Invariants of nonlinear evolution type equations and their exact solutions

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DECLARATION

I declare that the contents of this thesis are original except where due references have been made. It has not been submitted before for any degree to any other institution.

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Abstract

The role of invariants in obtaining exact solutions of differential equations is reviewed. The examples considered are nonlinear evolution type equations like the Fisher and Fitzhugh-Nagumo equations. Finally, we look directly at an equation (formulated) governing some non-Newtonian fluid in a rotating system.

Contents

Introduction

The importance of group theoretic approach and Painlevé analysis for solving non-linear differential equations of many physical and engineering systems has already been highlighted in a number of publications (for example see [1],[2]). The basis of this research is a consequence of the work of Sophus Lie in 1895 [3] and Emmy Noether in 1918 [4].

Lie's investigations were on problems of general integrability of differential equations by means of group theory. This is why many of his papers deal with ordinary differential equations or with linear partial differential equations of first order. In his paper on higher-order partial differential equations, Lie considered solutions invariant under one parameter groups admitted by these equations. With help of the invariant solutions, he returned to the problem of general integrability.

However, the majority of the partial differential equations that appear in mathematical physics are not completely integrable by means of group theoretic techniques. On the other hand, special types of exact solutions of differential equations were known and successfully used in mechanics and physics. These solutions were found by ad hoc methods. As the result of the efforts of many people, these special solutions are now identified as invariant or exact solutions in the sense of Lie. This connection with group theory led to the increased activity in group analysis of partial differential equations that occured after 1950 ([5], [6], [7], [8], [9] and [10]) and eventually to the explosion of activity starting in 1970.

For many years it has been known that the conservation laws of classical mechanics are connected with symmetry properties of the physical system. Although the given equation provides, in principle, a direct and exhaustive method for calculating conservation laws of finite order for any differential equation, it sheds no light on this connection. Noether was the first to combine the methods of variational calculus with the theory of Lie groups and to formulate a general approach for constructing conservation laws for Euler-Lagrange equations. It is this connection that is exhibited by Noether theorem.

The method of 'invariants' is now playing a significant role in analysing differential equations . The invariants referred to, in most cases, are a result of conserved forms of differential equations. These provide a way of reducing the differential equation which in the case of partial differential equations may mean a decrease in the number of independent variables or, as in the case of ordinary differential equations, refers to the order of the d.e. In [11] and [12], the author uses the invariant idea in conjunction with specific algorithms to analyse equations, inter alia, Korteweg de Vries, Nagumo and Fisher equations. All of these are evolution type equations which do not admit Lagrangians, hence the existence and knowledge of a Lagrangian which makes the task of finding invariants easier via Noether's Theorem is not applicable. Nevertheless, we show that some of these may still be analysed using the Lagrangian method.

An outline of this dissertation is as follows:

The first chapter entails the description of the notation that will be used. Also, the operators that will be used throughout are defined.

In the second chapter a detailed study on the Fisher equation is done.The Fisher equation is known to have only translational symmetries (in space and in time) but the its reduction is pursued using variational methods. Then, in a similar but brief way, the Firtzhugh-Nagumo equation is studied.

The third chapter is devoted to the formulatation of the nonlinear equation governing the steady state flow of a non-Newtonian fluid of differential type, which is of grade three, in an incompressible infinite plate. Furthermore, the formulated nonlinear equation is analysed using the group theoretic approach with emphasis on a Lagrangian formulation.

Finally, in chapter 4 concluding remarks are made.

Chapter 1

Preliminaries

1.1 Introduction

In this chapter, the notation that will be used in this report will be described, and the main operators will be defined.

1.2 Notation and Preliminaries

Einstein's repeated index summation convention is used.

Let $x = (x^1, x^2, \ldots, x^n) \in \mathbb{R}^n$ be the independent variable, and let $u = (u^1, u^2, \ldots, u^m) \in \mathbb{R}^m$ be the dependent variable, with co-ordinates x^i and u^α

respectively. Let $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ be the projection map $\pi(x, u) = x$. Let $s : \mathcal{X} \subset \mathbb{R}^n \to U \subset \mathbb{R}^{n+m}$ be a smooth map such that $\pi \circ s = 1_{\mathcal{X}}$, the identity map on $\mathcal{X}$.

Let u_i^α denote the partial derivative of u^α with respect to x^i, and similarly let $u_{i_1, i_2, \ldots, i_r}^\alpha$ denote the mixed partial derivatives of u^α with respect to $x^{i_1}, x^{i_2}, \ldots, x^{i_r}$. With one exception, namely the prolongation coefficients $\zeta_{i_1 \ldots i_n}^\alpha$ that will be reviewed later, subscripts will be used to indicate partial differentiation.

Let D_i be the operator of total differentiation with respect to x^i. Using the chain rule, we write

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \qquad i = 1, \ldots, n \tag{1.1}$$

We have

$$D_i(u^\alpha) = u_i^\alpha, \qquad D_j D_i(u^\alpha) = u_{ij}^\alpha, \cdots \tag{1.2}$$

Let $u_{(1)}$ denote the collection of all first-order derivatives, u_i^α. Similarly, let $u_{(2)}$, $u_{(3)}$, and so on, denote the collections of higher order drivatives. The r-jet bundle $J^r(U)$ is the equivalence class of sections of U, with co-ordinates $(x^i, u^\alpha, u_i^\alpha, u_{i_1,i_2}^\alpha, \ldots, u_{i_1,\cdots,i_r}^\alpha)$ where $1 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq n$. The r-jet bundle will be written $J^r(U) = \{(x, u, u_{(1)}, \ldots, u_{(r)}) \ : (x, u) \in U\}$. It will be expedient to work in $J^r(U)$, where u and its derivatives are treated as independent variables.

A *differential function of order k* is a locally analytic function $f(x, u, u_{(1)}, \ldots, u_{(k)})$. The universal space $\mathcal{A}$ is the vector space of all differential functions of all finite orders. Notice that $\mathcal{A}$ is closed under total differentiations. The space $\mathcal{A}$ was first introduced by Ibragimav [13].

We now briefly review some appropriate results regarding symmetries associated with differential equations that can be derived from a variational principle. The basics of the calculus of variations are not presented here; they are readily available from sources such as Olver [1] or Caratheodory [14].

DEFINITION 1.1. The *Euler-Lagrange* operator is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1\ldots i_s}}, \qquad \alpha = 1,\ldots,m \tag{1.3}$$

The Euler-Lagrange operator is the multi-dimensional version of the operator

$$\frac{\partial}{\partial u} - \frac{d}{dt}\frac{\partial}{\partial \dot{u}}$$

introduced by Euler as a means to solving one-dimensional variational problems. Lagrange extended it to the multi-dimensional case.

DEFINITION 1.2. The Lie-Bäcklund operator is given by

$$X = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha} \tag{1.4}$$

where the $\xi^i,\ \eta^\alpha \in \mathcal{A}$. This is an abreviation of the formal sum

$$X = \xi^i\frac{\partial}{\partial x^i} + \eta^\alpha\frac{\partial}{\partial u^\alpha} + \zeta^\alpha_i\frac{\partial}{\partial u^\alpha_i} + \zeta^\alpha_{ij}\frac{\partial}{\partial u^\alpha_{ij}} + \cdots, \tag{1.5}$$

where the coefficients are given by the prolongation formulae

$$\begin{aligned} \zeta_i^\alpha &= D_i(W^\alpha) + \xi^j u_{ij}^\alpha, \\ \zeta_{ij}^\alpha &= D_{i_1} D_{i_2}(W^\alpha) + \xi^j u_{ji_1 i_2}^\alpha, \\ &\vdots \end{aligned} \tag{1.6}$$

where W^α is the Lie characteristic function, is given by

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha. \tag{1.7}$$

As noted before, the subscripts of the $\zeta_{i_1 \ldots i_n}^\alpha$ do not indicate differentiation. Also, a Lie-Bäcklund operator is sometimes called a Lie-Bäcklund symmetry.

In the special cases where there are four or less independent variables (x_1, x_2, x_3, x_4), and one dependent variable u, the independent variables will be denoted (t, x, y, z) according to the more usual convention. Accordingly, the operators D_1, D_2, $\ldots$ will be denoted D_t, D_x, $\ldots$, and the Lie-Bäcklund operator $X = \xi^i \partial/\partial x^i + \eta^\alpha \partial/\partial u^\alpha$ will be written

$$X = \tau \frac{\partial}{\partial t} + \xi^x \frac{\partial}{\partial x} + \xi^y \frac{\partial}{\partial y} + \xi^z \frac{\partial}{\partial z} + \eta \frac{\partial}{\partial u}$$

with prolongation coefficients denoted by ζ_t, ζ_x, ζ_y, ζ_z, ζ_{tt}, ζ_{tx}, $\ldots$

In the case $\xi^i = \xi^i(x, u)$ and $\eta^\alpha = \eta^\alpha(x, u)$, X is called a Lie point symmetry.

DEFINITION 1.3. The Noether operator associated with the Lie-Bäcklund operator X is defined by

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s \geq 1} D_{i_1} \ldots D_{i_s}(W^\alpha) \frac{\delta}{\delta u_{i i_1 \cdots i_s}^\alpha}, \quad i = 1 \ldots n, \tag{1.8}$$

where the Euler-Lagrange operators $\delta/\delta u^{\alpha}_{ii_1\cdots i_s}$ are obtained by replacing u^{α} with the corresponding derivatives in (1.3).

Chapter 2

Analysis of some evolution type equations

2.1 Introduction

Evolution type equations arise frequently in mathematical physics and are used to model physical phenomena such as shock waves (Burgers equation), shallow water behaviour (Korteweg-deVries equation) and the Fisher equation (which models reaction-diffusion waves). The goal of this chapter is to provide the reader with a relatively quick and painless introduction to the study of non-linear evolution type equations concretely by considering examples like the Fisher and Fitzhugh-Nagumo equations. We will consider these equations from an analysis involving 'invariants'.

These invariants will be related to symmetry generators.

The first section deals with the reduction of the Fisher equation using the notion of invariants. Noether's theorem will be used to provide a conserved quantity, $I = I(y, w, w')$, corresponding to each Noether symmetry G (also the Lagrangian) and gauge function f. The second section is devoted to the straight analysis of the Fisher equation by finding the conserved vector of general diffusion equation. The last section is concerned with a similar but brief study on the Fitzhugh-Nagumo equation.

2.2 The Fisher equation

We consider, in detail, the Fisher equation

$$u_t = u_{xx} + \lambda u(1 - u) \tag{2.1}$$

which only admits point symmetries involving time and space translations. Also, (2.1) does not admit a Lagrangian.

A time translation reduction of the equation becomes the o.d.e ($y = x$, $w = u$)

$$w'' + \lambda w(1 - w) = 0$$

and a translation in x reduction yields ($y = t$, $w = u$)

$$w' = \lambda w(1 - w).$$

A travelling wave reduction ($y = x - ct$ and $w = u$) from a combination of these Lie symmetry generators yields

$$w'' + cw' + \lambda w(1 - w) = 0. \tag{2.2}$$

Equation (2.2) has a Painlevé property for $\lambda = \pm 6c^2/25$ (see [15]). We attempt an analysis and reduction of (2.1) using the notion of invariants. Firstly, we note that a Lagrangian of (2.2) is

$$L = e^{cy}[\frac{1}{2}w'^2 - \lambda(\frac{1}{2}w^2 - \frac{1}{3}w^3)]. \tag{2.3}$$

The Noether symmetries $G = \xi\partial/\partial y + \eta\partial/\partial w$, if any, are given by solving

$$GL + L\frac{\mathrm{d}\xi}{\mathrm{d}y} = \frac{\mathrm{d}f}{\mathrm{d}y}, \tag{2.4}$$

where $f = f(y, w)$ is some gauge term. Firstly , we compute the partial differentiation of L and we obtain the following

$$\begin{aligned}
\frac{\partial L}{\partial y} &= ce^{cy}[\frac{1}{2}w'^2 - \lambda(\frac{1}{2}w^2 - \frac{1}{3}w^3)],\\
\frac{\partial L}{\partial w} &= \lambda e^{cy}(w^2 - w),\\
\frac{\partial L}{\partial w'} &= e^{cy}w'.
\end{aligned}$$

The direct substitution of these into (2.4) yields

$\xi c e^{cy}[\frac{1}{2}w'^2 - \lambda(\frac{1}{2}w^2 - \frac{1}{3}w^3)] + \eta\lambda e^{cy}(w^2 - w) + e^{cy}w'[\eta_y + (\eta_w - \xi_y)w' - \xi_w w'^2]$

$+e^{cy}[\frac{1}{2}w'^2 - \lambda(\frac{1}{2}w^2 - \frac{1}{3}w^3)](\xi_y + w'\xi_w) = f_y + w' f_w.$

This is a cubic function in w'. It splits into the following four equations after one sets the coefficients of the various powers of w' to zero:

$$
\begin{aligned}
w'^3 &: \quad -\xi_w e^{cy} + \frac{1}{2}\xi_w e^{cy} = 0 \\
w'^2 &: \quad \frac{1}{2}c\xi e^{cy} + e^{cy}(\eta_w - \xi_y) + \frac{1}{2}e^{cy}\xi_y = 0 \\
w'^1 &: \quad e^{cy}\eta_y = f_w \\
w'^0 &: \quad c\xi\lambda(\frac{1}{3}w^3 - \frac{1}{2}w^2) + \eta\lambda(w^2 - w)e^{cy} + \lambda\xi_y e^{cy}(\frac{1}{3}w^3 - \frac{1}{2}w^2) = f_y.
\end{aligned}
$$

Manipulating and integrating the first of these with respect to w yields

$$\xi = a(y). \tag{2.5}$$

On substituting (2.5) into the second equation and some straightforward manipulation , including integrating with respect to w, we find that

$$\eta = \frac{1}{2}(a' - ca)w + b(y). \tag{2.6}$$

If we substitute the derivative of (2.6) with respect to y into the third equation, then, after integrating with respect to w, we obtain

$$f = e^{cy}[\frac{1}{4}a'' - ca'w^2 + b(y)'w] + d(y), \tag{2.7}$$

where $a(y)$, $b(y)$ and $d(y)$ are arbitrary functions. The substitution of (2.5), (2.6) and (2.7) (and their derivatives) into the last of these gives rise to

$ca\lambda(\frac{1}{3}w^3 - \frac{1}{2}w^2)+\lambda[\frac{1}{2}(a' - ca)w + b(y)](w^2 - w)e^{cy}+\lambda a'e^{cy}(\frac{1}{3}w^3 - \frac{1}{2}w^2)$

$=ce^{cy}[\frac{1}{4}(a'' - ca')w^2 + b(y)'w]+ d(y)'e^{cy}[\frac{1}{4}(a''' - ca'')w^2 + b(y)''w]$.

Splitting of coefficients in this by powers of w yields the following:

$$
\begin{aligned}
w^3 &: \frac{1}{3}\lambda ace^{cy} + \frac{1}{2}\lambda e^{cy}(a' - ca) + \frac{1}{3}\lambda a'e^{cy} = 0 \\
w^2 &: -\frac{1}{2}ac\lambda e^{cy} + \lambda e^{cy}b(y) - \frac{1}{2}\lambda e^{cy}(a' - ca) - \frac{1}{2}\lambda a'e^{cy} = c\frac{1}{4}e^{cy}(a'' - a'c) + \frac{1}{4}e^{cy}(a''' - a''c) \\
w^1 &: -\lambda b(y)e^{cy} = ce^{cy}b(y)' + e^{cy}b(y)'' \\
w^0 &: d(y)' = 0.
\end{aligned}
$$

Now, manipulating and integrating the first of these by the technique of separation of variables give rise to

$$a(y) = Ae^{cy/5} \tag{2.8}$$

where A is a constant. The second equation (after some work) gives

$$b(y) = \frac{1}{4\lambda}[a'' + a'(4\lambda - c^2)]. \tag{2.9}$$

Now, from (2.9) one can easily show that

$$b(y)' = \frac{c}{5}b(y), \quad b(y)'' = \frac{c^2}{5^2}b(y). \tag{2.10}$$

Thus, the third equation becomes $-\lambda b = \frac{c^2}{5}b + \frac{c^2}{5^2}b$, hence $\lambda = -6c^2/25$. Finally, from the last of these, we obtain $d(y) = D$, where D is a constant.

In summary we have

$$
\begin{aligned}
\xi &= Ae^{cy/5} \\
\eta &= Ae^{cy/5}[-\frac{2c}{5}w + \frac{1}{4\lambda}(\frac{c^3}{5^3} + \frac{c}{5}(4\lambda - c^2))] \\
f &= Ae^{6cy/5}[-\frac{C^2}{5^2}w^2 + \frac{1}{4\lambda}(\frac{c^4}{5^4} + \frac{c^2}{5^2}(4\lambda - c^2))] + D.
\end{aligned}
$$

Letting A=1 and D=0 in the above equation, with $\lambda = -6c^2/25$, we obtain

$$G = e^{cy/5}\{\partial/\partial y + 2c/51 - w\partial/\partial w\}, \quad f = \frac{c^2}{5^2}e^{6cy/5}w(2-w). \tag{2.11}$$

Noether's theorem then provides a conserved quantity, $I = I(y, w, w')$, corresponding to each Noether symmetry G and gauge f. It is known that the first-order equation $I = k$, k a constant, which is a reduced form of (2.2), is also invariant under G which allows us to reduce once more with the symmetry G and, hence, to find a solution to (2.2) by quadrature being an exact solution of the Fisher equation. The calculations (above) show that a Noether symmetry is $G = e^{cy/5}\{\partial/\partial y + 2c/5(1-w)\partial/\partial w\}$ $f = (2/5)^2 e^{6cy/5}w(2-w))$ coming from $\lambda = -6c^2/25$ which corresponds to the Painlevé property mentioned above.

The corresponding first-order o.d.e with G as Lie symmetry generator is obtained by solving

$$I = L\xi + (\eta - \xi w')\frac{\partial L}{\partial w'} - f, \tag{2.12}$$

where f is a gauge function. The direct substitution of ξ, η, f (as calculated above) and L in the right hand side of (2.12) leads to

$$I = e^{6cy/5}\{-\frac{1}{2}w'^2 + 2c^2/25w(-w^2 + 2w - 1) + 2c/5w'(1-w)\}. \tag{2.13}$$

Thus, by the law of conservation

$$e^{6cy/5}\{-\frac{1}{2}w'^2 - 2c^2/25w(1-w)^2 + 2c/5w'(1-w)\} = k, . \tag{2.14}$$

where k is a constant. Equation (2.14) can be mapped to an equation in Y and W,

$$\bar{I}(W, W') = \bar{k} \tag{2.15}$$

(a variables separable equation), i.e., with Lie symmetry $\bar{G} = \partial/\partial Y$ by solving the system of p.d.e.s $G(Y) = 1$ and $G(W) = 0$. We get $W = (1-w)e^{2cy/5}$ and $Y = (-5/c)e^{-cy/5}$. After some lengthy calculations, (2.14) has the transformed form

$$\frac{\mathrm{d}W}{\sqrt{(2c/5)^2W^3 + 2k}} = -\mathrm{d}Y. \tag{2.16}$$

Note. Equation (2.2) generates the two-dimensional Lie algebra of point symmetries $G_1 = \partial/\partial y$ and G (above), the latter for $\lambda = -6c^2/25$ with the Lie bracket $[G, G_1] = (-c/5)G$. Thus, a solution by quadratures is obtainable, without recourse to Lagrangians, by reducing (2.2) first by G and then by G_1 (see [16]).

2.3 The analysis of the Fisher equation

We now discuss a straight analysis of (2.1) by finding conserved vectors (T, S) on the general diffusion equation

$$u_t = u_{xx} + F(u). \tag{2.17}$$

If such a vector exists, (2.17) and, in particular, the Fisher equation may be analysed with possible potential symmetries (see [2]) . A conserved form of

$$\beta' = \alpha, \quad \alpha' + \frac{1}{2}\alpha^2 + c\alpha = 1 \tag{2.18}$$

is

$$D_tT + D_xS = 0|_{(2.17)}. \tag{2.19}$$

We now compute the conserved vectors given by solving (2.19). Next, we consider the general *total differentiation operator*

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots , \quad i = 1, \ldots, n. \tag{2.20}$$

Thus, from (1.1) it follows that

$$\begin{aligned} D_t &= \partial/\partial t + u_t\partial/\partial u + u_{tt}\partial/\partial u_t + u_{tx}\partial/\partial u_x + ... \\ D_x &= \partial/\partial x + u_x\partial/\partial u + u_{xx}\partial/\partial u_x + u_{xt}\partial/\partial u_t + ... \end{aligned}$$

After expansion into partial derivatives, (2.19) becomes

$T_t + u_tT_u + u_{tt}T_{u_t} + u_{tx}T_{u_x} + S_x + u_xS_u + u_{xx}S_{u_x} + u_{xt}S_{u_t} = 0,$

which after the replacement of u_t by $u_{xx} + F(u)$ becomes

$$T_t + (u_{xx} + F(u))T_u + u_{tt}T_{u_t} + u_{tx}T_{u_x} + S_x + u_xS_u + u_{xx}S_{u_x} + u_{xt}S_{u_t} = 0.$$

Assuming that T and S are independent of second and higher order derivatives of u, coefficients of second derivatives of u can split:

$$\begin{aligned} u_{tt} &: \; T_{u_t} = 0 \\ u_{tx} &: \; T_{u_x} + S_{u_t} = 0 \\ u_{xx} &: \; T_u + S_{u_x} = 0 \\ 1 &: \; T_t + F(u)T_u + S_x + u_xS_u = 0. \end{aligned}$$

The second of these gives

$$S = -T_{u_x}u_t + A(x,t,u,u_x). \tag{2.21}$$

Substituting the partial derivative of (2.21) with respect to u_x into the third equation yields

$$T_u - T_{u_x u_x} u_t + A_{u_x} = 0. \tag{2.22}$$

Next, we do the separation of coefficients by powers of u_t since T and A are independent of u_t. Thus we have

$$\begin{aligned} u_t &: \; T_{u_x u_x} = 0 \\ 1 &: \; T_u + A_{u_x} = 0. \end{aligned}$$

Integrating the first of these with respect to u_x gives

$$T = B(x,t,u)u_x + C(x,t,u). \tag{2.23}$$

Substituting the partial derivative of (2.23) with respect to u in the second of these results in $A_{u_x} = -B_u u_x - C_u$. If we intergrate this with respect to u_x we get

$$A = -\frac{1}{2}B_u u_x^2 - C_u u_x + \gamma(x,t,u), \tag{2.24}$$

where A, B, C and γ are arbitrary functions. After the substitution of (2.23) and (2.24) into (2.21) we have the following set of equations

$$T = B(x,t,u)u_x + C(x,t,u), \quad S = -B(x,t,u)u_t - \frac{1}{2}B_u {u_x}^2 - C_u u_x + \gamma(x,t,u). \tag{2.25}$$

Now, substituting (2.25) into the remaining term gives rise to

$$B_t + C_t + F(u)[B_u + C_u] - B_x u_t - \tfrac{1}{2}B_{ux}{u_x}^2 - C_{ux}u_x + \gamma_x + u_x[-B_u u_t - \tfrac{1}{2}B_{uu}{u_x}^2 - C_{uu}u_x + \gamma_u] = 0.$$

The separation of coefficients by the powers of u_x, u_t yields

$$\begin{aligned}
{u_x}^3 &:\quad -\frac{1}{2}B_{uu} = 0 \\
{u_x}^2 &:\quad -C_{uu} - \frac{1}{2}B_{ux} = 0 \\
u_x &:\quad B_t - C_{xu} + \gamma_u = 0 \\
u_t &:\quad -B_x = 0 \\
1 &:\quad C_t + F(u)C_u + \gamma_x = 0.
\end{aligned}$$

Solving the second of these equations gives

$$C = a(x,t)u + b(x,t). \tag{2.26}$$

On substituting the partial derivative of (2.26) with respect to u_x into the third equation, we obtain

$$\gamma = (a_x - B_t)u + d(x,t). \tag{2.27}$$

Also, we note that $B = B(t)$. Substituting the vaious partial derivatives of (2.26) and (2.27) into the last equation gives

$$a_t u + b_t + aF(u) + a_{xx}u + d_x = 0. \tag{2.28}$$

Finally, our conserved vector (T, S) is

$$T = B(t)u_x + a(x,t)u + b(x,t), \quad S = -B(t)u_t - au_x a_x u + d(x,t) \tag{2.29}$$

subject to $b_t + d_x = 0$ and $a_t u + F(u)a + ua_{xx} = 0$. It is clear that (T, S) is a nontrivial conserved vector only if $F = 0$ (heat equation) or $F = u$ (in which case we do not obtain the Fisher equation and a satisfies $a_t + a_{xx} + a = 0$). We conclude that the Fisher equation has no conservation laws (this implies that the analysis utilizing potential symmetries cannot be done on the Fisher equation). One may pursue higher-order conservation laws.

2.4 The Fitzhugh-Nagumo equation

We carry out a similar study of a generalized version of the Fitzhugh-Nagumo equation

$$u_t = u_{xx} + \lambda u(1-u)(u-a), \quad a \neq 1. \tag{2.30}$$

A Painlev$\acute{e}$ analysis has been done on (2.30) for $\lambda = 1$ in [15]. A travelling wave reduction $y = x - ct$ and $w = u$ yields the o.d.e

$$w'' + cw' + \lambda w(1-w)(w-a) = 0. \tag{2.31}$$

First, we note that a Lagrangian of (2.31) is

$$L = e^{cy}[\frac{1}{2}w'^2 - \lambda(-\frac{1}{4}w^4 - \frac{a}{2}w^2 + \frac{a+1}{3}w^3)]. \tag{2.32}$$

As mentioned in section 2, the Noether symmetries G, if any, are given by solving (2.4). Likewise, we start by partially differentiating the Lagrangian, resulting in

$$\frac{\partial L}{\partial y} \;\; = \;\; ce^{cy}[\frac{1}{2}w'^2 - \lambda(-\frac{1}{4}w^4 + \frac{a+1}{3}w^3 - \frac{a}{2}w^2)],$$

$$
\begin{aligned}
\frac{\partial L}{\partial w} &= \lambda e^{cy}[w^3 - (a+1)w^2 + ay], \\
\frac{\partial L}{\partial w'} &= e^{cy}w'.
\end{aligned}
$$

Thus, substituting this into (2.4) results in

$$
\begin{aligned}
&\xi c e^{cy}[\tfrac{1}{2}w'^2 - \lambda(-\tfrac{1}{4}w^4 + \tfrac{a+1}{3}w^3 - \tfrac{a}{2}w^2)] + \eta\lambda e^{cy}[w^3 - (a+1)w^2 + ay] \\
&+e^{cy}w'[\eta_y + (\eta_w - \xi_y)w' - \xi_w w'^2] + e^{cy}[\tfrac{1}{2}w'^2 - \lambda(-\tfrac{1}{4}w^4 - \tfrac{a}{2}w^2 + \tfrac{a+1}{3}w^3)](\xi_y + w'\xi_w) \\
&= f_y + w' f_w.
\end{aligned} \tag{2.33}
$$

Also, this is a cubic function in w', so it splits into the following four equations after one sets the coefficients of the various powers of w' to zero:

$$
\begin{aligned}
w'^3 \quad &: -\xi_w e^{cy} + \tfrac{1}{2}\xi_w e^{cy} = 0 \\
w'^2 \quad &: \tfrac{1}{2}c\xi e^{cy} + (\eta_w - \xi_y)e^{cy} + \tfrac{1}{2}\xi_y e^{cy} = 0 \\
w'^1 \quad &: \eta_y e^{cy} = f_w \\
w'^0 \quad &: -\xi c e^{cy}[-\tfrac{1}{4}w^4 + \tfrac{a+1}{3}w^3 - \tfrac{a}{2}w^2] + \eta\lambda e^{cy}[w^3 - (a+1)w^2 + ay] \\
&\quad -\lambda\xi_y e^{cy}[-\tfrac{1}{4}w^4 + \tfrac{a+1}{3}w^3 - \tfrac{a}{2}w^2] = c e^{cy}[\tfrac{1}{4}(\alpha'' - \alpha' c)w^2 + \beta(y)' w] \\
&\quad +\delta(y)' + e^{cy}[\tfrac{1}{4}(\alpha''' - \alpha'' c)w^2 + \beta(y)'' w].
\end{aligned} \tag{2.34}
$$

Solving the first of these with basic integration yields

$$\xi = \alpha(y). \tag{2.35}$$

Substituting (2.35) in the second of these and integrating with respect to w leads to

$$\eta = \frac{1}{2}(\alpha' - \alpha c)w + \beta(y). \tag{2.36}$$

After the substitution of the derivative of η with respect to y and then integrating with respect to w, the third equation becomes

$$f = e^{cy}[\frac{1}{4}(\alpha'' - \alpha' c)w^2 + \beta(y)' w] + \delta(y). \tag{2.37}$$

The separation of the coefficients in the last equation yields the following equations

$$\begin{aligned}
w^4 \quad &: \tfrac{1}{4}\lambda\alpha e^{cy} + \tfrac{1}{2}\lambda(\alpha' - \alpha c)e^{cy} + \tfrac{1}{4}\lambda\alpha' e^{cy} = 0 \\
w^3 \quad &: -\tfrac{a+1}{3}\lambda\alpha e^{cy} + \lambda\beta(y)e^{cy} - \tfrac{a+1}{2}\lambda(\alpha' - c\alpha)e^{cy} \\
&-\tfrac{a+1}{3}\lambda\alpha' e^{cy} = 0 \\
w^2 \quad &: \tfrac{a}{2}\lambda\alpha e^{cy} - \lambda\beta(y)e^{cy}(a+1) + \tfrac{a}{2}\lambda(\alpha' - c\alpha)e^{cy} + \tfrac{a}{2}\lambda\alpha' e^{cy} \\
&= \tfrac{c}{4}(\alpha'' - c\alpha')e^{cy} + \tfrac{1}{4}(\alpha''' - c\alpha'')e^{cy} \\
w^1 \quad &: \lambda\beta(y)ae^{cy} = c\beta(y)' e^{cy} + \beta(y)'' e^{cy} \\
w^0 \quad &: \delta(y)' = 0
\end{aligned} \tag{2.38}$$

Now, simplifying and integrating the first equation leads to

$$\alpha = Ae^{cy/3}, \tag{2.39}$$

After a few steps of manipulating the second equation, we obtain

$$\beta = \frac{(a+1)cA}{9}e^{cy/3}. \tag{2.40}$$

From (2.39) we see that

$$\begin{aligned}
\alpha' \quad &= \tfrac{c}{3}\alpha \\
\alpha'' \quad &= \tfrac{c^2}{3^2}\alpha \\
\alpha''' \quad &= \tfrac{c^3}{3^3}\alpha.
\end{aligned} \tag{2.41}$$

Thus, substituting this into the simplified form of the third equation, that is

$$\lambda(\frac{a}{2}\alpha' - (a+1)\beta(y)) = \frac{1}{4}(\alpha''' - c^2\alpha') \tag{2.42}$$

results in

$$\lambda_1 = \frac{2c^2}{3(a^2 - a + 1)} \tag{2.43}$$

Also, from (2.40) it is by no means obvious that

$$\begin{aligned} \beta' &= \tfrac{c}{3}\beta \\ \beta'' &= \tfrac{c^2}{3^2}\beta. \end{aligned} \tag{2.44}$$

Thus, substituting (2.44) in the fourth equation gives

$$\lambda_2 = \frac{4c^2}{9a}. \tag{2.45}$$

Lastly,

$$\delta(y) = B. \tag{2.46}$$

Thus the Lagrangian L has a single Noether symmetry $G = e^{cy/3}[\partial/\partial y + c/3(\frac{a+1}{3} - w)\partial/\partial w]$ for simultaneous forms of λ (as calculated above) from which we obtain $a = 1/2$ and $a = 2$ ($\lambda = 2\frac{4c^2}{9}$ and $\lambda = (1/2)\frac{4c^2}{9}$, respectively). Following from results regarding the Fisher equation, we propose that the equation (2.30) possesses Painlev*é* properties for these combinations of a and λ. Furthermore, the exact solutions for the equation are obtainable from a double reduction using the single point symmetry generator G.

2.5 Conclusion

In this chapter we have presented the role of invariants in obtaining exact solutions of differential equations. Invariants of reduced forms of a p.d.e are obtainable from a variational principle even though the p.d.e itself does not admit a Lagrangian. These reductions carry all the usual advantages regarding symmetries and further reductions as was seen in detail regarding the Fisher equation. The method can be applied to a large class of evolution type p.d.e.s, particularly diffusion type equations. The method proposes an alternative way of obtaining values of parameters for which equations may possess Painlevé properties.

In chapter 3 we will investigate some non-Newtonian fluid of differential type in a rotating system. A non-linear governing equaton will be formulated and then be analysed in detail using a variational principle.

Chapter 3

Some non-Newtonian fluid in a rotating system

3.1 Introduction

The analysis of the effects of rotation in fluid flows has been an active area of research in the area of geophysics, in particular, in the study of wind generated ocean currents on a rotating earth. The solutions of many rotating flow problems hinges on the understanding of the behaviour of the boundary layers. The literature on rotating flows of a viscous fluid is quite extensive and some important contributions include the works of Gupta [17], Murthy and Ram [18], Debnath [19], [20], [21], Loper [22], [23], Loper and Benton [24], Soundalgekar and Pop [25], Thornley [26], Deka et al

[27], Acheson [28] Mazumder [29], Ganapathy [30] and Singh [31].

There is another area in which rotating flows has specially drawn the attention of researchers, viz., the fluid dynamics of non-Newtonian fluids. There are manifestations of fluid behaviour which cannot be adequately expalained on the basis of the classical, linearly viscous model. Many materials such as drilling muds, clay coatings and other suspensions, certain oils and greases, polymer melts, clastomers and other emulsions have been treated as non-Newtonian fluids. Many non-Newtonian models or constitutive equations have been proposed and most of them are empirical or semiempirical. For general three-dimensional representations, the methods of continuum mechanics are needed. Amongst the many models which have been used to describe the non-Newtonian behaviour exhibited by certain fluids, the fluids of differential type have received special attention. One of the models for non-Newtonian fluids which form a subclass of the differential type fluids is the second grade fluid. The rotating flow of a second grade fluid has been examined by Hayat and Hutter [32] and Hayat et al [33].

Although a second grade model is able to predict the normal stress differences, it does not account for shear thinning and thickening phenomena which many non-Newtonian fluids display. The third grade fluid represents a further, although inconclusive, attempt towards a comprehensive description of the properties of viscoelastic fluids. With this in view, the model in the present work invloves a third grade fluid.

The main purpose of this chapter is to carry out an analytic approach for the steady rotating flow of a third grade fluid past an infinite plate. The governing non-linear equation is studied using the group theoretic approach with emphasis on a Lagrangian formulation.

In section 2, equations of continuity and motion along with constitutive equations of third grade fluids are given. Section 3 deals with the formulation of the problem. Section 4 is devoted to an analytic approach to the non-linear problem.

3.2 Basic equations

The non-Newtonian fluid model considered here is the third grade one whose constitutive equations consist of

$$\begin{aligned} &\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \\ &\mathbf{S} = \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + +\beta(tr\mathbf{A}_1^2)\mathbf{A}_1. \end{aligned} \tag{3.1}$$

The equations of motion and continuity for a non-rotating frame is

$$\begin{aligned} &\rho[\tfrac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\bigtriangledown)\mathbf{V}] = -\bigtriangledown p + div\mathbf{S}, \\ &div\mathbf{V} = 0. \end{aligned} \tag{3.2}$$

In these equations p is an arbitrary isotropic pressure, $\mathbf{T}$ is the stress tensor, $\mathbf{I}$ is the identity tensor, t is time, $\mathbf{V}$ is the velocity and $\mathbf{S}$ is the extra stress tensor. The Rivlin-Eriksen tensors $\mathbf{A}_1$ and $\mathbf{A}_2$ are defined as

$$\begin{aligned} &\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T, \\ &\mathbf{A}_2 = \tfrac{\partial \mathbf{A}_1}{\partial t} + (\mathbf{V}\cdot\bigtriangledown)\mathbf{A}_1 + \mathbf{A}_1\mathbf{L} + \mathbf{L}^T\mathbf{A}_1, \\ &\mathbf{L} = \bigtriangledown\mathbf{V}. \end{aligned} \tag{3.3}$$

The Clausius-Duhem inequality and the requirement that energy be a minimumin equilibrium imposes the following constraints on the dynamic μ, the normal stress moduli α_1 and α_2, and the coefficient β,

$$\mu \geq 0, \quad \alpha_1 \geq 0, \quad \beta \geq 0, |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta}. \tag{3.4}$$

A detailed description of the model has been given in a paper by Dunn and Rajagopal [34].

3.3 Formulation of the problem

An infinite plate (located at $z = 0$) and the fluid (which is in contact with the plate and occupies the region $z > 0$) are in a uniform rotation $\mathbf{\Omega}$. For the sake of simplicity, the angular velocity $\mathbf{\Omega}$ is taken parallel to the z-axis. The fluid considered is incompressible and third grade. We consider a uniform flow with velocity U_0 past an infinite plate which is at rest. With respect to the rotating frame of reference, the equation of continuity is (3.2b) and the momentum equation (3.2a) for steady flow becomes

$$(\mathbf{V} \cdot \bigtriangledown)\mathbf{V} + 2\mathbf{\Omega} \times \mathbf{V} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = -\frac{1}{\rho}\bigtriangledown p + \frac{1}{\rho} div\mathbf{S}, \tag{3.5}$$

where ρ is the density and r is the radial coordinate given by $r^2 = x^2 + y^2$ and velocity $\mathbf{V}$ is defined by

$$\mathbf{V} = [u, \theta, v]. \tag{3.6}$$

It follows from (3.2b) and (3.6) that for a rigid plate $v = 0$ and, hence, from equations (3.1) to (3.3) and (3.6), equation (3.5) becomes

$$\begin{aligned} -2\Omega\theta &= \nu\tfrac{\mathrm{d}^2 u}{\mathrm{d}z^2} + \tfrac{2\beta}{\rho}\tfrac{\mathrm{d}u}{\mathrm{d}z}[\tfrac{\mathrm{d}u}{\mathrm{d}z}((\tfrac{\mathrm{d}u}{\mathrm{d}z})^2 + (\tfrac{\mathrm{d}\theta}{\mathrm{d}z})^2)], \\ 2\Omega(u - U_0) &= \nu\tfrac{\mathrm{d}^2\theta}{\mathrm{d}z^2} + \tfrac{2\beta}{\rho}\tfrac{\mathrm{d}\theta}{\mathrm{d}z}[\tfrac{\mathrm{d}u}{\mathrm{d}z}((\tfrac{\mathrm{d}u}{\mathrm{d}z})^2 + (\tfrac{\mathrm{d}\theta}{\mathrm{d}z})^2)]. \end{aligned} \tag{3.7}$$

The boundary conditions necessary to find the solution are the following,

$$\begin{aligned} &u = \theta = 0 \quad at \quad z = 0, \\ &u \to U_0, \quad \theta \to 0 \quad as \quad z \to \infty. \end{aligned} \tag{3.8}$$

Before attempting the solution, we reduce equations (3.7) and boundary conditions (3.8) to non-dimensional form. With $\eta = zU_0/\nu$, $E = 2\Omega\nu/U_0^2$, $\gamma = 2\beta U_0^4/\nu^3$, $F(\eta) = -1 + (u + i\theta)/U_0$, equations (3.7) and boundary conditions (3.8) can be combined as

$$\frac{\mathrm{d}^2 F}{\mathrm{d}\eta^2} + \gamma\frac{\mathrm{d}}{\mathrm{d}\eta}[(\frac{\mathrm{d}F}{\mathrm{d}\eta})^2\frac{\mathrm{d}\bar{F}}{\mathrm{d}\eta}] - iEF = 0, \tag{3.9}$$

with

$$F(0) = -1, \quad F(\eta) \to 0 \quad as \quad \eta \to \infty, \tag{3.10}$$

in which

$$\bar{F} = -1 + (u - i\theta)/U_0. \tag{3.11}$$

3.4 Analysis of the main equation

We analyse (3.9) with a view to obtaining exact solutions. Firstly, we let $F = G + iH$, i.e., $G = -1 + u/U_0$ and $H = \theta/U_0$, so that the real and complex parts give rise to the system of odes

$$\begin{aligned} &G'' + \gamma\tfrac{\mathrm{d}}{\mathrm{d}\eta}[G'(-G'^2 + H'^2)] + EH = 0, \\ &H'' + \gamma\tfrac{\mathrm{d}}{\mathrm{d}\eta}[H'(G'^2 - H'^2)] - EG = 0. \end{aligned} \tag{3.12}$$

We attempt a variational analysis on (3.12). However, we begin considering a Lagrangian of type

$$L = \frac{1}{2}(G'^2 + H'^2) + \gamma(\frac{1}{2}G'^2H'^2 - \frac{1}{4}G'^4 - \frac{1}{4}H'^4) + \mathcal{F}(G, H) \tag{3.13}$$

whose Euler-Lagrange equations are

$$\begin{aligned} &G'' + \gamma\tfrac{\mathrm{d}}{\mathrm{d}\eta}[G'(-G'^2 + H'^2)] - \tfrac{\partial\mathcal{F}}{\partial G} = 0, \\ &H'' + \gamma\tfrac{\mathrm{d}}{\mathrm{d}\eta}[H'(G'^2 - H'^2)] - \tfrac{\partial\mathcal{F}}{\partial H} = 0. \end{aligned} \tag{3.14}$$

First we compute the Noether point symmetry generators (which keep invariant the functional) admitted by (3.13), which are obtained by solving

$$XL + L\frac{\mathrm{d}\tau}{\mathrm{d}\eta} = \frac{\mathrm{d}f}{\mathrm{d}\eta}, \tag{3.15}$$

where $f = f(\eta, G, H)$ is some gauge term and $X = \tau\frac{\partial}{\partial\eta} + \phi\frac{\partial}{\partial G} + \zeta\frac{\partial}{\partial H}$. Now, the partial derivatives of L are

$$\begin{aligned} \frac{\partial L}{\partial\eta} &= 0, \\ \frac{\partial L}{\partial G} &= \mathcal{F}_G, \\ \frac{\partial L}{\partial H} &= \mathcal{F}_H, \\ \frac{\partial L}{\partial G'} &= G' + \gamma(G'H'^2 - G'^3), \\ \frac{\partial L}{\partial H'} &= H' + \gamma(H'G'^2 - H'^3). \end{aligned}$$

If we substitute these into (3.15), we obtain

$\phi\mathcal{F}_G + \zeta\mathcal{F}_H + [\phi_\eta + (\phi_G - \tau_\eta)G' + H'\phi_H - G'^2\tau_G - \tau_H G'H'][G' + \gamma(G'H'^2 - G'^3)]$

$[\zeta_\eta + G'\zeta_G + H'(\zeta_H - \tau_\eta) - G'H'\tau_G - H'^2\tau_H][H' + \gamma(H'G'^2 - H'^3)] + [\frac{1}{2}(G'^2 + H'^2) +$

$\gamma(\frac{1}{2}G'^2H'^2 - \frac{1}{4}G'^4 - \frac{1}{4}H'^4) + \mathcal{F}(G,H)][\tau_\eta + G'\tau_G + H'\tau_H] = f_\eta + G'f_G + H'f_H.$

Splitting this by the powers of G' and H' yields

$$
\begin{aligned}
G'^5 &: \tau_G = 0 \\
H'^5 &: \tau_H = 0 \\
G'^4 &: \phi_G - \frac{3}{4}\tau_\eta = 0 \\
G'^3H' &: \zeta_G - \phi_H = 0 \\
G'^2H'^2 &: (\phi_G - \tau_\eta) + (\zeta_\eta - \tau_\eta) + \frac{1}{2}\tau_\eta = 0 \\
G'^3 &: \phi_\eta = 0 \\
H'^3 &: \zeta_\eta = 0 \\
G'^2 &: \phi_G = 0 \\
H'^2 &: \zeta_H = 0 \\
G'H' &: \phi_H + \zeta_G = 0 \\
G'H' &: f_\eta = \phi\mathcal{F}_G + \zeta\mathcal{F}_H \\
G' &: f_G = 0 \\
H' &: f_H = 0.
\end{aligned}
$$

Solving these leads to

$$\begin{aligned}
\tau &= A, \\
\phi &= B, \\
\zeta &= C, \\
f &= (B\mathcal{F}_G + C\mathcal{F}_H)\eta + D,
\end{aligned} \tag{3.16}$$

where A, B, C and D are constants. Thus (3.13) admits the following three Noether point symmetry generators

$$X_1 = \frac{\partial}{\partial G}, \quad X_2 = \frac{\partial}{\partial H}, \quad X_3 = \frac{\partial}{\partial \eta} \tag{3.17}$$

with gauge terms

$$f_1 = \eta\mathcal{F}_G, \quad f_2 = \eta\mathcal{F}_H, \quad f_3 = 0. \tag{3.18}$$

Correspondingly, with the use of Noethers theorem, the conserved quantities are

$$\begin{aligned}
I_1 &= G' + \gamma G'(H'^2 - G'^2) - \eta\mathcal{F}_G, \\
I_2 &= H' + \gamma H'(-H'^2 + G'^2) - \eta\mathcal{F}_H, \\
I_3 &= -\tfrac{1}{2}G'^2 - \tfrac{1}{2}H'^2 - 3\gamma(\tfrac{1}{2}G'^2H'^2 - \tfrac{1}{4}G'^4 - \tfrac{1}{4}H'^4) + \mathcal{F}.
\end{aligned} \tag{3.19}$$

The conservation laws obtained are

$$\begin{aligned}
\tfrac{\mathrm{d}I_1}{\mathrm{d}\eta} &= G'' + \gamma\tfrac{\mathrm{d}}{\mathrm{d}\eta}[G'(-G'^2 + H'^2)] - G'\mathcal{F}_{GG}\eta - H'\mathcal{F}_{GH}\eta - \mathcal{F}_G, \\
\tfrac{\mathrm{d}I_2}{\mathrm{d}\eta} &= H'' + \gamma\tfrac{\mathrm{d}}{\mathrm{d}\eta}[H'(G'^2 - H'^2)] - \mathcal{F}_{GH}G'\eta - \mathcal{F}_{HH}H'\eta - \mathcal{F}_H, \\
\tfrac{\mathrm{d}I_3}{\mathrm{d}\eta} &= -G'(G'' + \gamma\tfrac{\mathrm{d}}{\mathrm{d}\eta}[G'(-G'^2 + H'^2)] - \mathcal{F}_G) - H(H'' + \gamma\tfrac{\mathrm{d}}{\mathrm{d}\eta}[H'(G'^2 - H'^2)] - \mathcal{F}_H).
\end{aligned} \tag{3.20}$$

Now, from (3.20), considering the first two equations, it follows that

$$\begin{aligned}
G' + \gamma G'(H'^2 - G'^2) - \eta\mathcal{F}_G &= k \\
H' + \gamma H'(-H'^2 + G'^2) - \eta\mathcal{F}_H &= l,
\end{aligned} \tag{3.21}$$

where k and l are constants. We now work out G and H by solving (3.21) simultaneously. One can easily show that

$$G' = \frac{H'(k + \eta \mathcal{F}_G)}{2H' - l - \eta \mathcal{F}_H}, \tag{3.22}$$

$$H'^2 = \frac{H'^2(k + \eta \mathcal{F}_G)^2}{(2H' - (l + \eta \mathcal{F}_H))^2} + \frac{H' - (l + \eta \mathcal{F}_H)}{\gamma H'}. \tag{3.23}$$

For convenience we let $A = k + \eta \mathcal{F}_G$ and $B = l + \eta \mathcal{F}_H$ in (3.23). Multiplying by $(2H' - (l + \eta \mathcal{F}_H))^2$ on both sides of (3.23), we obtain

$$(2H' - B)^2 H'^2 = A^2 H^2 + \frac{(H' - B)(2H' - B)^2}{\gamma H'}. \tag{3.24}$$

After some straightforward manipulation (3.24) becomes

$$\gamma(2H' - B)^2 H'^3 = \gamma A^2 H^3 + (H' - B)(2H' - B)^2. \tag{3.25}$$

Expanding this results in

$$\gamma H'^3(4H'^2 - 4BH' + B^2) = \gamma A^2 H^3 + H'(4H'^2 - 4BH' + B^2) - B(4H'^2 - 4BH' + B^2). \tag{3.26}$$

After cancellation of a few terms (3.26) reduces to

$$4\gamma H'^5 - 4\gamma B H'^4 + (\gamma B^2 - \gamma A^2 - 4)H'^3 + 8BH'^2 - 5B^2 H' + B^3 = 0. \tag{3.27}$$

If we let $B = 0$, then (3.27) becomes

$$4\gamma H'^5 + (-\gamma A^2 - 4)H'^3 = 0, \tag{3.28}$$

which when solved (with the use of Mathematica) gives the following three solutions

$$\begin{aligned} H' &= 0 \\ H' &= -\tfrac{\sqrt{4+A^2\gamma}}{2\sqrt{\gamma}} \\ H' &= \tfrac{\sqrt{4+A^2\gamma}}{2\sqrt{\gamma}}. \end{aligned} \tag{3.29}$$

Integrating the second of these with respect to η, with $A = k + m\eta$, yields

$$H(\eta) = -\frac{\frac{(k+m\eta)\sqrt{4+\gamma(k+m\eta)^2}}{2m} + \frac{ArcSinh[\frac{1}{2}\sqrt{\gamma}(k+m\eta)]}{\sqrt{\gamma}m}}{2\sqrt{\gamma}} \tag{3.30}$$

(see figures 3.1 and 3.2 below).

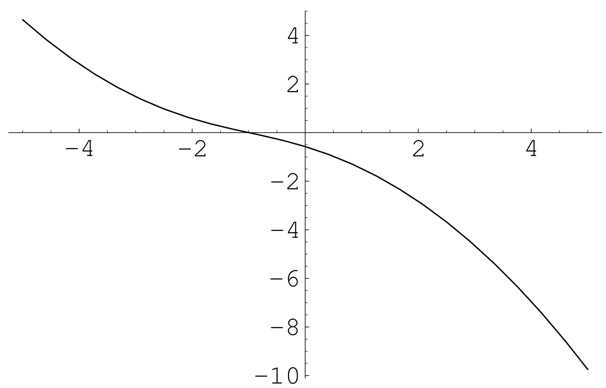

Figure 3.1: $H(\eta)$ - $m = 1$, $k = 1$, $\gamma = 4$

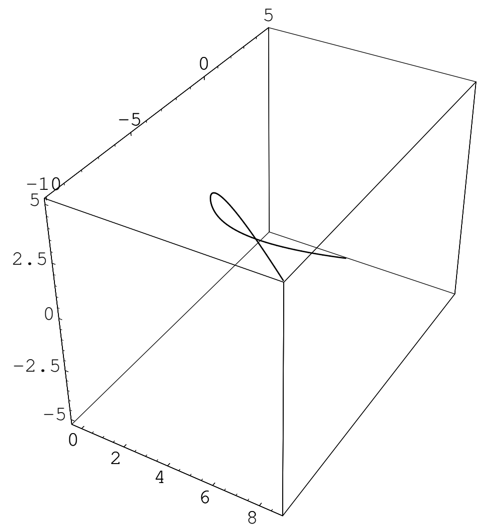

Figure 3.2: *$H-G$ paths, η parameter*

(vertical axis)

Now consider the case when $A=0$. Equation (3.27) becomes

$$4\gamma H'^5 - 4\gamma B H'^4 + (\gamma B^2 - 4)H'^3 + 8BH'^2 - 5B^2H' + B^3 = 0. \tag{3.31}$$

Solving this in a usual way gives the following real solutions

$$\begin{aligned} H' &= \tfrac{1}{2}B, \\ H' &= \frac{(\frac{2}{3})^{\frac{1}{3}}}{(-9B\gamma^2+\sqrt{3}\sqrt{-4\gamma^3+27B^2\gamma^4})^{\frac{1}{3}}} + \frac{(-9B\gamma^2+\sqrt{3}\sqrt{-4\gamma^3+27B^2\gamma^4})^{\frac{1}{3}}}{2^{\frac{1}{3}}3^{\frac{2}{3}}\gamma} \end{aligned} \tag{3.32}$$

However the second of these gives a very long and tedious solution - see (3.4) for the Mathematica version.

3.5 Conclusion

In this chapter we have presented the formulatation of the non-linear equation governing the steady state flow of a fluid , which is called the third order fluid or the fluid of the third grade, past an infinite plate which is at rest. We then did a variational analysis in the governing equation, with emphasis on the Lagrangian formulation. Also, some of the invariant solutions were obtained by making use of the software package, Mathematica.

$$
\begin{aligned}
H[\eta] \to -\frac{1}{6\gamma}\Bigg(& \frac{\left(\frac{\sqrt{-4\gamma^3+27b^2\gamma^4}}{2\,2^{1/3}\,3^{1/6}\,\gamma^2} - \frac{3}{2}\left(\frac{3}{2}\right)^{1/3}(1+n\eta)\right)\left(\sqrt{3}\,\sqrt{-4\gamma^3+27b^2\gamma^4}-9\gamma^2(1+n\eta)\right)^{1/3}}{n} + \\
& \Bigg((1+n\eta)\left(-9\gamma^2(1+n\eta)+\sqrt{3}\,\sqrt{-4\gamma^3+27\gamma^4(1+n\eta)^2}\right) \\
& \Bigg(\frac{108\gamma^6}{\left(-9\gamma^2(1+n\eta)+\sqrt{3}\,\sqrt{-4\gamma^3+27\gamma^4(1+n\eta)^2}\right)^{4/3}} + \\
& \frac{3}{8}\left(-9\gamma^2(1+n\eta)+\sqrt{3}\,\sqrt{-4\gamma^3+27\gamma^4(1+n\eta)^2}\right)^{8/3}\Bigg)\Bigg) \Bigg/ \\
& \Bigg(2\,2^{2/3}\,3^{1/3}\,n\,\gamma^2\left(-9\gamma^2+\frac{27\sqrt{3}\,\gamma^4(1+n\eta)}{\sqrt{-4\gamma^3+27\gamma^4(1+n\eta)^2}}\right) \\
& \left(-12\gamma^3-\left(-9\gamma^2(1+n\eta)+\sqrt{3}\,\sqrt{-4\gamma^3+27\gamma^4(1+n\eta)^2}\right)^2\right) \\
& \left(-\frac{1}{9\gamma^2}-\frac{-12\gamma^3-\left(-9\gamma^2(1+n\eta)+\sqrt{3}\,\sqrt{-4\gamma^3+27\gamma^4(1+n\eta)^2}\right)^2}{18\gamma^2\left(-9\gamma^2(1+n\eta)+\sqrt{3}\,\sqrt{-4\gamma^3+27\gamma^4(1+n\eta)^2}\right)^2}\right)\Bigg) - \\
& \Bigg(\sqrt{-4\gamma^3+27\gamma^4(1+n\eta)^2}\left(-12\gamma^3+\left(-9\gamma^2(1+n\eta)+\sqrt{3}\,\sqrt{-4\gamma^3+27\gamma^4(1+n\eta)^2}\right)^2\right) \\
& \Bigg(-288\gamma^6-96\gamma^3\left(-9\gamma^2(1+n\eta)+\sqrt{3}\,\sqrt{-4\gamma^3+27\gamma^4(1+n\eta)^2}\right)^2 + \\
& \left(-9\gamma^2(1+n\eta)+\sqrt{3}\,\sqrt{-4\gamma^3+27\gamma^4(1+n\eta)^2}\right)^4\Bigg)\Bigg) \Bigg/ \Bigg(96\,2^{2/3}\,3^{1/3}\,n\,\gamma^4 \\
& \left(-9\gamma^2+\frac{27\sqrt{3}\,\gamma^4(1+n\eta)}{\sqrt{-4\gamma^3+27\gamma^4(1+n\eta)^2}}\right)\left(-9\gamma^2(1+n\eta)+\sqrt{3}\,\sqrt{-4\gamma^3+27\gamma^4(1+n\eta)^2}\right)^{7/3} \\
& \sqrt{\frac{\left(-12\gamma^3+\left(-9\gamma^2(1+n\eta)+\sqrt{3}\,\sqrt{-4\gamma^3+27\gamma^4(1+n\eta)^2}\right)^2\right)^2}{\left(-9\gamma^2(1+n\eta)+\sqrt{3}\,\sqrt{-4\gamma^3+27\gamma^4(1+n\eta)^2}\right)^2}} \\
& \left(-\frac{1}{9\gamma^2}-\frac{-12\gamma^3-\left(-9\gamma^2(1+n\eta)+\sqrt{3}\,\sqrt{-4\gamma^3+27\gamma^4(1+n\eta)^2}\right)^2}{18\gamma^2\left(-9\gamma^2(1+n\eta)+\sqrt{3}\,\sqrt{-4\gamma^3+27\gamma^4(1+n\eta)^2}\right)^2}\right) \\
& \sqrt{-4\gamma^3+\frac{\left(-12\gamma^3-\left(-9\gamma^2(1+n\eta)+\sqrt{3}\,\sqrt{-4\gamma^3+27\gamma^4(1+n\eta)^2}\right)^2\right)^2}{12\left(-9\gamma^2(1+n\eta)+\sqrt{3}\,\sqrt{-4\gamma^3+27\gamma^4(1+n\eta)^2}\right)^2}}\Bigg)\Bigg) + C[1]
\end{aligned}
$$

Figure 3.3: *Solution to (3.32b)*

Chapter 4

Conclusion

In this report, the role of invariants in obtaing exact solutions of differential equations was presented. The results of this report can be summerized as follows.

In chapte 2, we discussed a straight analysis of (2.1). The Fisher equation is well known of only admitting symmetries involving time and space translations. The knowledge that the equations are by construction invariant under space-time symmetry groups ensures that we can always apply the symmetry reduction method to find exact particular solutions called group invariant solutions. Also, after a long exhaustive direct method of finding conserved vectors (T,S), we concluded that the Fisher equation has no conservation laws. Furthermore, we carried out a similar but brief study of a generalised version of the Fitzhugh-Nagumo equation.

Chapter 3 is devoted to the formulation of a nonlinear equation governing steady rotating flow of a differential type non-Newtonian fluid of grade three past an infinite plate. We then carried out an analytic approach on the governing equation using the group theoretic approach and a Lagrangian analysis.

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