International Journal of Geometric Methods in Modern Physics
Vol. 21, No. 9 (2024) 2450175 (30 pages)

DOI: 10.1142/S0219887824501755

Perfect fluid locally rotationally symmetric Bianchi Type-I spacetimes admitting concircular vector fields in $f(T)$ gravity

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Received 15 September 2023
Accepted 19 February 2024
Published 13 April 2024

We obtained the solutions of Einstein's Field Equations (EFEs) for locally rotationally symmetric (LRS) Bianchi type-I perfect fluid spacetimes through the concircular vector fields (CCVFs) in $f(T)$ gravity. It is shown that such metrics admit CCVFs of 4, 5, 6, 7, 8 and 15 dimensions. We also calculated the energy density, fluid pressure, torsion scalar T and the form of the function $f(T)$. We did not specify the form of $f(T)$ for the solution of EFEs but found a particular form of $f(T)$ during the process of finding the CCVFs. It is observed that energy density and fluid pressure of some solutions are related as $p = -\rho$, which means that the universe represented by these spacetime metrics behaves like dark energy models. For other solutions, it is observed that the fluid pressure and energy density can remain positive for particular choices of the constants involved. Thus, in these cases the rate of expansion can slow down due to positive attractive effect. We also calculated the Hubble's parameter (H), scalar expansion (θ), shear scalar (σ^2), anisotropy parameter (A_m) and the deceleration parameter (q) associated with each of the obtained exact LRS Bianchi type-I solutions. It is observed that the physical

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behavior of the spacetime changes with the number of admitting CCVFs. A constant anisotropic behavior for 15-dimensional CCVFs is observed to change to isotropic nature for seven-dimensional CCVFs.

Keywords: Concircular vector fields; conformal factors; $f(T)$ gravity.

1. Introduction

In general relativity (GR), spacetime symmetries are essential for solving Einstein's Field Equations (EFEs). Symmetry refers to a local smooth diffeomorphism which holds the geometric properties of spacetime manifold. This geometrical characteristic could refer specifically to tensors like metric tensor, Riemann tensor or Ricci tensor [1]. Therefore, one could derive any geometrical symmetry along a vector field by considering its Lie derivative along the vector field. Various symmetries have been discussed in the literature such as Killing, homothetic, conformal and concircular symmetries. Conformal symmetry means a vector field which maintains the spacetime metric up to a certain conformal factor. A conformal factor is a particular function of the spacetime coordinates. When the conformal factor becomes a nonzero constant, then the conformal Killing vector fields (CKVFs) reduce to homothetic vector fields (HVFs) and when the conformal factor vanishes, then CKVFs become the Killing vector fields (KVFs). Also, a concircular vector field (CCVF) X is the one in which concircular mapping is maintained along the local flow by the vector field X, i.e. a mapping which maintains the geodesic circles.

Conformal and concircular symmetries are crucial to examine because they play a significant role at the geometric, dynamical and kinematic levels. At the geometric level, these symmetries could make possible the choice of coordinates so that one of the metric components is singled out and the metric is simplified. The CKVFs have the geometric property of preserving the structure of the null cone by mapping null geodesics to null geodesics, while the CCVFs have the geometric property of preserving geodesic circles (i.e. preserving curves of constant geodesic curvature). These two symmetries could be helpful in imposing restrictions on the kinematic variables such as rotation, expansion and shear to produce interesting physical results. An important dynamical use of these two symmetries is to apply them for solving EFEs and obtain some new exact solutions with better physical properties. Conformal and concircular symmetries also play a key role in the classification and categorization of the known solutions of the field equations.

Many researchers have used conformal symmetry to analyze physical properties of our universe. Rahaman *et al.* [2, 3] studied the anisotropic and relativistic stars admitting CKVF. Esculpi and Aloma [4] worked on a charged fluid sphere with a linear equation of state and admitting CKVFs. The authors have also discussed the dynamical stability of the system. Usmani *et al.* [5] studied the charged gravastar admitting CKVFs. Some other researchers showed their interest in obtaining CKVFs admitted by particular classes of some well-known spacetimes. Maartens and Maharaj [6] obtained all 15 CKVFs of Robertson–Walker spacetimes. The same authors obtained the CKVFs of pp-waves [7]. They showed that nonconformally flat pp-waves always admit six CKVFs and one proper homothetic vector field, but in general, they do not admit special conformal vector fields. Moopanar and Maharaj [8] classified spherically symmetric spacetimes by their CKVFs without specifying the form of matter distribution. Rigidly rotating perfect fluid spacetimes admitting proper CKVFs were explored by Kramer and Carot [9]. Hussain *et al.* [10] obtained the CKVFs of locally rotationally symmetric (LRS) Bianchi type-V spacetimes. Khan *et al.* [11, 12] classified static and nonstatic plane symmetric spacetimes as per their CKVFs. Khan *et al.* [13] also investigated CKVFs of Bianchi type-I spacetimes. They showed that particular classes of Bianchi type-I spacetimes admit 5 or 15 CKVFs. Schouten [14] and Eisenhart [15] disclosed that all nonconformally flat spacetimes admit 15 CKVFs. Hall and Steele [16] carried out a study in which they proved that the maximum dimension of conformal algebra for nonconformally flat spacetimes is seven and it is 15 for conformally flat spacetimes. Maartens *et al.* obtained the CKVFs for conformally flat and nonconformally flat static spherically symmetric spacetimes [17]. Their results showed that when the spacetime is not conformally flat, it admits just two proper CKVFs. They also concluded that when the spacetime is conformally flat, 11 proper CKVFs exist. Shabbir and Ali studied the CKVFs of Bianchi types VIII and IX as well as spatially homogeneous rotating spacetimes [18, 19]. They proved that spatially homogeneous rotating spacetimes do not admit proper CKVFs.

Concircular symmetry is also considered as an important symmetry of spacetime which provides better insight of spacetime geometry and physics. The concept of concircular vector fields had been proposed by Fialkow [20] over Riemannian manifolds. CCVFs have numerous mathematical as well as physical applications in GR. Chen [21] categorized Ricci solitons with a concircular potential field and explored the uses of CCVFs for Ricci solitons. CCVFs can be utilized to create unique conformal Killing tensors, which are crucial in lots of different research fields. Chen [22] reported that a time-like CCVF will be admitted by a Lorentzian manifold if and only if it is a generalized Robertson–Walker spacetime. The existence of CCVFs on a Riemannian manifold modifies the topology and geometry of Riemannian manifolds. Chen and Deshmukh [23] used CCVFs to obtain new geometrical characterizations of spheres, Euclidean spaces and complex Euclidean spaces. From the physical perspective, CCVFs are useful while working in general relativity; for example, geodesics of time-like CCVFs in the de Sitter model govern the world lines of retreating or striking galaxies which satisfy Weyl hypothesis [21]. Many authors showed their interest to obtain the CCVFs of different spacetimes. Ali and Khan [24] explored the CCVFs of plane symmetric static spacetimes. Mahmood *et al.* [25] explored the CCVFs on the Lorentzian manifold of LRS Bianchi type-I spacetimes. Also, they [26] investigated the CCVFs for the LRS Bianchi type-V spacetimes. Khan *et al.* [27] investigated the CCVFs for Kantowski–Sachs and Bianchi type-III spacetimes.

The general theory of relativity has certain limitations. For example, it does not predict the expansion of the universe. Researchers are of the opinion that an unknown force having strong negative pressure is responsible for this expansion. This force is believed to exist due to an energy known as dark energy. Numerous modified theories have been proposed to identify the source of dark energy. These modified theories gain the attention of the researchers as some of them have the ability to explain the inflationary phase and the dark energy era. The $f(R)$ theories of gravity are the simplest generalization of GR, where the Einstein–Hilbert action is generalized by introducing a general function of R instead of the Ricci scalar. Gauss–Bonnet theories represent a class of modified theories of gravity, and a particular subclass of this theory is known as the $f(G)$ theory of gravity. $f(G)$ theory has been extended to $f(R,G)$ theory, where G stands for the Gauss–Bonnet invariant and R denotes the Ricci scalar. The $f(T)$ theory is another modified theory of gravity. $f(T)$ gravity is a generalization of the teleparallel equivalent of GR, where $f(T)$ is a function of the torsion scalar T. For a number of reasons, the $f(T)$ gravity is quite significant, e.g. it can identify the models of inflation [28, 29]. Another important modified theory of gravity is known as the $f(R,T)$ theory of gravity, where R is the Ricci scalar and T is the trace of the energy–momentum tensor. $f(R,T)$ theory of gravity is believed to be effective in cosmological studies.

Spacetime symmetries are not only considered in GR but also discussed widely in modified theories of gravity by different authors. Hussain *et al.* [30] investigated the CKVFs in $f(R)$ theory of gravity for Bianchi type-II spacetimes. They proved that Bianchi type-II spacetimes possess no proper CKVF. In the $f(R)$ theory of gravity, Shabbir *et al.* [31] categorized static spherically symmetric spacetimes according to their CKVFs by applying the direct integration technique. They investigated that these spacetime metrics admit no proper CKVF in the $f(R)$ theory of gravity. Hussain *et al.* [32] also studied CKVFs of perfect fluid nonstatic plane symmetric spacetimes. Qazi *et al.* [33] deduced several types of Bianchi type-I perfect fluid spacetime metrics from the field equations of $f(T)$ theory of gravity and then obtained the CKVFs for each metric. They concluded that 3, 4, 5, 6 or 15 CKVFs exist for Bianchi type-I perfect fluid spacetimes in the $f(T)$ theory of gravity. In $f(T)$ gravity, Qazi *et al.* [34] obtained the particular solutions of EFEs for the Kantowski–Sachs and Bianchi type-III spacetimes and explored the CKVFs for each spacetime metric. They found that such spacetime metrics admit CKVFs of dimensions 4, 5, 6 or 15. In the $f(T)$ gravity, Hussain *et al.* [35] classified the static spherically symmetric perfect fluid spacetimes by CKVFs. According to their analysis, these spacetimes admit proper CKVFs in the $f(T)$ gravity. Shah *et al.* [36] classified the perfect fluid plane symmetric static spacetimes according to their CCVFs in $f(T)$ theory. Their study revealed that in $f(T)$ gravity, perfect fluid static plane symmetric spacetimes admit CCVFs of 4, 5, 6, 7, 8 and 15 dimensions. They also obtained the form of $f(T)$. In some cases, they obtained a nonlinear function of the torsion scalar T. Malik *et al.* [37] solved the EFEs of static spherically symmetric perfect fluid spacetimes in $f(T,B)$ gravity and then obtained the CKVFs for each solution.

Conformal vector fields were investigated in $f(R, G)$ theory of gravity by Hussain *et al.* [38] for static spherically symmetric spacetimes. In this study, they deduced that there exist six cases which provide 4-, 6- and 15-dimensional CKVFs. In the $f(R,T)$ gravity, Mehmood *et al.* [39] explored the proper CKVFs of Bianchi type-I perfect fluid spacetimes. They came to the conclusion that Bianchi type-I spacetime metrics possess CKVFs of dimensions 3, 4, 5 and 15 in the $f(R,T)$ theory of gravity.

The aim of this study is to classify the perfect fluid LRS Bianchi type-I spacetimes as per their CCVFs in the $f(T)$ theory of gravity. The rest of the paper is organized as follows: In Sec. 2, we will obtain 20 differential equations of concircular symmetry for LRS Bianchi type-I spacetimes. Some algebraic techniques will also be applied to obtain the components of CVFs along with integrability conditions. In Sec. 3, we will derive the EFEs in $f(T)$ theory of gravity for the LRS Bianchi type-I spacetime by choosing a perfect fluid environment. In this section, we will also solve those EFEs and the concircular symmetry equations simultaneously. All the obtained results will be listed as different cases. In Sec. 4, some physical quantities are derived for the obtained results and are discussed briefly. A brief summary and a discussion of the results are given in the last section.

2. Concircular Vector Field Equations

In this section, our aim is to obtain the CCVF equations for LRS Bianchi type-I spacetimes. A vector field X is said to be CCVF if it satisfies the following equations:

$$\mathcal{L}_X \, g_{ij} = g_{ij,k} \, X^k + g_{jk} \, X^k_{,i} + g_{ik} \, X^k_{,j} = 2 \, \alpha \, g_{ij}, \tag{1}$$

$$\nabla_i \, \nabla_j \, \alpha = \alpha_{,ij} - \Gamma^k_{ij} \alpha_{,k} = \beta \, g_{ij}, \tag{2}$$

where g_{ij} is a metric tensor, Γ^k_{ij} denotes the Christoffel symbols of the second kind, $\mathcal{L}_X$ stands for the Lie derivative along X, $\nabla_i \nabla_j$ is the *Hessian* while α and β are conformal factors.

Nine types of Bianchi models are the best conventional cosmological models for describing the universe. Modern experimental data has shown that our universe is not completely flat but it has some curvature. As Bianchi models are anisotropic, they can provide us with a better insight of the universe. Some of the Bianchi models may show isotropic behavior at late time and this fact makes them the best choice for studying dark energy models. All the nine Bianchi spacetime models are spatially homogeneous spacetimes of dimension $1 + 3$ that have G_3 as the group of motions which acts on space-like hypersurfaces. The line element for LRS Bianchi type-I spacetime in the conventional coordinates (t, x, y, z) labeled by $\left(x^0, x^1, x^2, x^3\right)$, respectively, is given by [40]

$$ds^2 = -dt^2 + K^2(t) \, dx^2 + M^2(t) \, (dy^2 + dz^2), \tag{3}$$

such that K and M are functions of t only. The minimal KVFs which the above spacetime (3) admits are

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial y}, \quad V_3 = \frac{\partial}{\partial z}, \quad V_4 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}. \tag{4}$$

Equations (1) and (2) when written explicitly will generate 20 partial differential equations of first order. These equations will be solved through integrating directly. In the whole process, some algebraic techniques will also be applied and some equations will be solved by using Maple or Mathematica software. In this process, the conformal factors will also be obtained. These conformal factors will decide whether the obtained vector fields will be concircular, conformal, homothetic or Killing.

Using Eq. (3) in Eqs. (1) and (2), we obtain the following set of coupled equations:

$$X^0_{,0} = \alpha(t,x,y,z), \tag{5}$$

$$X^0_{,1} - K^2 X^1_{,0} = 0, \tag{6}$$

$$X^0_{,2} - M^2 X^2_{,0} = 0, \tag{7}$$

$$X^0_{,3} - M^2 X^3_{,0} = 0, \tag{8}$$

$$\dot{K}X^0 + KX^1_{,1} = K\alpha(t,x,y,z), \tag{9}$$

$$K^2 X^1_{,2} + M^2 X^2_{,1} = 0, \tag{10}$$

$$K^2 X^1_{,3} + M^2 X^3_{,1} = 0, \tag{11}$$

$$\dot{M}X^0 + MX^2_{,2} = M\alpha(t,x,y,z), \tag{12}$$

$$X^2_{,3} + X^3_{,2} = 0, \tag{13}$$

$$\dot{M}X^0 + MX^3_{,3} = M\alpha(t,x,y,z) \tag{14}$$

and

$$\alpha_{,00} + \beta(t,x,y,z) = 0, \tag{15}$$

$$K\alpha_{,01} = \dot{K}\alpha_{,1}, \tag{16}$$

$$M\alpha_{,02} = \dot{M}\alpha_{,2}, \tag{17}$$

$$M\alpha_{,03} = \dot{M}\alpha_{,3}, \tag{18}$$

$$\alpha_{,11} = K^2\beta + K\dot{K}\alpha_{,0}, \tag{19}$$

$$\alpha_{,12} = 0, \tag{20}$$

$$\alpha_{,13} = 0, \tag{21}$$

$$\alpha_{,22} = M^2\beta + M\dot{M}\alpha_{,0}, \tag{22}$$

$$\alpha_{,23} = 0, \tag{23}$$

$$\alpha_{,33} = M^2\beta + M\dot{M}\alpha_{,0}, \tag{24}$$

where "dot" represents a derivative with respect to t. Now our strategy is to find a solution for the above equations. We first solve Eqs. (5)–(14) by direct integration technique to find the components X^0, X^1, X^2 and X^3. We will then substitute the resulting components in Eqs. (15)–(24) to obtain the differential constraints, which when solved will give us the concircular vector fields. The process of obtaining these components is explained in the following.

Differentiating Eqs. (9), (10) and (12) with respect to z, y and x, respectively, and solving the resulting equations gives

$$X^1_{,23} = X^2_{,13} = X^3_{,12} = 0. \tag{25}$$

Also, differentiating Eqs. (6), (7) and (12) with respect to z, y and t, respectively, and solving the obtained equations among themselves, we get

$$X^0_{,23} = X^2_{,03} = X^3_{,02} = 0. \tag{26}$$

Following the same pattern for Eqs. (6), (7) and (13), we get

$$X^i_{,22} - X^i_{,33} = X^j_{,22} + X^j_{,33} = 0, \quad i = 0, 1,\ j = 2, 3. \tag{27}$$

Now, substitute the outcome of integration of Eqs. (24) and (25) into Eq. (26) and integrate the resulting equation. The last obtained equation can be used in Eqs. (6), (7), (9), (10) and (12) to obtain the following system:

$$\begin{cases} X^0 = H_0 + M^2[\dot{H}_2 y + \dot{H}_3 z + \dot{H}_4(y^2 + z^2)], \\ X^1 = H_1 + M^2K^{-2}[H_2' y + H_3' z + H_4'(y^2 + z^2)], \\ X^2 = H_2 + 2yH_4 + c_1 z + c_2(y^2 - z^2) + 2c_3 yz, \\ X^3 = H_3 + 2zH_4 - c_1 y - c_3(y^2 - z^2) + 2c_2 yz, \end{cases} \tag{28}$$

with the conformal factor $\alpha(t, x, y, z) = X^0_{,0}(t, x, y, z)$, where c_1, c_2 and c_3 are arbitrary constants and $H_i = H_i(t, x)$, $i = 0, 1, 2, 3, 4$, are given by

$$\begin{cases} H_0(t,x) = M(t)\left(G_1(x) + 2\displaystyle\int \frac{F_4(t)}{M(t)}dt + 2G_4(x)\int \frac{K(t)}{M^2(t)}dt\right), \\ H_1(t,x) = G_0(x) + G_1'(x)\displaystyle\int \frac{M(t)}{K^2(t)}dt + 2G_4'(x)\int \frac{M(t)}{K^2(t)}\left(\int \frac{K(t)}{M^2(t)}dt\right)dt, \\ H_k(t,x) = F_k(t) + \dfrac{K(t)G_k(x)}{M(t)}, \quad k = 2, 3, 4. \end{cases} \tag{29}$$

To find the integrability conditions, substituting the above equations in 10 conformal Killing equations, seven of these conformal Killing equations are satisfied directly and the remaining equations give the following integrability conditions:

$$\begin{cases} [M^2\ddot{K} + K\dot{M}^2 - M(K\ddot{M} + \dot{M}\dot{K})]G_k \\ \quad = M[2c_k - M(M\ddot{F}_k + \dot{M}\dot{F}_k)], \\ M[G_k'' + 2K\dot{M}\dot{F}_k + M(K\ddot{F}_k - \dot{K}\dot{F}_k)] \\ \quad = [K(K\ddot{M} - \dot{K}\dot{M}) - M(K\ddot{K} - \dot{K}^2)]G_k, \quad k = 2,3,4, \\ MK\left[G_0' + G_1''\int \frac{M}{K^2}dt + 2G_4''\int \frac{M}{K^2}\left(\int \frac{K}{M^2}dt\right)dt\right] \\ \quad + M(M\dot{K} - K\dot{M})\left[G_1 + G_4\int \frac{K}{M^2}dt + 2\int \frac{F_4}{M}dt\right] \\ \qquad = 2K\left(KG_4 + MF_4\right), \end{cases} \tag{30}$$

where $G_i = G_i(x)$, $F_k = F_k(t)$, $i = 0,1,2,3,4$ and $k = 2,3,4$.

3. EFEs and CCVFs in $f(T)$ Gravity

The $f(T)$ gravity is based upon the Weitzenböck connections whose geometry with the metric is defined by

$$ds^2 = g_{ab}dx^a dx^b, \tag{31}$$

where $g_{ab} = \eta_{ij}e^i_a e^j_b$ with $\eta_{ij} = \mathrm{diag}(-1,1,1,1)$ are the metric components and e^i_a represents the tetrad field. The mathematical formulation between the Weitzenböck connections and tetrad is

$$\Gamma^\alpha_{\mu\nu} = e^a{}_i\partial_\nu e^i{}_\mu = -e^i{}_\mu\partial_\nu e_i{}^\alpha. \tag{32}$$

This connection is the foundation upon which the basic geometrical objects of spacetime are built up. The antisymmetric portion of this connection serves to determine the elements of the torsion tensor, which is

$$T^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu} - \Gamma^\alpha_{\mu\nu} = e^\alpha{}_i(\partial_\mu e^i{}_\nu - \partial_\nu e^i{}_\mu). \tag{33}$$

Another important tensor known to be the contorsion tensor is defined as

$$K^{\mu\nu}_{\ \ \alpha} = -\frac{1}{2}(T^{\mu\nu}_\alpha - T^{\nu\mu}_\alpha - T^{\mu\nu}_\alpha). \tag{34}$$

To define the term torsion scalar, we need to define a new tensor $S^{\mu\nu}{}_\alpha$ which is called a superpotential tensor and is defined by

$$S^{\mu\nu}_{\ \ \alpha} = \frac{1}{2}(K^{\mu\nu}_\alpha + \delta^\mu_\alpha T^{\beta\nu}_{\ \ \beta} - \delta^\nu_\alpha T^{\beta\mu}_{\ \ \beta}). \tag{35}$$

In this section, first we will derive the EFEs for perfect fluid LRS Bianchi type-I spacetimes in $f(T)$ gravity and then we will solve those field equations. The procedure is to find the LRS Bianchi type-I solutions in the $f(T)$ gravity and then utilize those solutions to categorize via CCVFs. The EFEs in $f(T)$ gravity are

$$S_{\beta}^{\alpha\rho}\partial_\rho T f_{TT} + \left[e^{-1}e^{i}_{\beta}\partial_\rho\left(ee^{k}_{i}S_{k}^{\alpha\rho}\right) + T^{k}{}_{\eta\beta}S^{\alpha\eta}{}_{k}\right] f_T + \frac{1}{4}\delta^{\alpha}_{\beta} f \;=\; 4\pi E^{\alpha}_{\beta}, \tag{36}$$

where $f = f(T)$, f_T and f_{TT} denote the first and second derivatives of the function $f(T)$ with respect to T, $S_{\beta}^{\alpha\rho}$ is the spin tensor, e indicates the determinant of tetrad field $e^{k}{}_{i}$, E^{α}_{β} is the energy–momentum tensor and $T^{k}_{\eta\beta}$ represents the torsion tensor. It is to be mentioned here that in teleparallel gravity there is a larger class of Bianchi type-I solutions, see e.g. [41]. Here, we will study a subclass of all Bianchi type-I solutions and consider the existence of CCVFs in more general Bianchi type-I geometries. The chosen tetrad for the spacetimes (3) is: $e^{k}_{i} = \mathrm{diag}\big(1, K(t),\, M(t),\, M(t)\big)$. The following are the nonzero components of torsion tensor and superpotential tensor:

$$T^{1}_{01} = \frac{\dot K}{K}, \quad T^{2}_{02} = T^{3}_{03} = \frac{\dot M}{M}, \quad S^{10}_{1} = \frac{\dot M}{M}, \quad S^{20}_{2} = S^{30}_{3} = \frac{1}{2}\left[\frac{\dot K}{K} + \frac{\dot M}{M}\right].$$

The torsion scalar T is defined by the following contraction [44]:

$$T = T^{\alpha}_{\beta\gamma}\, S^{\beta\gamma}_{\alpha}. \tag{37}$$

Using Eq. (37), we will get the torsion scalar T for the above spacetimes (3) as

$$T = -2\left(\frac{2\,\dot K\,\dot M}{K\,M} + \frac{\dot M^2}{M^2}\right). \tag{38}$$

Taking perfect fluid as a source of energy–momentum tensor, Eq. (36) after using (3) yields the following equations:

$$f + 4\left[\frac{2\dot K\dot M}{KM} + \frac{\dot M^2}{M^2}\right] f_T(T) = -16\pi\rho, \tag{39}$$

$$f + 4\left[\frac{\dot M^2}{M^2} + \frac{\ddot M}{M} + \frac{\dot K\dot M}{KM}\right] f_T(T) + 4\dot T\frac{\dot M}{M} f_{TT}(T) = 16\pi p, \tag{40}$$

$$f + 2\left[\frac{\dot M^2}{M^2} + \frac{\ddot K}{K} + \frac{\ddot M}{M} + \frac{3\dot K\dot M}{KM}\right] f_T(T) + 2\dot T\left[\frac{\dot M}{M} + \frac{\dot K}{K}\right] f_{TT}(T) = 16\pi p, \tag{41}$$

where ρ denotes the energy density and p indicates the pressure of the fluid distribution. Now the objective is to solve Eqs. (39)–(41), which have three unknowns K, M and f_T. Here, one can use several approaches to find the solutions of Eqs. (39)–(41). One method is to specify $f(T)$ and then proceed for the values of K and M. Another approach is to place restrictions on the spacetime component derivatives and look for the function $f(T)$, which then yields the analytical solutions to the above Eqs. (39)–(41). Before setting such a process into place, we subtract Eqs. (40) and (41) from Eq. (39). This will provide two equations which yield the following after subtraction:

$$\left[\frac{\dot{K}\dot{M}}{KM} - \frac{\dot{M}^2}{M^2} + \frac{\ddot{K}}{K} - \frac{\ddot{M}}{M}\right] f_T(T) + \left(\frac{\dot{K}}{K} - \frac{\dot{M}}{M}\right)\dot{T} f_{TT}(T) = 0. \tag{42}$$

To solve the above equation, we discuss two cases:

(1) When $\dot{T} = 0 \Rightarrow f_T(T) = f_{TT}(T) = 0 \Rightarrow f(T) = 16\pi\lambda_0$, where λ_0 is an arbitrary constant. In this case we have $p(t) = \lambda_0 = -\rho(t)$.
(2) When $\dot{T} \neq 0$, Eq. (42) can be written in the following form:

$$f_{TT}(T) = \frac{\left[K\dot{M}^2 - M^2\ddot{K} + M\left(K\ddot{M} - \dot{K}\dot{M}\right)\right] f_T(T)}{M\left(M\dot{K} - K\dot{M}\right)\dot{T}}. \tag{43}$$

Integrating Eq. (43), we get

$$f_T(T) = \frac{\lambda_1}{M\left(M\dot{K} - K\dot{M}\right)}. \tag{44}$$

Again integrating Eq. (44), we get

$$f(T) = \lambda_0 + \int \frac{\lambda_1 \dot{T}(t)}{M\left(M\dot{K} - K\dot{M}\right)} dt, \tag{45}$$

where λ_0, λ_1 are arbitrary constants of integration.

Now, differentiating the first equation of (30) with respect to x, we obtain the following important equation:

$$\left[K\dot{M}^2 + M^2\ddot{K} - M\left(\dot{K}\dot{M} + K\ddot{M}\right)\right] G'_k(x) = 0 = L(t)G'_k(x), \quad k = 2,3,4. \tag{46}$$

We did not choose a specific form of T initially, rather its particular form will be obtained during the process of finding the CCVFs. The plan of finding CKVFs and then CCVFs is divided into two parts as follows.

3.1. *Part (A)*: $L(t) = 0$

Here, we consider $L(t) = 0$, which leads to

$$K(t) = \Psi(t)M(t), \quad \Psi(t) = \kappa_0 + \mu_0 \int \frac{dt}{M(t)}, \tag{47}$$

where κ_0 and μ_0 are arbitrary constants. Solving the first equation of (30), we get

$$F_k(t) = \sigma_k + \rho_k \int \frac{dt}{M(t)} + c_k \left(\int \frac{dt}{M(t)} \right)^2, \quad k = 2, 3, 4. \tag{48}$$

Again substituting $K(t)$ and $F_k(t)$ in the second equation of (30), we get the following ordinary differential equations:

$$G_k'' - \mu_0^2 G_k = \mu_0 \rho_k - 2\kappa_0 c_k, \quad k = 2, 3, 4. \tag{49}$$

The general solution of the above equation is

$$\begin{cases} G_k(x) = \sigma_{k+3} e^{\mu_0 x} + \rho_{k+3} e^{-\mu_0 x} + \dfrac{2\kappa_0 c_k - \mu_0 \rho_k}{\mu_0^2} & \text{when} \;\; \mu_0 \neq 0, \\ G_k(x) = \sigma_{k+3} + \rho_{k+3} x - \kappa_0 c_k x^2 & \text{when} \;\; \mu_0 = 0. \end{cases} \tag{50}$$

In the literature, it is known that not all Killing vectors are preserved as isometries of the teleparallel geometry, see e.g. [42, 43]. In this regard, a detailed study is required to investigate whether the CCVFs in teleparallel geometries impact the structure of the torsion tensor and its covariant derivatives in the same manner as their counterparts in Lorentzian geometries. We will address this point in our next work. To be concise, we write the final form of CCVFs directly as follows.

Case (I). When $\mu_0 \neq 0$, we obtained 15-dimensional CKVFs with the conformal factor as: $\alpha(t, x, y, z) = X^0_{,0}$. Now substituting the CKVF components into Eqs. (15)–(24), one can obtain the differential constraints for the CCVFs. We are not listing those constraints here, but after solving those constraints we obtain the final form of CCVFs as follows.

Concircular (I.1). In this case, all the 15 CKVFs are CCVFs which are given as follows:

$$\begin{cases} X^0 = \mu_0 M(t)[2\,(\delta_1 + \delta_2 y + \delta_3 z)\,\Psi(t) + (\delta_4 + \delta_5 y + \delta_6 z)\, e^{\mu_0 x} \\ \qquad\quad + \,(\delta_7 + \delta_8 y + \delta_9 z)\, e^{-\mu_0 x} + [\tilde{y}^2 + \tilde{z}^2 + \Psi^2(t)]\,(\delta_{10} e^{\mu_0 x} + \delta_{11} e^{-\mu_0 x})], \\ X^1 = \delta_{12} - \mu_0 \Psi^{-1}(t)[(\delta_4 + \delta_5 y + \delta_6 z)\, e^{\mu_0 x} - (\delta_7 + \delta_8 y + \delta_9 z)\, e^{-\mu_0 x} \\ \qquad\quad + [\tilde{y}^2 + \tilde{z}^2 - \Psi^2(t)]\,(\delta_{10} e^{\mu_0 x} - \delta_{11} e^{-\mu_0 x})], \\ X^2 = \delta_{13} + \delta_{14} z + 2\mu_0^2\,(\delta_1 + \delta_3 z)\, y + \delta_2[\kappa_0^2 + \tilde{y}^2 - \tilde{z}^2 + \Psi^2(t)] \\ \qquad\quad + \Psi(t)[\big(\delta_5 + 2\mu_0^2 \delta_{10} y\big)\, e^{\mu_0 x} + \big(\delta_8 + 2\mu_0^2 \delta_{11} y\big)\, e^{-\mu_0 x}], \\ X^3 = \delta_{15} - \delta_{14} y + 2\mu_0^2\,(\delta_1 + \delta_2 y)\, z + \delta_3[\kappa_0^2 - \tilde{y}^2 + \tilde{z}^2 + \Psi^2(t)] \\ \qquad\quad + \Psi(t)[\big(\delta_6 + 2\mu_0^2 \delta_{10} z\big)\, e^{\mu_0 x} + \big(\delta_9 + 2\mu_0^2 \delta_{11} z\big)\, e^{-\mu_0 x}], \end{cases} \tag{51}$$

where δ_i, $i = 1, \ldots, 15$, represent arbitrary constants while $\tilde{y} = \mu_0 y$ and $\tilde{z} = \mu_0 z$. Also in the above vector field, $M(t) = m_0$ where m_0 is a nonzero constant, $\Psi(t) = \kappa_0 + \mu_1 t$, $\mu_0 = m_0\mu_1$ and $\mu_1 \neq 0$. The metric which admits these 15-dimensional CCVFs is given as

$$ds^2 = -dt^2 + (\kappa_1 + \mu_0 t)^2 dx^2 + m_0^2 \left(dy^2 + dz^2\right), \tag{52}$$

where $\kappa_1 = m_0\kappa_0$. Also, the conformal factors can be listed as

$$\begin{cases} \alpha(t,x,y,z) = 2\mu_0^2\big(\delta_1 + \delta_2 y + \delta_3 z + (\kappa_0 + \mu_1 t)\left[\delta_{10}e^{\mu_0 x} + \delta_{11}e^{-\mu_0 x}\right]\big), \\ \beta(t,x,y,z) = 0. \end{cases} \tag{53}$$

In this case, we have

$$T = 0, \quad f(T) = 16\pi\lambda_0, \quad p(t) = \lambda_0 = -\rho(t), \tag{54}$$

$$T^1_{01} = \frac{\mu_0}{\kappa_1 + \mu_0 t}, \quad T^2_{02} = T^3_{03} = 0, \tag{55}$$

where λ_0 is an arbitrary constant.

Concircular (I.2). In this case, $\dot{M}(t) \neq 0$ and $M(t) = m_0 e^{m_1 t}$ and $K(t) = \kappa_1 e^{m_1 t} - \mu_1$, where m_0 and m_1 are arbitrary constants, $\kappa_1 = m_0\kappa_0$ and $\mu_0 = m_1\mu_1$. The metric becomes

$$ds^2 = -dt^2 + (\kappa_1 e^{m_1 t} - \mu_1)^2 dx^2 + m_0^2 e^{2m_1 t}(dy^2 + dz^2), \tag{56}$$

where μ_1, m_0 and m_1 are some nonzero constants. The above spacetime metric admits seven-dimensional CCVFs as follows:

$$\begin{cases} X^0 = 2m_1\mu_1(\delta_1 + \delta_2 y + \delta_3 z)(\kappa_1 e^{m_1 t} - \mu_1), \\ X^1 = \delta_{12}, \\ X^2 = \delta_{13} + \delta_{14} z + 2\mu_2^2(\delta_1 + \delta_3 z)y + \tilde{\delta}_2[\kappa_1^2 + \tilde{y}^2 - \tilde{z}^2 + (\kappa_1 - \mu_1 e^{-m_1 t})^2], \\ X^3 = \delta_{15} - \delta_{14} y + 2\mu_2^2(\delta_1 + \delta_2 y)z + \tilde{\delta}_3[\kappa_1^2 - \tilde{y}^2 + \tilde{z}^2 + (\kappa_1 - \mu_1 e^{-m_1 t})^2], \end{cases} \tag{57}$$

where $\mu_2 = m_1\mu_1$, $\delta_2 = m_0^2\tilde{\delta}_2$, $\delta_3 = m_0^2\tilde{\delta}_3$, $\tilde{y} = m_0\mu_2 y$ and $\tilde{z} = m_0\mu_2 z$, such that the following condition must be satisfied:

$$\kappa_1\mu_1\delta_j = 0, \quad j = 1, 2, 3.$$

Also, the conformal factors are given by $\alpha(t,x,y,z) = \beta(t,x,y,z) = 0$. In this case, we have

$$\begin{cases} T = 2m_1^2\left(\dfrac{2\mu_1}{\mu_1 - \kappa_1 e^{m_1 t}} - 3\right), \\ T^1_{01} = \dfrac{\kappa_1 m_1 e^{m_1 t}}{\kappa_1 m_1 e^{m_1 t} - \mu_1}, \quad T^2_{02} = T^3_{03} = m_1, \\ f(T) = \tilde{\lambda}_0 + \lambda_1\left(\mu_1 e^{-m_1 t} + \dfrac{\kappa_1\mu_1}{\kappa_1 e^{m_1 t} - \mu_1} + 2\kappa_1 \ln\left[\kappa_1 - \mu_1 e^{-m_1 t}\right]\right), \\ 16\pi p(t) = \tilde{\lambda}_0 + \lambda_1\left(2\mu_1 e^{-m_1 t} + 2\kappa_1 \ln\left[\kappa_1 - \mu_1 e^{-m_1 t}\right]\right), \\ 16\pi\rho(t) = \lambda_1\left(\dfrac{\mu_1^2 e^{-2m_1 t}}{\kappa_1} - 3\mu_1 e^{-m_1 t} + \dfrac{\kappa_1\mu_1}{\kappa_1 e^{m_1 t} - \mu_1}\right. \\ \qquad\qquad \left. - 2\kappa_1 \ln\left[\kappa_1 - \mu_1 e^{-m_1 t}\right]\right) - \tilde{\lambda}_0, \end{cases} \tag{58}$$

where $\tilde{\lambda}_0 = \lambda_0 + \lambda_1\kappa_1$ and λ_1 are arbitrary constants. Under the condition $\kappa_1\mu_1\delta_j = 0$, this case is divided into three cases as given in the following.

Concircular (I.2.A). In this case, when $\delta_1 = \delta_2 = \delta_3 = 0$ in Eq. (57), we will be left with only four CCVFs. The metric takes the form (56), where $\kappa_1\mu_1 \neq 0$. It is clear that the metric is not an Einstein spacetime metric.

Concircular (I.2.B). In this case, putting $\delta_j \neq 0$, $j = 1,2,3$, and $\kappa_1 = 0$ in Eq. (57), we will have seven-dimensional CCVFs. It is clear that the metric (56) with $\kappa_1 = 0$ is an Einstein metric satisfying $R_{ij} = -2m_1^2 g_{ij}$.

Concircular (I.2.C). In this case, putting $\delta_j \neq 0$, $j = 1,2,3$, $\mu_1 = 0$ and $\mu_2 = 0$ in the CCVF components equation (57), we will have four-dimensional CCVFs. Also, the metric (56) with $\mu_1 = 0$ is an Einstein metric satisfying $R_{ij} = -3m_1^2 g_{ij}$.

Concircular (I.3). In this case, $\dot{M}(t) \neq 0$, $M\ddot{M} \neq \dot{M}^2$, $\kappa_0 = 0$ and $\delta_i = 0$, $i = 2,\ldots,11$. Also, we have $M(t) = m_0(t+t_0)^{m_1}$ and $K(t) = \mu_1(t+t_0)$, where m_0 and m_1 are arbitrary nonzero constants and $\mu_0 = (1-m_1)\mu_1$. The metric becomes

$$ds^2 = -dt^2 + \mu_1^2(t+t_0)^2 dx^2 + m_0^2(t+t_0)^{2m_1}(dy^2 + dz^2), \tag{59}$$

where μ_1, m_0 are some nonzero constants and $m_1 \neq 1$. It admits six-dimensional concircular vector fields given in the following:

$$\begin{cases} X^0 = \tilde{\delta}_1 + \tilde{\delta}_2 t, & X^1 = \tilde{\delta}_3, \\ X^2 = \tilde{\delta}_4 + (1-m_1)\tilde{\delta}_2 y + \tilde{\delta}_5 z, & X^3 = \tilde{\delta}_6 + (1-m_1)\tilde{\delta}_2 z - \tilde{\delta}_5 y, \end{cases} \tag{60}$$

where $\tilde{\delta}_i$, $i = 1, \ldots, 6$, are arbitrary constants. Also, the conformal factors are given by $\alpha(t,x,y,z) = \tilde{\delta}_2$ and $\beta(t,x,y,z) = 0$. The form of conformal factors suggests that the obtained CCVFs are just HVFs with one proper homothetic vector field. In this case, we have

$$\begin{cases} T = -\dfrac{2m_1(2+m_1)}{(t+t_0)^2}, & f(T) = \lambda_0 + \lambda_1 (t+t_0)^{-2(1+m_1)}, \\ T^1_{01} = \dfrac{1}{t+t_0}, & T^2_{02} = T^3_{03} = \dfrac{m_1}{t+t_0}, \\ 16\pi p(t) = \lambda_0 + \lambda_1 (t+t_0)^{-2(1+m_1)}, & 16\pi\rho(t) = (1+2m_1)\lambda_1 \\ & \quad \times (t+t_0)^{-2(1+m_1)} - \lambda_0, \end{cases} \tag{61}$$

where λ_0 and λ_1 are arbitrary constants.

Concircular (I.4). In this case,

$$\dot{M}(t) \neq 0, \quad M\ddot{M} \neq \dot{M}^2, \quad \left(\frac{\kappa_0}{\mu_0} + \int \frac{dt}{M}\right)\left(\frac{\ddot{M}}{\dot{M}} - \frac{\dot{M}}{M}\right) + \frac{1}{M} \neq 0,$$

where $M(t)$ is an arbitrary function of t. The metric takes the form

$$ds^2 = -dt^2 + M^2(t)\left[\left(\kappa_0 + \mu_0 \int \frac{dt}{M(t)}\right)^2 dx^2 + dy^2 + dz^2\right], \tag{62}$$

and it admits four CCVFs which are the basic four KVFs given in Eq. (4). Also, both the conformal factors vanish. In this case, we have

$$\begin{cases} T = -\dfrac{4\mu_0\dot{M}}{MK} - \dfrac{6\dot{M}^2}{M^2}, & f(T) = \lambda_0 + \lambda_1 \int \dfrac{dT}{M^2}, \\ T^1_{01} = \dfrac{1 + \dot{M}\left(\kappa_0 + \mu_0 \int \frac{dt}{M}\right)}{M\left(\kappa_0 + \mu_0 \int \frac{dt}{M}\right)}, & T^2_{02} = T^3_{03} = \dfrac{\dot{M}}{M}, \\ 16\pi p(t) = f(T) & \\ \qquad + 4\lambda_1\left(\dfrac{2\mu_0\dot{M}}{M^3K} + \dfrac{\ddot{M}}{M^3}\right), & 16\pi\rho(t) = -f(T) - 4\lambda_1\left(\dfrac{2\mu_0\dot{M}}{M^3K} + \dfrac{3\dot{M}^2}{M^4}\right), \end{cases} \tag{63}$$

where λ_0 and λ_1 are arbitrary constants while $M(t)$ is an arbitrary function of t.

Case (II). In this case, we obtained $\mu_0 = 0$ then $K(t) = \kappa_0 M(t)$, where κ_0 is an arbitrary nonzero constant. The CCVFs in this case take the following forms:

Concircular (II.1). In this case, the components of concircular vector fields become

$$\begin{cases} X^0 = M(t)[\sigma_1 + \sigma_2\kappa_0\tilde{x} + \sigma_3 y + \sigma_4 z + 2\Theta(\sigma_5 + \sigma_6\kappa_0\tilde{x} + \sigma_7 y + \sigma_8 z) \\ \qquad\quad + \sigma_9(\Theta^2 + \tilde{x}^2 + y^2 + z^2)], \\ X^1 = \sigma_{10} + \sigma_2\Theta + \sigma_{11}y + \sigma_{12}z + 2x(\sigma_5 + \sigma_9 x + \sigma_7 y + \sigma_8 z) \\ \qquad\quad + \sigma_6(\Theta^2 + \tilde{x}^2 - y^2 - z^2), \\ X^2 = \sigma_{13} + \sigma_3\Theta - \sigma_{11}\kappa_0\tilde{x} + \sigma_{14}z + 2y(\sigma_5 + \sigma_9\Theta + \sigma_6\kappa_0\tilde{x} + \sigma_8 z) \\ \qquad\quad + \sigma_7(\Theta^2 - \tilde{x}^2 + y^2 - z^2), \\ X^3 = \sigma_{15} + \sigma_4\Theta - \sigma_{12}\kappa_0\tilde{x} - \sigma_{14}y + 2z(\sigma_5 + \sigma_9\Theta + \sigma_6\kappa_0\tilde{x} + \sigma_7 y) \\ \qquad\quad + \sigma_8(\Theta^2 - \tilde{x}^2 - y^2 + z^2), \end{cases} \tag{64}$$

such that $\Theta = -\frac{e^{-m_1 t}}{m_0 m_1}$, $\tilde{x} = \kappa_0 x$, $M(t) = m_0 e^{m_1 t}$ and σ_s, $s = 1, \ldots, 15$, are constants. The metric which admits the above 15 CCVFs is given as

$$ds^2 = -dt^2 + e^{2m_1 t}[\kappa_1^2 dx^2 + m_0^2(dy^2 + dz^2)], \tag{65}$$

where $m_0 \neq 0$ and m_1 are constants, while $\kappa_1 = m_0\kappa_0$ for $\kappa_0 \neq 0$. Also, the conformal factors are given by

$$\begin{cases} \alpha(t,x,y,z) = e^{m_1 t}(\tilde{\sigma}_1 + \kappa_0\tilde{\sigma}_2\tilde{x} + \tilde{\sigma}_3 y + \tilde{\sigma}_4 z + \tilde{\sigma}_9[\tilde{x}^2 + y^2 + z^2 - \Theta^2]), \\ \beta(t,x,y,z) = -m_1^2\alpha(t,x,y,z), \end{cases} \tag{66}$$

where $\tilde{\sigma}_i = m_0 m_1 \sigma_i$, $i = 1,2,3,4,9$. In this case, we have

$$T = -6m_1^2, \quad f(T) = 16\pi\lambda_0, \quad p(t) = \lambda_0 = -\rho(t), \quad T^1_{01} = T^2_{02} = T^3_{03} = m_1, \tag{67}$$

where λ_0 is an arbitrary constant.

Concircular (II.2). In this case, $M\ddot{M} \neq \dot{M}^2$, where $M(t)$ fulfills the following condition:

$$\big[\sigma_1 + 2\sigma_5\Theta\big]^2(M\ddot{M} - \dot{M}^2) = \kappa_1,$$

where κ_1 is an arbitrary constant while $\Theta = \int \frac{dt}{M(t)}$. The metric takes the form

$$ds^2 = -dt^2 + M^2(t)(\kappa_0^2 dx^2 + dy^2 + dz^2). \tag{68}$$

The above metric admits eight-dimensional CCVFs as follows:

$$\begin{cases} X^0 = M(t)[\sigma_1 + 2\sigma_5\Theta], & X^1 = \sigma_{10} + \sigma_{11}y + \sigma_{12}z + 2\sigma_5 x, \\ X^2 = \sigma_{13} - \sigma_{11}\kappa_0 x + \sigma_{14}z + 2\sigma_5 y, & X^3 = \sigma_{15} - \sigma_{12}\kappa_0\tilde{x} - \sigma_{14}y + 2\sigma_5 z. \end{cases} \tag{69}$$

The conformal factors are obtained as

$$\begin{cases} \alpha(t,x,y,z) = 2\sigma_5 + \big[\sigma_1 + 2\sigma_5\Theta\big]M(t), \\ M^2(t)\beta(t,x,y,z) = -\dot{M}^2(t)\alpha(t,x,y,z) - \dfrac{\kappa_1\dot{M}(t)}{\sigma_1 + 2\sigma_5\Theta}. \end{cases} \tag{70}$$

In this case, we have

$$\begin{cases} T = -\dfrac{6\dot{M}^2}{M^2}, \quad \text{where } f(T) \text{ is an arbitrary function of } T, \\ T^1_{01} = T^2_{02} = T^3_{03} = \dfrac{\dot{M}}{M}, \\ 16\pi p(t) = f(T) + \dfrac{12 f_T(T)\dot{M}^2}{M^2} + \dfrac{4\kappa_1\left[f_T(T)M^2 - 12 f_{TT}(T)\dot{M}^2\right]}{M^4\left(\sigma_1 + 2\sigma_5\Theta\right)}, \\ 16\pi\rho(t) = -f(T) - \dfrac{12 f_T(T)\dot{M}^2}{M^2}. \end{cases} \tag{71}$$

Important Remark. In general, Case (II) is a singular case, because $K(t) = \kappa_0 M(t)$ leads to

$$\frac{\dot{K}}{K} = \frac{\dot{M}}{M}, \quad \frac{\ddot{K}}{K} = \frac{\ddot{M}}{M}, \quad \frac{\dot{K}\dot{M}}{KM} = \frac{\dot{M}^2}{M^2}.$$

It is now easy to see that Eqs. (34) and (35) become a single equation and it implies that condition (36) does not exist [no condition for $f(T)$]. On the other hand, it means that Eq. (36) is satisfied directly, so $f(T)$ becomes an arbitrary function.

In the following, we will give a special case of the above case.

Concircular (II.2.A). In this case,

$$\sigma_5 = 0 \quad \Rightarrow \quad M(t) = m_1 e^{m_0 t} + m_2 e^{-m_0 t},$$

where m_1, m_2 and m_0 are arbitrary constants. The metric now becomes

$$ds^2 = -dt^2 + \left(m_1 e^{m_0 t} + m_2 e^{-m_0 t}\right)\left[\kappa_0^2 dx^2 + dy^2 + dz^2\right], \tag{72}$$

which admits the eight-dimensional concircular vector fields as follows:

$$\begin{cases} X^0 = \tilde{\sigma}_1 e^{m_0 t} + \tilde{\sigma}_2 e^{-m_0 t}, & X^1 = \sigma_{10} + \sigma_{11}y + \sigma_{12}z, \\ X^2 = \sigma_{13} - \sigma_{11}\kappa_0\tilde{x} + \sigma_{14}z, & X^3 = \sigma_{15} - \sigma_{12}\kappa_0\tilde{x} - \sigma_{14}y, \end{cases} \tag{73}$$

where $\tilde{\sigma}_1 = \sigma_1 m_1$ and $\tilde{\sigma}_2 = \sigma_1 m_2$. The conformal factors are given by

$$\left\{\alpha(t,x,y,z) = m_0(\tilde{\sigma}_1 e^{m_0 t} - \tilde{\sigma}_2 e^{-m_0 t}), \quad M^2(t)\beta(t,x,y,z) = -m_0^2\alpha(t,x,y,z). \right. \tag{74}$$

In this case, we have

$$\begin{cases} T \;=\; -6\,m_0^2\left(\dfrac{m_1 e^{2m_0 t} - m_2}{m_1 e^{2m_0 t} + m_2}\right)^2, \quad f(T)\,\text{is an arbitrary function of}\,T, \\[2ex] T^1_{01} \;=\; T^2_{02} \;=\; T^3_{03} \;=\; \dfrac{m_0(m_1 e^{m_0 t} - m_2 e^{-m_0 t})}{m_1 e^{m_0 t} + m_2 e^{-m_0 t}}, \\[2ex] 16\pi p(t) \;=\; f(T) + 4m_0^2\left[\dfrac{\left(3m_1^2 e^{4m_0 t} - 2m_1 m_2 e^{2m_0 t} + 3m_2^2\right) f_T(T)}{\left(m_1 e^{2m_0 t} + m_2\right)^2}\right. \\[2ex] \qquad\qquad \left. - \dfrac{48 m_1 m_2 m_0^2 \left(m_1 e^{3m_0 t} - m_2 e^{m_0 t}\right)^2 f_{TT}(T)}{\left(m_1 e^{2m_0 t} + m_2\right)^4}\right], \\[2ex] 16\pi\rho(t) \;=\; -f(T) - 12m_0^2 f_T(T)\left(\dfrac{m_1 e^{2m_0 t} - m_2}{m_1 e^{2m_0 t} + m_2}\right)^2. \end{cases} \tag{75}$$

Concircular (II.3). In this case, $M\ddot{M} \neq \dot{M}^2$, where $M(t)$ is an arbitrary function of t. The metric is given in (68) and it admits eight-dimensional CCVFs as follows:

$$\begin{cases} X^0 = 0, & X^1 = \sigma_{10} + \sigma_{11} y + \sigma_{12} z, \\ X^2 = \sigma_{13} - \sigma_{11}\kappa_0\tilde{x} + \sigma_{14} z, & X^3 \;=\; \sigma_{15} - \sigma_{12}\kappa_0\tilde{x} - \sigma_{14} y. \end{cases} \tag{76}$$

The conformal factors are given by $\beta(t,x,y,z) = \alpha(t,x,y,z) = 0$. In this case, we have

$$\begin{cases} T \;=\; -\dfrac{6\dot{M}^2}{M^2}, \quad f(T)\,\text{is an arbitrary function of}\,T, \\[2ex] T^1_{01} = T^2_{02} = T^3_{03} = \dfrac{\dot{M}}{M}, \\[2ex] 16\pi p(t) = f(T) + 4\left[f_T\left(\dfrac{2\dot{M}^2}{M^2} + \dfrac{\ddot{M}}{M}\right) + 12 f_{TT}\left(\dfrac{\dot{M}^4}{M^4} - \dfrac{\dot{M}^2\ddot{M}}{M^3}\right)\right], \\[2ex] 16\pi\rho(t) \;=\; -f(T) - \dfrac{12 f_T(T)\dot{M}^2}{M^2}. \end{cases} \tag{77}$$

3.2. *Part* (*B*): $L(t) \neq 0$

In this subsection, we consider $L(t) \neq 0 \Rightarrow G_k(x) = d_k$, where d_k are arbitrary constants for $k = 2,3,4$. Hence, substituting $G_k(x)$ in the first equation of (30), we obtain second-order ordinary differential equation of $F_k(t)$. The general solution of this equation is given by the following form:

$$F_k(t) = \sigma_k + \rho_k \int \frac{dt}{M(t)} + c_k \left(\int \frac{dt}{M(t)} \right)^2 - \frac{d_k K(t)}{M(t)}, \quad k = 2,3,4. \tag{78}$$

Again substituting $G_k(x)$ and $F_k(t)$ in the second equation of (30), we get the ordinary differential equations of the metric functions $M(t)$ and $K(t)$ as follows:

$$\left(\rho_k + 2c_k \int \frac{dt}{M(t)} \right) [M(t)\dot{K}(t) - K(t)\dot{M}(t)] = 2c_k K(t), \quad k = 2,3,4. \tag{79}$$

If we consider $\rho_k \neq 0$ and $c_k \neq 0$, then the general solution of the above equation is obtained as

$$K(t) = \kappa_0 \left(\rho_k + 2c_k \int \frac{dt}{M(t)} \right) M(t), \quad k = 2,3,4, \tag{80}$$

where κ_0 is an arbitrary nonzero constant of integration. This solution in turn implies that $L(t) = 0$, which is a contradiction. Therefore, we must consider $\rho_k = 0$ and $c_k = 0$ for all $k = 2,3,4$. For avoiding lengthy details, we write only the obtained results in the following cases.

Case (III). In this case,

$$M(t) = (\kappa_1 e^{\Phi[t]} + \kappa_2 e^{-\Phi[t]}) K(t), \quad \Phi(t) = \mu_0 \int \frac{dt}{K(t)},$$

where μ_0, κ_1 and κ_2 are arbitrary nonzero constants. The components of conformal Killing vector fields now become

$$\begin{cases} X^0 = (\epsilon_1 e^{\mu_0 x} + \epsilon_2 e^{-\mu_0 x}) \left(\kappa_1 e^{\Phi[t]} + \kappa_2 e^{-\Phi[t]} \right) K(t), \\ X^1 = \epsilon_3 + (\epsilon_1 e^{\mu_0 x} - \epsilon_2 e^{-\mu_0 x}) \left(\kappa_1 e^{\Phi[t]} - \kappa_2 e^{-\Phi[t]} \right), \\ X^2 = \epsilon_4 + \epsilon_5 z, \quad X^3 = \epsilon_6 - \epsilon_5 y; \end{cases} \tag{81}$$

also, the conformal factor takes the form: $\alpha(t,x,y,z) = X^0_{,0}$, where ϵ_i, $i = 1, \ldots, 6$, represent arbitrary constants. Thus, the metric

$$ds^2 = -dt^2 + K^2(t)[dx^2 + (\kappa_1 e^{\Phi[t]} + \kappa_2 e^{-\Phi[t]})(dy^2 + dz^2)], \tag{82}$$

where $K(t)$ is an arbitrary function depending upon t only. The above metric admits six conformal Killing vector fields (81). Substituting the vector field components (81) in Eqs. (15)–(24), one can obtain the differential constraints for the concircular vector fields. We are not listing those constraints here, but if we solve those constraints, we obtain the final form of concircular vector fields as different cases. We will list all those cases in the following.

Concircular (III.1). In this case,

$$\dot{K}^2 - K\ddot{K} = \mu_0^2 \Rightarrow K(t) = m_1 e^{m_0 t} - m_2 e^{-m_0 t} \Rightarrow M(t)$$
$$= -2\kappa_1(m_1 e^{m_0 t} + m_2 e^{-m_0 t}),$$

where κ_1, m_0 and m_1 are arbitrary constants while $m_2 = \frac{\mu_0^2}{4m_0^2 m_1}$ and $\kappa_2 = \kappa_1$. The metric becomes

$$ds^2 = -dt^2 + (m_1 e^{m_0 t} - m_2 e^{-m_0 t})^2 dx^2 + 4\kappa_1^2(m_1 e^{m_0 t} + m_2 e^{-m_0 t})^2(dy^2 + dz^2), \tag{83}$$

where m_0, m_1, μ_0 and κ_1 are arbitrary (nonzero) constants. This metric admits six-dimensional concircular vector fields as follows:

$$\begin{cases} X^0 = -2\kappa_1\left(m_1 e^{m_0 t} + m_2 e^{-m_0 t}\right)\left(\epsilon_1 e^{\mu_0 x} + \epsilon_2 e^{-\mu_0 x}\right), \\ X^1 = \epsilon_3 + \dfrac{2\mu_0}{m_0 m_1}\left(\dfrac{\epsilon_1 e^{\mu_0 x} - \epsilon_2 e^{-\mu_0 x}}{m_1 e^{m_0 t} - m_2 e^{-m_0 t}}\right), \\ X^2 = \epsilon_4 + \epsilon_5 z, \quad X^3 = \epsilon_6 - \epsilon_5 y, \end{cases} \tag{84}$$

where ϵ_i, $i = 1, \ldots, 6$, are arbitrary constants. Also, the conformal factors are given by

$$\begin{cases} \alpha(t,x,y,z) = -2m_0\kappa_1^2\left(m_1 e^{m_0 t} - m_2 e^{-m_0 t}\right)\left(\epsilon_1 e^{\mu_0 x} + \epsilon_2 e^{-\mu_0 x}\right), \\ \beta(t,x,y,z) = -m_0^2 \alpha(t,x,y,z). \end{cases} \tag{85}$$

In this case, we have

$$\begin{cases} T = \dfrac{-48 m_0^4 m_1^4 e^{2m_0 t} - 8m_0^2 m_1^2 \mu_0^2 - 3\mu_0^2 e^{-2m_0 t}}{8m_0^2 m_1^2\left(m_1 e^{m_0 t} + m_2 e^{-m_0 t}\right)^2}, \\ T^1_{01} = \dfrac{m_0(m_1 e^{m_0 t} + m_2 e^{-m_0 t})}{K}, \quad T^2_{02} = T^3_{03} = \dfrac{m_0(m_1 e^{m_0 t} - m_2 e^{-m_0 t})}{m_1 e^{m_0 t} + m_2 e^{-m_0 t}}, \\ f(T) = \lambda_0 + \tilde{\lambda}_1\left(m_1 e^{m_0 t} + m_2 e^{-m_0 t}\right)^{-3}, \\ 16\pi p(t) = \lambda_0 + \dfrac{\tilde{\lambda}_1\left(48 m_0^4 m_1^4 e^{2m_0 t} + 16 m_0^2 m_1^2 \mu_0^2 + 3\mu_0^2 e^{-2m_0 t}\right)}{24 m_0^2 m_1^2 \mu_0^2\left(m_1 e^{m_0 t} + m_2 e^{-m_0 t}\right)^3}, \\ 16\pi\rho(t) = -\lambda_0 - \dfrac{\tilde{\lambda}_1\left(144 m_0^4 m_1^4 e^{2m_0 t} + 48 m_0^2 m_1^2 \mu_0^2 + 9\mu_0^2 e^{-2m_0 t}\right)}{48 m_0^2 m_1^2 \mu_0^2\left(m_1 e^{m_0 t} + m_2 e^{-m_0 t}\right)^3}, \end{cases} \tag{86}$$

where λ_0 and $\tilde{\lambda}_1 = \dfrac{m_0\lambda_1}{\kappa_1^2}$ are arbitrary constants.

Concircular (III.2). In this case, $\dot{K}^2 - K\ddot{K} \neq \mu_0^2 \Rightarrow \epsilon_1 = \epsilon_2 = 0$. The metric is the same as given in (82). With the above condition, the same metric now admits only four CCVFs which are the four basic KVFs and are already listed in (4). Also, the conformal factors vanish. In this case, we have

$$\begin{cases} T \;=\; \dfrac{-2}{K^2(t)}\left[\mu_0+\dot{K}(t)-\dfrac{2\mu_0\kappa_2}{\kappa_1 e^{2\Phi[t]}+\kappa_2}\right]\left[\mu_0+3\dot{K}(t)-\dfrac{2\mu_0\kappa_2}{\kappa_1 e^{2\Phi[t]}+\kappa_2}\right], \\ T^1_{01}=\dfrac{\dot{K}}{K}, \quad T^2_{02} \;=\; T^3_{03}=\dfrac{\mu_0\big[\kappa_1 e^{\Phi[t]}-\kappa_2 e^{\Phi[t]}\big]}{K\big[\kappa_1 e^{\Phi[t]}-\kappa_2 e^{\Phi[t]}\big]}, \\ f(T) \;=\; \lambda_0+\lambda_1\int\dfrac{dT}{\mu_0 K^2(t)\left(\kappa_1^2 e^{2\Phi[t]}-\kappa_2^2 e^{-2\Phi[t]}\right)}, \\ 16\pi p(t) \;=\; f(T)+\dfrac{4\lambda_1}{\kappa_0}\left[\dfrac{4\mu_0\kappa_1\kappa_2\dot{K}(t)e^{2\Phi[t]}-\left(\kappa_1^2 e^{4\Phi[t]}-\kappa_2^2\right)K(t)\ddot{K}(t)}{K^4(t)\left(\kappa_1^2 e^{3\Phi[t]}-\kappa_2^2 e^{-\Phi[t]}\right)^2}\right], \\ 16\pi\rho(t)=-f(T)-\dfrac{2\lambda_1 T}{\mu_0 K^2(t)\left(\kappa_1^2 e^{2\Phi[t]}-\kappa_2^2 e^{-2\Phi[t]}\right)}, \end{cases} \tag{87}$$

where λ_0 and λ_1 are arbitrary constants.

Case (IV). In this case,

$$K(t)=\mu_0 M(t)\left(\kappa_1+\kappa_2\int\frac{dt}{M}\right)^{1-\frac{\kappa_4}{\kappa_2}},$$

where $\mu_0\neq 0$, κ_1, $\kappa_2\neq 0$ and $\kappa_4\neq 0$ are arbitrary constants such that $\kappa_4\neq\kappa_2$. The components of conformal Killing vector fields now become

$$\begin{cases} X^0=M(t)\left(\kappa_1+\kappa_2\int\dfrac{dt}{M}\right), & X^1=\kappa_3+\kappa_4 x, \\ X^2=\kappa_5+\kappa_2 y+\kappa_6 z, & X^3=\kappa_7+\kappa_2 z-\kappa_6 y; \end{cases} \tag{88}$$

also the conformal factor takes the form: $\alpha(t,x,y,z)=X^0_{,0}$, where κ_i, $i=1,\ldots,7$, represent arbitrary constants. Thus, the metric

$$ds^2=-dt^2+M^2(t)\left[\mu_0^2\left(\kappa_1+\kappa_2\int\frac{dt}{M}\right)^{2-\frac{2\kappa_4}{\kappa_2}}dx^2+dy^2+dz^2\right], \tag{89}$$

where $M(t)$ is an arbitrary function of t only, while μ_0, κ_1 and κ_2 are arbitrary (nonzero) constants, admits seven conformal Killing vector fields (81). Substituting the vector field components (81) in Eqs. (15)–(24), we obtain some differential constraints for the concircular vector fields. We are not listing those constraints here, but if we solve those constraints, we obtain the final form of concircular vector fields as follows.

Concircular (IV). In this case, $K(t)=n_0(t+t_0)^{n_1}$ and $M(t)=m_0(t+t_0)^{m_1}$, where n_0, n_1, m_0, m_1 and t_0 are arbitrary constants such that $n_1\neq m_1$. Thus, the metric

$$ds^2 = -dt^2 + n_0^2(t+t_0)^{2n_1}dx^2 + m_0^2(t+t_0)^{2m_1}(dy^2+dz^2) \tag{90}$$

admits five-dimensional concircular vector fields as follows:

$$\begin{cases} X^0 = \tilde{\kappa}_1(t+t_0), & X^1 = \tilde{\kappa}_2 + (1-n_1)\tilde{\kappa}_1 x, \\ X^2 = \tilde{\kappa}_3 + (1-m_1)\tilde{\kappa}_1 y + \tilde{\kappa}_4 z, & X^3 = \tilde{\kappa}_5 + (1-m_1)\tilde{\kappa}_1 z - \tilde{\kappa}_4 y, \end{cases} \tag{91}$$

where $\tilde{\kappa}_i$, $i = 1, \ldots, 5$, are arbitrary constants. Also, the conformal factors are given by: $\alpha(t,x,y,z) = \tilde{\kappa}_1$ and $\beta(t,x,y,z) = 0$. The form of conformal factors suggests that the obtained CCVFs are just the HVFs with one proper HVF. In this case, we have

$$\begin{cases} T = -\dfrac{2m_1(m_1+2n_1)}{(t+t_0)^2}, & f(T) = \lambda_0 + \lambda_1(t+t_0)^{-n_1-2m_1-1}, \\ T^1_{01} = \dfrac{n_1}{t+t_0}, & T^2_{02} = T^3_{03} = \dfrac{m_1}{t+t_0}, \\ 16\pi p(t) = \lambda_0 \\ \qquad + \lambda_1(t+t_0)^{-n_1-2m_1-1}, & 16\pi\rho(t) = (n_1+2m_1) \\ & \qquad \lambda_1(t+t_0)^{-n_1-2m_1-1} - \lambda_0, \end{cases} \tag{92}$$

where λ_0 and λ_1 are arbitrary constants.

Case (V). In this case, $K(t)$ and $M(t)$ are arbitrary functions of t [i.e. no conditions for the metric functions $K(t)$ and $M(t)$]. The metric that is given by the form (3), admits four-dimensional CKVFs and CCVFs given in Eq. (4). Also, the conformal factors are given by: $\beta(t,x,y,z) = \alpha(t,x,y,z) = 0$. In this case, we have

$$\begin{cases} T = -\dfrac{2\dot{M}^2}{M^2} - \dfrac{4\dot{K}\dot{M}}{KM}, \quad f(T) = \lambda_0 + \lambda_1 \displaystyle\int \frac{dT}{M^2\dot{K} - KM\dot{M}}, \\ T^1_{01} = \dfrac{\dot{K}}{K}, \quad T^2_{02} = T^3_{03} = \dfrac{\dot{M}}{M}, \\ 16\pi p(t) = f(T) - \dfrac{4\lambda_1\left[K\dot{K}\left(\dot{M}^2 - M\ddot{M}\right) - M\dot{M}\left(\dot{K}^2 - K\ddot{K}\right)\right]}{KM^2\left(M\dot{K} - K\dot{M}\right)^2}, \\ 16\pi\rho(t) = -f(T) + \dfrac{4\lambda_1\dot{M}}{KM^3}\left(\dfrac{2M\dot{K} + K\dot{M}}{K\dot{M} - M\dot{K}}\right), \end{cases} \tag{93}$$

where λ_0 and λ_1 are arbitrary constants.

4. Some Associated Cosmological Constraints for the Obtained Solutions

In this section, we are going to discuss the cosmological constraints, i.e. the Hubble's parameter (H), scalar expansion (θ), shear scalar (σ^2), anisotropy parameter (A_m) and the deceleration parameter (q) associated with each of the obtained exact LRS

Bianchi type-I solutions. These constraints will provide us better insight to predict the future of the obtained universe models. All these parameters are used to predict the physical behavior of the model. For example, Hubble's parameter is used to predict the rate of expansion of the universe. Also, the deceleration parameter (q) describes the physical behavior of the model. It will provide us the information on whether the universe is expanding with an accelerated rate or whether its rate is decelerating [45]. The value $q = 0$ predicts whether the model is in uniform motion or at rest. Similarly, the anisotropic parameter describes the shape of the model, i.e. whether the model has some anisotropy or whether it is isotropic everywhere. It is obviously interesting to see what kind of physical behavior our obtained models depict and how the concircular symmetry affects their physical nature. For the general LRS Bianchi type-I spacetime metric (3), these constraints are defined and obtained as follows:

$$\begin{cases} \Theta = u^i_{;i} = \dfrac{\dot{K}}{K} + \dfrac{2\dot{M}}{M}, \\ H = \dfrac{\Theta}{3} = \dfrac{1}{3}\left(\dfrac{\dot{K}}{K} + \dfrac{2\dot{M}}{M}\right), \\ \sigma^2 = \dfrac{1}{2}\sigma_{ij}\sigma^{ij} = \dfrac{1}{3}\left(\dfrac{\dot{K}}{K} - \dfrac{\dot{M}}{M}\right)^2, \\ A_m = \dfrac{1}{3}\displaystyle\sum_{i=1}^{3}\left(\dfrac{H_i}{H} - 1\right)^2 = \dfrac{2\sigma}{3H^2} = \dfrac{2\left(M\dot{K} - K\dot{M}\right)^2}{\left(M\dot{K} + 2K\dot{M}\right)^2}, \\ q = -\dfrac{\dot{H}}{H^2} - 1 = \dfrac{6K^2\left(\dot{M}^2 - M\ddot{M}\right) + 3M^2\left(\dot{K}^2 - K\ddot{K}\right)}{\left(M\dot{K} + 2K\dot{M}\right)^2} - 1. \end{cases} \tag{94}$$

Substituting the values of all the obtained exact solutions into Eqs. (94) and after some basic simplification, the obtained cosmological constraints are listed for each case as follows:

Concircular (I.1). $K(t) = \kappa_1 + \mu_0 t$ and $M(t) = m_0$. Here we have

$$\begin{cases} \Theta = \dfrac{\mu_0}{\kappa_1 + \mu_0 t}, \qquad H = \dfrac{\mu_0}{3\left(\kappa_1 + \mu_0 t\right)}, \\ \sigma^2 = \dfrac{\mu_0^2}{3\left(\kappa_1 + \mu_0 t\right)^2}, \quad A_m = 2, \quad q = 2. \end{cases} \tag{95}$$

In this case, we observe that all the derived parameters are functions of cosmic time t except the anisotropic parameter A_m and deceleration parameter q. These two parameters have constant values. When $t \to \infty$, the constraints H, θ and σ^2 vanish. The constant values of A_m and q indicate that this model has a fixed measure of anisotropy and is decelerating at constant speed. The vanishing nature of other parameters indicates that this LRS Bianchi type-I model might become a static model at late time. In the presence of 15-dimensional CCVFs, we can see how the geometry and physics of the spacetime are changing.

Concircular (I.2). $K(t) = \kappa_1 e^{m_1 t} - \mu_1$ and $M(t) = m_0 e^{m_1 t}$. Here we have

$$\begin{cases} \Theta = m_1 \left(3 + \dfrac{\mu_1}{\kappa_1 e^{m_1 t} - \mu_1}\right), \quad H = \dfrac{m_1}{3}\left(3 + \dfrac{\mu_1}{\kappa_1 e^{m_1 t} - \mu_1}\right), \\[2ex] \sigma^2 = \dfrac{\mu_1^2 m_1^2}{3\left(\kappa_1 e^{m_1 t} - \mu_1\right)^2}, \qquad A_m = \dfrac{2\mu_1^2}{3\left(3\kappa_1 e^{m_1 t} - 2\mu_1\right)^2}, \\[2ex] q = \dfrac{3\kappa_1 \mu_1 e^{m_1 t}}{\left(3\kappa_1 e^{m_1 t} - 2\mu_1\right)^2} - 1. \end{cases} \tag{96}$$

This case has three subcases: (i) $\delta_i = 0,\ \ i = 1, 2, 3$; (ii) $\mu_1 = 0$; and (iii) $\kappa_1 = 0$. In Case (i), the physical parameters do not depend upon δ_i. In this case, we observe that all the derived parameters are functions of cosmic time t. When $t \to \infty$, the constraints σ^2 and A_m become zero while the other quantities become constant. The zero value of A_m indicates that this model is isotropic. The value of $q = -1$ shows that the model is growing exponentially. The constant value of θ indicates that the model is expanding at constant rate at late time. In Case (ii), when $\mu_1 = 0$, the values of H and θ become constants while σ^2 and A_m become zero. A constant θ indicates that the model is expanding at constant rate and is unaffected by time, while the zero value of A_m indicates that this model is isotropic and this isotropy is also unaffected by time. The value of $q = -1$ shows that the model is growing exponentially. In Case (iii), when $\kappa_1 = 0$, then all the derived parameters become zero which indicates that this model has a constant shear and is decelerating at constant speed. The model has a constant anisotropy which is unaffected by the variation of time. This model is also growing exponentially due to $q = -1$ value.

Concircular (I.3). $K(t) = \mu_1(t + t_0)$ and $M(t) = m_0(t + t_0)^{m_1}$. Here we have

$$\begin{cases} \Theta = \dfrac{1 + 2m_1}{t + t_0}, \qquad H = \dfrac{1 + 2m_1}{3(t + t_0)}, \\[2ex] \sigma^2 = \dfrac{(1 - m_1)^2}{3(t + t_0)^2}, \quad A_m = 2\left(\dfrac{1 - m_1}{1 + 2m_1}\right)^2, \quad q = \dfrac{3}{2m_1 + 1} - 1. \end{cases} \tag{97}$$

In this case, we observe that all the derived parameters are functions of cosmic time t except the anisotropic parameter A_m and deceleration parameter q. These two parameters have constant values. One can see that when $m_1 = 1$, then σ^2, A_m and q become zero, but it is important to remind that in this case the model admits six-dimensional CCVFs and $m_1 \neq 1$. Thus, there will be a constant anisotropy at all the times and q cannot be zero due to the condition $m_1 \neq 1$. Moreover, q has negative value for $m_1 > 1$ and $m_1 < 0$ while it has a positive value when $0 \leq m_1 < 1$. Thus, we see that the model will face a deceleration if $0 \leq m_1 < 1$ and it will accelerate gradually or grow exponentially for the other two values of m_1. When $t \to \infty$, the constraints H, θ and σ^2 vanish. The vanishing nature of these parameters indicates that this LRS Bianchi type-I model might become a static model at late time.

Concircular (I.4). $K(t) = M(t)(\kappa_0 + \mu_0 \int \frac{dt}{M(t)})$. Here we have

$$\begin{cases} \Theta = \dfrac{3\dot{M}}{M} + \dfrac{\mu_0}{K}, \quad H = \dfrac{\dot{M}}{M} + \dfrac{\mu_0}{3K}, \quad \sigma^2 = \dfrac{\mu_0^2}{3K^2}, \\ A_m = 2\left(\dfrac{\mu_0 M}{\mu_0 M + 3K\dot{M}}\right)^2, \quad q = \dfrac{2\mu_0^2 M^2 - 3\mu_0 KM\dot{M} - 9MK^2\ddot{M}}{\left(\mu_0 M + 3K\dot{M}\right)^2}. \end{cases} \tag{98}$$

In this case, we did not obtain the particular forms of the metric functions. Thus, the obtained spacetime metric does not represent a physical model.

Concircular (II.1). $K(t) = \kappa_1 e^{m_1 t}$ and $M(t) = m_0 e^{m_1 t}$. Here we have

$$\{\Theta = 3m_1, \quad H = m_1, \quad \sigma^2 = 0, \quad A_m = 0, \quad q = -1. \tag{99}$$

In this case, the values of H and θ become constants while σ^2 and A_m become zero. A constant θ indicates that the model is expanding at a constant rate and is unaffected by time, while the zero value of A_m indicates that this model is isotropic and this isotropy is also unaffected by time. The value of $q = -1$ shows that the model is growing exponentially.

Concircular (II.2A). $K(t) = \kappa_0 M(t)$ and $M(t) = m_1 e^{m_0 t} + m_2 e^{-m_0 t}$. Here we have

$$\begin{cases} \Theta = -3m_0\left(\dfrac{m_2 - m_1 e^{2m_0 t}}{m_2 + m_1 e^{2m_0 t}}\right), \quad H = -m_0\left(\dfrac{m_2 - m_1 e^{2m_0 t}}{m_2 + m_1 e^{2m_0 t}}\right), \\ \sigma^2 = 0, \quad A_m = 0, \quad q = -\left(\dfrac{m_2 + m_1 e^{2m_0 t}}{m_2 - m_1 e^{2m_0 t}}\right)^2. \end{cases} \tag{100}$$

In this case, σ^2 and A_m are zero, which means that the model represents a shear-free and isotropic universe. Initially, i.e. at $t = 0$, the parameters H, θ and q have constant values for $m_2 > m_1$ and $m_0 < 0$ such that q has a negative value. At late times ($t \to \infty$), the values of H, θ and q remain constant. A constant θ indicates that the model is expanding at a constant rate while a negative value of q means that the model is growing exponentially.

Concircular (II.3). $K(t) = \kappa_0 M(t)$. Here we have

$$\left\{\Theta = \frac{3\dot{M}}{M}, \quad H = \frac{\dot{M}}{M}, \quad \sigma^2 = 0, \quad A_m = 0, \quad q = -\frac{M\ddot{M}}{\dot{M}^2}. \right. \tag{101}$$

In this case, we did not obtain particular forms of the metric functions. Thus, the obtained spacetime metric does not represent a physical model.

Concircular (III.1). $K(t) = m_1 e^{m_0 t} - m_2 e^{-m_0 t}$ and $M(t) = \kappa_3(m_1 e^{m_0 t} - m_2 e^{-m_0 t})$. Here we have

$$\begin{cases} \Theta = m_0\left(3 - \dfrac{2m_2\left(3m_2 + m_1 e^{2m_0 t}\right)}{m_2^2 - m_1^2 e^{4m_0 t}}\right), \quad H = m_0\left(1 - \dfrac{2m_2\left(3m_2 + m_1 e^{2m_0 t}\right)}{3\left(m_2^2 - m_1^2 e^{4m_0 t}\right)}\right), \\ \sigma^2 = \dfrac{1}{3}\left(\dfrac{4m_0 m_1 m_2}{m_1^2 e^{2m_0 t} - m_2^2 e^{-2m_0 t}}\right), \quad A_m = 2\left(\dfrac{4m_1 m_2}{3m_1^2 e^{2m_0 t} + 2m_1 m_2 + 3m_2^2 e^{-2m_0 t}}\right), \\ q = \dfrac{-9m_1^4 e^{4m_0 t} + 50 m_1^2 m_2^2 - 9m_2^4 e^{-4m_0 t}}{\left(3m_1^2 e^{2m_0 t} + 2m_1 m_2 + 3m_2^2 e^{-2m_0 t}\right)^2}. \end{cases} \tag{102}$$

In this case, we observe that all the derived parameters are functions of cosmic time t. Initially, the universe is created with constant values of H, θ, σ^2, A_m and q. This indicates that at the time of creation of this universe, it was expanding at a constant rate with a constant shear and the anisotropy was constant. At late time, when $t \to \infty$, the constraints H and θ have constant values while σ^2 and A_m vanish, whereas q becomes infinite. This infinite value shows that the universe will be decelerating rapidly.

Concircular (III.2). $M(t) = K(t)[\kappa_1 e^{\Phi(t)} + \kappa_2 e^{-\Phi(t)}]$ and $\Phi(t) = \mu_0 \int \frac{dt}{K(t)}$. Here we have

$$\begin{cases} \Theta = \dfrac{3\dot{K}}{K} + \dfrac{\mu_0}{K}\left(2 - \dfrac{4\kappa_2}{\kappa_1 e^{2\Phi(t)} + \kappa_2}\right), \quad H = \dfrac{\dot{K}}{K} + \dfrac{\mu_0}{3K}\left(2 - \dfrac{4\kappa_2}{\kappa_1 e^{2\Phi(t)} + \kappa_2}\right), \\ \sigma^2 = \dfrac{\mu_0^2\left(\kappa_1 e^{2\Phi(t)} - \kappa_2\right)^2}{3K\left(\kappa_1 e^{2\Phi(t)} + \kappa_2\right)^2}, \quad A_m = 2\mu_0^2\left[\dfrac{\kappa_1 e^{2\Phi(t)} - \kappa_2}{2\mu_0^2\left(\kappa_1 e^{2\Phi(t)} - \kappa_2\right) + 3\dot{K}\left(\kappa_1 e^{2\Phi(t)} + \kappa_2\right)}\right]^2, \\ q = -\dfrac{4\mu_0^2\left(\kappa_1^2 e^{4\Phi(t)} + 4\kappa_1\kappa_2 e^{2\Phi(t)} + \kappa_2^2\right) + 6\mu_0\dot{K}\left(\kappa_1^2 e^{4\Phi(t)} - \kappa_2^2\right) + 9K\ddot{K}\left(\kappa_1 e^{2\Phi(t)} + \kappa_2\right)^2}{\left[2\mu_0^2\left(\kappa_1 e^{2\Phi(t)} - \kappa_2\right) + 3\dot{K}\left(\kappa_1 e^{2\Phi(t)} + \kappa_2\right)\right]^2}. \end{cases} \tag{103}$$

In this case, we did not obtain particular forms of the metric functions. Thus, the obtained spacetime metric does not represent a physical model.

Concircular (IV). $K(t) = n_0(t+t_0)^{n_1}$ and $M(t) = m_0(t+t_0)^{m_1}$. Here we have

$$\begin{cases} \Theta = \dfrac{n_1+2m_1}{t+t_0}, \quad H = \dfrac{n_1+2m_1}{3(t+t_0)}, \quad \sigma^2 = \dfrac{(n_1-m_1)^2}{3(t+t_0)^2}, \\ A_m = 2\left(\dfrac{n_1-m_1}{n_1+2m_1}\right)^2, \quad q = \dfrac{3}{n_1+2m_1} - 1. \end{cases} \tag{104}$$

In this case, we observe that all the derived parameters except A_m and q are functions of cosmic time t. The two parameters A_m and q are unaffected by time and remain constant. Initially, the universe is created with constant values of H, θ, σ^2, A_m and q. This indicates that at the time of creation of this universe, it was expanding at a constant rate with constant shear and anisotropy. At late time, when $t \to \infty$, the constraints H, θ and σ^2 vanish which means that the universe will have zero expansion but constant anisotropy. Also, the deceleration parameter q will be positive when $n_1 + 2m_1 < 3$. This will then indicate that the universe will be decelerating. For $n_1 + 2m_1 > 3$, q will become negative, which means that the universe will face an accelerated expansion. For $n_1 + 2m_1 = 3$, q will vanish and the universe will either be in rest or in uniform motion. It is important to remind that H, θ and σ^2 will vanish at late time, thus we can conclude that $n_1 + 2m_1 = 3$ and $q = 0$, and the universe will be either in rest or in uniform motion.

Concircular (V). In this case, all the parametric values are given in Eqs. (94), as in this case we did not obtain particular forms of the metric functions. Thus, the obtained spacetime metric does not represent a physical model.

5. Summary and Discussion

In this paper, we obtained concircular vector fields of LRS Bianchi type-I spacetime in the presence of perfect fluid in $f(T)$ theory of gravity. The results are achieved by splitting the problem into two main parts. In the first part, we obtained EFEs for perfect fluid LRS Bianchi type-I spacetime in $f(T)$ theory of gravity. In the second part, we obtained the concircular vector field equations and solved those equations simultaneously along with the EFEs to obtain the metric functions, concircular vector field components and conformal factors. In this manner, different possibilities are generated and we solved all the possibilities in turn. All the obtained solutions are listed as different cases. In all the cases, we also calculated the energy density, fluid pressure, torsion scalar T and the form of the function $f(T)$.

In the literature, it is accustomed to specify the form of $f(T)$ first and then solve the EFEs and CCVF equations simultaneously. In this paper, we did not choose a specific form of $f(T)$ initially, rather we found the particular form of $f(T)$ during the process of finding the CCVFs. We started our process of finding the CCVFs by differentiating the first equation of (30) with respect to x and obtained Eq. (46). These equations produced two possibilities: either $L(t) = 0$ or $L(t) \neq 0$. We applied both of these possibilities in turn and listed all the obtained solutions in two parts, i.e. Parts (A) and (B).

In Part (A), we got two different cases and these cases are further divided into subcases. The total number of different solutions obtained in this part is 11. These 11 cases revealed that perfect fluid LRS Bianchi type-I spacetimes in $f(T)$ gravity admit CCVFs of dimensions 4, 6, 7 and 15. In Part (B), we got three different cases in which the obtained solutions admit CCVFs of dimensions 4, 5, 6 and 7.

In all the cases, we also calculated density, pressure, the form of T and $f(T)$. It is well known that an obtained solution will be a physically realistic model if its density and pressure are positive and decreasing functions of cosmic time t. As the volume of the space is interlinked to the density, thus a decrease in density will result in volume expansion [45]. Also, negative pressure may correspond to represent a dark energy model. In all the cases, energy density and fluid pressure are listed. In some cases, these terms are related as $p = -\rho$. It can thus be stated that the universe represented by these spacetime metrics behaves like dark energy model or can be vacuum energy. In other cases, the fluid pressure and energy density can remain positive for particular choices of the constants involved. Thus, in these cases the rate of expansion can slow down due to positive attractive effect. In Cases (I.1), (I.2.c), (I.3), (II.1) and (IV), both pressure and density are related as $p = -\rho$ for some particular values of the constants involved. In Case (III.1), the pressure and density are related as $p = -\frac{3}{2}\rho$. Cooperstock *et al.* [46] revealed that a singularity may exist for negative pressure. For the ratio $\frac{p}{\rho} > -\frac{1}{3}$, the singularity will be covered and it will be naked if the ratio $\frac{p}{\rho} \leq -\frac{1}{3}$. In Case (III.1), the ratio $\frac{p}{\rho} < -\frac{1}{3}$, therefore the obtained model may represent a naked singularity. In other cases, the pressure of the fluid and its density can remain positive for particular choices of the constants involved. Thus, in these cases the rate of expansion can slow down due to positive attractive effect.

In Sec. 4, we calculated the cosmological constraints, i.e. the Hubble's parameter (H), scalar expansion (θ), shear scalar (σ^2), anisotropy parameter (A_m) and the deceleration parameter (q) associated with each of the obtained exact LRS Bianchi type-I solutions. These constraints are important for the prediction of the future of the obtained universe models. Hubble's parameter is used to predict the rate of expansion of the universe. Also, the deceleration parameter (q) describes the physical behavior of the model. It provides the information on whether the universe is expanding with an accelerated rate or whether its rate is decelerating. The value $q = 0$ predicts whether the model is in uniform motion or at rest. Similarly, the anisotropic parameter describes the shape of the model, i.e. whether the model has some anisotropy or whether it is isotropic everywhere. Each obtained model depicted particular behavior. All the models are discussed in detail in this section. We can observe that concircular symmetry substantially affected the geometric and physical behaviors of the spacetime. It can be easily observed how the number of concircular vector fields affects the cosmological parameters. For example, in Case (I.1), the spacetime metric admits 15-dimensional CCVFs and we see that the anisotropic parameter A_m and deceleration parameter q are both constant. Also, when $t \to \infty$, then H, θ and σ^2 vanish. The constant values of A_m and q indicate that this model has a fixed measure of anisotropy and is decelerating at constant speed. In Case (I.2), when the LRS Bianchi type-I spacetime admits seven-dimensional CCVFs, then for $t \to \infty$ the constraints σ^2 and A_m become zero while the other quantities become constant. The zero value of A_m indicates that this model is isotropic. Thus, a constant anisotropic behavior for 15-dimensional CCVFs is now changed to isotropic nature for seven-dimensional CCVFs. Also, the value of $q = 2$ is now changed to $q = -1$ which shows that the model is growing exponentially. Furthermore, the value of θ does not vanish now. Thus, concircular symmetry of spacetime is crucial in understanding the physical and geometric properties of our spacetime.

Conflict of Interest

None of the authors have a conflict of interest to disclose.

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