ON THE BOUNDS FOR THE STRONG CHROMATIC INDEX FOR A SUBSET OF PLANAR GRAPHS

Submitted in fulfilment of the requirements for the degree
Master of Science
in
Mathematics
in the Faculty of Science, University of the Witwatersrand, Johannesburg, South Africa

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September 2022

Declaration of Authorship

I, Tejal Pramjeeth, declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other University.

Signature : T Pramjeeth

Signed on the 9th day of September 2022 at 15:32

Acknowledgements

First and foremost, I would like to offer thanks to God for blessing me with the knowledge and perseverance to see this dissertation to completion.

I want to extend my utmost appreciation to all my family and friends for their support and encouragement in this time.

To my wonderful mother, I am incredibly grateful to you for the great deal of sacrifices that you have made to bring me to where I am today. Thank you for always supporting me in the pursuit of my studies. I will always be indebted to you for the love and care that you have given me throughout my life.

To my late father, thank you for always watching over me and guiding me from Heaven. I am especially grateful for your encouragement and insistence that helped me persist through the challenging aspects of this dissertation.

To my loving partner, Keown Shanganlall, words cannot describe how thankful I am to have you in my life. Thank you for all your words of motivation and your comfort in my moments of despair. I am immensely grateful to you for the endless love and support that you have shown me over these years.

To my supervisor, Prof. Elizabeth Jonck, thank you for your patience, guidance and motivation over these two years. I am appreciative of all your wise counsel, for proofreading my work and all your helpful feedback on this dissertation.

To the Trustees of the Hillensberg Trust Fund, I would like to extend my thanks for all your financial support in this time, which has made this disseration possible.

Contents

Chapter 1

Introduction

In this chapter, we introduce some basic terminology used in the field of Graph Theory in Section 1.1. Thereafter, we introduce the concept of edge colouring of graphs in Section 1.2. Lastly, we provide an overview of the contents of this dissertation in Section 1.3.

1.1 Basic Terminology

A *graph* G is a finite non-empty set of *vertices* (singular *vertex*) along with a (possibly empty) set of *edges*. This mathematical structure is used to illustrate the relationships (shown via the edges) between objects (the vertices). In *Figure 1.1*, we see an example of a graph G_1. The vertices are the circular nodes and the edges are the lines between them.

The set of vertices in a graph G is known as the *vertex set* and is denoted by $V(G)$. For the graph in *Figure 1.1*, $V(G_1) = \{s, t, u, v, w, x, y, z\}$. The edge joining two vertices u and v in a graph G is usually labelled as uv (or vu), as can be seen in the graph in *Figure 1.1.* The set of edges in a graph G is the *edge set*, denoted $E(G)$. Furthermore, $E(G)$ is a subset of the set of 2-element subsets of $V(G)$. For the graph in *Figure 1.1*, $E(G_1) = \{us, uv, ux, vx, wt, wx, wy, xz, yz\}$.

If the sets $V(G)$ and $E(G)$ are finite sets, then the graph G is a *finite graph.* The graph in *Figure 1.1* is a finite graph. The *order* of a graph G is the number of vertices in the graph G, and is equivalent to the cardinality of the vertex set $V(G)$. The *size* of a graph G is the number of edges in the graph G, and is equivalent to the cardinality of the edge set $E(G)$. The order of the graph in *Figure 1.1* is 8 and the size of the graph is 9.

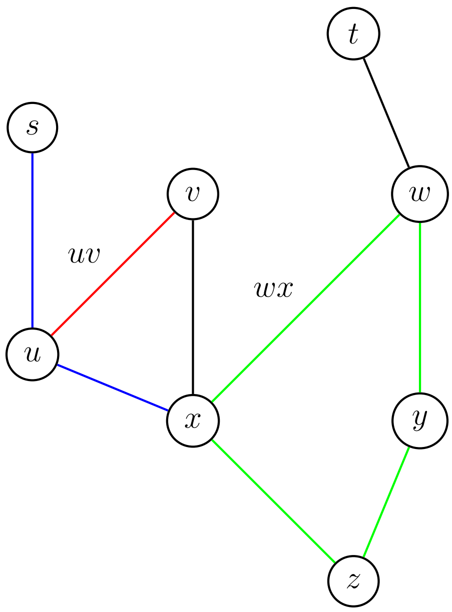

Figure 1.1: Example of a graph G_1

Two vertices u and v in a graph G are said to be *adjacent* if there is an edge between them. The vertex u (and v) in the graph G is said to be *incident* with the edge uv. Adjacent vertices are called *neighbours*. The *neighbourhood* of a vertex v in a graph G is the set of vertices that are adjacent to v. We denote the neighbourhood of the vertex v in the graph G as $N(v)$. In the graph in *Figure 1.1*, the vertices x and w are adjacent. The vertices x and w are both incident to the edge wx. Furthermore, the neighbours of the vertex x are the vertices u, v, w and z and we have that $N(x) = \{u, v, w, z\}$.

The *degree* of a vertex v in a graph G is the number of vertices in the graph G that v is adjacent to, and is denoted by $deg_G v$. The *maximum degree* of a graph G is the largest degree amongst the degrees of the vertices in the graph G, and is denoted by $\triangle(G)$. In this dissertation, we will use the notation $\triangle(G)$ and $\triangle$ interchangeably, where it is understood that the graph in consideration is the graph G. The *minimum degree* of a graph G is the smallest degree amongst the degrees of the vertices in the graph G, and is denoted by $\delta(G)$. As an example, in the graph in *Figure 1.1*, we have that $deg(w) = 3$, $\Delta(G_1) = 4$ and $\delta(G_1) = 1$.

A *uv path* in a graph G is a sequence of vertices and edges in the graph G, in which no vertices or edges are repeated, that begins at the vertex u in the graph G, called the *initial vertex*, and ends at the vertex v in the graph G, called the *terminal vertex*. A sx path in the graph in *Figure 1.1* is written as (s, u, x) and can be seen in blue in *Figure 1.1*. If the initial vertex and terminal vertex are the same, then we have a *cycle*. The *length of a cycle* can be determined by the number of edges in the cycle. In the graph in *Figure 1.1*, there is an $x - x$ cycle of length four, called a 4-cycle. This can be seen in green in *Figure 1.1*. The *girth* of a graph G, denoted g, is the length of the shortest cycle in the graph G. For the graph G_1 in *Figure 1.1*, we have that $g = 3$.

There can be multiple edges between two distinct vertices in a graph G, called *parallel edges*, as well as an edge joining a vertex to itself, called a *loop*. Graphs without these multiple edges and loops are known as *simple graphs*. Additionally, the edges between two vertices in a graph G may have a direction. If the edges in a graph G have no direction, then the graph G is an *undirected graph*. The graph in *Figure 1.1* is a simple, undirected graph.

A graph G is called a *planar graph* if it can be drawn in the plane without any edges crossing. The resultant graph from drawing a graph G in the plane without the edges crossing is called a *plane graph*. Hence, every plane graph is planar and every planar graph can be drawn as a plane graph. The graph in *Figure 1.1* depicts both a planar graph and a plane graph. Another example of a planar graph is shown in *Figure 1.2*. The graph in *Figure 1.2* is also a plane graph.

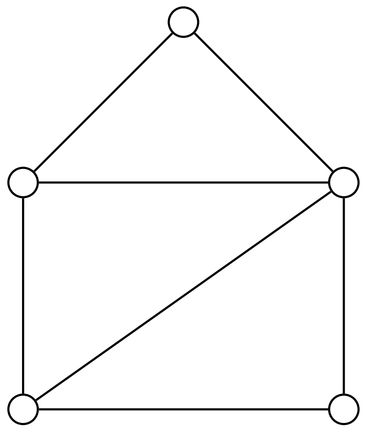

Figure 1.2: Example of a planar graph

The graph in *Figure 1.3* illustrates a non-planar graph, as there is no possible plane drawing of the graph without having two of the edges of the graph crossing.

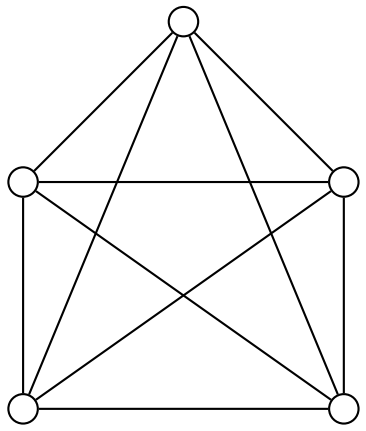

Figure 1.3: Example of a non-planar graph

The graph in *Figure 1.4* is a planar graph. Although the edges v_1v_4 and v_2v_3 cross, there still exists a plane drawing of the graph with no edges crossing. This is shown in *Figure 1.5*.

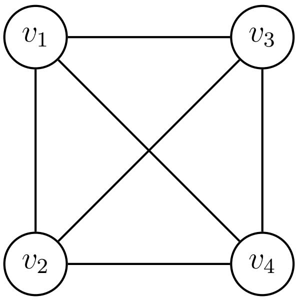

Figure 1.4: Example of a planar graph

The graph in *Figure 1.5* is a plane graph, unlike in *Figure 1.4*.

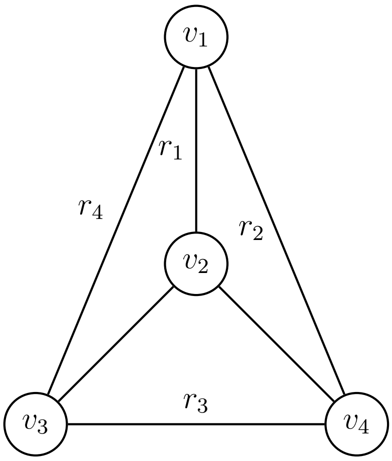

Figure 1.5: Plane drawing of Figure 1.4

A plane drawing of a graph G divides the graph into *regions*, called *faces*, including the *exterior region*. The set of the faces of a graph G is called its *face set*, and is denoted by $F(G)$. The regions of the graph in *Figure 1.5* have been labelled. The region r_1 is bounded by the edges v_1v_2, v_2v_3 and v_3v_1. The region r_2 is bounded by the edges v_1v_2, v_2v_4 and v_1v_4. The region r_3 is bounded by the edges v_2v_3, v_3v_4 and v_2v_4. The regions r_1, r_2 and r_3 are *interior regions*, as they lie on the inside of the graph. The region r_4 is an exterior region, as it lies outside of the entire graph. The exterior region is unbounded.

The *length* or *degree* of a face $f \in F(G)$, denoted $d(f)$, is the length of a boundary walk along the face f in a graph G. The degree of the face r_3 in the graph in *Figure 1.6* is 3, since the boundary walk of r_3 is $v_2 - v_3 - v_4 - v_2$. We say that $d(r_3) = 3$ and r_3 is a *3-face*. The 1-vertex v_5 is said to be *incident* to the face r_1. The vertex v_5 is also incident to the face r_2. Furthermore, $d(r_1) = 5$ since the boundary walk of r_1 is $v_1 - v_2 - v_3 - v_1 - v_5 - v_1$. For the graph in *Figure 1.6*, $F(G) = \{r_1, r_2, r_3, r_4\}$.

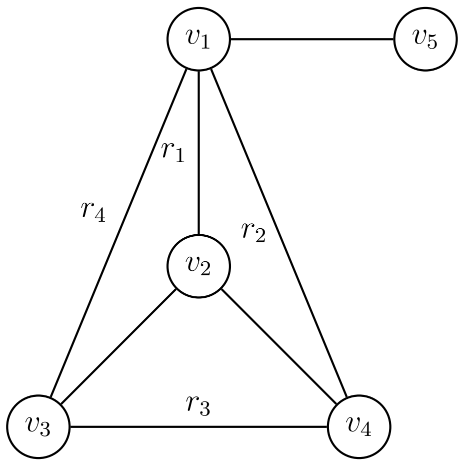

Figure 1.6: Example of a planar graph

The graphs considered in this dissertation are all finite, simple and undirected planar graphs. Additionally, some graphs may have restrictions on the girth and the maximum degree of the graph.

1.2 Edge Colouring in Graphs

Chromatic Graph Theory concerns itself with assigning the vertices or the edges of a graph with a label, called a colour.

A *vertex colouring* of a graph G is the assignment of colours to the vertices of G. A *proper vertex colouring* of a graph G is the assignment of colours to the vertices of G, such that no two adjacent vertices receive the same colour.

This dissertation deals with the colouring of the edges of a graph. An *edge colouring* of a graph G is the assignment of colours to the edges of G. A *proper edge colouring* of G is the assignment of colours to the edges of G, such that no two adjacent edges have the same colour. Two edges are said to be adjacent if they share a common vertex.

The *neighbourhood* of an edge e, denoted $N(e)$, is the set of edges that are adjacent to e. The *line graph* of a graph G, denoted $L(G)$, is a graph $L(G)$ in which the vertices of $L(G)$ are the edges of G and any two vertices in $L(G)$ are adjacent if and only if the two corresponding edges in G are adjacent. We remark that the neighbourhood of an edge e in G is equivalent to the neighbourhood of the vertex corresponding to edge e in $L(G)$.

A *proper k-edge colouring* of a graph G is a mapping

$$f : E(G) \to \{1, 2, 3, ..., k\}$$

such that $f(e_1) \neq f(e_2)$ for adjacent edges e_1 and e_2 in the graph G. That is, each edge of G is assigned one of k colours and no two adjacent edges in G have the same colour. A graph that has a proper k-edge colouring is said to be *proper k-edge colourable.*

A proper k-edge colouring of G partitions the set $E(G)$ into k edge colour classes. An *edge colour class* E_i is a set of edges defined as follows :

$$E_i = \{\ e \mid e \text{ is coloured } i\ \}$$

for a proper k-edge colouring of G, where $e \in E(G)$ and $i = 1, 2, ..., k$.

A *matching* is a set of edges such that no two edges in the set are adjacent. Each edge colour class induces a matching, since no two adjacent edges are assigned the same colour.

The graph in *Figure 1.7* has been proper edge coloured using 4 colours. The edge colour classes for the graph in *Figure 1.7* are $E_1 = \{us, vx, yz, tw\}, E_2 = \{uv, wx\}, E_3 = \{ux, wy\}$ and $E_4 = \{xz\}$. It is clear that these sets partition the edge set of the graph in *Figure 1.7* and each colour class induces a matching.

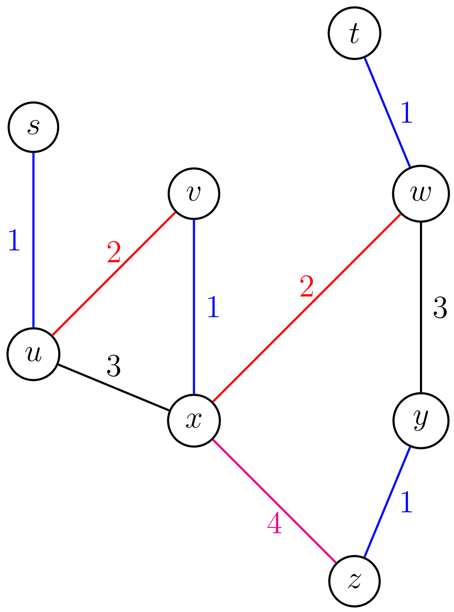

Figure 1.7: Proper edge colouring of G_1

A *strong edge colouring* of G is the assignment of colours to the edges of G, such that no two edges at a distance at most 2 have the same colour. Two edges are at distance 1 if they are adjacent edges (that is, they share a common vertex). Two edges are at distance 2 if they are not at distance 1 and they are both adjacent to a common edge. We note that every strong edge colouring is a proper edge colouring.

The *2-neighbourhood* of an edge e in the graph G is the set of edges in the graph G that are at distance at most 2 from the edge e. The 2-neighbourhood of e is sometimes called the *strong neighbourhood* of e and is denoted $N_2(e)$. The *square* of a graph G is obtained by adding edges between vertices at distance at most 2. Therefore, the square of the line graph of G, denoted $L^2(G)$, would have vertices adjacent to each other if and only if the corresponding edges are at distance at most 2 in G. We remark that the strong neighbourhood of e in G is equivalent to the neighbourhood of the vertex corresponding to edge e in $L^2(G)$.

A *strong k-edge colouring* is also a mapping f, where each edge is assigned one of k colours as follows :

$$f : E(G) \to \{1, 2, 3, ..., k\}$$

such that $f(e_1) \neq f(e_2)$ for edges e_1 and e_2 at a distance at most 2. Similarly, G is *strong k-edge colourable* if it can be strongly k-edge coloured.

A strong k-edge colouring of a graph will also partition $E(G)$ into k edge colour classes that induce a matching. Since no two edges in the matching are at a distance at most 2, we call the matching a *strong matching*.

The graph in *Figure 1.8* shows the strong edge colouring of the same graph in *Figure 1.7*, using 6 colours. It is notable that more colours are usually needed when applying a strong edge colouring to a graph G than when applying a proper edge colouring to the same graph G. We also see that the edge colour classes for the graph in *Figure 1.8* are
$E_1 = \{us, yz\}, E_2 = \{uv, wy\}, E_3 = \{ux, tw\}, E_4 = \{vx\}, E_5 = \{wx\}$ and $E_6 = \{xz\}$. Together, these colour classes form a partition and each colour class induces a strong matching.

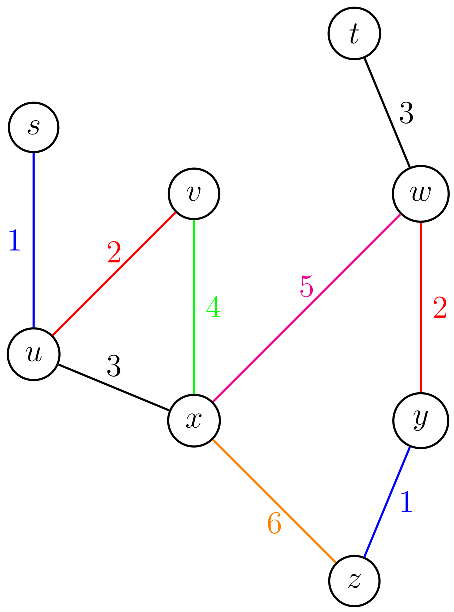

Figure 1.8: Strong edge colouring of G_1

Edge colouring has many applications in the real world. Some of these applications include scheduling problems, like scheduling players for sporting matches or scheduling production in factories. It is also useful in the telecommunications industry for assigning frequencies in fibre optic networks.

One of the main objectives in graph colouring is to minimize the number of colours needed to colour a graph. The *edge chromatic number* or *chromatic index* of a graph G is the smallest positive integer k, such that G can be proper k-edge coloured and is denoted by $\chi'(G)$. The *strong chromatic index* of G is the smallest integer k, such that G can be strongly k-edge coloured and is denoted by $\chi'_s(G)$. In this dissertation, we will investigate the bounds on the strong chromatic index for a subset of planar graphs.

For any additional defnitions, the reader is referred to the textbook entitled *Chromatic Graph Theory* by Gary Chartrand and Ping Zhang [5].

1.3 Overview of Dissertation

In this chapter, we introduced some basic definitions used in graph theory in Section 1.1. Thereafter, in Section 1.2, we discussed the subject of edge colouring in graphs. Now, we will outline the contents of this dissertation.

In Chapter 2, we will give a brief outline of the literature that has been published in the field. Then, in Chapter 3, we will present the research question of the dissertation and elaborate on the methodology that will be used to answer the research question. Thereafter, in Chapter 4, we will give proofs for the upper bound on the strong chromatic index for planar graphs. In Chapter 5, we will provide proofs for the upper bound on the strong chromatic index for a subset of planar graphs with the girth $g \geqslant 6$. Following this, in Chapter 6, we will present proofs for the upper bound on the strong chromatic index for a subset of planar graphs. Then, in Chapter 7, we will give proofs for the upper bound on the strong chromatic index for a subset of planar graphs with maximum degree $\Delta(G) \geqslant 5$ and girth $g \geqslant 5$. Furthermore, in Section 7.3, we will answer the research question of the dissertation. Finally, we will conclude the dissertation in Chapter 8.

Chapter 2

Literature Review

In this chapter, we give a brief summary of the literature that has been published in the research domain of strong edge colouring. In particular, we consider how the upper bound on the strong chromatic index has been improved by various authors.

Strong edge colouring was introduced by Fouquet and Jolivet in 1983 [8, 9] and since then, there have been several investigations into improving the upper bound on the strong chromatic index.

In 1985, Erdös [6] conjectured the following :

Conjecture 1. *For every graph G with maximum degree $\Delta(G)$,*
$\chi'_s \leqslant \frac{5}{4}\Delta(G)^2$*, if* $\Delta(G)$ *is even and*
$\chi'_s \leqslant \frac{1}{4}\left(5\Delta(G)^2 - 2\Delta(G) + 1\right)$*, if* $\Delta(G)$ *is odd.*

Conjecture 1 was shown to be true for $\Delta(G) \leqslant 3$ by Andersen [1] and Horák [11].

Thereafter, in 1997, Molloy and Reed [14] showed that for graphs G with sufficiently large maximum degree, $\chi'_s(G) \leqslant 1.998\Delta(G)^2$. This bound was improved 18 years later by Bruhn and Joos [4] and has now become the best known upper bound on the strong chromatic index, for a graph with sufficiently large maximum degree. The result is given in the following theorem.

Theorem 1. *If G is a graph of sufficiently large maximum degree $\Delta(G)$, then* $\chi'_s(G) \leqslant 1.93\Delta(G)^2$.

Theorem 1 relates the strong chromatic index of the graph G with the maximum degree of the graph. The authors Bruhn and Joos used a probabilistic colouring method, along with the concept of the square of a line graph, as defined in Section 1.2 of this dissertation, to show that the bound is true.

The following theorem by Faudree *et al* [7] relates the strong chromatic index of the planar graph G with the proper chromatic index of the graph.

Theorem 2. *If G is a planar graph with maximum degree $\Delta(G)$, then $\chi'_s(G) \leqslant 4\chi'(G) \leqslant 4\Delta(G) + 4$.*

Theorem 2 currently gives the best known upper bound for a planar graph G. The proof of Theorem 2 makes use of the Four Colour Theorem which states that every planar graph has a proper vertex colouring that uses at most 4 colours. The proof also makes use of Vizing's Theorem which states that the chromatic index of a graph G is at most $\Delta(G)+1$. In the same paper, the authors find a class of graphs that improve on the bound from Theorem 2. The result is shown below.

Theorem 3. *For every integer $\Delta \geqslant 2$, there exists a planar graph G with maximum degree $\Delta(G) = \Delta$ and $\chi'_s(G) = 4\Delta(G) - 4$.*

Several theorems that followed considered a subset of planar graphs with girth restrictions, for which there was an improvement on the upper bound on the strong chromatic index from Theorem 2. These theorems related the strong chromatic index of the graph with the maximum degree of the graph.

The following theorem by Hudák *et al* [12] obtains its proof by means of a contradiction. The result was published in the year 2014. The authors make use of a discharging method, detailed in Section 3.3 of this dissertation, to achieve this contradiction.

Theorem 4. *If G is a planar graph with maximum degree $\Delta(G) \geqslant 4$ and girth $g \geqslant 6$, then $\chi'_s(G) \leqslant 3\Delta(G) + 5$.*

The result of Theorem 4 was improved in the same year by Bensmail *et al* [2], using the same methodology as that of Hudák *et al* [12]. This result is given below.

Theorem 5. *If G is a planar graph with maximum degree $\Delta(G)$ and girth $g \geqslant 6$, then $\chi'_s(G) \leqslant 3\Delta(G) + 1$.*

In the same paper by Bensmail *et al* [2], the authors made a finding for smaller values of the girth, as shown below.

Theorem 6. *Let G be a planar graph with maximum degree $\Delta(G)$ and girth g. If G satisfies one of the following conditions, then $\chi'_s(G) \leqslant 4\Delta(G)$:*
(1) $\Delta(G) \geqslant 7$*;*
(2) $\Delta(G) \geqslant 5$ *and* $g \geqslant 4$*;*
(3) $g \geqslant 5$.

In 2017, Song *et al* [16] further improved the upper bound for (3) of Theorem 6 when $\Delta(G) \geqslant 5$. The result is given in the theorem below.

Theorem 7. *If G is a planar graph with maximum degree $\Delta(G) \geqslant 6$ and girth $g \geqslant 5$, then $\chi'_s(G) \leqslant 4\Delta(G) - 2$.*

In the same paper [16], the authors showed an improvement on the upper bound of the strong chromatic index, in the particular case when $\Delta(G) = 5$ and $g \geqslant 5$. The result is shown below.

Theorem 8. *If G is a planar graph with maximum degree $\Delta(G) = 5$ and girth $g \geqslant 5$, then $\chi'_s(G) \leqslant 19$.*

The authors used the same methodology as that of Hudák *et al* [12] to prove Theorem 7 and Theorem 8.

Lastly, we consider a question posed by Song *et al* [16] in their paper :

Let G be a planar graph with girth $g = 5$ and maximum degree $\Delta(G) \in \{5, 6\}$. Is it true that $\chi'_s(G) \leqslant 3\Delta(G) + 1$?

We will prove this statement to be true for $\Delta(G) = 5$ in Section 7.3 of this dissertation.

We will prove Theorem 1 to Theorem 3 in Chapter 4 of this dissertation. In Chapter 5, we will prove Theorem 4 and Theorem 5. Thereafter, in Chapter 6, we will prove Theorem 6. Finally, in Chapter 7, we will prove Theorem 7 and Theorem 8, as well as provide an answer to the open question posed by Song *et al* in their paper [16].

In this chapter, we gave an outline of the evolution of the upper bound on the strong chromatic index for a graph G. Additionally, we presented an open question in the field that will be answered in Section 7.3 of this dissertation.

Chapter 3

Research Question and Methodology

In this chapter, we list the aim and objectives of this dissertation in Section 3.1. In Section 3.2, we present the research question and state the hypothesis of the dissertation. Lastly, we describe the methodology that will be used in proving our hypothesis in Section 3.3.

3.1 Aim and Objectives

The aim of this study is to improve the upper bounds on the strong chromatic index for a planar graph G, with maximum degree $\Delta(G) = 5$ and girth $g = 5$, which is a subset of the graphs considered in Theorem 8 [16].

We will approach this by first carefully working through similar previous theorems and their proofs to aid us in constructing our own proof. Hence, our first objective would be to analyze and reconstruct, according to our own understanding, the proofs of Theorems 1 through 8, listed in Chapter 2 of this dissertation. Thereafter, our next objective will be to apply a similar methodology, described in Section 3.3 of this dissertation, in answering the open question proposed by Song *et al* [16]. Where needed, we will attempt to formulate the proof using our own approaches, or an adaptation of the previous methods used.

3.2 Research Question and Hypothesis

The research question of this dissertation was conjectured by Song *et al* [16] and is given below :

For a planar graph G with girth $g = 5$ and maximum degree $\Delta(G) = 5$, is it true that $\chi'_s(G) \leqslant 3\Delta(G) + 1$?

We hypothesise the following :

For a planar graph G with girth $g = 5$ and maximum degree $\Delta(G) = 5$, we have that $\chi'_s(G) \leqslant 3\Delta(G) + 1$.

We will prove this statement to be true in Section 7.3 of this dissertation.

3.3 Methodology

We will adopt part of the methodology used in the papers by Song *et al* [16], Bensmail *et al* [2] and Hudák *et al* [12] to prove our hypothesis. The approach used in these papers is a proof by contradiction. The general outline of the method used is given below.

1. Assume the statement is false. This means that there must exist a counterexample to the statement;
2. We investigate the structure of the minimal counterexample G;
3. We apply the discharging method, explained in Section 3.3.3 of this dissertation, on the graph G and obtain a contradiction. This means that no such counterexample exists, hence the statement must be true.

Further details on each step follows below.

3.3.1 Assuming the Contrary

We begin our approach by assuming that the statement is false. That is, for planar graphs with girth $g = 5$ and maximum degree $\Delta(G) = 5$, there exists a graph G, such that $\chi'_s(G) \geqslant 3\Delta(G) + 2 = 17$. The graph G is then a counterexample to our statement. Amongst these graphs, let H be the graph that minimizes $|E(H)| + |V(H)|$. Then, H is a minimal counterexample of the statement. Furthermore, we note that by the minimality on H, along with the condition on the girth, we can assume that H is connected. We will show that no such counterexample exists by means of a contradiction. We will achieve this contradiction using the method described in Section 3.3.3 of this dissertation.

3.3.2 Structure of Minimal Counterexample

We will investigate the properties that the minimal counterexample H needs to have. We will prove that each property needs to be present in the graph H. We outline the general procedure below.

- We first assume that the property is not present in the graph H.
- Then, we define the graph H' to be the graph H with one or more edges removed. Then, clearly H' can be strongly edge coloured using $3\Delta(H) + 1 = 16$ colours or less.
- Consider an edge $uv \in E(H)$ and the set of edges that are a distance at most 2 away from uv in H, denoted $N_2(uv)$. These edges would all have to be coloured differently for a strong edge colouring of the graph H.
- We let $H' = H - uv$ and consider the 2-neighbourhood of the edge uv in H'. We colour these edges using the colours from the set L, where L is the set of colours $1, 2, ..., 3\Delta(H) + 1$. This is possible since H' can be strongly edge coloured using at most 16 colours.
- We will show that after colouring the 2-neighbourhood of uv in H' that we will have a colour left over in L to colour uv in H.
- Then, we re-insert the edge uv to obtain the graph H and we can colour uv with the colour left over in L.
- If we removed multiple edges from H, then we continue to successively colour the missing edges in the same manner until we obtain the graph H.
- Then, we can extend the strong $3\Delta(H) + 1$-edge colouring from the graph H' to the graph H. This colouring is valid since any two edges separated by distance at least 3 in H' will still be separated by distance at least 3 in H.
- This will give us a contradiction, since H is supposed to be a counterexample to the statement.
- Then, we can conclude that the property must be present in H.

3.3.3 Discharging Method

In the discharging method, each face and vertex of a graph G is assigned an initial charge. The graph G is then subject to discharging rules which redistributes charges amongst the faces and vertices of the graph. The final charge of each face and vertex is then determined. The sum of the final charge should remain the same as the sum of the initial charges, as the charges are only redistributed and not removed from or added to the graph. This particular property of the discharging method will be used to obtain the contradiction and complete our proof.

Since H is a planar graph, H satisfies Euler's identity :

$$n - m + r = 2$$

where n is the number of vertices, m is the number of edges and r is the number of regions, or faces, in the graph H.

Let $k > 0$. We may rewrite Euler's identity as follows :

$$\begin{aligned} n - m + r &= 2 \\ kn - km + kr &= 2k \\ km - kn - kr &= -2k \\ (k-2)m + 2m - kn - kr &= -2k \\ (k-2)m - kn + 2m - kr &= -2k \end{aligned}$$

Now, we know that $\sum_{v \in V(H)} d(v) = \sum_{f \in F(H)} d(f) = 2|E(H)| = 2m$.

Hence, we have:

$$\sum_{v \in V(H)} \left(\frac{k-2}{2} d(v) - k \right) + \sum_{f \in F(H)} (d(f) - k) = -2k \tag{3.1}$$

We assign the initial charge of the vertices of H to be

$$ch(v) = \frac{k-2}{2} d(v) - k$$

for $v \in V(H)$ and we assign the initial charge of the faces of G to be

$$d(f) - k$$

for $f \in F(H)$. Therefore, the sum of the initial charges of the vertices and faces of H is -2$k < 0$.

Now, let the minimum girth for a set of planar graphs H be g. Consider $k = g$ and suppose that H has only g-faces. Then, we have that

$$\sum_{v \in V(H)} \left(\frac{g-2}{2} d(v) - g \right) + \sum_{f \in F(H)} (d(f) - g) = -2g$$

and we see that

$$d(v) = \frac{-2g}{g-2}.$$

Now, consider $k < g$ and suppose H has only g-faces. Suppose $k = g - 1$. Then, we have that

$$\sum_{v \in V(H)} \left(\frac{g-3}{2} d(v) - g + 1 \right) + \sum_{f \in F(H)} (g - g + 1) = -2g + 2$$

and we see that

$$d(v) = \frac{-2g}{g-3} < \frac{-2g}{g-2}.$$

Now, consider $k > g$ and suppose H has only g-faces. Then, $d(f) - k < 0$.

We see that in order to obtain our contradiction, we require that the charges of the vertices v and the faces f in H must be as large as possible. Then, from above, the best choice for k is $k = g$. In the proof of our hypothesis, we will use $k = g = 5$. So :

For a planar graph H with girth $g = 5$, we have that

$$\sum_{v \in V(H)} \left(\frac{3}{2} d(v) - 5 \right) + \sum_{f \in F(H)} (d(f) - 5) = -10$$

where $d(v)$ is the degree of vertex v and $d(f)$ is the degree of face f.

We assign the initial charge of the vertices of H to be

$$ch(v) = \frac{3}{2} d(v) - 5$$

for $v \in V(H)$ and we assign the initial charge of the faces of H to be

$$ch(f) = d(f) - 5$$

for $f \in F(H)$. Therefore, the sum of the initial charges of the vertices and faces of H is -10 < 0.

Thereafter, we will apply discharging rules on the faces and vertices of H and determine the final charges of the faces, $ch'(f)$, and of the vertices, $ch'(v)$, of H. We will construct the rules such that we obtain the final charges of each face and each vertex in the graph to be at least 0. Thus, the sum of the final charges of the vertices and faces of H will also be at least 0.

Hence, we will have that,

$$0 \leqslant \sum_{x \in V(H) \cup F(H)} ch'(x) = \sum_{x \in V(H) \cup F(H)} ch(x) < 0$$

which is a contradiction.

Thus, this would mean that no such graph H exists. That is, a counterexample does not exist and therefore, the statement must be true.

In this chapter, we stated the aim and objectives of this dissertation in Section 3.1. Following this, in Section 3.2, we proposed the research question of the dissertation and formulated our hypothesis. Finally, in Section 3.3, we detailed the methodology that will be used in proving our hypothesis to be true.

Chapter 4

Upper Bound for Planar Graphs

In this chapter, we will show that for a graph G with sufficiently large maximum degree $\Delta(G)$, $\chi'_s(G) \leqslant 1.93\Delta(G)^2$. Thereafter, we will prove that for a planar graph G with maximum degree $\Delta(G)$, $\chi'_s(G) \leqslant 4\chi'(G) \leqslant 4\Delta(G) + 4$. Finally, we will show that for a special class of planar graphs, we have that $\chi'_s(G) = 4\Delta(G) - 4$, for $\Delta(G) \geqslant 2$.

4.1 Proof of Theorem 1 ($\chi'_s(G) \leqslant 1.93\Delta(G)^2$ for Graphs with Large $\Delta(G)$)

We recall Theorem 1 [4] below :

Theorem 1. *If G is a graph of sufficiently large maximum degree $\Delta(G)$, then $\chi'_s(G) \leqslant 1.93\Delta(G)^2$.*

4.1.1 Preliminaries

As mentioned previously in Section 1.2 of this dissertation, a strong edge colouring of a graph G is the equivalent of a proper vertex colouring of the graph $L^2(G)$, the square of the line graph of G. To complete the proof of Theorem 1, we make use of this concept and two lemmas that are detailed below [4].

Lemma 9. *Let G be a graph of maximum degree $\Delta(G) \geqslant 1$, and let e be an edge of G. Let N_e^s denote the set of edges at distance at most 2 from e in G. Then, N_e^s induces in $L^2(G)$ a graph of at most $\frac{3}{2}\Delta(G)^4 + 5\Delta(G)^3$ edges.*

This first lemma shows that the neighbourhood of any vertex in $L^2(G)$ cannot be too dense. Thereafter, using probabilistic colouring methods, we obtain a colouring lemma for graphs with sparse neighbourhoods. The probabilistic colouring methods are explored in the paper by Molloy and Reed [14]. The colouring lemma is given below :

Lemma 10. *Let $\gamma, \delta \in (0, 1)$ be so that*

$$\gamma < \frac{\delta}{2(1-\gamma)} \exp^{-\frac{1}{1-\gamma}} - \frac{\delta^{\frac{3}{2}}}{6(1-\gamma^2} \exp^{-\frac{7}{8(1-\gamma)}} .$$

Then, there is an integer R so that for all $r \geqslant R$, it follows that $\chi(G) \leqslant (1-\gamma)r$ for every graph G with maximum degree at most r in which, for every vertex v, the neighbourhoods $N(v)$ induce graphs of at most $(1-\delta)\binom{r}{2}$ edges.

Moreover, we remark that, in a range of $\delta \in (0, 0.9]$, $\gamma = 0.1827\delta - 0.0778\delta^{\frac{3}{2}}$ satisfies the condition.

4.1.2 Proof

To prove the theorem, let G be a graph with maximum degree $\Delta(G)$ sufficiently large. With Lemma 9, we conclude that for every vertex v in $L^2(G)$, the neighbourhood of v induces a graph of at most $(\frac{3}{4} + o(1))$ $\binom{2\Delta(G)^2}{2}$ edges. Then, with $r = 2\Delta(G)^2$, $\delta = 0.24$ and $\gamma = 0.035$, we apply Lemma 10 on the graph $L^2(G)$ to get that $\chi(L^2(G)) \leqslant (0.965)(2\Delta(G)^2)$. Thus, we have that $\chi'_s(G) \leqslant 1.93\Delta(G)^2$.

This concludes the proof.

4.2 Proof of Theorem 2 ($\chi'_s(G) \leqslant 4\chi'(G)$ for Planar Graphs)

We recall Theorem 2 [7] :

Theorem 2. *If G is a planar graph with maximum degree $\Delta(G)$, then $\chi'_s(G) \leqslant 4\chi'(G) \leqslant 4\Delta(G) + 4$.*

We complete the proof below.

4.2.1 Preliminaries

We use $[\![n]\!]$ to indicate the set of integers $\{1, 2, \ldots, n\}$.

We state the **Four Colour Theorem** below :
The vertices of every planar graph can be coloured with at most four colours so that no two adjacent vertices receive the same colour.

We state **Vizing's Theorem** below :
Every simple, undirected graph G can be edge coloured using a number of colours that is at most one larger than the maximum degree, $\Delta(G)$, of the graph.

4.2.2 Proof

Let k be the proper edge chromatic index of graph G. Let M_i be the set of edges from graph G having colour i, where $i = [\![k]\!]$. This means that each set M_i is a matching and so any two edges in M_i are independent.

Now, define the graph $G(M_i)$ to be the graph with vertex set M_i and two vertices in $G(M_i)$ are adjacent if and only if the corresponding two edges in M_i do not form a strong matching in G. Therefore, the vertices of $G(M_i)$ are adjacent if and only if the corresponding edges in M_i are incident to a common edge in G.

Since $G(M_i)$ is a planar graph, we can colour the vertices of $G(M_i)$ using the Four Colour Theorem. So, we have that $\chi(G(M_i)) \leqslant 4$. After colouring $G(M_1), G(M_2), ..., G(M_k)$, we get a strong edge colouring of G since any two edges of M_i incident to a common edge receive distinct colours in G. Thus, we have that

$$\begin{aligned} \chi'_s &\leqslant \chi(G(M_1)) + \chi(G(M_2)) + ... + \chi(G(M_k)) \\ &= 4 + 4 + ... + 4 \\ &= 4k \\ &= 4\chi'(G) \end{aligned}$$

Thus,

$$\chi'_s \leqslant 4\chi'(G).$$

Now, using Vizing's Theorem, we see that

$$\chi'_s \leqslant 4\chi'(G) \leqslant 4\Delta(G) + 4.$$

This concludes the proof.

4.3 Proof of Theorem 3 ($\chi'_s(G) = 4\Delta(G) - 4$ for Class of Planar Graphs)

We recall Theorem 3 below [7].

Theorem 3. *For every integer $\Delta \geqslant 2$, there exists a planar graph G with maximum degree $\Delta(G) = \Delta$ and $\chi'_s(G) = 4\Delta(G) - 4$.*

4.3.1 Preliminaries

We use $[\![n]\!]$ to indicate the set of integers $\{1, 2, \ldots, n\}$.

An *antimatching* of a graph G is a maximal set of edges in G, such that each pair of edges in the set is incident to some edge of G.

A *bipartite graph* $K_{m,n}$ is a graph with vertex set $|V(K_{m,n})| = m + n$ and every vertex $v_i \in V(K_{m,n})$, for $i = [\![m]\!]$ is adjacent to each $v_j \in V(K_{m,n})$, for $j = [\![n]\!]$.

4.3.2 Proof

Let $m \geqslant 2$. Consider the bipartite graphs $K_{2,m}$ and $K_{m,2}$. If we take a fixed C_4 from $K_{2,m}$ and join the corresponding edges with a fixed C_4 from $K_{m,2}$, we will obtain a planar graph G, such that the set $E(G)$ is an antimatching.

The graph G has maximum degree m and $4m - 4$ edges. Since $m = \Delta$, we have that $\chi'_s(G) = 4\Delta(G) - 4$.

This concludes the proof.

Chapter 5

Upper Bound for Planar Graphs with Girth $g \geqslant 6$

In this chapter, we show that for planar graphs G with maximum degree $\Delta(G) \geqslant 4$ and girth $g \geqslant 6$, $\chi'_s(G) \leqslant 3\Delta(G) + 5$. Thereafter, we prove that for the same set of planar graphs, we can obtain an improved upper bound on the strong chromatic index. In particular, we have that $\chi'_s(G) \leqslant 3\Delta(G) + 1$.

5.1 Proof of Theorem 4 ($\chi'_s(G) \leqslant 3\Delta(G) + 5$ for Planar Graphs with $g \geqslant 6$)

We recall Theorem 4 [12] :

Theorem 4. *If G is a planar graph with maximum degree $\Delta(G) \geqslant 4$ and girth $g \geqslant 6$, then $\chi'_s(G) \leqslant 3\Delta(G) + 5$.*

5.1.1 Preliminaries

We define the following :

- We define $N_2(uv)$ as the set of edges at distance at most 2 from the edge uv, excluding the edge uv itself.
- We denote $SC(N_2(uv))$ as the set of colours used by the edges in the set $N_2(uv)$, given an edge colouring of G.
- For $k \in \mathbb{N}$, a *k-vertex* is a vertex of degree k.
  A *k^+-vertex* is a vertex of degree at least k.
  A *k^--vertex* is a vertex of degree at most k.

- A 4_2-*vertex* is a 4-vertex adjacent to at most two 2-vertices.
- A 4_3-*vertex* is a 4-vertex adjacent to exactly three 2-vertices.
- A *weak 2-vertex* is a 2-vertex with a 3^--neighbour.
- A *semi-weak 2-vertex* is a 2-vertex with a 4_3-neighbour.
- A *strong 2-vertex* is a 2-vertex that is neither a weak 2-vertex nor a semi-weak 2-vertex.
- We use $[\![n]\!]$ to indicate the set of integers $\{1, 2, \ldots, n\}$.

5.1.2 Proof

Assume that the theorem is false. Then, there exists a planar graph(s) G with girth $g \geqslant 6$ and $\Delta(G) \geqslant 4$ such that $\chi'_s(G) > 3\Delta(G) + 5$. That is, these graphs can be strongly edge coloured using 3 $\Delta(G)$ + 6 colours or more. Amongst these graphs, let H be the graph that minimizes $|E\,(H)\,| + |V\,(H)\,|$. Then, H is a minimal counterexample and by the minimality of H, along with the condition on the girth, we can assume that H is a connected graph. Furthermore, we have that $\chi'_s(H) \geqslant 3\Delta(H) + 6$.

Let $\Delta(H) = \Delta$ and let ϕ be a strong $(3\Delta(H) + 5)$-edge colouring. We define a set of $3\Delta(H) + 5$ colours as $L = [\![3\Delta(H) + 5]\!]$.

Now, we proceed to establish structural properties of the minimal counterexample H. We define and prove these properties in the lemmas below [12] :

Lemma 11. *Every* 1*-vertex in H is adjacent to a* 5^+*-vertex. Moreover, if it has a* 5*-neighbour u, then all the other neighbours of u have degree at least* 3*.*

Proof. Suppose H contains a 1-vertex v adjacent to a 4^--vertex u. Without loss of generality, we may assume that $d(u) = 4$, since the other possibilities for $d(u)$ would simply require fewer colours. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using $3\Delta + 5$ colours or less.

We have that $|N_2(uv)|$ in H is at most 3Δ. Then, $|L \backslash SC_\phi(N_2(uv))| \geqslant 3\Delta + 5 - 3\Delta = 5$. Thus, there is a colour available in L to colour uv in H, so we may extend the colouring ϕ to H. This means that H can be strongly edge coloured using at most 3 Δ + 5 colours, which is a contradiction.

Hence, every 1-vertex in H is adjacent to a 5^+-vertex.

Now, suppose that u is a 5-vertex and is adjacent to a 1-vertex v and at least one other 2^--vertex. Without loss of generality, we can assume that u is adjacent to exactly one 2^--vertex w with $d(w) = 2$. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using $3\Delta + 5$ colours or less.

We have that $|N_2(uv)|$ in H is at most $3\Delta + 2$ and we have that $|L\backslash SC_\phi(N_2(uv))| \geqslant 3\Delta + 5 - 3\Delta - 2 = 3$. This means that there is a colour available in L to colour uv in H, so we may extend the colouring ϕ to H, giving us a contradiction.

Therefore, if a 1-vertex in H has a 5-neighbour u, then all the other neighbours of u have degree at least 3.

□

Lemma 12. *Every 2-vertex in H has at least one 5^+-neighbour.*

Proof. Suppose H contains a 2-vertex u adjacent to two 4^--vertices v and w. Without loss of generality, we may assume that $d(v) = d(w) = 4$. Let H' = H - $\{uv\}$. Then, there is a colouring ϕ for H'.

We have that $|N_2(uv)|$ in H is at most 3Δ + 4. Then, $|L\backslash SC_\phi(N_2(uv))| \geqslant 3\Delta + 5 - 3\Delta - 4 = 1$. Thus, there is a colour available for uv in H, so we may extend the colouring ϕ from H' to H. This gives a contradiction.

Therefore, every 2-vertex has at least one 5^+-neighbour.

□

Lemma 13. *Every vertex in H has at least one 3^+-neighbour.*

Proof. Suppose H contains a k-vertex u, with $k \geqslant 1$, adjacent to only 2^--vertices. Without loss of generality, we may assume that these vertices all have degree exactly 2. Let one such vertex be v. Let $H' = H$ - $\{uv\}$. Then, there is a colouring ϕ for H'.

We see that $|N_2(uv)|$ in H is at most Δ + $2k$ - 2. Then, $|L\backslash SC_\phi(N_2(uv))| \geqslant 3\Delta + 5 - \Delta - 2k + 2 = 2\Delta - 2k + 7$. Since k is at most Δ, we have that $|L\backslash SC_\phi(N_2(uv))| \geqslant 7$. Thus, there is a colour available in L for uv in H, so we may extend the colouring ϕ to H, giving a contradiction.

Therefore, every vertex in H has at least one 3^+-neighbour.

□

Lemma 14. *Let u be a k-vertex in H, with $k \geqslant 5$. If u has exactly one 3^+-neighbour in H, then the remaining $k-1$ neighbours of u are all strong 2-neighbours.*

Proof. Let $k \geqslant 5$. Suppose H contains a k-vertex u with $N(u) = \{v_1, v_2, ..., v_{k-1}, v_k\}$. Suppose further that u is adjacent to exactly one 3^+-vertex, say v_k. Then, the remaining neighbours of u are 2^--vertices. Suppose that u has at least one 1-neighbour. Without loss of generality, we can assume that u has exactly one 1-neighbour, say v_1. Then, the vertices $v_2, v_3, ..., v_{k-1}$ are all 2-vertices and without loss of generality, we can assume that $d(v_k) = \Delta$. Let $H' = H$ - $\{uv_1\}$. Then, there is a colouring ϕ for H'.

We have that $|N_2(uv_1)|$ in H is at most Δ + $2k$ - 4 and so $|L \backslash SC_\phi(N_2(uv_1))| \geqslant 3\Delta + 5 - \Delta - 2k + 4 = 2\Delta - 2k + 9$. Since k is at most Δ, we have that $|L \backslash SC_\phi(N_2(uv_1))| \geqslant 9$. Thus, there is a colour available for uv_1 in H, so we may extend the colouring ϕ to H, giving a contradiction.

Therefore, for $k \geqslant 5$, every k-vertex in H having exactly one 3^+-neighbour has its remaining neighbours as 2-vertices.

Now, suppose H contains a k-vertex u with $N(u) = \{v_1, v_2, ..., v_{k-1}, v_k\}$. Suppose further that u is adjacent to exactly one 3^+-vertex, say v_k. Then, the remaining neighbours of u are all 2-vertices. Suppose that u has at least one 2-neighbour that is not a strong 2-neighbour. Without loss of generality, we can assume that u has exactly one 2-neighbour that is not a strong 2-neighbour, say v_1. Then, the vertices $v_2, v_3, ..., v_{k-1}$ are all strong 2-vertices and without loss of generality, we can assume that $d(v_k) = \Delta$. Let w be the neighbour of v_1 distinct from u. Then, w is either a 3^--vertex or a 4_3-vertex. Let $H' = H$ - $\{uv_1\}$. Then, there is a colouring ϕ for H'.

Suppose first that w is a 3^--vertex. Without loss of generality, we can assume that $d(w) = 3$.

We have that $|N_2(uv_1)|$ in H is at most Δ + $2k$ - 1 and so $|L \backslash SC_\phi(N_2(uv_1))| \geqslant 3\Delta + 5 - \Delta - 2k + 1 = 2\Delta - 2k + 6$. Since k is at most Δ, we have that $|L \backslash SC_\phi(N_2(uv_1))| \geqslant 6$. Thus, there is a colour available for uv_1 in H, so we may extend the colouring ϕ to H, giving a contradiction.

Now, suppose that w is a 4_3-vertex.

Then, $|N_2(uv_1)|$ is at most Δ + $2k$ and so
$|L\backslash SC_\phi(N_2(uv_1))| \geqslant 3\Delta + 5 - \Delta - 2k = 2\Delta - 2k + 5$. Since k is at most Δ, we have that $|L\backslash SC_\phi(N_2(uv_1))| \geqslant 5$. Thus, there is a colour available for uv_1 in H, so we may extend the colouring ϕ to H, giving a contradiction.

Therefore, for a k-vertex u in H, with $k \geqslant 5$, having exactly one 3^+-neighbour, has its remaining $k-1$ neighbours as strong 2-neighbours.

□

Lemma 15. *For $k \geqslant 5$, every k-vertex in H having exactly two 3^+-neighbours has at least three 2-neighbours that are not weak 2-vertices.*

Proof. Let $k \geqslant 5$. Suppose H contains a k-vertex u with $N(u) = \{v_1, v_2, ..., v_{k-1}, v_k\}$. Suppose further that u is adjacent to exactly two 3^+-vertices, say v_{k-1} and v_k. Then, the remaining neighbours of u are 2^--vertices. Without loss of generality, we can assume that $d(v_{k-1}) = d(v_k)$ $= \Delta$ and the remaining $k-2$ neighbours of u are all 2-vertices. Suppose that u is adjacent to at most two 2-neighbours that are not weak. Without loss of generality, we can assume that u is adjacent to exactly two 2-neighbours that are not weak, say v_{k-2} and v_{k-3}. Let the neighbour of v_{k-2} and v_{k-3}, distinct from u, be w_{k-2} and w_{k-3} respectively. Then, w_{k-2} and w_{k-3} cannot be 3^--vertices. Furthermore, the remaining $k-4$ neighbours of u are all weak 2-vertices. Let the neighbour of $v_1, v_2, ..., v_{k-4}$, distinct from u, be $w_1, w_2, ..., w_{k-4}$ respectively. Then, the vertices $w_1, w_2, ..., w_{k-4}$ are all 3^--vertices. Without loss of generality, we can assume that $d(w_1) = d(w_2) = ... = d(w_{k-4}) = 3$ and $d(w_{k-3}) = d(w_{k-2}) = \Delta$. Let $H' = H$ - $\{v_1, v_2, ..., v_{k-4}\}$. Then, there is a colouring ϕ for H'.

First, consider the 2-neighbourhood of the edge uv_1 in H. Then, $|N_2(uv_1)|$ in H is at most 2Δ + $2k$ - 3. However, if we omit the uncoloured edges $uv_2, ..., uv_{k-5}, uv_{k-4}, v_1w_1, v_2w_2, ..., v_{k-5}w_{k-5}, v_{k-4}w_{k-4}$ from this set, then we have $|N_2(uv_1)|$ in H is at most 2Δ + 6 and so
$|L\backslash SC_\phi(N_2(uv_1))| \geqslant 3\Delta + 5 - 2\Delta - 6 = \Delta - 1 \geqslant 3$. Thus, there is a colour available for uv_1 in H, so we may now colour uv_1.

Now, consider the 2-neighbourhood of the edge uv_2 in H. Then, $|N_2(uv_2)|$ in H is at most 2Δ + $2k$ - 3. However, if we omit the uncoloured edges $uv_3, ..., uv_{k-5}, uv_{k-4}, v_1w_1, v_2w_2, ..., v_{k-5}w_{k-5}, v_{k-4}w_{k-4}$ from this set, then we have $|N_2(uv_2)|$ in H is at most 2Δ + 7 and so

$|L \backslash SC_\phi(N_2(uv_2))| \geqslant 3\Delta + 5 - 2\Delta - 7 = \Delta - 2 \geqslant 2$. Thus, there is a colour available for uv_2 in H, so we may now colour uv_2.

We continue to colour the edges uv_i, for $i = 3, 4, 5, ..., k-5$ successively in this manner.

Now, we consider the 2-neighbourhood of uv_{k-4} in H. Then, $|N_2(uv_{k-4})|$ in H is at most 2Δ + $2k$ - 3. However, if we omit the uncoloured edges $v_1w_1, v_2w_2, ..., v_{k-5}w_{k-5}, v_{k-4}w_{k-4}$ from this set, then we have $|N_2(uv_{k-4})|$ in H is at most 2Δ + k + 1 and so $|L \backslash SC_\phi(N_2(uv_{k-4}))| \geqslant 3\Delta + 5 - 2\Delta - k - 1 = \Delta - k + 4$. Since, $k \leqslant \Delta$, we have that $|L \backslash SC_\phi(N_2(uv_{k-4}))| \geqslant 4$. Thus, there is a colour available for uv_{k-4} in H, so we may now colour uv_{k-4}.

Now, we proceed to colour the edge v_1w_1. Consider the 2-neighbourhood of the edge v_1w_1 in H. Then, $|N_2(v_1w_1)|$ in H is at most 2Δ + k and so $|L \backslash SC_\phi(N_2(v_1w_1))| \geqslant 3\Delta + 5 - 2\Delta - k = \Delta - k + 5$. Since $k \leqslant \Delta$, we have that $|L \backslash SC_\phi(N_2(v_1w_1))| \geqslant 5$. Thus, there is a colour available for v_1w_1 in H, so we may now colour v_1w_1.

Now, consider the 2-neighbourhood of the edges v_jw_j, with $j = 2, 3, ..., k-4$ in H. Given the condition on the girth, we see that no w_m, with $m = [\![k-4]\!]$, can be adjacent to w_l, with $l = [\![k-4]\!]; l \neq m$. This means that we can colour each v_jw_j with the same colour as the edge v_1w_1. This means that we can extend the colouring ϕ to H, giving a contradiction.

Thus, for $k \geqslant 5$, every k-vertex having exactly two 3^+-neighbours has at least three 2-neighbours that are not weak 2-vertices.

□

We now proceed to performing the discharging method on the graph H.

Discharging procedure We use the discharging method, as discussed in Section 3.3.3 of this dissertation. In particular, we use a multiple of $k = 6$ for Equation (3.1).

We define the discharging rules as follows:

(R1): Every face gives 2 to each incident 1-vertex.

(R2): Every 5^+-vertex gives 2 to every adjacent 1-vertex.

(R3): Every 5^+-vertex gives 2 to every adjacent weak 2-vertex.

(R4): Every 5^+-vertex gives $\frac{4}{3}$ to every adjacent semi-weak 2-vertex.

(R5): Every 5^+-vertex gives 1 to every adjacent strong 2-vertex.

(R6): Every 4_2-vertex gives 1 to each of its adjacent 2-vertices.

(R7): Every 4_3-vertex gives $\frac{2}{3}$ to each of its three adjacent 2-vertices.

Now, we will carry out these discharging rules on the graph H.

First, consider a k-face $f \in F(H)$. By the condition on the girth, we have that $k \geqslant 6$. Also, we have $ch(f) = k - 6$.

If f has α incident 1-vertices, then $k \geqslant 6 + 2\alpha$. By (R1), we have that $ch'(f) = ch(f) - 2\alpha = k - 6 - 2\alpha \geqslant 0$.

So, $ch'(f) \geqslant 0$.

Now, consider a k-vertex v in H.

Case $\underline{k = 1}$: We have that $ch(v) = -4$.

By (R1), v will receive 2 from its incident face. From Lemma 11, v is adjacent to a 5^+-vertex and thus by (R2), v receives 2 from its adjacent 5^+-vertex.

So, $ch'(v) = ch(v) + 2 + 2 = 0$.

Case $\underline{k = 2}$: We have that $ch(v) = -2$.

By Lemma 12, we have that v must be adjacent to at least one 5^+-neighbour w.

- Suppose v is a weak 2-vertex. Then, the neighbour of v distinct from w is a 3^--neighbour. So, by (R3), we have that the final charge on v is $ch'(v) = ch(v) + 2 = 0$.
- Suppose v is a semi-weak 2-vertex. Then, the neighbour of v distinct from w is a 4_3-vertex. So, by (R4) and (R7), we have that the final charge on v is $ch'(v) = ch(v) + \frac{4}{3} + \frac{2}{3} = 0$.

- Suppose v is a strong 2-vertex. Then, the neighbour of v distinct from w, say x, is neither a 3^--vertex nor a 4_3-vertex. By Lemma 13, the vertex x cannot be a 4_4-vertex. Therefore, x is either a 4_2-vertex or a 5^+-vertex. Then, by (R5) and (R6), the final charge on v is $ch'(v) = ch(v) + 1 + 1 = 0$.

So, $ch'(v) = 0$.

Case $k = 3$: We have $ch(v) = 0$.

The vertex v does not receive nor give any charge and so the final charge of v remains unchanged.

Thus, $ch'(v) = ch(v) = 0$.

Case $k = 4$: We have $ch(v) = 2$.

From Lemma 11, v is not adjacent to a 1-vertex. From Lemma 13, v is adjacent to at most three 2-vertices.

- Suppose v is a 4_2-vertex. Then, v is adjacent to at most two 2-vertices. Then, the remaining neighbours of v are 3^+-neighbours. So, by (R6), we have that the final charge on v is $ch'(v) \geqslant ch(v) - 1(2) = 0$.

- Suppose v is a 4_3-vertex. Then, v is adjacent to exactly three 2-vertices. The remaining neighbour of v is then a 3^+-neighbour. So, by (R7), we have that the final charge on v is $ch'(v) = ch(v) + \frac{2}{3}(3) = 0$.

- Suppose v does not have any 2^--neighbour. Then, the charge on v remains unchanged, and so the final charge on v is $ch'(v) = ch(v) > 0$.

So, $ch'(v) \geqslant 0$.

Case $k = 5$: We have that $ch(v) = 4$.

- Suppose v is adjacent to a 1-vertex. Then, by Lemma 11, v is not adjacent to any other 2^--vertices. Therefore, the remaining neighbours of v are all 3^+-vertices. So, by (R2), we have that the final charge on v is $ch'(v) = ch(v) - 2 > 0$.

- Suppose v is not adjacent to a 1-vertex. Then, by Lemma 13, we know that v is adjacent to a 3^+-vertex.

  - Suppose v has exactly one 3^+-neighbour. Then, by Lemma 14, the remaining 2-neighbours of v are strong. So, v is adjacent to four strong 2-vertices. Then, by (R5), the final charge on v is $ch'(v) = ch(v) - 1(4) = 0$.
  - Suppose v has exactly two 3^+-neighbours. Then, by Lemma 15, the remaining 2-neighbours of v are not weak 2-neighbours. From (R4), the final charge on v is $ch'(v) \geqslant ch(v) - \frac{4}{3}(3) = 0$.
  - Suppose v has exactly three 3^+-neighbours. Then, by (R3), the final charge on v is $ch'(v) \geqslant ch(v) - 2(2) = 0$.
  - Suppose v has exactly four 3^+-neighbours. Then, by (R3), the final charge on v is $ch'(v) \geqslant ch(v) - 2 > 0$.
  - Suppose v is adjacent to all 3^+-vertices. Then the charge on v remains unchanged and so the final charge on v is $ch'(v) = ch(v) > 0$.

So, $ch'(v) \geqslant 0$.

Case $k \geqslant 6$: We have that $ch(v) = 2k - 6$.

By Lemma 13, v is adjacent to at least one 3^+-vertex.

- Suppose that v is adjacent to exactly one 3^+-vertex. Then, by Lemma 14, v is adjacent to $k-1$ strong 2-vertices. So, by (R5), we have that the final charge on v is $ch'(v) = ch(v) - 1(k-1) = k-5$. Since $k \geqslant 6$, we have that $ch'(v) > 0$.

- Suppose v is adjacent to exactly two 3^+-vertices. Then, by Lemma 15, we know that v is adjacent to at least three 2-neighbours that are not weak. Then, each of these neighbours are either semiweak 2-vertices or strong 2-vertices. Without loss of generality, assume that these 2-neighbours are all semiweak 2-vertices, since by (R4) and (R5), this is the case that would require v to give more charge to its neighbours. Furthermore, we have at most $k-5$ neighbours of v, such that each of these $k-5$ neighbours is either a 1-vertex or a weak 2-vertex. Then, by (R2), (R3), and (R4), the final charge on v is $ch'(v) \geqslant ch(v) - \frac{4}{3}(3) - 2(k-5) = 0$.

- Suppose v has at least three 3^+-neighbours. Then, there are at most $k-3$ neighbours of v, such that each of the $k-3$ neighbours are either a 1-vertex, a weak 2-vertex, a semiweak 2-vertex or a strong 2-vertex. Then, by (R2), (R3), (R4) and (R5), the final charge of v is $ch'(v) \geqslant ch(v) - 2(k-3) = 0$.

So, $ch'(v) \geqslant 0$.

Now, we see that the total final charge of H is a positive value. The final charge of H is also equivalent to the initial charge of H. However, from Equation (3.1), the initial charge of H was $-2k = -12 < 0$. That is, :

$$0 \leqslant \sum_{x \in V(H) \cup F(H)} ch'(x) = \sum_{x \in V(H) \cup F(H)} ch(x) = -12 < 0,$$

giving a contradiction.

Therefore, the counterexample H does not exist.

Then, the theorem is true.

This concludes the proof.

5.2 Proof of Theorem 5 ($\chi'_s(G) \leqslant 3\Delta(G) + 1$ for Planar Graphs with $g \geqslant 6$)

We recall Theorem 5 [2] :

Theorem 5. *If G is a planar graph with maximum degree $\Delta(G)$ and girth $g \geqslant 6$, then $\chi'_s(G) \leqslant 3\Delta(G) + 1$.*

5.2.1 Preliminaries

We define the following :

- We define $N_2(uv)$ as the set of edges at distance at most 2 from the edge uv, excluding the edge uv itself.
- We denote $SC(N_2(uv))$ as the set of colours used by the edges in the set $N_2(uv)$, given an edge colouring of G.

- For $k \in \mathbb{N}$, a *k-vertex* is a vertex of degree k.
  A *k^+-vertex* is a vertex of degree at least k.
  A *k^--vertex* is a vertex of degree at most k.

- A *k_l-vertex* is a k-vertex adjacent to exactly l 2-vertices.

- A *bad 2-vertex* is a 2-vertex adjacent to another 2-vertex.

- We use $[\![n]\!]$ to indicate the set of integers $\{1, 2, \ldots, n\}$.

We also make use of the following theorem given by Hocquard *et al* [10] in the proof.

Theorem 16. *If G is a planar graph with $\Delta(G) \leqslant 3$ containing neither induced 4-cycles, nor induced 5-cycles, then $\chi'_s(G) \leqslant 9$.*

5.2.2 Proof

Assume that the theorem is false. Then, there exists a planar graph(s) G with girth $g \geqslant 6$ such that, $\chi'_s(G) > 3\Delta(G) + 1$. That is, these graphs can be strongly edge coloured using 3 $\Delta(G)$ + 2 colours or more. Amongst these graphs, let H be the graph that minimizes $|E\,(H)\,| + |V\,(H)\,|$. Then, the graph H is a minimal counterexample of Theorem 5 and by the minimality of H, along with the condition on the girth, we can assume that H is connected.

Since H is a counterexample, we have that $\chi'_s(H) \geqslant 3\Delta(H) + 2$. However, if $\Delta\,(H) = 3$, then we have that $\chi'_s(H) \geqslant 11$, which is a clear contradiction of Theorem 16. Hence, we must have that $\Delta\,(H) \geqslant 4$.

Let $\Delta(H) = \Delta$ and let L be the set of colours $[\![3\Delta + 1]\!]$. Let ϕ be a strong $3\Delta + 1$-edge colouring.

The following lemmas indicate the properties that must be satisfied in the counterexample H.

Lemma 17. *H does not contain a 1-vertex adjacent to a 4^--vertex.*

Proof. Suppose H contains a 1-vertex u adjacent to a 4^--vertex v. Without loss of generality, we may assume that $d(v) = 4$, since the other possibilities would simply require fewer colours. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using $3\Delta + 1$ colours or less.

We have that $|N_2(uv)|$ is at most 3Δ. Hence, $|L \backslash SC_\phi(N_2(uv))| \geqslant 3\Delta + 1 - 3\Delta = 1$. Thus, there is a colour available in L to colour uv in H, so we may extend the colouring ϕ to the graph H. This means that H can be strongly edge coloured using at most $3\ \Delta + 1$ colours, which is a contradiction.

Hence, H does not contain a 1-vertex adjacent to a 4^--vertex. $\square$

Lemma 18. *H does not contain a 2-vertex adjacent to two 3^--vertices.*

Proof. Suppose H contains a 2-vertex u adjacent to two 3^--vertices v and w. Without loss of generality, we may assume that $d(v) = d(w) = 3$. Let H' = H - $\{uv\}$. Then, we can colour the graph H' using $3\Delta + 1$ colours or less.

We have that $|N_2(uv)|$ is at most $2\Delta + 3$ and so $|L \backslash SC_\phi(N_2(uv))| \geqslant 3\Delta + 1 - 2\Delta - 3 = \Delta - 2 \geqslant 2$. Therefore, we may extend the colouring ϕ to H by colouring uv with a colour from L. This means that H can be strongly edge coloured using at most $3\ \Delta + 1$ colours, which gives a contradiction.

Hence, H does not contain a 2-vertex adjacent to two 3^--vertices. $\square$

Lemma 19. *H does not contain a 2-vertex adjacent to a 3^--vertex and either a 4_2-vertex or a 4_3-vertex.*

Proof. Suppose H contains a 2-vertex u adjacent to a 3^--vertex v and a 4-vertex w. Without loss of generality, we may assume that $d(v) = 3$. It suffices to consider only the case when w is a 4_2-vertex, since the remaining case would require fewer colours. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using $3\Delta + 1$ colours or less.

We have that $|N_2(uv)|$ is at most $2\Delta + 4$ and so $|L \backslash SC_\phi(N_2(uv))| \geqslant 3\Delta + 1 - 2\Delta - 4 = \Delta - 3 \geqslant 1$. Therefore, we may extend the colouring ϕ to H by colouring uv with a colour from L. This means that H can be strongly edge coloured using at most $3\ \Delta + 1$ colours, giving a contradiction.

Hence, H does not contain a 2-vertex adjacent to a 3^--vertex and either a 4_2-vertex or a 4_3-vertex. $\square$

Lemma 20. *H does not contain a 2-vertex adjacent to a 4_3-vertex and to a 4_2-vertex.*

Proof. Suppose H contains a 2-vertex u adjacent to a 4_3-vertex v and a 4_2-vertex w. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using $3\Delta + 1$ colours or less.

We have that $|N_2(uv)|$ is at most Δ + 8 and so $|L \backslash SC_\phi(N_2(uv))| \geqslant 3\Delta + 1 - \Delta - 8 = 2\Delta - 7 \geqslant 1$. Thus, we may extend the colouring ϕ to H by colouring uv with a colour from L. This means that H can be strongly edge coloured using at most 3 Δ + 1 colours, giving a contradiction.

Hence, H does not contain a 2-vertex adjacent to a 4_3-vertex and to a 4_2-vertex. □

Lemma 21. *H does not contain a 2-vertex adjacent to two 4_3-vertices.*

Proof. Suppose H contains a 2-vertex u adjacent to two 4_3-vertices v and w. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using $3\Delta + 1$ colours or less.

We have that $|N_2(uv)|$ is at most Δ + 8 and so $|L \backslash SC_\phi(N_2(uv))| \geqslant 3\Delta + 1 - \Delta - 8 = 2\Delta - 7 \geqslant 1$. Thus, we may extend the colouring ϕ to H by colouring uv with a colour from L. This means that H can be strongly edge coloured using at most 3 Δ + 1 colours, giving a contradiction.

Hence, H does not contain a 2-vertex adjacent to two 4_3-vertices. □

Lemma 22. *If $k \geqslant 4$, then H does not contain a k-vertex adjacent to $k-2$ 1-vertices. Furthermore, if the k-vertex is adjacent to $k-3$ 1-vertices, then it has no other 2^--neighbour.*

Proof. Let $4 \leqslant k \leqslant \Delta$. Suppose H contains a k-vertex u adjacent to $k-2$ 1-vertices $u_1, u_2, \ldots, u_{k-2}$. Let $H' = H$ - $\{uu_1\}$. Then, we can colour the graph H' using $3\Delta + 1$ colours or less.

We have that $|N_2(uu_1)|$ is at most 2Δ + $k - 3$ and so $|L \backslash SC_\phi(N_2(uu_1))| \geqslant 3\Delta + 1 - 2\Delta - k + 3 = \Delta - k + 4$. Since $k \leqslant \Delta$, we have that $|L \backslash SC_\phi(N_2(uu_1))| \geqslant 4$. Thus, we have a colour available in L for uu_1 in H. So, we may extend the colouring ϕ to H. This means that H can be strongly edge coloured using at most 3 Δ + 1 colours, giving a contradiction.

Hence, for $k \geqslant 4$, H does not contain a k-vertex adjacent to $k-2$ 1-vertices.

Now, suppose u is instead adjacent to $k-3$ 1-vertices $u_1, u_2, \ldots, u_{k-3}$ and a 2^--vertex x. If $d(x) = 1$, then the proof would be as above. Therefore, we assume $d(x) = 2$. Let $H' = H$ - $\{uu_1\}$. Then, we can colour the graph H' using $3\Delta + 1$ colours or less.

We have that $|N_2(uu_1)|$ is at most $2\Delta + k - 2$ and so $|L \backslash SC_\phi(N_2(uu_1))| \geqslant 3\Delta + 1 - 2\Delta - k + 2 = \Delta - k + 3$. Since, $k \leqslant \Delta$, we have that $|L \backslash SC_\phi(N_2(uu_1))| \geqslant 3$. Therefore, we have a colour available in L for uu_1 in H. Thus, we may extend the colouring ϕ to H. This means that H can be strongly edge coloured using at most 3 Δ + 1 colours, giving a contradiction.

Hence, for $k \geqslant 4$, if a k-vertex is adjacent to $k-3$ 1-vertices, then it has no other 2^--neighbour. □

Lemma 23. *If $k \geqslant 4$, then H does not contain a k-vertex adjacent to k 2^--vertices.*

Proof. Let $4 \leqslant k \leqslant \Delta$. Suppose H contains a k-vertex u adjacent to k 2^--vertices $u_1, u_2, \ldots, u_k$. Without loss of generality, we may assume that, for each $i \in [\![k]\!]$, the vertex u_i is a 2-vertex. Let $H' = H$ - $\{uu_1\}$. Then, we can colour the graph H' using $3\Delta + 1$ colours or less.

We have that $|N_2(uu_1)|$ is at most Δ + $2k - 2$ and so $|L \backslash SC_\phi(N_2(uu_1))| \geqslant 3\Delta + 1 - \Delta - 2k + 2 = 2\Delta - 2k + 3$. Since $k \leqslant \Delta$, we have that $|L \backslash SC_\phi(N_2(uu_1))| \geqslant 3$. This leaves a colour available in L for uu_1 in H. Thus, we may extend the colouring ϕ to H. This means that H can be strongly edge coloured using at most 3 Δ + 1 colours, giving a contradiction.

Therefore, for $k \geqslant 4$, H does not contain a k-vertex adjacent to k 2^--vertices. □

Lemma 24. *If $k \geqslant 5$, then H does not contain a k-vertex u with $N(u) = \{u_1, u_2, \ldots, u_{k-1}, x\}$ such that, each u_i, with $i \in [\![k-1]\!]$, is a 2^--vertex and u_1 is either a 1-vertex or a 2-vertex adjacent to either a 3^--vertex or a 4_3-vertex.*

Proof. Let $5 \leqslant k \leqslant \Delta$. Suppose H contains a k-vertex u adjacent to $k-1$ 2^--vertices $u_1, u_2, \ldots, u_{k-1}$. Let $H' = H$ - $\{uu_1\}$. Then, we can colour the graph H' using $3\Delta + 1$ colours or less.

We will consider two cases for $d(u_1)$. In both cases, without loss of generality, we may assume that the degree of the remaining $k-2$ 2^{-}-vertices $u_2, u_3, \ldots, u_{k-1}$ are of degree 2.

<u>**Case 1**</u>: Suppose u_1 is a 1-vertex.

We have that $|N_2(uu_1)|$ is at most $\Delta + 2k - 4$ and so $|L \backslash SC_\phi(N_2(uu_1))| \geqslant 3\Delta + 1 - \Delta - 2k + 4 = 2\Delta - 2k + 5$. Since $k \leqslant \Delta$, we have that $|L \backslash SC_\phi(N_2(uu_1))| \geqslant 5$. This leaves a colour available in L for uu_1 in H. Thus, we may extend the colouring ϕ to H. This means that H can be strongly edge coloured using at most 3 Δ + 1 colours, giving a contradiction.

<u>**Case 2**</u>: Suppose u_1 is a 2-vertex.

- Suppose u_1 is adjacent to u and to a 3^{-}-vertex v. Without loss of generality, we can assume $d(v) = 3$. In this case, we have that $|N_2(uu_1)|$ is at most $\Delta + 2k - 1$ and so $|L \backslash SC_\phi(N_2(uu_1))| \geqslant 3\Delta + 1 - \Delta - 2k + 1 = 2\Delta - 2k + 2$. Since $k \leqslant \Delta$, we have that $|L \backslash SC_\phi(N_2(uu_1))| \geqslant 2$. This leaves a colour available in L for uu_1 in H. Thus, we may extend the colouring ϕ to H. This means that H can be strongly edge coloured using at most 3 Δ + 1 colours, giving a contradiction.

- Suppose u_1 is adjacent to u and to a 4_3-vertex v. Here, $|N_2(uu_1)|$ is at most $\Delta + 2k$ and so $|L \backslash SC_\phi(N_2(uu_1))| \geqslant 3\Delta + 1 - \Delta - 2k = 2\Delta - 2k + 1$. Since $k \leqslant \Delta$, we have that $|L \backslash SC_\phi(N_2(uu_1))| \geqslant 1$. This leaves a colour available in L for uu_1 in H. Thus, we may extend the colouring ϕ to H. This means that H can be strongly edge coloured using at most 3 Δ + 1 colours, giving a contradiction.

Therefore, if $k \geqslant 5$, then H does not contain a k-vertex u with $N(u) = \{u_1, u_2, \ldots, u_{k-1}, x\}$ such that, each u_i, with $i \in [\![k-1]\!]$, is a 2^{-}-vertex and u_1 is either a 1-vertex or a 2-vertex adjacent to either a 3^{-}-vertex or a 4_3-vertex. □

Lemma 25. *If $k \geqslant 5$, then H does not contain a k-vertex adjacent to $k-2$ 2-vertices, $u_1, \ldots u_{k-2}$, such that, for $i \in [\![k-3]\!]$, each u_i is adjacent to either a 3^{-}-vertex or a 4_3-vertex.*

Proof. Let $5 \leqslant k \leqslant \Delta$. Suppose H contains a k-vertex u and $N(u) = \{u_1, u_2, \ldots, u_k\}$. Suppose further that u is adjacent to exactly $k-2$ 2-vertices, $u_1, u_2, \ldots, u_{k-2}$. By Lemma 23 and Lemma 24, the vertices u_{k-1} and u_k must both be 3^+-vertices. Without loss of generality, we can assume that $d(u_{k-1}) = d(u_k) = \Delta$. Let the neighbour of $u_1, u_2, \ldots, u_{k-2}$, distinct from u, be $v_1, v_2, \ldots, v_{k-2}$ respectively. Then, by Lemma 17, we have that $d(v_i) \geqslant 2$, for $i \in [\![k-2]\!]$. Suppose further that each v_j, for $j \in [\![k-3]\!]$, is either a 3^--vertex or a 4_3-vertex.

First, suppose that at least one v_j is a 3^--vertex. Without loss of generality, we may assume that exactly one v_j is a 3-vertex, say v_{k-3} and $d(v_{k-2}) = \Delta$. Then, the remaining $k-4$ vertices $v_1, v_2, \ldots, v_{k-4}$ are all 4_3-vertices. Let $H' = H$ - $\{u_1, u_2, \ldots, u_{k-3}\}$. Then, we can colour the graph H' using $3\Delta + 1$ colours or less.

First, we will colour the edges uu_l, for $l \in [\![k-4]\!]$. Consider the 2-neighbourhood of the edge uu_l in H. We have that $|N_2(uu_l)|$ in H is at most 2Δ + $2k$ - 2. However, if we omit the uncoloured edges $uu_{k-3}, u_1v_1, u_2v_2, ..., u_{k-4}v_{k-4}, u_{k-3}v_{k-3}$ from this set, then we have $|N_2(uu_l)|$ in H is at most 2Δ + k and so $|L \backslash SC_\phi(N_2(uu_l))| \geqslant 3\Delta + 1 - 2\Delta - k = \Delta - k + 1$. Since $k \leqslant \Delta$, we have that $|L \backslash SC_\phi(N_2(uu_l))| \geqslant 1$. Thus, there is a colour available for uu_l in H, so we may colour each uu_l.

Now, consider the 2-neighbourhood of the edge uu_{k-3} in H. Then, $|N_2(uu_{k-3})|$ in H is at most 2Δ + $2k$ - 3. However, if we omit the uncoloured edges $u_1v_1, u_2v_2, ..., u_{k-4}v_{k-4}, u_{k-3}v_{k-3}$ from this set, then we have $|N_2(uu_{k-3})|$ in H is at most 2Δ + k and so $|L \backslash SC_\phi(N_2(uu_{k-3}))| \geqslant 3\Delta + 1 - 2\Delta - k = \Delta - k + 1$. Since $k \leqslant \Delta$, we have that $|L \backslash SC_\phi(N_2(uu_{k-3}))| \geqslant 1$. Thus, there is a colour available for uu_{k-3} in H, so we may now colour uu_{k-3}.

Now, we proceed to colour the edge $u_{k-3}v_{k-3}$. Consider the 2-neighbourhood of the edge $u_{k-3}v_{k-3}$ in H. Then, $|N_2(u_{k-3}v_{k-3})|$ in H is at most 2Δ + k and so $|L \backslash SC_\phi(N_2(u_{k-3}v_{k-3}))| \geqslant 3\Delta + 1 - 2\Delta - k = \Delta - k + 1$. Since $k \leqslant \Delta$, we have that $|L \backslash SC_\phi(N_2(u_{k-3}v_{k-3}))| \geqslant 1$. Thus, there is a colour available for $u_{k-3}v_{k-3}$ in H, so we may now colour $u_{k-3}v_{k-3}$.

Now, consider the 2-neighbourhood of the edges u_lv_l, with $l \in [\![k-4]\!]$ in H. Then, $|N_2(u_lv_l)|$ in H is at most Δ + k + 4 and so

$|L \backslash SC_\phi(N_2(u_l v_l))| \geqslant 3\Delta + 1 - \Delta - k - 4 = 2\Delta - k - 3$. Since $k \leqslant \Delta$, we have that $|L \backslash SC_\phi(N_2(u_l v_l))| \geqslant \Delta - 3 \geqslant 2$. Thus, there is a colour available for $u_l v_l$ in H, so we may now colour $u_l v_l$. Furthermore, we see that no v_l, with $l \in [\![k-4]\!]$, can be adjacent to v_m, with $m \in [\![k-4]\!]; m \neq l$. This means that we can colour each $u_l v_l$ with the same colour. Thus, we can extend the colouring ϕ to H, giving a contradiction.

Now, suppose that each v_j, for $j \in [\![k-3]\!]$, is a 4_3-vertex. Let w be a 2-vertex adjacent to v_{k-3} and distinct from u_{k-3}. Let $H' = H$ - $\{u_1, u_2, \ldots, u_{k-3}\}$. Then, we can colour the graph H' using $3\Delta + 1$ colours or less.

We uncolour the edge $v_{k-3}w$. First, we will colour the edges uu_l, for $l \in [\![k-4]\!]$. Consider the 2-neighbourhood of the edge uu_l in H. We have that $|N_2(uu_l)|$ in H is at most 2Δ + $2k$ - 2. However, if we omit the uncoloured edges $uu_{k-3}, u_1v_1, u_2v_2, ..., u_{k-4}v_{k-4}, u_{k-3}v_{k-3}$ from this set, then we have $|N_2(uu_l)|$ in H is at most 2Δ + k and so $|L \backslash SC_\phi(N_2(uu_l))| \geqslant 3\Delta + 1 - 2\Delta - k = \Delta - k + 1$. Since $k \leqslant \Delta$, we have that $|L \backslash SC_\phi(N_2(uu_l))| \geqslant 1$. Thus, there is a colour available for uu_l in H, so we may colour each uu_l.

Now, consider the 2-neighbourhood of the edge uu_{k-3} in H. Then, $|N_2(uu_{k-3})|$ in H is at most 2Δ + $2k$ - 2. However, if we omit the uncoloured edges $u_1v_1, u_2v_2, ..., u_{k-4}v_{k-4}, u_{k-3}v_{k-3}, v_{k-3}w$ from this set, then we have $|N_2(uu_{k-3})|$ in H is at most 2Δ + k and so $|L \backslash SC_\phi(N_2(uu_{k-3}))| \geqslant 3\Delta + 1 - 2\Delta - k = \Delta - k + 1$. Since $k \leqslant \Delta$, we have that $|L \backslash SC_\phi(N_2(uu_{k-3}))| \geqslant 1$. Thus, there is a colour available for uu_{k-3} in H, so we may now colour uu_{k-3}.

Now, we colour $v_{k-3}w$. Consider the 2-neighbourhood of the edge $v_{k-3}w$ in H. Then, $|N_2(v_{k-3}w)|$ in H is at most 2Δ + 4. However, if we omit the uncoloured edge $u_{k-3}v_{k-3}$ from this set, then we have $|N_2(v_{k-3}w)|$ in H is at most 2Δ + 3 and so $|L \backslash SC_\phi(N_2(v_{k-3}w))| \geqslant 3\Delta + 1 - 2\Delta - 3 = \Delta - 2 \geqslant 3$. Thus, there is a colour available for $v_{k-3}w$ in H, so we may now colour $v_{k-3}w$.

Now, we proceed to colour the edge $u_{k-3}v_{k-3}$. Consider the 2-neighbourhood of the edge $u_{k-3}v_{k-3}$ in H. Then, $|N_2(u_{k-3}v_{k-3})|$ in H is at most Δ + k + 4 and so $|L \backslash SC_\phi(N_2(u_{k-3}v_{k-3}))| \geqslant 3\Delta + 1 - \Delta - k - 4 = 2\Delta - k - 3$. Since $k \leqslant \Delta$, we have that $|L \backslash SC_\phi(N_2(u_{k-3}v_{k-3}))| \geqslant \Delta - 3 \geqslant 2$. Thus, there is a colour available for $u_{k-3}v_{k-3}$ in H, so we may now colour $u_{k-3}v_{k-3}$.

Now, consider the 2-neighbourhood of the edges $u_l v_l$, with $l \in [\![k-4]\!]$, in H. Then, $|N_2(u_l v_l)|$ in H is at most $\Delta + k + 4$ and so $|L \backslash SC_\phi(N_2(u_l v_l))| \geqslant 3\Delta + 1 - \Delta - k - 4 = 2\Delta - k - 3$. Since $k \leqslant \Delta$, we have that $|L \backslash SC_\phi(N_2(u_l v_l))| \geqslant \Delta - 3 \geqslant 2$. Thus, there is a colour available for $u_l v_l$ in H, so we may now colour $u_l v_l$. Furthermore, we see that no v_l, with $l \in [\![k-4]\!]$, can be adjacent to v_m, with $m \in [\![k-4]\!]; m \neq l$. This means that we can colour each $u_l v_l$ with the same colour. Thus, we can extend the colouring ϕ to H, giving a contradiction.

Therefore, if k $\geqslant 5$, then H does not contain a k-vertex adjacent to $k-2$ 2-vertices, $u_1, \ldots u_{k-2}$, such that, for $i \in [\![k-3]\!]$, each u_i is adjacent to either a 3^--vertex or a 4_3-vertex. □

Lemma 26. *If $k \geqslant 5$ and $1 \leqslant \alpha \leqslant k-4$, then H does not contain a k-vertex adjacent to α 1-vertices and to $k-2-\alpha$ 2-vertices, $u_1, \ldots u_{k-2-\alpha}$, such that, for $i \in [\![k-3-\alpha]\!]$, each u_i is adjacent to either a 3^--vertex or a 4_3-vertex.*

Proof. Let $5 \leqslant k \leqslant \Delta$ and $1 \leqslant \alpha \leqslant k-4$. Suppose H contains a k-vertex u adjacent to α 1-vertices and $k-2-\alpha$ vertices $u_1, u_2, \ldots, u_{k-2-\alpha}$ of degree 2. Let the two remaining neighbours of u be u_{k-1} and u_k. By Lemma 23, at least one of the two remaining neighbours of u is not a 2^--vertex. Suppose u has exactly one 3^+-neighbour, say u_{k-1}. Then, by Lemma 24, the remaining neighbour u_k of u cannot be a 1-vertex or a 2-vertex adjacent to either a 3^--vertex or a 4_3-vertex. Without loss of generality, we can assume that $d(u_{k-1}) = d(u_k) = \Delta$. Let the neighbour of $u_1, u_2, \ldots, u_{k-2-\alpha}$, distinct from u, be $v_1, v_2, \ldots, v_{k-2-\alpha}$ respectively. Then, by Lemma 17, we have that $d(v_i) \geqslant 2$, for $i \in [\![k-2-\alpha]\!]$. Suppose further that each v_j, for $j \in [\![k-3-\alpha]\!]$, is either a 3^--vertex or a 4_3-vertex.

First, suppose that at least one v_j is a 3^--vertex. Without loss of generality, we may assume that exactly one v_j is a 3-vertex, say $v_{k-3-\alpha}$ and $d(v_{k-2-\alpha}) = \Delta$. Then, the remaining $k-4-\alpha$ vertices $v_1, v_2, \ldots, v_{k-4-\alpha}$ are all 4_3-vertices. Let $H' = H$ - $\{u_1, u_2, \ldots, u_{k-3-\alpha}\}$. Then, we can colour the graph H' using $3\Delta + 1$ colours or less.

First, we will colour the edges uu_l, for $l \in [\![k-4-\alpha]\!]$. Consider the 2-neighbourhood of the edge uu_l in H. We have that $|N_2(uu_l)|$ in H is at most 2Δ + $2k$ - α - 2. However, if we omit the uncoloured edges $uu_{k-3-\alpha}, u_1v_1, u_2v_2, ..., u_{k-4-\alpha}v_{k-4-\alpha}, u_{k-3-\alpha}v_{k-3-\alpha}$ from this set, then we have $|N_2(uu_l)|$ in H is at most 2Δ + k and so

$|L \backslash SC_\phi(N_2(uu_l))| \geqslant 3\Delta + 1 - 2\Delta - k = \Delta - k + 1$. Since $k \leqslant \Delta$, we have that $|L \backslash SC_\phi(N_2(uu_l))| \geqslant 1$. Thus, there is a colour available for uu_l in H, so we may colour each uu_l.

Now, consider the 2-neighbourhood of the edge $uu_{k-3-\alpha}$ in H. Then, $|N_2(uu_{k-3-\alpha})|$ in H is at most 2Δ + $2k$ - α - 3. However, if we omit the uncoloured edges $u_1v_1, u_2v_2, ..., u_{k-4-\alpha}v_{k-4-\alpha}, u_{k-3-\alpha}v_{k-3-\alpha}$ from this set, then we have $|N_2(uu_{k-3-\alpha})|$ in H is at most 2Δ + k and so $|L \backslash SC_\phi(N_2(uu_{k-3-\alpha}))| \geqslant 3\Delta + 1 - 2\Delta - k = \Delta - k + 1$. Since $k \leqslant \Delta$, we have that $|L \backslash SC_\phi(N_2(uu_{k-3-\alpha}))| \geqslant 1$. Thus, there is a colour available for $uu_{k-3-\alpha}$ in H, so we may now colour $uu_{k-3-\alpha}$.

Now, we proceed to colour the edge $u_{k-3-\alpha}v_{k-3-\alpha}$. Consider the 2-neighbourhood of the edge $u_{k-3-\alpha}v_{k-3-\alpha}$ in H. Then, $|N_2(u_{k-3-\alpha}v_{k-3-\alpha})|$ in H is at most 2Δ + k and so $|L \backslash SC_\phi(N_2(u_{k-3-\alpha}v_{k-3-\alpha}))| \geqslant 3\Delta + 1 - 2\Delta - k = \Delta - k + 1$. Since $k \leqslant \Delta$, we have that $|L \backslash SC_\phi(N_2(u_{k-3-\alpha}v_{k-3-\alpha}))| \geqslant 1$. Thus, there is a colour available for $u_{k-3-\alpha}v_{k-3-\alpha}$ in H, so we may now colour $u_{k-3-\alpha}v_{k-3-\alpha}$.

Now, consider the 2-neighbourhood of the edges u_lv_l, with $l \in [\![k-4-\alpha]\!]$ in H. Then, $|N_2(u_lv_l)|$ in H is at most Δ + k + 4 and so $|L \backslash SC_\phi(N_2(u_lv_l))| \geqslant 3\Delta + 1 - \Delta - k - 4 = 2\Delta - k - 3$. Since $k \leqslant \Delta$, we have that $|L \backslash SC_\phi(N_2(u_lv_l))| \geqslant \Delta - 3 \geqslant 2$. Thus, there is a colour available for u_lv_l in H, so we may now colour u_lv_l. Furthermore, we see that no v_l, with $l \in [\![k-4-\alpha]\!]$, can be adjacent to v_m, with $m \in [\![k-4-\alpha]\!]; m \neq l$. This means that we can colour each u_lv_l with the same colour. Thus, we can extend the colouring ϕ to H, giving a contradiction.

Now, suppose that each v_j, for $j \in [\![k-3-\alpha]\!]$, is a 4_3-vertex. Let w be a 2-vertex adjacent to $v_{k-3-\alpha}$ and distinct from $u_{k-3-\alpha}$. Let $H' = H$ - $\{u_1, u_2, \ldots, u_{k-3-\alpha}\}$. Then, we can colour the graph H' using $3\Delta + 1$ colours or less. We uncolour the edge $v_{k-3-\alpha}w$. First, we will colour the edges uu_l, for $l \in [\![k-4-\alpha]\!]$. Consider the 2-neighbourhood of the edge uu_l in H. We have that $|N_2(uu_l)|$ in H is at most 2Δ + $2k$ - α - 2. However, if we omit the uncoloured edges $uu_{k-3-\alpha}, u_1v_1, u_2v_2, ..., u_{k-4-\alpha}v_{k-4-\alpha}, u_{k-3-\alpha}v_{k-3-\alpha}$ from this set, then we have $|N_2(uu_l)|$ in H is at most 2Δ + k and so $|L \backslash SC_\phi(N_2(uu_l))| \geqslant 3\Delta + 1 - 2\Delta - k = \Delta - k + 1$. Since $k \leqslant \Delta$, we have that $|L \backslash SC_\phi(N_2(uu_l))| \geqslant 1$. Thus, there is a colour available for uu_l in H, so we may colour each uu_l.

Now, consider the 2-neighbourhood of the edge $uu_{k-3-\alpha}$ in H. Then,

$|N_2(uu_{k-3-\alpha})|$ in H is at most 2Δ + $2k$ - α - 2. However, if we omit the uncoloured edges $u_1v_1, u_2v_2, ..., u_{k-4-\alpha}v_{k-4-\alpha}, u_{k-3-\alpha}v_{k-3-\alpha}, v_{k-3-\alpha}w$ from this set, then we have $|N_2(uu_{k-3-\alpha})|$ in H is at most 2Δ + k and so $|L\backslash SC_\phi(N_2(uu_{k-3-\alpha}))| \geqslant 3\Delta + 1 - 2\Delta - k = \Delta - k + 1$. Since $k \leqslant \Delta$, we have that $|L\backslash SC_\phi(N_2(uu_{k-3-\alpha}))| \geqslant 1$. Thus, there is a colour available for $uu_{k-3-\alpha}$ in H, so we may now colour $uu_{k-3-\alpha}$.

Now, we colour $v_{k-3-\alpha}w$. Consider the 2-neighbourhood of the edge $v_{k-3-\alpha}w$ in H. Then, $|N_2(v_{k-3-\alpha}w)|$ in H is at most 2Δ + 4. However, if we omit the uncoloured edge $u_{k-3-\alpha}v_{k-3-\alpha}$ from this set, then we have $|N_2(v_{k-3-\alpha}w)|$ in H is at most 2Δ + 3 and so $|L\backslash SC_\phi(N_2(v_{k-3-\alpha}w))| \geqslant 3\Delta + 1 - 2\Delta - 3 = \Delta - 2 \geqslant 3$. Thus, there is a colour available for $v_{k-3-\alpha}w$ in H, so we may now colour $v_{k-3-\alpha}w$.

Now, we proceed to colour the edge $u_{k-3-\alpha}v_{k-3-\alpha}$. Consider the 2-neighbourhood of the edge $u_{k-3-\alpha}v_{k-3-\alpha}$ in H. Then, $|N_2(u_{k-3-\alpha}v_{k-3-\alpha})|$ in H is at most $\Delta + k + 4$ and so $|L\backslash SC_\phi(N_2(u_{k-3-\alpha}v_{k-3-\alpha}))| \geqslant 3\Delta + 1 - \Delta - k - 4 = 2\Delta - k - 3$. Since $k \leqslant \Delta$, we have that $|L\backslash SC_\phi(N_2(u_{k-3-\alpha}v_{k-3-\alpha}))| \geqslant \Delta - 3 \geqslant 2$. Thus, there is a colour available for $u_{k-3-\alpha}v_{k-3-\alpha}$ in H, so we may now colour $u_{k-3-\alpha}v_{k-3-\alpha}$.

Now, consider the 2-neighbourhood of the edges u_lv_l, with $l \in [\![k-4-\alpha]\!]$, in H. Then, $|N_2(u_lv_l)|$ in H is at most $\Delta + k + 4$ and so $|L\backslash SC_\phi(N_2(u_lv_l))| \geqslant 3\Delta + 1 - \Delta - k - 4 = 2\Delta - k - 3$. Since $k \leqslant \Delta$, we have that $|L\backslash SC_\phi(N_2(u_lv_l))| \geqslant \Delta - 3 \geqslant 2$. Thus, there is a colour available for u_lv_l in H, so we may now colour u_lv_l. Furthermore, we see that no v_l, with $l \in [\![k-4-\alpha]\!]$, can be adjacent to v_m, with $m \in [\![k-4-\alpha]\!]; m \neq l$. This means that we can colour each u_lv_l with the same colour. Thus, we can extend the colouring ϕ to H, giving a contradiction.

Therefore, if k $\geqslant$ 5 and $1 \leqslant \alpha \leqslant k-4$, then H does not contain a k-vertex adjacent to α 1-vertices and to $k-2-\alpha$ 2-vertices, $u_1, \ldots u_{k-2-\alpha}$, such that, for $i \in [\![k-3-\alpha]\!]$, each u_i is adjacent to either a 3^--vertex or a 4_3-vertex. □

Now that we have established the properties that H satisfies, we may proceed to the next part of the proof - performing a discharging procedure on the graph H.

Discharging procedure We use the discharging method as in Section 3.3.3 of this dissertation. We use a multiple of $k = 6$ for Equation (3.1) in Section 3.3.3.
We define the discharging rules as follows:

(R1): Every face gives 2 to each incident 1-vertex.

(R2): Every k-vertex, for $k \geqslant 5$, gives 2 to each adjacent 1-vertex.

(R3): Every 4_3-vertex gives $\frac{2}{3}$ to each adjacent 2-vertex.

(R4): Every 4_2-vertex gives 1 to each adjacent 2-vertex.

(R5): Every 4_1-vertex gives 2 to the adjacent 2-vertex.

(R6): Every k-vertex, for $k \geqslant 5$, gives:

- **(R6.1.):** 2 to each adjacent 2-vertex, if this 2-vertex is adjacent to a 3^{-}-vertex.
- **(R6.2.):** $\frac{4}{3}$ to each adjacent 2-vertex, if this 2-vertex is adjacent to a 4_3 -vertex.
- **(R6.3.):** 1 to each adjacent 2-vertex, if this 2-vertex is adjacent to a 4^{+}-vertex distinct from a 4_3-vertex.

Now, we will carry out these discharging rules on the graph H.

First, consider a k-face $f \in F(H)$. We have that $ch(f) = k - 6$. Since it is required for H that the girth $g \geqslant 6$, we know that $k \geqslant 6$.

If f has α incident 1-vertices, then $k \geqslant 6 + 2\alpha$. By (R1), we have that $ch'(f) = ch(f) - 2\alpha = k - 6 - 2\alpha \geqslant 0$.

So, $ch'(f) \geqslant 0$.

Now, consider a k-vertex v in H.

<u>Case $k = 1$</u> : We have that $ch(v) = -4$.

By (R1), v will receive 2 from its incident face. From Lemma 17, v is adjacent to a 5^{+}-vertex and thus by (R2), v receives 2 from its adjacent 5^{+}-vertex.

So, $ch'(v) = ch(v) + 2 + 2 = 0$.

<u>Case $k = 2$</u> : We have that $ch(v) = -2$.

Let v be adjacent to u and w. From Lemma 17, we see that u and w must be 2^+-vertices.

- Suppose u is a 3^--vertex. Then, by Lemma 18, w is a 4^+-vertex. If $d(w) = 4$, then, by Lemma 19 and Lemma 23, w must be a 4_1-vertex and so, by (R5), $ch'(v) = ch(v) + 2 = -2 + 2 = 0$. If $d(w) \geqslant 5$, then, by (R6.1.), $ch'(v) = ch(v) + 2 = -2 + 2 = 0$.

- Suppose $d(u) = d(w) = 4$. Then, the vertices u and w cannot be 4_4-vertices, by Lemma 23.

  - Suppose u is a 4_1-vertex. Then, w is either a 4_1-vertex, 4_2-vertex or a 4_3-vertex. So, by (R3), (R4) and (R5), we have $ch'(v) = ch(v) + 2 + min\{2, 1, \frac{2}{3}\} = -2 + 2 + \frac{2}{3} > 0$.
  - Suppose u is a 4_2-vertex. Then, w must be either a 4_1-vertex or a 4_2-vertex, by Lemma 20. So, by (R4) and (R5), $ch'(v) = ch(v) + 1 + min\{2, 1\} = -2 + 1 + 1 \geqslant 0$.
  - Suppose u is a 4_3-vertex. Then, w is a 4_1-vertex, by Lemma 20 and Lemma 21. So, by (R3) and (R5), $ch'(v) = ch(v) + \frac{2}{3} + 2 = -2 + \frac{2}{3} + 2 > 0$.

- Suppose $d(u) = 4$ and w is a 5^+-vertex. This is equivalent to the case of $d(w) = 4$ and u being a 5^+-vertex.

  - Suppose u is a 4_1-vertex. Then, by (R5) and R(6.3.), $ch'(v) = ch(v) + 2 + 1 = -2 + 2 + 1 > 0$.
  - Suppose u is a 4_2-vertex. Then, by (R4) and (R6.3.), $ch'(v) = ch(v) + 1 + 1 = -2 + 1 + 1 = 0$.
  - Suppose u is a 4_3-vertex. Then, by (R3) and (R6.2.), $ch'(v) = ch(v) + \frac{2}{3} + \frac{4}{3} = -2 + \frac{2}{3} + \frac{4}{3} = 0$.

- Suppose u and w are both 5^+-vertices. Then, by (R6.3.), $ch'(v) = ch(v) + 1 + 1 = -2 + 1 + 1 = 0$.

So, $ch'(v) \geqslant 0$.

Case $k = 3$: We have $ch(v) = 0$.

The charge on v remains unchanged.

So, $ch'(v) = ch(v) = 0$.

Case $k = 4$: We have $ch(v) = 2$.

From Lemma 17, v is not adjacent to a 1-vertex. From Lemma 23, v is adjacent to at most three 2-vertices.

- If v is a 4_1-vertex, then, by (R5), $ch'(v) = ch(v) - 2 = 2 - 2 = 0$.
- If v is a 4_2-vertex, then, by (R4), $ch'(v) = ch(v) - 1(2) = 2 - 2 = 0$.
- If v is a 4_3-vertex, then, by (R3), $ch'(v) = ch(v) - \frac{2}{3}(3) = 2 - 2 = 0$.

So, $ch'(v) = 0$.

Case $k \geqslant 5$: We have that $ch(v) = 2k - 6$.

- If v is not adjacent to a 1-vertex, then, v can be adjacent to at most $k - 1$ 2-vertices, by Lemma 23.
  - If v is adjacent to at most $k - 3$ 2-vertices, then, by (R6), $ch'(v) \geqslant ch(v) - 2(k - 3) = 2k - 6 - 2k + 6 = 0$.
  - If v is adjacent to exactly $k - 2$ 2-vertices, then, by Lemma 25, at most $k - 4$ 2-vertices can be adjacent to either a 3^--vertex or a 4_3-vertex. So, by (R6), $ch'(v) \geqslant ch(v) - 2(k - 4) - 1(2) = 2k - 6 - 2k + 8 - 2 = 0$.
  - If v is adjacent to exactly $k - 1$ 2-vertices, then, by Lemma 24, none of these 2-vertices can be adjacent to either a 3^--vertex or a 4_3-vertex. So, by (R6.3.), $ch'(v) = ch(v) - 1(k - 1) = 2k - 6 - k + 1 = k - 5 \geqslant 0$.
- If v is adjacent to α 1-vertices, with $\alpha \geqslant 1$, then, by Lemma 22, $\alpha \leqslant k - 3$.
  - If $\alpha = k - 3$, then, by Lemma 22, v has no other 2^--neighbour. So, by (R2), $ch'(v) = ch(v) - 2(\alpha) = 2k - 6 - 2(k - 3) = 0$.

- If $\alpha \leqslant k-4$ and the number of 2-vertices adjacent to v is at most $k-3-\alpha$, then, by (R2) and (R6), $ch'(v) \geqslant ch(v) - 2(\alpha) - 2(k-3-\alpha) = 2k-6-2\alpha-2k+6+2\alpha = 0$.
- If $\alpha \leqslant k-4$ and the number of 2-vertices adjacent to v is at least $k-2-\alpha$, then, the vertex v cannot be adjacent to $k-\alpha$ 2-vertices, by Lemma 23. Since $\alpha \geqslant 1$, v cannot be adjacent to $k-\alpha-1$ 2-vertices, by Lemma 24. Then, v has exactly $k-2-\alpha$ 2-neighbours. By Lemma 26, at most $k-4-\alpha$ 2-neighbours of v are adjacent to either a 3^--vertex or a 4_3-vertex. So, by (R2) and (R6), $ch'(v) \geqslant ch(v) - 2(\alpha) - 2(k-4-\alpha) - 1(2) = 2k-6-2\alpha-2k+8+2\alpha-2 = 0$.

So, $ch'(v) \geqslant 0$.

Now, we see that the sum of the final charges of the vertices and faces of H are positive. That is:

$$0 \leqslant \sum_{x \in V(H) \cup F(H)} ch'(x) = \sum_{x \in V(H) \cup F(H)} ch(x) = -12 < 0$$

Therefore, we have a contradiction.

Thus, the counterexample H does not exist and so the theorem is true.

This concludes the proof.

Chapter 6

Upper Bound for Subset of Planar Graphs

In this chapter, we prove that for planar graphs G with maximum degree $\Delta(G) \geqslant 7$, or for planar graphs G with maximum degree $\Delta(G) \geqslant 5$ and girth $g \geqslant 4$, or for planar graphs G with girth $g \geqslant 5$, we have that $\chi'_s(G) \leqslant 4\Delta(G)$.

6.1 Proof of Theorem 6 ($\chi'_s(G) \leqslant 4\Delta(G)$ for Subset of Planar Graphs)

We recall Theorem 6 [2] :

Theorem 6. *Let G be a planar graph with maximum degree $\Delta(G)$ and girth g. If G satisfies one of the following conditions, then $\chi'_s(G) \leqslant 4\Delta(G)$:*
(1) $\Delta(G) \geqslant 7$;
(2) $\Delta(G) \geqslant 5$ and $g \geqslant 4$;
(3) $g \geqslant 5$.

6.1.1 Preliminaries

We state the **Four Colour Theorem** below :
The vertices of every planar graph can be coloured with at most four colours so that no two adjacent vertices receive the same colour.

To complete the proof of Theorem 6, we will also make use of Theorem 2 [7]. We recall Theorem 2 as :

Theorem 2. *If G is a planar graph with maximum degree $\Delta(G)$, then* $\chi'_s(G) \leqslant 4\chi'(G) \leqslant 4\Delta(G) + 4$.

We also use the other theorems listed below.

Theorem 27. *[17] If G is a planar graph with maximum degree $\Delta(G) \geqslant 8$, then $\chi'(G) = \Delta(G)$.*

Theorem 28. *[15] If G is a planar graph with maximum degree $\Delta(G) = 7$, then $\chi'(G) = \Delta(G)$.*

Theorem 29. *[13] Let G be a planar graph with girth g and maximum degree $\Delta(G)$. Then, the following holds :*

- *if $g \geqslant 5$ and $\Delta(G) \geqslant 4$, then $\chi'(G) = \Delta(G)$;*
- *if $g \geqslant 4$ and $\Delta(G) \geqslant 5$, then $\chi'(G) = \Delta(G)$.*

Theorem 30. *[11] If G is a graph with maximum degree $\Delta(G) \leqslant 3$, then $\chi'_s(G) = 10$.*

Furthermore, we use $[\![n]\!]$ to indicate the set of integers $\{1, 2, \ldots, n\}$.

6.1.2 Proof

From the proof of Theorem 2, we can see that for every graph G, we can first perform a proper edge colouring of G and each colour class resulting from this will induce a matching M. Thereafter, we can create a new graph G_M for each matching M, with vertex set M, and having any two vertices in G_M adjacent if and only if the corresponding edges are incident to a common edge in G. Then, we proceed to colour the vertices of G_M for each matching M, and in doing so, we obtain a strong edge colouring of G. Therefore, we can see the following relationship :

$$\chi'_s(G) \leqslant \chi'(G)\chi(G_M) \tag{6.1}$$

where $\chi'(G)$ is the edge chromatic index of G and $\chi(G_M)$ is the chromatic number of G_M.

For planar graphs, we know that $\chi(G_M)$ is at most 4, by the Four Colour Theorem. Then, $\chi'_s(G) \leqslant 4\chi'(G)$.

Now, consider a graph G with girth $g \geqslant 5$ and $\Delta(G) \leqslant 3$. We divide this into three cases below.

Case 1 : $\Delta(G) = 1$

Here, the only possible graph is a graph G with $V(G) = \{u, v\}$ and $E(G) = \{uv\}$. This graph does not have a cycle and therefore does not satisfy the condition on the girth. Thus, we do not need to consider this case.

Case 2 : $\Delta(G) = 2$

Let $n \geqslant 5$ and let $V(G) = \{u_1, u_2, \ldots, u_{n-1}, u_n\}$. Then, G must be an n-cycle C_n.

For $n = 5, 6, 7, 8$, the graph G can be strongly edge coloured using 5, 6, 7, 8 colours respectively. Thus, $\chi'_s(G) \leqslant 8 = 4\Delta(G)$.

Now, consider $n \geqslant 9$.

Suppose $n = 3k$, for an integer $k \geqslant 3$. If we colour an edge e_i in G, for $i \in [\![k-1]\!]$ with colour α then, we can repeat the colour α for every edge e_{3i+1}. Then, we can colour the entire graph G using 3 colours.

Suppose $n = 3k + 1$. Then, we can colour the graph as above using 3 colours, and colour the remaining uncoloured edge with a 4th colour.

Suppose $n = 3k + 2$. Then, we can colour the graph as above using 3 colours, and colour the remaining two uncoloured edges using a 4th and 5th colour.

Then, for $n \geqslant 9$, the graph G can be strongly edge coloured using at most 5 colours.

Thus, $\chi'_s(G) \leqslant 5 < 8 = 4\Delta(G)$.

Case 3 : $\Delta(G) = 3$

From Theorem 30, we see that $\chi'_s(G) \leqslant 10 < 12 = 4\Delta(G)$.

Now, from above, along with Theorem 27, Theorem 28 and Theorem 29, we obtain Theorem 6.

This concludes the proof.

Chapter 7

Upper Bound for Planar Graphs with Maximum Degree $\Delta(G) \geqslant 5$ and Girth $g \geqslant 5$

In this chapter, we show that for a planar graph G with maximum degree $\Delta(G) \geqslant 6$ and girth g $\geqslant 5$, we have that $\chi'_s(G) \leqslant 4\Delta(G) - 2$. Thereafter, we show that for a subset of these planar graphs, with maximum degree $\Delta(G) = 5$ and girth $g \geqslant 5$, we can obtain an improved upper bound on the strong chromatic index. In particular, we have that $\chi'_s(G) \leqslant 19$. Finally, we then show for a subset of these planar graphs, with maximum degree $\Delta(G) = 5$ and girth $g = 5$, we have that $\chi'_s(G) \leqslant 3\Delta(G) + 1 = 16$. This proof is an answer to the research question posed in Section 3.2 of this dissertation.

7.1 Proof of Theorem 7 ($\chi'_s(G) \leqslant 4\Delta(G) - 2$ for Planar Graphs with $\Delta(G) \geqslant 6$ and $g \geqslant 5$)

Theorem 7 [16] is stated below.

Theorem 7. *If G is a planar graph with maximum degree $\Delta(G) \geqslant 6$ and girth $g \geqslant 5$, then $\chi'_s(G) \leqslant 4\Delta(G) - 2$.*

7.1.1 Preliminaries

We give the following definitions :

- We define $N_2(uv)$ as the set of edges at distance at most 2 from the edge uv, excluding the edge uv itself.

- We denote $SC\left(N_2\left(uv\right)\right)$ as the set of colours used by the edges in the set $N_2\left(uv\right)$, given an edge colouring of G.

- For $k \in \mathbb{N}$, a *k-vertex* is a vertex of degree k.
  A *k^+-vertex* is a vertex of degree at least k.
  A *k^--vertex* is a vertex of degree at most k.

- A *k_l-vertex* is a k-vertex adjacent to exactly l 2-vertices.

- A *bad 5-vertex* is a 5-vertex adjacent to three 2-vertices.

- A *good 5-vertex* is a 5-vertex that is not a bad 5-vertex.

- A *weak 2-path* is a path $v_1v_2v_3$ such that $d(v_1) = d(v_3) = 2$ and v_2 is a bad 5-vertex.

- A *bad 5-cycle* is a 5-cycle that is incident with a weak 2-path.

- We use $\llbracket n \rrbracket$ to indicate the set of integers $\{1, 2, \ldots, n\}$.

7.1.2 Proof

Assume that the theorem is false. Then, there exists a planar graph(s) G with $\Delta(G) \geqslant 6$ and $g \geqslant 5$ such that, $\chi'_s(G) > 4\Delta(G) - 2$. Hence, these graphs can be strongly edge coloured using $4\Delta(G) - 1$ colours or more. Amongst these graphs, let H be the graph that minimizes $|E\left(H\right)| + |V\left(H\right)|$. Therefore, the graph H is a minimal counterexample of Theorem 7. By the minimality on H, along with the condition on the girth, we can assume that H is connected. Furthermore, we have that $\chi'_s(H) \geqslant 4\Delta(H) - 1$.

Let $\Delta(H) = \Delta$ and let ϕ be a strong $(4\Delta(H) - 2)$-edge colouring. We define a set of $4\Delta(H) - 2$ colours as $L = \llbracket 4\Delta(H) - 2 \rrbracket$.

We list and prove the properties of the counterexample H that must be satisfied below.

Lemma 31. *H does not contain a 1-vertex adjacent to a 4^--vertex.*

Proof. Suppose H contains a 1-vertex u adjacent to a 4^--vertex v. Without loss of generality, we may assume that $d(v) = 4$, since the cases in which $1 \leqslant d(v) \leqslant 3$ would require fewer colours than when $d(v) = 4$. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using $4\Delta - 2$ colours or less.

We have that $|N_2(uv)|$ in H is at most 3Δ, and so H' can be strongly edge-coloured with at most 3Δ colours. Hence, $|L \backslash SC_\phi(N_2(uv))| \geqslant \Delta - 2 \geqslant 4$. Thus, there is a colour available in L to colour uv, so we may extend the colouring ϕ to H. This means that H can be strongly edge coloured using at most 4Δ - 2 colours, which is a contradiction.

Hence, H does not contain a 1-vertex adjacent to a 4^--vertex. □

Lemma 32. *H does not contain a 2-vertex adjacent to a* 3^-*-vertex.*

Proof. Suppose H contains a 2-vertex u adjacent to a 3^--vertex v. Without loss of generality, we can assume that $d(v) = 3$. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using $4\Delta - 2$ colours or less.

We see that $|N_2(uv)| \leqslant 3\Delta$, allowing H' to be strongly edge-coloured with at most 3Δ colours. Furthermore, $|L \backslash SC_\phi(N_2(uv))| \geqslant \Delta - 2 \geqslant 4$. This means that there is a colour available in L to assign to the edge uv in H, allowing us to extend the colouring ϕ to H. Therefore, H can be strongly edge coloured using at most 4Δ - 2 colours. This gives a contradiction.

Thus, H does not contain a 2-vertex adjacent to a 3^--vertex. □

Lemma 33. *H does not contain a 3-vertex adjacent to two* 3^-*-vertices.*

Proof. Suppose H contains a 3-vertex u adjacent to two 3^--vertices v and w. Without loss of generality, we may assume that $d(v) = d(w) = 3$. Let H' $= H$ - $\{uv\}$. Then, we can colour the graph H' using $4\Delta - 2$ colours or less.

Then, we have that, $|N_2(uv)| \leqslant 3\Delta + 3$ and $|L \backslash SC_\phi(N_2(uv))| \geqslant \Delta - 5 \geqslant 1$. Then, there is a colour available in L to assign to uv in H. This means that the colouring ϕ can be extended to H. Hence, H can be strongly edge coloured using at most 4Δ - 2 colours, giving a contradiction.

Thus, H does not contain a 3-vertex adjacent to two 3^--vertices. □

Lemma 34. *H does not contain a 4-vertex adjacent to a 2-vertex and a* 3^-*-vertex.*

Proof. Suppose H contains a 4-vertex u adjacent to a 2-vertex v and a 3^--vertex w. Without loss of generality, we can assume that $d(w) = 3$. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using $4\Delta - 2$ colours or less.

We see that $|N_2(uv)| \leqslant 3\Delta + 3$ and $|L\backslash SC_\phi(N_2(uv))| \geqslant \Delta - 5 \geqslant 1$. Therefore, we have a colour available in L to colour uv in H. This means that we can extend the colouring ϕ to H and so H can be strongly edge coloured using at most 4Δ - 2 colours, giving a contradiction.

Hence, H does not contain a 4-vertex adjacent to a 2-vertex and a 3^--vertex. $\square$

Lemma 35. *H contains neither a 5-vertex adjacent to four 2^--vertices nor a bad 5-cycle.*

Proof. Suppose H contains a 5-vertex u adjacent to four 2^--vertices v, w, x and y. Without loss of generality, we can assume $d(v) = d(w) = d(x) = d(y) = 2$. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using $4\Delta - 2$ colours or less.

Then, we have that $|N_2(uv)| \leqslant 2\Delta + 6$ and $|L\backslash SC_\phi(N_2(uv))| \geqslant 2\Delta - 8 \geqslant 4$. Hence, we may extend the colouring ϕ to H, since we will have a colour available in L for the edge uv in H. This means that H can be strongly edge coloured using at most 4Δ - 2 colours, giving a contradiction.

So, H cannot contain a 5-vertex adjacent to four 2^--vertices.

Now, consider a bad 5-cycle containing a bad 5-vertex u and two 2-vertices v and w. Without loss of generality, we can assume that $N(u) = \{v, w, x, y, z\}$, $N(v) = \{u, v_1\}$, $N(w) = \{u, w_1\}$, $N(y) = \{u, t\}$, $N(x) = \{u, x_1, x_2, ..., x_{\Delta-2}, x_{\Delta-1}\}$, $N(t) = \{y, t_1, t_2, ..., t_{\Delta-2}, t_{\Delta-1}\}$, $N(z) = \{u, z_1, z_2, ..., z_{\Delta-2}, z_{\Delta-1}\}$, $N(v_1) = \{v, w_1, {v_1}^1, {v_1}^2, ..., {v_1}^{\Delta-3}, {v_1}^{\Delta-2}\}$ and $N(w_1) = \{v_1, w, {w_1}^1, {w_1}^2, ..., {w_1}^{\Delta-3}, {w_1}^{\Delta-2}\}$. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using $4\Delta - 2$ colours or less.

We see that $|N_2(uv)| \leqslant 3\Delta + 4$ and $|L\backslash SC_\phi(N_2(uv))| \geqslant \Delta - 6$. If $\Delta \geqslant 7$, then we have that $|L\backslash SC_\phi(N_2(uv))| \geqslant 1$, which gives us an available colour in L for the edge uv in H. This will allow us to extend the colouring ϕ to H and so H can be strongly edge coloured using at most 4Δ - 2 colours, giving a contradiction.

If $\Delta(H) = 6$, then we do not have a colour available for uv, since $|L\backslash SC_\phi(N_2(uv))| = 0$. We will then recolour the edge uw. Let the colour of edge uw be θ. Since the colours of $v_1{v_1}^i$ and $w_1{w_1}^i$, for $i = [\![4]\!]$, are all different from each other, we can assign a colour of ${v_1}^i$, say α, to uw. This will still allow for a strong edge colouring of H', with the colour of uw being

α and the colour θ now unused. Hence, we can assign this unused colour θ to uv in H, allowing us to extend ϕ to H. This means that H can be strongly edge coloured using at most 4Δ - 2 colours, giving a contradiction.

Thus, H cannot contain a bad 5-cycle.

Therefore, H contains neither a 5-vertex adjacent to four 2^{-}-vertices nor a bad 5-cycle. □

Lemma 36. *H contains neither a bad 5-vertex adjacent to a 3^{-}-vertex nor a bad 5-vertex with a 2-neighbour adjacent to a $(\Delta(H)-1)^{-}$-vertex.*

Proof. Suppose H contains a bad 5-vertex u adjacent to three 2-vertices v, w and x and a 3^{-}-vertex y. Without loss of generality, we can assume that $d(y) = 3$. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using $4\Delta - 2$ colours or less.

We have that $|N_2(uv)| \leqslant 2\Delta + 7$ and $|L \backslash SC_{\phi}(N_2(uv))| \geqslant 2\Delta - 9 \geqslant 3$. Thus, we can extend the colouring ϕ to H, since we have a colour available in L for uv in H. Then, H can be strongly edge coloured using at most 4Δ - 2 colours, giving a contradiction.

Now, suppose H contains a bad 5-vertex u adjacent to three 2-vertices v, w and x. Suppose further that one of these 2-neighbours, say v, is adjacent to a $(\Delta(H)-1)^{-}$-vertex. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using $4\Delta - 2$ colours or less.

We see that $|N_2(uv)| \leqslant 3\Delta(H) + 3$ and $|L \backslash SC_{\phi}(N_2(uv))| \geqslant \Delta(H) - 5 \geqslant 1$. Hence, there is a colour available in L for uv in H, allowing us to extend the colouring ϕ to H. This means that H can be strongly edge coloured using at most 4Δ - 2 colours, giving a contradiction.

Therefore, H contains neither a bad 5-vertex adjacent to a 3^{-}-vertex nor a bad 5-vertex with a 2-neighbour adjacent to a $(\Delta(H)-1)^{-}$-vertex. □

Lemma 37. *H does not contain a 5_2-vertex adjacent to two other 3^{-}-vertices.*

Proof. Suppose H contains a 5-vertex u adjacent to exactly two 2-vertices v and w and two 3^{-}-vertices x and y. Without loss of generality, we may assume that $d(x) = d(y) = 3$. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using $4\Delta - 2$ colours or less.

We see that $|N_2(uv)| \leqslant 2\Delta + 8$ and $|L \backslash SC_\phi(N_2(uv))| \geqslant 2\Delta - 10 \geqslant 2$. Thus, we can extend the colouring ϕ to H, since we have a colour available in L for uv in H. This means that H can be strongly edge coloured using at most 4Δ - 2 colours, giving a contradiction.

Therefore, H does not contain a 5_2-vertex adjacent to two other 3^--vertices. $\square$

Lemma 38. *If $k \geqslant 5$, then H does not contain a k-vertex adjacent to $k-3$ 1-vertices. If the k-vertex is adjacent to $k-4$ 1-vertices, then it has no other 2^--neighbour. If a 5-vertex is adjacent to one 1-vertex, then it has no other 3^--neighbour.*

Proof. Let $k \geqslant 5$. Suppose H contains a k-vertex u adjacent to $k-3$ 1-vertices $u_1, u_2, ..., u_{k-3}$. Let $u_1 = v$. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using $4\Delta - 2$ colours or less.

Then, $|N_2(uv)| \leqslant 3\Delta + k - 4$ and $|L \backslash SC_\phi(N_2(uv))| \geqslant \Delta - k + 2 \geqslant 3$. Thus, we can extend the colouring ϕ to H, since we have a colour available in L for uv in H. This means that H can be strongly edge coloured using at most 4Δ - 2 colours, giving a contradiction.

Hence, H does not contain a k-vertex adjacent to $k-3$ 1-vertices.

Suppose now that u is instead adjacent to $k-4$ 1-vertices $u_1, u_2, ..., u_{k-4}$ and u_{k-3} is a 2^--vertex . Without loss of generality, we can assume that $d(u_{k-3}) = 2$. Let $u_1 = v$ and $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using $4\Delta - 2$ colours or less.

Then, $|N_2(uv)| \leqslant 3\Delta + k - 3$ and $|L \backslash SC_\phi(N_2(uv))| \geqslant \Delta - k + 1 \geqslant 2$. Thus, there is a colour available in L for uv in H. So, we can extend the colouring ϕ to H. This means that H can be strongly edge coloured using at most 4Δ - 2 colours, giving a contradiction.

Hence, if the k-vertex is adjacent to $k-4$ 1-vertices, then it has no other 2^--neighbour.

Now, suppose a 5-vertex u is adjacent to a 1-vertex v and a 3^--vertex w. Without loss of generality, we can assume that $d(w) = 3$. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using $4\Delta - 2$ colours or less.

Then, we have that $|N_2(uv)| \leqslant 3\Delta + 3$ and $|L \backslash SC_\phi(N_2(uv))| \geqslant \Delta - 5 \geqslant 1$. Thus, there is a colour available in L for uv in H. So, we can extend the colouring ϕ to H. This means that H can be strongly edge coloured using at most 4Δ - 2 colours, giving a contradiction.

Hence, if a 5-vertex is adjacent to one 1-vertex, then it has no other 3^--neighbour.

$\square$

Lemma 39. *If $k \geqslant 6$, then H does not contain a k-vertex adjacent to $k-1$ 2^--vertices. If the k-vertex is adjacent to $k-2$ 2^--vertices, then each 2^--vertex is not a 1-vertex. If the k-vertex is adjacent to $k-2$ 2-vertices, then it has no other 3^--neighbour.*

Proof. Let $k \geqslant 6$. Suppose H contains a k-vertex u adjacent to $k-1$ 2^--vertices $u_1, u_2, ..., u_{k-1}$. Without loss of generality, we can assume $d(u_1) = d(u_2) = ... = d(u_{k-1}) = 2$. Let $u_1 = v$. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using $4\Delta - 2$ colours or less.

We have that $|N_2(uv)| \leqslant 2\Delta + 2k - 4$ and $|L \backslash SC_\phi(N_2(uv))| \geqslant 2\Delta - 2k + 2 \geqslant 2$. Thus, we can extend the colouring ϕ to H, since we have a colour available in L for uv in H. This means that H can be strongly edge coloured using at most 4Δ - 2 colours, giving a contradiction.

Therefore, H does not contain a k-vertex adjacent to $k-1$ 2^--vertices.

Suppose now that u is adjacent to $k-2$ 2^--vertices $u_1, u_2, ..., u_{k-2}$ and u_1 is a 1-vertex. Let $u_1 = v$ and $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using $4\Delta - 2$ colours or less.

Then, $|N_2(uv)| \leqslant 2\Delta + 2k - 6$ and $|L \backslash SC_\phi(N_2(uv))| \geqslant 2\Delta - 2k + 4 \geqslant 4$. Therefore, there is a colour available in L for uv in H. So, we can extend the colouring ϕ to H. This means that H can be strongly edge coloured using at most 4Δ - 2 colours, giving a contradiction.

Hence, if the k-vertex is adjacent to $k-2$ 2^--vertices, then each 2^--vertex is not a 1-vertex.

Now, suppose that u is adjacent to $k-2$ 2-vertices $u_1 = v, u_2, ..., u_{k-2}$ and a 3^--vertex u_{k-1}. Without loss of generality, we can assume that $d(u_{k-1}) = 3$. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using $4\Delta - 2$ colours or less.

Then, $|N_2(uv)| \leqslant 2\Delta + 2k - 3$ and $|L \backslash SC_\phi(N_2(uv))| \geqslant 2\Delta - 2k + 1 \geqslant 1$. Thus, there is a colour available in L for uv in H. So, we can extend the colouring ϕ to H. This means that H can be strongly edge coloured using at most 4Δ - 2 colours, giving a contradiction.

Hence, if the k-vertex is adjacent to $k-2$ 2-vertices, then it has no other 3^--neighbour. $\square$

Lemma 40. *If $k \geqslant 6$ and $1 \leqslant \alpha \leqslant k-4$, then H does not contain a k-vertex adjacent to α 1-vertices and to $k-3-\alpha$ vertices of degree 2, such that this k-vertex is adjacent to another 3^--vertex.*

Proof. Let $k \geqslant 6$. Suppose H contains a k-vertex u adjacent to α 1-vertices and $k-3-\alpha$ 2-vertices, such that this k-vertex is adjacent to a 3^--vertex w. Without loss of generality, we can assume $d(w) = 3$. Let a 1-neighbour of u be v and let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using $4\Delta - 2$ colours or less.

Then, $|N_2(uv)| \leqslant 2\Delta + 2k - \alpha - 4$ and $|L \backslash SC_\phi(N_2(uv))| \geqslant 2\Delta - 2k + \alpha + 2 \geqslant 3$. Thus, we have a colour available in L for uv in H. So, we can extend the colouring ϕ to H. This means that H can be strongly edge coloured using at most 4Δ - 2 colours, giving a contradiction.

Hence, H does not contain a k-vertex adjacent to α 1-vertices and to $k-3-\alpha$ vertices of degree 2, such that this k-vertex is adjacent to another 3^--vertex. $\square$

Now, we proceed to use the discharging method on the graph H.

Discharging procedure We use the discharging method as in Section 3.3.3 of this dissertation. We use a multiple of $k = 5$ for Equation (3.1) in Section 3.3.3.
We define the discharging rules as follows:

(R1): Every face gives 2 to each incident 1-vertex.

(R2): Every k-face ($k \geqslant 6$) which is incident with a bad 5-vertex, gives $\frac{k-2p-5}{k-2p}$ to each incident 2^+-vertex, where p denotes the number of 1-vertices incident with the k-face.

(R3): Every 4^+-vertex gives 1 to each adjacent 2-vertex. In particular, if a 6^+-face contains a weak 2-path, then the bad 5-vertex in the 6^+-face gives $\frac{5}{6}$ to each adjacent 2-vertex which is in the weak 2-path.

(R4): Every 4^+-vertex gives $\frac{1}{4}$ to each adjacent 3-vertex.

(R5): Every 5^+-vertex gives $\frac{3}{2}$ to each adjacent 1-vertex.

Now, we will carry out these discharging rules on the graph H.

First, consider a k-face $f \in F(H)$. The initial charge on f is $ch(f) = d(f) - 5 = k - 5$. Since the graph H has girth $g \geqslant 5$, we have that $k \geqslant 5$. If f has p incident 1-vertices, then $k \geqslant 5 + 2p$.

- If f has p incident 1-vertices, then by (R1), the final charge on f will be $ch'(f) = k - 5 - 2p \geqslant 0$.
- If the face f has degree $k \geqslant 6 + 2p$ and is incident with a bad 5-vertex, then by (R2), the final charge on f will be $ch'(f) = k - 5 - 2p - \left(\frac{k-5-2p}{k-2p}\right)(k - 2p) \geqslant 0$.

So, $ch'(f) \geqslant 0$.

Now, consider a k-vertex $v \in V(H)$.

<u>**Case** $k = 1$</u> : We have that $ch(v) = \frac{3}{2}(1) - 5 = -\frac{7}{2}$.

By Lemma 31, v must be adjacent to a 5^+-vertex. Therefore, by (R1) and (R5), the final charge on v is $ch'(v) \geqslant -\frac{7}{2} + 2 + \frac{3}{2} = 0$.

So, $ch'(v) \geqslant 0$.

<u>**Case** $k = 2$</u> : We have that $ch(v) = \frac{3}{2}(2) - 5 = -2$.

By Lemma 32, v must be adjacent to two 4^+-vertices.

- Suppose both 4^+-vertices are not bad 5-vertices. Then, by (R3), the final charge on v is $ch'(v) = -2 + 1 + 1 = 0$.
- Now, suppose that one of these 4^+-vertices is a bad 5-vertex. Then, v is in a weak 2-path. We see that, by Lemma 35 and Lemma 36, the

weak 2-path is contained in a 6^+-face and the other neighbour of v must be a $\Delta(H)$-vertex. Thus, by (R2), the 6^+-face will give a charge of at least $\frac{1}{6}$ to v and by (R3), the bad 5-vertex will give a charge of $\frac{5}{6}$ to v and the $\Delta(H)$-vertex will give a charge of 1 to v. Thus, we have that the final charge of v is $ch'(v) \geqslant -2 + \frac{5}{6} + \frac{1}{6} + 1 = 0$.

So, $ch'(v) \geqslant 0$.

Case $\underline{k = 3}$: We have that $ch(v) = \frac{3}{2}(3) - 5 = -\frac{1}{2}$.

By Lemma 33, v must be adjacent to at least two 4^+-vertices. Therefore, by (R4), the final charge on v is $ch'(v) \geqslant -\frac{1}{2} + 2\left(\frac{1}{4}\right) = 0$.

So, $ch'(v) \geqslant 0$.

Case $\underline{k = 4}$: We have that $ch(v) = \frac{3}{2}(4) - 5 = 1$.

- Suppose that v is adjacent to a 2-vertex. By Lemma 34, if v is adjacent to a 2-vertex, then it cannot have any other 3^--neighbour. Therefore, by (R3), the final charge on v is $ch'(v) = 1 - 1 = 0$.

- Suppose that v is not adjacent to a 2-vertex. If v is not adjacent to a 2-vertex, then it could be adjacent to at most four 3-vertices. Then, by (R4), the final charge on v is $ch'(v) \geqslant 1 - \frac{1}{4}(4) = 0$.

So, $ch'(v) \geqslant 0$.

Case $\underline{k = 5}$: We have that $ch(v) = \frac{3}{2}(5) - 5 = \frac{5}{2}$.

- Suppose v is adjacent to a 1-vertex. Then, by Lemma 38, v cannot have any other 3^--neighbour. The other neighbours of v would therefore have to be 4^+-vertices. Thus, by (R5), the final charge of v is $ch'(v) = \frac{5}{2} - \frac{3}{2} > 0$.

- Suppose v is not adjacent to a 1-vertex. Then, by Lemma 35, v is adjacent to at most three 2-vertices.

  - Suppose v is not adjacent to a 2-vertex. Then, v can only be adjacent to 3^+-vertices. Therefore, by (R4), the final charge of v is $ch'(v) \geqslant \frac{5}{2} - \frac{1}{4}(5) > 0$.

- Suppose v is adjacent to exactly one 2-vertex. Then, v has 3^+-vertices as its remaining neighbours. Therefore, by (R3) and (R4), we have that the final charge of v is $ch'(v) \geqslant \frac{5}{2} - 1 - \frac{1}{4}(4) > 0$.
- Suppose v is adjacent to exactly two 2-vertices. By Lemma 37, v can only be adjacent to at most one 3-vertex. This means that the remaining neighbours of v are 4^+-vertices. So, by (R3) and (R4), we have that the final charge of v is $ch'(v) \geqslant \frac{5}{2} - 1(2) - \frac{1}{4}(1) > 0$.
- Suppose v is adjacent to exactly three 2-vertices. This means that v is a bad 5-vertex. Then, by Lemma 35, Lemma 36 and Lemma 37, we see that v must be incident with at least one 6^+-face which contains a weak 2-path. Furthermore, from Lemma 36, we see that the other neighbours of v must be 4^+-vertices. Therefore, by (R2) and (R3), v will receive a charge of at least $\frac{1}{6}$ from its incident 6^+-face and gives $\frac{5}{6}$ to each adjacent 2-vertex in the weak 2-path and 1 to the third 2-neighbour. Thus, the final charge of v is $ch'(v) \geqslant \frac{5}{2} + \frac{1}{6} - \frac{5}{6}(2) - 1 = 0$.

So, $ch'(v) \geqslant 0$.

<u>**Case** $k \geqslant 6$</u> : We have that $ch(v) = \frac{3}{2}(k) - 5$.

- Suppose v is not adjacent to a 1-vertex. By Lemma 39, v is adjacent to at most $k-2$ 2-vertices.

  - Suppose v is adjacent to at most $k-3$ 2-vertices. Then, the remaining three neighbours of v are 3^+-vertices. So, by (R3) and (R4), we have that the final charge of v is $ch'(v) \geqslant \frac{3}{2}k - 5 - 1(k-3) - \frac{1}{4}(3) = \frac{1}{2}k - \frac{11}{4} > 0$.
  - Suppose v is adjacent to exactly $k-2$ 2-vertices. Then, by Lemma 39, v cannot be adjacent to any other 3^--vertex. Thus, the remaining two vertices are 4^+-vertices. So, by (R3), the final charge of v is $ch'(v) \geqslant \frac{3}{2}k - 5 - 1(k-2) = \frac{1}{2}k - 3 \geqslant 0$.

- Suppose v is adjacent to α 1-vertices. From Lemma 38, we obtain the bounds on α as $1 \leqslant \alpha \leqslant k-4$.

  - Suppose v is adjacent to exactly $k-4$ 1-vertices. Then, by Lemma 38, v does not have any other 2^--neighbour. So, v is adjacent to at most four 3-vertices. So, by (R4) and (R5), the final charge of v is $ch'(v) \geqslant \frac{3}{2}k - 5 - \frac{1}{4}(4) - \frac{3}{2}(k-4) = 0$.

- Suppose v is adjacent to $\alpha \leqslant k-5$ 1-vertices and v is adjacent to at most $k-4-\alpha$ 2-vertices. Then, v is adjacent to at most four 3-vertices. So, by (R3), (R4) and (R5), we have that the final charge of v is
$ch'(v) \geqslant \frac{3}{2}k - 5 - \frac{3}{2}(\alpha) - 1(k-4-\alpha) - \frac{1}{4}(4) = \frac{1}{2}k - \frac{1}{2}\alpha - 2 > 0$.
- Suppose v is adjacent to $\alpha \leqslant k-5$ 1-vertices and there are at least $k-3-\alpha$ 2-vertices adjacent to v. Then, by Lemma 39, v cannot have $k-2-\alpha$ 2-neighbours nor can it have $k-1-\alpha$ 2-neighbours. Therefore, v has exactly $k-3-\alpha$ 2-neighbours. Furthermore, by Lemma 40, the remaining neighbours of v are 4^+-vertices. So, by (R3) and (R5), the final charge of v is
$ch'(v) \geqslant \frac{3}{2}k - 5 - \frac{3}{2}(\alpha) - 1(k-3-\alpha) = \frac{1}{2}k - \frac{1}{2}\alpha - 2 > 0$.

So, $ch'(v) \geqslant 0$.

Now, after performing the discharging rules on H, we see that the total final charge of H is a positive value. The final charge of H is also equivalent to the initial charge of H. However, from Equation (3.1), the initial charge of H was -$2k$ = -10 < 0. That is :

$$0 \leqslant \sum_{x \in V(H) \cup F(H)} ch'(x) = \sum_{x \in V(H) \cup F(H)} ch(x) = -10 < 0$$

giving a contradiction.

Therefore, the counterexample H does not exist and so the theorem is true.

This concludes the proof.

7.2 Proof of Theorem 8 ($\chi'_s(G) \leqslant 19$ for Planar Graphs with $\Delta(G) = 5$ and $g \geqslant 5$)

Theorem 8 [16] is stated below:

Theorem 8. *If G is a planar graph with maximum degree $\Delta(G) = 5$ and girth $g \geqslant 5$, then $\chi'_s(G) \leqslant 19$.*

7.2.1 Preliminaries

We give the following definitions :

- We define $N_2(uv)$ as the set of edges at distance at most 2 from the edge uv, excluding the edge uv itself.
- We denote $SC(N_2(uv))$ as the set of colours used by the edges in the set $N_2(uv)$, given an edge colouring of G.
- For $k \in \mathbb{N}$, a *k-vertex* is a vertex of degree k.
  A *k^+-vertex* is a vertex of degree at least k.
  A *k^--vertex* is a vertex of degree at most k.
- A k_l-vertex is a k-vertex adjacent to exactly l 2-vertices.
- A *bad* 5-*vertex* is a 5-vertex adjacent to three 2-vertices.
- A *good* 5-*vertex* is a 5-vertex that is not a bad 5-vertex.
- A *weak* 2-*path* is a path $v_1v_2v_3$ such that $d(v_1) = d(v_3) = 2$ and v_2 is a bad 5-vertex.
- A *bad* 5-*cycle* is a 5-cycle that is incident with a weak 2-path.
- We use $[\![n]\!]$ to indicate the set of integers $\{1, 2, \ldots, n\}$.

7.2.2 Proof

Assume that the theorem is false. Then, there exists a planar graph(s) G with $\Delta(G) = 5$ and $g \geqslant 5$, such that $\chi'_s(G) \geqslant 20$. Hence, these graphs can be strongly edge coloured using 20 or more colours. Amongst these graphs, let H be the graph that minimizes $|E(H)| + |V(H)|$. Then, H is a minimal counterexample and by the minimality of H, along with the condition on the girth, we can assume that H is connected. Furthermore, we have that $\chi'_s(H) \geqslant 20$.

Let $\Delta(H) = \Delta$ and let ϕ be a strong 19-edge colouring. We define a set of 19 colours as $L = [\![19]\!]$.

The lemmas below show the properties of the counterexample H that must be satisfied. We note that the proofs for Lemma 41 to Lemma 46 have already been proven in Section 7.1 of this dissertation, in the proof of Theorem 7. Therefore, we will only prove the remaining four properties, Lemma 47 to Lemma 50, below.

Lemma 41. *H does not contain a 1-vertex adjacent to a 4^--vertex.*

Lemma 42. *H does not contain a 2-vertex adjacent to a 3^--vertex.*

Lemma 43. *H does not contain a 3-vertex adjacent to two 3^--vertices.*

Lemma 44. *H does not contain a 4-vertex adjacent to a 2-vertex and a 3^--vertex.*

Lemma 45. *H does not contain a 5_2-vertex adjacent to two other 3^--vertices.*

Lemma 46. *H does not contain a 5-vertex adjacent to two 1-vertices. If the 5-vertex is adjacent to one 1-vertex, then it has no other 3^--neighbour.*

Lemma 47. *H does not contain a 2-neighbour adjacent to a bad 5-vertex and a 4^--vertex.*

Proof. Suppose H contains a 2-neighbour u adjacent to a bad 5-vertex v and a 4^--vertex w. Without loss of generality, we may assume that $d(w) = 4$, since the cases in which $1 \leqslant d(w) \leqslant 3$ would require fewer colours than when $d(w) = 4$. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using 19 colours or less.

Then, we have that $|N_2(uv)|$ is at most $2\Delta + 8$. Therefore, $|L \backslash SC_\phi(N_2(uv))| \geqslant 11 - 2\Delta(H) = 1$. Hence, there is a colour available in L to colour uv in H, so we may extend the colouring ϕ to H. This means that H can be strongly edge-coloured using at most 19 colours, which is a contradiction.

Thus, H does not contain a 2-neighbour adjacent to a bad 5-vertex and a 4^--vertex.

□

Lemma 48. *H does not contain a 2-vertex adjacent to two bad 5-vertices.*

Proof. Suppose that H contains a 2-vertex u adjacent to two bad 5-vertices v and w. Without loss of generality, we can assume that $N(u) = \{v, w\}$, $N(v) = \{u, v_1, v_2, v_3, v_4\}$, $N(w) = \{u, w_1, w_2, w_3, w_4\}$, $N(v_1) = \{v, v_1^1, v_1^2, v_1^3, v_1^4\}$, $N(v_2) = \{v, v_2^1, v_2^2, v_2^3, v_2^4\}$, $N(v_3) = \{v, x\}$, $N(v_4) = \{v, y\}$, $N(x) = \{v_3, x_1, x_2, x_3, x_4\}$, $N(y) = \{v_4, y_1, y_2, y_3, y_4\}$, $N(w_1) = \{w, w_1^1, w_1^2, w_1^3, w_1^4\}$, $N(w_2) = \{w, w_2^1, w_2^2, w_2^3, w_2^4\}$, $N(w_3) = \{w, z\}$, $N(w_4) = \{w, t\}$, $N(z) = \{w_3, z_1, z_2, z_3, z_4\}$ and $N(t) = \{w_4, t_1, t_2, t_3, t_4\}$.

If there is a colour available in the set $|L\backslash SC_{\phi}(N_2(uv))|$, then we may colour the edge uv in H with this colour. Then, H will have a strong edge-colouring using at most 19 colours. This will give a contradiction.

Assume, however, that $|L\backslash SC_{\phi}(N_2(uv))| = 0$. That is, we have no colour available from the set L to use to colour uv. Without loss of generality, assume that $\phi(uw) = 1$, $\phi(ww_1) = 2$, $\phi(ww_2) = 5$, $\phi(ww_3) = 3$, $\phi(ww_4) = 4$, $\phi(vv_1) = 6$, $\phi(vv_2) = 9$, $\phi(vv_3) = 7$, $\phi(vv_4) = 8$, $\phi(v_3x) = 10$, $\phi(v_4y) = 11$, $\phi(v_1v_1^1) = 12$, $\phi(v_1v_1^2) = 13$, $\phi(v_1v_1^3) = 14$, $\phi(v_1v_1^4) = 15$, $\phi(v_2v_2^1) = 16$, $\phi(v_2v_2^2) = 17$, $\phi(v_2v_2^3) = 18$ and $\phi(v_2v_2^4) = 19$. Then, we can try to recolour ww_3. If we cannot do this, then, without loss of generality, say, $\phi(w_3z) = 6$, $\phi(w_4t) = 7$, $\phi(zz_1) = 8$, $\phi(zz_2) = 9$, $\phi(zz_3) = 10$, $\phi(zz_4) = 11$, $\phi(w_1w_1^1) = 12$, $\phi(w_1w_1^2) = 13$, $\phi(w_1w_1^3) = 14$, $\phi(w_1w_1^4) = 15$, $\phi(w_2w_2^1) = 16$, $\phi(w_2w_2^2) = 17$, $\phi(w_2w_2^3) = 18$ and $\phi(w_2w_2^4) = 19$. So, we try to recolour ww_4. If we cannot do this, then, without loss of generality, say, $\phi(tt_1) = 8$, $\phi(tt_2) = 9$, $\phi(tt_3) = 10$ and $\phi(tt_4) = 11$. We now try to recolour vv_3 and vv_4. If this is possible, then we can obtain our contradiction by having a colour available in the set L to use for uv in H. Otherwise, without loss of generality, say, $\phi(xx_1) = \phi(yy_1) = 2$, $\phi(xx_2) = \phi(yy_2) = 3$, $\phi(xx_3) = \phi(yy_3) = 4$ and $\phi(xx_4) = \phi(yy_4) = 5$. Now, we can recolour vv_4 and ww_3 with 1, uw with 3 and uv with 8. This means that there is a strong edge-colouring of H that uses 19 colours, and therefore, we have a contradiction.

Thus, H does not contain a 2-vertex adjacent to two bad 5-vertices.

□

Lemma 49. *H does not contain a 2-vertex adjacent to a bad 5-vertex and a 5_2-vertex which is adjacent to a 3-vertex.*

Proof. Suppose H contains a 2-vertex u adjacent to a bad 5-vertex v and a 5_2-vertex w which is adjacent to a 3^--vertex w_3. Without loss of generality, we can assume that $d(w_3) = 3$. We can further assume that $N(u) = \{v, w\}$, $N(v) = \{u, v_1, v_2, v_3, v_4\}$, $N(w) = \{u, w_1, w_2, w_3, w_4\}$, $N(v_1) = \{v, v_1^1, v_1^2, v_1^3, v_1^4\}$, $N(v_2) = \{v, v_2^1, v_2^2, v_2^3, v_2^4\}$, $N(v_3) = \{v, x\}$, $N(v_4) = \{v, y\}$, $N(x) = \{v_3, x_1, x_2, x_3, x_4\}$, $N(y) = \{v_4, y_1, y_2, y_3, y_4\}$, $N(w_1) = \{w, w_1^1, w_1^2, w_1^3, w_1^4\}$, $N(w_2) = \{w, w_2^1, w_2^2, w_2^3, w_2^4\}$, $N(w_3) = \{w, w_3^1, w_3^2\}$, $N(w_4) = \{w, t\}$ and $N(t) = \{w_4, t_1, t_2, t_3, t_4\}$.

If there is a colour available in the set $|L\backslash SC_{\phi}(N_2(uv))|$, then we may colour the edge uv in H with this colour. Then, H will have a strong

19-edge-colouring, giving a contradiction.

Assume, however, that $|L \backslash SC_\phi(N_2(uv))| = 0$. Without loss of generality, assume that $\phi(uw) = 1$, $\phi(ww_1) = 2$, $\phi(ww_2) = 5$, $\phi(ww_3) = 3$, $\phi(ww_4) = 4$, $\phi(vv_1) = 6$, $\phi(vv_2) = 9$, $\phi(vv_3) = 7$, $\phi(vv_4) = 8$, $\phi(v_3x) = 10$, $\phi(v_4y) = 11$, $\phi(v_1v_1^1) = 12$, $\phi(v_1v_1^2) = 13$, $\phi(v_1v_1^3) = 14$, $\phi(v_1v_1^4) = 15$, $\phi(v_2v_2^1) = 16$, $\phi(v_2v_2^2) = 17$, $\phi(v_2v_2^3) = 18$ and $\phi(v_2v_2^4) = 19$. Then, we can try to recolour ww_4. If we cannot do this, then, without loss of generality, say, $\phi(w_3w_3^1) = 6$, $\phi(w_3w_3^2) = 7$, $\phi(w_4t) = 8$, $\phi(tt_1) = 9$, $\phi(tt_2) = 10$, $\phi(tt_3) = 11$, $\phi(tt_4) = 12$, $\phi(w_1w_1^1) = 13$, $\phi(w_1w_1^2) = 14$, $\phi(w_1w_1^3) = 15$, $\phi(w_1w_1^4) = 16$, $\phi(w_2w_2^1) = 17$, $\phi(w_2w_2^2) = 18$ and $\phi(w_2w_2^3) = 19$. Then, we try to recolour vv_3 and vv_4. If this is possible, then we can obtain our contradiction by having a colour available to use for uv in H. Otherwise, without loss of generality, say, $\phi(xx_1) = \phi(yy_1) = 2$, $\phi(xx_2) = \phi(yy_2) = 3$, $\phi(xx_3) = \phi(yy_3) = 4$ and $\phi(xx_4) = \phi(yy_4) = 5$. Now, we can recolour vv_4 and ww_4 with 1, uw with 4 and uv with 8. This can be done since the colour of $w_2w_4^2$ is neither 1 nor 4. Therefore, there is a strong edge-colouring of H that uses 19 colours, and hence, we have a contradiction.

Thus, H does not contain a 2-vertex adjacent to a bad 5-vertex and a 5_2-vertex which is adjacent to a 3-vertex.

□

Lemma 50. *H does not contain a bad 5-vertex adjacent to another 4^--vertex.*

Proof. Suppose H contains a bad 5-vertex u adjacent to another 4^--vertex w. Without loss of generality, we can assume that $d(w) = 4$. Let v be a 2-neighbour of u. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using 19 colours or less.

We have that $|N_2(uv)|$ is at most $2\Delta + 8$ and $|L \backslash SC_\phi(N_2(uv))| \geqslant 11 - 2\Delta(H) = 1$. Hence, there is a colour available in L to colour uv in H, so we may extend the colouring ϕ to H. This means that H can be strongly edge-coloured using at most 19 colours, which is a contradiction.

Thus, H does not contain a bad 5-vertex adjacent to another 4^--vertex. □

Now, we proceed to use the discharging method on the graph H.

Discharging procedure We use the discharging method as in Section 3.3.3 of this dissertation. We use a multiple of $k = 5$ for Equation (3.1) in Section 3.3.3.
We define the discharging rules as follows:

(R1): Every face gives 2 to each incident 1-vertex.

(R2): Every 4-vertex gives 1 to each adjacent 2-vertex.

(R3): Every 4-vertex gives $\frac{1}{4}$ to each adjacent 3-vertex.

(R4): Every good 5-vertex gives :

- **(R4.1.):** $\frac{3}{2}$ to each adjacent 1-vertex.
- **(R4.2.):** $\frac{7}{6}$ to each adjacent 2-vertex if this 2-vertex is adjacent to a bad 5-vertex, otherwise 1 to each adjacent 2-vertex.
- **(R4.3.):** $\frac{1}{4}$ to each adjacent 3-vertex.

(R5): Every bad 5-vertex gives $\frac{5}{6}$ to each adjacent 2-vertex.

First, consider the charge on a k-face $f \in F(H)$. The initial charge on f is $ch(f) = k - 5$. We have that $k \geqslant 5$, by the condition on the girth. If f has p incident 1-vertices, then $k \geqslant 5 + 2p$.

Then, by (R1), the final charge on f is $ch'(f) \geqslant k - 5 - 2p \geqslant 0$.

So, $ch'(f) \geqslant 0$.

Now, we consider the charge on a k-vertex $v \in V(H)$.

Case $k = 1$: We have that $ch(v) = \frac{3}{2}(1) - 5 = -\frac{7}{2}$.

By Lemma 41, v must be adjacent to a 5-vertex. Furthermore, by Lemma 46, this 5-vertex can only be a good 5-vertex. Therefore, by (R1) and (R4.1.), the final charge on v is $ch'(v) \geqslant -\frac{7}{2} + 2 + \frac{3}{2} = 0$.

So, $ch'(v) \geqslant 0$.

Case $k = 2$: We have that $ch(v) = \frac{3}{2}(2) - 5 = -2$.

By Lemma 42, v must be adjacent to two 4^+-vertices.

- Suppose v is adjacent to two 4-vertices. Then, by (R2), the final charge on v is $ch'(v) = -2 + 1(2) = 0$.
- Suppose v is adjacent to one 4-vertex and one good 5-vertex. Then, by (R2) and (R4.2.), the final charge on v is $ch'(v) = -2 + 1 + 1 = 0$.
- Suppose v is adjacent to one 4-vertex and one bad 5-vertex. By Lemma 47, this case is not possible.
- Suppose v is adjacent to two good 5-vertices. Then, by (R4.2.), the final charge on v is $ch'(v) = -2 + 1(2) = 0$.
- Suppose v is adjacent to two bad 5-vertices. By Lemma 48, this case is not possible.
- Suppose v is adjacent to one good 5-vertex and one bad 5-vertex. Then, by (R4.2.) and (R5), the final charge on v is $ch'(v) = -2 + \frac{7}{6} + \frac{5}{6} = 0$.

So, $ch'(v) = 0$.

<u>Case $k = 3$</u> : We have that $ch(v) = \frac{3}{2}(3) - 5 = -\frac{1}{2}$.

By Lemma 43, v is adjacent to at least two 4^+-vertices. Furthermore, by Lemma 50, if v is adjacent to a 5-vertex, then the 5-vertex must be a good 5-vertex.

- Suppose v is adjacent to at most three 4-vertices and v has no 5-neighbour. Then, by (R3), the final charge on v is $ch'(v) \geqslant -\frac{1}{2} + \frac{1}{4}(2) = 0$.
- Suppose v is adjacent to exactly two 4-vertices and one good 5-vertex. Then, by (R3) and (R4.3.), the final charge on v is $ch'(v) = -\frac{1}{2} + \frac{1}{4}(2) + \frac{1}{4} > 0$.
- Suppose v is adjacent to exactly one 4-vertex, one good 5-vertex and at most one other good 5-vertex. Then, by (R3) and (R4.3.), the final charge on v is $ch'(v) \geqslant -\frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 0$.
- Suppose v is adjacent to two good 5-vertices and at most one other good 5-vertex. Suppose further that v has no 4-neighbour. Then, by (R4.3.), the final charge on v is $ch'(v) \geqslant -\frac{1}{2} + \frac{1}{4}(2) = 0$.

So, $ch'(v) \geqslant 0$.

<u>**Case** $k = 4$</u> : We have that $ch(v) = \frac{3}{2}(4) - 5 = 1$.

- Suppose v is adjacent to a 2-vertex. Then, by Lemma 44, v is adjacent to only one 2-vertex and has no other 3^--neighbour. So, by (R2), the final charge on v is $ch'(v) = 1 - 1 = 0$.
- Suppose v is not adjacent to a 2-vertex. Then, v is adjacent to at most four 3-vertices. So, by (R3), the final charge on v is $ch'(v) \geqslant 1 - \frac{1}{4}(4) = 0$.

So, $ch'(v) \geqslant 0$.

<u>**Case** $k = 5$</u> : We have that $ch(v) = \frac{3}{2}(5) - 5 = \frac{5}{2}$.

- Suppose v is a good 5-vertex.
  - Suppose v is adjacent to a 1-vertex. Then, by Lemma 46, v does not have any other 3^--neighbour. So, by (R4.1.), the final charge on v is $ch'(v) = \frac{5}{2} - \frac{3}{2} > 0$.
  - Suppose v is not adjacent to a 1-vertex. Then, since v is a good 5-vertex, v is adjacent to at most two 2-vertices.
    * Suppose v is adjacent to exactly one 2-vertex. Then, the remaining neighbours of v are 3^+-vertices. Then, we determine the final charge of v using (R4.2.) and (R4.3.). If the 2-vertex is adjacent to a bad 5-vertex, then the final charge on v is $ch'(v) \geqslant \frac{5}{2} - \frac{7}{6} - \frac{1}{4}(4) > 0$. Otherwise, if the 2-vertex is adjacent to another good 5-vertex, then the final charge on v is $ch'(v) \geqslant \frac{5}{2} - 1 - \frac{1}{4}(4) > 0$.
    * Suppose v is adjacent to exactly two 2-vertices. Then, by Lemma 45, v can be adjacent to at most one 3-vertex. The remaining neighbours of v are then 4^+-vertices. If v is adjacent to a 3-vertex, then, by Lemma 49, the two 2-neighbours of v cannot be adjacent to a bad 5-vertex. Then, by (R4.2.) and (R4.3.), we have that the final charge on v is $ch'(v) = \frac{5}{2} - 1(2) - \frac{1}{4} > 0$. If v is not adjacent to a 3-vertex, then, we determine the final charge on v using (R4.2.). If the 2-neighbours of v are adjacent to good 5-vertices, then the final charge on v is $ch'(v) = \frac{5}{2} - 1(2) > 0$. If one of the 2-neighbours of v is adjacent to a good 5-vertex

and the other is adjacent to a bad 5-vertex, then the final charge on v is $ch'(v) = \frac{5}{2} - 1 - \frac{7}{6} > 0$. If both 2-neighbours of v are adjacent to bad 5-vertices, then the final charge on v is $ch'(v) = \frac{5}{2} - \frac{7}{6}(2) > 0$.

  ∗ Suppose v is adjacent only to 3^+-vertices. Then, v can be adjacent to at most five 3-vertices. So, by (R4.3.), the final charge on v is $ch'(v) \geqslant \frac{5}{2} - \frac{1}{4}(5) > 0$.

- Suppose v is a bad 5-vertex. Then, v is adjacent to three 2-vertices. Furthermore, by Lemma 50, v has no other 4^--neighbour. So, by (R5), the final charge on v is $ch'(v) = \frac{5}{2} - \frac{5}{6}(3) = 0$.

So, $ch'(v) \geqslant 0$.

Now, we see that the total final charge of H is a positive value. However, we obtain the following contradiction :

$$0 \leqslant \sum_{x \in V(H) \cup F(H)} ch'(x) = \sum_{x \in V(H) \cup F(H)} ch(x) = -10 < 0$$

Thus, the counterexample H does not exist and so the theorem is true.

This concludes the proof.

7.3 Proof of Hypothesis ($\chi'_s(G) \leqslant 16$ for Planar Graphs with $\Delta(G) = 5$ and $g = 5$)

We will prove the following statement, conjectured by Song *et al* [16] :

For a planar graph G with girth $g = 5$ and maximum degree $\Delta(G) = 5$, we have that $\chi'_s(G) \leqslant 3\Delta(G) + 1$.

7.3.1 Methodology

To construct the proof for the above statement, we first studied the literature in the field and examined the approaches used to prove similar theorems. An outline of the general methodology that was adopted for these proofs is detailed in Section 3.3 of this dissertation. We adopted a similar methodology to prove this statement.

The methodology used in Section 3.3 is a proof by contradiction. We obtain this contradiction using the discharging method, detailed in Section 3.3.3 of this dissertation. Therefore, when constructing the proof for this statement, it was imperative that we obtain a contradiction via the discharging method.

We began by first examining the initial charges of the faces and vertices of a counterexample H to the statement. It is here that we realized that the multiple for Equation (3.1) in Section 3.3.3 needed to be related to the girth $g = 5$. We required that the multiple needed to be 5 in order to ensure that the initial charge on the face f will always be at least zero. Furthermore, by examining the initial charges, this gave us insight into which vertices in H would be able to give charge and which vertices would need to receive charge to gain a final positive charge, in order to obtain our contradiction.

Thereafter, we determined the potential neighbours of the vertices in the graph H, using the method outlined in Section 3.3.2 of this dissertation. We then proceeded to define discharging rules between these neighbouring vertices. The rules were structured to ensure that all vertices with an initial negative charge receive enough charge from vertices with an initial positive charge, in order to obtain a final charge that is at least zero. However, this meant that the vertices with an initial positive charge would sometimes have to give more charge than it could afford to. We noted down the particular configurations that had this problem and denoted them as "problem scenarios" in H. Then, we examined these "problem scenarios" in detail and showed that we could always obtain a strong edge colouring using 16 colours or less for these configurations. This meant that we could exclude these "problem scenarios" from H, and so we could obtain our contradiction via the discharging method without issue.

7.3.2 Preliminaries

- We define $N_2(uv)$ as the set of edges at distance at most 2 from the edge uv, excluding the edge uv itself.
- We denote $SC(N_2(uv))$ as the set of colours used by the edges in the set $N_2(uv)$, given an edge colouring of G.
- For $k \in \mathbb{N}$, a *k-vertex* is a vertex of degree k.
A *k^+-vertex* is a vertex of degree at least k.
A *k^--vertex* is a vertex of degree at most k.

- We use $\llbracket n \rrbracket$ to indicate the set of integers $\{1, 2, \ldots, n\}$.

7.3.3 Proof

Assume that the theorem is false. Then, there exists a planar graph(s) G with $\Delta(G) = 5$ and $g = 5$, such that $\chi'_s(G) \geqslant 3\Delta + 2 = 17$. Hence, these graphs can be strongly edge coloured using 17 or more colours. Amongst these graphs, let H be the graph that minimizes $|E(H)| + |V(H)|$. Then, H is a minimal counterexample and by the minimality of H, along with the condition on the girth, we can assume that H is connected. Furthermore, we have that $\chi'_s(H) \geqslant 17$.

Let $\Delta(H) = \Delta = 5$ and let ϕ be a strong 16-edge colouring. We define a set of 16 colours as $L = \llbracket 16 \rrbracket$.

The lemmas below outline the properties of the counterexample H.

Lemma 51. *Every 1-vertex in H can only be adjacent to a 5-vertex.*

Proof. Suppose H contains a 1-vertex u adjacent to a 4^--vertex v. Without loss of generality, we may assume that $d(v) = 4$, since the other possibilities would simply require fewer colours. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using 16 colours or less.

We have that $|N_2(uv)|$ in H is at most 15. Then, $|L \backslash SC_\phi(N_2(uv))| \geqslant 16 - 15 = 1$. Thus, there is a colour available in L to colour uv in H, so we may extend the colouring ϕ to H. This means that H can be strongly edge coloured using at most 16 colours, which is a contradiction.

Hence, every 1-vertex in H can only be adjacent to a 5-vertex. $\square$

Lemma 52. *Every 2-vertex in H can only be adjacent to two 4^+-vertices.*

Proof. Suppose H contains a 2-vertex u adjacent to at most one 4^+-vertex. Let $N(u) = \{v, w\}$. Without loss of generality, we can assume that u is adjacent to exactly one 4^+-vertex, say w. Then, v must be a 3^--vertex. We can further assume that $d(v) = 3$ and $d(w) = \Delta = 5$. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using 16 colours or less.

We have that $|N_2(uv)|$ in H is at most 15. Then, $|L \backslash SC_\phi(N_2(uv))| \geqslant 16 - 15 = 1$. Thus, there is a colour available in L to colour uv in H, so we may extend the colouring ϕ to H. This means that H can be strongly edge coloured using at most 16 colours, which is a contradiction.

Hence, every 2-vertex in H can only be adjacent to two 4^+-vertices. $\square$

Lemma 53. *Every 3-vertex in H must have at least one 4^+-neighbour.*

Proof. Suppose H contains a 3-vertex u that is adjacent to only 3^--vertices. By Lemma 51 and Lemma 52, u cannot be adjacent to a 1-vertex or a 2-vertex. Then, u is adjacent to three 3-vertices v, w and x. Since the vertices v, w and x are also 3-vertices, the vertices v, w and x are all adjacent to three 3-vertices respectively. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using 16 colours or less.

We have that $|N_2(uv)|$ in H is at most 12. Then, $|L \backslash SC_\phi(N_2(uv))| \geqslant 16 - 12 = 4$. Thus, there is a colour available in L to colour uv in H, so we may extend the colouring ϕ to H. This means that H can be strongly edge coloured using less than 16 colours, which is a contradiction.

Hence, every 3-vertex in H must have at least one 4^+-neighbour. $\square$

Lemma 54. *Every 4-vertex u in H can be adjacent to at most two 2-vertices. Moreover, if u is adjacent to exactly two 2-vertices, then the remaining two neighbours of u must be a 5-vertex and a 4^+-vertex. If u is adjacent to exactly one 2-vertex, and u has no 5-neighbour, then u is adjacent to three 4-vertices. If u is adjacent to exactly one 2-vertex and a 5-vertex, then u has at most one 3-neighbour.*

Proof. Suppose H contains a 4-vertex u adjacent to at least three 2-vertices. Without loss of generality, we can assume that u is adjacent to exactly three 2-vertices and the remaining neighbour of u is a 5-vertex. Let v be a 2-neighbour of u. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using 16 colours or less.

We have that $|N_2(uv)|$ in H is at most 14. Then, $|L \backslash SC_\phi(N_2(uv))| \geqslant 16 - 14 = 2$. Thus, there is a colour available in L to colour uv in H, so we may extend the colouring ϕ to H. This means that H can be strongly edge coloured using less than 16 colours, which is a contradiction.

Hence, every 4-vertex u in H can be adjacent to at most two 2-vertices.

Now, suppose that u is adjacent to exactly two 2-vertices v and w. Suppose further that u does not have a 5-neighbour. Then, without loss of generality, we can assume that the two remaining neighbours of u are both 4-vertices. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using 16 colours or less.

We have that $|N_2(uv)|$ in H is at most 15. Then, $|L \backslash SC_\phi(N_2(uv))| \geqslant 16 - 15 = 1$. Thus, there is a colour available in L to colour uv in H, so we may extend the colouring ϕ to H. This means that H can be strongly edge coloured using at most 16 colours, which is a contradiction.

Hence, if a 4-vertex u in H is adjacent to exactly two 2-vertices, then the remaining two neighbours of u must be a 5-vertex and a 4^+-vertex.

Now, suppose that u is adjacent to exactly one 2-vertex v and u has no 5-neighbour. Suppose further that u is adjacent to at most two 4-vertices. Without loss of generality, we can assume that u is adjacent to exactly two 4-vertices, say u_1 and u_2, and the remaining neighbour of u is a 3-vertex, say u_3. Let the neighbour of v, distinct from u, be w. Then, without loss of generality, $d(w) = 5$.

Consider the 2-neighbourhood of the edge uv. Then, we can see that $|N_2(uv)|$ is at most 16. Then, $|L \backslash SC_\phi(N_2(uv))| \geqslant 16 - 16 = 0$. Thus, there is no colour available in L to colour uv.

Now, consider the vertex w. We see that w cannot be adjacent to any of the vertices u_i, for $i = [\![3]\!]$, due to the condition on the girth. However, w can have a neighbour adjacent to each of the vertices u_i. Suppose that w has a neighbour adjacent to a u_i. Let this neighbour be x. Let a neighbour of u_i, distinct from u and distinct from x, be y. Then, w cannot be adjacent to y, due to the condition on the girth. This means that w has at most one neighbour adjacent to any u_i. Let the colour of uu_i be $\alpha \in L$. Then, we can recolour an edge of w, distinct from vw and wx, with the colour α. Then, we will have a colour available in L to colour uv. This means that we will have a strong edge colouring using at most 16 colours, which is a contradiction.

Therefore, if a 4-vertex u is adjacent to exactly one 2-vertex, and u has no

5-neighbour, then u is adjacent to three 4-vertices.

Now, suppose that u is adjacent to exactly one 2-vertex v and u has a 5-neighbour. Suppose further that u is adjacent to at least two 3-vertices. Without loss of generality, we can assume that u is adjacent to exactly two 3-vertices, say u_1 and u_2. Let the 5-neighbour of u be u_3. Let the neighbour of v, distinct from u, be w. Then, without loss of generality, $d(w) = 5$.

Consider the 2-neighbourhood of the edge uv. Then, we can see that $|N_2(uv)|$ is at most 18.

Now, consider the vertex w. We see that w cannot be adjacent to any of the vertices u_i, for $i = [\![3]\!]$, due to the condition on the girth. However, w can have a neighbour adjacent to each of the vertices u_i. Suppose that w has a neighbour adjacent to a u_i. Let this neighbour be x. Let a neighbour of u_i, distinct from u and distinct from x, be y. Then, w cannot be adjacent to y, due to the condition on the girth. This means that w has at most one neighbour adjacent to any u_i. Then, we can recolour each of the three edges of w, distinct from vw and wx, with a colour from the set of colours assigned to uu_i. Then, we will be able to colour the 2-neighbourhood of uv using at most 15 colours. So, we will have a colour available in L to colour uv. This means that we will have a strong edge colouring using at most 16 colours, which is a contradiction.

Therefore, if a 4-vertex u is adjacent to exactly one 2-vertex and a 5-vertex, then u has at most one 3-neighbour. □

Lemma 55. *Every 4-vertex u in H, having no 2-neighbour, can be adjacent to at most three 3-vertices. Moreover, if u is adjacent to exactly three 3-vertices, then the remaining neighbour of u is a 5-vertex.*

Proof. Suppose that H contains a 4-vertex u adjacent to four 3-vertices. Let v be a 3-neighbour of u. Let the remaining neighbours of u be u_1, u_2 and u_3. Let the neighbours of v, distinct from u, be w_1 and w_2. Without loss of generality, we have that $d(w_1) = d(w_2) = \Delta = 5$.

Consider the 2-neighbourhood of the edge uv. Then, we can see that $|N_2(uv)|$ is at most 19.

Now, consider the vertices w_1 and w_2. We see that w_1 cannot be adjacent to the vertex w_2, due to the condition on the girth. Furthermore, a neighbour of w_1 cannot be adjacent to w_2, due to the condition on the girth. However, a neighbour of w_1, say x, distinct from v, can be adjacent to a neighbour of w_2, say y, distinct from v. Suppose that x is adjacent to y. Then, no other neighbour of w_1 can be adjacent to y, due to the condition on the girth. This means that each neighbour of w_1 must be adjacent to a distinct neighbour of w_2. Then, we can colour the edges of w_1, distinct from vw_1, and the edges of w_2, distinct from vw_2, with the same set of four colours. Then, we will be able to colour the 2-neighbourhood of uv with at most 15 colours. Then, we will have a colour available in L to colour uv. This means that we will have a strong edge colouring using at most 16 colours, which is a contradiction.

Therefore, every 4-vertex u in H, having no 2-neighbour, can be adjacent to at most three 3-vertices.

Now, suppose that H contains a 4-vertex u adjacent to exactly three 3-vertices and no 2-neighbour. Let v be a 3-neighbour of u. Let the remaining two 3-neighbours of u be u_1 and u_2. Suppose further that u is not adjacent to a 5-vertex. Then, the remaining neighbour of u, say u_3, must be a 4-vertex. Let the neighbours of v, distinct from u, be w_1 and w_2. Without loss of generality, we have that $d(w_1) = d(w_2) = \Delta = 5$.

Consider the 2-neighbourhood of the edge uv. Then, we can see that $|N_2(uv)|$ is at most 20.

Now, consider the vertices w_1 and w_2. We see that w_1 cannot be adjacent to the vertex w_2, due to the condition on the girth. Furthermore, a neighbour of w_1 cannot be adjacent to w_2, due to the condition on the girth. However, a neighbour of w_1, say x, distinct from v, can be adjacent to a neighbour of w_2, say y, distinct from v. Suppose that x is adjacent to y. Then, no other neighbour of w_1 can be adjacent to y, due to the condition on the girth. This means that each neighbour of w_1 must be adjacent to a distinct neighbour of w_2. Then, we can colour the edges of w_1, distinct from vw_1, and the edges of w_2, distinct from vw_2, with the same set of four colours. This means that we can colour the 2-neighbourhood of uv with at most 16 colours.

Now, consider the vertex w_1. We see that w_1 cannot be adjacent to any of the vertices u_i, for $i = [\![3]\!]$, due to the condition on the girth. However, w_1 can have a neighbour adjacent to each of the vertices u_i. Suppose that w_1 has a neighbour adjacent to a u_i. Let this neighbour be x. Let a neighbour of u_i, distinct from u and distinct from x, be y. Then, w cannot be adjacent to y, due to the condition on the girth. This means that w_1 has at most one neighbour adjacent to any u_i. Then, we can recolour an edge of w_1, distinct from vw_1 and w_1x, with the colour assigned to uu_i. Then, we will be able to colour the 2-neighbourhood of uv using at most 15 colours. So, we will have a colour available in L to colour uv. This means that we will have a strong edge colouring using at most 16 colours, which is a contradiction.

Therefore, if a 4-vertex u is adjacent to exactly three 3-vertices, then the remaining neighbour of u is a 5-vertex. □

Lemma 56. *Every 5-vertex u in H can be adjacent to at most two 1-vertices. Moreover, if u is adjacent to exactly two 1-vertices, then the remaining neighbours of u must all be 5-vertices.*

Proof. Suppose H contains a 5-vertex u adjacent to at least three 1-vertices. Without loss of generality, we can assume that u is adjacent to exactly three 1-vertices and the remaining two neighbours of u are both 5-vertices. Let v be a 1-neighbour of u. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using 16 colours or less.

We have that $|N_2(uv)|$ in H is at most 12. Then, $|L \backslash SC_\phi(N_2(uv))| \geqslant 16 - 12 = 4$. Thus, there is a colour available in L to colour uv in H, so we may extend the colouring ϕ to H. This means that H can be strongly edge coloured using less than 16 colours, which is a contradiction.

Hence, every 5-vertex u in H can be adjacent to at most two 1-vertices.

Now, suppose that u is adjacent to exactly two 1-vertices and u has at most two 5-neighbours. Without loss of generality, we can assume that u has exactly two 5-neighbours and we can further assume that the remaining neighbour of u is a 4-vertex. Let v be a 1-neighbour of u. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using 16 colours or less.

We have that $|N_2(uv)|$ in H is at most 15. Then, $|L \backslash SC_\phi(N_2(uv))| \geqslant 16 - 15 = 1$. Thus, there is a colour available in L to colour uv in H, so we may extend the colouring ϕ to H. This means that H can be strongly edge coloured using at most 16 colours, which is a contradiction.

Hence, if u is adjacent to exactly two 1-vertices, then the remaining neighbours of u must all be 5-vertices. □

Lemma 57. *If a 5-vertex u in H is adjacent to exactly one 1-vertex, then u has at most one 2-neighbour. Furthermore, if u is adjacent to exactly one 1-vertex and exactly one 2-vertex, then u must be adjacent to two 5-vertices and the remaining neighbour of u must be a 4^+-vertex.*

Proof. Suppose that H contains a 5-vertex u adjacent to exactly one 1-vertex v and u has at least two 2-neighbours. Without loss of generality, we can assume that u has exactly two 2-neighbours and we can further assume that the remaining two neighbours of u are both 5-vertices. Let H' = H - $\{uv\}$. Then, we can colour the graph H' using 16 colours or less.

We have that $|N_2(uv)|$ in H is at most 14. Then, $|L \backslash SC_\phi(N_2(uv))| \geqslant 16 - 14 = 2$. Thus, there is a colour available in L to colour uv in H, so we may extend the colouring ϕ to H. This means that H can be strongly edge coloured using less than 16 colours, which is a contradiction.

Hence, if a 5-vertex u in H is adjacent to exactly one 1-vertex, then u has at most one 2-neighbour.

Now, suppose that u is adjacent to exactly one 1-vertex v and u has exactly one 2-neighbour. Suppose further that u is adjacent to at most one 5-vertex. Then, without loss of generality, we can assume that u is adjacent to exactly one 5-vertex and the remaining two neighbours of u are both 4-vertices. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using 16 colours or less.

We have that $|N_2(uv)|$ in H is at most 15. Then, $|L \backslash SC_\phi(N_2(uv))| \geqslant 16 - 15 = 1$. Thus, there is a colour available in L to colour uv in H, so we may extend the colouring ϕ to H. This means that H can be strongly edge coloured using at most 16 colours, which is a contradiction.

Therefore, if u is adjacent to exactly one 1-vertex and exactly one 2-vertex, then u must be adjacent to two 5-vertices and the remaining neighbour of u must be a 4^+-vertex. □

Lemma 58. *If a 5-vertex u in H is adjacent to exactly one 1-vertex and u has no 2-neighbour, then u is adjacent to at most two 3-vertices. Moreover, if u is adjacent to exactly one 1-vertex, exactly two 3-vertices and u has no 2-neighbour, then the remaining neighbours of u must both be 5-vertices. Furthermore, if u is adjacent to exactly one 1-vertex, exactly one 3-vertex* *and u has no other 3^{-}-neighbour, then u is adjacent to at most two 4-vertices.*

Proof. Suppose that H contains a 5-vertex u adjacent to exactly one 1-vertex v and u has at least three 3-neighbours. Suppose further that u has no 2-neighbour. Without loss of generality, we can assume that u has exactly three 3-neighbours and we can further assume that the remaining neighbour of u is a 5-vertex. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using 16 colours or less.

We have that $|N_2(uv)|$ in H is at most 14. Then, $|L \backslash SC_{\phi}(N_2(uv))| \geqslant 16 - 14 = 2$. Thus, there is a colour available in L to colour uv in H, so we may extend the colouring ϕ to H. This means that H can be strongly edge coloured using less than 16 colours, which is a contradiction.

Hence, if a 5-vertex u in H is adjacent to exactly one 1-vertex and u has no 2-neighbour then, u is adjacent to at most two 3-vertices.

Now, suppose that u is adjacent to exactly one 1-vertex v and u has exactly two 3-neighbours. Suppose further that u does not have a 2-neighbour and is adjacent to at most one 5-vertex. Then, without loss of generality, we can assume that u is adjacent to exactly one 5-vertex and the remaining neighbour of u is a 4-vertex. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using 16 colours or less.

We have that $|N_2(uv)|$ in H is at most 15. Then, $|L \backslash SC_{\phi}(N_2(uv))| \geqslant 16 - 15 = 1$. Thus, there is a colour available in L to colour uv in H, so we may extend the colouring ϕ to H. This means that H can be strongly edge coloured using at most 16 colours, which is a contradiction.

Hence, if u is adjacent to exactly one 1-vertex, exactly two 3-vertices and u has no 2-neighbour then, the remaining two neighbours of u must both be 5-vertices.

Now, suppose that u is adjacent to exactly one 1-vertex v and u has exactly one 3-neighbour. Suppose further that u does not have a 2-neighbour and the remaining three neighbours of u are all 4-vertices. Let $H' = H$ - $\{uv\}$. Then, we can colour the graph H' using 16 colours or less.

We have that $|N_2(uv)|$ in H is at most 15. Then, $|L \backslash SC_\phi(N_2(uv))| \geqslant 16 - 15 = 1$. Thus, there is a colour available in L to colour uv in H, so we may extend the colouring ϕ to H. This means that H can be strongly edge coloured using at most 16 colours, which is a contradiction.

Hence, if u is adjacent to exactly one 1-vertex, exactly one 3-vertex and u has no other 3^--neighbour, then u is adjacent to at most two 4-vertices. □

Lemma 59. *If a 5-vertex u in H is adjacent to exactly one 1-vertex and u has no other 3^--neighbour, then, u is adjacent to at most three 4-vertices.*

Proof. Suppose H contains a 5-vertex u adjacent to exactly one 1-vertex v and u has no other 3^--neighbour. Suppose further that u is adjacent to four 4-vertices. Let a 4-neighbour of u be v_1 and let another 4-neighbour of u be v_2.

Consider the 2-neighbourhood of uv. We have that $|N_2(uv)|$ in H is at most 16.

Now, consider the vertices v_1 and v_2. We see that v_1 cannot be adjacent to v_2, nor can a neighbour of v_1, distinct from u, be adjacent to v_2, due to the condition on the girth. We can, however, have that a neighbour of v_1, say x, distinct from u, can be adjacent to a neighbour of v_2, say y, distinct from u. Suppose x is adjacent to y. Then, x cannot be adjacent to any other neighbour of v_2, distinct from u. This means that each neighbour of v_1 must be adjacent to a distinct neighbour of v_2. Then, we can recolour an edge of v_1, distinct from uv_1 and distinct v_1x, with a colour from an edge of v_2, distinct from uv_2 and distinct from v_2y. This means that we can colour the 2-neighbourhood of uv with at most 15 colours. So, we will have a colour available in L to colour uv. This means that H can be strongly edge coloured using at most 16 colours, which is a contradiction.

Hence, if a 5-vertex u in H is adjacent to exactly one 1-vertex and u has no other 3^--neighbour, then, u is adjacent to at most three 4-vertices. □

Lemma 60. *Every 5-vertex u in H that does not have a 1-neighbour can be adjacent to at most one 2-vertex. If u is adjacent to exactly one 2-vertex and u has no 1-neighbour, then u has no other 3^--neighbour and u has at most one 4-neighbour.*

Proof. Suppose H contains a 5-vertex u that does not have a 1-neighbour and is adjacent to at least two 2-vertices. Without loss of generality, we can assume that u is adjacent to exactly two 2-vertices and the remaining three neighbours of u are all 5-vertices, say u_1, u_2 and u_3. Let a 2-neighbour of u be v and let the remaining 2-neighbour of u be u_4. Let the neighbour of v, distinct from u, be w. Without loss of generality, we can assume that w is a 5-vertex.

Consider the 2-neighbourhood of uv. Then, we can see that $|N_2(uv)|$ is at most 22.

Now, consider the vertex w. We see that w cannot be adjacent to any of the vertices u_i, for $i = [\![4]\!]$, due to the condition on the girth. However, w can have a neighbour adjacent to each of the vertices u_i. Suppose that w has a neighbour adjacent to a u_i. Let this neighbour be x. Let a neighbour of u_j, for $j = [\![3]\!]$, distinct from u and distinct from x, be y. Then, w cannot be adjacent to y, due to the condition on the girth. This means that w has at most one neighbour adjacent to any u_i. Then, we can recolour each edge of w, distinct from vw, with a colour from the set of colours assigned to each uu_i. This means that we can colour the 2-neighbourhood of uv with at most 18 colours.

Now, consider the vertex u_4. We see that u_4 has only one neighbour distinct from u. Furthermore, we see that u_4 cannot be adjacent to any of the vertices u_j, for $j = [\![3]\!]$, nor can a neighbour of u_4 be adjacent to any u_j, due to the condition on the girth. However, u_4 can have a neighbour adjacent to a neighbour of each u_j. Suppose that u_4 has a neighbour t adjacent to a neighbour of each u_j. Then, t is adjacent to at most one neighbour of each u_j, due to the condition on the girth. This means that we can colour one edge of each u_j, distinct from uu_j and ut, with the colour u_4t. Then, we will need at most 15 colours to colour the 2-neighbourhood of uv. Therefore, we will have a colour available in L to colour the edge uv. This means that we will have a strong edge colouring using at most 16 colours, giving a contradiction.

Therefore, every 5-vertex u in H that does not have a 1-neighbour can be adjacent to at most one 2-vertex.

Now, suppose H contains a 5-vertex u that does not have a 1-neighbour and is adjacent to exactly one 2-vertex, say v. Suppose further that u has at least one 3^--neighbour. Without loss of generality, we can assume that u is adjacent to exactly one 3-vertex, say u_4, and the remaining three neighbours of u are all 5-vertices, say u_1, u_2 and u_3. Let the neighbour of v, distinct from u, be w. Without loss of generality, we can assume that w is a 5-vertex.

Consider the 2-neighbourhood of uv. Then, we can see that $|N_2(uv)|$ is at most 23.

Now, consider the vertex w. We see that w cannot be adjacent to any of the vertices u_i, for $i = [\![4]\!]$, due to the condition on the girth. However, w can have a neighbour adjacent to each of the vertices u_i. Suppose that w has a neighbour adjacent to a u_i. Let this neighbour be x. Let a neighbour of u_i, distinct from u and distinct from x, be y. Then, w cannot be adjacent to y, due to the condition on the girth. This means that w has at most one neighbour adjacent to any u_i. Then, we can recolour each edge of w, distinct from vw, with a colour from the set of colours assigned to each uu_i. This means that we can colour the 2-neighbourhood of uv with at most 19 colours.

Now, consider the vertex u_4. We see that u_4 has only two neighbours distinct from u. Furthermore, we see that u_4 cannot be adjacent to any of the vertices u_j, for $j = [\![3]\!]$, nor can a neighbour of u_4 be adjacent to any u_j, due to the condition on the girth. However, u_4 can have a neighbour adjacent to a neighbour of each u_j. Suppose that u_4 has a neighbour t adjacent to a neighbour of each u_j. Then, t is adjacent to at most one neighbour of each u_j, due to the condition on the girth. Suppose that this neighbour of u_j is z_j. Then, the neighbour of u_4, distinct from t and distinct from u, cannot be adjacent to z_j, due to the condition on the girth. This means that each neighbour of u_4 is adjacent to a distinct neighbour of u_j. Then, we can colour two edges of each u_j, distinct from uu_j, with a colour from the set of colours assigned to the edges of u_4, distinct from uu_4. Then, we will need no more than 15 colours to colour the 2-neighbourhood of uv. Therefore, we will have a colour available in L to colour the edge uv. This means that we will have a strong edge colouring using at most 16 colours, giving a contradiction.

Therefore, if u is adjacent to exactly one 2-vertex and u has no 1-neighbour, then u has no other 3^{-}-neighbour and u has at most one 4-neighbour. □

Lemma 61. *Every 5-vertex u in H that has no 2^{-}-neighbour is adjacent to at most three 3-vertices. Furthermore, if u is adjacent to exactly three 3-vertices then, u must be adjacent to at least one 5-vertex. If u is adjacent to at least one 3-vertex but no more than two 3-vertices, then u must be* *adjacent to at least two 5-vertices.*

Proof. Suppose H contains a 5-vertex u that has no 2^{-}-neighbour and is adjacent to at least four 3-vertices. Without loss of generality, we can assume that u is adjacent to exactly four 3-vertices and the remaining neighbour of u is a 5-vertex. Let a 3-neighbour of u be v. Then, let the remaining three 3-neighbours of u be u_1, u_2 and u_3. Let the 5-neighbour of u be u_4 and let the two neighbours of v, distinct from u, be w_1 and w_2. Without loss of generality, we can assume that each w_k, for $k = [\![2]\!]$, is a 5-vertex.

Consider the 2-neighbourhood of uv. Then, we can see that $|N_2(uv)|$ is at most 24.

Now, consider the vertices w_1 and w_2. We see that w_1 cannot be adjacent to the vertex w_2, due to the condition on the girth. Furthermore, a neighbour of w_1 cannot be adjacent to w_2, due to the condition on the girth. However, a neighbour of w_1, say x, distinct from v, can be adjacent to a neighbour of w_2, say y, distinct from v. Suppose that x is adjacent to y. Then, no other neighbour of w_1 can be adjacent to y, due to the condition on the girth. This means that each neighbour of w_1 must be adjacent to a distinct neighbour of w_2. Then, we can colour the edges of w_1, distinct from vw_1, and the edges of w_2, distinct from vw_2, with the same set of four colours. Then, we will be able to colour the 2-neighbourhood of uv with at most 20 colours.

Now, consider the vertex w_k, for $k = [\![2]\!]$. We see that w_k cannot be adjacent to any of the vertices u_i, for $i = [\![4]\!]$, due to the condition on the girth. However, w_k can have a neighbour adjacent to each of the vertices u_i. Suppose that w_k has a neighbour adjacent to a u_i. Let this neighbour be x. Let a neighbour of u_i, distinct from u and distinct from x, be y. Then, w_k cannot be adjacent to y, due to the condition on the girth. This means that w_k has at most one neighbour adjacent to any u_i. Then, we can recolour each edge of w_k, distinct from vw_k, with a colour from the set of colours assigned to each uu_i. This means that we can colour the 2-neighbourhood of uv with at most 16 colours.

Now, consider the vertex u_1. We see that u_1 has only two neighbours distinct from u. Furthermore, we see that u_1 cannot be adjacent to any of the vertices u_j, for $j = [\![3]\!]$, nor can a neighbour of u_1 be adjacent to any u_j, due to the condition on the girth. However, u_1 can have a neighbour adjacent to a neighbour of each u_j. Suppose that u_1 has a neighbour t adjacent to a neighbour of each u_j. Then, t is adjacent to at most one neighbour of each u_j, due to the condition on the girth. Suppose that this neighbour of u_j is z_j. Then, the neighbour of u_1, distinct from t and distinct from u, cannot be adjacent to z_j, due to the condition on the girth. This means that each neighbour of u_1 is adjacent to a distinct neighbour of u_j. Then, we can colour an edge of u_4, distinct from uu_4, with a colour from the set of colours assigned to the edges of u_1, distinct from uu_1. Then, we will need no more than 15 colours to colour the 2-neighbourhood of uv. Therefore, we will have a colour available in L to colour the edge uv. This means that we will have a strong edge colouring using at most 16 colours, giving a contradiction.

Therefore, every 5-vertex u in H that has no 2^--neighbour is adjacent to at most three 3-vertices.

Now, suppose that u has no 2^- neighbour and is adjacent to exactly three 3-vertices and has no 5-neighbour. Then, the remaining two neighbours of u must be 4-vertices. Let one 3-neighbour of u be v. Consider the configuration in the above case. If we remove an edge from u_4, distinct from uu_4, and add this edge to u_3 then, we have the same configuration as the one in this current case. Then, we can colour the 2-neighbourhood of uv as in the previous case. This means that we will have a strong edge colouring using at most 16 colours, giving a contradiction.

Therefore, if u is adjacent to exactly three 3-vertices and has no 2^--neighbour then, u must be adjacent to at least one 5-vertex.

Now, suppose H contains a 5-vertex u that has no 2^--neighbour and is adjacent to exactly three 3-vertices. Suppose further that u has no 5-neighbour. Without loss of generality, we can assume that the remaining neighbours of u are both 4-vertices. Let a 3-neighbour of u be v. Then, let the two 4-neighbours of u be u_1 and u_2 and let the remaining two 3-neighbours of u be u_3 and u_4. Let the two neighbours of v, distinct from u, be w_1 and w_2. Without loss of generality, we can assume that each w_k, for $k = [\![2]\!]$, is a 5-vertex.

Consider the 2-neighbourhood of uv. Then, we can see that $|N_2(uv)|$ is at most 24.

Now, consider the vertices w_1 and w_2. We see that w_1 cannot be adjacent to the vertex w_2, due to the condition on the girth. Furthermore, a neighbour of w_1 cannot be adjacent to w_2, due to the condition on the girth. However, a neighbour of w_1, say x, distinct from v, can be adjacent to a neighbour of w_2, say y, distinct from v. Suppose that x is adjacent to y. Then, no other neighbour of w_1 can be adjacent to y, due to the condition on the girth. This means that each neighbour of w_1 must be adjacent to a distinct neighbour of w_2. Then, we can colour the edges of w_1, distinct from vw_1, and the edges of w_2, distinct from vw_2, with the same set of four colours. Then, we will be able to colour the 2-neighbourhood of uv with at most 20 colours.

Now, consider the vertex w_k, for $k = [\![2]\!]$. We see that w_k cannot be adjacent to any of the vertices u_i, for $i = [\![4]\!]$, due to the condition on the girth. However, w_k can have a neighbour adjacent to each of the vertices u_i. Suppose that w_k has a neighbour adjacent to a u_i. Let this neighbour be x. Let a neighbour of u_i, distinct from u and distinct from x, be y. Then, w_k cannot be adjacent to y, due to the condition on the girth. This means that w_k has at most one neighbour adjacent to any u_i. Then, we can recolour each edge of w_k, distinct from vw_k, with a colour from the set of colours assigned to each uu_i. This means that we can colour the 2-neighbourhood of uv with at most 16 colours.

Now, consider the vertex u_4. We see that u_4 has only two neighbours distinct from u. Furthermore, we see that u_4 cannot be adjacent to any of the vertices u_j, for $j = [\![3]\!]$, nor can a neighbour of u_4 be adjacent to any u_j, due to the condition on the girth. However, u_4 can have a neighbour adjacent to a neighbour of each u_j. Suppose that u_4 has a neighbour t adjacent to a neighbour of each u_j. Then, t is adjacent to at most one neighbour of each u_j, due to the condition on the girth. Suppose that this neighbour of u_j is z_j. Then, the neighbour of u_4, distinct from t and distinct from u, cannot be adjacent to z_j, due to the condition on the girth. This means that each neighbour of u_4 is adjacent to a distinct neighbour of u_j. Then, we can colour an edge of u_1, distinct from uu_1, with a colour from the set of colours assigned to the edges of u_4, distinct from uu_4. Then, we will need no more than 15 colours to colour the 2-neighbourhood of uv. Therefore, we will have a colour available in L to colour the edge uv. This means that we will have a strong edge colouring using at most 16 colours, giving a contradiction.

Therefore, if a 5-vertex u in H that has no 2^--neighbour is adjacent to exactly three 3-vertices then, u must be adjacent to at least one 5-vertex.

Now, suppose H contains a 5-vertex u that has no 2^--neighbour and is adjacent to at least one 3-vertex. Suppose further that u has at most one 5-neighbour. Without loss of generality, we can assume that u is adjacent to exactly one 3-vertex, say v and exactly one 5-vertex, say u_1. Let the remaining three 4-neighbours of u be u_2, u_3 and u_4. Let the neighbours of v, distinct from u, be w_1 and w_2. Then, without loss of generality, we may assume that each w_k, for $k = [\![2]\!]$, is a 5-vertex.

Consider the 2-neighbourhood of uv. Then, we can see that $|N_2(uv)|$ is at most 27.

Now, consider the vertices w_1 and w_2. We see that w_1 cannot be adjacent to the vertex w_2, due to the condition on the girth. Furthermore, a neighbour of w_1 cannot be adjacent to w_2, due to the condition on the girth. However, a neighbour of w_1, say x, distinct from v, can be adjacent to a neighbour of w_2, say y, distinct from v. Suppose that x is adjacent to y. Then, no other neighbour of w_1 can be adjacent to y, due to the condition on the girth. This means that each neighbour of w_1 must be adjacent to a distinct neighbour of w_2. Then, we can colour the edges of w_1, distinct from vw_1, and the edges of w_2, distinct from vw_2, with the same set of four colours. Then, we will be able to colour the 2-neighbourhood of uv with at most 23 colours.

Now, consider the vertex w_k, for $k = [\![2]\!]$. We see that w_k cannot be adjacent to any of the vertices u_i, for $i = [\![4]\!]$, due to the condition on the girth. However, w_k can have a neighbour adjacent to each of the vertices u_i. Suppose that w_k has a neighbour adjacent to a u_i. Let this neighbour be x. Let a neighbour of u_i, distinct from u and distinct from x, be y. Then, w_k cannot be adjacent to y, due to the condition on the girth. This means that w_k has at most one neighbour adjacent to any u_i. Then, we can recolour each edge of w_k, distinct from vw_k, with a colour from the set of colours assigned to each uu_i. This means that we can colour the 2-neighbourhood of uv with at most 19 colours.

Now, consider the vertex u_4. We see that u_4 has only two neighbours distinct from u. Furthermore, we see that u_4 cannot be adjacent to any of the vertices u_j, for $j = [\![3]\!]$, nor can a neighbour of u_4 be adjacent to any u_j, due to the condition on the girth. However, u_4 can have a neighbour adjacent to a neighbour of each u_j. Suppose that u_4 has a neighbour t adjacent to a neighbour of each u_j. Then, t is adjacent to at most one neighbour of each u_j, due to the condition on the girth. Suppose that this neighbour of u_j is z_j. Then, the neighbour of u_4, distinct from t and distinct from u, cannot be adjacent to z_j, due to the condition on the girth. This means that each neighbour of u_4 is adjacent to a distinct neighbour of u_j. Then, we can recolour two edges of u_j, distinct from uu_j, with a colour from the set of colours assigned to the edges of u_4, distinct from uu_4. Then, we will need no more than 15 colours to colour the 2-neighbourhood of uv. Therefore, we will have a colour available in L to colour the edge uv. This means that we will have a strong edge colouring using at most 16 colours, giving a contradiction.

Therefore, if a 5-vertex u having no 2^--neighbour is adjacent to at least one 3-vertex, but no more than two 3-vertices, then, u must be adjacent to at least two 5-vertices. □

Lemma 62. *If a 5-vertex u in H has no 3^--neighbour, then u is adjacent to at most two 4-vertices.*

Proof. Suppose H contains a 5-vertex u with no 3^--neighbour that is adjacent to at least three 4-vertices. Without loss of generality, we can assume that u is adjacent to exactly three 4-vertices and the remaining two neighbours of u are both 5-vertices. Let a 4-neighbour of u be v. Let the remaining two 4-neighbours of u be u_1 and u_2. Let the two 5-neighbours of u be u_3 and u_4 and let the neighbours of v, distinct from u, be w_1, w_2 and w_3. Without loss of generality, we can assume that each w_k, for $k = [\![3]\!]$, is a 5-vertex.

Consider the 2-neighbourhood of uv. Then, we can see that $|N_2(uv)|$ is at most 33.

Now, consider the vertices w_k. We see that w_k cannot be adjacent to the vertex w_m, for $m = [\![3]\!]; m \neq k$, due to the condition on the girth. Furthermore, a neighbour of w_k cannot be adjacent to w_m, due to the condition on the girth. However, a neighbour of w_k, say x, distinct from v, can be adjacent to a neighbour of w_m, say y, distinct from v. Suppose that x is adjacent to y. Then, no other neighbour of w_k can be adjacent to y, due to the condition on the girth. This means that each neighbour of w_k must be adjacent to a distinct neighbour of w_m. Then, we can colour the edges of w_k, distinct from vw_k, and the edges of w_m, distinct from vw_m, with the same set of four colours. Then, we will be able to colour the 2-neighbourhood of uv with at most 25 colours.

Now, consider the vertex w_k, for $k = [\![3]\!]$. We see that w_k cannot be adjacent to any of the vertices u_i, for $i = [\![4]\!]$, due to the condition on the girth. However, w_k can have a neighbour adjacent to each of the vertices u_i. Suppose that w_k has a neighbour adjacent to a u_i. Let this neighbour be x. Let a neighbour of u_i, distinct from u and distinct from x, be y. Then, w_k cannot be adjacent to y, due to the condition on the girth. This means that w_k has at most one neighbour adjacent to any u_i. Then, we can recolour each edge of w_k, distinct from vw_k, with a colour from the set of colours assigned to each uu_i. This means that we can colour the 2-neighbourhood of uv with at most 21 colours.

Now, consider the vertex u_1. We see that u_1 has only three neighbours distinct from u. Furthermore, we see that u_1 cannot be adjacent to any of the vertices u_j, for $j = 2, 3, 4$, nor can a neighbour of u_1 be adjacent to any u_j, due to the condition on the girth. However, u_1 can have a neighbour adjacent to a neighbour of each u_j. Suppose that u_1 has a neighbour t adjacent to a neighbour of each u_j. Then, t is adjacent to at most one neighbour of each u_j, due to the condition on the girth. Suppose that this neighbour of u_j is z_j. Then, the neighbour of u_1, distinct from t and distinct from u, cannot be adjacent to z_j, due to the condition on the girth. This means that each neighbour of u_1 is adjacent to a distinct neighbour of u_j. Then, we can recolour two edges of each u_j, distinct from uu_j, with a colour from the set of colours assigned to the edges of u_1, distinct from uu_1. Then, we will need at most 15 colours to colour the 2-neighbourhood of uv. Therefore, we will have a colour available in L to colour the edge uv. This means that we will have a strong edge colouring using at most 16 colours, giving a contradiction.

Therefore, a 5-vertex u having no 3^--neighbour is adjacent to at most two 4-vertices. □

Now, we proceed to use the discharging method on the graph H.

Discharging procedure We use the discharging method as in Section 3.3.3 of this dissertation. We use a multiple of $k = 5$ for Equation (3.1), as detailed in Section 3.3.3.

We define the discharging rules as follows:

(R1): Every face gives 2 to each incident 1-vertex.

(R2): Every 1-vertex gives $\frac{1}{2}$ to each adjacent 5-vertex.

(R3): Every 4-vertex gives 1 to each adjacent 2-vertex.

(R4): Every 4-vertex gives $\frac{1}{2}$ to each adjacent 3-vertex.

(R5): Every 5-vertex gives 1 to each adjacent 2-vertex.

(R6): Every 5-vertex gives $\frac{1}{2}$ to each adjacent 3-vertex.

(R7): Every 5-vertex gives 1 to each adjacent 4-vertex.

First, consider the charge on a k-face $f \in F(H)$. The initial charge on f is $ch(f) = k - 5$. We have that $k \geqslant 5$, by the condition on the girth. If f has p incident 1-vertices, then $k \geqslant 5 + 2p$.

Then, by (R1), the final charge on f is $ch'(f) \geqslant k - 5 - 2p \geqslant 0$.

So, $ch'(f) \geqslant 0$.

Now, we consider the charge on a k-vertex $v \in V(H)$.

Case $k = 1$: We have that $ch(v) = \frac{3}{2}(1) - 5 = -\frac{7}{2}$.

By Lemma 51, v must be adjacent to a 5-vertex. Furthermore, by Lemma 56, this 5-vertex has at most one other 1-neighbour. Then, every 1-vertex in H must be incident to at least two 5^+-faces. Therefore, by (R1) and (R2), the final charge on v is $ch'(v) \geqslant -\frac{7}{2} + 2(2) - \frac{1}{2}(1) = 0$.

So, $ch'(v) \geqslant 0$.

Case $k = 2$: We have that $ch(v) = \frac{3}{2}(2) - 5 = -2$.

By Lemma 52, v must be adjacent to two 4^+-vertices.

- Suppose v is adjacent to two 4-vertices. Then, by (R3), the final charge on v is $ch'(v) = -2 + (1)2 = 0$.
- Suppose v is adjacent to one 4-vertex and one 5-vertex. Then, by (R3) and (R5), the final charge on v is $ch'(v) = -2 + 1 + 1 = 0$.
- Suppose v is adjacent to two 5-vertices. Then, by (R5), the final charge on v is $ch'(v) = -2 + (1)2 = 0$.

So, $ch'(v) = 0$.

Case $\underline{k = 3}$: We have that $ch(v) = \frac{3}{2}(3) - 5 = -\frac{1}{2}$.

By Lemma 53, v is adjacent to at least one 4^+-vertex. Then, by (R4) and (R6), the final charge on v is $ch'(v) \geqslant -\frac{1}{2} + \frac{1}{2} = 0$.

So, $ch'(v) \geqslant 0$.

Case $\underline{k = 4}$: We have that $ch(v) = \frac{3}{2}(4) - 5 = 1$.

- Suppose v is adjacent to a 2-vertex. Then, by Lemma 54, v is adjacent to at most two 2-vertices.

  - Suppose v is adjacent to one 2-vertex and v has no 5-neighbour. Then, by Lemma 54, v is adjacent to three 4-vertices. So, by (R3), the final charge on v is $ch'(v) = 1 - 1 = 0$. Suppose v is adjacent to one 2-vertex and v has a 5-neighbour. Then, by Lemma 54, v is adjacent to at most one 3-vertex. So, by (R3), (R4) and (R7), the final charge on v is $ch'(v) \geqslant 1 - 1 - \frac{1}{2}(1) + 1 = \frac{1}{2} > 0$.
  - Suppose v is adjacent to two 2-vertices. Then, by Lemma 54, v must be adjacent to at least one 5-vertex. So, by (R3) and (R7), the final charge on v is $ch'(v) \geqslant 1 - 1(2) + 1 = 0$.

- Suppose v is not adjacent to a 2-vertex. Then, v is adjacent to at most three 3-vertices, by Lemma 55.

  - Suppose v is adjacent to exactly one 3-vertex. Then, by Lemma 55, v has at most three 5-neighbours. So, by (R4) and (R7), the final charge on v is $ch'(v) \geqslant 1 - \frac{1}{2}(1) = \frac{1}{2} > 0$.
  - Suppose v is adjacent to exactly two 3-vertices. Then, by Lemma 55, v has at most two 5-neighbours. So, by (R4) and (R7), the final charge on v is $ch'(v) \geqslant 1 - \frac{1}{2}(2) = 0$.
  - Suppose v is adjacent to exactly three 3-vertices. Then, by Lemma 55, the remaining neighbour of v is a 5-vertex. So, by (R4) and (R7), the final charge on v is $ch'(v) = 1 - \frac{1}{2}(3) + 1 = \frac{1}{2} > 0$.

So, $ch'(v) \geqslant 0$.

<u>**Case** $k = 5$</u> : We have that $ch(v) = \frac{3}{2}(5) - 5 = \frac{5}{2}$.

- Suppose v is adjacent to a 1-vertex. Then, by Lemma 56, v is adjacent to at most two 1-vertices.
    - Suppose v is adjacent to exactly one 1-vertex. Then, by Lemma 57, v has at most one 2-neighbour.
        * Suppose v is adjacent to exactly one 2-vertex. Then, by Lemma 57, v is adjacent to at most one 4-vertex and v has no 3-neighbour. So, by (R2), (R5) and (R7), the final charge on v is $ch'(v) \geqslant \frac{5}{2} + \frac{1}{2}(1) - 1 - 1 = 1 > 0$.
        * Suppose v has no 2-neighbour. Then, by Lemma 58, v is adjacent to at most two 3-vertices. If v is adjacent to exactly two 3-vertices and v has no 2-neighbour then, by Lemma 58, v has no 4-neighbour. So, by (R2) and (R6), we have that the final charge on v is $ch'(v) = \frac{5}{2} + \frac{1}{2}(1) - \frac{1}{2}(2) = 2 > 0$. If v is adjacent to exactly one 3-vertex and v has no 2-neighbour, then, by Lemma 58, v has at most two 4-neighbours. Then, by (R2), (R6) and (R7), we have that the final charge on v is $ch'(v) \geqslant \frac{5}{2} + \frac{1}{2}(1) - \frac{1}{2}(1) - 1(2) = \frac{1}{2} > 0$. If v is adjacent to a 1-vertex and no other 3^{-}-neighbour, then, by Lemma 59, v is adjacent to at most three 4-vertices. So, by (R2) and (R7), we have that the final charge on v is $ch'(v) \geqslant \frac{5}{2} + \frac{1}{2}(1) - 1(3) = 0$.
    - Suppose v is adjacent to exactly two 1-vertices. Then, by Lemma 56, the remaining neighbours of v are all 5-vertices. So, by (R2), we have the final charge on v is $ch'(v) = \frac{5}{2} + \frac{1}{2}(2) = \frac{7}{2} > 0$.
- Suppose v is adjacent to a 2-vertex and v has no 1-neighbour. Then, by Lemma 60, v can only be adjacent to one 2-vertex. Furthermore, v has no 3-neighbour and v has at most one 4-neighbour. Then, by (R5) and (R7), we have that the final charge on v is $ch'(v) \geqslant \frac{5}{2} - 1 - 1 = \frac{1}{2} > 0$.
- Suppose v is adjacent to a 3-vertex and has no 2^{-}-neighbour. Then, by Lemma 61, v is adjacent to at most three 3-vertices. Furthermore, by Lemma 61, v has at most one 4-neighbour. So, by (R6) and (R7), we have that the final charge on v is $ch'(v) \geqslant \frac{5}{2} - \frac{1}{2}(3) - 1 = 0$.

- Suppose v is adjacent to a 4-vertex and has no 3^{-}-neighbour. Then, by Lemma 62, v is adjacent to at most two 4-vertices. So, by (R7), we have that the final charge on v is $ch'(v) \geqslant \frac{5}{2} - 1(2) = \frac{1}{2} > 0$.

Now, we see that the total final charge of H is a positive value. However, we obtain the following contradiction :

$$0 \leqslant \sum_{x \in V(H) \cup F(H)} ch'(x) = \sum_{x \in V(H) \cup F(H)} ch(x) = -10 < 0$$

Thus, the counterexample H does not exist and so the theorem is true.

This concludes the proof.

Chapter 8

Conclusion and Open Questions

Strong edge colouring of a graph G is a labeling of colours to the edges of G, such that no two edges at distance at most 2 have the same colour. The smallest integer k for which G can be strongly edge coloured is known as the strong chromatic index, denoted $\chi'_s(G)$. This dissertation considered the upper bound on the strong chromatic index for planar graphs.

In this dissertation, the concept of strong edge colouring, along with basic terminology associated with graphs, was introduced in Chapter 1. Thereafter, in Chapter 2, we presented important theorems that gave an upper bound for the strong chromatic index. Following this, in Chapter 3, we discussed the aims of the dissertation and proposed the research question and hypothesis of the dissertation. In particular, we hypothesised that the conjecture posed by Song *et al* [16] was indeed true. The conjecture hypothesised that *for planar graphs G with maximum degree $\Delta(G) = 5$ and girth $g = 5$, the strong chromatic index of G, $\chi'_s(G)$, is at most 16.* Additionally, in Chapter 3, we discussed the methodology that was used to prove this statement to be true. In Chapter 4, we proved Theorem 1 to Theorem 3, as listed in Chapter 2 of this dissertation. In particular, we showed that for a graph G with sufficiently large maximum degree $\Delta(G)$, $\chi'_s(G) \leqslant 1.93\Delta(G)^2$. We also proved that for a planar graph G with maximum degree $\Delta(G)$, $\chi'_s(G) \leqslant 4\chi'(G) \leqslant 4\Delta(G) + 4$. Finally, we showed that for a special class of planar graphs, $\chi'_s(G) = 4\Delta(G) - 4$, for $\Delta(G) \geqslant 2$. In Chapter 5, we proved Theorem 4 and Theorem 5, as listed in Chapter 2 of this dissertation. More specifically, we showed that for planar graphs G with maximum degree $\Delta(G) \geqslant 4$ and girth $g \geqslant 6$, $\chi'_s(G) \leqslant 3\Delta(G) + 5$. Furthermore, we proved that for the same set of planar graphs, $\chi'_s(G) \leqslant 3\Delta(G) + 1$. Following this, in Chapter 6, we proved Theorem 6, as listed in Chapter 2 of this dissertation. In particular, we proved that for planar graphs G with maximum degree $\Delta(G) \geqslant 7$, or for planar graphs G with maximum degree $\Delta(G) \geqslant 5$ and girth $g \geqslant 4$, or for planar graphs G with girth $g \geqslant 5$, $\chi'_s(G) \leqslant 4\Delta(G)$. Finally, in Chapter 7, we proved Theorem 7 and Theorem 8, as listed in Chapter 2 of this dissertation. More specifically, we showed that for a planar graph G with maximum degree $\Delta(G) \geqslant 6$ and girth g $\geqslant 5$, $\chi'_s(G) \leqslant 4\Delta(G) - 2$. Then, we proved that for planar graphs with maximum degree $\Delta(G) = 5$ and girth $g \geqslant 5$, $\chi'_s(G) \leqslant 19$. In Chapter 7, we also gave a proof for our hypothesis that was proposed in Section 3.2 of this dissertation. To be specific, we proved that for a planar graph G, with maximum degree $\Delta(G) = 5$ and girth $g = 5$, $\chi'_s(G) \leqslant 3\Delta(G) + 1 = 16$.

We give a summary of the results of the theorems in the table below :

	Δ(G) = 4	Δ(G) = 5	Δ(G) = 6	Δ(G) ≥ 7
No girth restriction	4Δ(G) + 4	4Δ(G) + 4	4Δ(G) + 4	4Δ(G)
g ≥ 4	4Δ(G) + 4	4Δ(G)	4Δ(G)	4Δ(G)
g = 5	4Δ(G)	3Δ(G) + 1	4Δ(G) - 2	4Δ(G) - 2
g ≥ 6	3Δ(G) + 1	3Δ(G) + 1	3Δ(G) + 1	3Δ(G) + 1

Table 8.1: Table showing strong chromatic index for planar graphs G with particular values for girth g and particular values for maximum degree $\Delta(G)$

We conclude the dissertation by noting the following open questions for further work :

Question 1 [2] For planar graphs G, with girth $g = 4$ and maximum degree $\Delta(G) = 4$, can we have that $\chi'(G) = \Delta(G)$?

If we can prove that we can indeed have that $\chi'(G) = \Delta(G)$ for Question 1, then we can conclude that for planar graphs G with girth $g \geqslant 4$ and maximum degree $\Delta(G) \geqslant 4$, $\chi'_s(G) \leqslant 4\Delta(G)$.

Question 2 [16] For planar graphs G, with girth $g = 5$ and maximum degree $\Delta(G) = 6$, can we have that $\chi'_s(G) \leqslant 3\Delta(G) + 1$?

We also propose two questions of our own :

Question 3 For planar graphs G, with girth $g = 5$ and maximum degree

$\Delta(G) = 4$, can we have that $\chi'_s(G) \leqslant 3\Delta(G) + 1$?

Question 4 For planar graphs G, with girth $g = 5$ and maximum degree $\Delta(G) \geqslant 7$, can we have that $\chi'_s(G) \leqslant 3\Delta(G) + 1$?

If we can prove the upper bounds for the strong chromatic index in Question 2, Question 3 and Question 4 to be true, then we can conclude that for planar graphs G, with girth $g \geqslant 5$ and maximum degree $\Delta(G) \geqslant 4$, $\chi'_s(G) \leqslant 3\Delta(G) + 1$.

Bibliography

[1] Lars Døvling Andersen. The strong chromatic index of a cubic graph is at most 10. *Discrete Mathematics*, 108(1-3):231–252, 1992.

[2] Julien Bensmail, Ararat Harutyunyan, Hervé Hocquard, and Petru Valicov. Strong edge-colouring of sparse planar graphs. *Discrete Applied Mathematics*, 179:229–234, 2014.

[3] John Adrian Bondy, Uppaluri Siva Ramachandra Murty, et al. *Graph theory with applications*, volume 290. Macmillan London, 1976.

[4] Henning Bruhn and Felix Joos. A stronger bound for the strong chromatic index. *Electronic Notes in Discrete Mathematics*, 49:277–284, 2015.

[5] Gary Chartrand and Ping Zhang. *Chromatic graph theory*. CRC press, 2019.

[6] Paul Erdös. Problems and results in combinatorial analysis and graph theory. In *Annals of discrete mathematics*, volume 38, pages 81–92. Elsevier, 1988.

[7] Ralph J Faudree, Richard H Schelp, András Gyárfás, and Zsolt Tuza. The strong chromatic index of graphs. *Ars Combinatoria*, 29:205–211, 1990.

[8] Jean-Luc Fouquet and Jean-Loup Jolivet. Strong edge-colorings of graphs and applications to multi-k-gons. *Ars Combinatoria A*, 16:141–150, 1983.

[9] JL Fouquet and JL Jolivet. Strong edge-coloring of cubic planar graphs, progress in graph theory, 1984.

[10] Hervé Hocquard, Mickaël Montassier, André Raspaud, and Petru Valicov. On strong edge-colouring of subcubic graphs. *Discrete Applied Mathematics*, 161(16-17):2467–2479, 2013.

[11] Peter Horák, He Qing, and William T Trotter. Induced matchings in cubic graphs. *Journal of Graph Theory*, 17(2):151–160, 1993.

[12] Dávid Hudák, Borut Lužar, Roman Soták, and Riste Škrekovski. Strong edge-coloring of planar graphs. *Discrete Mathematics*, 324:41–49, 2014.

[13] Xuechao Li and Rong Luo. Edge coloring of embedded graphs with large girth. *Graphs and Combinatorics*, 19(3):393–401, 2003.

[14] Michael Molloy and Bruce Reed. A bound on the strong chromatic index of a graph. *Journal of Combinatorial Theory, Series B*, 69(2):103–109, 1997.

[15] Daniel P Sanders and Yue Zhao. Planar graphs of maximum degree seven are class i. *Journal of Combinatorial Theory, Series B*, 83(2):201–212, 2001.

[16] Wen-Yao Song and Lian-Ying Miao. Strong edge-coloring of planar graphs. *Discussiones Mathematicae Graph Theory*, 37(4):845–857, 2017.

[17] Vadim G Vizing. Critical graphs with given chromatic class (in russian). *Metody Discret. Analiz.*, 5:9–17, 1965.