ULTRAFILTER SEMIGROUPS

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Declaration

I declare that this thesis was composed by myself while registered as a student at The University of the Witwatersrand, and that I have acknowledged all sources of information contained herein. One of the main results of this thesis was presented at the annual congress of the South African Mathematical Society in 2009 and another will be presented at the same congress in November 2010. Both results were also presented at the seminars of The School of Mathematics, University of the Witwatersrand.

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Abstract

(1) Given a local left topological group X with a distinguished element 0, denote by $Ult(X)$ the subsemigroup of βX consisting of all nonprincipal ultrafilters on X converging to 0. Any two countable nondiscrete zero-dimensional local left topological groups X and Y with countable bases are locally isomorphic and, consequently, the subsemigroups $Ult(X) \subseteq \beta X$ and $Ult(Y) \subseteq \beta Y$ are isomorphic. However, not every two homeomorphic zero-dimensional local left topological groups are locally isomorphic. In the first result of this thesis it is shown that for any two homeomorphic direct sums X and Y, the semigroups $Ult(X)$ and $Ult(Y)$ are isomorphic.

(2) Let S be a discrete semigroup, let βS be the Stone-Čech compactification of S, and let T be a closed subsemigroup of βS. The second and main result of this thesis consists of characterizing ultrafilters from the smallest ideal $K(T)$ of T and from its closure $c\ell\ K(T)$, and showing that for a large class of closed subsemigroups T of βS, $c\ell\ K(T)$ is not an ideal. This class includes the subsemigroups $0^+ \subset \beta\mathbb{R}_d$ and $\mathbb{H}_\lambda \subset \beta(\bigoplus_\lambda \mathbb{Z}_2)$.

Introduction

Historical background

$\beta\mathbb{N}$, the Stone-Čech compactification of the discrete space $\mathbb{N}$, can be described as follows.

(a) The elements of $\beta\mathbb{N}$ are the ultrafilters on $\mathbb{N}$, with elements of $\mathbb{N}$ being identified with principal ultrafilters.

(b) The base for the topology consists of subsets of the form

$$\overline{A} = \{p \in \beta\mathbb{N} : A \in p\},$$

where $A \subseteq \mathbb{N}$.

Being the Stone-Čech compactification of $\mathbb{N}$, every (continuous) function $f : \mathbb{N} \to Y$ into a compact Hausdorff topological space Y factors uniquely through $\beta\mathbb{N}$, i.e. there exists a unique continuous function $\bar{f} : \beta\mathbb{N} \to Y$ such that $\bar{f}_{|\mathbb{N}} = f$. In particular, every compactification of $\mathbb{N}$ is a quotient of $\beta\mathbb{N}$. The semigroup operation $+$ on $\mathbb{N}$ can be extended to $\beta\mathbb{N}$ as follows. For any $p, q \in \beta\mathbb{N}$, $p + q$ is defined by

$$A \in p + q :\Longleftrightarrow \{x \in \mathbb{N} : -x + A \in q\} \in p,$$

where for every $x \in \mathbb{N}$ and every $A \subseteq \mathbb{N}$, $-x+A = \{y \in \mathbb{N} : x+y \in A\}$. Then for each $p \in \beta\mathbb{N}$, the translation $\rho_p : \beta\mathbb{N} \ni q \mapsto q + p \in \beta\mathbb{N}$ is continuous, and for each $n \in \mathbb{N}$, the translation $\lambda_n : \beta\mathbb{N} \ni q \mapsto n + q \in \beta\mathbb{N}$ is continuous. Although the same extension can be done for $(\mathbb{N}, \cdot)$, unless otherwise stated, when we write $\beta\mathbb{N}$, we mean $(\beta\mathbb{N}, +)$.

This extension works for any discrete semigroup S, making the Stone-Čech compactification βS of S a right topological semigroup with S contained in its topological center. That means for each $p \in \beta S$, the right translation $\rho_p : \beta S \ni q \mapsto qp \in \beta S$ is continuous, and for each $x \in S$, the left translation $\lambda_x : \beta S \ni q \mapsto xq \in \beta S$ is continuous, respectively. This extension was explicitly established by Day [10] in 1957 using the methods of Arens [1].

For any semigroup S with a smallest ideal, we use $K(S)$ to denote that smallest ideal. As any compact right topological semigroup, βS has a smallest ideal $K(\beta S)$. In addition, since S is dense in βS and S is a subset of the topological center of βS (the set of all ultrafilters $p \in \beta S$ such that $\lambda_p : \beta S \to \beta S$ is continuous), the topological center of βS is dense in βS and, thus, the closure $c\ell\, K(\beta S)$ is also an ideal of βS.

The semigroup βS has an algebraic structure of extraordinary complexity which makes βS interesting for its own sake. The semigroup βS is also interesting for its important applications to Ramsey Theory and topological dynamics.

A typical problem in Ramsey Theory can be thought of as follows. One has a set X and a collection $\mathcal{G}$ of certain subsets of X. One then asks whether, whenever X is partitioned into finitely many pieces, one piece must contain a member of $\mathcal{G}$. Viewed in this way one can see why we prefer the construction of $\beta\mathbb{N}$ with ultrafilters on $\mathbb{N}$ as points. If p is an ultrafilter on $\mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{n} A_i$ is a finite partition of $\mathbb{N}$, then $A_i \in p$ for exactly one $i \in \{1, \cdots, n\}$.

The Graham-Rothschild conjecture, now known as the Finite Sum Theorem, is one of the concepts that connect $\beta\mathbb{N}$ and Ramsey Theory. This theorem says the following.

Theorem 0.0.1. *Let $r \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^{r} A_i$ be a partition of $\mathbb{N}$. Then there exists $i \in \{1, 2, \cdots, r\}$ and a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $FS((x_n)_{n\in\mathbb{N}}) \in A_i$.*

For any nonempty subset $B \subseteq \mathbb{N}$,

$$FS(B) = \{\Sigma F : F \text{ is a nonempty finite subset of } B\}$$

and

$$FP(B) = \{\prod F : F \text{ is a nonempty finite subset of } B\}.$$

More generally, let S be a semigroup, let D be a linearly ordered set, and let $(x_s)_{s\in D}$ be a D-sequence in S.

(a) For $F \in \mathcal{P}_f(S)$, $\prod_{s\in F} x_s$ is the product in the increasing order of indices.

(b) $FP((x_s)_{s\in D}) = \{\prod_{s\in F} x_s : F \in \mathcal{P}_f(S)\}$.

(c) The D-sequence $(x_s)_{s\in D}$ is said to satisfy the *distinct finite products property* if and only if whenever $F, G \in \mathcal{P}_f(S)$ and $\prod_{s\in F} x_s = \prod_{s\in G} x_s$, one has $F = G$.

(Given a set X, we use $\mathcal{P}(X)$ to denote the powerset of X, and $\mathcal{P}_f(X)$ to denote the collection of all nonempty finite subsets of X.)

The Graham-Rothschild conjecture had been open for some decades, even though several mathematicians (including Hilbert) worked on it. Fred Galvin worked on this conjecture around the year 1970. At some point Galvin wanted to know whether there could exist $p \in \beta\mathbb{N}$ such that whenever $A \in p$, $\{x \in \mathbb{N} : -x + A \in p\} \in p$. Galvin called such ultrafilters "almost downward translation invariant ultrafilters". In current terminologies Galvin wanted to know if there could exist an idempotent in $\beta\mathbb{N}$, i.e. $p \in \beta\mathbb{N}$ such that $p + p = p$. He had a construction showing that any element of such an ultrafilter contains $FS(B)$ for some infinite subset $B \subseteq \mathbb{N}$. In fact, using current terminologies, what Galvin knew amounts to the following fact: The set

$$\Omega = \{p \in \beta\mathbb{N} : \text{for all } A \in p \text{ there exists an infinite } B \subseteq A \text{ with } FS(B) \subseteq A\}$$

is the closure of the set $\{p \in \beta\mathbb{N} : p + p = p\}$ in $\beta\mathbb{N}$.

Around that time Steven Glazer was one of the few mathematicians who knew about the extension of the semigroup operation from a discrete semigroup S to βS, and who were also aware of the following theorem of Ellis [15]: Every compact right topological semigroup contains an idempotent. These mathematicians also viewed the underlying set of βS as the set of all ultrafilters on S. So, one day in 1975 Galvin learned from Glazer that idempotents exist in $\beta\mathbb{N}$ and together, using this information, they devised a simple and elegant proof of the Finite Sum Theorem.

Going back a little, around 1972, while trying to answer the question of Galvin about the existence of almost downward translation invariant ultrafilters in $\beta\mathbb{N}$, Neil Hindman established the following result, where

$$\Gamma = \{A \subseteq \mathbb{N} : \text{there exists an infinite } B \subseteq A \text{ with } FS(B) \subseteq A\}.$$

Theorem 0.0.2. *The Graham-Rothschild conjecture holds if and only if there is an ultrafilter p on $\mathbb{N}$ such that $p \subseteq \Gamma$.*

The ultrafilter produced in Theorem 0.0.2 does not necessarily answer Galvin's question even if the Graham-Rothschild conjecture is assumed to be true. It is in fact possible to produce, under this assumption, an ultrafilter contained in Γ which has an element A such that $-x+A \notin \Gamma$ for every x in A. However Hindman established the following result.

Theorem 0.0.3. *Assume the continuum hypothesis. The Graham-Rothschild conjecture implies, and therefore is equivalent to, the existence of an almost downward translation invariant ultrafilter on $\mathbb{N}$.*

In fact, Galvin also asked whether what he called "almost upward translation invariant ultrafilters" exist. This question was answered in the negative by Hindman. Soon thereafter, Hindman showed directly, using a combinatorial argument, that the Graham-Rothschild conjecture is true. However, the proof was of enormous complexity.

Since the proof by Galvin and Glazer, there have been numerous applications of the algebraic structure of $\beta\mathbb{N}$ to Ramsey Theory, various strong combinatorial results have been obtained using the algebraic structure of $\beta\mathbb{N}$, and the Finite Sum Theorem has been extended using the combination of addition and multiplication in $\beta\mathbb{N}$. Other famous combinatorial theorems that were initially established using combinatorial methods, such as the van der Waerden's Theorem, were reestablished elegantly using the algebraic structure of $\beta\mathbb{N}$. In the process more came to be known about the algebraic structure of βS, for a general semigroup S.

Among other interesting results and questions about the structure of $\beta\mathbb{N}$, we mention the following.

(1) The center of $\beta\mathbb{N}$ is $\mathbb{N}$. Even more than that, $\beta\mathbb{N}$ contains copies of the free group on 2^{2^ω} generators. (We use ω to denote the first infinite cardinal, i.e. $\omega = \{0, 1, 2, 3, \cdots\}$.)

(2) $\beta\mathbb{N}$ has 2^{2^ω} minimal left ideals and 2^{2^ω} minimal right ideals.

(3) It is not yet known whether $\beta\mathbb{N}$ contains any element of finite order, other than the idempotents.

(4) Only recently it was shown that there are no finite nontrivial groups in $\beta\mathbb{N}$. See [63] and [64].

(5) Let

$$\mathbb{H} = \bigcap_{n\in\omega} c\ell_{\beta\mathbb{N}}\, 2^n\mathbb{N} = \bigcap_{n\in\omega} \overline{2^n\mathbb{N}}\ ,\ \mathbb{T} = \bigcap_{n\in\mathbb{N}} c\ell_{\beta\mathbb{N}}\, n\mathbb{N} = \bigcap_{n\in\mathbb{N}} \overline{n\mathbb{N}}.$$

Then,

(a) $\mathbb{T}$ and $\mathbb{H}$ are subsemigroups of $\beta\mathbb{N}$ with $\mathbb{T} \subset \mathbb{H}$.

(b) All idempotents of $\beta\mathbb{N}$ are in $\mathbb{T}$.

(c) If $\mathcal{A}$ is a left ideal, a right ideal, a minimal left ideal, a minimal right ideal, or the smallest ideal of $\beta\mathbb{N}$, then $\mathcal{A} \cap \mathbb{T}$ is the corresponding object in $\mathbb{T}$.

(c) There are functions $f : \mathbb{N} \to S$, where S is a compact Hausdorff right topological semigroup, such that f is not a homomorphism but $\bar{f}(p+q) = \bar{f}(p)+\bar{f}(q)$ for every $p \in \beta\mathbb{N}$ and every $q \in \mathbb{H}$ (where $\bar{f} : \beta\mathbb{N} \to S$ is the continuous extension on f). In particular the restriction of $\bar{f}$ to $\mathbb{H}$ is a homomorphism.

We now look at the applications of βS to topological dynamics. A *topological dynamical system* is a pair $(X, (T_s)_{s\in S})$ where

(1) X is a compact Hausdorff space,

(2) S is a semigroup,

(3) for each $s \in S$, T_s is a continuous function from X to X, and

(4) for each $s, t \in S$, $T_s \circ T_t = T_{st}$.

For our introductory purpose we consider the case when $S = (\omega, +)$. Therefore, unless otherwise stated, by a topological dynamical system we will mean a topological dynamical system of the form $(X, (T_n)_{n\in\omega})$ with T_0 equal to the identity function of X. In this case, let $T = T_1$. Then, for any $n \geq 1$, $T_n = T_{1+\cdots+1} = T_1 \circ \cdots \circ T_1 = T^n$, i.e. T_n is the n^{th} iterate of T. We define

$T^0 = T_0$, the identity function of X. We will write (X, T) for $(X, (T_n)_{n\in\omega})$. Topological dynamics is concerned with the behavior of the iterates of T. To study the limiting behavior of the iterates of T for large n, one defines for each ultrafilter $\mathcal{U} \in \beta\omega$,

$$T^{\mathcal{U}} : X \to X \ , \ T^{\mathcal{U}}(x) = \mathcal{U}\text{-}\lim_{n\in\mathbb{N}} T^n(x),$$

(see Chapter 1, Definition 1.1.13). It is clear that if $\mathcal{U}$ is an ultrafilter generated by m, then $T^{\mathcal{U}} = T^m$. We also have $T^{\mathcal{U}}(T^{\mathcal{V}}(x)) = T^{\mathcal{U}+\mathcal{V}}(x)$ for every $x \in X$ and for every $\mathcal{U}, \mathcal{V} \in \beta\omega$. Regarded as a function of $\mathcal{U} \in \beta\omega$, for a fixed $x \in X$, this is the continuous extension to $\beta\omega$ of the function $\omega \ni n \mapsto T^n(x) \in X$. It follows that $\{T^{\mathcal{U}}(x) : \mathcal{U} \in \beta\omega\}$ is the closure of the forward orbit $\{T^n(x) : n \in \omega\}$ of the point x. More generally, a semigroup S of continuous functions acting on a compact Hausdorff space X has a closure in X^X called the enveloping semigroup of S, [15]. The enveloping semigroup is a compact right topological semigroup and, therefore, it is a quotient of βS. In some important cases it is equal to βS. The algebraic properties of βS have implications for the dynamical behavior of the topological dynamical system.

The topological space $\beta\omega$ with the function $P : \beta\omega \ni \mathcal{U} \mapsto 1 + \mathcal{U} \in \beta\omega$ is a topological dynamical system. One observes that addition in $\beta\omega$ is just the ultrafilter iteration of P, i.e. for any $\mathcal{U}, \mathcal{V} \in \beta\omega$, $P^{\mathcal{U}}(\mathcal{V}) = \mathcal{U} + \mathcal{V}$. For any topological dynamical system (X, T) and for any $x \in X$, the function

$$f : \beta\omega \to X, \ \ \mathcal{U} \mapsto T^{\mathcal{U}}(x)$$

is the unique continuous function from $\beta\omega$ to X with $f \circ P = T \circ f$ and $f(0) = x$.

$$\begin{tikzcd}
\beta\omega \arrow[r, "f"] \arrow[d, "P"'] & X \arrow[d, "T"] \\
\beta\omega \arrow[r, "f"] & X.
\end{tikzcd}$$

Therefore $(\beta\omega, P)$ may be regarded as the free topological dynamical system on one generator.

Let (X, T) be a topological dynamical system and let $x \in X$. Then,

(1) x is called *recurrent* if for each neighborhood G of x there are infinitely many $n \in \omega$ such that $T^n(x) \in G$, and

(2) x is called *uniformly recurrent* if for each neighborhood G of x there exists $M \in \omega$ such that for any $n \in \omega$ there is $k < M$ such that $T^{n+k}(x) \in G$.

Theorem 0.0.4. *Let (X, T) be a topological dynamical system.*

(1) A point $x \in X$ is recurrent if and only if $T^{\mathcal{U}}(x) = x$ for some nonprincipal ultrafilter $\mathcal{U}$ on ω.

(2) A point $x \in X$ is uniformly recurrent if and only if for every ultrafilter $\mathcal{V}$ on ω there is an ultrafilter $\mathcal{U}$ on ω such that $T^{\mathcal{U}}(T^{\mathcal{V}}(x)) = x$.

The interaction with topological dynamics works both ways. Several combinatorial notions which originated in topological dynamics such as *syndetic* and *piecewise syndetic sets*, are important in describing the algebraic structure of βS. In a semigroup S a subset $A \subseteq S$ is said to be *syndetic* if there is a finite subset $G \subseteq S$ such that $S = G^{-1}A$, where

$$G^{-1}A = \{x \in S : tx \in A \text{ for some } t \in G\}.$$

For example, one can easily verify that in a topological dynamical system (X, T), a point $x \in X$ is uniformly recurrent if and only if for every neighborhood G of x, the set $\{n \in \omega : T^n(x) \in G\}$ is a syndetic subset of ω. A is said to be *piecewise syndetic* if there is a finite subset $G \subseteq S$ such that for every finite subset $F \subseteq S$, there exists $x \in S$ with $Fx \subseteq G^{-1}A$. We have the following very important theorem.

Theorem 0.0.5. *Let S be a semigroup, let $K(\beta S)$ be the smallest ideal of βS and let $p \in \beta S$. The following statements hold.*

(1) $p \in K(\beta S)$ if and only if for every $A \in p$, $\{x \in S : x^{-1}A \in p\}$ is syndetic.

(2) $p \in c\ell\, K(\beta S)$ if and only if every $A \in p$ is piecewise syndetic.

These combinatorial characterizations of members of $K(\beta S)$ and $c\ell\, K(\beta S)$ have been extended from βS to certain important closed subsemigroup of βS, namely, to $\mathbb{H}$ and $0^+ \subseteq \beta\mathbb{R}_d$. Here $\mathbb{R}_d$ is the additive group of real numbers reendowed with the discrete topology and $0^+ = \bigcap_{n \in \mathbb{N}} c\ell_{\beta\mathbb{R}_d}(0, \frac{1}{n})$. As mentioned earlier, $\mathbb{H}$ contains most of the information about $\beta\mathbb{N}$ and one can expect it to 'behave' in the same way as $\beta\mathbb{N}$. Surprisingly, as distinguished from $c\ell\ K(\beta S)$, $c\ell\ K(\mathbb{H})$ is not a left ideal of $\mathbb{H}$ [22]. The question of whether $c\ell\ K(0^+)$ is not a left ideal of 0^+ remained open [25]. We have now answered this question and this answer is the main result in this thesis.

In [56], J. Pym noticed that certain important subsemigroups of βS depend very little on the operation of S and they can be defined in terms of direct sums of sets with a distinguished element. We have obtained a result which says that if X and Y are homeomorphic direct sums, then the subsemigroups of βX and βY consisting of all nonprincipal ultrafilters converging to the distinguished element are isomorphic.

Synopsis

In Chapter 1 we review the elementary concepts about ultrafilters and semigroups. One of the important results presented in this chapter is the structure of a completely simple semigroup. We then present the construction of the Stone-Čech compactification βS of a discrete semigroup S and the extension of the semigroup operation from S to βS. The elements of $K(\beta S)$ and $c\ell\ K(\beta S)$ are given combinatorial characterizations as stated in Theorem 0.0.5.

Chapter 2 deals with ultrafilter semigroups. These are defined as follows. Let G be a group and let $\mathcal{T}$ be a topology on G such that the left shifts $\lambda_a : G \ni x \mapsto ax \in G$, where $a \in G$, are continuous. The nonprincipal ultrafilters on G that converge to the identity of G in this topology form a closed subsemigroup of the Stone-Čech compactification of the discrete semigroup G. We call this subsemigroup *the ultrafilter semigroup of* $(G, \mathcal{T})$. Indeed, this can be defined for any local left topological group. Ultrafilter semigroups proved useful in studying the structure of the semigroups βS. They are also a main tool in the proof of our main result. Homomorphisms of ultrafilter semigroups are induced by what are called local homomorphisms. We also present two ways of constructing left invariant topologies on a group G, using idempotents from $\beta G \setminus G$. In Section 2.5 we prove our first result, that the ultrafilter semigroups of any two homeomorphic direct sums are isomorphic.

Chapter 3 starts by presenting old results about the smallest ideals of $\mathbb{T}$ and 0^+ and the closures of these smallest ideals. We extend the combinatorial characterizations of elements of $K(\beta S)$ and $c\ell\ K(\beta S)$ to an arbitrary closed subsemigroup T of βS. We then move on to our main result, showing that for a large class of closed subsemigroups T of βS, $c\ell\ K(T)$ is not a left ideal of T. This class includes the semigroups 0^+ and $\mathbb{H}_\lambda$. For every infinite cardinal λ, $\mathbb{H}_\lambda$ is the ultrafilter semigroup of the Boolean group $\bigoplus_\lambda \mathbb{Z}_2$ in the topology having as the basic neighborhood of the identity the subgroups of the form

$$H_\alpha = \{x \in \bigoplus_\lambda \mathbb{Z}_2 : x(\gamma) = 0 \text{ for all } \gamma < \alpha\},$$

where $\alpha < \lambda$.

Chapter 4 deals with finite ultrafilter semigroups. The ultrafilter semigroup of a left topological group G reflects quite good topological properties of G, especially when the ultrafilter semigroup is finite. A topological space X is called *almost maximal* if it is without isolated points and for every element $x \in X$, there are only finitely many ultrafilters on X converging to x. We look at almost maximal spaces, almost maximal left topological semigroups, projectives in the category of finite semigroups and almost maximal topological groups.

Chapter 1

The Semigroup βS

1.1 Ultrafilters

Definition 1.1.1. Let X be a set and let $\mathcal{F}, \mathcal{B} \subseteq \mathcal{P}(X)$. $\mathcal{F}$ is called a *filter* on X if the following properties are satisfied:

(a) $\emptyset \notin \mathcal{F}$ and $X \in \mathcal{F}$,

(b) for every $A, B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$, and

(c) for any $A \in \mathcal{F}$ and any $B \subseteq X$ with $A \subseteq B$, $B \in \mathcal{F}$.

$\mathcal{B}$ is called a *filter base* on X if the following conditions are satisfied:

(i) $\emptyset \notin \mathcal{B}$ and $\emptyset \neq \mathcal{B}$, and

(ii) for any $A, B \in \mathcal{B}$, there exists $C \in \mathcal{B}$ such that $C \subseteq A \cap B$.

An *ultrafilter* on X is a filter on X that is not properly contained in any other filter on X.

When dealing with filters, the following condition will serve the same purpose as condition (ii) in the definition of a filter base.

$$\text{For any } A, B \in \mathcal{B},\ A \cap B \in \mathcal{B}.$$

For any filter base $\mathcal{B}$ on X, there is a smallest filter on X that contains $\mathcal{B}$. It is defined as follows.

$$\mathcal{F}_{\mathcal{B}} := \{A \subseteq X \mid B \subseteq A \text{ for some } B \in \mathcal{B}\}.$$

We call $\mathcal{F}_{\mathcal{B}}$ *the filter generated by* $\mathcal{B}$ and we call $\mathcal{B}$ *a filter base* (or simply *a base*) *for* $\mathcal{F}_{\mathcal{B}}$.

Let X be a topological space. For each $x \in X$, let $\mathcal{N}_x$ denotes the family of neighborhoods of x. It is easy to check, using the definition of a neighborhood and the definition of an open set, that the family $\{\mathcal{N}_x : x \in X\}$ satisfies the following conditions.

(N1) For every $x \in X$ and $U \in \mathcal{N}_x$, $x \in U$.

(N2) For every $x \in X$ and $U \in \mathcal{N}_x$, $\{y \in X \ : \ U \in \mathcal{N}_y\} \in \mathcal{N}_x$.

(N3) For every $x \in X$ and $U, V \in \mathcal{N}_x$, $U \cap V \in \mathcal{N}_x$.

(N4) For every $x \in X$ and $U \in \mathcal{N}_x$, $U \subseteq V \subseteq X$ implies $V \in \mathcal{N}_x$.

Remark 1.1.2.

(1) Condition (N2) is equivalent to the following condition.

(N2)$'$ For every $x \in X$ and for every $U \in \mathcal{N}_x$, there exists $V \in \mathcal{N}_x$ such that $U \in \mathcal{N}_y$ for every $y \in V$.

(2) (N1), (N3) and (N4) imply that $\mathcal{N}_x$ is a filter on X. $\mathcal{N}_x$ is called *the neighborhood filter at* (or *of*) x and $\{\mathcal{N}_x : x \in X\}$ is called *the neighborhood system for* X.

Proposition 1.1.3. *Let X be a set and let $\{\mathcal{N}_x : x \in X\}$ be a family of filters on X satisfying (N1) and (N2). Then there exists a topology $\mathcal{T}$ on X such that $\mathcal{N}_x$ is precisely the neighborhood filter at x for each $x \in X$.*

Proof. Let $\mathcal{T} = \{U \subseteq X : \text{for every } x \in U \text{ there is } W \in \mathcal{N}_x \text{ with } W \subseteq U\}$. Clearly $\emptyset \in \mathcal{T}$. Since for any $x \in X$, $\mathcal{N}_x \neq \emptyset$, $X \in \mathcal{T}$. Let $U, V \in \mathcal{T}$ and let $x \in U \cap V$. Pick $W_1, W_2 \in \mathcal{N}_x$ such that $W_1 \subseteq U$ and $W_2 \subseteq V$. Then $W := W_1 \cap W_2 \in \mathcal{N}_x$ and $W \subseteq U \cap V$. Therefore $U \cap V \in \mathcal{T}$. Let $(U_i)_{i \in I}$ be a family in $\mathcal{T}$ and let $x \in \bigcup_{i \in I} U_i$. Pick $j \in I$ such that $x \in U_j$ and pick $W \in \mathcal{N}_x$ with $W \subseteq U_j$. Then $W \subseteq \bigcup_{i \in I} U_i$ and, hence, $\bigcup_{i \in I} U_i \in \mathcal{T}$. It

follows that $\mathcal{T}$ is a topology on X.
Let $x \in X$ and denote by $\mathcal{F}_x$ the neighborhood filter at x in $\mathcal{T}$. Let $U \in \mathcal{N}_x$. Then $x \in U$. By (N2), $W := \{y \in X : U \in \mathcal{N}_y\} \in \mathcal{N}_x$. By (N1) $W \subseteq U$. Therefore $U \in \mathcal{T}$. Hence $U \in \mathcal{F}_x$. It follows that $\mathcal{N}_x \subseteq \mathcal{F}_x$. Now let $U \in \mathcal{F}_x$. There exists $V \in \mathcal{T}$ such that $x \in V \subseteq U$. Pick $W \in \mathcal{N}_x$ such that $W \subseteq V$. Then $W \subseteq U$. Since $\mathcal{N}_x$ is a filter, $U \in \mathcal{N}_x$. It follows that $\mathcal{F}_x \subseteq \mathcal{N}_x$. Hence $\mathcal{F}_x = \mathcal{N}_x$.

$\square$

Recall that a family $\mathcal{A}$ of sets is said to *have the finite intersection property* if for every nonempty finite subset $\mathcal{F}$ of $\mathcal{A}$, $\bigcap \mathcal{F} \neq \emptyset$, and that a topological space X is compact if and only if for every family $\mathcal{A}$ of closed subsets of X such that $\mathcal{A}$ has the finite intersection property, $\bigcap \mathcal{A} \neq \emptyset$. A subset $\mathcal{A} \subseteq \mathcal{P}(X)$ is said to be *maximal with respect to the finite intersection property* if $\mathcal{A}$ has the finite intersection property and $\mathcal{A}$ is not properly contained in any other subset of $\mathcal{P}(X)$ having the finite intersection property.

Theorem 1.1.4. *[[29], Theorem 3.6] Let X be a set and let $p \subseteq \mathcal{P}(X)$. The following statements are equivalent.*

(a) p is an ultrafilter on X.

(b) p has the finite intersection property and for every $A \subseteq X$, $A \in p$ if and only if $A \cap B \neq \emptyset$ for every $B \in p$.

(c) p is maximal with respect to the finite intersection property.

(d) p is a filter on X and for every finite $\mathcal{A} \subseteq \mathcal{P}(X)$, if $\bigcup \mathcal{A} \in p$, then $\mathcal{A} \cap p \neq \emptyset$.

(e) p is a filter on X and for every $B \subseteq X$, either $B \in p$ or $X \setminus B \in p$.

Proof. (a)$\Rightarrow$ (b): Let p be an ultrafilter on X. Since for each $A, B \in p$, $A \cap B \in p$ and $\emptyset \notin p$, $A \cap B \neq \emptyset$. Therefore p has the finite intersection property. Let $A \subseteq X$. If $A \in p$, then clearly $A \cap B \neq \emptyset$ for every $B \in p$. Conversely, suppose $A \cap B \neq \emptyset$ for every $B \in p$. Put $q = \{C \subseteq X : A \cap B \subseteq C$ for some $B \in p\}$. It is clear that q is a filter on X with $A \in q$ and $p \subseteq q$. Since p is an ultrafilter, $p = q$. Thus $A \in p$.
(b)$\Rightarrow$ (c): Suppose (b) holds. Let $\mathcal{U} \subseteq \mathcal{P}(X)$ such that $p \subseteq \mathcal{U}$ and $\mathcal{U}$ has the finite intersection property. Let $A \in \mathcal{U}$ and let $B \in p$. Then $A, B \in \mathcal{U}$ and, therefore, $A \cap B \neq \emptyset$. By (b), $A \in p$. Since A was an arbitrary element of $\mathcal{U}$,

$\mathcal{U} \subseteq p$. Therefore $p = \mathcal{U}$.
(c)$\Rightarrow$ (d): Suppose (c) holds. It is clear that $\emptyset \notin p$. Let $A, B \in p$ and let $C \subseteq X$ with $A \subseteq C$. It is easy to see that the families $p \cup \{X\}$, $p \cup \{A \cap B\}$ and $p \cup \{C\}$ have the finite intersection property and contain p. Therefore these families are equal to p. Hence p is a filter on X. Let $A, B \subseteq X$ such that $A \cup B \in p$ and suppose $A \notin p$ and $B \notin p$. Since $p \subsetneqq p \cup \{A\}$ and $p \subsetneqq p \cup \{B\}$, $p \cup \{A\}$ and $p \cup \{B\}$ do not have the finite intersection property. Pick $C_1, \cdots, C_n, D_1, \cdots, D_m \in p$, for some $n, m \in \mathbb{N}$, such that $A \cap C_1 \cap \cdots \cap C_n = \emptyset = D_1 \cap \cdots \cap D_m \cap B$. Then $\mathcal{F} := \{C_1, \cdots, C_n, D_1, \cdots, D_m, A \cup B\} \subseteq p$ with $\bigcap \mathcal{F} = \emptyset$, a contradiction. Therefore $A \in p$ or $B \in p$.
(d)$\Rightarrow$(e): Suppose (d) holds. Let $B \subseteq X$. Take $\mathcal{A} = \{B, X \setminus B\}$. Then $\bigcup \mathcal{A} = X \in p$. Therefore $B \in p$ or $X \setminus B \in p$.
(e)$\Rightarrow$(a): Suppose (e) holds. Let q be a filter on X with $p \subseteq q$. Assume $p \neq q$. Pick $A \in q \setminus p$. Then $X \setminus A \in p \subseteq q$ and, therefore, $\emptyset = A \cap (X \setminus A) \in q$, a contradiction. It follows that $p = q$ and, thus, p is an ultrafilter on X. $\square$

Theorem 1.1.5. *[[29], Theorem 3.8] Let X be a set and let $\mathcal{A} \subseteq \mathcal{P}(X)$ such that $\mathcal{A}$ has the finite intersection property. Then there is an ultrafilter p on X such that $\mathcal{A} \subseteq p$.*

Proof. Let

$$\Gamma = \{\mathcal{B} \subseteq \mathcal{P}(X) : \mathcal{A} \subseteq \mathcal{B} \text{ and } \mathcal{B} \text{ has the finite intersection property}\}.$$

Since $\mathcal{A} \in \Gamma$, $\Gamma \neq \emptyset$. Let $\mathfrak{C}$ be a nonempty chain in Γ. Then $\mathcal{A} \subseteq \bigcup \mathfrak{C}$. Let $\emptyset \neq \mathcal{F} \subseteq \bigcup \mathfrak{C}$ be finite. Since $\mathfrak{C}$ is a chain, $\mathcal{F} \subseteq \mathcal{B}$ for some $\mathcal{B} \in \mathfrak{C}$. Since $\mathcal{B}$ has the finite intersection property, $\bigcap \mathcal{F} \neq \emptyset$. Therefore, $\bigcup \mathfrak{C}$ has the finite intersection property. It is clear that $\bigcup \mathfrak{C}$ is an upper bound of $\mathfrak{C}$ in Γ. By Zorn's Lemma, Γ has a maximal element $\mathcal{U}$. Let $\mathcal{V} \subseteq \mathcal{P}(X)$ such that $\mathcal{U} \subseteq \mathcal{V}$ and $\mathcal{V}$ has the finite intersection property. Then $\mathcal{A} \subseteq \mathcal{V}$ and, therefore, $\mathcal{V} \in \Gamma$. Since $\mathcal{U}$ is a maximal element of Γ, $\mathcal{U} = \mathcal{V}$. Hence $\mathcal{U}$ is maximal with respect to the finite intersection property. Thus $\mathcal{U}$ is an ultrafilter on X, with $\mathcal{A} \subseteq \mathcal{U}$. $\square$

Since every filter has the finite intersection property, we obtain the following.

Corollary 1.1.6. *Let X be a set. Every filter on X is contained in an ultrafilter.*

Example 1.1.7. Given $x \in X$, the set $\{A \subseteq X : x \in A\}$ is an ultrafilter on X. It is called the *principal ultrafilter generated by x*. More generally, for every nonempty subset A of X, $\{B \subseteq X : A \subseteq B\}$ is a filter on X. However, if A is not a singleton, then $\{B \subseteq X : A \subseteq B\}$ is not an ultrafilter since for any $a \in A$, $\{B \subseteq X : A \subseteq B\} \subsetneq \{B \subseteq X : a \in B\}$.

A nonprincipal ultrafilter is also called a *free ultrafilter*. Note that an ultrafilter p is free if and only if $\bigcap p = \emptyset$.

Definition 1.1.8. Let X be a topological space and let $\mathcal{F}$ be a filter on X. A point $x \in X$ is called a *cluster point* of $\mathcal{F}$ if x belongs to the closure of every element of $\mathcal{F}$. A point $x \in X$ is called a *limit* of $\mathcal{F}$ if $\mathcal{F}$ contains the neighborhood filter of x. If x is a limit of $\mathcal{F}$, we say that $\mathcal{F}$ *converges to* x, and we write $\mathcal{F} \to x$.

Remark 1.1.9. Let X be a topological space, let $x \in X$ and let $\mathcal{F}$ and $\mathcal{G}$ be filters on X.

(a) Every cluster point of an ultrafilter is a limit of that ultrafilter.

(b) If $\mathcal{F} \to x$ and $\mathcal{G} \subseteq \mathcal{F}$, then x is a cluster point of $\mathcal{G}$. In particular if every ultrafilter is convergent, then every filter has a cluster point.

(c) If we combine the remarks (a) and (b), we obtain the following.

$$\text{Every ultrafilter is convergent} \iff \text{every filter has a cluster point.}$$

Proof.

(a) It suffices to recall that for an ultrafilter $\mathcal{F}$ and $A \subseteq X$, $A \in \mathcal{F}$ if and only if $A \cap B \neq \emptyset$ for all $B \in \mathcal{F}$.

(b) Let U be a neighborhood of x and let $B \in \mathcal{G}$. Then $U, B \in \mathcal{F}$. Hence $U \cap B \neq \emptyset$. Therefore, $x \in c\ell\ B$.

(c) This is clear from (a) and (b).

□

Remark 1.1.10. Let X be a topological space and let $x \in X$. Denote the neighborhood filter at x by $\mathcal{F}_x$. It is easy to see that

$$\mathcal{F}_x = \bigcap\{\mathcal{F} \mid \mathcal{F} \text{ is a filter on } X \text{ and } \mathcal{F} \to x\}.$$

In other words, $\mathcal{F}_x$ is the smallest filter on X converging to x.

Theorem 1.1.11. *Let X be a topological space.*

(1) X is Hausdorff if and only if every ultrafilter on X has at most one limit.

(2) X is compact if and only if every ultrafilter on X has at least one limit.

Proof. (1) Suppose X is Hausdorff and let p be an ultrafilter on X such that p has a limit. Suppose x and y are distinct limits of p. Pick neighborhoods U and V of x and y respectively such that $U \cap V = \emptyset$. Then $U, V \in p$ and, therefore, $\emptyset = U \cap V \in p$, a contradiction.
Conversely, suppose every ultrafilter on X has at most one limit. Let x and y be distinct points of X. Let $\mathcal{N}_x$ and $\mathcal{N}_y$ be neighborhood filters of x and y respectively. Assume for every $U \in \mathcal{N}_x$ and every $V \in \mathcal{N}_y$, $U \cap V \neq \emptyset$. Then $\mathcal{N}_x \cup \mathcal{N}_y$ has the finite intersection property. Pick an ultrafilter $\mathcal{F}$ on X such that $\mathcal{N}_x \cup \mathcal{N}_y \subseteq \mathcal{F}$. Then $\mathcal{F}$ converges to x and $\mathcal{F}$ converges to y, a contradiction. It follows that $U \cap V = \emptyset$ for some $U \in \mathcal{N}_y$ and some $V \in \mathcal{N}_y$. Hence X is Hausdorff.
(2) Suppose X is compact. Let p be an ultrafilter on X. Then the intersection of all closed sets in p is nonempty, since every finite subintersection is nonempty. Note that

$$\{A : A \in p \text{ and } A \text{ is closed}\} = \{c\ell\ A : A \in p\}.$$

Pick $x \in \bigcap\{c\ell\ A : A \in p\}$ and let U be a neighborhood of x. Then $U \cap A \neq \emptyset$ for every $A \in p$. Since p is an ultrafilter, $U \in p$. Therefore p contains the neighborhood filter at x and, hence, p converges to x.
Conversely, suppose every ultrafilter on X has a limit. Let $\mathcal{A}$ be a collection of closed subsets of X satisfying the finite intersection property. Pick an ultrafilter p on X such that $\mathcal{A} \subseteq p$ and let x be a limit of p. Suppose there exists $A \in \mathcal{A}$ such that $x \notin A$. Then $x \in X \setminus A$ and $X \setminus A$ is open. It follows that $X \setminus A \in p$, a contradiction. Therefore $x \in \bigcap \mathcal{A}$ and, hence, $\bigcap \mathcal{A} \neq \emptyset$. Thus X is compact. □

Theorem 1.1.11 shows that, if $(X, \mathcal{T})$ is a compact Hausdorff topological space, then the operation of taking limits of ultrafilters defines a function ξ from the set of all ultrafilters on X to X. Moreover, the proof of Theorem 1.1.11 shows that, for any ultrafilter p on X, $\bigcap\{c\ell\ A : A \in p\} = \{\xi(p)\}$. Since X is regular, it follows that $\xi(p) \in U$ for some open set U if and only if there is an open set $V \in p$ such that $c\ell\ V \subseteq U$. Thus,

$$\xi^{-1}(U) = \{p : c\ell\ V \subseteq U \text{ for some open } V \in p\}.$$

In Section 1.5 the set of all ultrafilters on X will be endowed with a topology having sets of the form $\{p : p \in V\}$, where $V \subseteq X$, as basic open sets. It follows that ξ is continuous with respect to that topology.

Let X and Y be topological spaces, let p be an ultrafilter on X and let $f : X \to Y$ be a function. Then $\{A \subseteq Y : f^{-1}(A) \in p\}$ is an ultrafilter on Y (we will prove this in Theorem 1.5.7), and we have the following lemma.

Lemma 1.1.12. *Let X and Y be compact Hausdorff topological spaces. A function $f : X \to Y$ is continuous if and only if for every ultrafilter p on X, $f(\xi(p)) = \xi(\{A \subseteq Y : f^{-1}(A) \in p\})$ (i.e. f preserves limits of ultrafilters).*

We have the following, more general, concept.

Definition 1.1.13. Let X be a topological space, let p be an ultrafilter on X, let $f : X \to Y$ be a function into a topological space Y and let $y \in Y$. y is called a *p-limit* of f if for every neighborhood U of y, $f^{-1}(U) \in p$.

If a p-limit y of f is unique we denote it by

$$p\text{-}\lim_{x \in X} f(x) \quad \text{or} \quad p\text{-}\lim f(x) \quad \text{or} \quad p\text{-}\lim f \quad \text{or} \quad \lim_{x \to p} f(x).$$

Theorem 1.1.14. *Let $f : X \to Y$ be a function from a topological space X into a topological space Y and let p be an ultrafilter on X.*

(1) If Y is Hausdorff and a p-limit of f exists, then it is unique.

(2) If Y is compact, then f has a p-limit.

Proof. (1) Suppose Y is Hausdorff and suppose y_1 and y_2 are p-*limits* of f in Y. Suppose $y_1 \neq y_2$. Pick two disjoint open subsets $U, V \subseteq Y$ such that $y_1 \in U$ and $y_2 \in V$. Then $\emptyset = f^{-1}(U) \cap f^{-1}(V) \in p$, a contradiction. It follows $y_1 = y_2$.
(2) Suppose Y is compact. Let $\mathcal{A}_p = \{c\ell\, f(A) : A \in p\}$. Let $A, B \in p$. Then $c\ell\, f(A) \cap c\ell\, f(B) \supseteq c\ell\, f(A \cap B) \supseteq f(A \cap B) \neq \emptyset$. Therefore $\mathcal{A}_p$ has the finite intersection property. Since Y is compact, $\bigcap \mathcal{A}_p \neq \emptyset$. Pick $y \in \bigcap \mathcal{A}_p$ and let U be a neighborhood of y. Then, $U \cap f(A) \neq \emptyset$ for every $A \in p$. Therefore $f^{-1}(U) \cap A \neq \emptyset$ for every $A \in p$. Since p is an ultrafilter, $f^{-1}(U) \in p$. It follows that y is a p-limit of f. □

Theorem 1.1.15. *Let $f : X \to Y$ and $g : Y \to Z$ be functions between topological spaces with Y and Z satisfying the Hausdorff separation property. Let p be an ultrafilter on X. If g is continuous and y is a p-limit of f, then $g(y)$ is a p-limit of $g \circ f$.*

Proof. Let W be a neighborhood of $g(y)$. By continuity of g, $g^{-1}(W)$ is a neighborhood of y. Therefore, $f^{-1}(g^{-1}(W)) \in p$. $\square$

Remark 1.1.16. Let X be a topological space, let p be an ultrafilter on X and let $y \in X$. Then

$$y \text{ is a limit of } p \text{ in } X \text{ if and only if } y \text{ is a } p\text{-limit of } id_X,$$

where id_X is the identity function on X.

A filter $\mathcal{F}$ on a topological space X is called *open* (respectively, *closed*) if $\mathcal{F}$ has a base consisting of open (respectively, closed) subsets of X.

Theorem 1.1.17. *Let X be a set and let Y be a nonempty subset of X. There is a bijective correspondence between filters on X containing Y and filters on Y. This correspondence restricts to a bijection between ultrafilters on X that contain Y and ultrafilters on Y.*

Proof. Consider the correspondence

$$\mathcal{F} \longmapsto \mathcal{F} \cap Y := \{A \cap Y : A \in \mathcal{F}\},$$

where $\mathcal{F}$ ranges over filters on X containing Y;

$$\mathcal{G} \longmapsto \mathcal{G}^* := \{A \subset X : U \subseteq A \text{ for some } U \in \mathcal{G}\},$$

where $\mathcal{G}$ ranges over filters on Y.

It is easy to see that this correspondence satisfies the required conditions. $\square$

Note that the correspondence in the proof of Theorem 1.1.17 sends principal ultrafilters to principal ultrafilters.

Definition 1.1.18. Let α be a cardinal and let p be an ultrafilter on α. The *norm* of p, denoted $\|p\|$, is defined by

$$\|p\| = \min\{|A| : A \in p\}.$$

It is clear that $1 \leq \|p\| \leq \alpha$ for any ultrafilters p on α.

Definition 1.1.19. Let κ and α be cardinals and let p be an ultrafilter on α. p is called *κ-uniform* if $\|p\| \geq \kappa$. We denote the set of κ-uniform ultrafilters on α by $U_\kappa(\alpha)$ and simply use $U(\alpha)$ if $\alpha = \kappa$. The elements of $U(\alpha)$ are simply called *uniform ultrafilters*, instead of α-uniform ultrafilters.

Note that an ultrafilter p on α is nonprincipal if and only if p is ω-uniform. It follows that $\beta\alpha \setminus \alpha = U_\omega(\alpha)$. In particular, $\beta\alpha \setminus \alpha = \emptyset$ if and only if $\alpha < \omega$.

Definition 1.1.20. Let κ and α be cardinals and let $\mathcal{F} \subseteq \mathcal{P}(\alpha)$. $\mathcal{F}$ is said to *have the κ-uniform finite intersection property* if $\bigcap \mathcal{F} \neq \emptyset$ and

$$|\bigcap_{i \leq n} A_i| \geq \kappa,$$

whenever $n < \omega$ and $A_i \in \mathcal{F}$ for $i \leq n$. If $\alpha = \kappa$, we say $\mathcal{F}$ has the uniform finite intersection property, instead of saying $\mathcal{F}$ has the α-uniform finite intersection property.

Definition 1.1.21. Let κ and α be cardinals. A filter $\mathcal{F}$ on α is *κ-complete* if $\bigcap \mathcal{G} \in \mathcal{F}$ whenever $\mathcal{G} \subseteq \mathcal{F}$ with $|\mathcal{G}| < \kappa$.

It follows that every filter is ω-complete.

Remark 1.1.22. Let α be a cardinal and let $\mathcal{F} \subseteq \mathcal{P}(\alpha)$ such that $\bigcap \mathcal{F} \neq \emptyset$. Then, $\mathcal{F}$ has the finite intersection property if and only if $\mathcal{F}$ has the 1-uniform finite intersection property.

Definition 1.1.23. Let α and κ be cardinals. Then we define

$$\mathcal{P}^\kappa(\alpha) = \{A \in \mathcal{P}(\alpha) : |\alpha \setminus A| < \kappa\}.$$

If $\omega \leq \kappa \leq \alpha$, then $\mathcal{P}^\kappa(\alpha)$ is a filter on α, called the *(generalised) Fréchet filter on α.*

Lemma 1.1.24. *Let $\omega \leq \kappa \leq \alpha$. Then*

(a) an ultrafilter p on α is κ-uniform if and only if $\mathcal{P}^\kappa(\alpha) \subseteq p$;

(b) there is a κ-uniform ultrafilter on α; and

(c) each family of subsets of α with the κ-uniform finite intersection property is contained in a κ-uniform ultrafilter on α.

Proof. (a) Suppose p is κ-uniform. Let $A \in \mathcal{P}^{\kappa}(\alpha)$. Then $|\alpha \setminus A| < \kappa$ and hence $\alpha \setminus A \notin p$. Thus $A \in p$. Conversely, suppose $\mathcal{P}^{\kappa}(\alpha) \subseteq p$. Let $B \in p$ and suppose $|B| < \kappa$. Then $|\alpha \setminus B| \geq \kappa$ and, therefore, $\alpha \setminus B \in p$. Then $\emptyset = B \cap (\alpha \setminus B) \in p$, a contradiction. It follows that $|B| \geq \kappa$.
(b) Since $\omega \leq \kappa \leq \alpha$, $\mathcal{P}^{\kappa}(\alpha)$ is a filter on α. Pick an ultrafilter p on α with $\mathcal{P}^{\kappa}(\alpha) \subseteq p$. Then, by (a), p is κ-uniform.
(c) Let $\mathcal{F}$ be a family of subsets of α with the κ-uniform finite intersection property. Let $m, n \in \mathbb{N}$ and let $\{A_k : k \leq n\} \subseteq \mathcal{F}$ and $\{B_i : i \leq m\} \subseteq \mathcal{P}^{\kappa}(\alpha)$. Then $|\bigcap_{k \leq n} A_k| \geq \kappa$ and $|\alpha \setminus \bigcap_{i \leq m} B_i| < \kappa$. Therefore, $(\bigcap_{k \leq n} A_k) \cap (\bigcap_{i \leq m} B_i) \neq \emptyset$. It follows that $\mathcal{F} \cup \mathcal{P}^{\kappa}(\alpha)$ has the finite intersection property. Pick an ultrafilter p on α such that $\mathcal{F} \cup \mathcal{P}^{\kappa}(\alpha) \subseteq p$. Then, by (a), p is κ-uniform. □

Definition 1.1.25. Let α be a limit ordinal. The cofinality of α, denoted $cf(\alpha)$, is the least ordinal ξ for which there is an order-preserving function $f : \xi \to \alpha$ such that f is unbounded, i.e. $\sup_{\eta < \xi} f(\eta) = \alpha$.

An infinite cardinal α is called *regular* if $\alpha = cf(\alpha)$. Otherwise, α is called *singular*.

1.2 Semigroups

Definition 1.2.1. A *semigroup* $(S, \cdot)$ is a nonempty set S equipped with a binary operation $\cdot$ such that

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

for all $x, y, z \in S$; i.e. $\cdot$ is associative. We will often write xy instead of $x \cdot y$ and S instead of $(S, \cdot)$, unless if we would like to emphasize the binary operation in question. For any $A, B \subseteq S$ we define

$$AB := \{ab \mid a \in A, b \in B\},$$

with $aB := \{a\}B$ and $Ab := A\{b\}$ for all $a, b \in S$. We will often write A^2 for AA. If $A \subseteq S$ and $x \in S$, we denote by $x^{-1}A$ the set $\{y \in S : xy \in A\}$. It follows from the associative law of the binary operation on S that for any $x, y \in S$ and any $A \subseteq S$, $(xy)^{-1}A = y^{-1}(x^{-1}A)$. A semigroup S is said to be *commutative* if $\cdot$ is commutative (i.e. $xy = yx$ for all $x, y \in S$.)

Definition 1.2.2. Let S be a semigroup and let $0, e \in S$.

(a) e is called a *left identity* of S if $ex = x$ for every $x \in S$. e is called a *right identity* of S if $xe = x$ for every $x \in S$. e is called an *identity* of S if e is both a left and a right identity of S; i.e. $ex = x = xe$ for every $x \in S$.

(b) 0 is called a *left zero* of S if $0x = 0$ for every $x \in S$. 0 is called a *right zero* of S if $x0 = 0$ for every $x \in S$. 0 is called a *zero* of S if 0 is both a left and a right zero of S; i.e. $0x = 0 = x0$ for every $x \in S$.

(c) Let $x, y, z \in S$. x is called a left z-inverse of y (and y is called a right z-inverse of x) if $xy = z$. If e is an identity of S, then x is called an inverse of y if $xy = e = yx$.

Proposition 1.2.3. *Let S be a semigroup. If S has a left identity (left zero) and a right identity (right zero) then they are equal. In particular every semigroup has at most one identity (zero).*

Proof. Let e_L be a left identity of S and e_R be a right identity of S. Then, $e_L = e_L e_R = e_R$. Similarly, if 0_L is a left zero and 0_R is a right zero of S, then $0_L = 0_L 0_R = 0_R$. Uniqueness of an identity (a zero) follows easily from these equations since an identity (a zero) is both a left identity (left zero) and a right identity (right zero). $\square$

If a semigroup S does not have an identity element, we can adjoin an extra element e to S to obtain the set $S \cup \{e\}$ and extend multiplication on S to $S \cup \{e\}$ by

$$xe = ex = x \text{ and } ee = e,$$

for every $x \in S$. This makes $S \cup \{e\}$ a semigroup and we denote this semigroup by S^1.

Similarly, if a semigroup S does not have a zero, we can adjoin an extra element 0 to S to obtain the semigroup $S^0 := S \cup \{0\}$ with multiplication on S extended to S^0 by

$$x0 = 0 = 0x \text{ and } 00 = 0,$$

for every $x \in S$. If G is a group, then G^0 is called a 0-*group*. Indeed G^0 is not a group.

Unless otherwise stated, we will not assume that a given semigroup has an identity or a zero.

Definition 1.2.4. Let S be a semigroup and let $a \in S$. We say that a is *(an) idempotent* if $a^2 = a$. We define $E(S) := \{x \in S \mid x^2 = x\}$.

Definition 1.2.5. Let S be a semigroup. S is called a *left zero semigroup* if for every $x, y \in S$, $xy = x$. S is called a *right zero semigroup* if for every $x, y \in S$, $xy = y$.

Example 1.2.6.

(a) Let S be a nonempty set. Define multiplication on S by $xy := y$. Then S with this multiplication is a right zero semigroup.

(b) Let S be a semigroup with a left identity and let

$$T = \{x \in S \mid xy = y \text{ for all } y \in S\},$$

i.e. T is the set of all left identities of S. Then T is a right zero semigroup. Similarly, in a semigroup S with a right identity,

$$P = \{y \in S \mid xy = x \text{ for all } x \in S\},$$

i.e. the set of all right identities of S, is a left zero semigroup. Note that T and P are subsets of $E(S)$. In particular, if every idempotent is a left (respectively, right) identity, then $E(S)$ is a right (respectively, left) zero semigroup.

Let A and B be nonempty sets. Define a binary operation on the cartesian product $A \times B$ by

$$(a_1, b_1)(a_2, b_2) = (a_1, b_2).$$

This operation is clearly associative. $A \times B$ equipped with this operation is called a *rectangular band*. If S is a rectangular band, we sometimes refer to S as a *rectangular semigroup*. Note that if $|A| = 1$, then $A \times B$ is a right zero semigroup, while if $|B| = 1$ then $A \times B$ is a left zero semigroup. In particular, every left(right) zero semigroup can be regarded as a rectangular band (what we mean here will become clear after we have defined semigroup isomorphisms, see Definition 1.2.19).

Definition 1.2.7. A *monoid* is a semigroup which has an identity element. Let M be a monoid with identity e. It is clear that every element of M has at most one inverse. If $x \in M$ has an inverse we call x *invertible* and we will denote the inverse of x by x^{-1}. A *group* is a monoid in which every element is invertible. If S is a monoid, we will denote the set of its invertible elements by $I(S)$.

Proposition 1.2.8. *Let S be a semigroup.Then*

$$S \text{ is a group} \iff aS = Sa = S \text{ for every } a \in S.$$

Proof. $\Rightarrow$: Suppose S is a group. Let e be the identity of S. We need to show that $S \subseteq aS \cap Sa$ for every $a \in S$. Let $a, x \in S$. $x = ex = (aa^{-1})x = a(a^{-1}x) \in aS$ and $x = xe = x(a^{-1}a) = (xa^{-1})a \in Sa$. Hence $S \subseteq aS \cap Sa$. It follows that $aS = S = Sa$.
$\Leftarrow$: Suppose $aS = Sa = S$ for every $a \in S$. Let $a \in S$. There exists $b, c \in S$ such that $ab = a = ca$. We will show that b is a right identity of S and c is a left identity of S and hence we have an identity. Let $z \in S$. There exists $x, y \in S$ such that $ax = z = ya$. Then, $cz = c(ax) = (ca)x = ax = z$ and $zb = (ya)b = y(ab) = ya = z$. Thus we have an identity $e := b(= c)$. Now we show that every element of S has an inverse. So, let $t \in S$. Since $e \in S = tS = St$, there exist $g, h \in S$ such that $tg = e = ht$. Then $g = eg = (ht)g = h(tg) = he = h$. Hence we have $t^{-1} := h(= g)$. It follows that S is a group. □

Proposition 1.2.9. *Let S be a semigroup. The following statements are equivalent.*

(a) S is a group

(b) S has a left identity e and every element of S has a left e-inverse.

(c) S has a right identity e and every element of S has a right e-inverse.

Proof. The implications (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) are obvious. We prove (b) $\Rightarrow$ (a) and then (c) $\Rightarrow$ (a) follows by a left-right switch. So, suppose (b) holds. Let $x \in S$ and let y be a left e-inverse of x. Let z be a left e-inverse of y. Then $xy = e(xy) = (zy)(xy) = z(y(xy)) = z((yx)y) = z(ey) = zy = e$. Hence y is also a right e-inverse for x. In addition, we get: $xe = x(yx) = (xy)x = ex = x$. Hence e is also a right identity for S. It follows that S is a group. □

Definition 1.2.10. Let S be a semigroup and let T be a nonempty subset of S. T is called a *subsemigroup* of S if $T^2 \subseteq T$, (i.e. T is closed under the binary operation). T is called a *submonoid* of S if T is a subsemigroup of S and T has an identity element. T is called a *subgroup* of S if T is a submonoid of S and every element of T has an inverse in T.

Example 1.2.11.

(a) Let S be a semigroup. For every $x \in S$, xS and Sx are subsemigroups of S.

(b) Let S be a semigroup and let $x \in S$. $\{x\}$ is a subsemigroup (in fact a subgroup) of S if and only if $x^2 = x$. In particular, a singleton consisting of a left(right) identity or a left (right) zero of S is a subgroup of S.

(c) For any monoid M, $I(M)$ is a subgroup of M (where $I(M)$ is as defined in Definition 1.2.7) .

We will call a subsemigroup T of a semigroup S *proper* if $T \neq S$. As a corollary to Proposition 1.2.8 we have the following.

Corollary 1.2.12. *Every semigroup containing more than one element has a proper subsemigroup.*

Proof. Let S be a semigroup containing more than one element and suppose S has no proper subsemigroup. Then for every $a \in S$, $aS = S = Sa$. Then, by Proposition 1.2.8, S is a group. Let e be the identity of S. Then $\{e\}$ is a proper subsemigroup of S. This is a contradiction. Therefore, S has a proper subsemigroup. □

A subsemigroup T of a semigroup S is said to be *left saturated in S* if

$$(S \setminus T)T \cap T = \emptyset.$$

Remark 1.2.13. Let S be a semigroup.

(a) Every left (right) zero and every left (right) identity of S is an idempotent.

(b) If S is commutative and $E(S) \neq \emptyset$, then $E(S)$ is a subsemigroup of S.

(c) If S has a left identity e, then every idempotent which has a right e-inverse is a left identity.

(d) If S has a right identity e, then every idempotent which has a left e-inverse is a right identity.

(e) If S has an identity, then the only invertible idempotent of S is the identity. In particular, every group has exactly one idempotent.

Proof. (a) is obvious, (d) is similar to (c), and (e) follows from (c) and (d). We will prove (b) and (c).
(b): Suppose S is commutative and $E(S) \neq \emptyset$. Let $x, y \in E(S)$. $x^2 = x$ and $y^2 = y$. Then $(xy)(xy) = x(yx)y = x(xy)y = (xx)(yy) = xy$. Hence $xy \in E(S)$.Thus $E(S)$ is a subsemigroup of S.
(c): Let $x \in E(S)$ and suppose x has a right e-inverse z. Let $y \in S$. Then $xy = xey = x(xz)y = (xx)zy = (xz)y = ey = y$. Hence x is a left identity. □

A semigroup in which every element is idempotent is called a *band.* A *rectangular component* of a band is a maximal rectangular subsemigroup. As suggested by the name, distinct rectangular components are disjoint. The rectangular components of a band are partially ordered by the following relation, [[43], Theorem 1]:

$$P \leq Q \textit{ if and only if } PQ \subseteq P, \textit{ equivalently } QP \subseteq P.$$

For any two semigroups S and T, the *direct product of S and T* is the cartesian product $S \times T$, equipped with the componentwise operation:

$$(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2).$$

At this point it is reasonable to ask the following question. Given a semigroup S and $e \in E(S)$, is there a subgroup of S in which e is the identity? There are two canonical answers to this question. We call a subgroup of S *maximal* if it is maximal with respect to the inclusion order $\subseteq$ among subgroups of S.

Definition 1.2.14. Let S be a semigroup and let $e \in E(S)$. We define

$$H(e) := \bigcup \{G \mid G \text{ is a subgroup of S and } e \in G\}.$$

Equivalently, $H(e) := \bigcup \{G \mid G \text{ is a subgroup of S and } e \text{ is the identity of } G\}$.

Theorem 1.2.15. *[[29], Theorem 1.18] Let S be a semigroup and let $e \in E(S)$. Then $\{e\}$ is the smallest subgroup of S with e as the identity and $H(e)$ is the largest subgroup of S with e as the identity. In addition, $H(e)$ is a maximal subgroup of S.*

Proof. It is obvious that $\{e\}$ is the smallest subgroup of S with e as the identity and that every subgroup of S with e as the identity is a subgroup of H(e). It is also obvious that any subgroup of S which contains $H(e)$ contains

e and, hence, is contained in $H(e)$. It remains only to show that $H(e)$ is a subgroup of S. In this regard the only property that is not immediately clear is whether $H(e)$ is closed under multiplication. Let $a, b \in H(e)$. $a \in G_1$ and $b \in G_2$ for some subgroups G_1 and G_2 of S with e as their identity. Let

$$G = \{\prod_{i=1}^{n} x_i \mid n \in \mathbb{N} \text{ and } x_1, \cdots, x_n \in G_1 \cup G_2\}.$$

G is obviously closed under multiplication and contains e. Let $x = \prod_{i=1}^{n} x_i \in G$. Then, $y := \prod_{i=1}^{n} x_{n-i+1}^{-1} \in G$ and $xy = e = yx$. It follows that G is a subgroup of S with identity e and $ab \in G$. Therefore, $ab \in H(e)$. Hence $H(e)$ is a subgroup of S. $\square$

We have just seen that $H(e)$ is a maximal subgroup of S. In fact, as we will see shortly, all maximal subgroups of S are of this form. We will refer to them as *maximal groups*.

Proposition 1.2.16. *Let S be a semigroup. Every subgroup of S is contained in a group $H(e)$ for some idempotent e of S.*

Proof. Let G be a subgroup of S and let e be the identity of G. Then e is an idempotent and $G \subseteq H(e)$. $\square$

Corollary 1.2.17. *Let S be a semigroup. The maximal subgroups of S are exactly the elements of the set $\{H(e) \mid e \in E(S)\}$.*

At this point we would like to answer another question arising from what we have seen before. Let S be a semigroup with a left identity e such that every element of S has a right e-inverse. Is S necessarily a group? The answer is negative, but we will see that S has a special structure.

Example 1.2.18. Let S be a right zero semigroup. The statement: $xy = y$ for every $x, y \in S$, implies that every element of S is a left identity and for every $x, y \in S$, y is a right y-inverse of x. Now fix $e \in S$ (as a left identity). Let $x \in S$. If $x \neq e$, then x has no left inverse since for every $y \in S$, $yx = x \neq e$. Hence, if $|S| > 1$, then S is not a group. This provides a counterexample to our question. It follows from this example that a right zero semigroup S has a unique left identity if and only if S is a singleton, if and only if S is a group.

We now show that the above example is essentially the only instance of a semigroup with a left identity e such that every element has a right e-inverse. First we need the following definition.

Definition 1.2.19. Let S and T be semigroups and let $f : S \to T$ be a function. f is called a *semigroup homomorphism* if $f(xy) = f(x)f(y)$ for every $x, y \in S$. A *semigroup isomorphism* is a bijective semigroup homomorphism. Two semigroups are *isomorphic* if there is a semigroup isomorphism between them. If S and T are isomorphic, we write $S \approx T$.

Theorem 1.2.20. *[[29], Theorem 1.40] Let S be a semigroup with a left identity e such that every element of S has a right e-inverse. Then $E(S)$ is a right zero semigroup, Se is a group and $S = Se \cdot E(S) \approx Se \times E(S)$.*

Proof. We know that if an idempotent has a right e-inverse, then it is a left identity (see Remark 1.2.13 (c)). Therefore the right zero property $xy = y$ is satisfied in $E(S)$. (In fact, $xy = y$ for every $x \in E(S)$ and every $y \in S$). Let $x, y \in E(S)$. Then, $(xy)(xy) = yy = y = xy$. Therefore $E(S)$ is closed with respect to multiplication. It follows that $E(S)$ is a right zero semigroup. Since e is an idempotent, e is a right identity for Se. Let $x, y \in S$. $(xe)(ye) = (xey)e = xye \in Se$. Therefore, Se is closed under multiplication. Let $x \in S$. Let y be a right e-inverse of x. Then, $(xe)(ye) = x(ey)e = xye = (xy)e = ee = e$. It follows that Se is a group. Now define $\varphi : Se \times E(S) \to S$ by $\varphi(g, y) = gy$. Let $(g_1, y_1), (g_2, y_2) \in Se \times E(S)$. Then, $\varphi(g_1g_2, y_1y_2) = g_1g_2y_1y_2 = g_1g_2y_2 = g_1(y_1g_2)y_2 = (g_1y_1)(g_2y_2) = \varphi(g_1, y_1)\varphi(g_2, y_2)$. Therefore, φ is a homomorphism. Let $a \in S$ and let b be the inverse of ae in Se. Then, $baba = ba(eb)a = b(aeb)a = bea = ba$. Therefore $ba \in E(S)$. We will show that $\varphi^{-1}(a) = \{(ae, ba)\}$, thus, showing that φ is bijective. $\varphi(ae, ba) = (ae)(ba) = (aeb)a = ea = a$. Therefore, φ is surjective and, thus, $S = Se \cdot E(S)$. Now let $(g, y) \in Se \times E(S)$ such that $gy = a$. Then, $g = ge = gye = ae$ and $y = ey = baey = bgyey = bgyy = bgy = ba$. Therefore, φ is injective. It follows that φ is bijective. $\square$

(Note that we have already seen that if every idempotent is a left identity, then $E(S)$ is a right zero semigroup. Thus, the proof of Theorem 1.2.20 can be shortened.)

Corollary 1.2.21. *Any semigroup S with a unique left identity e such that every element of S has a right e-inverse is a group.*

Proof. In Theorem 1.2.20, if e is the only left identity of S, then $E(S) = \{e\}$ and, hence, $S \approx Se \times \{e\}$ is a group. $\square$

Definition 1.2.22. Let S be a semigroup and let $a \in S$. We say that a is *left (right) cancelable* if whenever $x, y \in S$ are such that $ax = ay$ (respectively,

$xa = ya$), then $x = y$. S is *left (right) cancellative* if every element of S is left (respectively, right) cancelable. S is *cancellative* if it is both left cancellative and right cancellative.

Remark 1.2.23. Let S be a semigroup and let $e \in E(S)$. If S is left (right) cancellative, then e is a left (right) identity of S. If S is cancellative, then e is an identity of S.

Proof. Suppose S is left cancellative. Let $x \in S$. Since $ex = eex$, $x = ex$. Hence e is a left identity. Similarly, suppose S is right cancellative. Let $x \in S$. Since $xe = xee$, $x = xe$. Hence e is a right identity. This suffices. □

Remark 1.2.24. Let S and T be semigroups and let N be a subsemigroup of S and M a subsemigroup of T. Let $f : S \to T$ be a semigroup homomorphism. We have the following.

(a) If $f^{-1}(M) \neq \emptyset$, then $f^{-1}(M)$ is a subsemigroup of S.

(b) $f(N)$ is a subsemigroup of T. In particular, the image of any idempotent under a semigroup homomorphism is an idempotent.

(c) If N has an identity e, then $f(e)$ is an identity for $f(N)$.

(d) If N has an identity, then $f(x^{-1}) = f(x)^{-1}$ for every $x \in I(N)$.

We have the following observation.

Proposition 1.2.25. *A semigroup is a rectangular band if and only if it is a direct product of a left zero semigroup and a right zero semigroup.*

Remark 1.2.26. Let S be a semigroup with an idempotent a and let T be a semigroup. Then, there exists a semigroup homomorphism from S to T if and only if T has an idempotent.

Proof. $\Rightarrow$: (This is actually part of Remark 1.2.24 (c).) Let $f : S \to T$ be a semigroup homomorphism and let $b := f(a)$. Then $b^2 = f(a)f(a) = f(aa) = f(a) = b$. Therefore b is an idempotent.
$\Leftarrow$: Suppose T has an idempotent b. Define $f : S \to T$ by $f(x) := b$ for every $x \in S$. Then for every $x, y \in S$, $f(x)f(y) = bb = b = f(xy)$. Therefore f is a semigroup homomorphism. □

Definition 1.2.27. Let S be a semigroup and let I be a nonempty subset of S. I is called a *left ideal* of S if $SI \subseteq I$. I is called a *right ideal* of S if $IS \subseteq I$. I is called an *ideal* (or *a two sided ideal*) of S if I is a left ideal and a right ideal of S.

Example 1.2.28.

(a) For any $a \in S$, Sa is a left ideal of S, aS is a right ideal of S and SaS is an ideal of S. More generally, we have the following.

(b) Let L be a left ideal of S, let R be a right ideal of S and let $a \in S$. Then, La is a left ideal of S, aR is a right ideal of S and LaR is an ideal of S.

Remark 1.2.29. Let S be a semigroup, let $L \subseteq S$ and let $a \in L$.

(a) If L is a left ideal of S, then $Sa \subseteq L$.

(b) If L is a right ideal of S, then $aS \subseteq L$.

(c) If L is an ideal of S, then $SaS \subseteq L$.

Note that the relation " T is a left ideal of L" among the subsemigroups of a semigroup S is not transitive. However, if L and T are left ideals of S such that $T \subseteq L$, then T is a left ideal of L. These facts hold for the right ideals and ideals as well.

Proposition 1.2.30. *Let $(J_\alpha)_{\alpha\in\Gamma}$ be a nonempty family of ideals of a semigroup S. Define $J := \bigcap\limits_{\alpha\in\Gamma} J_\alpha$. If $J \neq \emptyset$, then J is an ideal of S.*

Proof. The statement is obvious. □

Corollary 1.2.31. *Let S be a semigroup and let $\emptyset \neq A \subseteq S$. Consider*

$$\mathcal{A} := \{I \subseteq S \mid I \textit{ is an ideal and } A \subseteq I\} \textit{ and } \langle A\rangle := \bigcap \mathcal{A}.$$

Then, $\langle A\rangle$ is the smallest ideal of S containing A. We call $\langle A\rangle$ the ideal generated by A.

For any $a \in S$, we call $\langle\{a\}\rangle$ the *principal ideal* generated by a. In a similar way we obtain the left ideal generated by A and the right ideal generated by A. Note that for any $a \in S$, the principal left ideal generated by a

is $Sa \cup \{a\}$, the principal right ideal generated by a is $aS \cup \{a\}$ and the principal ideal generated by a is $Sa \cup aS \cup SaS \cup \{a\}$. If S is a monoid, then $Sa \cup aS \cup SaS \cup \{a\} = SaS$. If $a \in E(S)$, then $a \in Sa$ and $a \in aS$, in particular, $Sa \cup \{a\} = Sa$ and $aS \cup \{a\} = aS$. Let I be a left ideal of a semigroup S. I is said to be a *proper left ideal of* S if $I \neq S$. Similarly, we define a *proper right ideal* and a *proper ideal* of S.

1.3 The structure of a completely simple semigroup

Definition 1.3.1. Let S be a semigroup, let L be a left ideal of S, let R be a right ideal of S, and let I be an ideal of S.

(a) L is a *minimal left ideal* of S if whenever $J \subseteq L$ for some left ideal J of S, then $J = L$. R is a *minimal right ideal* of S if whenever $J \subseteq R$ for some right ideal J of S, then $R = J$.

(b) S is *left simple* if S has no proper left ideal. S is *right simple* if S has no proper right ideal. S is *simple* if S has no proper ideal. If S has a zero 0, then S is called 0-*simple* if its only ideals are S itself and $\{0\}$, and $S^2 \neq \{0\}$.

(c) I is a *minimal ideal of* S if whenever $J \subseteq I$ for some ideal J of S, then $I = J$.

As we will see later in Theorem 1.3.21, a semigroup S can have at most one minimal ideal and, as mentioned in the introduction, it is denoted by $K(S)$.

Remark 1.3.2. Let S be a semigroup and let L and J be left ideals of S and let R be a right ideal of S. We have the following.

(i) $S(L \cap J) \subseteq L \cap J$. In particular, if $L \cap J \neq \emptyset$, then $L \cap J$ is a left ideal of S.

(ii) LR is an ideal of S.

(iii) $RL \subseteq L \cap R$. In particular, $L \cap R \neq \emptyset$.

Note that every left, right or two sided ideal of S is a subsemigroup of S but the converse is false.

Remark 1.3.3. Let S be a semigroup and let $e \in E(S)$.

(a) $Se = \{x \in S \mid xe = x\}$

(b) $eS = \{x \in S \mid ex = x\}$

(c) $eSe = \{x \in S \mid xe = x = ex\}$

In particular, e is a left identity for eS, a right identity for Se and an identity for eSe. Note that in this case Se is a subsemigroup of S with a right identity, eS is a subsemigroup of S with a left identity and eSe is a monoid. If S is left simple then e is a right identity for S and, if S is right simple, then e is a left identity for S.

Remark 1.3.4. Let S be a semigroup and let $e \in E(S)$. Se is the smallest left ideal of S containing e.

Proof. Since $e = ee$, $e \in Se$. Now let L be a left ideal of S with $e \in L$. Then, $Se \subseteq SL \subseteq L$. □

Definition 1.3.5. Let S be a semigroup. We define the relations $\leq_L$, $\leq_R$ and $\leq$ on $E(S)$ as follows.

(1) $f \leq_L e$ if and only if $f = fe$.

(2) $f \leq_R e$ if and only if $f = ef$.

(3) $f \leq e$ if and only if $f \leq_L e$ and $f \leq_R e$.

Proposition 1.3.6. *Let S be a semigroup and let $e, f \in E(S)$.*

(a) $f \leq_L e$ if and only if for any left ideal L with $e \in L$, $f \in L$.

(b) $f \leq_R e$ if and only if for any right ideal R with $e \in R$, $f \in R$.

(c) $f \leq e$ if and only if for any ideal I with $e \in I$, $f \in I$.

Proof. (a) Suppose $f = fe$ and let L be a left ideal with $e \in L$. Then $f = fe \in SL \subseteq L$. Conversely, suppose for any left ideal L with $e \in L$, $f \in L$. Take $L = \{x \in S : x = xe\}$. Since $ee = e$, $e \in L$. Thus $L \neq \emptyset$. Let $x \in L$ and let $a \in S$. Then $ax = a(xe) = (ax)e$. Therefore $ax \in L$. It follows that L is a left ideal. Since $e \in L$, $f \in L$; i.e $f = fe$. Thus $f \leq_L e$.
(b) This is similar to (a).

(c) This follows easily from (a) and (b), using the fact that an intersection of a left ideal and a right ideal is nonempty, and it contains an ideal (see Remark 1.3.2 (iii)). $\square$

Remark 1.3.7. Let S be a semigroup, let L be a minimal left ideal of S and let $e, f \in E(S)$. If $e, f \in L$, then $Se = L = Sf$. In particular, if $e, f \in L$, then $e \leq_L f$ and $f \leq_L e$. The converse is also true but obvious.

Proof. Suppose $e, f \in L$. Then $Se \subseteq SL \subseteq L$. Since Se is a left ideal and L is a minimal left ideal, $L = Se$. Similarly, $Sf = L$. Now let J be a left ideal with $f \in J$. Then $Se = L = Sf \subseteq J$. Since $e = ee \in Se$, $e \in J$. Hence $e \leq_L f$. Similarly, $f \leq_L e$. $\square$

Remark 1.3.8. Let S be a semigroup. The relations $\leq_L, \leq_R$ and $\leq$ on $E(S)$ are reflexive and transitive. In addition, $\leq$ is antisymmetric.

Definition 1.3.9. Let S be a semigroup, let $\preceq$ be a relation on $E(S)$ and let $e \in E(S)$. We say that e is *minimal with respect to* $\preceq$ if and only if for every $f \in E(S)$, if $f \preceq e$ then $e \preceq f$.

Proposition 1.3.10. *[[29], Theorem 1.36] Let S be a semigroup and let $e \in E(S)$. Then e is minimal with respect to $\leq_L$ if and only if e is minimal with respect to $\leq_R$ if and only if e is minimal with respect to $\leq$.*

Proof. Suppose e is minimal with respect to $\leq_L$ and let $f \leq e$. Then $f \leq_L e$ and $f \leq_R e$. Since e is minimal with respect to $\leq_L$, $e \leq_L f$. We obtain: $e = ef = e(ef) = (ee)f = ef = f$. Hence $f = e$. Hence e is minimal with respect to $\leq$. Now suppose e is minimal with respect to $\leq$ and let $f \leq_L e$, i.e. $f = fe$. Let $g = ef$. Then $gg = efef = e(fe)f = eff = ef = g$, i.e. $g \in E(S)$. Also $g = ef = efe$. Therefore $ge = efee = efe = g = eefe = eg$. So, $g \leq e$. Hence $g = e$; i.e. $ef = g = e$. Thus $e \leq_L f$. Therefore that e is minimal with respect to $\leq_L$. It follows that e is minimal with respect to $\leq$ if and only if e is minimal with respect to $\leq_L$. By a left-right switch in the above arguments we obtain: e is minimal with respect to $\leq$ if and only if e is minimal with respect to $\leq_R$. $\square$

We will say that an idempotent is minimal if it is minimal with respect to any of the three relations defined above. As a corollary to Remark 1.3.3 and Proposition 1.3.6, we have the following.

Corollary 1.3.11. *Let S be a semigroup and let $e \in E(S)$. If $e \in L$ for some minimal left ideal L of S (equivalently, if Se is a minimal left ideal of S), then e is a minimal idempotent.*

Proof. Suppose Se is a minimal left ideal of S. Let $f \in E(S)$ such that $f \leq_L e$. Then $f \in Se$. Since Se is minimal, $Sf = Se$. Then, $ef = e$ and, thus, $e \leq_L f$. It follows that e is minimal. $\square$

Definition 1.3.12. A semigroup S is called *completely simple* if S is simple and contains a minimal idempotent. A semigroup S with a zero 0 is called *completely* 0*-simple* if S is 0-simple and has a minimal idempotent different from 0.

Theorem 1.3.13. *Let S be a semigroup and let $L \subseteq S$ be a left ideal. L is a minimal left ideal of S if and only if $Sa = L$ for every $a \in L$.*

Proof. Suppose L is a minimal left ideal. Let $a \in L$. Since Sa is a left ideal and $Sa \subseteq L$, $Sa = L$. Conversely, suppose $Sa = L$ for every $a \in L$. Let $J \subseteq S$ be a left ideal with $J \subseteq L$. Let $a \in J$. Then $L = Sa \subseteq J$. Hence $J = L$. It follows that L is a minimal left ideal. $\square$

Corollary 1.3.14. *Let S be a semigroup. Minimal left ideals of S are exactly left ideals of S of the form Sa, for some $a \in S$, satisfying the property: for every $b \in S$, $Sba = Sa$.*

Proof. Let L be a minimal left ideal of S and let $a \in L$. By Theorem 1.3.13, $L = Sa$. Let $b \in S$. Then $ba \in L$. Therefore, $Sba \subseteq SL \subseteq L$. Since Sba is a left ideal of S, $Sba = L$.
Conversely, let $a \in S$ such that $Sba = Sa$ for every $b \in S$. Let $x \in Sa$. $x = ba$ for some $b \in S$. Then $Sx = Sba = Sa$. By Theorem 1.3.13, Sa is a minimal left ideal of S. $\square$

Note that in a right zero semigroup every left ideal of the form Sa is minimal and it is a singleton. Since left ideals, right ideals and ideals are semigroups, we can apply to them what we already know about semigroups.

Theorem 1.3.15. *[[29], Theorem 1.42] Let S be a semigroup and assume that there is a minimal left ideal L of S which has an idempotent e. Then $X := E(L)$ is a left zero semigroup, $G := eL = eSe$ is a group and $L = XG = X \times G$.*

Proof. Since $Le = L$, e is a right identity for L. To see that $eL = eSe$, let $x \in L$. Then $ex = e(xe) \in eSe$. Therefore, $eL \subseteq eSe$. Conversely, let $x \in S$. Then $xe \in L$ and, therefore, $exe \in eL$. Thus $eSe \subseteq eL$.
Let $x \in L$. Since $Lx = L$, there exists $y \in L$ such that $yx = e$. Thus, L is a semigroup with a right identity e and every element of L has a left e-inverse. Therefore, by the right-left switch, Theorem 1.2.20 is applicable. $\square$

It is routine to show that maximal subgroups of $X \times G$ are of the form $\{x\} \times G$, where $x \in X$.

Proposition 1.3.16. *Let S be a semigroup and let L be a left ideal of S. L is a minimal left ideal of S if and only if L is left simple.*

Proof. $\Rightarrow$: Suppose L is a minimal left ideal of S and let T be a left ideal of L. Let $t \in T$. Then Lt is a left ideal of S, $Lt \subseteq T \subseteq L$. Therefore, $L = Lt \subseteq T$. It follows that $L = T$. Hence L is left simple.
$\Leftarrow$: Suppose L is left simple. Let T be a left ideal of S with $T \subseteq L$. Then T is a left ideal of L. Hence $L = T$. It follows that L is a minimal left ideal of S. $\square$

Recall that we remarked that in a given semigroup S, a left ideal of a small subsemigroup is not necessarily a left ideal of a bigger subsemigroup. For example, a subsemigroup of S is a left ideal of itself but not necessarily a left ideal of S. However the following proposition gives a sufficient condition for a left ideal of a small subsemigroup to be a left ideal of a bigger subsemigroup.

Proposition 1.3.17. *Let S be a semigroup, let L be a left ideal of S and let T be a minimal left ideal of L. Then T is a left ideal of S.*

Proof. Let $t \in T$. Then Lt is a left ideal of L and $Lt \subseteq T$. Since T is minimal, $Lt = T$. Then, $ST = S(Lt) = (SL)t \subseteq Lt \subseteq T$. Hence T is a left ideal of S. $\square$

Lemma 1.3.18. *Let S be a semigroup and let L be a minimal left ideal of S. Then, L is contained in every ideal of S.*

Proof. Let I be an ideal of S. Then IL is a left ideal of S and $IL \subseteq I \cap L$. Since L is minimal, $IL = L$. Hence, $L = IL \subseteq I$. $\square$

Theorem 1.3.19. *Let S be a semigroup and let L be a minimal left ideal of S. Then the minimal left ideals of S are exactly members of the family*

$$\{La \mid a \in S\}.$$

Proof. Let $a \in S$. We already know that La is a left ideal of S. Now, let T be a left ideal of S with $T \subseteq La$. Consider $A = \{s \in L \mid sa \in T\}$. Since $T \subseteq La$, $A \neq \emptyset$. If $sa \in T$, then $tsa \in T$ for any $t \in S$. Hence A is a left ideal of S. Since $A \subseteq L$ and L is minimal, $A = L$. Thus $La \subseteq T$. It follows that La is a minimal left ideal of S. Conversely, let T be a minimal left ideal of S. Let $a \in T$. Then, La is a left ideal of S and $La \subseteq T$. Therefore, $T = La$. $\square$

Corollary 1.3.20. *Let S be a semigroup. If S has a minimal left ideal, then every left ideal of S contains a minimal left ideal.*

Proof. Let L be a minimal left ideal of S and let J be a left ideal of S. Pick $a \in J$. Then La is a minimal left ideal of S and $La \subseteq J$. □

Theorem 1.3.21. *[[29], Theorem 1.51] Let S be a semigroup. If S has a minimal left ideal, then $K(S)$ exists and $K(S) = \bigcup\{L : L$ is a minimal left ideal of $S\}$.*

Proof. Let $I = \bigcup\{L : L \text{ is a minimal left ideal of } S\}$. Then, by Lemma 1.3.18, $I \subseteq J$ for every ideal J of S. Therefore it suffices to show that I is an ideal of S. By assumption $I \neq \emptyset$. Let $x \in I$ and let $a \in S$. Pick a minimal left ideal L of S such that $x \in L$. Then $ax \in L \subseteq I$. Also, by Theorem 1.3.18, La is a minimal left ideal of S. Therefore, $La \subseteq I$. It follows that I is an ideal of S. □

Similarly, one can show that if $K(S)$ exists, then

$$K(S) = \bigcup\{R : R \text{ is a minimal right ideal of } S\}.$$

Some examples of semigroups which do not have a smallest ideal are $(\mathbb{N}, +)$ and $(\mathbb{N}, .)$.

We note that any two distinct minimal left (right) ideals are disjoint, see Remark 1.3.2 (i). We also note that the semigroups of the form $K(S)$ are exactly the completely simple semigroups. Therefore we have the following.

Corollary 1.3.22. *Every completely simple semigroup is a disjoint union of its minimal left ideals and a disjoint union of its minimal right ideals.*

Lemma 1.3.23. *Let S be a semigroup, let L be a left ideal of S, and let I be an ideal of S.*

(1) L is minimal if and only if $Lx = L$ for every $x \in L$.

(2) I is the smallest ideal of S if and only if $IxI = I$ for every $x \in I$.

Proof. (1) Suppose L is minimal. Let $x \in L$. Then, by Theorem 1.3.19, Lx is a minimal left ideal of S. Since L is a left ideal, $Lx \subseteq L$. It follows that

$L = Lx$. Conversely assume that $Lx = L$ for every $x \in L$. Let J be a left ideal of S with $J \subseteq L$. Let $y \in J$. Then $L = Ly \subseteq LJ \subseteq J$. Thus $L = J$. It follows that L is minimal.
(2) Suppose I is the smallest ideal of S. Let $x \in I$. Since I is a left ideal of S, $IxI \subseteq I$. Since IxI is an ideal of S, it follows that $IxI = I$. Conversely suppose $IxI = I$ for every $x \in I$. Let J be an ideal of S, with $J \subseteq I$. Let $y \in J$. Then $I = IyI \subseteq IJI \subseteq J$. Thus $I = J$. It follows that I is the smallest ideal of S. □

Theorem 1.3.24. *[[29], Theorem 1.53] Let S be a semigroup. If L is a minimal left ideal of S and R is a minimal right ideal of S, then $K(S) = LR$.*

Proof. It is clear that LR is an ideal of S. Let $x \in LR$. By Lemma 1.3.23, it suffices to show that $LRxLR = LR$. Now $LRxL$ is a left ideal of S which is contained in L. So $LRxL = L$ and, thus, $LRxLR = LR$. □

Theorem 1.3.25. *[[29], Theorem 1.54] Let S be a semigroup and assume that $K(S)$ exists and $e \in E(S)$. The following statements are equivalent.*

(a) $e \in K(S)$.

(b) $K(S) = SeS$.

Proof. $(a) \Rightarrow (b)$: Since SeS is an ideal, then $K(S) \subseteq SeS$. Since, by assumption, $e \in K(S)$, then $SeS \subseteq K(S)$.
$(b) \Rightarrow (a)$: By assumption, $e = eee \in SeS = K(S)$. Thus, $e \in K(S)$ as required. □

Theorem 1.3.26. *[[29], Theorem 1.56] Let S be a semigroup and assume that there is a minimal left ideal of S which has an idempotent. Then every minimal left ideal of S has an idempotent.*

Proof. Let L be a minimal left ideal with an idempotent e and let J be a minimal left ideal. $J = La$ for some $a \in S$. Then, by Theorem 1.3.15, $eL = eSe$ is a group. Let $b \in L$ such that $b = ebe$ is the inverse of eae in eSe, i.e. $(eae)(ebe) = e = (ebe)(eae)$. Then $(ba)(ba) = (be)a(eb)a = b(eae)ba = ba$. Hence ba is an idempotent of J. □

Let S be a semigroup and let $e \in E(S)$. It can be shown that Se is left simple if and only if eS is right simple, see for example [[29], Theorem 1.48].

Lemma 1.3.27. *[[29], Theorem 1.57] Let S be a semigroup and assume that there is a minimal left ideal of S which has an idempotent. Then there is a minimal right ideal of S which has an idempotent.*

Proof. Let L be a minimal left ideal of S with an idempotent e. Then $Se = L$ is a minimal left ideal of S and, thus, eS is a minimal right ideal of S and e is an idempotent of eS. □

Theorem 1.3.28 ([29], Theorem 1.58)**.** *Let S be a semigroup and assume that there is a minimal left ideal of S which has an idempotent. Let $T \subseteq S$.*

(1) T is a minimal left ideal of S if and only if there is some $e \in E(K(S))$ such that $T = Se$.

(2) T is a minimal right ideal of S if and only if there is some $e \in E(K(S))$ such that $T = eS$.

Proof. Let L be a minimal left ideal with an idempotent $f \in L$.
(1) Necessity. Since Sf is a left ideal contained in L, $Sf = L$. Therefore, by Theorem 1.3.15 fSf is a group with identity f. Pick any $a \in T$. Then $faf \in fSf$. Pick $x \in fSf$ such that $x(faf) = f$. Then $xaxa = (xf)a(fx)a = (xfaf)xa = fxa = xa$. Hence xa is idempotent. Since $xa \in T \subseteq K(S)$, $xa \in E(K(S))$. Since Sxa is a left ideal with $Sxa \subseteq T$ and T is a minimal left ideal, $Sxa = T$.
Sufficiency. Since $e \in K(S)$, there is a minimal left ideal I of S with $e \in I$. Then $T = Se = I$. Therefore T is a minimal left ideal of S.
(2) By Lemma 1.3.27, there is a minimal right ideal of S which has an idempotent. The result then follows from (a) by left-right switch. □

The following theorem summarizes some of the results we have so far in this section.

Theorem 1.3.29. *[[29], Theorem 1.59] Let S be a semigroup, assume that there is a minimal left ideal of S which has an idempotent, and let $e \in E(S)$. The following statements are equivalent.*

(1) Se is a minimal left ideal.

(2) Se is left simple.

(3) eSe is a group.

(4) $eSe = H(e)$.

(5) eS *is a minimal right ideal.*

(6) eS *is right simple*

(7) e *is a minimal idempotent.*

(8) $e \in K(S)$.

(9) $K(S) = SeS$.

Proof. See [[29], Theorem 1.59] □

Theorem 1.3.30. *[[29], Theorem 1.60] Let* S *be a semigroup, assume that there is a minimal left ideal of* S *which has an idempotent, and let* e *be an idempotent in* S*. There is a minimal idempotent* f *of* S *such that* $f \leq e$*.*

Proof. Se is a left ideal of S. Pick a minimal left ideal L of S and an idempotent g such that $g \in L \subseteq Se$. Since e is a right identity for Se, $g = ge$. Let $f = eg$. Then $ff = egeg = egg = eg = f$. Thus f is an idempotent. In addition, $ef = eeg = eg = f$ and $fe = ege = eg = f$. Hence $f \leq e$. □

Theorem 1.3.31. *Let* S *be a semigroup and assume that there is a minimal left ideal of* S *which has an idempotent. Given any minimal left ideal* L *of* S *and any minimal right ideal* R *of* S*, there is an idempotent* $e \in R \cap L$ *such that* $R \cap L = RL = eSe$ *and* eSe *is a group.*

Proof. See [[29], Theorem 1.61] □

Lemma 1.3.32. *Let* S *be a semigroup and assume that there is a minimal left ideal of* S *which has an idempotent. Then all minimal left ideals of* S *are isomorphic.*

Proof. See [[29], Lemma 1.62] □

Definition 1.3.33. Let G be a group. Let I and Λ be non-empty sets and let P be a matrix with entries in the 0-group G^0 which is 'regular' in the sense that each row and each column contains at least one non-zero entry. The $I \times \Lambda$ *Rees matrix semigroup over* G^0 *with sandwich matrix* P, which we denote by $\mathcal{M}^0(G, I, \Lambda, P)$, is the set $I \times G \times \Lambda \cup \{0\}$, together with the binary operation given by:

$$0(i, g, \lambda) = 0 = (i, g, \lambda)0 \text{ and}$$

$$(i, g, \lambda)(j, h, \mu) = \begin{cases} (i, gp_{\lambda j}h, \mu) & \text{if } p_{\lambda j} \neq 0 \\ 0 & \text{if } p_{\lambda j} = 0, \end{cases}$$

for every $(i, g, \lambda), (j, h, \mu) \in I \times G \times \Lambda$.

We define a *Rees matrix semigroup without* 0 simply by deleting all reference to 0 in the above definition, dropping the insistence on P being regular and defining multiplication simply by $(i, g, \lambda)(j, h, \mu) = (i, gp_{\lambda j}h, \mu)$.

Theorem 1.3.34. *Let S be a Rees matrix semigroup over a* 0*-group with a regular sandwich matrix. Then S is completely* 0*-simple. Conversely, every completely* 0*-simple semigroup is isomorphic to such a Rees matrix semigroup.*

Proof. See [33]. □

For clarity, we state the completely simple version of this theorem separately.

Theorem 1.3.35. *Let S be a Rees matrix semigroup without* 0*. Then S is completely simple. Conversely, every completely simple semigroup is isomorphic to such a Rees matrix semigroup.*

Proof. See [33]. □

Theorem 1.3.36. *Let X be a left zero semigroup, let Y be a right zero semigroup, and let G be a group. Let e be the identity of G, fix $u \in X$ and $v \in Y$ and let $[\ ,\ ] : Y \times X \to G$ be a function such that $[y, u] = [v, x] = e$ for all $y \in Y$ and all $x \in X$. Let $S = X \times G \times Y$ and define an operation $\cdot$ on S by $(x, g, y) \cdot (x', g', y') = (x, g[y, x']g', y')$. Then S is a simple semigroup (so that $K(S) = S = X \times G \times Y$) and each of the following statements holds.*

(a) *Idempotents of S are exactly elements of the form $(x, [y, x]^{-1}, y)$, where $(x, y) \in X \times Y$. In particular, the idempotents in $X \times G \times \{v\}$ are of the form (x, e, v) and the idempotents in $\{u\} \times G \times Y$ are of the form (u, e, y).*

(b) *For every $y \in Y$, $X \times G \times \{y\}$ is a minimal left ideal of S and all minimal left ideals of S are of this form.*

(c) *For every $x \in X$, $\{x\} \times G \times Y$ is a minimal right ideal of S and all minimal right ideals of S are of this form.*

(d) *For every $(x, y) \in X \times Y$, $\{x\} \times G \times \{y\}$ is a maximal group in S and all maximal groups in S are of this form.*

(e) *The minimal left ideal $X \times G \times \{v\}$ is the direct product of X, G and $\{v\}$ and the minimal right ideal $\{u\} \times G \times Y$ is the direct product of $\{u\}, G$ and Y.*

(f) *All maximal groups in S are isomorphic to G.*

(g) *All minimal left ideals of S are isomorphic to $X \times G$ and all minimal right ideals of S are isomorphic to $G \times Y$.*

Proof. See [[29], Theorem 1.63] □

Therefore, we have the following result.

Theorem 1.3.37. *(The Structure Theorem) Let S be a semigroup and assume that there is a minimal left ideal of S which has an idempotent. Let R be a minimal right ideal of S, let L be a minimal left ideal of S, let $X = E(L)$, let $Y = E(R)$, and let $G = RL$. Define an operation $\cdot$ on $X \times G \times Y$ by $(x, g, y) \cdot (x', g', y') = (x, gyx'g', y')$. Then $X \times G \times Y$ satisfies the conclusions of Theorem 1.3.36 (where $[y, x] = yx$) and $K(S) \approx X \times G \times Y$. In particular,*

(1) *The minimal right ideals of S partition $K(S)$ and the minimal left ideals of S partition $K(S)$.*

(2) *The maximal groups in $K(S)$ partition $K(S)$.*

(3) *All minimal right ideals of S are isomorphic and all minimal left ideals of S are isomorphic.*

(4) *All maximal groups in $K(S)$ are isomorphic.*

Proof. See [[29], Theorem 1.64] □

Theorem 1.3.38. *Let S be a semigroup and assume that there is a minimal left ideal of S which has an idempotent. Let T be a subsemigroup of S and assume also that T has a minimal left ideal with an idempotent. If $K(S) \cap T \neq \emptyset$, then $K(T) = K(S) \cap T$.*

Proof. See [[29], Theorem 1.65] □

Theorem 1.3.39. *Let S be a semigroup and assume that there is a minimal left ideal of S which has an idempotent. Let $e, f \in E(K(S))$. If g is the inverse of efe in eSe, then the function $\phi : eSe \to fSf$ defined by $\phi(x) = fxgf$ is an isomorphism.*

Proof. See [[29], Theorem 1.66] □

1.4 Right topological semigroups

Definition 1.4.1. Let S be a semigroup. Each $a \in S$ gives two functions $\lambda_a, \rho_a : S \to S$ defined as follows.

$$\lambda_a(x) := ax \ , \ \rho_a(x) := xa \text{ for every } x \in S.$$

λ_a is called the *left shift (by a)* and ρ_a is called the *right shift (by a)*. Shifts are also called *translations*.

Definition 1.4.2. Let S be a semigroup and let $\mathcal{T}$ be a topology on S. $(S, \mathcal{T})$ is called a *right topological semigroup* if for every $a \in S$, the right shift ρ_a is continuous. $(S, \mathcal{T})$ is called a *left topological semigroup* if for every $a \in S$, the left shift λ_a is continuous. We often write S instead of $(S, \mathcal{T})$ if it is understood which topology we are referring to.

Remark 1.4.3. Let S be a semigroup.

(a) If S is commutative, then $\lambda_a = \rho_a$ for every $a \in S$.

(b) For any $a \in S$ and $U \subseteq S$, we will often write $a^{-1}U$ for $\lambda_a^{-1}(U)$. Thus

$$a^{-1}U = \{x \in S \mid ax \in U\}.$$

One should note that in this case a^{-1} is simply a notation, it is not necessarily the inverse of a. In fact the inverse of a need not exist. Similarly, we will often write Ua^{-1} for $\rho_a^{-1}(U)$.

(c) Let $a \in S$ and let $U \subseteq S$. If a is invertible, then we have the following.

$$\begin{aligned} \lambda_a^{-1}(U) = a^{-1}U &= \{x \in S \mid ax \in U\} \\ &= \{a^{-1}u \mid u \in U\} \\ &= a^{-1}U \\ &= \lambda_{a^{-1}}(U) \end{aligned}$$

where a^{-1} is the last three expressions stands for the inverse of a. Thus $\lambda_a^{-1} = \lambda_{a^{-1}}$ and our notation in (b) is consistent with the invertible case. Similarly, if a is invertible, then $\rho_a^{-1} = \rho_{a^{-1}}$.

(d) Let S be a left (respectively, a right) topological semigroup. If S is a group, then we also call S a *left* (respectively, a *right*) *topological group.*

Definition 1.4.4. Let T be a right topological semigroup. The topological center of T, denoted $\Lambda(T)$, is defined as

$$\Lambda(T) := \{a \in T \mid \lambda_a \text{ is continuous}\}.$$

Similarly, one defines the topological center of a left topological semigroup.

Remark 1.4.5.

(a) Since for an invertible a, $\lambda_a^{-1} = \lambda_{a^{-1}}$, λ_a is continuous if and only if $\lambda_{a^{-1}}$ is open. Similarly, ρ_a is continuous if and only if $\rho_{a^{-1}}$ is open. Thus we have the following.

(b) A group G (equipped with a topology) is right (respectively, left) topological if and only if for each $a \in G$, ρ_a (respectively, λ_a) is open.

(c) Let T be a right topological semigroup. If $\Lambda(T) \neq \emptyset$, then $\Lambda(T)$ is a subsemigroup of T.

Of particular importance in our study are compact right topological semigroups.

Theorem 1.4.6. *(Ellis's Theorem)[[29], Theorem 2.5] Every compact Hausdorff right topological semigroup has an idempotent.*

Proof. Let T be a compact Hausdorff right topological semigroup. Let

$$\mathcal{A} = \{A \subseteq T : AA \subseteq A,\ A \neq \emptyset, \text{ and } A \text{ is compact}\},$$

i.e. $\mathcal{A}$ is the family of all compact subsemigroups on T. Since $T \in \mathcal{A}$, $\mathcal{A} \neq \emptyset$. Let $\mathcal{C}$ be a chain in $\mathcal{A}$. Let $x, y \in \bigcap\mathcal{C}$. Then $xy \in C$ for every $C \in \mathcal{C}$. Therefore, $xy \in \bigcap\mathcal{C}$. Thus $(\bigcap\mathcal{C}) \cdot (\bigcap\mathcal{C}) \subseteq \bigcap\mathcal{C}$. Since $\mathcal{C}$ is a chain of nonempty sets, $\mathcal{C}$ has the finite intersection property. Since T is compact, $\bigcap\mathcal{C} \neq \emptyset$. Since $\bigcap\mathcal{C}$ is closed and T is compact, $\bigcap\mathcal{C}$ is compact. It follows that $\bigcap\mathcal{C}$ is a lower bound for $\mathcal{C}$ in $\mathcal{A}$. By Zorn's Lemma, $\mathcal{A}$ has a minimal element A. Let $x \in A$ and let $B = Ax$. Then $B \neq \emptyset$ and $B = \rho_x(A)$. Since

$\rho_x : T \to T$ is continuous and A is compact, B is compact. In addition, $BB = AxAx \subseteq AAAx \subseteq AAx \subseteq Ax = B$. Therefore $B \in \mathcal{A}$. Since $B = Ax \subseteq AA \subseteq A$ and A is minimal in $\mathcal{A}$, $B = A$. Hence $A = Ax$ for every $x \in A$. Let $D = \{y \in A : yx = x\}$. Since $x \in A$ and $Ax = A$, $x = yx$ for some $y \in A$. In particular, $D \neq \emptyset$. Since $D = A \cap \rho_x^{-1}(\{x\})$, D is compact. For any $y, z \in D$, $(yz)x = y(zx) = yx = x$. Therefore, $DD \subseteq D$. It follows that $D \in \mathcal{A}$. Since $D \subseteq A$, $D = A$. Thus $Ax = \{x\}$. In particular, $xx = x$. Hence x is an idempotent of T. □

Corollary 1.4.7. *Let T be a compact Hausdorff right topological semigroup. Every left ideal of T contains a minimal left ideal. Minimal left ideals are closed, and each minimal left ideal has an idempotent.*

Theorem 1.4.8. *[[29], Theorem 2.7] Every compact Hausdorff right topological semigroup has a smallest ideal.*

Proof. This follows from Corollary 1.4.7 and Theorem 1.3.21. □

Theorem 1.4.9. *[[29], Theorem 2.9] Let T be a compact right topological semigroup and let e be an idempotent in T. Then e is minimal with respect to any (and hence to all) of the orders $\leq_R, \leq_L$ or $\leq$ if and only if e belongs to the smallest ideal of T.*

Proof. We know that $K(T) = \bigcup\{L : L \text{ is a minimal left ideal of } T\}$.
Necessity. Since Te is a left ideal of T, there exists a minimal left ideal L of T with $L \subseteq Te$. By Corollary 1.4.7, L is closed. Therefore there exists an idempotent $f \in L$. Since $f \in Te$, $f = ge$ for some $g \in T$. Then $fe = gee = ge = f$. Thus, $f \leq_L e$ and, hence, $f = e$. It follows hat $e \in K(T)$.
Sufficiency. This follows from Corollary 1.3.11. □

Theorem 1.4.10. *Let T be a compact right topological semigroup and let e be an idempotent in T. There is a minimal idempotent f such that $f \leq e$.*

Proof. As in the proof of "necessity" of Theorem 1.4.9, pick a minimal left ideal L of T with $L \subseteq Te$ and pick an idempotent $f \in L$. Then $f = fe$. Take $r = ef$. Then $r \in L$ and we have $rr = efef = eff = ef = r$. Therefore r is a minimal idempotent. In addition, $er = eef = ef = r$ and $re = efe = ef = r$, so $r \leq e$. □

Theorem 1.4.11. *Let T be a right topological semigroup and let R be a right ideal of T. Then $c\ell(R)$ is a right ideal of T.*

Proof. Let $x \in c\ell(R)$, let $a \in T$ and let U be an open neighborhood of xa. Since ρ_a is continuous, there exists an open neighborhood V of x such that $Va \subseteq U$. Since $x \in c\ell(R)$, $V \cap R \neq \emptyset$. Pick $y \in V \cap R$. Then, $ya \in Va \cap Ra \subseteq U \cap R$. Therefore $U \cap R \neq \emptyset$. Thus $xa \in c\ell(R)$. It follows that $c\ell(R)$ is a right ideal of T. □

Remark 1.4.12. Let T be a right topological semigroup and let L be a left ideal of T. The closure of L need not be a left ideal, as illustrated by the following example.

Example [[4], Example V.1.1]: Let X be a compact Hausdorff space with a non-closed subset $D \subseteq X$ such that $c\ell D \neq X$. Define a binary operation on X as follows.

$$xy = \begin{cases} y & \text{if } y \in D, \\ x & \text{if } y \notin D. \end{cases}$$

Let $x, y, z \in X$. If $z \notin D$, then $(xy)z = xy = x(yz)$. If $z \in D$, then $(xy)z = z = xz = x(yz)$. Therefore, X with this binary operation is a semigroup. Let $x \in X$ and let U be an open subset of X. If $x \notin D$, then $\rho_x^{-1}(U) = U$. If $x \in U \cap D$, then $\rho_x^{-1}(U) = X$. If $x \in D \cap (X \setminus U)$, then $\rho_x^{-1}(U) = \emptyset$. Therefore $\rho_x : X \to X$ is continuous. Thus X with this binary operation is a right topological semigroup. In addition $\Lambda(T) = \emptyset$.
Since $XD = D$, D is a left ideal. Let $x \in D$ and let $y \in X$. If $y \in D$, then $xy = y \in D$. If $y \notin D$, then $xy = x \in D$. Therefore D is also a right ideal. Hence D is an ideal. Given any ideal J of X, $D = JD \subseteq J$. It follows that D is the smallest ideal of X.
Let I be a proper ideal of X and suppose $I \neq D$. Pick $x \in I \setminus D$ and pick $y \in X \setminus I$. Then $y = yx \in I$, a contradiction. It follows that $I = D$. Hence D is the only proper ideal of X. Since $D \neq c\ell(D) \neq X$, then $c\ell(D)$ is not an ideal. By Theorem 1.4.11, $c\ell(D)$ is a right ideal. Therefore, $c\ell(D)$ is not a left ideal.

However, we have the following theorem.

Theorem 1.4.13. *Let T be a right topological semigroup such that the topological center $\Lambda(T)$ is dense in T. Then, the closure of every left ideal of T is a left ideal of T.*

Proof. Let L be a left ideal of T, let $x \in c\ell(L)$, let $a \in T$ and let U be an open neighborhood of ax. Then $V := Ux^{-1}$ is an open neighborhood of a with $Vx \subseteq U$. Since $\Lambda(T)$ is dense in T, there exists $b \in V \cap \Lambda(T)$. Then $bx \in U$. Since λ_b is continuous, $W := b^{-1}U$ is a neighborhood of x with

$bW \subseteq U$. Then $W \cap L \neq \emptyset$. Pick $c \in W \cap L$. Then, $bc \in bW \subseteq U$ and $bc \in L$. Therefore, $U \cap L \neq \emptyset$. Hence $ax \in c\ell(L)$. It follows that $c\ell(L)$ is a left ideal of T. $\square$

Definition 1.4.14. A *topological semigroup* is a semigroup S equipped with a topology $\mathcal{T}$ such that multiplication

$$S \times S \to S \ , \ (x, y) \mapsto xy$$

is continuous.

Definition 1.4.15. A *topological group* is a group G equipped with a topology $\mathcal{T}$ such that multiplication and inversion are continuous functions, i.e. the functions

$$G \times G \to G \ , \ (x, y) \mapsto xy \ \text{ and } \ G \to G \ , \ x \mapsto x^{-1}$$

are continuous. In particular every topological group is a topological semigroup. If $(G, \mathcal{T})$ is a topological group, we say that $\mathcal{T}$ is a *group topology.*

Note that continuity of the group multiplication implies continuity in each variable separately. Therefore, every topological group is a left(right) topological semigroup.

We recall that a topological space X is *homogeneous* if for every $x, y \in X$, there is a homeomorphism $f : X \to X$ such that $f(x) = y$.

Proposition 1.4.16. *Every topological group is homogeneous.*

Proof. Let G be a topological group and let $a, b \in G$. Consider the homeomorphism $\lambda_{ab^{-1}} : G \to G$. Then, $\lambda_{ab^{-1}}(b) = (ab^{-1})b = a(b^{-1}b) = a$. $\square$

In the next proposition and in the next theorem the topological groups are assumed to be Hausdorff.

Proposition 1.4.17. *Let G be a topological group and let $\mathcal{F}$ be the neighborhood filter of the identity $e \in G$. For every $x \in G$,*

$$x\mathcal{F} := \{xU \ : \ U \in \mathcal{F}\}$$

is the neighborhood filter of x. Thus, the topology is completely determined by the neighborhood filter of the identity.

Proof. Let $x \in G$. Let $U \in \mathcal{F}$. There exists an open set V such that $e \in V \subseteq U$. Then xV is open, $x = xe \in xV$ and $xV \subseteq xU$. Hence xU is a neighborhood of x. Conversely, let W be a neighborhood of x. Then $x^{-1}W \in \mathcal{F}$ and $W = x(x^{-1}W)$. Thus, the first statement holds. Since a topology is determined by the neighborhood filters at all the points of the space, the second statement holds. □

The next theorem characterizes the neighborhood filter of the identity of a topological group.

Theorem 1.4.18. *Let G be a group and let e be the identity of G. Then for every group topology $\mathcal{T}$ on G, the neighborhood filter $\mathcal{F}$ of e in $\mathcal{T}$ satisfies the following conditions.*

(1) $\bigcap \mathcal{F} = \{e\}$,

(2) for very $U \in \mathcal{F}$*, there exists* $V \in \mathcal{F}$ *such that* $V^2 \subseteq U$,

(3) for every $U \in \mathcal{F}$*, there exists* $V \in \mathcal{F}$ *such that* $V^{-1} \subseteq U$,

(4) for every $U \in \mathcal{F}$ *and for every* $x \in G$*, there exists* $V \in \mathcal{F}$ *such that* $xVx^{-1} \subseteq U$.

Conversely, for every filter $\mathcal{F}$ on G satisfying conditions (1)-(4)*, there is a unique group topology $\mathcal{T}$ on G in which $\mathcal{F}$ is the neighborhood filter at e.*

Proof. Necessity: (1) follows by the Hausdorff property. (2) follows from the fact that for every $U \in \mathcal{F}$, $ee \in U$. (3) follows from the fact that for every $U \in \mathcal{F}$, $e^{-1} \in U$. (4) follows from the fact that for every $x \in G$, $\lambda_x \circ \lambda_{x^{-1}}$ is continuous and $\lambda_x \circ \lambda_{x^{-1}}(e) = e$.
Sufficiency: Consider the system of filters $\mathcal{F}_x := x\mathcal{F}$, $x \in G$. We have

(i) $\bigcap \mathcal{F}_x = \{x\}$,

(ii) for every $U \in \mathcal{F}_x$, there exists $V \in \mathcal{F}_x$ such that $U \in \mathcal{F}_y$ for all $y \in V$.

Therefore, $(\mathcal{F}_x)_{x \in G}$ determines a topology on G with $\mathcal{F}_x$ being the neighborhood filter at x. □

1.5 The Stone-Čech compactification of a discrete semigroup

Definition 1.5.1. Let D be a discrete topological space. We define the following.

(a) $\beta D := \{p \subseteq \mathcal{P}(D) \ : \ p \text{ is an ultrafilter on } D\}$.

(b) Given $A \subseteq D$, $\overline{A} := \{p \in \beta D \ : A \in p\}$.

We have a function $e : D \to \beta D$ defined by $e(x) = \{A \subseteq D \ : \ x \in A\}$, i.e. $e(x)$ is the principal ultrafilter generated by x. It is clear that e is injective.

Lemma 1.5.2. *[[29], Lemma 3.17] Let D be a discrete topological space. For any $A, B \subseteq D$ the following statements hold.*

(1) $\overline{A \cap B} = \overline{A} \cap \overline{B}$.

(2) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(3) $\overline{D \setminus A} = \beta D \setminus \overline{A}$.

(4) $\overline{A} = \emptyset$ *if and only if* $A = \emptyset$.

(5) $\overline{A} = \beta D$ *if and only if* $A = D$.

(6) $\overline{A} = \overline{B}$ *if and only if* $A = B$.

Proof. The statements follow easily from the definition of $\overline{A}$ and properties of ultrafilters. □

As a consequence of Lemma 1.5.2 (1), the set $\mathfrak{B} := \{ \ \overline{A} \ : \ A \subseteq D \ \}$ forms a basis for a topology $\mathcal{T}$ on βD. From now on, unless otherwise stated, we will always assume that βD is equipped with this topology. In addition, for each $x \in D$, we will identify $e(x)$ with x.

Theorem 1.5.3. *[[29], Theorem 3.18] Let D be a discrete topological space.*

(1) βD is a compact Hausdorff space.

(2) For every $A \subseteq D$, $\overline{A}$ is a clopen subset of βD.

(3) *For every $A \subseteq D$, $\overline{A} = c\ell_{\beta D}\ e(A)$ or, which is the same, for any $A \subseteq D$ and any $p \in \beta D$, $p \in c\ell_{\beta D}\ e(A)$ if and only if $A \in p$.*

(4) *$e(D)$ is a dense subset of βD whose points are precisely the isolated points of βD.*

(5) *If U is an open subset of βD, then $c\ell_{\beta D}\ U$ is also open.*

Proof. Verification of these statements is an easy common routine. See for example [[29], Theorem 3.18.] □

In particular, it follows that $e : D \to \beta D$ is an embedding. Since we identify $e(x)$ by x, we will often write $c\ell_{\beta D}\ A$ instead of $c\ell_{\beta D}\ e(A)$, where $A \subseteq D$. Sometimes, to avoid confusion, we use e_D instead of e.

Lemma 1.5.4. *Let D be a discrete topological space. For every $p \in \beta D$, $\lim\limits_{x \to p} e(x) = p$.*

Proof. The statement to be proven is equivalent to the statement: For any $p \in \beta D$, and for any $U \subseteq \beta D$, if U is a neighborhood of p then $e^{-1}(U) \in p$. So, let U be a neighborhood of p. There exists $A \subseteq D$ such that $p \in \overline{A} \subseteq U$, i.e. $A \in p$ and for any $q \in \beta D$ if $A \in q$ then $q \in U$. Therefore $e(x) \in U$ for all $x \in A$ and, thus, $A \subseteq e^{-1}(U)$. Since p is a filter, $e^{-1}(U) \in p$. □

Lemma 1.5.5. *Let D be a discrete topological space and let $f : D \to Y$ be a function into a compact Hausdorff topological space. For every $p \in \beta D$, let $\mathcal{A}_p = \{c\ell_Y f(A) : A \in p\}$. Then,*

$$\bigcap \mathcal{A}_p = \{ \lim_{x \to p} f(x) \}.$$

In particular for every $t \in D$,

$$\lim_{x \to e(t)} f(x) = f(t).$$

Proof. Let $p \in \beta D$ and let $y \in \bigcap \mathcal{A}_p$. Let W be an open neighborhood of y. Since $W \cap f(A) \neq \emptyset$ for all $A \in p$, $f^{-1}(W) \cap A \neq \emptyset$ for all $A \in p$. Since p is an ultrafilter, $f^{-1}(W) \in p$. It follows that $y = \lim_{x \to p} f(x)$. To show that $\bigcap \mathcal{A}_p \neq \emptyset$, it suffices to show that $\mathcal{A}_p$ has the finite intersection property. Let $A_1, \cdots, A_n \in p$ for some $n \in \mathbb{N}$. We have $c\ell_Y f(A_1) \cap \cdots \cap c\ell_Y f(A_n) \supseteq c\ell_Y(f(A_1) \cap \cdots \cap f(A_n)) \supseteq c\ell_Y(f(A_1 \cap \cdots \cap A_n)) \supseteq f(A_1 \cap \cdots \cap A_n) \neq \emptyset$. Therefore $\mathcal{A}_p$ has the finite intersection property. Since Y is compact,

$\bigcap \mathcal{A}_p \neq \emptyset$. Hence the equality follows (see also Theorem 1.1.14 (2)).
For any $t \in D$ we have $\bigcap \mathcal{A}_{e(t)} \subseteq c\ell_Y f(\{t\}) = c\ell_Y\{f(t)\} = \{f(t)\}$. It follows that $\bigcap \mathcal{A}_{e(t)} = \{f(t)\}$. □

Theorem 1.5.6. *Let D be a discrete topological space. βD is the Stone-Čech compactification of D.*

Proof. It remains to show that for any function f from D into a compact Hausdorff topological space Y there exists a unique continuous function $\overline{f} : \beta D \to Y$ such that the diagram

$$\begin{tikzcd} D \arrow[r, hook, "e"] \arrow[dr, "f"'] & \beta D \arrow[d, dotted, "\overline{f}"] \\ & Y \end{tikzcd}$$

commutes, i.e. $\overline{f} \circ e = f$.
Let Y be a compact Hausdorff space and consider a function $f : D \to Y$. Define $\overline{f} : \beta D \to Y$ by

$$\overline{f}(p) = \lim_{x \to p} f(x).$$

If follows from Lemma 1.5.5 that $\overline{f}(e(t)) = f(t)$ for all $t \in D$. For continuity of $\overline{f}$, let $p \in \beta D$ and let U be a neighborhood of $\overline{f}(p)$. Since Y is regular, pick a neighborhood V of $\overline{f}(p)$ such that $c\ell_Y\ V \subseteq U$ and let $A = f^{-1}(V)$. Suppose $D \setminus A \in p$. Then $\overline{f}(p) \in c\ell_Y\ f(D \setminus A)$. Then $V \cap f(D \setminus A) \neq \emptyset$, contradicting the fact that $A = f^{-1}(V)$. Thus $\overline{A}$ is a neighborhood of p. Let $q \in \overline{A}$ and suppose that $\overline{f}(q) \notin U$. Then $Y \setminus c\ell_Y\ V$ is a neighborhood of $\overline{f}(q)$ and $\overline{f}(q) \in c\ell_Y\ f(A)$. Then, $(Y \setminus c\ell_Y\ V) \cap f(A) \neq \emptyset$, again contradicting the fact that $A = f^{-1}(V)$. Hence $\overline{f}$ is continuous. The uniqueness of $\overline{f}$ follows from its definition. □

As mentioned earlier, we will identify principal ultrafilters on D with points of D, and $D^* := \beta D \setminus D$, i.e. the elements of D^* are the *free ultrafilters* on D.

Lemma 1.5.7. *Let D and E be discrete topological spaces and let $f : D \to E$ be a function. For every $p \in \beta D$, $\{A \subseteq E : f^{-1}(A) \in p\} \in \beta E$. If, in addition, f is injective, then for every $p \in D^*$, $\{A \subseteq E : f^{-1}(A) \in p\} \in E^*$.*

Proof. Let $p \in \beta D$ and let $q = \{A \subseteq E : f^{-1}(A) \in p\}$. Since $f^{-1}(\emptyset) = \emptyset \notin p$, $\emptyset \notin q$. Since $f^{-1}(E) = D \in p$, $E \in q$. Let $A \in q$ and let $B \subseteq E$ with $A \subseteq B$.

Since $f^{-1}(B) \supseteq f^{-1}(A) \in p$, $f^{-1}(B) \in p$. Therefore $B \in q$. Let $A, B \in q$ Since $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B) \in p$, $A \cap B \in q$. It follows that q is a filter on E. Let $\mathcal{G}$ be a filter on E such that $q \subseteq \mathcal{G}$ and assume $q \neq \mathcal{G}$. Pick $B \in \mathcal{G} \setminus q$. Then $f^{-1}(B) \notin p$. Pick $U \in p$ such that $f^{-1}(B) \cap U = \emptyset$. Since $U \subseteq f^{-1}(f(U))$, $f^{-1}(f(U)) \in p$ and, therefore, $f(U) \in q$. On the other hand $f(U) \cap B = \emptyset$. This is not possible since $f(U), B \in \mathcal{G}$. Therefore $q = \mathcal{G}$. It follows that $q \in \beta E$. Now suppose f is injective. Let $p \in D^*$ and assume $\{A \subseteq E : f^{-1}(A) \in p\} \in E$. Then $f^{-1}(B) \in p$ for some finite subset B of E. Since f is injective, $|f^{-1}(B)| \leq |B|$ and, therefore, $f^{-1}(B)$ is finite. This contradicts the fact that $f^{-1}(B) \in p \in D^*$. Therefore, $\{A \subseteq E : f^{-1}(A) \in p\} \in E^*$. □

Proposition 1.5.8. *Let D and E be discrete topological spaces and let $f : D \to E \subseteq \beta E$ be a function. The continuous extension $\overline{f} : \beta D \to \beta E$ of $e_E \circ f : D \to \beta E$ is given by*

$$\overline{f}(p) = \{A \subseteq E : f^{-1}(A) \in p\}$$

for every $p \in \beta D$.

Proof. Let $g : \beta D \to \beta E$, $g(p) := \{A \subseteq E : f^{-1}(A) \in p\}$. It remains to show that g is a continuous extension of $e_E \circ f$. Let $x \in D$. $g(e_E(x)) = \{A \subseteq E : f^{-1}(A) \in e_E(x)\} = \{A \subseteq E : x \in f^{-1}(A)\} = e_E(f(x))$. Therefore, $g_{|D} = e_E \circ f$. Now to show that g is continuous. Let $A \subset E$ and consider the basic open subset $\overline{A} \subseteq \beta E$. $g^{-1}(\overline{A}) = \{p \in \beta D : g(p) \in \overline{A}\} = \{p \in \beta D : A \in g(p)\} = \{p \in \beta D : f^{-1}(A) \in p\} = \overline{f^{-1}(A)}$ is open in βD. It follows that g is the continuous extension of $e_E \circ f$ and, hence, $g = \overline{f}$. □

Note that in Proposition 1.5.8 we used $\overline{f}$ but, formally, the correct notation is $\overline{e_E \circ f}$, as one can see from the following diagram.

$$\begin{tikzcd} D \arrow[r, hook, "e_D"] \arrow[d, "f"'] & \beta D \arrow[d, dotted, "\overline{e_E \circ f}"] \\ E \arrow[r, hook, "e_E"] & \beta E \end{tikzcd}$$

Furthermore, since β is a functor, one usually use the notation βf instead of $\overline{e_E \circ f}$.

Proposition 1.5.9. *Let D and E be discrete topological spaces and let $f : D \to E \subseteq \beta E$ be a function. The continuous extension $\overline{f} : \beta D \to \beta E$ is injective if f is injective, surjective if f is surjective and a homeomorphism if f is bijective.*

Proof. Suppose f is injective. Let $p, q \in \beta D$ with $p \neq q$. There exists $A \in p \setminus q$. Since $A = f^{-1}(f(A)) \in p \setminus q$, $f(A) \in \bar{f}(p) \setminus \bar{f}(q)$. Therefore, $\bar{f}(p) \neq \bar{f}(q)$. Hence, $\bar{f}$ is injective. Suppose f is surjective. Let $r \in \beta E$ and consider

$$p := \{A \subseteq D : f^{-1}(X) \subseteq A \text{ for some } X \in r\}.$$

It is easy to see that p is a filter on D and, therefore, p is contained in some ultrafilter q on D. Let $X \in r$. Then, $f^{-1}(X) \in p \subseteq q$. Therefore, $r \subseteq \bar{f}(q)$. Conversely, let $X \in \bar{f}(q)$ and let $Y \in r$. Then, $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y) \neq \emptyset$. Therefore, $X \cap Y \neq \emptyset$. It follows that $X \in r$. Thus $\bar{f}(q) \subseteq r$. It follows that $\bar{f}(q) = r$. Hence, $\bar{f}$ is surjective. Now, suppose f is bijective. It remains to show that $\bar{f}^{\,-1}$ is continuous or, equivalently, that $\bar{f}$ is open. Let $A \subset D$. It is easy to see that $\bar{f}(\overline{A}) = \overline{f(A)}$, where $\bar{f}(\overline{A}) \subseteq \overline{f(A)}$, follows from injectivity while $\bar{f}(\overline{A}) \supseteq \overline{f(A)}$ follows from surjectivity. Therefore, $\bar{f}$ is open.

□

Definition 1.5.10. Let D be a discrete topological space and let $\mathcal{F}$ be a filter on D. We define

$$\overline{\mathcal{F}} := \{p \in \beta D \ : \ \mathcal{F} \subseteq p\},$$

equivalently

$$\overline{\mathcal{F}} := \bigcap\{\overline{A} \ : \ A \in \mathcal{F}\}.$$

$\overline{\mathcal{F}}$ is called *the subspace (or subset) induced by* $\mathcal{F}$.

Remark 1.5.11. Note that

$$\bigcap_{p \in \beta S} p = \{S\} \text{ and, hence, } \beta S = \overline{\{S\}}.$$

Theorem 1.5.12. *Let D be a discrete topological space. Given a filter $\mathcal{F}$ on D, $\overline{\mathcal{F}}$ is a nonempty closed subset of βD. Conversely, any nonempty closed subset of βD can be uniquely represented as $\overline{\mathcal{F}}$ for some filter $\mathcal{F}$ on D. In particular, for any two filters $\mathcal{F}$ and $\mathcal{G}$ on D, $\overline{\mathcal{G}} \subseteq \overline{\mathcal{F}}$ if and only if $\mathcal{F} \subseteq \mathcal{G}$.*

Proof. Let $\mathcal{F}$ be a filter on D. Since every filter is contained in an ultrafilter, $\overline{\mathcal{F}} \neq \emptyset$. Let $q \in \beta D \setminus \overline{\mathcal{F}}$. There exists $A \in \mathcal{F}$ such that $A \notin q$. Then $q \in \overline{D \setminus A}$ and $\overline{D \setminus A} \subseteq \beta D \setminus \overline{\mathcal{F}}$. Thus $\beta D \setminus \overline{\mathcal{F}}$ is open and $\overline{\mathcal{F}}$ is closed.
(One can also see that $\overline{\mathcal{F}}$ is closed from $\overline{\mathcal{F}} = \bigcap\{\overline{A} \mid A \in \mathcal{F}\}$, since $\overline{X}$ is closed for any $X \subseteq D$.) Conversely, let $\emptyset \neq \mathfrak{F} \subseteq \beta D$. Put $\mathcal{F} = \bigcap \mathfrak{F}$. Since $\mathcal{F}$ is an intersection of filters, $\mathcal{F}$ is a filter. It then follows, from the first part, that

$\overline{\mathcal{F}}$ is closed and, hence, $\overline{\mathcal{F}} \supseteq cl_{\beta D}\mathfrak{F}$. Now, let $q \in \overline{\mathcal{F}}$ and let $A \in q$. Then $D \setminus A \notin q$. There exists $p \in \mathfrak{F}$ such that $D \setminus A \notin p$. Then $A \in p$. Thus $p \in \overline{A} \cap \mathfrak{F}$. Therefore $q \in cl_{\beta D}\mathfrak{F}$. Hence $\overline{\mathcal{F}} \subseteq cl_{\beta D}\mathfrak{F}$. It now follows that $\overline{\mathcal{F}} = cl_{\beta D}\mathfrak{F}$. In particular, if $\mathfrak{F}$ is closed then $\mathfrak{F} = \overline{\mathcal{F}}$, i.e. $\mathfrak{F} = \overline{\bigcap \mathfrak{F}}$.
For uniqueness let $\mathcal{F}$ and $\mathcal{G}$ be filters on D with $\overline{\mathcal{F}} = \overline{\mathcal{G}}$ and assume $\mathcal{F} \nsubseteq \mathcal{G}$. Pick $A \in \mathcal{F} \setminus \mathcal{G}$. Then $\mathcal{G} \cup \{D \setminus A\} \subseteq \mathcal{U}$ for some $\mathcal{U} \in \beta D$. But then $A \in \mathcal{F} \subseteq \mathcal{U}$, a contradiction. It follows that $\mathcal{F} \subseteq \mathcal{G}$. Similarly, $\mathcal{G} \subseteq \mathcal{F}$. Hence $\mathcal{F} = \mathcal{G}$.
To prove the last part, suppose $\overline{\mathcal{G}} \subseteq \overline{\mathcal{F}}$. Then $\mathcal{F} = \bigcap \overline{\mathcal{F}} \subseteq \bigcap \overline{\mathcal{G}} = \mathcal{G}$. Conversely, suppose $\mathcal{F} \subseteq \mathcal{G}$. Then it is clear that $\overline{\mathcal{G}} \subseteq \overline{\mathcal{F}}$.

□

Now we would like to extend a binary operation on a discrete topological space D to βD.

Theorem 1.5.13. *[[29], Theorem 4.1] Let D be a discrete topological space and let $\cdot$ be a binary operation on D. There is a binary operation $*$ on βD satisfying the following conditions.*

(a) $*$ extends $\cdot$; i.e. $s * t = s \cdot t$ for all $s, t \in D$.

(b) For each $q \in \beta D$, the function from βD into βD, $p \mapsto p * q$ is continuous.

(c) For each $s \in D$, the function from βD into βD, $p \mapsto s*p$ is continuous.

Proof. Given $s \in D$, s defines the function $\lambda_s : D \to D \subseteq \beta D$, $\lambda_s(t) = st$. By the universal property of βD, there exists a unique continuous function $l_s : \beta D \to \beta D$ such that $l_s \circ e = \lambda_s$, i.e. such that the diagram

$$\begin{tikzcd} D \arrow[r, hook, "e"] \arrow[dr, "\lambda_s"'] & \beta D \arrow[d, dotted, "l_s"] \\ & \beta D \end{tikzcd}$$

commutes.
For $s \in D$ and $p \in \beta D$, we define

$$s * p := l_s(p) = \lim_{t \to p} st = \{A \subseteq D : s^{-1}A \in p\}.$$

Then for every $B \subseteq D$ we have

$$(s * (-))^{-1}(\overline{B}) = l_s^{-1}(\overline{B}) = \overline{s^{-1}B}.$$

In particular condition (c) is satisfied. Since $l_s \circ e = \lambda_s$, $s * t = l_s(t) = \lambda_s(t) = st$ for every $t \in S$. Therefore, condition (a) is satisfied. Since D is dense in βD and l_s is a continuous extension of λ_s, l_s is unique. Now, we would like to extend $*$ to $\beta D \times \beta D$. Given $q \in \beta D$, define $R_q : D \to \beta D$ by $R_q(s) = s * q = l_s(q)$. Then, there is a unique continuous extension $r_q : \beta D \to \beta D$, such that $r_q \circ e = R_q$.

$$\begin{tikzcd} D \arrow[r, hook, "e"] \arrow[dr, "R_q"'] & \beta D \arrow[d, dotted, "r_q"] \\ & \beta D \end{tikzcd}$$

For $p \in \beta D \setminus D$ we define $p * q := r_q(p)$. Note that if $t \in D$, then $r_q(t) = R_q(t) = t * q$. Thus, for all $p \in \beta D$, $r_q(p) = p * q$. Hence condition (b) is satisfied. By the uniqueness of a continuous extension of a function defined on a dense subset, r_q is unique. Hence we have the required binary operation $*$. $\square$

For any binary operation on a discrete topological space D we adopt the custom of denoting its extension to βD by the same symbol.

Remark 1.5.14. Let $\cdot$ be a binary operation on a discrete topological space D.

(a) For every $s \in D$ and every $q \in \beta D$, $sq = \lim\limits_{t \to q} st$.

(b) For every $p, q \in \beta D$, $pq = \lim\limits_{s \to p} \lim\limits_{t \to q} st$.

Proof. For $s \in D$ and for $q \in \beta D$ let the functions λ_s, l_s R_q and r_q be as in the proof of Theorem 1.5.13

(a) Let $s \in D$ and let $q \in \beta D$.

$$\begin{tikzcd} D \arrow[r, hook, "e"] \arrow[d, "\lambda_s"'] & \beta D \arrow[d, "l_s"] \\ D \arrow[r, hook, "e"'] & \beta D \end{tikzcd}$$

Let U be an open neighborhood of sq. Since l_s is continuous, $l_s^{-1}(U)$ is an open neighborhood of q. Therefore, there exists $A \subseteq S$ such that

$q \in \overline{A} \subseteq l_s^{-1}(U)$. Then, $A \in q$ and for any $t \in A$, $l_s(e(t)) \in U$. Since $l_s \circ e = \lambda_s \circ e$, for any $t \in A$, $(\lambda_s \circ e)(t) \in U$, i.e. $st \in U$. Hence $A \subseteq \{t \in D : st \in U\}$. Since $A \in q$ and q is a filter, $\{t \in D : st \in U\} \in q$.

(b) Let $p, q \in \beta D$ and let U be an open neighborhood of pq.

$$\begin{tikzcd} D \arrow[r, hook, "e"] \arrow[dr, "R_q"'] & \beta D \arrow[d, "r_q"] \\ & \beta D \end{tikzcd}$$

Recall that $R_q(s) = l_s(q)$. By (a), $\lim\limits_{t \to q} st = sq$. Therefore, we would like to show that $pq = \lim\limits_{s \to p} sq$. So, let U be an open neighborhood of $pq(= r_q(p))$. Since r_q is continuous, there exists $A \subseteq D$ such that $p \in \overline{A} \subseteq {r_q}^{-1}(U)$. Then $A \in p$ and for every $s \in A$, $sq = r_q(s) \in U$. Therefore, $A \subseteq \{s \in D : sq \in U\}$. Since p is a filter, $\{s \in D : sq \in U\} \in p$.

□

Statement (b) in Remark 1.5.14 can be generalized as follows.

Remark 1.5.15. Let D be a discrete topological space and let $p, q \in \beta D$. Let $P \in p$ and let $Q \in q$. The following statement holds.

$$pq = p\text{-}\lim_{s \in P}(\, q\text{-}\lim_{t \in Q} st).$$

In particular, for any compact Hausdorff topological space Y and for any function $f : D \to Y$,

$$\lim_{s \to p} f(y) = p\text{-}\lim_{s \in D} f(s).$$

Proof. Let U be an open neighborhood of pq. We would like to show that $\{s \in P : \{t \in Q : st \in U\} \in q\} \in p$. From the previous remark we obtain the following:

$$\{t \in Q : st \in U\} = Q \cap \{t \in D : st \in U\} \in q$$

and

$$\{s \in P : \{t \in Q : st \in U\} \in q\} = P \cap \{s \in D : \{t \in Q : st \in U\} \in q\} \in p.$$

Thus the first statement holds. Now let $f : D \to Y$ be a function into a compact Hausdorff space Y, let $z = \lim_{s \to p} f(s)$ and let U be a neighborhood of z. Then $f^{-1}(U) \in p$ and, therefore, $\{s \in B : f(s) \in U\} = B \cap f^{-1}(U) \in p$. Thus the second statement follows. □

As a result of the last two remarks above, we have the following corollary.

Corollary 1.5.16. *Let D be a discrete topological space endowed with a binary operation and let $p, q \in \beta D$. The following statements hold.*

(i) For every $U \subseteq D$,

$$U \in pq \iff \{s \in D : \{t \in D : st \in U\} \in q\} \in p.$$

(ii) pq has a base consisting of subsets of D of the form

$$\bigcup \{xBx : x \in A\},$$

where $A \in p$ and $B_x \in q$.

Proof. (i) Let $U \subseteq D$. Since $\lim_{t \to q} st = sq$ and, hence, $pq = \lim_{s \to p} \lim_{t \to q} st = \lim_{s \to p} sq$,

$$sq \in U \iff \{t \in D : st \in U\} \in q.$$

Therefore,

$$U \in pq \iff \{s \in D : sq \in U\} \in p \iff \{s \in D : \{t \in D : st \in U\} \in q\} \in p.$$

(ii) Let $p, q \in \beta S$, let $A \in p$ and, for each $x \in A$, let $B_x \in q$. For each $s \in S$ define V_s by

$$V_s = \{t \in S : st \in \bigcup_{x \in A} xB_x\},$$

and let

$$W = \{s \in S : V_s \in q\}.$$

We would like to show that $W \in p$. Let $s \in S$. Then $B_s \subseteq V_s$. Since $B_s \in q$, $V_s \in q$. Therefore, $A \subseteq W$. Since $A \in p$, $W \in p$. It follows that $\bigcup_{x \in A} xB_x \in pq$. Now let $U \in pq$ and let $B \in q$. Then $V := \{x \in A : \{y \in B : xy \in U\} \in q\} \in p$. In particular, $V \neq \emptyset$. Pick $x \in V$ and let $B_x = \{y \in B : xy \in U\}$. Then, $B_x \in q$ and $xB_x \subseteq U$. Hence $\bigcup_{x \in V} xB_x \subseteq U$. It follows that the family

$$\left(\bigcup_{x \in A} xB_x \right)_{A \in p}$$

form a base for pq.

□

Remark 1.5.17. In Corollary 1.5.16 (i) if we take p to be a principal ultrafilter x, then the statement becomes

$$U \in xq \iff \{t \in D : xt \in U\} \in q.$$

On the other hand, if we take q to be a principal ultrafilter y, then the statement becomes

$$U \in py \iff \{s \in D : sy \in U\} \in p.$$

Now we would like to look at the extended binary operation on βS where S is a semigroup, regarded as a discrete space (we will call such a semigroup a *discrete semigroup*). So, let S be a discrete semigroup. As we have seen above, the binary operation on S can be extended to βS as follows. For $p, q \in \beta S$

$$pq = \lim_{s \to p} \lim_{t \to q} st,$$

where $s, t \in S$.

Lemma 1.5.18. *[[29], Theorem 4.5] Let S be a discrete semigroup, let $f : S \to X$ be a function into a Hausdorff space, and let $p, q \in \beta S$. If all limits involved exist, then $\lim_{v \to pq} f(v) = \lim_{s \to p} \lim_{t \to q} f(st)$.*

Proof. Suppose all limits concerned exist. Let $x_0 = \lim_{s \to p} \lim_{t \to q} f(st)$ and let U be a neighborhood of x_0. Then, $\{s \in S : \lim_{t \to q} f(st) \in U\} \in p$. On the other hand,

$$\begin{aligned}
\lim_{t \to q} f(st) \in U \quad &\Rightarrow \quad \{t \in S : f(st) \in U\} \in q \\
&\Leftrightarrow \quad \{t \in S : st \in f^{-1}(U)\} \in q \\
&\Leftrightarrow \quad \{t \in S : t \in s^{-1}(f^{-1}(U))\} \in q \\
&\Leftrightarrow \quad s^{-1}(f^{-1}(U)) \in q.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\{s \in S : \lim_{t \to q} f(st) \in U\} \in p \quad &\Leftrightarrow \quad \{s \in S : s^{-1}(f^{-1}(U)) \in q\} \in p \\
&\Leftrightarrow \quad f^{-1}(U) \in pq.
\end{aligned}$$

It follows that $x_0 = \lim_{v \to pq} f(v)$. $\square$

Theorem 1.5.19. *Let S be a discrete semigroup. Then, with the extended binary operation, βS is a semigroup.*

Proof. Let $p, q, r \in \beta S$.

$$\begin{aligned}
(pq)r &= \lim_{x\to pq}\lim_{y\to r} xy \\
&= \lim_{s\to p}\lim_{t\to q}\left(\lim_{y\to r}(st)y\right) \\
&= \lim_{s\to p}\lim_{t\to q}\left(\lim_{y\to r} s(ty)\right) \\
&= \lim_{s\to p}\left(\lim_{v\to qr} sv\right) \\
&= p(qr)
\end{aligned}$$

□

For any semigroup S, unless otherwise stated, when we refer to the semigroup βS, we are considering the extension of the binary operation of S.

Theorem 1.5.20. *Let S be a discrete semigroup. Then, βS is a compact right topological semigroup, with S contained in its topological center.*

Proof. It remains to show that βS is right topological and that $S \subseteq \Lambda(\beta S)$, but these follow from Theorem 1.5.13 (b) and (c) respectively. □

Remark 1.5.21. Let S be a discrete semigroup. Let $p, q \in \beta S$ and let $A \subseteq S$.

(a) Recall that $x^{-1}A = \{y \in S : xy \in A\}$. Therefore, using these notations, we have the following.

$$A \in pq \iff \{x \in S \ : \ x^{-1}A \in q\} \in p.$$

In particular for any $p \in \beta S$ and for any $x \in S$,

$$A \in xp \iff x^{-1}A \in p$$

and

$$A \in px \iff Ax^{-1} \in p,$$

where $Ax^{-1} = \{y \in S : yx \in A\}$.

(b) $A \in xq$ for every $x \in S$ if and only if $A \in rq$ for every $r \in \beta S$. To see this, suppose $A \in xq$ for every $x \in S$. Then $x^{-1}A \in q$ for every $x \in S$. Let $r \in \beta S$. Since $\{x \in S : x^{-1}A \in q\} = S \in r$, $A \in rq$. The converse is obvious.

(c) We will use $A^*(p)$ to denote the set $\{x \in A : x^{-1}A \in p\}$. Thus, if $A^*(q) \in p$ then $A \in pq$. In particular, for $A \in p$, $A^*(q) \in p$ if and only if $A \in pq$.

(d) It follows that $pp = p$ if and only if $\{A^*(p) : A \in p\} \subseteq p$.

Lemma 1.5.22. *[[29], Lemma 4.14] Let S be a semigroup, let $p \in \beta S$ such that $pp = p$, and let $A \subseteq S$. For each $x \in A^*(p)$, $x^{-1}(A^*(p)) \in p$.*

Proof. Let $x \in A^*(p)$. Then $B := x^{-1}A \in p$. Since $pp = p$, $B^*(p) \in p$. We show that $B^*(p) \subseteq x^{-1}(A^*(p))$. So let $y \in B^*(p)$. Then $y \in B$, i.e. $xy \in A$, and $(xy)^{-1} = y^{-1}(x^{-1}A) = y^{-1}B \in p$. Therefore, $xy \in A^*(p)$ and, hence, $y \in x^{-1}(A^*(p))$. It follows that $B^*(p) \subseteq x^{-1}(A^*(p))$. Since $B^*(p) \in p$, $x^{-1}(A^*(p)) \in p$.

□

Theorem 1.5.23. *Let S be a discrete cancellative semigroup. Then, S^* is an ideal of βS.*

Proof. Let $p \in \beta S$ and let $q \in S^*$. Suppose $pq \in S$, say $pq = \{A \subset S : a \in A\}$ for some $a \in S$. Since $\{a\} \in pq$, $\{s \in S : s^{-1}a \in q\} \in p$. In particular, $\{s \in S : s^{-1}a \in q\} \neq \emptyset$. Pick $s \in S$ such that $s^{-1}a \in q$. Then, the equation $sb = a$ has infinitely many solutions. This is a contradiction, since S is cancellative. Therefore $pq \in S^*$. It follows that S^* is a left ideal of βS. Similarly, S^* is a right ideal of βS. It follows that S^* is an ideal of βS. □

Theorem 1.5.24. *[[29], Theorem 4.20] Let S be a discrete semigroup and let $\mathcal{A} \subseteq \mathcal{P}(S)$ such that $\mathcal{A}$ has the finite intersection property. If for each $A \in \mathcal{A}$ and each $x \in A$, there exists $B_x \in \mathcal{A}$ such that $xB_x \subseteq A$, then*

$$\bigcap_{A \in \mathcal{A}} \overline{A}$$

is a subsemigroup of βS.

Proof. Let $T = \bigcap_{A \in \mathcal{A}} \overline{A}$. Since $\mathcal{A}$ has the finite intersection property, there is $p \in \beta S$ such that $\mathcal{A} \subseteq p$. Then $p \in T$ and, therefore, $T \neq \emptyset$. Let $p, q \in T$

and let $A \in \mathcal{A}$. By assumption, for each $x \in A$ there exists $B_x \in \mathcal{A}$ such that $xB_x \subseteq A$. Since $\bigcup\{xB_x : x \in A\}$ belongs to a base for pq, $A \in pq$. It follows that $pq \in T$. Therefore, T is a subsemigroup of βS. □

Example 1.5.25. Let S be a discrete semigroup and let $(x_i)_{i\in\mathbb{N}}$ be a sequence in S satisfying the distinct finite products property. For each $m \in \mathbb{N}$, let

$$A_m = FP((x_i)_{i\geq m}).$$

Since $A_{n+1} \subseteq A_n$ and $A_n \neq \emptyset$ for all $n \in \mathbb{N}$, the family $(A_n)_{n\in\mathbb{N}}$ has the finite intersection property (see page 2). Let $m \in \mathbb{N}$, let $F \subseteq \mathbb{N}$ be finite with $\min(F) \geq m$, and let $x = \Pi_{i\in F}\, x_i$. Let $n \in \mathbb{N}$ such that $n > \max(F)$. Let $E \subset \mathbb{N}$ with $\min(E) \geq n$ and let $G = F \cup E$. Then, G is a finite subset of $\mathbb{N}$, $\min(G) \geq m$ and $x \cdot \Pi_{i\in E}\, x_i = \Pi_{i\in F}\, x_i \cdot \Pi_{i\in E}\, x_i = \Pi_{i\in G}\, x_i$. Therefore, $x \cdot A_n \subseteq A_m$. It follows that

$$T := \bigcap_{m\in\mathbb{N}} c\ell_{\beta S}\, FP((x_i)_{i\geq m})$$

is a subsemigroup of βS.

1.6 $K(\beta S)$ and its closure

It follows from Theorem 1.4.8 that for any semigroup S, βS has a smallest ideal $K(\beta S)$, which is a completely simple semigroup, and that $c\ell\, K(\beta S)$ is also an ideal of βS. We would like to present the combinatorial characterizations of these ideals.

Definition 1.6.1. Let S be a semigroup and let $A \subseteq S$.

(a) A is called *syndetic* if there exists a finite subset G of S such that for every $x \in S$, $tx \in A$ for some $t \in G$, i.e.

$$G^{-1}A = \bigcup_{t\in G} t^{-1}A = S.$$

(b) A is called *piecewise syndetic* if there exists a finite subset G of S such that the family

$$\left(y^{-1}\left(\bigcup_{t\in G} t^{-1}A\right)\right)_{y\in S}$$

has the finite intersection property.

Remark 1.6.2. Let S be a semigroup and let $A \subseteq S$.

(1) It is clear that every syndetic subset of S is piecewise syndetic.

(2) Let $A, G \subseteq S$ and let $x, y \in S$.

$$x \in y^{-1}\left(\bigcup_{t\in G} t^{-1}A\right) \iff yx \in \bigcup_{t\in G} t^{-1}A \iff tyx \in A \text{ for some } t \in G.$$

Therefore, $\left(y^{-1}\left(\bigcup_{t\in G} t^{-1}A\right)\right)_{y\in S}$ has the finite intersection property if and only if for any finite subset $\{y_1, \cdots, y_n\}$ of S there exists $x \in S$ such that $x \in y_1^{-1}\left(\bigcup_{t\in G} t^{-1}A\right) \cap \cdots \cap y_n^{-1}\left(\bigcup_{t\in G} t^{-1}A\right)$, if and only if there exists a finite subset $\{t_1, \cdots, t_n\}$ of G such that $t_1y_1x, \cdots, t_ny_nx \in A$.

(3) It follows easily from (2) that the following statement holds. A subset A of a semigroup S is piecewise syndetic if and only if there is a finite $G \subseteq S$ such that for every finite $F \subseteq S$ there is some $x \in S$ with

$$Fx \subseteq \bigcup_{t\in G} t^{-1}A.$$

Elements of $K(\beta(S))$ are characterized by the following theorem.

Theorem 1.6.3. *[[29], Theorem 4.39] Let S be a semigroup and let $p \in \beta S$. The following statements are equivalent.*

(a) $p \in K(\beta S)$.

(b) For every $A \in p$, $\{x \in S : x^{-1}A \in p\}$ is syndetic.

(c) For every $q \in \beta S$, $p \in \beta Sqp$.

Proof. $(a) \Rightarrow (b)$: Let $p \in K(\beta S)$ and let $A \in p$. Put $B = \{x \in S : x^{-1}A \in p\}$. Let L be the minimal left ideal of βS with $p \in L$. For every $q \in L$, we have $p \in \beta Sq = c\ell_{\beta S}(Sq)$. Since $\overline{A}$ is a neighborhood of p in βS, $tq \in \overline{A}$ for some $t \in S$ and so $q \in \overline{t^{-1}A}$. Thus the sets of the form $\overline{t^{-1}A}$ cover the compact set L and hence $L \subseteq \bigcup_{t\in G} \overline{t^{-1}A}$ for some finite subset G of S. We show that $S \subseteq \bigcup_{t\in G} t^{-1}B$. Let $a \in S$. Then $ap \in L$. Pick $t \in G$ such that $ap \in \overline{t^{-1}A}$. Then $t^{-1}A \in ap$ so that $(ta)^{-1}A = a^{-1}(t^{-1}A) \in p$ and so $ta \in B$ and thus $a \in t^{-1}B$.

$(b) \Rightarrow (c)$: Let $q \in \beta S$ and suppose that $p \notin \beta Sqp$. Pick $A \in p$ such that $\overline{A} \cap Sqp = \emptyset$. Let $B = \{x \in S : x^{-1}A \in p\}$ and pick $G \in \mathcal{P}_f(S)$ such that

$S = \bigcup_{t\in G} t^{-1}B$. Pick $t \in G$ such that $t^{-1}B \in q$. Then $B \in tq$. That is, $\{x \in S : x^{-1}A \in p\} \in tq$. So $A \in tqp$, a contradiction.
$(c) \Rightarrow (a)$: Take $q \in K(\beta S)$. Then $\beta Sqp \subseteq K(\beta S)$. Hence $p \in K(\beta S)$. □

Theorem 1.6.4. *[[29], Theorem 4.40] Let S be a semigroup and let $A \subseteq S$. Then,*

$$A \text{ is piecewise syndetic} \iff \overline{A} \cap K(\beta S) \neq \emptyset.$$

Proof. $\Leftarrow$: Let $p \in \overline{A} \cap K(\beta S)$ and let $B = \{x \in S : x^{-1}A \in p\}$. Then B is syndetic. Pick a finite subset G of S such that $S = \bigcup_{t\in G} t^{-1}B$. For each $a \in S$ pick $t \in G$ such that $a \in t^{-1}B$, i.e. $ta \in B$. Then $a^{-1}(t^{-1}A) = (ta)^{-1}A \in p$. Then $a^{-1}(\bigcup_{t\in G} t^{-1}A) = \bigcup_{t\in G} a^{-1}(t^{-1}A) \in p$. It follows that $\{a^{-1}(\bigcup_{t\in G} t^{-1}A) : a \in S\}$ has the finite intersection property. Hence A is piecewise syndetic.
$\Rightarrow$: Suppose A is piecewise syndetic. Pick a finite subset $G \subseteq S$ such that $\{a^{-1}(\bigcup_{t\in G} t^{-1}A) : a \in S\}$ has the finite intersection property. Pick $q \in \beta S$ such that $\{a^{-1}(\bigcup_{t\in G} t^{-1}A) : a \in S\} \subseteq q$. Then $\bigcup_{t\in G} t^{-1}A \in aq$ for every $a \in S$ and, hence, $\bigcup_{t\in G} t^{-1}A \in pq$ for every $p \in \beta S$, i.e. $(\beta S)q \subseteq \overline{\bigcup_{t\in G} t^{-1}A}$. Pick $y \in K(\beta S) \cap (\beta S \cdot q)$. Then, $y \in \overline{t^{-1}A}$ for some $t \in G$ and so $ty \in \overline{A} \cap K(\beta S)$. Hence $\overline{A} \cap K(\beta S) \neq \emptyset$. □

Corollary 1.6.5. *Let S be a semigroup and let $A \subseteq S$. Then,*

$$p \in c\ell\, K(\beta S) \iff \text{every } A \in p \text{ is piecewise syndetic.}$$

Definition 1.6.6. Let S be a semigroup and let $A \subseteq S$. A is said to be *central* in S if there is some idempotent $p \in K(\beta S)$ such that $A \in p$.

Theorem 1.6.7. *[[29], Theorem 4.43] Let S be a semigroup and let $A \subseteq S$. The following statements are equivalent.*

(1) A is piecewise syndetic.

(2) The set $\{x \in S : x^{-1}A$ is central$\}$ is syndetic.

(3) There is some $x \in S$ such that $x^{-1}A$ is central.

Proof. (a)$\Rightarrow$(b): Since A is piecewise syndetic, there exists $p \in K(\beta S)$ such that $A \in p$. Pick a minimal left ideal L of βS with $p \in L$ and pick an idempotent $e \in L$. Then $A \in p = pe$. So $\{y \in S : y^{-1}A \in e\} \in p$. In particular $\{y \in S : y^{-1}A \in e\} \neq \emptyset$. Pick $y \in S$ such that $y^{-1}A \in e$. Since $e \in K(\beta S)$, $B := \{z \in S : z^{-1}(y^{-1}A) \in e\}$ is syndetic. Pick a finite $G \subseteq S$ such

that $S = \bigcup_{t \in G} t^{-1}B$. Let $D = \{x \in S : x^{-1}A \text{ is central}\}$. We claim that $S = \bigcap_{t \in yG} t^{-1}D$. Let $x \in S$. Pick $t \in G$ such that $tx \in B$. Then $(tx)^{-1}(y^{-1}A) \in e$. Thus $(tx)^{-1}(y^{-1}A)$ is central. But $(tx)^{-1}(y^{-1}A) = (ytx)^{-1}A$. Thus $ytx \in D$. Hence $x \in (yt)^{-1}D$. It follows that $S = \bigcap_{t \in yG} t^{-1}D$.
(b)⇒(c): Since $\emptyset$ is not syndetic, $\{x \in S : x^{-1}A \text{ is central}\} \neq \emptyset$.
(c)⇒(a): Pick $x \in S$ such that $x^{-1}A$ is central and pick an idempotent $p \in K(\beta S)$ such that $x^{-1}A \in p$. Then $A \in xp$ and $xp \in K(\beta S)$. Thus $\overline{A} \cap c\ell K(\beta S) \neq \emptyset$. Hence A is piecewise syndetic. $\square$

Chapter 2

Ultrafilter semigroups and local homomorphisms

Let G be an infinite group with identity e and let $\mathcal{T}$ be a left invariant topology on G. We obtain a very important subsemigroup $Ult(\mathcal{T})$ of βG consisting of all nonprincipal ultrafilters on G converging to e in $\mathcal{T}$. In particular, $Ult(\mathcal{T})$ carries a lot of information about $\mathcal{T}$. In Section 2.1 we present the basic information about $Ult(\mathcal{T})$. In Section 2.2 we look at left invariant topologies on a group G determined by idempotents in βG. In Section 2.3 we look at local homomorphisms. These give rise to homomorphisms of ultrafilter semigroups. In Section 2.5 we prove that for any two homeomorphic direct sums X and Y, $Ult(X)$ and $Ult(Y)$ are isomorphic. In Section 2.4 and in Section 2.6 we present two very important ultrafilter semigroups.

2.1 Definition and basic properties

Although some concepts in the following definition have been defined before, we state them again here for easy reference.

Definition 2.1.1. Let G be a group and let $\mathcal{T}$ be a topology on G. $\mathcal{T}$ is said to be *left invariant* (respectively, *right invariant*) if for every $a \in G$, and every $U \in \mathcal{T}$, $aU \in \mathcal{T}$ (respectively, $Ua \in \mathcal{T}$). $\mathcal{T}$ is said to be *inverse invariant* if for every $U \in \mathcal{T}$, $U^{-1} \in \mathcal{T}$. $\mathcal{T}$ is said to be *invariant* if it is left invariant, right invariant and inverse invariant. A group endowed with a left invariant topology is called a *left topological group*.

We have the following easy observation.

Remark 2.1.2. Let G be a group and let $\mathcal{T}$ be a topology on G. The following are equivalent.

(a) $\mathcal{T}$ is left invariant.

(b) λ_a is open for every $a \in G$.

(c) λ_a is continuous for every $a \in G$.

(d) G is a left topological (semi)group.

(e) λ_a is a homeomorphism for every $a \in G$.

Proof. The implications follow directly from the definitions and the fact that for any $a \in G$, $\lambda_a^{-1} = \lambda_{a^{-1}}$ $\square$

Remark 2.1.3. Let G be a group.

(a) Since G is a cancellative semigroup, G^* is an ideal of βG (see Theorem 1.5.23).

(b) For any $A \subseteq G$ and for any $x \in G$,

$$x^{-1}A = \{y \in G : xy \in A\} = \{x^{-1}a : a \in A\},$$

where x^{-1} in $x^{-1}a$ stands for the inverse of x in G.

Note that a left topological group is homogeneous. Also note that a homogeneous space X is regular if and only if there exists a point $x \in X$ such that the neighborhood filter at x is closed. In particular, a left topological group is regular if and only if the neighborhood filter at the identity is closed. Unless otherwise stated, we will be using e to denote the identity of a group. In addition, unless otherwise stated, all topologies are assumed to satisfy the T_1 separation axiom.

Proposition 2.1.4. *Let G be a group, let $\mathcal{T}$ be a left invariant topology on G and let $\mathcal{F}$ be the neighborhood filter of e in $\mathcal{T}$. $\mathcal{F}$ satisfies the following conditions:*

(1) $\bigcap \mathcal{F} = \{e\}$, *and*

(2) for every $U \in \mathcal{F}$, there exists $V \in \mathcal{F}$ such that for every $x \in V$, $x^{-1}U \in \mathcal{F}$.

Conversely, suppose there is a filter $\mathcal{F}$ on G satisfying conditions (1) and (2). Then, there is a left invariant topology $\mathcal{T}$ on G such that $\mathcal{F}$ is the neighborhood of e in $\mathcal{T}$.

Proof. Condition (1) follows from the T_1 property. We show (2). Note that for every $y \in G$, $y\mathcal{F}$ is the neighborhood filter of y in $\mathcal{T}$. Let $U \in \mathcal{F}$. There exists $V \in \mathcal{F}$ such that $U \in x\mathcal{F}$ for every $x \in V$. One then notices that $U \in x\mathcal{F}$ if and only if $x^{-1}U \in \mathcal{F}$.
Conversely, let $\mathcal{F}$ be a filter on G satisfying conditions (1) and (2). Then, for every $x \in G$, $x\mathcal{F}$ is a filter on G with $\bigcap x\mathcal{F} = \{x\}$. In particular, $x \in U$ for every $U \in x\mathcal{F}$. Let $x \in G$ and let $U \in x\mathcal{F}$. Then, $x^{-1}U \in \mathcal{F}$. Therefore, there exists $V \in \mathcal{F}$ such that $z^{-1}U \in \mathcal{F}$ for every $z \in V$. Then, $xV \in x\mathcal{F}$. Let $y \in xV$. Then $x^{-1}y \in V$. Then $y^{-1}U = y^{-1}xx^{-1}U = (x^{-1}y)^{-1}(x^{-1}U) \in \mathcal{F}$. Hence $U \in y\mathcal{F}$. Therefore $U \in y\mathcal{F}$ for every $y \in xV$. It follows that there is a topology $\mathcal{T}$ on G such that $x\mathcal{F}$ is precisely the neighborhood filter at x for each $x \in G$. To see that $\mathcal{T}$ is left invariant, let $a, x \in G$ and let U be a neighborhood of ax. U can be written as $U = axU_0$ for some $U_0 \in \mathcal{F}$. Then $xU_0 \in x\mathcal{F}$ and $a(xU_0) = U$. It follows that the left shift λ_a is continuous. □

Remark 2.1.5. Condition (2) of Proposition 2.1.4 can be restated in the following stronger form:

(2)′ For every $U \in \mathcal{F}$, there is a $V \in \mathcal{F}$ such that $V \subseteq U$ and for all $x \in V$,
$$x^{-1}V \in \mathcal{F}.$$

Proof. Assume (2) holds. For every $U \in \mathcal{F}$, let

$$U^0 = \{x \in U : x^{-1}U \in \mathcal{F}\}.$$

Then $U^0 \subseteq U$. By (2), there exists $A \in \mathcal{F}$ such that $x^{-1}U \in \mathcal{F}$ for all $x \in A$. Then $A \cap U \in \mathcal{F}$. Since $A \cap U \subseteq U^0$, $U^0 \in \mathcal{F}$. We claim that for every $x \in U^0$, $x^{-1}U^0 \in \mathcal{F}$, i.e. $(U^0)^0 \supseteq U^0$ and, hence, $(U^0)^0 = U^0$. Let $x \in U^0$. Then $x^{-1}U \in \mathcal{F}$. Let $V = x^{-1}U$. Then $V \in \mathcal{F}$ and $xV \subseteq U$. By (2), $V^0 \in \mathcal{F}$. So, it suffices to show that $xV^0 \subseteq U^0$. Let $y \in V^0$ and let $W = y^{-1}V$. Then $W \in \mathcal{F}$ and $yW \subseteq V$. Consequently, $xyW \subseteq xV \subseteq U$, which implies that $W \subseteq (xy)^{-1}U$ and, thus, $(xy)^{-1}U \in \mathcal{F}$. Hence, $xy \in U^0$. It follows that $xV^0 \subseteq U^0$.
Conversely, it is easy to see that (2)′ implies (2). □

Definition 2.1.6. Let G be a group, let $\mathcal{T}$ be a left invariant topology on G and let e be the identity of G. The *ultrafilter semigroup of* $\mathcal{T}$, denoted $Ult(\mathcal{T})$, is defined as follows.

$$Ult(\mathcal{T}) := \{p \in G^* \mid p \text{ converges to } e \text{ in } \mathcal{T}\}.$$

Remark 2.1.7. Let G be a group, let $\mathcal{T}$ be a left invariant topology on G and let $\mathcal{F}$ be the neighborhood filter of the identity in $\mathcal{T}$. It is clear that

$$Ult(\mathcal{T}) = \overline{\mathcal{F}} \setminus \{e\}.$$

Since $\overline{\mathcal{F}}$ is a closed subset of βG and e is an isolated point of βG, $\overline{\mathcal{F}} \setminus \{e\}$ is a closed subset of G^*. Sometimes we will also write $Ult(G)$ for $Ult(\mathcal{T})$ if it is clear which topology we are referring to.

Remark 2.1.8. Let G be a group. If $\mathcal{T}$ and $\mathcal{T}'$ are distinct left invariant topologies on G, the ultrafilter semigroups $Ult(\mathcal{T})$ and $Ult(\mathcal{T}')$ are distinct.

Proof. Let $\mathcal{F}$ and $\mathcal{F}'$ be the neighborhood filters of e in $\mathcal{T}$ and $\mathcal{T}'$ respectively. Since $\mathcal{F} \neq \mathcal{F}$, $\overline{\mathcal{F}} \neq \overline{\mathcal{F}'}$ (see the proof of Theorem 1.5.12). Therefore, $Ult(\mathcal{T}) = \overline{\mathcal{F}} \setminus \{e\} \neq \overline{\mathcal{F}'} \setminus \{e\} = Ult(\mathcal{T}')$. □

Proposition 2.1.9. *[[62], Proposition 2.5] Let G be a group, let $\mathcal{T}$ be a left invariant topology on G and let $S = Ult(\mathcal{T})$. Then, $\mathcal{T}$ is Hausdorff if and only if for every $a \in G \setminus \{e\}$, $aS \cap S = \emptyset$.*

Proof. Let $\mathcal{F}$ be the neighborhood filter of e, i.e. $S^1 = \overline{\mathcal{F}}$, where S^1 denotes $Ult(\mathcal{T}) \cup \{e\}$. Note that the topology $\mathcal{T}$ is Hausdorff if and only if for every $a \in G \setminus \{e\}$, there exists $U \in \mathcal{F}$ such that $aU \cap U = \emptyset$.
Suppose $\mathcal{T}$ is Hausdorff. Let $a \in G \setminus \{e\}$ and let $U \in \mathcal{F}$ such that $aU \cap U = \emptyset$. Since $a\overline{U} = \overline{aU}$ and βG is extremally disconnected, $a\overline{U} \cap \overline{U} = \emptyset$. Hence, $aS^1 \cap S^1 = \emptyset$ and so $aS \cap S = \emptyset$.
Conversely, suppose $aS \cap S = \emptyset$ for every $a \in G \setminus \{e\}$. Let $a \in G \setminus \{e\}$. Then $aS^1 \cap S^1 = \emptyset$. Hence, $a\overline{U} \cap \overline{U} = \emptyset$ for some $U \in \mathcal{F}$, and so $aU \cap U = \emptyset$. Thus, $\mathcal{T}$ is Hausdorff. □

Lemma 2.1.10. *[[62], Lemma 2.9] Let G be a group, let $\mathcal{T}$ be a left invariant topology on G and let $U \subseteq G$. Then U is open if and only if $\overline{U} \cdot Ult(\mathcal{T}) \subseteq \overline{U}$.*

Proof. Suppose U is open. Let $p \in \overline{U}$ and let $q \in Ult(\mathcal{T})$. It suffices to show that U contains a subset of the form $\bigcup\{xB_x : x \in A\}$ for some $A \in p$ such that $B_x \in q$ for each $x \in A$. Take $A = U$. Since the left shifts

$G \ni y \mapsto xy \in G$ are continuous and $x = xe \in U$, for each $x \in U$ choose an open neighborhood B_x of e such that $xB_x \subseteq U$. Then $\bigcup\{xB_x : x \in U\} \subseteq U$ and $B_x \in q$ for each $x \in U$, as q converges to e. Hence $U \in pq$. It follows that $pq \in \overline{U}$. Therefore $\overline{U} \cdot Ult(\mathcal{T}) \subseteq \overline{U}$.
Conversely, suppose $\overline{U} \cdot Ult(\mathcal{T}) \subseteq \overline{U}$ and let $x \in U$. By continuity of the left shift $\beta G \ni q \mapsto xq \in \beta G$, for every $q \in Ult(\mathcal{T})$ pick $B_q \in q$ such that $x\overline{B_q} \subseteq \overline{U}$. Put $V = \bigcup_{q \in S} B_q \cup \{e\}$. Then V is a neighborhood of e and $xV \subseteq U$. Therefore U is a neighborhood of x. It follows that U is open. □

Corollary 2.1.11. *Let G be a left topological group with the neighborhood filter $\mathcal{F}$ of e and let $\mathcal{R}$ be a filter on G with a base of open sets such that $\mathcal{F} \subseteq \mathcal{R}$. Then $\overline{\mathcal{R}} \cdot \overline{\mathcal{F}} \subseteq \overline{\mathcal{R}}$.*

Proof. Let $\mathcal{B}$ be a base of $\mathcal{R}$. Then

$$\overline{\mathcal{R}} = \bigcap_{U \in \mathcal{B}} \overline{U}.$$

If $\mathcal{B}$ is open, then

$$\overline{\mathcal{R}} \cdot \overline{\mathcal{F}} \subseteq \bigcap_{U \in \mathcal{B}} \overline{U} = \overline{\mathcal{R}}.$$

□

Proposition 2.1.12. *[[64], Lemma 2.4] Let G be a group and let $\mathcal{T}$ be a left invariant topology on G. Then $Ult(\mathcal{T})$ is a closed subsemigroup of G^*.*

Proof. It remains to show that $Ult(\mathcal{T})$ is a subsemigroup of G^*. Let $p, q \in Ult(\mathcal{T})$. Since G^* is an ideal of βG, $pq \in G^*$. Let U be an open neighborhood of e. Then, $Ult(\mathcal{T}) \cdot Ult(\mathcal{T}) \subseteq \overline{U} \cdot Ult(\mathcal{T}) \subseteq \overline{U}$. Therefore, $U \in pq$. Since pq is a filter, pq contains all neighborhoods of e. It follows that $pq \in Ult(\mathcal{T})$. Thus $Ult(\mathcal{T})$ is a subsemigroup of G^*. □

Proposition 2.1.13. *[[62], Proposition 2.4] Let G be a group and let S be a finite subsemigroup of G^*. Then there is a left invariant topology $\mathcal{T}$ on G with $S = Ult(\mathcal{T})$.*

Proof. Let $\mathcal{F} = \bigcap S^1$, so that $S^1 = \overline{\mathcal{F}}$. By proposition 2.1.4 it suffices to show that $\mathcal{F}$ satisfies the following conditions (noting that condition (2) implies condition (2) of Proposition 2.1.4):

(1) $\bigcap \mathcal{F} = \{e\}$, and

(2) for every $V \in \mathcal{F}$ there exists $W \in \mathcal{F}$ and $W \ni x \mapsto W_x \in \mathcal{F}$ such that $xW_x \subseteq V$.

Since $\mathcal{F} \subseteq e$, $e \in \bigcap \mathcal{F}$. Conversely, suppose $x \in \bigcap \mathcal{F}$ for some $x \in G$. Then $\mathcal{F} \subseteq x$. Therefore $x \in \overline{\mathcal{F}} = S^1 = S \cup \{e\}$. Since $S \subseteq G^*$, $x = e$. It follows that $\{e\} = \bigcap \mathcal{F}$.
Let $V \in \mathcal{F}$. For every $p, q \in S^1$, $V \in pq$, i.e. $pq \in \overline{V}$. Since $\rho_q : \beta G \to \beta G$ is continuous, there exists $A_{p,q} \in p$ such that $\overline{A_{p,q}}q \subseteq \overline{V}$. Let

$$A_p = \bigcap_{q \in S^1} A_{p,q}.$$

Then $A_p \in p$ (since S^1 is finite) and $\overline{A_p}S^1 \subseteq \overline{V}$. Put

$$W = \bigcup_{p \in S^1} A_p.$$

Then $W \in \mathcal{F}$ and $\overline{W}S^1 \subseteq \overline{V}$. Next, for every $x \in W$ and $q \in S^1$, there exists $B_{x,q} \in q$ such that $x\overline{B_{x,q}} \subseteq \overline{V}$, so $xB_{x,q} \subseteq V$. Put

$$W_x = \bigcup q \in S^1 B_{x,q}.$$

Then $W_x \in \mathcal{F}$ and $xW_x \subseteq V$. □

Lemma 2.1.14. *[[64], Lemma 2.6]Let $(G, \mathcal{T})$ be a left topological group. If $Ult(\mathcal{T})$ has only one minimal right ideal, then $\mathcal{T}$ is extremally disconnected.*

Proof. Assume $Ult(\mathcal{T})$ has only one minimal right ideal and that $\mathcal{T}$ is not extremally disconnected. Then there are two disjoint open subsets U and V such that

$$e \in c\ell(U) \cap c\ell(V).$$

But then, by Lemma 2.1.10, $\overline{U} \cap Ult(\mathcal{T})$ and $\overline{V} \cap Ult(\mathcal{T})$ are two disjoint right ideals of $Ult(\mathcal{T})$, which gives a contradiction. □

Lemma 2.1.15. *[[64], Lemma 2.8] Let $(G, \mathcal{T})$ be a left topological group and let $S = Ult(\mathcal{T})$. If $\mathcal{T}$ is regular, then S^1 is left saturated in βG.*

Proof. It is clear that S^1 is a subsemigroup of βG. Let $p \in \beta G \setminus S^1$. There is a neighborhood U of e such that $U \notin p$. Since $\mathcal{T}$ is regular there exists a closed neighborhood V of e with $V \subseteq U$. Then $V \notin p$. Let $C = G \setminus V$. Then C is open and $C \in p$. Since every element of S^1 contains V, none of the elements of S^1 contains C, i.e. $\overline{C} \cap S^1 = \emptyset$. Since C is open, we have $\overline{C} \cdot S^1 = (\overline{C} \cdot S) \cup (\overline{C} \cdot \{e\}) \subseteq \overline{C} \cup \overline{C} = \overline{C}$. Since, $pS^1 \subseteq \overline{C} \cdot S^1 \subseteq \overline{C}$, $pS^1 \cap S^1 = \emptyset$. Hence S^1 is left saturated in βG. □

Since $S \subseteq S^1$, for any $p \in \beta G$ we have $pS \cap S \subseteq pS^1 \cap S^1$. Therefore, if S^1 is left saturated in βG then S is left saturated in G^*. In particular, by Lemma 2.1.15, if $\mathcal{T}$ is regular, then S is left saturated in G^*.

Proposition 2.1.16. *[[64], Proposition 2.10] Let G be an infinite group and let S be a closed subsemigroup of G^*. Suppose that S^1 is left saturated in βG and that S has a finite left ideal. Then there is a regular left invariant topology $\mathcal{T}$ on G with $S = Ult(\mathcal{T})$.*

Proof. Since $\{e\} = \overline{\{e\}}$ is closed, $S^1 = S \cup \{e\}$ is a union of two closed sets and, hence, closed. Therefore $S^1 = \overline{\mathcal{F}}$ for some filter $\mathcal{F}$ on G (specifically, $\mathcal{F} = \bigcap S^1$). We will show that $\mathcal{F}$ satisfies the conditions for generating (defining) a left invariant topology on G (in which $\mathcal{F}$ will be a system of neighborhoods of e). It will then follow automatically that $S = Ult(\mathcal{T})$, and we will have to show that $\mathcal{T}$ is regular.

(i) Since $\mathcal{F} \subseteq e$, $e \in \bigcap \mathcal{F}$. Conversely, suppose $x \in \bigcap \mathcal{F}$ for some $x \in G$. Then $\mathcal{F} \subseteq x$. Therefore $x \in \overline{\mathcal{F}} = S^1 = S \cup \{e\}$. Since $S \subseteq G^*$, $x = e$. Therefore $\{e\} = \bigcap \mathcal{F}$.

(ii) Now we show that for every $U \in \mathcal{F}$, there exists $V \in \mathcal{F}$ such that for all $x \in V$, $x^{-1}U \in \mathcal{F}$.
Let $U \in \mathcal{F}$ and let L be a finite left ideal of S. Then $\overline{G \setminus U} \subseteq \beta G \setminus S^1$. Since S^1 is left saturated in βG, $\overline{G \setminus U} \cdot S^1 \cap S^1 = \emptyset$. For every $q \in L$, choose $W_q \in \mathcal{F}$ such that $\overline{G \setminus U} \cdot q \cap \overline{W_q} = \emptyset$, and put

$$W = \bigcap_{q \in L} W_q.$$

Then $W \in \mathcal{F}$, as L is finite, and $(\overline{G \setminus U} \cdot L) \cap \overline{W} = \emptyset$. Next, since L is a left ideal of S, it follows that $S^1 \cdot L \subseteq L$. For every $q \in L$, choose $V_q \in \mathcal{F}$ such that $\overline{V_q} \cdot q \subseteq \overline{W}$ (This is possible since ρ_q is continuous), and put

$$V = \bigcap_{q \in L} V_q.$$

Then $V \in \mathcal{F}$ and $\overline{V} \cdot L \subseteq \overline{W}$.
We claim that for all $x \in V$, $x^{-1}U \in \mathcal{F}$. Indeed, otherwise $xp \in \overline{G \setminus U}$ for some $x \in V$ and $p \in S$. Take any $q \in L$. Then, on one hand

$$xpq = xp \cdot q \in \overline{G \setminus U} \cdot L \subseteq \beta G \setminus \overline{W},$$

and on the other hand,

$$xpq = x \cdot pq \in \overline{V} \cdot L \subseteq \overline{W},$$

which is a contradiction.

It follows from (i)-(ii) that there is a left invariant topology $\mathcal{T}$ on G in which $\mathcal{F}$ is the neighborhood filter of e, and so $Ult(\mathcal{T}) = S$. We now show that $\mathcal{T}$ is regular. Assume the contrary. Then there is a neighborhood, U, of e such that for every neighborhood V of e, $c\ell(V) \setminus U \neq \emptyset$. For every open neighborhood V of e, choose $x_V \in c\ell(V) \setminus U$. Since $x_V \in c\ell(V)$, there is an ultrafilter on G containing V and converging to x_V. Consequently, there is a $p_V \in S$ such that $V \in x_V p_V$. Take $q \in L$. By Lemma 2.1.10, $V \in x_V p_V q$, and $p_V q \in L$, as L is a left ideal. Since L is finite, it follows that there exists $q \in L$ and $p \in \overline{G \setminus U}$ such that for every neighborhood V of e, $V \in pq$, and so $pq \in S$. We have obtained that S^1 is not left saturated in βG, which is a contradiction. $\square$

2.2 Left invariant topologies defined by idempotents

In this section we will present two ways of generating left invariant topologies on an infinite group G using idempotents of G^*. To give a complete picture, in [40], A.Markov asked whether every infinite group admits a non-discrete Hausdorff group topology. For Abelian groups and many other groups (see [6]) the answer is positive, but the problem for general groups was eventually solved in the negative in [58], [44]. On the other hand, for every infinite group G and for every idempotent $p \in \beta G$, the largest topology on G in which p converges to the identity $e \in G$ is zero-dimensional and Hausdorff. In [71], it was shown that in fact every infinite group admits a non-discrete zero-dimensional Hausdorff topology with continuous left and right translations and inversion.

Definition 2.2.1. [51] Let G be a group with identity e, let φ be a filter on G, and let $A \subseteq G$. The subset

$$c\ell(A, \varphi) = \{x \in G : xF \cap A \neq \emptyset \text{ for every } F \in \varphi\}$$

is called *the closure of A in the direction of φ* (or *the closure of A by φ*).

The subset

$$int(A, \varphi) = \{x \in G : xF \subseteq A \text{ for some } F \in \varphi\}$$

is called *the interior of A with respect to φ.*

Note that $int(A, \varphi) \subseteq c\ell(A, \varphi)$ for every $A \subseteq G$.

Remark 2.2.2. Let G be a left topological group with the neighborhood filter τ of the identity. Then $c\ell(A, \tau) = c\ell(A)$ and $int(A, \tau) = int(A)$.

Lemma 2.2.3. *[[51], Lemma 1.1] Let G be a group, let $A \subseteq G$ and let $p \in \beta G$. Then,*

$$c\ell(A, p) = \{x \in G : A \in xp\} = int(A, p).$$

Proof. This is obvious from the definitions, recalling that $A \in xp$ if and only if $x^{-1}A \in p$. $\square$

Lemma 2.2.4. *[55] Let G be a left topological group with identity e, let $A \subseteq G$ and let $x \in G$. Then $x \in c\ell(A)$ if and only if there exists $p \in Ult(G) \cup \{e\}$ such that $xF \subseteq A$ for some $F \in p$. Thus*

$$c\ell(A) = \bigcup\{c\ell(A, p) : p \in Ult(G) \cup \{e\}\ \}.$$

Proof. Necessity. Let $x \in c\ell(A)$. Then for every neighborhood U of x, $U \cap A \neq \emptyset$. Since neighborhoods of x are exactly sets of the form xW where W is a neighborhood of e, there exists an ultrafilter q on G such that

$$\{xW : W \text{ is a neighborhood of } e\} \cup \{A\} \subseteq q.$$

Then $x^{-1}q = \{x^{-1}B : B \in q\}$ is an ultrafilter on G converging to e. In addition, $x^{-1}A \in x^{-1}q$ and $x(x^{-1}A) = A$. Therefore we take $p = x^{-1}q$ and $F = x^{-1}A$.
Sufficiency. Let U be a neighborhood of x. Since $F \in p$ and $p \in Ult(G) \cup \{e\}$, $F \cap V \neq \emptyset$ for every neighborhood V of e. Therefore, $xF \cap W \neq \emptyset$ for every neighborhood W of x. In particular $xF \cap U \neq \emptyset$. Since $xF \subseteq A$, $A \cap U \neq \emptyset$. Therefore, $x \in c\ell(A)$. $\square$

Lemma 2.2.5. *[[55], Lemma 2.1] Let G be a group, let $A \subseteq G$ and let φ be a filter on G. Then*

$$c\ell(A, \varphi) = \bigcap\{AF^{-1} : F \in \varphi\}.$$

Proof. Let $x \in G$.

$$\begin{aligned} x \in c\ell(A,\varphi) &\iff xF \cap A \neq \emptyset \text{ for all } F \in \varphi \\ &\iff x \in AF^{-1} \text{ for every } F \in \varphi. \end{aligned}$$

□

Lemma 2.2.6. *[[55], Lemma 2.2] Let G be a group, let $A \subseteq G$ and let $p \in \beta G$. Then*

$$c\ell(A,p) \subseteq AA^{-1}x \textit{ for every } x \in c\ell(A,p).$$

Proof. Let $x \in c\ell(A,p)$. Pick a subset $F_1 \in p$ such that $xF_1 \subseteq A$. Let $y \in c\ell(A,p)$. Let $F_2 \in p$ such that $yF_2 \subseteq A$. Take $F = F_1 \cap F_2$. Then $F \in p$, $F \subseteq F_1$ and $yF \subseteq yF_2 \subseteq A$. We have $y \in AF^{-1} \subseteq AF_1^{-1} \subseteq AA^{-1}x$. □

Lemma 2.2.7. *[[51], Lemma 1.1] Let G be a group and let φ be a filter on G. The following statements hold.*

(i) $c\ell(A \cup B,\varphi) = c\ell(A,\varphi) \cup c\ell(B,\varphi)$ *for all subsets* $A, B \subseteq G$.

(ii) $c\ell(\emptyset,\varphi) = \emptyset$ *and* $int(\emptyset,\varphi) = \emptyset$.

(iii) $c\ell(A,\varphi) = \bigcup\{c\ell(A,p) : p \in \beta G \textit{ with } \varphi \subseteq p\}$ *for every* $A \subseteq G$.

(iv) $int(A,\varphi) = \bigcap\{int(A,p) : p \in \beta G \textit{ with } \varphi \subseteq p\}$ *for every* $A \subseteq G$.

Proof. (i) $\supseteq$: Let $x \in c\ell(A,\varphi) \cup c\ell(B,\varphi)$ and let $F \in \varphi$. Then $xF \cap A \neq \emptyset$ or $xF \cap B \neq \emptyset$. Hence $xF \cap (A \cup B) = (xF \cap A) \cup (xF \cap B) \neq \emptyset$. It follows that $c\ell(A \cup B,\varphi) \supseteq c\ell(A,\varphi) \cup c\ell(B,\varphi)$.
$\subseteq$: Let $x \in c\ell(A \cup B,\varphi)$ and let $F \in \varphi$. Then $xF \cap (A \cup B) \neq \emptyset$. Therefore $xF \cap A \neq \emptyset$ or $xF \cap B \neq \emptyset$. We claim that if $xF \cap B = \emptyset$ then $xE \cap A \neq \emptyset$ for any other $E \in \varphi$. This will imply that $x \in c\ell(A,\varphi)$. So suppose $xF \cap B = \emptyset$ and let $E \in \varphi$. Since $x \in c\ell(A \cup B,\varphi)$ and $E \cap F \in \varphi$, $x(E \cap F) \cap (A \cup B) \neq \emptyset$. Therefore $\emptyset \neq x(E \cap F) \cap (A \cup B) = (x(E \cap F) \cap A) \cup (x(E \cap F) \cap B)$. Since $x(E \cap F) \cap B \subseteq xF \cap B = \emptyset$, $x(E \cap F) \cap B = \emptyset$. Therefore $x(E \cap F) \cap A \neq \emptyset$. Since $x(E \cap F) \cap A \subseteq xE \cap A$, $xE \cap A \neq \emptyset$. It follows that if $x \notin c\ell(B,\varphi)$, then $x \in c\ell(A,\varphi)$. Hence $c\ell(A \cup B,\varphi) \subseteq c\ell(A,\varphi) \cup c\ell(B,\varphi)$

(ii) Since $xF \cap \emptyset = \emptyset$ for any $F \in \varphi$ and for any $x \in G$, $c\ell(\emptyset,\varphi) = \emptyset$ and $int(\emptyset,\varphi) = \emptyset$.

(iii) $\subseteq$: Let $x \in c\ell(A, \varphi)$. For every $F \in \varphi$, $xF \cap A \neq \emptyset$. It follows that the family $\{xF : F \in \varphi\} \cup \{A\}$ has the finite intersection property. Pick $q \in \beta G$ such that $\{xF : F \in \varphi\} \cup \{A\} \subseteq q$. Then $x^{-1}q \in \beta G$ and $\varphi \cup \{x^{-1}A\} \subseteq x^{-1}q$. For any $F \in \varphi$, $x(F \cap x^{-1}A) = xF \cap A \subseteq A$ and $F \cap x^{-1}A \in x^{-1}q$. Hence $x \in int(A, x^{-1}q) = c\ell(A, x^{-1}q)$.
$\supseteq$: Let $p \in \beta G$ with $\varphi \subseteq p$ and let $x \in c\ell(A, p)$. Let $F \in \varphi$. Since $\varphi \subseteq p$, $F \in p$. Therefore $xF \cap A \neq \emptyset$. Thus $x \in c\ell(A, \varphi)$.

(iv) $\subseteq$: Let $x \in int(A, \varphi)$ and let $p \in \beta G$ with $\varphi \subseteq p$. There exists $F \in \varphi$ such that $xF \subseteq A$. Since $\varphi \subseteq p$, $F \in p$ and hence $x \in int(A, p)$.
$\supseteq$: Let $x \in G$ such that for every $p \in \beta G$ with $\varphi \subseteq p$, $xF \cap A \neq \emptyset$ for any $F \in p$. Suppose $x \notin int(A, \varphi)$. Then for every $E \in \varphi$, $xE \cap (G \setminus A) \neq \emptyset$. Pick $q \in \beta G$ with $\varphi \cup \{x^{-1}(G \setminus A)\} \subseteq q$. Then $x(x^{-1}(G \setminus A)) \cap A \neq \emptyset$. We obtain $\emptyset \neq x(x^{-1}(G \setminus A)) \cap A = (G \setminus A)) \cap A = \emptyset$, a contradiction. □

Definition 2.2.8. [51] Let G be group with identity e and let φ be a filter on G. φ is called *pointed* if $e \in F$ for every $F \in \varphi$.

Lemma 2.2.9. *[[51], Lemma 1.1] Let G be a group, let $A \subseteq G$ and let φ be a pointed filter on G. Then $A \subseteq c\ell(A, \varphi)$ and $int(A, \varphi) \subseteq A$.*

Proof. If $A = \emptyset$, then $c\ell(A, \varphi) = \emptyset = A$ and $int(A, \varphi) = \emptyset = A$. So, we may assume that $A \neq \emptyset$. Let $a \in A$ and let $F \in \varphi$. Since φ is pointed, $a \in aF$. Therefore, since $A \neq \emptyset$, $aF \cap A \neq \emptyset$. Hence $A \subseteq c\ell(A, \varphi)$. Now let $x \in int(A, \varphi)$. Pick $F \in \varphi$ such that $xF \subseteq A$. Then $x = xe \in xF \subseteq A$. Hence $int(A, \varphi) \subseteq A$. □

Lemma 2.2.10. *[[55], Lemma 2.3] Let G be a left topological group, let $A \subseteq G$ and let $p \in \beta G$. Then $c\ell(A, p)$ is a neighborhood of the identity of G if and only if $A \in qp$ for every $q \in Ult(G) \cup \{e\}$.*

Proof. Suppose $c\ell(A, p)$ is a neighborhood of the identity and suppose $A \notin qp$ for some $q \in Ult(G) \cup \{e\}$. Then $G \setminus A \in qp$. Choose a subset $B \in q$ and for each $b \in B$ choose $C_b \in p$ such that $\bigcup\{bC_b : b \in B\} \subseteq G \setminus A$. Then $bC_b \cap A = \emptyset$ for every $b \in B$. Since $q \in Ult(G)$, $c\ell(A, p) \in q$ and, therefore, $c\ell(A, p) \cap B \neq \emptyset$. Pick $b \in c\ell(A, p) \cap B$. Since $b \in c\ell(A, p)$, $bC_b \cap A \neq \emptyset$, a contradiction.
Conversely, suppose $A \in pq$ for every $q \in Ult(G) \cup \{e\}$ and suppose that $c\ell(A, p) \notin Ult(G) \cup \{e\}$. Then in each neighborhood of the identity we may choose an element x such that $xF \cap A = \emptyset$ for some $F \in p$. Consequently, there is an ultrafilter $q \in Ult(G) \cup \{e\}$ such that $G \setminus A \in pq$. Then $\emptyset = (G \setminus A) \cap A \in pq$, a contradiction. □

Definition 2.2.11. [51] Let G be a group and let φ and ψ be filters on G. We define the product $\varphi * \psi$ as follows. For every $A \subseteq G$,

$$A \in \varphi * \psi \iff int(A, \psi) \in \varphi.$$

The following lemma shows that this product generalizes the product on βG.

Lemma 2.2.12. *Let G be a group. For every $p, q \in \beta G$, $p * q = pq$.*

Proof. Let $A \in pq$. It suffices to show that $A \in p * q$. Put $C = int(A, q) = \{x \in G : xF \subseteq A \text{ for some } F \in q\}$ and put $B = \{x \in G : x^{-1}A \in q\}$. Then $B \in p$. Let $x \in B$. Then $x^{-1}A \in q$. Put $F = x^{-1}A$. Then $xF = x(x^{-1}A) = A$. It follows that $x \in C$. Therefore, $B \subseteq C$. Since $B \in p$, $C \in p$. Hence $A \in p * q$. Since A was an arbitrary element of pq, $pq \subseteq p * q$ and, thus $pq = p * q$. □

Remark 2.2.13. [51] The family

$$\{\bigcup_{x \in F} xH_x : F \in \varphi, H_x \in \psi \text{ for every } x \in F\}$$

forms a base for the filter $\varphi * \psi$.

Proof. Let $A \in \varphi * \psi$. Then $int(A, \psi) \in \varphi$. Take $F = int(A, \psi)$. For each $x \in F$, there exists $H_x \in \psi$ such that $xH_x \subseteq A$. Therefore $\bigcup_{x \in F} xH_x \subseteq A$. □

Lemma 2.2.14. *[[51], Lemma 1.2] Let G be a group, let ψ and φ be filters on G and let $A \subseteq G$. The following statements hold.*

(i) $int(int(A, \psi), \varphi) = int(A, \varphi * \psi)$.

(ii) $c\ell(c\ell(A, \psi), \varphi) = c\ell(A, \varphi * \psi)$.

Proof. (i) Let $y \in int(int(A, \psi), \varphi)$. Take $F \in \varphi$ such that $yF \subseteq int(A, \psi)$. For every $x \in F$ pick $H_x \in \psi$ such that $yxH_x \subseteq A$. Put $P = \bigcup_{x \in F} xH_x$. Then $P \in \varphi * \psi$ and $yP \subseteq A$. Therefore $y \in int(A, \varphi * \psi)$. It follows that $int(int(A, \psi), \varphi) \subseteq int(A, \varphi * \psi)$.
Conversely, let $y \in int(A, \varphi * \psi)$. Choose $F \in \varphi$ and for each $x \in F$ choose $H_x \in \psi$ such that $yxH_x \subseteq A$. Then $yF \subseteq int(A, \psi)$ and so $y \in int(int(A, \psi), \varphi)$. It follows that $int(int(A, \psi), \varphi) \supseteq int(A, \varphi * \psi)$.
(ii) $\supseteq$: Let $x \in c\ell(A, \varphi * \psi)$ and assume that $x \notin c\ell(c\ell(A, \psi), \varphi)$. There exists

$F \in \varphi$ such that $xF \cap c\ell(A, \psi) = \emptyset$. For every $y \in F$ take $H_y \in \psi$ such that $xyH_y \cap A = \emptyset$. Put $B = \bigcup_{y \in F} yH_y$. Then $B \in \varphi * \psi$ and $xB \cap A = \emptyset$. This implies that $x \notin c\ell(A, \varphi * \psi)$, a contradiction.
$\subseteq$: Let $x \in c\ell(c\ell(A, \psi), \varphi)$. Let $F \in \varphi$. Then $xF \cap c\ell(A, \psi) \neq \emptyset$. Therefore, there exists $y \in F$ such that for every $E \in \psi$, $xyE \cap A \neq \emptyset$. For each $z \in F$ pick $H_z \in \psi$. Then $x(\bigcup_{z \in F} zH_z) \cap A \neq \emptyset$. Since F was an arbitrary element of ψ, $xB \cap A \neq \emptyset$ for every basic element B of $\varphi * \psi$ in the base described in Remark 2.2.13. Hence $xC \cap A \neq \emptyset$ for every $C \in \varphi * \psi$. □

Corollary 2.2.15. *Let G be a group. For any filters ϕ, φ and ψ on G,*

$$\phi * (\varphi * \psi) = (\phi * \varphi) * \psi.$$

Proof. Using Lemma 2.2.14 (a), we have the following.

$$\begin{aligned} A \in \phi * (\varphi * \psi) &\iff int(A, \varphi * \psi) \in \phi \\ &\iff int(int(A, \psi), \varphi) \in \phi \\ &\iff int(A, \psi) \in \phi * \varphi \\ &\iff A \in (\phi * \varphi) * \psi. \end{aligned}$$

□

Definition 2.2.16. Let G be a group and let τ be a filter on G. τ is called *left topological* if τ is the filter of neighborhoods of the identity of G for some left invariant topology on G.

Lemma 2.2.17. *Let G be a group with identity e and let φ be a filter on G. Then $\{int(A, \varphi) : A \in \varphi\}$ is a filter base on G.*

Proof. Let $A \in \varphi$. Then $e \in int(A, \varphi)$. Therefore $int(A, \varphi) \neq \emptyset$. It is easy to see that for any $A, B \subseteq G$, $int(A, \varphi) \cap int(B, \varphi) = int(A \cap B, \varphi)$. □

Definition 2.2.18. Let G be a group. Given any filter φ on G,

(i) we denote by $int(\varphi)$ the filter with the base $\{int(A, \varphi) : A \in \varphi\}$, and

(ii) we denote by $C\ell_\varphi$ the mapping determined by the rule

$$C\ell_\varphi(A) = c\ell(A, \varphi), \ A \subseteq G.$$

Remark 2.2.19. Let G be a group and let φ be a filter on G. The following statements hold.

(i) $int(\varphi)$ is pointed.

(ii) If φ is pointed, then $\varphi \subseteq int(\varphi)$.

Proof. (i) Let $A \in \varphi$. Then $e \in int(A, \varphi)$. It follows that $e \in F$ for every $F \in int(\varphi)$.
(ii) Suppose φ is pointed. Then, by Lemma 2.2.9, $int(A, \varphi) \subseteq A$ for every $A \subseteq G$. In particular, $\varphi \subseteq int(\varphi)$. □

Theorem 2.2.20. *[[51], Theorem 1.3] Let G be a group and let φ be a filter on G. The following statements are equivalent.*

(i) φ is left topological,

(ii) φ is pointed and $\varphi * \varphi = \varphi$,

(iii) $int(\varphi) = \varphi$,

(iv) $C\ell_\varphi$ is a closure operator.

Proof. $(i) \Rightarrow (ii)$: Suppose φ is left topological. Then clearly φ is pointed. Let $A \in \varphi * \varphi$. Then $int(A, \varphi) \in \varphi$. Since φ is pointed, by Lemma 2.2.9, $int(A, \varphi) \subseteq \varphi$. Therefore, $A \in \varphi$ and, hence, $\varphi * \varphi \subseteq \varphi$. Now let $A \in \varphi$. Choose an open neighborhood of the identity in (G, φ) such that $U \subseteq A$. For each $x \in U$, pick $V_x \in \varphi$ with $xV_x \subseteq U$. Put $V = \bigcup_{x \in U} xV_x$. Then $V \in \varphi * \varphi$ and $V \subseteq A$. Therefore $A \in \varphi * \varphi$ and, hence, $\varphi \subseteq \varphi * \varphi$. It follows that $\varphi = \varphi * \varphi$.

$(ii) \Rightarrow (iii)$: Suppose φ is pointed and $\varphi * \varphi = \varphi$. Then for every $A \subseteq G$, $int(A, \varphi) \subseteq A$. Hence $\varphi \subseteq int(\varphi)$. Now let $B \in int(\varphi)$. Pick $A \in \varphi$ such that $int(A, \varphi) \subseteq B$. Since $\varphi * \varphi = \varphi$ and $A \in \varphi$, $int(A, \varphi) \in \varphi$. Therefore $B \in \varphi$ and, hence, $int(\varphi) \subseteq \varphi$. It follows that $int(\varphi) = \varphi$.

$(iii) \Rightarrow (ii)$: Suppose $int(\varphi) = \varphi$. Since $int(\varphi)$ is pointed, φ is pointed. Let $A \in \varphi$. Then $int(A, \varphi) \in int(\varphi) = \varphi$. Therefore $A \in \varphi * \varphi$. Thus $\varphi \subseteq \varphi * \varphi$. Conversely, let $A \in \varphi * \varphi$. Then $int(A, \varphi) \in \varphi$. Since φ is pointed, $int(A, \varphi) \subseteq A$. Therefore $A \in \varphi$. Thus $\varphi * \varphi \subseteq \varphi$. It follows that $\varphi * \varphi = \varphi$.

$(ii) \Rightarrow (iv)$: Suppose φ is pointed and $\varphi * \varphi = \varphi$. By Lemma 2.2.9, $A \subseteq c\ell(A, \varphi) = c\ell_\varphi(A)$ for every $A \subseteq G$. It follows from the definition that $c\ell_\varphi(A) = c\ell(A, \varphi) \subseteq c\ell(B, \varphi) = c\ell_\varphi(B)$ for every $A, B \subseteq G$ with $A \subseteq B$. By Lemma 2.2.14 (ii), since $\varphi * \varphi = \varphi$, we have $c\ell_\varphi(c\ell_\varphi(A)) = c\ell(c\ell_\varphi(A), \varphi) = c\ell(c\ell(A, \varphi), \varphi) = c\ell(A, \varphi * \varphi) = c\ell(A, \varphi) = c\ell_\varphi(A)$. It follows that $c\ell_\varphi$ is a closure operator.

$(iv) \Rightarrow (i)$: Let $\mathcal{T}$ be the topology on G determined by the closure operator $c\ell_\varphi$. Let $F \subseteq G$ with $c\ell_\varphi(F) = F$ and let $x \in G$. We show that $c\ell_\varphi(xF) = xF$. So let $y \in c\ell_\varphi(xF)$ and let $E \in \varphi$. Then $yE \cap xF \neq \emptyset$. Then $x^{-1}yE \cap F \neq \emptyset$. Since E was an arbitrary element of φ, we conclude that $x^{-1}y \in F$. Therefore $y \in xF$. Hence $c\ell_\varphi(xF) \subseteq xF$. Since $c\ell_\varphi$ is a closure operator, $c\ell_\varphi(xF) \supseteq xF$. Thus $c\ell_\varphi(xF) = xF$. It follows that $\mathcal{T}$ is left invariant.
We now show that φ is the neighborhood filter of the identity $e \in G$ in $\mathcal{T}$. Let $F \subseteq G$ such that $c\ell_\varphi(F) = F$ and $e \notin F$. Put $U = G \setminus F$. Since $e \in U$ and $int(U, \varphi) = U$, we get $U \in \varphi$. Therefore, every open neighborhood of e is an element of φ. Hence, every neighborhood of e is an element of φ. Let $V \in \varphi$. Since $eV \cap (G \setminus V) = \emptyset$, we have $e \notin c\ell(G \setminus V, \varphi)$. Hence, V is a neighborhood of e in $\mathcal{T}$. It follows that φ is the neighborhood filter of e in the topology $\mathcal{T}$. □

Given a group G and a filter φ on G, one can construct two left topological filters on G using φ. We now look at these two left topological filters.

Lemma 2.2.21. *[[51], Lemma 1.4] Let G be a group and let φ be a filter on G. Then*
$$\overline{int(\varphi)} = \{p \in \beta G : \varphi \subseteq p * \varphi\}.$$
In particular, if $p \in \beta G$, then $\overline{int(p)} = \{q \in \beta G : p = qp\}$.

Proof. Let $p \in \beta G$ such that $int(\varphi) \subseteq p$. Since $\{int(A, \varphi) : A \in \varphi\} \subseteq int(\varphi)$, $\{int(A, \varphi) : A \in \varphi\} \subseteq p$. It follows that $\varphi \subseteq p * \varphi$.
Conversely, let $p \in \beta G$ with $\varphi \subseteq p * \varphi$. Let $A \in \varphi$. Then $A \in p * \varphi$ and, therefore, $int(A, \varphi) \in p$. Hence $int(\varphi) \subseteq p$. □

Theorem 2.2.22. *[[51], Theorem 1.5] Let G be a group. For every filter φ on G, $int(\varphi)$ is left topological.*

Proof. We will show that $int(int(\varphi)) = int(\varphi)$. Since $int(\varphi)$ is pointed, $int(\varphi) \subseteq int(int(\varphi))$. For the other inclusion, let $B \in int(\varphi)$ and choose

$A \in \varphi$ such that $int(A, \varphi) \subseteq B$. We have the following

$$\begin{aligned} int(int(A,\varphi), int(\varphi)) &= \bigcap\{int(int(A,\varphi),p) : p \in \beta G \text{ with } int(\varphi) \subseteq p\} \\ &= \bigcap\{int(int(A,\varphi),p) : p \in \beta G \text{ with } \varphi \subseteq \varphi * p\} \\ &= \bigcap\{(int(A, p * \varphi) : p \in \beta G \text{ with } \varphi \subseteq \varphi * p\} \end{aligned}$$

By Lemma 2.2.21, $int(A,\varphi) \subseteq int(A, p * \varphi)$ for every $p \in \beta G$ with $\varphi \subseteq p$. Hence,

$$int(A,\varphi) \subseteq int(int(A,\varphi), int(\varphi)) \subseteq int(B, int(\varphi))$$

and $int(B, int(\varphi)) \in int(\varphi)$. $\square$

Theorem 2.2.23. *For every ultrafilter p on a group G, the left topological group $(G, int(p))$ is Hausdorff and zero-dimensional. If $pp = p$, then $(G, int(p))$ is extremally disconnected.*

Proof. [[29], Theorem 9.13] $\square$

Let G be a group and let $p \in G^*$ be an idempotent. In addition to Theorem 2.2.23, the topology of $(G, int(p))$ is the same as the topology on G induced by the restriction of the mapping $\rho_p : \beta G \to G^* \subseteq \beta G$ to G, see [[29], Lemma 7.6 and Theorem 9.13]. These functions can be seen in the commutative diagram

$$\begin{tikzcd} \beta G \arrow[r, "\rho_p"] & G^* \\ G \arrow[u, hook, "e"] \arrow[r, "(\rho_p)_{|G}"'] & Gp, \arrow[u, hook', "i"'] \end{tikzcd}$$

where i is the set inclusion function.

Let G be a group with identity e and let φ be a filter on G. Let Γ be the family of all left invariant (respectively, invariant) topologies on G in which φ converges to e. Since $\{\emptyset, G\} \in \Gamma$, $\Gamma \neq \emptyset$. Let $\mathcal{G} = \bigcup \Gamma$ and let $\mathcal{T}$ be the topology generated by $\mathcal{G}$ on G. Then, clearly, $\mathcal{T}$ is left invariant (respectively, invariant) and φ converges to e in $\mathcal{T}$. In addition, $\mathcal{T}$ is the largest left invariant (respectively, invariant) topology on G in which φ converges to e. In particular, the following definition is justified.

Definition 2.2.24. Let G be an infinite group with identity e and let φ be a filter on G. We denote by $\mathcal{T}_l(\varphi)$ (respectively, $\mathcal{T}(\varphi)$) the largest left invariant (respectively, invariant) topology on G in which φ converges to e.

Definition 2.2.25. Let G be a group and let φ be a filter on G. We denote by $hull(\varphi)$ the finest left topological filter on G contained in φ.

It is clear that $hull(\varphi)$ is the neighborhood filter of e in $\mathcal{T}_l(\varphi)$.

Definition 2.2.26. [[71], Definition 3] Let G be a group. For every map $M : G \to \mathcal{P}(G)$ and every $x_0 \in G$, define $[M]_{x_0} \in \mathcal{P}(G)$ by

$$[M]_{x_0} = \{x_0 x_1 \cdots x_n \ : \ x_{i+1} \in M(x_0 \cdots x_i),\ i < n < \omega\}.$$

The following lemma gives an explicit description of $hull(\varphi)$.

Lemma 2.2.27. *[[71], Lemma 1] Let G be a group with identity e and let φ be a filter on G. The subsets of the form $[M]_{x_0}$, where $M : G \to \varphi$, are precisely the open neighborhoods of x_0 in $\mathcal{T}_l(\varphi)$.*

Proof. Let U be a neighborhood of $x_0 \in G$ in $\mathcal{T}_l(\varphi)$. Define $M : G \to \varphi$ by

$$M(x) = \begin{cases} x^{-1}U & \text{if } x \in U, \\ G & \text{otherwise.} \end{cases}$$

Then clearly $[M]_{x_0} = U$.
Conversely, the subsets $[M]_{x_0}$ form a base of open neighborhoods of x_0 for a topology $\mathcal{T}$ on G since

(1) if $y \in [M]_{x_0}$ then $[M]_y \subseteq [M]_{x_0}$, and

(2) $[M]_{x_0} \cap [N]_{x_0} \supseteq [K]_{x_0}$, where $K : G \to \varphi$ is defined by

$$K(x) = M(x) \cap N(x).$$

The topology $\mathcal{T}$ is left invariant as

(3) for every $a \in G$, $a[M]_{x_0} = [N]_{ax_0}$, where $N : G \to \varphi$ is defined by $N(x) = M(a^{-1}x)$.

The filter φ converges to e in $\mathcal{T}$ since $M(e) \in \varphi$ and $M(e) \subseteq [M]_e$. It follows that $\mathcal{T} \subseteq \mathcal{T}_l(\varphi)$, and so $\mathcal{T} = \mathcal{T}_l(\varphi)$. □

We recall that a topological space X is said to be *strongly extremally disconnected* if for every open nonclosed subset $U \subset X$, there exists $x \in X \setminus U$ such that $U \cup \{x\}$ is open. Notice that every strongly extremally disconnected space is extremally disconnected.

Theorem 2.2.28. *For every infinite group G and every ultrafilter p on G, the left topological group $(G, hull(p))$ is strongly extremally disconnected.*

Proof. [[51], Theorem 4.12] □

Theorem 2.2.29. *[[29], Theorem 9.14] Let G be an infinite discrete group with identity e and let $p \in G^*$. Let*

$$\mathcal{N} = \{A \cup \{e\} : A \in p\} \text{ and } \mathcal{T} = \{U \subseteq G \ : \text{ for all } a \in U,\ a^{-1}U \in p\}.$$

The following statements hold.

(i) $\mathcal{T}$ is a left invariant topology on G.

(ii) $\mathcal{N}$ contains the neighborhood filter of e in $\mathcal{T}$.

(iii) $\mathcal{T}$ is the finest topology on G satisfying (i) and (ii).

(iv) The neighborhood filter of e in $\mathcal{T}$ is equal to $\mathcal{N}$ if and only if p is idempotent. In this case $\mathcal{T}$ is Hausdorff.

Proof. (i) It is clear that $\emptyset, G \in \mathcal{T}$. Let $U, V \in \mathcal{T}$. Since for any $a \in G$, $a^{-1}(U \cap V) = a^{-1}U \cap a^{-1}V$, $U \cap V \in \mathcal{T}$. Let $(U_i)_{i \in I}$ be a family of elements of $\mathcal{T}$. Since for any $a \in G$, $a^{-1}(\bigcup_{i \in I} U_i) = \bigcup_{i \in I} a^{-1}U_i$, $\bigcup_{i \in I} U_i \in \mathcal{T}$. It follows that $\mathcal{T}$ is a topology on G. Now let $U \in \mathcal{T}$ and let $x \in G$. Let $a \in x^{-1}U$. Then $xa \in U$. Therefore $a^{-1}(x^{-1}U) = (xa)^{-1}U \in p$. It follows that $x^{-1}U \in \mathcal{T}$. Hence $\mathcal{T}$ is left invariant.
(ii) Let W be a neighborhood of e in $\mathcal{T}$. There exists $U \in \mathcal{T}$ such that $e \in U \subseteq W$. Then $U = e^{-1}U \in p$ and thus $W \in p$. Since $e \in W$, $W = W \cup \{e\} \in \mathcal{N}$.
(iii) Let $\mathcal{T}'$ be a topology on G satisfying (i) and (ii). Let $U \in \mathcal{T}'$ and let $a \in U$. Then $e \in a^{-1}U$. Since $\mathcal{T}'$ is left invariant, $a^{-1}U \in \mathcal{T}'$. Therefore, $a^{-1}U$ is a neighborhood of e in $\mathcal{T}'$. It follows that $a^{-1}U = A \cup \{e\}$ for some $A \in p$. Therefore, $a^{-1}U \in p$. Hence $U \in \mathcal{T}$ and thus $\mathcal{T}' \subseteq \mathcal{T}$.
(iv) Let $\mathcal{M}$ be the neighborhood filter at e in $\mathcal{T}$. Suppose $\mathcal{M} = \mathcal{N}$. Let $A \in p$. Pick $U \in \mathcal{T}$ such that $e \in U \subseteq A \cup \{e\}$. Since $a^{-1}U \in p$ for any $a \in U$, $U = e^{-1}U \in p$. Now let $a \in U$. Since $a^{-1}U \subseteq a^{-1}(A \cup \{e\}) = a^{-1}A \cup \{a^{-1}\}$

and $a^{-1}U \in p$, $a^{-1}A \cup a^{-1}\{e\} \in p$. Since $p \in G^*$, $a^{-1}A \in p$. Therefore $U \subseteq \{a \in G : a^{-1}A \in p\}$ and, hence, $\{a \in G : a^{-1}A \in p\} \in p$. It follows that $pp = p$.
Conversely, assume $pp = p$. Let $V \in \mathcal{N}$. Then $V \in p$. Let $B = \{x \in V : x^{-1}V \in p\} = V \cap \{x \in G : x^{-1}V \in p\}$. Then $e \in B$. Since $pp = p$, $B \in p$. It follows (see Lemma 1.5.21) that $x^{-1}B \in p$ for every $x \in B$. Therefore $e \in B \cup \{e\} = B \in \mathcal{T}$. Since $e \in B \subseteq V$, $V \in \mathcal{M}$. Hence $\mathcal{N} = \mathcal{M}$.
Finally, assume $pp = p$ and let $a \neq e$. Then, by [[29], Lemma 6.28], $ap \neq ep$. Pick $B \in p \setminus ap$. Let $A = B \setminus \{a^{-1}B \cup \{a, a^{-1}\}\}$. Then $aA \cup \{a\}$ and $A \cup \{e\}$ are disjoint neighborhoods of a and e respectively. □

Corollary 2.2.30. *Let G be an infinite discrete group with identity e and let $p \in G^*$ such that p is idempotent. Put $\varphi = \{A \cup \{e\} : A \in p\}$. Then $\varphi = hull(p)$.*

Proof. This follows directly from Theorem 2.2.29. □

Corollary 2.2.31. *Let G be an infinite discrete group with identity e and let $p \in G^*$ such that p is idempotent. Put $\varphi = \{A \cup \{e\} : A \in p\}$. Then φ satisfies the following conditions:*

(i) $\bigcap \varphi = \{e\}$, and

(ii) for every $A \in p$, there exists $B \in p$ such that for every $x \in B$,

$$x^{-1}(A \cup \{e\}) \in \varphi.$$

Proof. This follows directly from Proposition 2.1.4 and Theorem 2.2.29. □

Given an infinite group G and an idempotent $p \in G^*$, we denote $(G, int(p))$ by $G[p]$ and we denote $(G, hull(p))$ by $G(p)$.

Lemma 2.2.32. *Let G be an infinite discrete group with identity e and let $p \in G^*$ be an idempotent. Then $Ult(G(p)) = \{p\}$.*

Proof. It is clear that p converges to e. Let $q \in Ult(G(p))$. Since $\{A \cup \{e\} : A \in p\} \subseteq q$ and $q \in G^*$, $A \cap B \neq \emptyset$ for every $A \in p$ and every $B \in q$. It follows that $q = p$. □

2.3 Local homomorphisms

Definition 2.3.1. A topological space X with a partial binary operation $\cdot$ and a distinguished element e is called a *local left topological group* if for every $x \in X$ there is an open neighborhood U_x of e such that

(1) $x \cdot y$ is defined for all $y \in U_x$, $x \cdot e = x$, xU_x is an open neighborhood of x, and the mapping $U_x \ni y \mapsto x \cdot y \in xU_x$ is a homeomorphism, and

(2) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ whenever $y \in U_x$, $z \in U_{x \cdot y} \cap U_y$ and $y \cdot z \in U_x$.

For a local left topological group, from this point on, when we write $x \cdot y$ we assume that $y \in U_x$ and when we write xU, where U is a neighborhood of e, we assume that $U \subseteq U_x$. We will write xy for $x \cdot y$. The definition of a local left topological group includes the following important basic example.

Example 2.3.2. Let G be a left topological group. Every open neighborhood X of e in G is a local left topological group. To see this, for every $x \in X$ take an open subset W of G such that $x \in W \subseteq X$ and take $U_x = x^{-1}W \cap X$.

Given a local left topological group X with a distinguished element e, as for left topological groups, we let $Ult(X)$ be the set of all nonprincipal ultrafilters on X converging to e. We recall that a *partial semigroup* is a nonempty set S equipped with an associative partial binary operation (i.e. xy is not necessarily defined for all $x, y \in S$, but $(xy)z = x(yz)$ whenever the products that are involved are all defined).

Definition 2.3.3. A mapping $f : X \to S$ from a local left topological group X into a partial semigroup S is called a *local homomorphism* if for every $x \in X \setminus \{e\}$, there is a neighborhood U_x of e such that $f(xy) = f(x)f(y)$ for all $y \in U_x \setminus \{e\}$. In case when S is also a local left topological group, we in addition assume that $f(e_X) = e_S$. A bijection $f : X \to Y$ such that both f and f^{-1} are local homomorphisms is called a *local isomorphism*. By a *topological local homomorphism (isomorphism)* between local left topological groups we will mean a local homomorphism (isomorphism) $f : X \to Y$ such that f is continuous (a homeomorphism).

Example 2.3.4.

1. Let $\mathbb{B} = \bigoplus_\omega \mathbb{Z}_2$ be the countable Boolean group endowed with the topology induced by the product topology on $\Pi_\omega \mathbb{Z}_2$, i.e. the topology

on $\mathbb{B}$ having the neighborhood base at zero consisting of the subgroups

$$H_\alpha = \{x \in \mathbb{B} : \operatorname{supp}(x) \cap \alpha = \emptyset\},$$

where $\alpha < \omega$. As usual, for any $b = (a_n) \in \mathbb{B}$, $\operatorname{supp}(b) = \{n < \omega : a_n \neq 0\}$. Observe that each nonzero element $b \in \mathbb{B}$ has a unique *canonical representation* in the form $b = b_0 + \cdots + b_k$, where

(a) for every $l \leq k$, $\operatorname{supp}(b_l)$ is a nonempty interval in ω.

(b) for every $l \leq k-1$, $\max \operatorname{supp}(b_l) + 2 \leq \min \operatorname{supp}(b_{l+1})$.

If $\operatorname{supp}(b)$ is a nonempty interval in ω, we say that b is basic. It follows that a mapping f_0 from the set of basic elements of $\mathbb{B}$ into a semigroup S can be extended to the mapping $f : \mathbb{B} \to S$ by

$$f(b) = f(b_0) \cdots f(b_k),$$

where $b = b_0 + \cdots + b_k$ is the canonical representation of b.
Let $x, y \in \mathbb{B}$ such that $\max \operatorname{supp}(x) + 1 < \min \operatorname{supp}(y)$ and let

$$x = x_0 + \cdots + x_k \text{ and } y = y_0 + \cdots + y_m$$

be the canonical representations of x and y respectively. Then,

$$x + y = x_0 + \cdots + x_k + y_0 + \cdots + y_m$$

is the canonical representation of $x + y$. Therefore,

$$\begin{aligned} f(x+y) &= f(x_0 + \cdots + x_k + y_0 + \cdots + y_m) \\ &= f(x_0) \cdots f(x_k) f(y_0) \cdots f(y_m) \\ &= (f(x_0) \cdots f(x_k))(f(y_0) \cdots f(y_m)) \\ &= f(x) f(y). \end{aligned}$$

It follows that f is a local homomorphism.

2. Let κ be an infinite cardinal. For every ordinal $\alpha < \kappa$, let G_α be a nontrivial group and let

$$G = \bigoplus_{\alpha < \kappa} G_\alpha.$$

Let $\mathcal{T}_0$ denote the group topology on G with the neighborhood base of e consisting of the subgroups

$$H_\alpha = \{x \in G : \operatorname{supp}(x) \cap \alpha = \emptyset\},$$

where $\alpha < \kappa$. Let

$$D = \{x \in G : |\text{supp}(x)| = 1\}.$$

It is clear that every nonzero $x \in G$ has a canonical representation $x = x_1 \cdots x_n$ where $x_1, \cdots, x_n \in D$ are such that if $\text{supp}(x_1) = \{\alpha_1\}, \cdots,$ $\text{supp}(x_n) = \alpha_n$, then $\alpha_1 < \cdots < \alpha_n$. It is then clear that if $x = x_1 \cdots x_n$ and $y = y_1 \cdots y_m$ are such representations with max $\text{supp}(x) <$ min supp(y), then $x + y$ has the representation $x + y = x_1 \cdots x_n y_1 \cdots y_m$. Therefore, every mapping $f_0 : D \to S$ into a semigroup S can be extended to the local homomorphism $f : G \to S$ defined by

$$f(x_1 \cdots x_n) = f(x_1) \cdots f(x_n).$$

Remark 2.3.5. (1) Let X be a local left topological group. The partial operation on X can be extended naturally to βX by

$$pq = \lim_{x \to p} \lim_{y \to q} xy,$$

where $x, y \in X$, making βX a compact right topological partial semigroup. The product pq is defined if and only if

$$\{x \in X : \{y \in X : xy \text{ is defined}\} \in q\} \in p.$$

To see the necessity, suppose pq is defined. Since pq is a filter on X, $X \in pq$. By the definition of pq,

$$\begin{aligned} X \in pq \quad &\Longleftrightarrow \quad \{x \in X : x^{-1}X \in q\} \in p \\ &\Longleftrightarrow \quad \{x \in X : \{y \in X : xy \text{ is defined }\} \in q\} \in p \end{aligned}$$

Conversely, suppose $\{x \in X : \{y \in X : xy \text{ is defined }\} \in q\} \in p$. Let

$$\mathcal{F} = \{A \subseteq X : \{x \in X : x^{-1}A \in q\} \in p\}.$$

It is clear that $\emptyset \notin \mathcal{F}$. The equality $x^{-1}X = \{y \in X : xy \text{ is defined}\}$ together with the hypothesis implies that $X \in \mathcal{F}$. Let $A, B \in \mathcal{F}$. Since for any $x \in X$, $x^{-1}(A \cap B) = x^{-1}A \cap x^{-1}B$, $A \cap B \in \mathcal{F}$. For any $A \in \mathcal{F}$ and any $B \subseteq X$ with $A \subseteq B$, $x^{-1}A \subseteq x^{-1}B$. Since p is a filter, $B \in \mathcal{F}$. It follows that $\mathcal{F}$ is a filter. Hence, pq is defined.

In particular, let $p \in \beta X$ and let $q \in Ult(X)$. Then for every $x \in X$, $U_x \in q$. Since xy is defined for every $y \in U_x$,

$$U_x \subseteq \{y \in X : xy \text{ is defined}\}.$$

Therefore $\{y \in X : xy \text{ is defined}\} \in q$. It follows that

$$\{x \in X : \{y \in X : xy \text{ is defined}\} \in q\} = X \in p.$$

Hence pq is defined. In particular, $Ult(X)$ is a (closed) subsemigroup of X^*.

(2) Recall that for any topological space X and any nonempty subset Y of X, there is a bijective correspondence between filters on X containing Y and filters on Y (see Theorem 1.1.17). This correspondence sends ultrafilters to ultrafilters and principal ultrafilters to principal ultrafilters. Now, let G be a left topological group and let X be an open neighborhood of e in G. Let $p \in Ult(G)$. Then $X \in p$. Therefore, for every neighborhood U of e in G, $X \cap U \in p$. The sets $X \cap U$, where U is a neighborhood of e in G, are exactly the neighborhoods of e in X. It follows that $p \cap X := \{A \cap X : A \in p\} \in Ult(X)$. Thus $Ult(X)$ can be identified with $Ult(G)$. In particular, if a local left topological group X is an open neighborhood of the identity in a left topological group G, then we will identify $Ult(X)$ with $Ult(G)$.

Theorem 2.3.6. *[[29], Theorem 4.21, [64], Lemma 3.2] Let S be a discrete semigroup and let $\mathcal{A} \subseteq \mathcal{P}(S)$ have the finite intersection property. Let T be a compact right topological semigroup and let $\varphi : S \to T$ be a function such that $\varphi(S) \subseteq \Lambda(T)$. Assume that there is some $A_0 \in \mathcal{A}$ such that for each $x \in A_0$, there is $B_x \in \mathcal{A}$ such that $\varphi(xy) = \varphi(x)\varphi(y)$ for each $y \in B_x$. Then, for every $p \in \beta S$ and every $q \in \bigcap_{A\in\mathcal{A}} \overline{A}$, $\overline{\varphi}(pq) = \overline{\varphi}(p)\overline{\varphi}(q)$. In particular,*

$$\varphi^* := \overline{\varphi}_{|\bigcap_{A\in\mathcal{A}} \overline{A}} : \bigcap_{A\in\mathcal{A}} \overline{A} \to T$$

is a homomorphism. Furthermore, if for every $A \in \mathcal{A}$ $\varphi(A)$ is dense in T, then φ^* is surjective.

Proof. Let $p \in \beta S$ and let $q \in \bigcap_{A\in\mathcal{A}} \overline{A}$. Then for each $x \in A_0$, $B_x \in q$. Therefore,

$$\lim_{y\to q} \varphi(xy) = q\text{-}\lim_{y\in B_x} \varphi(xy) = q\text{-}\lim_{y\in B_x} \varphi(x)\varphi(y) = \lim_{y\to q} \varphi(x)\varphi(y).$$

Therefore, we obtain the following.

$$\begin{aligned}
\overline{\varphi}(pq) &= \overline{\varphi}(\lim_{x\to p}\lim_{y\to q} xy) \\
&= \lim_{x\to p}\lim_{y\to q} \varphi(xy) \\
&= \lim_{x\to p}\lim_{y\to q} \varphi(x)\varphi(y) \\
&= \lim_{x\to p} \varphi(x)\overline{\varphi}(q) \qquad\qquad \text{because } \varphi(x) \in \Lambda(T) \\
&= \overline{\varphi}(p)\overline{\varphi}(q).
\end{aligned}$$

Now suppose for every $A \in \mathcal{A}$, $\varphi(A)$ is dense in T. Let $t \in T$. Then, for each neighborhood V of t and for each $A \in \mathcal{A}$, $V \cap \varphi(A) \neq \emptyset$ and, therefore, $\varphi^{-1}(V) \cap A \neq \emptyset$. This implies that the family

$$\mathcal{B} := \{\varphi^{-1}(V) \cap A : V \text{ is a neighborhood of } t \text{ and } A \in \mathcal{A}\}$$

has the finite intersection property. Therefore, $\mathcal{B} \subseteq p$ for some ultrafilter p on S. It is clear that $\mathcal{A} \subseteq \mathcal{B}$. Therefore $\mathcal{A} \subseteq p$, i.e. $p \in \bigcap_{A\in\mathcal{A}} \overline{A}$. In addition since for every neighborhood V of t and every $A \in \mathcal{A}$, $\varphi^{-1}(V) \supseteq \varphi^{-1}(V) \cap A$, $\varphi^{-1}(V) \in p$. It follows that $\overline{\varphi}(p) = t$. Hence, φ^* is onto. □

Corollary 2.3.7. *Let S be a discrete semigroup and let $\varphi : S \to T$ be a homomorphism into a compact right topological semigroup T such that $\varphi(S) \subseteq \Lambda(T)$. Then, the continuous extension $\overline{\varphi} : \beta S \to T$ is a homomorphism.*

Proof. Let $\mathcal{A} = \{S\}$. Then $\overline{\mathcal{A}} = \overline{\{S\}} = \beta S$ and the result follows easily from Theorem 2.3.6. □

Corollary 2.3.8. *Let $f : X \to T$ be a local homomorphism from a local left topological group into a compact right topological semigroup such that $f(X) \subseteq \Lambda(T)$ and let $\overline{f} : \beta X \to T$ be the continuous extension of f. Then $f^* = \overline{f}_{|Ult(X)} : Ult(X) \to T$ is a homomorphism. Furthermore, if for every neighborhood U of e $f(U \setminus \{e\})$ is dense in T, then f is surjective.*

Proof. The first part is obtained by taking

$$\mathcal{A} = \{U \setminus \{e\} : U \text{ is a neighborhood of } e\}$$

in Theorem 2.3.6 and noting that in that theorem it suffices if xy is defined for $x \in A_0$ and $y \in B_x$. The second statement follows easily from the last part of Theorem 2.3.6. □

Example 2.3.9. Consider the topological group G and $D \subseteq G$ in Example 2.3.4 (2). Let λ be a cardinal such that $|G_\alpha| \geq \lambda$ for all $\alpha < \kappa$ and let $\mu = \max\{\kappa, \lambda\}$. Let $\mathbf{H}$ be the ultrafilter semigroup of $\mathcal{T}_0$, i.e.

$$\mathbf{H} = \bigcap_{\alpha<\kappa} c\ell_{\beta G}\{x \in G \setminus \{e\} : \min \operatorname{supp}(x) \geq \alpha\}.$$

Let T be a compact right topological semigroup containing a dense subset A such that $|A| \leq \mu$ and $A \subseteq \Lambda(T)$. Choose a partition $\mathcal{D} = \{D_\gamma : \gamma < \mu\}$ of D such that $\overline{D_\gamma} \cap \mathbf{H} \neq \emptyset$ and choose any surjection $g : \mu \to A$. Define $f_0 : D \to A$ by

$$f_0(x) = g(\gamma) \ \text{ if } x \in D_\gamma,$$

and let $f : G \to T$ be a local homomorphism extending f_0. Then $f^* : \mathbf{H} \to T$ is a surjective homomorphism.

Corollary 2.3.10. *Let $f : X \to T$ be a local homomorphism from a local left topological group into a finite discrete semigroup and let $\overline{f} : \beta X \to T$ be the continuous extension of f. Then $f^* = \overline{f}_{|Ult(X)} : Ult(X) \to T$ is a homomorphism.*

Proof. A finite discrete semigroup is a compact right topological semigroup which is equal to its topological center. □

Corollary 2.3.11. *For every local homomorphism $f : X \to S$ from a local left topological group into a partial semigroup, the continuous extension $\overline{f} : \beta X \to \beta S$ induces a homomorphism $f^* = \overline{f}_{|Ult(X)} : Ult(X) \to \beta S$.*

Proof. The composition "f followed by the embedding of S into βS" is a local homomorphism into the compact right topological semigroup βS, and the image of X is contained in the topological center. □

Theorem 2.3.12. *Let X and Y be local left topological groups and let $f : X \to Y$ be a topological local isomorphism. Then, $\overline{f}(Ult(X)) = Ult(Y)$ and*

$$\widetilde{f} := \overline{f}_{|Ult(X)} : Ult(X) \to Ult(Y)$$

is a topological isomorphism.

Proof. It remains to show that $\overline{f}(Ult(X)) = Ult(Y)$ and that $\widetilde{f}$ is a homeomorphism. Let $p \in Ult(X)$ and let V be a neighborhood of e_Y. Since f is continuous and $f(e_X) = e_Y$, $f^{-1}(V)$ is a neighborhood of e_X and, therefore, $f^{-1}(V) \in p$. Hence $V \in \overline{f}(p)$. It follows that $\overline{f}(p) \in Ult(Y)$. Therefore,

$\overline{f}(Ult(X)) \subseteq Ult(Y)$. Conversely, let $q \in Ult(Y)$. Since $\overline{f}$ is surjective, $q = \overline{f}(p)$ for some $p \in \beta X$. Let U be a neighborhood of e_X. Since $f(U)$ is a neighborhood of e_Y, $f(U) \in q$. Then, $U = f^{-1}(f(U)) \in p$. Hence, $p \in Ult(X)$. Therefore, $Ult(Y) \subseteq \overline{f}(Ult(X))$. Thus $\overline{f}(Ult(X)) = Ult(Y)$. Since $\overline{f}$ is bijective, $\widetilde{f}$ is a bijective. Let $A \subseteq \beta X$ and $B \subseteq \beta Y$ be open. Then, $\widetilde{f}(Ult(X) \cap A) = \overline{f}(Ult(X) \cap A) = \overline{f}(Ult(X)) \cap \overline{f}(A) = Ult(Y) \cap \overline{f}(A)$ is an open subset of $Ult(Y)$ and $\widetilde{f}^{-1}(Ult(Y) \cap B) = \overline{f}^{-1}(Ult(Y)) \cap \overline{f}^{-1}(B) = Ult(X) \cap \overline{f}^{-1}(B)$ is an open subset of $Ult(X)$. Therefore, $\widetilde{f}$ is a homeomorphism. □

The induced homomorphisms $f^* : Ult(X) \to T$ and $\widetilde{f} : Ult(X) \to Ult(Y)$ in Corollary 2.3.10 and Theorem 2.3.12, respectively, are called *proper*. In particular, in Theorem 2.3.12, we say that $Ult(X)$ is proper isomorphic to $Ult(Y)$.

Theorem 2.3.13. *[[62], Theorem 3.1] Let X be a countable nondiscrete regular left topological group, let S be a partial semigroup, and let $f : X \to S$ be a local homomorphism. Then there is a continuous local isomorphism $g : X \to \mathbb{B}$ such that $g(xy) = g(x)g(y)$ and $f(xy) = f(x)f(y)$ whenever max supp$(g(x)) + 2 \leq$min supp$(g(y))$. If X is first countable, g can be chosen to be a homeomorphism.*

Proof. Let F be the set of words on the letters 0 and 1 with empty word $\emptyset$. For any $w = a_0 \cdots a_n$ and $v = b_0 \cdots b_m$, define $w + v = c_0 \cdots c_k$ by $k = max\{n, m\}$ and

$$c_l = \begin{cases} a_l, & \text{if } l \leq n, \\ b_l, & \text{otherwise.} \end{cases}$$

Notice that each nonempty $w \in F$ has a unique canonical representation in the form $= w_0 + \cdots + w_k$, where

(a) for every $l \leq k$, $w_l = 0^{i_l}1^{j_l}$ with $i_l, j_l < \omega$ and $i_l + j_l > 0$,

(b) for every $l \leq k - 1$, $j_l > 0$,

(c) for every $l \leq k - 1, i_l + j_l < i_{l+1}$.

Words of the form 0^i1^j , where $i, j < \omega$ and $j > 0$, are called basic. Words of the form 0^i , where $i < \omega$, are called *zero*. From now on when we write $w = w_0 + \cdots + w_k$, we mean that this is the canonical representation. For any $w \in F$, $|w|$ denotes the length of w. Since X is countable, enumerate X as $\{e, x_1, x_2, \cdots\}$. We shall assign to each $w \in F$ a point $x(w) \in X$ and a clopen neighborhood $X(w)$ of $x(w)$ such that

(1) $x(0^n) = e$ and $X(\emptyset) = X$,

(2) $X(w^\frown 0) \cap X(w^\frown 1) = \emptyset$ and $X(w^\frown 0) \cup X(w^\frown 1) = X(w)$,

(3) $x(w) = x(w_0) \cdots x(w_k)$ and $X(w) = x(w_0) \cdots x(w_{k-1}) X(w_k)$, where $w = w_0 + \cdots + w_k$,

(4) $f(x(w)y) = f(x(w))f(y)$ for all $y \in X(0^{|w|+1}) \setminus \{e\}$ (for nonzero w),

(5) $x_n \in \{x(w) : |w| = n\}$.

We take as $X(0)$ a clopen neighborhood of e such that $x_1 \notin X(0)$. Put $X(1) = X \setminus X(0)$, $x(0) = e$ and $x(1) = x_1$. Fix $n > 1$ and suppose that $X(w)$ and $x(w)$ have been constructed for all w with $|w| < n$ such that conditions (1)-(5) hold. Notice that the subsets $X(w)$, $|w| = n - 1$, form a partition of X. So, one of them, say $X(u)$, contains x_n. Let $u = u_0 + \cdots + u_r$. Then $X(u) = x(u_0) \cdots x(u_{r-1})(X(u_r))$ and $x_n = x(u_0) \cdots x(u_{r-1}) y_n$ for some $y_n \in X(u_r)$. If $y_n = x(u_r)$, we take as $X(0^n)$ a clopen neighborhood of e such that for each basic w with $|w| = n - 1$, $X(w) \setminus x(w)X(0^n) \neq \emptyset$ and $f(x(w)z) = f(x(w))f(z)$ for all $z \in X(0^n) \setminus \{e\}$. For each such w, put $x(w^\frown 0) = x(w)$, $X(w^\frown 0) = x(w)X(0^n)$ and $X(w^\frown 1) = X(w) \setminus X(w^\frown 0)$, and take as $x(w^\frown 1)$ any element of $X(w^\frown 1)$. If $y_n \neq x(u_r)$, we take $X(0^n)$ in addition such that $y_n \notin x(u_r)X(0^n)$, and put $x(u_r^\frown 1) = y_n$. For all nonbasic nonzero w with $|w| = n$, we define $X(w)$ and $x(w)$ by condition (3). Then

$x(w) = x(w_0) \cdots x(w_k) \in x(w_0) \cdots x(w_{k-1}) X(w_k) = X(w)$,
for all $y \in X(0^{n+1}) \setminus \{e\}$,

$$\begin{aligned} f(x(w)y) &= f(x(w_0) \cdots x(w_k) y) \\ &= f(x(w_0) \cdots x(w_{k-1})) f(x(w_k) y) \\ &= f(x(w_0)) \cdots f(x(w_{k-1}) f(x(w_k)) f(y) \\ &= f(x(w)) f(y), \end{aligned}$$

and if $x_n \notin \{x(w) : |w| = n - 1\}$, $x_n = x(u_0) \cdots x(u_{r-1}) x(u_r^\frown 1) = x(u^\frown 1) \in \{x(w) : |w| = n\}$.

To check (2), let $|w| = n - 1$. Then

$$\begin{aligned}
x(w^\frown 0) &= x(w_0 + \cdots + w_k + 0^n) \\
&= x(w_0)\cdots x(w_k)x(0^n) \\
&= x(w_0)\cdots x(w_k) \\
&= x(w), \\
X(w^\frown 0) &= x(w)X(0^n) \\
&= x(w_0)\cdots x(w_k)X(0^n) \\
&= x(w_0)\cdots x(w_{k-1})X(w_k^\frown 0), \\
X(w^\frown 1) &= x(w_0)\cdots x(w_{k-1})X(w_k^\frown 1),
\end{aligned}$$

so,

$$\begin{aligned}
X(w^\frown 0) \cup X(w^\frown 1) &= x(w_0)\cdots x(w_{k-1})[X(w_k^\frown 0) \cup X(w_k^\frown 1)] \\
&= x(w_0)\cdots x(w_{k-1})X(w_k) \\
&= X(w), \\
X(w^\frown 0) \cap X(w^\frown 1) &= \emptyset.
\end{aligned}$$

Now for every $x \in X$, there is $w \in F$ with nonzero last letter such that $x = x(w)$, so $\{v \in F : x = x(v)\} = \{w^\frown 0^n : n < \omega\}$. It follows that we can define $g : X \to \mathbb{B}$ by putting for every $w = a_0 \cdots a_n \in F$,

$$g(x(w)) = \overline{w} = (a_0, \cdots, a_n, 0, 0, \cdots).$$

It is clear that g is a bijection with $g(e) = 0$. Since for every $w = a_0 \cdots a_n \in F$, $X(w)$ consists of all elements $x \in X$ such that $g(x) = (a_0, \cdots, a_n, \cdots)$, g is continuous. To check that $g(xy) = g(x)g(y)$ and $f(xy) = f(x)f(y)$ whenever $\max \operatorname{supp}(g(x)) + 2 \leq \min \operatorname{supp}(g(y))$, let w and v be words with nonzero last letters such that $x = x(w)$ and $y = x(v)$, and let $w = w_0 + \cdots + w_k$ and $v = v_0 + \cdots + v_t$. We have $y \in X(0^{|w|+1})$, so $w + v = w_0 + \cdots + w_k + v_0 + \cdots + v_t$ and $xy = x(w_0)\cdots x(w_k)x(v_0)\cdots x(v_t) = x(w + v)$. Hence,

$$g(xy) = g(x(w + v)) = \overline{w + v} = \overline{w} + \overline{v} = g(x(w)) + g(x(v)) = g(x) + g(y)$$

and

$$\begin{aligned} f(xy) &= f(x(w+v)) \\ &= f(x(w_0)\cdots x(w_k)x(v_0)\cdots x(v_t)) \\ &= f(x(w_0))\cdots f(x(w_k))f(x(v_0))\cdots f(x(v_t)) \\ &= f(x(w))f(x(v)) \\ &= f(x)f(y). \end{aligned}$$

If X is first-countable, we can choose $\{X(0^n) : n < \omega\}$ to be a neighborhood base of e, and then g will be a homeomorphism.

□

Corollary 2.3.14. *For every countable nondiscrete regular left topological group X, there is a continuous local isomorphism $g : X \to \mathbb{B}$. If X is first countable, then g can be chosen to be a homeomorphism.*

Proof. Take $S = X$ and $f = id_X$ in Theorem 2.3.13 □

Corollary 2.3.15. *Let X be a countable nondiscrete regular local left topological group, let T and Q be finite semigroups, let $g : T \to Q$ be a surjective homomorphism, and let $f : X \to Q$ be a local homomorphism. Then there is a local homomorphism $h : X \to T$ such that $f = g \circ h$.*

Proof. Let $k : X \to \mathbb{B}$ be a local isomorphism guaranteed by Theorem 2.3.13. For every basic $b \in \mathbb{B}$, choose $l(b) \in g^{-1} \circ f \circ k^{-1}(b) \subseteq T$. Extend the mapping $b \mapsto l(b)$ to the local homomorphism $l : \mathbb{B} \to T$ by $l(b) = l(b_0)\cdots l(b_k)$, where $b = b_0 + \cdots + b_k$ is the canonical representation. Then $g \circ l = f \circ k^{-1}$. Put $h = l \circ k$. □

Corollary 2.3.16. *All countably infinite nondiscrete regular left topological groups are locally isomorphic.*

Corollary 2.3.17. *Let X and Y be countably infinite nondiscrete regular local left topological groups. Then $Ult(X)$ is proper isomorphic to $Ult(Y)$.*

Proof. This follows immediately from Theorem 2.3.12 and Corollary 2.3.16.
□

Corollary 2.3.18. *For every countable nondiscrete regular left topological group X, there is a proper injective homomorphism $f : Ult(X) \to Ult(\mathbb{B})$. If X is first countable, f can be chosen to be an isomorphism.*

Corollary 2.3.19. *Let X be a countable nondiscrete regular local left topological group and let $S = Ult(X)$. Let T and Q be finite semigroups, let $g : T \to Q$ be a surjective homomorphism, and let $f : S \to Q$ be a proper homomorphism. Then there is a proper homomorphism $h : S \to T$ such that $f = g \circ h$.*

We have the following strengthened version of Theorem 2.3.13.

Theorem 2.3.20. *[[65], Theorem 2.1] Let G be a countable nondiscrete regular left topological group and let D be a discrete subset of G such that $e \notin D$ and $c\ell\ D \setminus D \subseteq \{e\}$. Then there is a continuous bijection $h : G \to \mathbb{B}$ with $h(e) = 0$ such that*

1. *$h(xy) = h(x)h(y)$ whenever max supp$(h(x)) + 2 \leq$ min supp$(h(y))$,*
2. *for every $x \in D$, $h(x)$ is basic.*

If G is first countable then h can be chosen to be a homeomorphism.

Proof. Let W be the set of words on the letters 0 and 1 with empty word $\emptyset$. For any $w = a_0 \cdots a_n$ and $v = b_0 \cdots b_m$, define $w + v = c_0 \cdots c_k$ by $k = max\{n, m\}$ and

$$c_l = \begin{cases} a_l, & \text{if } l \leq n, \\ b_l, & \text{otherwise.} \end{cases}$$

Notice that each nonempty $w \in W$ has a unique canonical representation in the form $= w_0 + \cdots + w_k$, where

(a) for every $l \leq k$, $w_l = 0^{i_l}1^{j_l}$ with $i_l, j_l < \omega$ and $i_l + j_l > 0$,

(b) for every $l \leq k - 1$, $j_l > 0$,

(c) for every $l \leq k - 1, i_l + j_l < i_{l+1}$.

Words of the form 0^i1^j , where $i, j < \omega$ and $j > 0$, are called basic. Words of the form 0^i , where $i < \omega$, are called *zero*. From now on when we write $w = w_0 + \cdots + w_k$, we mean that this is the canonical representation. For any $w \in F$, $|w|$ denotes the length of w. Since G is countable, enumerate $G \setminus \{e\}$ as $\{x_n : n < \omega\}$. We shall assign to each $w \in W$ a point $x(w) \in G$ and a clopen neighborhood $X(w)$ of $x(w)$ such that

(1) $x(0^n) = e$ and $X(\emptyset) = G$,

(2) $X(w^\frown 0) \cap X(w^\frown 1) = \emptyset$ and $X(w^\frown 0) \cup X(w^\frown 1) = X(w)$,

(3) $x(w) = x(w_0) \cdots x(w_k)$ and $X(w) = x(w_0) \cdots x(w_{k-1}) X(w_k)$, where $w = w_0 + \cdots + w_k$,

(4) $X(w^\frown 0) \cap D \subseteq \{x(w)\}$ for nonzero w,

(5) $x_n \in \{x(w) : |w| = n + 1\}$.

We take as $X(0)$ a clopen neighborhood of e such that $x0 \notin X(0)$. Put $X(1) = G \setminus X(0)$, $x(0) = e$ and $x(1) = x_0$. Fix $n \geq 1$ and suppose that $X(w)$ and $x(w)$ have been constructed for all w with $|w| \leq n$ such that conditions (1)-(5) hold. Notice that the subsets $X(w)$, $|w| = n$, form a partition of X. So, one of them, say $X(u)$, contains x_n. Let $u = u_0 + \cdots + u_r$. Then $X(u) = x(u_0) \cdots x(u_{r-1})(X(u_r))$ and $x_n = x(u_0) \cdots x(u_{r-1}) y_n$ for some $y_n \in X(u_r)$. If $y_n = x(u_r)$, we take as $X(0^n)$ a clopen neighborhood of e such that for each basic w with $|w| = n$, $X(w) \setminus x(w)X(0^{n+1}) \neq \emptyset$ and, following (4), for each nonzero w with $|w| = n$, $x(w) \cdot X(0^{n+1}) \cap D \subseteq \{x(w)\}$. For each such w, put $x(w^\frown 0) = x(w)$, $X(w^\frown 0) = x(w)X(0^n)$ and $X(w^\frown 1) = X(w) \setminus X(w^\frown 0)$, and take as $x(w^\frown 1)$ any element of $X(w^\frown 1)$. If $y_n \neq x(u_r)$, we take $X(0^n)$ in addition such that $y_n \notin x(u_r)X(0^n)$, and put $x(u_r^\frown 1) = y_n$. For all nonbasic nonzero w with $|w| = n+1$, we define $X(w)$ and $x(w)$ by condition (3). Then

$x(w) = x(w_0) \cdots x(w_k) \in x(w_0) \cdots x(w_{k-1}) X(w_k) = X(w)$,
and if $x_n \notin \{x(w) : |w| = n\}$, then

$$x_n = x(u_0) \cdots x(u_{r-1}) x(u_r^\frown 1) \in \{x(w) : |w| = n + 1\}.$$

To check (2), let $|w| = n - 1$. Then

$$\begin{aligned}
x(w^\frown 0) &= x(w_0 + \cdots + w_k + 0^n) \\
&= x(w_0) \cdots x(w_k) x(0^n) \\
&= x(w_0) \cdots x(w_k) \\
&= x(w), \\
X(w^\frown 0) &= x(w) X(0^n) \\
&= x(w_0) \cdots x(w_k) X(0^n) \\
&= x(w_0) \cdots x(w_{k-1}) X(w_k^\frown 0), \\
X(w^\frown 1) &= x(w_0) \cdots x(w_{k-1}) X(w_k^\frown 1),
\end{aligned}$$

so,

$$\begin{aligned} X(w^\frown 0) \cup X(w^\frown 1) &= x(w_0)\cdots x(w_{k-1})[X(w_k^\frown 0) \cup Y(w_k^\frown 1)] \\ &= x(w_0)\cdots x(w_{k-1})X(w_k) \\ &= X(w), \\ X(w^\frown 0) \cap X(w^\frown 1) &= \emptyset. \end{aligned}$$

Now for every $x \in G \setminus \{e\}$, there is $w \in W$ with nonzero last letter such that $x = x(w)$, so $\{v \in W : x = x(v)\} = \{w^\frown 0^n : n < \omega\}$. It follows that we can define $h : G \to \mathbb{B}$ by putting, for every $w = a_0 \cdots a_n \in W$,

$$h(x(w)) = \overline{w} = (a_0, \cdots, a_n, 0, 0, \cdots).$$

It is clear that h is a bijection with $h(e) = 0$. Since for every $w = a_0 \cdots a_n \in W$, $X(w)$ consists of all elements $x \in G$ such that $h(x) = (a_0, \cdots, a_n, \cdots)$, h is continuous. To check that $h(xy) = h(x)h(y)$ whenever $\max \operatorname{supp}(h(x)) + 2 \leq \min \operatorname{supp}(h(y)$, let $x = x(w)$ and $y = x(v) \in X(0^{|w|+1})$. Then

$$h(x(w)x(v)) = h(x(w+v)) = \overline{w+v} = \overline{w} + \overline{v} = h(x(w)) + h(x(v)).$$

To check that $h(x)$ is basic for every $x \in D$, let $x = x(w) \in D$ with $w = w_0 + \cdots + w_k$. Then

$$x \in x(w_0)X(0^{|w|+1}) \text{ and } x(w_0)X(0^{|w|+1}) \cap D \subseteq \{x(w_0)\},$$

so $x = x(w_0)$. Hence $h(x) = \overline{w_0}$ is basic.

If G is first countable, we can choose $\{X(0^n) : n < \omega\}$ to be a neighborhood base of e, and then h will be a homeomorphism. □

Corollary 2.3.21. *Let G be a countable regular left topological group and let D be a discrete subset of G such that $e \notin D$ and $c\ell\ D \setminus D \subset \{e\}$. Then every mapping from D into a semigroup S can be extended to a local homomorphism $f : G \to S$.*

Proof. Let $h : G \to \mathbb{B}$ be a local homomorphism guaranteed by Theorem 2.3.16 and let f_0 be a mapping from D into S. Define a local homomorphism $g : \mathbb{B} \to S$ as follows. Let $x \in \mathbb{B}$. If $h^{-1}(x) \in D$, put $g(x) = f_0(h^{-1}(x))$. For any other basic $x \in \mathbb{B}$, pick $g(x)$ arbitrarily. If x is nonbasic, put $g(x) = g(x_0) \cdots g(x_k)$, where $x = x_0 + \cdots + x_k$ is the canonical representation of x. We then define $f = g \circ h$. Obviously, $f_{|D} = f_0$. To see that f is a local homomorphism, let $x \in G \setminus \{e\}$. Put

$$U_x = \{y \in G : \text{minn} \operatorname{supp}(h(y)) \geq \max \operatorname{supp}(h(x)) + 2\} \cup \{e\}.$$

Then U_x is a neighborhood of e because h is continuous, and for every $y \in U_x \setminus \{e\}$,

$$f(xy) = g(h(x) + h(y)) = g(h(x)) \cdot g(h(y)) = f(x)f(y).$$

□

Corollary 2.3.22. *Every countable regular left topological group containing a discrete subset with exactly one accumulation point admits a local homomorphism onto* $\mathbb{N}$.

Proof. Let G be a countable regular left topological group and let D be a discrete subset of G with $c\ell\ D \setminus D = \{e\}$. For every $x \in G$, put $f_0(x) = 1 \in \mathbb{N}$. By Corollary 2.3.17, the mapping $f_0 : D \to \mathbb{N}$ can be extended to a local homomorphism $f : G \to \mathbb{N}$. Since $e \in c\ell\ D$ and $f(D) = \{1\}$, f is onto. □

Corollary 2.3.16 suggests the question whether any two homeomorphic zero-dimensional local left topological groups are locally isomorphic. The following example shows that the answer to this question is negative.

Example 2.3.23. The topological groups $G = \prod_\omega \mathbb{Z}_2$ and $H = \prod_\omega \mathbb{Z}_3$ are homeomorphic but not locally isomorphic.
To show this, assume on the contrary that there is a local isomorphism $f : G \to H$. Construct inductively a decreasing sequence $(U_n)_{n<\omega}$ of clopen neighborhoods of $0 \in G$ and a sequence $(x_n)_{n<\omega}$ of nonzero elements of G such that

(i) $\{U_n : n < \omega\}$ is a neighborhood base at 0,

(ii) for every $n < \omega$, $x_n + U_{n+1} \subseteq U_n \setminus U_{n+1}$, and

(iii) for every $n < \omega$ and $y \in U_{n+1}$, $f(x_0 + \cdots + x_n + y) = f(x_0 + \cdots + x_n) + f(y)$.

Then whenever $m < \omega$, one has $x_n + \cdots + x_{n+m} \in U_n \setminus U_{n+1}$. Since G is compact, it follows that the series $\sum_{n=0}^{\infty} x_n$ is convergent. For every $n < \omega$, let

$$y_n = \sum_{m=n}^{\infty} x_m.$$

Note that $y_n \in U_n \setminus U_{n+1}$. Pick $n > 0$ such that $f(y_0 + y) = f(y_0) + f(y)$ for all $y \in U_n$. Then

$$f(y_0 + y_n) = f(y_0) + f(y_n).$$

Since also

$$f(y_0) = f(x_0 + \cdots + x_{n-1} + y_n) = f(x_0 + \cdots + x_{n-1}) + f(y_n),$$

we obtain that

$$f(y_0 + y_n) = f(x_0 + \cdots + x_{n-1}) + f(y_n) + f(y_n) \neq f(x_0 + \cdots + x_{n-1}).$$

But on the other hand

$$f(y_0 + y_n) = f(x_0 + \cdots + x_{n-1} + y_n + y_n) = f(x_0 + \cdots + x_{n-1}),$$

- a contradiction.

2.4 The ultrafilter semigroup $\mathbb{H}_\lambda$

Given a semigroup S, certain important subsemigroups of βS depend very little on the algebraic structure of S. In fact, they can be defined in terms of direct sums of sets with a distinguished element [56] (see also [29], p. 111). Let λ be an infinite cardinal and for every ordinal $\alpha < \lambda$, let A_α be a set with a distinguished element $0_\alpha \in A_\alpha$. Let X be the direct sum $X = \bigoplus_{\alpha<\lambda} A_\alpha$. We recall that the direct sum (strong direct sum) topology on X is generated by taking as a neighborhood base at $x \in X$ the subsets of the form $\{y \in X : y(\alpha) = x(\alpha) \text{ for all } \alpha \in F\}$ where F is a finite subset of λ (respectively, $\{y \in X : y(\alpha) = x(\alpha) \text{ for all } \alpha < \gamma\}$ where $\gamma < \lambda$). There is a natural partial semigroup operation $+$ on X: for any $x, y \in X$ with $\mathrm{supp}(x) \cap \mathrm{supp}(y) = \emptyset$, $x + y$ is defined by

$$\mathrm{supp}(x+y) = \mathrm{supp}(x) \cup \mathrm{supp}(y) \text{ and } (x+y)(\alpha) = \begin{cases} x(\alpha) & \text{if } \alpha \in \mathrm{supp}(x) \\ y(\alpha) & \text{if } \alpha \in \mathrm{supp}(y). \end{cases}$$

Lemma 2.4.1. $(X, +)$ *equipped with the direct sum (strong direct sum) topology is a local left topological group.*

Proof. We prove the case of the direct sum topology. The strong direct sum topology case is similar. For every $x \in X$, let

$$\begin{aligned} U_x &= \{y \in X : y(\alpha) = 0_\alpha \text{ for all } \alpha \in \mathrm{supp}(x)\} \\ &= \{y \in X : \mathrm{supp}(y) \cap \mathrm{supp}(x) = \emptyset\}. \end{aligned}$$

Then U_x is a neighborhood of $0 \in X$, $x + U_x$ is defined, with $x + 0 = x$, and the mapping $U_x \ni y \mapsto x + y \in x + U_x$ is clearly surjective. From the definition of $+$, it is easy to see that the mapping $U_x \ni y \mapsto x + y \in x + U_x$ is injective. It is also clear that the associative rule $x + (y + z) = (x + y) + z$ holds whenever the sums involved are defined. To see that this mapping is continuous, let $z = x + y$ for some $y \in U_x$ and let $W \subseteq X$ be open with $z \in (x + U_x) \cap W$. We may assume that there is a finite subset F of λ such that $W = \{w \in X : w(\alpha) = z(\alpha) \text{ for all } \alpha \in F\}$. Take $V = \{v \in X : v(\alpha) = y(\alpha) \text{ for all } \alpha \in F\}$. Then V is open and $y \in U_x \cap V$. In addition, for every $v \in U_x \cap V$, $(x + v)(\alpha) = z(\alpha)$ for every $\alpha \in F$, i.e. $x + v \in (x + U_x) \cap W$. It follows that the mapping $U_x \ni y \mapsto x + y \in x + U_x$ is continuous. Now let $V \subset X$ be open and let $z = x + y$ for some $y \in U_x \cap V$. Since $U_x \cap V \subseteq X$ is open, there is a basic neighborhood W of y with $W \subseteq U_x \cap V$. Since W is a basic neighborhood of y, there exists a finite subset E of λ such that $W = \{w \in X : w(\alpha) = y(\alpha) \text{ for all } \alpha \in E\}$. Then $z \in x + W \subseteq x + (U_x \cap V)$ and $(x + w)(\alpha) = z(\alpha)$ for every $\alpha \in E \cup \operatorname{supp}(x)$. Therefore, $x + (U_x \cap V)$ is open. It follows that the mapping $U_x \ni y \mapsto x + y \in x + U_x$ is open. Hence $U_x \ni y \mapsto x + y \in x + U_x$ is a homeomorphism. □

Corollary 2.4.2. *The operation $+$ on X can be extended to βX.*

Proof. This follows from Remark 2.3.5 (1). □

We will denote the ultrafilter semigroup of X with respect to the discrete sum (respectively, the strong discrete sum) topology by $Ult_0(X)$ (respectively, $Ult(X)$). It is easy to see that $Ult(X) \subseteq Ult_0(X)$.

Corollary 2.4.3. *For every $p \in \beta X$ and $q \in Ult_0(X)$, $p + q$ has a base consisting of subsets of the form $\bigcup_{x \in A} x + B_x$ where $A \in p$ and for every $x \in A$, $B_x \in q$ with* $\operatorname{supp}(x) \cap \operatorname{supp}(y) = \emptyset$ *for all $y \in B_x$.*

Proof. This follows from Remark 2.3.5 (1). □

Alternatively, for each $\alpha < \lambda$, pick a semigroup operation on A_α with identity 0_α, and let $\cdot$ denote the corresponding semigroup operation on X. Then for all $x, y \in X$ with $\operatorname{supp}(x) \cap \operatorname{supp}(y) = \emptyset$, $x + y = x \cdot y$, and for all $p \in \beta X$ and $q \in Ult_0(X)$, $p + q = p \cdot q$.

Remark 2.4.4. Let $D = \{x \in X : |\operatorname{supp}(x)| = 1\}$, and let S be a semigroup. Then any mapping $f_0 : D \to S$ can be extended to a local homomorphism $f : X \to S$ by

$$f(x) = \prod_{\alpha \in \operatorname{supp}(x)} f_0(x(\alpha)).$$

It follows that if $\min\{|\bigoplus_{\gamma\leq\alpha<\lambda} A_\alpha| : \gamma < \lambda\} \geq |S|$, then $Ult(X)$ admits a continuous homomorphism onto βS.

Remark 2.4.5. If for each $\alpha < \lambda$, A_α is a group, then the direct sum topology and the strong direct sum topology on X are group topologies (with respect to the componentwise group operation on X).

Proof. We consider the direct sum topology. The case for the strong direct sum topology is similar. Let $f : X\times X \to X$, $f(x, y) = xy$. Let $(a, b) \in X\times X$ and let U be a basic neighborhood of ab. There is a finite subset F of λ such that $U = \{z \in X : z(\alpha) = ab(\alpha) \text{ for all } \alpha \in F\}$. Consider $V = \{z \in X : z(\alpha) = a(\alpha) \text{ for all } \alpha \in F\}$ and $W = \{z \in X : z(\alpha) = b(\alpha) \text{ for all } \alpha \in F\}$. V is a basic neighborhood of a and W is a basic neighborhood of b and for any $(x, y) \in V \times W$ and for any $\alpha \in F$, $xy(\alpha) = x(\alpha)y(\alpha) = a(\alpha)b(\alpha) = ab(\alpha)$. Therefore, $f(V \times W) \subseteq U$. It follows that f is continuous.
Now let $h : X \to X$, $h(x) = x^{-1}$. Let $a \in X$ and let U be a basic neighborhood of a^{-1}. Then U^{-1} is a basic neighborhood of a with $h(U^{-1}) \subseteq U$. Hence, h is continuous.

□

Note that in Section 2.3 the direct sums were equipped with the strong direct sum topology. Again, from now on, unless otherwise stated, all direct sums are assumed to be equipped with the strong direct sum topology.

Recall that for each $n \in \mathbb{N}$, the *support of* n is a finite subset of ω defined by the unique expression of n as

$$n = \sum_{i\in\mathrm{supp}(n)} 2^i.$$

For each $n \in \omega$, consider the set $2^n\mathbb{N}$. Note the following.

(i) $x \in 2^n\mathbb{N}$ if and only if $\min \mathrm{supp}(x) \geq n$.

(ii) The family $\mathcal{A} := (2^n\mathbb{N})_{n\in\omega}$ has the finite intersection property.

(iii) For any $n, m \in \mathbb{N}$ with $n \leq m$ and for any $x \in 2^n\mathbb{N}$, $x + 2^m\mathbb{N} \subseteq 2^n\mathbb{N}$.

Therefore, the set

$$\mathbb{H} = \bigcap_{n\in\omega} c\ell_{\beta\mathbb{N}}(2^n\mathbb{N})$$

is a subsemigroup of $\beta\mathbb{N}$. Now let T be a compact right topological semigroup and let $\varphi : \mathbb{N} \to T$ with $\varphi[\mathbb{N}] \subseteq \Lambda(T)$. Assume that there is some $k \in \mathbb{N}$ such that whenever $x, y \in \mathbb{N}$ and $\max \mathrm{supp}(x) + k < \min \mathrm{supp}(y)$, one has $\varphi(x + y) = \varphi(x)\varphi(y)$. Let $n \in \mathbb{N}$ and let $A = 2^n\mathbb{N}$. For each $x \in A$, take

$$B_x = 2^{(\max \mathrm{supp}(x)+k+1)}\ \mathbb{N}.$$

Then, for each $y \in B_x$, $\min \mathrm{supp}(y) \geq \max \mathrm{supp}(x)+k+1 > \max \mathrm{supp}(x)+k$. Therefore, for each $n \in \mathbb{N}$ and each $x \in 2^n\mathbb{N}$, there is $m \in \mathbb{N}$, such that for all $y \in 2^m\mathbb{N}$, $\varphi(xy) = \varphi(x)\varphi(y)$. Therefore the restriction of the continuous extension $\overline{\varphi} : \beta\mathbb{N} \to T$ to $\mathbb{H}$ is a homomorphism.

Remark 2.4.6. Recall that $\mathbb{T} = \bigcap_{n\in\mathbb{N}} c\ell_{\beta\mathbb{N}}(n\mathbb{N})$. Note the following.

(a) As in the case of $\mathbb{H}$, it can easily be verified that $\mathbb{T}$ is a subsemigroup of $\beta\mathbb{N}$.

(b) Since for any $n \in \omega$, $2^n\mathbb{N} \in \{m\mathbb{N} : m \in \mathbb{N}\}$, it follows that $\mathbb{T} \subseteq \mathbb{H}$.

Proposition 2.4.7. *[[29], Lemma 6.6] All idempotents of $\beta\mathbb{N}$ are in $\mathbb{T}$.*

Proof. For each $n \in \mathbb{N}$, let φ_n be the natural homomorphism from $\mathbb{N}$ to the group of integers modulo n:

$$\varphi_n : \mathbb{N} \to \mathbb{Z}_n, \quad \varphi_n(x) = x (\mathrm{mod}\, n).$$

Since $\mathbb{Z}_n$ is finite, $\varphi(\mathbb{N}) = \mathbb{Z}_n = \Lambda(\mathbb{Z}_n)$ and, therefore, the continuous extension $\overline{\varphi_n} : \beta\mathbb{N} \to \mathbb{Z}_n$ is a homomorphism. Let $p \in \beta\mathbb{N}$. Then,

$$p \in \ker(\overline{\varphi_n}) \Leftrightarrow \overline{\varphi_n}(p) = 0(\mathrm{mod}\, n) \Leftrightarrow \lim_{x\to p} \varphi_n(x) = 0(\mathrm{mod}\, n) \Leftrightarrow n\mathbb{N} = [0] \in p,$$

where $[0]$ is the equivalence class of 0 modulo n. It follows that the kernel of $\overline{\varphi_n}$ is $c\ell_{\beta\mathbb{N}}(n\mathbb{N})$. Let $p \in \beta\mathbb{N}$ be any idempotent. Since $\overline{\varphi_n}(p) = \overline{\varphi_n}(p + p) = \overline{\varphi_n}(p) + \overline{\varphi_n}(p)$, then $\overline{\varphi_n}(p) = 0(mod\ n)$. Therefore, $p \in c\ell_{\beta\mathbb{N}}(n\mathbb{N})$. Since this holds for every $n \in \mathbb{N}$, $p \in \bigcap_{n\in\mathbb{N}} c\ell_{\beta\mathbb{N}}(n\mathbb{N}) = \mathbb{T}$. $\square$

Lemma 2.4.8. *Define $\phi : \mathbb{N} \to \omega$ and $\theta : \mathbb{N} \to \omega$ by $\phi(n) = \max(supp(n))$ and $\theta(n) = \min(supp(n))$. Then, for any $p \in \beta\mathbb{N}$ and any $q \in \mathbb{H}$,*

$$\overline{\phi}(p + q) = \overline{\phi}(q) \text{ and } \overline{\theta}(p + q) = \overline{\theta}(p).$$

Proof. Let $p \in \beta\mathbb{N}$ and let $q \in \mathbb{H}$. Given any $m \in \mathbb{N}$, choose $r \in \mathbb{N}$ such that $\phi(m) < r$. Let $n \in 2^r\mathbb{N}$. Then $\phi(m) < r \leq \theta(n)$. Therefore, $\mathrm{supp}(m+n) = \mathrm{supp}(m) \cup \mathrm{supp}(n)$, $\phi(m+n) = \phi(n)$ and $\theta(m+n) = \theta(m)$. Since $2^r\mathbb{N} \in q$,

$$\begin{aligned} \overline{\phi}(m+q) &= \text{q-}\lim_{n \in 2^r\mathbb{N}} \phi(m+n) \\ &= \text{q-}\lim_{n \in 2^r\mathbb{N}} \phi(n) \\ &= \overline{\phi}(q). \end{aligned}$$

It follows that $\overline{\phi}(p+q) = \overline{\phi}(q)$. Similarly, $\overline{\theta}(p+q) = \overline{\theta}(p)$. □

Using Lemma 2.4.8, one can show that $\mathbb{N}$ is the center of $(\beta\mathbb{N}, +)$ and of $(\beta\mathbb{N}, \cdot)$. It is known (see [[27], Theorem 3.9]) that if p is an idempotent in $K(\beta\mathbb{N})$, then $(p + \beta\mathbb{N} + p) \cap \mathbb{H}$ contains a copy of the free group on 2^{2^ω} generators. It is also known (see [[29], Theorem 6.32]) that copies of $\mathbb{H}$ can be found in S^* for any infinite cancellative semigroup S.

Proposition 2.4.9. *[[29], Theorem 6.4] Any compact right topological semigroup T with a countable dense topological center is an image of $\mathbb{H}$ under a continuous homomorphism.*

Proof. Enumerate $\Lambda(T)$ as $\Lambda(T) = \{t_i : i \in I\}$, where I is either ω or $\{0, 1, 2, \cdots, k\}$ for some $k \in \omega$. We then choose a disjoint partition $\{A_i : i \in I\}$ of $\{2^n : n \in \omega\}$, with each A_i being infinite. We define a mapping $\tau : \mathbb{N} \to T$ by first defining τ on $\{2^n : n \in \omega\}$ by

$$\tau(2^n) = t_i \ \text{ if } \ 2^n \in A_i.$$

We then extend τ to $\mathbb{N}$ by

$$\tau(n) = \Pi_{i \in \mathrm{supp}(n)}\, \tau(2^i),$$

with the terms in this product occurring in the order of increasing i. It is easy to see that for any $m, n \in \mathbb{N}$, if $\max\mathrm{supp}(x) < \min\mathrm{supp}(m)$, then $\tau(n+m) = \tau(n) + \tau(m)$. Therefore, the continuous extension $\overline{\tau} : \beta\mathbb{N} \to T$ is a homomorphism on $\mathbb{H}$. Now let $i \in I$ and let $x \in A_i^*$. Then $x \in \mathbb{H}$ and $\overline{\tau}(x_i) = t_i$. It follows that $\overline{\tau}(\mathbb{H}) \supseteq \Lambda(T)$ and thus $\overline{\tau}(\mathbb{H}) \supseteq c\ell(\Lambda(T)) = T$.

□

Corollary 2.4.10. *Every finite discrete semigroup is the image of $\mathbb{H}$ under a continuous homomorphism.*

Proof. A finite discrete semigroup is a compact right topological semigroup which is equal to its topological center. $\square$

Definition 2.4.11. Given any cardinals $\lambda \geq \omega$ and $\mu \geq 2$,

$$\mathbb{H}_{\lambda,\mu} = Ult(\bigoplus_{\lambda} \mu) \text{ and } \mathbb{H}_\lambda = \mathbb{H}_{\lambda,2}.$$

Given $\lambda \geq \omega$, we will often denote the group $\bigoplus_{\alpha<\lambda} 2$ by $\mathbb{B}_\lambda$ and for each $\alpha < \lambda$, e_α is that member of $\mathbb{B}_\lambda$ such that $\mathrm{supp}(e_\alpha) = \{\alpha\}$. Note that $\mathbb{B}_\omega = \mathbb{B}$, one of the groups we have seen in section 2.3, and $\mathbb{H}_\omega = Ult(\mathbb{B})$. Therefore, from Section 2.3, we make the following conclusion.

Proposition 2.4.12. *Let X be a countably infinite nondiscrete regular local left topological group. Then, $Ult(X)$ is isomorphic to $\mathbb{H}_\omega$.*

Proposition 2.4.13. *$\mathbb{H}_\omega$ and $\mathbb{H}$ are topologically and algebraically isomorphic.*

Proof. Define $\varphi : \mathbb{N} \to \mathbb{B}$ by

$$\varphi(x) = \begin{cases} 0, & \text{if } i \notin \mathrm{supp}(x) \\ 1, & \text{if } i \in \mathrm{supp}(x). \end{cases}$$

Clearly, for every $x, y \in \mathbb{N}$ with $\max \mathrm{supp}(x) < \min \mathrm{supp}(y)$, $\varphi(x+y) = \varphi(x)\varphi(y)$. It is easy to see that φ is injective and that for any nonzero $a \in \mathbb{B}$, $a = \varphi(x)$, where $x \in \mathbb{N}$ with $\mathrm{supp}(x) = \mathrm{supp}(a)$. Hence φ maps $\mathbb{N}$ bijectively onto $\mathbb{B} \setminus \{0\}$. It is also easy to see that $\mathrm{supp}(\varphi(x)) = \mathrm{supp}(x)$. In particular for every $n \in \mathbb{N}$, since $x \in 2^n\mathbb{N}$ if and only if $\min \mathrm{supp}(x) \geq n$, $\varphi(2^n\mathbb{N}) = H_n$.Therefore the continuous extension $\overline{\varphi}$ satisfies $\overline{\varphi}(\mathbb{H}) = \mathbb{H}_\omega$. Since, φ is injective, so is $\overline{\varphi}$. Thus $\overline{\varphi}_{|\mathbb{H}}$ is a homeomorphism onto $\mathbb{H}_\omega$. Now let $p, q \in \mathbb{H}$. Then

$$\overline{\varphi}(p+q) = \overline{\varphi}(p\text{-}\lim_{x\in\mathbb{N}}\ q\text{-}\lim_{y\in\mathbb{N}}(x+y)) = p\text{-}\lim_{x\in\mathbb{N}}\ q\text{-}\lim_{y\in\mathbb{N}}\varphi(x+y).$$

Given any $x \in \mathbb{N}$, in $n = \max(\mathrm{supp}(x)) + 1$, then for all $y \in 2^n\mathbb{N}$, $\varphi(x+y) = \varphi(x)\varphi(y)$, so that, since $q \in \mathbb{H}$ and $\lambda_{\varphi(x)}$ and $\overline{\varphi}$ are continuous,

$$q\text{-}\lim_{y\in\mathbb{N}}\varphi(x+y) = q\text{-}\lim_{y\in2^n\mathbb{N}}\varphi(x+y) = q\text{-}\lim_{y\in2^n\mathbb{N}}(\varphi(x)\varphi(y)) = \varphi(x)\varphi(y).$$

Thus $\overline{\varphi}(p+q) = p\text{-}\lim_{x\in\mathbb{N}}(\varphi(x)\overline{\varphi}(q)) = \overline{\varphi}(p)\overline{\varphi}(q)$. $\square$

Indeed this proposition is an example of the general statement as stated in [[29], Theorem 6.15].

Proposition 2.4.14. *Let S be a countably infinite discrete group. Then $\beta S \setminus S$ contains a topological and algebraic copy T of $\mathbb{H}$ such that $K(T) = K(\beta S) \cap T$.*

Proof. [[30], Theorem 2.3]. □

Proposition 2.4.15. *[[30], Theorem 2.5] Let λ be an infinite cardinal. Every compact right topological semigroup T containing a dense subset A such that $|A| \leq \lambda$ and $A \subseteq \Lambda(T)$ is a continuous homomorphic image of $\mathbb{H}_\lambda$.*

Proof. Enumerate A as $\{t_\alpha : \alpha < \lambda\}$, with repetition if $|A| < \lambda$. Let $\{I_\gamma : \gamma < \lambda\}$ be a partition of λ into subsets of size λ. Define $h : \mathbb{B}_\lambda \to T$ by first agreeing that for each $\alpha < \lambda$, $h(e_\alpha) = t_\gamma$, where $\alpha \in I_\gamma$. Then for a finite $F \subset \lambda$, define

$$h\left(\sum_{\alpha \in F} e_\alpha\right) = \Pi_{\alpha \in F}\, h(e_\alpha),$$

where the product is taken in increasing order of indices. Define $h(0)$ arbitrarily. Let $\overline{h} : \beta\mathbb{B}_\lambda \to T$ be the continuous extension of h and let h^* be the restriction of $\overline{h}$ to $\mathbb{H}_\lambda$. To see that $h(\mathbb{H}_\lambda) = T$, it suffices to show that $A \subseteq h(\mathbb{H}_\lambda)$. Given $\gamma < \lambda$ we have that $|I_\gamma| = \lambda$. Pick a λ-uniform ultrafilter p on $\{e_\alpha \ : \ \alpha \in I_\gamma\}$. Then $p \in \mathbb{H}_\lambda$ and $h(p) = t_\gamma$ because f is constantly equal to t_γ on $\{e_\alpha : \alpha \in I_\gamma\}$. It is easy to see that whenever x, y are nonzero elements of $\mathbb{B}_\lambda$ with max supp$(x) <$ min supp(y), then $h(x + y) = h(x)h(y)$. Therefore h^* is a homomorphism. □

Proposition 2.4.16. *[[30], Theorem 2.7] Let λ be an infinite cardinal and let S be an infinite cancellative discrete semigroup with cardinality λ. Then $\beta S \setminus S$ contains a topological and algebraic copy of $\mathbb{H}_\lambda$.*

Proof. Choose a λ-sequence $(t_\kappa)_{\kappa<\lambda}$ in S with distinct finite products. Let $T = FP((t_\kappa)_{\kappa<\lambda})$. For each $\gamma < \lambda$, let $T_\gamma = FP((t_\kappa)_{\gamma<\kappa<\lambda})$. Put

$$\widetilde{T} = \bigcap_{\gamma<\lambda} c\ell_{\beta S}(T_\gamma) = \bigcap_{\gamma<\lambda} \overline{T_\gamma}.$$

$\widetilde{T}$ is a subsemigroup of βS. Define $\theta : T \to \mathbb{B}_\lambda$ by taking, for each finite $F \subseteq \lambda$

$$\theta\left(\Pi_{\kappa \in F}\, t_\kappa\right) = \sum_{\kappa \in F} e_\kappa.$$

Since $(t_\kappa)_{\kappa\in\lambda}$ has distinct finite products, θ is well defined. Since the operation on T is that given by concatenation, it follows that the restriction θ^* of the continuous extension $\overline{\theta} : \widetilde{T} :\to \beta\mathbb{B}_\lambda$ to $\widetilde{T}$ is a homomorphism. It is clear that θ and, hence, $\overline{\theta}$ are injective. Since for each $\gamma < \lambda$, $\overline{\theta}(\overline{T}_\gamma) = \{x \in \mathbb{B}_\lambda \ : \ \min \operatorname{supp}(x) > \gamma\}$, $\overline{\theta}$ maps $\widetilde{T}$ onto $\mathbb{H}_\lambda$. $\square$

2.5 Local isomorphisms of direct sums

Let λ be an infinite cardinal and for every ordinal $\alpha < \lambda$ let A_α be a set with a distinguished element 0_α, and let $\kappa = \min\{|\bigoplus_{\gamma\leq\alpha<\lambda} A_\alpha| : \gamma < \lambda\}$. In this section we show that $Ult(\bigoplus_{\alpha<\kappa} A_\alpha)$ is topologically and algebraically isomorphic either to $\mathbb{H}_\kappa$ or to $\mathbb{H}_{\lambda,\kappa}$. Furthermore, we show that $\mathbb{H}_{\lambda,\mu}$ and $\mathbb{H}_\kappa$ are topologically and algebraically isomorphic if either $\mu \leq \lambda = \kappa = \omega$ or $\mu < \lambda = \kappa$ or $\lambda \leq \mu = \kappa$ and $\operatorname{cf}(\lambda) = \operatorname{cf}(\kappa) < \kappa$. This section forms the main part of [61].

Definition 2.5.1. Let $X = \bigoplus_{\alpha<\kappa} A_\alpha$. Then

(1) $\Delta(X) = \min\{|\bigoplus_{\gamma\leq\alpha<\lambda} A_\alpha| : \gamma < \lambda\}$,

(2) X is *narrow* if there is $\gamma < \lambda$ such that $|\bigoplus_{\gamma\leq\alpha<\delta} A_\alpha| < \Delta(X)$ for all $\delta < \lambda$,

(3) X is *broad* if for every $\gamma < \lambda$ there is $\delta < \lambda$ such that $|\bigoplus_{\gamma\leq\alpha<\delta} A_\alpha| = \Delta(X)$, and

(4) X is *reduced* if $\Delta(X) = |X|$ and either $|\bigoplus_{\alpha<\gamma} A_\alpha| < |X|$ for all $\gamma < \delta$ or for every $\gamma < \lambda$ there is $\delta < \lambda$ such that $|\bigoplus_{\gamma\leq\alpha<\delta} A_\alpha| = |X|$.

Lemma 2.5.2. *For every $X = \bigoplus_{\alpha<\kappa} A_\alpha$ there is $\gamma < \lambda$ such that $Y = \bigoplus_{\gamma\leq\alpha<\lambda} A_\alpha$ is reduced.*

Proof. Let $\kappa = \Delta(X)$. Pick $\gamma_0 < \lambda$ such that $|\bigoplus_{\gamma_0\leq\alpha<\lambda} A_\alpha| = \kappa$. If for every $\epsilon \in [\gamma_0, \lambda)$ there is $\delta < \lambda$ such that $|\bigoplus_{\epsilon\leq\alpha<\delta} A_\alpha| = \kappa$, put $\gamma = \gamma_0$. Otherwise pick $\gamma \in [\gamma_0, \lambda)$ such that $|\bigoplus_{\gamma\leq\alpha<\delta} A_\alpha| < \kappa$ for all $\delta < \lambda$. $\square$

Note that for every $\gamma < \lambda$, $Ult(\bigoplus_{\gamma\leq\alpha<\lambda} A_\alpha)$ is topologically and algebraically isomorphic to $Ult(\bigoplus_{\alpha<\lambda} A_\alpha)$.

Lemma 2.5.3. *Let* $X = \bigoplus_{\alpha<\kappa} A_\alpha$*, let* $\nu = \mathrm{cf}(\lambda)$*, and let* $(\lambda_\xi)_{\xi<\nu}$ *be an increasing cofinal sequence in* λ *such that* $\lambda_0 = 0$ *and* $\lambda_\xi = \sup_{\eta<\xi} \lambda_\eta$ *if* ξ *is a limit ordinal. Then* X *is locally isomorphic to* $Y = \bigoplus_{\xi<\nu} \kappa_\xi$ *where* $\kappa_\xi = |\bigoplus_{\lambda_\xi \le \alpha < \lambda_{\xi+1}} A_\alpha|$.

Proof. For every $\xi < \nu$, let

$$X_\xi = \{x \in X : \emptyset \neq \mathrm{supp}(x) \subseteq [\lambda_\xi, \lambda_{\xi+1})\}$$

and pick a bijection $f_\xi : X_\xi \to \kappa_\xi \setminus \{0\}$. Define $f : X \to Y$ as follows. Put $f(0) = 0$. Now let $0 \neq x \in X$. Then x can be written in the from $x = x_1 + \cdots + x_n$ where, for each $i \in [1, n]$, $x_i \in X_{\xi_i}$ for some $\xi_i < \nu$ and $\xi_1 < \cdots < \xi_n$. Define $y = f(x)$ by

$$\mathrm{supp}(y) = \{\xi_1, \cdots, \xi_n\} \ \text{ and } \ y(\xi_i) = f_{\xi_i}(x_i).$$

Clearly, f is bijective. Furthermore,

(i) if $x, y \in X \setminus \{0\}$ and $\max \mathrm{supp}(x) < \lambda_\xi \le \min \mathrm{supp}(y)$ for some $\xi < \nu$, then $f(x + y) = f(x) + f(y)$, and

(ii) for every $\xi < \nu$,

$$f(\{x \in X : \min \mathrm{supp}(x) \ge \lambda_\xi\}) = \{y \in Y : \min \mathrm{supp}(y) \ge \xi\}.$$

Hence, $f : X \to Y$ is a local isomorphism.

□

Proposition 2.5.4. *Let* $X = \bigoplus_{\alpha<\kappa} A_\alpha$ *be a reduced and let* $\kappa = |X|$*. Then* X *is locally isomorphic to* $\bigoplus_\kappa 2$ *if* $\kappa = \omega$ *or* X *is narrow and to* $\bigoplus_\lambda \kappa$ *otherwise.*

Proof. If $\kappa = \omega$, then X is locally isomorphic to $\bigoplus_\kappa 2$ by Corollary 2.3.14. Suppose $\kappa > \omega$ and let $\nu = \mathrm{cf}(\lambda)$. Consider two cases.

Case 1: X is narrow. Then $\mathrm{cf}(\kappa) = \nu$. Construct inductively increasing cofinal sequences $(\lambda_\xi)_{\xi<\nu}$ in λ and $(\kappa_\xi)_{\xi<\nu}$ in κ such that

(i) $\lambda_0 = \kappa_0 = 0$,

(ii) if $\xi < \nu$ is limit, then $\lambda_\xi = \sup_{\eta<\xi} \lambda_\eta$ and $\kappa_\xi = \sup_{\eta<\xi} \kappa_\eta$, and

(iii) for every $\xi < \nu$, $|\bigoplus_{\lambda_\xi \le \alpha < \lambda_{\xi+1}} A_\alpha| = |\bigoplus_{\kappa_\xi \le \beta < \kappa_{\xi+1}} 2|$.

Then by Lemma 2.5.3, both X and $\bigoplus_\kappa 2$ are locally isomorphic to $\bigoplus_{\xi<\nu} \kappa_{\xi+1}$.

Case2: X is broad. Construct inductively an increasing cofinal sequence $(\lambda_\xi)_{\xi<\nu}$ in λ such that

(i) $\lambda_0 = 0$,

(ii) if $\xi < \nu$ is limit, then $\lambda_\xi = \sup_{\eta<\xi} \lambda_\eta$ and

(iii) for every $\xi < \nu$, $|\bigoplus_{\lambda_\xi \leq \alpha < \lambda_{\xi+1}} A_\alpha| = \kappa$.

Then by Lemma 2.5.3, both X and $\bigoplus_\lambda \kappa$ are locally isomorphic to $\bigoplus_\nu \kappa$.

□

Corollary 2.5.5. *Let $X = \bigoplus_{\alpha<\kappa} A_\alpha$ and let $\kappa = \Delta(X)$. Then $Ult(X)$ is topologically and algebraically isomorphic to $\mathbb{H}_\kappa$ if $\kappa = \omega$ or X is narrow and to $\mathbb{H}_{\lambda,\kappa}$ otherwise.*

It follows from the next lemma that if $\kappa > \omega$ is regular, then $\bigoplus_\kappa 2$ and $\bigoplus_\lambda \kappa$ are not homeomorphic.

Lemma 2.5.6. *Let $\kappa > \omega$ be regular. Then every family of pairwise disjoint nonempty open subsets of $\bigoplus_\kappa 2$ has cardinality $< \kappa$.*

Proof. Let $\{U_\alpha : \alpha < \kappa\}$ be a family of nonempty open subsets of $\bigoplus_\kappa 2$. For every $\alpha < \kappa$, pick $x_\alpha \in U_\alpha$. Applying the Δ-system lemma [[36], Chapter II, Theorem 1.6], we obtain a subset $S \subseteq \kappa$ of cardinality κ and a finite subset $F \subset \kappa$ such that for any distinct $\alpha, \gamma \in S$, one has $\mathrm{supp}(x_\alpha) \cap \mathrm{supp}(x_\gamma) = F$ and $x_\alpha(\delta) = x_\gamma(\delta)$ for each $\delta \in F$. Now let $\alpha \in S$. It then follows that $|\{\gamma \in S : x_\gamma \notin U_\alpha\}| < \kappa$. Consequently, there is $\beta \in S$ distinct from α such that $x_\beta \in U_\alpha$, and so $U_\alpha \cap U_\beta \neq \emptyset$. □

We now come to the main result of this section.

Theorem 2.5.7. *Let $\lambda, \kappa \geq \omega$ and let $\mu \geq 2$. Then the following statements are equivalent:*

(1) $\bigoplus_\lambda \mu$ *and* $\bigoplus_\kappa 2$ *are locally isomorphic,*

(2) $\bigoplus_\lambda \mu$ *and* $\bigoplus_\kappa 2$ *are homeomorphic, and*

(3) *either* $\mu \leq \lambda = \kappa = \omega$ *or* $\mu < \lambda = \kappa$ *or* $\lambda \leq \mu = \kappa$ *and* $\mathrm{cf}(\lambda) = \mathrm{cf}(\kappa) < \kappa$.

The crucial moment in the proof of Theorem 2.5.7 is contained in the following proposition.

Proposition 2.5.8. *If* $\lambda \leq \kappa$ *and* $\mathrm{cf}(\lambda) = \mathrm{cf}(\kappa) < \kappa$*, then* $\bigoplus_\lambda \kappa$ *and* $\bigoplus_\kappa 2$ *are locally isomorphic.*

The proof of Proposition 2.5.8 involves some auxiliary constructions. First of all, by Lemma 2.5.3, one may suppose that λ is regular, and so

$$\lambda = \mathrm{cf}(\kappa) < \kappa.$$

Pick a cofinal λ-sequence $(\kappa_\alpha)_{\alpha<\lambda}$ in κ such that

(1) $\kappa_0 > \lambda$, and

(2) for every $\alpha > 0$, $\kappa_\alpha > \sup_{\beta<\alpha} \kappa_\beta$.

Then $\bigoplus_\kappa 2$ is locally isomorphic to $\bigoplus_{\alpha<\lambda} \kappa_\alpha$. To see this, define inductively a ν-sequence $(\lambda_\xi)_{\xi<\nu}$ in κ by

(i) $\lambda_0 = 0$,

(ii) $\lambda_\xi = \sup_{\eta<\xi} \lambda_\eta$ if ξ is limit, and

(iii) $\lambda_{\xi+1} = \lambda_\xi + \kappa_\xi$,

and apply Lemma 2.5.3. We shall prove that $\bigoplus_\lambda \kappa$ and $\bigoplus_{\alpha<\lambda} \kappa_\alpha$ are locally isomorphic.

Definition 2.5.9. (a) $X = \bigoplus_\lambda \kappa$.

(b) For every $\alpha < \lambda$, $X_{\alpha,\alpha} = \{x \in X : \mathrm{supp}(x) = \{\alpha\}$ and $x(\alpha) \in \kappa_\alpha\}$.

(c) For every $\alpha < \gamma < \lambda$, $X_{\alpha,\gamma}$ is the subset of all $x \in X$ such that

   (i) $\mathrm{supp}(x) \subseteq [\alpha, \gamma]$,

   (ii) for each $\beta \in \mathrm{supp}(x)$, $x(\beta) \in \kappa_\gamma$, and

   (iii) for each $\delta \in [\alpha, \gamma)$, there is $\beta \in [\alpha, \delta]$ such that $x(\beta) \notin \kappa_\delta$.

(d) For every $\alpha \leq \gamma < \lambda$ and $x \in X_{\alpha,\gamma}$, $l(x) = \alpha$ and $r(x) = \gamma$.

Lemma 2.5.10. *Every nonzero $x \in X$ can be uniquely written in the form*

$$x = x_1 + \cdots + x_n$$

where for each $i \in [1,n]$, $x_i \in X_{\alpha,\gamma}$ for some $\alpha \leq \gamma < \lambda$, and for each $i \in [1, n-1]$, $r(x_i) < l(x_{i+1})$.

Proof. Let $\alpha = \min \operatorname{supp}(x)$ and $\gamma = \min\{\delta \geq \alpha : x(\beta) \in \kappa_\delta \text{ for all } \beta \leq \delta\}$. If $\operatorname{supp}(x) \subseteq [\alpha, \gamma]$, then $x \in X_{\alpha,\gamma}$. Otherwise define $x_1, y \in X$ by

$$\operatorname{supp}(x_1) = \{\beta \in \operatorname{supp}(x) : \beta \leq \gamma\}, \ \ \operatorname{supp}(y) = \{\beta \in \operatorname{supp}(x) : \beta > \gamma\},$$

$x_1(\beta) = x(\beta)$ for all $\beta \in \operatorname{supp}(x_1)$, and $y(\beta) = x(\beta)$ for all $\beta \in \operatorname{supp}(y)$. Then $x_1 \in X_{\alpha,\gamma}$, $r(x_1) < \min \operatorname{supp}(y)$ and $x = x_1 + y$. Continuing this process, in $\leq |\operatorname{supp}(x)|$ steps we obtain the required decomposition.
To see the uniqueness, let $x = x'_1 + \cdots + x'_m$ be another such decomposition. It suffices to show that $x'_1 = x_1$. We have that $x'_1 \in X_{\alpha',\gamma'}$ for some $\alpha' \leq \gamma' < \lambda$. Then $\alpha' = \min \operatorname{supp}(x) = \alpha$ and $\gamma' = \min\{\delta \geq \alpha : x(\beta) \in \kappa_\delta \text{ for all } \beta \leq \delta\} = \gamma$. Consequently, $x'_1 = x_1$. □

Definition 2.5.11. (a) $Y = \bigoplus_{\alpha<\lambda} \kappa_\alpha$.

(b) For every $\alpha < \lambda$, $Y_{\alpha,0} = \{y \in Y : \operatorname{supp}(y) = \{\alpha\} \text{ and } y(\alpha) \in \kappa_\alpha \setminus \lambda\}$.

(c) For every $\alpha < \lambda$ and nonzero $\xi < \lambda$,

$$Y_{\alpha,\xi} = \{y \in Y : \operatorname{supp}(y) \subseteq [\alpha, \alpha + \xi] \text{ and } y(\alpha) = \xi\}.$$

(d) For every $\alpha, \xi < \lambda$ and $y \in Y_{\alpha,\xi}$, $l(y) = \alpha$ and $r(y) = \alpha + \xi$.

Lemma 2.5.12. *Every nonzero $y \in Y$ can be uniquely written in the form*

$$y = y_1 + \cdots + y_n$$

where for each $i \in [1,n]$, $y_i \in Y_{\alpha,\xi}$ for some $\alpha, \xi < \lambda$, and for each $i \in [1, n-1]$, $r(y_i) < l(y_{i+1})$.

Proof. The proof is similar to that of Lemma 2.5.10. We restrict ourselves to showing the existence of such a decomposition.
Let $\alpha = \min \operatorname{supp}(y)$. If $y(\alpha) \in \kappa_\alpha \setminus \lambda$, put $\xi = 0$. Otherwise $\xi = y(\alpha)$. If $\operatorname{supp}(y) \subseteq [\alpha, \alpha + \xi]$, then $y \in Y_{\alpha,\alpha+\xi}$. Otherwise define $y_1, z \in Y$ by

$$\operatorname{supp}(y_1) = \{\beta \in \operatorname{supp}(y) : \beta \leq \alpha+\xi\}, \ \ \operatorname{supp}(z) = \{\beta \in \operatorname{supp}(y) : \beta > \alpha+\xi\},$$

$y_1(\beta) = y(\beta)$ for all $\beta \in \operatorname{supp}(y_1)$, and $z(\beta) = y(\beta)$ for all $\beta \in \operatorname{supp}(z)$. Then $y_1 \in Y_{\alpha,\alpha+\xi}$, $r(y_1) < \min \operatorname{supp}(z)$ and $y = y_1 + z$. Continuing this process, in $\leq |\operatorname{supp}(y)|$ steps we obtain the required decomposition. □

As in Example 2.3.4, we call the decompositions in Lemma 2.5.10 and Lemma 2.5.12 *canonical*.

Proof of Proposition 2.5.8. We show that X and Y are locally isomorphic. For every $\alpha, \xi < \lambda$, both $|X_{\alpha,\alpha+\xi}| = \kappa_{\alpha+\xi}$ and $|Y_{\alpha,\xi}| = \kappa_{\alpha+\xi}$, so there is a bijection $f_{\alpha,\xi} : X_{\alpha,\alpha+\xi} \to Y_{\alpha,\xi}$. Define $f : X \to Y$ as follows. Put $f(0) = 0$. Now let $0 \neq x \in X$ and let

$$x = x_1 + \cdots + x_n$$

be a canonical decomposition. For each $i \in [1, n]$, let $\alpha_i = l(x_i)$, $\gamma_i = r(x_i)$, and write γ_i as $\gamma_i = \alpha_i + \xi_i$ for some $\xi_i < \lambda$. Put

$$f(x) = f_{\alpha_1,\xi_1}(x_1) + \cdots + f_{\alpha_n,\xi_n}(x_n).$$

Note that this is a canonical decomposition. It then follows that f is a bijection. Furthermore,

(i) if $x, y \in X \setminus \{0\}$, $x = x_1 + \cdots + x_n$ is the canonical decomposition and $r(x_n) < \min \operatorname{supp}(y)$, then $f(x + y) = f(x) + f(y)$, and

(ii) for every $\alpha < \lambda$,

$$f(\{x \in X : \min \operatorname{supp}(x) \geq \alpha\}) = \{y \in Y : \min \operatorname{supp}(y) \geq \alpha\}.$$

Hence $f : X \to Y$ is a local isomorphism. □

Now we are in a position to prove Theorem 2.5.7.

Proof of Theorem 2.5.7. (1) ⇒ (2) is obvious.
(2) ⇒ (3): If $\kappa = \omega$, then $\mu \leq \lambda = \omega$. Let $\kappa > \omega$. If $\mu < \lambda$, then $\lambda = \kappa$. Suppose that $\mu \geq \lambda$. Then $\mu = \kappa$ and clearly $\operatorname{cf}(\lambda) = \operatorname{cf}(\kappa)$. Since $\bigoplus_\lambda \kappa$ can be partitioned into κ nonempty open subsets, it follows from Lemma 2.5.6 that $\operatorname{cf}(\kappa) < \kappa$.
(3) ⇒ (1): If either $\mu \leq \lambda = \kappa = \omega$ or $\mu < \lambda = \kappa$, then $\bigoplus_\lambda \mu$ and $\bigoplus_\kappa 2$ are locally isomorphic by Proposition 2.5.4. And if $\lambda \leq \mu = \kappa$ and $\operatorname{cf}(\lambda) = \operatorname{cf}(\kappa) < \kappa$, then they are locally isomorphic by Proposition 2.5.8. □

Corollary 2.5.13. *Let $\lambda, \kappa \geq \omega$ and $\mu \geq 2$. Suppose that either $\mu \leq \lambda = \kappa = \omega$ or $\mu < \lambda = \kappa$ or $\lambda \leq \mu = \kappa$ and $\operatorname{cf}(\lambda) = \operatorname{cf}(\kappa) < \kappa$. Then $\mathbb{H}_{\lambda,\mu}$ and $\mathbb{H}_\kappa$ are topologically and algebraically isomorphic.*

2.6 The ultrafilter semigroup 0^+

Unless, otherwise stated, we consider the semigroup $(\mathbb{R}, +)$.

Definition 2.6.1. [25] Let S be a subsemigroup of $(\mathbb{R}, +)$. We define the following, where $\mathbb{R}$ is equipped with the usual topology.

1. $\alpha : \beta S \to [-\infty, \infty]$ is the continuous extension of the identity function.
2. $B(S) = \alpha^{-1}(-\infty, \infty)$.
3. Given $x \in \mathbb{R}$,
   $x^+ = \alpha^{-1}(x) \cap \overline{(x, \infty) \cap S}$ and $x^- = \alpha^{-1}(x) \cap \overline{(-\infty, x) \cap S}$.
4. $U = \bigcup_{x \in \mathbb{R}} x^+$ and $D = \bigcup_{x \in \mathbb{R}} x^-$

It follows that an ultrafilter p on S is in $B(S)$ if and only if there is some $n \in \mathbb{N}$ such that $[-n, n] \cap S \in p$.

Lemma 2.6.2. *[25] Let S be a subsemigroup of $\mathbb{R}$.*

(a) Let $x \in \mathbb{R}$ and let $p \in \beta S$. Then $p \in x^+$ if and only if for every $\epsilon > 0$, $(x, x + \epsilon) \cap S \in p$. Also $p \in x^-$ if and only if for every $\epsilon > 0$, $(x - \epsilon, x) \cap S \in p$.

(b) Let $x \in \mathbb{R}$. Then $x^+ \neq \emptyset$ if and only if $x \in c\ell_{\mathbb{R}}((x, \infty) \cap S)$ and $x^- \neq \emptyset$ if and only if $x \in c\ell_{\mathbb{R}}((-\infty, x) \cap S)$.

(c) Let $p, q \in B(S)$, let $x = \alpha(p)$, and let $y = \alpha(q)$. If $p \in x^+$, then $p + q \in (x + y)^+$. If $p \in x^-$, then $p + q \in (x + y)^-$.

(d) $B(S) \setminus S = U \cup D$. If U and D are nonempty, they are disjoint right ideals of $B(S)$. In particular, $B(S)$ is not commutative.

(e) If $0^+ \neq \emptyset$, then 0^+ is a compact subsemigroup of $B(S)$.

(f) If $x \in S$ and $p \in \beta S$, then $x + p = p + x$.

Proof. (a) Let $p \in x^+$ and let $\epsilon > 0$. Then $(x - \epsilon, x + \epsilon)$ is a neighborhood of x in $\mathbb{R}$ and, therefore, $(x - \epsilon, x + \epsilon) \cap S \in p$. It follows that $(x, x + \epsilon) \cap S = ((x - \epsilon, x + \epsilon) \cap S) \cap ((x, \infty) \cap S) \in p$. Conversely, suppose $(x, x + \epsilon) \cap S \in p$ for every $\epsilon > 0$. Since $(x, x + \epsilon) \cap S \subset (x, \infty) \cap S$, $(x, \infty) \cap S \in p$. In addition, since for every neighborhood U of x in $[-\infty, \infty]$, $U \cap S \supset (x, x + \epsilon) \cap S$ for some $\epsilon > 0$, $\alpha(p) = x$. Therefore, $p \in x^+ \cap \overline{(x, \infty) \cap S}$. Similarly the second statement follows.

(b) Let $x \in \mathbb{R}$. Suppose $x^+ \neq \emptyset$. Pick $p \in x^+$ and let $U \subseteq \mathbb{R}$ be a neighborhood of x in $\mathbb{R}$. Then, $(x+\epsilon, x-\epsilon) \subseteq U$ for some $\epsilon > 0$. Then, $(x, x+\epsilon) \cap S \in p$. In particular $(x, x+\epsilon) \cap S \neq \emptyset$. Therefore, $\emptyset \neq (x, x+\epsilon) \cap S \subseteq U \cap (x, \infty) \cap S$. It follows that $x \in c\ell_{\mathbb{R}}((-\infty, x) \cap S)$. Conversely, suppose $x \in c\ell_{\mathbb{R}}((-\infty, x) \cap S)$. Then, there exists an ultrafilter p on S containing the family $\{U \cap ((x, \infty) \cap S) \ : \ U$ is a neighborhood of x in $\mathbb{R}\}$. It then follows that $\alpha(p) = x$ and, for example taking $U = (x-\epsilon, \infty)$ for some $\epsilon > 0$, we see that $(x, \infty) \cap S \in p$. Similarly one establishes the second statement.

(c) Assume first that $p \in x^+$. To see that $p + q \in (x+y)^+$, let $\epsilon > 0$ be given and let $A = (x+y, x+y+\epsilon) \cap S$. T see that $A \in p+q$, we show that $(x, x+\epsilon) \cap S \subseteq \{z \in S \ : \ -z+A \in q\}$. So let $z \in (x, x+\epsilon) \cap S$. Let $\delta = \min\{z-x, x+\epsilon-z\}$. Since $\alpha(q) = y$, we have $(y-\delta, y+\delta) \cap S \in q$ and $(y-\delta, y+\delta) \cap S \subseteq -z+A$, so $-z+A \in q$ as required. The proof that $p+q \in (x+y)^-$ is nearly identical.

(d) Let $p \in B(S) \setminus S$. Let $x = \alpha(p)$. Then $x \in (-\infty, \infty)$ and $x \notin S$. Since p is an ultrafilter, $(-\infty, x) \in p$ or $(x, \infty) \in p$. Assume that $(x, \infty) \in p$. Then $p \in x^+$. Suppose $p \in y^-$ for some $y \in \mathbb{R}$. Then, $x = \alpha(p) = y$ and $(-\infty, x) \cap S = (-\infty, y) \cap S \in p$. This implies that $\emptyset \in p$, which is a contradiction. It follows that $D \cap U = \emptyset$. It follows easily from (c) that D and U are right ideals of $B(S)$. In particular, $B(S)$ is not commutative.

(e) Suppose $0^+ \neq \emptyset$. It follows easily from (c) that 0^+ is a subsemigroup of $B(S)$. Since $0^+ = \alpha^{-1}(0) \cap \overline{(0, \infty) \cap S}$ is closed in βS, 0^+ is compact.

(f) It suffices to note that the left shift λ_x and the right shift ρ_x are continuous functions agreeing on S, hence on βS_d.

$\square$

0^+ is a subsemigroup of $\beta\mathbb{R}$, while for any other $x \in \mathbb{R}$, x^+ and x^- are not subsemigroups of $\beta\mathbb{R}$. In addition, the following theorem shows that 0^+ holds all of the algebraic structure of $B(\mathbb{R})$ not already revealed by $\mathbb{R}$.

Theorem 2.6.3. *[25] Let $S = \mathbb{R}$.*

(a) 0^+ *and* 0^- *are isomorphic.*

(b) The function $\varphi : \mathbb{R}_d \times (\{0\} \cup 0^+ \cup 0^-) \to B(\mathbb{R})$ *defined by* $\varphi(x, p) = x + p$ *is a continuous isomorphism onto* $B(\mathbb{R})$.

Proof. (a) Define $\tau : 0^+ \to 0^-$ by $\tau(p) = -p$, where $-p = \{-A : A \in p\}$. Let $p, q \in 0^+$ with $\tau(p) = \tau(q)$. Then, $p = -(-p) = -(-q) = q$. Therefore τ is injective. Since $S = \mathbb{R}$, $p \in 0^+$ if and only if $\alpha(p) = 0$ and $(0, \infty) \in p$ if and only if $\alpha(p) = 0$ and $(-\infty, 0) \in p$. Since 0 has the neighborhood base

$$\{(-\epsilon, \epsilon) \ : \ \epsilon > 0\}$$

and since $-(-\epsilon, \epsilon) = (-\epsilon, \epsilon)$, i.e. 0 has a neighborhood base consisting of symmetric subsets of $\mathbb{R}$, $\alpha(p) = 0$ if and only if $\alpha(-p) = 0$. It follows that $p \in 0^+$ if and only if $-p \in 0^-$. Hence τ is surjective. It follows that τ is bijective.

Let $p, q \in 0^+$. To see that $\tau(p+q) = \tau(p)+\tau(q)$ it suffices, since $\tau(p+q)$ and $\tau(p)+\tau(q)$ are both ultrafilters, to show that $\tau(p+q) \subseteq \tau(p)+\tau(q)$. So let $A \in \tau(p+q)$. Then $-A \in p+q$ so $B = \{x \in \mathbb{R} \ : \ -x + (-A) \in q\} \in p$ and hence $-B \in \tau(p)$. Then $-B \subseteq \{x \in \mathbb{R} \ : \ -x + A \in \tau(q)\}$ so $A \in \tau(p) + \tau(q)$ as required.

(b) To see that φ is a homomorphism, let (x, p) and (y, q) be in $\mathbb{R}_d \times (\{0\} \cup 0^+ \cup 0^-)$. Since $p+y = y+p$, we have $\varphi(x, p)+\varphi(y, q) = \varphi(x+y, p+q)$. To see that φ is injective, assume we have $\varphi(x, p) = \varphi(y, q)$. By Lemma 2.51 (c) we have $x = \alpha(x + p) = \alpha(y + q) = y$. Then $x + p = x + q$ so $p = -x + x + p = -x + x + q = q$. To see that φ is surjective, let $q \in B(\mathbb{R})$ and let $x = \alpha(q)$. Let $p = -x + q$. Then $q = x + p = \varphi(x, p)$. To see that φ is continuous, let $(x, p) \in \mathbb{R}_d \times (\{0\} \cup 0^+ \cup 0^-)$ and let $A \in x+p$. Then $-x+A \in p$ so $\{x\} \times (c\ell_{\beta S}(-x+A))$ is a neighborhood of (x, p) contained in $\varphi^{-1}(c\ell_{\beta S} A)$.

□

0^+ has yielded useful combinatorial results. Therefore the algebra of 0^+ is of great interest. Since $\{x \in \mathbb{R} : x \leq 0\}$ does not contribute to 0^+ at all, we will assume that 0^+ is defined in terms of some semigroup $S \subseteq (0, \infty)$. Since we want $0^+ \neq \emptyset$, we need $0 \in c\ell_{\mathbb{R}} S$. In addition, this suffices to imply that S is dense in $(0, \infty)$. Furthermore, 0^+ is far from being an ideal of $B(S)$. In fact, 0^+ is the inverse image of 0 under the homomorphism $\beta S \to [0, \infty]$ and, thus, 0^+ is a prime subsemigroup of βS. It follows that no general results apply to 0^+ beyond those that apply to any compact right topological semigroup.

Remark 2.6.4. Let $\mathcal{T}$ be the translation invariant topology on $(\mathbb{R},+)$ with the following neighborhood base at 0:

$$\mathcal{B} = \{[0, \frac{1}{n}) : n \in \mathbb{N}\},$$

i.e. $\mathcal{T}$ is the Sorgenfrey topology. Let S be a dense subsemigroup of $(0,\infty)$. Let us still denote by $\mathcal{T}$, the topology induced on S by the Sorgenfrey topology. Then 0^+ is the ultrafilter semigroup of $((0,\infty)\cap S, \mathcal{T})$, i.e.

$$0^+ = Ult(\mathcal{T}) = \bigcap_{n\in\mathbb{N}} \overline{I_n},$$

where $I_n = (0, \frac{1}{n}) \cap S$.

Proof. Let $p \in 0^+$. Then $\alpha(p) = 0$ and $(0,\infty)\cap S \in p$. $\alpha(p) = 0$ implies that for every neighborhood U of 0 in $\mathcal{T}$, $U \cap (0,\infty) \cap S \in p$. In particular, $(0, \frac{1}{n}) \cap S \in p$ for every $n \in \mathbb{N}$. Therefore $0^+ \subseteq Ult(\mathcal{T})$. Conversely, let $p \in Ult(\mathcal{T})$. Since $\{(0, \frac{1}{n}) : n \in \mathbb{N}\}$ is a base for the neighborhood filter at 0, it follows easily that $p \in 0^+$. $\square$

Remark 2.6.5. It is worth noting that $(\mathbb{H}, \cdot)$ is an ideal of $(\beta\mathbb{N}, \cdot)$ (see for example [[23], Theorem 7.5]) and, similarly, $((0^+), \cdot)$ is an ideal of $(\beta(0,1), \cdot)$. Therefore, much is known about the structures of $(\mathbb{H}, \cdot)$ and $((0^+), \cdot)$.

Chapter 3

The smallest ideal and its closure

3.1 Combinatorial characterizations of $K(\mathbb{T})$ and its closure

In this section we present the extension of the combinatorial characterizations of $K(\beta S)$ and its closure to $K(\mathbb{T})$ and its closure. Most of the materials in this section come from [22].

Let S be a semigroup. Recall the following.

(a) A subset $A \subseteq S$ is said to be syndetic if there exists a finite $G \subseteq S$ such that $S = G^{-1}A$.

(b) A subset $A \subseteq S$ is said to be piecewise syndetic if there exists a finite $G \subseteq S$ such that for every finite $F \subseteq S$ there exists $x \in S$ such that $Fx \subseteq G^{-1}A$.

(c) $K(\beta S)$ can be characterized as :

$$K(\beta S) = \{p \in \beta S : \text{for all } A \in p,\ \{x \in S : x^{-1}A \in p\} \text{ is syndetic}\}.$$

(d) $c\ell\, K(\beta S)$ can be characterized as:

$$c\ell\, K(\beta S) = \{p \in \beta S : \text{for all } A \in p,\ A \text{ is piecewise syndetic}\}.$$

Considering the semigroup $(\mathbb{N}, +)$ we observe that statements (a) and (b) can be written as follow.

(a') A subset $A \subseteq \mathbb{N}$ is syndetic if and only if there exists $l \in \mathbb{N}$ such that for every $x \in \mathbb{N}$ there is $t \in \mathbb{N}$ with $1 \leq t \leq l$ such that $t + x \in A$ if and only if there exists $l \in \mathbb{N}$ such that $\mathbb{N} = \bigcup_{t=1}^{l}(-t + A)$.

(b') A subset $A \subseteq S$ is piecewise syndetic if and only if there exists $l \in \mathbb{N}$ such that for each $n \in \mathbb{N}$ there exists $x \in \mathbb{N}$ with $\{x+1, \cdots, x+n\} \subseteq \bigcup_{t=1}^{l}(-t + A)$.

In statement (b') sufficiency follows easily from the fact that for every nonempty $F \in \mathcal{P}_f(\mathbb{N})$ and every $x \in \mathbb{N}$, $x + F \subseteq \{x+1, x+2, \cdots, x + \max F\}$.

Proposition 3.1.1. *For every discrete semigroup S, $c\ell\ K(\beta S)$ is an ideal.*

Proof. Since βS is a right topological semigroup and $K(\beta S)$ is a right ideal, $c\ell\ K(\beta S)$ is a right ideal (see Section 1.4). Since βS is a right topological semigroup with a dense topological center, $c\ell\ K(\beta S)$ is a left ideal. It follows that $c\ell\ K(\beta S)$ is an ideal. □

$\mathbb{T}$ is a compact right topological semigroup and, therefore, $\mathbb{T}$ has the smallest ideal $K(\mathbb{T})$. We would like to present the characterizations of $K(\mathbb{T})$ and its closure and then show that $c\ell_{\mathbb{T}}\ K(\mathbb{T})$ is not an ideal.

Lemma 3.1.2. *[[27], Lemma 1.5] If $\mathcal{A}$ is a left ideal, right ideal, a minimal left ideal, a minimal right ideal, or the smallest ideal of $\beta\mathbb{N}$, then $\mathcal{A} \cap \mathbb{T}$ is the corresponding object in $\mathbb{T}$.*

Proof. Let $\mathcal{A}$ be a left ideal or right ideal of $\beta\mathbb{N}$. Then $\mathcal{A}$ contains an idempotent. Therefore, $\mathcal{A} \cap \mathbb{T} \neq \emptyset$. It is clear that $\mathcal{A} \cap \mathbb{T}$ is a left ideal or a right ideal of $\mathbb{T}$. Observe that if $p \in \mathbb{T}$, $q \in \beta\mathbb{N} \setminus \mathbb{N}$, and $p + q \in \mathbb{T}$ or $q + p \in \mathbb{T}$, then $q \in \mathbb{T}$. Consequently, $\mathcal{A} \cap \mathbb{T}$ is minimal in $\mathbb{T}$ if $\mathcal{A}$ is minimal in $\beta\mathbb{N}$. □

As a consequence of Lemma 3.1.2, we have the following description of $K(\mathbb{T})$.

Corollary 3.1.3. *$K(\mathbb{T}) = K(\beta\mathbb{N}) \cap \mathbb{T}$.*

In fact, $K(\mathbb{H}) = K(\beta\mathbb{N}) \cap \mathbb{H}$, see [[29], Theorem 1.65]

Definition 3.1.4. [[22], Definition 2.1 (b)] Let $A \subseteq \mathbb{N}$. A is said to be *piecewise* $\mathbb{T}$*-syndetic* if there exists a sequence $(l_m)_{\mathbb{N}}$ in $\mathbb{N}$ such that for each $n \in \mathbb{N}$ there exists $x \in \mathbb{N}$ such that whenever $m \in \mathbb{N}$ with $m \leq n$,

$$\{x+1, \cdots, x+n\} \subseteq \bigcup_{t=1}^{l_m}(-t + (A \cap m\mathbb{N})).$$

Remark 3.1.5. Let $A \subseteq \mathbb{N}$.

(a) If A is piecewise $\mathbb{T}$-syndetic, then A is piecewise syndetic.

(b) The converse of (a) is false.

Proof. (a): Suppose A is piecewise $\mathbb{T}$-syndetic. Pick a sequence $(l_m)_{\mathbb{N}}$ in $\mathbb{N}$ satisfying the condition in Definition 3.1.4 and take $l = l_1$. Let $n \in \mathbb{N}$. By assumption, there exists $x \in \mathbb{N}$ such that

$$\{1+x, \cdots, n+x\} \subseteq \bigcup_{t=1}^{l}(-t + (A \cap 1\mathbb{N})) = \bigcup_{t=1}^{l}(-t + A).$$

Thus A is piecewise syndetic.

(b): Let A be a set of odd positive integers, which is syndetic in $\mathbb{N}$. Since $A \cap 2\mathbb{N} = \emptyset$, then A can not satisfy the condition in Definition 3.1.4 for any sequence $(l_m)_{\mathbb{N}}$ in $\mathbb{N}$. Hence A is piecewise syndetic but not piecewise $\mathbb{T}$-syndetic. $\square$

The following lemma gives a stronger version of Definition 3.1.4.

Lemma 3.1.6. *[[22], Lemma 2.6] Let $A \subseteq \mathbb{N}$. A is piecewise $\mathbb{T}$-syndetic if and only if there exists a sequence $(l_m)_{\mathbb{N}}$ in $\mathbb{N}$ such that for each $n \in \mathbb{N}$ there exists $x \in n!\mathbb{N}$ such that whenever $m \in \mathbb{N}$ with $m \leq n$, then*

$$\{x+1, \cdots, x+n\} \subseteq \bigcup_{t=1}^{l_m}(-t + (A \cap m!\mathbb{N})).$$

Proof. $(\Leftarrow)$ Suppose A satisfies the given condition. Since $k!\mathbb{N} \subseteq \mathbb{N}$ and $k!\mathbb{N} \subseteq k\mathbb{N}$ for every $k \in \mathbb{N}$, it follows easily that A is piecewise $\mathbb{T}$-syndetic.

($\Rightarrow$) Suppose A is piecewise $\mathbb{T}$-syndetic. Pick a sequence $(l_m)_{\mathbb{N}}$ such that for each $n \in \mathbb{N}$ there exists $x \in \mathbb{N}$ such that whenever $m \in \mathbb{N}$ with $m \leq n$, then $\{x+1, \cdots, x+n\} \subseteq \bigcup_{t=1}^{l_m}(-t+(A \cap m\mathbb{N}))$. Form a sequence $(l'_m)_{\mathbb{N}}$ in $\mathbb{N}$ by taking $l'_m = l_{m!}$ for each $m \in \mathbb{N}$. Let $n \in \mathbb{N}$ be given and let $r = n! + n$. Pick $y \in \mathbb{N}$ such that for $m \leq r$, $\{y+1, \cdots, y+r\} \subseteq \bigcup_{t=1}^{l_m}(-t+(A \cap m\mathbb{N}))$. Pick $x \in \{y+1, \cdots, y+n!\} \cap n!\mathbb{N}$. Let $m \leq n$ so that $m! \leq n! < n! + n = r$. Then

$$\begin{aligned}
\{x+1, \cdots, x+n\} &\subseteq \{y+1, \cdots, y+n!+n\} \\
&= \{y+1, \cdots, y+r\} \\
&\subseteq \bigcup_{t=1}^{l_{m!}}(-t+(A \cap m!\mathbb{N})) \\
&= \bigcup_{t=1}^{l'_m}(-t+(A \cap m!\mathbb{N})).
\end{aligned}$$

□

Lemma 3.1.7. *[[22], Lemma 2.7] The following statements hold.*

(a) $\mathbb{N}$ is piecewise $\mathbb{T}$-syndetic and $\emptyset$ is not piecewise $\mathbb{T}$-syndetic.

(b) If $A \subseteq B \subseteq \mathbb{N}$ and A is piecewise $\mathbb{T}$-syndetic, then B is piecewise $\mathbb{T}$-syndetic.

(c) If $A, B \subseteq \mathbb{N}$ such that $A \cup B$ is piecewise $\mathbb{T}$-syndetic then, A is piecewise $\mathbb{T}$-syndetic or B is piecewise $\mathbb{T}$-syndetic.

Proof. (a) For every $n \in \mathbb{N}$ and every $x \in \mathbb{N}$, $\{x+1, \cdots, x+n\} \neq \emptyset$. On the other hand, for every $t \in \mathbb{N}$ and every $m \in \mathbb{N}$, $-t+(\emptyset \cap m\mathbb{N}) = -t+\emptyset = \emptyset$. It follows that $\emptyset$ is not piecewise $\mathbb{T}$-syndetic.
It is very easy to see that for every $m \in \mathbb{N}$ and every $n \in \mathbb{N}$, $\{2, 3, \cdots, 1+n\} \subseteq \bigcup_{t=1}^{m}(-t+m\mathbb{N})$. Therefore, take $(l_m)_{\mathbb{N}} = (m)_{\mathbb{N}}$ and given any $n \in \mathbb{N}$, take $x = 1$. It follows that $\mathbb{N}$ is piecewise $\mathbb{T}$-syndetic.

(b) Since $A \subseteq B$ implies $A \cap m\mathbb{N} \subseteq B \cap m\mathbb{N}$, the statement follows.

(c) Let $A, B \subseteq \mathbb{N}$ such that $A \cup B$ is piecewise $\mathbb{T}$-syndetic. Pick a sequence $(l_m)_{\mathbb{N}}$ as guaranteed by Lemma 3.1.6. Since the union $\bigcup_{t=1}^{l_m}(-t+((A \cup B) \cap m\mathbb{N}))$ does not depend on the order of the terms $l_1, \cdots, l_m$, we may assume that $l_m \leq l_{m+1}$ for every $m \in \mathbb{N}$. For each $n \in \mathbb{N}$ pick

$x_n \in n!\mathbb{N}$ such that whenever $m \in \mathbb{N}$ with $m \leq n$, $\{x_n+1, \cdots, x_n+n\} \subseteq \bigcup_{t=1}^{l_m}(-t+(m!\mathbb{N} \cap A))$. As in [[24], Lemma 3.4] we may obtain a subsequence $(y_n)_{\mathbb{N}}$ of $(x_n)_{\mathbb{N}}$ such that

(1) If $t \leq r \leq n$ then $y_r + t \in A$ if and only if $y_n + t \in A$.

(2) $y_n \in n!\mathbb{N}$ and $\{y_n+1, \cdots, y_n+n\} \subseteq \bigcup_{t=1}^{l_m}(-t+(m!\mathbb{N} \cap (A \cup B)))$ whenever $m \in \mathbb{N}$ with $m \leq n$.

Let $F = \{n \in \mathbb{N} : y_n + n \in A\}$. Then

(3) for $t \in \{1, 2, \cdots, n\}$, $y_n + t \in A$ if and only if $t \in F$.

We assume first that there is some $a \in \mathbb{N}$ such that for each k there is some z with $\{z+1, z+2 \cdots, z+k\} \cap F \cap a!\mathbb{N} = \emptyset$. In this case we define $k_m = l_m$ if $m \geq a$ and $k_m = l_a$ if $m < a$. Let $s \in \mathbb{N}$ be given. We will produce $x \in \mathbb{N}$ such that $\{x+1, x+2, \cdots, x+s\} \subseteq \bigcup_{t=1}^{k_m}(-t+(B \cap m!\mathbb{N}))$ whenever $m \in \mathbb{N}$ with $m \leq s$. This will mean that B is piecewise $\mathbb{T}$-syndetic.

Let $b = k_s + s$ and pick $r \in \mathbb{N}$ such that $\{r+1, \cdots, r+b\} \cap F \cap a!\mathbb{N} = \emptyset$. Let $n = \max\{r+b, a\}$ and let $x = y_n + r$. Let $m \in \{1, 2, \cdots, s\}$. To see that $\{x+1, x+2, \cdots, x+s\} \subseteq \bigcup_{t=1}^{k_m}(-t+(B \cap m!\mathbb{N}))$, let $i \in \{1, 2, \cdots, s\}$ and note that $x + i = y_n + r + i \leq y_n + r + b \leq y_n + n$.

Case 1: $m < a$. Then $x + i \in \{y_n+1, y_n+2, \cdots, y_n+n\}$ and $n \geq a$ so $x + i \in \bigcup_{t=1}^{l_a}(-t+((A \cup B) \cap a!\mathbb{N}))$. Pick $t \in \{1, 2, \cdots, l_a\}$ such that $x+i+t \in (A \cup B) \cap a!\mathbb{N}$. Now $t \leq l_a = k_m$ and $x+i+t \in a!\mathbb{N} \subseteq m!\mathbb{N}$ so it suffices to show that $x+i+t \in B$. Suppose instead $x+i+t \in A$. Then $y_n+r+i+t \in A$ and $r+i+t \leq r+s+l_a \leq r+s+k_s = r+b \leq n$ so, by (3), $r+i+t \in F$. But also $y_n \in n!\mathbb{N} \subseteq a!\mathbb{N}$ and $y_n+r+i+t \in a!\mathbb{N}$ so $r+i+t \in a!\mathbb{N}$. But $i+t \leq s+l_a \leq s+k_s = b$ so $r+i+t \notin F \cap a!\mathbb{N}$, a contradiction.

Case 2: $m \geq a$. Since $m \leq s \leq n$ and $x+i \in \{y_n+1, y_n+2, \cdots, y_n+n\}$, we have $x+i \in \bigcup_{t=1}^{l_m}(-t+((A \cup B) \cap m!\mathbb{N}))$. Pick $t \in \{1, 2, \cdots, l_m\}$ such that $x+i+t \in (A \cup B) \cap m!\mathbb{N}$. Then $t \leq l_m = k_m$ and $x+i+t \in m!\mathbb{N}$ so it suffices to show that $x+i+t \in B$. Suppose instead $x+i+t \in A$. Then $y_n+r+i+t \in A$ and $r+i+t \leq r+s+k_m \leq r+s+k_s = r+b \leq n$ so $r+i+t \in F$. Also $y_n \in n!\mathbb{N} \subseteq a!\mathbb{N}$ and $y_n+r+i+t \in m!\mathbb{N} \subseteq a!\mathbb{N}$ so $r+i+t \in a!\mathbb{N}$. But $i+t \leq s+k_s = b$ so $r+i+t \in F \cap a!\mathbb{N}$, a contradiction.

Now assume that for each a, there exists k such that for each z, $\{z+1, z+2 \cdots, z+k\} \cap F \cap a!\mathbb{N} \neq \emptyset$. Pick for each m some k_m such that, for each $r \in \mathbb{N}$, $\{r+1, r+2, \cdots, r+k_m\} \cap F \cap m!\mathbb{N} \neq \emptyset$, and if $m > 1$, $k_m \geq k_{m-1}$. Let $s \in \mathbb{N}$ be given. We will produce $x \in \mathbb{N}$ such that $\{x+1, x+2, \cdots x+s\} \subseteq \bigcup_{t=1}^{k_m}(-t+(A \cap m!\mathbb{N}))$ for all $m \in \{1, 2, \cdots, s\}$. Let $n = s+k_s$ and let $x = y_n$. Let $m \in \{1, 2, \cdots, s\}$. To see that $\{x+1, x+2, \cdots, x+s\} \subseteq \bigcup_{t=1}^{k_m}(-t+(A \cap m!\mathbb{N}))$, let $i \in \{1, 2, \cdots, s\}$. Pick $t \in \{1, 2, \cdots, k_m\}$ such that $i + t \in F \cap m!\mathbb{N}$. Then $i + t \leq s + k_m \leq s + k_s = n$. Thus $y_n + i + t \in m!\mathbb{N}$. Thus $y_n + i + t \in A$, by (3). But also $y_n \in n!\mathbb{N} \subseteq m!\mathbb{N}$ so $y_n + i + t \in m!\mathbb{N}$ so $x + i + t \in A \cap m!\mathbb{N}$ as required.

□

Corollary 3.1.8. *Let $\mathcal{G} \subseteq \mathcal{P}(\mathbb{N})$ such that $\mathcal{G}$ is closed under finite intersections and for every $A \in \mathcal{G}$, A is piecewise $\mathbb{T}$-syndetic. Then there exists $p \in \beta\mathbb{N}$ such that $\mathcal{G} \subseteq p$ and for every $B \in p$, B is piecewise $\mathbb{T}$-syndetic.*

Proof. This follows immediately from Lemma 3.1.7. □

Lemma 3.1.9. *[[22], Lemma 2.2]*

$$K(\mathbb{T}) \subseteq \{p \in \beta\mathbb{N} : \textit{for all } A \in p,\ A \textit{ is piecewise } \mathbb{T}\textit{-syndetic}\}.$$

Proof. Let $p \in K(\mathbb{T})$ and let $A \in p$. Then given $m \in \mathbb{N}$, $A \cap m\mathbb{N} \in p$, since $p \in \mathbb{T}$. Let

$$F_m = \{x \in \mathbb{N} : -x + (A \cap m\mathbb{N}) \in p\}.$$

Since $p \in K(\beta\mathbb{N})$, F_m is syndetic. Therefore, pick $l_m \in \mathbb{N}$ such that

$$\mathbb{N} = \bigcup_{t=1}^{l_m}(-t + F_m).$$

For each $y \in \mathbb{N}$ and $m \in \mathbb{N}$ pick $t(y,m) \in \{1, 2, \cdots, l_m\}$ such that $y + t(y,m) \in F_m$. Now let $n \in \mathbb{N}$. Then for any $y, m \in \{1, 2, \cdots, n\}$, $-(y + t(y,m)) + (A \cap m\mathbb{N}) \in p$. Pick

$$x \in \bigcap_{y=1}^{n} \bigcap_{m=1}^{n} -(y + t(y,m)) + (A \cap m\mathbb{N}).$$

Let $m \leq n$ be given. To see that $\{x+1, \cdots, x+n\} \subseteq \bigcap_{t=1}^{l_m}$, let $y \in \{1, 2, \cdots, n\}$. Then $x + y + t(y,m) \in A \cap m\mathbb{N}$. So $x + y \in (-t(y,m) + (A \cap m\mathbb{N})) \subseteq \bigcup_{t=1}^{l_m}(-t + (A \cap m\mathbb{N}))$. □

Theorem 3.1.10. *[[22], Theorem 2.3]*
$c\ell_{\mathbb{T}}\, K(\mathbb{T}) = \{p \in \beta\mathbb{N} : \text{for all } A \in p,\ A \text{ is piecewise } \mathbb{T}\text{-syndetic}\}$.

Proof. Let $\mathcal{H} = \{p \in \beta\mathbb{N} : \text{for all } A \in p,\ A \text{ is piecewise } \mathbb{T}\text{-syndetic}\}$. Let $q \in \beta\mathbb{N} \setminus \mathcal{H}$ and let $B \in q$ such that B is not piecewise $\mathbb{T}$-syndetic. Then $\overline{B} \cap \mathcal{H} = \emptyset$. Therefore $q \in \overline{B} \subseteq \beta\mathbb{N} \setminus \mathcal{H}$. It follows that $\mathcal{H}$ is closed. Since $K(\mathbb{T}) \subseteq \mathcal{H}$, $c\ell\, K(\mathbb{T}) \subseteq c\ell\, \mathcal{H} = \mathcal{H}$.

For the other inclusion, let $p \in \mathcal{H}$ and let $A \in p$. We would like to show that $\overline{A} \cap K(\mathbb{T}) = \emptyset$. Choose a sequence $(l_m)_{\mathbb{N}}$ in $\mathbb{N}$ such that for each $n \in \mathbb{N}$ there exists $x \in \mathbb{N}$ such that whenever $m \in \mathbb{N}$ with $m \leq n$,

$$\{x+1, \cdots, x+n\} \subseteq \bigcup_{t=1}^{l_m} (-t + (A \cap m\mathbb{N})).$$

For each $n \in \mathbb{N}$ choose $x_n \in \mathbb{N}$ such that whenever $m \in \mathbb{N}$ with $m \leq n$,

$$\{x_n+1, \cdots, x_n+n\} \subseteq \bigcup_{t=1}^{l_m} (-t + (A \cap m\mathbb{N})).$$

Let $D = \{x_n+i : i, n \in \mathbb{N} \text{ with } 2i \leq n\}$ and let $E = \{x_n+i : i, n \in \mathbb{N} \text{ with } i \leq n\}$. We observe that for every $m \in \mathbb{N}$, $\{x_{2m}+1, \cdots, x_{2m}+m\} \subseteq D$ and, therefore, $m\mathbb{N} \cap D \neq \emptyset$. In addition, $m_1\mathbb{N} \cap \cdots \cap m_k\mathbb{N} = m_1 \cdots m_k\mathbb{N}$ for every $m_1, m_2, \cdots, m_k \in \mathbb{N}$. Therefore the family $D \cup \{m\mathbb{N} : m \in \mathbb{N}\}$ has the finite intersection property. Pick $q \in \beta\mathbb{N}$ such that $D \cup \{m\mathbb{N} : m \in \mathbb{N}\} \subseteq q$. Then $q \in \mathbb{T}$.

Next we observe that $q + \beta\mathbb{N} \subseteq \overline{E}$. Indeed, let $r \in \beta\mathbb{N}$. To see that $E \in q + r$, we show that $\{x \in \mathbb{N} : -x + E \in q\} = \mathbb{N}$. Let $m \in \mathbb{N}$. To see that $-m + E \in q$, we show that $\{x_n + i : i, n \in \mathbb{N}, 2i \leq n, \text{ and } 2m \leq n\} \subseteq -m + E$. (Since all but finitely members of D are in $-m + E$, we then have $-m + E \in q$.) Let $i, n \in \mathbb{N}$ with $2i \leq n$ and $2m \leq n$. Then $i + m \leq n$ so $x_n + i + m \in E$ as required. Now $q + \mathbb{T}$ is a right ideal of $\mathbb{T}$ so $(q + \mathbb{T}) \cap K(\mathbb{T}) \neq \emptyset$. Pick $r \in (q + \mathbb{T}) \cap K(\mathbb{T})$. Since $q + \mathbb{T} \subseteq \overline{E}$, $E \in r$. We claim that $\{\{x \in \mathbb{N} : -x + A \in r\}\} \cup \{m\mathbb{N} : m \in \mathbb{N}\}$ has the finite intersection property. As before, it suffices to let $m \in \mathbb{N}$ and show that $\{x \in \mathbb{N} : -x+A \in r\} \cap m\mathbb{N} \neq \emptyset$. Now $E \in r$ so, since it includes all but finitely members of E, $\{x_n+i : i, n \in \mathbb{N} \text{ and } i \leq n \text{ and } m \leq n\} \in r$. By the choice of the $x_n' s$, $\{x_n+i : i, n \in \mathbb{N} \text{ and } i \leq n \text{ and } m \leq n\} \subseteq \bigcup_{t=1}^{l_m}(-t+(A \cap m\mathbb{N}))$, so $\bigcup_{t=1}^{l_m}(-t+(A \cap m\mathbb{N})) \in r$. Pick $t \in \{1, 2, \cdots, l_m\}$ such that $-t+(A \cap m\mathbb{N}) \in r$. Now $r \in \mathbb{T}$ so $m\mathbb{N} \in r$ so pick $x \in (-t+(A \cap m\mathbb{N})) \cap m\mathbb{N}$. Then $x \in m\mathbb{N}$ and $x+t \in m\mathbb{N}$ so $t \in m\mathbb{N}$. Also $-t+A \in r$. Thus $\{x \in \mathbb{N} : -x+A \in r\} \cap m\mathbb{N} \neq \emptyset$.

Pick $s \in \beta\mathbb{N}$ such that $\{\{x \in \mathbb{N} : -x + A \in r\}\} \cup \{m\mathbb{N} : m \in \mathbb{N}\} \subseteq s$. Then $s \in \mathbb{T}$ and $A \in r+s$. Since $r \in K(\mathbb{T})$, $r+s \in K(\mathbb{T})$ and hence $r+s \in K(\mathbb{T}) \cap \overline{A}$ as required. □

Corollary 3.1.11. *Let $\mathcal{G} \subseteq \mathcal{P}(\mathbb{N})$ such that $\mathcal{G}$ is closed under finite intersections and for every $A \in \mathcal{G}$, A is piecewise $\mathbb{T}$-syndetic. Then there exists $p \in c\ell K(\mathbb{T})$ such that $\mathcal{G} \subseteq p$.*

Proof. This follows immediately from Corollary 3.1.8 and Theorem 3.1.10. □

Theorem 3.1.12. *[[22], Theorem 2.8] $c\ell_{\mathbb{T}}\, K(\mathbb{T})$ is not a left ideal of $\mathbb{T}$.*

Proof. Take any bijection $\tau : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ and define a doubly indexed sequence $(u_{(a,b)})_{(a,b)\in\mathbb{N}\times\mathbb{N}}$ recursively as follows.

$$u_{(\tau(n))} = 1 + \max\{u_{(a,b)} + a! + b + a \ : \ (a, b) = \tau(k) \text{ for some } k < n\}.$$

Then $(u_{(a,b)})_{(a,b)\in\mathbb{N}\times\mathbb{N}}$ is a one-to-one sequence and for each $(a, b) \in \mathbb{N} \times \mathbb{N}$ we have

$$u_{(a,b)} + a! + b + a < \min\{u(c, d) \ : \ u(c, d) > u(a, b)\}.$$

For each $a \in \mathbb{N}$ let

$$E_a = \bigcup_{b\in\mathbb{N}} \{u(a, b) + 1, U(a, b) + 2, \cdots, u(a, b) + b\} \cap a!\mathbb{N}$$

and for each $k \in \mathbb{N}$ let

$$A_k = \bigcup_{a\geq k} E_a.$$

Since $A_k \supseteq A_{k+1}$ for every $k \in \mathbb{N}$, $\{A_k \ : \ k \in \mathbb{N}\}$ is closed under finite intersections. We claim that for every $k \in \mathbb{N}$, A_k is not piecewise $\mathbb{T}$-syndetic. Let $k \in \mathbb{N}$ be given. Define a sequence $(l_m)_{\mathbb{N}}$ in $\mathbb{N}$ as follows.

$$l_m = \begin{cases} m! & \text{if } m > k \\ k! & \text{if } m \leq k. \end{cases}$$

Let $n \in \mathbb{N}$ be given and let $d = \max\{n, k\}$. Let $x = u_{(k,d!+d)}$ and let $m \leq n$ be given. We would like to show that $\{x+1, \cdots, x+n\} \subseteq \bigcup_{t=1}^{l_m}(-t+(A_k \cap m\mathbb{N}))$. We look at two cases.

Case 1: $m \leq k$. Pick $t \in \{1, 2, \cdots, k!\}$ such that $x + i + t \in k!\mathbb{N}$. Then $x + i + t \in m\mathbb{N}$. Also $i + t \leq n + k! \leq d + d!$ so $x + i + t \in \{u_{(k,d+d!)} + 1, u_{(k,d+d!)} + 2, \cdots, u_{(k,d+d!)} + d + d!\} \cap k!\mathbb{N}$. Thus $x + i + t \in m\mathbb{N}$ as required.

Case 2: $m > k$. Pick $t \in \{1, 2, \cdots, m!\}$ such that $x + i + t \in m!\mathbb{N}$. Then $x + i + t \in m\mathbb{N} \cap k!\mathbb{N}$. Also $i + t \leq n + m! \leq d + d!$ so $x + i + t \in A_k \cap m\mathbb{N}$ as required.

Therefore we can pick $p \in c\ell\ K(\mathbb{T})$ such that $\{A_k : k \in \mathbb{N}\} \subseteq p$. Pick $q \in \mathbb{T}$ such that $\{k! : k \in \mathbb{N}\} \in q$. Let $B = \bigcup_{k \in \mathbb{N}}(-(k-1)! + A_k)$. Then $B \in q+p$. We show that B is not piecewise $\mathbb{T}$-syndetic, so that $q+p \notin c\ell\ K(\mathbb{T})$. Suppose instead that B is piecewise $\mathbb{T}$-syndetic. Pick a sequence $(l_m)_{\mathbb{N}}$ in $\mathbb{N}$ such that for each $n \in \mathbb{N}$ there exists $x \in n!\mathbb{N}$ such that whenever $m \in \mathbb{N}$ with $m \leq n$, then $\{x+1, \cdots, x+n\} \subseteq \bigcup_{t=1}^{l_m}(-t+(B \cap m!\mathbb{N}))$. Pick $m > l_1$ and pick $\alpha \in \mathbb{N}$ such that $\alpha! - (\alpha - 1)! > l_1$. Let $n = l_m + (\alpha - 1)l_1 + m$. Pick $x \in n!\mathbb{N}$ such that, for each $r \in \mathbb{N}$ with $r \leq n$, $\{x + 1, \cdots, x + n\} \subseteq \bigcup_{t=1}^{l_r}(-t + (B \cap r!\mathbb{N}))$. Pick $t_0 \in \mathbb{N}$ with $t_0 \leq l_m$ such that $x + 1 + t_0 \in B \cap m!\mathbb{N}$. Recursively given $i \in \{1, 2, \cdots, \alpha\}$, $t_0 \in \{1, 2, \cdots, l_m\}$, and $\{t_j : j \in \{1, 2, \cdots, i - 1\}\} \subseteq \{1, 2, \cdots, l_1\}$, we have

$$x + 1 + \sum_{j=0}^{i-1} t_j \leq x + 1 + l_m + (\alpha - 1)l_1 \leq x + n.$$

Therefore pick $t_i \in \{1, 2, \cdots, l_1\}$ such that $x + 1 + \sum_{j=0}^{i} t_j \in B$. For each $i \in \{0, 1, \cdots, \alpha\}$ pick $k_i \in \mathbb{N}$ such that $x + 1 + \sum_{j=0}^{i} t_j \in A_{k_i} + (k_i - 1)!$ and, since $x + 1 + \sum_{j=0}^{i} t_j - (k_i - 1)! \in A_{k_i}$, pick $a_i \geq k_i$ and $b_i \in \mathbb{N}$ such that

$$x + 1 + \sum_{j=0}^{i} t_j - (k_i - 1)! \in \{u_{(a_i,b_i)} + 1, u_{(a_i,b_i)} + 2, \cdots, u_{(a_i,b_i)} + b_i\} \cap a!\mathbb{N}.$$

We show first that $a_0 > m$. Suppose instead $a_0 \leq m$. Then since $x+1+t_0 \in m!\mathbb{N}$, $a_0!$ divides $x + 1 + t_0$. But also $x + 1 + t_0 - (k_0 - 1)! \in a_0!\mathbb{N}$ so that $a_0!$ divides $(k - 1)!$ while $a_0 \geq k_0$, a contradiction.
We now show that each $i \in \{1, 2, \cdots, \alpha\}$, $(a_i, b_i) = (a_0, b_0)$. Suppose not and pick the first i such that $(a_i, b_i) \neq (a_0, b_0)$. Since $(u_{(a,b)})_{(a,b) \in \mathbb{N} \times \mathbb{N}}$ is a one-to-one sequence, $u_{(a_i,b_i)} \neq u_{(a_0,b_0)}$.

Case 1: $u_{(a_i,b_i)} < u_{(a_0,b_0)}$. Then

$$\begin{aligned}
u_{(a_0,b_0)} &> u_{(a_i,b_i)} + a_i! + b_i + a_i > u_{(a_i,b_i)} + b_i + (k_i-1)! \\
&\geq \left(x + 1 + \sum_{j=0}^{i} t_j - (k_i - 1)! \right) + (k_i-1)! \\
&= x + 1 + \sum_{j=0}^{i} t_j > x + 1 + t_0 > x + i + t_0 - (k_0-1)! > u_{(a_0,b_0)},
\end{aligned}$$

a contradiction.

Case 2: $u_{(a_i,b_i)} > u_{(a_0,b_0)}$. Note that $a_0 > m > l_1$. Thus

$$\begin{aligned}
u_{(a_i,b_i)} &> u_{(a_0,b_0)} + a_0! + b_0 + a_0 > u_{(a_0,b_0)} + b_0 + a_0! + l_1 \\
&= u_{(a_{i-1},b_{i-1})} + b_{i-1} + l_1 + a_{i-1}! \\
&> \left(x + 1 + \sum_{j=0}^{i-1} t_j - (k_{i-1}-1)! \right) + t_i + (k_{i-1}-1)! \\
&= x + 1 + \sum_{j=0}^{i} t_j \\
&> x + 1 + \sum_{j=0}^{i} t_j - (k_j - 1)! \\
&> u_{(a_i,b_i)},
\end{aligned}$$

a contradiction.

Let $(a,b) = (a_0,b_0)$, so that for all $i \in \{0,1,\cdots,\alpha\}$, $(a,b) = (a_i,b_i)$. We have established that $a > m$. Let i be any member of $\{1,2,\cdots,\alpha\}$. Then $a!$ divides $x+1+\sum_{j=0}^{i} t_j - (k_i-1)!$ and $a!$ divides $x+1+\sum_{j=0}^{i-1} t_j - (k_{i-1}-1)!$. Therefore, $a!$ divides $t_i - (k_i-1)! + (k_{i-1}-1)!$. Pick an integer d such that $da! = t_i - (k_i-1)! + (k_{i-1}-1)!$. Then $da! < t_i + (k_{i-1}-1)! \leq t_i + (a-1)! \leq l_1 + (a_i-1)! < m + (a-1)! < 2(a-1)!$. Thus $(a-1)!(da-2) < 0$ so $da < 2$. Since $a > m > l_1$, $a > 2$ so $d \leq 0$. But also $da! = t_i - (k_i-1)! + (k_{i-1}-1)! \geq -(k_i-1)! \geq -(a-1)!$. Thus $(a-1)!(da+1) > 0$ so $da+1 > 0$ so $d \geq 0$ and hence $d = 0$.
Consequently, for each $i \in \{1,2,\ldots,\alpha\}$, $t_i = (k_i-1)! - (k_{i-1}-1)!$. Since $t_i \geq 1$, $k_i \geq k_{i-1}$. Since $k_0 \geq 1$, $k_i \geq i+1$. Thus $t_\alpha = (k_\alpha-1)! - (k_{\alpha-1}-1)! \geq (k_\alpha-1)! - (k_\alpha-2)! \geq \alpha! - (\alpha-1)! > l_1 \geq t_\alpha$, a contradiction.

□

Corollary 3.1.13.
$K(\mathbb{T})$ is not closed. In particular, $c\ell_{\mathbb{T}}\ K(\mathbb{T}) \neq (c\ell_{\beta\mathbb{N}}\ K(\beta\mathbb{N})) \cap \mathbb{T}$.

Proof. Since $c\ell_{\mathbb{T}}\ K(\mathbb{T})$ is not an ideal of $\mathbb{T}$, $c\ell_{\mathbb{T}}\ K(\mathbb{T}) \neq K(\mathbb{T})$. In particular $K(\mathbb{T})$ is not closed. Since $c\ell_{\beta\mathbb{N}}\ K(\beta\mathbb{N})$ is an ideal of $\beta\mathbb{N}$, by Lemma 3.1.2 $c\ell_{\beta\mathbb{N}}\ K(\beta\mathbb{N}) \cap \mathbb{T}$ is an ideal of $\mathbb{T}$. It follows that $c\ell_{\mathbb{T}}\ K(\mathbb{T}) \neq (c\ell_{\beta\mathbb{N}}\ K(\beta\mathbb{N})) \cap \mathbb{T}$. □

3.2 Combinatorial characterizations of $K(0^+)$ and its closure

In this section we present the extension of the combinatorial characterizations of $K(\beta S)$ and its closure to $K(0^+)$ and its closure. Most of the materials in this section come from [25].

Since 0^+ is a compact right topological semigroup, $K(0^+)$ exists. However, since 0^+ is not an ideal of $\beta\mathbb{R}$, a description of $K(0^+)$ does not follow from known results about arbitrary discrete semigroups. We would like to present the characterization of the members of $K(0^+)$ and its closure.

Definition 3.2.1. Let S be a dense subsemigroup of $(0,\infty)$. A subset $A \subseteq S$ is said to be *syndetic near* 0 if and only if for every $\epsilon > 0$ there exist some finite $F \subseteq (0,\epsilon) \cap S$ and some $\delta > 0$ such that $S \cap (0,\delta) \subseteq \bigcup_{t\in F}(-t + A)$.

The following theorem characterizes $K(0^+)$.

Theorem 3.2.2. *[[25], Theorem 3.3] Let S be a dense subsemigroup of $(0,\infty)$ and let $p \in 0^+$. The following statements are equivalent.*

(a) $p \in K(0^+)$.

(b) *For all $A \in p$, $\{x \in S : -x + A \in p\}$ is syndetic near 0.*

(c) *For all $r \in 0^+$, $p \in 0^+ + r + p$.*

Proof. $(a) \Rightarrow (b)$. Suppose $p \in K(0^+)$, let $A \in p$, let $B = \{x \in S : -x + A \in p\}$ and suppose B is not syndetic near 0. Pick $\epsilon > 0$ such that for all $F \in \mathcal{P}_f((0,\epsilon) \cap S)$ and all $\delta > 0$, $(S \cap (0,\delta)) \cap (S \setminus F^{-1}B) \neq \emptyset$.
Let $\mathcal{G} = \{(S \cap (0,\delta)) \cap (S \setminus F^{-1}B) \ : \ F \in \mathcal{P}_f((0,\epsilon) \cap S)$ and $\delta > 0\}$.

Then $\mathcal{G}$ has the finite intersection property. Pick $r \in \beta S$ with $\mathcal{G} \subseteq r$. Since $\{S \cap (0,\delta) : \delta > 0\} \subseteq r$ we have $r \in 0^+$. Pick a minimal left ideal L of 0^+ with $L \subseteq 0^+ + r$. Since $K(0^+)$ is the union of all the minimal right ideals of 0^+, pick a minimal right ideal R of 0^+ with $p \in R$. Then $L \cap R$ is a group. Let q be the identity of $L \cap R$. Then $R = q + 0^+$, so $p \in q + 0^+$ so $p = q + p$ and therefore $B \in q$. Also $q \in 0^+ + r$ so pick $w \in 0^+$ such that $q = w + r$. Then $(0,\epsilon) \cap S \in w$ and $\{t \in S : -t + B \in r\} \in w$ so pick $t \in (0,\epsilon) \cap S$ such that $-t + B \in r$. But $(S \cap (0,1)) \setminus (-t + B) \in \mathcal{G} \subseteq r$, a contradiction.

$(b) \Rightarrow (c)$. Let $r \in 0^+$. For each $A \in p$, let $B(A) = \{x \in S : -x + A \in p\}$ and let $C(A) = \{t \in S : -t + B(A) \in r\}$. It is easy to see that for any $A_1, A_2 \in p$, one has $B(A_1 \cap A_2) = B(A_1) \cap B(A_2)$ and $C(A_1 \cap A_2) = C(A_1) \cap C(A_2)$. We claim that for every $A \in p$ and every $\epsilon > 0$, $C(A) \cap (0,\epsilon) \neq \emptyset$. To see this, let $A \in p$ and let $\epsilon > 0$ be given and pick $F \in \mathcal{P}_f((0,\epsilon) \cap S)$ and $\delta > 0$ such that $(0,\delta) \cap S \subseteq F^{-1}B(A)$. Since $(0,\delta) \cap S \in r$ we have $F^{-1}B(A) \in r$ and hence there is some $t \in F$ with $-t + B(A) \in r$. Then $t \in C(A) \cap (0,\epsilon)$ and, therefore, $C(A) \cap (0,\epsilon) \neq \emptyset$.
Thus $\{(0,\epsilon) \cap C(A) : \epsilon > 0 \text{ and } A \in p\}$ has the finite intersection property so pick $q \in \beta S$ with $\{(0,\epsilon) \cap C(A) : \epsilon > 0 \text{ and } A \in p\} \subseteq q$. Then $q \in 0^+$. We claim that $p = q + r + p$. It suffices to show that $p \subseteq q + r + p$. Let $A \in p$ be given. Then $\{t \in S : -t + B(A) \in r\} = C(A) \in q$ so $B(A) \in q + r$, so $A \in q + r + p$ as required.

$(c) \Rightarrow (a)$. Pick $r \in K(0^+)$. □

Definition 3.2.3. Let S be a dense subsemigroup of $(0,\infty)$. A subset $A \subseteq S$ is said to be *piecewise syndetic near* 0 if and only if there exist sequences $(F_n)_{n\in\mathbb{N}}$ and $(\delta_n)_{\mathbb{N}}$ such that

(a) For each $n \in \mathbb{N}$, F_n is a finite subset of $(0,\frac{1}{n}) \cap S$ and $0 < \delta_n < \frac{1}{n}$ and

(b) for all finite $G \subseteq S$ and all $\mu > 0$ there is some $x \in (0,\mu) \cap S$ such that for all $n \in \mathbb{N}$, $(G \cap (0,\delta_n)) + x \subseteq \bigcup_{t\in F_n}(-t + A)$.

Theorem 3.2.4. *[[25], Theorem 3.5] Let S be a dense subsemigroup of $(0,\infty)$ and let $A \subseteq S$. The following statements are equivalent.*

(a) $K(0^+) \cap \overline{A} \neq \emptyset$.

(b) A is piecewise syndetic near 0.

Proof. (a) $\Rightarrow$ (b): Suppose $K(0^+) \cap \overline{A} \neq \emptyset$. Let $p \in K(0^+) \cap \overline{A}$ and let $B = \{x \in S : -x + A \in p\}$. Then B is syndetic near 0. Inductively for $n \in \mathbb{N}$ pick $F_n \in \mathcal{P}_f((0, \frac{1}{n}) \cap S)$ and $\delta_n \in (0, \frac{1}{n})$ (with $\delta_n \leq \delta_{n-1}$ if $n > 1$) such that $S \cap (0, \delta_n) \subseteq -F_n + B$.
Let $G \in \mathcal{P}_f(S)$ be given. If $G \cap (0, \delta_1) = \emptyset$, the conclusion is trivial, so assume $G \cap (0, \delta_1) \neq \emptyset$ and let $H = G \cap (0, \delta_1)$. For each $y \in H$ let $m(y) = \max\{n \in \mathbb{N} : y < \delta_n\}$. For each $n \in \{1, 2, \cdots, m(y)\}$, we have $y \in -F_n + B$ so pick $t(y, n) \in F_n$ such that $y \in -t(y, n) + B$. Then given $y \in H$ and $n \in \{1, 2, \cdots, m(y)\}$, one has $-(t(y, n) + y) + A \in p$. Now let $\mu > 0$ be given. Then $(0, \mu) \in p$ so pick

$$x \in (0, \mu) \cap \bigcap_{y \in H} \bigcap_{n=1}^{m(y)} \left(-(t(y, n) + y) + A\right).$$

Then given $n \in \mathbb{N}$ and $y \in G \cap (0, \delta_n)$, one has $y \in H$ and $n \leq m(y)$ so $t(y, n) + y + x \in A$ so

$$y + x \in -t(y, n) + A \subseteq -F_n + A$$

(b) $\Rightarrow$ (a): Suppose A is piecewise syndetic near 0. Pick sequences $(F_n)_{n \in \mathbb{N}}$ and $(\delta_n)_{\mathbb{N}}$ such that

1. For each $n \in \mathbb{N}$, F_n is a finite subset of $(0, \frac{1}{n}) \cap S$ and $0 < \delta_n < \frac{1}{n}$ and
2. for all finite $G \subseteq S$ and all $\mu > 0$ there is some $x \in (0, \mu) \cap S$ such that for all $n \in \mathbb{N}$, $(G \cap (0, \delta_n)) + x \subseteq -F_n + A$.

Given $G \in \mathcal{P}_f(S)$ and $\mu > 0$, let

$$C(G, \mu) = \{x \in (0, \mu) \cap S : \text{for all } n \in \mathbb{N}, (G \cap (0, \delta_n)) + x \subseteq -F_n + A\}.$$

By assumption each $C(G, \mu) \neq \emptyset$. Further, given G_1 and G_2 in $\mathcal{P}_f(S)$ and $\mu_1, \mu_2 > 0$, one has

$$C(G_1 \cup G_2, \min\{\mu_1, \mu_2\}) \subseteq C(G_1, \mu_1) \cap C(G_2, \mu_2).$$

Therefore, $\{C(G, \mu) : G \in \mathcal{P}_f(S) \text{ and } \mu > 0\}$ has the finite intersection property. Pick $p \in \beta S$ with $\{C(G, \mu) : G \in \mathcal{P}_f(S) \text{ and } \mu > 0\} \subseteq p$. Since $C(G, \mu) \subseteq (0, \mu)$, one has $p \in 0^+$. We claim that

$$0^+ + p \subseteq \bigcap_{n \in \mathbb{N}} \overline{(-F_n + A)}.$$

Let $n \in \mathbb{N}$ and let $q \in 0^+$. To show that $-F_n + A \in q + p$, we show that

$$(0, \delta_n) \cap S \subseteq \{y \in S : -y + (-F_n + A) \in p\}.$$

So let $y \in (0, \delta_n) \cap S$. Then $C(\{y\}, \delta_n) \in p$ and $C(\{y\}, \delta_n) \subseteq -y + (-F_n + A)$. Now pick $r \in (0^+ + p) \cap K$ (since $0^+ + p$ is a left ideal of 0^+). Given $n \in \mathbb{N}$, $-F_n + A \in r$ so pick $t_n \in F_n$ such that $-t_n + A \in r$. Now for each $n \in \mathbb{N}$, $t_n \in F_n \subseteq (0, \frac{1}{n})$ so $\lim_{n\to\infty} t_n = 0$ so pick $q \in 0^+ \cap \overline{\{t_n : n \in \mathbb{N}\}}$. Then $q + r \in K(0^+)$ and $\{t_n : n \in \mathbb{N}\} \subseteq \{t \in S : -t + A \in r\}$ so $A \in q + r$. $\square$

Therefore we have the following characterization of $c\ell K(0^+)$.

Corollary 3.2.5. *Let S be a dense subsemigroup of $(0, \infty)$ and let $p \in \beta S$. The following statements are equivalent.*

(a) $p \in c\ell K(0^+)$.

(b) Every $A \in p$ is piecewise syndetic near 0.

It follows that:
$c\ell_{0^+}\ K(0^+) = \{p \in \beta S : \text{for all } A \in p,\ A \text{ is piecewise syndetic near } 0\}$.

Definition 3.2.6. Let S be a dense subsemigroup of $(0, \infty)$.

(a) A subset $A \subseteq S$ is said to be *central near* 0 if and only if there is some idempotent $p \in K(0^+)$ with $A \in p$.

(b) A collection $\mathcal{A} \subseteq \mathcal{P}(S)$ is said to be *collectionwise piecewise syndetic* near 0 if and only if there exist functions

$$F : \mathcal{P}_f(\mathcal{A}) \to \prod_{n\in\mathbb{N}} ((0, \frac{1}{n}) \cap S)$$

and

$$\delta : \mathcal{P}_f(\mathcal{A}) \to \prod_{n\in\mathbb{N}} (0, \frac{1}{n})$$

such that for every $\mu > 0$, every $G \in \mathcal{P}_f(S)$, and every $\mathcal{H} \in \mathcal{P}_f(\mathcal{A})$, there is some $t \in (0, \mu) \cap S$ such that for every $n \in \mathbb{N}$ and every $\mathcal{F} \in \mathcal{P}_f(\mathcal{H})$,

$$(G \cap (0, \delta(\mathcal{F})_n)) + t \subseteq \bigcup_{x\in F(\mathcal{F})_n} (-x + \bigcap \mathcal{F}).$$

Theorem 3.2.7. *Let S be a dense subsemigroup of $(0, \infty)$ and let $\mathcal{A} \subseteq \mathcal{P}(S)$. There exists $p \in K(0^+)$ such that $\mathcal{A} \subseteq p$ if and only if $\mathcal{A}$ is collectionwise piecewise syndetic near* 0.

3.3 Generalization of combinatorial characterizations

At this stage we would like to generalize even further the definitions of syndetic and piecewise syndetic subsets of a semigroup. This time we will consider an arbitrary closed subsemigroup of βS. Recall that every closed subsemigroup T of βS is of the form $\overline{\mathcal{F}}$ for some filter $\mathcal{F}$ on S. The materials in this section and in the next section are the contents of [60].

Definition 3.3.1. Let S be a semigroup and let $\mathcal{F}$ and $\mathcal{G}$ be filters on S. A subset $A \subset S$ is $(\mathcal{F}, \mathcal{G})$-*syndetic* if for every $V \in \mathcal{F}$, there is a finite $F \subseteq V$ such that $F^{-1}A \in \mathcal{G}$. If $\mathcal{F} = \mathcal{G}$, we say $\mathcal{F}$-syndetic instead of $(\mathcal{F}, \mathcal{G})$-syndetic.

Remark 3.3.2. Let S be a semigroup, let $\mathcal{F}$ and $\mathcal{G}$ be filters on S and let $A \subseteq S$.

(a) If $\mathcal{F} \subseteq \mathcal{G}$ and A is either $\mathcal{F}$-syndetic or $\mathcal{G}$-syndetic, then A is $(\mathcal{F}, \mathcal{G})$-syndetic.

(b) A is syndetic if and only if A is $\{S\}$-syndetic. Hence, $(\mathcal{F}, \mathcal{G})$-syndetic is a generalization of syndetic.

Proof.

(a) Suppose $\mathcal{F} \subseteq \mathcal{G}$. Let $V \in \mathcal{F}$. Since $\mathcal{F} \subset \mathcal{G}$, $V \in \mathcal{G}$. If A is $\mathcal{F}$-syndetic, there exists a finite $F \subseteq V$ such that $F^{-1}A \in \mathcal{F} \subseteq \mathcal{G}$. If A is $\mathcal{G}$-syndetic, there exists a finite $E \subseteq V$ such that $E^{-1}A \in \mathcal{G}$. In both cases A is $(\mathcal{F}, \mathcal{G})$-syndetic.

(b) Suppose A is syndetic. Let $V \in \{S\}$. Then $V = S$. By assumption, there exists a finite $F \subseteq S$ such that $F^{-1}A = S \in \{S\}$. Therefore, A is $\{S\}$-syndetic. The converse is clear.

□

Lemma 3.3.3. *Let S be a semigroup, let T be a closed subsemigroup of βS, let L be a minimal left ideal of T, let $\mathcal{F}$ and $\mathcal{G}$ be filters on S such that $\overline{\mathcal{F}} = T$ and $\overline{\mathcal{G}} = L$, and let $A \subseteq S$. The following statements are equivalent.*

(1) $\overline{A} \cap L \neq \emptyset$.

(2) *A is $\mathcal{G}$-syndetic.*

(3) *A is $(\mathcal{F}, \mathcal{G})$-syndetic.*

Proof. (First note that since L is a nonempty closed subset of T (being a minimal left ideal), L is a nonempty closed subset of βS. In particular $\mathcal{G}$ such that $\overline{\mathcal{G}} = L$ exists.)

(1) $\Rightarrow$ (2): Suppose $\overline{A} \cap L \neq \emptyset$. Pick $p \in \overline{A} \cap L$. For every $q \in L$, one has $p \in L = Lq$. Thus for each $q \in L$, there exists $r_q \in L$ such that $p = r_q q$. Now to show that A is $\mathcal{G}$-syndetic, let $V \in \mathcal{G}$. Since $L = \overline{\mathcal{G}}$, $V \in r_q$ for all $q \in L$. Since $A \in p = r_q q$, $\{x \in S : x^{-1}A \in q\} \in r_q$ for each $q \in L$. Since r_q is a filter, $V \cap \{x \in S : x^{-1}A \in q\} \neq \emptyset$ for every $q \in L$. For each $q \in L$, choose $x_q \in V$ such that $x_q^{-1}A \in q$, i.e. such that $q \in \overline{x_q^{-1}A}$. Then the set $\{ \overline{x_q^{-1}A} : q \in L \}$ is an open cover of L. Since L is compact, there exists a finite $\{q_1, \cdots, q_n\} \subseteq L$ such that

$$L \subseteq \overline{x_{q_1}^{-1}A} \cup \cdots \cup \overline{x_{q_n}^{-1}A} = \overline{x_{q_1}^{-1}A \cup \cdots \cup x_{q_n}^{-1}A} = \overline{F^{-1}A},$$

where $F = \{x_{q_1}, \cdots, x_{q_n}\} \subseteq V$. $\overline{\mathcal{G}} \subseteq \overline{F^{-1}A}$ implies that $F^{-1}A \in \mathcal{G}$. It follows that A is $\mathcal{G}$-syndetic.
(2) $\Rightarrow$ (3): Suppose A is $\mathcal{G}$-syndetic. Since $\overline{\mathcal{G}} = L \subseteq T = \overline{\mathcal{F}}$, $\mathcal{F} \subseteq \mathcal{G}$. By Remark 3.3.2 (a), A is $(\mathcal{F}, \mathcal{G})$-syndetic.
(3) $\Rightarrow$ (1): Suppose A is $(\mathcal{F}, \mathcal{G})$-syndetic. Pick $q \in L$. For every $V \in \mathcal{F}$, take a finite $F = \{x_1, \cdots, x_n\} \subseteq V$ such that $F^{-1}A \in \mathcal{G} \subseteq q$. Then, $x_1^{-1}A \cup \cdots \cup x_n^{-1}A = F^{-1}A \in q$. Since q is an ultrafilter, $x_i^{-1}A \in q$ for some $1 \leq i \leq n$. Therefore, for every $V \in \mathcal{F}$ choose $x_V \in V$ such that $A \in x_V q$. Pick $r \in T \cap c\ell_{\beta S}\{x_V : V \in \mathcal{G}\}$. Since $\{t \in S : t^{-1}A \in q\} \supseteq \{x_V : V \in \mathcal{G}\} \in r$, $\{t \in S : t^{-1}A \in q\} \in r$. Therefore $A \in rq$. Since L is a left ideal of T, $rq \in L$. It follows that $\overline{A} \cap L \neq \emptyset$. $\square$

The next theorem characterizes ultrafilters from $K(T)$, where T is a closed subsemigroup of βS.

Theorem 3.3.4. *Let S be a semigroup, let T be a closed subsemigroup of βS, let $\mathcal{F}$ be a filter on S such that $\overline{\mathcal{F}} = T$, and let $p \in T$. Then $p \in K(T)$ if and only if for every $A \in p$, $\{x \in S : x^{-1}A \in p\}$ is $\mathcal{F}$-syndetic.*

Proof. Suppose that $p \in K(T)$. Let $L = Tp$. Then L is a left ideal of T and, since T is a subsemigroup of S, L is a left ideal of S. Since $K(T) \subseteq Tp$, $p \in Tp$. Let $\mathcal{G}$ be a filter on S such that is $L = \overline{\mathcal{G}}$. Now let $A \in p$, let $B = \{x \in S : x^{-1}A \in p\}$ and let $V \in \mathcal{F}$. Then $p \in \overline{A} \cap L$. It follows that

A is $\mathcal{G}$-syndetic. Pick a finite subset $F \subseteq V$ such that $F^{-1}A \in \mathcal{G}$. Since $L = Tp$, there is $W \in \mathcal{F}$ such that $Wp \subseteq \overline{f^{-1}A}$. We claim that $W \subseteq F^{-1}B$. Indeed, let y W. Then there is $x \in F$ such that $yp \in \overline{x^{-1}A}$. It follows that $(xy)^{-1}A \in p$. Hence, $xy \in B$, and then $y \in x^{-1}B$.
Conversely, suppose that $p \notin K(T)$. Pick $q \in K(T)$. Then $Tqp \subseteq K(T)$ and, therefore, $p \notin Tqp$. It follows that there is $A \in p$ such that $\overline{A} \cap Tqp = \emptyset$. Now let $B = \{x \in S : x^{-1}A \in p\}$. We claim that B is not $\mathcal{F}$-syndetic. To show this, pick a minimal left ideal L of T contained in Tp and let $\mathcal{G}$ be the filter on S such that $\overline{\mathcal{G}} = L$. Assume on the contrary that B is $\mathcal{F}$-syndetic. Then B is also $(\mathcal{F}, \mathcal{G})$-syndetic. Hence by Lemma 3.3.3, $\overline{B} \cap L \neq \emptyset$. Consequently, $B \in rq$ for some $r \in T$, and so $\{x \in S : x^{-1}A \in p\} \in rq$. But then $A \in rqp$, which is a contradiction. □

Let S be a dense subsemigroup of $((0, \infty), +)$, let $\mathcal{F}$ be the filter on S with $\overline{\mathcal{F}} = 0^+$ and let $A \subseteq S$ (equivalently let $(\mathbb{R}, \mathcal{T})$ be the Sorgenfrey topological space, let $\mathcal{F}$ denote the neighborhood filter at 0 and let $A \subseteq \mathbb{R}$.). It follows form Theorem 3.2.2 and Theorem 3.3.4 that A is syndetic near 0 if and only if A is $\mathcal{F}$-syndetic.

Now we characterize ultrafilters from $c\ell\ K(T)$.

Definition 3.3.5. Let S be a semigroup and let $\mathcal{F}$ be a filter on S. A subset $A \subseteq S$ is *piecewise $\mathcal{F}$-syndetic* if for every $V \in \mathcal{F}$, there is a finite $F_V \subseteq V$ and $W_V \in \mathcal{F}$ such that the family

$$\{(x^{-1}F_V^{-1}A) \cap V \ : \ V \in \mathcal{F}, x \in W_V\}$$

has the finite intersection property.

Remark 3.3.6.

1. Let S be a semigroups and let $A \subseteq S$. A is piecewise syndetic if and only if A is piecewise $\{S\}$-syndetic. Therefore piecewise $\mathcal{F}$-syndetic is a generalization of piecewise syndetic.

2. The family $\{(x^{-1}F_V^{-1}A) \cap V \ : \ V \in \mathcal{F}, x \in W_V\}$ has the finite intersection property if and only if whenever $V_1, \cdots, V_n \in \mathcal{F}$ and $H_1, \cdots, H_n$ are finite subsets of $W_{V_1}, \cdots, W_{V_n}$, respectively, there is $y \in V_1 \cap \cdots \cap V_n$ such that $H_i y \subseteq F_{V_i}^{-1}A$ for each $i = 1, \cdots, n$.

Proof.

1. This is clear from the definitions.

2. The necessity is easy to see, by taking each H_i to be a singleton. Now to prove the sufficiency. Note that we may assume that each H_i is nonempty. For each $1 \leq i \leq i$, pick $x_i \in H_i \subset W_{V_i}$. Pick
$$y \in \bigcap_{i=1}^{n} (x_i^{-1} F_{V_i}^{-1} A \cap V_i).$$
Then $x_i y \in F_{V_i}^{-1} A$ for every $i \in \{1, 2, \cdots, n\}$. Since x_i was an arbitrary element of H_i, we conclude that $H_i y \subseteq F_{V_i}^{-1} A$.

□

Theorem 3.3.7. *Let T be a closed subsemigroup of βS, let $\mathcal{F}$ be a filter on S such that $T = \overline{\mathcal{F}}$, and let $A \subseteq S$. Then $\overline{A} \cap K(T) \neq \emptyset$ if and only if A is piecewise $\mathcal{F}$-syndetic.*

Proof. Suppose $\overline{A} \cap K(T) \neq \emptyset$. Pick $p \in \overline{A} \cap K(T) \neq \emptyset$. Let $L = Tp$ and let $\mathcal{G}$ denote the filter on S such that $\overline{\mathcal{G}} = L$. Then by Lemma 3.3.3, A is $(\mathcal{F}, \mathcal{G})$-syndetic. Consequently, for every $V \in \mathcal{F}$, there is a finite $F_V \subseteq V$ such that $F_V^{-1} A \in \mathcal{G}$. Since $L = Tp$, it follows that for every $V \in \mathcal{F}$, there is $W_V \in \mathcal{F}$ such that $W_V p \subseteq \overline{F_V^{-1} A}$, and so $x^{-1} F_V^{-1} A \in p$ for all $x \in W_V$. Hence, the family $\{(x^{-1} F_V^{-1} A) \cap V : V \in \mathcal{F}, x \in W_V\}$ has the finite intersection property. It follows that A is piecewise $\mathcal{F}$-syndetic.
Conversely, suppose that A is piecewise $\mathcal{F}$-syndetic. Then for every $V \in \mathcal{F}$,there is a finite $F_V \subseteq V$ and $W_V \in \mathcal{F}$ such that $\{(x^{-1} F_V^{-1} A) \cap V : V \in \mathcal{F}, x \in W_V\}$ has the finite intersection property. Pick an ultrafilter q on S. extending this family. Then $q \in T$ and for every $V \in \mathcal{F}$ and $x \in W_V$, $x_q \in \overline{F_V^{-1} A}$. It follows that for every $V \in \mathcal{F}$, $Tq \subseteq \overline{F_V^{-1} A}$. Let L be the minimal left ideal of T contained in Tq and let $\mathcal{G}$ denote the filter on S such that $\overline{\mathcal{G}} = L$. Then for every $V \in \mathcal{F}$, one has $F_V^{-1} A \in \mathcal{G}$. Hence, A is $(\mathcal{F}, \mathcal{G})$-syndetic. Therefore, by Lemma 3.3.3, $\overline{A} \cap L \neq \emptyset$.

□

Corollary 3.3.8. *Let T be a closed subsemigroup of βS, let $\mathcal{F}$ be a filter on S such that $T = \overline{\mathcal{F}}$, and let $p \in T$. Then $p \in c\ell\ K(T)$ if and only if every $A \in p$ is piecewise $\mathcal{F}$-syndetic.*

From Theorem 3.1.10, Theorem 3.2.5 and Theorem 3.3.7 we can see that "piecewise $\mathbb{T}$-syndetic" and "piecewise syndetic near 0" are examples of "piecewise $\mathcal{F}$- syndetic".

3.4 $c\ell\ K(T)$ is not an ideal

We have seen that in a right topological semigroup the closure of a right ideal is a right ideal, while the closure of a left ideal is not necessarily a left ideal. In this section we present our main result: $c\ell\ K(Ult(\mathcal{T}))$ is not a left ideal for any nondiscrete regular left invariant topology $\mathcal{T}$ on an arbitrary infinite group G satisfying some extra condition as stated in Lemma 3.4.1 below. So, throughout this section G is an arbitrary infinite group.

Recall the following, where X is a topological space.

(1) Let $x \in X$. The *character of* x, denoted $\chi_x(X)$, is the minimum of cardinalities of neighborhood bases at x.

(2) The *character of* X, denoted $\chi(X)$, is defined as

$$\chi(X) = \sup\{\chi_x(X) : x \in X\}.$$

It is easy to see that for a left topological group $(G, \mathcal{T})$ and any $x \in G$, $\chi_x(G) = \chi_e(G) = \chi(G)$, where e is the identity of G.

Lemma 3.4.1. *Let $\mathcal{T}$ be a left invariant topology on G and let κ denote the character of $\mathcal{T}$. Then the following conditions are equivalent:*

(1) there is a neighborhood base $\{W_\alpha : \alpha < \kappa\}$ at the identity of $(G, \mathcal{T})$ such that $W_\alpha \subseteq W_\gamma$ if $\gamma < \alpha < \kappa$, and $W_\alpha = \bigcap_{\gamma<\alpha} W_\gamma$ if α is a limit ordinal,

(2) there is a neighborhood base $\{W_\alpha : \alpha < \kappa\}$ at the identity of $(G, \mathcal{T})$ such that $W_\alpha \subseteq W_\gamma$ if $\gamma < \alpha < \kappa$, and

(3) for every family $\mathcal{A} \subseteq \mathcal{T}$ of cardinality less that κ, $\bigcap \mathcal{A} \in \mathcal{T}$.

If $\mathcal{T}$ satisfies these conditions, then κ is regular.

Proof. suppose $\kappa < \omega$. Let $\mathcal{N} = \{U_1, \cdots, U_\kappa\}$ be a neighborhood base at the identity of G. Then $U := U_1 \cap \cdots \cap U_\kappa \in \mathcal{N}$ and, hence, $\{U\}$ is a neighborhood base at the identity. It follows that $\kappa = 1$ and, in this case, all three statements are clearly equivalent. Therefore we may assume that $\kappa \geq \omega$.

(1) $\Rightarrow$ (2): This implication is clear, since condition (2) is part of condition (1).
(2) $\Rightarrow$ κ is regular: Suppose (2) holds. Let γ be an ordinal such that there exists an order-preserving function $f : \gamma \to \kappa$ with $\sup\{f(\eta) : \eta < \gamma\} = \kappa$. Then, for each $\alpha < \kappa$ there exists $\eta < \gamma$ such that $\alpha < f(\eta)$ and, therefore, $W_\alpha \supseteq W_{f(\eta)}$. It follows that $\{W_{f(\eta)} : \eta < \gamma\}$ is a neighborhood base at the identity. Therefore, $\gamma = |\{W_{f(\eta)} : \eta < \gamma\}| \geq \kappa$. Hence κ is a regular cardinal.
(2) $\Rightarrow$ (3): Suppose (2) holds. Let $\mathcal{A} \subseteq \mathcal{T}$ be a family of cardinality less than κ and let $W = \bigcap \mathcal{A}$. Let $x \in W$. Then $\{xW_\alpha : \alpha < \kappa\}$ is a neighborhood base at x. Therefore for every $V \in \mathcal{A}$ choose $\alpha(V) < \kappa$ such that $xW_{\alpha(V)} \subseteq V$. Let $\gamma = \sup\{\alpha(V) : V \in \mathcal{A}\}$. Then $\gamma < \kappa$, since κ is regular, and $xW_\gamma \subseteq W$. Hence $W \in \mathcal{T}$.
(3) $\Rightarrow$ (1): Suppose (3) holds. Let $\{V_\alpha : \alpha < \kappa\}$ be a neighborhood base at the identity of $(G, \mathcal{T})$. For every $\alpha < \kappa$, put

$$W_\alpha = \begin{cases} \bigcap_{\gamma \leq \alpha} V_\gamma & \text{if } \alpha \text{ is not limit} \\ \\ \bigcap_{\gamma < \alpha} V_\gamma & \text{if } \alpha \text{ is limit.} \end{cases}$$

By (3), $\{W_\alpha : \alpha < \kappa\} \subseteq \mathcal{T}$. Hence $\{W_\alpha : \alpha < \kappa\}$ is the required neighborhood base at the identity. □

Therefore, our main theorem is as follows.

Theorem 3.4.2. *Let $\mathcal{T}$ be a nondiscrete regular left invariant topology on G and let $T = Ult(\mathcal{T})$. Suppose that $\mathcal{T}$ satisfies the conditions from Lemma 3.4.1. Then $c\ell\ K(T)$ is not a left ideal of T.*

As an immediate consequence we obtain that

Corollary 3.4.3. *Both $c\ell\ K(0^+)$ and $c\ell\ K(\mathbb{H}_\lambda)$ are not left ideals.*

In the rest of this section we prove Theorem 3.4.2. Let $|G| = \lambda$ and let κ denote the character of $\mathcal{T}$.

Lemma 3.4.4. *There is a function $\phi : G \to \lambda$ such that*

(a) for every $X \subseteq G$ with $|X| = \lambda$, $|\phi(X)| = \lambda$,

(b) if $\lambda = \omega$, then for every $x \in G$, there is a neighborhood V of $e \in G$ such that $\phi(xy) = \phi(y)$ for all $y \in V \setminus \{e\}$,

(c) if $\lambda > \omega$, then for every $x \in G$, there is a subset $Y \subseteq G$ with $|G \backslash Y| < \lambda$ such that $\phi(xy) = \phi(yx) = \phi(y)$ for all $y \in Y$, and

(d) if $\kappa < \lambda$, then for every $\alpha < \lambda$, $e \in c\ell_{\mathcal{T}}\phi^{-1}(\alpha + 1)$.

Proof. Consider two cases.

Case 1 : $\lambda = \omega$. Then $\kappa \le \omega$. Since our topologies are assumed to satisfy the T_1 separation axiom and $\mathcal{T}$ is nondiscrete, $\kappa \not< \omega$. Therefore $\kappa = \omega$. In particular, G is first-countable. Recall that we denote by $\mathbb{B}$ the countably infinite Boolean group $\bigoplus_\omega \mathbb{Z}_2$ endowed with the topology with a neighborhood base at zero consisting of the subgroups $H_n = \{x \in \bigoplus_\omega \mathbb{Z}_2 : x(m) = 0 \text{ for all } m < n\}$, where $n < \omega$ (see Example 2.3.4). By Theorem 2.3.13, there is a homeomorphism $h : G \to \mathbb{B}$ such that $h(e) = 0$ and $h(xy) = h(x)+h(y)$ whenever $\max\ \mathrm{supp}(h(x))+2 \le \min\ \mathrm{supp}(h(y))$. Define $\phi : G \to \omega$ by

$$\phi(x) = \begin{cases} \max\ \mathrm{supp}(h(x)) & \text{if } x \neq e \\ \\ 0 & \text{if } x = e. \end{cases}$$

Since for any $x \in G$ and for any $0 \le i \le \max \mathrm{supp}\, h(x)$, $(h(x))(i)$ has only a finite number of possible values (two in this case), ϕ is a finite-to-one function. In addition, h is injective. Therefore, for any $X \subseteq G$ with $|X| = \omega$, $|\phi(X)| = \omega$. Thus (a) is satisfied. To check (b), let $x \in G$. If $x = e$, then $xy = ey = y$ and, therefore, $\phi(xy) = \phi(y)$ for all $y \in G$. Therefore, we may assume that $x \neq e$. Let $m = \max\ \mathrm{supp}(h(x)) + 2$ and let

$$V = \{y \in G : m \le \min\ \mathrm{supp}(h(y))\} \cup \{e\}.$$

Then $V = h^{-1}(H_m) \cup \{e\} = h^{-1}(H_m \cup \{0\}) \in \mathcal{T}$. It follows that V is an open neighborhood of e. Furthermore, for every $y \in V \setminus \{e\}$ one has

$$\begin{aligned} \phi(xy) &= \max \mathrm{supp}(h(xy)) \\ &= \max \mathrm{supp}(h(x) + h(y)) \\ &= \max\ \mathrm{supp}(h(y)) \\ &= \phi(y). \end{aligned}$$

Case 2 : $\lambda > \omega$. Construct inductively a λ-sequence $(G_\alpha)_{\alpha<\lambda}$ of subgroups of G with $G_0 = \{e\}$ such that

(i) for every $\alpha < \lambda$, $|G_\alpha| < \lambda$,

(ii) for every $\alpha < \lambda$, $G_\alpha \subseteq G_{\alpha+1}$,

(iii) for every limit ordinal $\alpha < \lambda$, $G_\alpha = \bigcup_{\gamma<\alpha} G_\gamma$,

(iv) $\bigcup_{\alpha<\lambda} G_\alpha = G$, and

(v) if $\kappa < \lambda$, then for every $\alpha < \lambda$, $e \in c\ell_{\mathcal{T}}(G_{\alpha+1} \setminus G_\alpha)$.

Note that $G = \bigcup_{\alpha<\lambda}(G_{\alpha+1} \setminus G_\alpha) \cup G_0$ is a disjoint union of sets. Define $\phi : G \to \lambda$ by

$$\phi(x) = \begin{cases} \alpha + 1 & \text{if } x \in G_{\alpha+1} \setminus G_\alpha \\ \\ 0 & \text{if } x \in G_0. \end{cases}$$

Clearly ϕ satisfies (a) and (d). To check (c), let $x \in G$. One may assume that $x \neq e$. Then $x \in G_{\alpha+1} \setminus G_\alpha$ for some $\alpha < \lambda$. Now let $Y = G \setminus G_{\alpha+1}$ and let $y \in Y$. Then $y \in G_{\gamma+1} \setminus G_\gamma$ for some $\gamma \geq \alpha+1$. Assume $xy \in G_\gamma$. Then $y = x^{-1}(xy) \in G_\gamma$, a contradiction. Therefore $xy \in G_{\gamma+1} \setminus G_\gamma$. Similarly, $yx \in G_{\gamma+1} \setminus G_\gamma$. Hence, $\phi(xy) = \phi(yx) = \gamma + 1 = \phi(y)$.

□

Let $\phi : G \to \lambda$ be a function guaranteed by Lemma 3.4.4 and let $\overline{\phi} : \beta G \to \beta\lambda$ denote the continuous extension of ϕ. Recall that for any $p \in \beta G$, $\overline{\phi}(p) = \{E \subseteq \lambda : \phi^{-1}(E) \in p\}$.

Corollary 3.4.5.

(1) If $\lambda = \omega$, then $\overline{\phi}(qp) = \overline{\phi}(p)$ for all $q \in \beta G$ and $p \in T$.

(2) If $\lambda > \omega$, then $\overline{\phi}(qp) = \overline{\phi}(p)$ for all $q \in \beta G$ and $p \in U(G)$.

Proof. (1) Let $q \in \beta G$. To see that $\overline{\phi}(qp) = \overline{\phi}(p)$, let $R \in p$. For every $x \in G$, there is a neighborhood V_x of $e \in G$ such that $\phi(xy) = \phi(y)$ for all $y \in V_x \setminus \{e\}$. Since $p \in T$, $V_x \setminus \{e\} \in p$. Put $P_x = R \cap V_x \setminus \{e\}$. Then $\bigcup_{x\in G} xP_x \in qp$ and, whenever $x \in G$ and $y \in P_x$, one has $\phi(xy) = \phi(y) \in \phi(R)$. It follows that for every $A \in q$, every $x \in A$ and every $y \in P_x$, $\phi(xy) = \phi(y)$. Therefore, for every $A \in q$, $\phi(\bigcup_{x\in A} xP_x) \subseteq \phi(R) \in \overline{\phi}(p)$. Since the sets $\bigcup_{x\in A} xP_x$, where $A \in q$, form a base for qp, the sets $\phi(\bigcup_{x\in A} xP_x)$, where $A \in q$, form a base for $\overline{\phi}(qp)$. It follows that $\overline{\phi}(qp) \subseteq \overline{\phi}(p)$ and, hence, $\overline{\phi}(qp) = \overline{\phi}(p)$.

(2) Let $q \in \beta G$ and $p \in U(G)$. To see that $\overline{\phi}(qp) = \overline{\phi}(p)$, let $R \in p$. For every $x \in G$, there is $Y_x \subseteq G$ with $|G \setminus Y_x| < \lambda$ such that $\phi(xy) = \phi(y)$ for all $y \in Y_x$. Since p is a uniform ultrafilter, $Y_x \in p$. Put $P_x = R \cap Y_x$. Then $\bigcup_{x \in G} xP_x \in qp$ and, whenever $x \in G$ and $y \in P_x$, one has $\phi(xy) = \phi(y) \in \phi(R)$. As in (1), it follows that $\overline{\phi}(qp) = \overline{\phi}(p)$.

□

Corollary 3.4.6. *There is $D \subseteq \lambda$ with $|D| = |\lambda|$ such that whenever $D_0 \subseteq D$ and $|D_0| = |\lambda|$, $\overline{\phi}^{-1}(U(D_0)) \cap T$ is a left ideal of T.*

Proof. Consider two cases.

Case 1: $\kappa = \lambda$. Let $\{V_\alpha : \alpha < \lambda\}$ be a decreasing neighborhood base at e in $\mathcal{T}$. Inductively, for each $\alpha < \lambda$, pick $x_\alpha \in V_\alpha \setminus \{e\}$ such that $\phi(x_\alpha) \notin \{\phi(x_\gamma) : \gamma < \alpha\}$. We have that $(x_\alpha)_{\alpha<\lambda}$ is a λ-sequence converging to e and such that $\phi(x_\alpha) \neq \phi(x_\gamma)$ if $\alpha \neq \gamma$. Put $D = \{\phi(x_\alpha) : \alpha < \lambda\}$. To see that D is as required, let $D_0 \subseteq D$ and $|D_0| = \lambda$. Then $\{x_\alpha : \phi(x_\alpha) \in D_0\}$ is a λ-subsequence of $(x_\alpha)_{\alpha<\lambda}$, and so is converging to e. It follows that $L = \overline{\phi}^{-1}(U(D_0)) \cap T$ is nonempty. To check that L is a left ideal of T, let $p \in L$ and $q \in T$. Since T is a subsemigroup of βG, $qp \in T$. By Corollary 3.4.5, $\overline{\phi}(qp) = \overline{\phi}(p) \in U(D_0)$. Hence, $qp \in L$.

Case 2: $\kappa < \lambda$. Put $D = \{\alpha + 1 : \alpha < \lambda\}$. To see that D is as required, let $D_0 \subseteq D$ and $|D_0| = \lambda$. For every $\alpha \in D_0$, $e \in c\ell_{\mathcal{T}}\phi^{-1}(\alpha)$. It follows that $L = \overline{\phi}^{-1}(U(D_0)) \cap T$ is nonempty. Then, applying Corollary 3.4.5, we obtain that L is a left ideal of T.

□

Construct inductively an open neighborhood base $\{W_\alpha : \alpha < \kappa\}$ at the identity of G and a κ-sequence $(a_\alpha)_{\alpha<\kappa}$ of elements of G such that $W_0 = G$ and for every $\alpha < \kappa$, the following conditions are satisfied:

(i) $a_\alpha \in W_\alpha$,

(ii) $c\ell_{\mathcal{T}} W_{\alpha+1} \subseteq W_\alpha$,

(iii) $a_\alpha W_{\alpha+1} \subseteq W_\alpha \setminus W_{\alpha+1}$,

(iv) for every $y \in W_{\alpha+1} \setminus \{e\}$, one has $\phi(a_\alpha y) = \phi(y)$, and

(v) $W_\alpha = \bigcap_{\gamma<\alpha} W_\gamma$ if α is a limit ordinal.

Let D be a subset of λ guaranteed by Corollary 3.4.6. Pick any family $\{D_\alpha : \alpha < \kappa\}$ of pairwise disjoint subsets of D of cardinality λ. For every $\alpha < \lambda$, put

$$E_\alpha = \bigcup_{\alpha\leq\gamma<\kappa} D_\gamma,\ A_\alpha = \phi^{-1}(D_\alpha) \cap W_\alpha \text{ and } B_\alpha = \bigcup_{\alpha\leq\gamma<\kappa} A_\gamma.$$

Define $N \subseteq \beta\lambda, L \subseteq T$ and $M \subseteq L$ by

$$N = \bigcap_{\alpha<\kappa} \overline{E_\alpha}\ ,\ \ L = \overline{\phi}^{-1}(N) \cap T, \text{ and } M = \bigcap_{\alpha<\kappa} \overline{B_\alpha}.$$

Lemma 3.4.7. *$M \cap c\ell\ K(T)) \neq \emptyset$.*

Proof. It follows from Corollary 3.4.6 that for each $\alpha < \kappa$, $\overline{\phi}^{-1}(U(D_\alpha)) \cap T$ is a left ideal of T. Consequently, there is $p_\alpha \in \overline{\phi}^{-1}(U(D_\alpha)) \cap K(T)$, so $p_\alpha \in \overline{A_\alpha} \cap K(T)$. Let

$$p \in \bigcap_{\alpha<\kappa} c\ell_{\beta G}\{p_\gamma : \alpha \leq \gamma < \kappa\}.$$

Then $p \in M \cap (c\ell K(T))$. □

Lemma 3.4.8. *For every $q \in \beta G \setminus \{e\}$ and $r \in L$, $qr \notin \overline{B_0}$.*

Proof. It suffices to show that for every $x \in G \setminus \{e\}$, $xr \notin \overline{B_0}$. There is $\alpha < \kappa$ such that $x \in W_\alpha \setminus W_{\alpha+1}$. Since $r \in T$ and $c\ell_{\mathcal{T}} W_{\alpha+2} \subseteq W_{\alpha+1}$, $xr \in \overline{W_\alpha \setminus W_{\alpha+2}}$. Assume that $xr \in \overline{B_0}$. Then $xr \in \overline{\bigcup_{\gamma\leq\alpha+1} A_\gamma}$, so $\overline{\phi}(r) = \overline{\phi}(xr) \in \overline{\bigcup_{\gamma\leq\alpha+1} D_\gamma}$. Hence, $\overline{\phi}(r) \notin \overline{E_{\alpha+2}}$, a contradiction. □

Corollary 3.4.9. *$M \cap T^2 = \emptyset$ and so $M \cap K(T) = \emptyset$.*

Proof. Assume on the contrary that $p = qr$ for some $p \in M$ and $q, r \in T$. Then $\phi(p) = \phi(qr) = \phi(r)$, so $r \in \overline{\phi}^{-1}(N) \cap T = L$. But by Lemma 3.4.8, $p = qr \notin \overline{B_0}$, which is a contradiction. Therefore, $M \cap T^2 = \emptyset$. Since $K(T)^2 = K(T) \subseteq T^2$, $M \cap K(T) = \emptyset$. □

Let $A = \{a_\alpha : \alpha < \kappa\}$. Since $(a_\alpha)_{\alpha<\kappa}$ converges to e in $\mathcal{T}$, $\{W_\alpha : \alpha < \kappa\} \cup \{A\}$ has the finite intersection property. Therefore there exists $p \in \beta G$ with $\{W_\alpha : \alpha < \kappa\} \cup \{A\} \subseteq p$. It follows that $p \in \overline{A} \cap T$ and, therefore, $\overline{A} \cap T \neq \emptyset$.

On the other hand, whenever $q \in U(A)$, one has $\{x \in G : x^{-1}A \in q\} = \{e\}$, consequently by Theorem 3.3.4, $\overline{A} \cap K(T) = \emptyset$.
Now let

$$C = \bigcup_{\alpha<\kappa} a_\alpha B_{\alpha+1}.$$

Note that $\overline{\phi}(\overline{C} \cap T) \subseteq \overline{\phi}(M)$. Indeed, if $x \in C \cap W_\alpha$, then $x = a_\gamma y$ for some $\gamma \geq \alpha$ and $y \in W_{\gamma+1}$, so $\phi(x) = \phi(a_\gamma y) = \phi(y) \in W_{\gamma+1}$. It is clear also that $\overline{A}M \subseteq \overline{C}$. Since $M \cap (c\ell\ K(T)) \neq \emptyset$ (Lemma 3.4.7), it follows that $(T(c\ell\ K(T))) \cap \overline{C} \neq \emptyset$. Therefore (since $\overline{C}$ is open) in order to prove that $c\ell\ K(T)$ is not a left ideal, it suffices to show that $\overline{C} \cap K(T) = \emptyset$. To this end, let $q \in K(T)$ and let r be the identity of the maximal subgroup in $K(T)$ containing q. Then $q = qr$ and, since $K(T) \cap \overline{A} = \emptyset$, $q \notin \overline{A}$. Consequently by the next lemma, $q \notin \overline{C}$.

Lemma 3.4.10. *Let $qr \in \overline{C}$ for some $q, r \in T$. Then $q \in \overline{A}$ and $r \in M$.*

Proof. Since $\overline{\phi}(r) = \overline{\phi}(qr) \in \overline{\phi}$ and $\overline{\phi}(\overline{C} \cap T) \subseteq \overline{\phi}(M)$, one has $r \in L$. To see that $q \in \overline{A}$, assume the contrary. Then there is $x \in G \setminus (A \cup \{e\})$ such that $xr \in \overline{C}$. It follows that $xr \in \overline{a_\alpha B_{\alpha+1}}$ for some $\alpha < \kappa$, and so $a_\alpha^{-1}xr \in \overline{B_{\alpha+1}}$, which contradicts Lemma 3.4.8. Now to see that $r \in M$, pick $Q \in q$ such that $Q \subseteq A$ and $Qr \subseteq \overline{C}$. Then for every $\alpha < \kappa$ such that $a_\alpha \in Q$, one has $B_{\alpha+1} \in r$, and the set of all such α is cofinal in κ. Hence $r \in M$. □

Chapter 4

Finite ultrafilter semigroups

Given a (left) topological group G, $Ult(G)$ is either finite or of cardinality greater than or equal to 2^{2^ω}. In fact, this is true for any closed subsemigroup of G^*. $Ult(G)$ reflects quite good topological properties of G. In [66] it was shown that every countable regular homogeneous space admits a structure of a left topological group. This allows us to define $Ult(X)$ for any countable regular space X. In [19], Hewitt studied maximal spaces. He gave a general way of constructing them and he used this construction to produce irresolvable spaces. If the topology of a (left) topological group is maximal, then $Ult(G)$ is finite. Projectives in the category of finite semigroups were studied in [67, 73, 62]. It turned out that for every countable regular left topological group G, if $Ult(G)$ is finite, then $Ult(G)$ is projective. In Section 4.1 we look at resolvable spaces, maximal spaces and spaces X satisfying the condition that for every $x \in X$, there is only a finite number of ultrafilters converging to x. In Section 4.2, we look at projectives. In section 4.3 we present results on topological groups G with $Ult(G)$ finite. We denote by **FinS** the category of finite semigroups and semigroup homomorphisms between them, and we denote by **KHausRS** the category of compact Hausdorff right topological semigroups and semigroup homomorphisms between them.

4.1 Almost maximal spaces and almost maximal left topological groups

Definition 4.1.1. Let X be a topological space and let $x \in X$. The *dispersion character* $\Delta(x, X)$ of X at the point x is the minimum of the cardinalities of open subsets of X containing x. The cardinal number

$$\Delta(X) := \min\{\Delta(x, X) : x \in X\}$$

is called the *dispersion character* of X.

Lemma 4.1.2. *Let X be a topological space.*

(1) If $\Delta(X)$ is infinite, then X is dense-in-itself (i.e. X has no isolated point).

(2) If X is T_0 and dense-in-itself, then $\Delta(X)$ is infinite.

Proof. (1) Suppose $\Delta(X)$ is infinite. Then every nonempty open subset of X is infinite. In particular X has no isolated point.
(2) Suppose X is T_0 and dense-in-itself. Assume $\Delta(X)$ is finite. Pick a subset $U \subseteq X$ such that $|U| = \Delta(X)$ and pick $a \in U$. Since X is dense-in-itself, there exists $x \in U \setminus \{a\}$. Pick an open subset $V \subseteq X$ such that $|V \cap \{a, x\}| = 1$. Then, $U \cap V$ is an open subset of X with $|U \cap V| < \Delta(X)$, a contradiction. It follows that $\Delta(X)$ is infinite. □

We call two subsets, A and B, of a set X *complementary* if $A \cap B = \emptyset$ and $A \cup B = X$.

Definition 4.1.3. Let X be a nonempty topological space. X is said to be *resolvable* if X can be partitioned into two dense subsets.

If $X = A \cup B$ where A and B are disjoint dense subsets of X, we say that $\{A, B\}$ is a *resolution* of X.

Remark 4.1.4. Let X be a topological space with an isolated point c and let A and B be dense subsets of X. Then $c \in A \cap B$. Therefore $A \cap B \neq \emptyset$. In particular X has no resolution. Therefore, when we want to know which spaces are resolvable, we need only to consider spaces without isolated points.

Theorem 4.1.5. *[[19], Theorem 19] The following statements about a topological space X are equivalent.*

(a) X is resolvable;

(b) X contains two disjoint dense subsets;

(c) X contains a dense set with empty interior;

(d) X contains two complementary subsets with empty interiors.

Proof. These implications are obvious. $\square$

Definition 4.1.6. Let X be a topological space which is dense-in-itself. X is called *irresolvable* if X is not resolvable.

Remark 4.1.7. It is clear from Theorem 4.1.5 (c) that a space X which is dense-in-itself is irresolvable if and only if every dense subset of X has a nonempty interior.

Lemma 4.1.8. *Let X be a topological space. Let $\mathcal{S}$ be the family of all ordered pairs $\{A, B\}$ of disjoint subsets of X. $\mathcal{S}$ has the partial order $\leq$ defined by $\{A_1, B_1\} \leq \{A_2, B_2\}$ if and only if $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$, and maximal elements of $\mathcal{S}$ are exactly the pairs $\{A, X \setminus A\}$, where $A \subseteq X$.*

Proof. It is clear that $\leq$ is a partial order. Since $\{X, \emptyset\} \in \mathcal{S}$, $\mathcal{S} \neq \emptyset$. Every chain $(\{A_i, B_i\})_{i\in I}$ is $\mathcal{S}$ has an upper bound $\{\bigcup_{i\in I} A_i, \bigcup_{i\in I} B_i\}$. By Zorn's Lemma $\mathcal{S}$ has a maximal element. It is easy to see that maximal elements of $\mathcal{S}$ are of the form $\{A, X \setminus A\}$, where $A \subseteq X$. $\square$

Corollary 4.1.9. *Let X be a resolvable space. The maximal resolutions of X with respect to the order in Lemma 4.1.8 are exactly pairs of the form $\{A, X \setminus A\}$, where $A \subset X$ with $c\ell\ A = X = c\ell\ (X \setminus A)$.*

Proof. Immediate from Lemma 4.1.8. $\square$

Let X be a nonempty topological space. Following the concept of Hewitt [19] , we will call a dense subset of X whose complement is also dense a *CD-set*. Thus a maximal resolution of X is a pair consisting of a CD-set and its complement.

Theorem 4.1.10. *[[19], Theorem 21] A topological space X containing a dense subset A which is resolvable in its relative topology is also resolvable.*

Proof. Let A be a dense subset of X which is resolvable in its relative topology with a resolution of CD-sets $\{B, C\}$. It is easy to see that $\{B, C \cup (X \setminus A)\}$ is a resolution of X. $\square$

Theorem 4.1.11. *A topological space X is resolvable if and only if every nonempty open subset of X contains a nonempty subset which is resolvable in its relative topology.*

Proof. Necessity. Suppose X is resolvable. Let U be a nonempty subset of X and let $\{A, B\}$ be a resolution of X. It is easy to see that $\{A \cap U, B \cap U\}$ is a resolution of U.
Sufficiency. Let U_o be an arbitrary nonempty open subset of X and let A_0 be a subset of U_0 which is resolvable in its relative topology with a pair of CD-sets $\{D_0, E_0\}$. Let $U_1 = X \setminus c\ell(A_0)$. If $U_1 = \emptyset$, then $c\ell(A_0) = X$ and $\{D_0, E_0 \cup (X \setminus A_0)\}$ is a resolution of X. If $U_1 \neq \emptyset$, it must, as an open subset of X, contain a set A_1 which has a resolution of CD-sets $\{D_1, E_1\}$ in its relative topology. Suppose that sets A_β with resolutions of CD-sets $\{D_\beta, E_\beta\}$ have been selected for every ordinal $\beta < \alpha$, α also being an ordinal. Take $U_\alpha = X \setminus c\ell\ (\bigcup_{\beta<\alpha} A_\beta)$. Then U_α is an open subset of X. If $U_\alpha \neq \emptyset$, we select a subset A_α of U_α with a resolution, in its relative topology, of CD-sets $\{D_\alpha, E_\alpha\}$. By induction, it follows that there is some ordinal number α_0 such that $c\ell\ (\bigcup_{\alpha<\alpha_0} A_\alpha) = X$. It can then be shown that the sets $\bigcup_{\alpha<\alpha_0} D_\alpha$ and $\bigcup_{\alpha<\alpha_0} E_\alpha$ are disjoint and dense in X. By Theorem 4.1.5 (b), X is resolvable. □

Proposition 4.1.12. *[[62], Lemma 2.3] If a homogeneous space X has a resolvable subspace, then X itself is also resolvable.*

Proof. Suppose X has a resolvable subspace Y_0. Let $\{A_0, B_0\}$ be a partition of Y_0 into dense subsets. Consider the family $\mathcal{S}$ of all pairs $\{A, B\}$ of disjoint subsets of X such that $A_0 \subseteq A$, $B_0 \subseteq B$, and $c\ell\ A = c\ell\ B$. Since $\{A_0, B_0\} \in \mathcal{S}$, $\mathcal{S} \neq \emptyset$. Define a relation $\leq$ on $\mathcal{S}$ by $\{A_1, B_1\} \leq \{A_2, B_2\}$ if and only if $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$. It is easy to see that $\leq$ is a partial order on $\mathcal{S}$. Let $(\{A_i, B_i\})_{i\in I}$ be a chain in $\mathcal{S}$. Let $A = \bigcup_{i\in I} A_i$ and let $B = \bigcup_{i\in I} B_i$. Then $\{A, B\}$ is an upper bound for $(\{A_i, B_i\})_{i\in I}$. By Zorn's Lemma $\mathcal{S}$ has a maximal element.
Let $\{A, B\}$ be a maximal element of $\mathcal{S}$. We claim that $\{A, B\}$ is a partition of X into dense subsets. Already $A \cap B = \emptyset$. It is clear that $Y = A \cup B$ is closed. Assume that $Y \neq X$. Pick $x \in X \setminus Y$ and $y \in Y$. Let $f : X \to Y$ be a homeomorphism with $f(y) = x$. Choose an open neighborhood U of y such that $f(U) \cap Y = \emptyset$. Put $A_1 = A \cup f(U \cap A)$ and $B_1 = B \cup f(U \cap B)$. Then $\{A_1, B_1\} \in \mathcal{S}$ and $\{A, B\} < \{A_1, B_1\}$, a contradiction. It follows that $Y = X$. Since $c\ell\ A = c\ell\ B$ and $A \cup B = X$, $c\ell\ A = c\ell\ B = c\ell\ A \cup c\ell\ B = c\ell\ (A \cup B) = c\ell\ X = X$. □

Theorem 4.1.13. *[[19], Theorem 7] Let X be a set. Any family of topologies on X is partially ordered by inclusion. The family of all topologies on X form a complete lattice under this partial order.*

Proof. It is fairly obvious that inclusion is a partial order. Let $(\mathcal{T}_i)_{i\in I}$ be a nonempty family of topologies on X. Let $\mathcal{G} = \bigcup_{i\in I} \mathcal{T}_i$ and let $\mathcal{T}$ be the topology on X generated by $\mathcal{G}$. Then $\mathcal{T}_i \subseteq \mathcal{T}$ for every $i \in I$ and $\mathcal{T}$ is the smallest topology on X with this property. Thus $\mathcal{T}$ is the least upper bound for $(\mathcal{T}_i)_{i\in I}$. Now let $\tau = \bigcap_{i\in I} \mathcal{T}_i$. Then τ is the greatest lower bound for $(\mathcal{T}_i)_{i\in I}$. □

Let X be a set and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X. We will call $\mathcal{T}_2$ *an expansion of* $\mathcal{T}_1$ (and $\mathcal{T}_1$ *a contraction of* $\mathcal{T}_2$) if $\mathcal{T}_2$ is finer than $\mathcal{T}_1$, i.e. $\mathcal{T}_1 \subseteq \mathcal{T}_2$. An expansion or contraction is said to be proper or improper as the inclusion $\mathcal{T}_1 \subseteq \mathcal{T}_2$ is proper or improper.

Lemma 4.1.14. *[[19], Theorem 2] Let X be a set and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X with $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Let $A \subseteq X$ and let $x \in X$. Then $int_{\mathcal{T}_1}(A) \subseteq int_{\mathcal{T}_2}(A)$ and $c\ell_{\mathcal{T}_2}(A) \subseteq c\ell_{\mathcal{T}_1}(A)$. If x is an accumulation point of A in $\mathcal{T}_2$, then x is an accumulation point of A in $\mathcal{T}_1$.*

Proof. The statements follow immediately from the fact that $\mathcal{T}_1 \subseteq \mathcal{T}_2$. □

It is also an easy observation that any expansion of a T_0, T_1, Hausdorff, or Urysohn topological space satisfies the same separation axiom.

Lemma 4.1.15. *Every nonempty finite open set in a T_0 space contains an isolated point. In particular, every nonempty finite T_0 space has an isolated point.*

Proof. Let X be a T_0 space. It is clear that any open singleton contains an isolated point. Let $n \in \mathbb{N}$ and suppose every nonempty open subset consisting of at most n elements contains an isolated point. Let $U = \{x_1, \cdots, x_{n+1}\} \subseteq X$ be open. Pick an open subset $V \subseteq X$ such that $|V \cap \{x_1, x_2\}| = 1$. Then $U \cap V$ is an open subset of X consisting of at most n elements. By assumption, $U \cap V$ contains an isolated point. Hence U contains an isolated point. By induction, every nonempty finite open subset of X contains an isolated point. The second statement follows at once. □

It follows from Theorem 4.1.15 that every T_0-space has dispersion character infinite or equal to 1.

Theorem 4.1.16. *[[19], Theorem 9] Every T_0-space X has an expansion Y which is a T_1 space and which has the same dispersion character as X.*

Proof. Let $(X, \mathcal{T})$ be a T_0-space and let

$$\mathcal{T}' = \{U \subseteq X : |X \setminus U| < \omega\} \cup \{\emptyset\},$$

i.e. the cofinite topology on X. Let $\mathcal{T}''$ be the topology generated by $\mathcal{T} \cup \mathcal{T}'$. Since $\mathcal{T}'$ is a T_1-space, $\mathcal{T}''$ is a T_1-space. Suppose $\Delta(X, \mathcal{T}) = 1$. Since $\mathcal{T} \subseteq \mathcal{T}''$, $\Delta(X, \mathcal{T}'') = 1$. On the other hand, if $\Delta(X, \mathcal{T})$ is infinite, the intersection of any open set $U \in \mathcal{T}$ with an open set $V \in \mathcal{T}'$ must have the same cardinal number as U, since the removal of a finite number of points from an infinite set does not change the cardinal number of that set. Hence $\Delta(X, \mathcal{T}) = \Delta(X, \mathcal{T}'')$. □

Theorem 4.1.17. *Every Hausdorff space X has an expansion Y which is a Urysohn space, whose isolated points are exactly the isolated points in X, and which has the dispersion character equal to the dispersion character of X.*

Proof. [[19], Theorem 10]. □

Theorem 4.1.18. *Let X be a set and let $(\mathcal{T}_i)_{i \in I}$ be a nonempty family of topologies on X completely ordered by inclusion. If $\mathcal{T}_j$ is T_0, T_1, Hausdorff or Urysohn, for some $j \in I$, then the least upper bound of $(\mathcal{T}_i)_{i \in I}$ satisfies the same separation property. If $\mathcal{T}_i$ is regular (completely regular) for every $i \in I$, then the least upper bound of $(\mathcal{T}_i)_{i \in I}$ is regular (completely regular).*

Proof. [[19], Theorem 11]. □

Definition 4.1.19. Let κ be a cardinal number and let P be a property intelligible for topological spaces. Let $(X, \mathcal{T})$ be a topological space. $(X, \mathcal{T})$ is called *κ-maximal with respect P* if $\mathcal{T}$ is maximal among topologies on X satisfying the following properties.

(a) $\Delta(X, \mathcal{T}) \geq \kappa$,

(b) $(X, \mathcal{T})$ has the property P.

Definition 4.1.20. Let κ be a cardinal number and let $(X, \mathcal{T})$ be a topological space. $(X, \mathcal{T})$ is called *κ-maximal* if $\mathcal{T}$ is maximal among topologies on X with dispersion character greater than or equal to κ. An ω-maximal space is simply said to be *maximal*.

Let P be a property intelligible of topological spaces. If every expansion of a space with the property P has the property P, then P is called *expansion-invariant.* In [[19], Theorems 12-16], Hewitt showed that certain topologies has κ-maximal expansions with respect to some properties. Further, he proved that every maximal space is irresolvable. In [14], van Douwen proved that there exists a countable regular maximal space in ZFC. In [38], Malykhin proved, under Martin's Axiom (to be stated later), that there exists a nondiscrete Hausdorff topological group whose underlying space is maximal. In particular, maximal spaces exist.

In the sequel our maximal topological spaces are Hausdorff. The following lemma gives two characterizations of maximal spaces among Hausdorff spaces.

Proposition 4.1.21. *[[62], Proposition 2.1] Let $(X, \mathcal{T})$ be a Hausdorff topological space. The following statements are equivalent.*

(1) X is maximal,

(2) X is without isolated points, extremally disconnected and has no resolvable subspaces and no nonclosed nowhere dense subsets,

(3) for each $x \in X$ there is exactly one nonprincipal ultrafilter on X converging to x.

Proof. Let $X = (X, \mathcal{T})$.
(1) $\Rightarrow$ (2): It is clear that X has no isolated points. Assume that X is not extremally disconnected. Let V be an open subset of X with a nonopen closure $c\ell\ V$. Then $\mathcal{T} \cup \{U \cap c\ell\ V : U \in \mathcal{T}\}$ is a base for a topology $\mathcal{T}_1$ on X which refines $\mathcal{T}$. Notice that $c\ell\ V \subseteq X$ has no isolated points in $\mathcal{T}$. Consequently, if $U \cap c\ell\ V \neq \emptyset$, then $U \cap c\ell\ V$ is infinite. Hence, $(X, \mathcal{T}_1)$ has no isolated points, a contradiction.
Assume that X has a resolvable subspace Y. Let A be a subset of Y such that A and $Y \setminus A$ are dense in Y. Then $A \subseteq X$ is a nonopen subset without isolated points. Hence, one can refine $\mathcal{T}$ to the topology $\mathcal{T}_1$ without isolated points by taking as a base $\mathcal{T} \cup \{U \cap A : U \in \mathcal{T}\}$, a contradiction.
Assume now that X has a nonclosed nowhere dense subset B. Let $x \in c\ell\ B \setminus B$ and let $Z = (X \setminus c\ell\ B) \cup \{x\}$. Then $Z \subseteq X$ is a nonopen subset without isolated points. Hence, one can refine $\mathcal{T}$ to the topology $\mathcal{T}_1$ without isolated points by taking as a base $\mathcal{T} \cup \{U \cap Z : U \in \mathcal{T}\}$, a contradiction.

(2) $\Rightarrow$ (3): First we prove that every nonprincipal ultrafilter p on X converging to a point $x \in X$ has a base of open subsets. To this end, let $A \in p$ and suppose that $int\ A \notin p$. Let $B = A \setminus ((int\ A) \cup \{x\})$. Then $B \in p$ and $int\ B = \emptyset$. Put $U = int\ c\ell\ B$. We claim that $U = \emptyset$. Indeed, otherwise B is nowhere dense and, since $x \in c\ell\ B$, nonclosed. Put $C = U \cap B$. Then C is dense in U and $int\ C = \emptyset$, hence $U \setminus C$ is also dense in U, so U is resolvable, a contradiction. Now suppose that there are two different nonprincipal ultrafilters p and q on X converging to a point $x \in X$. Choose disjoint open subsets $U_p \in p$ and $U_q \in q$. Then $x \in c\ell\ U_p \cap c\ell\ U_q$, so X is not extremally disconnected, a contradiction.
(3) $\Rightarrow$ (1) Let $X = (X, \mathcal{T})$ and let $\mathcal{T}_1$ be any refinement of $\mathcal{T}$. Then there exists a nonprincipal ultrafilter p on X which is convergent to a point $x \in X$ in $\mathcal{T}$ but is not convergent in $\mathcal{T}_1$. But then x is an isolated point in $\mathcal{T}_1$. Hence, $\mathcal{T}$ is maximal. □

Corollary 4.1.22. *A homogeneous space is maximal if and only if it is nondiscrete, extremally disconnected, irresolvable and has no nonclosed nowhere dense subsets.*

Proof. It is immediate from Proposition 4.1.21 and Proposition 4.1.12. □

Definition 4.1.23. A topological (left topological) group G is called *maximal* if its underlying space is maximal.

Proposition 4.1.24. *Let G be an infinite group with identity e and let $p \in G^*$ be an idempotent. Then $G(p)$ is maximal.*

Proof. Recall that the topology $\mathcal{T}$ of $G(p)$ is

$$\mathcal{T} = \{U \subseteq G : a^{-1}U \in p \text{ for every } a \in U\}$$

with the neighborhood filter at e equal to the family $\{A \cup \{e\} : A \in p\}$, see Section 2.2. Since $pp = p$, $\mathcal{T}$ is Hausdorff. Suppose there exists a topology $\mathcal{T}'$ on G such that $\mathcal{T} \subsetneq \mathcal{T}'$ and G has no isolated points in $\mathcal{T}'$. Pick $V \in \mathcal{T}' \setminus \mathcal{T}$ and pick $a \in V$ such that $a^{-1}V \notin p$. Then $G \setminus a^{-1}V \in p$. Therefore $(G \setminus a^{-1}V) \cup \{e\}$ is a neighborhood of e in $\mathcal{T}$, and thus $a((G \setminus a^{-1}V) \cup \{e\}) = (G \setminus V) \cup \{a\}$ is a neighborhood of a in $\mathcal{T}$. Then $(G \setminus V) \cup \{a\}$ is a neighborhood of a in $\mathcal{T}'$. But then $(G \setminus V) \cup \{a\}) \cap V = \{a\}$ is a neighborhood of a in $\mathcal{T}'$, a contradiction. □

Conversely we have the following.

Proposition 4.1.25. *Let $(G, \mathcal{T})$ be an infinite maximal left topological group. There exists an idempotent $p \in G^*$ such that $(G, \mathcal{T}) = G(p)$.*

Proof. [[53], Theorem 1.1] □

Corollary 4.1.26. *Let G be an infinite group. There is a bijection between the maximal left invariant topologies on G and the idempotents in G^*.*

We recall that in a semigroup S, an idempotent $p \in S$ is said to be *strongly right maximal* if the equation $xp = p$ has the unique solution $x = p$.

Theorem 4.1.27. *Let G be an infinite group with identity e and let $p \in G^*$. Let $\mathcal{T}$ be the topology for $G(p)$. Then the following are equivalent.*

1. *$\mathcal{T}$ is regular.*
2. *The idempotent p is strongly right maximal in G^*.*
3. *The map $(\rho_p)_{|G}$ is a homeomorphism from $G(p)$ onto Gp.*
4. *$\mathcal{T}$ is regular and extremally disconnected.*
5. *$\mathcal{T}$ is zero dimensional.*

Proof. [[29], Theorem 9.15] □

Corollary 4.1.28. *Let G be an infinite group with identity e and let $p \in G^*$. The topologies of the groups $G(p)$ and $G[p]$ coincide if and only if the underlying space of $G(p)$ is regular.*

Corollary 4.1.29. *Let G be an infinite group with identity e and let $p \in G^*$ be an idempotent. If p is strongly right maximal, then $G(p)$ is homogeneous, zero dimensional, Hausdorff, extremally disconnected, and maximal among all topologies without isolated points. Distinct strongly right maximal idempotents q and r give rise to distinct topologies $G(q)$ and $G(r)$.*

Proof. [[29], Corollary 9.17] □

Definition 4.1.30. A topological space X is called *almost maximal* if it is without isolated points and for every $x \in X$ there are only finitely many nonprincipal ultrafilters on X converging to x.

Note that as a consequence of Proposition 4.1.21, every maximal space is almost maximal. By *an almost maximal topological (left topological) group* we will mean a topological (respectively, left topological) group whose underlying space is almost maximal.

Note that every almost maximal left topological group G gives a finite semigroup $Ult(G)$. Conversely, by Proposition 2.1.13, if every finite semigroup can be realized as a subsemigroup of G^* for some left topological group G, then every finite semigroup is an ultrafilter semigroup. We will return to this remark in the next section.

Proposition 4.1.31. *[[62], Proposition 2.6] Let G be an Abelian torsion free group. Then every almost maximal left invariant topology on G is Hausdorff.*

Proof. Let $\mathcal{T}$ be an almost maximal left invariant topology on G and let $S = Ult(\mathcal{T})$. Then S is finite. Assume on the contrary that $\mathcal{T}$ is not Hausdorff. Then there is $a \in G \setminus \{0\}$ such that $(a + S) \cap S \neq \emptyset$. It follows that $\{q \in S : a + q \in S\}$ is a nonempty and, obviously, a subsemigroup of S (and even a right ideal), and consequently, as any finite semigroup, has an idempotent. Thus, there is an idempotent $p \in S$ with $a + p \in S$. Since p is an idempotent and G is Abelian, $(a + p) + \cdots + (a + p) = (a + \cdots + a) + p$. Hence $na + p \in S$ for all $n \in \mathbb{N}$. Since G is torsion free, all elements na are different. But then all elements $na + p$ are different as well, so S is infinite, a contradiction. It follows that $\mathcal{T}$ is Hausdorff. □

Corollary 4.1.32. *Let G be an Abelian torsion free group and let $\mathcal{T}$ be an almost maximal left invariant topology on G. Then $|Ult(G)| = 1$ if and only if $(G, \mathcal{T})$ is maximal.*

Proof. By Proposition 4.1.31, $(G, \mathcal{T})$ is Hausdorff. Suppose $|Ult(G)| = 1$. Since $(G, \mathcal{T})$ is homogeneous, for every $x \in G$ there is exactly one nonprincipal ultrafilter on G converging to x. Therefore, by Proposition 4.1.21, $(G, \mathcal{T})$ is maximal. The converse follows directly from Proposition 4.1.21. □

In [63] it was proved that there are no nontrivial finite groups in $\beta\mathbb{N}$. In fact, it was proved that for any countable group G and any finite group F in βG, if the subgroup $\{x \in G : xF = F\}$ is trivial, the group F is trivial as well. The following theorem follows from this result and Proposition 2.1.9.

Theorem 4.1.33. *[[62], Proposition 2.7] For every countable Hausdorff left topological group G, the ultrafilter $Ult(G)$ has no nontrivial finite subgroups.*

Corollary 4.1.34. *Let G be a countable Abelian group and let $\mathcal{T}$ be an almost maximal left invariant topology on G. Then $\mathcal{T}$ is Hausdorff if and only if $Ult(\mathcal{T})$ has no nontrivial subgroups.*

Proof. $\Rightarrow$: This implication follows directly from Theorem 4.1.33.
$\Leftarrow$: Let $S = Ult(\mathcal{T})$. Suppose S has no nontrivial finite subgroups and assume that $\mathcal{T}$ is not Hausdorff. Then there exists $a \in G \setminus \{0\}$ such that $(a + S) \cap S \neq \emptyset$. Then, as in the proof of Proposition 4.1.31, there is an idempotent $p \in S$ such that $\langle a \rangle + p \subseteq S$, where $\langle a \rangle = \{na : n \in \mathbb{N}\}$. Clearly $\langle a \rangle + p$ is a subsemigroup isomorphic to $\langle a \rangle$. Since S is finite, $\langle a \rangle + p$ is a finite subgroup, a contradiction. $\square$

Let φ be a filter on a topological space X. We recall the following. φ is called *open* (respectively, *closed*) if φ has a base consisting of open (respectively, closed) subsets. We will denote by $c\ell\ \varphi$ the largest closed filter on X contained in φ and by φ° the largest open filter (possibly improper) containing φ.

Lemma 4.1.35. *[[62], Lemma 2.10] Let G be an almost maximal left topological group, let $S = Ult(G)$, and let ψ be a filter on G. Then*

(a) ψ is closed if and only if for every $p \in \beta G \setminus \overline{\psi}$, $pS \cap \overline{\psi} = \emptyset$,

(b) ψ is open if and only if $\overline{\psi} S \subseteq \overline{\psi}$.

Proof. (a) Suppose ψ is closed. Let $p \in \beta G \setminus \overline{\psi}$. Then $\psi \nsubseteq p$. Pick a closed $F \in \psi \setminus p$ and let $U = G \setminus F$. Then $p \in \overline{U}$ and $\overline{U} \cap \overline{\psi} = \emptyset$. Since U is open, $\overline{U} \cdot S \subseteq \overline{U}$. Since $pS \cap \overline{\psi} \subseteq \overline{U} \cdot S \cap \overline{\psi} = \emptyset$, $pS \cap \overline{\psi} = \emptyset$.
Conversely, assume for every $p \in \beta G \setminus \overline{\psi}$, $pS \cap \overline{\psi} = \emptyset$ and assume that ψ is not closed. Then there is $U \in \psi$ such that U is not closed. Given $V \in \psi$, pick $x_V \in c\ell\ V \setminus U$. Then $e \in c\ell\ x_V^{-1} V$. Then the family $\{N : N$ is a neighborhood of the identity $\} \cup \{x_V^{-1} V\}$ has the finite intersection property and, therefore, is contained in some ultrafilter q_V. Then $q_V \in S$ and $x_V^{-1} V \in q_V$, or equivalently, $V \in x_V q_V$. Since S is finite, one may suppose that $q_V = q$ is the same filter. Pick an ultrafilter p on G such that for every $V \in \psi$, $\{x_W : W \in \psi$ and $W \subseteq V\} \in p$. Then $pq \in \overline{\psi}$ and $G \setminus U \in p$, so $p \in \beta G \setminus \overline{\psi}$.

(b) Suppose ψ is open. Then by Corollary 2.1.11, $\overline{\psi} S \subseteq \overline{\psi}$.
Conversely, suppose $\overline{\psi} S \subseteq \overline{\psi}$ and let $U \in \psi$. For every $p \in \overline{\psi}$ and $q \in S$, $U \in pq$, so there exists $A_{p,q} \in p$ such that $\overline{A_{p,q}} \cdot q \subseteq \overline{U}$. Denote

$V_q = \bigcup_{p\in\overline{\psi}} A_{p,q}$. Then $V_q \in \overline{\psi}$ and $\overline{V}_q \cdot q \subseteq \overline{U}$. Let $V = \bigcap_{q\in S} V_q$. Then $V \in \psi$ and $\overline{V} \cdot S \subseteq \overline{U}$. Finally, put $W = \{x \in U : xS \subseteq \overline{U}\}$. Clearly, W is open and, since $V \subseteq W$, $W \in \psi$. Hence, ψ is open.

□

Corollary 4.1.36. *Let G be an almost maximal left topological group, let $S = Ult(G)$, and let ψ be a filter on G. Then*

(a) $\overline{c\ell\,\psi} = \overline{\psi} \cup \{p \in \beta G : pS \cap \overline{\psi} \neq \emptyset\}$,

(b) $\overline{\psi^\circ} = \{p \in \overline{\psi} : pS \subseteq \overline{\psi}\}$.

Proposition 4.1.37. *[[62], Proposition 2.12] Let $(G, \mathcal{T})$ be an almost maximal left topological group and let $S = Ult(\mathcal{T})$. Then*

(1) $\mathcal{T}$ is regular if and only if S^1 is left saturated in βG,

(2) if $\mathcal{T}$ is Hausdorff, $\mathcal{T}$ is regular if and only if S is left saturated in G^*.

Proof. (1) ($\Rightarrow$): This implication is Lemma 2.1.15.
($\Rightarrow$): Note that $\mathcal{T}$ is regular if and only if the neighborhood filter of the identity is closed. Now suppose S^1 is left saturated in βG. Let $\mathcal{N}$ be the neighborhood filter at the identity, so that $S^1 = \overline{\mathcal{N}}$. Let $p \in \beta G \backslash \overline{\mathcal{N}}$. Since G is cancellative, $pS \subseteq G^*$. Therefore, $pS \cap \overline{\mathcal{N}} = pS \cap S^1 = pS \cap S \subseteq pS^1 \cap S^1 = \emptyset$. It follows that $pS \cap \overline{\mathcal{N}} = \emptyset$. By Lemma 4.1.13 (a), $\mathcal{N}$ is closed. Hence $\mathcal{T}$ is regular.
(2) Assume $\mathcal{T}$ is regular. Then, by (1), S^1 is left saturated in βG and, thus, S is left saturated in G^*. Conversely, suppose S is left saturated in βG. By (1) it suffices to show that S^1 is left saturated in βG. Let $p \in \beta G \setminus S^1$. If $p \notin G$, then $pS^1 \cap S^1 = (pS \cup S) \cap S^1 = (pS \cup S) \cap S = \emptyset$. Suppose $p \in G$. Since $\mathcal{T}$ is Hausdorff, $pS \cap S = \emptyset$ by Proposition 2.1.9. Then $pS^1 \cap S^1 = (pS \cup \{p\}) \cap S^1 = pS \cap S^1 = pS \cap S = \emptyset$. □

The following is a stronger version of Proposition 4.1.37.

Proposition 4.1.38. *[[62], Proposition 2.13] Let G be an almost maximal left topological group, let X be an open neighborhood of the identity, and let $S = Ult(G)$. Then*

(a) X is regular iff for every $x \in X$ and $p \in \overline{X} \setminus xS^1$, $pS \cap xS = \emptyset$,

(b) if G is Hausdorff, X is regular iff for every $x \in X$ and $p \in X^* \setminus xS$, $pS \cap xS = \emptyset$, where $X^d = \overline{X} \setminus X$,

(c) if G is Hausdorff and X is a subsemigroup such that for every $x \in X$, $|x^{-1}X \setminus X| < \omega$, then X is regular iff S is left saturated in X^*.

Proof. First we note the following. For every $x \in G$, the neighborhood filter at x is given $\phi_x = x\mathcal{N}$, where $\mathcal{N}$ is the neighborhood filter at the identity. Thus $\overline{\phi}_x = xS^1$. X is regular if and only if for every $x \in X$, ϕ_x is closed.
(a) $\Rightarrow$: Suppose X is regular. Then xS^1 is closed. Let $x \in X$ and let $p \in \overline{X} \setminus xS^1$. By Lemma 4.1.35 (a), $pS \cap xS^1 = \emptyset$. Since $pS \cap xS \subseteq pS \cap xS^1$, $pS \cap xS = \emptyset$.
$\Leftarrow$: Conversely, suppose the condition is satisfied. Let $x \in X$ and let $p \in \beta G \setminus \overline{\phi_x}$. By assumption, $pS \cap xS = \emptyset$. It remains to show that $x \notin pS$. Since G is cancellative, G^* is an ideal of βG. Therefore, $pS \subseteq G^*$ and, thus, $x \notin pS$. It follows that ϕ_x is closed and, hence, X is regular.
(b) Suppose G is Hausdorff. Then for every $x \in G \setminus \{e\}$, $xS \cap S = \emptyset$. Then the result follows from (a).
(c) We can rewrite condition (b) as follows: for every $x \in X$ and $q \in x^{-1}X^* \setminus S$, $qS \cap S = \emptyset$. Note that in this case $x^{-1}X^* = X^*$. Then the results follows. $\square$

Corollary 4.1.39. *Let S be a finite subsemigroup in $\mathbb{N}^* \subseteq \mathbb{Z}^*$ and let $\mathcal{T}$ be a left invariant topology on $\mathbb{Z}$ with $Ult(\mathcal{T}) = S$. Then $X = \mathbb{N} \cup \{0\}$ is an open neighborhood of $0 \in \mathbb{Z}$ in $\mathcal{T}$ and X is regular if and only if S is left saturated in $\mathbb{N}^*$.*

Proof. Let ϕ be the neighborhood filter of 0 in $\mathcal{T}$. Then $\overline{\phi} = S^1$. Since $S \subseteq \mathbb{N}^*$, $\mathbb{N} \in p$ for every $p \in S$. It follows that X is a neighborhood of 0 in $\mathcal{T}$. For every $x \in X$, $x + X = \{n \in \mathbb{N} : n \geq x\} \subseteq X$ and $x + X$ is a neighborhood of x. Therefore, X is open. Since $(\mathbb{Z}, +)$ is Abelian, $\mathcal{T}$ is Hausdorff, by Proposition 4.1.31. For any $x \in \mathbb{N}$, $|(-x + X) \setminus X| = |\{-x, -x+1, \cdots, -1\}| = x < \omega$ and $|(-0 + X) \setminus X| = |X \setminus X| = 0 < \omega$. Therefore, the remaining statement follows from Proposition 4.1.38 (c). $\square$

Proposition 4.1.40. *[[62], Proposition 2.15] Let $(G, \mathcal{T})$ be an almost maximal left topological group and let $S = Ult(G)$. Then*

(a) $\mathcal{T}$ is extremally disconnected if and only if $K(S)$ consists of only one minimal right ideal,

(b) $\mathcal{T}$ is irresolvable if and only if $K(S)$ is a left zero semigroup,

(c) *for every $p \in S$, p does not contain nowhere dense subsets if and only if $p \in K(S)$,*

(d) *$(G, \mathcal{T})$ has no nonclosed nowhere dense subsets if and only if $S = K(S)$.*

Proof. (a) We note that the implication from the right to the left is Lemma 2.1.14, but we give a complete proof here.
$\mathcal{T}$ is extremally disconnected if and only if there are no two disjoint open sets having e as an accumulation point, consequently by Lemma 4.1.35 (b), if and only if there are no two disjoint right ideals in S, so if and only if $K(S)$ has only one minimal right ideal (since distinct minimal right ideals are disjoint).
(b) $\mathcal{T}$ is irresolvable if and only if there is an open ultrafilter on G converging to e, consequently, by Lemma 4.1.35 (b), if and only if there is a singleton right ideal in S, so if and only if $K(S)$ is a left zero semigroup.
(c) p does not contain nowhere dense subsets if and only if $(\overline{c\ell\, p})^\circ \neq \emptyset$. By Corollary 4.1.36, $\overline{c\ell p} = \{p\} \cup \{q \in \beta G : p \in qS\}$ and $(\overline{c\ell\, p})^\circ = \{q \in \overline{c\ell\, p} : qS \subseteq \overline{c\ell\, p}\}$. If $p \notin K(S)$, $\overline{c\ell\, p} \cap K(S) = \emptyset$ and then $(\overline{c\ell\, p})^\circ = \emptyset$. If $p \in K(S)$, $\overline{c\ell\, p} \supseteq pS$ and then $p \in (\overline{c\ell\, p})^\circ$.
(d) $(G, \mathcal{T})$ has no nonclosed nowhere dense subsets if and only if for every $p \in S$, p does not contain nowhere dense subsets (if $A \in p$ is nowhere dense, then $A \setminus \{e\}$ is nonclosed nowhere dense), consequently, by (c), if and only if $S = K(S)$. □

4.2 Projectives in the category of finite semigroups

We will call an object S in some category a *projective* if for every morphism $f : S \to Q$ and every surjective morphism $g : T \to Q$ there exists a morphism $h : S \to Q$ such that the diagram

$$\begin{tikzcd} & S \arrow[dl, dashed, "h"'] \arrow[d, "f"] \\ T \arrow[r, two heads, "g"'] & Q \end{tikzcd}$$

commutes, i.e. $g \circ h = f$. We will call S an *absolute coretract* if for every surjective morphism $f : T \to S$ there exists a morphism $g : S \to T$ such that $f \circ g = id_S$. It is clear from the following diagram that if S is a projective, then S is an absolute coretract.

$$\begin{tikzcd} & S \arrow[dl, dashed, "g"'] \arrow[d, "id_S"] \\ T \arrow[r, two heads, "f"'] & S \end{tikzcd}$$

In some categories these notions coincide.

Theorem 4.2.1. *For every countable regular almost maximal left topological group G, $Ult(G)$ is a projective in the category of finite semigroups.*

Proof. Proof. Let $S = Ult(G)$, let Q be a finite semigroup, and let $f : S \to Q$ be a homomorphism. Since S is finite, it suffices to prove that f is proper by Corollary 2.3.15. For each $p \in S$, choose $C_p \in p$ such that $C_p \cap C_q = \emptyset$ if $p \neq q$, and choose $A_p \in p$ such that $A_p \subseteq C_p$, $X = \bigcup_{p \in U} A_p \cup \{e\}$ is open in G, and $\overline{A_p} \cdot q \subseteq \overline{Cpq}$ for all $p, q \in S$. Then for every $p \in S$ and $x \in A_p$, there exists a neighborhood V_x of $e \in X$ such that $x(V_x \cap A_q) \subseteq A_{pq}$ for all $q \in S$. Define $f_0 : X \to Q$ putting for every $p \in S$ and $x \in A_p$, $f_0(x) = f(p)$. Then f_0 is a local homomorphism inducing f, i.e. $\overline{f_0}_{|S} = f$. □

Projectives in **FinS** were described in [67, 73, 62]. In [67] the author came to projectives in **FinS** from projectives in **KHausRS**. As we remarked in the previous section, if every finite semigroup can be realized as a subsemigroup of G^* for some left topological group G, then every finite semigroup is an ultrafilter semigroup. The absolute coretracts in **KHausRS** arose in solving the following two questions. The first question is: Which finite semigroups are ultrafilter semigroups of topological groups? The second question is concerned with the semigroup $\beta\mathbb{N}$. Until now it is not known whether $\beta\mathbb{N}$ has elements of finite order other than idempotents. Which finite bands exist in $\beta\mathbb{N}$? It is well known that $\beta\mathbb{N}$ contains closed subsemigroups admitting a continuous homomorphism onto any finite semigroup. Hence, $\beta\mathbb{N}$ contains isomorphic copies of any finite absolute coretract in **KHausRS**.

In [67] the author described finite idempotent absolute coretracts in **KHausRS** and proved that these are precisely idempotent absolute coretracts in **FinS**. All projectives in **FinS** turned out to be bands decomposing into a certain chain of rectangular components.

Denote by W the semigroup of words of the form $a_1 \cdots a_n u_n \cdots u_1$, where

$a_k, u_k \in \omega$, $1 \leq k \leq n < \omega$, with the operation

$$a_1 \cdots a_n v_n \cdots v_1 \cdot b_1 \cdots b_m v_m \cdots v_1 = \begin{cases} a_1 \cdots a_n u_n \cdots u_1 & \text{if } n = m, \\ a_1 \cdots a_n u_n \cdots u_{m+1} v_m \cdots v_1 & \text{if } n > m, \\ a_1 \cdots a_n b_{n+1} \cdots b_m v_m \cdots v_1 & \text{if } n < m. \end{cases}$$

It is clear that W is a band. In fact, it is clear that W is a rectangular band. For every $n \in \mathbb{N}$, denote by W_n the subset of W consisting of all words of length $2n$. Then W_n is clearly a rectangular band. Let $x = a_1 \cdots a_n u_n \cdots u_1 \in W_n$ and let $y = b_1 \cdots b_m v_m \cdots v_1$ for some $m \neq n$. Then $xy \neq a_1 \cdots a_n v_m \cdots v_1$. In particular W_n is not contained in another proper rectangular subsemigroup of W. Hence W_n is a rectangular component. In addition, it is easy to see that for any $n \in \mathbb{N}$, $W_{n+1} \subseteq W_{n+1}W_n$. Hence W decomposes into a decreasing chain of the rectangular components W_n. (See the paragraph just after the proof of Remark 1.2.13 for the partial order on the set of rectangular components of a band).

Next denote by $\mathcal{M}$ the set of all matrices $M = (m_{i,j})_{i,j=0}^{n}$ without the main diagonal $(m_{i,i})$, where $n \in \mathbb{N}$ and $m_{i,j} \in \omega$, satisfying the following conditions for every $k \in \mathbb{N}$ with $1 \leq k \leq n$:

(a) $m_{0,k} \leq m_{1,k} \cdots \leq m_{k-1,k} \in \mathbb{N}$ and $m_{k,0} \leq m_{k,1} \leq \cdots \leq m_{k,k-1} \in \mathbb{N}$,

(b) either $m_{k-1,k} = 1$ and $m_{k-1,k+1} = \cdots = m_{k-1,n} = 0$ or $m_{k,k-1} = 1$ and $m_{k+1,k-1} = \cdots = m_{n,k-1} = 0$.

These are precisely matrices of the form:

$$
\begin{pmatrix}
\blacksquare & 1 & 0 & \cdots & 0 & 0 & & & & & & & & & & & & & \\
+ & \blacksquare & 1 & \cdots & 0 & 0 & & & & & & & & & & & & & \\
\ast & + & \blacksquare & \cdots & 0 & 0 & & & 0 & & & & & 0 & & & & & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & & & & & & & & & & \\
 & \ast & \ast & \cdots & \blacksquare & 1 & & & & & & & & & & & & & \\
\ast & \ast & \ast & \cdots & + & \blacksquare & + & \ast & \cdots & \ast & \ast & \ast & & & & & & & \\
\ast & \ast & \ast & \cdots & \ast & 1 & \blacksquare & + & \cdots & \ast & \ast & \ast & & & & & & & \\
 & & & & & 0 & 1 & \blacksquare & \cdots & \ast & \ast & \ast & & & & & & & \\
 & & & 0 & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & 0 & & & & & 0 \\
 & & & & & 0 & 0 & 0 & \cdots & \blacksquare & + & \ast & & & & & & & \\
 & & & & & 0 & 0 & 0 & \cdots & 1 & \blacksquare & 1 & 0 & \cdots & 0 & 0 & & & \\
 & & & & & & & & & \ast & + & \blacksquare & 1 & \cdots & 0 & 0 & & & \\
 & & & & & & & & & & \ast & + & \blacksquare & \cdots & 0 & 0 & & & \\
 & & & 0 & & & & & 0 & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & 0 \\
 & & & & & & & & & & \ast & \ast & \ast & \cdots & \blacksquare & 1 & & & \\
 & & & & & & & & & & \ast & \ast & \ast & \cdots & + & \blacksquare & + & \ast & \cdots \\
 & & & & & & & & & & \ast & \ast & \ast & \cdots & \ast & 1 & \blacksquare & + & \cdots \\
 & & & & & & & & & & & & & & & 0 & 1 & \blacksquare & \cdots \\
 & & & 0 & & & & & 0 & & & & & 0 & & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}
$$

and their transposes, where $+$ is a positive integer, $\ast$ is a nonnegative integer, and all rows and columns are nondecreasing up to the main diagonal.
For every $M = (m_{i,j})_{i,j=0}^{n} \in \mathcal{M}$, denote by $W(M)$ the subsemigroup of $\bigcup_{i=1}^{n} W_i$ which consists of all words $a_1 \cdots a_k u_k \cdots u_1 \in W_k$, $1 \le k \le n$, satisfying the following conditions:

(a) both $a_k \neq 0$ and $u_k \neq 0$,

(b) for every $j < i \le k$, if $a_{j+i} = \cdots = a_{i-1} = 0$, then $a_i \le m_{j,i}$, and dually, if $u_{j+1} = \cdots = u_{i-1} = 0$, then $u_i \le m_{i,j}$.

It can be shown that for different matrices $M \in \mathcal{M}$, the semigroups $W(M)$ are nonisomorphic. The following theorem has been obtained.

Theorem 4.2.2. *[[62], Theorem 4.2] For every finite semigroup S the following statements are equivalent.*

1. *S is isomorphic to $W(M)$ for some $M \in \mathcal{M}$,*

2. *S is a projective in* **FinS**,

3. *S is an absolute coretract in* **FinS***,*

4. *S is a projective in* **KHausRS***,,*

5. *S is an absolute coretract in* **KHausRS***.*

Proof. See [62, 73] □

As we will see in the next section, under Martin's Axiom each of the statements in Theorem 4.2.2 is equivalent to the following statement.

> S is isomorphic to $Ult(G)$ for some countable almost maximal topological group G.

Corollary 4.2.3. *For each countable regular almost maximal left topological group G, the semigroup $Ult(G)$ is isomorphic to $W(M)$ for some $M \in \mathcal{M}$.*

Proof. This is immediate from Theorem 4.2.1 and Theorem 4.2.2. □

Theorem 4.2.4. *Let G be a countable group and let S be a finite subsemigroup in G^* such that S^1 is left saturated in βG. Then S is isomorphic to $W(M)$ for some $M \in \mathcal{M}$.*

Proof. Since S is a finite subsemigroup of G^*, $S = Ult(\mathcal{T})$ for some left invariant topology $\mathcal{T}$ on G. Since $Ult(\mathcal{T})$ is finite and $(G, \mathcal{T})$ is homogeneous, $(G, \mathcal{T})$ is almost maximal. Since S^1 is left saturated in βG, $\mathcal{T}$ is regular. Therefore, by Theorem 4.2.1, S is a projective in **FinS**. The result follows by Theorem 4.2.2. □

4.3 Almost maximal topological groups

We start this section by stating Martin's Axiom (MA) and some related concepts.

Definition 4.3.1. Let $(P, \leq)$ be a partially ordered set.

(a) Two elements $a, b \in P$ are said to be *compatible* if there exists $c \in P$ such that $c \leq a$ and $c \leq b$. Otherwise a and b are *incompatible.*

(b) P is called a *c.c.c. partially ordered set* if whenever D is a subset of P consisting of pairwise incompatible elements (i.e. D is an antichain), one has that D is countable.

(c) A subset $D \subseteq P$ is *dense* in P if for every $a \in P$, there exists $d \in D$ with $d \leq a$.

(d) A *filter* on P is a nonempty subset $G \subseteq P$ satisfying the following conditions.

  (i) For every elements $a, b \in G$, there is $c \in G$ with $c \leq a$ and $c \leq b$.

  (ii) For every $a \in G$ and $b \in P$, if $a \leq b$, then $b \in G$.

(c.c.c. stands for countable chain condition.)

When we say that a topological space X is c.c.c., we mean the topology on X with set inclusion is a c.c.c. partially ordered set. Thus a topological space is c.c.c. if and only if there is no uncountable family of pairwise disjoint nonempty open subsets of X.

Lemma 4.3.2. *Every separable topological space is c.c.c.*

Proof. Let X be separable. Pick a countable dense subset D of X. Let $\mathcal{G}$ be a family of pairwise disjoint nonempty open subsets of X. For each $U \in \mathcal{G}$, pick $x_U \in U \cap D$. Then $x_U \neq x_V$ for $U \neq V$. Therefore, the correspondence $\mathcal{G} \ni U \mapsto x_U \in D$ is injective. It follows that $\mathcal{G}$ is countable. $\square$

Theorem 4.3.3. *Let $(X_i)_{i \in I}$ be a family of c.c.c. topological spaces such that for any finite $F \subseteq I$, the product $\prod_{i \in F} X_i$ is c.c.c.. Then $\prod_{i \in I} X_i$ is c.c.c.*

Proof. [[36], Theorem 1.9]. $\square$

Definition 4.3.4. Let κ be an infinite cardinal.

(a) MA(κ) is the assertion that whenever P is a c.c.c. partially ordered set and $\mathcal{D}$ is a family of at most κ dense subsets of P, there is a filter G on P such that $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.

(b) Martin's Axiom is the assertion that for all cardinals κ with $\omega \leq \kappa < 2^\omega$, MA($\kappa$).

(c) P(κ) is the following assertion: Whenever $\mathcal{A}$ is a family of subsets of $\mathbb{N}$ such that $|\mathcal{A}| < \kappa$ and $|A_0 \cap \cdots \cap A_n| = \omega$ for any $A_0, \cdots, A_n \in \mathcal{A}$, then there is an infinite subset $I \subseteq \mathbb{N}$ such that $|I \setminus A| < \omega$ for every $A \in \mathcal{A}$.

Theorem 4.3.5.

(a) If $\kappa < \kappa'$ and MA(κ') is true, then MA(κ) is true.

(b) MA(ω) is true.

(c) P($(2^\omega)^+$) is false.

Proof. (a) This statement is obvious.
(b) Let P be a c.c.c. partially ordered set and let $\mathcal{D}$ be a family of at most ω dense subsets of P. Enumerate $\mathcal{D}$ as $\mathcal{D} = \{D_0, D_1, D_2, \cdots\}$. Pick any $d_0 \in D_0$. Since D_1 is dense, we can pick $d_1 \in D_1$ such that $d_1 \leq d_0$. Since D_2 is dense, pick $d_2 \in D_2$ such that $d_2 \leq d_1$. Continuing in this way, we obtain a decreasing sequence $(d_n)_{n<\omega}$ with $d_n \in D_n$ for all $n < \omega$. Let

$$G = \{b \in P : d_n \leq b \text{ for some } n < \omega\}.$$

Then G is a filter and $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.
(c) This is obvious, consider any nonprincipal ultrafilter $\mathcal{A}$ on $\mathbb{N}$. $\square$

Theorem 4.3.6. *MA(2^ω) is false.*

As a proof to this theorem, we look at the following example.

Example 4.3.7. Let $\mathbb{P} = \{p \subseteq \omega \times 2 : |p| < \omega \text{ and } p \text{ is a function}\}$. Define a partial order $\leq$ on $\mathbb{P}$ by $p \leq q$ if and only if $q \subseteq p$. Then p and q are compatible if and only if they agree on $dom(p) \cap dom(q)$. If p and q are compatible, then $p \cup q$ is a function and it is the greatest lower bound of p and q. Since $|\mathbb{P}| = \omega$, $\mathbb{P}$ is a c.c.c. partially ordered set. Let G be a filter on $\mathbb{P}$. Since elements of G are compatible, we can form $f_G = \bigcup G$. For each $n \in \omega$, let $D_n = \{p \in \mathbb{P} : n \in dom(p)\}$. It is clear that any function in $\mathbb{P}$ can be extended to a function in D_n. Therefore D_n is dense. If $G \cap D_n \neq \emptyset$ for all $n \in \omega$, then $dom(f_G) = \omega$.
Given any function $h : \omega \to 2$, let

$$E_h = \{p \in \mathbb{P} : p(n) \neq h(n) \text{ for some } n \in dom(p)\}.$$

Then E_h is dense. If $G \cap E_h \neq \emptyset$, then $f_G \neq h$.
Let

$$\mathcal{D} = \{D_n : n \in \omega\} \cup \{E_h : h \text{ is a function from } \omega \text{ to } 2\}.$$

Then $|\mathcal{D}| = 2^\omega$. If $G \cap D \neq \emptyset$ for every $D \in \mathcal{D}$, then f_G is a function from ω to 2 which is different from every function from ω to 2, which is impossible. Hence $MA(2^\omega)$ is false.

We can then observe that MA can be stated as follows: For every κ, MA(κ) if and only if $\kappa < 2^\omega$. One of the important consequences of MA is the following result, known as the *combinatorial principle*: *For any cardinal κ, $P(\kappa)$ if and only if $\kappa \leq 2^\omega$.*

Theorem 4.3.8. *For any $\kappa \geq \omega$, the following statements are equivalent.*

(a) MA(κ).

(b) If X is a compact c.c.c. Hausdorff space and $(U_\alpha)_{\alpha<\kappa}$ is a family of dense open subsets of X, then $\bigcap_{\alpha<\kappa} U_\alpha \neq \emptyset$.

Definition 4.3.9. Let X be a topological space and let $x \in X$. x is called a *P-point* if the intersection of any family of open neighborhoods of x is a (not necessarily open) neighborhood of x.

Definition 4.3.10. Let X be a set and let $\mathcal{A} \subseteq \mathcal{P}(X)$.

(a) $\mathcal{A}$ is called an almost disjoint family if whenever A and B are distinct members of $\mathcal{A}$, $|A \cap B| < \omega$.

(b) $\mathcal{A}$ is called a maximal almost disjoint family if it is an almost disjoint family which is not properly contained in any other almost disjoint family.

We list some main results about almost maximal topological groups. These results solved several questions in general topology. In particular the question of whether every irresolvable topological group is extremally disconnected.

(1) As mentioned earlier, under Martin's Axiom, a countable maximal topological group has been constructed, [38].

(2) Every maximal topological group contains a countable open Boolean subgroup, [39].

(3) The existence of a maximal topological group implies the existence of a P-point, [52].

(4) In [72], under Martin's Axiom, there were constructed some almost maximal topological groups.

(5) In [70] it was proved that every almost maximal Abelian topological group contains a countable open Boolean subgroup and its existence implies the existence of a P-point in $\mathbb{N}^*$.

Next we present an exhaustive construction of countable almost maximal topological groups, which was initially done in [62].

Lemma 4.3.11. *[[62], Lemma 5.1] Let G be an infinite group and let $S = \{p_i : i < m\} = \overline{\varphi}$ be a finite semigroup in G^*. Then for any pairwise disjoint $A_i \in p_i$, $i < m$, and for any $G \ni x \mapsto B_x \in \varphi$, there exists a sequence $(x_n)_{n<\omega}$ in G such that*

(a) $x_n \in A_{i_n}$ where $i_n \equiv n(mod\, m)$,

(b) for every $\kappa < \omega$ and $n_0 < \cdots \leq n_k < \omega$, one has $x_{n_0} \cdots x_{n_k} \in A_i$, where i is defined by $p_i = p_{i_{n_0}} \cdots p_{i_{n_k}}$,

(c) for every $0 < k < \omega$, $FP((x_n)_{k \leq n < \omega}) \subseteq \bigcap\{B_x : x \in FP((x_n)_{n<k})\}$.

Proof. We will construct a sequence $(x_n)_{n<\omega}$ satisfying (a)-(c) and, in addition,

(d) for all $k < \omega$, $n_0 < \cdots < n_k < \omega$ and $p \in S$, one has $x_{n_0} \cdots x_{n_k} p \subseteq \overline{A_i}$, where $p_i = p_{i_{n_0}} \cdots p_{i_{n_k}} p$,

(e) for all $0 < k < \omega$ and $p \in S$, $FP((x_n)_{k \leq n < \omega}) \cdot p \subseteq \overline{\bigcap\{B_x : x \in FP((x_n)_{n<k})\}}$.

Choose $C_0 \in p_0$ such that $C_0 \subseteq A_0$ and for all $p \in S$, $\overline{C_0} p \subseteq \overline{A_i}$, where $p_0 p = p_i$. Then fix $0 < l < \omega$ and assume that we have constructed a sequence $(x_n)_{n<l}$ satisfying (a) - (e). Let $j < m$ and $j \equiv l(mod\, m)$. Choose $C_l \in p_j$ such that the following conditions are satisfied.

(1) $C_l \subseteq A_j$.

(2) For all $k < \omega$ and $n_0 < \cdots < n_k < l$, one has $x_{n_0} \cdots x_{n_k} \subseteq \overline{C_l} \subseteq \overline{A_i}$, where $p_i = p_{i_{n_0}} \cdots p_{i_{n_k}} p_j$.

(3) For all $0 < k < l$, $FP((x_n)_{k\leq n<l})\overline{C_l} \subseteq \overline{\bigcap\{B_x : x \in FP((x_n)_{n<k})\}}$.

(4) For all $p \in S$, $\overline{C_l}p \subseteq \overline{A_i}$, where $p_i = p_j p$.

(5) For all $k < \omega$, $n_0 < \cdots < n_k < l$ and $p \in S$, one has $x_{n_0} \cdots x_{n_k}\overline{C_l}p \subseteq \overline{A_i}$, where $p_i = p_{i_{n_0}} \cdots p_{i_{n_k}} p_j p$.

(6) For all $p \in S$, $\overline{C_l}p \subseteq \overline{\bigcap\{B_x : x \in FP((x_n)_{n<l})\}}$.

(7) For all $0 < k < \omega$ and $p \in S$, $FP((x_n)_{k\leq n<l})\overline{C_l}p \subseteq \overline{\bigcap\{B_x : x \in FP((x_n)_{n<k})\}}$.

Then choose $x_l \in C_l$. □

Theorem 4.3.12. *[[62], Theorem 5.2] Under MA, for each finite absolute coretract S in* **KHausRS**, *there is a group topology $\mathcal{T}$ on the countable Boolean group $\mathbb{B}$ with $Ult(\mathcal{T})$ isomorphic to S.*

Proof. Enumerate subsets of $\mathbb{B}$ as $\{Z_\alpha : \alpha < 2^\omega\}$ and let $Y = \{y_n : n < \omega\}$ be the standard base for $\mathbb{B}$, that is

$$(y_n)_m = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{if } n \neq m. \end{cases}$$

and let

$$U = \bigcap_{m<\omega} \overline{FS((y_n)_{m\leq n<\omega})}.$$

Then U is the ultrafilter semigroup for the strong direct sum topology on $\mathbb{B}$. Take any mapping $f_0 : Y \to S$ with $|f_0^{-1}(p)| = \omega$ for all $p \in S$. Extend f_0 to the local homomorphism $f : \mathbb{B} \to S$ by $f(y_{n_0}+\cdots+y_{n_k}) = f_0(y_{n_0}) \cdots f_0(y_{n_k})$, where $n_0 < \cdots < n_k$. Let $\pi : U \to S$ be the proper homomorphism induced by f. For every $p \in S$, put

$$B_p = \{y_{n_0} + \cdots + y_{n_k} : k < \omega,\ n_0 < \cdots < n_k < \omega,\ f(y_{n_0}) \cdots f(y_{n_k}) = p\}.$$

We will call a sequence $(x_n)_{n<\omega}$ in $\mathbb{B}$ correct if for every $n < \omega$, $n \neq 0$ and $\max\operatorname{supp}(x_n) < \min\operatorname{supp}(x_{n+1})$. For every $\alpha < 2^\omega$, we shall construct a correct sequence $(x_{\alpha,n})_{n<\omega}$ in $\mathbb{B}$ with $X_\alpha = \{x_{\alpha,n} : n < \omega\}$ and $V_\alpha = FS((x_{\alpha,n})_{n<\omega})$ such that

(1) for every $\beta < \alpha$, $|X_\alpha \setminus V_\beta| < \omega$,

(2) for every $p \in S$, $|X_\alpha \cap B_p| = \omega$, and

(3) for every $p \in S$, either $V_\alpha \cap B_p \subseteq Z_\alpha$ or $V_\alpha \cap B_p \subseteq \mathbb{B} \setminus Z_\alpha$.

Since S is an absolute coretract in **KHausRS**, there is an embedding $\varepsilon_0 : S \to U$ such that $\pi \circ \varepsilon_0 = id_S$. For every $p \in S$, choose $A_{0,p} \in \varepsilon_0(p)$ such that either $A_{0,p} \subseteq Z_0$ or $A_{0,p} \subseteq \mathbb{B} \setminus Z_0$, and put $A_0 = \bigcup_{p\in S} A_{0,p}$. Using Lemma 4.3.11, choose a correct sequence $(x_{n,0})_{n<\omega}$ such that for every $p \in S$, $|X_0 \cap B_p| = \omega$ and $V_0 \subseteq A_0$.
Fix $\alpha < 2^\omega$ and assume that we have constructed sequences $(x_{\beta,n})_{n<\omega}$, $\beta < \alpha$, satisfying conditions (1)-(3). Using the combinatorial principle, choose a correct sequence $(y_{\alpha,n})_{n<\omega}$ such that for every $\beta < \alpha$, $|Y_\alpha \setminus V_\beta| < \omega$ and for every $p \in S$, $|Y_\alpha \cap B_p| = \omega$, where $Y_\alpha = \{y_{\alpha,n} : n < \omega\}$. Put

$$U_\alpha = \bigcap_{m<\omega} \overline{FS((y_{\alpha,n})_{m\le n<\omega})}$$

and let $\varepsilon_\alpha : S \to U_\alpha$ be an embedding such that $\pi \circ \varepsilon_\alpha = id_S$. For every $p \in S$, choose $A_{\alpha,p} \in \varepsilon_\alpha(p)$ such that either $A_{\alpha,p} \subseteq Z_\alpha$ or $A_{\alpha,p} \subseteq \mathbb{B} \setminus Z_\alpha$, and put $A_\alpha = \bigcup_{p\in S} A_{\alpha,p}$. Then, using Lemma 4.3.11, choose a correct sequence $(x_{\alpha,n})_{n<\omega}$ satisfying (2) and with $V_\alpha \subseteq A_\alpha$.
The subgroups $V_\alpha \cup \{0\}$, $\alpha < 2^\omega$, form a neighborhood base of 0 in a group topology $\mathcal{T}$ on $\mathbb{B}$. For every $p \in S$, the subsets $V_\alpha \cap B_p$, $\alpha < 2^\omega$, form a base of an ultrafilter p' on $\mathbb{B}$. Put $S' = \{p' : p \in S\}$. Then $S' = Ult(\mathcal{T})$ and $\pi_{|S'} : S' \ni p' \mapsto p \in S$ is an isomorphism. □

Recall from the previous section that for every $M = (m_{i,j})_{i,j=0}^n \in \mathcal{M}$, $W(M)$ denotes the subsemigroup of $\bigcup_{i=1}^n W_i$ consisting of all words $a_1 \cdots a_k u_k \cdots u_1 \in W_k$, $1 \le k \le n$, satisfying the following conditions:

(a) both $a_k \neq 0$ and $u_k \neq 0$,

(b) for every $j < i \le k$, if $a_{j+i} = \cdots = a_{i-1} = 0$, then $a_i \le m_{j,i}$, and dually, if $u_{j+1} = \cdots = u_{i-1} = 0$, then $u_i \le m_{i,j}$.

Applying Theorem 4.3.12 and Proposition 4.1.40, for each combination of the properties of being extremally disconnected, irresolvable and having no nonclosed nowhere dense subsets, besides the combination (-,-,+), we indicate under MA the corresponding almost maximal topological group.

(1) Let $M = \begin{pmatrix} \blacksquare & 1 \\ 1 & \blacksquare \end{pmatrix}$. Then $W(M)$ is a 1-element semigroup. By Theorem 4.3.12, $W(M)$ is isomorphic to the ultrafilter semigroup for some

group topology $\mathcal{T}$ on $\mathbb{B}$. In particular $\mathcal{T}$ is maximal. By Proposition 4.1.40, $\mathcal{T}$ is extremally disconnected, irresolvable and has no nonclosed nowhere dense subsets.

(2) The semigroup determined by the matrix $M = \begin{pmatrix} \blacksquare & 1 \\ 2 & \blacksquare \end{pmatrix}$ is a 2-element right zero semigroup. It is an ultrafilter of a topological group satisfying the combination (+,-,+).

(3) The semigroup determined by the matrix $M = \begin{pmatrix} \blacksquare & 2 \\ 1 & \blacksquare \end{pmatrix}$ is a 2-element left zero semigroup. It is an ultrafilter of a topological group satisfying the combination (-,+,+).

(4) The semigroup determined by the matrix $M = \begin{pmatrix} \blacksquare & 1 & 0 \\ 1 & \blacksquare & 1 \\ 0 & 1 & \blacksquare \end{pmatrix}$ is a 2-element chain of idempotents. It is an ultrafilter of a topological group satisfying the combination (+,+,-).

(5) The semigroup determined by the matrix $M = \begin{pmatrix} \blacksquare & 1 & 0 \\ 1 & \blacksquare & 1 \\ 0 & 2 & \blacksquare \end{pmatrix}$ is a 3-element semigroup consisting of two components. Its second component is a 2-element right zero semigroup. $W(M)$ is an ultrafilter of a topological group satisfying the combination (+,-,-).

(6) The semigroup determined by the matrix $M = \begin{pmatrix} \blacksquare & 1 & 0 \\ 1 & \blacksquare & 2 \\ 0 & 2 & \blacksquare \end{pmatrix}$ is a 3-element semigroup consisting of two components. Its second component is a 2-element left zero semigroup. $W(M)$ is an ultrafilter of a topological group satisfying the combination (-,+,-).

(7) The semigroup determined by the matrix $M = \begin{pmatrix} \blacksquare & 1 & 0 \\ 1 & \blacksquare & 2 \\ 1 & 1 & \blacksquare \end{pmatrix}$ is a 5-element semigroup consisting of two components, where the second component is the 2×2-rectangular semigroup. $W(M)$ is an ultrafilter of a topological group satisfying the combination (-,-,-).

For the combination (-,-,+), there is not even a corresponding countable regular homogeneous almost maximal space.

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