# Behavioural Profiling:
# A Learning-Augmented Approach to Pricing Risk

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# DECLARATION

I, Jacques Peeperkorn, declare that this research report is my own unaided work. It is submitted in partial fulfilment of the requirements for the degree of Master of Commerce in Finance at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at this or any other university.

Jacques Peeperkorn

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## Definitions of Terms and Abbreviations

**Adaptive market hypothesis (AMH)**: An alternate market description to the efficient market hypothesis where efficiency is not seen as a singular equilibrium state. Market efficiency is dependent upon the evolutionary nature of interactions between market participants (Lo, 2004; 2005).

**Autocorrelation (or serial correlation)**: The dependency of one observation at a point in time to another observation of the same variable preceding point in time.

**ARCH:** Autoregressive conditional heteroskedasticity.

**Bayes theorem**: A theorem in statistics that links the degree of belief in an outcome before and after accounting for evidence (Bayes & Price, 1973).

**CAPM:** The capital asset pricing model (Sharpe, 1964; Lintner, 1965), as build upon the theory of modern portfolio theory.

**Contrarian investing**: investing against the flow of the market majority. Profit to contrarian investing comes through correctly identifying shares that are inexpensive relative to fundamental risk.

**Diffuse prior:** A diffuse prior is a prediction is the probability distribution when directly observable evidence is not available, formed to come as close as possible to representing the expectant prior states.

**EMH**: Efficient market hypothesis.

**Game theory**: Describing the field of strategic decision making, game theory describes how individuals interact so as to maximise utility through decision making under uncertainty. "the study of mathematical models of conflict and cooperation between intelligent rational decision-makers" (Schelling, 2010).

**GARCH:** Generalised autoregressive conditional heteroskedasticity.

**Gaussian distribution**: Also referred to as a normal distribution, this is a continuous distribution which has a “bell-shaped” probability density function. It was first introduced by (Gauss, 1857).

**Heteroskedastic:** A collection of random variables where formative sub-populations have non-uniform variances.

**Homoskedastic:** A collection of random variables with a uniform finite variance across all observations.

**JSE**: Johannesburg Securities Exchange Ltd.

**Kalman filter**: The Kalman is a recursive filter which estimates the true dynamic evolution of a system based off of noisy inputs through a recursive updating state space representation.

**MPT**: Modern portfolio theory (Markowitz, 1952).

**Probability belief**: Referring to choice under uncertainty, it is the rational choice when objective probabilities are unknown. It can also be thought of as a subjective expectation.

**Prospect theory**: A descriptive theory by Kahneman and Tversky (1979) which tries to reconcile real world actions and behaviour of investors with that of utility theory.

**Rational expectations**: A presumption of the efficient market hypothesis where agents' predictions of the future value of economically relevant variables are not systematically wrong - the errors are random (Muth, 1961).

**Self-organised stable state:** A spontaneous state, the stability of which is dictated by the interactions of the components which make up the whole system. A famous example of this is a pile of sand, to which more sand is added, maintaining a conical shape. The system dynamically changes through continuous avalanches, yet the overall state dictates conical stability through the interactive forces of the grains of sand (Bak, 1996).

**State space**: A set of mathematical equations which track the dynamic evolution of a process which relates system inputs, outputs and state variables through a set of first order differential equations.

**Student's t distribution**: Similar to the Gaussian distribution, it is symmetrical and bell shaped. The t-distribution has heavier tails than a Gaussian distribution. The student's t distribution was proposed by statistician William Sealy Gosset in 1908, writing under the pseudonym "Student" (Student, 1908).

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## Abstract

To imagine that asset pricing is not dependant on a complex form of behavioural heuristics and interactive game theory it is requisite that we reduce the definition of the participants to that of traditionally defined utility maximising risk-averse uniform automata. This study tackles this statement directly through an application of behavioural theory which speaks to the individual ability of investors to perceive risk, as well as the interactive effects of game theory to distort the perception of risk from exogenous variables to that of endogenous probability beliefs. The result is an asset pricing model which tracks the evolution of investor probability beliefs as traders learn to adapt to their market position captured through the application of a Kalman filter. The behaviourally inspired asset pricing model shows marked improvement over a traditional OLS CAPM in light of evidence that the volatility of equity returns vary over time, finding that estimated pricing errors are reduced by as much as 41%. In comparisons of various behavioural designs, evidence presented suggests that investors tend to price risk towards long run-risk whilst being notably influenced by exposure to lagged market performance. Together, these findings lend support to the hypothesis that investors tend to price risk as a dynamic learning process which is informed by both internal behavioural heuristics of cognisance, as well as position in the market place as a matter of game theory placement. The findings of this study provide a strong basis for the further development of asset pricing in the state-space modelling environment in order to build on a theory of asset pricing through risk perception as a behaviourally dependent self-organised dynamic system.

## Contents

## List of Tables

# 1. Introduction

Are we, as a financial research community sufficiently aware, and considerate of behavioural considerations to the asset pricing problem? This is the central question upon which the following research report has been built. Through application of state-space modelling, the asset pricing model is re-explored as a dynamically evolving model which is sensitive to statistical and behavioural considerations of a market place which is described to resemble a self-organised stable market, as opposed to a static equilibrium market. The result is a model which interprets the evolution of risk, as a probability belief, as one which evolves along a time-series according to a learning algorithm. The expansive literature presented covers cross-disciplinary material which defines risk variation as a dynamically varying singular coefficient which represents an endogenously defined causal model, based on exogenous inputs leading to outputs defined by subjective beliefs and active interactions of market participants. The result is a single factor asset pricing model, the evolution of which is captured through the use of a Kalman filter.

In order to assess the viability of the proposed learning-augmented approach to asset pricing in a manner which is relevant to existing research into asset pricing, the learning-augmented model is tested against share portfolios sorted according to size and value. This is done in light of a well documented inability of the CAPM, as well as its extensions, to effectively and completely price risk in light of portfolios sorted according to these firm specific characteristics in both an international setting, as well as in the South African market (Fama & French, 1992; van Rensburg & Robertson, 2003; Basiewicz & Auret, 2009). The dynamic linear regression approach to pricing these stock market features used in this study considers a single factor asset pricing model which uses a Kalman Filter to model risk exposure as a state-space function. The benefit of this approach allows for the separation of an observable price evolution from that of a "harder to observe" quantitative risk coefficient evolution. The resultant model considers two equations which, through an interactive learning algorithm, allow the model to predict the risk coefficient as a linearly dependent function which improves its prediction through time. The treatment of risk as a probability belief which is dictated by behavioural heuristics as opposed to traditional rational expectations theory, whilst being allowed to be time-variant, presents as a significant divergence from traditional pricing theory. It is this conceptual divergence which forms the central theme and line of enquiry for this research. As a precursor to the development of an asset pricing model concerned with pricing risk on the JSE, this study begins with the establishment of a theoretical behavioural profile intended to describe how investors are likely to perceive risk. Understanding how a typical investor perceives risk will allow one to identify what it is that causes demand and supply fluctuations in the market, and hence price fluctuations. Thus, in order for an asset pricing model to accurately capture the risk-return relationship, it must be sensitive to the behavioural nature of the perception of causation. An expansive body of literature is presented in order to establish a new hypothesis for defining expectant investor behaviour characteristics with specific consideration to the perception of risk. The literature presented provides a strong theoretical rationale as to why a learning-augmented model might be considered a relevant avenue of investigation as it highlights the importance of considering the effect of human behaviour at both the individual level, as well as a group level in order to understand how risk in an asset pricing context can be expected to be defined. The application of the Kalman filter neatly applies this theoretical basis into a conceptually easy to understand regression function and how it can be expected to capture the behavioural profile of a typical investor actively trading on the JSE.

In order to proceed from the research basis outlined above, reading towards this topic has involved the consideration of three questions fundamental to asset pricing in the context of current applications of asset pricing models being subject to pervasive anomalies:

1. How are risks identified, processed, and thus priced by individuals?
2. What risks are relevant to asset pricing? Are all non-systematic risks diversified?
3. Are both homogenous and heterogeneous behavioural predispositions to investors appropriately considered?

The results of this investigation, presented in the literature review below, dictate that an appropriate model should follow on from the one factor systematic risk factor model design of the traditional capital asset pricing model (CAPM). The fundamental divergence from the traditional CAPM comes in how, as a cross-sectional collective, the risk representative slope coefficient of the linear regression model is specified, as well as how it is described to evolve over time, tailored to be considerate of specific individual and interactive behavioural characteristics which are postulated to be indicative of a typical investor. The result is a form of linear regression which is predicted through the use of a recursive learning-augmented algorithm by making use of state-space mathematics. The qualitative basis for this research is derived from diverse cross-disciplinary sources, whilst the quantitative description takes direction from the successful application of the state-space routed Kalman filter to equity markets as dynamic systems by Adrian and Franzoni (2009) as well as Huang and Hueng (2009).

The remainder of the study is organised as follows: A qualitative exploration of stock market dynamics as a feature of human interaction and behavioural anomalies is presented across 3 chapters in the literature review before being collectively packaged as an overall behavioural framework, appearing under section 2; A presentation of proposed hypotheses based on a qualitative investor model is then considered under section 3; The data selection for a quantitative evaluation is described under section 4; a description of the quantitative model and state-space modelling methodology is presented for testing of the time-variant nature of the behavioural description presented in the literature review is detailed in section 5; results and interpretation thereof are presented under section 6, and discussed in light of the model under section 7; finally a conclusion and a review of the research is presented in section 8.

## 2. Literature review: A qualitative analysis

The perseverance of pricing errors to traditional asset pricing models remains at the forefront of financial literature today, with particular prevalence observed where portfolios are sorted by size and book-to-market ratio. The extent to which these, afore mentioned, pricing anomalies are observed across markets are considered to be too common to be ignored or explained away as outliers to traditional pricing theory. Noted by Fama and French (1993) on American data, Chiao and Hueng (2005) on Japanese data and, more relevantly to a South African investigation, by Basiewicz and Auret (2010) on South African data, the extent to which traditional asset pricing models fail to capture relevant market risk in a complete description is clearly evident. An extremely large body of work in financial literature has explored these pricing inefficiencies through extensive cross-sectional exploration, looking to capture these unexplained risk events by isolating and regressing returns on fundamental risk drivers (Chen, Roll, & Ross, 1986; van Rensburg, 1999). This approach is generally developed to be in line with a rational expectations model whereby the central premise of risk pricing of the CAPM is maintained by considering non-diversifiable systematic risk as the only rational exposure for which returns should be expected to serve as compensation, whilst addressing the practical limitations of a single risk factor to appropriately represent a complete cross-section of market variables (Roll, 1977). The results of this avenue of research have however been mixed, and whilst there has been notable evidence of the success of this multifactor approach to improve on reducing prediction errors, none of these models have achieved an unequivocal triumph of the asset pricing problem. In contrast to the traditional efficient market view in which asset prices are not considered to be linearly dependent on past observations, studies into the perseverance of momentum returns in excess of those predicted by a traditional CAPM model have drawn time-series explorations into the fore as possible limitations to traditional asset pricing theory (Jegadeesh, 1990; Jegadeesh & Titman, 1993; Carhart, 1997). Extending the consideration of time-series dependent return observations, the success of conditional extensions of the CAPM to consider cross-sections of fundamental risk is shown to add little to no value when tested in light of time-series risk variation (Lewellen & Nagel, 2006).

The results of these studies have failed to prove asset pricing to be a definitively exogenous risk phenomenon which simply requires extended cross-sectional analysis, setting the stage for an alternate pursuit of asset pricing which deviates from a strictly exogenous cross-sectional investigation, and follows a time-series based time variant pursuit with a strong behavioural flavour. Support for a time-series approach to considering the asset pricing problem has had strong empirical support as extensions of the traditional CAPM framework having been documented by authors such as (Harvey, 1989), Ferson and Harvey (1991, 1993) and Jagannathan and Wang (1996). In more recent extensions to the time-series approach to asset pricing, the success of research such as Adrian and Franzoni (2009), Huang and Hueng (2009) and Trecroci (2013) in exploiting this volatility through state-space modelling forms the basis from which this study builds its empirical research design. The point of departure from the afore mentioned authors' considerations of time variant model design rests with the definition of risk itself, a point which is detailed through the diverse body of literature to follow.

The literature review is structured as three distinct discussions. The first topic considered, which is presented under chapter one of the literature review, is that of risk from an isolated behavioural perspective, focusing on investors as individuals. This section considers studies from finance, philosophy and neuroscience in order to build an understanding of how individuals are expected to perceive risk. The second leg of the literature review, presented under chapter two of the literature review, considers the interactive effect of market participants and how the market can be seen to behave according to behavioural expectations of groups. This section considers game theory, the adaptive market hypothesis (AMH) and the notion of feedback models. The final section of the literature review, presented under chapter three of the literature review, deals with these implications from a statistical design perspective. As a lead into the methodology, the behavioural perspectives considered in chapters one and two are realised from a distributional perspective, as well as a look into the treatment of observable correlation as an indicator for establishing a causal model.

## 2.1 Chapter 1: A behavioural description of the individual

### 2.1.1 Establishing probability beliefs

The movement to consider an alternate risk exposure measure to that of a singular systematic risk variable is not new, nor is it unfounded in economic literature. For a long time, asset pricing models have used the assumption of a complete asset universe, or at least a proxy thereof in order to price assets as calculated by their risk exposures. This has been done by calculating their correlation to a "complete/accurate proxy thereof" market portfolio (Roll, 1977). Under the assumption of a perfect proxy sample, the market portfolio correlations should be approximately causative, given an adoption of traditional rational market efficiency as an appropriate descriptive theory of investor behaviour. The major drawback to reality however, is highlighted in Roll (1977) which points out with unforgiving assertion that a complete market portfolio is as unlikely to define as it is to test. This leaves the ability to perceive market correlation as an effective proxy for fundamental risk as potentially undermined, and practically unreliable. Bearing this consideration in mind, this study considers an alternative approach which looks to price risk based upon the premise of decision making under uncertainty, being relevant to both systematic and non-systematic risks, captured through investor probability beliefs. Instead of defining the relevance of risk by a theoretical description of rational expectations, the behaviour of investors is developed through observation of the effect of demand and supply which is used to inform the true nature of investor probability belief. The rationale behind this takes its direction from a large body of work into the psychology of investors, and the possibility that behavioural heuristics, in conjunction with social interactions provide a sufficient deviation from rational efficient market theories so as to consider that risk exposure, whilst informed by exogenous factors, manifests as endogenous output of a feedback model.

The background to establishing the concept of risk presented in this pricing model lies in an exploration into the trader rather than the assets traded. The perspective it taken that, whilst assets must have value to be traded, it is only the market dynamic which ultimately dictates a share's value as a traded entity, and it is the probability belief surrounding the perspective of future value which determines that market dynamic, and therefore the risk of the asset. Indeed, in a market, it is only what another will pay which determines the liquid value of an asset and the perception of future value which determines that asset's price stability and hence risk. The perspective of market dynamics shifts the focus of an observation of risk from cross-sectional exogenous risk variables to a time-series variation of investor demand and supply. Whilst the focus is shifted, it is not to say that exogenous risks are not relevant, but instead they are considered to be too difficult to directly observe as their effects are not expected to add unbiased informational value in isolation once the confounding effects of a behavioural market theory are taken into account (a concept built upon in chapter 2). The cross-section of risk drivers are considered to be represented in the market's collective probability belief, the development of which is subject to specific behavioural biases at both the individual and market level. The effect of the probability belief collective is to shift the focus of risk drivers from exogenous inputs to an individual's decision criteria, to an observation of endogenously determined outputs of that process. A resultant feature of the shifted focus is that the probability belief considered in this study is not representative of exogenous systematic risks only, but rather is expected to be representative of all variables which are considered to be fundamentally relevant to the individual investor at the time of investment. Fundamental risk is thus collectively referred to as variation around the mean objective probability, probability which is generally underlying and unknown by the investor therefore leading to the requisite formation and use of probability belief (Markowitz, 1991).

An investigation into the manner in which investors learn about risk, and therefore form probability beliefs themselves is thus required. Specifically, the question at hand, and one which is central to developing a behaviourally defined pricing model, is how individuals infer a causal relationship between news and events which they observe and the effect to an asset in which they invest, as it is this which initialises the mechanisms of demand and supply and therefore, risk and return.

### 2.1.2 Human nature

The following subject of investigation forms the basis for all literature to follow, as it builds the principle probability belief as a definition of causal risk from the perspective of individual behaviour. The point of the investigation is to conceptualise a framework upon which investors' decision making abilities are expected to be predisposed to, based on the philosophy and science of human nature, which serve to instruct how best to design a model which can quantitatively improve on existing asset pricing theories.

Causation and Inductive Inference: "Reasoning's... It is constantly supposed that there is a connection between the present fact and that which is inferred from it" (Hume, 1975b). Hume (1975b) notes an important connection between an observed event and its cause by presenting the ambiguity of inexperienced reasoning. This is established by defining effects as being distinct events from their causes, that is to say, following the observation of an event, the human mind is equally capable of reasoning whether a particular outcome may transpire, or whether it may not transpire, the formation of a relationship between which can be equally rational. The result of this distinction, of ability to reason a potential belief before a time-series of observation is considered, presents as prescriptive for why causal reasoning cannot be considered *a priori* reasoning. Thus, cause and effect are discovered *ex post*, and not *ex ante*. This presents as direct opposition to the concept of prices being driven *a priori* through reasoning of potential outcomes. What this means in terms of research into the pricing of assets is that a forward looking (exclusively) practitioner has no tangible ability to reason outcomes based on cause.

It would appear that by our very human nature we are destined to rely on past information to help guide our future interpretation and reasoning. The extent to which this may, or may not undermine the concept of the efficient market hypothesis (EMH) is up to a degree of interpretation, and whilst that is a topic for debate, it is not of primary focus at this point. What this interpretation of cause and inductive inference suggests to the practitioner is that whilst prices might be forward looking, the ability of a practitioner to use this information is bound by his requirement to understand past market events. It is important to note that the issue at hand is that of the perception of risk, and how it is established. Thus the point that reliance on past observation is required in order to make a forward looking causative inference is not to say that asset prices are necessarily autoregressive in their development, but rather the formation of a probability belief begins to take on an autoregressive description.

The extent to which observed experience is central to causal reasoning warrants an extension into the theory behind which causal reasoning through experience is expected to be formed. Hume (1975b) finds that causal expectation is produced by experience of cause, and expands on a theory of causal reasoning through an exploration of causal inference through observation. Through repeated exposure to *ex ante* cause and effect we establish a custom, and it is the strength of this customary observation of cause and effect which drives our confidence in causal expectation. The description of causal expectation and its emphatic relation to past events is not complete here however, as our reasoning at this stage of explanation remains hypothetical. One can hypothesise the likelihood of an event transpiring by reasoning the likelihood that previously observed causal custom might occur. Thus you might say here that rational behaviour is bounded by experience. In this sense, the ability to reason is based on a time-series of observation subject to progressive understanding through learning.

To this consideration, Hume (1975b) adds to the requirement of causal reasoning, the subjective belief that the event will transpire. This belief is extended to a tangible experience, a fact present to the individual's memory. As nothing in life should ever be taken for granted, this extension to the development of causal inference to be reliant on memory necessarily brings to the fore recent research into human memory function and its functional form. Evidence from the field of neuroscience has found that the functional design of the hippocampus (the part of the brain tasked with memory storage and recall), is likely not, by design, to be used as a purely recall function, but rather a form of learning predictive function-Thus the memory of belief may be necessarily biased itself, and therefore may not present as strictly objective or scientifically rational (Sharot, 2011). Hume’s philosophy of belief is not contradictory to this sentiment, as it is reduced to a sensory rather than cognitive part of human nature (Hume, 1975a). Hume presents a work which thus limits human reasoning (causal) to one which is “independent of all the laboured deductions of the understanding” (Hume, 1975a).

The proposed theory presents as a fundamental departure from statistics when considering the human mind and its construct of reasoned case, in essence, this contention should really be no contention at all, but rather a case of returning statistics to the business of observation as opposed to assumption governed mathematics. “There is nothing in a number of instances, different from every single instance which is supposed to be exactly similar” (Hume, 1975b). This statement is comparable to the event of a toss of a fair coin. In repeated tosses, previous tosses, which result in their particular outcome, do not influence the outcome the forthcoming coin toss. However, repetition of a single event's outcome is seen to form a habit in one’s mind. Hume (1975b) states how although the sensory experience of each event is separate and non-compounding, the mind tends to over reflect, and compound the sensation of a particular observation. This compounding of sensation illustrates its self through enhanced belief due to observed cause forming habit/custom. This behavioural observation provides a clear description of the, at times seemingly irrational, way in which the human mind is predisposed to process information, whilst being empirically perceivable as an optimism bias of cognitive memory recall (Sharot, Riccardi, Candace, & Phelps, 2007).

Hume, through his studies of cause produced two definitions of cause, one of which (the first) is statistical in its nature, and the second of which is the behaviourally influenced cousin. The first describes the nature of cause necessary to statistically observe and test data. The second, importantly, provides insight into how such raw data might best be interpreted.

1)”An object, followed by another, and where all objects similar to the first are followed by objects similar to the second”; 2)” An object followed by another, and whose appearance always conveys the thought to that other” (Hume, 1975b).

This work on causative reasoning further provides just reasoning for proceeding with a methodology which uses past causative trends to inform expectancy surrounding predictable future price performance. Specifically, the above work needs to be considered in the learning context, and thus this philosophy needs to be applied to the development of the learning process, the separation of which is effectively handled by the state-space form of the Kalman filter, all the while maintaining necessary observatory relevance through a separate pricing equation.

This leaves us with an interesting prospect regarding asset pricing. Essentially pricing is expected to still operate in an inherently forward looking manner. However, the only way to understand how one is to price an event forward is to understand how one has done so in the past. When considering the many confounding directions and opinions on the stock markets, this requires that the aggregate reaction to any given event is considered, considering the strength and persistence of a particular custom. Further to the concept of custom, it is necessary to consider that varying investment horizons are likely to have varying customs, and thus an asset pricing model based on this premise of sensory driven causal reasoning cannot be expected to apply globally, but rather in order to be accurate (as possible), it should be specified to a particular market. Evidence to the importance of cross market heteroskedasticity is clear when comparing what appears to some to be "quasi-rational" trading in eastern markets as compared to the market dynamics of the west (Fama & French, 1998). A further caveat in this analysis of pricing models is to be mindful of the fact that causal reasoning is by no means expected to be consistently rational and free from behavioural biases. Trends and momentum are indeed expected to be prevalent, as well as pricing deviations from purely rationally deduced fundamental risk factors. An interesting artefact of statistical design becomes apparent not only for its importance in economic model design and the testing thereof, but in proper learning required for accurate interpretation of cause and effect for a practitioner. This is the requirement to train habit and sensory experience over a sufficiently large period of past data so as to assimilate the effects of all reasonably foreseeable causative possibilities so as not to skew the learning/training period.

#### 2.1.3 Risk in light of the human predisposition

In order to price assets one needs to consider the long standing risk-reward relationship, where-by one needs to be compensated for excess risk exposure. One might say that as risk levels vary, necessarily so too do prices. The problem is identifying the risk, and thereby predicting associated returns, a premise by which many asset pricing models are established. Risk is an abstract prediction of the likelihood of stability; it is how likely one's exposure might lead to a drop in returns due to a drop in asset price. In a sense then, risk drives price because variability of the outcome is personified as risk affects price. The two are circular in a sense. Thus what, one might ask, is risk? Risk is the perception of change and thus the abstract cause defined above. In order to price asset returns, one must understand the perception of risk, and the perception of risk is the product of causal reasoning.

Fundamental movements change the value of assets and prices move; there is no risk here, simply a cause and effect. Risk is only present when an opinion is created regarding said movements. One might argue the question: If an event happens in the absence of opinion, is there risk or simply result? Risk is introduced when there is doubt in belief that the event may happen. Thus risk is a manifestation of sensory interpretation of past experience leading to reasoned interpretation of causation. By this argument, at least a part of risk has to be attributed, if not all of risk, to causal reasoning which has been shown to be a feature of a non-rational backward learning forward reasoning, and confounded sensory experience of the human mind. It is important to understand here that this theory is based on the treatment of the equity market in a sense that whilst fundamentals are informative, cognitive bias skews market behaviour from a purely efficient form to a type of secondary asset market whereby prices are driven through the type of behavioural anomalies described above. A separation is therefore made between the quantity of loss experienced through the fundamental drop of prices, and that of the confounding effect of cognitive bias and associated trends in the movement of prices on the asset market of equity stocks abstract of the direct effect of fundamental variation causal drivers. It is this expansive movement which exists only in the presence of opinion. One might therefore consider in light of traditional pricing models that the risk attributable to endogenously driven equity returns is a product of opinion, and only that which is endogenous to the supply and demand of equity markets is by definition a fundamental risk. Any consideration of endogenously driven risk must be a product of some or other form of bias in the process of reasoning and inference. Importantly, these factors are necessarily informed by fundamental movements, and thus must be considered in the presence of a model which prices assets based on exposure to fundamental risk in light of the human predisposition.

Thus asset pricing needs to operate in a fashion which considers the input based importance of exogenous risk asset factors, whilst pricing based on the development of their subjective interpretations as dictated by behavioural heuristics of the market and market participants in question in order to perceive, as best as possible in such an expansive context/model, relevant risk and therefore price. The behavioural description of subjective interpretation of exogenous movement leading to an endogenous probability belief is easily extended beyond the individual when considered at the market level. Where market movement is defined as a result of varying game theory due to variant probability belief expectations, risk takes on the following form: The perception that something will change unexpectedly is based on perceived cause as viewed by market segments. The expectance is still based on causal reason; however it is also based on the subjective interpretation of other market participants who themselves are subject to the very same behavioural and neurological predisposition. The investigation is thus confounded further from initial fundamental exogenous risk factors, and the necessity of the learning from market participants as a feedback model becomes a necessary factor in risk pricing. Thus a probability belief is a product of causal reason both as individual isolation, as well as a diverse market, and risk presents as its failure to perceive a particular outcome. That is, if risk is the chance that an outcome deviates from the expected, then the cause of the expected outcome causes risk in its absence or impression. Thus risk is a derivative of probability beliefs of market participants which is directly represented in the returns to assets as a function of variation in demand and supply levels.

Considering that market prices are driven by investor demand and supply (as the JSE operates as an auction type market), what drives stock prices is ultimately the cumulative market perception of risk and return. That is to say, whilst investors may consider forward looking expectations, the causative inference ascribed to forward looking data is necessarily informed by past observations of correlations, and continually updated in a dynamic learning framework. In addition, and in support of this, modern research in the field of neuroscience, as presented in a description of human nature above, is considered for findings into the functional form of human memory (Sharot, et al., 2007; Sharot, 2011) and the effects that these findings may have on the development of a description of risk as a product of perception. These papers establish a hypothesis for hippocampus function which is contrary to the layman opinion of the purpose of memory. The memory function of the brain is found to be descriptive of predictive functionality, as opposed to that of direct recall. As a result, under this hypothesis, an asset pricing model which looks to establish a risk-return relationship through unbiased past correlative observation is unlikely to adequately profile the behaviour of an investor who biases interpretation of past events in order to make a predictive causative inference.

Reading the above as a behavioural description of investor mechanisms which ultimately drive demand and supply establishes the first model construct condition. In order to be considerate of potential biases in the way in which investors arrive at a basis for causative inference, a dynamic state-space asset pricing model which incorporates a learning functionality is required. This leads to the consideration of a Bayesian type model base[1]. The Bayesian approach provides a mathematical rule to explain how you should change your existing beliefs in the light of new evidence. The theorem can be viewed as a mathematical description of how probability beliefs about the future are updated through updating past observations. This form of inference updating system is agreeable to the kind of causative inference described by Hume (1975a; 1975b) by which individual belief systems are influenced and updated through observation of patterns and habits. Where the Bayesian model falls down though, in light of the behavioural structure presented, is its unbiased probability updates which do not allow for dynamic learning conditions. In order to consider a Bayesian statistics based model which allows for observation importance to be influenced by learning about predictive accuracy, a Kalman filter is natural model progression from a recursive Bayesian model.

#### 2.1.4 Diversification and allocative efficiency to a market utility theory

The behavioural description of probability beliefs and the nature in which it is expected to evolve through learning and thus dictate investor behaviour presents a contrast to traditional rational expectations, through which a theoretical framework is presented whereby investor returns are related to risk exposure beyond that of systematic market variability. Specifically, the CAPM and many of its extensions base the pricing of risk on, amongst others, the assumption that investors are myopic risk averse traders who hold assets in accordance with modern portfolio theory (MPT) which suggests that investors hold a portfolio of stocks to diversify idiosyncratic risk, a concept on which theory assumes that investors hold a minimum variance market portfolio in equilibrium. The result is that popular asset pricing models including the CAPM and many of its extensions only price systematic risk, as it is argued that idiosyncratic risk can and should be diversified away. As a result investors should not be able to earn excess return over the market portfolio for taking on excess diversifiable risk. This section of the individual investor model contends this, by presenting evidence to suggest that these assumptions are non-descriptive of individual investor behaviour, and as such, a model which does not consider the relevance of risk beyond that of systematic risk exposure is very likely expected to be miss-specified. Evidence presented here suggests that, in contrast to traditional asset pricing theory, investors may in fact not tend to perfectly diversify their equity holdings. This behavioural tendency of investor portfolio theory is presented in direct contrast to the risk-averse theoretical basis of the CAPM. As such, an

[1] A recursive Bayesian model is a predictive model which updates future predictions based on Bayes' Theorem in a time-series environment.

alternate expectation of risk appetite is considered under a behaviourally inspired utility theory. This descriptive model of how decisions are made under uncertainty is presented as a basis for why probability beliefs are expected to price risk as an endogenous collective which considers asset exposures beyond systematic risk considerations alone.

Goetzmann and Kumar (2004) find that in 62 000 American households considered, less than 10% of the portfolios contained more than 10 stocks, a number which is considered to be far too low for perfect portfolio diversification when considering research by Campbell, Lettau, Malkiel, and Xu (2001) which finds that in the modern investment universe, the number of stocks required in order to diversify a portfolio has increased from previously documented levels, suggesting a requirement in the region of 50 stocks in order to achieve portfolio diversification (Fu, 2009). Further research by Goetzmann and Kumar (2008) details the extent of under diversification across wide ranging investment profiles, citing various explanations including behavioural bias and superior information, among others. Irrespective of the reason, where under diversification is widespread, these portfolios are clearly exposed to idiosyncratic risk variation, which impresses two points on expected portfolio theory and associated asset pricing: first is that investors are not absolutely risk averse, as they are seen to accept exposure beyond that which is undiversifiable; second, risk exposure beyond systematic market exposure is required to be relevant to investors. Importantly, these findings of large stock holding requirements for effective diversification are not unique to the American market. Evidence from Bradfield and Kgomari (2004) (as cited in Kruger & van Rensburg, 2008) presented on the JSE, suggests that as many as 30 to 45 stocks are required in order to diversify a portfolio when considering the ALSI. These findings, when read in light of evidence to suggest that as many as 60% of South African investors active on the JSE hold portfolios composed of 10 shares or less (Firer, 1988) suggests that risk appetite and associated considerations presented on the US study above are relevant to a South African enquiry. This suggests that there is scope to consider the relevance of a risk factor which accounts, in a more focal manner, for behavioural bias and trends in equity markets beyond the restrictive assumptions of MPT and the CAPM. As investors are evidenced to hold portfolios which are not isolated from firm specific risk considerations, the probability belief upon which individuals base their investment activity needs to be necessarily dependent upon both systematic and non-systematic risk, the interpretation of which is expected to be subject to the bounded rationality of the human behavioural predisposition described above. This perceptive requirement of risk consideration is further endorsed through a survey of South

African investors (Firer, Oliver, & Farrelly, 2011). Importantly, Firer et al., (2011) find that these risk perceptions are felt by both individual and institutional investors alike, aiding to suggest that the importance of perception of risk is important to informing investment decisions across all types of investors on the JSE. Furthermore their study explicitly displays the inability of a market beta to account for this perceived risk.

Beyond this argument of necessity to price risks beyond systematic market risk, the evidence which displays that investors tend to hold a modest number of stocks demonstrates an implied risk appetite at odds with traditional risk aversion, as investors clearly appear to display from evidence of under diversification. This behaviour is further emphasised as a likely feature of an informed investor community when quoting Warren Buffett, who had the following to say about diversification: “Diversification is protection against ignorance. It makes little sense if you know what you are doing.” (Buffett, Date unknown). Where divergence from a traditional rational expectations model under MPT is found to be wide spread and common amongst investors, the effect on supply and demand is unlikely to be dictated from the singular systematic risk consideration, as is proposed by the CAPM.

#### *2.1.4.1 Towards a utility theory*

Various theories assuming under-diversification predict that idiosyncratic risk is positively related to expected stock returns in the cross-section, that is, investors are expected to be compensated for bearing idiosyncratic risk, contrary to traditional portfolio and pricing theory (Markowitz, 1952). The concept is certainly not new, and previous notable advocates of this theory of idiosyncratic relevance include (Levy, 1978; Merton, 1987; Malkiel & Xu, 2002) amongst many more. Recent confirmation by Fu (2009) suggests that the concepts of risk and return borne out of MPT (Markowitz, 1952) are incomplete due to their assumptions about a uniform rational investor universe. Evidence to suggest that investors are rewarded for taking on idiosyncratic risk through excess returns suggest that any complete multifactor asset pricing model must necessarily provide scope to price assets based on their exposure to a combination of systematic macroeconomic risk factors, as well as microeconomic idiosyncratic risk factors, all of which are expected to be subject to individual interpretation and the interaction driven feedback effects of game theory.

Building upon empirical evidence presented above referring to a behavioural description of investors with respect portfolio theory, the assumption of risk-aversion amongst investors becomes somewhat questionable. As such, alternate utility theory is explored in order to define a description of investor perspective which is more appropriate in the explanation of investors' risk appetite than a traditional risk-aversion model. Myopic loss aversion is considered as fitting this requirement as a more appropriate descriptive model, in which agents are considered to be risk averse over individual bets, but tend towards risk neutrality as bets are lumped together (Rabin & Thaler, 2001). This application is relevant for the equity market investor who takes on (generally speaking) multiple market positions. According to Arrow (1971), expected utility maximisers are arbitrarily close to risk neutral when stakes are arbitrarily small (Rabin & Thaler, 2001). Risk neutrality of expected utility is however not restricted to particular contexts, particular functional forms, or negligible stakes. "Loss aversion says that people are significantly more averse to losses relative to the status quo than they are attracted by gains, and more generally that people's utilities are determined by changes in their wealth rather than absolute levels" (Rabin, 2000), where loss aversion is a term attributed to Kahneman and Tversky (1979) as a descriptive part of behaviour under prospect theory which is the basis upon which the investor probability belief system is built.

In extensions of Rabin's (2000) theory, the theory of concave utility functions tends to result in the marginal utility of wealth diminishing at absurd rates, Rabin and Thaler (2001) moves to suggest that people will not be averse to risk involving monetary gains and losses that do not alter lifetime wealth enough to affect significantly the marginal utility one derives from that lifetime wealth. This provides a possible behavioural rationale for markets which may experience some fluctuations which deviate from absolute efficiency in spite of an environment with perfect informational efficiency. Certain pricing errors are expected to exist, which is in line with an adaptive market, and fluctuation around perfect allocative equilibrium is descriptive of a stable self-organised system. Building upon this description of non-equilibrium stability, (Kandel & Stambaugh, 1991) find, in an adaption of the equity premium puzzle, that due to the varying circumstances of individuals and their degree of intent when investing in equity markets, the coefficient of relative risk aversion cannot be expected to be consistent. Based on this, the conjecture that allowance be made for the consideration of idiosyncratic risk factors in a general asset pricing model should not be unreasonable as a function of utility theory. Importantly, these findings are indicative of a model which is definitive of a dynamic risk factor, a finding which dictates that any asset pricing model which is appropriately descriptive of expectant human behaviour must be necessarily dynamic by nature. As investor circumstances vary, and the interpretation of risk is thus non-uniform across investors and time, the description of a risk factor takes on a time-variant nature of a dynamic probability belief model.

#### 2.1.5 An empirical description of idiosyncratic risk relevance

Evidence to support a theory of idiosyncratic risk relevance is expanded upon here through a comparison to liquidity. As demand and supply directly affect liquidity, if idiosyncratic risk is influential to motivating investor demand and supply, it should encompass risk as defined by a liquidity factor. In evidence to support this notion, Spiegel and Wang (2006) find that idiosyncratic volatility fully encompassed liquidity in explaining the cross-sectional variation of average returns, but not vice versa (as cited in Fu, 2009). Liquidity is a highly motivated idiosyncratic risk factor in its own right, in that it is firm specific, and is a concern which, in its absence, would generally attenuate demand. Whilst liquidity compounds on a low demand issue, it surely cannot be a root cause. Liquidity results rather from a particular characteristic of an asset which makes it more (or less) desirable to own. The risk attributable to the term liquidity itself is no more than a by-product of the probability beliefs system by which individuals adjudge desirability of an asset, and in turn exercise their demand upon. Thus liquidity presents as a by product of market interactions, and when probability belief is specified correctly, it is expected to present no value. Spiegel and Wang's (2006) paper aids to suggest that market movement and liquidity are linearly dependent, a conjecture which this paper makes through probability beliefs.

A fundamental problem with the concept of diversification and it's appropriateness in finance is that whilst it is possible to effectively diversify a portfolio with a modest number of stocks, a significant number of authors find that generally expanding pricing models to include idiosyncratic risk factors consistently provide universal and robust improvement to asset pricing models (Fama & French, 1993; van Rensburg & Robertson, 2003; Lakonishok, Shleifer, & Vishny, 1994). This suggests that on aggregate, the risk factors of idiosyncratic risk are not being arbitraged out of the market through complete diversification strategies.

A central argument as to why this may be the case is a consideration of the fundamental premise of the investor. This is achieved through a comparison between investment and insurance as functions of alternate pursuit (investment being that of returns, and insurance being that of limitation of loss). The diversification principle is described in its success in defining a risk-return principle in the insurance industry whereby returns are maximised through increased diversification (Burmeister, Roll, & Ross, 2003). This comparison to risk-aversion in the insurance industry helps to place into stark contrast the concept of complete risk elimination, and that which is likely to be expected on the stock market. The insurance industry makes its money when risk is eliminated, maximising towards complete risk elimination. With equity investing on the other hand movements towards complete diversification, whilst eliminating losses, also eliminates large potential for returns. This effectively creates juxtaposition for the investor whereby (complete) diversification and total risk exposure should be viewed as relative extreme scenarios, and investors should be expected to, on average, occupy the space in between. Because investors as a collective are not expected to pursue complete diversification, asset prices should be expected to be influenced to some extent by idiosyncratic risk factors, in contrast to the position taken by the CAPM under the assumptions of MPT (Sharpe, 1964; Lintner, 1965; Markowitz, 1952), as well as systematic multifactor expansions of asset pricing models such as that of Chen et al. (1986).

Considering the scarce evidence of perfect diversification as considered by MPT (Markowitz, 1952) (the premise that investors do not follow diversification strategies), it seems unreasonable that said investors would, or should, follow the asset pricing of a model which bases its ability on the assumption of perfect diversification of all assets in a complete asset universe. This is not a new concern, and is one clearly considered for its lack of justifiable use in that proof of its legitimacy is considered as being infeasible by Roll (1977). Other authors have implicitly drawn into question this concept of efficient diversification through their practical tests of idiosyncratic risk minded CAPM expansions, not to mention the consistent success of said models (Fama & French, 1993; Lakonishok et al., 1994; Carhart, 1997), amongst many others, in finding a clear market risk-return relationship between diversifiable risk and excess market returns. Considering the results of the empirical findings of the authors above, it appears to be clear that the assumption that the market as a risk proxy for systematic risk, and the irrelevance of idiosyncratic risk as a model argument is infeasible leading to an investigation into information efficiency through descriptive behaviour as opposed to informational and allocative efficiency through strict rationality.

The under diversification effects discussed above serve to provide empirical evidence to the behavioural assumptions which make up prospect theory, whereby investors are willing to accept degrees of idiosyncratic risk exposure all the while still behaving in a rational utility maximising manner. The implication is that a relevant asset pricing model should not be focused on allocative efficiency, but rather informational efficiency, the risk appetite of investors is not expected to result in all assets uniformly being arbitraged to eliminate all idiosyncratic risk. As a result, a singular systematic risk factor which is defined to capture only undiversifiable market risk and ignore idiosyncratic and behavioural variables is submitted as inappropriate in favour of the collectively descriptive linear dependence model of risk pricing presented in this investigation. This argument of utility theory and evidence to describe a risk appetite informed by wealth considerations as utility maximisation, read in light of theories surrounding the human predisposition to learning and memory function presented above are descriptive of a model which views risk as being a linearly dependent representation of both systematic and non-systematic risks where they influence individual probability beliefs.

## 2.2 Chapter 2: A behavioural expansion to the market

Chapter 2 expands upon the behavioural outline presented above, which focused specifically on an individual market participant perspective. Here the behavioural perspective is expanded to consider the effects of market interactions, and how the behaviour of individuals in groups can be significant to that which might be expected of an individual in isolation. Considering the effects of game theory and it's evolutionary description as an AMH, a dynamic model is presented which builds upon probability beliefs as not only being a linear learning function of time, but a cross-sectional learning function of market participants, which too is expected to develop as a time-series in its own right. This description of market dynamics ultimately leads to a unified description of the interactive effect through a feedback model which is used to describe the information processing behaviour which is proposed for better understanding risk perception, and thus  asset pricing.

A large body of research into the time-varying effects of risk exists to support the notion of a time-varying investigation as pointed out in chapter 1 above. The question then is, if probability beliefs are indicative of investor behaviour and the market is considered as a collective trading body, why then has momentum not formed the basis for a time-series dependent risk variation explanation. The use of momentum has been tested to great extent in capital asset pricing (Jegadeesh 1990; Carhart, 1997; Jegadeesh and Titman 1993, 2002) and whilst these authors note the potential strength of a momentum factor in explaining returns, it has not been found to solve the asset pricing problem. The fact is, the time-varying treatment of risk appears to be a less vanilla problem to solve than unified market trending in the form of momentum. Momentum trading follows a generally singular perspective of a majority rule, one which for the most part conforms to a uniform beliefs system. This considers that all traders conform to a singular majority opinion regarding probability beliefs and thus undermines the individuality of human beings. The effects of heterogeneous beliefs systems is explored through consideration of opposing views through the effect of minority rule, leading to a perpetual example of the AMH. The economic value of heterogeneous beliefs is empirically demonstrated at the individual level by Malkiel and Xu (2002) which notes the importance of considering risk exposure in excess of systematic risk, the value of which is shown as a feature of greater market interactions through the contrarian strategy study of Lakonishok et al. (1994).

### 2.2.1 Minority rule

Game theory is a complex discipline which observes the behaviour of individuals as they interact in various environments. The consideration of game theory in its simplest form leads to the establishment of best expected equilibrium behaviour, most often presenting a solution of how the persuasions of the majority come to rest in light of the likely expected behaviour of each other participant. Whilst this is a bold simplification of this extensive subject, the study of how behavioural patterns come to equilibrium is a useful simplification for an economic study of asset pricing. In this section, the concept of minority rule is considered for its relevance in undermining majority rule momentum in supply and demand, and how the behavioural contrast is relevant to a dynamic model specification. That is to say, the concept of a self-organised stable market which does not settle into a static behavioural equilibrium is explored. Market participants are expected to dynamically shift their positions according to the utility theory of loss-aversion as opposed to risk-aversion, leading to a market which has freedom to shift dynamically between positions of variant risk profiles, governing itself as opposing market positions change, causing the market to dynamically shift as the favourability of various game theory strategies interchange.

Economist Brian Arthur observed a version of the minority problem when studying complex systems at the Santa Fe Institute in New Mexico. Dubbed the El Farol problem (Arthur, 1994), describes how utility maximising individuals will often, in certain circumstances, find it to be beneficial to always be in the minority. This is much like considering the benefit and desire to be a buyer in a seller’s market, and vice versa. In his model, the problem presents a form of game theory which rewards individuals who can best guess majority intentions. The model considers how individuals use various rules of thumb to try and best guess the probable movements of the masses.

This presents a complication, as the decisions of the individuals are not, in the circumstance of strategically shifting between minority and majority behaviour, based solely on the fundamentals of the consumption variables in question. These fundamentals are considered to determine whether the market (game) is entered, beyond which the decision making process is further complicated as value is expected to be greatly influenced by the interactive interpretation of market participants and their various interpretations of expected probability beliefs. Only once the merits of the consumption variables in question have been established and are considered to generate sufficient utility to the market will the game be entered. Once the market as a whole establishes consensus of the benefits of the variables, the decision becomes more complex, as the focal point of decision drivers changes to be dictated by a form of game theory. Whilst the notion of directing investments in directions other than what the fundamentals dictate may be considered to be counter intuitive and possibly irrational, one must consider that whenever there is a question of uncertainty of multiple individual participants, the prospects of game theory need to be considered. Whilst the aim of this research is not directly consider the validity of theories such as the EMH, it should go without saying that, the EMH aside, the often perplexing nature of the stock market's movements, as well as varying liquidity in stocks should suggest that individual investors are not investing in the exact same way. There are constantly opposing views and thus demands for various financial assets. This simple truth should prove adequate to suggest that a level of game theory is a plausibly relevant consideration necessary to the valuation of risk and return to an asset market, which is ultimately dictated by human interactions and diverse behavioural and circumstantial predispositions.

Challet and Zhang (1997) refined the El Farol problem to establish a more complete minority game theory. Their study draws some interesting parallels to the AMH and how minority rule is expected to affect market dynamics. Under the AMH, research has found less than complete support for persistent adaption towards efficiency in equity markets compared to derivative markets. In an option pricing world the AMH appears to hold well for the reason that the instruments can be priced to their theoretical fair value (Potters, Cont, & Bouchaud, 1998). The success of the AMH in option markets is also expected to be linked to the leverage effect, allowing bad traders to be quickly priced out of the market due to expansive downside exposure, allowing it to adapt to equilibrium. A possible explanation for why this effect does not prevail in equity markets is presumed to be due to the limited liability of equity markets compared with the leverage of derivative markets - downside penalties to equity investors are capped at their initial investment value. The result is that consecutive poor trades in derivatives markets are likely to wipe out and eliminate poor traders faster than in equity markets where losses are limited to investments. The parallel to be drawn is that, when Challet and Zhang (1997) tested their model, they were only able to considerably improve market efficiency in light of a minority game where they allowed participants to evolve in a Darwinian manner, whereby poor traders are eliminated and the most successful are allowed to multiply. The equity market, on the other hand, offers a resilient opposition to Darwinian evolution due to the equity market's feature of limited liability. Traders are not wiped out so easily, that is to say that the inefficient rounds of a minority game are likely to perpetuate, resulting in a constant adaption to market conditions as multiple rounds of games perpetuate. The effect of this is that a simple majority rule is not expected to clear minority traders and in this manner the market is expected to take on the form of a self-organised dynamic market which does not reach a singular equilibrium solution.

The application of game theory to the economy is an interesting and complex one, as the economy can never be considered to be a pure minority game, nor should it be assumed to be void of minority game considerations. Indeed both minority and majority game theory should be considered to be interweaving. An interesting result of defining the market as dynamic and self-organised is that it implies that traders are required to continuously adapt to their environment. Investors are required to continue learning and whilst trends in behaviour may persist over periods, they are not expected to be stable in perpetuity. Thus no model which is designed to price risk in a static market description can be expected to succeed. This conjecture is not an unfamiliar one to many financial authors, the support of which is particularly evident in the findings in favour of the AMH by (Neely, Weller & Ulrich, 2009). In their research, Neely et al. (2009) display that whilst particular technical trading rules can be seen to generate genuine value, excess returns to these market positions are diminishing over time as markets adapt. The adaption of market positions in this study are considered by the authors to be too slow to be indicative of an efficient market, and the range of time and studies considered bodes to suggest that the adaptive market description is expected to perpetuate.

### 2.2.2 Behavioural evolution

The relevance of behaviourally induced trading and associated market driven price reversals is documented clearly across empirical literature, notably so by Coval and Shumway (2005), finding that behavioural biases are rife even within arguably an example of some of the most sophisticated of trading circles as noted in their investigation into trading tendencies amongst the Chicago Board of Traders. The presence of behavioural anomalies leading to high risk taking, as well as the associated risks of market reversals to counter the trading behaviour is clearly demonstrated. The consideration of this behaviour in a sophisticated market suggests that this is not an anomaly, but rather likely a feature of financial markets. Kogan, Ross, Wang and Westerfield (2006) add to this robust econometric evidence to suggest that the pricing effect of these so called "irrational traders" is likely to be persistent in stock market prices, despite efforts of so called "rational market forces" to correct the pricing effect of behavioural anomalies. Daniel and Titman (1999) further add to this line of dynamic market definition, finding evidence to suggest that the market does not only deviate from an efficient form in the traditional sense, but it fails to arbitrage out pricing anomalies even under less restrictive efficiency theories. They show, with 30 years of data tested to confirm the suggestion, that while it may be possible to price out the anomalies in hindsight, it does not seem plausible in reality (Daniel & Titman, 1999).

The pricing anomalies described in Daniel and Titman's (1999) research on market efficiency do not conform to traditional risk measures and are described as being endogenous to the system. It is therefore unrealistic to assume that all possible endogenous risk drivers can be diversified away and thereby be excluded from a multifactor pricing model as is suggested by Chen et al. (1986) when referring to microeconomic factors. Clearly, if risk measures fail to price assets correctly, based on behavioural heuristics, it is necessary to include a set of internal risk factors designed to capture these risks. Whilst some authors, including Coval and Shumway (2005), present evidence that certain behavioural heuristics leading to pricing anomalies are priced out by rational traders in a short time, evidence presented by Daniel and Titman (1999), amongst others, suggests that the potential presence of a behavioural influence on asset pricing can be expected to be persistent.

Stepping aside from a financial view point, the following social digression is felt to highlight the point that an expectation of a theory which rejects endogenous variability of risk is a decidedly non-human expectation, and thus, one which cannot be reasonably rejected in any study where individual and group behaviour are to be considered. What can be likened to irrational behaviour described in financial/economic text is seen to be common place in society and notoriously difficult to control, where issues of health are concerned. A human experience which is presumably more important and sensitive to positive decision making than financial gain. The human predisposition to entertain variant opinions and behaviour in the face of fundamental risks should highlight the importance of this consideration. If we cannot entertain such strict rationality where our lives are concerned, how can we expect the same degree of resistance to second hand information and peer pressure on the financial markets. To extend the digression of persistence to expectant irrational behaviour, the following abstract consideration is presented:

Markets behave as collectives and the behaviour of the market cannot be deduced from the study of an individual. The significance and prevalence of crime, smoking and suicide is given as an example of the suggestive sensitivity of humans (Ball, 2004). Add to this the broken telephone effect of information (and its notorious inability to remain intact) and sensationalism of news and a seemingly rational *a priori* thought spreads like wild fire to the end of creating an unexplainable anomaly of *ex post* market perturbations. Thus, the thought of entertaining deviations from perfect rationality might be a force of human nature destined to persist despite our best *ex ante* rational intentions.

These brief extensions to a behavioural description raise two important concepts with respect this research. Firstly, it appears that to continue forward with the assumption that traders are rational, and thus continue forward only considering pricing models which ignore any form of non-rational trading appears to be a decidedly non-progressive avenue of research. Secondly, with the importance of an appreciation of behavioural heuristics, and the importance of the role they play in asset pricing must be read with the appreciation that it is by its very nature a non-exact science, and whilst it should be expected to improve pricing models through its inclusion, it cannot be expected to produce a perfect pricing model.

### 2.2.3 The adaptive market hypothesis

A key defining feature of economic agents and one which uniquely defines the economic market from strict natural science is that, as humans, the decisions of economic agents depend upon their expectations of beliefs about the future (Hommes, 2001). Hommes (2001) adds to this introductory statement by quoting Keynes (1936) which questions objective rational valuation of assets in favour of an investor belief system which considers mass market psychology not unlike that described in the economic study of game theory. Keynes

(1936) makes his point through a parable which describes a beauty contest used to describe the financial markets. The parable suggests that in order to predict the outcome of a beauty contest, objective beauty is not at all important, but knowledge or prediction of others' perceptions of beauty is much more relevant. To extend this parable to a single factor asset pricing framework, it would suggest that in order to best predict the value of an asset, one must understand the market opinion of that asset. In order to do this, a probability belief needs to be established. To quote Keynes further:

> *Investment based on genuine long-term expectation is so difficult as to be scarcely predictable. He who attempts it must surely lead much more laborious days and run greater risks than he who tries to guess better than the crowd how the crowd will behave; and, given equal intelligence, he may make more disastrous mistakes* (Keynes, 1936, p. 157).

The ability to price financial assets accurately is often assumed in literature as a matter of financial course, an assumption where the EMH has dominated financial research. Potters et al. (1998) label the pursuit of asset pricing as being, in general, difficult to assess quantitatively as the "true" value of a stock is difficult to determine. The authors go so far as to suggest that the concept of efficient pricing may be an empty concept altogether (Potters et al., 1998). Whilst this opinion may be somewhat extreme, the mass of research over the past 40 to 50 years into asset pricing based on identifying specific causal risk factors both microeconomic and macroeconomic in nature suggests that whilst there is definite value to the pursuit, it cannot be viewed as a complete description of the market, nor does it fully capture how investors price assets.

One of the limiting factors to this testing is the ability to observe the amount of wealth controlled by irrational investors and what their specific tastes are. This is by definition expected to be a varying figure which is only observable in hindsight. Further, even when seen in hindsight, Daniel and Titman (1999) argue that a model developed based on rational expectations would fail to adapt and observe these heuristics and price them into the market accordingly. The second issue is that, based on this dynamic nature of influential behavioural heuristics, pricing models which price the heuristics with the benefit of hindsight will improve on traditional pricing models.

These considerations to heterogeneous beliefs lead to the pursuit of a market model which considers the question of informational efficiency, as opposed to allocative efficiency. As agents in these markets are expected to follow probability beliefs in lieu of difficult to observe fundamental risks, a test of whether assets are priced to reflect efficiently their fundamental risk exposure is not relevant. Rather, informational efficiency, which suggests that market returns should be difficult to forecast due to a lack of arbitrage opportunities is considered. By this extension, an appropriate model design is one which, when correctly specified, observes no excess returns to a collective market belief of risk, where all risk are plausibly relevant (based upon the behavioural perception of investors).

Traditional long run beta would be suggestive of a traditionally defined rational agent market place if this risk measure fully explains asset prices. However evidence suggests that there are unexplained returns statistically observed as prediction errors to the OLS regression. This study tests whether a probability belief inspired framework can reduce regression prediction errors and provide evidence to support probability belief dynamic market behaviour over a traditional rational expectations model. If a probability beliefs model is not a better description of market dynamics, it should add little to no value over a traditional OLS CAPM model under the assumption that markets actively move to arbitrage pricing inefficiencies. The assumption that excess returns are actively priced out under an AMH framework is support by Neely et al. (2009) which concludes that excess returns are eroded over time, finding that markets do adapt to evolutionary selection pressures. Whilst their paper was concerned with testing of technical trading rules in the foreign exchange market, their results build on the premise that market participants' behaviour adapt over time and that the market is actively learning from observation of its participants (Neely et al., 2009). The result is that the investment tendencies of the market dynamically adjust over time resulting in fluctuations in relative demand and supply and thus in the perceived risk coefficient of assets.

#### 2.2.4 The survival of all in free markets

Milton Friedman describes market efficiency by an argument which suggests that irrational traders will continue to lose money, and will not survive. The effect of this argument being that those traders will not have a marked or persistent influence on long-run asset prices (Friedman, 1953). Kogan et al. (2006) show how this is not necessarily the case, and that the effect of irrational traders on market prices is not so easily explained away.

In establishing the resilient nature of traders, Kogan et al. (2006) provide evidence to suggest that even where survival of irrational traders is threatened through a diminishment of their wealth, these traders can continue to have a significant impact on asset prices. In the case of momentum strategies and resilience to short term or immediate reversals, as appears to be the case when one considers the short term reversals in (Jegadeesh, 1990) in light of the long term persistence of momentum strategies over longer horizons as documented in (Chan, Jegadeesh, & Lakonishok, 1996), rational trading may involve following irrational trading, that is, to support view of (De Longe, Shleifer, Summers, & Waldman, 1991) who suggest that rational traders may amplify irrational trading by buying ahead of feedback traders, thus profiting off of the momentum. This makes rational sense from a game theory perspective, especially if one ascribes superior trading knowledge to these "rational traders", who would then be able to back out of their positions before the prices reverse.

Shiller (2003) echoes these perspectives in a depiction of a market composed of smart versus ordinary investors. In an alternate perspective, substantiation to explaining the persistence of market price inefficiency through research into behavioural heuristics, whereby extrapolation of trends and a lack of trading on fundamentals is considered, finds support for behavioural theory which suggests that non-rational risk perception and associated momentum style trading can be considered to be inherently human and not expectantly uncommon (Kahneman & Tversky, 1979; Daniel, Hirshleifer, & Subrahmanyam, 2001; Andreasson & Kraus, 1988). The persistence of this sort of trading, based on cross-market participant effects is not unreasonable when the theory of causal inference as proposed by David Hume is considered (Hume, 1975b) (Hume, 1975a). So long as there are past observations on which habitual understanding can be built, future trading is expected to be informed by past patterns of behaviourally influenced trading, irrespective of rational market theories. This sort of reasoning lends support towards a probabilistic understanding of risk, as opposed to a fundamental exogenous treatment of risk. This view of risk as a probability belief is supported by Markowitz (1991) as a justifiable explanation of expectant risk perception, the expansions of linear dependence being inferred from the behavioural model of causal inference.

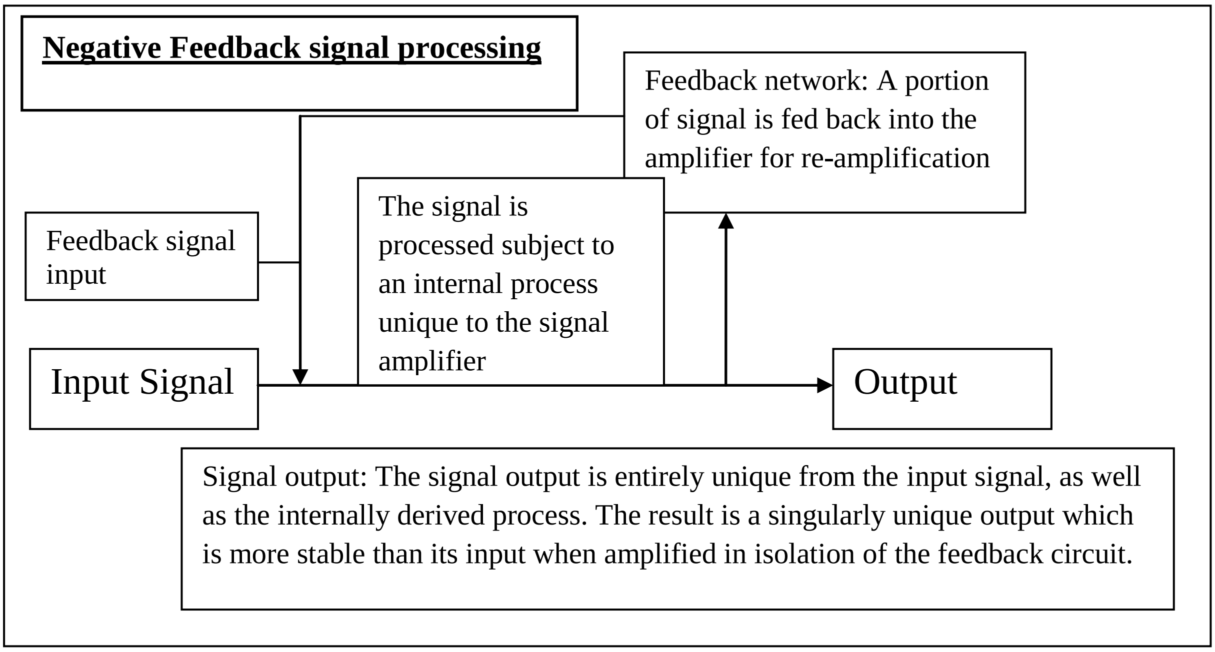
De Longe et al. (1991) presents a complimentary view to this theory of heterogeneous persistence of market position, suggesting that whilst the effect of irrational traders is diminished as their total wealth diminishes towards zero, this state is merely transitional. De Longe et al. (1991) shows that as prices recover to be based on rational beliefs, the wealth of irrational traders can grow at a faster rate than that of rational traders, allowing them to recover and survive. Kogan et al. (2006) suggests that this is based on unrealistic premises, yet offer explanation and economic proof to show how irrational price influence can survive in spite of diminishing wealth. Kogan et al. (2006) pay specific attention to proving that their premise that irrational traders can have significant price impact even with little wealth is practically relevant. They do this using a conical model in similar vein to that of the successful practically relevant option pricing model of Black and Scholes (Black & Scholes, 1973). The model considers final wealth of a trader as being a fundamental point of reference (as opposed to interim movement). This is consistent with modern literature whereby investors are expected to be loss averse (as opposed to traditionally defined risk averse). Through a logarithmic preference investigation Kogan et al. submit that irrational traders influence on prices does not decay as quickly as his relative wealth share, and thus influence on market prices are expected to be persistent beyond the theory proposed by Friedman (1953).

#### 2.2.5 A single risk factor: Feedback to cybernetics

As per the discussion into probability risk, a singular risk coefficient (beta) is suggested as a proxy for the true risk drivers, be they stock fundamentals, unique non-systematic variables and their undulations, behavioural heuristics and less than rational trading, or systematic macro economic variables. Here the behavioural effects of heterogeneous market considerations are formalised into a model which retains the probability belief definition as a singular evolutionary factor through a feedback model design of cognisance.

The treatment and distinction of economic variables, as described by Chen et al. (1986), is to be explored and built upon in light of probability beliefs; the concept that no variables considerable are truly exogenous to the system. Liberty is taken with this perspective, and the concept that no risk variable is purely exogenous has been expanded upon greatly. In this description of market dynamics, even macroeconomic drivers form part of the financial market’s feedback loop. Specifically, this market endogeneity is comparable to a confounding negative feedback model. The relevance of how a negative feedback loop operates, generating an output which, whilst founded upon a variety of fundamental inputs, is specifically and singularly unique to its input can be explained by considering its application in electronic circuitry, and specifically negative feedback in amplifiers. The inputs can be thought of as fundamental market variants, signal noise can be interpreted as heterogeneous information and subjective interpretation thereof, and the feedback as participant interactions subject to interpretative noise. This is broken down further below, following a detailed description of negative feedback.

Negative feedback loops: A common feature of amplifier circuits, where by a portion of the output of the preamp is fed back into the input section to be reamplified before being sent to the power amp section. This compounding of the raw signal layers trades off input gain for higher degrees of linearity in its output, resulting in a cleaner complete audible signal which presents as a complete audio with its own signal dynamics which are influenced but ultimately unique as compared with the original signal dynamics. This principle is depicted below:




Building on an economic interpretation of this, decision making agents in the market can be thought of as converting noisy raw information into clean probability beliefs which are observable in the market through the effects of supply and demand. The initial interpretation of raw fundamental information serves as a model input from which an initial probability belief can be built. The interactive effects of agents in a market described by game theory and associated AMH considerations already presented serves to impress the effect of heterogeneous beliefs based on the status quo of the market. This effect of participant beliefs is seen being fed back into the model to affect an endogenously informed probability based on individual and interactive factors. In this way, information processing done by market participants can be thought of as being indicative of amplifier circuitry in that they receive raw fundamental information regarding market assets; they then interpret this by establishing a probability belief, subject to behavioural heuristics. Thereafter, the effect of these probability beliefs presents as an important consideration for interpreting best responses by other investors in the market as the effects of game theory define the social interactions of economic agents. This is all fed back into the decision making system of rational economic agents, resulting in a negative feedback loop which ultimately drives demand and dictates output. In this way, the output of a market as observed through share price returns is the result of a complex negative feedback system of information and the interpretation thereof.

This extension of negative feedback control system has been applied as a control system governance used to define the behaviour of society through the study of cybernetics. Norbert Wiener defines feedback in the subject of cybernetics as the chain of the transmission and return of information, a system of physiological and neurological feedback in humans which Norbert Wiener applied to, amongst other fields, time series economics (Mindell, 2000). The effect of this on the research at hand serves to support the notion that behavioural heuristics and cognitive predisposition of humans have the distinct theoretical characteristic of a self governed control system. The point of departure from negative feedback as a topic of electrical engineering is the heteroskedastic nature of market participants which amplify the information at hand, as evidenced qualitatively by the theory of game theory, and minority rule there within. This departure however only serves to reinforce the effect of the negative feedback comparison and thus the importance of a model which considers the prediction of an asset pricing risk coefficient as one which is defined as a evolutionary time-series. The effect is thus to consider an evolutionary observation system which is cognisant of the endogenous feedback effects of the observer (the investor), which confirms the study path to the subject of Cybernetics in which observation based systems are updated to be observer based systems.

#### 2.2.6 Cybernetics

The subject of cybernetics helps to formalise the formation of the feedback model as a definitively behavioural exploration of observational enquiry. The term Cybernetics was derived from the Greek word for steersman, which has a common route meaning to governance. The subject deals specifically with the importance of the observer to the description of model dynamics, while maintaining a foundation in feedback (adapted from source), (Pangaro, 2006). The magnitude of the importance of the cybernetics approach to observation based systems is to marry the observation based nature of the Kalman filter with the interpretation of a singular risk variable which represents subjective probability beliefs. The endogenously dependent probability belief of market participants is allowed to evolve in a stepwise autoregressive manner through the state equations, whilst this is updated using an observation system via the Kalman filter's signal equation. All the while these two equations are allowed to evolve in manner which is sensitive to each other's result through the algorithmic design of state space mathematics and the Kalman filter's gain function, whereby observation disturbance is fed back into future prediction in the same way that a negative feedback amplifier feeds back gain in order to stabilize output.

An important point to note is that, under the following research design, it is not felt to be necessary nor useful to attempt to deconstruct a cross-section for inputs to the negative feedback loop. It is considered that the negative feedback of information, and the effect to creating clean probability beliefs, creates a singular signal which is representative of risk impressions of the investor universe. The feedback effects in world markets can be quantitatively seen to have a robust dominating effect on investor behaviour in the paper Filimonov and Sornette (2012). This leads then to the necessary consideration of the interactive market effects which define the feedback system's state update perspective, and how these intermarket effects are expected to vary in a time-series manner which individualises the correlative investigation from a signal observation perspective.

## 2.3 Chapter 3: Into model design

Chapter 3 leads on from the behavioural description presented in chapters 1 and 2, and considers the practical considerations of the model design from an empirical and statistical design point of view. Beginning with considerations of interactive effects and the importance of perspective in probability distributions, importantly the concept of correlation versus causation is revisited from a statistical perspective. From these supporting arguments, an argument for linear dependence is reconsidered from empirical evidence as this is a fundamentally testing contention to traditional pricing theory.

### 2.3.1 The sum of its parts make the whole

The principle of this point of research is to establish the relevance of clear and deliberate causal factors to explain stock price movements which otherwise might appear to be random. Specifically, the issue of contrasting evidence in favour of and against returns being considered to be random and unpredictable is considered through a matter of comparative perspective of the individual and the whole. In physics related concepts presented here, and supported by behavioural investigations already presented, the stance is taken to suggest that return distributions are not inherently random, but rather are causally linked to movements in demand and supply. In this way, it is proposed that, whilst return distributions may take on a random appearance, the mechanisms which dictate those returns are based upon probability beliefs developed by market participants in an inherently non-random fashion. As the probability beliefs of individuals are cumulatively descriptive of the relevant asset risk coefficient, it is this which is non-random, and by definition, attributes a degree of serial correlation to the investigation of a risk coefficient. Thus, the inquest to developments of stock return distributions are still treated within the methodology as conforming to the assumption of random return distributions whilst the development of the probability belief is allowed to develop in an autoregressive fashion. The two somewhat opposing points of contact to the investigation at hand are likened to an analysis of Brownian motion as an observable whole, as well as sum of its unobservable parts simultaneously.

The inspiration for the perspective outlined above lies in an analogy between individual stock movements and gas particles as they follow Brownian motion. On an individual particle level, the movement of gas particles is inherently non-random, each particle moves as a direct consequence of a collision with another particle, the interactive motions are thus autoregressive. However, when an unaided perspective is taken, the effect of these collisions is not clear at all, the movement of the particles is viewed as a singular collective of gas which appears to move seemingly at random. To extend this matter of perspective to the stock market, individual trades can be thought of as individual particles and the price effect is directly caused by demand or supply which is effected through an individual's will (a will which is informed by that individual's probability belief). Taking a step back and considering the market as a whole, it is seemingly impossible to expect to directly observe the individual causal effects on returns as described above.

This analogy of individual market movements behaving as the parts, which describe the whole in a return distribution to that of gas particles behaving in Brownian motion, is inspired by the work on fractals presented in Mandelbrot (1963b) as well as Mandelbrot and Van Ness, (1968). Whilst Brownian motion appears to be random, it is due to the confounding effects of multiple causal (heteroskedastic) impacts of multiple particles. Mandelbrot and van Ness's (1968) work on fractional Brownian motion explains how, whilst Brownian motion is a name given to the apparent random movement of gas particles, if one decomposes the motion far enough, particles can be seen to behave in a clear non-random fashion, as particle motion is defined by impacts from other particles, and the distribution of movements, albeit for short periods, displays serial correlation as previous particle movement and interaction defines subsequent movement. The random nature of Brownian motion can there for be accounted for as a matter of perspective (Mandelbrot & Van Ness, 1968). Whilst the long term movement of the market may be difficult to predict, much like that of particle motion, the reduction of an observation thereof to a purely Gaussian one is felt to misunderstand the underlying movement which drives the market. Fractal mathematics presents a basis for a more fundamental analysis of motion as a description of the sum of its parts as opposed to the observation of its whole. The assertion to consider the development of the whole as a causal evolution of the sum of its parts is compounded by the subjective nature of human behaviour, of which a descriptive and complimentary subject matter has already been presented in chapters above. When contrasted with the automata behaviour of a gas particle, fractal level causal drivers of the market, described as the period to period probability belief of asset value, are assumed to have a profound impact on the development of an asset pricing theory and empirical model. It is this separation of perspective which the proposed behavioural model is built upon. The fractal nature of causal market impact is developed through a probability belief which, like the description of particle motion above, is allowed to evolve in an autoregressive manner. This compliments the expected behavioural bias by which investors are expected to extrapolate causal inference from past observations. The return distribution however is expected to approximate a normal distribution through the collective effects of a probability belief.

Evidence to suggest the stock market is a generally accepted approximation of a normal distribution which is descriptive of a highly non-random interaction of participants can be seen through stock distribution analysis across time-series. Fama (1965) notes a degree of leptokurtosis in market returns, as does Mandelbrot (1963a), and whilst the degree of market leptokurtosis is generally assumed over longer sampling frequencies to be conformable to a parametric student's t investigation (Haas & Christian, 2011), authors concerned with behavioural tendencies of human interaction have found leptokurtosis to be indicative of behavioural interaction or endogeneity (Cont & Bouchaud, 2000). The importance of the consideration of interactive effects to fatter tails of distributions is noted beyond the realm of financial time-series. In work to investigate root causes of heavy tailed distributions as products of exogenous factors versus endogenous factors routed in behavioural dynamics, the behavioural effect of individual interaction remains as inextricably linked to heavy tailed distributions. In an empirical investigation which considers the effect of observation frequency, exogenous heterogeneity can be explained away as a cause for heavy tailed distribution, but it is submitted that endogenous heterogeneity is a feature of human behavioural dynamics (Zhou, Zhao, Yang, & Zhou, 2012).

Whilst the fractional nature of the causal factors underlying the stock distributions leads to the reasoning behind specifying the asset pricing model to account for these causal drivers, general econometric testing proceeds with a random t-distribution assumption for statistical simplicity, an approach felt to be appropriate in light of the stock market's apparent non-normal stable distribution (Fama, 1965).

Due to the enormity of modern equity markets and the vast dispersion of individual perspectives, the observation of fractal forces is not reduced to a trivial observation and, for this reason, a simple period to period observation is not expected to be sufficient to explain returns. It is felt to be necessary for investors to learn and update their predictions through updating their probability beliefs. This is practically specified through the use of a Kalman filter's stepwise learning function which uses maximum likelihood measures to develop a predictive belief which minimises variance. In this way, the evolution of probability beliefs are allowed to evolve in an autoregressive manner, but once the validity of these beliefs are compared to the market collective, the individual is expected to learn and update their probability beliefs. Thus the development of the individual's risk is expected to be observant of the microstructure environment in which the investor operates, whilst being mindful of where that places the individual's position in the greater market, thus being aware of the interactive effects of groups and thus the implications of game theory. The specification therefore does not allow the prediction of returns to be autoregressive, but rather the prediction of risk is based on an autoregressive structure and it is this measure of risk which is then updated in a stepwise fashion.

### 2.3.2 A view of risk through correlation

#### *2.3.2.1 Correlation versus causation: A qualitative description*

Whilst the behavioural model specifications of chapter 1 are compelling from a behavioural point of view, the idea of causation as an implication of observed risk is considered to be worthy of a more scientific evaluation. Here, the prospect of correlation and causation as descriptive necessities for an empirical financial investigation of asset pricing are considered for susceptibility to statistical bias, as well as necessity for establishing relationships between observations.

Empirically observed covariation is a necessary, but not sufficient condition for causality (Aldrich, 1995). Whilst correlation serves as a commonly observable feature of regression analysis in financial time-series, caution must be observed when using it to define risk exposure, as correlation neither guarantees an implied causation, nor a common causal driver or link. Statistically, investigations into causation which are based on observing correlation are naturally susceptible to type 1 error, in that the causation can be erroneously implied based on a statistical observation of correlation. Thus, using correlation to dictate investment profiles opens one up to the possibility of miss-specifying variable interactions, and thus failing to adequately capture, in the present case of financial asset pricing, the risk-return relationship. The question then begs: Why does the culture of correlation driven methodology continue to be so common? The simple answer to this is that correlations are easily observable through regression analysis, where as causal interactions require a qualitative justification along with a quantitative correlation observation in order to infer their existence due to environmental difficulty in isolating causal event observations. Under current asset pricing theory, the majority of literature is focused on just this; the CAPM and its conditional variable expansions, as well as arbitrage pricing theory (APT) models, extend this investigation into preselecting cross-sections of qualitatively justified risk variables and then consider the correlative implications.

Sensitive to the limitations to describing causative relationships through correlation based regression analysis, an alternative perspective to qualitative explanation is taken. Specifically, the approach in this study shifts the point of causal analysis from the root state and idiosyncratic variables which are argued to influence asset value by traditional pricing theories to the point of economic demand and supply. This approach places regression analysis at the focal point of that which directly drives asset prices, and as a matter of economic course, returns and the variations thereof. Thus asset price movement and associated risks surrounding equity investment which are driven by market ebbs and flows are observable by a traditional correlation measure in so far as it tracks the influence of traders and how they enforce their will and belief on the market. In so far as this represents the market condition, correlation of a risk coefficient which is singular in value and time-series evolutionary as a feature of dynamic market movements tends towards an appropriate proxy for causation. This design is both extremely similar, and completely different to traditional theory. Whilst a singular risk variable is considered; it is not a collective cross-section of risk exogenous exposures, but rather dependant on the time-series evolution of endogenous market risk through probability beliefs. The separation of probability belief evolution from market observations is theoretically based in behavioural assumptions and practically handled through state-space regression analysis, extended upon in methodology.

The issue in question here is the extent to which such a model has potential to succeed considering real world applications and further, not simply where there are limitations to the accuracy, but where these limitations may have the potential to obscure observations of genuine causally driven correlation.

Filimonov and Sornette (2012), provide compelling evidence which notes the growing importance of endogenously defined risk. As markets become more integrated they appear to be less reliant on exogenous news about fundamentals and rather are seen to increasingly react to endogenous triggers. Filimonov and Sornette (2012) find over the period 1998 to 2010 on the Standard & Poor 500 that endogeneity as a driving market force of market activity increased significantly, with the amount of market activity which could be linked to exogenous information falling from 70% in 1998 to less than 30% by 2007 (Filimonov & Sornette, 2012). The trouble with endogenously defined risk is how to identify and price it. The proposed solution to this problem rests with probability beliefs built through the use of a Kalman Filter to incorporate a dynamic learning model. The correlative implication to this is explained as a single risk factor which varies linearly in a stepwise fashion.

#### *2.3.2.2 Correlation versus causation: Statistical considerations*

The traditional CAPM falls prey to a concern surrounding model design when calculating the correlation between beta and the market. As beta represents a collective of all systematic risk variables in cross-section, there is a potential that correlations may be biased due to an ecological effect. That is, where individual variables are not isolated for observation, the combined (or ecological) effect is to bias the observations thereof, and thus undermine theoretical inferences. Ecological correlations tend to be numerically larger than that of actual individual correlations and the conjecture of scenarios where they can be considered to be mathematically equivalent is rejected (Robinson, 2009). A higher biased projection of ecological correlation to beta might serve as a possible explanation for its inability to capture risk (as it would be downwardly biased). Higher bias in correlations is expected to result in a reduced ability to pick up tail risk, as well as idiosyncratic risk not otherwise arbitraged away, resulting in unexplainable excess returns. Thus it is submitted, for models which base exposure to risk as interpretations of observed correlations, every effort must be made to achieve, or at least approximate, a study of individual correlations.

When ecological correlation over estimates correlation, variation from the mean is by implication underestimated (Robinson, 2009). Many forms of extension to the traditional

CAPM model design can be viewed to circumvent the errors associated with ecological correlation bias. Modestly, through conditional extensions to the CAPM, and then more broadly into the realm of APT and multifactor models, the premise of which is to isolate theoretically justifiable causal drivers of risk. A point of contention for this approach from a correlation-causation perspective is still drawn however in that the identification and isolation of these extended risk variables poses complications to model specification, all the while implicitly imposing a continued degree of ecological bias as it is impossible to specify all risk individually. This issue of complication can be viewed as a corollary to the market proxy specification problem of Roll (1977) in that if one cannot be expected to compile an appropriately inclusive collective systematic risk proxy, how can one expect to efficiently isolate individual considerations.

Under the traditional CAPM framework whereby all non-systematic risk is diversified away and the market represents as a proxy for collective systematic risk, this lumping together of market factors which may influence the fundamental value of an asset will have the result of correlations being biased away from their true values, thus complicating the asset pricing model. Many studies into multifactor analysis by design move to correct for this correlation bias implicitly, but as noted above, not only are these models extremely difficult to specify, they have a tendency to improve, but not solve the asset pricing problem. This study therefore takes an alternate route to dealing with this correlation issue, counter intuitively, by moving back towards a single factor model. The method in which this is attempted views correlations as linearly dependent on observation and learning in a time-series manner, as individuals are presumed to form probability beliefs which serve as singular risk measures used to influence demand, which as a collective market directly causes returns. Considering risk as a time-varying probability belief allows the model in question to be sensitive to interactions of market participants and thus endogenous influence to demand and supply. Speaking directly to the concern of ecological bias, the observation of correlation is individualised by considering probability belief and the evolution thereof on a period by period basis, representing an individual period-isolated opinion of probability belief at each time-series observation. This treatment of correlation observation certainly takes some liberties with the interpretations of ecological bias and whilst it is hoped that this simplified approach to identifying causative correlations improves accuracy thereof, it cannot guarantee to do so. The worst case scenario is that ecological bias is unimproved and equally comparable to that of the CAPM.

A further attempt to individualise correlations, particularly where the probability beliefs model is concerned, is done through an attempt to individualise the assets being analysed. This is considered as assets of similar idiosyncratic description are assumed to be viewed in a similar light by the market and thus be subject to comparable risk description as an endogenous probability belief. This is done by subjecting assets to the known characteristic sorts of size and book-to-market value, proven to have unique return profiles (Fama & French, 1993; Basiewicz & Auret, 2010). In this manner, the formation of a probability belief is unique to the risks experienced by a particular asset type. This treatment of analysing asset behaviour by collective asset class based on empirically borne out fundamental characteristics is informed by Barra’s Newsletter from September 1988: “What’s new about beta?” (Adrian & Franzoni, 2009). Further to this, the extensive use of characteristic sorts in financial analysis and their ability to pick mispricing under traditional asset pricing models presents as justification for their use in considering an alternate approach to asset pricing by serving as a common benchmark

To round out the correlation-causation description considerations presented, empirical evidence from Lewellen and Nagel (2006) is considered to suggest that conditional CAPM extensions as exogenous time-series models do not prove to improve on asset pricing. These findings underline further the limits to correlative model testing from an exogenous risk variable perspective, further supporting the need for an alternate investigation into the risk-return relationship of assets in the equity market. This time-series treatment under behaviourally considerate learning based evolution, whilst theoretically justified on unique grounds, is not dissimilar to the learning models successfully implemented in similar studies of time variant Bayesian risk models (Adrian & Franzoni, 2009; Huang & Hueng, 2009).

#### 2.3.3 A linear argument of dependence

Fama (1965) describes acceptable independence to be where dependence of observations is not sufficient to be effected. Thus, if stock prices are correlated, they are only considered so, statistically speaking, if the dependence is relative to the study in question. If one could profitably trade on the dependence, the observations are not considered to be statistically independent. On the other hand, where dependence is irrelevant to the trade, the distribution is considered to be stationary.

Whilst there is evidence to support a degree of autocorrelation in the stock market, as evidenced by Fama (1970), the degree to which past prices actually cause future prices should be a tempered consideration. Prices are based on risk (as far as risk represents variation driven by market forces of supply and demand) and are inherently forward looking, an argument which should render slight autocorrelation a mere artefact of stochastic processes involved in stock returns. Of course, this theory is based on rational assumptions, and therefore does not consider the psychological effect of price movements on less informed traders behaving according to bounded rationality and associated momentum.

Thus, in so far as pricing is concerned with forward looking best expectations of rational market participants, stock markets can be considered to follow a random walk, as per Fama (1965), rendering an analysis of past prices as irrelevant. This conclusion begs the question then, what about the evidence of linear dependence in momentum and contrarian strategies? Empirical evidence finding autocorrelation and long term memory effects are not uncommon to the asset pricing debate, a feature decidedly contrary to a random process (Lo & MacKinlay, 1988; Lo & MacKinlay, 1990; Ding, Granger, & Engle, 1993).

To reconcile these competing arguments, attention needs to be drawn to the question of causation and how past prices might cause future prices to take on a particular demeanour. Momentum is a clear case where current and past price movement serves to represent a particular market sentiment. Where this sentiment starts to take on a self-fulfilling prophecy, as opposed to serving as a proxy for actual fundamental risk factors, prices themselves can be considered as a direct risk factor in that they represent market participant behaviour and its causal effect on price momentum. An extreme indicator that this type of behavioural phenomenon is a relevant hypothesis to consider presents itself in the form of stock market bubbles. Under these conditions, prices can be seen to be speculatively pushed far beyond any rational consideration of fundamental value as momentum from endogenous interaction takes over from fundamental risk analysis, seemingly completely.

Carhart (1997) finds in an analysis of mutual fund performance, that the one year momentum in stock returns found by Jegadeesh and Titman's (1993) paper accounts for Hendricks, Patal, and Zeckhauser (1993) hot hands effect in mutual fund performance. Wermers (1996) finds that momentum strategies themselves generate short-term persistence of performance (Carhart, 1997), supporting the concept that market participants' behaviour and interpretations of stock movements can themselves generate returns which can, as a risk driven return, be understood to be severable from fundamental risk and in portion identified as inferred risk. As Filimonov and Sornette (2012) suggests, this is likely a non-negligible effect. The finding that investment strategies which follow market momentum strategies can realise performance greater than non-momentum strategies warrants the pursuit of a linearly dependent model (Grinblatt, Titman, & Wermers, 1995).

Importantly (Jegadeesh & Titman, 1993) find that the momentum effect is robust to time period considerations and further to country specification as found by Asness, Liew, and Stevens (1996) (as cited in Carhart, 1997). Thus, whilst momentum may present as a form of market inefficiency, it is not due to a slow reaction to information, and is therefore rather more likely to be a function of market participants displaying a momentum driving behavioural tendency rather than a limitation to timeous reaction and interpretation to exogenous risk factors as is implied to be the case by (Chan et al. 1996) with their hypothesis that momentum is a market inefficiency born out of a limit to information.

Placing the evidence of equity returns momentum into the context of the behavioural theories presented in chapters one and two above, the importance of heterogeneity in linearly dependence is found to be of significant economic relevance. As the strength of a momentum strategy becomes more persistent and pronounced, the more likely traders will find a contrarian investment strategy to be profitable, an area of consideration which found traction in Lakonishok et al. (1994) in what the term “exploitation of suboptimal behaviour of the typical investor”. Lakonishok et al. (1994) present the success of their contrarian investment investigation to represent a strategy which is not subject to additional fundamental risk, suggesting that their findings are representative of endogenous influence as investors pursue a minority game theory approach. Thus momentum, as it builds, implies an ever greater chance that events will occur other than that which momentum dictates. In this regard, momentum becomes increasingly risky as the probability of aggregate market turn around builds. Thus momentum serves to build on a foundation of endogenously generated risk based on investor perspectives. More importantly, the finding that there is widespread evidence of both momentum and contrarian pricing relevance in the market adds significantly to an adaptive market description based on dynamically evolving game theory, and a resultant self-organised market. A note here is drawn to evidence by Lo and MacKinlay (1990) which displays contrarian profits to be beyond that of reaction to overreaction of momentum strategies, adding further evidence to minority rule behaviour as being part of the market’s interaction status quo. Momentum is considered by many to be a product of cognitive bias (Barberis, Shleifer, & Vishny, 1998; Daniel, Hirshleifer, & Subrahmanyam, 1998; Hong & Stein, 1999). This concept of behavioural deviation from traditional rational expectations models is in agreement with Hume's work on cognitive ability to predict the outcomes of events (Hume, 1975b; Hume, 1975a), as well as work on biological explanations for optimism bias through a neurological predisposition for optimism (Sharot et al., 2007; Sharot, 2011).

The learning feature of the Kalman filter is expected to capture the momentum effect as a feature of causative inference to feedback modelling through its autoregressive nature and more specifically through the learning factor which will allow the model to incorporate momentum as it learns about pricing trends. This expectation is born out in the research of Huang and Hueng (2009) who test for momentum in light of a Kalman filter learning algorithm specification and find that the otherwise important risk factor is completely encompassed by the learning model which mimics the investors' evolutionary learning process. To this, the advice of Fama (1970) is heeded and the value attributed to autocorrelation is indeed tempered in the use of a Kalman filter, in that past stock prices are not used to predict future stock performance. Instead, only the evolution of the probability belief surrounding variable risk exposure is assumed to depend linearly on past events as an autocorrelative process which learns based on prediction error minimization.

### 2.3.4 Empirical developments in treating risk

#### *2.3.4.1 The systematic specification of risk*

In establishing a choice of systematic risk variables, the question comes down to whether a single market factor type model or a macroeconomic multifactor model is more appropriate for capturing risk innovations on the stock market. In order to establish this model condition, an investigation into understanding the effects of correlation and when correlation can be implied to be indicative of a causal relationship is required.

Informed by Robinson (2009), which outlines how ecological correlations are subject to a positive result bias, and considering that a single market risk factor is indicative of a cross-sectional ecology of collective underlying risk factors, multifactor analysis was considered as a potential model specification. Whilst macroeconomic multifactor models have long been considered for their potential strength over single market factor models, (Chen et al., 1986; van Rensburg, 1999), this investigation led to the consideration of another set of constraints which bias correlations upwards from their actual interactions. The concern of data mining and model over-fitting has been at the forefront of criticism to multifactor analysis and in light of research by Clark (2004) which finds that out of sample testing alone is not sufficient to rule out model misspecification, new literature was considered in order to assess the appropriateness of a multifactor model.

Gupta and Modise (2013) find that, when considering a multifactor macroeconomic model, both the in-sample and out-of sample test statistics are susceptible to data mining biases. The results of Gupta and Modise (2013) tend to suggest that macroeconomic and financial variables do not seem to contain much information in predicting South African stock returns in a linear predictive regression, concluding that a lagged stock return variable is all one needs to forecast future stock returns.

These results suggest that, from a macroeconomic point of view, JSE investors tend to efficiently price macroeconomic risk factors into stock market prices, contrary to much of the international evidence into multifactor macroeconomic analysis (Rapach, Wohar, & Rangvid, 2005). Based on these findings it is felt that it is prudent to pursue further research into behavioural and idiosyncratic risk variables using a single market risk factor to capture systematic risk as innovations to lagged market returns.

From an idiosyncratic and behavioural perspective, in so far as a linear framework is appropriate, the findings of South African research papers into stock return characteristics as captured through size and value sorted portfolio analysis (Basiewicz & Auret, 2009; 2010), are not likely to be the product of a miss-specified systematic risk variable from a macroeconomic cross sectional perspective. That is to say that the value and size effects on the JSE are not expected to be attributed to incidental correlation with systematic risks which are not captured in market prices.

Thus, the question of efficiency remains. Whilst Gupta and Modise (2013) cite their findings as market efficiency, due to the inability of macroeconomic variables to improve on a singular market risk variable, these findings also highlight the inability to price the market based off of systematic risk exposure alone in a South African context. This interpretation derived empirical observations finding that macroeconomic specification is found to be efficiently represented by a singular market variable (Gupta & Modise, 2013), yet a singular market risk variable is found to be consistently unable to explain returns to value and size sorted portfolios on the JSE (Basiewicz & Auret, 2009; van Rensburg & Robertson, 2003). Whilst systematic effects may be efficiently proxied for through the estimation of an equity-returns derived market factor, correct equity pricing is felt necessary to be representative of relevant idiosyncratic and behavioural induced characteristics.

#### *2.3.4.2 Non-systematic anomalies*

The evidence to suggest the persistent relevance of idiosyncratic risk considerations in light of single systematic risk factor models is widespread, consistently pricing assets in contradiction to traditional arbitrage assumptions (Fama & French, 1992; Fraser & Page, 2000; van Rensburg & Robertson, 2003; Basiewicz & Auret, 2010). This leads to one of two explanations; either the concept of non-systematic risk diversification does not hold water practically, or the asset pricing models currently employed are not specified correctly, and as a result, unable to account for systematic risk variation effectively.

Research into investor behaviour finds that individuals do not on average fully diversify their own investments (Goetzmann & Kumar, 2004; Goetzmann & Kumar, 2008; Fu, 2009). Reading this in light of the risk tolerant views of a loss-averse utility theory as presented in chapter one, suggests that non-systematic risk considerations are to be considered to be relevant to asset pricing, as they are not expected to, nor are they evidenced to be, diversified away. The persistence of idiosyncratic anomalies, read alongside evidence to support a single systematic risk factor model from a linear as well as a cross-sectional behavioural perspective, serves as justification for a new consideration to the asset pricing problem by speaking to both explanations for non-systematic anomalies suggested above.

The empirical evidence presented here serves to build upon the suggested model description for asset pricing offered in the above literature in that a singular probability belief factor which is sensitive to idiosyncratic and behavioural implications through the negative feedback mechanics described to represent interactive game theory is suggested to better test for informational efficiency. The Kalman filter and its state-space description handles this well as the development of the model, which is tested as an evolutionary risk perception, considers the risk variable as a linearly dependent process which is estimated separately to, yet informed by, a linear regression model.

## 2.4 The behavioural framework: A consolidation of chapters 1, 2 and 3

The proposed study is not new to financial literature from enquiry perspective, however, the combined theoretical approach and resultant methodological considerations do entertain a new line of questioning for the South African stock market. Whilst the theoretical question at hand is one which tests for the relevance of a behaviourally sensitive asset pricing model, the literature presented is structured so as to outline the question of whether the drawbacks to market pricing efficiency exist due to a misspecification of existing popular pricing models, specifically where this is due to a misunderstanding of the investor universe from a behavioural perspective. The proposed model rejects absolute risk aversion in favour of loss aversion as a utility theory, which broadens expectant investment behaviour beyond perfect diversification, an assumption which finds great support in diverse literature. The width of cross-sectional risk consideration is expanded still further through the consideration of negative feedback and how market interactions are relevant to establishing probability beliefs. These compounding effects make an investigation directly into the cross-section of returns a task which is likely too complex to be considered practically viable, instead the simpler deductions of Keynes (1936) are adopted and the task of establishing the true objective value of a stock is rejected in favour of a singular probability belief which is dependent on endogenous interactions to determine investor behaviour. The singular time-variant nature of probability beliefs is based on variant heterogeneous beliefs which investors are expected to learn about in a manner which is subject to bias and not expected to be *ex post* perfectly rational. In order to capture these model specifications, a Kalman filter is applied to the asset pricing for its ability to allow for an unknown variable such as a probability belief to evolve as an autoregressive function, whilst allowing the prediction of this risk variable to be updated from period to period through the application of an observation equation which tests the prediction. Future steps of the procedure use the observation step to analyse the strength of probability belief prediction and learn from any miss-measurement accordingly. As the causative inference principles of behavioural learning described in chapter one are applied as universal governance to the behavioural model, it is predicted that this learning function will help to reduce pricing errors, whilst being necessary in perpetuity as the market is not described to be one which will settle in an equilibrium state, as per the theory of a non-Darwinian AMH presented in chapter two.

The use of the Kalman filter is still relatively new territory for South African financial research, with little published work on the methodology to date. More to this, on an international front, existing applications of the Kalman filter to financial literature have focussed on learning principles surrounding the prediction of beta as a systematic risk variable. The probability beliefs model is tested through the use of the Kalman filter as a definitively behaviourally driven evolution of a collective risk variable which represents self-organised stability through non-singular equilibrium market interactions as they vary across time. Specifically, the non-Darwinian evolution of minority rule presents a theory of AMH which does not evolve to a singular state model, but rather a dynamic stable model description. It is felt that the state-space asset pricing model proposed by the behavioural framework presented in literature review above presents a less restrictive model design which should serve as fertile ground upon which behaviourally defined asset pricing research can be built upon and refined. In application to human cognitive processes, the state represents the system environment at any point in time, and behaviour represents the change over time in the state description. The state space then represents the totality which the states can be in, capturing the evolutionary dynamic process of human cognition (van Gelder & Port, 1995). Below, the testing hypotheses are presented, and in the sections which follow, the behaviourally inspired asset pricing approach is empirically tested on a South African data set.

## 3. Research hypotheses

**Primary null hypothesis:** In light of the theoretical and statistical treatments, a dynamic systematic risk pricing model is unable to subsume the effects of idiosyncratic and behavioural anomalies as defined through size and value on the JSE over the period July 2001 to June 2012.

**Secondary null hypothesis:** Time-series variation in Market dynamics and the progressive learning based prediction thereof does not improve a one factor asset pricing model's ability to price risk.

**Tertiary null hypothesis:** Risk is strictly short-run dynamic. Investors do not predict risk exposure dynamically, based on probability beliefs, towards an overall unobservable long-run measure of risk exposure.

## 4. Data

The data testing period covers a 10 year and 11 month period, from end July 2001 to end June 2012; however, it is assumed that investors start learning about risk exposures long before this. Thus, the entire data sample covers a 16 year period from end of July 1996 to the end of June 2012. The choice of sample period is taken to coincide with the introduction of an order driven, centralised, automated trading system being introduced to replace the open outcry trading floor. Named the JSE Equities Trading system (JET), the newer system design provides an increased level of price information dissemination efficiency which is expected to be relevant to the behaviour of investors. Considering the entire JSE listing over the sample period, firms are excluded where accounting data is missing or stated solely in a foreign currency; market data is not available for all or part of the required testing period; the stock is listed for less than 24 months prior to portfolio selection; the listed firm represents a pure real estate investment trust; the listing represents a pure cash company (Basiewicz & Auret, 2009; 2010). All returns are presented for regression analysis as excess returns which are calculated to be the difference between monthly asset portfolio or market returns and the prevailing equivalent monthly risk free rate. This is done following the logic that returns represent compensation for risk bearing positions. The risk free rate is taken to be the 3 month Treasury bill rate as published on the South African Reserve Bank's web site (The South African Reserve Bank, 2013) converted to the monthly equivalent rate.

In order to keep with a study which is practically minded, liquidity and price restrictions are imposed in order to consider the significance of premia which a typical investor would be able to capture. Thus all return premia of equity listings which satisfy a 100 cent price restriction and a 0.1% volume restriction on liquidity is considered for the reason that the price and liquidity restrictions proxy for trading costs (Basiewicz & Auret, 2009). The liquidity variable to be used to filter stock selection is in the design of Hou and Moskowitz (2005), where-by a twelve month moving average of trading volume scaled by the number of shares in issue is used. Following the method employed by Basiewicz and Auret (2009), informed by Chan, Jegadeesh, and Lakonishok (1995), all data is adjusted for name changes in order to create a continuous string of data. Monthly price data, as well firm specific data such as volume traded (for liquidity sorts), size, and published book-to-market value is captured for the sample selection from FinData (a database compiled by the University of the Witwatersrand) and McGregor BFA. Across the full sample period of 16 years, an average of 497 listed stocks were considered in each year (the highest being 682 listed entities in June 1999, and the lowest being 379 entities in 2010). Following application of trade, liquidity and price filters described above, the average sample selection considered to be relevant to a practical market description fell by 59% of listed entities with an average yearly market listing of just less than 291 entities. A year by year breakdown of the filter application for market formation is available in Table 17 of appendix 1.

# 5. Description of research methodology

The basic methodological design can be seen as a descriptive analysis into asset pricing, with the intent of understanding better how investors price risk in a time series context. In order to do this, a one factor learning-augmented capital asset pricing model (LA-CAPM) is tested against portfolio sorts based on size and value, factors consistently mispriced by traditional CAPM design. The firm specific pricing anomalies considered as the basis for portfolio sorts are done so for their consistency in being empirically proven as being deterministic for asset pricing in an international (Fama & French, 1993), as well as a South African context (Basiewicz & Auret, 2010; van Rensburg & Robertson, 2003). In order to extend testing methodology to consider what is argued to be a relevant consideration of investor behaviour (Hume, 1975a; Hume 1975b; Sharot et al., 2007 Sharot, 2011) and the associated effects on asset pricing through the influence of endogenous market dynamics, the use of a Kalman filter in the applied design of Adrian and Franzoni (2009) and Hamilton (1994) is considered for its ability to predict the risk exposure factor, beta, as a time-series learning predictive factor. The measurement of the effect or relevance of the LA-CAPM is done so by forming and comparing the results of the LA-CAPM against those of a traditional Sharpe-Lintner CAPM across the various portfolio sorts.

In order to test the learning hypothesis of how individuals make decisions under uncertainty, the CAPM is tested as a learning augmented model considered with, and without conditioning variables in the method of Adrian and Franzoni (2009). The methodology is based on an argument, presented by Adrian and Franzoni (2009), which states that the wedge between investors' *ex ante* expectations of beta and *ex post* OLS estimates can account for a large fraction of the unconditional alpha in standard OLS time-series regressions. The Kalman filter is used to estimate betas of size and book-to-market sorted portfolios as opposed to individual stocks. This is in line with style investing behavioural specifications described in Barberis and Shleifer (2003). Further, it makes the results comparable with studies on South African data such as Fraser and Page (2000) and Basiewicz and Auret (2009; 2010) which find anomalous short comings of the CAPM when tested against size and book-to-market sorted portfolios.

### 5.1.1 The learning augmented CAPM

The Kalman filter is a set of mathematical equations that provides an efficient computational means to estimate the state of a process, in a way that minimises the mean of the squared error. The major strength as a dynamic model being used to approximate, in this setting, investor learning in the complex game theory environment of financial trade markets is that the model is powerful even when the precise nature of the modelled system is unknown (Welch & Bishop, 2004).

The nature of the Kalman filter applied to this study is always out-of-sample in beta prediction through the learning effect, regardless of the coefficient specification method chosen. The filter recursively updates to predict at each month a forward looking estimation based solely on past observations. The model is specifically AR (1) in this nature which has the implicit feature of representing all past information in the *t-1* estimate, augmented by an *ex post* realisation of the model's prediction error at time, *t-1* (Commandeur, Koopman, & Ooms, 2011). The AR (1) nature of the Kalman Filter satisfies the time series efficiency of market pricing, in that all historical information relevant to the prediction of a future state performance is already priced into the preceding observation. The residual updating effect of the Kalman gain adds to this the dynamic nature of human learning expected to be persistent in markets.

The Kalman filter is further specified under two specifications in order to assess the relevance of long run risk as an unobservable risk variable which investors strive to learn about through a dynamically varying risk coefficient variable. The first model is inspired by Adrian and Franzoni (2009) by specifying a model in which expectant probability risk is considered in light of a long-run unobservable beta which whilst not directly determinate of return prediction, is assumed to be factored into risk prediction by investors through the state equations. The second specification considered entertains the idea that investors are either not aware of long-run unobservable risk or they are unconcerned with it the determination of probability risks. This simpler specification assumes that the risk exposure variable, $\beta_{t+1|t}^{ie}$, is determined exclusively through an AR (1) process updated through the Kalman filter's period to period learning process. The model specification follows the Kalman filter design basis as outlined in Hamilton (1994).

In addition to the two model specifications outlined above, each one is considered in light of a conditioning variable in the form of market returns lagged by one period. This is inspired by the success of market conditioning in Adrian and Franzoni's (2009) study into Kalman filter modelling applied to the asset pricing problem. Further to this, the consideration of research into cross-sectional variable selection for multi-factor asset pricing by Gupta and Modise (2013) is considered in which only market returns lagged by one period are considered to add any significant pricing ability to a macroeconomic multifactor model of asset pricing on the JSE over a comparable data period. The addition of a conditioning variable further brings the model comparison in line with traditional asset pricing theory of OLS regression based CAPM in that predicted returns are fitted to an observation of market return movement. The unconditional Kalman filter is by comparison a strictly endogenous interpretation of exogenous risk, as the risk exposure variable evolves through an observer dependent observation based iterative process of learning.

### 5.1.2 The learning model

The LA-CAPM model to be estimated using Kalman filtering is specified as follows:

$$E_t\left[R_{t+1}^i\right] = \beta_{t+1|t}^{ie} E_t[R_{t+1}^M] \tag{1}$$

$R_{t+1}^i$ = Excess return to portfolio, $i$, in period $t + 1$.

$R_{t+1}^M$ = Excess return to the market in period $t + 1$.

$\beta_{t+1|t}^{ie}$ = The expected ($e$) risk-factor loading of the asset (portfolio) exposure to the market.

$$\beta_{t+1|t}^{ie} = E_t\left[\beta_{t+1}^i\right] = Cov_t(R_{t+1}^i, R_{t+1}^M)/Var_t(R_{t+1}^M) \tag{2}$$

$E_t\left[\beta_{t+1}^i\right]$ = Time, $t$, expectation of the time, $t + 1$, risk-factor loading, $\beta_{t+1}^i$.

$Cov_t(R_{t+1}^i, R_{t+1}^M)$ = Covariance between the asset/portfolio returns and the market returns.

$Var_t(R_{t+1}^M)$ = Variance of the market returns series.

The expected risk-factor loading, $\beta_{t+1|t}^{ie}$, is a conditional expectation and its evolution depends on the stochastic specification of the evolution of the unobserved long-run beta. Following the methodological specification of Adrian and Franzoni (2009), $\beta_{t+1}^i$ evolves according to an autoregressive process on a vector of stationary exogenous state variables $y_t$. The feature of the model which allows the beta prediction to be updated as a learning process simulates the endogenous behaviour of market participants as they update predictions based on past observations, as is suggested to be the case by Hume (1975a, 1975b). The updating formula for predicting beta is presented below:

$$\beta^i_{t+1} = (1 - F^i)B^i + F^i\beta^i_t + \emptyset^{iT}y_t + u^i_{t+1} \tag{3}$$

F = Constant scaling variable.

$B^i$ = Unobservable long term risk-factor, the long-run mean of the factor loading, $\beta^i_{t+1}$.

$\beta^i_t$ = Time, *t*, risk-factor exposure, beta.

$\emptyset y_t$ = Conditioning variable, for exogenous state variables, $y_t$.

$u^i_{t+1}$ = An independent normally distributed shock variable.

Practically the above equations evolve in a dynamic state-space model, where the signal equation defines the predicted evolution of asset returns as informed by the evolution of the state equations as per the Kalman filter's autoregressive, step-wise updating algorithm. The state equation set out above for the Adrian and Franzoni (2009) model specification is practically interpreted as two separate equations whereby long run exposure, $B^i$, evolves as an independent function which is calculated in parallel to the main state space equation which predicts period to period observed probability risk, $\beta^i_t$ . This specific model design is set out below in equations 4 and 5:

$$B^{ie}_t = F^i B^{ie}_{t-1} + u^i_{t+1} \tag{4}$$

$$\beta^i_{t+1} = (1 - F^i)B^i + F^i\beta^i_t + \emptyset^{iT}y_t + u^i_{t+1} \tag{5}$$

The Hamilton (1994) inspired model of Kalman filter follows the same basic specification of Adrian and Franzoni (2009) as detailed above and further below with the exception of long run risk factor, $B^i$, leading trivially to the following single state equation specification:

$$\beta^i_{t+1} = F^i\beta^i_t + \emptyset^{iT}y_t + u^i_{t+1} \tag{6}$$

The remainder of methodological specification proceeds with the more general form of Kalman specification which includes a long run risk consideration. When running the model under the Hamilton (1994) based specification however, instead of simply setting long run risk, $B^i$, equal to zero, this second state equation is removed altogether.

#### *5.1.2.1 Conditioning variable*

As a consequence of the model assumption that $u_{t+1}^{i}$ is an independent normal shock, innovations to the factor loadings are uncorrelated with expected risk premia. This would limit the prediction of beta to that of a purely iterative procedure which is independent of the equity premium, which would be counterfactual to traditional asset pricing theory. As such, a conditioning variable is included in order to capture the correlative interaction of beta in light of expected market return premia.

The conditioning variable chosen is done so from the perspective of maintaining time-series relevance, ensuring that the evolution of beta is dependent on the equity premium through exposure to variation in market excess returns. In order to keep the research comparable to results of tests which use a traditional CAPM, the conditioning variable considered is excess market return, calculated as the log excess return to the weighted (value/equally dependent on the test) JSE market portfolio above the 3-month treasury rate (risk-free rate). Gupta and Modise (2013) provide strong evidence to suggest that market excess returns are expected to constitute an appropriate and sufficient conditioning variable upon which risk exposure, beta, is to be predicted.

### 5.1.3 Applied model specification

To test the one factor time-variant asset pricing model in light of long term beta estimation, the Kalman filter technique as presented by Adrian and Franzoni (2009) is applied to beta estimates in order to establish a model with time-varying coefficients to the following state space:

$$R_{t+1}^{i} = \beta_{t+1}^{i} R_{t+1}^{M} + \eta_{t+1} \tag{7}$$

$R_{t+1}^{i}$ = Return to asset (portfolio), $i$.

$\beta_{t+1}^{i}$ = Risk-factor exposure of asset (portfolio), $i$, to the market.

$R_{t+1}^{M}$ = Return to the market.

$\eta_{t+1}$ = Conditionally normal shock variable, assumed to be orthogonal to $R_{t+1}^{M}$, defined as follows:

$$\eta_{t+1} = \left(\beta_{t+1|t}^{ie} - \beta_{t+1}^{i}\right) E_t[R_{t+1}^{M}] + \varepsilon_{t+1}^{i} \tag{8}$$

$\varepsilon_{t+1}^{i}$ = Idiosyncratic shock to $R_{t+1}^{i}$.

Following the premise that changes in the expectation of the factor loading are determined by Bayes' rule[2], Adrian and Franzoni (2009) show that the conditional expectation of $\beta_{t+1}^{i}$ evolves according to the Kalman filter. The evolution is based on time-series, learning conditioned, expectations. The dynamics of investors' expectations are thus assumed to take on the following form:

$$\beta_{t+1|t}^{ie} = \left(1 - F^{i}\right)B_{t-1}^{ie} + F^{i}\beta_{t|t-1}^{ie} + \emptyset^{iT}y_t + k_t^{i}(R_t^{i} - E_{t-1}\left[R_t^{i}\right]) \tag{9}$$

$B_{t-1}^{ie}$ = $t-1$ forecast of $B^{i}$, made using the Kalman filter.

$\beta_{t|t-1}^{ie}$ = $t-1$ forecast of $\beta_t^{i}$ made using the Kalman filter.

$\emptyset^{iT}y_t$ = Conditioning variable, for exogenous state variables, $y_t$.

$k_t^{i}$ = Kalman gain, a time-varying regression coefficient.

The model follows the optimal rule which uses the unexpected part of the current return realisation to update the previous period's estimate of the factor loading (Adrian & Franzoni, 2009). The updating equation for expectations about the long-run risk factor exposure, $B^{i}$, is therefore given as follows:

$$B_t^{ie} = B_{t-1}^{ie} + K_t^{i}\left(R_t^{i} - E_{t-1}\left[R_t^{i}\right]\right), \tag{10}$$

The autoregressive parameter, $F^{i}$, and standard deviations of error terms $(\sigma_\eta^{i})^2$ and $(\sigma_\mu^{i})^2$ and the loadings on the conditioning variables, $\emptyset^{i}$, are estimated using maximum likelihood methods on the whole history of portfolio and market returns. The estimation of $F^{i}$ from the whole sample period corresponds to the hypothesis that agents know $F^{i}$, which is based on the assumption that this parameter is stable over time. In order to investigate the importance of these model specifications to the Kalman filter, as well as to consider a model specification which is strictly specified out-of-sample, the filter tests are repeated using an out-of-sample estimation of autoregressive parameter, $F^{i}$, standard deviations of error terms $(\sigma_\eta^{i})^2$ and $(\sigma_\mu^{i})^2$

[2] Bayes' rule: $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$. The theorem provides a mathematical description of how subjective beliefs should be updated to account for observed evidence. In this context, observations of beta prediction strength given actual observations are updated to account for unexpected innovations.

and the loadings on the conditioning variables, $\emptyset^{i}$. For the out-of-sample tests, these parameters are again estimated using maximum likelihood estimation techniques; however, the data from which these estimates are made is limited to the 5 year period, July 1996 to June 2001. This period is chosen to be out-of-sample for comparative reasons, as the OLS regression output begins from June 2001.

In order to consider the event in which investors are unconcerned or unable to predict a long term risk variable, a simpler specification which is built, like Adrian and Franzoni's (2009) model, off of the Kalman filter model presented in Hamilton (1994). This model simply removes the long term beta prediction from the dynamic risk estimation state equation, resulting in the following:

$$\beta_{t+1|1}^{ie} = F^{i}\beta_{t|t-1}^{ie} + \emptyset^{iT}y_t + k_t^i(R_t^i - E_{t-1}[R_t^i]) \tag{11}$$

$\beta_{t|t-1}^{ie}$ = $t-1$ forecast of $\beta_t^i$ made using the Kalman filter.

$\emptyset^{iT}y_t$ = Conditioning variable, for exogenous state variables, $y_t$.

$k_t^i$ = Kalman gain, a time-varying regression coefficient.

#### 5.1.4 Model justification

The decision to proceed with parametric testing is motivated by two factors, the first being a qualitative motivation and the second a quantitative justification. The first is in order to keep the findings of this research comparable with respect to investigation into the relevance of risk factors of asset returns in equilibrium. All base research into time-varying risk models, as well as those focusing on variable long-term risk such as the CAPM, have been tested using parametric methodology. Thus it is felt prudent to follow similar methodology in order to benchmark research results where extensions and variations of these models are concerned. The second justification for proceeding with a parametric methodology is cited from McCurdy and Stengos (1992). The concern regarding a linear framework, where modelling of the conditional mean is concerned, is that a linear function of risk variables will ignore predictable nonlinear components (McCurdy & Stengos, 1992). Based on this concern, McCurdy and Stengos (1992) set out to investigate and compare parametric versus nonparametric methodology prowess in describing predictable returns. The results of the investigation suggest that nonparametric models may tend to result in over-fitting, as discovered when comparing in-sample to out-of-sample testing. Parametric specification on the other hand is shown to be well-specified, producing unbiased forecasts with lower Mean Squared Error (MSE) than nonparametric specification, performing consistently in out-of-sample tests (McCurdy & Stengos, 1992).

## 5.2 Model evaluation

### 5.2.1 Calculation of mispricing

#### *5.2.1.1 Individual portfolio measures*

Mispricing is measured, as in Adrian and Franzoni (2009), whereby the pricing error is defined as the excess portfolio return which cannot be explained by the proposed risk factors (Huang & Hueng, 2009). The calculation is presented below in equation 12. Each period, an alpha is calculated for each portfolio, *i*, as:

$$\hat{\alpha}_{t+1}^{i} = R_{t+1}^{i} - \hat{\beta}_{t+1|t}^{i\,kf} R_{t+1}^{M} \tag{12}$$

$\hat{\alpha}_{t+1}^{i}$ = Pricing error of returns by which the proposed model fails to capture the risk to explain.

$R_{t+1}^{i}$= Returns to the idiosyncratic risk sorted portfolio.

$R_{t+1}^{M}$= Returns to the entire market as specified in this text.

$\hat{\beta}_{t+1|t}^{i\,kf}$ = One period ahead forecast of beta made at time, *t*, using the Kalman filter.

For OLS regression, the same method is applied, where-by $\hat{\beta}_{t+1|t}^{i\,kf}$ is replaced by $\hat{\beta}_{t+1|t(5yr\ rolling)}^{i\,OLS}$. This allows for the final estimate of the pricing error for portfolio *i*, $\hat{\alpha}_i$, to equal the time-series mean of $\hat{\alpha}_{t+1}^{i}$:

$$\hat{\alpha}^{i} = \frac{1}{N}\sum_{t=1}^{N} \hat{\alpha}_{t}^{i} \tag{13}$$

The standard deviation of $\hat{\alpha}^{i}$ is calculated as follows:

$$\sigma^{i} = \sqrt{\frac{1}{N-1}\sum_{t=1}^{N} (\hat{\alpha}_{t}^{i} - \hat{\alpha}^{i})^{2}} \tag{14}$$

The significance of portfolio mispricing is calculated using a student t-test under the null hypothesis of an insignificant difference from zero. As these tests are interested in whether the model specification in question is able to predict returns with a pricing error as close to zero as possible, as should be assumed given market arbitrage, the significance of the pricing errors are considered for significant difference from zero.

#### *5.2.1.2 Overall model performance*

The second set of tests applied in order to gauge to relative performance of the various model specifications considered in this study have to do with overall performance of a model across all portfolios. In order to achieve this, two tests of overall performance are considered in order to compare the level of joint pricing errors for a given model specification. These estimates follow the methodology of Adrian and Franzoni (2009), although their use is widespread across literature which investigates time varying risk investigations.

The first summary measure is the square root of the mean-squared pricing errors (RMSE), which gives equal weight to the pricing errors of each individual portfolio:

$$RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N} \hat{\alpha}^{i^2}} \qquad (15)$$

The second test statistic which tests for overall performance is the composite pricing error (CPE) of Campbell and Vuolteenaho (2004), which places less weight on more volatile portfolios through the interaction of a weighting matrix, $\hat{\Omega}^{-1}$ :

$$\hat{\alpha}'\hat{\Omega}^{-1}\hat{\alpha} \qquad (16)$$

$\hat{\alpha}$ = An N-dimensional vector of the individual portfolio pricing errors, $\hat{\alpha}_i$.

$\hat{\Omega}$ = An N x N diagonal matrix with estimated variances, $\sigma^{i^2}$'s on the main diagonal.

The performance of the learning augmented CAPM is compared with the performance of an unconditional CAPM estimated using a 5 year rolling window for the calculation of its beta. The pricing errors of the unconditional CAPM are the intercepts from time-series regressions, and their t-statistics are computed and presented accordingly. The pricing errors and t-statistics for the learning CAPM follow the statistics described above.

### 5.2.2 Comparative pricing

The performance of the Kalman filter is further compared to that of a traditional unconditional CAPM through an investigation as to whether pricing anomalies can be explained by time varying beta whilst still maintaining the description of beta of a systematic risk exposure coefficient which describes all relevant non-diversifiable risk to an asset. Lewellen and Nagel (2006) describe a method for considering a comparison of CAPM models which allows for two individual comparative tests which are appropriate in this specification. Lewellen and Nagel's (2006) model is designed to consider conditioning variables, added to the CAPM framework, as time varying in their description of market beta. The direction of their investigation comes from empirical evidence to support the conditional CAPM as being able to significantly reduce pricing errors, (citing Jensen, 1968; Dybvig and Ross, 1985 and Jagannathan & Wang, 1996). These successes are attributed to a cross sectional design, which ignores the time-series effect of variance in beta (Lewellen & Nagel, 2006). The following description as to why a time varying consideration is relevant is offered: "a stock's unconditional alpha will differ from zero if its beta covaries with the market risk premium" (Lewellen & Nagel, 2006). Whilst the investigation presented in this study does not consider any conditioning variables, with the exception of a lagged market variable, the nature in which beta is calculated under the Kalman filter specification is not explicitly done based on the covariance between beta and the market's risk premium. Thus the test proposed for consideration of the time-variable beta in light of conditional variables is extended to the model specification described in this study with the purpose of discerning whether time-varying covariance between beta and the market excess return can explain beta as being descriptive of a systematic risk variable, or whether the time-variance of beta fails to support a traditional market description in favour of a probability risk time-variable model proposed in this study.

A second test of the Kalman filter's performance which is adapted from Lewellen and Nagel (2006) is considered in which the effect of learning is isolated by observing the difference in investors' *ex ante* expectation of beta and *ex post* estimations of beta through OLS regression. This difference in expectations and realisations surrounding the prediction of risk exposure can account for a large amount of the alpha as seen in a typical OLS regression. This is empirically tested by taking the difference between the period mean Kalman filter estimated beta and the period mean OLS beta, and multiplying the difference by the period mean excess market return. The result of this calculation is interpreted as the fraction of unconditional alpha which is explained by the learning process of the Kalman Filter.

Together the two tests of CAPM performance outlined in this section, as described by Lewellen and Nagel (2006) and applied to a Kalman Filter learning model specification by Adrian and Franzoni (2009) results in an alpha decomposition as described by the equation below:

$$\alpha_{OLS}^{i} = E\left[\beta_{t+1}^{i} - \beta_{OLS}^{i}\right]\left[R_{t+1}^{M}\right] + Cov\left[\beta_{t+1,}^{i} R_{t+1}^{M}\right] \tag{18}$$

$E\left[\beta_{t+1}^{i} - \beta_{OLS}^{i}\right]\left[R_{t+1}^{M}\right]$ = Learning component

$Cov\left[\beta_{t+1,}^{i} R_{t+1}^{M}\right]$ = Covariance component

The decomposition of alpha as described above is detailed in the Appendix B. For a full breakdown of the methodological rationale behind this line of testing, the reader is directed to Lewellen and Nagel (2006).

### 5.3 Potential issues and considerations

Adrian and Franzoni (2009) note a limited ability of the Kalman filter to improve on CAPM mispricing in respect of small-growth portfolios is due to a decreasing trend in beta. Campbell and Vuolteenaho (2004) along with Fama and French (1993) found similar issues in pricing small-growth portfolios (as cited in Adrian & Franzoni, 2009). It is suggested that pricing errors on these stocks may be due to a liquidity premium implication associated with small stocks, in that their low liquidity imposes a separate account of their returns (Adrian & Franzoni, 2009). This issue is foreseen to be of particular concern when testing stocks on the JSE, as an illiquid equity market (Basiewicz & Auret, 2010). For this reason, the liquidity filter applied by Basiewicz and Auret (2010) is applied in order to limit any negative impact from excessively small and illiquid stocks found on the JSE. Further, this treatment allows for this study to be comparable to existing literature concerning asset pricing on the JSE.

The use of non-parametric testing has been suggested as a possible extension to asset pricing tests due to potential limitations of parametric asset pricing models ability to capture market dynamics (Gupta & Modise, 2013; Adrian & Franzoni, 2009). A decision has been made against this based on evidence that non-parametric model specification can lead to issues of model over-fitting (McCurdy & Stengos, 1992). Further to this, literature by Andersen, Bollerslev, Diebold, and Labys (2003) and Andersen, Bollerslev, and Huang (2007) (as cited in Kunal, 2011) have empirically shown that linear models can oftentimes predict future volatility more accurately than non-linear models.

Another concern surrounding the Kalman filter and its application to economic time-series problems is due to its required assumption that data series errors are normally distributed. Whilst this assumption is unlikely to hold perfectly for financial time-series, the Kalman filter is still felt to be the correct model for its dynamic learning structure. As long as the model is a descriptively appropriate model, in that financial time-series data is indicative of a dynamic linear system, the Kalman filter will be a best least squared error estimator "While the Kalman filter forecasts need no longer be optimal for systems that are not normal, no other forecast based on a linear function of $\zeta_t$ will have a smaller mean squared error [see Anderson and Moore (1979, pp. 92-98) or Hamilton (1994, Section 13.2)]" (Hamilton, 1994). This is suggested by Hamilton (1994) to be in parallel to the Gauss-Markov theorem[3] for ordinary least squares regression where the regression model can considered to be optimal as a best linear unbiased estimator for application even in the presence of certain limitations, such as non-normal errors in this case (Drygas, 1983).

### 5.3.1 Idiosyncratic sort rationale

Book-to-market value is found to successfully capture a firm's distress risk, riskiness of a firm's cash flows and financial risk (Chen & Zhang, 1998). Vassalou and Xing (2004) find the book-to-market ratio to be largely a proxy for default risk. These findings confirm the book-to-market ratio as a theoretically relevant variable considering idiosyncratic risks of a firm. This variable consideration is relevant for size as well. Whilst size may proxy for other fundamental risk factors, where those factors are practically unobservable, or sufficiently complicated for traders to consider, size presents as a reasonable proxy. An alternate size effect explanation considered by Banz (1981) as well as Zeghal (1984) considers an information cost or risk hypothesis which states that due to the lack of information surrounding small firms. From a probability beliefs perspective, the relationship between the amount of information available for a firm and firm size makes this a sorting criteria one which is of interest to this study. Further to this, the effect of size on asset pricing has also found great empirical support both locally and abroad as a relevant consideration of asset

[3] In a regression model where E$\{\epsilon_i\}$= 0 and variance $\sigma^2\{\epsilon_i\} = \sigma^2 < \infty$ and $\epsilon_i$ and $\epsilon_j$ are uncorrelated for all $i$ and $j$ the least squares estimators $b_0$ and $b_1$ are unbiased and have minimum variance among all unbiased linear estimators (Wood, 2013).

pricing (Lakonishok & Shapiro, 1986; van Rensburg & Robertson, 2003; Fama & French, 1993; Basiewicz & Auret, 2010).

#### 5.3.2 Portfolio sort methodology

Portfolio sorts are carried out following the method of the Fama and French (1993) model, considering the characteristic factors; size and value which are decomposed into thirds. The portfolio sorts are tested as being equally-weighted, as well as value-weighted. Due to mixed opinions amongst academics regarding the best weighting scheme, both will be considered (Basiewicz & Auret, 2009).

The sorting of stocks by idiosyncratic criteria, whilst behaviourally justified and necessary for this asset pricing investigation, raises a concern surrounding portfolio size. Given the relatively small size of the JSE, which is greatly limited by liquidity and size restrictions imposed, the characteristic based equity sorts result in portfolios containing between 20 and 30 stocks each. The small portfolio size, by number of stocks, is expected to be a practically appropriate description of a behaviourally inspired model; small undiversified portfolios are consistent with both local and international research into investor behaviour (Firer, 1988; Fu, 2009). Whilst behaviourally justified, small expectant portfolio size will bring into question the statistical robustness of the regression tests, potentially biasing any conclusions to be drawn. To address this concern, an independent parallel sort in the vein of Basiewicz and Auret (2009), is considered in order to keep sample size as high as possible. Both variables are sorted simultaneously as opposed to sequentially, as no *a priori* justification can be given for preferring either sort variable over the other.

The result is that testing of the LA-CAPM is carried out on a cross-section of nine portfolios which comprise three size sorted portfolios; small, medium and large, as well as thee value sorted portfolios defined by book-to-market ratio; low, medium and high. In order to alleviate look ahead bias as documented by Banz and Breen (1986), the accounting ratios are only considered to “come into effect” with regard to their influence on asset pricing six months after the financial year end, ensuring that the effect of the accounting data being released into the market is reflected in the various idiosyncratic indicators. All the portfolios are rebalanced annually with the appeal that, in addition to reducing the effects of short term reversals (Jegadeesh, 1990; Carhart, 1997), it is expected to mimic the experience of an average investor; as established by Barberis and Thaler (2003), who note that annual portfolio evaluation is a reasonable behavioural approximation.

## 6. Results

### Table1: OLS regression

OLS Performance: The table reports the regression estimated alphas for OLS calculated over a 5 year rolling window for the period, July 2001 to June 2012. The alpha values are calculated as the intercepts to the regression by subtracting the *ex ante* regression predicted returns from *ex post* observed returns. The alpha values presented are all multiplied by 100 in order to ease interpretation as well as being accompanied by a t-statistic. The parametric test compares the return series for which the presented alphas represent the mean excess returns to a vector of equivalent rows, comprised of zeros, and a two-tailed t-test for zero difference of means was calculated. Estimated statistical significance (statistical difference from zero) is indicated in the respective superscripts where applicable: *significant at the 10% level, **significant at the 5% level, ***significant at the 1% level. Panel A and B report the mean OLS prediction errors for JSE listed stocks subject to liquidity and price filters of 0.1% and 100 cents per share. At the bottom of each panel are the results of overall significance. The root mean squared error (RMSE) and the composite pricing error (CPE) are presented on the combined performance of each model specification across the 9 portfolios specifications. Panel A reports on share returns to the market, and portfolios as weighted equally across shares. Panel B reports the same information for the market and portfolio returns calculated according to market weight.

**Panel A**: Alphas of OLS regression on Portfolios with Equally-weighted Returns

| | | B/M | | |
|---|---|---|---|---|
| | | Low | Medium | High |
| Size | Small | **-0.358** | **0.385** | **0.356**$^{*}$ |
| | | (-0.975) | (1.581) | (1.798) |
| | Medium | **-0.702**$^{***}$ | **-0.174** | **0.963**$^{**}$ |
| | | (-2.870) | (-1.031) | (2.059) |
| | Large | **-0.422**$^{*}$ | **-0.134** | **0.103** |
| | | (-1.885) | (-0.681) | (0.453) |
| | RMSE | **0.478** | CPE | **0.185** |

**Panel B**: Alphas of OLS regression on Portfolios specified with Value-weighted Returns

| | | B/M | | |
|---|---|---|---|---|
| | | Low | Medium | High |
| Size | Small | **0.889** | **1.529**$^{***}$ | **1.165**$^{***}$ |
| | | (1.417) | (3.935) | (5.449) |
| | Medium | **0.266** | **0.645**$^{**}$ | **0.749**$^{***}$ |
| | | (0.824) | (2.141) | (4.787) |
| | Large | **-0.172** | **-0.053** | **0.591**$^{*}$ |
| | | (-1.121) | (-0.323) | (1.925) |
| | RMSE | **0.811** | CPE | **0.609** |

**Table 2: Long-run beta on equally-weighted portfolios (in-sample)**

Results below compare the relative performance of the Kalman filter specified according to Adrian and Franzoni (2009), by inclusion of a long-run risk estimation variable, $B_t^{ie}$, for the prediction of the state equation predicting the probability risk, $\beta_{t+1|t}^{ie}$. The results pertain to models estimated on equally-weighted return series, with the Kalman filter's diffuse priors estimated based on the entire data set. Panel A contains the results to this Kalman specification without the inclusion of a conditioning variable. Panel B contains results for the model specification in the case of probability risk variable, $\beta_{t+1|t}^{ie}$, being predicted in the presence of a lagged market return conditioning variable. For each of the panels, mean estimated regression prediction errors are reported in bold (multiplied by 100). Estimated statistical significance (statistical difference from zero) is indicated in the respective superscripts where applicable: *significant at the 10% level, **significant at the 5% level, ***significant at the 1% level. Below the reported estimated pricing errors of the portfolios are t-statistics, reported in parentheses. At the bottom of each panel are the results of overall significance. The root mean squared error (RMSE) and the composite pricing error (CPE) are presented on the combined performance of each model specification across the 9 portfolios specifications. Below these statistics are comparisons of performance to OLS regression. A positive percentage represents the reduction of pricing error as compared with the 5 year rolling OLS regression. A negative percentage reports that the model in question does not reduce prediction errors when compared with an OLS regression.

| | **Panel A:** Unconditional Adrian &Franzoni Specification (long-run) | | | | **Panel B:** Adrian &Franzoni Specification (short-run) Conditioned on Lagged Market Returns | | | |
|---|---|---|---|---|---|---|---|---|
| | | | **B/M** | | | | | |
| | | Low | Medium | High | | Low | Medium | High |
| | Small | **-0.421** | **0.341** | **0.323** | Small | **-0.375** | **0.332** | **0.300** |
| | | (-1.155) | (1.419) | (1.632) | | (-1.021) | (1.355) | (1.516) |
| **Size** | Medium | **-0.442*** | **-0.188** | **0.952**** | Medium | **-0.437*** | **-0.193** | **0.917**** |
| | | (-1.906) | (-1.133) | (2.125) | | (-1.886) | (-1.160) | (2.073) |
| | Large | **-0.298** | **-0.107** | **0.072** | Large | **-0.260** | **-0.106** | **0.020** |
| | | (-1.395) | (-0.555) | (0.318) | | (-1.210) | (-0.552) | (0.088) |
| | RMSE | **0.427** | CPE | **0.135** | RMSE | **0.407** | CPE | **0.122** |
| | | | | **Reduction in estimated pricing errors as compared with OLS regression** | | | | |
| | RMSE | **11%** | CPE | **27%** | RMSE | **15%** | CPE | **34%** |

**Table 3 Long-run beta on equally-weighted portfolios (out-of-sample)**

Results below compare the relative performance of the Kalman filter specified according the Adrian and Franzoni (2009), by inclusion of a long-run risk estimation variable, $B_t^{ie}$, for the prediction of the state equation predicting the probability risk, $\beta_{t+1|t}^{ie}$. The results pertain to models estimated on equally-weighted return series, with the Kalman filter's diffuse priors estimated based on an out of sample period, July 1996 to June 2001. Panel A contains the results to this Kalman specification without the inclusion of a conditioning variable. Panel B contains results for the model specification in the case of probability risk variable, $\beta_{t+1|t}^{ie}$, being predicted in the presence of a lagged market return conditioning variable. For each of the panels, mean estimated regression prediction errors are reported in bold (multiplied by 100). Estimated statistical significance (a significant difference from zero) is indicated in the respective superscripts where applicable: *significant at the 10% level, **significant at the 5% level, ***significant at the 1% level. Below the reported estimated pricing errors of the portfolios are t-statistics, reported in parentheses. At the bottom of each panel are the results of overall significance. The root mean squared error (RMSE) and the composite pricing error (CPE) are presented on the combined performance of each model specification across the 9 portfolios specifications. Below these statistics are comparisons of performance to OLS regression. A positive percentage represents the reduction of pricing error as compared with the 5 year rolling OLS regression. A negative percentage reports that the model in question does not reduce prediction errors when compared with an OLS regression.

| | | **Panel A:** Unconditional Adrian &Franzoni Specification (long-run) | | | | **Panel B:** Adrian &Franzoni Specification (short-run) Conditioned on Lagged Market Returns | | |
|---|---|---|---|---|---|---|---|---|
| | | | | **B/M** | | | | |
| | | Low | Medium | High | | Low | Medium | High |
| | Small | **-0.427** | **0.333** | **0.318** | Small | **-0.396** | **0.298** | **0.300** |
| | | (-1.162) | (1.383) | (1.601) | | (-1.073) | (1.148) | (1.511) |
| **Size** | Medium | **-0.619**$^{**}$ | **-0.232** | **0.806**$^{*}$ | Medium | **-0.549**$^{**}$ | **-0.186** | **0.765** |
| | | (-2.607) | (-1.401) | (1.722) | | (-2.293) | (-1.119) | (1.603) |
| | Large | **-0.299** | **-0.095** | **0.070** | Large | **-0.092** | **-0.106** | **-0.029** |
| | | (-1.397) | (-0.487) | (0.308) | | (-0.417) | (-0.543) | (-0.125) |
| | RMSE | 0.419 | CPE | 0.150 | RMSE | 0.377 | CPE | 0.108 |
| | | | **Reduction in estimated pricing errors as compared with OLS regression** | | | | | |
| | RMSE | 12% | CPE | 19% | RMSE | 21% | CPE | 41% |

**Table 4 Short-run beta on equally-weighted portfolios (in-sample)**

Results below compare the relative performance of the Kalman filter specified according the Hamilton (1994), for the prediction of the state equation predicting the probability risk, $\beta^{ie}_{t+1|t}$. The results pertain to models estimated on equally-weighted return series, with the Kalman filter's diffuse priors estimated based on the entire data set. Panel A contains the results to this Kalman specification without the inclusion of a conditioning variable. Panel B contains results for the model specification in the case of probability risk variable, $\beta^{ie}_{t+1|t}$, being predicted in the presence of a lagged market return conditioning variable. For each of the panels, mean estimated regression prediction errors are reported in bold (multiplied by 100). Estimated statistical significance (a significant difference from zero) is indicated in the respective superscripts where applicable: *significant at the 10% level, **significant at the 5% level, ***significant at the 1% level. Below the reported estimated pricing errors of the portfolios are t-statistics, reported in parentheses. At the bottom of each panel are the results of overall significance. The root mean squared error (RMSE) and the composite pricing error (CPE) are presented on the combined performance of each model specification across the 9 portfolios specifications. Below these statistics are comparisons of performance to OLS regression. A positive percentage represents the reduction of pricing error as compared with the 5 year rolling OLS regression. A negative percentage reports that the model in question does not reduce prediction errors when compared with an OLS regression.

| | | **Panel A:** Unconditional Hamilton Specification (short-run) | | | | **Panel B:** Hamilton Specification (short-run) Conditioned on Lagged Market Returns | | |
|---|---|---|---|---|---|---|---|---|
| | | | | **B/M** | | | | |
| | | Low | Medium | High | | Low | Medium | High |
| **Size** | Small | **-0.414** | **0.341** | **0.299** | Small | **-0.375** | **0.361** | **0.305** |
| | | (-1.126) | (1.417) | (1.547) | | (-1.021) | (1.505) | (1.582) |
| | Medium | **-0.348** | **-0.213** | **1.004**** | Medium | **-0.395*** | **-0.203** | **1.012**** |
| | | (-1.501) | (-1.283) | (2.095) | | (-1.714) | (-1.248) | (2.106) |
| | Large | **-0.169** | **-0.090** | **0.088** | Large | **-0.150** | **-0.095** | **0.072** |
| | | (-0.799) | (-0.463) | (0.390) | | (-0.706) | (-0.492) | (0.318) |
| | RMSE | 0.421 | CPE | 0.113 | RMSE | 0.424 | CPE | 0.118 |
| | | | | **Reduction in estimated pricing errors as compared with OLS regression** | | | | |
| | RMSE | 12% | CPE | 39% | RMSE | 11% | CPE | 36% |

**Table 5 Short-run beta on equally-weighted portfolios (out-of-sample)**

Results below compare the relative performance of the Kalman filter specified according the Hamilton (1994), for the prediction of the state equation predicting the probability risk, $\beta^{ie}_{t+1|t}$. The results pertain to models estimated on equally-weighted return series, with the Kalman filter's diffuse priors estimated based on an out of sample period, July 1996 to June 2001. Panel A contains the results to this Kalman specification without the inclusion of a conditioning variable. Panel B contains results for the model specification in the case of probability risk variable,$\beta^{ie}_{t+1|t}$, being predicted in the presence of a lagged market return conditioning variable. For each of the panels, mean estimated regression prediction errors are reported in bold (multiplied by 100). Estimated statistical significance (a significant difference from zero) is indicated in the respective superscripts where applicable: *significant at the 10% level, **significant at the 5% level, ***significant at the 1% level. Below the reported estimated pricing errors of the portfolios are t-statistics, reported in parentheses. At the bottom of each panel are the results of overall significance. The root mean squared error (RMSE) and the composite pricing error (CPE) are presented on the combined performance of each model specification across the 9 portfolios specifications. Below these statistics are comparisons of performance to OLS regression. A positive percentage represents the reduction of pricing error as compared with the 5 year rolling OLS regression. A negative percentage reports that the model in question does not reduce prediction errors when compared with an OLS regression.

| | **Panel A:** Unconditional Hamilton Specification (short-run) | | | | **Panel B:** Hamilton Specification (short-run) Conditioned on Lagged Market Returns | | | |
|---|---|---|---|---|---|---|---|---|
| | | | **B/M** | | | | | |
| | | Low | Medium | High | | Low | Medium | High |
| **Size** | Small | **-0.415** | **0.313** | **0.320** | Small | **-0.061** | **0.392** | **0.357** |
| | | (-1.111) | (1.283) | (1.632) | | (-0.121) | (1.597) | (1.589) |
| | Medium | **-0.426**$^{*}$ | **-0.218** | **0.906**$^{**}$ | Medium | **-0.604**$^{**}$ | **-0.186** | **0.938**$^{**}$ |
| | | (-1.829) | (-1.316) | (2.023) | | (-2.374) | (-1.108) | (2.074) |
| | Large | **-0.162** | **-0.090** | **0.098** | Large | **-0.112** | **-0.012** | **0.091** |
| | | (-0.761) | (-0.464) | (0.430) | | (-0.522) | (-0.047) | (0.403) |
| | RMSE | 0.404 | CPE | 0.119 | RMSE | 0.420 | CPE | 0.126 |
| | **Reduction in estimated pricing errors as compared with OLS regression** | | | | | | | |
| | RMSE | 16% | CPE | 36% | RMSE | 12% | CPE | 32% |

**Table 6 Long-run beta on value-weighted portfolios (in-sample)**

Results below compare the relative performance of the Kalman filter specified according the Adrian and Franzoni (2009), by inclusion of a long-run risk estimation variable, $B_t^{ie}$, for the prediction of the state equation predicting the probability risk, $\beta_{t+1|t}^{ie}$. The results pertain to models estimated on value-weighted return series, with the Kalman filter's diffuse priors estimated based on the entire data set. Panel A contains the results to this Kalman specification without the inclusion of a conditioning variable. Panel B contains results for the model specification in the case of probability risk variable, $\beta_{t+1|t}^{ie}$, being predicted in the presence of a lagged market return conditioning variable. For each of the panels, mean estimated regression prediction errors are reported in bold (multiplied by 100). Estimated statistical significance (a significant difference from zero) is indicated in the respective superscripts where applicable: *significant at the 10% level, **significant at the 5% level, ***significant at the 1% level. Below the reported estimated pricing errors of the portfolios are t-statistics, reported in parentheses. At the bottom of each panel are the results of overall significance. The root mean squared error (RMSE) and the composite pricing error (CPE) are presented on the combined performance of each model specification across the 9 portfolios specifications. Below these statistics are comparisons of performance to OLS regression. A positive percentage represents the reduction of pricing error as compared with the 5 year rolling OLS regression. A negative percentage reports that the model in question does not reduce prediction errors when compared with an OLS regression.

| | **Panel A:** Unconditional Adrian &Franzoni Specification (long-run) | | | | **Panel B:** Adrian &Franzoni Specification (short-run) Conditioned on Lagged Market Returns | | | |
|---|---|---|---|---|---|---|---|---|
| | **B/M** | | | | | | | |
| | | Low | Medium | High | | Low | Medium | High |
| | Small | **0.860** | **1.392**$^{***}$ | **0.949**$^{***}$ | Small | **0.846** | **1.386**$^{***}$ | **0.959**$^{***}$ |
| | | (1.365) | (3.613) | (5.070) | | (1.344) | (3.598) | (5.093) |
| **Size** | Medium | **0.208** | **0.498**$^{*}$ | **0.570**$^{***}$ | Medium | **0.154** | **0.508**$^{*}$ | **0.525**$^{***}$ |
| | | (0.675) | (1.709) | (4.410) | | (0.493) | (1.726) | (4.013) |
| | Large | **-0.144** | **-0.040** | **0.521**$^{*}$ | Large | **-0.126** | **-0.068** | **0.491**$^{*}$ |
| | | (-0.935) | (-0.243) | (1.752) | | (-0.820) | (-0.407) | (1.657) |
| | RMSE | 0.706 | CPE | 0.511 | RMSE | 0.697 | CPE | 0.482 |
| | **Reduction in estimated pricing errors as compared with OLS regression** | | | | | | | |
| | RMSE | 13% | CPE | 16% | RMSE | 14% | CPE | 21% |

**Table 7 Long-run beta on value-weighted portfolios (out-of-sample)**

Results below compare the relative performance of the Kalman filter specified according the Adrian and Franzoni (2009), by inclusion of a long-run risk estimation variable, $B_t^{ie}$, for the prediction of the state equation predicting the probability risk, $\beta_{t+1|t}^{ie}$. The results pertain to models estimated on value-weighted return series, with the Kalman filter's diffuse priors estimated based on an out of sample period, July 1996 to June 2001. Panel A contains the results to this Kalman specification without the inclusion of a conditioning variable. Panel B contains results for the model specification in the case of probability risk variable,$\beta_{t+1|t}^{ie}$, being predicted in the presence of a lagged market return conditioning variable. For each of the panels, mean estimated regression prediction errors are reported in bold (multiplied by 100). Estimated statistical significance (a significant difference from zero) is indicated in the respective superscripts where applicable: *significant at the 10% level, **significant at the 5% level, ***significant at the 1% level. Below the reported estimated pricing errors of the portfolios are t-statistics, reported in parentheses. At the bottom of each panel are the results of overall significance. The root mean squared error (RMSE) and the composite pricing error (CPE) are presented on the combined performance of each model specification across the 9 portfolios specifications. Below these statistics are comparisons of performance to OLS regression. A positive percentage represents the reduction of pricing error as compared with the 5 year rolling OLS regression. A negative percentage reports that the model in question does not reduce prediction errors when compared with an OLS regression.

| | **Panel A:** Unconditional Adrian &Franzoni Specification (long-run) | | | | **Panel B:** Adrian &Franzoni Specification (short-run) Conditioned on Lagged Market Returns | | | |
|---|---|---|---|---|---|---|---|---|
| | | | **B/M** | | | | | |
| | | Low | Medium | High | | Low | Medium | High |
| | Small | **0.916** | **1.391**$^{***}$ | **0.935**$^{***}$ | Small | **0.902** | **1.390**$^{***}$ | **0.948**$^{***}$ |
| | | (1.460) | (3.612) | (5.014) | | (1.436) | (3.610) | (5.038) |
| **Size** | Medium | **0.138** | **0.537**$^{*}$ | **0.570**$^{***}$ | Medium | **0.137** | **0.523**$^{*}$ | **0.474**$^{***}$ |
| | | (0.439) | (1.840) | (4.387) | | (0.432) | (1.777) | (3.690) |
| | Large | **-0.143** | **-0.065** | **0.526**$^{*}$ | Large | **-0.117** | **-0.086** | **0.521**$^{*}$ |
| | | (-0.932) | (-0.382) | (1.765) | | (-0.755) | (-0.505) | (1.744) |
| | RMSE | 0.714 | CPE | 0.510 | RMSE | 0.704 | CPE | 0.464 |

**Reduction in estimated pricing errors as compared with OLS regression**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| RMSE | 12% | CPE | 16% | RMSE | 13% | CPE | 24% |

**Table 8 Short-run beta on value-weighted portfolios (in-sample)**

Results below compare the relative performance of the Kalman filter specified according the Hamilton (1994), for the prediction of the state equation predicting the probability risk, $\beta^{ie}_{t+1|t}$. The results pertain to models estimated on value-weighted return series, with the Kalman filter's diffuse priors estimated based on the entire data set. Panel A contains the results to this Kalman specification without the inclusion of a conditioning variable. Panel B contains results for the model specification in the case of probability risk variable, $\beta^{ie}_{t+1|t}$, being predicted in the presence of a lagged market return conditioning variable. For each of the panels, mean estimated regression prediction errors are reported in bold (multiplied by 100). Estimated statistical significance (a significant difference from zero) is indicated in the respective superscripts where applicable: *significant at the 10% level, **significant at the 5% level, ***significant at the 1% level. Below the reported estimated pricing errors of the portfolios are t-statistics, reported in parentheses. At the bottom of each panel are the results of overall significance. The root mean squared error (RMSE) and the composite pricing error (CPE) are presented on the combined performance of each model specification across the 9 portfolios specifications. Below these statistics are comparisons of performance to OLS regression. A positive percentage represents the reduction of pricing error as compared with the 5 year rolling OLS regression. A negative percentage reports that the model in question does not reduce prediction errors when compared with an OLS regression.

| | **Panel A:** Unconditional Hamilton Specification (short-run) | | | | **Panel B:** Hamilton Specification (short-run) Conditioned on Lagged Market Returns | | | |
|---|---|---|---|---|---|---|---|---|
| | **B/M** | | | | | | | |
| | | Low | Medium | High | | Low | Medium | High |
| | Small | **0.880** | **1.397**$^{***}$ | **0.961**$^{***}$ | Small | **0.978** | **1.429**$^{***}$ | **1.132**$^{***}$ |
| | | (1.399) | (3.634) | (4.693) | | (1.569) | (3.721) | (4.806) |
| **Size** | Medium | **0.224** | **0.531**$^{*}$ | **0.622**$^{***}$ | Medium | **0.196** | **0.584**$^{**}$ | **0.575**$^{***}$ |
| | | (0.726) | (1.824) | (4.421) | | (0.630) | (2.021) | (4.472) |
| | Large | **-0.132** | **-0.060** | **0.483** | Large | **-0.131** | **-0.001** | **0.486** |
| | | (-0.828) | (-0.363) | (1.611) | | (-0.821) | (-0.004) | (1.625) |
| | RMSE | 0.717 | CPE | 0.485 | RMSE | 0.763 | CPE | 0.509 |
| | **Reduction in estimated pricing errors as compared with OLS regression** | | | | | | | |
| | RMSE | 12% | CPE | 20% | RMSE | 6% | CPE | 16% |

**Table 9 Short-run beta on value-weighted portfolios (out-of-sample)**

Results below compare the relative performance of the Kalman filter specified according the Hamilton (1994), for the prediction of the state equation predicting the probability risk, $\beta^{ie}_{t+1|t}$. The results pertain to models estimated on value-weighted return series, with the Kalman filter's diffuse priors estimated based on an out of sample period, July 1996 to June 2001. Panel A contains the results to this Kalman specification without the inclusion of a conditioning variable. Panel B contains results for the model specification in the case of probability risk variable,$\beta^{ie}_{t+1|t}$, being predicted in the presence of a lagged market return conditioning variable. For each of the panels, mean estimated regression prediction errors are reported in bold (multiplied by 100). Estimated statistical significance (a significant difference from zero) is indicated in the respective superscripts where applicable: *significant at the 10% level, **significant at the 5% level, ***significant at the 1% level. Below the reported estimated pricing errors of the portfolios are t-statistics, reported in parentheses. At the bottom of each panel are the results of overall significance. The root mean squared error (RMSE) and the composite pricing error (CPE) are presented on the combined performance of each model specification across the 9 portfolios specifications. Below these statistics are comparisons of performance to OLS regression. A positive percentage represents the reduction of pricing error as compared with the 5 year rolling OLS regression. A negative percentage reports that the model in question does not reduce prediction errors when compared with an OLS regression.

| | | **Panel A:** Unconditional Hamilton Specification (short-run) | | | | **Panel B:** Hamilton Specification (short-run) Conditioned on Lagged Market Returns | | |
|---|---|---|---|---|---|---|---|---|
| | | **B/M** | | | | | | |
| | | Low | Medium | High | | Low | Medium | High |
| | Small | **1.006** | **1.399*** ** | **0.968*** ** | Small | **1.146*** | **1.590*** ** | **1.953*** ** |
| | | (1.599) | (3.641) | (4.545) | | (1.685) | (4.034) | (4.904) |
| **Size** | Medium | **0.156** | **0.566*** | **0.619*** ** | Medium | **0.209** | **0.758**** | **0.580*** ** |
| | | (0.496) | (1.941) | (4.407) | | (0.647) | (2.354) | (4.335) |
| | Large | **-0.136** | **-0.063** | **0.555*** | Large | **-0.114** | **-0.079** | **0.639*** |
| | | (-0.798) | (-0.344) | (1.844) | | (-0.659) | (-0.422) | (1.978) |
| | RMSE | 0.743 | CPE | 0.485 | RMSE | 1.002 | CPE | 0.549 |

**Reduction in estimated pricing errors as compared with OLS regression**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| RMSE | 8% | CPE | 20% | RMSE | -24% | CPE | 10% |

The results of the Kalman filter regression across both long-run and short-run model designs, based on both regression training periods, both in- and out-of-sample, are extremely promising for a model design which aims to capture the dynamic investor model described in the literature. With the exception of table 9, where the RMSE shows a reduced performance of the Kalman filter specification of 24% as compared with OLS prediction errors, every specification offers a clear reduction of estimated pricing errors. Specifically, to this end even when considering the short-run out-of-sample conditional specification of table 9, the CPE test statistic of model performance, which accounts for disparate variance amongst composite portfolios in order to attenuate the possibility of results being skewed by potentially “outlier” portfolio variance, reports improved model specification as compared with the OLS CAPM. In this respect, the CPE statistic is favoured as an indication of model performance as it weights performance as a feature of variance, reducing any bias in results due to the unique variance structure of the various asset portfolio constructs, and resultantly so too is the LA-CAPM favoured.

Looking at the specific portfolio performance, it is clear that whilst the Kalman filter specification is an improvement over OLS CAPM as an asset pricing model specification based on the RMSE and CPE complete specification statistics, the performance of asset pricing in a South African context is very much dependant on portfolio composition. Across all model specifications regressed on value-weighted returns (tables 6 through 9) there is still a clear value and size affect, as statistically significant estimated pricing errors appear in the upper right diagonal of each table reporting on regression alpha. These findings are in line with prior research into the value and size affect on the JSE. The statistically significant estimated pricing errors to value sorted portfolios are directly comparable to the positive alpha found with small and high book-to-market (value) portfolios seen in previous investigations into South African returns data (Basiewicz & Auret, 2010). Further, these results are in line with international findings of CAPM based models (Fama & French, 1998), as well as being directly comparable to the observation structure seen in the results of Adrian and Franzoni (2009). It appears from these results that when results are compared with traditional OLS CAPM regressions, the Kalman Filter is seen to consistently offer a notable reduction in overall prediction errors as noted by RMSE and CPE statistics, whilst the effects of statistically significant estimated pricing errors are seen where small and value portfolio formation is concerned.

An interesting feature of this research is evident when comparing the results to value-weighted portfolios with those in respect of equally-weighted portfolio returns as seen in tables 2 through 5. The results to equally-weighted portfolios formation appear to be far superior to that of value-weighted portfolios. Whilst the effect of the LA-CAPM is clear to be seen where overall portfolio performance is concerned, the most notably difference is where reduction of estimated pricing errors across specific portfolios is concerned. The results are more comparable to those of (Huang & Hueng, 2009), finding no clear value of size effect in portfolio estimated pricing errors. The Kalman Filter applied to equally-weighted portfolios has almost no sign of statistically significant estimated pricing errors, finding modestly significant estimated pricing errors amongst portfolios of medium size only, with alphas being both nominally small and never statistically significant beyond the 5% level. These findings present as a resounding success of the model design to reduce estimated pricing errors, suggesting that the Kalman Filter as applied to equally-weighted portfolio specification offers a model design which is likely very close to an appropriate description of investor behaviour.

The limitation to value sorted portfolio performance is thought to be attributable to the top heavy nature of JSE equity market capitalisation distribution (Kruger & van Rensburg, 2008), which is expected to increase time-series variance as portfolios are skewed by heavy stock effect. This concern is tested and validated, as portfolio variance is compared in table 18 in Appendix A, finding the variance to equally-weighted portfolio formation to be significantly lower than that of value-weighted portfolios.

Comparing the Kalman filter regression results further, the conditioning criteria offers an interesting result, which seems to be specifically linked to the model design being based on long-run beta or short-run beta. Where the Adrian and Franzoni (2009) model design is followed to include a long-run beta specification, it appears that a lagged market variable inclusion offers a significant improvement of overall model performance as noted through reduction to overall portfolio estimated pricing errors. This effect appears to reverse where the short-run, Hamilton (1994) inspired, model specification is concerned. Estimated pricing errors to the short-run specification are consistently lowest where the lagged market return conditioning variable is excluded. It appears from these results that, where long-run beta is not considered, probability beliefs are formed as a purely iterative learning process which is concerned with period to period variance minimization as opposed to basing risk expectations on market performance. On the other hand, where long-run beta is a relevant concern to probability beliefs formation, the overall market performance is an important information criterion. This dispersion of results across model form also brings about a sense of variability in asset return volatility. This finding underscores the importance of the dynamic pricing model and serves to add explanation to the improved performance of the Kalman filter as a method for pricing risk over that of the traditional CAPM.

Baring these results in mind, the most successful model specification is evident in panel B of table 3, where the long-run beta model inclusive of a lagged market return conditioning variable shows overall reduction of estimated pricing errors of 41% according the CPE statistic, as well as boasting only one portfolio which has a statistically significant pricing error. With an estimated pricing error of -0.549% which is statistically significant at the 5% level attributed to a medium size, growth firm as the only statistically significant pricing error, the overall impression is that the Kalman filter with long-run beta consideration as well as market performance consideration is a successful description of investor behaviour when forming probability beliefs.

Below, the results are expanded to consider the appropriateness of a learning based model of behavioural description as compared with traditional CAPM expectations when time variance is considered. This is achieved through a decomposition of regression alphas in the design of Lewellen and Nagel (2006). The effect here is to consider the results as a feature of learning or time-variance. If the results to the LA-CAPM are based on capturing a time variance effect of prediction beta and the market, then beta prediction and market portfolio performance should be covariant, and the failures of the CAPM would be attributed to a static model design, as opposed to a breakdown of its behavioural expectations of homogeneous risk-aversion. The results below display a test of this, finding that there is no covariance to speak of between *ex ante* beta estimation and *ex post* market returns, regardless of the model specification or regression and model specification period. This result suggests that the CAPM based model of beta as a market risk coefficient which is seen to breakdown under traditional 5 year rolling OLS, as evidenced by statistically significant estimated pricing errors in table 1, is not due to the model being insufficiently time-variant, but rather that the specification itself is inappropriate as a description of how investors price risk. The fact that the only explanatory feature of the alpha decomposition of estimated pricing errors is attributable to the learning function lends support to the behavioural approach to asset pricing, as seen in its success in reducing estimated pricing errors detailed above. Specifically, this test does not consider the statistical weight of the observation of estimated pricing errors, but rather the nominal size of alpha as decomposed through covariance and the effect of the learning algorithm of the Kalman filter design. Whilst the nominal value of the Learning algorithm's effect to the reduction of estimated pricing errors appears small in these findings, what is important to note is the comparative effect to that of covariance between *ex ante* beta and *ex post* market returns. The effect of learning is consistently, across all specifications, a greater explanatory feature of pricing error reduction than covariance by a factor of between 10 and 100. These results are in line with international findings (Adrian & Franzoni, 2009) and serve to distinctly underline the importance of the learning effect to capturing probability risk, and thus the importance to asset pricing model specification.

**Table 10 Summary statistics for beta on equally-weighted portfolios (out-of-sample)**

Panel A reports beta estimates and their standard deviations for unconditional OLS regression. Panel B reports beta estimates and their standard deviations for the unconditional Kalman filter in the specification of Adrian and Franzoni inclusive of long-run beta estimation. Panel C reports beta estimates and their standard deviations for the Kalman filter conditioned on lagged market excess returns in the specification of Adrian and Franzoni inclusive of long-run beta estimation. Panel D reports beta estimates and their standard deviations for the unconditional Kalman filter in the specification of Hamilton following short-run period to period estimation. Panel E reports beta estimates and their standard deviations for the Kalman filter in the specification of Hamilton following short-run period to period estimation, conditioned on lagged excess market returns. The sample covers the period: End July 2001-End June 2012. Portfolios are equally-weighted, and the Kalman filter is trained on a purely out of sample period: July 1996 to June 2001.

| | **B/M** | | | | | |
|---|---|---|---|---|---|---|
| | Low | Medium | High | Low | Medium | High |
| **Size** | Panel A: Traditional unconditional OLS CAPM | | | | | |
| | OLS Beta | | | Std. Dev of Beta | | |
| Small | 0.919 | 0.863 | 0.742 | 0.046 | 0.056 | 0.095 |
| Medium | 1.072 | 0.979 | 1.163 | 0.149 | 0.069 | 0.280 |
| Large | 1.126 | 1.081 | 1.135 | 0.144 | 0.070 | 0.061 |
| | Panel B: Unconditional Probabilistic Pricing (Long-run) | | | | | |
| | OLS Beta | | | Std. Dev of Beta | | |
| Small | 0.989 | 0.881 | 0.693 | 0.202 | 0.014 | 0.037 |
| Medium | 1.135 | 0.999 | 1.271 | 0.070 | 0.013 | 0.375 |
| Large | 1.178 | 1.104 | 1.132 | 0.116 | 0.019 | 0.029 |
| | Panel C: Probabilistic Pricing Conditional on Market Excess Return (Long-run) | | | | | |
| | OLS Beta | | | Std. Dev of Beta | | |
| Small | 0.960 | 0.875 | 0.696 | 0.051 | 0.186 | 0.052 |
| Medium | 1.091 | 0.933 | 1.327 | 0.254 | 0.030 | 0.459 |
| Large | 1.030 | 1.095 | 1.168 | 0.327 | 0.040 | 0.155 |
| | Panel D: Unconditional Probabilistic Pricing (Short-run) | | | | | |
| | OLS Beta | | | Std. Dev of Beta | | |
| Small | 0.843 | 0.966 | 0.730 | 0.165 | 0.050 | 0.047 |
| Medium | 1.022 | 0.989 | 1.058 | 0.211 | 0.022 | 0.146 |
| Large | 1.003 | 1.068 | 1.091 | 0.261 | 0.029 | 0.029 |
| | Panel E: Probabilistic Pricing Conditional on Market Excess Return (Short-run) | | | | | |
| | OLS Beta | | | Std. Dev of Beta | | |
| Small | -0.021 | 0.749 | 0.496 | 0.345 | 0.198 | 0.043 |
| Medium | 1.292 | 0.941 | 0.908 | 0.197 | 0.212 | 0.085 |
| Large | 0.989 | 0.621 | 1.108 | 0.287 | 0.227 | 0.021 |

**Table 11 Alpha decomposition for beta on equally-weighted portfolios (out-of-sample)**

The table below reports the alpha decomposition for the portfolio and model specifications of table 10, above. Panel A reports unconditional pricing error, $\alpha^{i}_{OLS}$ for the 9 portfolio sorts. Panel B reports the learning component as a description of pricing error which is reduced under the Kalman filter specifications. Panel C reports the covariance component. Results in panel are sample covariances which have been multiplied by 100 for interpretation, as the sample covariances between predicted beta, and observed market returns were exceptionally small across all specifications.

**Panel A: Unconditional Probabilistic Pricing (Long-run)**

| | B/M | | |
|---|---|---|---|
| **Size** | Low | Medium | High |
| Panel A: Unconditional Alphas (OLS) | | | |
| Small | -0.358 | 0.385 | 0.356 |
| Medium | -0.702 | -0.174 | 0.963 |
| Large | -0.422 | -0.134 | 0.103 |
| Panel B: Learning Component | | | |
| Small | 0.0271 | 0.0072 | -0.0192 |
| Medium | 0.0243 | 0.0075 | 0.0419 |
| Large | 0.0199 | 0.0091 | -0.0011 |
| Panel C: Covariance Component | | | |
| Small | 0.0002 | -0.0001 | -0.0001 |
| Medium | 0.0001 | -0.0001 | -0.0004 |
| Large | 0.0000 | 0.0002 | 0.0001 |

**Panel B: Probabilistic Pricing Conditional on Market Excess Return (Long-run)**

| | B/M | | |
|---|---|---|---|
| **Size** | Low | Medium | High |
| Panel A: Unconditional Alphas (OLS) | | | |
| Small | -0.358 | 0.385 | 0.356 |
| Medium | -0.702 | -0.174 | 0.963 |
| Large | -0.422 | -0.134 | 0.103 |
| Panel B: Learning Component | | | |
| Small | 0.016 | 0.005 | -0.018 |
| Medium | 0.007 | -0.018 | 0.063 |
| Large | -0.037 | 0.005 | 0.013 |
| Panel C: Covariance Component | | | |
| Small | 0.000 | 0.000 | 0.000 |
| Medium | 0.000 | 0.000 | 0.000 |
| Large | -0.002 | 0.000 | 0.001 |

**Panel C: Unconditional Probabilistic Pricing (Short-run)**

| | B/M | | |
|---|---|---|---|
| **Size** | Low | Medium | High |
| Panel A: Unconditional Alphas (OLS) | | | |
| Small | -0.358 | 0.385 | 0.356 |
| Medium | -0.702 | -0.174 | 0.963 |
| Large | -0.422 | -0.134 | 0.103 |
| Panel B: Learning Component | | | |
| Small | -0.029 | 0.040 | -0.005 |
| Medium | -0.019 | 0.004 | -0.041 |
| Large | -0.048 | -0.005 | -0.017 |
| Panel C: Covariance Component | | | |
| Small | 0.0006 | -0.0002 | -0.0003 |
| Medium | -0.0014 | -0.0002 | -0.0006 |
| Large | -0.0007 | 0.0003 | 0.0000 |

**Panel D: Probabilistic Pricing Conditional on Market Excess Return (Short-run)**

| | B/M | | |
|---|---|---|---|
| **Size** | Low | Medium | High |
| Panel A: Unconditional Alphas (OLS) | | | |
| Small | -0.358 | 0.385 | 0.356 |
| Medium | -0.702 | -0.174 | 0.963 |
| Large | -0.422 | -0.134 | 0.103 |
| Panel B: Learning Component | | | |
| Small | -0.364 | -0.044 | -0.095 |
| Medium | 0.085 | -0.015 | -0.099 |
| Large | -0.053 | -0.178 | -0.010 |
| Panel C: Covariance Component | | | |
| Small | 0.0004 | -0.0002 | 0.0002 |
| Medium | -0.0007 | -0.0003 | -0.0003 |
| Large | -0.0011 | 0.0012 | 0.0000 |

**Table 12 Summary statistics for beta on equally-weighted portfolios (in-sample)**

Panel A reports beta estimates and their standard deviations for unconditional OLS regression. Panel B reports beta estimates and their standard deviations for the unconditional Kalman filter in the specification of Adrian and Franzoni inclusive of long-run beta estimation. Panel C reports beta estimates and their standard deviations for the Kalman filter conditioned on lagged market excess returns in the specification of Adrian and Franzoni inclusive of long-run beta estimation. Panel D reports beta estimates and their standard deviations for the unconditional Kalman filter in the specification of Hamilton following short-run period to period estimation. Panel E reports beta estimates and their standard deviations for the Kalman filter in the specification of Hamilton following short-run period to period estimation, conditioned on lagged excess market returns. The sample covers the period: End July 2001-End June 2012. Portfolios are equally-weighted, and the Kalman filter is trained on the entire data set of June 1996 to June 2012.

| | B/M | | | | | |
|---|---|---|---|---|---|---|
| | Low | Medium | High | Low | Medium | High |
| **Size** | Panel A: Traditional unconditional OLS CAPM | | | | | |
| | OLS Beta | | | Std. Dev of Beta | | |
| Small | 0.919 | 0.863 | 0.742 | 0.046 | 0.056 | 0.095 |
| Medium | 1.072 | 0.979 | 1.163 | 0.149 | 0.069 | 0.280 |
| Large | 1.126 | 1.081 | 1.135 | 0.144 | 0.070 | 0.061 |
| | Panel B: Unconditional Probabilistic Pricing (Long-run) | | | | | |
| | OLS Beta | | | Std. Dev of Beta | | |
| Small | 0.988 | 0.865 | 0.700 | 0.150 | 0.015 | 0.033 |
| Medium | 1.081 | 0.952 | 0.909 | 0.207 | 0.033 | 0.145 |
| Large | 1.179 | 1.126 | 1.135 | 0.119 | 0.088 | 0.029 |
| | Panel C: Probabilistic Pricing Conditional on Market Excess Return (Long-run) | | | | | |
| | OLS Beta | | | Std. Dev of Beta | | |
| Small | 0.948 | 0.870 | 0.705 | 0.045 | 0.071 | 0.053 |
| Medium | 1.078 | 0.947 | 0.914 | 0.205 | 0.028 | 0.152 |
| Large | 1.178 | 1.127 | 1.159 | 0.127 | 0.086 | 0.073 |
| | Panel D: Unconditional Probabilistic Pricing (Short-run) | | | | | |
| | OLS Beta | | | Std. Dev of Beta | | |
| Small | 0.941 | 0.865 | 0.821 | 0.033 | 0.015 | 0.068 |
| Medium | 1.011 | 0.989 | 0.685 | 0.247 | 0.027 | 0.546 |
| Large | 1.040 | 1.069 | 1.142 | 0.245 | 0.029 | 0.035 |
| | Panel E: Probabilistic Pricing Conditional on Market Excess Return (Short-run) | | | | | |
| | OLS Beta | | | Std. Dev of Beta | | |
| Small | 0.845 | 0.847 | 0.815 | 0.079 | 0.090 | 0.071 |
| Medium | 1.019 | 0.966 | 0.687 | 0.220 | 0.116 | 0.544 |
| Large | 1.036 | 1.062 | 1.154 | 0.255 | 0.068 | 0.060 |

**Table 13 Alpha decomposition for beta on equally-weighted portfolios (in-sample)**

The table below reports the alpha decomposition for the portfolio and model specifications of table 12, above. Panel A reports unconditional pricing error, $\alpha_{OLS}^{i}$ for the 9 portfolio sorts. Panel B reports the learning component as a description of pricing error which is reduced under the Kalman filter specifications. Panel C reports the covariance component. Results in panel are sample covariances which have been multiplied by 100 for interpretation, as the sample covariances between predicted beta, and observed market returns were exceptionally small across all specifications.

**Panel A: Unconditional Probabilistic Pricing (Long-run)**

| | B/M | | |
|---|---|---|---|
| **Size** | Low | Medium | High |
| Panel A: Unconditional Alphas (OLS) | | | |
| Small | -0.358 | 0.385 | 0.356 |
| Medium | -0.702 | -0.174 | 0.963 |
| Large | -0.422 | -0.134 | 0.103 |
| Panel B: Learning Component | | | |
| Small | 0.0267 | 0.0007 | -0.0162 |
| Medium | 0.0033 | -0.0106 | -0.0983 |
| Large | 0.0203 | 0.0177 | 0.0003 |
| Panel C: Covariance Component | | | |
| Small | 0.0001 | -0.0001 | -0.0002 |
| Medium | -0.0015 | -0.0003 | -0.0005 |
| Large | 0.0000 | 0.0002 | 0.0000 |

**Panel B: Probabilistic Pricing Conditional on Market Excess Return (Long-run)**

| | B/M | | |
|---|---|---|---|
| **Size** | Low | Medium | High |
| Panel A: Unconditional Alphas (OLS) | | | |
| Small | -0.358 | 0.385 | 0.356 |
| Medium | -0.702 | -0.174 | 0.963 |
| Large | -0.422 | -0.134 | 0.103 |
| Panel B: Learning Component | | | |
| Small | 0.011 | 0.003 | -0.014 |
| Medium | 0.002 | -0.012 | -0.096 |
| Large | 0.020 | 0.018 | 0.009 |
| Panel C: Covariance Component | | | |
| Small | -0.0002 | 0.0000 | 0.0000 |
| Medium | -0.0015 | -0.0003 | -0.0001 |
| Large | -0.0004 | 0.0002 | 0.0005 |

**Panel C: Unconditional Probabilistic Pricing (Short-run)**

| | B/M | | |
|---|---|---|---|
| **Size** | Low | Medium | High |
| Panel A: Unconditional Alphas (OLS) | | | |
| Small | -0.358 | 0.385 | 0.356 |
| Medium | -0.702 | -0.174 | 0.963 |
| Large | -0.422 | -0.134 | 0.103 |
| Panel B: Learning Component | | | |
| Small | 0.009 | 0.001 | 0.030 |
| Medium | -0.024 | 0.004 | -0.185 |
| Large | -0.033 | -0.005 | 0.003 |
| Panel C: Covariance Component | | | |
| Small | 0.0003 | -0.0001 | -0.0004 |
| Medium | -0.0022 | -0.0002 | -0.0001 |
| Large | -0.0008 | 0.0003 | -0.0001 |

**Panel D: Probabilistic Pricing Conditional on Market Excess Return (Short-run)**

| | B/M | | |
|---|---|---|---|
| **Size** | Low | Medium | High |
| Panel A: Unconditional Alphas (OLS) | | | |
| Small | -0.358 | 0.385 | 0.356 |
| Medium | -0.702 | -0.174 | 0.963 |
| Large | -0.422 | -0.134 | 0.103 |
| Panel B: Learning Component | | | |
| Small | -0.029 | -0.006 | 0.028 |
| Medium | -0.021 | -0.005 | -0.184 |
| Large | -0.035 | -0.007 | 0.007 |
| Panel C: Covariance Component | | | |
| Small | 0.0002 | -0.0002 | -0.0005 |
| Medium | -0.0017 | -0.0002 | -0.0002 |
| Large | -0.0009 | 0.0004 | 0.0000 |

**Table 14 Summary statistics for beta on value-weighted portfolios (out-of-sample)**

Panel A reports beta estimates and their standard deviations for unconditional OLS regression. Panel B reports beta estimates and their standard deviations for the unconditional Kalman filter in the specification of Adrian and Franzoni inclusive of long-run beta estimation. Panel C reports beta estimates and their standard deviations for the Kalman filter conditioned on lagged market excess returns in the specification of Adrian and Franzoni inclusive of long-run beta estimation. Panel D reports beta estimates and their standard deviations for the unconditional Kalman filter in the specification of Hamilton following short-run period to period estimation. Panel E reports beta estimates and their standard deviations for the Kalman filter in the specification of Hamilton following short-run period to period estimation, conditioned on lagged excess market returns. The sample covers the period: End July 2001-End June 2012. Portfolios are value-weighted by market cap., and the Kalman filter is trained on a purely out of sample period: July 1996 to June 2001.

| | **B/M** | | | | | |
|---|---|---|---|---|---|---|
| | Low | Medium | High | Low | Medium | High |
| **Size** | Panel A: Traditional unconditional OLS CAPM | | | | | |
| | OLS Beta | | | Std. Dev of Beta | | |
| Small | 0.651 | 0.533 | 0.488 | 0.083 | 0.122 | 0.105 |
| Medium | 0.583 | 0.600 | 0.595 | 0.120 | 0.099 | 0.097 |
| Large | 1.077 | 0.952 | 0.944 | 0.054 | 0.075 | 0.091 |
| | Panel B: Unconditional Probabilistic Pricing (Long-run) | | | | | |
| | OLS Beta | | | Std. Dev of Beta | | |
| Small | 0.720 | 0.596 | 0.542 | 0.102 | 0.026 | 0.104 |
| Medium | 0.673 | 0.616 | 0.659 | 0.046 | 0.020 | 0.036 |
| Large | 1.029 | 0.995 | 0.972 | 0.053 | 0.050 | 0.068 |
| | Panel C: Probabilistic Pricing Conditional on Market Excess Return (Long-run) | | | | | |
| | OLS Beta | | | Std. Dev of Beta | | |
| Small | 0.716 | 0.596 | 0.536 | 0.068 | 0.026 | 0.090 |
| Medium | 0.675 | 0.646 | 0.677 | 0.059 | 0.029 | 0.093 |
| Large | 1.026 | 1.010 | 0.973 | 0.073 | 0.063 | 0.045 |
| | Panel D: Unconditional Probabilistic Pricing (Short-run) | | | | | |
| | OLS Beta | | | Std. Dev of Beta | | |
| Small | 0.564 | 0.584 | 0.493 | 0.226 | 0.025 | 0.173 |
| Medium | 0.632 | 0.597 | 0.627 | 0.053 | 0.070 | 0.021 |
| Large | 0.978 | 0.917 | 0.935 | 0.190 | 0.153 | 0.037 |
| | Panel E: Probabilistic Pricing Conditional on Market Excess Return (Short-run) | | | | | |
| | OLS Beta | | | Std. Dev of Beta | | |
| Small | 0.099 | 0.333 | 0.031 | 0.255 | 0.135 | 0.205 |
| Medium | 0.422 | 0.267 | 0.636 | 0.154 | 0.108 | 0.132 |
| Large | 0.971 | 0.897 | 0.708 | 0.202 | 0.153 | 0.096 |

**Table 15 Alpha decomposition for beta on value-weighted portfolios (out-of-sample)**

The table below reports the alpha decomposition for the portfolio and model specifications of table 14, above. Panel A reports unconditional pricing error, $\alpha_{OLS}^{i}$ for the 9 portfolio sorts. Panel B reports the learning component as a description of pricing error which is reduced under the Kalman filter specifications. Panel C reports the covariance component. Results in panel are sample covariances which have been multiplied by 100 for interpretation, as the sample covariances between predicted beta, and observed market returns were exceptionally small across all specifications.

**Panel A: Unconditional Probabilistic Pricing (Long-run)**

| | B/M | | |
|---|---|---|---|
| **Size** | Low | Medium | High |
| Panel A: Unconditional Alphas (OLS) | | | |
| Small | 0.889 | 1.529 | 1.165 |
| Medium | 0.266 | 0.645 | 0.749 |
| Large | -0.172 | -0.053 | 0.591 |
| Panel B: Learning Component | | | |
| Small | 0.0474 | 0.0432 | 0.0370 |
| Medium | 0.0618 | 0.0113 | 0.0438 |
| Large | -0.0330 | 0.0295 | 0.0189 |
| Panel C: Covariance Component | | | |
| Small | -0.0002 | -0.0001 | -0.0002 |
| Medium | -0.0001 | -0.0001 | -0.0001 |
| Large | -0.0001 | 0.0002 | 0.0000 |

**Panel B: Probabilistic Pricing Conditional on Market Excess Return (Long-run)**

| | B/M | | |
|---|---|---|---|
| **Size** | Low | Medium | High |
| Panel A: Unconditional Alphas (OLS) | | | |
| Small | 0.889 | 1.529 | 1.165 |
| Medium | 0.266 | 0.645 | 0.749 |
| Large | -0.172 | -0.053 | 0.591 |
| Panel B: Learning Component | | | |
| Small | 0.044 | 0.044 | 0.033 |
| Medium | 0.063 | 0.032 | 0.057 |
| Large | -0.035 | 0.040 | 0.020 |
| Panel C: Covariance Component | | | |
| Small | 0.000 | 0.000 | 0.000 |
| Medium | 0.000 | 0.000 | 0.000 |
| Large | 0.000 | 0.000 | 0.000 |

**Panel C: Unconditional Probabilistic Pricing (Short-run)**

| | B/M | | |
|---|---|---|---|
| **Size** | Low | Medium | High |
| Panel A: Unconditional Alphas (OLS) | | | |
| Small | 0.889 | 1.529 | 1.165 |
| Medium | 0.266 | 0.645 | 0.749 |
| Large | -0.172 | -0.053 | 0.591 |
| Panel B: Learning Component | | | |
| Small | -0.060 | 0.035 | 0.004 |
| Medium | 0.034 | -0.002 | 0.022 |
| Large | -0.068 | -0.024 | -0.006 |
| Panel C: Covariance Component | | | |
| Small | 0.0000 | -0.0001 | 0.0012 |
| Medium | 0.0000 | -0.0002 | -0.0002 |
| Large | 0.0002 | 0.0007 | 0.0000 |

**Panel D: Probabilistic Pricing Conditional on Market Excess Return (Short-run)**

| | B/M | | |
|---|---|---|---|
| **Size** | Low | Medium | High |
| Panel A: Unconditional Alphas (OLS) | | | |
| Small | 0.889 | 1.529 | 1.165 |
| Medium | 0.266 | 0.645 | 0.749 |
| Large | -0.172 | -0.053 | 0.591 |
| Panel B: Learning Component | | | |
| Small | -0.380 | -0.137 | -0.314 |
| Medium | -0.111 | -0.229 | 0.028 |
| Large | -0.073 | -0.038 | -0.162 |
| Panel C: Covariance Component | | | |
| Small | 0.0018 | -0.0003 | -0.0005 |
| Medium | 0.0009 | 0.0001 | -0.0005 |
| Large | 0.0000 | 0.0010 | 0.0007 |

**Table 16 Summary statistics for beta on value-weighted portfolios (in-sample)**

Panel A reports beta estimates and their standard deviations for unconditional OLS regression. Panel B reports beta estimates and their standard deviations for the unconditional Kalman filter in the specification of Adrian and Franzoni inclusive of long-run beta estimation. Panel C reports beta estimates and their standard deviations for the Kalman filter conditioned on lagged market excess returns in the specification of Adrian and Franzoni inclusive of long-run beta estimation. Panel D reports beta estimates and their standard deviations for the unconditional Kalman filter in the specification of Hamilton following short-run period to period estimation. Panel E reports beta estimates and their standard deviations for the Kalman filter in the specification of Hamilton following short-run period to period estimation, conditioned on lagged excess market returns. The sample covers the period: End July 2001-End June 2012. Portfolios are value-weighted, and the Kalman filter is trained on the entire data set of June 1996 to June 2012.

| | B/M | | | | | |
|---|---|---|---|---|---|---|
| | Low | Medium | High | Low | Medium | High |
| **Size** | Panel A: Traditional unconditional OLS CAPM | | | | | |
| | OLS Beta | | | Std. Dev of Beta | | |
| Small | 0.651 | 0.533 | 0.488 | 0.083 | 0.122 | 0.105 |
| Medium | 0.583 | 0.600 | 0.595 | 0.120 | 0.099 | 0.097 |
| Large | 1.077 | 0.952 | 0.944 | 0.054 | 0.075 | 0.091 |
| | Panel B: Unconditional Probabilistic Pricing (Long-run) | | | | | |
| | OLS Beta | | | Std. Dev of Beta | | |
| Small | 0.752 | 0.595 | 0.544 | 0.034 | 0.026 | 0.135 |
| Medium | 0.644 | 0.638 | 0.660 | 0.090 | 0.083 | 0.031 |
| Large | 1.029 | 0.976 | 0.975 | 0.044 | 0.015 | 0.080 |
| | Panel C: Probabilistic Pricing Conditional on Market Excess Return (Long-run) | | | | | |
| | OLS Beta | | | Std. Dev of Beta | | |
| Small | 0.723 | 0.600 | 0.539 | 0.143 | 0.027 | 0.129 |
| Medium | 0.656 | 0.647 | 0.663 | 0.072 | 0.063 | 0.059 |
| Large | 1.026 | 0.998 | 0.985 | 0.055 | 0.043 | 0.090 |
| | Panel D: Unconditional Probabilistic Pricing (Short-run) | | | | | |
| | OLS Beta | | | Std. Dev of Beta | | |
| Small | 0.710 | 0.587 | 0.511 | 0.056 | 0.025 | 0.160 |
| Medium | 0.538 | 0.589 | 0.626 | 0.090 | 0.149 | 0.022 |
| Large | 1.026 | 0.948 | 0.982 | 0.138 | 0.069 | 0.174 |
| | Panel E: Probabilistic Pricing Conditional on Market Excess Return (Short-run) | | | | | |
| | OLS Beta | | | Std. Dev of Beta | | |
| Small | 0.604 | 0.551 | 0.424 | 0.080 | 0.056 | 0.137 |
| Medium | 0.533 | 0.537 | 0.649 | 0.085 | 0.116 | 0.048 |
| Large | 1.021 | 0.909 | 0.987 | 0.139 | 0.032 | 0.104 |

**Table 17 Alpha decomposition for beta on value-weighted portfolios (in-sample)**

The table below reports the alpha decomposition for the portfolio and model specifications of table 16, above. Panel A reports unconditional pricing error, $\alpha_{OLS}^{i}$ for the 9 portfolio sorts. Panel B reports the learning component as a description of pricing error which is reduced under the Kalman filter specifications. Panel C reports the covariance component. Results in panel are sample covariances which have been multiplied by 100 for interpretation, as the sample covariances between predicted beta, and observed market returns were exceptionally small across all specifications.

**Panel A: Unconditional Probabilistic Pricing (Long-run)**

| | B/M | | |
|---|---|---|---|
| **Size** | Low | Medium | High |
| Panel A: Unconditional Alphas (OLS) | | | |
| Small | 0.889 | 1.529 | 1.165 |
| Medium | 0.266 | 0.645 | 0.749 |
| Large | -0.172 | -0.053 | 0.591 |
| Panel B: Learning Component | | | |
| Small | 0.0698 | 0.0429 | 0.0385 |
| Medium | 0.0422 | 0.0264 | 0.0444 |
| Large | -0.0330 | 0.0164 | 0.0215 |
| Panel C: Covariance Component | | | |
| Small | 0.0001 | -0.0001 | -0.0003 |
| Medium | -0.0006 | 0.0002 | -0.0001 |
| Large | -0.0001 | 0.0001 | 0.0000 |

**Panel B: Probabilistic Pricing Conditional on Market Excess Return (Long-run)**

| | B/M | | |
|---|---|---|---|
| **Size** | Low | Medium | High |
| Panel A: Unconditional Alphas (OLS) | | | |
| Small | 0.889 | 1.529 | 1.165 |
| Medium | 0.266 | 0.645 | 0.749 |
| Large | -0.172 | -0.053 | 0.591 |
| Panel B: Learning Component | | | |
| Small | 0.049 | 0.046 | 0.035 |
| Medium | 0.050 | 0.032 | 0.047 |
| Large | -0.035 | 0.031 | 0.028 |
| Panel C: Covariance Component | | | |
| Small | 0.000 | 0.000 | 0.000 |
| Medium | 0.000 | 0.000 | 0.000 |
| Large | 0.000 | 0.000 | 0.000 |

**Panel C: Unconditional Probabilistic Pricing (Short-run)**

| | B/M | | |
|---|---|---|---|
| **Size** | Low | Medium | High |
| Panel A: Unconditional Alphas (OLS) | | | |
| Small | 0.889 | 1.529 | 1.165 |
| Medium | 0.266 | 0.645 | 0.749 |
| Large | -0.172 | -0.053 | 0.591 |
| Panel B: Learning Component | | | |
| Small | 0.041 | 0.037 | 0.016 |
| Medium | -0.031 | -0.007 | 0.022 |
| Large | -0.035 | -0.003 | 0.026 |
| Panel C: Covariance Component | | | |
| Small | 0.0002 | -0.0001 | 0.0010 |
| Medium | -0.0001 | 0.0002 | -0.0002 |
| Large | -0.0002 | 0.0005 | 0.0004 |

**Panel D: Probabilistic Pricing Conditional on Market Excess Return (Short-run)**

| | B/M | | |
|---|---|---|---|
| **Size** | Low | Medium | High |
| Panel A: Unconditional Alphas (OLS) | | | |
| Small | 0.889 | 1.529 | 1.165 |
| Medium | 0.266 | 0.645 | 0.749 |
| Large | -0.172 | -0.053 | 0.591 |
| Panel B: Learning Component | | | |
| Small | -0.032 | 0.013 | -0.044 |
| Medium | -0.034 | -0.043 | 0.037 |
| Large | -0.038 | -0.030 | 0.030 |
| Panel C: Covariance Component | | | |
| Small | 0.0000 | -0.0002 | 0.0012 |
| Medium | 0.0002 | 0.0000 | -0.0003 |
| Large | -0.0002 | 0.0001 | 0.0003 |

## 7. Discussion

The results presented above prove to show general support for the proposed state-space representation of regression. Under a probability beliefs risk interpretation which is allowed to vary at a monthly frequency through an algorithm which updates through a process of minimising variance using maximum likelihood techniques, the model simulates the stepwise learning process of an investor who displays myopic loss aversion and predicts probability beliefs regarding the expected risk attributable to equity movement, all the while adhering to a stepwise learning process. The strength of the model is felt to lie in that the autoregressive and updating Kalman gain feature of it rests with the prediction and evolution of a probability, and not with returns prediction directly, unlike GARCH type models which provide for volatility as presented directly through heteroskedasticity in return series themselves.

Under the Adrian and Franzoni (2009) specification of risk prediction which predicts probability beliefs towards a long-run unobservable market exposure, the addition of a lagged market return variable is found to consistently reduce prediction errors, when compared to the strictly unconditional model, regardless of the sampling of diffuse priors, or the manner in which stocks were sorted. In direct contrast, the Hamilton (1994) inspired "short run only" Kalman filter specification, attracted a greater estimation error across all returns specifications where the lagged market variable was included to the model specification, regardless of the sampling of diffuse priors or the manner in which the stocks were sorted. Comparing the two models, the Long-run exposure model of Adrian and Franzoni (2009) and the Hamilton (1994) inspired short term risk model, the Adrian and Franzoni (2009) model provided estimation error reduction which is on average better when comparing the best of each specification, with overall estimation error being reduced by as much as 41% from OLS estimation. An important interpretation to these results is found in the variability of reduction of estimated pricing errors across the applied models. This non-uniformity (by absolute value) seems to suggest that volatility is changing over time, which speaks to dynamic variability of feedback in the model. From these results it is felt that investors are not only likely to base their investment decisions on a single probability type risk interpretation which varies linearly across time, but they appear to do so in a manner which suggests that they are aware of a long run risk factor which although is generally unobservable, the aspiration of which informs local level probability belief estimation.

Interestingly, when considering both general model forms, no matter which sampling period was used to estimate the diffuse priors, or whether a lagged market exposure conditioning variable was included, equally-weighted portfolio specification resulted in greater reduction in estimation errors, and thus greater improvement over the OLS CAPM. The CPE bears out this distinction clearly, highlighting a potentially unique limitation to value sorted portfolios in the South African setting. The JSE displays a relatively small liquid tradable stock universe ranging from a maximum of 408 shares to a minimum of 213 shares available in a given year after the trade; price and liquidity filters are applied. This leads to an average stock portfolio selection of between 22 and 34 stocks per portfolio after the characteristic sort based on size and book-to-market ratio are applied. This degree of limited stock selection, whilst line with evidence surrounding investor behaviour, (Firer, 1988; Fu, 2009), leads to the potential of excessively high idiosyncratic volatility independent of those associated with the selection criteria where portfolios are dominated by relatively large cap stocks, thereby undermining the very purpose of estimating returns based on portfolios as opposed to individual listings. Over and above this, overall market variance for the value portfolio sort is well in excess of that seen on the equally-weighted sort, which may well stand to explain reduced model performance where value-weighted portfolios are concerned. The CPE statistic, whilst designed to attribute less weight to alphas of portfolios with higher volatility, still exposes the value sorted portfolios for explaining significantly higher portfolio return variance. This is expected to be attributable to the top heavy nature of the JSE in which more than 60% of total market cap lies with the 10 largest stocks (Kruger & van Rensburg, 2008). The effects of a top heavy distribution of share value by market capitalisation are seen to have greater effect to value-weighted returns, which may prove to explain why the Kalman filter performance is significantly more pronounced for equally-weighted portfolios. Tables 18 in Appendix A compares the volatility across equally and value-weighted portfolios for the entire sample, as well as the purely out-of-sample periods. Whilst the Kalman filter specification of probability beliefs estimation is designed to, and clearly does, improve equity return estimation, the excess volatility of value sorted portfolios in the South African market place is seen to disrupt the ability of the model to efficiently learn about risk exposure variability. The question of whether investors are evident and sensitive to the market capitalisation of equity listings is brought somewhat into question here, and raises an interesting direction for future behavioural analysis.

Speaking to the issue of specification of diffuse priors, it is evident that the Kalman filter in the setting tested is fairly insensitive to whether the priors are estimated off of 5 years of out-of-sample data, or the full 16 year period. This said, in running the tests, the increased data set of 16 years generally lead to a faster and more stable specification, with the specification of diffuse priors for the out-of-sample test period taking more estimation attempts on average before model convergence was achieved. To this, a notable feature of the results show that once a converging model was achieved, the estimation strength is clearly in the model specification. The Kalman filter design itself does not appear to be significantly biased based on the period over which it was specified being in or out of sample.

Overall, it is felt that the model could do well to be estimated on a longer time-series from which a market place description can be established in so far as the predicted estimates for the market and asset error terms, $(\sigma_{\eta}^{i})^2$ and $(\sigma_{\mu}^{i})^2$ , the autoregressive parameter, $F^i$, and the conditioning variables, $\emptyset^i$are concerned. It is felt that a longer period is also theoretically justified, as Adrian and Franzoni (2009) suggest that market participants start learning about the market over an extended period before they make any inferences regarding the relative risk of an asset.

The results of the alpha decomposition serve to highlight the importance of endogenous learning in an asset pricing framework. Looking at the covariance analysis, whilst there is significant variance in beta estimates across time, the exceptionally small covariance between these risk coefficients and realised market returns provides no evidence that a CAPM specification can hold to explain unconditional mispricing as a feature of time-variance. Instead, it appears that beta estimates vary independently of excess market returns over time, suggesting that market participants base their expectations of risk on probability beliefs which are not directly related or comparable to that which is predicted under a traditional CAPM framework. That is to say that market participants do not, from the evidence presented here, appear to base their expectations of risk exposure on systematic risk as proxied through market excess return. This serves to underscore the results pertaining to reduced estimated pricing errors in the Kalman filter probability beliefs models presented in this study, in that the reduction in pricing error is independent of covariance of risk and market excess returns over time. Whilst the amount of improvement in the asset pricing when considering the alpha decomposition is modest, it is generally significantly more efficient at capturing pricing errors than covariance, suggesting a better model specification for pricing risk than a traditional unconditional CAPM. These results are in line with those of Adrian and Franzoni (2009) who find that on the NYSE the learning component forms the only significant feature of alpha decomposition to explaining unconditional OLS alpha.

The Kalman filter, whilst sensitive to autoregressive regression of unobservable variables, is unlike an ARCH or GARCH regression model (both of which are typically used to model time-varying data under linear regression), as it does not apply the autoregressive principle to asset prediction directly. Rather, only the probability belief of risk exposure is specified to evolve in an autoregressive manner in order to mimic the stepwise nature of causal inference described by Hume (1975a, 1975b) to define human learning. This evolution is further evolved from observations in prediction error which introduces the learning component of the model as investors are expected to dynamically update their beliefs as suggested by the AMH as presented in light of an equity market which is specifically non-Darwinian in its game theory evolution. For these reasons, the Kalman filter is expected to appropriately model dynamic probability beliefs, subject to behavioural consideration in a market place which can be described as being a self organised dynamic state.

The complete success of all model specifications to improve upon the CAPM presents a description of behaviour which is in contradiction to homoskedastic myopic risk-aversion of traditional pricing theory as risk exposure is seen to be clearly better specified as a function of dynamic risk pricing which is based on an updating opinion of market risk through the learning of the Kalman filter. Based on the superior model performance of the Adrian and Franzoni (2009) inspired model which is inclusive of an unobservable long-run risk exposure term to price out estimation errors of an OLS CAPM, it is submitted that investors tend to price towards an unobservable long-run risk, suggesting that investors are pricing exposure towards long-run overall wealth, which is in line with the predicted utility theory of myopic risk neutrality as presented in Chapter 1 of the Literature Review. In addition to this, the long-run Kalman filter specification consistently performed better where the evolution of probability beliefs were seen to be subject to lagged market returns, suggesting that exposure as compared with that of the market improved risk assessment, possibly speaking to enhance considerations of game theory. In contrast, where the Hamilton (1994) inspired short-run exposure model is concerned, the model performed consistently better where no conditioning term was included, suggesting that where short run probability beliefs are considered in isolation of explicit long-run concerns, the effect of lagged market returns, and possibly by implication, game theory considerations are second to the period by period learning process.

## 7.1 Review of the research hypotheses

Returning to the research hypotheses in light of the results and discussion above, the hypotheses are recapped below, and the interpretation is presented below.

> **Primary null hypothesis:** In light of the theoretical and statistical treatments, a dynamic systematic risk pricing model is unable to subsume the effects of idiosyncratic and behavioural anomalies as defined through size and value on the JSE over the period July 2001 to June 2012.
>
> **Secondary null hypothesis:** Time-series variation in Market dynamics and the progressive learning based prediction thereof does not improve a one factor asset pricing model's ability to price risk.
>
> **Tertiary null hypothesis:** Risk is strictly short-run dynamic. Investors do not predict risk exposure dynamically, based on probability beliefs, towards an overall unobservable long-run measure of risk exposure.

The primary hypothesis is rejected in favour of the null, albeit with necessary qualification. Specifically, this reads to say that in light of the theoretical and statistical treatments, a dynamic systematic risk pricing model is seen to be able to subsume, at least in part, the effects of idiosyncratic and behavioural anomalies as defined through size, and value. Whilst this interpretation is limited by the results to value-sorted portfolios, when the notable improvements to equally-weighted portfolios are observed in light of limitations to value-sorting portfolios on the JSE, the opinion taken is one to reject the null in favour of the alternated in respect of hypothesis one. The improvement to asset pricing suggests that a behaviourally inspired dynamic model is a promising avenue of asset pricing theory from which further improvements are likely to develop.

The second research hypothesis is firmly rejected, as the Kalman filter improves asset pricing as a dynamic learning model across all model specifications. This result is firmly up held when considering the time variant alpha decomposition, which supports the effects of the dynamic learning model of Kalman filtering over a misspecification of time-variance in OLS beta prediction as an explanation for excess returns to the traditional OLS CAPM.

The third and final research hypothesis is rejected based on the comparative results attributable to the model specifications based on Adrian and Franzoni (2009) and Hamilton (1994) respectively. The Adrian and Franzoni (2009) specification, which is inclusive of long-run risk as a factor in probability beliefs formation, is seen have superior reduction in overall returns prediction error when compared with the results attributable to the strictly short-run dynamic model specification based on Hamilton (1994). This speaks directly to the significance of time varying volatility and the relevance this has on asset pricing. An interesting addition to this interpretation is the relevance of a market returns conditioning variable. For reasons already discussed, the relevance of a market variable appears to be tied to long-run risk prediction, finding that it is more important to investors who are assumed to be aware of a long-run risk factor than those who are not cognisant of long-run risk.

## 8. Conclusion

To imagine that the equity market is not a form of complex game theory, it is requisite that we reduce the definition of the participants to that of traditionally defined utility maximising risk-averse uniform automata. This sort of definition is built on the shoulders of economic giants, yet equally so almost every part of it has been brought into question, if not completely dismissed by the triumph of evolving research. We know from authors concerned with the human mind and its economic interpretation that rationality is at the very least bounded, if not in some respects an altogether intangible notion. We too understand that the concept of risk-aversion is a theory on less than steady footing as much research defines investors as loss-averse, or a mixture of the two, dependant on circumstance. In fact, going as far back as Harry Markowitz's own lecturer at Chicago University, Leonard J. Savage describes risk as being defined by two broad formats, the first as an exogenous intangible variance of fundamentals, and the second, endogenous, probabilistic in nature and defined by the market participants themselves (Markowitz, 1991). Read this in the light of philosopher David Hume's work on understanding causation more than 250 years on (Hume, 1975a, 1975b), and the setting of how market participants define casual inference and risk presents the rules for a case of complex economic game theory. Evidence supports the fact that in the immediate, the time frame of practical participation, the market is not likely to be defined by a traditional definition of efficiency. The endogenously driven portion of the market presents a semi-independent level of asset market governed by behaviourally biased functions of demand and supply which is informed and influenced, but not fully dictated to, by underlying fundamental variation of the businesses' these equity tradables are supposed to represent.

This study disseminates the concepts of correlation and causation for its overlooked capacity and complexity in much of the prevalent finance literature. It considers risk for its two functions based on the concept of fundamental variation as a root cause, as well as its function as a by-product of cognitive bias in memory and its role in defining reasoned causal inference. Added to this consideration of endogenous probability risk is the consideration of its relevance in light of modern work on utility functions for their independence of risk functions. Finally a model is considered which aims to investigate the extent to which asset pricing is significantly influenced through exposure beyond that of macro-economic multifactor model analysis, and to this known firm specific relevant risk factors are considered. These extended risk considerations are considered first in individual cognitive isolation, and then in a group environment in order to appreciate asset pricing for its exposure to a form of secondary asset market dynamics driven by what would traditionally be referred to as market inefficiencies, but is well defined through an adaptive form of game theory.

From the theory and results presented, it is felt very strongly that the use of state-space mathematics to separate observation from observer based evolution, where risk and the interpretation thereof are concerned, is a robust and progressive substitute for describing markets which have outcomes inseparably influenced by human nature and its associated heuristics and diversities. The results presented speak for themselves in this respect, finding that, particularly where equally-weighted portfolio sorting is concerned, the learning augmented approach to asset pricing displays a reduction of estimated pricing errors to the traditional OLS predicted CAPM which appears to be indicative of volatility changing over time, whilst almost completely eliminating the observation of statistical significance across regression estimated pricing errors to portfolios sorted by size and book-to-market value. The finding of statistically significant reductions in portfolio specific estimated pricing errors speaks to a semblance of time varying informational efficiency which the behavioural model presented in this study predicts to be an appropriate description of expectant investor behaviour. These findings add support to the opinion of this conclusion, which submits that a behavioural model which prices risk as an evolutionary probability belief subject to behavioural bias in individual cognisance presented in isolation as well as the interactive effects to behaviour when decisions are made out of groups, presents as an appropriate model for pricing the perception of risk in equity markets.

Limitations to the study and results inferred thereof include the definition of a few methodological choices. These include the simplicity of the filters used to proxy for the effect of transaction costs which may not have been sufficiently robust. The consideration of liquidity and cost filters could be improved through an extensive study into security specific trading costs, as well as the appropriate liquidity level as a time variant constraint considering the variability of the JSE stock listings across the test period. The question of linearity of the model is a natural progression to the empirical question and there is space to expand the investigation to a non-linear space, particularly where the state equation prediction of the risk variable is concerned. Extending on this development of the learning process, the learning ability of a Kalman filter may not sufficiently capture the effects of a behavioural interpretation of causal inference through the maximum likelihood minimisation of variance. Perhaps an extension into a bounded variance minimisation could proxy for variant ability of traders on the market as well as other behavioural considerations which may limit the learning curve's efficiency. In this way any potential limiting effects of behavioural bias, which may undermine the effects of the Kalman filter's ability to effectively proxy for learning, can be potentially addressed.

Building on the line of reasoning presented in this study whilst being mindful of the potential limitations explored above, extensions to the research presented could do well to investigate further time-series linear progression in the learning component and evolution of risk as a form of probability beliefs. This investigation extension is suggested to include exploration of higher and (or) mixed autoregressive specifications, as well as a non-linear evolution of the state equation. This is suggested to cover non-linearity to the learning curve, whilst maintaining a parametric description of market observables. In essence, this is to suggest that further research should focus on *observer* based explorations rather than *observation* based models, encompassing the field of cybernetics for progressive learning algorithms to best fit behavioural anomalies. An additional avenue considered for future research into this topic on JSE data speaks to a broader consideration of liquidity as a limitation. The results of this study could do well to be considered alongside a smaller sample of the most liquid shares on the JSE, in order to consider the effects of dynamic behavioural considerations to asset pricing. Whilst this may bring into question limitations of small sample size, it will serve to provide perspective to a study routed in dynamic behavioural adjustment and is thus suggested as a valuable consideration for further research on this topic.

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## Appendix A: Portfolio statistics

### Table 17: Firm attrition

For each year of the sample selection, the total number of forms listed is presented as Firms Listed. Each subsequent row presented in the table below shows the attrition of firms due to each filter applied by displaying the reduced number of firms listed which meets each requirement. The Final row, labelled "Listings Excluded" calculates the percentage of firms excluded from the market portfolio listings for failing to meet the requirements of the 3 filters applied in respect of length of time listed, liquidity requirements, and minimum price requirement respectively.

| **Date** | **'96** | **'97** | **'98** | **'99** | **'00** | **'01** | **'02** | **'03** | **'04** | **'05** | **'06** | **'07** | **'08** | **'09** | **'10** | **'11** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Firms Listed | 608 | 612 | 637 | 682 | 668 | 591 | 520 | 449 | 419 | 388 | 385 | 385 | 423 | 415 | 379 | 391 |
| Listing Filter | 529 | 532 | 512 | 488 | 512 | 545 | 500 | 431 | 408 | 369 | 349 | 323 | 330 | 345 | 377 | 379 |
| Liquidity Filter | 485 | 511 | 488 | 450 | 463 | 471 | 420 | 356 | 339 | 310 | 301 | 294 | 293 | 307 | 337 | 336 |
| Price Filter | 432 | 448 | 420 | 355 | 325 | 281 | 261 | 238 | 234 | 228 | 233 | 243 | 245 | 222 | 240 | 249 |
| Listings Excluded | **29%** | **27%** | **34%** | **48%** | **51%** | **52%** | **50%** | **47%** | **44%** | **41%** | **39%** | **37%** | **42%** | **47%** | **37%** | **36%** |

**Table 18: Variance-covariance decomposition**

A comparison of variance and covariance is presented between portfolios formed by equal weighting share returns, and those formed by value weighting share returns. Panel A presents the same comparison over the 6 year out of sample period: July 1996 to June 2001, whereas Panel B displays these comparative statistics calculated over the entire 16 year sample period.

**Panel A: June 1996-July 2001**

| | Equally-weighted | | | Value-weighted | | |
|---|---|---|---|---|---|---|
| | **Market Return Variance** | **Portfolio Return Variance** | **Covariance** | **Market Return Variance** | **Portfolio Return Variance** | **Covariance** |
| **High-Large** | 0.00307 | 0.00600 | 0.00315 | 0.00486 | 0.00651 | 0.00432 |
| **High-Medium** | 0.00307 | 0.00342 | 0.00270 | 0.00486 | 0.00361 | 0.00319 |
| **High-Small** | 0.00307 | 0.00223 | 0.00207 | 0.00486 | 0.00475 | 0.00307 |
| **Low-Large** | 0.00307 | 0.00600 | 0.00381 | 0.00486 | 0.00625 | 0.00514 |
| **Low-Medium** | 0.00307 | 0.00521 | 0.00360 | 0.00486 | 0.00514 | 0.00407 |
| **Low-Small** | 0.00307 | 0.00465 | 0.00298 | 0.00486 | 0.00589 | 0.00340 |
| **Medium-Large** | 0.00307 | 0.00430 | 0.00330 | 0.00486 | 0.00514 | 0.00470 |
| **Medium-Medium** | 0.00307 | 0.00417 | 0.00307 | 0.00486 | 0.00448 | 0.00349 |
| **Medium-Small** | 0.00307 | 0.00426 | 0.00264 | 0.00486 | 0.00643 | 0.00308 |
| **Average** | 0.00307 | 0.00447 | 0.00304 | 0.00486 | 0.00535 | 0.00383 |

**Panel B: June 1996-June2012**

| | Equally-weighted | | | Value-weighted | | |
|---|---|---|---|---|---|---|
| | **Market Return Variance** | **Portfolio Return Variance** | **Covariance** | **Market Return Variance** | **Portfolio Return Variance** | **Covariance** |
| **High-Large** | 0.00203 | 0.00375 | 0.00222 | 0.00339 | 0.00463 | 0.00319 |
| **High-Medium** | 0.00203 | 0.00441 | 0.00217 | 0.00339 | 0.00238 | 0.00209 |
| **High-Small** | 0.00203 | 0.00175 | 0.00152 | 0.00339 | 0.00293 | 0.00187 |
| **Low-Large** | 0.00203 | 0.00346 | 0.00232 | 0.00339 | 0.00434 | 0.00364 |
| **Low-Medium** | 0.00203 | 0.00320 | 0.00221 | 0.00339 | 0.00288 | 0.00222 |
| **Low-Small** | 0.00203 | 0.00354 | 0.00191 | 0.00339 | 0.00618 | 0.00229 |
| **Medium-Large** | 0.00203 | 0.00285 | 0.00215 | 0.00339 | 0.00347 | 0.00322 |
| **Medium-Medium** | 0.00203 | 0.00260 | 0.00203 | 0.00339 | 0.00271 | 0.00213 |
| **Medium-Small** | 0.00203 | 0.00266 | 0.00176 | 0.00339 | 0.00376 | 0.00189 |
| **Average** | 0.00203 | 0.00314 | 0.00203 | 0.00339 | 0.00370 | 0.00250 |

## Appendix B: Alpha decomposition

In this appendix, the decomposition of alpha into a learning component designed to assess the performance of the Kalman filter in its ability to reduce mispricing of risk from unconditional OLS CAPM, as described by $\alpha^i_{OLS}$, due to changes in the descriptive specification of beta, as well as the dynamic learning ability thereof. The second function is to assess whether the principals of the CAPM, as a description of risk through exposure to systematic risk variation, hold in light of a time variant betas or whether the time variance of beta is unrelated to the market, effectively undermining the traditional assumption of risk specification through beta of the CAPM. This specification of alpha decomposition is adapted from Adrian and Franzoni (2009) as an interpretation of Lewellen and Nagel (2006).

$$
\begin{aligned}
\alpha^i_{OLS} &= E\left[R^i_{t+1}\right] - \beta^i_{OLS}[R^M_{t+1}] \\
&= E\left[\beta^i_{t+1}\right][R^M_{t+1}] - \beta^i_{OLS}[R^M_{t+1}] \\
&= E\left[\beta^i_{t+1}\right][R^M_{t+1}] + Cov\left[\beta^i_{t+1,}R^M_{t+1}\right] - \beta^i_{OLS}[R^M_{t+1}] \\
&= E\left[\beta^i_{t+1} - \beta^i_{OLS}\ \right][R^M_{t+1}] + Cov\left[\beta^i_{t+1,}R^M_{t+1}\right]
\end{aligned}
$$