finished drawing graphs they digressed and started exploring other irrelevant things in the computers. One learner complained about this and had this to say:

> … sometimes my group mates were disturbing me. They were not focussed at all. Instead of looking at the given questions for discussion, they started looking at photos, taking pictures and playing games using the computer. This made us to miss some of the explanations and illustrations given by the teacher.

This scenario limited the learners' ability to identify patterns, relationships between concepts and making conjectures. To them, the drawing of graphs was the objective of the activity. To counter this constraint, there is need for the teacher to closely monitor and control all the interactions going on in the classroom, be they technical or social. The teacher has to constantly remind the learners and always draw their attention to what they have to achieve by doing that activity.

Some constraints were *technical* in nature. What came out under the discussion on technical interaction (see Section 8.5.3.1) was that the learners faced a number of technical challenges during the first two or three lessons. They lacked sufficient technical competency in using the Autograph software. This was a major constraint to some of the learners as they expressed their frustration,

anxiety and lack of confidence. This was particularly critical in the context of this study because learners had to physically interact with the technological tool at an individual level. Thus, technical constraints may be reduced in cases where the teacher operates the technological tool and learners simply visualize the display. Whichever way the lesson is planned, one indisputable lesson we can draw from this is that technical competency on the part of the teacher or the learner is a critical consideration in preparing for an ICT-mediated lesson.

The way the teacher orchestrates all pedagogical components in an ICT-mediated classroom may present pedagogical constraints to the learners. It has already been noted that the use of ICTs further exposes the learners' unique challenges and individual differences. The challenge this presents to the teacher is to try and meet the needs of each individual learner and at the same time ensure that the intended objective of the pedagogical activity is met. For example, in the case of this study, some learners complained that the pace of the lessons was too high. One learner had this to say:

> The main problem that I faced was that I found the lessons too fast sometimes. You would find that while I am still busy trying to draw the graph on my computer, the teacher is already explaining something and I would miss that. As I said, some of us it was our first time to use computer so we were very slow in using the key board and drawing the graphs.

Such concerns were mostly raised by those learners who experienced technical challenges in using the computer, and in some cases the problem was compounded by their lack of visual and connectional fluency (see Section 8.5.2). On the other hand, other learners were much faster and completed the activity more quickly. If these fast learners were left waiting and idle, they would quickly occupy themselves with computer games and other irrelevant computer stuff. If the teacher fails to cater for and manage these differential learners' needs, pedagogical constraints arise, which may stifle the ICT affordances discussed above.

In my view, constraints should not be viewed as barriers or inhibiters to ICT integration in the classroom. They are an integral component of ICT-mediated instruction and they compliment ICT

affordances. While affordances provide potential for action, constraints provide structure and guidance for the course of actions. As the teacher encounters constraints and strives to overcome them, s/he develops efficient pedagogical practices that yield better outcomes.

8.6. Conclusion of the chapter

This chapter established the affordances and/or enablers for knowledge construction provided to the learners by the ICT-mediated intervention lessons. The enablers emerged from the learners' experiences during the intervention lessons and their own interpretation of these experiences. This approach was premised on the socio-constructivist perspective that though knowledge construction is an internal process, it is enhanced and facilitated by shared external experiences.

Literature reviewed generally established that ICT integration in the classroom enriches the pedagogical setting, but this study went deeper, more specifically to find out what learning opportunities were presented to the learners and what constraints were associated with these opportunities. The affordances were discussed under the themes: motivational enablers, representational competency and class interactivity. ICT constraints form an integral part of these affordances and they help to develop more effective pedagogical practices. These components of ICT-mediated instruction are highly situational and contextual, and they form the basis of the ICT-mediated instructional model that this study proposes.

CHAPTER 9: SUMMARY OF FINDINGS, CONCLUSION AND RECOMMENDATIONS

9.1. Introduction

This chapter summarises the study. The purpose of this study was threefold. Firstly, the study was motivated by the view that learners' errors and misconceptions point to some important issues for further exploration in the learning of mathematics. After studying yearly examination and diagnostic reports, I realised that that there are persistent errors that learners made, particularly, with regard to the topic of functions. These recurring errors and misconceptions are clear testimony of the epistemological challenges learners face in their learning of functions. Research in mathematics education in South Africa suggests that teaching of mathematics is largely traditional and procedural (Valero et al.,2012; Makonye, 2014; Pournara et al., 2015). Given the complexity and abstract nature of concepts under functions, procedural pedagogical approaches are not helping learners to understand function concepts.

Secondly, we are in the fourth industrial revolution where the fusion of ICTs is revolutionising and radically transforming operations both in the private and public sectors. Unfortunately, the use of ICTs in education, particularly in teaching and learning, is lagging behind (Ndlovu & Moll, 2016; Nkula & Krauss, 2014).

The DBE, in partnership with other stakeholders in the private and public sectors, are putting ICT resources in schools in terms of hardware and software. Though the resources are far from being adequate and are unevenly provided, the utilisation of the available resources for teaching and learning is uncoordinated and problematic. I witnessed cases where computers, tablets and overhead projectors were securely locked in strong-rooms, while teachers were struggling with their old and archaic teaching methodologies which are no longer appealing to this technological generation of learners. In isolated cases where ICTs are integrated, the usage is either administrative or representational (Wilson-Strydom & Thomson, 2015; Hokanson & Hooper, 2000).

Representational usage re-presents information in another form or medium. For example, the projecting of notes or diagrams on a whiteboard using overhead projectors is representational use. It does not promote knowledge construction and higher order cognitive skills (Nkula & Krauss, 2014; Mdlongwa, 2012; Ndlovu & Mji, 2011). Another concerning observation was that ICT integration is confined to schools in more economically advantaged communities. Even research studies on ICT integration are clustered around such communities (Tustin et al., 2012; Gudmunsndsdottir, 2010). The situation in more economically disadvantaged schools is direr and little is known about them.

In light of this background, this study was undertaken to provide answers to three research questions phrased as follows:

- What errors and misconceptions do learners hold on hyperbolic functions?
- What impact does ICT-mediated instruction have in enhancing the conceptual understanding of concepts under hyperbolic functions?
- What pedagogical model for ICT-mediated instruction could this study recommends in deepening learner conceptual understanding and minimise misconceptions on hyperbolic functions?

The chapter begins by explaining how the study answered each of the research questions above. It then moves on to give concluding remarks and recommendations based on the findings of the study. It also includes a discussion on the limitations of the study and the researcher's reflections on his personal experiences during the study.

9.2. Summary of findings

9.2.1. Errors and misconceptions under hyperbolic functions

The first research question sought to establish the errors that learners make and misconceptions they hold in dealing with hyperbolic functions and functions in general. The focus on hyperbolic functions was informed by the view that hyperbolic functions encompass some features which do not exist in

other types of functions. Notwithstanding, the findings of this study ultimately apply to functions in general. The findings recorded here were based on a sample of thirteen grade 11 learners conveniently sampled on the basis that they were studying pure mathematics at the school where the study was carried out. The learners wrote a pre-test on hyperbolic functions. The pre-test was broad enough to cover all the concepts and skills under hyperbolic functions as spelt out in the CAPS document. To ensure that the test catered for all cognitive levels, the question items were drawn according to the APOS conception levels and guided by the genetic decomposition model developed as the analytical framework for this study (see Section 3.3.1). Task-based interviews were carried out during analysis of learners' written responses. The interviews gave learners an opportunity to explain, clarify or justify the thinking behind their written responses.

An analysis of learners' written responses was initially deductive as it started off with an error categorisation protocol. During the data analysis stage, some misconceptions emerged which could not fit into the predetermined categories and thus a new set of misconceptions and errors emerged. The central focus was not only on identifying and classifying errors made, but to infer on possible misconceptions underlying those errors. The following misconceptions emerged from the analysis:

- **Algebraic Misconceptions**

Dealing with hyperbolic functions involves algebraic equations with fractions. Learners displayed a number of shortcomings in their execution of algebraic manipulations. Though algebraic misconceptions may not be directly linked to the understanding of function concepts, they negatively influence the learners' ability to solve a given problem.

The following error categories were identified under algebraic misconceptions:

- **Executive errors** – A particular example of this error was noted when learners calculated the x- and y-intercepts of the given hyperbolic function. They managed to pick the correct

approach by substituting $y = 0$ and $x = 0$ respectively but failed to execute algebraic procedures to solve the equation.

- **Procedural-extrapolation errors** – A case in point was when learners applied cross multiplication over addition. The learners erroneously extrapolated that if we can cross multiply: $\frac{6}{x-2} = \mathbf{3}$, we can do the same for $\frac{6}{x-2} + 3$. This further underscored the danger of teaching procedures with no understanding of the mathematical justification behind the procedures. Learners who made this error saw nothing wrong with their working and were even surprised why they could not obtain the correct answer.

- **Operational extrapolation in relation to the number 0** – This was an unexpected error whereby learners wrongly applied the additive identity property of the number 0 over multiplication and division. In the first instance, the learner got the equation $-3x = 0$ and went on to say $\therefore x = -3$. Clearly, she divided both sides by -3 and got $x = -3$. In the second instance, the learner failed to expand: $-3y\ (0 - 2) = 0$ and got $-3y + 6y = 6$**.** This finding illustrated the cumulative nature of misconceptions. This misconception might have been picked in earlier grades and carried over to the end of the learner's school mathematics.

➢ **Conceptual Misconceptions**

One prevalent cause of the learners' errors was their lack of conceptual understanding of concepts under hyperbolic functions. It emerged during data analysis that learners had muddled understanding of concepts such as intercepts, asymptotes, domain, range, axes and the axis of symmetry. They demonstrated that they only had a mechanical understanding of how to determine these features of the function but lacked an understanding of what they represent. This conceptual

shortcoming exposed them to a number of errors when confronted with problem situations. Examples are explained below:

- **Intercepts given as equations not a point** – In answering questions that required the x- and y-intercepts, some learners attempted to derive the equation of a line. During task-based interviews, it came out that learners could not identify the intercepts, asymptotes and axes from a given hyperbolic graph.
- **Learners failed to justify why they substituted $x = 0$ for y-intercept and $y = 0$ for x-intercept** – In responding to the question, which required the coordinates of the x- and y-intercepts, nearly all the learners demonstrated some algebraic understanding that in determining the x-intercept, we substitute $y = 0$ and for y-intercept we substitute $x = 0$. When each of them was then asked to explain or justify the approach, none of the learners gave a sensible mathematical explanation. Clearly the procedure was used as a given rule and the learners used it without an understanding of why they substituted $x = 0$ or $y = 0$.

- **Failure to link domain and range to the asymptote** – The inability by the learners to connect the domain and range of a hyperbolic function to its asymptotes further exposed their lack of conceptual understanding. They only had a memorized understanding that when $f(x) = \frac{a}{x-p} + q$, then the domain is written as: $x\epsilon R, x \neq p$ and the range: $y\varepsilon R, y \neq q$. When the algebraic equation was slightly twisted or transformed, the majority of the learners failed to give the domain and range. On the other hand, when given the domain and range of a hyperbolic function, the learners failed to extract its asymptotes.

- **Asymptotic Misconceptions**

The learners' conception of an asymptote was either flawed or incomplete. In the first place, the majority of the learners defined an asymptote as a dotted line on a graph. Given a hyperbolic graph

where asymptotes were not shown, the majority of the learners either did not attempt the question or gave wrong responses. Those who did not attempt to answer the question gave the reason that they thought the asymptote was not there. They could not figure out the asymptotes from the shape of the graph. Other learners presented the asymptotes as a point and not an equation. The learners' responses simply reflected that they lacked a correct concept image of what an asymptote is and how it relates to other concepts like domain, range and axis of symmetry.

- **Graphical Misconceptions**

One of the most prevalent weaknesses that emerged from the analysis of the learners' responses was that their conception of functions was confined to the algebraic representation. They struggled to represent a function given in symbolic terms as a graph or to identify the critical features of a function from a given graph. They could not appreciate that the algebraic equation of a function and its graph are representations of one thing. This restricted conception of a function manifested itself in the following errors:

- **Representational errors -** No link or incoherence between algebraic equation and graph. Sometimes the graph was drawn cutting the asymptotes.
- **Meta-cognition errors** – Learners failed to reflect on the sensibility of calculations done. For example, given $\boldsymbol{f(x) = \frac{a}{x+p} + q}$, the sign of $\boldsymbol{a}$ indicates the quadrants in which the graph should lie. If learners erroneously calculated the x- and y-intercepts and found them in the wrong quadrants, they could not reflect on their working to correct the error.
- **Logically - Invalid Inferences** – In the case where both the x- and y-intercept was the origin, O, most of the learners wrongly concluded that the graph would have one curve instead of two symmetrical curves. They could not give any reasonable justification for their inference, except that they were used to cases where the intercepts were two distinct points.

- **Hybridization error –** Learners hybridised the graph of a hyperbolic function and that of the quadratic function. After showing the vertical and horizontal asymptotes, they still went on to draw a parabolic curve. This confirmed that the learners had a limited understanding of what the graphs were showing.

➢ **Notational Misconceptions**

Finally, data analysis revealed that some misconceptions emanated from failure by the learners to correctly read the meaning of functional notations and symbols. Mathematical language is largely symbolic and notational. A wrong conception of the symbolic representation of a mathematical idea is a barrier to their understanding of concepts. The following errors resulted from notational misconceptions:

- **Interpretation Errors –** learners failed to correctly interpret questions like, 'For what values of x is $\boldsymbol{f(x) \geq 0}$?'
- **Decoding and Encoding Errors –** Learners failed to understand the meaning of symbols and terminology resulting in answering unintended questions. For example, learners struggled to see the link between $\boldsymbol{f(x+3)}$; $\boldsymbol{f(x)+h}$; $\boldsymbol{f(-x)}$ and the original function f(x).

9.2.2. The impact of ICT-mediated instructional intervention

The second research question of this study sought to establish the impact of the ICT-mediated instructional intervention in deepening learner conceptualisation of hyperbolic functions. After identifying the learners' errors and misconception in the pre-test, the next phase of the study was to take learners through an ICT-mediated intervention programme on hyperbolic functions. As detailed in Chapter 7 of this study, the intervention lessons were based on a mathematical software called Autograph, supported with other technological tools su the overhead projector and the whiteboard. Each learner had a computer desktop installed with Autograph. After the lessons, the learners wrote a post-test.

A qualitative comparative analysis of learners' responses in the pre- and post-test was done. The analysis was qualitative in the sense that no statistical comparative tests were used, but the analysis was based on an analytical framework built on the APOS theory. Proponents of the APOS theory identified four mental constructions that depict the depth of understanding of a concept. The mental constructions are namely *action*, *process*, *object* and *schema* (see Chapter 3).

Based on the mental constructions, a genetic decomposition (see Section 3.3.1) was developed on hyperbolic functions which stipulated observable indicators for each of the conception levels. The deepening of conceptual understanding was assessed on the basis of whether there was any indication of learner conceptual progression from one level of conception to a higher level. In other words, by analysing learners' written and oral responses, it was possible to identify what learners could not do before the intervention lessons and what they could do after the lessons.

The APOS analysis of the learners' written and oral responses in the pre-test revealed that the majority of the learners were operating at the *action*-conception level. The major indicator characterising this conception level is the use of procedures without reason. Given the standard equation of a hyperbolic as: $f(x) = \frac{a}{x+p} + q$, learners had a memorised understanding that when calculating the x- and y-intercept we substitute $y = 0$ and $x = 0$, respectively, but they failed to justify why they substituted 0 in both cases. Similarly, the equations of asymptotes and the domain and range were remembered verbatim as they were given in class. This level of conception is limited and restricts learners to lower order questions. Any question that deviated from the standard equation or required application of knowledge was either not attempted or wrongly answered.

In the post-test, there were noticeable improvements in the quality of the learners' responses. The improvements were explained in terms of whether there was evidence of interiorization (transition from *action* to *process* conception), encapsulation (transition from *process* to *object* conception) and thematization (transition from *object* to *schema* conception). The findings on these mental processes were as follows:

- **Indications of Interiorization**

Interiorization was viewed as a change from a procedural activity to doing the same activity again demonstrating some understanding of the processes involved. In other words, learners began to reflect on an action thereby forming a mental structure that performs the same operation as the action, but wholly in the mind of the individual.

During the intervention lessons, learners were engaged in a series of activities involving the drawing of graphs using the graphical software and identifying and comparing their graphical features. It was evident from the findings of this study that these activities enabled learners to develop mental prototypes of features such as intercepts, asymptotes and axis of symmetry. Learners were able to explain their calculation of intercepts in relation to their coordinates. Asymptotes were explained in terms of the function being undefined and connected to the domain and range. The graphical representation of the function improved significantly though some minor errors still occurred.

- **Indications of encapsulation**

Encapsulation condenses the processes into a cognitive object that one can change, manipulate or transform. In determining whether learners had encapsulated hyperbolic functions or not, I checked for the learners' ability and flexibility in switching from symbolic to graphical representation or vice-versa and their ability to understand transformations on hyperbolic functions. Unlike in the pre-test, the analysis of the post-test showed an improvement in the graphical representation of the given hyperbolic function. The curves were placed in the correct quadrants and the learners demonstrated a clearer understanding of the connection between the graph and its asymptotes. There was evidence that the learners were now able to link the critical points of a function to its graph. I interpreted this as a conceptual shift from a '*process*-orientated' conception, where critical points were calculated and viewed as isolated features of the function, to a more '*object*-oriented' conception where the critical points were seen in totality, as integrated parts of one object.

- **Indications of thematization**

In APOS terms, thematization is viewed as the transition from an *object*-conception to the *schema*-conception level. *Schema* is the highest level of understanding and is connected to other existing schemas. Once the development of a concept has matured into a *schema* in the learner's mind, he/she becomes more flexible in approaching a problem situation. In this study, I expected a learner who had thematized the hyperbolic function schema to be able to link the critical features located on the drawn graph and the ones calculated using the algebraic equation. I expected such a learner to be able to comprehend that critical points connect the graph and its function.

When confronted with a problem and grappling to solve it, the learner must be able to flexibly switch from equation to graph and from graph to equation. In the pre-test, none of the learners demonstrated any trace of the *schema*-conception. It was only after the intervention lessons that three of the learners proved beyond any reasonable doubt that they had acquired the hyperbolic schema. Not only did these learners present perfectly correct responses in the post-test, their explanations during interviews demonstrated that they had crystallized the concepts into a malleable concept image. These learners were able to connect, coordinate and synthesize the critical features of a hyperbola and use them in totality to tackle higher order problem situations.

The findings that emerged from the comparative analysis of learners' written and oral responses in the pre- and post-test revealed that learners had a deeper conceptual understanding of hyperbolic functions after the intervention lessons. This improvement could only be attributed to the use of ICTs as mediating tools. The question one would ask is: 'what exactly, in the ICT-mediated lessons, might have facilitated or enhanced the deepening of the learners' conceptual understanding?' The answer to this question constitutes the findings to the third research question of this study.

9.2.3. My proposed ICT-mediated Instructional Model

The study has confirmed that ICT-instructional intervention deepened learner conceptualization of concepts under hyperbolic functions. However, the study aimed to find out what aspects of the ICT-mediated instruction facilitated learner understanding. Admittedly, while the use of ICTs in teaching and learning enriches the pedagogical setting, it complicates it. It is for this reason that most teachers are reluctant to integrate ICTs in their classrooms. This study has identified affordances and constraints that teachers need to consider in their planning and preparation for ICT-mediated instruction. These affordances and constraints make up the components of an ICT-mediated pedagogical model that this study developed. The details of the pedagogical model are discussed under recommendations section of this chapter. In the next section, I present the conclusions based on the findings of this study.

9.3. Conclusions

This study was premised on the assumption that detecting and diagnosing learners' errors and misconceptions is a critical step in developing conceptual understanding. One thing I learnt in doing this study was that diagnosing misconceptions is a deep and involving process which goes far beyond mere detection of errors and mistakes in learners' responses. Usually the visible errors or mistakes that we detect are manifestations of underlying alternative conceptions that might have developed over time.

The approach used in this study to unearth these underlying misconceptions was to ask learners to explain and give reasons for their written responses. Understandably, teachers may argue that this is a mammoth and time-consuming exercise, but I do maintain that it is a doable and worthwhile exercise. There are several opportunities available to teachers that can be used to interact with learners, probing them and get to understand the thinking behind what they say or write. The key point here is that misconceptions should be viewed from the learners' perspective in order to be in a better position to assist the learners to overcome them. I used to think that correcting misconceptions is simply by 'telling' the learner that this is wrong, but my experiences in this study have shown that

some misconceptions are stubborn and resistant. The stubbornness or resistance is anchored on the fact that the misconception makes sense from the learner's perspective. To overcome this, some strategy is needed to create a cognitive conflict in the learners' mind which may lead to reflection and possibly self-correction.

Another observation to note from the findings on errors and misconceptions was that the learners' misconceptions were not confined to functions. Rather, other misconceptions were carried over from other related topics. Algebraic misconceptions featured predominantly as one the major reasons why learners could not obtain correct responses (see analysis of item 1.1 of the pre-test). In the majority of cases, learners picked the right approach to the question but failed to execute algebraic procedures correctly. This illustrates that misconceptions are cumulative and have a compound effect.

Some misconceptions are vertically transferred from earlier concepts making it difficult for the learner to develop a mature or complete cognitive structure (or schema) of a concept (Dubinsky & McDonald, 2002). On the other hand, some misconceptions are horizontally or laterally transferred from other related topics and impede on learner's understanding of the new concept. Either way, misconceptions make it difficult for learners to assimilate and accommodate new knowledge (Kilpatrick, 1987). It is therefore imperative for teachers that before introducing a new concept, they need to check the learners' understanding of background concepts and the related algebraic skills or procedures.

The discussion on what causes the misconceptions may have been outside the scope of this study but from what I picked in this study, the major causes are pedagogically related. The learners who participated in this study were initially taught hyperbolic functions using the traditional approach (without the use of ICTs). The findings on the misconceptions exposed a number of deficiencies in this approach, particularly in the teaching of functions. Its major limitation is the lack of multiple representations and as a way of bridging this gap teachers tend to stress more on the algebraic (symbolic) form of functions. Consequently, learners' conception of a function was restricted to the

algebraic form. Problems that could easily be addressed using the graphical approach were attempted through lengthy algebraic methods. A typical example was when learners were given the graph of a hyperbolic function $g(x)$ and were asked to write down the equations of the asymptotes of $g(x+3)$. None of the learners viewed $g(x+3)$ as a horizontal translation of $g(x)$. Instead, they preferred to substitute $x+3$ for x in the standard equation, which exposed them to several procedural errors. Any problem situation that required learners to switch from the symbolic to the graphical form of a function gave the learners serious challenges.

The ICT-mediated instructional intervention was effective in deepening the learners' conceptual understanding of concepts under hyperbolic functions. In line with the vision of the White Paper on e-education (DBE, 2013), the lessons were designed and crafted in a way that learner-computer interaction advanced cognitive skills such as reasoning, problem-solving and deductive reasoning (see Section 7.4.2).

Drawing from the Activity theory (see Section 3.3), each pedagogical activity engaged learners in guided discovery targeting a particular concept or exploring relationship between concepts. It is noteworthy that this calls for careful and contextualised planning and preparation on the part of the teacher. It is for this reason that teachers cite time constraints as one of the barriers for ICT integration (Bialobrzaska & Cohen, 2005; Nkula & Krauss, 2014). However, the good thing is that once the lesson preparations have been developed, they can be re-used or simply adjusted to suit prevailing circumstances.

Another consideration worth mentioning here is the technical competency of the learners in using the Autograph software. This was particularly a challenge in the context of this study because most of the learner participants had never used a computer before. The first two lessons were devoted to learning about the basics of Autograph but apparently, they were not enough as some learners still struggled during the first two lessons of the actual lessons. However, with the interest and motivation aroused by the use of computers, learners quickly learn. What is critical is that learners should be

given sufficient basic technological skills needed to perform the learning activity. What I also picked is that once exposed to a technological gadget or tool, learners are very explorative and they learn most of the things on their own.

The finding of this study demonstrated that, if properly integrated, ICT-mediated instruction deepens conceptual understanding of concepts under functions. As other researchers have established, proper use of ICTs involves using ICTs as instruments for exploration, rather than instruments for recall, recording or administrative purposes (Hardman, 2005). In the same vein, Wilson-Strydom and Thomson (2015) gave a distinction between representational use and generative use of ICTs in teaching and learning (see Section 4.5.2). They argued that representational use reinforces recall and procedural knowledge while generative use promotes problem-solving and creative thinking.

The literature that was reviewed for the purposes of this study on ICT integration in South Africa generally established that it concentrates on learning about ICTs instead of learning with or through ICTs. The findings of this study have shown that ICT integration in the teaching and learning of functions enriches the didactical context and it provides opportunities for learners to be engaged in guided discovery activities.

Through multiple representations, ICTs affords learners an opportunity to identify patterns, discover relationships or connections and develop correct mental prototypes of the abstract concepts under functions. This study further proposes an ICT-mediated instructional model that can be used by teachers using ICTs in teaching functions.

The analytical framework used to analyse and establish the impact of the instructional intervention was based on the APOS theory, whose main proponent is Dubinsky (see Section 3.4). Insights from this theory were relevant because its proponents argue that the understanding of a concept occurs at four different conception levels. The lowest level is the *action*-conception level, followed by the *process*-conception level, followed by the *object*-conception level before reaching the *schema*-conception level. In light of these conception levels, the deepening of conceptual understanding was

viewed as the development from one conception level to a higher level. In other words, the impact of the instructional intervention was assessed on the basis of whether it enabled learners to move from one conception level to another.

The weakness of this conceptual framework is that these mental constructions are internal and not easily observable. To circumvent this challenge, APOS theorists propose that the concept must be broken down into observable indicators for each of the conception level. They call this genetic decomposition (see Section 3.3.1). The genetic decomposition has the limitation that it is subjective. It is influenced by factors such as the concept itself, the perceptions of the person developing it and the level of the learners (or students).

The findings of this study also revealed that before the intervention lessons, most of the learners were stuck at the *action*-conception level of hyperbolic functions. They mechanically followed procedures with no understanding of the meaning behind them. Their understanding of hyperbolic functions was restricted to the symbolic or algebraic representation and they struggled to connect a function to its graph. This shallow and limited conception confined the learners to lower order questions and exposed them to several errors and misconceptions.

After the intervention lessons, there was convincing evidence that learners had deepened and broadened their understanding. Most were now in the *process*-conception level where they could explain and justify their procedures and begin to see the connection between a given function and its graph. The learners eventually managed to correctly identify concepts like intercepts and asymptotes on the graph and explained them in terms of their connection to the graph. However, not all learners developed to the *object*- and *schema*-conception levels. Possibly others needed more time or lacked some fundamental background concepts on functions. On the overall, each of the learner participants registered some improvement in the quality of their written responses and oral responses during the task-based interviews. What I picked was that as learners develop their conceptual understanding,

their level of confidence and motivation grows as well, and ultimately errors and misconceptions are drastically minimised.

The other question that this study had to deal with was: What exactly caused the improvement in learners' conceptual understanding? The answer to this question was grounded on the learners' experiences during the intervention lessons and their interpretation of the experiences. In an open-ended post interview session, learners were given a platform to freely share their experiences in terms of what they found helpful and what challenges they encountered during the ICT-mediated intervention lessons. The central focus was on finding out what affordances and constraints the use of ICTs presented to the learners. I found the learners' perspective very central because any instructional programme is targeted towards the learner.

The success or failure of an instructional programme depends on how learners receive and perceive it. My role, as the designer and implementer of the intervention programme, was to actively observe, reflect on and make sense of what learners said or did. The interview data and the observation notes were inductively analysed, and four themes emerged which constituted the components of the ICT-instructional model that this study is recommending. The four themes are: motivation enablers, representational competency, class interactivity and constraints.

9.4. Limitations of the study

In doing my study, a number of factors constrained the findings of the study. I explain here these constraints and how I minimised their effect on the trustworthiness and credibility of the findings. Firstly, this study was highly contextualised and the generalizability of its findings to other contexts is limited.

The study was carried out at a rural secondary school with a sample of thirteen learners conveniently sampled on the basis of the fact that they were studying pure mathematics amongst the grade 11 learners. Furthermore, the school is located in an economically disadvantaged community whereby

the bulky of the learners came from low-income families. The majority of the learner participants had never used a computer before.

In terms of the availability of ICT resources, the school initially had three desktop computers which were used for administrative purposes only and later on, it received a donation of forty computers, two overhead projectors and two whiteboards. Such unique contextual factors make it difficult to simply extend the findings to other contexts in South Africa or beyond. However, in my view, the focus of the study was not on generalising the findings but on presenting an in-depth understanding of how learning could be enhanced through integration of ICTs within the given context. My take is that while the findings of this study may not be generalised, they provide valuable insights to teachers in similar contexts.

Another constraint placed on this study was that it was confined to hyperbolic functions, and not extended to other types of functions. After looking around extensively on previous studies, I noted that most of the studies on functions focussed on quadratic functions and very little was done on hyperbolic functions, yet hyperbolic functions possess some unique features that are problematic to learners.

As stipulated in the mathematics CAPS documents, the hyperbolic functions covered at FET level only include those with a vertical and a horizontal asymptote. Their asymptotic nature makes them unique and presents some conceptual obstacles to the learners. Despite these slight variations, the concepts covered, and the skills needed under functions are basically the same in all types of functions.

Focussing on one type of function allowed the researcher to have an in-depth investigation of the issues under study, which would not have been achieved if the content coverage was broader. I therefore believe that the findings on errors and misconceptions under hyperbolic functions can be extended to other types of functions. Similarly, the proposed ICT-mediated instructional model is applicable in the teaching and learning of functions in general.

My role in this study was that of a researcher, developer, evaluator and a teacher to the participant learners. This was challenged by other research colleagues who felt that if one designs, implements and evaluates their intervention programme, there are bound to be biases. Their argument was that since I was the designer and implementer of the ICT-mediated instructional intervention programme, it would be hard for me to maintain objectivity needed in research. To minimise researcher bias I tried as much as possible to maintain an open mind throughout the study. I kept a reflexive journal where I recorded observations, critique and reflections at every stage of the study.

I was open to opinions from my colleagues, other specialists and of course my supervisor. For example, in developing the lessons for the instructional intervention, I solicited input from one of my colleagues (an experienced mathematics educator) and from the District Curriculum Advisor (who facilitated in one of the district workshops on ICT integration). I also had the privilege of videotaping all the intervention lessons, with the consent of the learner participants. After each of the lessons, I would go over the lesson together with my colleague, re-examining and critiquing everything that occurred during the lesson, with the aim of improving the next lesson.

In this study, an assumption was made that learners were sincere in their written and oral responses. In soliciting the learners' consent to take part in this study, I clearly explained to them that neither the pre-test nor the post-test was going to be used as part of their formal assessment mark. I however clarified that the content covered during the programme was part of their syllabus and they stood to benefit more, provided they took the programme with the seriousness it deserved. The reader might view this as a limitation in the sense that once learners were aware that the tests were not part of their formal assessment, they would not take them seriously and would not give sincere responses. I considered it differently and my observations throughout the data collection process corroborated my viewpoint. Learners were very excited and cooperative throughout the programme such that none of them missed a single lesson from day one until the last. Even the task-based interviews after the pre- and post-tests confirmed that learners were earnest and sincere in their written responses.

Actually, the relaxed and friendly atmosphere relieved learners of the pressure and tension that real examinations exert on them. It made them free to express their thoughts without fear of being labelled right or wrong. Even during the open-ended interviews, after the intervention lessons, it was critical to get the learners' honest opinion on what they liked or did not like and what worked well for them and what did not work well for them. The only way to achieve this was to create a friendly and relaxed, but goal-oriented environment that eliminated any trace of fear or mistrust. I tried my best to achieve this.

In light of the above arguments, I argue that this study was done in a way that minimised its limitations so that the validity and reliability of this research were not compromised. As a qualitative research, I need to underscore that context shaped a number of things in this study. The generalizability of the findings in this study has to be done in light of the context in which the study was undertaken.

9.5. Reflections on the proposed ICT-mediated Instructional Model

The Mathematics curriculum documents in South Africa recommend that available technological tools should be used in the teaching of functions but there is no clear guidance on how teachers should go about this. Available policies on ICT in education (see Section 4.4) provide the vision and expectations of the DBE but they still do not provide specific guidelines to teachers on the quality use of ICTs in the classroom. It is not surprising that the actual use of ICTs in teaching and learning in South Africa is lagging behind and uncoordinated, particularly in schools located in marginalised communities. This study is interventionist and proposes an ICT-mediated instructional model that teachers can use in the teaching of functions and other related topics.

9.5.1. The role of context in the model

I begin by reiterating that the proposed model is developing from a study that is highly contextualised. As explicated under the limitations of the study, context shaped a number of things in this study. Thus, the adaption or adoption of this model should be based on a clear understanding of the context

under which the study was done. My point of departure is the argument that context needs to be at the forefront in ICT integration. It makes ICT integration more relevant, meaningful and applicable. The Cambridge English dictionary defines context as circumstances that form the setting for an event, statement or idea, and in terms of which it can be fully understood. In the case of teaching and learning, context refers to the circumstances surrounding a pedagogical setting. Drawing from the Social constructivist perspective, learning cannot be separated from the social context (Vygotsky, 1978). The point here is that during planning and preparation for an ICT-mediated lesson, there are contextual factors that the teacher has to carefully consider.

One obvious contextual factor is the availability of the ICT resources. These include the hardware, for example, computers, projectors, whiteboard, whiteboard markers and softwares if needed. The availability of these tools has to be considered in terms of their quantities versus the number of learners and whether they are functional or not. For example, in this study, each learner participant had his/her own computer.

The learners may share one computer in pairs or in groups depending on availability or the objective of the pedagogical activity. Closely linked to availability of ICT resources is the availability of infrastructure at the school. These include desks, classrooms, and computer lab or mathematics room. These have implications on class organisation and management which subsequently affect class interactivity. The absence of special rooms such as computer lab or mathematics classrooms may mean that the ICT gadgets have to be carried from one class to another and this movement has to be carefully managed to ensure their safety.

Admittedly, the list of contextual factors can never be exhausted. Without being prescriptive, other factors include the learners themselves, the concept/s to be taught and the school administration. At the centre of all these factors is the teacher who is expected to orchestrate and coordinate all these factors towards the intended goal. The assumption here is that we are dealing with a teacher who is professionally qualified, progressive and abreast with technological and pedagogical trends in

mathematics education. If the teacher's beliefs, perceptions and theoretical orientations are not well aligned to the demands of this technological era, s/he may be an impediment to ICT integration (Ertmer, 1999).

9.5.2. My proposed ICT-mediated Instructional model

The model is grounded on data collected from the learner participants' experiences and perspectives throughout their participation in this study. My experiences, observations and reflective judgements as the researcher throughout the study were valuable in developing the model. To minimise researcher bias and uphold objectivity, I received valuable input from my research colleagues, other experienced mathematics educators, subject specialists and my supervisor. Literature reviewed provided critical insights on what other researchers and scholars have done and on what gaps still exist with regard to ICT integration in mathematics education. The model is premised on the principles of the Social Constructivism theory and it is contextualised to suit the infusion of ICTs and other prevailing circumstances. Figure 9.1 below is a diagrammatic representation of the model.

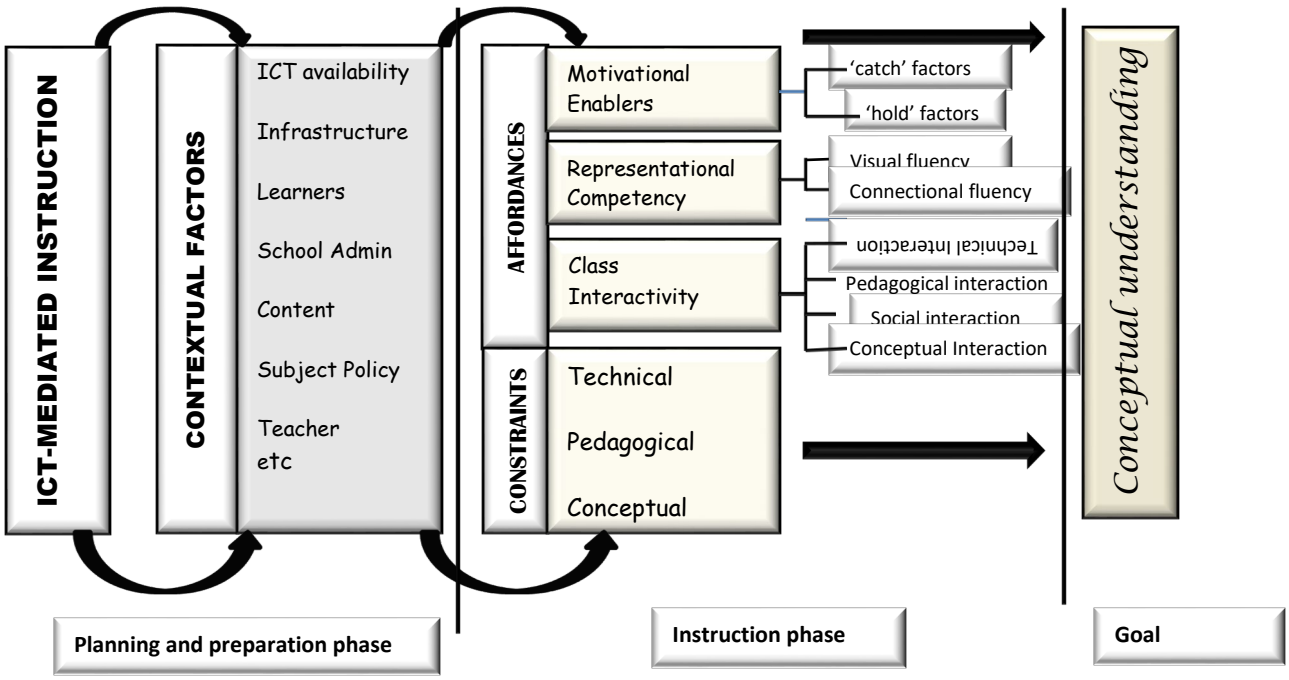


Figure 9. 1. ICT- Mediated Instructional Model

Effective ICT-mediated instruction requires careful planning and preparation. It is at this stage that the teacher must consider the prevailing contextual factors, as alluded to in the preceding section. Time constraints are commonly cited by teachers as one of the barriers to ICT integration (Ertmer, 1999; Ndlovu & Moll, 2016; Vandeyar, 2015). Admittedly, the use of ICTs for teaching inevitably demands more time for thoughtful and innovative planning.

In my view, the reason why teachers complain about time is that they are hesitant to step out of their comfort zone. The traditional pedagogical practices are generally instructivist in nature (Steyn & Van Greunen, 2014), whereby knowledge is transmitted from the teacher to the learners. Proper ICT integration in the classroom negates this traditional view of teaching and calls for teachers to radically transform their pedagogical beliefs and develop a new philosophical perspective.

What I picked during this study was that in planning and preparing for an ICT-mediated lesson, the teacher cannot just 'copy and paste' another lesson developed elsewhere or repeat a lesson developed in another setting. Every lesson has its unique experiences and the teacher has to be flexible and adaptive to the prevailing circumstances. Sometimes even the software in use may have to be adjusted to suit the learners' needs. The proposed model suggests that during the planning and preparation phase, the teacher should depart from the 'business as usual' mentality, and critically look at the contextual factors surrounding the pedagogical setting.

9.5.2.1. ICT Affordances

It is of paramount importance for the teacher to have a clear understanding of the affordances provided by the technological tool in use. In the context of this model, affordances are viewed as the attributes of the pedagogical tool, which provide potential for action Kennewell (2001). In designing didactical activities based on a technological tool, the teacher has to know the potentialities imbedded in that tool and how to unlock them to enhance learner understanding of the targeted content.

This study was mainly based on a mathematical software called Autograph and other supporting ICT tools. Though this study had a special focus on functions, the software can be used in teaching several other mathematical topics. Three broad affordances emerged from this study which, in my view, can be extended to other ICT tools, namely, motivational enablers, representational competency and class interactivity.

- **Motivational enablers** – It emerged from this study that ICTs are motivation enablers. This finding corroborates several other research findings (Ogbonnaya, 2010; Padayachee, 2017; Mooketsi & Chigona, 2014; Mdlongwa, 2012). The view of motivation adopted in this model was drawn from the social cognitivist perspective who perceived motivation as situational, contextual and domain-specific. What this means is that what motivates a learner in one context may not be motivating to another learner in a different setting. What this study established was that motivation occurs at two levels. The first level is what Mitchell (1993) described as situational interest, which is stimulated or aroused by what I can call 'catch' factors. The technological garget itself and the software used were 'catch' factors as they stimulated learners' interest. Situational interest has two weaknesses. Firstly, it is extrinsic and short-term because once the learners get used to the technological tool they will lose interest. Secondly, it may create contradictions as learners focus on the tool instead of the targeted concept.

  The second level of motivation is based on 'hold' factors such as content mastery and improved academic achievement, which eventually raises learners' self-efficacy (Bandura, 1997). Once learners begin to realise some improvement in their performance of a task, it propels them to want to achieve more and ultimately results in higher levels of effort. The central argument here is that introducing and exposing learners to ICT tools may stimulate the learners' interest to learn and what cultivates and sustains the interest is when ICT tools are gainfully used to enhance and promote learner understanding.

- **Representational Competency** – Another ICT affordance that emerged from the learners' narrations after the intervention lessons was that of representational competency. Learners stressed how they benefited from 'seeing' or 'visualising' the graphical representation of the function and its transformations. They explained that this helped them to have a better connectional understanding of hyperbolic functions in their algebraic and graphical forms (see Section 8.5.2). Literature reviewed indicated that the ability to represent the same thing in different representations and flexibility in moving from one representation to another allows one to see rich relationships and confers deeper understanding of concepts (Even, 1998; Ainsworth, 1999). However, learners' experiences revealed that multiple representations may be confusing to learners and fail to yield desired outcomes. Learners need to be equipped with representational competencies to make sense of the multiple representations. Matthew and Rau (2017) identified two representational competencies that I found felicitous to this model. These are *visual* fluency and *connectional* fluency (see Section 8.5.2).

  Visual fluency is intra-representational. It involves the ability to connect a visual representation to concepts and 'seeing' meaning in them. For example, given the graph of a function, learners must be able to identify features like asymptotes, x- and y – intercepts and the axes of symmetry. Such an activity may be extended to include two or more graphs of the same type of functions, whereby learners identify similarities and differences, patterns and relationships. Visual fluency empowers learners to extract meaning and make sense of any external representation.

  On the other hand, connectional fluency is trans-representational. It entails the ability to connect multiple representations and flexibly switch between them. It allows learners to navigate between different representations and see their interconnectedness. In the context of functions, ICT tools can simultaneously represent a function algebraically, using ordered pairs, a flow diagram and graphically. This allows learners to appreciate that these are different representations of one and the same thing. The point that I want to stress is that for

learners to benefit from multiple representations, they need to be actively engaged in activities that promote visual and connectional fluency.

- **Class interactivity** – This study has established that the introduction of ICTs enrich the pedagogical setting but adds another complicated component onto the already complex classroom interactions. The question that this model attempts to answer is how ICTs can influence various classroom interactions to enhance learning more effectively and more efficiently. To have a more holistic perspective, findings on class interactivity were discussed under the sub-categories: *technical* interaction; *pedagogical* interaction; *conceptual* interaction and *social* interaction (see Section 8.5.3 for more details).

  Technical interaction is the interaction between the learner and the technological tool. Depending on the nature of the technological tool (or software) and the pedagogical setting, this study found that learners should have sufficient technical competency to use the tool. If they struggle to use the tool, it becomes counter-productive and an impediment to learning. Another important finding was that learner-tool interaction requires close monitoring from the teacher to constantly direct the learners' focus on the targeted goal.

  Pedagogical interaction is the interaction between the teacher and the learners. One observation I made throughout the study was that the more learners are actively engaged in their learning, the more individual attention they need. For example, during learner-tool interaction, learners exposed their unique challenges and needed attention at individual level. The point I want to make is that the teacher should allow learners to explore and lead them through guided discovery. To achieve this, the teacher's role has to be that of directing, monitoring and facilitating. In cases where whole-class teaching has to be used, it should be favoured with technological tools like overheard projectors and the interactive whiteboard to provide visual images and capture learners' interest.

Conceptual interaction involves the interaction between the learner and the content. It is the ultimate goal of the teaching and learning process. In other words, in the absence of conceptual interaction no learning would have taken place. This study concluded that any didactical activity involving ICTs should be designed and targeted to promote learner – concept interaction. As Bialobrzska and Cohen (2005) put it, the focus is not learning about ICTs, but learning through and with ICTs.

Social interaction was viewed as learner-learner interaction. It was found that focussed and controlled learner-learner interaction is very critical in ICT-mediated lessons. For example during learner-tool interaction, learners who were technically more fluent assisted others who were struggling to use the software. During group discussions, observations revealed that learners assisted each other in interpreting the graphs and making deductions. This has its theoretical backing in Vygotsky's notion of ZPD. He argues that the 'more knowledgeable ones' are not only the teachers, but other learners as well (Thomson, 2002; Enerst, 1999) (see Section 2.4.1).

9.5.2.2. ICT constraints

In everyday life, we view constraints as conditions that make it difficult to make progress or to achieve a set goal. In the context of this study, constraints were viewed as not divorced from affordances (see Section 8.5.4). As Kennewell (2001) puts it, affordances provide potential for action while constraints provide the structure and guidance for the course of actions. In practical terms, when trying to unlock the potentialities or affordances discussed above, challenges are bound to arise, and a course of action has to be taken to overcome them.

Taking motivation as an example, it has been noted that ICTs are motivation enablers. However, this study has found that if the motivation is *situational interest* stimulated by 'catch' factors, it may create

contradictions. Learners would focus their attention on the technological tool and lose sight of the targeted content. This condition becomes a constraint because it compromises learner-content interaction. In the same way, constraints may be technical or pedagogical in nature. It is important to stress that constraints are an integral component of ICT integration in the classroom. They should not be viewed as barriers or impediments to the use of ICTs but rather they should be seen as opportunities for the development of effective pedagogical practices.

To sum up my reflections on the proposed ICT-mediated instructional model I reiterate that this model is not prescriptive but rather it aims to provide a guideline and some valuable insights to teachers on effective and relevant ICT integration. The DBE has laid out its vision and expectations through white papers and policies on ICT integration in education in South Africa. The government and other stakeholders are pumping in large sums of money on technological resources and infrastructure development in schools, yet there is still a worrying disparity between government expectations and the practices of teachers. Teachers still raise the issue that there is lack of clarity and clear guidance on *how* ICTs should be used in the classroom. I believe such an instructional model, grounded on classroom-based research may help to close the gap.

9.6. Recommendations

As I highlighted in Section 9.4 on limitations of this study, this study was highly contextualised making it difficult to extend its findings to other contexts in South Africa. I however admitted that the focus of the study was not on the generalizability of its findings but on developing a deeper understanding of the scenario within the context in which the study was done. Having taken all reasonable steps to ensure the findings of this research are authentic, I believe I can make some valuable recommendations to mathematics teachers (particularly those in marginalised communities) and the DBE on the basis of this study.

9.6.1. Mathematics teachers

I recommend the following to mathematics teachers:

- It is critical for mathematics teachers to help learners understand and appreciate that it is acceptable to make errors in doing mathematics. Learners' errors help teachers to understand their level of conception and their thinking patterns. Errors should be viewed as aids to the formulation of proper mathematical ideas.
- Teachers should always have time to carefully listen to learners' contributions, be they oral or written. This helps to diagnose the misconceptions underlying learners' errors. If possible, it may be necessary to probe or interview the learners to elicit their thinking.
- Misconceptions are part of the learner's existing schema and they can be resistant. To correct the learner's misconception, I recommend that the teacher should create a cognitive conflict in the learners' mind using explorative and discovery oriented strategies.
- This study found out that learner conception of functions is restricted to the algebraic (or symbolic) form and they struggle with graphical representation. To overcome this, ICT tools and relevant software should be used as much as possible to expose learners to multiple external representations of functions. To benefit from these multiple representations, didactical activities should be designed to develop learners' visual and connectional fluency.
- Proper ICT integration should promote knowledge construction and not be limited to *representing* information, like projecting notes or diagrams on the whiteboard. Knowledge construction is achieved when learners are engaged in explorative activities, identifying patterns and making deductions. For example when using ICTs to teach functions, learners should be guided to discover the effect of parameters on their own and make deductions on graphical transformations.
- Literature reviewed revealed that ICT integration in schools located in marginalised communities is minimal. This study was carried out in such a context and the findings have shown that ICTs are powerful motivation enablers and deepen learners' conception level of hyperbolic functions. With careful planning and preparation, the limited ICT resources available can be used to make an impact on the teaching and learning of functions.

Mathematics educators should be prepared to step out of their 'comfort zone' and be ready to explore the technologically driven methodologies. Agreeably, ICT resources in marginalised schools are more limited than in more affluent schools, but the little technological resources available should be taken out of strong rooms and be utilised.

- Finally, mathematics teachers should align their philosophical and pedagogical perspectives to the dictates of this new technological era. Indisputably, the fourth industrial revolution is fast approaching and is revolutionising business practices both in the private and public sector. Education is no exception in this regard. I do understand that introduction of ICTs 'de-skills' the teacher and sometimes threatens their professional position as the 'more knowledgeable others', but failure to keep abreast with technological advancements may eventually render them ineffective and irrelevant.

9.6.2. The Department of Basic Education

I recommend the following to the Department of Basic Education:

- This study, together with several other previous studies, have noted that curriculum and other policy documents lack clarity on how teachers can integrate ICTs in the teaching and learning process. The Mathematics CAPS document only states that available technological tools should be used in teaching functions but there is no specific guideline on how this can be done. The White Paper on e-education clearly spells out government's vision and expectations on the infusion of ICTs in education, but it is silent on practical steps teachers can take to achieve those expectations. I therefore recommend that policies and curriculum documents should provide clear implementation guidelines on how teachers can use ICTs in their teaching.
- Provision of ICT resources and infrastructure should be biased towards schools in marginalised communities to re-dress the current imbalance. Currently, ICT resources are clustered around schools in more affluent communities while those in marginalised

communities are lagging behind in terms of the availability of resources and ICT integration. The net impact of this digital divide is evident even in the performance of learners, particularly in mathematics and science subjects. I should however note that provision of resources should be followed up with monitoring and supervision to ensure that the resources are properly used.

- Capacity building of teachers should be a key priority if ICT integration has to yield some positive results. From my experience in this study and my interaction with other mathematics teachers, teachers are hesitant to explore ICT integration mainly because they lack the technological pedagogical competency to do so. Most of the experienced teachers acquired their academic and professional qualifications before this technological era. Their resistance and negative attitude towards using ICTs in the classroom is fuelled by their lack of technological competency. Capacity building can be achieved by encouraging teachers to form communities of practice. These are platforms where teachers can meet and share technological pedagogical practices at school or circuit level.
- Policy makers and curriculum designers should reconsider the exclusion of transformation as a topic in the FET CAPS mathematics syllabus. I understand that transformation of geometrical shapes is covered at GET but at this level, learners would be mathematically immature that they cannot really comprehend abstract concepts like translation, rotation and reflection. The next time they encounter these concepts is in Grade 11 under the complex topic of functions. In my view, teaching transformations in the context of functions presents an epistemological obstacle to the learners. If the concepts of translation and reflection were properly developed within the context of transformations, learners would find it easier to understand them under functions.

9.7. Areas for further research

This study makes the following recommendations for further study:

- In what ways do learners' misconceptions in algebra affect their learning of functions?

- Exploring the technological pedagogical competency of mathematics teachers and their readiness to integrate ICTs in their teaching.
- Exploring the affordances and constraints provided by the use of Autograph software in the teaching and learning of statistics and probability.
- A survey on the practicability of the ICT-mediated instructional model proposed from this study as a guide to effective ICT integration in teaching functions.

9.8. Concluding Remarks

As I reflect on my Phd journey, it was characterised by both moments of frustrations and excitement. As a family man and doing my studies on a part-time basis, the Phd journey was generally not easy. I had to balance work, family responsibilities and my studies. This meant going through some very difficult moments and to overcome, I had to develop new life-skills like time management and ability to work under pressure. However, what made this difficult journey exciting was that I encountered new insights at every stage of my study. New interesting things emerged that challenged my conventional way of looking at learners and my perspective about mathematics teaching.

Interacting with learners to an extent of understanding how they think and why they think the way they do is a very complex but worthwhile exercise. Our education system has presented teachers as 'judges' to the learners, who are there to *tell* them what is wrong or right and acceptable or not acceptable. Naturally, learners become very careful on how they present themselves and what they say before the 'judge'. To break this shell and get access into the learner's social and cognitive space, it takes conscious efforts and patience. As mathematics teachers, as long as we do not get access into the learners' cognitive space, we may never be able to develop them to their fullest potential.

The complexity of exposing learners to computers in a classroom context exceeded my expectations. Though learners might lack basic computer skills, they interact with various ICT gargets in their social space. Their social usage of ICTs is in an uncontrolled and unmonitored space whereby they are free to explore whatever they want. When it comes to ICT classroom usage, it is target-oriented

and more controlled. This may induce contradictions within the pedagogical setting and the way it is managed is very critical. I had classroom rules but what I picked was that enforcing those rules is not simple. Learners have their own unique ways of perceiving and interpreting their social context and their *real* actions are usually dictated by *their* perceptions. In most cases we ignore their perspectives and expect them to see things the way we do. This study taught me that once we gain access into the learners' social and cognitive space, it gives us a better vantage point to correct their perceptions and misconceptions. Once the learners' perceptions are aligned to proper use of technological tools for educational purposes, it makes educational resources available to them beyond classroom, school or community boundaries.

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11. APPENDICES

11.1 APPENDIX A: The University of the Witwatersrand clearance certificate.

Wits School of Education

WITS UNIVERSITY

27 St Andrews Road, Parktown, Johannesburg, 2193 Private Bag 3, Wits 2050, South Africa. Tel: +27 11 717-3064 Fax: +27 11 717-3100 E-mail: enquiries@educ.wits.ac.za Website: www.wits.ac.za

11 August 2017

Student Number: 1261401

Protocol Number: 2017ECE023D

Dear Fanuel Matindike

Application for Ethics Clearance: Doctor of Philosophy

Thank you very much for your ethics application. The Ethics Committee in Education of the Faculty of Humanities, acting on behalf of the Senate, has considered your application for ethics clearance for your proposal entitled:

Resourcing learners' misconceptions on functions through mediation of ICTs at a rural school.

The committee recently met and I am pleased to inform you that **clearance was granted**.

Please use the above protocol number in all correspondence to the relevant research parties (schools, parents, learners etc.) and include it in your research report or project on the title page.

The Protocol Number above should be submitted to the Graduate Studies in Education Committee upon submission of your final research report.

All the best with your research project.

Yours sincerely,

M Makgeth

Wits School of Education

011 717-3416

cc Supervisor – Dr Judah Makonye

11.2 APPENDIX B: Letter to the Limpopo Department of Education (LDoE) seeking approval to collect data at the school.

REF: MATINDIKE, F.
CELL: 078 423 2854 / 071 346 2858
EMAIL: fmatindike@yahoo.com

xxxxx Sec School
P O Box xxxx
Senwabarwana
0790

02 May 2017

The HOD
Limpopo Department of Education
Private Bag X9489
Polokwane
0700

Dear Sir / Madam

RE : REQUEST FOR APPROVAL TO UNDERTAKE A RESEARCH STUDY AT XXXXX SECONDARY SCHOOL.

I write to kindly seek your approval to carry out a research study at xxxxx Secondary school in xxxxx District, xxxxx West Circuit. I am an educator at this school and doing part-time studies with University of Witwatersrand. The research study is mainly undertaken in fulfilment to the requirements of the PHD study but will also benefit the participating learners academically and contribute towards improving teaching and learning of Mathematics

Basically, the study aims to explore the use of Information and Communication Technologies (ICTs) in the teaching and learning of Mathematics at FET level. The study will focus on teaching of the topic of functions as it is one of the problematic topics to learners. The participants in this study will be Grade 11 mathematics learners. The learners will attend about 10 intervention lessons on functions using computers in the afternoon, after normal lessons. The learners will be expected to write a pre- and post-diagnostic test to assess the effectiveness of the intervention lessons. I intend to carry out the study during the third-quarter in August.

I commit myself to ensuring that all necessary ethical considerations are strictly adhered to, making sure that the participating learners are not affected or disadvantaged in any way. The participation of learners will be absolutely voluntary and their informed consent, and that of their parents, will be sought.

I believe this study will help the learners as the content taught is in line with their syllabus. The findings of the research study will also go a long way in improving the quality of teaching and learning in mathematics at FET level.

I am hopefully looking forward to your assistance in this regard.

Yours faithfully,



Matindike Fanuel

11.3 APPENDIX C: LDoE approval letter to collect data

LIMPOPO
PROVINCIAL GOVERNMENT
REPUBLIC OF SOUTH AFRICA

DEPARTMENT OF
EDUCATION

Ref: 2/2/2 Enq: MC Makola PhD Tel No: 015 290 9448 E-mail: MakolaMC@edu.limpopo.gov.za

Matindike F
P O Box 1718
Senwabarwana
0790

RE: REQUEST FOR PERMISSION TO CONDUCT RESEARCH

1. The above bears reference.

The Department wishes to inform you that your request to conduct research has been approved. Topic of the research proposal: **"TEACHING OF THE TOPIC OF FUNCTIONS".**

2. The following conditions should be considered:

3.1 The research should not have any financial implications for Limpopo Department of Education.

3.2 Arrangements should be made with the Circuit Office and the schools concerned.

3.3 The conduct of research should not anyhow disrupt the academic programs at the schools.

3.4 The research should not be conducted during the time of Examinations especially the fourth term.

3.5 During the study, applicable research ethics should be adhered to; in particular the principle of voluntary participation (the people involved should be respected).

3.6 Upon completion of research study, the researcher shall share the final product of the research with the Department.

4 Furthermore, you are expected to produce this letter at Schools/ Offices where you intend conducting your research as an evidence that you are permitted to conduct the research.

REQUEST FOR PERMISSION TO CONDUCT RESEARCH: MATINDIKE F

CONFIDENTIAL

Cnr. 113 Biccard & 24 Excelsior Street, POLOKWANE, 0700, Private Bag X9489, POLOKWANE, 0700
Tel: 015 290 7600, Fax: 015 297 6920/4220/4494

The heartland of southern Africa - development is about people!

11.4 APPENDIX D: Letter to the Principal and School Governing Body (SGB) seeking approval to conduct research at the school.

Wits School of Education
University of the Witwatersrand
Private Bag 3
Wits
2050

The Principal and SGB
xxxxx Sec School
P O Box xxxx
xxxxxxxxxxx
xxxxx

26 May 2017

Dear Sir / Madam

RE : PERMISSION TO CONDUCT A RESEARCH STUDY AT YOUR SCHOOL

I, Mr Fanuel Matindike, request your permission that Grade 11 mathematics learners at your school take part in the research study entitled: '*Resourcing learners' misconceptions on functions through mediation of ICTs*'. I am undertaking this study as a partial fulfillment of a PHD at the University of the Witwatersrand, Johannesburg.

The research study is on how the use of technological tools like computers can help remedy and address the errors and misconceptions that learners have under the topic of functions. For the purpose of this research, my focus is on the Grade 11 mathematics class. The learners will be expected to write a pre-diagnostic test. From the test, the learners' errors and misconceptions will be identified and noted. Some of the learners will be interviewed to seek clarification on the reasons for those misconceptions. The interviews will be audio-taped and later transcribed for data analysis. A teaching intervention will be administered to the learners on functions using computers, Autograph software, projectors, whiteboard and other technological tools. The intervention lessons will be video-taped. After the teaching intervention, a post-diagnostic test will be administered to establish the impact of the interventions.

The participants' participation is completely voluntary. The learners will not be disadvantaged in any way. They can withdraw their participation at any time during this study without any penalty. There

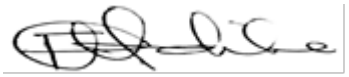
are no foreseeable risks in participating in the study. Rather, the learners may benefit in that their participation in this project may enhance their understanding of concepts under functions and raise their level of competency in computers.

Anonymity and confidentiality of the learners' responses to test and interviews is uncompromisingly assured. No names will be used on the tests to identify the learners. The names of the research participants and identity of the school will be kept confidential at all times and in all academic writing about the study. Your individual privacy will be maintained in all published and written data resulting from the study.

The research data will be used for academic purposes and possibly shared at an academic conference presentation and/or published as a general paper in a journal. The research data will be kept safely and destroyed 5 years after completion of the study. The results and findings of the study will be presented as a research report to Wits School of Education.

Please let me know if you require any further information. I am hopefully forward to your response as soon as is convenient.

Yours sincerely

SIGNATURE : 

NAME : FANUEL MATINDIKE
E- MAIL : fmatindike@yahoo.com
CELL NO : 078 423 2854 / 071 346 2858
NAME OF SUPERVISOR : Dr J. MAKONYE
E-MAIL : Judah.Makonye@wits.ac.za

11.5 APPENDIX E: Information sheet for the learner participants

Wits School of Education
University of the Witwatersrand
Private Bag 3
Wits
2050

Learner…….
xxxxx Sec School
P O Box xxxx
Xxxxxxxxxxx
xxxx

26 May 2017

Dear Learner

RE: Request for your Permission to Participate in a Research Study

I, Mr F. Matindike is doing Phd studies in the School of Education at Witwatersrand University in Johannesburg. I do hereby kindly request your permission to take part in my research study entitled: *'Resourcing learners' misconceptions on functions through mediation of ICTs'.* I am undertaking this study as a partial fulfillment of my studies.

In this research, I want to investigate how the use of technological tools like computers can help remedy and address the errors and learning challenges that learners have under the topic of functions. For the purpose of this research, my focus is on the Grade 11 mathematics class. You will be expected to write a pre-diagnostic test on the topic of functions. The purpose of the test is not for marks but it will enable me to identify the common mistakes and difficulties that you might be having in this topic. After identifying these challenges, I may then request to have an interview with you. The reason for the interview is to try and find out the cause of these challenges and how best we can overcome them. With your permission, the interview will be audio-taped for further analysis so that I may be able to plan some intervention lessons to address the mistakes identified in the test. You will then be requested to attend 10 one hour intervention lessons on functions. These lessons aim to address the mistakes and challenges identified in the pre-test. The lessons will be conducted outside your normal school timetable so that they don't interfere with your daily lessons. With your permission, the lessons will be video-taped for further analysis of the learning activities. After the teaching intervention, you will then be requested to write a post-diagnostic test to check the effectiveness of the intervention lessons.

Your participation in this research is completely voluntary. You have the right to withdraw your participation at any time during this study without any penalty. There are no foreseeable risks in participating in the study. Rather, you may benefit in that your participation in this project may enhance your understanding of concepts under functions and may raise your level of competency in computers.

I will not be using your real name throughout the research. However I will make up a name that will identify you in this study. The reason for this is to protect your privacy and confidentiality of your responses in tests and interviews. All information about you will be kept secret in all my writing about the study. All information collected will be stored safely and destroyed between 3-5 years after I have completed my research.

Your parents have also been given an information sheet and consent form, but at the end of the day it is your decision to join us in this study.

I look forward to working with you.

Please let me know if you require any further information.

Thank you!

SIGNATURE : 

NAME : FANUEL MATINDIKE
E- MAIL : fmatindike@yahoo.com
CELL NO : 078 423 2854 / 071 346 2858
NAME OF SUPERVISOR : Dr J. MAKONYE
E-MAIL : Judah.Makonye@wits.ac.za

11.6 APPENDIX F: Learner consent form

- **Permission for pre- and post-testing**

Please fill in the reply slip below if you agree to participate in my study called:

RESOURCING LEARNERS' MISCONCEPTIONS ON FUNCTIONS THROUGH THE MEDIATION OF ICT TOOLS.

My name is: ……

Permission for testing	**Circle one**
I agree to write a pre- and post-diagnostic test for this study	YES/NO

Informed Consent
I understand that:

- my name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- I do not have to answer every question and can withdraw from the study at any time.
- all the data collected during this study will be destroyed within 3-5 years after completion of my project.

Signature :…….. Date: …….

Contact No : ……

- **Permission to be interviewed**

Please fill in the reply slip below if you agree to participate in my study called:

RESOURCING LEARNERS' MISCONCEPTIONS ON FUNCTIONS THROUGH THE MEDIATION OF ICT TOOLS.

My name is: ……

Permission to be interviewed	**Circle one**
I agree to be interviewed for this study.	YES/NO
I know that I can stop the interview at any time and don't have to answer all the questions asked.	YES / NO

Informed Consent

I understand that:

- my name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- I do not have to answer every question and can withdraw from the study at any time.
- all the data collected during this study will be destroyed within 3-5 years after completion of my project.

Signature: …….. Date: …….

Contact No: ……

- **Permission to be audiotaped**

Please fill in the reply slip below if you agree to participate in my study called:

RESOURCING LEARNERS' MISCONCEPTIONS ON FUNCTIONS THROUGH THE MEDIATION OF ICT TOOLS.

My name is: ……

Permission to be audiotaped	**Circle one**
I agree to be audiotaped during the interview.	YES / NO
I know that audiotapes will be used for this project only.	YES / NO

Informed Consent:

I understand that:

- my name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- I do not have to answer every question and can withdraw from the study at any time.
- all the data collected during this study will be destroyed within 3-5 years after completion of my project.

Signature: …….. Date: …….

Contact No: ……

- **Permission to be videotaped**

Please fill in the reply slip below if you agree to participate in my study called:

RESOURCING LEARNERS' MISCONCEPTIONS ON FUNCTIONS THROUGH THE MEDIATION OF ICT TOOLS.

My name is: ……

Permission to be videotaped	**Circle one**
I agree to be videotaped during the intervention lessons	YES/NO
I know that videotapes will be used for this project only.	YES / NO

Informed Consent:

I understand that:

- my name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- I do not have to answer every question and can withdraw from the study at any time.
- all the data collected during this study will be destroyed within 3-5 years after completion of my project.

Signature: …….. Date: …….

Contact No: ……

11.7 APPENDIX G: Information Sheet for Learner's Parent or Guardian

Wits School of Education
University of the Witwatersrand
Private Bag 3
Wits
2050

The Parent or Guardian
xxxxx Sec School
P O Box xxxx
xxxxxxxxxxxx
xxxx

26 May 2017

Dear Sir / Madam

RE : REQUEST FOR PERMISSION FOR YOUR CHILD TO TAKE PART IN A RESEARCH STUDY

My name is Mr Fanuel Matindike and I am doing some studies with the School of Education at the University of Witwatersrand in Johannesburg. I am doing a research entitled: '*Resourcing learners' misconceptions on functions through mediation of ICTs'*. I am undertaking this study as a partial fulfillment of the studies I am undertaking.

I humbly request your permission for your child to take part in this research study, together with other learners in his/her class. The reason why I have chosen your child's class is that learners' errors and misconceptions hinder their successful learning in mathematics. Thus the study aims to address the learners' mistakes and challenges in their understanding of functions in mathematics.

The research study is on how the use of technological tools like computers can help remedy and address the errors and misconceptions that learners have under the topic of functions. For the purpose of this research, my focus is on the Grade 11 mathematics class. Your child will be expected to write a pre-diagnostic test. From the test, the child's errors and misconceptions will be identified and noted. Your child may then be interviewed to seek clarification on the reasons for those misconceptions. The interviews will be audio-taped and later transcribed for data analysis. A teaching intervention will be administered to the whole class on functions using computers, Autograph software, projectors, whiteboard and other technological tools. The intervention lessons will be video-taped. After the teaching intervention, a post-diagnostic test will be administered to the class to establish the impact of the interventions.

Your child's participation is completely voluntary. The child will not be disadvantaged in any way. He/ she can withdraw his/her participation at any time during this study without any penalty. There are no foreseeable risks in participating in the study. Rather, your child may benefit in that his/her participation in this project may enhance his/her understanding of concepts under functions and raise their level of competency in computers. If your child has a good understanding of functions, he/she stands a better chance of doing well in mathematics.

Anonymity and confidentiality of your child's responses to tests and interviews is uncompromisingly assured. No names will be used on the tests to identify your child. The name of your child, identity of the school and that of the principal will be kept confidential at all times and in all academic writing about the study.

The research data will be used for academic purposes only and may possibly be shared an academic conference presentations and/or published as a general paper in a journal. The research data will be kept safely and destroyed 5 years after completion of the study. The results and findings of the study will be presented as a research report to Wits School of Education.

Please let me know if you require any further information. I am hopefully forward to your response as soon as is convenient.

Yours sincerely



SIGNATURE :
NAME : FANUEL MATINDIKE
E- MAIL : fmatindike@yahoo.com
CELL NO : 078 423 2854 / 071 346 2858
NAME OF SUPERVISOR : Dr J. MAKONYE
E-MAIL : Judah.Makonye@wits.ac.za

11.8 APPENDIX H: Parent or Guardian Consent Form

- **Permission to test**

Please fill in and return the reply slip below indicating your willingness to allow your child to participate in the research project called:
RESOURCING LEARNERS' MISCONCEPTIONS ON FUNCTIONS THROUGH THE MEDIATION OF ICT TOOLS.

I …… the parent of ……

Permission to test **Circle one**

I agree that my child may write a pre- and post-diagnostic test
to be used for this study only. YES/NO

Informed Consent
I understand that:

- My child's name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- He/she does not have to answer every question and can withdraw from the study at any time.
- He/she can ask not to be audiotaped, photographed and/or videotape
- All the data collected during this study will be destroyed within 3-5 years after completion of my project.

Signature: …….. Date: ……

Contact No: ……

- **Permission to take part in the intervention lessons**

Please fill in and return the reply slip below indicating your willingness to allow your child to participate in the research project called:
RESOURCING LEARNERS' MISCONCEPTIONS ON FUNCTIONS THROUGH THE MEDIATION OF ICT TOOLS.

I…… the parent of……

Permission to take part in intervention lessons	**Circle one**
I agree that my child may take part in the intervention lessons for purposes of this study.	YES / NO

Informed Consent

I understand that:

- My child's name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- He/she does not have to answer every question and can withdraw from the study at any time.
- He/she can ask not to be audiotaped, photographed and/or videotape
- All the data collected during this study will be destroyed within 3-5 years after completion of my project.

Signature: …….. Date: ……

Contact No: ……

- **Permission to audiotape**

Please fill in and return the reply slip below indicating your willingness to allow your child to participate in the research project called:
RESOURCING LEARNERS' MISCONCEPTIONS ON FUNCTIONS THROUGH THE MEDIATION OF ICT TOOLS.

I …… the parent of ……

Permission to audiotape	**Circle one**
I agree that my child may be audiotaped during interviews	YES/NO
I know that the audiotapes will be used for this research only	YES/NO

Informed Consent

I understand that:

- My child's name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- He/she does not have to answer every question and can withdraw from the study at any time.
- He/she can ask not to be audiotaped, photographed and/or videotape
- All the data collected during this study will be destroyed within 3-5 years after completion of my project.

Signature: …….. Date: ……

Contact No: ……

- **Permission to videotape**

Please fill in and return the reply slip below indicating your willingness to allow your child to participate in the research project called:
RESOURCING LEARNERS' MISCONCEPTIONS ON FUNCTIONS THROUGH THE MEDIATION OF ICT TOOLS.

I …… the parent of ……

Permission to videotape	**Circle one**
I agree that my child may be videotaped during intervention lessons.	YES/NO
I know that the videotapes will be used for this research only	YES/NO

Informed Consent

I understand that:

- My child's name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- He/she does not have to answer every question and can withdraw from the study at any time.
- He/she can ask not to be audiotaped, photographed and/or videotape
- All the data collected during this study will be destroyed within 3-5 years after completion of my project.

Signature: …….. Date: ……

Contact No: ……

- **Permission to be interviewed**

Please fill in and return the reply slip below indicating your willingness to allow your child to participate in the research project called:
RESOURCING LEARNERS' MISCONCEPTIONS ON FUNCTIONS THROUGH THE MEDIATION OF ICT TOOLS.

I …… the parent of ……

Permission to be interviewed	**Circle one**
I agree that my child may be interviewed for this study	YES / NO
I know that my child can stop the interview at any time and doesn't have to answer all the questions asked.	YES / NO

Informed Consent

I understand that:

- My child's name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- He/she does not have to answer every question and can withdraw from the study at any time.
- He/she can ask not to be audiotaped, photographed and/or videotape
- All the data collected during this study will be destroyed within 3-5 years after completion of my project.

Signature: …….. Date: ……

Contact No: ……

11.9 APPENDIX I: CAPS Specification of Content under Functions from Grade 10 to 12

FUNCTIONS		
GRADE 10	**GRADE 11**	**GRADE 12**
Work with relationships between variables in terms of numerical, graphical, verbal and symbolic representations of functions and convert flexibly between these representations (tables, graphs, words and formulae). Include linear and some quadratic polynomial functions, exponential functions, some rational functions and trigonometric functions.	Extend Grade 10 work on the relationships between variables in terms of numerical, graphical, verbal and symbolic representations of functions and convert flexibly between these representations (tables, graphs, words and formulae). Include linear and quadratic polynomial functions, exponential functions, some rational functions and trigonometric functions. Gen	Introduce a more formal definition of a function and extend Grade 11 work on the relationships between variables in terms of numerical, graphical, verbal and symbolic representations of functions and convert flexibly between these representations (tables, graphs, words and formulae). Include linear, quadratic and some cubic polynomial functions, exponential and logarithmic functions, and some rational functions.
Generate as many graphs as necessary, initially by means of point-by-point plotting, supported by available technology, to make and test conjectures and hence generalise the effect of the parameter which results in a vertical shift and that which results in a vertical stretch and /or a reflection about the x axis.	Generate as many graphs as necessary, initially by means of point-by-point plotting, supported by available technology, to make and test conjectures and hence generalise the effects of the parameter which results in a horizontal shift and that which results in a horizontal stretch and/or reflection about the y axis	The inverses of prescribed functions and be aware of the fact that, in the case of many-to-one functions, the domain has to be restricted if the inverse is to be a function.
Problem solving and graph work involving the prescribed functions.	Problem solving and graph work involving the prescribed functions. Average gradient between two points.	Problem solving and graph work involving the prescribed functions (including the logarithmic function).

Source: Curriculum and Assessment Policy Statement (CAPS) Grades 10 – 12: Mathematics

11.10 APPENDIX J: Pre-test

Pre-Test : Hyperbolic Functions

Pseudonym: …….. Grade: 11

QUESTION 1:

1. Consider the graph defined by $f(x) = \frac{-9}{x-1} - 2$.

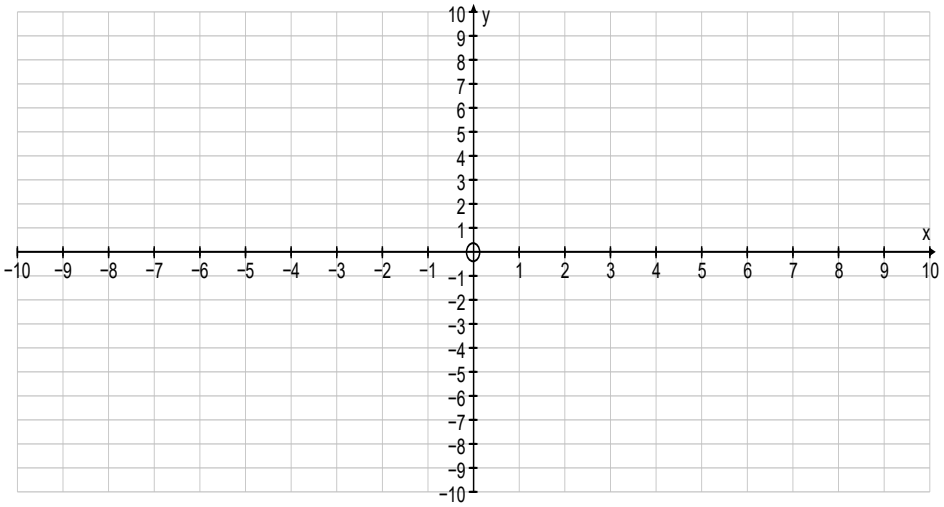
1.1. Calculate the coordinates of the *x*- and *y*-intercepts of $f(x)$

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1.2. Write down the equations of the asymptotes of $f(x)$

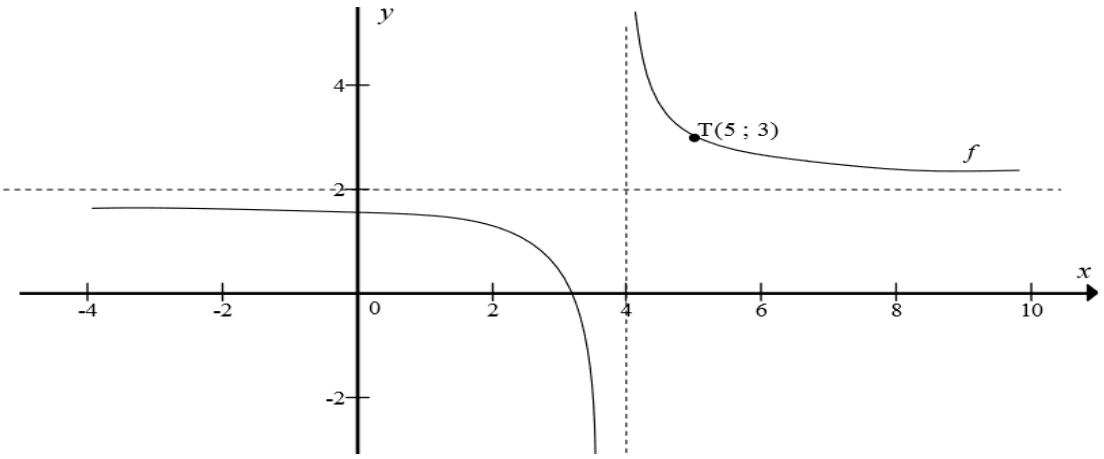
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1.3. Sketch the graph of $f(x)$ showing all intercepts with the axes as well as the asymptotes. Use the set of axes provided below.



QUESTION 2

2. The graph of $\boldsymbol{h}(x) = \frac{a}{x-p} + q$ is given below. Use the graph to answer the following questions.



2.1. Write down the equations of the asymptotes of $\boldsymbol{h(x)}$.

……
……

2.2.Write down the equations of the asymptotes of $\boldsymbol{h}(x-2)$.

……
……
……

2.3. A new function $\boldsymbol{f}(x)$ is obtained by shifting $\boldsymbol{h}(x)$ **2** units downwards. Determine the domain and range of $\boldsymbol{f}(\boldsymbol{x})$.

Domain:

……

Range:

……

2.4. By using the graph of $\boldsymbol{h}$ above, determine the values of $\boldsymbol{x}$ for which: $\boldsymbol{h(x) < 2:}$

……
……

QUESTION 3

3.1. The graph of a hyperbola with equation $y = g(x)$ has the following Properties:

- Domain: $x \in R,\ x \neq 5$
- Range: $y \in R,\ y \neq 1$
- Passes through the point $(2; 0)$

Determine the equation of$g(x)$.

……
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……
……

3.2 A hyperbola of the form: $\boldsymbol{k(x) = \frac{a}{x+p} + p}$ passes through the point$(\mathbf{4};\ \mathbf{3})$. If the axes of

symmetry of $k(x)$ intersect at $(-\mathbf{1};\ \mathbf{2})$, determine the equation of $k(x)$.

……

APPENDIX K: Post-test

Post-Test : Hyperbolic Functions

Pseudonym : …….. Grade: 11

QUESTION 1

1. Given: $\boldsymbol{g(x) = \frac{-3}{x-2} + 1}$

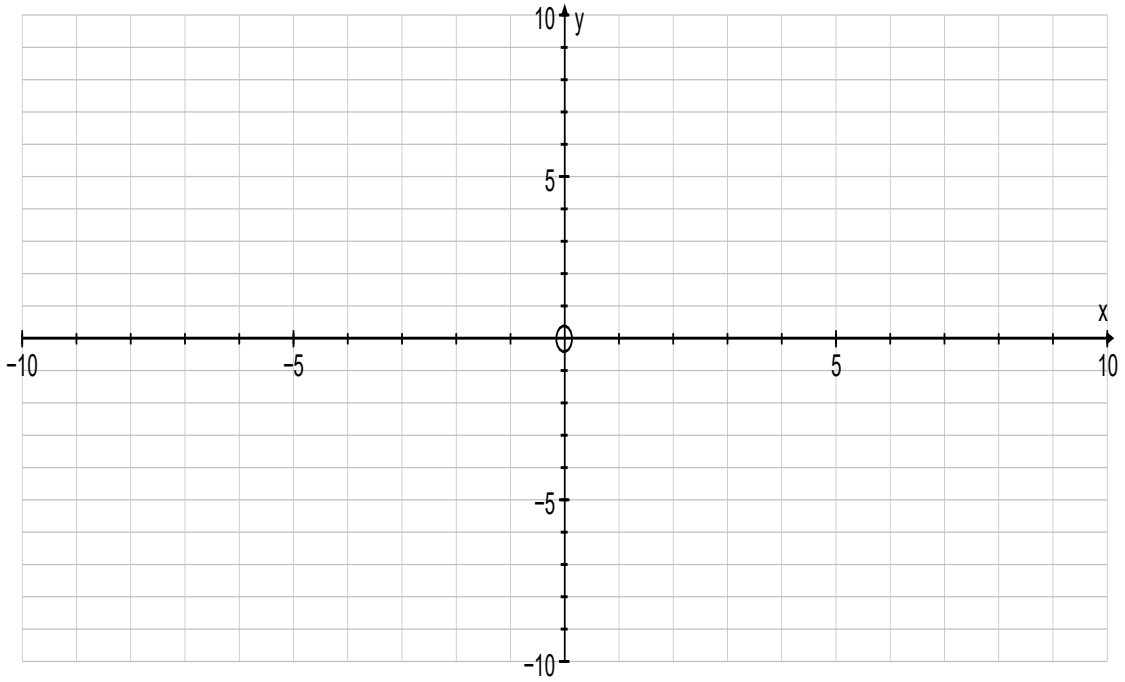
1.1. Calculate the coordinates of the x- and y-intercept of ***g.***

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1.2. Write down the equations of the asymptotes of ***g***.

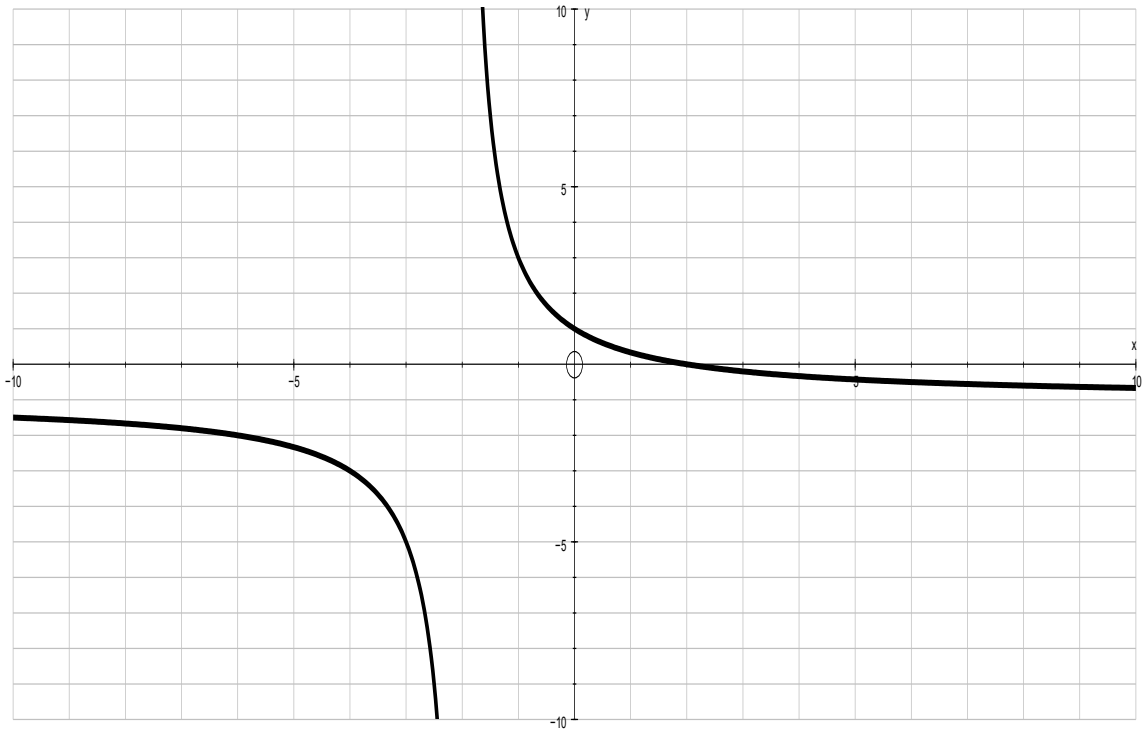
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……

1.3. Sketch the graph of g(x) on the grid below, clearly showing the asymptotes and the intercepts with the axes.



QUESTION 2

2. The graph of a hyperbolic function $f(x) = \frac{a}{x+p} + q$ is shown below. Use the graph to answer the questions that follow.



2.1. Write down the equations of asymptotes of $f(x)$.

……

……

2.2. Determine the equations of asymptote of $f(x+1)$.

……

……

……

2.3. Another function, $\boldsymbol{h(x)}$, is obtained by translating $\boldsymbol{f(x)}$ 3 units to the right and 4 units down. Determine the domain and range of $\boldsymbol{h(x)}$.

Domain:

……

……

Range:

……

……

2.4.For which values of x is $(x) \leq -4$?

……

……

……

QUESTION 3

3.1. A hyperbolic function, $f(x)$, has the following features:

- The point $B(-1;0)$ is the y-intercept
- The domain of f is real numbers, but $x \neq 2$
- The range of f is real numbers, but $y \neq 3$
- f is a decreasing function

Determine the equation of $f(x)$.

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3.2. The axes of symmetry of a hyperbolic function, $k(x)$, intersect at the turning point of a quadratic function, $h(x) = \frac{1}{2}(x-2)^2 - 9$. If the y-intercept of $k(x)$ is $(0;-11)$, determine its equation.

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APPENDIX L: Task-based Interview Guide

Date: ……

Time: ……

Venue: ……

Interviewer: ……

Interviewee: …… (not real name)

Introductory Remarks

Once again I thank you for participating in the study. The aim of the study is to explore learners' errors and misconceptions on hyperbolic functions and how ICT-mediated instruction may assist learners to overcome and address their misconceptions. I have some questions with me here that I would want us to talk about. Basically the questions are based on your responses to the pre-test question items and seek clarification on how you answered some of the questions. Feel very free to share your honest opinion based on your understanding of hyperbolic functions.

Let me remind you and assure you that this interview is strictly confidential and your responses will only be used for the purposes of this study. As far as language is concerned, we will use English but you are free to switch to your mother language, Sepedi, if you feel it necessary to do so. The interview will take approximately 30 minutes.

Question Item	Interview Question	Interviewee's Response	Observation Notes
Question 1			
1.1.	▪ Let's look at question 1.1 which required you to calculate the x-and y-intercepts. You started by saying: for x-intercept, $y = 0$ and for y-intercept, $x = 0$, can you explain why you said so? ▪ Why did you choose to equate $x = 0$ or $y = 0$? ▪ What is your understanding of the x-and y-intercept of a function?		
1.2.	▪ Lets move on to the next question item which required you to write down the equations of the asymptotes of $f(x)$. You responded by equating $x - 2 = 0$, which		

	gave you $x = 2$ and $y = 0$, can you explain how you got this. ▪ Why did you choose 0, and not any other number? ▪ What is your understanding of an asymptote?		
1.3.	▪ Now moving on to item 1.3, you were required to sketch the graph of $f(x)$. Can you share what you considered to come with your graph? ▪ I notice your graph is in the 1st and 3rd quadrants, can you explain why it is like that?		
Question 2			
2.1	▪ Can you share with me how you identified the asymptotes from the graph?		
2.2	▪ What is your understanding of the notation $g(x + 3)$? ▪ I notice that you equated $x + 3$ to 0, can you explain why you approached the question that way?		
2.3	▪ I liked the way you responded to question 2.2. May you clarify how you determined the domain and range of the new function? ▪ From your understanding, how does the vertical translation of a hyperbolic function affects its domain and range?		
2.4.	▪ Moving on to question item 2.4, may you briefly explain how you determined your values of x for which $g(x) \leq 0$? ▪ What is your understanding of $g(x) \leq 0$?		
Question 3			
3.1.	▪ Given the three properties of a hyperbolic function in this equation, can you shed more light on how you came up with the equation of the function?		
3.2.	▪ Coming to the last question item, you were given the point of intersection of the axes of symmetry of a hyperbolic function and one point that lie on the graph. May you clarify how you came up with the equations of asymptotes and the equation of the hyperbola?		

Closing remarks:

- Thank you very much for your time and the valuable information you have shared in this interview. I reassure you that this information will be kept and used with strict confidentiality.
- I also kindly request your permission to make follow up if ever I need further clarification from you on information shared in this interview.

APPENDIX M: Lesson Observation Guide

Activity Theory Dimensions	Questions to ask when analysing evaluative episodes	Observation notes
Object	▪ What was the focus of the activity / episode? ▪ Was the focus on transmitting practical (or technical) skills? ▪ Was there any attempt to develop and reinforce learners' content knowledge? ▪ Were there any probing questions to develop metacognitive skills and higher mental functions?	
Mediating ICT tools	▪ Was the use of ICTs representational (to represent work, notes or content covered)? ▪ Was the use of ICTs generative (to promote mathematical thinking and problem solving)? ▪ Was the use of ICTs linked to the targeted content or concept? ▪ Did the use of ICT tools facilitate learner-content interaction? ▪ Did the use of ICT tool promote learner-learner interaction and learner-teacher interaction?	
Division of labour	▪ What were the major roles played by the teacher? Eg controlling learners, offering technical skills, elaborating maths concepts, probing and asking questions, ▪ What were the major roles played by the learners? Eg listening to the teacher, engaged in group and individual activities, answering question orally, asking questions.	
Rules	▪ What kind of instructional rules were observable? ▪ How did learners respond to teacher's questions? ▪ How did the teacher respond to learners' contributions? ▪ How did learners interact with each other? ▪ How were learners with technical challenges assisted? ▪ Was the order / discipline during the episode?	
Community	▪ Was there productive and orderly interaction between the learners? ▪ Was the teacher working together with learners towards the object? ▪ Was the video recorder always well positioned – not obstructing or drawing learners' attention?	

APPENDIX N: Open-ended Interview Guide

Date: ……

Time: ……

Venue: ……

Interviewer: ……

Interviewee: …… (not real name)

Introductory Remarks

Once again thank you for availing yourself for this third round of our interview sessions. As I explained from the beginning, the aim of this study is to explore learners' errors and misconceptions under hyperbolic functions and how ICT-mediated instruction may assist learners to overcome and address their misconceptions. Basically the focus of this interview session is to talk about your experiences during the intervention programme and your views on those experiences. Your honest views and opinion is critical for the attainment of the objectives of this study and for the development of similar intervention programmes in future.

I remind you and reassure you that this interview is strictly confidential and your responses will only be used for the purposes of this study. As far as language is concerned, we will use English but you are free to switch to your mother language, Sepedi, if you feel it necessary to do so. The interview session will take approximately 30 minutes.

1. **Can you share with me your experiences and how you felt during the lessons on hyperbolic functions using the computers?**

2. **In what way would you say the use of computers assisted you to better understand the content taught under hyperbolic functions?**

3. **What form of support and assistance did you get from the teacher or from your fellow learners and do you think it was adequate?**

4. **Can you share with me some of the challenges you encountered throughout the intervention lessons?**

5. **If the lessons were to be repeated or the computers were to be used in teaching another topic, what do you think should be done differently in your view?**

Closing remarks:

- Thank you very much for your time and the valuable information you have shared in this interview. I reassure you that this information will be kept and used with strict confidentiality.
- I also kindly request your permission to make follow up if ever I need further clarification from you on information shared in this interview.