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Inertial Halpern-type Tseng's method for approximating a solution to monotone inclusion problems with fixed point constraint

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Abstract
In this paper, we introduce an iterative method for approximating the common solution of a finite family of monotone inclusion problems and fixed point problem of a finite family of relatively nonexpansive mappings in reflexive Banach spaces. Our proposed method contains the inertial technique and the Halpern method. We obtain a strong convergence result without the prior knowledge of the Lipschitz constants of the mappings. Finally, we present some numerical experiments to demonstrate the applicability of our method. Several existing results in the literature are improved, extended, and generalized by our result.

1 Introduction

Let $\mathcal{X}$ be a real Banach space and $\mathcal{X}^*$ its dual. The fixed point problem is the problem of finding a point $x \in \mathcal{X}$ such that

$$x = Tx, \tag{1}$$

where $T : \mathcal{X} \to \mathcal{X}$ is a nonlinear mapping. We denote the set of fixed points of the nonlinear mapping T by $F(T)$. Several iterative methods have been proposed for solving the fixed point problem. In 1967, Halpern [19] introduced the following iterative method for approximating

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the fixed points of a nonexpansive mapping $T : \mathcal{C} \to \mathcal{C}$ and defined it as follow:

$$x_{k+1} = \alpha_k x_0 + (1 - \alpha_k) T x_k, \quad k \geq 1, \tag{2}$$

where $\mathcal{C}$ is a closed convex subset of a real Hilbert space $\mathcal{H}$, $\{\alpha_k\} \subset (0, 1)$ and $x_0 \in \mathcal{C}$. Halpern obtained a strong convergence result. Several researchers have studied the convergence analysis of Halpern iterative process with various additional conditions for approximating optimization and fixed point problems (see [18, 22, 50, 58] and other references therein).

The monotone inclusion problem (MIP) is the problem of finding a point $x \in \mathcal{X}$ such that

$$0 \in (N + M)x, \tag{3}$$

where $N : \mathcal{X} \to \mathcal{X}^*$ is a monotone operator and $M : \mathcal{X} \to 2^{\mathcal{X}^*}$ is a maximal monotone operator. Throughout this work, we will denote the set of solution of (3) by $(N + M)^{-1}0$. The MIP has attracted the interest of several researchers since it lies at the heart of many mathematical problems, including convex programming, variational inequality problem, split feasibility problem, minimization problem, among others (see [15–17, 24, 31, 35, 36, 40–44] and other references therein). The MIP has wide applications in several fields such as signal processing, machine learning, computer vision, image recovery, statistical regression, among others (see [15, 16, 25, 35] and other references therein).

Several authors have studied and proposed iterative algorithms for solving (3). A simple and effective method for solving (3) was introduced and studied in a Hilbert space $\mathcal{H}$ by Lions and Merciers [23]. They called the proposed method the forward-backward splitting method and defined it as follows:

$$x_{k+1} = J^M_{\lambda_k}(x_k - \lambda_k N x_k), \ \ \forall\, k \geq 1, \tag{4}$$

where $J^M_{\lambda_k} := (I + \lambda_k M)^{-1}$ is the resolvent of M, $\{\lambda_k\}$ is a positive real sequence and I is the identity mapping on $\mathcal{H}$. In this setting, N is the forward operator and M is the backward operator. The authors obtained a weak convergence result of the proposed method under the assumption that the operator N is β-inverse strongly monotone with $\beta > 0$. Precisely, the authors require the operator N to satisfy the condition

$$\langle Nx - Ny, x - y\rangle \geq \beta \|Nx - Ny\|^2, \ \ \forall\, x, y \in \mathcal{H}.$$

The forward-backward splitting method have been studied by several authors (see [35, 47, 52, 53] and other references therein). The limitation to this forward-backward splitting method is the strict condition placed on the operator N. To weaken this assumption, Tseng [54] proposed a Tseng's splitting algorithm for solving (3) and defined it as follows:

$$\begin{cases} x_1 \in \mathcal{H}, \\ y_k = J^M_{\lambda_k}(x_k - \lambda_k N x_k), \\ x_{k+1} = y_k - \lambda_k (N y_k - N x_k), \quad \forall\, k \geq 1, \end{cases} \tag{5}$$

where $N : \mathcal{H} \to \mathcal{H}$ is monotone and L-Lipschitz continuous, and $\{\lambda_k\} \in \left(0, \frac{1}{L}\right)$. Tseng also obtained a weak convergence result. The limitation of this method lies in the fact that the method requires the prior knowledge of the Lipschitz constant which can be very difficult and sometimes impossible to estimate. Several authors have studied and modified the Tseng's splitting method (see [53, 56, 57] and other references therein).

In 2019, Shehu [47] extended the result of Tseng [54] from Hilbert spaces to Banach spaces. Shehu [47] studied and introduced an iterative method for approximating a solution of (3) in a 2-uniformly convex Banach space $\mathcal{X}$ which is also uniformly smooth. Shehu [47] defined the proposed method as follows:

$$\begin{cases} x_1 \in \mathcal{X}, \\ y_k = J_{\lambda_k}^M J^{-1}(Jx_k - \lambda_k N x_k), \\ x_{k+1} = Jy_k - \lambda_k(Ny_k - Nx_k), \ \forall\, k \geq 1, \end{cases} \tag{6}$$

where $N : \mathcal{X} \to \mathcal{X}^*$ is monotone and L-Lipschitz continuous, $J : \mathcal{X} \to \mathcal{X}^*$ is the duality mapping,

$$J_{\lambda_k}^M := (J + \lambda_k M)^{-1} J \tag{7}$$

is the resolvent of M and $\lambda_k \in \left(0, \frac{1}{\sqrt{2\rho r L}}\right)$. Under certain conditions, the author obtained a weak convergence result. Furthermore, the author modified Algorithm 6 by replacing stepsize with a linesearch for solving (3).

The main aim of this work is to find a common solution to a finite family of MIP and FPP of a finite family of relatively nonexpansive mappings. That is the problem of finding a point x in a Banach space $\mathcal{X}$ such that

$$x \in \Gamma = \cap_{i=1}^m F(T_i) \bigcap \cap_{i=1}^m (N_i + M_i)^{-1} 0, \tag{8}$$

where for each $i = 1, 2, \ldots, m$, $M_i : \mathcal{X} \to 2^{\mathcal{X}^*}$ are maximal monotone operators, $N_i : \mathcal{X} \to \mathcal{X}$ be monotone and L_i-Lipschitz continuous operators, and $T_i : \mathcal{X} \to \mathcal{X}$ are relatively nonexpansive mappings. Due to the applications of this common solution problems to mathematical models whose constraints can be stated as FPP and MIP, many methods have been proposed and investigated for approximating the common solution to FPP and MIP. Common solution problem, in particular, have applications in network resource allocation, image recovery, and signal processing, machine learning, computer vision, data compression, among other areas.

Manaka and Takahashi [26] studied and introduced an iterative method for approximating a common solution to the FPP and MIP in a real Hilbert space $\mathcal{H}$ and defined it as follows:

$$\begin{cases} x_1 \in \mathcal{H}, \\ x_{k+1} = \alpha_k x_k + (1 - \alpha_k) T J_{\lambda_k}^M (x_k - \lambda_k N x_k), \quad \forall\ k \geq 1, \end{cases} \tag{9}$$

where $N : \mathcal{H} \to \mathcal{H}$ is α-inverse strongly monotone, $T : \mathcal{H} \to \mathcal{H}$ is a nonspreading mapping, $\{\alpha_k\} \in (0, 1)$ and $\{\lambda_k\} \in (0, 2\alpha)$. The authors proved that the sequence $\{x_k\}$ generated by the proposed algorithm (9) under certain conditions converges weakly to a point in $F(T) \cap (N + M)^{-1}0$. Very recently, Sunthrayuth et al. [48] proposed algorithms for approximating a common solution to the FPP (for a relatively nonexpansive mapping) and MIP (for a monotone operator) in a reflexive Banach space. The authors under certain conditions obtained weak convergence results. It is well known that the strong convergence is more desirable than the weak convergence.

The inertial method is one of the effective techniques often employed by authors to speed up the convergence of iterative methods when solving optimization problems. This method was introduced by Polyak [34] and it is based on a discrete form of a second order dissipative dynamical system in time. The inertial approach has the advantage of increasing the speed of convergence of iterative techniques. There has recently been a surge in interest in researching inertial type algorithms for solving optimization problems (see [4, 5, 11, 21, 27, 32, 38, 39, 51] and other references therein).

Motivated by the above mentioned works in literature, we propose an inertial iterative method in a reflexive Banach space for approximating the common solution to a finite family of monotone inclusion problems and fixed point problems of a finite family of relatively nonexpansive mappings. Our proposed method combines the inertial technique, a Halpern method and a self-adaptive step-size. We establish a strong convergence result. Furthermore, we present some numerical experiments to demonstrate the applicability of our method.

The remainder of the paper is organized as follows: In Sect. 2, we review some basic definitions and results that are essential for our convergence study. Section 3 introduces and examines the features of our suggested technique. Section 4 investigates the method's convergence. In Sect. 5, we apply our result to approximate the common solution of a finite family of variational inequality problems (VIP) and FPP of a finite family of relatively nonexpansive mappings in Banach spaces. We conduct numerical experiments to demonstrate the applicability of our method in Sect. 6 and we conclude in Sect. 7.

2 Preliminaries

In this section, we provide some basic definitions and results that are essential in the subsequent sections. Throughout this work, we denote the real Banach space and its dual by $\mathcal{X}$ and $\mathcal{X}^*$ respectively, and the duality pairing between $\mathcal{X}$ and $\mathcal{X}^*$ by $\langle \cdot, \cdot \rangle$. We denote the strong and weak convergence by $\rightarrow$ and $\rightharpoonup$ respectively.

If $M : \mathcal{X} \to 2^{\mathcal{X}^*}$ is a set valued operator, the domain, range and graph of the operator is defined respectively as follows: $\text{dom M} = \{x \in \mathcal{X} : \ Mx \neq \emptyset\}, \quad \text{ran M} = \bigcup\{Mx : x \in \text{dom M}\}$ and $G(M) = \{(x, y) \in \mathcal{X} \times \mathcal{X}^* : y \in Mx\}$.

Definition 2.1 The function $g : \text{dom } g \subseteq \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ is said to be

(a) *Proper*, if dom $g \neq \emptyset$;

(b) *Lower semicontinuous at a point* $x \in D(g)$, if

$$g(x) \leq \liminf_{x_k \to x} g(x_k);$$

(c) *Convex*, if

$$g(\lambda x + (1 - \lambda)y) \leq \lambda g(x) + (1 - \lambda)g(y), \quad \forall \lambda \in (0, 1), \ x, y \in D(g).$$

Definition 2.2 Let $\mathcal{C}$ be a nonempty subset of $\mathcal{X}$. Then, a mapping $M : \mathcal{C} \to \mathcal{X}^*$ is said to be

(i) *Monotone*, if

$$\langle x - y, Mx - My \rangle \geq 0, \ \ \forall x, y \in \mathcal{C};$$

(ii) *L-Lipschitz continuous,* if there exists a constant $L > 0$ such that

$$\|Mx - My\| \leq L\|x - y\|, \ \ \forall x, y \in \mathcal{C};$$

(iii) *Hemicontinuous,* if for each $x, y \in \mathcal{C}$, the mapping $g : [0, 1] \to \mathcal{X}^*$ defined by $g(t) = M(tx + (1 - t)y)$ is continuous with respect to the weak topology of $\mathcal{X}^*$.

A monotone operator is said to be maximal if its graph is not contained in the graph of any other monotone operator.

Definition 2.3 The Fénchel conjugate function of g is the convex function $g^* : \mathcal{X} \to \mathbb{R}$ defined by

$$g^*(y) = \sup\{\langle y, x\rangle - g(x) : \ \forall\, x \in \mathcal{X}\}.$$

The function g on $\mathcal{X}$ is said to be cofinite if $\operatorname{dom} g^* = \mathcal{X}^*$ and g is said to be strongly coercive if

$$\lim_{\|x\|\to\infty} \frac{g(x)}{\|x\|} = \infty.$$

Definition 2.4 The function $g : \mathcal{C} \to (-\infty, \infty]$ is said to be Legendre if and only if the following hold:
(i) $\operatorname{int}(\operatorname{dom} g) \neq \emptyset$ and ∂g is single-valued on its domain;
(ii) $\operatorname{int}(\operatorname{dom}\ g^*) \neq \emptyset$ and ∂g^* is single-valued on its domain.

Let $g : \mathcal{X} \to (-\infty, \infty]$ be a function. Let $x \in \operatorname{int}(\operatorname{dom} g)$, for any $y \in \mathcal{X}$, the directional derivative of g at x is defined by

$$g^{'}(x, y) = \lim_{l\to 0^+} \frac{g(x + ly) - g(x)}{l}. \tag{10}$$

The function g is said to be Gâteaux differentiable at x if the limit as $l \to 0^+$ in (10) exists. In this scenario, the gradient of g at x is the linear function $\nabla g(x)$, defined by $\langle \nabla g(x), y\rangle = g^{'}(x, y),\ \forall\, y \in \mathcal{X}$. It is well known that if g is continuous at x and $\partial g(x)$ is single valued, then g is Gâteaux differentiable at x. The mapping ∇g is said to be weakly sequentially continuous if for any sequence $\{x_k\} \subset \mathcal{X},\ x_k \rightharpoonup x$ implies that $\nabla g(x_k) \overset{*}{\rightharpoonup} \nabla g(x)$.

The subdifferential set of g at a point x, denoted by $\partial g(x)$ is given by

$$\partial g(x) = \{z \in \mathcal{X} : g(y) - g(x) \geq \langle z, y - x\rangle,\ \ \forall\, y \in \mathcal{X}\}.$$

An element $z \in \partial g(x)$ is called the subgradient of g.

The function g is said to be Fréchet differentiable at x if the limit $l \to 0^+$ in (10) is attained uniformly in $\|y\| = 1$.
Let $g : \mathcal{X} \to (-\infty, \infty]$ be a Gâteaux differentiable function. The function $P_g : \operatorname{dom} g \times \operatorname{int}(\operatorname{dom} g) \to [0, \infty)$ defined by

$$P_g(x, y) = g(x) - g(y) - \langle x - y, \nabla g(y)\rangle \tag{11}$$

is called the Bregman distance with respect to g. It is well known that the Bregman distance is not a metric because it fails to be symmetric and does not satisfy the triangle inequality property. However, the Bregman distance posses the following properties (see [10]):

(i) Three-point identity given by

$$P_g(x, y) + P_g(y, z) - P_g(x, z) = \langle \nabla g(z) - \nabla g(y), x - y\rangle; \tag{12}$$

(ii) Four-point identity given by

$$P_g(x, y) - P_g(x, z) - P_g(w, y) + P_g(w, z) = \langle x - w, \nabla g(z) - \nabla g(y)\rangle. \tag{13}$$

A Gâteaux differentiable function g is said to be η-strongly convex if there exists a constant $\eta > 0$ such that

$$g(x) \geq g(y) + \langle x - y, \nabla g(y)\rangle + \frac{\eta}{2}\|x - y\|^2,\ \forall\, x \in \operatorname{dom} g,\ \ y \in\ \operatorname{int}(\operatorname{dom} g).$$

Let $\mathcal{C}$ be a nonempty, closed and convex subset of $\mathrm{int}(\mathrm{dom}\, g)$. If g is η-strongly convex function on $\mathcal{C}$, then

$$P_g(x, y) \geq \frac{\eta}{2}\|x - y\|^2, \quad \forall\, x, y \in \mathcal{C}. \tag{14}$$

It is well known that if $g : \mathcal{X} \to \mathbb{R}$ is Gâteaux differentiable, strictly convex and cofinite. Then M is maximal monotone if and only if $\mathrm{ran}(\nabla g + \lambda N) = \mathcal{X}^*, \quad \lambda > 0$, (see [9]).

Definition 2.5 Let $g : \mathcal{X} \to (-\infty, \infty]$ be a Fréchet differentiable function which is bounded on bounded subsets of $\mathcal{X}$ and M be a maximal monotone operator, then the resolvent of M for $\lambda > 0$ defined by

$$J_\lambda^M(x) = \left(\nabla g + \lambda M\right)^{-1} \nabla g(x), \quad \forall\, x \in \mathcal{X}$$

is a single valued Bregman quasi nonexpansive mapping from $\mathcal{X}$ onto $\mathrm{dom}(M)$ with $F(J_\lambda^M) = M^{-1}0$.

An example of a Legendre function is $g(x) := \frac{1}{\lambda}\|x\|^\lambda \quad (1 < \lambda < \infty)$ when $\mathcal{X}$ is a smooth and strictly convex Banach space. In this case, the gradient ∇g of g coincides with the generalized duality mappings of $\mathcal{X}$, that is, $\nabla g = J_\lambda \ (1 < \lambda < \infty)$. Particularly, $\nabla g = I$ is the identity mapping in a real Hilbert space.

Definition 2.6 The Bregman projection with respect to g of $x \in \mathrm{int}(\mathrm{dom}\, g)$ onto a nonempty, closed and convex set $\mathcal{C} \subset \mathrm{dom}\, g$ is the unique vector $\Pi_{\mathcal{C}}(x) \in \mathcal{C}$ satisfying

$$P_g(\Pi_{\mathcal{C}}(x), x) := \inf\{P_g(y, x) : y \in \mathcal{C}\}.$$

Lemma 2.7 *[12] Let $\mathcal{C}$ be a nonempty, closed and convex subset of $\mathcal{X}$. Let $g : \mathcal{X} \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and $x \in \mathcal{X}$. The following statements hold:*

(i). $z = \Pi_{\mathcal{C}}^g(x)$ if and only if $\langle y - z, \nabla g(x) - \nabla g(z)\rangle \leq 0, \quad \forall\ y \in \mathcal{C}$.
(ii). $P_g(y, \Pi_{\mathcal{C}}^g(x)) + P_g(\Pi_{\mathcal{C}}^g(x), x) \leq P_g(y, x), \quad \forall\, y \in \mathcal{C}$.

Let $g : \mathcal{X} \to \mathbb{R}$ be a Legendre function. We also require the functional $Q_g : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ defined by:

$$Q_g(x, y) = g(x) - \langle x, y\rangle + g^*(y), \quad \forall\, x \in \mathcal{X}, \quad \forall\, y \in \mathcal{X}^*. \tag{15}$$

Then, Q_g is nonnegative and

$$Q_g(x, y) = P_g(x, \nabla g^*(y)). \tag{16}$$

The function Q_g also satisfies the following inequality:

$$Q_g(x, y) \leq Q_g(x, y + z) - \langle z, \nabla g^*(y) - x\rangle. \tag{17}$$

Let $g : \mathcal{X} \to (-\infty, \infty]$ be a proper, convex and lower semicontinuous function. Then, $g^* : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ is a proper, convex and weakly lower semicontinuous function (see [33]). It is known that P_g is convex in the second variable, that is

$$P_g\Big(x, \nabla g^*\Big(\sum_{i=1}^{m} t_i \nabla g(\xi_i)\Big)\Big) \leq \sum_{i=1}^{m} t_i P_g(x, \xi_i), \ \ \forall \ x \in \mathcal{X}, \tag{18}$$

where $\{\xi_i\}_{i=1}^m \subset \mathcal{X}$ and $\{t_i\}_{i=1}^m \subset (0, 1)$ with $\sum_{i=1}^{m} t_i = 1$.

Definition 2.8 [30] Let $\mathcal{X}$ be a Banach space and let $B_r := \{z \in \mathcal{X} : \|z\| \leq r\}$ for all $r > 0$. Then a function $g : \mathcal{X} \to \mathbb{R}$ is said to be uniformly convex on bounded subsets of $\mathcal{X}$ if $\delta_r(t) > 0$ for all $r, t > 0$, where $\delta_r : [0, +\infty) \to [0, \infty]$ is defined by

$$\delta_r(t) = \inf_{x,y \in B_r, \|x-y\|=t, \alpha \in (0,1)} \frac{\alpha g(x) + (1-\alpha)g(y) - g(\alpha x + (1-\alpha)y)}{\alpha(1-\alpha)},$$

for all $t \geq 0$. The function δ_r is called the gauge of uniform convexity of g.

Proposition 2.9 *[29] The gauge function $\delta_r : [0, \infty) \to [0, \infty]$ is a continuous function.*

Lemma 2.10 *[30] Let $\mathcal{X}$ be a Banach space, let $r > 0$ be a constant and $g : \mathcal{C} \to \mathbb{R}$ be a uniformly convex function on bounded subsets of $\mathcal{X}$. Then*

$$g\left(\sum_{k=0}^{m} \lambda_k x_k\right) \leq \sum_{k=0}^{m} \lambda_k g(x_k) - \lambda_i \lambda_j \delta_r \left(\|x_i - x_j\|\right)$$

for all $i, j \in \{0, 1, 2, \ldots, m\}$, $x_k \in B_r$, $\lambda_k \in (0, 1)$ for $k = 0, 1, 2, \ldots, m$ with $\sum_{k=0}^{m} \lambda_k = 1$, where δ_r is the gauge of uniform convexity of g.

Definition 2.11 [13] The mapping $T : \mathcal{C} \to \mathrm{int}(\mathrm{dom}\, g)$ is said to be relatively nonexpansive if it satisfies the following conditions:

(i) $F(T) \neq \emptyset$;
(ii) $P_g(y, Tx) \leq P_g(y, x), \ \ \forall\, x \in F(T)$ and $y \in \mathcal{C}$;
(iii) $I - T$ is demi-closed at zero, that is, whenever a sequence $\{x_k\} \in \mathcal{C}$ such that $x_k \rightharpoonup x$ and $\lim_{n\to\infty} \|x_k - Tx_k\| = 0$, it follows that $x \in F(T)$.

Remark 2.12 [48] If T satisfies (i) and (ii), then T is called Bregman quasi-nonexpansive. From the definition above, it is known that if $\mathcal{X}$ is a Hilbert space and $g(x) = \frac{1}{2}\|x\|^2$, then T is quasi-nonexpansive and $I - T$ is demi-closed at zero.

Lemma 2.13 *[45] Let $g : \mathcal{X} \to (-\infty, \infty]$ be a Legendre function. Let $\mathcal{C}$ be a nonempty, closed and convex subset of* $\mathrm{int}(\mathrm{dom}\, g)$ *and $T : \mathcal{C} \to \mathcal{C}$ be a Bregman quasi-nonexpansive mapping. Then $F(T)$ is closed and convex.*

Lemma 2.14 *[37] Let $g : \mathcal{X} \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. Suppose that $x \in \mathcal{X}$, if $\{P_g(x, x_k)\}$ is bounded, then the sequence $\{x_k\}$ is bounded.*

Lemma 2.15 *[8] Let $N : \mathcal{X} \to \mathcal{X}^*$ be a monotone, hemicontinuous and bounded operator, and $M : \mathcal{X} \to 2^{\mathcal{X}^*}$ be a maximal monotone operator. Then $N + M$ is maximal monotone.*

Lemma 2.16 *[30] Let $\mathcal{X}$ be a Banach space and $g : \mathcal{X} \to \mathbb{R}$ be a Gâteaux differentiable function which is uniformly convex and bounded subsets of $\mathcal{X}$. Suppose that $\{x_k\}$ and $\{y_k\}$ are two sequences in $\mathcal{X}$. Then $\lim_{n\to\infty} P_g(x_k, y_k) = 0$ if and only if $\lim_{n\to\infty} \|x_k - y_k\| = 0$.*

Lemma 2.17 *[6] Let $\{a_k\}$, $\{b_k\}$ and $\{\alpha_k\}$ be sequences in $[0, \infty)$ such that*

$$a_{k+1} \leq a_k + \alpha_k(a_k - a_{k-1}) + b_k,$$

for all $k \geq 1$, $\sum_{n=1}^{\infty} b_k < \infty$, and there exists a real number α with $0 \leq \alpha_k \leq \alpha < 1$, for all $k \in \mathbb{N}$. Then, the following hold:

(i) $\sum_{n\geq 1}[a_k - a_{k-1}]_+ < \infty$, where $[t]_+ = \max\{t, 0\}$;
(ii) there exists $a^* \in [0, \infty)$ such that $\lim_{k\to\infty} a_k = a^*$.

Lemma 2.18 *[49] Let $\{a_k\}$ be a sequence of nonnegative real numbers, $\{\alpha_k\}$ be sequence of real numbers in $(0, 1)$ such that $\sum_{k=1}^{\infty} \alpha_k = \infty$ and $\{b_k\}$ be a sequence of real numbers. Assume that*

$$a_{k+1} \leq (1 - \alpha_k)a_k + \alpha_k b_k, \quad \forall \; k \geq 1.$$

If $\limsup_{j\to\infty} b_{k_j} \leq 0$ for every subsequence $\{a_{k_j}\}$ of $\{a_k\}$ satisfying the condition

$$\liminf_{j\to\infty}(a_{k_{j+1}} - a_{k_j}) \geq 0,$$

then $\lim_{k\to\infty} a_k = 0$.

3 Proposed method

In this section, we introduce a Halpern inertial algorithm for approximating the common solution to a finite family of MIP and FPP of a finite family of relatively nonexpansive mappings. We begin with the following assumptions under which our strong convergence result is obtained.

Assumption 3.1 Suppose that the following conditions hold:

Assumption A

(A1) Let $g : \mathcal{X} \to \mathbb{R}$ be a η-strongly convex, strongly coercive Legendre function that is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of $\mathcal{X}$. Let ∇g is weakly sequentially continuous on $\mathcal{X}$.
(A2) For each $i = 1, 2, \ldots, m$, the mapping $N_i : \mathcal{X} \to \mathcal{X}^*$ is a finite family of monotone and L_i-Lipschitz continuous mappings with $L_i > 0$, $\forall\, i = 1, 2, \ldots, m$.
(A3) Let $M_i : \mathcal{X} \to 2^{\mathcal{X}^*}$ be a finite family of maximal monotone mappings for each $i = 1, 2, \ldots, m$.
(A4) The mapping $T_i : \mathcal{X} \to \mathcal{X}$ is a finite family of relatively nonexpansive mappings.
(A5) The solution set $\Gamma = \cap_{i=1}^m F(T_i) \bigcap \cap_{i=1}^m (N_i + M_i)^{-1}0 \neq \emptyset$.

Assumption B

(B1) Let $\alpha_k \in (0, 1)$ such that $\sum_{k=1}^{\infty} \alpha_k = \infty$, $\lim_{k\to\infty} \alpha_k = 0$.
(B2) $\liminf_{k\to\infty} \beta_{k,0}\beta_{k,i} > 0$, $\; i = 1, 2, \ldots, m$.

We now present our proposed method.

Algorithm 3.2 Inertial Halpern-type Tseng's method with self-adaptive stepsize

Initialization: Choose $x_0, x_1 \in \mathcal{C}$, $\lambda_1 > 0$, $\tau \in (0, 1)$ and $\mu \in (0, 1)$. Set $k = 1$.
Step 0: Given x_k, x_{k-1}, choose τ_k such that $0 < \tau_k \leq \bar{\tau}_k$ where

$$\bar{\tau}_k = \begin{cases} \min\left\{\tau, \frac{\epsilon_k}{\delta_r\|\nabla g(x_k)-\nabla g(x_{k-1})\|}, \frac{\epsilon_k}{P_g(x_k, x_{k-1})}\right\}, & \text{if } x_k - x_{k-1} \neq 0, \\ \tau, & \text{otherwise} \end{cases} \tag{19}$$

and $\{\epsilon_k\} \subset (0, 1)$ is a sequence of nonnegative numbers such that $\{\epsilon_k\} = \circ(\alpha_k)$, that is

$$\lim_{k\to\infty} \frac{\epsilon_k}{\alpha_k} = 0. \tag{20}$$

Step 1: Given the kth iterates x_k and x_{k-1}, compute

$$w_k = \nabla g^*\left[\nabla g(x_k) + \tau_k(\nabla g(x_k) - \nabla g(x_{k-1}))\right].$$

Step 2: Compute

$$y_{k,i} = J_{\lambda_k}^{M_i} \nabla g^*\left(\nabla g(w_k) - \lambda_k N_i w_k\right)$$

and

$$z_{k,i} = \nabla g^*\Big(\nabla g(y_{k,i}) - \lambda_k(N_i y_{k,i} - N_i w_k)\Big),$$

where

$$\lambda_{k+1} = \begin{cases} \min\left\{\min\limits_{1\leq i\leq m}\left\{\frac{\mu\left(P_g(z_{k,i},\, y_{k,i})+P_g(y_{k,i}, w_k)\right)}{\langle z_{k,i}-y_{k,i},\, N_i w_k - N_i y_{k,i}\rangle}\right\}, \lambda_k\right\} \\ \qquad\qquad\qquad \text{if } \langle z_{k,i} - y_{k,i}, N_i w_k - N_i y_{k,i}\rangle \neq 0, \\ \lambda_k, \text{ otherwise.} \end{cases} \tag{21}$$

If $w_k = y_{k,i}$, then Stop: w_k is a solution of the problem (8). Otherwise, go to **Step 3**.
Step 3: Compute

$$u_k = \nabla g^*\left(\beta_{k,0}\nabla g(w_k) + \sum_{i=1}^{m} \beta_{k,i}\nabla g(T_i z_{k,i})\right)$$

with $\beta_{k,i} \in (0, 1)$ such that $\sum\limits_{i=0}^{m} \beta_{k,i} = 1$.
Step 4: Compute

$$x_{k+1} = \nabla g^*\left(\alpha_k\nabla g(x_0) + (1 - \alpha_k)\nabla g(u_k)\right),$$

Set $k := k + 1$ and go to **Step 1**.

Remark 3.3 We note that Algorithm 3.2 is a generalization of the method proposed in [1] for solving monotone inclusion problem, while we have considered a common solution to a finite family of monotone inclusion problems and fixed point problem for a finite family of relatively nonexpansive mappings in reflexive Banach spaces.

4 Convergence analysis

Lemma 4.1 *Let $\{\lambda_k\}$ be the sequence generated by Algorithm 3.2. Then, the sequence $\{\lambda_k\}$ is nonincreasing and*

$$\lim_{k\to\infty}\lambda_k=\lambda\geq \min\left\{\lambda_1,\frac{\mu\eta}{L_i}\right\},$$

where $\eta>0,\ \mu\in(0,1)$ and L_i is the Lipschitz constant associated with the mapping N_i.

Proof From (21), it is obvious that $\lambda_{k+1}\leq\lambda_k$ for all $k\geq 1$. Now, since N_i is L_i-Lipschitz continuous, we have in the case where $\langle z_{k,i}-y_{k,i}, N_iw_k-N_iy_{k,i}\rangle\neq 0$ and from (21) and (14), that

$$\begin{aligned}\frac{\mu\big(P_g(z_{k,i},\ \ y_{k,i})+P_g(y_{k,i},w_k)\big)}{\langle z_{k,i}-y_{k,i},N_iw_k-N_iy_{k,i}\rangle}&\geq\frac{\frac{\mu\eta}{2}\Big[\|z_{k,i}-y_{k,i}\|^2+\|y_{k,i}-w_k\|^2\Big]}{\|z_{k,i}-y_{k,i}\|\|N_iw_k-N_iy_{k,i}\|}\\&\geq\frac{\mu\eta\Big[\|z_{k,i}-y_{k,i}\|\cdot\|w_k-y_{k,i}\|\Big]}{\|z_{k,i}-y_{k,i}\|\|N_iw_k-N_iy_{k,i}\|}\\&\geq\frac{\mu\eta\Big[\|z_{k,i}-y_{k,i}\|\cdot\|w_k-y_{k,i}\|\Big]}{L_i\|z_{k,i}-y_{k,i}\|\|w_k-y_{k,i}\|}\\&=\frac{\mu\eta}{L_i}.\end{aligned}$$

Clearly, we have

$$\lambda_{k+1}\geq\min\left\{\lambda_k,\frac{\mu\eta}{L_i}\right\}.$$

By induction, we obtain that $\lambda_k\geq\min\left\{\lambda_1,\frac{\mu\eta}{L_i}\right\}$. Thus, $\lim_{k\to\infty}\lambda_k=\lambda\geq\min\left\{\lambda_1,\frac{\mu\eta}{L_i}\right\}$ which completes the proof. □

Remark 4.2 Since the limit

$$\lim_{k\to\infty}\left(1-\frac{\mu\lambda_k}{\lambda_{k+1}}\right)=1-\mu>0, \tag{22}$$

there exists $k_0>0$ such that for all $k>k_0$, we have $\left(1-\frac{\lambda_k\mu}{\lambda_{k+1}}\right)>0$.

Lemma 4.3 *Let $\{x_k\}$ be the sequence generated by Algorithm 3.2 satisfying Assumption 3.1. Then, the following inequality holds*

$$P_g(x,z_{k,i})\leq P_g(x,w_k)-\left(1-\mu\frac{\lambda_k}{\lambda_{k+1}}\right)\Big[P_g(z_{k,i},y_{k,i})+P_g(w_k,y_{k,i})\Big].$$

Proof Let $x\in\Gamma$, see (A5). Then, from the definition of $z_{k,i}$ in **Step 2** and the Bregman distance (11) we have

$$\begin{aligned}P_g\Big(x,z_{k,i}\Big)&=P_g\Big(x,\nabla g^*(\nabla g(y_{k,i})-\lambda_k(N_iy_{k,i}-N_iw_k))\Big)\\&=g(x)-g(z_{k,i})-\langle x-z_{k,i},\nabla g(y_{k,i})-\lambda_k(N_iy_{k,i}-N_iw_k)\rangle\end{aligned}$$

$$
\begin{aligned}
&= g(x) - g(z_{k,i}) - \langle x - z_{k,i}, \nabla g(y_{k,i})\rangle + \lambda_k \langle x - z_{k,i}, N_i y_{k,i} - N_i w_k\rangle \\
&= g(x) - g(y_{k,i}) + g(y_{k,i}) - \langle x - z_{k,i}, \nabla g(y_{k,i})\rangle - g(z_{k,i}) \\
&\quad + \langle x - y_{k,i}, \nabla g(y_{k,i})\rangle - \langle x - y_{k,i}, \nabla g(y_{k,i})\rangle \\
&\quad + \lambda_k \langle x - z_{k,i}, N_i y_{k,i} - N_i w_k\rangle \\
&= g(x) - g(y_{k,i}) + \langle x - y_{k,i}, \nabla g(y_{k,i})\rangle + g(y_{k,i}) - g(z_{k,i}) \\
&\quad + \langle z_{k,i} - y_{k,i}, \nabla g(y_{k,i})\rangle + \lambda_k \langle x - z_{k,i}, N_i y_{k,i} - N_i w_k\rangle \\
&= P_g(x, y_{k,i}) - P_g(z_{k,i}, y_{k,i}) + \lambda_k \langle x - z_{k,i}, N_i y_{k,i} - N_i w_k\rangle. \qquad (23)
\end{aligned}
$$

From (12), we have

$$
P_g(x, y_{k,i}) = P_g(x, w_k) - P_g(y_{k,i}, w_k) + \langle x - y_{k,i}, \nabla g(w_k) - \nabla g(y_{k,i})\rangle. \qquad (24)
$$

Substituting (24) into (23), we have

$$
\begin{aligned}
P_g\Big(x, z_{k,i}\Big) &= P_g(x, w_k) - P_g(y_{k,i}, w_k) + \langle x - y_{k,i}, \nabla g(w_k) - \nabla g(y_{k,i})\rangle \\
&\quad - P_g(z_{k,i}, y_{k,i}) + \lambda_k \langle x - z_{k,i}, N_i y_{k,i} - N_i w_k\rangle \\
&= P_g(x, w_k) - P_g(y_{k,i}, w_k) - P_g(z_{k,i}, y_{k,i}) \\
&\quad + \lambda_k \langle z_{k,i} - y_{k,i}, N_i w_k - N_i y_{k,i}\rangle \\
&\quad - \langle y_{k,i} - x, \nabla g(w_k) - \nabla g(y_{k,i}) - \lambda_k (N_i w_k - N_i y_{k,i})\rangle. \qquad (25)
\end{aligned}
$$

From the definition of $y_{k,i}$ in **Step 2** we have $\nabla g(w_k) - \lambda_k N_i w_k \in \nabla g(y_{k,i}) + \lambda_k M_i y_{k,i}$. By the maximal monotonicity of M_i, there exists $d_k \in M_i y_{k,i}$ such that $\nabla g(w_k) - \lambda_k N_i w_k = \nabla g(y_{k,i}) + \lambda_k d_k$. It follows from this that

$$
\frac{1}{\lambda_k}\Big(\nabla g(w_k) - \lambda_k N_i w_k - \nabla g(y_{k,i})\Big) = d_k. \qquad (26)
$$

Also, from Lemma 2.15, the fact that $0 \in (N_i + M_i)x$ (see (3)) and $N_i y_{k,i} + d_k \in (N_i + M_i)y_{k,i}$, we have that $N_i + M_i$ is maximal monotone. Therefore,

$$
\langle y_{k,i} - x, N_i y_{k,i} + d_k\rangle \geq 0, \qquad (27)
$$

see Definition 2.2. From (26) and (27), we have

$$
\frac{1}{\lambda_k}\Big\langle y_{k,i} - x, \lambda_k N_i y_{k,i} + \nabla g(w_k) - \lambda_k N_i w_k - \nabla g(y_{k,i})\Big\rangle \geq 0
$$

which implies that

$$
\Big\langle y_{k,i} - x, \nabla g(w_k) - \nabla g(y_{k,i}) - \lambda_k \Big(N_i w_k - N_i y_{k,i}\Big)\Big\rangle \geq 0. \qquad (28)
$$

Substituting (28) into (25) and using (21), we have

$$
\begin{aligned}
P_g\Big(x, z_{k,i}\Big) &\leq P_g(x, w_k) - P_g(y_{k,i}, w_k) - P_g(z_{k,i}, y_{k,i}) \\
&\quad + \lambda_k \langle z_{k,i} - y_{k,i}, N_i w_k - N_i y_{k,i}\rangle \\
&\leq P_g(x, w_k) - P_g(y_{k,i}, w_k) - P_g(z_{k,i}, y_{k,i}) \\
&\quad + \frac{\mu\lambda_k}{\lambda_{k+1}}\Big[P_g(z_{k,i}, y_{k,i}) + P_g(y_{k,i}, w_k)\Big] \\
&= P_g(x, w_k) - \Big(1 - \mu\frac{\lambda_k}{\lambda_{k+1}}\Big)\Big[P_g(z_{k,i}, y_{k,i}) + P_g(y_{k,i}, w_k)\Big], \qquad (29)
\end{aligned}
$$

which is the required result. □

Applying (22) to (29), we have

$$P_g\Big(x, z_{k,i}\Big) \le P_g(x, w_k). \tag{30}$$

Lemma 4.4 *Let $\{x_k\}$ be a sequence generated by Algorithm 3.2 satisfying Assumption 3.1. Then $\{x_k\}$ is bounded.*

Proof Let $x \in \Gamma$ and let $b_k = \nabla g(x_k) + \tau_k\Big(\nabla g(x_k) - \nabla g(x_{k-1})\Big)$. Then, w_k can be written as $w_k = \nabla g^*(b_k)$. Hence, we have from (15) to Lemma 2.10 that

$$\begin{aligned}
P_g(x, w_k) &= P_g(x, \nabla g^*(b_k)) \\
&= g(x) + g^*(b_k) - \langle x,\ b_k\rangle \\
&= g(x) + g^*\Big(\nabla g(x_k) + \tau_k\Big(\nabla g(x_k) - \nabla g(x_{k-1})\Big)\Big) \\
&\quad -\Big\langle x, \nabla g(x_k) + \tau_k\Big(\nabla g(x_k) - \nabla g(x_{k-1})\Big)\Big\rangle \\
&\le g(x) - \langle x, \nabla g(x_k)\rangle - \Big\langle x, \tau_k\Big(\nabla g(x_k) - \nabla g(x_{k-1})\Big)\Big\rangle \\
&\quad +\Big[(1+\tau_k)g^*(\nabla g(x_k)) - \tau_k g^*(\nabla g(x_{k-1})) \\
&\quad +\tau_k(1+\tau_k)\delta_r\Big(\|\nabla g(x_k) - \nabla g(x_{k-1})\|\Big)\Big] \\
&= g(x) - \langle x, \nabla g(x_k)\rangle + g^*(\nabla g(x_k)) - \Big\langle x, \tau_k\Big(\nabla g(x_k) - \nabla g(x_{k-1})\Big)\Big\rangle \\
&\quad +\Big[\tau_k g^*(\nabla g(x_k)) - \tau_k g^*(\nabla g(x_{k-1})) \\
&\quad +\tau_k(1+\tau_k)\delta_r\Big(\|\nabla g(x_k) - \nabla g(x_{k-1})\|\Big)\Big] \\
&= P_g(x, x_k) - \Big\langle x, \tau_k\Big(\nabla g(x_k) - \nabla g(x_{k-1})\Big)\Big\rangle \\
&\quad +\Big[\tau_k g^*(\nabla g(x_k)) - \tau_k g^*(\nabla g(x_{k-1})) \\
&\quad +\tau_k(1+\tau_k)\delta_r\Big(\|\nabla g(x_k) - \nabla g(x_{k-1})\|\Big)\Big] \\
&= P_g(x, x_k) - \tau_k\Big(g^*(\nabla g(x_{k-1})) - g^*(\nabla g(x_k))\Big) \\
&\quad -\Big\langle x, \tau_k\Big(\nabla g(x_k) - \nabla g(x_{k-1})\Big)\Big\rangle \\
&\quad +\tau_k(1+\tau_k)\delta_r\Big(\|\nabla g(x_k) - \nabla g(x_{k-1})\|\Big) \\
&\le P_g(x, x_k) - \tau_k\Big\langle x_k, \nabla g(x_{k-1}) - \nabla g(x_k)\Big\rangle - \tau_k\Big\langle x, \nabla g(x_k) - \nabla g(x_{k-1})\Big\rangle \\
&\quad +\tau_k(1+\tau_k)\delta_r\Big(\|\nabla g(x_k) - \nabla g(x_{k-1})\|\Big) \\
&= P_g(x, x_k) + \tau_k\Big\langle x - x_k, \nabla g(x_{k-1}) - \nabla g(x_k)\Big\rangle \\
&\quad +\tau_k(1+\tau_k)\delta_r\Big(\|\nabla g(x_k) - \nabla g(x_{k-1})\|\Big) \\
&= P_g(x, x_k) + \tau_k\Big(P_g(x, x_k) - P_g(x, x_{k-1})\Big) + \tau_k P_g(x_k, x_{k-1})
\end{aligned}$$

$$+\tau_k(1+\tau_k)\delta_r\Big(\|\nabla g(x_k)-\nabla g(x_{k-1})\|\Big). \tag{31}$$

From the definition of u_k in **Step 3**, Lemma 2.10, (15) and (16), we have

$$\begin{aligned}
P_g\Big(x,u_k\Big) &= P_g\left(x,\nabla g^*\left(\beta_{k,0}\nabla g(w_k)+\sum_{i=1}^m \beta_{k,i}\nabla g(T_i z_{k,i})\right)\right)\\
&= Q_g\left(x,\beta_{k,0}\nabla g(w_k)+\sum_{i=1}^m \beta_{k,i}\nabla g(T_i z_{k,i})\right)\\
&= g(x)-\beta_{k,0}\Big\langle x,\nabla g(w_k)\Big\rangle-\sum_{i=1}^m \beta_{k,i}\Big\langle x,\nabla g(T_i z_{k,i})\Big\rangle\\
&\quad +g^*\Big(\beta_{k,0}\nabla g(w_k)+\sum_{i=1}^m \beta_{k,i}\nabla g(T_i z_{k,i})\Big)\\
&\le g(x)-\beta_{k,0}\Big\langle x,\nabla g(w_k)\Big\rangle-\sum_{i=1}^m \beta_{k,i}\Big\langle x,\nabla g(T_i z_{k,i})\Big\rangle\\
&\quad +\beta_{k,0}g^*(\nabla g(w_k))+\sum_{i=1}^m \beta_{k,i}g^*\Big(\nabla g(T_i z_{k,i})\Big)\\
&\quad -\sum_{i=1}^m \beta_{k,0}\beta_{k,i}\delta_r\Big(\|\nabla g(w_k)-\nabla g(T_i z_{k,i})\|\Big)\\
&= g(x)+\beta_{k,0}g(x)-\beta_{k,0}g(x)-\beta_{k,0}\Big\langle x,\nabla g(w_k)\Big\rangle-\sum_{i=1}^m \beta_{k,i}\Big\langle x,\nabla g(T_i z_{k,i})\Big\rangle\\
&\quad +\beta_{k,0}g^*(\nabla g(w_k))+\sum_{i=1}^m \beta_{k,i}g^*\Big(\nabla g(T_i z_{k,i})\Big)\\
&\quad -\sum_{i=1}^m \beta_{k,0}\beta_{k,i}\delta_r\Big(\|\nabla g(w_k)-\nabla g(T_i z_{k,i})\|\Big)\\
&= \beta_{k,0}g(x)-\beta_{k,0}\Big\langle x,\nabla g(w_k)\Big\rangle+\beta_{k,0}g^*(\nabla g(w_k))\\
&\quad +(1-\beta_{k,0})g(x)-\sum_{i=1}^m \beta_{k,i}\Big\langle x,\nabla g(T_i z_{k,i})\Big\rangle+\sum_{i=1}^m \beta_{k,i}g^*\Big(\nabla g(T_i z_{k,i})\Big)\\
&\quad -\sum_{i=1}^m \beta_{k,0}\beta_{k,i}\delta_r\Big(\|\nabla g(w_k)-\nabla g(T_i z_{k,i})\|\Big)\\
&= \beta_{k,0}Q_g\Big(x,\nabla g(w_k)\Big)+\sum_{i=1}^m \beta_{k,i}Q_g\Big(x,\nabla g(T_i z_{k,i})\Big)\\
&\quad -\sum_{i=1}^m \beta_{k,0}\beta_{k,i}\delta_r\Big(\|\nabla g(w_k)-\nabla g(T_i z_{k,i})\|\Big) \qquad (32)\\
&= \beta_{k,0}P_g\Big(x,w_k\Big)+\sum_{i=1}^m \beta_{k,i}P_g\Big(x,T_i z_{k,i}\Big)
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{i=1}^{m}\beta_{k,0}\beta_{k,i}\delta_r\Big(\|\nabla g(w_k)-\nabla g(T_iz_{k,i})\|\Big)\\
&\le \beta_{k,0}P_g\Big(x,w_k\Big)+\sum_{i=1}^{m}\beta_{k,i}P_g\Big(x,w_k\Big)\\
&-\sum_{i=1}^{m}\beta_{k,0}\beta_{k,i}\delta_r\Big(\|\nabla g(w_k)-\nabla g(T_iz_{k,i})\|\Big)\\
&=P_g\Big(x,w_k\Big)-\sum_{i=1}^{m}\beta_{k,0}\beta_{k,i}\delta_r\Big(\|\nabla g(w_k)-\nabla g(T_iz_{k,i})\|\Big)\\
&\le P_g(x,w_k).
\end{aligned}\tag{33}
$$

Now, from the definition of x_{k+1} in **Step 4**, (31) and (33), we have

$$
\begin{aligned}
P_g\Big(x,x_{k+1}\Big)&=P_g\Big(x,\nabla g^*(\alpha_k\nabla g(x_0)+(1-\alpha_k)\nabla g(u_k))\Big)\\
&\le \alpha_kP_g(x,x_0)+(1-\alpha_k)P_g(x,u_k)\\
&\le \alpha_kP_g(x,x_0)+(1-\alpha_k)\Big(P_g(x,x_k)+\tau_k\Big(P_g(x,x_k)-P_g(x,x_{k-1})\Big)\\
&\quad+\tau_kP_g(x_k,x_{k-1})+\tau_k(1+\tau_k)\delta_r\Big(\|\nabla g(x_k)-\nabla g(x_{k-1})\|\Big)\Big)\\
&\le \max\Big\{P_g(x,x_0),\,P_g(x,x_k)+\tau_k\Big(P_g(x,x_k)-P_g(x,x_{k-1})\Big)\\
&\quad+\tau_kP_g(x_k,x_{k-1})+\tau_k(1+\tau_k)\delta_r\Big(\|\nabla g(x_k)-\nabla g(x_{k-1})\|\Big)\Big\}.
\end{aligned}
$$

The conclusion follows trivially in the case when $P_g(x,x_0)$ is the maximum. Otherwise, there exists $k_0\in\mathbb{N}$ such that $\forall\ \ k\ge k_0$ we have

$$
\begin{aligned}
P_g(x,x_{k+1})\le P_g(x,x_k)&+\tau_k\Big(P_g(x,x_k)-P_g(x,x_{k-1})\Big)\\
&+\tau_kP_g(x_k,x_{k-1})+\tau_k(1+\tau_k)\delta_r\Big(\|\nabla g(x_k)-\nabla g(x_{k-1})\|\Big).
\end{aligned}
$$

Then by Lemma 2.17, we have that $\{P_g(x,x_k)\}$ is convergent. This implies that $\{P_g(x,x_k)\}$ is bounded. From Lemma 2.14 we have that $\{x_k\}$ bounded. Consequently, $\{w_k\}$ is bounded. From the properties of $J_{\lambda_k}^M$ (see Definition 2.5), the continuity of ∇g and ∇g^*, we have that $\{y_{k,i}\}$ is bounded. Consequently, $\{z_{k,i}\}$ and $\{u_k\}$ are bounded. □

Lemma 4.5 *Assume that $\{w_k\}$ and $\{y_{k,i}\}$ are sequences generated by Algorithm 3.2 satisfying Assumption 3.1 and $\lim_{j\to\infty}\|w_{k_j}-y_{k_{j,i}}\|=0$. Suppose there exists a subsequence $\{w_{k_j}\}$ of $\{w_k\}$ such that $\{w_{k_j}\}$ converges weakly to some $p\in\mathcal{X}$ as $j\to\infty$. Then, $p\in\cap_{i=1}^m(N_i+M_i)^{-1}(0)$.*

Proof Let $(u,v)\in G(N_i+M_i)$. Then, $v-N_iu\in M_iu$. From the definition of $y_{k,i}$, we have $\nabla g(w_{k_j})-\lambda_{k_j}N_iw_{k_j}\in\nabla g(y_{k_{j,i}})+\lambda_{k_j}M_iy_{k_{j,i}}$ which implies that

$$
\frac{1}{\lambda_{k_j}}\Big(\nabla g(w_{k_j})-\nabla g(y_{k_{j,i}})-\lambda_{k_j}N_iw_{k_j}\Big)\in M_iy_{k_{j,i}}.
$$

Since M_i is maximal monotone we have

$$
\Big\langle u-y_{k_{j,i}},\,v-N_iu-\frac{1}{\lambda_{k_j}}\Big(\nabla g(w_{k_j})-\nabla g(y_{k_{j,i}})-\lambda_{k_j}N_iw_{k_j}\Big)\Big\rangle\ge 0.
$$

By the monotonicity of N_i, we have

$$
\begin{aligned}
\langle u - y_{k_{j,i}}, v\rangle &\geq \Big\langle u - y_{k_{j,i}}, N_i u + \frac{1}{\lambda_{k_j}}\Big(\nabla g(w_{k_j}) - \nabla g(y_{k_{j,i}}) - \lambda_{k_j} N_i w_{k_j}\Big)\Big\rangle \\
&= \Big\langle u - y_{k_{j,i}}, N_i u - N_i w_{k_j}\Big\rangle + \frac{1}{\lambda_{k_j}}\langle u - y_{k_{j,i}}, \nabla g(w_{k_j}) - \nabla g(y_{k_{j,i}})\Big\rangle \\
&= \Big\langle u - y_{k_{j,i}}, N_i u - N_i y_{k_{j,i}}\Big\rangle + \Big\langle u - y_{k_{j,i}}, N_i y_{k_{j,i}} - N_i w_{k_j}\Big\rangle \\
&\quad + \frac{1}{\lambda_{k_j}}\Big\langle u - y_{k_{j,i}}, \nabla g(w_{k_j}) - \nabla g(y_{k_{j,i}})\Big\rangle \\
&\geq \Big\langle u - y_{k_{j,i}}, N_i y_{k_{j,i}} - N_i w_{k_j}\Big\rangle + \frac{1}{\lambda_{k_j}}\Big\langle u - y_{k_{j,i}}, \nabla g(w_{k_j}) - \nabla g(y_{k_{j,i}})\Big\rangle.
\end{aligned}
$$

From the hypothesis of the lemma we have $\lim_{j\to\infty} \|w_{k_j} - y_{k_{j,i}}\| = 0$. Also, since N_i is Lipschitz continuous we have that $\lim_{j\to\infty} \|N_i w_{k_j} - N_i y_{k_{j,i}}\| = 0$. Since g is bounded and uniformly smooth on bounded subsets of $\mathcal{X}$, it follows that ∇g is uniformly continuous on bounded subsets of $\mathcal{X}$. Hence, we have

$$
\lim_{j\to\infty} \|\nabla g(w_{k_j}) - \nabla g(y_{k_{j,i}})\| = 0. \tag{34}
$$

Since N_i is Lipschitz continuous and $y_{k_{j,i}} \rightharpoonup p$, it follows from the hypothesis of the lemma and (34) that

$$
\Big\langle u - p, v\Big\rangle \geq 0.
$$

By the monotonicity of $N_i + M_i$, we obtain that $0 \in \cap_{i=1}^m (N_i + M_i)p$, that is, $p \in \cap_{i=1}^m (N_i + M_i)^{-1}0$. □

Lemma 4.6 *Let $\{x_k\}$ be the sequence generated by Algorithm 3.2. Then, the following inequality holds*

$$
\begin{aligned}
P_g\Big(x, x_{k+1}\Big) &\leq (1-\alpha_k) P_g\Big(x, x_k\Big) + \alpha_k s_k \\
&\quad -(1-\alpha_k)\Big[\sum_{i=1}^m \beta_{k,i}\Big(1 - \mu\frac{\lambda_k}{\lambda_{k+1}}\Big)\Big[P_g(z_{k,i}, y_{k,i}) + P_g(y_{k,i}, w_k)\Big] \\
&\quad - \sum_{i=1}^m \beta_{k,0}\beta_{k,i}\delta_r\Big(\|\nabla g(w_k) - \nabla g(T_i z_{k,i})\|\Big)\Big]
\end{aligned}
$$

where

$$
\begin{aligned}
s_k &= \Big\langle \nabla g(x_0) - \nabla g(x), x_{k+1} - x\Big\rangle \\
&\quad + \frac{\tau_k}{\alpha_k}\Big[\Big(P_g(x, x_k) - P_g(x, x_{k-1})\Big) + P_g(x_k, x_{k-1}) \\
&\quad + (1+\tau_k)\delta_r\Big(\|\nabla g(x_k) - \nabla g(x_{k-1}\|)\Big)\Big].
\end{aligned}
$$

Proof Let $x \in \Gamma$. Then, from the definition of x_{k+1} in **Step 4**, (11), (16), (17), (29), (31) and (32). we have

$$
P_g\Big(x, x_{k+1}\Big) = P_g\Big(x, \nabla g^*(\alpha_k \nabla g(x_0) + (1-\alpha_k)\nabla g(u_k))\Big)
$$

$$
\begin{aligned}
&= Q_g\Big(x, \alpha_k \nabla g(x_0) + (1-\alpha_k)\nabla g(u_k)\Big)\\
&\le Q_g\Big(x, \alpha_k \nabla g(x_0) + (1-\alpha_k)\nabla g(u_k) - \alpha_k(\nabla g(x_0) - \nabla g(x))\Big)\\
&\quad +\alpha_k \langle \nabla g(x_0) - \nabla g(x), x_{k+1} - x\rangle\\
&= Q_g\Big(x, \alpha_k \nabla g(x) + (1-\alpha_k)\nabla g(u_k)\Big)\\
&\quad +\alpha_k \langle \nabla g(x_0) - \nabla g(x), x_{k+1} - x\rangle\\
&\le \alpha_k Q_g(x, \nabla g(x)) + (1-\alpha_k) Q_g(x, \nabla g(u_k))\\
&\quad +\alpha_k \langle \nabla g(x_0) - \nabla g(x), x_{k+1} - x\rangle\\
&= \alpha_k P_g(x, x) + (1-\alpha_k) P_g(x, u_k) + \alpha_k \langle \nabla g(x_0) - \nabla g(x), x_{k+1} - x\rangle\\
&= (1-\alpha_k) P_g(x, u_k) + \alpha_k \langle \nabla g(x_0) - \nabla g(x), x_{k+1} - x\rangle\\
&\le (1-\alpha_k)\Big[\beta_{k,0} P_g\Big(x, w_k\Big) + \sum_{i=1}^{m} \beta_{k,i} P_g\Big(x, T_i z_{k,i}\Big)\\
&\quad -\sum_{i=1}^{m} \beta_{k,0}\beta_{k,i}\delta_r\Big(\|\nabla g(w_k) - \nabla g(T_i z_{k,i})\|\Big)\Big]\\
&\quad +\alpha_k \langle \nabla g(x_0) - \nabla g(x), x_{k+1} - x\rangle\\
&\le (1-\alpha_k)\Big[\beta_{k,0} P_g\Big(x, w_k\Big) + \sum_{i=1}^{m} \beta_{k,i} P_g\Big(x, z_{k,i}\Big)\\
&\quad -\sum_{i=1}^{m} \beta_{k,0}\beta_{k,i}\delta_r\Big(\|\nabla g(w_k) - \nabla g(T_i z_{k,i})\|\Big)\Big]\\
&\quad +\alpha_k \langle \nabla g(x_0) - \nabla g(x), x_{k+1} - x\rangle\\
&\le (1-\alpha_k)\Big[\beta_{k,0} P_g\Big(x, w_k\Big) + \sum_{i=1}^{m} \beta_{k,i} P_g(x, w_k)\\
&\quad -\sum_{i=1}^{m} \beta_{k,i}\Big(1 - \mu\frac{\lambda_k}{\lambda_{k+1}}\Big)\Big[P_g(z_{k,i}, y_{k,i}) + P_g(y_{k,i}, w_k)\Big]\\
&\quad -\sum_{i=1}^{m} \beta_{k,0}\beta_{k,i}\delta_r\Big(\|\nabla g(w_k) - \nabla g(T_i z_{k,i})\|\Big)\Big]\\
&\quad +\alpha_k \langle \nabla g(x_0) - \nabla g(x), x_{k+1} - x\rangle\\
&= (1-\alpha_k) P_g\Big(x, w_k\Big)\\
&\quad -(1-\alpha_k)\sum_{i=1}^{m} \beta_{k,i}\Big(1 - \mu\frac{\lambda_k}{\lambda_{k+1}}\Big)\Big[P_g(z_{k,i}, y_{k,i}) + P_g(y_{k,i}, w_k)\Big]\\
&\quad -(1-\alpha_k)\sum_{i=1}^{m} \beta_{k,0}\beta_{k,i}\delta_r\Big(\|\nabla g(w_k) - \nabla g(T_i z_{k,i})\|\Big)\\
&\quad +\alpha_k \langle \nabla g(x_0) - \nabla g(x), x_{k+1} - x\rangle\\
&\le (1-\alpha_k)\Big[P_g(x, x_k) + \tau_k\Big(P_g(x, x_k) - P_g(x, x_{k-1})\Big)\\
&\quad +\tau_k P_g(x_k, x_{k-1}) + \tau_k(1+\tau_k)\delta_r\Big(\|\nabla g(x_k) - \nabla g(x_{k-1})\|\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\alpha_k\langle\nabla g(x_0)-\nabla g(x),x_{k+1}-x\rangle\\
&-(1-\alpha_k)\Big[\sum_{i=1}^{m}\beta_{k,i}\Big(1-\mu\frac{\lambda_k}{\lambda_{k+1}}\Big)\Big[P_g(z_{k,i},y_{k,i})+P_g(y_{k,i},w_k)\Big]\\
&+\sum_{i=1}^{m}\beta_{k,0}\beta_{k,i}\delta_r\Big(\|\nabla g(w_k)-\nabla g(T_iz_{k,i})\|\Big)\Big]\\
=\;&(1-\alpha_k)P_g(x,x_k)+\alpha_k\Big(\langle\nabla g(x_0)-\nabla g(x),x_{k+1}-x\rangle\\
&+\frac{\tau_k}{\alpha_k}\Big[\Big(P_g(x,x_k)-P_g(x,x_{k-1})\Big)+P_g(x_k,x_{k-1})\\
&+(1+\tau_k)\delta_r\Big(\|\nabla g(x_k)-\nabla g(x_{k-1}\|)\Big)\Big]\Big)\\
&-(1-\alpha_k)\Big[\sum_{i=1}^{m}\beta_{k,i}\Big(1-\mu\frac{\lambda_k}{\lambda_{k+1}}\Big)\Big[P_g(z_{k,i},y_{k,i})+P_g(y_{k,i},w_k)\Big]\\
&+\sum_{i=1}^{m}\beta_{k,0}\beta_{k,i}\delta_r\Big(\|\nabla g(w_k)-\nabla g(T_iz_{k,i})\|\Big)\Big].
\end{aligned}\tag{35}
$$

□

Now, we are in the position to present our main result.

Theorem 4.7 *Let $\{x_k\}$ be the sequence generated by Algorithm 3.2 satisfying Assumption 3.1. Then the sequence $\{x_k\}$ converges strongly to a point $\bar{x}\in\Gamma$, where $\bar{x}=\Pi_\Gamma(x_0)$.*

Proof Let $\bar{x}\in\Gamma$. Then it follows from Lemma 4.6 that

$$
P_g\Big(\bar{x},x_{k+1}\Big)\le(1-\alpha_k)P_g\Big(\bar{x},x_k\Big)+\alpha_k s_k,\tag{36}
$$

where

$$
\begin{aligned}
s_k=&\Big\langle\nabla g(x_0)-\nabla g(\bar{x}),x_{k+1}-\bar{x}\Big\rangle\\
&+\frac{\tau_k}{\alpha_k}\Big[\Big(P_g(\bar{x},x_k)-P_g(\bar{x},x_{k-1})\Big)+P_g(x_k,x_{k-1})\\
&+(1+\tau_k)\delta_r\Big(\|\nabla g(x_k)-\nabla g(x_{k-1}\|)\Big)\Big].
\end{aligned}
$$

Let $P_g(\bar{x},x_{k+1})=a_{k+1}$. Then (36) becomes

$$
a_{k+1}\le(1-\alpha_k)a_k+\alpha_k s_k.
$$

Now, we claim that the sequence a_k converges to zero. To show this, by Lemma 2.18 it sufficies to show that $\limsup_{j\to\infty}s_{k_j}\le 0$ for every subsequence $\{a_{k_j}\}$ of $\{a_k\}$ satisfying

$$
\liminf_{j\to\infty}\Big(a_{k_{j+1}}-a_{k_j}\Big)\ge 0.\tag{37}
$$

Suppose that $\{a_{k_j}\}$ is a subsequence $\{a_{k_j}\}$ such that (37) holds. Then, from Lemma 4.6, we have

$$
(1-\alpha_{k_j})\Big[\sum_{i=1}^{m}\beta_{k_j,i}\Big(1-\mu\frac{\lambda_k}{\lambda_{k+1}}\Big)\Big[P_g(z_{k_j,i},y_{k_j,i})+P_g(y_{k_j,i},w_{k_j})\Big]
$$

$$+\sum_{i=1}^{m}\beta_{k_j,0}\beta_{k_j,i}\delta_r\Big(\|\nabla g(w_{k_j})-\nabla g(T_i z_{k_j,i})\|\Big)\Big]$$
$$\leq (1-\alpha_{k_j})a_{k_j}-a_{k_{j+1},i}+\alpha_{k_j}s_{k_j}.$$

Applying (22), (37) and the conditions on α_{k_j}, we obtain

$$\lim_{j\to\infty}\Big[(1-\alpha_{k_j})\Big[\sum_{i=1}^{m}\beta_{k_j,i}\Big(1-\mu\frac{\lambda_k}{\lambda_{k+1}}\Big)\Big[P_g(z_{k_j,i},y_{k_j,i})+P_g(y_{k_j,i},w_{k_j})\Big]$$
$$+\sum_{i=1}^{m}\beta_{k_j,0}\beta_{k_j,i}\delta_r\Big(\|\nabla g(w_{k_j})-\nabla g(T_i z_{k_j,i})\|\Big)\Big]\Big]=0,$$

which implies

$$\lim_{j\to\infty}P_g(y_{k_{j,i}},w_{k_j})=\lim_{j\to\infty}P_g(z_{k_{j,i}},y_{k_{j,i}})=0$$

and

$$\lim_{j\to\infty}\|\nabla g(w_{k_j})-\nabla g(T_i z_{k_j,i})\|=0.$$

Since ∇g^* is continuous on bounded subset of $\mathcal{X}^*$ we have,

$$\lim_{j\to\infty}\|w_{k_j}-T_i z_{k_{j,i}}\|=0. \tag{38}$$

By Lemma 2.16, we have

$$\lim_{j\to\infty}\|w_{k_j}-y_{k_{j,i}}\|=\lim_{j\to\infty}\|z_{k_{j,i}}-y_{k_{j,i}}\|=0. \tag{39}$$

From (38) and (39), we have

$$\lim_{j\to\infty}\|w_{k_j}-z_{k_{j,i}}\|=\lim_{j\to\infty}\|z_{k_{j,i}}-T_i z_{k_{j,i}}\|=\lim_{j\to\infty}\|y_{k_{j,i}}-T_i z_{k_{j,i}}\|=0. \tag{40}$$

From the definition of u_{k_j}, we have

$$P_g(z_{k_{j,i}},u_{k_j})=P_g\Big(z_{k_{j,i}},\nabla g^*\Big(\beta_{k_{j,0}}\nabla g(w_{k_j})+\sum_{i=1}^{m}\beta_{k_{j,i}}\nabla g(T_i z_{k_{j,i}})\Big)\Big)$$
$$\leq \beta_{k_{j,0}}P_g(z_{k_{j,i}},w_{k_j})+\sum_{i=1}^{m}\beta_{k_{j,i}}P_g(z_{k_{j,i}},T_i z_{k_{j,i}})\to 0,\quad j\to\infty.$$

Hence,

$$\lim_{j\to\infty}P_g(z_{k_{j,i}},u_{k_j})=0,\quad \text{for each } i=1,2,\ldots,m$$

which implies that

$$\lim_{j\to\infty}\|z_{k_{j,i}}-u_{k_j}\|=0,\quad i=1,2,\ldots,m. \tag{41}$$

Also, we have from Assumption 3.1 and (11) that

$$P_g(u_{k_j}, x_{k_{j+1}}) \leq \alpha_{k_j} P_g(u_{k_j}, x_0) + (1 - \alpha_{k_j}) P_g(u_{k_j}, u_{k_j}) \to 0, \quad \text{as} \quad j \to \infty,$$

which implies that

$$\lim_{j\to\infty} \|u_{k_j} - x_{k_{j,i}}\| = 0. \tag{42}$$

From (41) and (42), we have

$$\lim_{j\to\infty} \|z_{k_{j,i}} - x_{k_{j,i}}\| = 0. \tag{43}$$

From (40) and (43), we have

$$\lim_{j\to\infty} \|w_{k_j} - x_{k_{j+1}}\| = 0. \tag{44}$$

Note from (19) and (20) that

$$\begin{aligned}
\delta_r \|\nabla g(w_{k_j}) - \nabla g(x_{k_j})\| &= \tau_{k_j} \delta_r \|\nabla g(x_{k_j}) - \nabla g(x_{k_{j-1}})\| \\
&= \alpha_{k_j} \frac{\tau_{k_j}}{\alpha_{k_j}} \delta_r \|\nabla g(x_{k_j}) - \nabla g(x_{k_{j-1}})\| \\
&\leq \alpha_{k_j} \frac{\epsilon_{k_j}}{\alpha_{k_j}} \to 0, \ \ j \to \infty.
\end{aligned}$$

By the continuity of δ_r we have

$$\lim_{k\to\infty} \|\nabla g(w_{k_j}) - \nabla g(x_{k_j})\| = 0.$$

Since ∇g^* is continuous on bounded subset of $\mathcal{X}^*$ we have,

$$\lim_{j\to\infty} \|w_{k_j} - x_{k_j}\| = 0. \tag{45}$$

Replacing $\bar{x}$ with x_{k_j} in (35), we have

$$\begin{aligned}
P_g(x_{k_j}, x_{k_{j+1}}) \leq (1 - \alpha_{k_j}) P_g(x_{k_j}, x_{k_j}) + \alpha_{k_j} \Big(&\langle \nabla g(x_0) - \nabla g(x_{k_j}), x_{k_{j+1}} - x_{k_j} \rangle \\
&+ \frac{\tau_{k_j}}{\alpha_{k_j}} \Big[\Big(P_g(x_{k_j}, x_{k_j}) - P_g(x_{k_j}, x_{k_{j-1}}) \Big) + P_g(x_{k_j}, x_{k_{j-1}}) \Big) \\
&+ (1 + \tau_{k_j}) \delta_r \Big(\|\nabla g(x_{k_j}) - \nabla g(x_{k_{j-1}})\| \Big) \Big] \Big) \\
&- (1 - \alpha_{k_j}) \sum_{i=1}^{m} \beta_{k_{j,i}} \Big(1 - \mu \frac{\lambda_k}{\lambda_{k+1}} \Big) \Big[P_g(y_{k_{j,i}}, w_{k_j}) + P_g(z_{k_{j,i}}, y_{k_{j,i}}) \Big] \\
&- \sum_{i=1}^{m} \beta_{k_{j,0}} \beta_{k_{j,i}} \delta_r \Big[\|\nabla g(w_{k_j}) - \nabla g(T_i z_{k_{j,i}})\| \Big].
\end{aligned}$$

Taking limits on both sides, we have

$$\lim_{j\to\infty} P_g(x_{k_j}, x_{k_{j+1}}) = 0 \tag{46}$$

which implies from Lemma 2.16 that

$$\lim_{j\to\infty} \|x_{k_j} - x_{k_{j+1}}\| = 0. \tag{47}$$

By the boundedness of $\{x_k\}$, there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that $x_{k_j} \rightharpoonup p \in \mathcal{C}$. Since $\lim_{j\to\infty} \|x_{k_j} - z_{k_{j,i}}\| = 0$, we have that $z_{k_{j,i}} \rightharpoonup p$ as $k \to \infty$. Also, since $I - T_i$ is demiclosed at zero (see Definition (2.11)), it follows from (40) that $p \in \cap_{i=1}^m F(T_i)$. From Lemma 4.5, we have that $p \in \cap_{i=1}^m (N_i + M_i)^{-1}0$. Hence, $p \in \Gamma$.
Next, since $\{x_{k_j}\}$ is bounded, it follows that there exists a subsequence $\{x_{k_{j_l}}\}$ of $\{x_{k_j}\}$ which converges weakly to some $p \in \mathcal{X}$ such that we have

$$\lim_{l\to\infty} \langle \nabla g(x_0) - \nabla g(\bar{x}), x_{k_{j_l}} - \bar{x} \rangle = \limsup_{j\geq 0} \langle \nabla g(x_0) - \nabla g(\bar{x}), x_{k_j} - \bar{x} \rangle. \tag{48}$$

From (48) and the variational characterization of the metric projection, we have

$$\begin{aligned} \limsup_{j\geq 0} \langle \nabla g(x_0) - \nabla g(\bar{x}), x_{k_j} - \bar{x} \rangle &= \lim_{l\to\infty} \langle \nabla g(x_0) - \nabla g(\bar{x}), x_{k_{j_l}} - \bar{x} \rangle \\ &= \langle \nabla g(x_0) - \nabla g(\bar{x}), p - \bar{x} \rangle. \end{aligned} \tag{49}$$

Since $\bar{x} = \Pi_\Gamma(x_0)$, we obtain from (49) that

$$\begin{aligned} \limsup_{j\to\infty} \langle \nabla g(x_0) - \nabla g(\bar{x}), x_{k_{j+1}} - \bar{x} \rangle &= \limsup_{j\geq 0} \langle \nabla g(x_0) - \nabla g(\bar{x}), x_{k_j} - \bar{x} \rangle \\ &= \langle \nabla g(x_0) - \nabla g(\bar{x}), p - \bar{x} \rangle \\ &\leq 0, \end{aligned}$$

which implies that $\limsup_{j\to\infty} s_{k_j} \leq 0$. Applying Lemma 2.18 to (36) and using Lemma 2.16, we conclude that the sequence $\{x_k\}$ converges strongly to $\bar{x}$. □

Let $\mathcal{C}$ be a nonempty, closed and convex subset of a real Hilbert space $\mathcal{H}$. If $g(x) = \frac{1}{2}\|x\|^2$, then $\nabla g(x) = \|x\|$. Hence, Algorithm 3.2 reduces to the following result:

Algorithm 4.8 Inertial Halpern-type Tseng's method with self-adaptive stepsize

Initialization: Choose $x_0, x_1 \in \mathcal{C}$, $\lambda_1 > 0$, $\tau \in (0, 1)$ and $\mu \in (0, 1)$. Set $k = 1$.
Step 0: Given x_k, x_{k-1}, choose τ_k such that $0 < \tau_k \leq \bar{\tau}_k$ where

$$\bar{\tau}_k = \begin{cases} \min\left\{\tau, \dfrac{\epsilon_k}{\left\|\|x_k\|-\|x_{k-1}\|\right\|}\right\}, & \text{if} \quad x_k - x_{k-1} \neq 0, \\ \tau, \quad \text{otherwise} \end{cases} \tag{50}$$

and $\{\epsilon_k\} \subset (0, 1)$ is a sequence of nonnegative numbers such that $\{\epsilon_k\} = \circ(\alpha_k)$, that (20) holds.
Step 1: Given the kth iterates x_k and x_{k-1}, compute

$$w_k = x_k + \tau_k(x_k - x_{k-1}).$$

Step 2: Compute

$$y_{k,i} = J_{\lambda_k}^{M_i}(w_k - \lambda_k N_i w_k)$$

and

$$z_{k,i} = y_{k,i} - \lambda_k(N_i y_{k,i} - N_i w_k).$$

If $w_k = y_{k,i}$, then Stop: w_k is a solution of the problem (8). Otherwise, go to **Step 3**.
Step 3: Compute

$$u_k = \beta_{k,0} w_k + \sum_{i=1}^{m} \beta_{k,i} T_i z_{k,i}$$

with$\beta_{k,i} \in (0, 1)$ such that $\sum_{i=0}^{m} \beta_{k,i} = 1$.
Step 4: Compute

$$x_{k+1} = \alpha_k x_0 + (1 - \alpha_k) u_k,$$

where

$$\lambda_{k+1} = \begin{cases} \min \left\{ \min_{1 \le i \le m} \left\{ \frac{\mu\left(\|z_{k,i} - y_{k,i}\|^2 + \|y_{k,i} - w_k\|^2\right)}{\langle z_{k,i} - y_{k,i},\ N_i w_k - N_i y_{k,i} \rangle} \right\}, \lambda_k \right\}, \\ \qquad\qquad \text{if} \langle z_{k,i} - y_{k,i}, N_i w_k - N_i y_{k,i} \rangle \neq 0 \\ \lambda_k, \quad \text{otherwise.} \end{cases} \tag{51}$$

Set $k := k + 1$ and go to **Step 1**.

Corollary 4.9 *Let $\{x_k\}$ be the sequence generated by Algorithm 4.8 satisfying Assumption 3.1. Then the sequence $\{x_k\}$ converges strongly to a point $x \in \Gamma$, where $\bar{x} = \Pi_\Gamma(x_0)$.*

5 Application

In this section, we apply our results to approximate the common solution of a finite family of variational inequality problem (VIP) and FPP of a finite family of relatively nonexpansive mappings in Banach spaces. Let $\mathcal{C}$ be a nonempty, closed and convex subset of a real reflexive Banach space $\mathcal{X}$. Let $g : \mathcal{X} \to (-\infty, \infty]$ be Legendre and totally convex function and $N_i : \mathcal{C} \to \mathcal{X}^*$ be a finite family of monotone mapping. The variational inequality problem (VIP) is a problem of finding a point $x \in \mathcal{C}$ such that

$$\langle y - x, \quad N_i x \rangle \geq 0, \quad \forall\ y \in \mathcal{C}. \tag{52}$$

We denote the solution set of (52) by $\cap_{i=1}^{m} VI(\mathcal{C}, N_i)$. The solution set $VI(\mathcal{C}, N_i)$ is closed and convex if N_i is a continuous and hemicontinuous mapping (see [28]).
The indicator function of $\mathcal{C}$ is defined by

$$i_{\mathcal{C}}(x) = \begin{cases} 0, & \text{if} \quad x \in \mathcal{C}, \\ \infty, & \text{if} \quad x \notin \mathcal{C}. \end{cases}$$

It is well known (see [46]) that $i_{\mathcal{C}}$ is convex, proper convex and lower semicontinuous function and its subdifferential $\partial_{i_{\mathcal{C}}}$ is maximal monotone. However, we know from [3] that

$$\partial_{i_{\mathcal{C}}} = \mathcal{N}_{\mathcal{C}}(w) = \left\{ v \in \mathcal{X}^* : \langle y - w, v \rangle \leq 0, \quad \forall\ y \in \mathcal{C} \right\},$$

where $\mathcal{N}_{\mathcal{C}}$ is the normal cone for $\mathcal{C}$ at a point w.

Algorithm 5.1 Inertial Halpern-type Tseng's method with self adaptive stepsize for solving VIP and FPP

Initialization: Choose $x_0, x_1 \in \mathcal{C}$, $\lambda_1 > 0$ and $\mu \in (0,1)$. Set $k = 1$.
Step 0: Given x_k, x_{k-1}, choose τ_k such that $\tau_k \in [0, \bar{\tau}_k]$ where

$$\bar{\tau}_k = \begin{cases} \min\left\{\tau, \frac{\epsilon_k}{\delta_r^* \|\nabla g(x_k) - \nabla g(x_{k-1})\|}, \frac{\epsilon_k}{P_g(x_k, x_{k-1})}\right\}, & \text{if } \nabla g(x_k) - \nabla g(x_{k-1}) \neq 0, \\ \tau, & \text{otherwise} \end{cases} \tag{53}$$

and $\{\epsilon_k\}$ is a sequence of nonnegative numbers such that $\{\epsilon_k\} = \circ(\alpha_k)$, that (20) holds.
Step 1: Given the kth iterates x_k and x_{k-1}, compute

$$w_k = \nabla g^* \left[\nabla g(x_k) + \tau_k(\nabla g(x_k) - \nabla g(x_{k-1}))\right].$$

Step 2: Compute

$$y_{k,i} = P_{\mathcal{C}}^{g} \nabla g^* \left(\nabla g(w_k) - \lambda_k N_i w_k\right)$$

and

$$z_{k,i} = \nabla g^* \Big(\nabla g(y_{k,i}) - \lambda_k (N_i y_{k,i} - N_i w_k)\Big).$$

If $w_k = y_{k,i}$, then Stop: w_k is a solution of the problem (8). Otherwise, go to **Step 3**.
Step 3: Compute

$$u_k = \nabla g^* \left(\beta_{k,0} \nabla g(w_k) + \sum_{i=1}^{m} \beta_{k,i} \nabla g(T_i z_{k,i})\right)$$

with $\beta_{k,i} \in (0,1)$ such that $\sum\limits_{i=0}^{m} \beta_{k,i} = 1$.
Step 4: Compute

$$x_{k+1} = \nabla g^* \left(\alpha_k \nabla g(x_0) + (1 - \alpha_k) \nabla g(u_k)\right),$$

where

$$\lambda_{k+1} = \begin{cases} \min\left\{ \min\limits_{1 \le i \le m} \left\{ \frac{\mu\left(P_g(z_{k,i}, \; y_{k,i}) + P_g(y_{k,i}, w_k)\right)}{\langle z_{k,i} - y_{k,i}, \; N_i w_k - N_i y_{k,i} \rangle} \right\}, \lambda_k \right\}, \\ \qquad \text{if } \langle z_{k,i} - y_{k,i}, N_i w_k - N_i y_{k,i} \rangle \neq 0, \\ \lambda_k, \quad \text{otherwise.} \end{cases} \tag{54}$$

Set $k := k + 1$ and go to **Step 1**.

Theorem 5.2 *Let $\mathcal{X}$ be a real reflexive Banach space and $\mathcal{C}$ be a nonempty closed and convex subset of $\mathcal{X}$. Let $g : \mathcal{C} \to \mathbb{R}$ be a η-strongly convex, strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of $\mathcal{X}$, $N_i : \mathcal{X} \to \mathcal{X}^*$ is a finite family of monotone and L_i-Lipschitz continuous mapping, and $T_i : \mathcal{X} \to \mathcal{X}^*$ be finite family of relatively nonexpansive mappings. Assume that $\Gamma := \cap_{i=1}^m F(T_i) \bigcap \cap_{i=1}^m VI(\mathcal{C}, N_i) \neq \emptyset$ and Assumption 3.1 is satisfied. Let $\{x_k\}$ be the sequence generated by Algorithm 5.1. Then $\{x_k\}$ converges strongly to an element in Γ.*

Proof From [55], we have that $J_\lambda^{\partial i_{\mathcal{C}}} = P_{\mathcal{C}}^g$. Setting $M_i = \mathcal{N}_{\mathcal{C}}$ in Theorem 4.7, we have that $(N_i + \mathcal{N}_{\mathcal{C}})^{-1} 0 = VI(\mathcal{C}, N_i)$. Following similar line of argument as in Theorem 4.7, we obtain the desired result.

□

6 Numerical experiments

In this section, we present numerical experiments to illustrate the applicability of our methods. We discuss the numerical behavior of our proposed methods.
We perform all implementations using Matlab 2016 (b), installed on a personal computer with Intel(R) Core(TM) i5-2600 CPU@2.30GHz and 8.00 Gb-RAM running on Windows 10 operating system. We choose the following values for our parameters: $\mu = 0.6,\ \beta^{k,0} = \frac{2}{100k+1},\ \beta^{k,1} = \frac{5}{500k+7},\ \beta^{k,2} = 1 - \beta^{k,0} - \beta^{k,1},\ \alpha_k = \frac{1}{5k+1},\ \epsilon^k = \frac{1}{(2k+1)^3}$.

Example 6.1 Firstly, we consider an example in a Banach space ℓ_3 defined by

$$\ell_3 := \left\{ x = (x_1, x_2, \dots), x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x|^3 < \infty \right\}$$

with the induced norm $\|\cdot\|_{\ell_3} : \ell_3 \to [0, \infty)$ defined by $\|x\|_{\ell_3} = \left(\sum_{i=1}^{\infty} |x_i|^3 \right)^{\frac{1}{3}}$ and inner product $\langle \cdot, \cdot \rangle : \ell_3 \times \ell_3 \to \mathbb{R}$ defined by $\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i$ for arbitrary $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots) \in \ell_3$.
Let

$$\mathcal{C} := \left\{ x \in \mathcal{X} : \|x\|_{\ell_3} \leq 1 \right\}$$

and $N_i : \mathcal{C} \to (\ell_3)^*$ be defined by

$$N_i x = \frac{2x}{i} + (1, 1, 1, 0, 0, 0, 0, \dots)$$

and $M_i : \mathcal{C} \to (\ell_3)^*$ be defined by

$$M_i x = \frac{5x}{i}.$$

It is easy to see that N_i is a finite family family of monotone mappings and M_i is the finite family of maximal monotone mappings. By direct calculation we have for $s > 0$,

$$\begin{aligned} y_{k,i} &= (I + sM_i)^{-1}(w_k - sN_i w_k) \\ &= \frac{(1 - \frac{2}{i}s)w_k}{(1 + \frac{5}{i}s)} - \frac{s}{(1 + \frac{5}{i}s)}(1, 1, 1, 0, 0, 0, \dots). \end{aligned}$$

Let $g : \ell_3 \times \ell_3 \to \mathbb{R}$ be defined by $g(x) = \frac{1}{2}\|x\|_{\ell_3}^2,\ \ \forall\, x \in \ell_3$. Let $T_i : \ell_3 \to \ell_3$ be a finite family of mapping defined by $T_i x = \frac{x}{2i},\ \ i = 1, 2, \dots, m,\ \ x \in \ell_3$.
For this experiment, we test with different initial values of x_0 and x_1 as follows:
(**Case 1**:) $x_0 = (0.07, 0.02, 0.3, 0, 0, 0, 0, \dots)$, $x_1 = (0.21, 0.53, 0, 0, 0, 0, 0, \dots)$;
(**Case 2**): $x_0 = (-0.56, -0.34, 0, 0, 0, 0, 0, \dots)$, $x_1 = (-0.21, 0.12, 0, 0, 0, 0, 0, \dots)$;
(**Case 3**): $x_0 = (1.06, 2.4, 0, 0, 0, 0, 0, \dots)$, $x_1 = (-0.25, 1.3, 0, 0, 0, 0, 0, \dots)$;
(**Case 4**): $x_0 = (-0.06, -0.04, -0.02, 0, 0, 0, 0, \dots)$,
$x_1 = (-0.05, -0.03, -0.01, 0, 0, 0, 0, \dots)$.

We present the report of this example in Fig. 1.

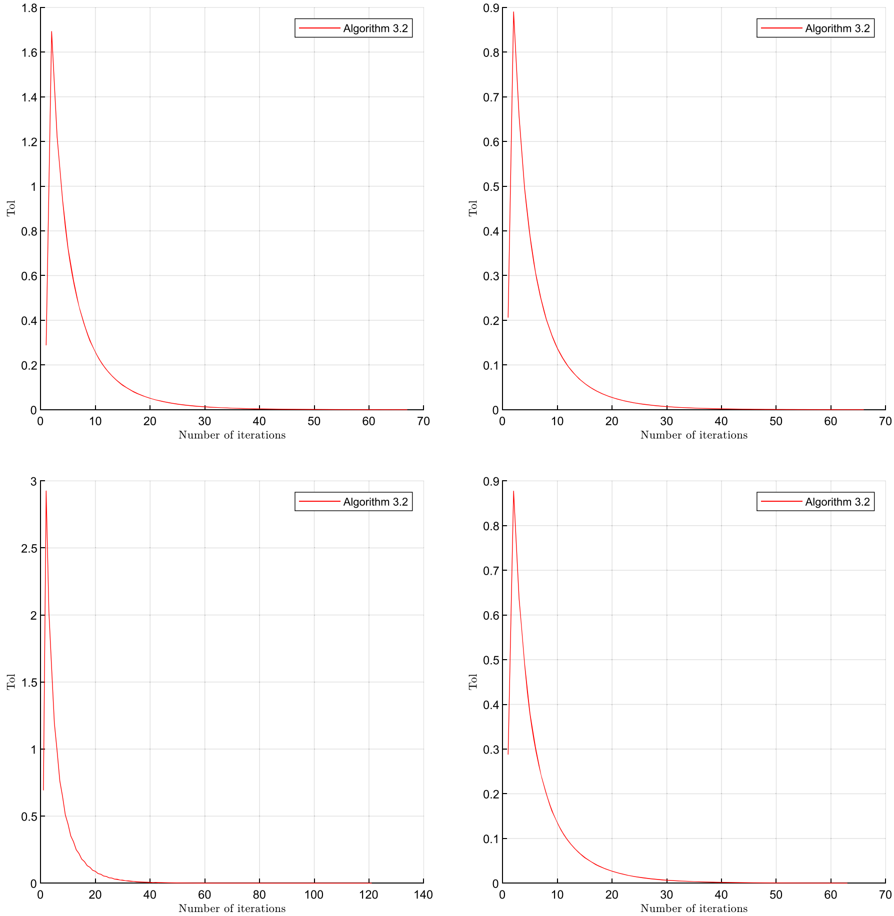

Fig. 1 Example 6.1. Top left: Case 1, Top right: Case 2, Bottom left: Case 3, Bottom right: Case 4

Example 6.2 Let $\mathcal{X} = L_2([0, 2\pi])$ and $\mathcal{C} = \left\{x \in L_2([0, 2\pi]) : \|x(t) - \sin t\|^2 \leq \sqrt{t}\right\}$ for some $t \in [0, 2\pi]$. To compare our method with the ones proposed by Shehu [47] and Manaka and Takahashi [26], we set $i = 1$. Let $M_1 = M = P_{\mathcal{C}}$ and the operator N be defined by $N(x) = x$. Let $T = T_1 : L_2([0, 2\pi]) \to L_2([0, 2\pi])$ be a finite family of mapping defined by $Tx = \frac{x}{2}$ with $x \in L_2([0, 2\pi])$. For this example, we choose $\alpha_k = \frac{1}{100k+3}$ and $\beta_{k,0} = (1 - \beta_{k,1}) = \beta_k = \frac{1}{2k+3}$. We also choose $\mu = 0.5$ and $\tau_k = \frac{1}{5}$. We illustrate this example some initial points as x_0 and x_1 and terminate the execution of the method with a stopping criterion $E_k = \|x_{k+1} - x_k\| < 10^{-4}$ or when $k <= 100$.

Case A: $x_0 = \sin(t)$ and $x_1 = 3\cos(3t)$;
Case B: $x_0 = \sin(t)$ and $x_1 = 10t$;
Case C: $x_0 = t^2 + 5t$ and $x_1 = \frac{50t^3+1}{124}$;
Case D: $x_0 = \cos(3t) + 5$ and $x_1 = sin(3\pi t)$.

The result of this experiment is given in Fig. 2.

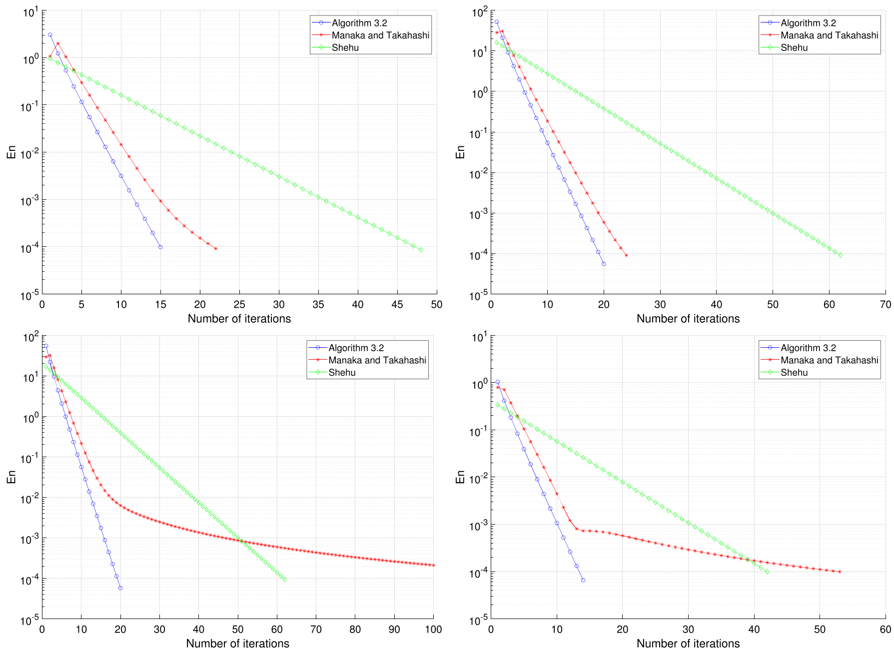

Fig. 2 Example 6.2. Top Left: Case A, Top Right: Case B, Bottom Left: Case C, Bottom Right: Case D

7 Conclusion

In this work, we proposed an inertial Halpern-type method for approximating a common solution to a finite family of monotone inclusion problem and fixed point problem for a finite family of relatively nonexpansive mappings. Our proposed method uses the self adaptive step-size which limits the reliance of our method on a scaling parameter. Under certain conditions, strong convergence result of our proposed method was obtained. Furthermore, we applied our result to approximate a common solution to a finite family of monotone inclusion problem and fixed point problem for a finite family of relatively nonexpansive mappings. Finally, we presented some numerical experiments to illustrate the convergence of our result.

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Data availability Not applicable.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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