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Symmetry algebra classification of scalar nth-order ordinary differential equations

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We obtain a complete classification of scalar nth-order ordinary differential equations for all subalgebras of vector fields in the real plane. While softwares like Maple can compute invariants of a given order, our results are for a general n. The $n = 1, 2, 3$ cases are well-known in the literature. Further, it is known that there are three types of nth-order equations depending upon the point symmetry algebra they possess, namely, first-order equations which admit an infinite dimensional Lie algebra of point symmetries, second-order equations possessing the maximum 8-point symmetries, and higher-order, $n \geq 3$, admitting the maximum $n+4$ dimensional point symmetry algebra. We show that scalar nth-order equations for $n > 5$ do not admit maximally an $n+3$ dimensional real Lie algebra of point symmetries. Moreover, we prove that for $n > 4$, equations can admit two types of $n+2$ dimensional real Lie algebra of point symmetries: one type resulting in nonlinear equations which are not linearizable via a point transformation and the second type yielding linearizable (via point transformation) equations. Furthermore, we present the types of maximal real n dimensional and higher than n-dimensional point symmetry algebras admissible for equations of order $n \geq 4$ and their canonical forms. The types of lower-dimensional point symmetry algebras which can be admitted are shown, and the equations are constructible as well. We state the relevant results in tabular form and in theorems.

KEYWORDS

invariants, Lie symmetry classification of ODEs, symmetry Lie algebras

MSC CLASSIFICATION

34C14, 34C20

1 | INTRODUCTION

We obtain a complete classification of scalar nth-order ordinary differential equations for all subalgebras of vector fields in the real plane. While softwares like Maple can compute invariants of a given order, our results are for a general n. The precise results are given in Section 7 as theorems (7.1 to 7.5).

Symmetry Lie algebras of scalar nth-order ordinary differential equations (ODEs) have been extensively studied over several years since the initial ground breaking works of Lie [1–3]. Lie [3], inter alia, provided all continuous groups of transformations in the complex plane. He emphasized that this can form the basis of classification as well as reduction of scalar nth-order ODEs which he implicitly performed.

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The classification of Lie algebras in terms of vector fields is essential in the algebraic study of scalar nth-order ODEs which possess infinitesimal symmetries, both symmetry classification and reduction algorithms. After the works of Lie, there have been much interest in this area. Lie algebras of vector fields in the real plane are completely classified in González-López et al. [4]. Recently, a proof of Lie's classification of solvable Lie algebras of vector fields in the plane is presented (see Azad et al. [5]). Of essence also is the contribution [6] which refers to contemporary works on Lie algebra realizations as well as equivalence of realizations.

In recent years, there have been much focus on the Lie algebra classification of ODEs in several works as we refer to henceforth. Equations of order 1 have infinite Lie point symmetries and are equivalent to each other via point transformation. For scalar higher-order ODEs, Lie [1] proved that the maximum dimension of the point symmetry Lie algebra for a scalar second-order ODE is eighth-dimensional and occurs for linear and linearizable by point transformation equations. Lie [2] obtained a complex classification of second-order ODEs in terms of their point symmetry Lie algebras. Mahomed and Leach [7] derived the real classification and showed that a second-order equation can admit 0, 1, 2, 3, or the maximum 8 dimension real point symmetry Lie algebra. The original Lie classification and the classification in the real domain [7] for second-order ODEs are, inter alia, compared in Ibragimov and Mahomed [8]. Algebraic linearizability criteria were initiated by Lie himself [1], who showed that such second-order equations possessing a Lie algebra of dimension 2 and of rank 1 are linearizable via point transformations. This falls under Types II and IV Lie canonical forms in Lie's classification. The Types I and III cases with focus on linearizability were achieved in Sarlet et al. [9] and Mahomed and Leach [10]. The reader is also referred to the survey [11].

In the study of scalar linear ODEs of order n, $n \geq 3$, Mahomed and Leach [12] (see also earlier studies [8, 11] as well as the contribution by Krause and Michel [13]) demonstrated that the point symmetry algebra can be $n+1, n+2$ or $n+4$. Thus, for $n \geq 3$, scalar linear ODEs are not necessarily equivalent to each other via point transformation. Moreover, for $n \geq 3$, there exist linear as well as nonlinearizable ODEs with $n+2$ and $n+3$ symmetry algebras [12]. It is important to remark that second-order ODEs are quite different to higher-order equations $n \geq 3$ as per the point symmetry algebras they admit. Apart from the maximum dimension of the point symmetry algebra being 2+6 for second-order ODEs and that for higher-order, $n \geq 3$, equations $n+4$ (see Lie [1, 2]), there are two more notable differences. Secondly, that all linear second-order ODEs are equivalent to the free particle equation, whereas a linear higher-order $n \geq 3$ ODE has three equivalence classes depending upon whether it has $n+1, n+2$ or the maximum number $n+4$ of point symmetries [12]. Thirdly, the complete or full algebra of point symmetries of a second-order ODE is a subalgebra of its maximum algebra $sl(3, R)$, whereas the full algebra of a higher-order $n \geq 3$ ODE is not necessarily a subalgebra of its maximum Lie point symmetry algebra [12].

The point symmetry Lie algebra classification of third-order ODEs as well as linearization by point transformation have been investigated in a number of relevant publications (see earlier studies [12–22]). Furthermore, integrability and reductions for third-order ODEs were investigated in Ibragimov and Nucci [23].

Scalar fourth-order ODEs were considered in recent works from the point of view of Lie point symmetry classification in terms of four-dimensional algebras, canonical forms, as well as integrability (see previous research [24–26]). A complete Lie point symmetry classification of fourth-order ODEs and algebraic linearization were also attempted [27]. Linearization criteria, by point transformation, for such ODEs, were found as well [28].

The classification of scalar nth-order ODEs which possess nontrivial irreducible contact Lie symmetry algebras was completed in the work of Wafo Soh et al [29]. For scalar third-order ODEs, the contact symmetry algebra is a subalgebra of the 10-dimensional contact symmetry algebra of $y''' = 0$ except for linear equations that admit four- and five-dimensional point symmetry algebras [29]. It is shown in this work that fourth-order scalar ODEs do not admit irreducible contact symmetry algebras. Further, it was proved in this paper [29] that there are three types of contact symmetry algebras (of dimensions 6, 7, and 10) admissible for nth-order ODEs for $n > 4$, up to local contact transformations.

The present work is a natural generalization of previous contributions on the classification of scalar second- and third-order ODEs. Here we present the complete classification of nth-order, $n \geq 4$, ODEs according to the Lie point symmetry algebras they admit. Previously, for order $n > 3$, only linearizability of nth-order ODEs was performed into three classes [12]. We explicitly present the new maximal n dimensional point symmetry Lie algebras admissible and their new representative or canonical equations as well as higher-dimensional point symmetry algebras and new canonical ODEs. We also have shown how one can obtain lower-dimensional point symmetry algebras and determine the corresponding ODEs. All these results are stated as main theorems in Section 7. Furthermore, scalar higher-order ODEs arise in various applications, including non-Newtonian fluids, when the governing high-order PDEs are reduced to ODEs by similarity transformations, as well as in Euler–Bernoulli beam theory. The algebraic analysis presented here enables researchers in these and allied areas to search for Lie symmetry reductions and solutions of the scalar higher-order ODEs.

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Section 2 deals with notation which are well-known in books (see, e.g., earlier studies [30–34]) which we have utilized in Shah et al. [27] as well as provide an overview of the methods used herein. In Sections 3 and 4, we classify all equations of order n which admit $n+1$ and $n+2$ dimensional Lie symmetry algebras, respectively, including those that are linearizable by point transformation. Then in Section 5, we find all the fundamental invariants of order n and $n-1$ for five and higher dimensions and present the representative ODEs as well. In Section 6, we discuss admissibility for $n+3$ dimensional algebras. Thereafter, in Section 7, we obtain a complete classification for $n \geq 4$ for the maximal n, lower as well as higher symmetry algebra cases. We finally present concluding remarks.

2 | NOTATION AND METHODOLOGY

We utilize the vector fields in the plane presented in the important work [4] for the real Lie symmetry classification of scalar nth-order ODEs performed in this seminal study.

By (m, n), we denote the real algebra realization, where m is the type of algebra in González-López et al. [4] and n is the dimension of the Lie algebra. One writes a general vector field or generator in the real plane as

$$X_i = \xi_i(x, y)\partial_x + \eta_i(x, y)\partial_y, \qquad i = 1, \ \dots \ , n.$$

Here $(x, y) \in \mathbb{R}^2$ and ∂_x denotes $\partial/\partial x$. Note also that n is the dimension of a Lie algebra of which X_is are the generators. Other notations will be introduced as they arise in the sequel.

Now let L be an m dimensional Lie subalgebra of vector fields in the real plane. To find an invariant equation of order n, we as usual consider the normal form of an nth-order ODE

$$y^{(n)} = H\left(x, y, y', y'', \ \dots \ , y^{(n-1)}\right). \tag{2.1}$$

For a generator X to be a symmetry of (2.1), the symmetry condition

$$X^{[n]}\left(y^{(n)} - H\right)|_{y^{(n)}=H} = 0 \tag{2.2}$$

must hold on the equation, where $X^{[n]}$ denotes the nth prolongation. If condition (2.2) is satisfied for every X_i, $i = 1, 2, \ \dots \ , m$, then the resulting nth-order equation is said to be invariant under L and L is called the symmetry Lie algebra of the equation.

Further, let L be an m dimensional Lie subalgebra of vector fields defined on a subspace $\mathbb{D} \subset \mathbb{R}^2$. Then the nth-order prolonged Lie algebra is defined on a subspace $\mathbb{D}^{(n)} \subset \mathbb{R}^{n+2}$. Suppose that r_0 is the rank of L in $\mathbb{D}$. Then r_n will be the rank of the prolonged L in $\mathbb{D}^{(n)}$. The rank here means the rank of the matrix whose rows are coefficients of the m generating vector fields of L.

If we denote d_n to be the number of differential invariants of order n, then we have

$$d_n = n + 2 - r_n, \ n \geq 0.$$

Example 2.1. Consider simply $X = \partial_x$. Here $n = 0$ and $r_0 = 1$. Thus, $d_0 = 0 + 2 - 1 = 1$, so we have one zeroth-order differential invariant which is $u = y$.

Example 2.2. Now we take $X_1 = \partial_x, X_2 = \partial_y$. Again, $n = 0, r_o = 2$, and $d_0 = 0+2-2 = 0$, and there is no zeroth-order invariant. Now for the first prolongations of X_1 and X_2, $n = 1$, $r_1 = 2$, and $d_1 = 1 + 2 - 2 = 1$. Thus, there is a first-order differential invariant $u = y'$.

We now mention invariant differentiation and its operator. If u, v are invariants, then by Lie's theorem, $D_x v/D_x u$ is also an invariant. This process is also called invariant differentiation (see, e.g., Ibragimov [31]). Recall that D_x is the total differentiation operator. We can write this as

$$\begin{aligned}\frac{D_x v}{D_x u} &= (D_x u)^{-1} D_x v \\ &= \lambda D_x v.\end{aligned}$$

We call $\lambda D_x = \mathcal{D}$ the invariant differentiation operator once we know λ.

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Suppose we have an unknown $\lambda(x, y, y', \ldots, y^{(n)})$. We require $\lambda D_x v$ to be invariant, and therefore, we need

$$X^{[n]}(\lambda D_x v) = 0. \tag{2.3}$$

If we refer to $X^{[n]}$ as X by ignoring the prolongation sign, then we can simply write

$$X = \overline{X} + \xi D_x,$$

where $\overline{X} = w\partial_y + D_x w\partial_{y'} + D_x^2 w\partial_{y''} + \ldots + D_x^n w\partial_{y^{(n)}}$ is called the canonical operator and $w = \eta - y'\xi$. Equation (2.3) then becomes

$$\begin{aligned}\left(\overline{X} + \xi D_x\right)(\lambda D_x v) = 0 &\Rightarrow \xi D_x(\lambda D_x v) + \overline{X}(\lambda) D_x v + \lambda \overline{X}(D_x v) = 0\\ &\Rightarrow \xi D_x \lambda D_x v + \xi \lambda D_x^2 v + (X\lambda - \xi D_x \lambda) D_x v + \lambda \overline{X}(D_x v) = 0\\ &\Rightarrow \xi D_x \lambda D_x v + \lambda \xi D_x^2 v + X(\lambda) D_x v - \xi D_x \lambda D_x v + \lambda \overline{X}(D_x v) = 0\\ &\Rightarrow \xi \lambda D_x^2 v + X(\lambda) D_x v + \lambda D_x(Xv - \xi D_x v) = 0\\ &\Rightarrow (X(\lambda) - \lambda D_x \xi) D_x v = 0.\end{aligned}$$

This yields the known result (see Ibragimov [31])

$$X(\lambda) = \lambda D_x \xi. \tag{2.4}$$

Hence, λ satisfies the nonhomogenous linear PDE (2.4). One only requires one nontrivial solution for λ which can be a constant as well.

Example 2.3. Let $X_1 = \partial_x$ and $X_2 = \partial_y$. We know that $u = y'$ is a first-order differential invariant of X_1 and X_2. Applying condition (2.4), we have

$$X_1 \lambda = 0, X_2 \lambda = 0.$$

This clearly shows that we can set $\lambda = 1$, and therefore, the invariant differentiation operator can be taken as $\mathcal{D} = (1)D_x = D_x$. Thus,

$$\mathcal{D}y' = D_x y' = y''$$

is a second-order differential invariant.

We briefly consider Lie determinants.

Definition 2.1. Consider an m-dimensional Lie subalgebra of vector fields whose generators are given as

$$X_k = \xi_k \partial_x + \eta_k \partial_y, k = 1, 2, ..., m.$$

Then the determinant of the following matrix M

$$\begin{pmatrix} \xi_1 & \eta_1 & \eta_1^{[1]} & \eta_1^{[2]} & \cdots & \eta_1^{[m-2]} \\ \xi_2 & \eta_2 & \eta_2^{[1]} & \eta_2^{[2]} & \cdots & \eta_2^{[m-2]} \\ \vdots & \vdots & \ddots & \vdots & & \\ \xi_m & \eta_m & \eta_m^{[1]} & \eta_m^{[2]} & \cdots & \eta_m^{[m-2]} \end{pmatrix}$$

is called the Lie determinant which corresponds to the m dimensional Lie algebra for $m \geq 2$. We denote the Lie determinant by Λ_L.

Lie proved that for an m dimensional Lie subalgebra of vector fields L, the Lie determinant gives rise to all the invariant equations of order $\leq m - 2$, [2]. Similarly, it can be noticed that the rank of the prolonged algebra is m, that is, maximal unless the Lie determinant vanishes in which case the rank of L diminishes. Here the algebra is prolonged up to order $m - 2$. These invariant equations are called the singular invariant equations of L. The fundamental differential invariants which are not singular must be of order $m - 1$ and m and the higher-order differential invariants can then be found from the fundamental invariants.

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Note here that once we find the $(m-1)$th-order differential invariant say ϕ and the invariant differentiation operator $\mathcal{D}=\lambda D_x$, we can determine the mth-order differential invariant as $\mathcal{D}\phi$.

We commence our classification for $n+1$ dimensions below, in Section 3, and then $n+2$ dimensions in Section 4. These include linearization. Thereafter, in Section 5, we find all the fundamental invariants for five and higher dimensions and provide the canonical scalar ODEs. Then we proceed to $n+3$ dimensions, Section 6, which have a paucity of three classes. Finally, in Section 7, we obtain the complete Lie symmetry classification of scalar ODEs of any higher-order $n>3$ and encapsulate the main results in the stated theorems.

3 | $n+1$ DIMENSIONAL ALGEBRAS AND ODEs

3.1 | Nonlinear equations

It is easy to observe that $(20,n+1), r=n-1, n>2: X_1=\partial/\partial y, X_2=x\partial/\partial y, X_3=\xi_1(x)\partial/\partial y, \ldots, X_{n-1}=\xi_{n-1}(x)\partial/\partial y$, where $\xi_i(x)$s are not linear in x and are linearly independent arbitrary functions, is not admitted by an equation as there can only be n solution symmetries. The linearizable case $(21,n+1)$ is considered in subsection 3.2. Also, $(22,n+1)$ and $(23,n+1)$ are not possessed as maximal Lie algebras. We consider the other $n+1$ dimensional algebras. These are as below.

$$(24,n+1), r=n-2, n\geq 3: X_1=\partial_x, X_2=\partial_y, X_3=x\partial_x+\alpha y\partial_y, X_4=x\partial_y, X_5=x^2\partial_y, \ldots, X_{n+1}=x^{n-2}\partial_y.$$

The generators, except X_3, imply that an nth-order equation of the form (2.1) admitting these generators must be of the form $y^{(n)}=H\left(y^{(n-1)}\right)$. The nth prolongation of X_3 is as follows: $x\partial_x+\alpha y\partial_y+(\alpha-1)y'\partial_{y'}+, \ldots, +(\alpha-n)y^{(n)}\partial_{y^{(n)}}$. By applying this to the resultant equation and solving, we find the general form of an nth-order equation admitting this algebra to be

$$y^{(n)}=K\left(y^{(n-1)}\right)^{\frac{\alpha-n}{\alpha-n+1}}, K\neq 0, \tag{3.1}$$

where $K\neq 0$ is an arbitrary constant and $\alpha\neq n-1$. For $K=0$ or $\alpha=n-1$, it is easy to see that the general form of such an equation admitting this algebra must be $y^{(n)}=0$, and we know from Lie that this simplest equation admits the maximal $n+4$ dimensional algebra of which such an algebra is a subalgebra.

$$(25,n+1), r=n-1, n\geq 2: X_1=\partial_x, X_2=\partial_y, X_3=x\partial_y, X_4=x^2\partial_y, \ldots, X_n=x^{n-2}\partial_y, X_{n+1}=x\partial_x+(ry+x^r)\partial_y.$$

The nth prolongation of X_{n+1} is as follows: $x\partial_x+(ry+x^r)\partial_y+(rx^{r-1}+(r-1)y')\partial_{y'}+\ldots+((r(r-1)\ldots(r-n+1))x^{r-n}+(r-n)y^{(n)})\partial_{y^{(n)}}$.

The invariant equation is

$$y^{(n)}=K\exp\left(\frac{-y^{(n-1)}}{(n-1)!}\right), K\neq 0. \tag{3.2}$$

$$(26,n+1), r=n-3, n\geq 4: X_1=\partial_x, X_2=\partial_y, X_3=x\partial_x, X_4=y\partial_y, X_5=x\partial_y, \ldots, X_{n+1}=x^{n-3}\partial_y.$$

Here the equation turns out to be (K is constant)

$$y^{(n)}=K\frac{\left(y^{(n-1)}\right)^2}{y^{(n-2)}}, K\neq 0, n/(n-1). \tag{3.3}$$

If $K=n/(n-1)$, then there is one more symmetry $X=x^2\partial x+(n-3)xy\partial y$ as discussed in Section 4 as the type $(28,n+2)$.

$$(27,n+1), r=n-3, n\geq 4: X_1=\partial_x, X_2=\partial_y, X_3=2x\partial_x+ry\partial_y, X_4=x^2\partial_x+rxy\partial_y, X_5=x\partial_y, \ldots, X_{n+1}=x^{n-3}\partial_y.$$

The nth prolongation of X_3 and X_4 is as follows: $X_3+\sum_{k=1}^{n}(r-2k)y^{(k)}\partial_{y^{(k)}}$ and $X_4+\sum_{k=1}^{n}(k(r-k+1)y^{(k-1)}+x(r-2k)y^{(k)})\partial_{y^{(k)}}$, respectively.

The equation is (K is constant)

$$y^{(n)}=\frac{n}{n-1}\frac{\left(y^{(n-1)}\right)^2}{y^{(n-2)}}+K\left(y^{(n-2)}\right)^{\frac{n+3}{n-1}}, K\neq 0. \tag{3.4}$$

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If $K = 0$, then there is one more symmetry $X = y\partial y$ as pursued in Section 4.

$$(28, n+1), r = n-4, n \geq 5 : X_1 = \partial_x, X_2 = \partial_y, X_3 = x\partial_x, X_4 = y\partial_y, X_5 = x^2\partial_x + (n-4)xy\partial_y, X_6 = x\partial_y, \ \dots \ , X_{n+1} = x^{n-4}\partial_y.$$

The nth-order invariant equation is

$$y^{(n)} = \left(y^{(n-2)}\right)^3 \left(y^{(n-3)}\right)^{-2} \left[\frac{n(3(n-2)K_1 - 2n + 2)}{(n-2)^2} + K((n-2)K_1 - (n-1))^{\frac{3}{2}}\right], \tag{3.5}$$

where $K_1 = y^{(n-3)}y^{(n-1)}\left(y^{(n-2)}\right)^{-2}$ and K constant.

3.2 | Linearizable equations

Here we consider linearization for higher-order $n \geq 3$ equations. We demonstrate how one obtains the linear form. The reader is referred to Mahomed and Leach [12] for details.

We have the algebra with realization $(21, n+1), r = n-1$: $X_1 = \partial_y, X_2 = y\partial_y, X_3 = \xi_1(x)\partial_y, ..., X_{n+1} = \xi_r(x)\partial_y$ which can be simplified by introducing coordinates: $\overline{x} = \xi_1(x)$, $\overline{y} = y$. Ignoring the bars, the generators of this algebra can be transformed to $X_1 = \partial_y, X_2 = y\partial_y, X_3 = x\partial_y, X_4 = \xi_2(x)\partial_y, ..., X_{n+1} = \xi_r(x)\partial_y$.

Thus, we consider

$$(21, n+1), r = n-2 : X_1 = \partial_y, X_2 = y\partial_y, X_3 = x\partial_y, X_4 = \xi_1(x)\partial_y, \ \dots \ , X_{n+1} = \xi_r(x)\partial_y.$$

We have the result from Mahomed and Leach [12] which we state as follows.

Proposition 3.1 (see Mahomed & Leach [12]). *$(21, n+1)$ is the symmetry algebra of the nth, $n \geq 3$, order linear homogenous equation*

$$y^{(n)} = \sum_{i=2}^{n-2} A_i(x)y^{(i+1)}, \tag{3.6}$$

such that each ξ_i for $i = 1, 2, \dots, n-2$, form independent solutions of this equation, that is, the ξ_is satisfy the system of homogenous equations

$$\xi_k^{(n)} = \sum_{i=1}^{n-2} A_i(x)\xi_k^{(i+1)}, \qquad k = 1, ..., n-2. \tag{3.7}$$

Note that if A_is are constant, then a further symmetry ∂_x arises and one gets the algebra $(23, n+2)$ as discussed in the next Section 4.2. Furthermore, if the A_is satisfy conditions (3.20) and (3.21) in Mahomed and Leach [12], then one has the maximal algebra $(28, n+4)$ admitted as also looked at in Section 4.2.

4 | $n+2$ DIMENSIONAL ALGEBRAS AND REPRESENTATIVE ODEs

4.1 | Nonlinear equations

For the $n+2$ dimensional algebras, $(20, n+2)$, $(21, n+2)$, and $(22, n+2)$ are clearly not admissible algebras. The linearization case $(23, n+2)$ is looked at in Section 4.2.

$$(24, n+2), r = n-1, n \geq 2 : X_1 = \partial_x, X_2 = \partial_y, X_3 = x\partial_x + \alpha y\partial_y, X_4 = x\partial_y, X_5 = x^2\partial_y, \ \dots \ , X_{n+2} = x^{n-1}\partial_y.$$

$$\Lambda_L = 1 \cdot 2! \cdot 3! ... (n-1)!(\alpha - n)y^{(n)},$$

where Λ_L here and in what follows is the Lie determinant of Definition 2.1.

$$(25, n+2), r = n, n \geq 1 : X_1 = \partial_x, X_2 = \partial_y, X_3 = x\partial_y, X_4 = x^2\partial_y, \ \dots \ , X_{n+1} = x^{n-1}\partial_y, X_{n+2} = x\partial_x + (ry + x^r)\partial_y.$$

$$\Lambda_L = 1 \cdot 2! \cdot 3! .... (n-2)! \cdot (n-1)! \cdot n!$$

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$$(26, n+2), r = n-2, n \geq 3 : X_1 = \partial_x, X_2 = \partial_y, X_3 = x\partial_x, X_4 = y\partial_y, X_5 = x\partial_y, \ \dots \ , X_{n+2} = x^{n-2}\partial_y.$$

$$\Lambda_L = 1 \cdot 2! \cdot 3! \ldots (n-3)! \cdot (n-2)! y^{(n-1)} y^{(n)}.$$

$$(27, n+2), r = n-2, n \geq 3 : X_1 = \partial_x, X_2 = \partial_y, X_3 = 2x\partial_x + ry\partial_y, X_4 = x^2\partial_x + rxy\partial_y, X_5 = x\partial_y, \ \dots \ , X_{n+2} = x^{n-2}\partial_y.$$

$$\Lambda_L = 1 \cdot 2! \cdot 3! \ldots (n-2)! \cdot n^2 \left(y^{(n-1)}\right)^2.$$

$$(28, n+2), r = n-3, n \geq 4 : X_1 = \partial_x, X_2 = x\partial_x, X_3 = y\partial_y, X_4 = x^2\partial_x + rxy\partial_y, X_5 = \partial_y, X_6 = x\partial_y, \ \dots \ , X_{n+2} = x^{n-3}\partial_y.$$

The equation is

$$y^{(n)} = \frac{n}{n-1} \frac{\left(y^{(n-1)}\right)^2}{y^{(n-2)}}. \tag{4.1}$$

Here we have only one case, the last, which constitute an invariant equation with maximal $n+2$ dimensional real symmetry algebra. The rest do not form an equation or are singular equations with the maximal $n+4$ dimension algebra.

Next, we need to again deal with linearization. We also mention $(28, n+4)$ as it results in linearization. This is stated at the end of Section 4.2.

4.2 | Linearizable equations for $n+2$ symmetries

We obtain the form for the reduced linear equation that is a consequence of linearizability (see Mahomed & Leach [12]).

$$(23, n+2), r = n : X_1 = \eta_1(x)\partial_y, X_2 = \eta_2(x)\partial_y, \ \dots \ , X_n = \eta_n(x)\partial_y, X_{n+1} = y\partial_y, X_{n+2} = \partial_x.$$

We sate the following proposition (see Mahomed and Leach [12] for details).

Proposition 4.1 (see Mahomed & Leach [12]). *The generators given in* $(23, n+2)$ *form a Lie symmetry algebra of the homogenous constant coefficient equation*

$$y^{(n)} = \sum_{i=0}^{n-1} A_i y^{(i)}, \tag{4.2}$$

if the η_i, $i = 1, \ \dots \ , n$, *form a fundamental set of solutions of the constant coefficient equation itself.*

We conclude by considering the following algebra which is maximal.

$$(28, n+4), r = n-1, n \geq 3 : X_1 = \partial_x, X_2 = x\partial_x, X_3 = y\partial_y, X_4 = x^2\partial_x + rxy\partial_y, X_5 = \partial_y, X_6 = x\partial_y, \ \dots \ , X_{n+4} = x^{n-1}\partial_y.$$

This algebra has as representative, the simplest, nth, $n \geq 3$, order ODE

$$y^{(n)} = 0, \tag{4.3}$$

which admits the maximal $n+4$ dimensional algebra as is well-known from the landmark works of Lie. Further, if (4.2) has A_is satisfying the conditions (3.20) and (3.21) as in Mahomed and Leach [12], then the maximal algebra $(28, n+4)$ is admitted by the equation.

5 | n DIMENSIONAL ALGEBRAS AND CORRESPONDING EQUATIONS

In this section, we determine all the fundamental invariants (invariants of order n and $n-1$) for five and higher dimensions as well as present the representative ODEs. Note that the algebras $(20, n)$ and $(21, n)$ are not admissible as maximal symmetry algebras.

$(5, 5)$: The generators of this algebra are $X_1 = \partial_x, X_2 = \partial_y, X_3 = x\partial_x - y\partial_y, X_4 = y\partial_x, X_5 = x\partial_y$.

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This is a five-dimensional algebra; hence, $r = 5$. We find the third prolongations of the generators of this algebra. Then the Lie determinant, which is the determinant of the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & x & 1 & 0 & 0 \\ x & -y & -2y' & -3y'' & -4y''' \\ y & 0 & -y'^2 & -3y'y'' & -\left(3y''^2 + 4y'y'''\right), \end{pmatrix}$$

is $\Lambda_L = 9y''^3$. This shows that $y'' = 0$ is the only equation of order ≤ 3 invariant under this algebra.

The fundamental differential invariants can be found in the usual way. For the given algebra, we already know a fourth-order invariant

$$\phi_1 = 3y^{(4)}y''^{\frac{-5}{3}} - 5y'''^2y''^{-8/3}.$$

Thus, a fourth-order ODE $\phi_1 = 3K$, K a constant, has (5,5).

To find a fifth-order differential invariant, we can solve the determining system of linear partial differential equations $X_i^{[5]}\left(y^{(5)} - H\right)|_{eq} = 0, \quad \text{for} i = 1, 2, \ldots, 5$. The solution easily results in the fifth-order equation

$$y^{(5)} = \frac{-40}{9}y'''^3y''^{-2} + 5y'''y^{(4)}y''^{-1} + y''^2H(\phi_1) \tag{5.1}$$

invariant under this algebra, where ϕ_1 is the fourth-order invariant of the algebra and H is an arbitrary function of its argument. We can thus write the fifth-order differential invariant to be

$$\phi_2 = \left(y''^2y^{(5)} + \frac{40}{9}y'''^3 - 5y''y'''y^{(4)}\right)/y''^4.$$

Here ϕ_1 and ϕ_2 are the fundamental differential invariants of this algebra. All higher-order differential invariants can be deduced by Lie's invariant differentiation $\phi_{n+1} = D_x(\phi_n)/D_x(\phi_{n-1})$ or by using the invariant derivative operator $\mathcal{D} = y''^{-1/3}D_x$ so that higher-order equations possessing this algebra are

$$\mathcal{D}^{n-4}\phi_1 = H\left(\phi_1, \ldots, \mathcal{D}^{n-5}\phi_1\right), n \geq 5, \tag{5.2}$$

where D_x is the total derivative operator.

$$(15,5) : X_1 = \partial_x, X_2 = \partial_y, X_3 = x\partial_x, X_4 = y\partial_y, X_5 = x^2\partial_x.$$

The generators X_1 to X_3 and X_5 give rise to the third-order invariant

$$K_1 = y'^{-3}y''' - \frac{3}{2}y'^{-4}y''^2,$$

and the invariant differentiation operator is $y'^{-1}D_x$.

Now writing X_4 in terms of K_1, $K_2 = y'^{-1}D_xK_1$ and $K_3 = y'^{-1}D_xK_2$ results in

$$\tilde{X}_4 = -2K_1\partial/\partial K_1 - 3K_2\partial/\partial K_2 - 4K_3\partial/\partial K_3.$$

This provides the invariants

$$\phi_1 = K_2K_1^{-3/2}, \ \phi_2 = y'^{-1}K_1^{-2}D_xK_2.$$

We can evaluate K_2 as

$$K_2 = y'^{-4}y^{(4)} - 6y'^{-5}y''y''' + 6y'^{-6}y''^3.$$

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Also, similarly D_xK_2. Thus, one has the fundamental invariants

$$\phi_1 = \frac{y'^2y^{(4)} - 6\left(y'y''y''' - y''^3\right)}{\left(y'y''' - \frac{3}{2}y''^2\right)^{\frac{3}{2}}}$$

and

$$\phi_2 = \frac{y'^3y^{(5)} - 10y'^2y''y^{(4)} - 6y'^2y'''^2 + 48y'y''^2y''' - 36y''^4}{\left(y'y''' - \frac{3}{2}y''^2\right)^2}$$

with the fifth-order equation as

$$\frac{y'^3y^{(5)} - 10y'^2y''y^{(4)} - 6y'^2y'''^2 + 48y'y''^2y''' - 36y''^4}{\left(y'y''' - \frac{3}{2}y''^2\right)^2} = H(\phi_1), \tag{5.3}$$

and with $\mathcal{D} = y'^{-1}K_1^{-1/2}D_x$, we can invoke (5.2) for higher-order invariant equations.

Note that a fourth-order ODE admitting (15,5) is $\phi_1 = K$, K constant.

(6, 6): $X_1 = \partial_x, X_2 = \partial_y, X_3 = x\partial_x, X_4 = y\partial_x, X_5 = x\partial_y, X_6 = y\partial_y$.

The generators X_1, X_2, X_5, and X_6 result in the invariants

$$K_1 = 3y''^{-1}y^{(4)} - 5y'''^2y''^{-2},$$

$$K_2 = 9y''^{-1}y^{(5)} - 45y''^{-2}y'''y^{(4)} + 40y''^{-3}y'''^3.$$

Writing X_4 in terms of K_1 and K_2, we find (up to scaling)

$$\tilde{X}_4 = 2K_1\partial_{K_1} + 3K_2\partial_{K_2}$$

which provides the fifth-order invariant of $(6, 6)$

$$\begin{aligned}\phi_1 &= K_1^{-3/2}K_2,\\ &= \frac{9y''^2y^{(5)} - 45y''y'''y^{(4)} + 40y'''^3}{\left(3y''y^{(4)} - 5y'''^2\right)^{\frac{3}{2}}}.\end{aligned}$$

The invariant differentiation operator is $\mathcal{D} = K_1^{-1/2}D_x$, and hence, one can derive the sixth-order invariant $K_1^{-3}\left(-\frac{3}{2}K_2D_xK_1 + K_1D_xK_2\right)$.

The fifth-order invariant equation is

$$K_1^{-3/2}K_2 = \frac{9y''^2y^{(5)} - 45y''y'''y^{(4)} + 40y'''^3}{(3y''y^{(4)} - 5y'''^2)^{\frac{3}{2}}} = K, \tag{5.4}$$

where K is constant, and with $\mathcal{D} = y''(3y''y^{(4)} - 5y'''^2)^{-1/2}D_x$, we have

$$\mathcal{D}^{n-5}\phi_1 = H(\phi_1, \ldots, \mathcal{D}^{n-6}\phi_1),\ n \geq 6, \tag{5.5}$$

for the sixth- and higher-order equations. Thus, a sixth-order invariant ODE is

$$K_1^{-3}\left(-\frac{3}{2}K_2D_xK_1 + K_1D_xK_2\right) = H\left(K_1^{-3/2}K_2\right). \tag{5.6}$$

Note that $K_1 = 0$ is the fourth-order singular invariant equation having this algebra.

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(16, 6): $X_1 = \partial_x, X_2 = \partial_y, X_3 = x\partial_x, X_4 = y\partial_y, X_5 = x^2\partial_x, X_6 = y^2\partial_y$.
Here the invariants, using X_1, X_2, X_4 and X_6, are

$$K_1 = y'''y^{-1} - \frac{3}{2}y'^{-2}y''^2,$$

as well as $K_2 = D_xK_1$ and $K_3 = D_xK_2$.

Utilizing X_5 in (K_1, K_2, K_3) space, we end up with, up to scaling,

$$\tilde{X}_5 = 2xK_1\partial_{K_1} + (3xK_2 + 2K_1)\partial_{K_2} + (4xK_3 + 5K_2)\partial_{K_3}.$$

Hence, a fifth-order invariant transpires as $\phi_1 = K_1^{-3}\left(5(D_xK_1)^2 - 4K_1D_x^2K_1\right)$ and invariant fifth-order ODE is (K constant)

$$K_1^{-3}\left(5(D_xK_1)^2 - 4K_1D_x^2K_1\right) = K. \tag{5.7}$$

The invariant differentiation operator is $\mathcal{D} = K_1^{-1/2}D_x$, and one can deduce the sixth-order invariant. Higher-order invariant equations are obtained as in (5.2). The sixth-order invariant ODE is therefore

$$K_1^{-1/2}D_x\left(K_1^{-3}\left(5(D_xK_1)^2 - 4K_1D_x^2K_1\right)\right) = H\left(K_1^{-3}\left(5(D_xK_1)^2 - 4K_1D_x^2K_1\right)\right). \tag{5.8}$$

We remark that $K_1 = 0$ is the third-order singular invariant equation admitting this algebra.
(7, 6): $X_1 = \partial_x, X_2 = \partial_y, X_3 = x\partial_x + y\partial_y, X_4 = y\partial_x - x\partial_y, X_5 = (x^2 - y^2)\partial_x + 2xy\partial_y, X_6 = 2xy\partial_x + (y^2 - x^2)\partial_y$.
The generators X_1 to X_4 give rise to the invariant

$$K_1 = \left(1 + y'^2\right)y''^{-2}y''' - 3y',$$

as well as the fourth- and fifth-order invariants

$$K_2 = y''^{-1}\left(1 + y'^2\right)D_xK_1,\ K_3 = y''^{-1}\left(1 + y'^2\right)D_xK_2.$$

Now X_5 in (K_1, K_2, K_3) space yields

$$\begin{aligned}\tilde{X}_5 = \left(1 + y'^2\right)y''^{-1}\big[&-4y'K_1\partial_{K_1} + \left(4y'K_1^2 - 6y'K_2 - 4K_1\right)\partial_{K_2}\\ &+ \left(14y'K_1K_2 + 4y'K_1 - 4y'K_1^3 - 8y'K_3 - 10K_2 + 8K_1^2\right)\partial_{K_3}\big].\end{aligned}$$

The invariant deduced is of order 5 (which is admitted by X_6 as well since $[X_4, X_5] = X_6$) given by $\phi_1 = K_1^{-3}\left(2K_1K_3 + 4K_1^2K_2 - \frac{5}{2}K_2^2 + 2K_1^4 - 2K_1^2\right)$. The fifth-order invariant ODE hence is

$$K_1^{-3}\left(2K_1K_3 + 4K_1^2K_2 - \frac{5}{2}K_2^2 + 2K_1^4 - 2K_1^2\right) = K. \tag{5.9}$$

The operator of invariant differentiation is $\mathcal{D} = \left(1 + y'^2\right)y''^{-1}K_1^{-1/2}D_x$ and the sixth- and higher-order equations are given by (5.5). One has the sixth-order invariant ODE

$$\begin{aligned}&\left(1 + y'^2\right)y''^{-1}K_1^{-1/2}D_x\left(K_1^{-3}\left(2K_1K_3 + 4K_1^2K_2 - \frac{5}{2}K_2^2 + 2K_1^4 - 2K_1^2\right)\right)\\ &= H(K_1^{-3}\left(2K_1K_3 + 4K_1^2K_2 - \frac{5}{2}K_2^2 + 2K_1^4 - 2K_1^2\right).\end{aligned} \tag{5.10}$$

Here, $K_1 = 0$ is the third-order singular invariant equation.
(8, 8): $X_1 = \partial_x, X_2 = \partial_y, X_3 = x\partial_y, X_4 = y\partial_y, X_5 = y\partial_x, X_6 = x\partial_x, X_7 = x^2\partial_x + xy\partial_y, X_8 = xy\partial_x + y^2\partial_y$.

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The operators X_1 to X_6 result in the fifth-order invariant of the algebra (6,6), namely,

$$J_1 = K_1^{-3/2} K_2,$$

where K_1 and K_2 are as in (6,6).

The operator X_7 in terms of J_1 and its invariant derivatives $J_2 = K_1^{-1/2} D_x J_1$ and $J_3 = K_1^{-1/2} D_x J_2$ turns out to be

$$\begin{aligned}\tilde{X}_7 = &-9K_1^{-1}y''^{-1}y'''J_1\partial/\partial_{J_1} + \left(-12K_1^{-1}J_2y''^{-1}y''' + 3J_1^2K_1^{-1}y''^{-1}y''' - 3J_1K_1^{-1/2}\right)\partial/\partial_{J_2}\\ &+ \left(10J_1J_2K_1^{-1}y''^{-1}y''' - J_1^3K_1^{-1}y''^{-1}y''' + J_1K_1^{-1}y''^{-1}y'''\right.\\ &\left.- 15J_3K_1^{-1}y''^{-1}y''' - 7J_2K_1^{-1/2} + \frac{3}{2}J_1^2K_1^{-1/2}\right)\partial/\partial_{J_3}\end{aligned}$$

which yields the invariant of order 7 of this algebra (8,8) (note $[X_5, X_7] = X_8$) as $\phi_1 = 12J_1^{-2/3}J_2 - 28J_1^{-8/3}J_2^2 + 24J_1^{-5/3}J_3 + J_1^{4/3} - 4J_1^{-2/3}$, and thus, the seventh-order invariant ODE is (K is a constant)

$$12J_1^{-2/3}J_2 - 28J_1^{-8/3}J_2^2 + 24J_1^{-5/3}J_3 + J_1^{4/3} - 4J_1^{-2/3} = K. \tag{5.11}$$

The invariant differential operator of this algebra is $\mathcal{D} = K_2^{-1/3} D_x$.

Therefore, eighth-order ODEs admitting this eight dimensional algebra is given by

$$K_2^{-1/3} D_x \left(12J_1^{-2/3}J_2 - 28J_1^{-8/3}J_2^2 + 24J_1^{-5/3}J_3 + J_1^{4/3} - 4J_1^{-2/3}\right) = H(\phi_1) \tag{5.12}$$

and higher-order by invariant differentiation as $\mathcal{D}^{n-7}\phi_1 = H\left(\phi_1, \dots, \mathcal{D}^{n-8}\phi_1\right)$, $n \geq 8$.

For this algebra, the singular invariant equation is $K_2 = 0$ which is fifth-order. This is also given in Section 6.

$$(22, n) : r = n - 1, n \geq 2 : X_1 = \partial_x, X_2 = \eta_1(x)\partial_y, \dots, X_n = \eta_{n-1}(x)\partial_y.$$

The nth-order equation is

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' = H\left(y^{(n-1)} + a_1 y^{(n-2)} + \cdots + a_{n-1}y\right), \tag{5.13}$$

where the a_is are constants and the η_is satisfy the linear $(n-1)$th-order equation

$$\eta_i^{(n-1)} + a_1\eta_i^{(n-2)} + \cdots + a_{n-1}\eta_i = 0, i = 1, \dots, n-1.$$

$(23, n) : r = n - 2, n \geq 3$: $X_1 = \partial_x, X_2 = y\partial_y, X_3 = \eta_1(x)\partial_y, \dots, X_n = \eta_{n-2}(x)\partial_y$.

The nth-order equation that admits this algebra is

$$\frac{D_x^2(y^{(n-2)} + a_1 y^{(n-3)} + \cdots + a_{n-2}y)}{y^{(n-2)} + a_1 y^{(n-3)} + \cdots + a_{n-2}y} = H\left(D_x \ln|y^{(n-2)} + a_1 y^{(n-3)} + \cdots + a_{n-2}y|\right), \tag{5.14}$$

where the a_is are constants and the η_is satisfy the linear $(n-2)$th-order equation

$$\eta_i^{(n-2)} + a_1\eta_i^{(n-3)} + \cdots + a_{n-2}\eta_i = 0, i = 1, \dots, n-2.$$

$(24, n) : r = n - 3, n \geq 4$: $X_1 = \partial_x, X_2 = \partial_y, X_3 = x\partial_x + \alpha y\partial_y, X_4 = x\partial_y, \dots, X_n = x^{n-3}\partial_y$.

Fundamental invariants, if $\alpha \neq n-1$, are

$$\phi_1 = \left(y^{(n-1)}\right)^{\alpha-n+2}\left(y^{(n-2)}\right)^{-(\alpha-n+1)}, \qquad \phi_2 = y^{(n)}\left(y^{(n-1)}\right)^{-\left(\frac{\alpha-n}{\alpha-n+1}\right)},$$

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and for $\alpha = n-1$

$$\phi_1 = y^{(n-1)}, \qquad \phi_2 = y^{(n)}y^{(n-2)}.$$

Thus, the invariant equation is

$$y^{(n)}\left(y^{(n-1)}\right)^{-\left(\frac{\alpha-n}{\alpha-n+1}\right)} = H\left(\left(y^{(n-1)}\right)^{\alpha-n+2}\left(y^{(n-2)}\right)^{-(\alpha-n+1)}\right), \alpha \neq n-1. \tag{5.15}$$

If $\alpha = n-1$, the invariant equation simply is

$$y^{(n)} = \left(y^{(n-2)}\right)^{-1} H\left(y^{(n-1)}\right). \tag{5.16}$$

$(25, n)$: $r = n-2$, $n \geq 3$: $X_1 = \partial_x, X_2 = \partial_y, X_3 = x\partial_y, X_4 = x^2\partial_y, \ldots, X_{n-1} = x^{n-3}\partial_y, X_n = x\partial_x + \left((n-2)y + x^{(n-2)}\right)\partial_y$. The fundamental differential invariants are

$$\phi_1 = y^{(n-1)} \exp \frac{y^{(n-2)}}{(n-2)!}, \ \phi_2 = y^{(n)} \exp \frac{2y^{(n-2)}}{(n-2)!}$$

and invariant equation

$$y^{(n)} \exp \frac{2y^{(n-2)}}{(n-2)!} = H\left(y^{(n-1)} \exp \frac{y^{(n-2)}}{(n-2)!}\right). \tag{5.17}$$

$(26, n)$: $r = n-4$, $n \geq 5$: $X_1 = \partial_x, X_2 = \partial_y, X_3 = x\partial_x, X_4 = y\partial_y, X_5 = x\partial_y, \cdots, X_n = x^{n-4}\partial_y$.
The fundamental differential invariants:

$$\phi_1 = \frac{y^{(n-3)}y^{(n-1)}}{\left(y^{(n-2)}\right)^2}, \ \phi_2 = \frac{\left(y^{(n-3)}\right)^2 y^{(n)}}{\left(y^{(n-2)}\right)^3}$$

and invariant ODE

$$\frac{\left(y^{(n-3)}\right)^2 y^{(n)}}{\left(y^{(n-2)}\right)^3} = H\left(\frac{y^{(n-3)}y^{(n-1)}}{\left(y^{(n-2)}\right)^2}\right). \tag{5.18}$$

$(27, n)$: $r = n-4$, $n \geq 5$: $X_1 = \partial_x, X_2 = \partial_y, X_3 = 2x\partial_x + ry\partial_y, X_4 = x^2\partial_x + rxy\partial_y, X_5 = x\partial_y, \cdots, X_n = x^{n-4}\partial_y$. The fundamental invariants are as follows:

$$\phi_1 = y^{(n-1)}\left(y^{(n-3)}\right)^{\frac{n+2}{2-n}} - \frac{n-1}{n-2}\left(y^{(n-2)}\right)^2\left(y^{(n-3)}\right)^{\frac{-2n}{n-2}}$$
$$\phi_2 = y^{(n)}\left(y^{(n-3)}\right)^{\frac{n+4}{2-n}} + \frac{2n(n-1)}{(n-2)^2}\left(y^{(n-2)}\right)^3\left(y^{(n-3)}\right)^{\frac{3n}{2-n}} + \frac{3n}{2-n}y^{(n-1)}y^{(n-2)}\left(y^{(n-3)}\right)^{\frac{2(n+1)}{2-n}}.$$

The invariant equation here is

$$y^{(n)}\left(y^{(n-3)}\right)^{\frac{n+4}{2-n}} + \frac{2n(n-1)}{(n-2)^2}\left(y^{(n-2)}\right)^3\left(y^{(n-3)}\right)^{\frac{3n}{2-n}} + \frac{3n}{2-n}y^{(n-1)}y^{(n-2)}\left(y^{(n-3)}\right)^{\frac{2(n+1)}{2-n}} = H(\phi_1). \tag{5.19}$$

$(28, n)$: $r = n-5$, $n \geq 6$: $X_1 = \partial_x, X_2 = \partial_y, X_3 = x\partial_y, \ldots, X_{n-3} = x^{n-5}\partial_y, X_{n-2} = x\partial_x, X_{n-1} = y\partial_y, X_n = x^2\partial_x + rxy\partial_y$.
The fundamental invariants are as follows:

$$\phi_1 = \frac{(n-3)^2K_2 - 3(n-1)(n-3)K_1 + 2(n-1)(n-2)}{(n-3)^2((n-3)K_1 - (n-2))^{\frac{3}{2}}},$$
$$\phi_2 = \frac{(n-3)^3K_3 - 4n(n-3)^2K_2 + 6n(n-1)(n-3)K_1 - 3n(n-1(n-2)}{(n-3)^3((n-3)K_1 - (n-2))^2},$$

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where

$$K_i = \frac{y^{(n-3+i)}\left(y^{(n-4)}\right)^i}{\left(y^{(n-3)}\right)^{i+1}},\ i = 1, 2, 3.$$

The invariant ODE is

$$\frac{(n-3)^3K_3 - 4n(n-3)^2K_2 + 6n(n-1)(n-3)K_1 - 3n(n-1(n-2)}{(n-3)^3((n-3)K_1 - (n-2))^2} = H(\phi_1). \tag{5.20}$$

6 | $n+3$ DIMENSIONAL ALGEBRAS AND EQUATIONS

Scalar first-order equations admit infinite number of point symmetries. A second-order equation does not admit a five-dimensional symmetry algebra, as is well-known. The only third-order equations that admit six-dimensional algebras are

$$y''' = \frac{3y''^2}{2y'} \text{ and } y''' = \frac{3y'y''^2}{1+y'^2}, \tag{6.1}$$

where the algebras are $\mathfrak{sl}(2,R) \oplus \mathfrak{sl}(2,R)$, and $\mathfrak{so}(3,1)$, respectively. These are the algebras (16,6) and (7,6) as stated in the previous Section 5. These are also known from the initial seminal works of Lie.

In Shah et al. [27], it was shown that a fourth-order equation does not admit a maximal seven-dimensional algebra.

For fifth-order equations, the only equation which admits an eighth-dimensional algebra is

$$y^{(5)} = \frac{5y'''y^{(4)}}{y''} - \frac{40y'''^3}{9y''^2}, \tag{6.2}$$

whose symmetry algebra is $\mathfrak{sl}(3,R)$ (also referred to as (8,8)) as stated in the previous section.

We check all the possible algebras of dimension $n+3$ for $n \geq 5$. Since an nth-order linear equation cannot have more than n independent solutions, the algebras $(m, n+3)$ for $m = 20, \ldots, 25$ are not admitted. We discuss the remaining possible algebras:

$(26, n+3)$: $X_1 = \partial_x, X_2 = \partial_y, X_3 = x\partial_x, X_4 = y\partial_y, X_5 = x\partial_y, \cdots, X_{n+3} = x^{n-1}\partial_y$.

The generators X_1, X_2, and $X_5, \cdots, X_{n+3}$ imply that the equation must be of the form $y^{(n)} = K$. The X_4 then implies that K must vanish. Then X_3 is automatically admitted by this equation. However, the maximal symmetry algebra of this equation is $n+4$. In the same way, it can be shown easily that the algebras $(27, n+3)$ and $(28, n+3)$ are admitted by an nth-order equation if and only if the equation is equivalent to $y^{(n)} = 0$ and hence these algebras are admitted by the equation but are not maximal. We state the general result on admission of $(n+3)$ dimensional algebra in the next section.

In the following section, we review the classification for $n = 4$ and discuss $n \geq 5$ in some detail. Then we present the results on how one can obtain a complete classification for $n \geq 4$. We state relevant theorems of our main results.

7 | CLASSIFICATION OF HIGHER, $n \geq 4$, ORDER ODEs

In this section, we review the main aspects on the classification of the fourth-order ODEs and then discuss $n \geq 5$. We thus consider the Lie algebraic classification of ODEs of any high order.

In general, we can classify scalar ODEs into three subclasses as follows:

Subclass (1): nth-order equations admitting $n+1$, $n+2$, $n+3$ and the maximal $n+4$ dimensional algebras. These are higher symmetries admitted by a scalar ODE.

Subclass (2): nth-order equations admitting n dimensional algebras. For this, we have found the fundamental invariants of n dimensional algebras which are of order $n-1$ and n.

Subclass (3): nth-order equations admitting algebras of dimension lower than n.

All these subclasses (1), (2), and (3) are completed herein for $n \geq 4$ and the algebra of dimension $n+4$ is already a well-known algebra since the initial work of Lie with corresponding equation $y^{(n)} = 0$ that possesses this maximal dimension Lie algebra. This is the algebra (28,$n+4$) as stated in Section 4.2. Also, for all subclasses, we provide the

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TABLE 1 The nth-order equations admitting one- and two-dimensional algebras for $n \geq 3$.

Algebra	Generators	nth-order invariant ODE
$(9, 1)$	∂_x	$y^{(n)} = H\left(y, y', y'', \ldots, y^{(n-1)}\right)$
$(10, 2)$	$\partial_x, x\partial_x$	$y^{(n)} = y'^n H\left(y, y''y'^{-2}, \ldots, y^{(n-1)}y'^{1-n}\right)$
$(20, 2)$	$\partial_y, x\partial_y$	$y^{(n)} = H\left(x, y'', \ldots y^{(n-1)}\right)$
$(22, 2)$	∂_x, ∂_y	$y^{(n)} = H\left(y', y'', \ldots, y^{(n-1)}\right)$
$A_{2,1}$	$\partial_x, x\partial_x + y\partial_y$	$y^{(n)} = y^{1-n}H\left(y', y''y, y'''y^2, \ldots, y^{(n-1)}y^{n-2}\right)$

Abbreviation: ODE, ordinary differential equation.

TABLE 2 Scalar fourth-order equations admitting real three-dimensional algebras.

Algebra	λ	Invariants ϕ_i	Fourth-order ODE
$(1, 3)$	$a^{-1/2}\exp(-\alpha \arctan y')$	$\phi_2 = y''a^{-3/2}\exp(-\alpha \arctan y')$	$\phi_4 = H(\phi_2, \phi_3)$
$\alpha \geq 0$	$a = 1 + y'^2$	$\phi_3 = y''^{-2}y'''a - 3y'$	
$(2, 3)$	$xa^{-1/2}$	$\phi_2 = a^{-3/2}(xy'' - ay')$	$xa^{-1/2}D_x\phi_3 = H(\phi_2, \phi_3)$
	$a = 1 + y'^2$	$\phi_3 = a^{-3}x^2(ay''' - 3y'y''^2)$	
$(3, 3)$	$ra^{-1/2}, a = 1 + y'^2$	$\phi_2 = a^{-3/2}(ry'' + 2a(y - xy'))$	$ra^{-1/2}D_x\phi_3$
	$r = 1 + x^2 + y^2$	$\phi_3 = r^2a^{-3}(ay''' - 3y'y''^2)$	$= H(\phi_2, \phi_3)$
$(11,3)$	y'^{-1}	$\phi_0 = y$	$y^{(4)}y'^{-4} + 6y'^{-6}y''^3$
		$\phi_3 = y'^{-4}(2y'y''' - 3y''^2)$	$-6y'^{-5}y''y''' = H(\phi_0, \phi_3)$
$(12,3)$	$y'^{1/(\alpha-1)}$	$\phi_2 = y''y'^{\frac{2-\alpha}{\alpha-1}}$	$y^{(4)} = y'''^{\frac{\alpha-4}{\alpha-3}}H(\phi_2, \phi_3)$
$\lvert\alpha\rvert < 1$	$\alpha \neq 0$	$\phi_3 = y'''y''^{\frac{3-\alpha}{\alpha-2}}$	
$\alpha = 1$	y''^{-1}	$\phi_1 = y', \phi_3 = y''^{-2}y'''$	$y^{(4)} = y'''^{3/2}H(\phi_1, \phi_3)$
$(17,3)$	$xb^{-1/2}$	$\phi_2 = (xy'' - by')b^{-3/2}$	$xb^{-1/2}D_x\phi_3 = H(\phi_2, \phi_3)$
	$b = 1 - y'^2$	$\phi_3 = x^2b^{-3}(by''' + 3y'y''^2)$	
$(18,3)$	y^2	$\phi_2 = y^3y''$	$y^7y^{(4)} + 8y'y^6y'''$
		$\phi_3 = y^5y''' + 3y'y''y^4$	$+12y^5y'^2y'' = H(\phi_2, \phi_3)$
$(20,3)$	1	$\phi_0 = x, \phi_3 = y''' - y''\frac{\xi'''}{\xi''}$	$y^{(4)} = y'''\frac{\xi^{(4)}}{\xi'''} + H(\phi_0, \phi_3)$
$(21,3)$	1	$\phi_0 = x, \phi_3 = y'''/y''$	$y^{(4)} = y'''H(\phi_0, \phi_3)$
$(22,3)$	1	$\phi_2 = E(y) = y'' + a_1y' + a_2y$	$D_x\phi_3 = H(\phi_2, \phi_3)$
		$\phi_3 = D_x\phi_2$, η_i satisfy	
		$E(\eta_i) = 0, i = 1, 2, a_i$ const	
$(23,3)$	1	$\phi_2 = y''/y', \phi_3 = y'''/y''$	$y^{(4)} = y'''H(\phi_2, \phi_3)$
$(25,3)$	$\exp y'$	$\phi_2 = y''\exp y', \phi_3 = y''^{-2}y'''$	$y^{(4)} = y'''^{3/2}H(\phi_2, \phi_3)$

Abbreviation: ODE, ordinary differential equation.

procedure to obtain the representative ODEs. These are presented in tabular forms and the main results in theorems at the end of this section.

For algebras of dimensions 1 and 2, the algebras and canonical equation are given in Table 1.

The nth-order equations admitting one- and two-dimensional algebras for $n \geq 3$ are easy to determine as in Table 1.

For any three-dimensional algebra, we already know the invariants of orders 2, 3, and 4 from previous works [7, 8, 15, 16] as well as in Shah et al. [27]. Then by invariant differentiation, we can find higher-order invariants up to the required order. Lie's recursive formula may also be used.

We provide the types of 3D algebras that are possessed by fourth-order ODEs. They are as follows:

$(1, 3)$: $\alpha \geq 0, \partial_x, \partial_y, (\alpha x + y)\partial_x + (\alpha y - x)\partial_y,$
$(2, 3)$: $\partial_y, x\partial_x + y\partial_y, 2xyp_x + (y^2 - x^2)\partial_y,$
$(3, 3)$: $y\partial_x - x\partial_y, (1 + x^2 - y^2)\partial_x + 2xy\partial_y, 2xy\partial_x + (1 + y^2 - x^2)\partial_y,$
$(11, 3)$: $\partial_x, x\partial_x, x^2\partial_x,$
$(12, 3)$: $0 < \alpha \leq 1, \partial_x, \partial_y, x\partial_x + \alpha y\partial_y,$
$(17, 3)$: $\partial_y, x\partial_x + y\partial_y, 2xy\partial_x + (x^2 + y^2)\partial_y,$
$(18, 3)$: $\partial_y, x\partial_x + y\partial_y, 2xyp_x + y^2\partial_y,$
$(20, 3)$: $\partial_y, x\partial_y, \xi(x)\partial_y,$ ξ is not linear in x and arbitrary,
$(21, 3)$: $\partial_y, y\partial_y, x\partial_y,$
$(22, 3)$: $\partial_x, \eta_1(x)\partial_y, \eta_2(x)\partial_y, \eta_i$s are linearly independent arbitrary functions,
$(23, 3)$: $\partial_x, y\partial_y, \partial_y,$
$(25, 3)$: $\partial_x, \partial_y, x\partial_x + (x + y)\partial_y,$

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TABLE 3 The fourth-order equations admitting four-dimensional algebras.

Algebra	λ	Invariant/s	Fourth-order ODE
(4, 4)	$\left(1+y'^2\right)y''^{-1}$	$y''^{-2}y'''\left(1+y'^2\right)-3y'$	$y^{(4)}=y'''^3\left(1+y'^2\right)^{-2}(15y'^2+10\phi y'+H(\phi))$
(13, 4)	$y'y''^{-1}$	$y'y''^{-2}y'''$	$y^{(4)}=y'^{-2}y'''^3H(\phi),H\neq 0$
(14, 4)	y'^{-1}	$y'''y'^{-3}-\frac{3}{2}y''^2y'^{-4}$	$y^{(4)}=y'^4H(\phi)+6y'^2y''\phi+3y'^{-2}y''^3$
(19, 4)	$y^{1/2}y''^{-1/2}$	$y^{1/2}y''^{-3/2}y'''+3y'y^{-1/2}y''^{-1/2}$	$y^{(4)}=\frac{4}{3}y''^{-1}y'''^2+y^{-1}y''^2H(\phi),H\neq 0$
(22,4)	1	$E(y)\equiv y'''+a_1y''+a_2y'+a_3y$ η_i solves $E(\eta_i)=0$ for a_is const.	$D_xE(y)=H(E(y)),H\neq 0$
(23,4)	1	$D_x\ln\lvert E(y)\rvert\equiv D_x\ln\lvert y''+a_1y'+a_2y\rvert$ η_i satisfy $E(\eta_i)=0$ for a_i const.	$D_x^2E(y)=E(y)H(\phi),$ $H\neq 0$
(24,4)	$y''^{1/(\alpha-2)}$, y'' y'''^{-1}	$y'''^{\alpha-2}y''^{3-\alpha},\alpha\neq 3$ $y''',\alpha=3$ $\phi=y'',y^{(4)}/y'''^2,\alpha=2$	$y^{(4)}=y'''^{\frac{\alpha-4}{\alpha-3}}H(\phi),H\neq 0$ $y^{(4)}=y'''^{-1}H(\phi),H\neq 0$ $y^{(4)}=y'''^2H(\phi),H\neq 0$
(25,4)	$\exp(y''/2)$	$y'''\exp(y''/2)$	$y^{(4)}=\exp(-y'')H(\phi),H\neq 0$

Abbreviation: ODE, ordinary differential equation.

TABLE 4 The fourth-order invariant ODEs that correspond to their five- and higher-dimensional symmetry algebras.

Higher algebra	Fourth-order ODE
(5, 5)	$y^{(4)}=Ky''^{\frac{5}{3}}+\frac{5}{3}y'''^2y''^{-1},K\neq 0,$
(15, 5)	$\frac{y'^2y^{(4)}-6(y'y''y'''-y''^3)}{\left(y'y'''-\frac{3}{2}y''^2\right)^{\frac{3}{2}}}=K,$
(21,5)	$y^{(4)}=\sum_{i=1}^{2}A_i(x)y^{(i+1)},\ \xi_k$ satisfy $\xi_k^{(4)}=\sum_{i=1}^{2}A_i(x)\xi_k^{(i+1)},k=1,2,$
(24, 5)	$y^{(4)}=Ky'''^{\frac{\alpha-4}{\alpha-3}},\alpha\neq 3\ K\neq 0,$
(25, 5)	$y^{(4)}=K\exp(-y'''/6),K\neq 0,$
(26, 5)	$y^{(4)}=Ky'''^2y''^{-1},K\neq 0,4/3,5/3,$
(27, 5)	$y^{(4)}=Ky''^{7/3}+\frac{4}{3}y'''^2y''^{-1},K\neq 0$
(6, 6)	$y^{(4)}=\frac{5}{3}y'''^2y''^{-1}$
(23, 6)	$y^{(4)}=\sum_{i=0}^{3}A_iy^{(i)},\ A_i$ const., η_i satisfy same equation
(28, 6)	$y^{(4)}=\frac{4}{3}y'''^2y''^{-1}$
(28, 8)	$y^{(4)}=0$

Note: Note that K is a constant and for (21,5) as well as (23, 6); the A_is do not satisfy the maximal symmetry conditions of Mahomed and Leach [12] as mentioned in Sections 3.2 and 4.2.
Abbreviation: ODE, ordinary differential equation.

The algebras (2,3), (17,3), and (18,3) above are equivalent to the realizations as given in González-López et al. [4] via the transformations $\bar{x}=y,\bar{y}=x,\bar{x}=x+y,\bar{y}=y-x$, and $\bar{x}=y,\bar{y}=|x|^{1/2}$, respectively.

We compactly review scalar fourth-order ODEs which admit real three-dimensional Lie algebras as alluded to above. The reader is also referred to Shah et al. [27] for further details. This is important for higher than fourth-order symmetry classification of scalar ODEs. We provide the main results in Table 2.

Remark. In Table 2, the ODEs for four types of algebras are as follows.

$$
\begin{aligned}
&(1,3):\ \phi_4=y^{(4)}a^2y''^{-3}-2a^2y''^{-4}y'''^2+2ay'y''^{-2}y'''-3a,\\
&(2,3):\ xa^{-1/2}D_x\phi_3=x^2a^{-5/2}\left(2y'''+xy^{(4)}-10xy'y''y'''a^{-1}-6y'y''^2a^{-1}-3xy''^3a^{-1}+18xy'^2y''^3a^{-2}\right),\\
&(3,3):\ ra^{-1/2}D_x\phi_3=r^2a^{-5/2}\left(4(x+yy')y'''-10ra^{-1}y'y''y'''+ry^{(4)}-3ra^{-1}y''^3-12a^{-1}y'y''^2(x+yy')+18ra^{-2}y'^2y''^3\right),\\
&(17,3):xb^{-1/2}D_x\phi_3=x^2b^{-5/2}\left(2y'''+xy^{(4)}+10xb^{-1}y'y''y'''+6y'y''^2b^{-1}+3xy''^3b^{-1}+18xb^{-2}y'^2y''^3\right)
\end{aligned}
$$

The a, b, and r are as in Table 2. There are 12 types of real three-dimensional Lie algebras admitted by scalar fourth-order ODEs as listed in Table 2. One can utilize the operator of invariant differentiation $\mathcal{D}=\lambda D_x$ for fourth- and higher-order ODEs possessing three-dimensional algebras. Therefore, nth-order, $n\geq 4$, scalar ODEs with three-dimensional symmetry algebras have the form $\mathcal{D}^{n-3}\phi_3=H(\phi,\phi_3,\ \dots\ ,\mathcal{D}^{n-4}\phi_3),\ n\geq 4$, where ϕ is ϕ_0, ϕ_1, or ϕ_2 as given in Table 2. Hence, for

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$n = 4$, we have the fourth-order ODE given by $\mathcal{D}\phi_3 = H(\phi, \phi_3)$ as in Table 2 with two arguments in H. In general, for higher-order ODEs, there are $n - 2$ arguments.

We present the list of four-dimensional algebra types as used in Shah et al. [27]. These are

(4, 4): ∂_x, ∂_y, $x\partial_x + y\partial_y$, $y\partial_x - x\partial_y$,
(13, 4): ∂_x, ∂_y, $x\partial_x$, $y\partial_y$,
(14, 4): ∂_x, ∂_y, $x\partial_x$, $x^2\partial_x$,
(19, 4): ∂_x, $x\partial_x$, $y\partial_y$, $x^2\partial_x + xy\partial_y$,

Also, the algebras (22,4), (23,4), (24,4), and (25,4) are easily found by setting $n = 4$ in the general n dimensional cases discussed in Section 5.

We now briefly review fourth-order equations that possess real four-dimensional algebras in a compact way than discussed in Shah et al. [27]. This is presented in Table 3.

Remark. There are multiple cases of (22,4) in Shah et al. [27]. These are considered as a single case in Table 3. We observe that there are eight types of four-dimensional algebras admissible by fourth-order ODEs, and also, there are seven types of five-dimensional algebras as listed in Table 4, namely, (5,5), (15,5) (Section 5), (21,5) (linear homogeneous ODE, Proposition 3.1), (24,5), (25,5), (26,5), and (27,5) (these four are given in Section 3.1 for $n = 4$). There are as well three types of six-dimensional algebras (as in Table 4) which are (6,6) (singular invariant ODE stated in Section 5), (23,6) (linear constant coefficient ODE, Proposition 4.1), and (28,6) (Section 4.1). Further, there is the maximum eighth-dimensional case (28,8) (simplest linear ODE, Section 4.2 end). Thus, altogether, there are 11 types of higher than four-dimensional algebra types. All these 11 types are mentioned in the previous Sections 3–5 as indicated and summarized in Table 4.

Note that invariant differentiation of the invariants in Table 3 will give fifth- and higher-order invariants. It is also important to include the algebras (20,m) and (21,m) for higher than mth-order ODEs. We now focus on these two algebra types. The algebras (20,3) and (21,3) are stated in Table 2 and included here.

For (20,m): $\partial/\partial y, x\partial/\partial y, \xi_1\partial/\partial y, \ldots, \xi_{m-2}\partial/\partial y$, $3 \leq m < n$, one has $\lambda = 1$ and the invariants $\phi = x$ and $\phi_m = y^{(m)} - a_1 y^{(m-1)} - \ldots - a_{m-2}y''$, where ξ_i, $i = 1, \ldots, m-2$ satisfy $\xi_i^{(m)} - a_1\xi_i^{(m-1)} - \ldots - a_{m-2}\xi_i'' = 0$. The nth-order ODE with (20,m) is

$$\mathcal{D}^{(n-m)}\phi_m = H\left(\phi, \phi_m, \ldots, \mathcal{D}^{(n-m-1)}\phi_m\right) \tag{7.1}$$

with $\mathcal{D} = D_x$.

In the case (21,m): $\partial/\partial y, y\partial/\partial y, x\partial/\partial y, \xi_1\partial/\partial y, \ldots, \xi_{m-3}\partial/\partial y$, $3 \leq m < n$, we have $\lambda = 1$ and invariants $\phi = x$, $\phi_m = D_x \ln|\phi_{m-1}|$, where $\phi_{m-1} = y^{(m-1)} - a_1 y^{(m-2)} - \ldots - a_{m-3}y''$ and $\xi_i, i = 1, \ldots, m-3$ satisfy $\xi_i^{(m-1)} - a_1\xi_i^{(m-2)} - \ldots - a_{m-3}\xi_i'' = 0$. The nth-order ODE with (21,m) is of the form (7.1) with appropriate ϕ, ϕ_m and $\mathcal{D} = D_x$.

In the case of higher than fourth-order ODEs admitting four-dimensional symmetry algebras, we resort to Table 3 and also have (20, 4) and (21, 4). For (20,4), the invariants as above are $\phi = x$, $\phi_4 \equiv E(y) = y^{(4)} - a_1 y''' - a_2 y''$, where ξ_i, $i = 1, 2$ satisfy $E(\xi_i) = 0$ for a_i constants. One thus obtains the fifth-order ODE $D_x\phi_4 = H(\phi, \phi_4)$ which has (20,4). In the case (21,4), the invariants are $\phi = x$ and $\phi_4 = D_x \ln|y''' - y''\xi'''/\xi''|$ and one has again the form $D_x\phi_4 = H(\phi, \phi_4)$. For higher than 4, nth-order ODEs with four-dimensional algebra, we have the form (7.1) with $m = 4$. Therefore, for higher nth, $n \geq 5$, order ODEs with four-dimensional algebras, there are 10 types. These follow from Table 3 and inclusion of (20,4) and (21,4). One deduces these by invariant differentiation as (H has $n - 3$ arguments) $\mathcal{D}^{n-4}\phi_4 = H(\phi, \phi_4, \ldots, \mathcal{D}^{n-5}\phi_4)$, $n \geq 5$.

We now consider fifth-order invariant equations admitting one or more dimensional algebras. The one- (single type) and two-symmetry algebras (four types) are known as in Table 1. In the case of three-dimensional algebras, there are 12 types as a consequence of Table 2 and the ensuing discussions. Thus, a fifth-order ODE possessing 3D algebras has the form $\mathcal{D}^2\phi_3 = H(\phi, \phi_3, \mathcal{D}\phi_3)$, where $\mathcal{D} = \lambda D_x$ for each of the λs as stated in Table 2.

For four-dimensional algebras, there are 10 types as in Table 3 and immediate deliberations which included (20,4) and (21,4). Hence, a fifth-order ODE which admits a 4D algebra is of the form $\mathcal{D}\phi_4 = H(\phi, \phi_4)$ for each λ in Table 3. One can use each of the λs in Table 3 to find $\mathcal{D}\phi_4$ for all the types.

Now we focus on algebras of dimension 5 and greater. These are discussed and follow from our deliberations in the previous sections as well.

The fifth-order ODEs admitting a five-dimensional algebra are presented in Table 5. Higher symmetries admitted by such ODEs are listed in Table 6.

The fifth-order ODEs for algebras (5,5) and (15,5) are Equations (5.1) and (5.3). Those for the types (22,5) to (27,5) are the seven Equations (5.13) to (5.19).

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TABLE 5 The fifth-order invariant ODEs that correspond to their five-dimensional symmetry algebras.

Algebra	Fifth-order ODE
(5, 5)	$y^{(5)} = \frac{-40}{9} y'''^3 y''^{-2} + 5y''' y^{(4)} y''^{-1} + y''^2 H(\phi), H \neq 0,$ $\phi = 3y^{(4)} y''^{\frac{-5}{3}} - 5y'''^2 y''^{-8/3},$
(15, 5)	$\frac{y'^3 y^{(5)} - 10y'^2 y'' y^{(4)} - 6y'^2 y'''^2 + 48y' y''^2 y''' - 36y''^4}{\left(y' y''' - \frac{3}{2} y''^2\right)^2} = H(\phi)$ $\phi = \frac{y'^2 y^{(4)} - 6(y' y'' y''' - y''^3)}{\left(y' y''' - \frac{3}{2} y''^2\right)^{\frac{3}{2}}},$
(22,5)	$D_x(E_4(y)) = H(E_4(y)), H \neq 0,$ η_i is solution of $E_4(\eta_i) = 0, i = 1, \ldots, 4$ for a_i consts.
(23,5)	$D_x^2(E_3(y)) = E_3(y) H(D_x \ln \lvert E_3(y)\rvert), H \neq 0,$ η_i is solution of $E_3(\eta_i) = 0, i = 1, \ldots, 3$ for a_i consts.
(24, 5)	$y^{(5)} = \left(y^{(4)}\right)^{\frac{\alpha-5}{\alpha-4}} H\left(\left(y^{(4)}\right)^{\alpha-3}, \left(y^{(3)}\right)^{4-\alpha}\right), \alpha \neq 4, H \neq 0,$ $y^{(5)} = (y''')^{-1} H\left(y^{(4)}\right), \alpha = 4, H \neq 0,$
(25, 5)	$y^{(5)} = \exp\left(-\frac{2y'''}{3!}\right) H(y^{(4)} \exp(y'''/3!)), H \neq 0,$
(26, 5)	$y^{(5)} y''^2 y'''^{-3} = H(y'' y'''^{-2} y^{(4)}), H \neq 0,$
(27, 5)	$y^{(5)} y''^{-3} + \frac{40}{9} y'''^3 y''^{-5} - 5y^{(4)} y''' y''^{-4} = H(\phi), H \neq 0,$ $\phi = y^{(4)} y''^{-7/3} - \frac{4}{3} y'''^2 y''^{-10/3}, H \neq C\phi^{3/2}, C$ const.

Note: Note that $E_n(z) \equiv z^{(n)} + a_1 z^{(n-1)} + \ldots + a_n z$ and in (27,5) if $H = C\phi^{3/2}$ for C constant, then the algebra (28, 6) occurs.

Abbreviation: ODE, ordinary differential equation.

TABLE 6 The fifth-order invariant ODEs that correspond to their higher symmetry algebras.

Higher algebra	Fifth-order equation
(6, 6)	$\frac{9y''^2 y^{(5)} - 45y'' y''' y^{(4)} + 40y'''^3}{(3y'' y^{(4)} - 5y'''^2)^{\frac{3}{2}}} = K, K \neq 0,$
(16,6)	$K_1^{-3}(5(D_x K_1)^2 - 4K_1 D_x^2 K_1) = K,$ $K_1 = y^{-1} y''' - \frac{3}{2} y'^{-2} y''^2,$
(7,6)	$K_1^{-3}\left(2K_1 K_3 + 4K_1^2 K_2 - \frac{5}{2} K_2^2 + 2K_1^4 - 2K_1^2\right) = K,$ $K_1 = \left(1 + y'^2\right) y''^{-2} y''' - 3y', K_2 = y''^{-1}\left(1 + y'^2\right) D_x K_1,$ $K_3 = y''^{-1}\left(1 + y'^2\right) D_x K_2,$
(21,6)	$y^{(5)} = \sum_{i=1}^3 A_i(x) y^{(i+1)}$, ξ_k satisfy $\xi_k^{(5)} = \sum_{i=1}^3 A_i(x) \xi_k^{(i+1)}, k = 1, 2, 3,$
(24,6)	$y^{(5)} = K\left(y^{(4)}\right)^{\frac{\alpha-5}{\alpha-4}}, \alpha \neq 4, K \neq 0,$
(25, 6)	$y^{(5)} = K \exp(-y^{(4)}/4!), K \neq 0,$
(26, 6)	$y^{(5)} = K y'''^{-1}\left(y^{(4)}\right)^2, K \neq 0, 5/4,$
(27, 6)	$y^{(5)} = K\left(y^{(3)}\right)^2 + \frac{5}{4}\left(y^{(4)}\right)^2 y'''^{-1}, K \neq 0,$
(28, 6)	$\left(\frac{y^{(5)} y''^2}{y'''^3} - \frac{5(9K_1 - 8)}{9}\right)/(3K_1 - 4)^{3/2} = K, K \neq 0,$ $K_1 = y^{(4)} y'' y'''^{-2}$
(23,7)	$y^{(5)} = \sum_{i=0}^4 A_i y^{(i)}$, A_i const., η_i satisfy the same equation.
(28, 7)	$y^{(5)} = \frac{5}{4}\left(y^{(4)}\right)^2 y'''^{-1}$
(8,8)	$9y''^2 y^{(5)} - 45y'' y''' y^{(4)} + 40y'''^3 = 0$
(28, 9)	$y^{(5)} = 0$

Note: Note that K is a constant and in (21,6) (A_is not constant) and (23,7), the A_is do not satisfy the maximal symmetry condition as in Mahomed and Leach [12]; see Sections 3.2 and 4.2.

Abbreviation: ODE, ordinary differential equation.

The fifth-order equations for (6,6), (16,6), and (7,6) are (5.4), (5.7), and (5.9). Also for (24,6) to (28,6), the ODEs are (3.1) to (3.5) for $n = 5$. The linear case for (21,6) follows from Proposition 3.1 by setting $n = 5$. Moreover, the ODE for (8,8) is the singular invariant equation $K_2 = 0$; (23,7) is the linear ODE (4.2); for (28,7), the ODE is (4.1) and for (28,9), one has the simplest Equation (4.3).

One can proceed to classify sixth-order ODEs as follows. For one and two dimensions, these are in Table 1. As for three-dimensional algebras, there are 12 types and equations are of the form $\mathcal{D}^3\phi_3 = H(\phi, \phi_3, \mathcal{D}\phi_3, \mathcal{D}^2\phi_3)$. In the case of four dimensions, we have 10 types with ODEs $\mathcal{D}^2\phi_4 = H(\phi, \phi_4, \mathcal{D}\phi_4)$. Five dimensions result in 10 types with equations $\mathcal{D}\phi_5 = H(\phi, \phi_5)$ (Table 5 and (20,5), (21,5)). Note here that $\lambda = 1/D_x\phi$ in $\mathcal{D} = \lambda D_x$.

The algebras for dimension 6 and ODEs are as follows. One has the algebra (6,6) with equation given by (5.6), (16,6) with Equation (5.8), and (7,6) with representative ODE (5.10). Each of these three types is presented in Section 5.

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Also, one can easily deduce the algebras and forms for the equations for the six-dimensional algebras (22,6) to (28,6) from Section 5 by setting $n = 6$ for each type. The equations are (5.13) to (5.20) (there are two equations for (24.6)). Altogether, there are hence 10 maximal algebra types for the sixth-order ODEs admitting six-dimensional algebras. Moreover, for higher symmetries $n+1, n+2, n+4$ for $n = 6$, there are nine types given by (21,7) (linear ODE in Proposition 3.1 with $n = 6$), (24,7) to (28,7) (Section 3.1 for $n = 6$ with corresponding Equations (3.1) to (3.5)) as well as (23,8) (linear Equation (4.2)), (28,8) ((4.1) for $n = 6$), and (28,10) simplest linear ODE with maximal $n+4 = 10$ symmetries, as in (4.3).

Likewise, if we investigate the seventh-order ODEs, one has the same number of types of lower-dimensional algebras as for the sixth-order ODEs, namely, one-, two-, three-, four-, and five-dimensional algebras. For six dimensions, we have 12 types.

The number of types of maximal algebras of dimension 7 are seven given by (22,7) to (28,7) in Section 5 with ODEs (5.13) to (5.20) with $n = 7$. Moreover, for higher symmetries, we end up with 10 types, namely, (21,8) (Proposition 3.1), (24,8) to (28,8) with Equations (3.1) to (3.5) with $n = 7$, (23,9) (linear, Proposition 4.1), (28,9) having (4.1), (28,11) with maximal symmetry equation, and (8,8) given by the ODE (5.11), namely,

$$12J_1^{-2/3}J_2 - 28J_1^{-8/3}J_2^2 + 24J_1^{-5/3}J_3 + J_1^{4/3} - 4J_1^{-2/3} = K,$$

where K is constant.

Now proceeding to the eighth-order ODEs, we have similar one to six lower dimension types as for the seventh-order ODEs. For dimension 7, we have nine types. One also has the eight maximal algebra types (22,8) to (28,8) with corresponding ODEs (5.13) to (5.20) for $n = 8$ as well as (8,8) given by Equation (5.12).

There are also nine types of higher symmetries, namely, (21,9) (linear), (24,9) to (28,9), (23,10) (linear), (28,10), and (28,12) (linear maximal case).

We have the generalization to order 9 and greater as follows:

For $n \geq 9$, there are the lower-dimension types similar to the eighth-order ODEs with dimension 8 as lower-dimension algebra having 10 types. Moreover, seven maximal types for nth-order ODEs admit n symmetries. These are $(22, n)$ to $(28, n)$ with representative ODEs (5.13) to (5.20) with two cases for (24,n).

Also, for higher symmetries for nth-order equations, $n \geq 9$, there are nine types, namely, $(21, n+1)$ (Proposition 3.1, linear), $(24, n+1)$ to $(28, n+1)$ with ODEs (3.1) to (3.5), $(23, n+2)$ (Proposition 4.1, linear), $(28, n+2)$ with ODE (4.1) and $(28, n+4)$ which is the maximal algebra with simplest ODE $y^{(n)} = 0$.

We now state the following important results on maximal n-dimensional and higher-dimensional symmetry algebras admitted by a scalar higher-order ODEs for $n \geq 4$. These are consequences of our prior discussions.

Theorem 7.1. *The number of types of maximal n dimensional Lie symmetry algebras admitted by scalar nth-order ODEs for $n \geq 4$ are as follows:*

8 for $n = 4, 5$ with algebras and canonical ODEs given in Tables 3 and 5,

10 for $n = 6$ with algebras and representative equations

Algebra	*sixth-order equation*
(6,6)	$K_1^{-3}\left(-\frac{3}{2}K_2D_xK_1 + K_1D_xK_2\right) = H(K_1^{-3/2}K_2)$
	$K_1 = 3y''^{-1}y^{(4)} - 5y'''^2y''^{-2}, K_2 = 9y''^{-1}y^{(5)} - 45y''^{-2}y'''y^{(4)} + 40y''^{-3}y'''^3$
(16,6)	$K_1^{-1/2}D_x\phi = H(\phi),\ \phi = K_1^{-3}(5(D_xK_1)^2 - 4K_1D_x^2K_1)$
	$K_1 = y'''y^{-1} - \frac{3}{2}y'^{-2}y''^2$
(7,6)	$\left(1+y'^2\right)y''^{-1}K_1^{-1/2}D_x\phi = H(\phi),\ \phi = K_1^{-3}\left(2K_1K_3 + 4K_1^2K_2 - \frac{5}{2}K_2^2 + 2K_1^4 - 2K_1^2\right),$
	$K_1 = \left(1+y'^2\right)y''^{-2}y''' - 3y', K_2 = y''^{-1}\left(1+y'^2\right)D_xK_1, K_3 = y''^{-1}\left(1+y'^2\right)D_xK_2$
(22,6)	$D_xE_5(y) = H(E_5(y)), E_5(\eta_i) = 0, i = 1, \ldots, 5, H \neq 0,$
(23,6)	$D_x^2E_4(y) = E_4(y)H(D_x \ln \lvert E_4(y)\rvert), E_4(\eta_i) = 0, i = 1, \ldots, 4, H \neq 0$
(24,6)	$y^{(6)} = (y^{(5)})^{\frac{\alpha-6}{\alpha-4}}H\left((y^{(5)})^{\alpha-4}\left(y^{(4)}\right)^{5-\alpha}\right),\ \alpha \neq 5, H \neq 0,$
	$y^{(6)} = \left(y^{(4)}\right)^{-1}H(y^{(5)}), \alpha = 5, H \neq 0,$
(25,6)	$y^{(6)} = \exp(-y^{(4)}/12)H\left(y^{(5)}\exp(y^{(4)}/24)\right), H \neq 0,$
(26,6)	$y^{(6)} = \left(y^{(4)}\right)^3\left(y^{(3)}\right)^{-2}H\left(y^{(5)}y^{(3)}\left(y^{(4)}\right)^{-2}\right), H \neq 0,$
(27,6)	$y^{(6)} + \frac{15}{4}\left(y^{(4)}\right)^3\left(y^{(3)}\right)^{-2} - \frac{9}{2}(y^{(5)}y^{(4)}\left(y^{(3)}\right)^{-1} = \left(y^{(3)}\right)^{5/2}H(\phi)$
	$\phi = y^{(5)}\left(y^{(3)}\right)^{-2} - \frac{5}{4}\left(y^{(4)}\right)^2\left(y^{(3)}\right)^{-3}, H \neq C\phi^{3/2},$
(28,6)	$K_3 - 8K_2 + 20K_1 - \frac{40}{3} = (3K_1 - 4)^2H\left(\frac{9K_2 - 45K_1 + 40}{9(3K_1-4)^{3/2}}\right)$
	$K_1 = y^{(4)}y^{(2)}\left(y^{(3)}\right)^{-2}, K_2 = y^{(5)}\left(y^{(2)}\right)^2\left(y^{(3)}\right)^{-3}, K_3 = y^{(6)}\left(y^{(2)}\right)^3\left(y^{(3)}\right)^{-4}$

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7 for $n = 7$ with algebras and canonical ODEs given below in (7.2) by letting $n = 7$
8 for $n = 8$ with algebras and equations

$$K_2^{-1/3} D_x \left(12J_1^{-2/3} J_2 - 28J_1^{-8/3} J_2^2 + 24J_1^{-5/3} J_3 + J_1^{4/3} - 4J_1^{-2/3} \right) = H(\phi),$$

where $\phi = 12J_1^{-2/3} J_2 - 28J_1^{-8/3} J_2^2 + 24J_1^{-5/3} J_3 + J_1^{4/3} - 4J_1^{-2/3}$
as well as by letting $n = 8$ in the seven cases (7.2).
7 for $n \geq 9$ with algebras and representative equations

Algebra	*nth-order Equation*
(22,n)	$D_x E_{n-1}(y) = H(E_{n-1}(y)),\ E_{n-1}(\eta_i) = 0, i = 1, \dots, n-1, H \neq 0,$
(23,n)	$D_x^2 E_{n-2}(y) = E_{n-2}(y) H(D_x \ln \lvert E_{n-2}(y) \rvert),\ E_{n-2}(\eta_i) = 0, i = 1, \dots, n-2, H \neq 0$
(24,n)	$y^{(n)} = \left(y^{(n-1)}\right)^{\frac{\alpha-n}{\alpha-n+1}} H\left(\left(y^{(n-1)}\right)^{\alpha-n+2}\left(y^{(n-2)}\right)^{-(\alpha-n+1)}\right),\ \alpha \neq n-1, H \neq 0,$
	$y^{(n)} = \left(y^{(n-2)}\right)^{-1} H\left(y^{(n-1)}\right),\ \alpha = n-1, H \neq 0,$
(25,n)	$y^{(n)} \exp \frac{2y^{(n-2)}}{(n-2)!} = H\left(y^{(n-1)} \exp \frac{y^{(n-2)}}{(n-2)!}\right),\ H \neq 0$
(26,n)	$\frac{\left(y^{(n-3)}\right)^2 y^{(n)}}{\left(y^{(n-2)}\right)^3} = H\left(\frac{y^{(n-3)} y^{(n-1)}}{\left(y^{(n-2)}\right)^2}\right),\ H \neq 0,$
(27,n)	$y^{(n)} + \frac{2n(n-1)}{(n-2)^2}\left(y^{(n-2)}\right)^3\left(y^{(n-3)}\right)^{-2} + \frac{3n}{2-n} y^{(n-1)} y^{(n-2)}\left(y^{(n-3)}\right)^{-1}$
	$= \left(y^{(n-3)}\right)^{\frac{n+4}{n-2}} H(\phi),\ H \neq C\phi^{3/2},$
	$\phi = y^{(n-1)}\left(y^{(n-3)}\right)^{\frac{n+2}{2-n}} - \frac{n-1}{n-2}\left(y^{(n-2)}\right)^2\left(y^{(n-3)}\right)^{\frac{-2n}{n-2}}$
(28,n)	$K_3 - \frac{4n}{n-3} K_2 + \frac{6n(n-1)}{(n-3)^2} K_1 - \frac{3n(n-1)(n-2)}{(n-3)^3}$
	$= ((n-3)K_1 - (n-2))^2) H(\phi)$
	$\phi = \frac{(n-3)^2 K_2 - 3(n-1)(n-3)K_1 + 2(n-1)(n-2)}{(n-3)^2((n-3)K_1 - (n-2))^{\frac{3}{2}}}$
	$K_i = \frac{y^{(n-3+i)}\left(y^{(n-4)}\right)^i}{\left(y^{(n-3)}\right)^{i+1}},\ i = 1, 2, 3,$ (7.2)
Remark:	*Note that in the above, $E_n(z) \equiv z^{(n)} + a_1 z^{(n-1)} + \dots + a_n z$.*
	If $H = C\phi^{3/2}$, then (28,7) arises for the sixth-order ODEs or (28,n + 1) for (27, n).

Theorem 7.2. *The number of types of maximal $n + 1$ dimensional Lie symmetry algebras admitted by scalar nth-order ODEs for $n \geq 4$ are as follows:*
7 for $n = 4$, with algebras and canonical ODEs given in Table 4
9 for $n = 5$, with algebras and representative equations in Table 6
6 for $n = 6$, with algebras and equations given in (7.3) by setting $n = 6$
7 for $n = 7$, with algebras and canonical ODEs
(8,8), $12J_1^{-2/3} J_2 - 28J_1^{-8/3} J_2^2 + 24J_1^{-5/3} J_3 + J_1^{4/3} - 4J_1^{-2/3} = K$, where $J_1 = K_1^{-3/2} K_2$, with K_1 and K_2 as in (6,6), $J_2 = K_1^{-1/2} D_x J_1$ and $J_3 = K_1^{-1/2} D_x J_2$
and the 6 algebras in (7.3) by setting $n = 7$.
6 for $n \geq 8$, with algebras and representative ODEs

Higher algebra	*nth-order equation*
(24, n + 1)	$y^{(n)} = K\left(y^{(n-1)}\right)^{\frac{\alpha-n}{\alpha-n+1}},\ K \neq 0$
(25, n + 1)	$y^{(n)} = K \exp\left(\frac{-y^{(n-1)}}{(n-1)!}\right),\ K \neq 0$
(26, n + 1)	$y^{(n)} = K \frac{\left(y^{(n-1)}\right)^2}{y^{(n-2)}},\ K \neq 0, n/(n-1)$
(27, n + 1)	$y^{(n)} = \frac{n}{n-1} \frac{\left(y^{(n-1)}\right)^2}{y^{(n-2)}} + K\left(y^{(n-2)}\right)^{\frac{n+3}{n-1}},\ K \neq 0$
(28, n + 1)	$y^{(n)} = \left(y^{(n-2)}\right)^3\left(y^{(n-3)}\right)^{-2}\left[\frac{n(3(n-2)K_1 - 2n+2)}{(n-2)^2} + K((n-2)K_1 - (n-1))^{\frac{3}{2}}\right]$
(21, n + 1)	$y^{(n)} = \sum_{i=2}^{n-2} A_i(x) y^{(i+1)},$
	ξ_k, *satisfy* $\xi_k^{(n)} = \sum_{i=1}^{n-2} A_i(x) \xi_k^{(i+1)},\quad k = 1, \dots, n-2.$ (7.3)

Remark. For $(21, n + 1)$, the A_is are not constant or satisfy the maximal conditions as stated in section 3.2.

Theorem 7.3. *The number of types of $n + 2$ dimensional maximal symmetry algebras admitted by an nth-order ODEs for $n \geq 5$ are two, one for linear class of equations, namely, $(23, n + 2)$, with Equation (4.2) and one for nonlinear*

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Equations (4.1) with algebra $(28, n+2)$. *For* $n = 4$, *there are three types, (23,6) results in a linear equation as well as (6,6) and (28,6) which give rise to nonlinear classes of equations.*

Theorem 7.4. *The only higher-order* $n \geq 3$, *ODEs which admit real* $n + 3$ *dimensional algebras are the two ODEs for* $n = 3$, *namely, (6.1) possessing* $\mathfrak{sl}(2, R) \oplus \mathfrak{sl}(2, R)$ *and* $\mathfrak{so}(3, 1)$, *respectively, and the fifth-order equation which is (6.2) having* $\mathfrak{sl}(3, R)$ *as maximal algebra.*

Note that the maximum real Lie algebra admissible is $(28, n + 4)$ as is well-known from Lie for $y^{(n)} = 0$, $n \geq 3$ and all scalar linear higher-order ODEs (3.20) and (3.21) as shown in Mahomed and Leach [12].

In order to recall, for lower-dimensional algebras admitted by higher-order ODEs $n \geq 4$, one can refer to the one- and two-dimensional algebras listed in Table 1 as well as the 12 types of real 3D algebra realizations given in Table 2. Moreover, one should note that for lower dimensions, there also appears the two nonmaximal Lie algebras $(20, m)$ and $(21, m)$, when one classifies scalar ODEs $n \geq 4$ as discussed. Therefore, we can state the following theorem for lower dimensions.

Theorem 7.5. *The number of types of lower* $m < n$ *dimensional Lie algebras possessed by scalar nth-order ODEs for* $n \geq 4$ *are given by the following:*

1,4 for $m = 1, 2$ *dimensional algebras with the algebras and ODEs as in Table 1 and*

12 for $m = 3$ *dimension algebras with algebras and ODEs given in Table 2 for the fourth-order equations and for higher* $n > 4$ *order equations obtained by (with algebras as in Table 2) invariant differentiation from (7.1) with* $m = 3$, *10 for dimensions* $m = 4, 5$ *with 8 algebras in Tables 3 and 5 as well as the two algebras* $(20, m)$ *and* $(21, m)$ *with ODEs deduced from (7.1) via invariant derivatives. The equations are* $\phi_4 = H(\phi)$ *and* $\phi_5 = H(\phi)$ *in Tables 3 and 5.*

Twelve for dimension $m = 6$ *with 10 algebras as in Theorem 7.1 together with* $(20, m)$ *and* $(21, m)$ *and equations determined by (7.1). The sixth-order ODEs* $\phi_6 = H(\phi)$ *are stated in Theorem 7.1*

Nine for dimension $m = 7$ *with seven algebras in Theorem 7.1 as well as the two algebras* $(20, m)$ *and* $(21, m)$ *and ODEs derived by (7.1). The equations of the form* $\phi_7 = H(\phi)$ *are known.*

Ten for dimension $m = 8$ *with eight algebras in Theorem 7.1 and two algebras* $(20, m)$ *and* $(21, m)$ *and ODEs deduced by (7.1). ODEs of the form* $\phi_8 = H(\phi)$ *are stated.*

Nine for dimension $m \geq 9$ *with seven algebras in Theorem 7.1 as well as the algebras* $(20, m)$ *and* $(21, m)$ *and ODEs deduced by (7.1); nth-order equations are given.*

Thus, we have a full understanding of the real symmetry Lie algebras admissible by scalar nth-order ODEs for $n \geq 2$ keeping in mind that the second- and third-order ODEs are well-known from the literature as referenced.

8 | CONCLUDING REMARKS

We have completely classified scalar nth, $n \geq 4$ order ODEs according to the real Lie algebras they possess by using the classification of Lie algebras in the plane as in the seminal contribution [4].

The classification for the second- and third-order equations are well-known in a number of influential works [2, 7, 8, 15, 16, 23].

In the case of fourth-order equations, there have been progress in a number of papers, namely, in previous research [24–27]. Here we have relooked at the maximal four-dimensional algebras in a compact form to enhance further classification of higher-order ODEs which admit four-dimensional algebras as subalgebras.

We have shown that a higher nth-order equation cannot admit maximally $n+3$ point symmetries with the exception of third- and fifth-order ODEs. For third-order equations, six symmetries occur for two classes having algebras $so(3, 1)$ and $so(2, 2)$ which are famous from the pioneering contribution of Lie [2]. In the case of the fifth-order ODEs, one has $sl(3, R)$ admitted for one class of equations which is quite interesting as usually this algebra is often alluded to as the maximal Lie algebra of scalar second-order ODEs.

The results on the number of $n + 1$ dimensional symmetry algebras for scalar higher-order ODEs are provided with theorems on higher symmetries.

Further, it is important to mention that $n+2$ dimensional algebras occur for the third-order equations only in the linear class having 3+2=5 symmetries (see, e.g., Ibragimov & Mahomed [8]); for the fourth-order, there are two further classes apart from the linear class as discussed (see also Shah et al. [27]) and for the fifth- and higher-order ODEs, there is one more class besides the linear class. These can be deduced from the discussions and results stated relating to the fourth- and higher-order ODEs.

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We have derived all the maximal n dimensional symmetry algebras and their representative equations for $n \geq 4$. Furthermore, one can easily extract the higher symmetry classification as well as pointed out. These are presented as main results. For lower-dimensional algebras, one can continue by considering further the two types $(20, m)$ and $(21, m)$ for $m < n$, where n is the order of the ODE, as well as the familiar two- and three-dimensional algebra types which are recalled in tabular form. This is stated as a theorem as well on when higher-order ODEs admit lower-dimensional Lie symmetry algebras and the ODEs that result.

The theorems on linearization appear in two cases which are mentioned here as propositions with reference to the initial works [11, 12].

We have therefore achieved a complete classification of scalar nth, $n \geq 4$, order ODEs in terms of the real Lie algebras of point symmetries they admit.

AUTHOR CONTRIBUTIONS

Said Waqas Shah: Conceptualization; investigation; writing—original draft; methodology; validation; software; formal analysis; writing—review and editing. **F. M. Mahomed:** Conceptualization; investigation; funding acquisition; writing—original draft; methodology; validation; writing—review and editing; formal analysis; supervision; resources. **H. Azad:** Conceptualization; investigation; funding acquisition; methodology; validation; writing—review and editing; formal analysis; supervision; resources.

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CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.

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