Symmetries and Conservation Laws of Certain Classes of Nonlinear Schrödinger Partial Differential Equations

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A dissertation submitted to the Faculty of Science, University of the Witwatersrand, in fulfilment of the requirements for the Master of Science.

Johannesburg, 2012.

DECLARATION

I declare that the contents of this thesis are original except where due references have been made. It is being submitted for the degree of Master of Science at the University of the Witwatersrand in Johannesburg. It has not been submitted before for any degree to any other institution.

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This 24 day of April 2012.

Acknowledgements

I gratefully acknowledge the continued support and advice of my supervisor, Professor A.H. Kara. I really appreciate the time he has taken to share his guidance and technical assistance.

This work has been made possible in part by the University of the Witwatersrand as well as the National Research Foundation (NRF) of South Africa (grant number FA - 2007041200006).

I would also like to say a special thanks to my family and friends for their moral support, advice and encouragement while compiling my thesis.

Abstract

A number of versions of the Nonlinear Schrödinger Equation (NLSE) are used to model a range of physical phenomena. In particular, the physics of the NLSE with damping and driving terms have been discussed in a number of papers. We present the invariance, exact solutions, conservation laws and double reductions of the underlying equation. Furthermore, we obtain solutions to a specific case of the density dependent diffusion Nagumo equation and Fisher equation using the method of double reduction.

Contents

Introduction

A number of versions of the nonlinear Schrödinger equation (NLSE) are used to model a range of physical phenomena. In particular, the physics of the NLSE with damping and driving terms are discussed in a number of papers such as [16, 17, 69] and references therein. In this dissertation, we study the invariance, exact solutions, conservation laws and double reductions of the NLSE, with damping and driving terms, by decomposing the respective equation into real and imaginary parts.

When a Lie point or Lie-Bäcklund symmetry generator is applied to a conserved vector, one of two things may happen. We may find that the conservation law is associated with the symmetry or we may find another conservation law which could either be trivial, known or new. We will illustrate that when the generated conserved vector is the null vector, i.e. the symmetry is associated with the conserved vector, a double reduction is possible for partial differential equations (PDE) with two independent variables. In the double reductions that follow, the PDE of order q is reduced to an ordinary differential equation (ODE) of order $q-1$. Therefore, the use of one symmetry which is associated with a conservation law leads to two reductions, the first being a reduction of the number of independent variables and the second being a reduction of the order of the differential equation. The paper [69] goes into great detail about this metho of ouble reduction.

Next, we consider the solutions of the density dependent Nagumo equation. Travelling wave solutions have been discussed in the following papers [65, 66]. However, for the purpose of this dissertation we shall investigate whether the conservation laws and their corresponding multipliers can be used to construct equivalent systems to the original ones. We shall apply the generalized double reduction theorem to specific cases of the density dependent Nagumo equation via associated symmetries, to obtain exact solutions. Similarly the same procedure will be applied to the Fisher equation.

Chapter 1

Preliminaries

1.1 Introduction

In this chapter, we present the following definitions and theorems of the fundamental concepts that will be required to conduct our research throughout this dissertation.

1.2 Fundamental Concepts

A function $f(x, u, u_{(1)}, \ldots, u_{(k)})$ of a finite number of variables is called a differential function of order k, where $u_{(1)}, u_{(2)}, \ldots, u_{(k)}$ denotes the collections of all first, second,..., k^{th} order partial derivatives, that is, $u_i^\alpha = D_i(u^\alpha), u_{ij}^\alpha = D_j D_i(u^\alpha)$,... respectively, with the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \; i = 1, \ldots, n \; , \tag{1.1}$$

in which the summation convention is used whenever appropriate.

Consider a k^{th} order system of partial differential equations (pde's) of n independent variables $x = (x^1, x^2, \dots, x^n)$ and m dependent variables $u = (u^1, u^2, \dots, u^m)$

$$G^{\mu}(x, u, u_{(1)}, \dots, u_{(k)}) = 0, \quad \mu = 1, \dots, \tilde{m}. \tag{1.2}$$

The Lie-Bäcklund or generalised operator is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^{\alpha} \frac{\partial}{\partial u^{\alpha}}, \xi^i, \ \eta^{\alpha} \in \mathcal{A}, \tag{1.3}$$

where $\mathcal{A}$ is the vector space of differential functions. The operator (1.3) is an abbreviated form of the infinite formal sum

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^{\alpha} \frac{\partial}{\partial u^{\alpha}} + \sum_{s \geq 1} \zeta^{\alpha}_{i_1 i_2 \dots i_s} \frac{\partial}{\partial u^{\alpha}_{i_1 i_2 \dots i_s}}, \tag{1.4}$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{aligned} \zeta^{\alpha}_i &= D_i(W^{\alpha}) + \xi^j u^{\alpha}_{ij} \\ \zeta^{\alpha}_{i_1 \dots i_s} &= D_{i_1} \dots D_{i_s}(W^{\alpha}) + \xi^j u^{\alpha}_{j i_1 \dots i_s}, \ s > 1. \end{aligned} \tag{1.5}$$

In (1.5), W^{α} is the Lie characteristic function

$$W^{\alpha} = \eta^{\alpha} - \xi^j u^{\alpha}_j. \tag{1.6}$$

One of the more important aspects of nonlinear pde's are the conservation laws. Without these conserved quantities (integrals of motion), an understanding of the problem would be incomplete [34].

A current $T = (T^1, \dots, T^n)$ is conserved if it satisfies

$$D_i T^i = 0 \tag{1.7}$$

along the solutions of (1.2).

It can be shown that every admitted conservation law arises from multipliers $Q_\mu(x, u, u_{(1)}, \ldots)$ such that

$$Q_\mu G^\mu = D_i T^i \tag{1.8}$$

holds identically (i.e., off the solution space) everywhere, not just on solutions for some current T^i.

When the pde system is variational, multipliers are variational symmetries. There is a determining system for finding multipliers (and hence conservation laws) for any given pde system. We resort to the invariance and multiplier approach based on the well known result that the Euler-Lagrange operator annihilates a total divergence.

The conserved flow is then determined directly or by a homotopy formula (see [40, 46]).

Definition 1.1 [44] A Lie-Bäcklund symmetry generator X of the form (1.4) is associated with a conserved vector T of the system (1.2) if X and T satisfy the relations

$$X(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i) = 0,\ i = 1, \ldots, n. \tag{1.9}$$

Definition 1.2 The Euler-Lagrange operator, for each α, is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 \cdots i_s}}, \quad \alpha = 1, \ldots, m. \tag{1.10}$$

Theorem 1.3 [43, 45] Suppose that X is any Lie-Bäcklund symmetry of (1.2) and $T^i, i = 1, \ldots, n$ are the components of the conserved vector of (1.2). Then

$$T^{*i} = [T^i, X] = X(T^i) + T^i D_j \xi^j - T^j D_j \xi^i, \; i = 1, \ldots, n, \tag{1.11}$$

constitute the components of a conserved vector of (1.2) , i.e.,
$D_i T^{*i} \mid_{(1.2)} = 0$.

Theorem 1.4 [21] Suppose that $D_i T^i = 0$ is a conservation law of the pde system (1.2). Then under a contact transformation, there exists functions $\tilde{T}^i$ such that $J \; D_i T^i = \tilde{D}_i \tilde{T}^i$, where $\tilde{T}^i$ is given as

$$\begin{pmatrix} \tilde{T}^1 \\ \tilde{T}^2 \\ \vdots \\ \tilde{T}^n \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix}, \quad J \begin{pmatrix} T^1 \\ T^2 \\ \vdots \\ T^n \end{pmatrix} = A^T \begin{pmatrix} \tilde{T}^1 \\ \tilde{T}^2 \\ \vdots \\ \tilde{T}^n \end{pmatrix} \tag{1.12}$$

in which

$$A = \begin{pmatrix} \tilde{D}_1 x_1 & \tilde{D}_1 x_2 & \cdots & \tilde{D}_1 x_n \\ \tilde{D}_2 x_1 & \tilde{D}_2 x_2 & \cdots & \tilde{D}_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_n x_1 & \tilde{D}_n x_2 & \cdots & \tilde{D}_n x_n \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} D_1 \tilde{x}_1 & D_1 \tilde{x}_2 & \cdots & D_1 \tilde{x}_n \\ D_2 \tilde{x}_1 & D_2 \tilde{x}_2 & \cdots & D_2 \tilde{x}_n \\ \vdots & \vdots & \vdots & \vdots \\ D_n \tilde{x}_1 & D_n \tilde{x}_2 & \cdots & D_n \tilde{x}_n \end{pmatrix} \tag{1.13}$$

and $J = \det(A)$.

Theorem 1.5 [21] **(fundamental theorem on double reduction)** Suppose that $D_i T^i = 0$ is a conservation law of the pde system (1.2). Then under a similarity transformation of a symmetry X of the form (1.4) for the pde, there exist functions

$\tilde{T}^i$ such that X is still a symmetry for the pde $\tilde{D}_i\tilde{T}^i = 0$ and

$$\begin{pmatrix} X\tilde{T}^1 \\ X\tilde{T}^2 \\ \vdots \\ X\tilde{T}^n \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} [T^1, X] \\ [T^2, X] \\ \vdots \\ [T^n, X] \end{pmatrix}, \tag{1.14}$$

where

$$A = \begin{pmatrix} \tilde{D}_1 x_1 & \tilde{D}_1 x_2 & \cdots & \tilde{D}_1 x_n \\ \tilde{D}_2 x_1 & \tilde{D}_2 x_2 & \cdots & \tilde{D}_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{D}_n x_1 & \tilde{D}_n x_2 & \cdots & \tilde{D}_n x_n \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} D_1 \tilde{x}_1 & D_1 \tilde{x}_2 & \cdots & D_1 \tilde{x}_n \\ D_2 \tilde{x}_1 & D_2 \tilde{x}_2 & \cdots & D_2 \tilde{x}_n \\ \vdots & \vdots & \vdots & \vdots \\ D_n \tilde{x}_1 & D_n \tilde{x}_2 & \cdots & D_n \tilde{x}_n \end{pmatrix} \tag{1.15}$$

and $J = \det(A)$.

Chapter 2

Conservation Laws for the nonlinear Schrödinger Partial Differential Equation

2.1 Introduction and background

In theoretical physics, the nonlinear Schrödinger equation is a nonlinear version of the Schrödinger equation. It is a classical field equation with applications to optics and water waves. Unlike the Schrödinger equation, it never describes the time evolution of a quantum state. It is an example of an integrable model. In quantum mechanics, it is a special case of the nonlinear Schrödinger field, and when canonically quantized, it describes bosonic point particles with delta-function interactions (the particles either attract or repel when they are at the same point). The nonlinear Schrödinger equation is integrable when the particles move in one dimension of space. In the limit of infinite strength repulsion, the nonlinear Schrödinger equation bosons are equivalent to one dimensional free fermions.

2.2 Conserved Vectors

We first consider an illustrative example via a version of the nonlinear Schrödinger equation

$$iq_t + aq_{xx} + b|q|^2 q = 0, \tag{2.1}$$

where the dependent variable q represents the excitations corresponding to the dipole intra-peptide vibrations while the independent variables x and t are equivalent to space and time. The second term represents the dispersion term that arises from the effective mass of the exciton while the third term is the non-linear term. If we write $q = u + iv$, (2.1) becomes

$$i(u + iv)_t + a(u + iv)_{xx} + b(u^2 + v^2)(u + iv) = 0. \tag{2.2}$$

Separating (2.2) into real and imaginary parts leads to the following second-order system of pdes

$$\begin{aligned} G^1 &= -v_t + au_{xx} + b(u^2 + v^2)u = 0, \\ G^2 &= u_t + av_{xx} + b(u^2 + v^2)v = 0, \end{aligned} \tag{2.3}$$

where G^1 and G^2 are functions satisfying (2.1).

The multiplier (f, g) will satisfy the 'joint' Euler operator

$$\frac{\delta}{\delta(u, v)}(fG^1 + gG^2) = 0. \tag{2.4}$$

Equation (2.4) is equivalent to the action of the Euler operator on each dependent variable, given by

$$\frac{\delta}{\delta u}(fG^1 + gG^2) = 0,$$
$$\frac{\delta}{\delta v}(fG^1 + gG^2) = 0. \tag{2.5}$$

Thus by (2.5)a, we have

$$fG^1 + gG^2 = D_x S + D_t T.$$

It turns out that, inter alia, we get the following pairs of multipliers

$$\begin{aligned}(f_1, g_1) &= (v_x, u_x),\\ (f_2, g_2) &= (u, -v).\end{aligned} \tag{2.6}$$

We begin with the case (f_1, g_1).

Our determining equation for (T, S) in (2.5)a :

$$\begin{aligned} & v_x(u_t + av_{xx} + b(u^2+v^2)v) + u_x(-v_t + au_{xx} + b(u^2+v^2)u) \\ = \;& T_t + u_t T_u + u_{tt} T_{u_t} + u_{tx} T_{u_x} + v_t T_v + v_{tt} T_{v_t} + v_{tx} T_{v_x} + S_x + u_x S_u \\ & + u_{xx} S_{u_x} + u_{xt} S_{u_t} + v_x S_v + v_{xx} S_{v_x} + v_{xt} S_{v_t}. \end{aligned} \tag{2.7}$$

The governing equations are obtained from (2.7) by the separation of the monomials,

$$
\begin{aligned}
&av_x = S_{v_x},\\
&au_x = S_{u_x},\\
&T_{u_t} = 0,\\
&T_{u_x} + S_{u_t} = 0,\\
&T_{v_t} = 0\\
&T_{v_x} + S_{v_t} = 0,\\
&v_x u_t + bv_x v(u^2+v^2) - u_x v_t + bu_x u(u^2+v^2) =\\
&T_t + u_t T_u + v_t T_v + S_x + u_x S_u + v_x S_v.
\end{aligned}
\tag{2.8}
$$

From (2.8)c and (2.8)e, we have T is independent of u_t and v_t.

From (2.8)a, we get

$$S = \frac{a}{2}v_x^2 + A(x,t,u,v,u_t,u_x,v_t),$$

and from (2.8)b,

$$au_x = A_{u_x},$$

so that

$$A = \frac{a}{2}u_x^2 + B(x,t,u,v,u_t,v_t).$$

Thus, we have

$$S = \frac{a}{2}v_x^2 + \frac{a}{2}u_x^2 + B(x,t,u,v,u_t,v_t).$$

From (2.8)d, we obtain

$$T_{u_x} + B_{u_t} = 0,$$

so that

$$T = -u_x B_{u_t} + C(x, t, u, v, v_x).$$

From (2.8)f,we obtain

$$C_{v_x} + B_{v_t} = 0,$$

Thus we get

$$C = -v_x B_{v_t} + D(x, t, u, v).$$

Now from (2.8)g i.e.

$$v_x u_t + b v_x v(u^2 + v^2) - u_x v_t + b u_x u(u^2 + v^2) = T_t + u_t T_u + v_t T_v + S_x + u_x S_u + v_x S_v.$$

Substituting our summarised versions of T and S above we have:

$$\begin{aligned} & v_x u_t + b v_x v(u^2 + v^2) - u_x v_t + b u_x u(u^2 + v^2) \\ = \quad & -u_x B_{t,u_t} - v_x B_{t,v_t} + D_t + u_t(-u_x B_{u,u_t} - v_x B_{u,v_t} + D_u) \\ & + v_t(-u_x B_{v,u_t} - v_x B_{v,v_t} + D_v) + B_x + u_x B_u + v_x B_v, \end{aligned}$$

The governing equations are obtained from the equation above by the separation of the monomials,

$$\begin{aligned} & u_t + bv(u^2 + v^2) = -B_{t,v_t} - u_t B_{u,v_t} - v_t B_{v,v_t} + B_v, \\ & -v_t + bu(u^2 + v^2) = -B_{t,u_t} - u_t B_{u,u_t} - v_t B_{v,u_t} + B_u, \\ & D_t + u_t D_u + v_t D_v + B_x = 0. \end{aligned} \tag{2.9}$$

From (2.9)c we see that we can separate by u_t and v_t since D is not dependent on either.

$$\begin{aligned} & D_u = 0 \\ & D_v = 0 \\ & D_t + B_x = 0. \end{aligned} \tag{2.10}$$

So D is independent of u and v. Therefore

$$T = -u_x B_{u_t} - v_x B_{v_t} + D(x,t).$$

From (2.10)c, differentiate partially with respect to u_t

$$D_{t,u_t} = -B_{x,u_t},$$

we get

$$-B_{x,u_t} = 0$$

.

Also differentiate partially with respect to v_t

$$D_{t,v_t} = -B_{x,v_t},$$

we get,

$$0 = -B_{x,v_t}.$$

So we can substitute these two results into (2.9)a and (2.9)b.
From (2.9)a we get

$$B = vu_t + \frac{b}{2}u^2v^2 + \frac{b}{4}v^4 + E(x,t,u,u_t,v_t),$$

and (2.9)b we get,

$$B = -uv_t + \frac{b}{4}u^4 + \frac{b}{2}u^2v^2 + F(x,t,v,u_t,v_t).$$

So using the two forms of B we have the following:

$$\begin{aligned} vu_t + uv_t - \frac{b}{4}(u^4 - v^4) &= F(x,t,v,u_t,v_t) - E(x,t,u,u_t,v_t). \\ \text{Then, } B &= 2vu_t + uv_t + \frac{b}{4}(v^4 - u^4) + \frac{b}{2}u^2v^2 + \frac{b}{4}v^4, \end{aligned}$$

so that

$$\begin{aligned} T &= -u_x B_{u_t} - v_x B_{v_t} \\ &= -u_x v + v_x u \\ &= \overline{q} q_x - q \overline{q_x} \end{aligned}$$

and

$$\begin{aligned} S &= \frac{a}{2} v_x^2 + \frac{a}{2} u_x^2 + 2vu_t - 2uv_t + \frac{b}{4}(v^4 + u^4) + \frac{b}{2} u^2 v^2 + \frac{b}{4} v^4 \\ &= -\frac{1}{4} b q \overline{q} |q|^2 + \frac{1}{2} a |q_x|^2 + 2(q \overline{q_t} - \overline{q} q_t). \end{aligned}$$

Our next case is when

$$(f_2, g_2) = (u, -v).$$

The determining equation becomes:

$$\begin{gathered} u(u_t + av_{xx} + b(u^2 + v^2)v) - v(-v_t + au_{xx} + b(u^2 + v^2)u) = T_t + u_t T_u + u_{tt} T_{u_t} + u_{tx} T_{u_x} \\ + v_t T_v + v_{tt} T_{v_t} + v_{tx} T_{v_x} + S_x + u_x S_u + u_{xx} S_{u_x} + u_{xt} S_{u_t} + v_x S_v + v_{xx} S_{v_x} + v_{xt} S_{v_t}. \end{gathered} \tag{2.11}$$

The governing equations are obtained from (2.11) by the separation of the monomials,

$$\begin{aligned} & au = S_{v_x}, \\ & -av = S_{u_x}, \\ & T_{u_t} = 0, \\ & T_{ux} + S_{ut} = 0, \\ & T_{v_t} = 0, \\ & T_{v_x} + S_{v_t} = 0, \\ & uu_t + vv_t = T_t + u_t T_u + v_t T_v + S_x + u_x S_u + v_x S_v. \end{aligned} \tag{2.12}$$

From (2.12)a

$$S = v_x ua + A(x,t,u,v,u_x,u_t,v_t)$$

From (2.12)b, we get

$$A = -avu_x + B(x,t,u,v,u_t,v_t).$$

Thus we have

$$S = v_x ua - avu_x + B(x,t,u,v,u_t,v_t).$$

From (2.12)d, we obtain

$$T = -u_x B_{ut} + C(x,t,u,v,v_x).$$

From (2.12)f, we have

$$C_{vx} + B_{vt} = 0,$$

therefore

$$C = -v_x B_{vt} + D(x,t,u,v).$$

Thus we have

$$T = -u_x B_{ut} - B_{vt} + D(x,t,u,v).$$

From (2.12)g, i.e.

$$uu_t + vv_t = T_t + u_t T_u + v_t T_v + S_x + u_x S_u + v_x S_v$$

The governing equations are obtained by the separation of the monomials,

$$\begin{aligned} u &= T_u, \\ v &= T_v, \\ T_t &+ S_x + u_x S_u + v_x S_v = 0. \end{aligned} \tag{2.13}$$

From (2.13)a, we have

$$u = D_u,$$

thus

$$D = \frac{1}{2}u^2 + E(x,t,v),$$

and from (2.13)b

$$v = E_v$$

we get

$$E = \frac{1}{2}v^2 + F(x,t).$$

Thus we have

$$T = -u_x B_{ut} - B_{vt} + \frac{1}{2}u^2 + \frac{1}{2}v^2 + F(x,t).$$

From(2.13)c,

$$F_t + B_x + au_x v_x + u_x B_u - au_x v_x + v_x B_v = 0.$$

The governing equations are obtained by the separation of the monomials,

$$\begin{aligned} B_u &= 0, \\ B_v &= 0, \\ F_t + B_x &= 0. \end{aligned} \tag{2.14}$$

From (2.14)a and (2.14)b we see that B is independent of u and v. Furthermore, when we compute $D_tT + D_xS$, by (1.4) we have that $B = F = 0$. Therefore,

$$\begin{aligned} T &= \frac{1}{2}(u^2 + v^2), \\ &= \frac{1}{2}|q|, \end{aligned}$$

and

$$\begin{aligned} S &= -avu_x + auv_x, \\ &= -ia(\overline{q}q_x - q\overline{q_x}). \end{aligned}$$

2.3 Discussion and Conclusion

We have obtained new conservation laws for the nonlinear Schrödinger equation, by using the multiplier approach. Corresponding to each multiplier there is a unique conserved flow. In the next chapter, we study the damped-driven nonlinear Schrödinger equation.

Chapter 3

Conservation Laws, Double Reductions and Exact Solutions of the nonlinear damped-driven Schrödinger Equation

3.1 Introduction and Background

This chapter investigates the dynamics of Davydov solitons (self reinforcing wave that maintains its shape while it travels at a constant speed) in α helix proteins that is governed by the nonlinear Schrödinger equation (NLSE). The α helical structure is found in those protein molecules where the transduction of energy from one end of a molecule to the other occurs, [1-10]. For example, bacteriorhodospin molecule incorporated into the membranes of halo-bacteria that live in salty lakes and basins, intersects the cell membrane seven times in the form of α helix segments. Besides motion, helix segments transfer energy and information from one cite in a cell to another by means of propogation of soliton exactations, [6]. Solitons arise as the solutions of a widespread class of weakly nonlinear partial differential equations. The nonlinear dynamics of solitons can provide a key for the understanding of the mechanism of transduction of information from the exterior to the interior of a cell. We proceed to calculate exact solutions of the damped-driven Schrödinger equation using double reduction.

3.2 The Double Reduction of the Driven-damped Schrödinger System of Equations via Conservation Laws

We start with the equation given below, it is given by the form

$$iq_t + q_{xx} + 2|q|^2 q - q = h\bar{q} - i\gamma q, \tag{3.1}$$

where $\gamma \geq 0$ is the damping coefficient, h is the amplitude of the parametric driver (which can be assumed to be positive) and q is complex valued.

We also assume q to be of the form $q = u + iv$ so that separating (3.1) into real and imaginary parts results in the system of pdes

$$\begin{aligned} u_t + v_{xx} + 2(u^2+v^2)v + (h-1)v + \gamma u &= 0, \\ -v_t + u_{xx} + 2(u^2+v^2)u + (h-1)u - \gamma v &= 0, \end{aligned} \tag{3.2}$$

which is the version we will consider for our analysis.

3.2.1 Case 1: $\gamma \neq 0, h \neq 0$

The equation (3.2) admits the following four Lie point symmetries

$$
\begin{aligned}
X_1 &= -v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}, \\
X_2 &= \frac{\partial}{\partial t} - (h-1)v\frac{\partial}{\partial u} + (h-1)u\frac{\partial}{\partial v}, \\
X_3 &= \frac{\partial}{\partial x}, \\
X_4 &= 2t\frac{\partial}{\partial x} - xv\frac{\partial}{\partial u} + xu\frac{\partial}{\partial v}.
\end{aligned} \tag{3.3}
$$

Equation (3.2) admits the following conservation law

$$
\begin{aligned}
T_1 = & \left[\frac{1}{2}e^{2\gamma t}(uv_x - vu_x), \frac{1}{2}e^{2\gamma t}((u^2+v^2)^2 + (h-1)(u^2+v^2)\right. \\
& \left. + vu_t - uv_t + u_x^2 + v_x^2)\right]
\end{aligned} \tag{3.4}
$$

with the corresponding multiplier

$$
Q_1 = [e^{2\gamma t}v_x, e^{2\gamma t}u_x]. \tag{3.5}
$$

Equation (3.2) is equivalent to

$$
sys_1 = \begin{cases} {q_1}^1 G^1 + {q_1}^2 G^2 = 0, \\ {q_1}^1 G^1 - {q_1}^2 G^2 = 0. \end{cases} \tag{3.6}
$$

(3.6) system can be rewritten as

$$
\begin{aligned}
D_t {T_1}^t + D_x {T_1}^x &= 0, \\
{q_1}^1 G^1 - {q_1}^2 G^2 &= 0.
\end{aligned} \tag{3.7}
$$

We now show that X_1 and X_3 are associated with $T_1 = (T_1^t, T_1^x)$.

We obtain

$$\begin{aligned}
\begin{pmatrix} T_1^{*t} \\ T_1^{*x} \end{pmatrix} &= X_1^{[1]}\begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} + (0)\begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}e^{2\gamma t}(-vv_x - uu_x + vv_x + uu_x) \\ \frac{1}{2}e^{2\gamma t}\Big(- 4uv(u^2+v^2) - 2(h-1)uv + vv_t + 4uv(u^2+v^2) \\ \quad + 2(h-1)uv + uu_t - vv_t - 2u_xv_x - uu_t + 2u_xv_x\Big) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\begin{pmatrix} T_1^{*t} \\ T_1^{*x} \end{pmatrix} &= X_3^{[1]}\begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} + (0)\begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial}{\partial x}\Big[\frac{1}{2}e^{2\gamma t}(uv_x - vu_x)\Big] \\ \frac{\partial}{\partial x}\Big[\frac{1}{2}e^{2\gamma t}((u^2+v^2)^2 + (h-1)(u^2+v^2) + vu_t \\ \quad - uv_t + u_x^2 + v_x^2)\Big] \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{aligned}$$

where

$$X_1^{[1]} = -v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v} - v_t\frac{\partial}{\partial u_t} - v_x\frac{\partial}{\partial u_x} + u_t\frac{\partial}{\partial v_t} + u_x\frac{\partial}{\partial v_x}, \quad X_3^{[1]} = \frac{\partial}{\partial x}.$$

Thus, X_1 and X_3 are both associated with T_1.

A Double Reduction of the Damped-driven Schrödinger Equation by $< X_1, X_3 >$

We can get a reduced conserved form for the first equation of the first system, sys_1 from (3.6) since X_1 and X_3 are both associated symmetries of T_1.

We consider a linear combination of X_1 and X_3, i.e., of the form $X = X_3 + cX_1$, and transform this generator to its canonical form $Y = \frac{\partial}{\partial s}$, where we assume that this generator is of the form $Y = 0\frac{\partial}{\partial r} + \frac{\partial}{\partial s} + 0\frac{\partial}{\partial w} + 0\frac{\partial}{\partial p}$ (as in subsection 3.2.1).

From $X(r) = 0$, $X(s) = 1$, $X(w) = 0$ and $X(p) = 0$, we obtain

$$\frac{dt}{0} = \frac{dx}{1} = \frac{du}{-cv} = \frac{dv}{cu} = \frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0} = \frac{dp}{0}. \tag{3.8}$$

We solve (3.8) using the method of invariance, where the results are summarized in the following table

$\frac{dt}{0}$	$b_1 = t$
$\frac{du}{-cv} = \frac{dv}{cu}$	$b_2 = u^2 + v^2$
$\frac{dx}{1} = \frac{dv}{cu}$	$b_3 = \arctan\left(\frac{v}{u}\right) - cx$
$\frac{dr}{0}$	$b_4 = r$
$\frac{dx}{1} = \frac{ds}{1}$	$b_5 = s - x$
$\frac{dw}{0}$	$b_6 = w$
$\frac{dp}{0}$	$b_7 = p$

b_4, b_5, b_6 and b_7 are arbitrary functions all dependent on b_1, b_2 and b_3.

By choosing $b_4 = b_1$, $b_5 = 0$, $b_6 = \sqrt{b_2}$ and $b_7 = b_3$, we obtain the canonical coordinates

$$r = t, \quad s = x, \quad w = \sqrt{u^2 + v^2}, \quad p = \arctan\left(\frac{v}{u}\right) - cs, \tag{3.9}$$

where $w = w(r)$ and $p = p(r)$ since $Y = \frac{\partial}{\partial s}$.

From (3.9), the inverse canonical coordinates are given by

$$t = r, \quad x = s, \quad u = w\cos(p + cs), \quad v = w\sin(p + cs). \tag{3.10}$$

From (1.13), we compute A and $(A^{-1})^T$

$$A = \begin{pmatrix} D_r t & D_r x \\ D_s t & D_s x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$A^{-1} = \begin{pmatrix} D_t r & D_t s \\ D_x r & D_x s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (A^{-1})^T,$$

where $J = \det(A) = 1$.

We compute the first and second-order partial derivatives of u and v from (3.10), viz.,

$$\begin{aligned}
u_t &= w_r \cos(p + cs) - wp_r \sin(p + cs), \\
u_x &= -cw \sin(p + cs), \\
v_t &= w_r \sin(p + cs) + wp_r \cos(p + cs), \\
v_x &= cw \cos(p + cs), \\
u_{xx} &= -c^2 w \cos(p + cs), \\
v_{xx} &= -c^2 w \sin(p + cs).
\end{aligned} \tag{3.11}$$

We now apply the formula from (1.14) with $i = 1, 2$ to obtain the reduced conserved form

$$\begin{pmatrix} T_1^r \\ T_1^s \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix}. \tag{3.12}$$

By substituting (3.10) and (3.11) into (3.12), we obtain

$$\begin{pmatrix} T_1^r \\ T_1^s \end{pmatrix} = \begin{pmatrix} \frac{1}{2} c e^{2\gamma r} w^2 \\ \frac{1}{2} e^{2\gamma r} \left[w^4 + (h-1) w^2 - p_r w^2 + c^2 w^2 \right] \end{pmatrix}, \tag{3.13}$$

where the reduced conserved form is also given by

$$D_r T_1^r = 0. \tag{3.14}$$

The second step of double reduction can be given as

$$e^{2\gamma r} w^2 = k \tag{3.15}$$

or equivalently

$$w^2 = k e^{-2\gamma r}, \tag{3.16}$$

where k is a constant.

Differentiating (3.15) implicitly with respect to r and then taking out a common factor of $2e^{2\gamma r} w$ results in

$$\gamma w + w_r = 0. \tag{3.17}$$

The second equation of sys_1 (3.6) is given by

$$\begin{aligned} &e^{2\gamma t} v_x \Big[u_t + v_{xx} + 2(u^2+v^2)v + (h-1)v + \gamma u\Big] \\ &\quad - e^{2\gamma t} u_x \Big[-v_t + u_{xx} + 2(u^2+v^2)u + (h-1)u - \gamma v\Big] = 0. \end{aligned} \tag{3.18}$$

After transforming (3.18) using (3.10) and (3.11), we obtain

$$\begin{aligned} &-2ce^{2\gamma r} w^2 \Big[p_r + c^2 - 2w^2 - (h-1)\Big] \cos(p+cs) \sin(p+cs) \\ &\quad + ce^{2\gamma r} w(\gamma w + w_r) \cos 2(p+cs) = 0. \end{aligned} \tag{3.19}$$

We now substitute (3.15) and (3.17) into (3.19).

After dividing both sides by $-2ck$, this results in the ode

$$p_r = 2w^2 + h - 1 - c^2. \tag{3.20}$$

If we substitute (3.16) into (3.20) and then integrate both sides with respect to r, we obtain

$$p(r) = \frac{-k}{\gamma} e^{-2\gamma r} + (h - 1 - c^2)r + m, \tag{3.21}$$

where m is an integration constant.

Using (3.11), we obtain the final solution to our original system as

$$\begin{aligned} u(t,x) &= \sqrt{k} e^{-\gamma t} \cos\left(\frac{-k}{\gamma} e^{-2\gamma t} + (h - 1 - c^2)t + m + cx\right) \\ v(t,x) &= \sqrt{k} e^{-\gamma t} \sin\left(\frac{-k}{\gamma} e^{-2\gamma t} + (h - 1 - c^2)t + m + cx\right). \end{aligned} \tag{3.22}$$

3.2.2 Case 2: $\gamma \neq 0, h = 0$

In this case, the equation (3.2) also admits four Lie point symmetries

$$\begin{aligned} X_1 &= -v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}, \\ X_2 &= \frac{\partial}{\partial t} + v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}, \\ X_3 &= \frac{\partial}{\partial x}, \\ X_4 &= 2t\frac{\partial}{\partial x} - xv\frac{\partial}{\partial u} + xu\frac{\partial}{\partial v}. \end{aligned} \tag{3.23}$$

Equation (3.2) admits the following two conserved vectors

$$\begin{aligned} T_1 &= \left[-\frac{1}{2}e^{2\gamma t}(x(u^2+v^2) + 2t(vu_x - uv_x)),\right. \\ &\quad e^{2\gamma t}(tu^4 - tv^2 + tv^4 + tu^2(-1+2v^2)) + v(tu_t + xu_x) \\ &\quad \left. - u(tv_t + xv_x) + t(u_x^2 + v_x^2)\right], \\ T_2 &= \left[\frac{1}{2}e^{2\gamma t}(u^2+v^2), e^{2\gamma t}(-vu_x + uv_x)\right] \end{aligned} \tag{3.24}$$

with the corresponding multipliers

$$\begin{aligned} Q_1 &= \left[-xue^{2\gamma t}v_x + 2te^{2\gamma t}v_x, xve^{2\gamma t} + 2te^{2\gamma t}u_x\right], \\ Q_2 &= \left[e^{2\gamma t}u, -e^{2\gamma t}v\right]. \end{aligned} \tag{3.25}$$

Our original system (3.2) is equivalent to

$$sys_i = \begin{cases} {q_i}^1 G^1 + {q_i}^2 G^2 = 0, \\ {q_i}^1 G^1 - {q_i}^2 G^2 = 0, \end{cases} \tag{3.26}$$

for $i = 1, 2$.

This system (3.26) can be rewritten as

$$\begin{aligned} D_t {T_i}^t + D_x {T_i}^x &= 0, \\ {q_i}^1 G^1 - {q_i}^2 G^2 &= 0. \end{aligned} \tag{3.27}$$

We now show that X_4 is associated with $T_2 = (T_2^t, T_2^x)$.

We obtain

$$\begin{aligned} \begin{pmatrix} T_2^{*t} \\ T_2^{*x} \end{pmatrix} &= X_4^{[1]} \begin{pmatrix} T_2^t \\ T_2^x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} T_2^t \\ T_2^x \end{pmatrix} + (0) \begin{pmatrix} T_2^t \\ T_2^x \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} e^{2\gamma t}(-2xuv + 2xuv) \\ e^{2\gamma t}(- xvv_x - xuu_x + xvv_x + v^2 + xuu_x + u^2 - u^2 - v^2) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} X_4^{[1]} &= 2t\frac{\partial}{\partial x} - xv\frac{\partial}{\partial u} + xu\frac{\partial}{\partial v} - (xv_t + 2u_x)\frac{\partial}{\partial u_t} - (xv_x + v)\frac{\partial}{\partial u_x} \\ &\quad + (xu_t - 2v_x)\frac{\partial}{\partial v_t} + (xu_x + u)\frac{\partial}{\partial v_x}. \end{aligned}$$

Thus, X_4 is associated with T_2.

A Double Reduction of the Damped-driven Schrödinger Equation by $< X_4 >$

We can get a reduced conserved form for the first equation of the first system, sys_1 from (3.26) since X_4 is an associated symmetry of T_2.

We again transform the generator X_4 to its canonical form $Y = \frac{\partial}{\partial s}$, where we assume that this generator is also of the form $Y = 0\frac{\partial}{\partial r} + \frac{\partial}{\partial s} + 0\frac{\partial}{\partial w} + 0\frac{\partial}{\partial p}$.

From $X_4(r) = 0$, $X_4(s) = 1$, $X_4(w) = 0$ and $X_4(p) = 0$, we obtain

$$\frac{dt}{0} = \frac{dx}{2t} = \frac{du}{-xv} = \frac{dv}{xu} = \frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0} = \frac{dp}{0}. \tag{3.28}$$

We solve (3.28) using the method of invariance, where the results are summarized in the following table

$\frac{dt}{0}$	$b_1 = t$
$\frac{du}{-xv} = \frac{dv}{xu}$	$b_2 = u^2 + v^2$
$\frac{dx}{2t} = \frac{dv}{xu}$	$b_3 = \arctan\left(\frac{v}{u}\right) - \frac{x^2}{4t}$
$\frac{dr}{0}$	$b_4 = r$
$\frac{dx}{2t} = \frac{ds}{1}$	$b_5 = s - \frac{x}{2t}$
$\frac{dw}{0}$	$b_6 = w$
$\frac{dp}{0}$	$b_7 = p$

b_4, b_5, b_6 and b_7 are arbitrary functions all dependent on b_1, b_2 and b_3.

By choosing $b_4 = b_1$, $b_5 = 0$, $b_6 = \sqrt{b_2}$ and $b_7 = b_3$, we obtain the canonical coordinates

$$r = t, \quad s = \frac{x}{2t}, \quad w = \sqrt{u^2 + v^2}, \quad p = \arctan\left(\frac{v}{u}\right) - \frac{x^2}{4t}, \tag{3.29}$$

where $w = w(r)$ and $p = p(r)$ since $Y = \frac{\partial}{\partial s}$.

From (3.29), the inverse canonical coordinates are given by

$$t = r, \quad x = 2rs, \quad u = w\cos(p + rs^2), \quad v = w\sin(p + rs^2). \tag{3.30}$$

From (1.13), we compute A and $(A^{-1})^T$

$$A = \begin{pmatrix} D_r t & D_r x \\ D_s t & D_s x \end{pmatrix} = \begin{pmatrix} 1 & 2s \\ 0 & 2r \end{pmatrix}$$

and

$$(A^{-1})^T = \begin{pmatrix} D_t r & D_x r \\ D_t s & D_x s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\dfrac{s}{r} & \dfrac{1}{2r} \end{pmatrix}$$

where $J = \det(A) = 2r$.

We compute the first and second-order partial derivatives of u and v from (3.30)

$$\begin{aligned}
u_t &= w_r \cos(p + rs^2) - wp_r \sin(p + rs^2) \\
&\quad + s^2 w \sin(p + rs^2) \\
u_x &= -swr \sin(p + rs^2) \\
v_t &= w_r \sin(p + rs^2) + wp_r \cos(p + rs^2) \\
&\quad - s^2 w \cos(p + rs^2) \\
v_x &= sw \cos(p + rs^2) \\
u_{xx} &= -\frac{w \sin(p + rs^2)}{2r} - s^2 w \cos(p + rs^2) \\
v_{xx} &= \frac{w \cos(p + rs^2)}{2r} - s^2 w \sin(p + rs^2).
\end{aligned} \tag{3.31}$$

We now apply the formula (1.14) for $i = 1, 2$ to obtain the reduced conserved form

$$\begin{pmatrix} T_2^r \\ T_2^s \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} T_2^t \\ T_2^x \end{pmatrix}. \tag{3.32}$$

By substituting (3.30) and (3.31) into (3.32), we obtain

$$\begin{pmatrix} T_2^r \\ T_2^s \end{pmatrix} = \begin{pmatrix} re^{2\gamma r} w^2 \\ 0 \end{pmatrix}, \tag{3.33}$$

where the reduced conserved form is also given by

$$D_r T_2^r = 0. \tag{3.34}$$

The second step of double reduction can be given as

$$re^{2\gamma r}w^2 = k \tag{3.35}$$

or equivalently

$$w^2 = \frac{ke^{-2\gamma r}}{r}, \tag{3.36}$$

where k is a constant.

Differentiating (3.35) implicitly with respect to r and then taking out a common factor of $e^{2\gamma r}w$ results in

$$2rw_r + (2\gamma r + 1)w = 0 \tag{3.37}$$

or equivalently after dividing both sides by $2r$

$$w_r + \frac{w}{2r} + \gamma w = 0. \tag{3.38}$$

The second equation of sys_2 from (3.27) is given by

$$e^{2\gamma t}\Big[uu_t - vv_t + uv_{xx} + vu_{xx} + 2uv(2(u^2+v^2) - 1) + \gamma(u^2 - v^2)\Big] = 0. \tag{3.39}$$

After transforming (3.39) using (3.30) and (3.31), we obtain

$$\begin{aligned} &e^{2\gamma r}(-2wp_r + 4w^3 - 2w)\cos(p + rs^2)\sin(p + rs^2) \\ &\quad + e^{2\gamma r}\left(w_r + \frac{w}{2r} + \gamma w\right)\cos 2(p + rs^2) = 0. \end{aligned} \tag{3.40}$$

After multiplying both sides of (3.40) by $-\frac{w}{2}$ and then substituting (3.36) and (3.38), this results in the ode

$$p_r = 2\frac{ke^{-2\gamma r}}{r} - 1. \tag{3.41}$$

If we substitute (3.36) into (3.41) and then integrate both sides with respect to r

$$p(r) = 2k \int \frac{e^{-2\gamma r}}{r} dr - r, \tag{3.42}$$

where $\int \frac{e^{-2\gamma r}}{r} dr = \ln r + \sum_{j=1}^{\infty} \frac{(-1)^j (2\gamma r)^j}{jj!} + m$. Where m is the integration constant. Using (3.30), we obtain the final solution to our original system as

$$\begin{aligned} u(t,x) &= \sqrt{\frac{k}{t}} e^{-\gamma t} \cos\left(p(r) + \frac{x^2}{4t}\right) \\ v(t,x) &= \sqrt{\frac{k}{t}} e^{-\gamma t} \sin\left(p(r) + \frac{x^2}{4t}\right), \end{aligned} \tag{3.43}$$

where $p(r) = 2k\Big(\ln t + \sum_{j=1}^{\infty} \frac{(-1)^j (2\gamma t)^j}{jj!} + m\Big) - t$.

When $h \neq 0$, the system (3.2) admits $\frac{\partial}{\partial t}$ which is built into the symmetry X_2. However, when $h = 1$ the symmetry X_2 is strictly $\frac{\partial}{\partial t}$. Considering X_1 and X_2 we could replace X_2 by $\bar{X}_2 = \frac{\partial}{\partial t}$

3.3 Discussion and Conclusion

We showed that conservation laws and their corresponding multipliers can be used to construct equivalent systems to the original ones. By applying the generalized double reduction theorem to these systems via associated symmetries, one can construct exact solutions. This procedure was applied to some second-order Schrödinger systems of equations such as the parametrically driven-damped Schrödinger equation.

Chapter 4

The Double Reduction of The Nagumo and Fisher Equation

4.1 Introduction and Background

The motivation for the FitzHugh-Nagumo model was to isolate, conceptually, the essentially mathematical properties of excitation and propagation from the electrochemical properties of sodium and potassium ion flow. The equation is a simplification of the Hodgkin-Huxley model devised in 1952. The Hodgkin-Huxley has four variables and FitzHugh-Nagumo equation is a reduction of that model. For details, the reader is referred to [67, 68]. The Nagumo equation has various applications in the fields of logistic population growth, flame propogation, neurophysiology, autocatalytic chemical reaction, branching Brownian motion process and nuclear reactor theory. We investigate exact solutions to a specific case of the density dependent Nagumo equation [65, 66].

Another closely related equation is the well known Fisher equation also known as Fisher-Kolmogorov equation derived for the stimulation of propagation of a gene in a population. It is perhaps the simplest model problem for reaction-diffusion equations. As above we consider a more general density dependent Fisher equation.

The symmetry, conservation laws and double reduction methods used in obtaining the solutions may be found in, inter alia, the references [21, 43, 44, 45]

4.2 The density dependent Nagumo equation

In this section, we analyse the Nagumo equation

$$u_t = D_x(u^p u_x) + u(1-u)(u-\alpha), \tag{4.1}$$

where α is the density and $u(1-u)(u-\alpha)$ represents the reaction term [65, 66] - $p = 0$ represents the standard Nagumo equation. In general, equation (4.1) is only invariant under translations in time and space leading to traveling wave solutions and has no conservation laws. The algebra of symmetries, however, is extended and a nontrivial conservation law exists for the special case $p = 2, \alpha = -1$ leading to an additional exact solution which may be determined by direct double reduction or variational methods on the reduced ordinary differential equation (ode).

Direct double reduction

For $p = 2, \alpha = -1$, (4.1) becomes

$$u_t = 2uu_x^2 + u^2 u_{xx} - u^3 + u. \tag{4.2}$$

The equation admits a 3-dimensional Lie algebra of Lie point symmetries spanned by

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = \frac{-1}{2} e^{-2t} \frac{\partial}{\partial t} - \frac{1}{2} e^{-2t} u \frac{\partial}{\partial u} \tag{4.3}$$

and the conserved vectors

$$\begin{aligned} T_1 &= \left(e^{-\sqrt{3}x - t} u,\, e^{-\sqrt{3}x - t} \left(-u_x u^2 - \frac{\sqrt{3}}{3} u^3 \right) \right), \\ T_2 &= \left(e^{\sqrt{3}x - t} u,\, e^{\sqrt{3}x - t} \left(-u_x u^2 - \frac{\sqrt{3}}{3} u^3 \right) \right) \end{aligned} \tag{4.4}$$

with associated multipliers

$$Q_1 = e^{-\sqrt{3}x - t}, \qquad Q_2 = e^{\sqrt{3}x - t}. \tag{4.5}$$

X_3 is associated with $T_1 = (T_1^t, T_1^x)$ since

$$\begin{aligned} \begin{pmatrix} T_1^{*t} \\ T_1^{*x} \end{pmatrix} &= X_3^{[1]} \begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} - \begin{pmatrix} e^{-2t} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} + (e^{-2t}) \begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

where

$$X_3^{[1]} = \frac{-1}{2} e^{-2t} \frac{\partial}{\partial t} - \frac{1}{2} e^{-2t} u \frac{\partial}{\partial u} - \frac{1}{2} e^{-2t} u_x \frac{\partial}{\partial u_x}.$$

Thus X_3 is associated with T_1.

We can get a reduced conserved form for the equation (4.2) since X_3 is associated with T_1.

We transform the generator X_3 to its canonical form $Y = \frac{\partial}{\partial s}$, where we assume that this generator is of the form $Y = 0\frac{\partial}{\partial r} + \frac{\partial}{\partial s} + 0\frac{\partial}{\partial w}$. From $X(r) = 0$, $X(s) = 1$ and $X(w) = 0$, we obtain

$$\frac{-2dt}{e^{-2t}} = \frac{dx}{0} = \frac{-2du}{ue^{-2t}} = \frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0}. \tag{4.6}$$

We solve (4.6) using the method of invariance, where the results are summarized in the following table

$\dfrac{-2dt}{e^{-2t}} = \dfrac{-2du}{ue^{-2t}}$	$b_1 = ue^{-t}$
$\dfrac{dx}{0}$	$b_2 = x$
$\dfrac{dr}{0}$	$b_3 = r$
$\dfrac{-2dt}{e^{-2t}} = \dfrac{ds}{1}$	$b_4 = s + e^{2t}$
$\dfrac{dw}{0}$	$b_5 = w$

b_3, b_4, and b_5 are arbitrary functions all dependent on b_1 and b_2. By choosing $b_3 = b_2$, $b_4 = 0$, and $b_5 = b_1$, we obtain the canonical coordinates

$$r = x, \quad s = -e^{2t}, \quad w = ue^{-t}, \tag{4.7}$$

where $w = w(r)$ and since $Y = \frac{\partial}{\partial s}$. We compute A and $(A^{-1})^T$

$$A = \begin{pmatrix} D_r t & D_r x \\ D_s t & D_s x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{2s} & 0 \end{pmatrix}$$

and

$$A^{-1} = \begin{pmatrix} D_t r & D_t s \\ D_x r & D_x s \end{pmatrix} = \begin{pmatrix} 0 & 2s \\ 1 & 0 \end{pmatrix}.$$

So

$$(A^{-1})^T = \begin{pmatrix} 0 & 1 \\ 2s & 0, \end{pmatrix}$$

where $J = \det(A) = \frac{-1}{2s}$. From (4.7), the inverse canonical coordinates are given by

$$t = \frac{\ln(-s)}{2}, \quad u = w(-s)^{\frac{1}{2}}, \tag{4.8}$$

The first and second-order partial derivatives of u are

$$u_t = w(-s)^{\frac{1}{2}}, \quad u_x = w_r(-s)^{\frac{1}{2}}, \quad u_{xx} = w_{rr}(-s)^{\frac{1}{2}}. \tag{4.9}$$

We now apply the formula from with $i = 1, 2$ to obtain the reduced conserved form

$$\begin{pmatrix} T_1^r \\ T_1^s \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix}. \tag{4.10}$$

By substituting (4.8) and (4.9) into (4.10), we obtain

$$\begin{pmatrix} T_1^r \\ T_1^s \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \tag{4.11}$$

where

$$U_1 = \frac{-1}{2} e^{-\sqrt{3}r} w^2 \left(w_r + \frac{\sqrt{3}}{3} w \right)$$

and

$$U_2 = -T_1^t,$$

where the reduced conserved form is also given by

$$D_r T_1^r = 0. \tag{4.12}$$

The second step in the double reduction can be given as

$$k = \frac{-1}{2} e^{-\sqrt{3}r} w^2 \left(w_r + \frac{\sqrt{3}}{3} w \right), \tag{4.13}$$

so that

$$w = \frac{e^{-\sqrt{3}r}}{2} \left(\left(e^{2\sqrt{3}r} 4k\sqrt{3} + 8p \right) e^{2\sqrt{3}r} \right)^{\frac{1}{3}}, \tag{4.14}$$

where p is a constant of integration. Substituting (4.14) into (4.8), we obtain the exact solution

$$u = \frac{e^{-\sqrt{3}x+t}}{2} \left(\left(e^{2\sqrt{3}x} 4k\sqrt{3} + 8p \right) e^{2\sqrt{3}x} \right)^{\frac{1}{3}}.$$

4.3 The density dependent Fisher equation

A generalized version of the density dependent Fisher equation is given by

$$u_t = D_x(u^s u_x) + u(1-u). \tag{4.15}$$

It turns out that as in the case of the Nagumo equation, conservation laws do not exist in general. However, conservation laws and additional corresponding symmetries do exist for the nontrivial case $s = 1$, i.e.,

$$u_t = u_x^2 + uu_{xx} - u^2 + u. \tag{4.16}$$

The equation admits a 3-dimensional Lie algebra of Lie point symmetries spanned by

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = e^{-t}\frac{\partial}{\partial t} - e^{-t}u\frac{\partial}{\partial u} \tag{4.17}$$

and the conserved vectors

$$\begin{aligned} T_1 &= \left(e^{-\sqrt{2}x-t}u, e^{-\sqrt{2}x-t}\left(-u_x u - \frac{\sqrt{2}}{2}u^2\right)\right), \\ T_2 &= \left(e^{\sqrt{2}x-t}u, e^{\sqrt{2}x-t}\left(-u_x u - \frac{\sqrt{2}}{2}u^2\right)\right) \end{aligned} \tag{4.18}$$

with associated multipliers

$$Q_1 = e^{-\sqrt{2}x-t}, \quad Q_2 = e^{\sqrt{2}x-t}. \tag{4.19}$$

The association of X_3 with $T_1 = (T_1^t, T_1^x)$ is a consequence of

$$\begin{aligned} \begin{pmatrix} T_1^{*t} \\ T_1^{*x} \end{pmatrix} &= X_3^{[1]}\begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix}, - \begin{pmatrix} e^{-t} & 0 \\ 0 & 0, \end{pmatrix}\begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} + (e^{-t})\begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

where

$$X_3^{[1]} = e^{-t}\frac{\partial}{\partial t} - e^{-t}u\frac{\partial}{\partial u} - e^{-t}u_x\frac{\partial}{\partial u_x}.$$

Thus X_3 is associated with T_1.

We can get a reduced conserved form for the equation (4.16) since X_3 is associated with T_1.

We transform the generator X_3 to its canonical form $Y = \frac{\partial}{\partial s}$, where we assume that this generator is of the form $Y = 0\frac{\partial}{\partial r} + \frac{\partial}{\partial s} + 0\frac{\partial}{\partial w}$. From $X(r) = 0$, $X(s) = 1$ and $X(w) = 0$, we obtain

$$\frac{-dt}{e^{-t}} = \frac{dx}{0} = \frac{-du}{ue^{-t}} = \frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0}. \tag{4.20}$$

We solve (4.20) using the method of invariance, where the results are summarized in the following table

$\frac{-dt}{e^{-t}} = \frac{-du}{ue^{-t}}$	$b_1 = ue^{-t}$
$\frac{dx}{0}$	$b_2 = x$
$\frac{dr}{0}$	$b_3 = r$
$\frac{-dt}{e^{-t}} = \frac{ds}{1}$	$b_4 = s + e^t$
$\frac{dw}{0}$	$b_5 = w$

b_3, b_4, and b_5 are arbitrary functions all dependent on b_1 and b_2. By choosing $b_3 = b_2$, $b_4 = 0$, and $b_5 = b_1$, we obtain the canonical coordinates

$$r = x, \quad s = -e^t, \quad w = ue^{-t}, \tag{4.21}$$

where $w = w(r)$ and since $Y = \frac{\partial}{\partial s}$. We compute A and $(A^{-1})^T$

$$A = \begin{pmatrix} D_r t & D_r x \\ D_s t & D_s x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{s} & 0 \end{pmatrix}$$

and

$$A^{-1} = \begin{pmatrix} D_t r & D_t s \\ D_x r & D_x s \end{pmatrix} = \begin{pmatrix} 0 & s \\ 1 & 0. \end{pmatrix}$$

So

$$(A^{-1})^T = \begin{pmatrix} 0 & 1 \\ s & 0, \end{pmatrix}$$

where $J = \det(A) = \frac{-1}{s}$. From (4.21), the inverse canonical coordinates are given by

$$t = \frac{\ln(-s)}{2}, \quad u = -ws, \tag{4.22}$$

Thus,

$$u_t = -ws, \quad u_x = -w_r s, \quad u_{xx} = -w_{rr}. \tag{4.23}$$

We now apply the formula from with $i = 1, 2$ to obtain the reduced conserved form

$$\begin{pmatrix} T_1^r \\ T_1^s \end{pmatrix} = J(A^{-1})^T \begin{pmatrix} T_1^t \\ T_1^x \end{pmatrix}. \tag{4.24}$$

By substituting (4.22) and (4.23) into (4.24), we obtain

$$\begin{pmatrix} T_1^r \\ T_1^s \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \tag{4.25}$$

where

$$U_1 = e^{\sqrt{2}r} w \left(-w_r + \frac{\sqrt{2}}{2} w \right)$$

and

$$U_2 = -T_1^t,$$

where the reduced conserved form is also given by

$$D_r T_1^r = 0. \tag{4.26}$$

Further reduction leads to

$$k = e^{-\sqrt{2}r} w \left(-w_r + \frac{\sqrt{2}}{2} w \right) \tag{4.27}$$

and, therefore,

$$w = \frac{-1}{2} \left(2e^{-\sqrt{2}r} k\sqrt{2} + 4ze^{\sqrt{2}r} \right)^{\frac{1}{2}}, \quad w = \frac{1}{2} \left(2e^{-\sqrt{2}r} k\sqrt{2} + 4ze^{\sqrt{2}r} \right)^{\frac{1}{2}} \tag{4.28}$$

where z is a constant of intergration. Substituting (4.28) into (4.22),we obtain

$$u = \frac{-e^t}{2} \left(2e^{-\sqrt{2}x} k\sqrt{2} + 4ze^{\sqrt{2}x} \right)^{\frac{1}{2}}, \quad u = \frac{e^t}{2} \left(2e^{-\sqrt{2}x} k\sqrt{2} + 4ze^{\sqrt{2}x} \right)^{\frac{1}{2}}.$$

4.4 Discussion and Conclusion

We showed that conservation laws and their corresponding multipliers can be used to construct equivalent systems to the original ones. By applying the generalized double reduction theorem to these systems via associated symmetries, one can construct exact solutions. This procedure was applied to specific cases of the density dependent Nagumo and Fisher equations for which we obtained new solutions; the well known, previous solutions of the density dependent Nagumo are traveling wave type ones.

Note. Alternatively, the equations (4.2) and (4.16) may be reduced to ordinary differential equations which, in turn, may be doubly reduced using variational methods and Noether's theorem.

Chapter 5

Conclusion

We have shown that by utilizing the interplay between symmetries and conservation laws, one can obtain a large class of exact solutions to the damped-driven NLSE, Nagumo and Fisher equations.This method would take into account solutions like the traveling wave ones, inter alia, if required.

In paticular, we considered PDE's with two independent variables of order q which admit a Lie point symmetry with generator X and have a conservation law with conserved vector T. We have shown that an ODE of order $q-1$ can be constructed for the solutions of the PDE invariant under X. That is, solutions of the PDE invariant under X also solve this ODE. Furthermore, if X is associated with T by the relation then the invariant solutions of the PDE satisfy an ODE of order at most $q-1$. Thus we have affected a double reduction of the PDE.

The generalisation of this theory to PDE's of higher dimensions is still an open problem and is intimately connected with the definition of non-local variables. The definition of these variables are known in theory, but it is not clear how to overcome the problem that not all derivatives of the non-local variables sre known explicitly. Calculations for higher dimensions become quite involved. One only has to consider the lack of examples of potential symmetries and symmetries with associated conservation laws for higher dimensional PDE's to demonstrate the complexity of this problem.

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