Multi-Trace Operators and the Gauge-Gravity Correspondence

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Declaration

I hereby declare that all work contained in this dissertation is my own unless stated otherwise. It is being submitted for the Degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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Date

Abstract

In this dissertation we explore restricted Schur polynomials as a means of changing basis from the fields from $\mathcal{N} = 4$ super Yang-Mills theory to the states of type IIB string theory. This is of significant import to the AdS/CFT correspondence. We explore the correlators of the restricted Schurs as well as their $N \to \infty$ and finite N counting. We also illuminate the relationship between restricted Schurs and the operators created by Brown, Heslop and Ramgoolam.

Acknowledgments

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Contents

List of Figures

List of Tables

1 Introduction

In this dissertation we will be studying the correlators of restricted Schur polynomials in $\mathcal{N} = 4$ super Yang-Mills theory. The AdS/CFT correspondence (first proposed by Maldacena in 1997 in [1], see also [2, 3]) proposes an equivalence between $\mathcal{N} = 4$ SYM and Type IIB string theory on the $\mathrm{AdS}_5 \times \mathrm{S}^5$ spacetime. If true it would enable us to use $\mathcal{N} = 4$ SYM to study quantum gravity on the $\mathrm{AdS}_5 \times \mathrm{S}^5$ spacetime. Specifically the $\mathcal{N} = 4$ SYM theory is a 3+1 dimensional theory (since it resides on the $\mathcal{M}_4$ manifold i.e. Minkowski spacetime.) The type IIB string theory is a 9+1 dimensional theory (since it resides on the $\mathrm{AdS}_5 \times \mathrm{S}^5$ manifold.) The boundary of $\mathrm{AdS}_5 \times \mathrm{S}^5$ is $\mathcal{M}_4$. We will elaborate upon this conjecture more in section 2.

In section 3 we will review relevant literature [4] on giant gravitons. This will be necessary since Corley, Jevicki and Ramgoolam [5] have proposed Schur polynomials as being dual to giant gravitons in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ spacetime. Balasubramanian [6] et al. first described what a giant graviton was and originally proposed subdeterminant operators which are in fact equivalent to Schur polynomials $\chi_R(Z)$ for R an antisymmetric representation i.e. in terms of Young diagrams this representation would be labelled by a single column. These authors came to this conclusion using the cutoff on angular momentum for giant gravitons and properties of subdeterminant operators. We discuss this more in section 4.

In sections 5 and 6 we will discuss two different operator candidates for the change of basis from the fields of $\mathcal{N} = 4$ SYM to the states of Type IIB string theory. These operators are respectively those put forward by Brown, Heslop and Ramgoolam [7] and those utilising restricted Schur polynomials we studied in [8]. For the BHR operators we will review their two-point functions, counting, derivation etc. Section 6 is completely novel and is based on work we published earlier this year [8]. In section 6 we will elaborate upon this with discussions about the restricted Schur polynomials orthogonality, counting, numerical checks etc.

In section 7 we elucidate various relationships between the aforementioned operators.

A brief aside on nomenclature. Throughout this dissertation we will use $\circ$ to denote the *outer* tensor product [9] of two representations. By $\otimes$ we will denote the *direct* or *inner* tensor product [10, 11] of two representations.

2 The AdS/CFT Correspondence in a Nutshell

The AdS/CFTcorrespondence - conjectured by Maldacena in [1] - states an equivalence between $\mathcal{N} = 4$ super Yang-Mills (SYM) theory on four dimensional Minkowski spacetime ($\mathcal{M}_4$) and type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$. In fact $\mathcal{M}_4$ is the boundary of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ spacetime.

It is known by the *operator/state correspondence* that operators of $\mathcal{N} = 4$ SYM on $\mathbb{R}^4$ are related to states of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$. And since $\mathbb{R} \times S^3$ is the boundary of $\mathrm{AdS}_5 \times \mathrm{S}^5$ we see the emergent link between the operators of $\mathcal{N} = 4$ SYM on the Minkowski spacetime and the states of type IIB string theory on the $\mathrm{AdS}_5 \times \mathrm{S}^5$ spacetime. In this dissertation we will be focused on studying the mappings between the states of type IIB string theory and the operators of $\mathcal{N} = 4$ SYM and seeing if they yield the correct counting, form a complete basis for gauge invariant operators, diagonalise the two point function etc.

We will now briefly discuss the following

- A brief heuristic motivation of the AdS/CFT correspondence.
- How we can deduce the fact that the operators in $\mathcal{N} = 4$ SYM and states in type IIB string theory are related.
- Why we need to diagonalise the two point function $< \mathcal{O}_1 \mathcal{O}_2^\dagger >$.
- And why Schur polynomials are being used as a mapping between operators and states.

2.1 A Naive Motivation for the AdS/CFT Conjecture

It is believed that the following two physical descriptions are equivalent [12].

i. The first is comprised of D3 branes in $\mathcal{M}_{10}$ - 10 dimensional Minkowski spacetime. The dynamics of this theory consists of the following:

(a) Closed strings moving in the $\mathcal{M}_{10}$ spacetime.

(b) Open strings attached to the D3 branes – this includes strings stretched between two D3 branes.

(c) Interactions between the closed and open strings.

ii. The second description consists of a p-brane in an asymptotically $\mathcal{M}_{10}$ spacetime. A p-brane is like a black hole except its not a point, it is in fact p-dimensional and not 0-dimensional. In this description we would have closed strings moving in the deformed geometry around the p-brane. Thus the dynamics of this description consists of the following:

(a) Closed strings moving in the $\mathcal{M}_{10}$ spacetime

(b) Closed strings moving in the deformed geometry around the p-brane.

(c) Interactions between the closed strings in $\mathcal{M}_{10}$ and the closed strings in the deformed geometry due to the p-brane.

What Maldacena [1] did was consider the above two descriptions at low energy. For case i and ii above, at low energy, we have that their dynamics are now respectively transformed into the following.

I. The dynamics of case i above, at low energy:

(a) The dynamics of the closed strings moving in $\mathcal{M}_{10}$ becomes supergravity in $\mathcal{M}_{10}$.

(b) The dynamics of the open strings attached to the D3 branes turns into $\mathcal{N} = 4$ SYM theory in 3+1 dimensions.

(c) And the interactions between the closed and open strings tend to zero.

II. The dynamics of case ii above, at low energy:

(a) Supergravity in the $\mathcal{M}_{10}$ spacetime.

(b) String theory in the near horizon geometry the p-brane – note this includes all modes of the closed strings.

(c) The interaction between the p-brane and the closed strings tends to zero.

Let us elaborate on the last two entries in case II above. Since we are considering a low energy theory why then are all the string modes included in the last point b above? This is because in the near horizon geometry, in a Newtonian gravity language, the large negative potential due to the p-brane compensates for any string excitations and thus leaves us with a massless theory. In other words all energies are red-shifted due to the p-brane geometry. This is the crux of the correspondence since it allows us to include all string modes.

And the last point. The interaction between the p-brane and the closed strings drops to zero because at low energy the wavefunctions of the closed strings have a large wavelength due to de Broglie's relation

$$p = \frac{h}{\lambda}.$$

Thus we expect the closed string to pass by the p-brane without scattering from it i.e. no interaction would take place. This argument can also be used to explain why the interactions between the closed and open strings in case I above drops to zero – the wavelengths of the closed strings become so large that they too do not scatter/interact with the D3 branes. And since the open strings are attached to the D3 branes the closed and open strings don't interact.

Now comparing cases I and II we can draw the same conclusion that Maldacena did – $\mathcal{N} = 4$ SYM in 3+1 dimensions is equivalent to type IIB string theory on the $AdS_5 \times S^5$ geometry.

2.2 Deduction that $\mathcal{N} = 4$ SYM Operators are Related to States in Type IIB String Theory

$\mathcal{N} = 4$ SYM resides on a four dimensional manifold $\mathcal{M}_4$. The metric on this spacetime is given by

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2.$$

If we perform a Wick rotation $t \to it = w$ then we have $\mathcal{N} = 4$ SYM on $\mathbb{R}^4$

$$\begin{aligned} ds^2 &= dw^2 + dx^2 + dy^2 + dz^2 \\ &= dr^2 + r^2 d\Omega_3^2, \end{aligned}$$

where $d\Omega_3^2$ is the metric on an S^3 of unit radius. If we let $r = e^\tau$ then we must have that $dr = e^\tau d\tau$ thus yielding

$$ds^2 = e^{2\tau}\left(d\tau^2 + d\Omega_3^2\right). \tag{2.2.1}$$

But multiplying by $e^{2\tau}$ is just a conformal transformation i.e. it preserves angles and the dot product, so we can simply write

$$ds^2 = d\tau^2 + d\Omega_3^2, \tag{2.2.2}$$

and if we perform a Wick rotation again we have

$$ds^2 = -d\tau^2 + d\Omega_3^2. \tag{2.2.3}$$

But this is just the metric on $\mathbb{R} \times S^3$ which is the boundary of $AdS_5 \times S^5$ in global coordinates. So we are now considering $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$. If we consider a time translation in these new coordinates i.e.

$$\tau \to \tau + a, \tag{2.2.4}$$

then in the original coordinates we have that

$$\begin{aligned} dr = e^{\tau} d\tau \to e^{\tau + a} d\tau &= e^{a}(e^{\tau} d\tau) \\ &= e^{a} dr. \end{aligned}$$

So a time translation in the new coordinates corresponds to a scaling in the old coordinates. We have illustrated this pictorially in figure 1. The image on the left of figure 1 represents the coordinates of the metric $ds^2 = dr^2 + r^2 d\Omega_3^2$ in $\mathbb{R}^4$ while the figure on the right represents the coordinates in the metric $ds^2 = -d\tau^2 + d\Omega_3^2$ in $\mathbb{R} \times S^3$. We can see in figure 1 that when $r = 0$ we are at the origin in the figure on the left which depicts $\mathbb{R}^4$. When $r = 0$ then $\tau = -\infty$. Thus we can see that when we are at a specific point in $\mathcal{N} = 4$ SYM on $\mathbb{R}^4$ we are not at a specific point in $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$. This corresponds, respectively, to an operator which can be defined at a specific point and a state/wavefunction which is spread out over a region.

This is where another interesting result of the operator/state correspondence emerges. It can be shown that the dimension of an operator in $\mathcal{N} = 4$ SYM on $\mathbb{R}^4$ corresponds to energy levels of states in $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$. We can see this by considering the following. Finite time translations are generated by the operator $i\hbar\frac{\partial}{\partial t}$ – which also yields the energy of some state $|\psi\rangle$ when acting on this particular ket. Similarly, finite scalings are generated by the operator $r\frac{\partial}{\partial r}$ which is related to the conformal dimension. Thus if we consider the fact that time translations in $\mathcal{N} = 4$ SYM on $\mathbb{R}^4$ are related to scalings in $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ then we are lead by the above to the connection between conformal dimensions of operators in $\mathcal{N} = 4$ SYM on $\mathbb{R}^4$ and energy levels of states in $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$.

Finally, the boundary of $\mathrm{AdS}_5 \times \mathrm{S}^5$ is $\mathbb{R} \times S^3$ with $\mathbb{R}$ representing the time

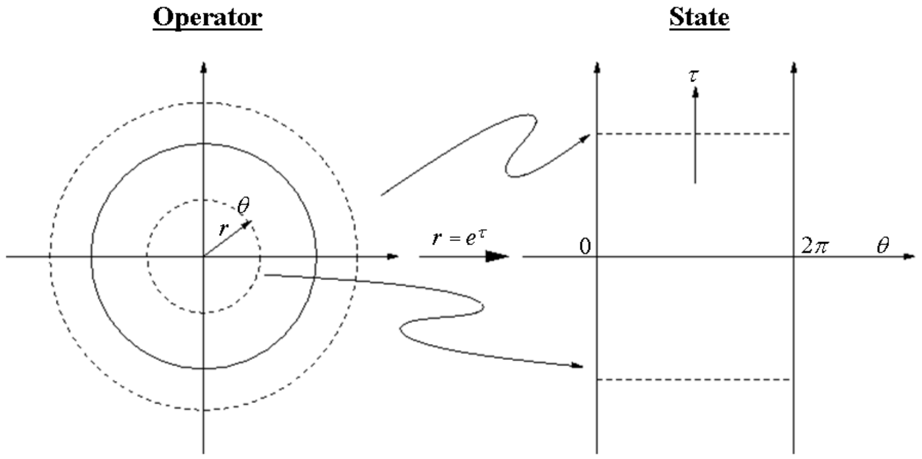


Figure 1: This is an illustration of how the operators of $\mathcal{N} = 4$ SYM on $\mathbb{R}^4$ (left) are mapped to the states of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ (right). On the left the metric is $ds^2 = dr^2 + r^2 d\Omega_3^2$ while on the right it $ds^2 = -d\tau^2 + d\Omega_3^2$. The two are related by $r = e^\tau$. Thus when $r = 0$ we are located at the origin in the figure on the left i.e. at a specific point – like an operator which can be defined at a specific point. However on the right $r = 0$ implies $\tau = -\infty$ and we are no longer located at a specific point but spread out like a wavefunction/state.

coordinate. This allows us to identify the time of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ with the time coordinate of the string theory, and hence allows us to identify their Hilbert spaces. Thus, operators in $\mathcal{N} = 4$ SYM on $\mathcal{M}_4$ are equivalent to states of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ which are in turn equivalent to states of IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$.

2.3 Explanation of the Large N Limit

Here we will shortly discuss what is meant by the large N limit. Yang-Mills theory has a $U(N)$ gauge group. The rank of the gauge group Ncorresponds to the number of D3 branes in our theory. At large N, it is hypothesised that, Yang-Mills theory behaves like string theory. This was put forward by 't Hooft [13].

String theory has two parameters viz. the string length l_s and a coupling constant g_s (or $\hbar_s$). In Yang-Mills there are also two parameters viz. g^2_{YM} (or $\hbar_f$) and the rank of the gauge group N.

String Theory	**Yang-Mills Theory**
l_s (String length)	g_{YM} (The Yang-Mills coupling constant.)
g_s (String coupling constant)	N (The rank of the gauge group $U(N)$.)

Table 2.1: String theory has two parameters as does Yang-Mills, however a connection only becomes apparent at large N. This is explained below.

It is known [14] that the string length, the Yang-Mills coupling constant and the rank of the gauge group are all related by the equation

$$\frac{R^4}{l_s^4} = g^2_{\mathrm{YM}} N. \tag{2.3.1}$$

In the 't Hooft limit we have that

$$g^2_{\mathrm{YM}} N = \lambda = \text{fixed}, \tag{2.3.2}$$

and

$$g_s = g_{\rm YM}^2, \tag{2.3.3}$$

and thus

$$g_{\rm YM}^2 = \frac{\lambda}{N} = O(\frac{1}{N}). \tag{2.3.4}$$

Thus at large N we have

$$\hbar_s = g_s = \frac{\lambda}{N} \to 0. \tag{2.3.5}$$

So we can see that the theory becomes classical at large N. Another way of seeing this is as follows. In the matrix model we combine the 6 Higgs fields ϕ_i where $i = 1, 2, \dots 6$ into three complex matrices as follows.

$$X = \phi_1 + i\phi_2,$$
$$Y = \phi_3 + i\phi_4,$$
$$Z = \phi_5 + i\phi_6.$$

The two point function of mixed fields is zero (e.g. $\langle (X)^i_j (Y^\dagger)^k_l \rangle = 0$) whilst

$$\langle (X)^i_j (X^\dagger)^k_l \rangle = \langle (Y)^i_j (Y^\dagger)^k_l \rangle = \langle (Z)^i_j (Z^\dagger)^k_l \rangle = \delta^i_l \delta^k_j.$$

A brief explanation of these correlators will be given in section 6.3. Thus we have that

$$\langle {\rm Tr}(XX^\dagger) \rangle = \langle X^i_j (X^\dagger)^j_i \rangle = \delta^i_i \delta^j_j = N^2.$$

Similarly we have that

$$\begin{aligned}\langle \mathrm{Tr}(XX^\dagger)\mathrm{Tr}(XX^\dagger)\rangle &= N^4 + N^2 \\ &\approx N^4 \\ &= \langle \mathrm{Tr}(XX^\dagger)\rangle \langle \mathrm{Tr}(XX^\dagger)\rangle .\end{aligned}$$

In the second line we used the fact that $N \to \infty$. Thus as $N \to \infty$ we obtain a classical limit where the expectation value of the product of operators is given by the product of the expectation values of the operators. Thus for some arbitrary n we would have that the expectation value of the product of the operators O_1, O_2, $\cdots$ O_n is equivalent to the product of the expectation values of the operators.

$$\langle O_1 O_2 \cdots O_n\rangle = \langle O_1\rangle \langle O_2\rangle \cdots \langle O_n\rangle . \tag{2.3.6}$$

The expectation value of an operator O is of course given by

$$\langle O\rangle = \sum_i \mu(i) O(i), \tag{2.3.7}$$

where $\mu(i)$ is the probability of O being in i^{th} state and $O(i)$ is the value of O in the i^{th} state. So equation (2.3.6) can be rewritten as follows

$$\sum_i \mu(i) O_1(i) O_2(i) \cdots O_n(i) = \sum_i \mu(i) O_1(i) \sum_j \mu(j) O_2(j) \cdots \sum_k \mu(k) O_n(k).$$

The only way this is possible is if

$$\mu(i) = \begin{cases} 1 & \text{if } i = i^* \\ 0 & \text{if } i \neq i^*, \end{cases}$$

where i^* denotes a particular state. So $\mu(i)$ is zero for all but one state. Thus we have a classical limit since only a single state can be occupied at a time and not a *mixture* of states as in a quantum theory.

2.4 Why Diagonalise the Two Point Function?

Since we have that operators in quantum field theory are related to states in type IIB string theory we can infer that correlators such as $< \mathcal{O}_1 \mathcal{O}_2^\dagger >$ in QFT are related to an overlap of states in type IIB string theory. So for instance the correlator $< \mathcal{O}_1 \mathcal{O}_2^\dagger >$ could be related to the overlap $\langle \mathcal{O}_1 | \mathcal{O}_2 \rangle$ where $|\mathcal{O}_1\rangle$ and $|\mathcal{O}_2\rangle$ are states in string theory.

States of dissimilar particles are of course orthogonal. For instance the states $|\text{photon}\rangle$ and $|\text{electron}\rangle$ must be orthogonal

$$\langle \text{photon} | \text{electron} \rangle = 0$$
$$\langle \text{photon} | \text{photon} \rangle = 1$$
$$\langle \text{electron} | \text{electron} \rangle = 1,$$

since a particle cannot be an electron and a photon simultaneously. Thus since the states are orthogonal the overlap of states and hence the correlator is diagonal. If we can diagonalise the 2 point correlators it is natural to think that the operators correspond to distinct objects.

2.5 Why Schur Polynomials?

First let us provide the definition of a Schur polynomial $\chi_R(Z)$.

$$\chi_R(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \text{Tr}(\sigma Z^{\otimes n}), \tag{2.5.1}$$

where S_n is the permutation group of $n!$ elements, $\chi_R(\sigma)$ is the character of the S_n element σ in the representation R (which is labelled by Young diagrams - something we will arrive at shortly) and $\text{Tr}(\sigma Z^{\otimes n})$ denotes the trace of the n copies of the complex Z matrices once we have permuted their lower indices. σ permutes the lower indices of $Z^{\otimes n}$ as follows (note that $Z^{\otimes n}$ is shorthand for n copies of the complex matrix Z i.e. $Z^{\otimes n} =$

$\underbrace{Z \otimes Z \otimes \cdots \otimes Z}_{n \text{ copies}})$

$$\sigma Z^{i_1}_{j_1} Z^{i_2}_{j_2} \cdots Z^{i_2}_{j_2} = Z^{i_1}_{j_{\sigma}(1)} Z^{i_2}_{j_{\sigma}(2)} \cdots Z^{i_2}_{j_{\sigma}(2)}. \tag{2.5.2}$$

Schur polynomials can be related to integer partitions. To illustrate this relation let us consider n copies of some field Z. How many ways are there of taking traces of these n copies. In general, we would have something like this

$$\mathrm{Tr}\,(Z^{p_1})^{q_1}\,\mathrm{Tr}\,(Z^{p_2})^{q_2}\cdots\mathrm{Tr}\,(Z^{p_k})^{q_k}\,. \tag{2.5.3}$$

So for example for 4 copies of Z we could have

$$\mathrm{Tr}\left(Z^2\right)\mathrm{Tr}\,(Z)^2\,. \tag{2.5.4}$$

Here we see that the sum of the exponents (2+1+1) gives us a partition of the number of fields viz. 4. In general the sum of the exponents is going to be a partition of the number of fields n. Thus from (2.5.3) we see that

$$p_1 q_1 + p_2 q_2 + \cdots + p_k q_k = n. \tag{2.5.5}$$

Thus Schur polynomials provide an injective map between these two spaces. But of significant importance is that Schur polynomials are orthogonal to each other. By orthogonal we mean that the correlator of two Schur polynomials, call them $\chi_R(Z)$ and $\chi_S(Z)$, is orthogonal i.e.

$$\langle \chi_R(Z) \chi_S^{\dagger}(Z) \rangle = \delta_{RS}.$$

This makes them perfect for describing orthonormal states. Furthermore Schur polynomials are gauge invariant since they are comprised of multitraces of copies of some matrix Z. Their gauge invariance is necessary if they are to describe something physical. Note that the Schur polynomials give a map from the set of non-singular matrices to the reals i.e.

$$\chi_R : \mathrm{GL}(N, \mathbb{C}) \to \mathbb{R}.$$

The Schur polynomials contain a projector (example given in equation (2.5.9)) which maps non-singular matrices to some irreducible representation R of the symmetric group i.e.

$$P_R : \mathrm{GL}(N, \mathbb{C}) \to R. \tag{2.5.6}$$

Thus the Schur polynomials are given by

$$\chi_R(Z) = \frac{1}{d_R} \mathrm{Tr}(P_R Z^{\otimes n}). \tag{2.5.7}$$

The one property that all projectors must satisfy is

$$P_R P_S \propto \delta_{RS} P_R. \tag{2.5.8}$$

This observation was emphasized and exploited to compute correlators in [15]. Consider the following two projectors

$$P_R = \frac{d_R}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma)\sigma, \tag{2.5.9}$$

and

$$P_S = \frac{d_R}{n!} \sum_{\tau \in S_n} \chi_S(\tau)\tau, \tag{2.5.10}$$

If either of these operators acts on $Z^{\otimes n}$ and we then take the trace we will obtain a Schur polynomial. Let us check to see if these projectors satisfy equation (2.5.8).

$$
\begin{aligned}
P_R P_S &= \left(\frac{d_R}{n!}\right)^2 \sum_\sigma \chi_R(\sigma)\sigma \sum_\tau \chi_S(\tau)\tau \\
&= \left(\frac{d_R}{n!}\right)^2 \sum_\sigma \sum_\tau \chi_R(\sigma)\chi_S(\tau)\sigma\tau \\
&= \left(\frac{d_R}{n!}\right)^2 \sum_\sigma \sum_\eta \chi_R(\sigma)\chi_S(\sigma^{-1}\eta)\eta \\
&= \left(\frac{d_R}{n!}\right)^2 \sum_\eta \left(\sum_\sigma [\Gamma_R(\sigma)]_{ii}[\Gamma_S(\sigma^{-1})]_{jk}[\Gamma_S(\eta)]_{kj}\right)\eta \\
&= \left(\frac{d_R}{n!}\right)^2 \sum_\eta \left(\sum_\sigma \frac{n!}{d_R}\delta_{RS}\delta_{ij}\delta_{ik}[\Gamma_S(\eta)]_{kj}\right)\eta \\
&= \delta_{RS}\frac{d_R}{n!}\sum_\eta \chi_S(\eta)\eta \\
&= \delta_{RS}P_S, \qquad (2.5.11)
\end{aligned}
$$

where we have exploited the orthogonality relation

$$
\sum_\sigma [\Gamma_R(\sigma)]_{ij}[\Gamma_S(\sigma^{-1})]_{kl} = \frac{n!}{d_R}\delta_{RS}\delta_{il}\delta_{jk}, \qquad (2.5.12)
$$

to obtain the desired result. The fact that the correlator inherits the orthogonality of the projector is explained in section 6 (see equation (6.3.5)).

The two point function of two Schur polynomials was found in [5] to be

$$
\langle \chi_R(Z)\chi_S^\dagger(Z)\rangle = \delta_{RS}\frac{\mathrm{Dim}(R)n_R!}{d_R}, \qquad (2.5.13)
$$

where $\mathrm{Dim}(R)$ is the dimension of the representation R if it labels an irreducible representation of the unitary group and d_R is the dimension of the representation R if it labels an irreducible representation of the symmetric group. A derivation of equation (2.5.13) can be found in [5]. Similarly we can also find the multipoint function for several Schur polynomials [16]. This can be achieved if we use the Littewood-Richardson rule to split up the product of two Schur polynomials into a sum as follows

$$\chi_R \chi_S = \sum_T f_{RST} \chi_T,$$

where f_{RST} denotes the Littlewood-Richardson coefficient. Thus to work out the multipoint function one would just split up any product of Schur polynomials using the above sum and find the corresponding two point functions. Let us consider the following three point function as an example.

$$\begin{aligned}
\langle \chi_R(Z)\chi_S(Z)\chi_T^\dagger(Z)\rangle &= \langle \sum_K f_{RSK}\chi_K(Z)\chi_T^\dagger(Z)\rangle \\
&= \sum_K f_{RSK}\,\langle \chi_K(Z)\chi_T^\dagger(Z)\rangle \\
&= \sum_K f_{RSK}\delta_{KT}\frac{\mathrm{Dim}(T)n_T!}{d_T} \\
&= f_{RST}\frac{\mathrm{Dim}(T)n_T!}{d_T}.
\end{aligned}$$

One can extend this method to determine any other multi-point functions. Maldacena's conjecture relates IIB string theory on $\mathrm{AdS}_5\times\mathrm{S}^5$ to $\mathcal{N}=4$ SYM with gauge group $\mathrm{SU}(N)$. The correlators we have described in this section have all been computed for gauge group $\mathrm{U}(N)$. For an extension of the $\mathrm{U}(N)$ results to $\mathrm{SU}(N)$ see [15], [17] and [18].

3 The Stringy Exclusion Principle

One manifestation of the stringy exclusion principle [19, 20, 21] describes how the number of states accessible to a giant graviton residing in the spherical component of an AdS×S space is bounded above. In [4] Susskind, McGreevy and Toumbas offer an interpretation of this. They show that as the graviton gains momentum in the spherical component of the AdS×S space it expands, until it's as large as the sphere itself. Once this happens it can no longer grow; thus establishing a limit on the number of accessible states.

In the following sections we will proceed as follows to illustrate the above

- First we will discuss a useful toy model which can be used to help explain the cutoff on the states of a giant graviton. This toy model is that of an electric dipole moving on a sphere, with a uniform magnetic flux radiating radially from the center of the sphere. We will show that an increase in momentum of the dipole results in it expanding and that this implies that there is a maximum momentum the dipole can attain.
- Next we will derive the Lagrangian for a D3 brane in the S^5 component of the $AdS_5{\times}S^5$ space and show that its increasing momentum also results in an expansion of the brane in accord with the point above.
- Finally we will show that the maximum value for the momentum is N - the total flux permeating S^5.

3.1 Electric Dipole in a Magnetic Field

An electic dipole moving on a plane, with a uniform magnetic field perpendicular to the plane, forms the basis of a non-commutative field theory. In this particular non-commutative field theory measurements along orthogonal axes do not commute. We'll use it, as in [4], to show that there is a maximum number of states accessible to the graviton in the spherical component of the AdS×S space in question. As the dipole rotates, with increasing speed, in the uniform magnetic field it starts to stretch. Thus as its angular momentum increases the relative position of the two charged particles also increases.

The dipole serves as a toy model for the giant graviton. The overall D3 brane charge of a giant graviton is zero, as is the *overall* electric charge of a dipole. All pairs of antipodal points on the giant graviton can be modeled by dipoles of D3 brane charge. As the angular momentum of the giant increases - and thus the angular momentum of the dipoles comprising the brane increases - the brane starts to expand in the field permeating it.

Now we will show that the increase in angular momentum does indeed serve to stretch the dipole. Let us consider a dipole moving on a plane which has a uniform magnetic field perpendicular to it. Let the vectors $\vec{a}$ and $\vec{b}$ denote the respective positions of the two charged particles comprising the dipole. Then the Lagrangian of the dipole would be of the form

$$\mathcal{L} = \frac{1}{2}m\left(|\dot{\vec{a}}|^2 + |\dot{\vec{b}}|^2\right) + e\vec{A}\cdot\left(\dot{\vec{a}} - \dot{\vec{b}}\right) - \frac{1}{2}K\left(|\vec{a}|^2 - |\vec{b}|^2\right), \qquad (3.1.1)$$

where the term $\frac{1}{2}K\left(|\vec{a}|^2 - |\vec{b}|^2\right)$ is the potential energy introduced to describe a spring coupling. Note that the particles have opposite charge - this gives rise to the two particle velocities being subtracted in the second term.

Since the magnetic field $\vec{B}$ is constant perpendicular to the plane we must have

$$\vec{B} = \nabla \times \vec{A} = \text{const.} \qquad (3.1.2)$$

Let us establish the cartesian coordinate system (x_1, x_2, x_3). Let $\hat{x}_1$ and $\hat{x}_2$ be orthonormal vectors on the plane in which the dipole moves and let $\hat{x}_3$ be the vector perpendicular to the plane (and orthonormal to the aforementioned vectors.) Then clearly the magnetic field in question is given by $\vec{B} = (0, 0, B^3) = (0, 0, B)$ where B is the constant magnitude of the magnetic field. Therefore we must have the following

$$\begin{aligned} B^3 &= \epsilon^{3jk}\partial_j A_k \\ &= \epsilon^{312}\partial_1 A_2 + \epsilon^{321}\partial_2 A_1 \\ &= \partial_1 A_2 - \partial_2 A_1 \end{aligned} \tag{3.1.3}$$

To get $B^3 = B$ we could set $A_2 = \frac{x_1}{2}$ and $A_1 = -\frac{x_2}{2}$. If we write the coordinates of the charged particles as follows in our cartesian coordinate system (x_1, x_2, x_3)

$$\begin{aligned} \vec{a} &= (a^1, a^2, 0) \\ \vec{b} &= (b^1, b^2, 0), \end{aligned}$$

then we obtain

$$\begin{aligned} e\dot{\vec{a}} \cdot \vec{A} &= \frac{B}{2}(-\dot{a}^1 a^2 + \dot{a}^2 a^1) \\ -e\dot{\vec{b}} \cdot \vec{A} &= \frac{B}{2}(-\dot{b}^1 b^2 + \dot{b}^2 b^1). \end{aligned}$$

Finally we can rewrite the Lagrangian in (3.1.1) as follows:

$$\mathcal{L} = \frac{1}{2}m\left(|\dot{\vec{a}}|^2 + |\dot{\vec{b}}|^2\right) + \frac{1}{2}B\epsilon_{ij}\left(\dot{a}^i a^j - \dot{b}^i b^j\right) - \frac{1}{2}K\left(|\vec{a}|^2 - |\vec{b}|^2\right).$$

We can assume that the mass ($m \approx 0$) is negligible so that the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}B\epsilon_{ij}\left(\dot{a}^i a^j - \dot{b}^i b^j\right) - \frac{1}{2}K\left(|\vec{a}|^2 - |\vec{b}|^2\right). \tag{3.1.4}$$

Next we rewrite the above Lagrangian in terms of centre of mass and relative position coordinates by defining the relative position and centre of mass vectors as follows

$$\vec{\Delta} = \frac{\vec{a} - \vec{b}}{2}$$
$$\vec{X} = \frac{\vec{a} + \vec{b}}{2},$$

yielding the Lagrangian as follows

$$\mathcal{L} = B\epsilon_{ij}\dot{X}^i\Delta^j - 2K\Delta^2. \tag{3.1.5}$$

The above equation is reminiscent of the Legendre transform

$$\mathcal{L} = \vec{P} \cdot \vec{v} - \mathcal{H}, \tag{3.1.6}$$

where $\mathcal{H}$ is the hamiltonian. Now the Hamiltonian is just the total energy of the system. Since we have assumed that $m \approx 0$ there is no kinetic energy term - just a potential term viz. $2K\Delta^2$. So intuitively we make the identification

$$\vec{P} \cdot \vec{v} = B\epsilon_{ij}\dot{X}^i\Delta^j. \tag{3.1.7}$$

Clearly we mean

$$\dot{P}_i = B\epsilon_{ij}\Delta^j. \tag{3.1.8}$$

We will duplicate the results of [4] where they showed that

$$|\vec{\Delta}| = \frac{|\vec{P}|}{B},$$

i.e. as the momentum increases the distance between the endpoints of the dipole also increases.

However to show that the definition of $\vec{P}$ in equation (3.1.8) is valid we will need to use Dirac's constrained quantisation. To quantise the system we first need to find the momenta Π_X and Π_Δ which are conjugate to X and Δ respectively. We obtain these as follows

$$
\begin{aligned}
(\Pi_X)_i &= \frac{\partial \mathcal{L}}{\partial \dot{X}^i} \\
&= B\epsilon_{ij}\Delta^j,
\end{aligned}
$$

and

$$
\begin{aligned}
(\Pi_\Delta)_i &= \frac{\partial \mathcal{L}}{\partial \dot{\Delta}^i} \\
&= 0.
\end{aligned}
$$

Clearly Π_X can be identified with $\vec{P}$. This is where we need Dirac's constrained quantisation. We should have

$$
[X^i, (\Pi_X)_i] = i \qquad \text{and} \qquad [\Delta^i, (\Pi_\Delta)_i] = i. \tag{3.1.9}
$$

The first commutation relation leads to

$$
[X^i, \Delta^j] = i\frac{\epsilon^{ij}}{B}, \tag{3.1.10}
$$

which they obtained in [4]. The second commutation relation leads to a contradiction, since we have shown that $\Pi_\Delta = 0$. In Dirac's notation $\Pi_\Delta = 0$ is a second class constraint and thus we should quantise use Dirac commutators.

Dirac's commutator rectifies the fact that the commutator $[\Pi_\Delta, \Delta] \neq 0$ seems to yield an inconsistency. It removes the inconsistency in the constraints but does not affect the dynamics.

From the result of careful analysis, we are permitted to make the following identification (as they did in [4]) without fear of contradiction

$$
|\vec{\Delta}| = \frac{|\vec{P}|}{B}, \tag{3.1.11}
$$

so that as the momentum $|\vec{P}|$ increases the dipole stretches i.e. $|\vec{\Delta}|$ increases.

We can carry this analogy even further. Suppose the dipole lives on a 2-sphere with the *centre of mass* moving along the equator. (Of course a centre of mass cannot be defined on curvilinear coordinates so what we mean by the centre of mass is the following. Consider a curve on the sphere stretching between the two end points of the dipole. The centre of this line will stand as the *centre of mass* – we will use this language throughout this section.) We will show that yet again as the momentum increases the dipole will expand, but it will reach a maximum size once it spans the diameter of the 2-sphere on which it resides.

Let N be the magnetic flux permeating the 2-sphere on which the dipole is moving. Then the quantisation of flux requires that

$$2\pi N = \Omega_2 B R^2, \tag{3.1.12}$$

where $\Omega_2 = 4\pi$ corresponds to the solid angle which spans a sphere. Suppose the potential of the magnetic field is given by

$$A_\phi = N \frac{1 - \sin\theta}{2R\cos\theta}, \tag{3.1.13}$$

where we are working in spherical coordinates (r, θ, ϕ) - note my notation deviates here from that of [4]. $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is the *azimuthal* angle i.e. it measures the angle from the equator towards the poles. $\phi \in [0, 2\pi]$ is the *longitudinal* angle i.e. it measures angular distance along the equator.

We can assume $A_r = 0$ since we want a *hedgehog* solution - i.e. the magnetic field lines should emit radially from the centre of the sphere and since $\vec{B} = \nabla \times \vec{A}$ there should be no A_r component. Let us check that (3.1.13) indeed offers the correct *hedgehog* or radial solution. In spherical coordinates we have

$$\begin{aligned}\nabla \times \vec{A} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\sin\theta A_\varphi - \frac{\partial A_\theta}{\partial\varphi}\right]\hat{r} \\ &+ \left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi)\right]\hat{\theta} \\ &+ \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\varphi}.\end{aligned} \tag{3.1.14}$$

We make the assumption that $A_r = A_\theta = 0$. Thus the only term from the above to contribute to $\vec{B}$ is the following

$$\begin{aligned}\vec{B} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta N\frac{1-\sin\theta}{2R\cos\theta})\right]\hat{r} \\ &= \frac{N}{2Rr}\left[\csc\theta\sec^2\theta - \sec^2\theta - 1\right]\hat{r}.\end{aligned} \tag{3.1.15}$$

Thus as required our magnetic field $\vec{B}$ is radial.

Returning to the problem at hand, we find that the Lagrangian $\mathcal{L}_A$ due to coupling with the magnetic field is given by

$$\begin{aligned}\mathcal{L}_A &= \vec{A}\cdot\vec{v}_{\mathrm{CM}} \\ &= A_\phi v_\phi \qquad\qquad \text{All other } \vec{A} \text{ components are zero.} \\ &= N\frac{1-\sin\theta}{2R\cos\theta}R\cos\theta\dot{\phi} \\ &= N\frac{1-\sin\phi}{2}\dot{\phi}.\end{aligned} \tag{3.1.16}$$

What we have been studying thus far is just the motion of the centre of mass of the dipole along the equator. What we really have is one point particle located at (ϕ, θ) and another at $(\phi, -\theta)$. And since $A_\phi = N\frac{1-\sin\theta}{2R\cos\theta}$ we can rewrite (3.1.16) as

$$\begin{aligned}\mathcal{L}_A &= N\left(\frac{1-\sin\theta}{2}\right)\dot{\phi} - N\left(\frac{1+\sin\theta}{2}\right)\dot{\phi} \\ &= -N\sin\theta\dot{\phi}.\end{aligned} \tag{3.1.17}$$

Therefore we can rewrite the total Lagrangian as

$$\mathcal{L} = -2R^2 \sin^2\theta \underbrace{-N\sin\theta\dot{\phi}}_{\mathcal{L}_A}, \tag{3.1.18}$$

where the term $-2R^2 \sin^2\theta$ is just the spring coupling between the two particles of the dipole. We get the angular momentum as follows

$$\begin{aligned} L &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \\ &= -N\sin\theta. \end{aligned}$$

Therefore the maximum angular momentum is N i.e. $L_{\max}$ is the same as the total magnetic flux. This is obviously attained at the poles where $\theta = \pm\frac{\pi}{2}$.

3.2 Relation to $\mathbf{AdS_5{\times}S^5}$

In [4] Susskind, McGreevy and Toumbas use the Lagrangian, to derive the maximum angular momentum and the stable minimum energy for an M5 brane wrapped in the spherical component of the $\mathrm{AdS_4{\times}S^7}$ space. Here we will explicitly derive these for a D3 brane wrapped in the spherical component of the $\mathrm{AdS_5{\times}S^5}$ space filling in the intermediate steps left out in [4]. But most importantly we will show that when the brane reaches its maximum size it also attains its maximum angular momentum, thus ensuring a cutoff on the number of states.

To determine the Lagrangian and hence the other quantities specified above we will need to calculate the metric on the S^5 space. Let us suppose, for the sake of computation, that the S^5 space is embedded in $\mathbb{R}^6$. We can rewrite the coordinates of the embedding space in terms of the five angles parametrising S^5 as follows (note that R is the radius of the S^5 sphere)

$$
\begin{aligned}
X^1 &= R\cos\varphi^1 \\
X^2 &= R\sin\varphi^1\cos\varphi^2 \\
X^3 &= R\sin\varphi^1\sin\varphi^2\cos\varphi^3 \\
X^4 &= R\sin\varphi^1\sin\varphi^2\sin\varphi^3\cos\varphi^4 \\
X^5 &= R\sin\varphi^1\sin\varphi^2\sin\varphi^3\sin\varphi^4\cos\varphi^5 \\
X^6 &= R\sin\varphi^1\sin\varphi^2\sin\varphi^3\sin\varphi^4\sin\varphi^5,
\end{aligned}
$$

where $\varphi^i \in [0,\pi]$ for $i = 1,2,3,4$ and $\varphi^5 \in [0,2\pi]$ i.e. φ^5 corresponds to the longitudinal angle and all the others are like azimuthal angles. It is a trivial matter to check that the above satisfy

$$(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 + (X^5)^2 + (X^6)^2 = R^2. \tag{3.2.1}$$

The graviton in S^5 will expand into a D3 brane - as explained using the dipole as a toy model. Therefore we can consider the D3 brane (which is 3 dimensional) to be moving around in the plane spanned by the X^1 and X^2 coordinates. Therefore if r is the radius of the D3 brane, we can describe it in terms of the other coordinates (viz. X^3, X^4, X^5 and X^6) as follows

$$(X^3)^2 + (X^4)^2 + (X^5)^2 + (X^6)^2 = r^2. \tag{3.2.2}$$

Thus condition (3.2.1) becomes

$$r^2 = R^2 - (X^1)^2 - (X^2)^2. \tag{3.2.3}$$

Clearly the radius of the D3 brane reaches a maximum (i.e. $r = R$) when $X^1 = X^2 = 0$. This coincides with the case of the M5 brane reaching its maximum size in the $AdS_4 \times S^7$ space at the origin in the X^1-X^2 plane, as determined explicitly in [4]. This is also similar to the case of the dipole, in the previous section; when both charges were located at the poles ($X^1 = X^2 = 0$) the dipole also reached its maximum size.

The D3 brane can be interpreted as a 3-sphere moving on an X^1-X^2 disk. Thus we can simplify things by separately determining the contributions of D^2 and S^3 to the S^5 metric. Condition (3.2.2) yields the metric $d\Omega_3^2$ of a 3-sphere which we will determine shortly.

First we determine the contributions of X^1 and X^2 to the metric. Since the D3 brane moves around on a circle of radius $\sqrt{R^2 - r^2}$ on the X^1-X^2 plane we can paramterise X^1 and X^2 as follows

$$\begin{aligned} X^1 &= \sqrt{R^2 - r^2} \sin \phi \\ X^2 &= \sqrt{R^2 - r^2} \cos \phi. \end{aligned}$$

This yields

$$(dX^1)^2 + (dX^2)^2 = \frac{r^2 dr^2}{R^2 - r^2} + \left(R^2 - r^2\right) d\phi^2. \tag{3.2.4}$$

Therefore the total metric becomes

$$ds^2 = \frac{r^2 dr^2}{R^2 - r^2} + \left(R^2 - r^2\right) d\phi^2 + r^2 d\Omega_3^2 \tag{3.2.5}$$

where $d\Omega_3$ is just the metric of a 3-sphere - and a function of φ^3, φ^4 and φ^5. The metric for a 3-sphere is found trivially if we consider that it is defined by the set of coordinates (in $\mathbb{R}^4$) as follows

$$\begin{aligned} \xi^1 &= r \cos \varphi^3 \\ \xi^2 &= r \sin \varphi^3 \cos \varphi^4 \\ \xi^3 &= r \sin \varphi^3 \sin \varphi^4 \cos \varphi^5 \\ \xi^4 &= r \sin \varphi^3 \sin \varphi^4 \sin \varphi^5. \end{aligned}$$

Note these are the same as X^3, X^4, X^5 and X^6 respectively, since $r = R\sin\varphi^1\sin\varphi^2$. We obtain the S^3 metric as follows

$$\begin{aligned} d\Omega_3^2 &= (d\xi^1)^2 + (d\xi^2)^2 + (d\xi^3)^2 + (d\xi^4)^2 \\ &= \left(\frac{d\xi^1}{d\varphi^3}\right)^2 (d\varphi^3)^2 + \left(\frac{\partial\xi^2}{\partial\varphi^3}\right)^2 (d\varphi^3)^2 + \left(\frac{\partial\xi^2}{\partial\varphi^4}\right)^2 (d\varphi^4)^2 + \dots \\ &= r^2[(d\varphi^3)^2 + \sin^2\varphi^3(d\varphi^4)^2 + \sin^2\varphi^3\sin^2\varphi^4(d\varphi^5)^2], \end{aligned} \tag{3.2.6}$$

where in the second line we have, for the sake of brevity, just shown how the pattern of derivatives continues and not written them out in full. Thus our metric for this $\mathbb{R}\times S^5$ space becomes

$$\begin{aligned} ds^2 =& dt^2 - \frac{r^2}{R^2 - r^2}dr^2 - (R^2 - r^2)d\phi^2 \\ & - r^2[(d\varphi^3)^2 + \sin^2\varphi^3(d\varphi^4)^2 + \sin^2\varphi^3\sin^2\varphi^4(d\varphi^5)^2]. \end{aligned} \tag{3.2.7}$$

Note the inclusion of dt^2 in the above and c has been set to 1. We can already make a tentative guess about the form of the Lagrangian for the above metric. If we assume that the radius of the brane does not vary much (i.e. $\dot{r} = 0$) then the brane only has an angular velocity of $(R^2 - r^2)\dot{\phi}^2$ - so in accordance with special relativity where the lagrangian of a free particle is of the form $\sqrt{1 - v^2}$ we can predict that

$$\mathcal{L} \propto \sqrt{1 - (R^2 - r^2)\dot{\phi}^2}. \tag{3.2.8}$$

The reason we can draw this conclusion is that the D3 brane is spherically symmetric and at the low energy approximation we can treat it like a point particle.

To get $\mathcal{L}$ explicitly we consider the usual definition of the action

$$S = \int \mathcal{L}dt. \tag{3.2.9}$$

So what we will do is integrate over the world volume of the D3 brane to obtain an expression similar to (3.2.9) and then just read off the Lagrangian. But first we need to compute the induced metric on the 3-brane as follows

$$\begin{aligned}
\sigma^0 &= t \\
\sigma^1 &= \varphi^3 \\
\sigma^2 &= \varphi^4 \\
\sigma^3 &= \varphi^5,
\end{aligned}$$

thus yielding

$$\begin{aligned}
ds^2 =& (d\sigma^0)^2 - (R^2 - r^2)\left(\frac{\partial \phi}{\partial \sigma^0}\right)^2 (d\sigma^0)^2 \\
&- r^2[(d\sigma^1)^2 + \sin^2\sigma^1 (d\sigma^2)^2 + \sin^2\sigma^1 \sin^2\sigma^2 (d\sigma^3)^2].
\end{aligned} \tag{3.2.10}$$

Note that ϕ is assumed to only be a function of σ_0 (i.e. time) and that the term involving $\frac{\partial r}{\partial \sigma_0}$ has been omitted since we are assuming the brane remains constant in size. The world volume of the D3 brane is obtained from

$$\int d\sigma^0 d\sigma^1 d\sigma^2 d\sigma^3 \sqrt{G}, \tag{3.2.11}$$

where $G = \det G_{\alpha\beta}$ i.e. the determinant of the induced metric on the world volume. We can read off the induced metric $G_{\alpha\beta}$ from equation (3.2.10) to obtain

$$G_{\alpha\beta} = \begin{pmatrix} \left(1 - (R^2 - r^2)\dot{\phi}^2\right) & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2\sin^2\sigma^1 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\sigma^1\sin^2\sigma^2 \end{pmatrix} \tag{3.2.12}$$

Thus we have

$$\begin{aligned}\int d\sigma^0 d\sigma^1 d\sigma^2 d\sigma^3 \sqrt{G} &= \int d\sigma^0 d\sigma^1 d\sigma^2 d\sigma^3 \left[r^3 \sin^2\sigma^3 \sin\sigma^4\right] \\ &\times \sqrt{1-(R^2-r^2)\dot{\phi}^2} \\ &= \int d\sigma^0 r^3 \Omega_3 \sqrt{1-(R^2-r^2)\dot{\phi}^2}. \end{aligned} \tag{3.2.13}$$

This is in accord with our previous prediction. Thus our Lagrangian $\mathcal{L}$ is

$$\mathcal{L} = -T_{D_3}\Omega_3 r^3 \sqrt{1-(R^2-r^2)\dot{\phi}^2}, \tag{3.2.14}$$

where

$$T_{D_3} = \frac{1}{(2\pi)^2 l_s^4 g_s}, \tag{3.2.15}$$

is the tension of the D3 brane. Note that l_s is the string length and g_s is the string coupling constant. The factor $-T_{D_3}\Omega_3 r^3$ can be interpreted as the mass of the D3 brane.

To get the complete Lagrangian we also need a Chern-Simons term which couples the background five form field strength to the D3 brane. The contribution of the Chern-Smons term to the action is provided by

$$S_{CS} = \int_{\text{W.V.}} A = \int_{\Sigma} F_{\mu\nu\rho\sigma\tau} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \wedge dx^\tau, \tag{3.2.16}$$

where A is given by $A = A_{\nu\rho\sigma\tau} dx^\nu \wedge dx^\rho \wedge dx^\sigma \wedge dx^\tau$ ($A_{\nu\rho\sigma\tau}$[1] is the potential) which we are integrating over the world volume (W.V.) of the D3 brane. $F_{\mu\nu\rho\sigma\tau}$ is the five form field strength tensor which we are integrating over Σ. Note that the boundary of Σ is the world volume. $F_{\mu\nu\rho\sigma\tau}$ is related to the potential $A_{\nu\rho\sigma\tau}$ by

[1]As an aside, the existence of a 3 brane is evinced by the 4 indices of the potential. For instance, when we are dealing with a point particle the potential takes the form A_μ. For a string we have $A_{\mu\nu}$. And of course for a 3 brane we have $A_{\nu\rho\sigma\tau}$.

$$F = dA, \tag{3.2.17}$$

where $A = A_{\nu\rho\sigma\tau} dx^\nu \wedge dx^\rho \wedge dx^\sigma \wedge dx^\tau$ and d is the exterior derivative. So in its entirety (3.2.17) reads

$$F_{\mu\nu\rho\sigma\tau} = \left(\frac{\partial}{\partial x^\mu} A_{\nu\rho\sigma\tau} \right) dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \wedge dx^\tau. \tag{3.2.18}$$

However we have that $F = B(d\text{vol})$ where $d$vol is a volume element of the manifold Σ over which we are integrating $F_{\mu\nu\rho\sigma\tau}$ in equation (3.2.16). Therefore the Chern-Simons term in the action, given by equation (3.2.16), becomes

$$S_{\text{CS}} = B\text{vol}(\Sigma) \tag{3.2.19}$$

Therefore the Chern-Simons term in the Lagrangian becomes

$$\mathcal{L}_{\text{CS}} = \frac{S_{\text{CS}}}{T} = B\text{vol}(\Sigma)\frac{\dot{\phi}}{2\pi}, \tag{3.2.20}$$

where T denotes the period of the orbit of the D3 brane. The volume of Σ is given by

$$\begin{aligned} \text{vol}(\Sigma) &= R \int d\Omega_3 \int_0^{2\pi} d\phi \int_0^r \tilde{r}^3 d\tilde{r} \\ &= \frac{1}{2}\pi\Omega_3 R r^4. \end{aligned}$$

Therefore we have

$$\mathcal{L}_{\text{CS}} = \frac{\dot{\phi}}{2\pi}\Omega_5 R r^4 \tag{3.2.21}$$

where $\Omega_5 = \frac{1}{2}\pi\Omega_3$ - this is easy to see from the equation

$$\Omega_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}. \tag{3.2.22}$$

Using the quantisation of charge

$$2\pi N = B\Omega_5 R^5, \tag{3.2.23}$$

we can rewrite $\mathcal{L}_{\text{CS}}$ as

$$\mathcal{L}_{\text{CS}} = \dot{\phi} N \frac{r^4}{R^4}. \tag{3.2.24}$$

Thus finally we have the effective lagrangian

$$\mathcal{L} = -T_{D3}\Omega_3 r^3 \sqrt{1 - (R^2 - r^2)\dot{\phi}^2} + \dot{\phi} N \frac{r^4}{R^4}. \tag{3.2.25}$$

Now we can calculate the angular momentum of the D3 brane.

$$\begin{aligned} L &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \\ &= \frac{T_{D3}\Omega_3 r^3 (R^2 - r^2)\dot{\phi}}{\sqrt{1 - (R^2 - r^2)\dot{\phi}^2}} + N \frac{r^4}{R^4} \\ &= \frac{m(R^2 - r^2)\dot{\phi}}{\sqrt{1 - (R^2 - r^2)\dot{\phi}^2}} + N \frac{r^4}{R^4}, \end{aligned}$$

where we have made the identification $m = T_{D3}\Omega_3 r^3$. It is quite clear that as $r \to R$ the first term in the last line tends to zero and the second term tends to N. Thus since an increase in angular momentum increases the radius r of the giant we have that at maximum $r = R$ and the maximum angular momentum is

$$|L_{\max}| = N, \tag{3.2.26}$$

in accordance with the result for the dipole as well as what was obtained for the M2 brane in [4].

If we determine the Hamiltonian for the D3 brane we have also effectively determined the energy. Thus we have

$$H = pv - \mathcal{L}, \tag{3.2.27}$$

by the Legendre transform. Thus, using the angular momentum L and angular velocity $\dot{\phi}$ we obtain

$$\begin{aligned} E &= \dot{\phi}L - \mathcal{L} \\ &= \frac{m\dot{\phi}^2(R^2 - r^2)}{\sqrt{1 - \dot{\phi}^2(R^2 - r^2)}} + N\dot{\phi}\frac{r^4}{R^4} + m\sqrt{1 - (R^2 - r^2)\dot{\phi}^2} - N\dot{\phi}\frac{r^4}{R^4} \\ &= \frac{m}{\sqrt{1 - (R^2 - r^2)\dot{\phi}^2}}. \end{aligned} \tag{3.2.28}$$

To find the minimum stable energy in terms of the angular momentum L we rewrite L as follows

$$L = \frac{m(R^2 - r^2)\dot{\phi}}{\sqrt{1 - (R^2 - r^2)\dot{\phi}^2}} + N\frac{r^4}{R^4} \tag{3.2.29}$$

we can rewrite this as

$$\left(L - N\frac{r^4}{R^4}\right)^2 \left(1 - (R^2 - r^2)\dot{\phi}^2\right) = \left(m(R^2 - r^2)\dot{\phi}\right)^2, \tag{3.2.30}$$

after regrouping terms we obtain

$$\dot{\phi}^2(R^2 - r^2) = \frac{\left(L - \frac{r^4}{R^4}\right)}{m^2(R^2 - r^2) + (L - N\frac{r^4}{R^4})^2}. \tag{3.2.31}$$

Substituting this result into our equation for the energy we get

$$E = \sqrt{N^2\frac{r^6}{R^8} + \frac{\left(L - N\frac{r^4}{R^4}\right)^2}{(R^2 - r^2)}}. \tag{3.2.32}$$

The stable minimum energy occurs when $\frac{dE}{dr} = 0$

$$\begin{aligned}\frac{dE}{dr} &= \frac{1}{E}\frac{1}{(R^2-r^2)^2}\big[3N^2\frac{r^5}{R^8}(R^2-r^2)^2 \\ &+ \left(L - N\frac{r^4}{R^4}\right)\left(-4N\frac{r^3}{R^4}\right)(R^2-r^2) + r\left(L - N\frac{r^4}{R^4}\right)^2\big] \\ &= \frac{r}{E(R^2-r^2)^2}\left[\left(L - N\frac{r^2}{R^2}\right)\left(L - 3N\frac{r^2}{R^2} + 2N\frac{r^4}{R^4}\right)\right]. \end{aligned} \tag{3.2.33}$$

Thus, from the first factor in the square brackets, we can see that the minimum energy occurs when

$$r^2 = \frac{L}{N}R^2. \tag{3.2.34}$$

Thus the minimum energy is given by

$$\begin{aligned} E_{\min} &= \frac{1}{R}\sqrt{\underbrace{\frac{L^3}{N}}_{\approx 0} + \frac{L^2\left(1-\frac{L}{N}\right)^2}{1-\frac{L}{N}}} \\ &= \frac{L}{R}. \end{aligned} \tag{3.2.35}$$

In summary what we have shown in this section is that

i. The Lagrangian is given by

$$\mathcal{L} = -T_{D3}\Omega_3 r^3\sqrt{1-(R^2-r^2)\dot{\phi}^2} + \dot{\phi}N\frac{r^4}{R^4}.$$

ii. The angular momentum is given by

$$L = \frac{m(R^2-r^2)\dot{\phi}}{\sqrt{1-(R^2-r^2)\dot{\phi}^2}} + N\frac{r^4}{R^4}.$$

And the maximum angular momentum is given by

$$|L_{\max}| = N. \tag{3.2.36}$$

This is probably the most important point since it is one manisfestation of the stringy exclusion principle i.e. that there is a cutoff on the number of states for a graviton. Note that this is a semiclassical result so L is not quantised.

iii. The minimum energy occurs when

$$r^2 = \frac{L}{N} R^2.$$

iv. And the minimum energy is given by

$$E_{\min} = \frac{L}{R}.$$

3.3 An Aside on $O(1)$, $O(\sqrt{N})$ etc.

In $AdS_5 \times S^5$ we have that the radius of S^5 is equal to the radius of curvature of AdS_5

$$R = (4\pi g_s N)^{\frac{1}{4}} l_s, \tag{3.3.1}$$

or more succintly for our purposes

$$R \propto N^{\frac{1}{4}} l_s. \tag{3.3.2}$$

In the proceeding section we showed that the minimum energy of a D3 brane in the spherical component of the $AdS_5 \times S^5$ space occurs when

$$r^2 = \frac{L}{N} R^2. \tag{3.3.3}$$

Lets determine the relative size of the entity we are studying if the angular momentum is of the order of $\sqrt{N}$ i.e. $L \sim O(\sqrt{N})$. We then have that

$$\begin{aligned} r^2 &= \frac{\sqrt{N}}{N} R^2 \\ &\propto \frac{1}{\sqrt{N}} \sqrt{N} l_s^2, \end{aligned}$$

Order of L	**Corresponding Entity**
$O(1)$	Graviton
$O(\sqrt{N})$	String
$O(N)$	Brane
$O(N^2)$	Spacetime

Table 3.1: The order of the angular momentum in the equation $r^2 = \frac{L}{N}R^2$ versus the corresponding entity it engenders. Note that r is the size of the entity we are dealing with in the S^5 space of radius R. For example if $L \sim O(\sqrt{N})$ then it can be shown that $r \sim l_s$ and hence that we are dealing with a string.

where in the last line we used the proportionality given in (3.3.2). Thus we have that

$$r \sim l_s, \tag{3.3.4}$$

and we are dealing with a string since the size of its radial dimension corresponds to the length of a string.

What if $L \sim O(N)$? Then $\frac{L}{N} \sim O(1)$, so $r^2 \sim R^2$, so we are dealing with a membrane. And obviously if $\frac{L}{N} \sim O(N)$ (i.e. $L \sim O(N^2)$) then we are dealing with a whole new entity i.e. a new spacetime emerges.

We have summarised these results in table 3.1.

4 Relation between Giant Gravitons and Operators

In this section we will give a brief discussion of how certain operators can be interpreted as the states of giant gravitons. See [5] and [6] for a more in depth discussion. Firstly, we will give a brief description of the different gravitons

Sphere Giants: These are giant gravitons which expand in the spherical component of the $\text{AdS}_5 \times \text{S}^5$ space we are interested in. In the previous section we showed that there is a cutoff on the number of states accessible to a sphere giant. This is due to the limited volume of the sphere in which they may expand – resulting in them having a maximum angular momentum. Spherical giants wrap themselves on an S^3 subspace of the S^5 component of the $\text{AdS}_5 \times \text{S}^5$ spacetime.

AdS Giants: These are giant gravitons which expand in the AdS component of the AdS×S space under investigation. Sphere giants wrap themselves on an S^3 subspace of the AdS_5 component of the $\text{AdS}_5 \times \text{S}^5$ spacetime. Unlike the sphere giants these do not have a restriction imposed on their angular momentum and hence they don't have a cutoff on the number of accessible states. This is because the AdS space is not positively curved like the spherical component of the AdS×S space so the AdS giants can grow without pause. However, as we will explain shortly, there is a limit on the number of AdS giants that may exist.

Kaluza-Klein Gravitons: Also known as KK modes are just point-like gravitons.

The states of sphere giants are described by Schur polynnomials of the following form

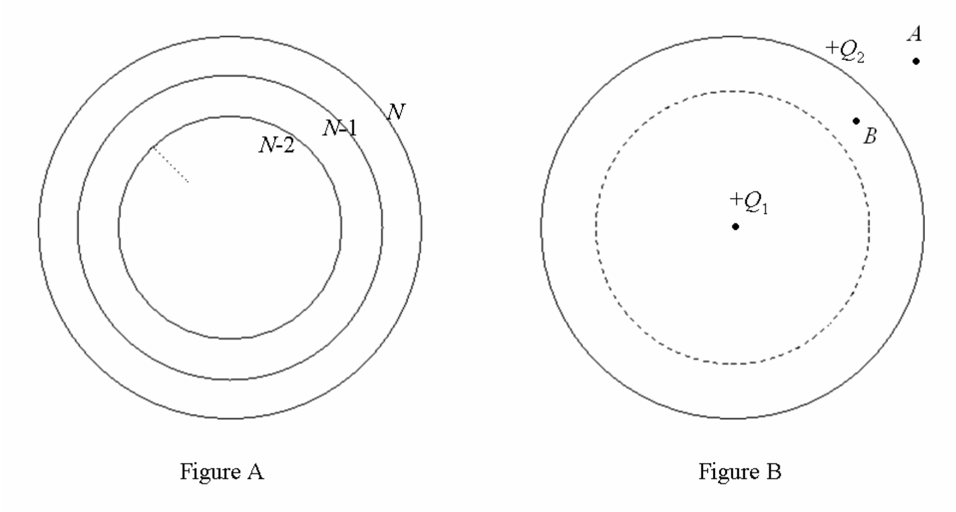
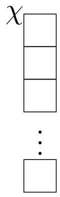


Figure 2: In Figure A there are N spherical giant gravitons. The innermost giant is a point. Each successive outer giant expands due to the D3 charge flux radiating from the giants contained within it. Let us consider figure B for an explanation as to why there are at most N AdS giants. The field at point A is due to Q_1+Q_2 units of flux whilst the field at B is, by Gauss' law, only due to Q_1 units of flux. Thus the innermost graviton in figure A won't expand due to the absence of flux from inner giants. Note that this picture is not absolutely correct. We have depicted the giants as a monopole spread over a surface whilst it should in fact be represented as a dipole spread over a surface. Of course the monopole case is just easier to visualise. Antipodal points on a spherical D3 brane form a dipole.



(4.0.5)

Note that there is one column and at most N boxes in the column labelling the state. Thus the representation which labels the state of a sphere giant is purely antisymmetric. This corresponds to the fact that the momentum of a sphere giant has a cutoff at N - as explained in the previous section. Each box corresponds to a unit of angular momentum.

Similarly the AdS giants have wavefunctions given by Schur polynomials of the form

$$\chi_{\square\square\square\ldots} \tag{4.0.6}$$

We can now append as many boxes as we want to a row - this means that the giant graviton can grow without limit on the AdS component of the AdS×S space. However we can still only have at most N rows - corresponding to the fact that we can only have N giants in the AdS space. Thus for 2 gravitons the state would be labeled as follows

$$\chi_{\begin{smallmatrix}\square\square\square \\ \square\square\square\ldots\end{smallmatrix}}$$

with as many boxes each respective row as necessary – note the number of boxes in the top row must be greater than or equal to the number of boxes in the second row. The reason that there can be at most N AdS giants is as follows. Consider figure 2B. There are two charges Q_1 and Q_2; and for the sake of argument they are both positive. The Q_1 charge consists of a point while the Q_2 charge has been spread out in a sphere around Q_1. Now at the point labelled A the field is due to both the Q_1 and Q_2 charges i.e. we have $Q_1 + Q_2$ units of flux. However at the point B the electric field is due to the Q_1 charge only - this follows simply from Gauss' law. Thus at point B we only have Q_1 units of flux.

Now let us consider figure 2A. A graviton expands due to the field radiating from inside of it. Thus, if like the 2 charges in figure 2B, we proceed just inside the graviton labelled N then the field inside this giant is only due to the gravitons contained in it viz. the gravitons labelled 1 through $N - 1$. If you continue in this fashion until you reach the innermost graviton then, by the above inductive reasoning, it contains no other gravitons inside itself – it is a point and the field inside of it is zero. Thus it won't expand. Note that the giants are not like monopoles spread across a spherical surface they are in fact dipoles spread across a sphere. Antipodal points on the giant form a dipole. The monopole-like explanation above is just easier to conceptualise. For a more detailed examination of the limit on the number of giants see [22] and [23]

5 The Brown-Heslop-Ramgoolam Operators

In [7] Brown, Heslop and Ramgoolam seek a non-redundant, complete, orthonormal basis for gauge invariant operators. Suppose we have M complex matrices, which represent fields, labeled X_1, X_2, ... X_M. Let these matrices (or fields) be subject to $U(M)$ transformations i.e. if X is a field and $U \in U(M)$ then

$$X \rightarrow UX \text{ or equivalently } X_a \rightarrow U_{ab}X_b. \tag{5.0.7}$$

$U(M)$ is the global symmetry group. Suppose that $U(N)$ is the local symmetry group, due to a redundancy in our description of the system. Thus besides a $U(M)$ index each field will also carry $U(N)$ indices. In $(X_a)^i_j$, a is the $U(M)$ index and i and j are the $U(N)$ indices. The $U(N)$ indices transform as follows

$$X \rightarrow UXU^\dagger \text{ or equivalently } X^i_j \rightarrow U_{ik}X^k_l U^*_{lj}. \tag{5.0.8}$$

Note that $U^*_{lj} = (U_{jl})^\dagger$. We say that the fields X transform in the fundamental of $U(M)$ and in the adjoint of $U(N)$. Thus, in general, we have operators of the form

$$(X_1^{\mu_1})^{i_1}_{j_1} \otimes (X_2^{\mu_2})^{i_2}_{j_2} \otimes \ldots \otimes (X_M^{\mu_M})^{i_M}_{j_M}, \tag{5.0.9}$$

where μ_i denotes the number of copies we have of the i^{th} field. Thus if we have n fields in total then we must have that

$$\sum_{i=1}^{M} \mu_i = n.$$

These operators provide a map between between vector spaces of the form $V^{\otimes n} = \underbrace{V \otimes V \otimes \ldots \otimes V}_{n \text{ copies}}$. To make the above operator gauge invariant we have to trace over the $U(N)$ to produce a $U(N)$ scalar. We are interested in gauge invariant operators because these are the physical observables of the theory.

Thus all gauge invariant operators can be built up from operators of the form

$$\mathrm{Tr}\left(X_1^{\mu_1} \otimes X_2^{\mu_2} \otimes \ldots \otimes X_M^{\mu_M}\right), \tag{5.0.10}$$

where Tr denotes the trace over the $U(N)$ indices – note that there are several ways of taking the trace. For example if we have 2 copies of the field labeled X_1 and 1 copy of the field labeled X_2 we can have $\mathrm{Tr}(X_1X_1X_2)$, $\mathrm{Tr}(X_1)\mathrm{Tr}(X_1X_2)$ etc.

The problem is trying to find a complete, orthonormal, non-redundant basis for all these gauge invariant operators. Brown, Heslop and Ramgoolam achieve this as follows

- They consider operators of the form $\mathrm{Tr}\left(\alpha X_1^{\mu_1} \otimes X_2^{\mu_2} \otimes \ldots \otimes X_M^{\mu_M}\right)$[2] where α is an element of the permutation\symmetry group S_n which permutes the lower indices of the operator i.e. it rearranges the order in which the fields act on the states in our system and thus also alters the way in which the trace is taken.

- Next they perform a *transformation* with regards to the α in the above operator. The transformation results in an operator which is essentially a projection operator. Projection operators are orthogonal and this is clearly a natural step if you want to show that the two-point function is orthogonal. This also negates the α dependence.

- Several of the operators are in fact equivalent (consider for example a rearrangement of the fields X in the operator $\mathrm{Tr}(XXXY)$). Thus

[2]Actually they consider operators of the form $\mathrm{Tr}\left(\alpha\sigma X_1^{\mu_1} \otimes X_2^{\mu_2} \otimes \ldots \otimes X_M^{\mu_M}\sigma^{-1}\right)$ where $\sigma \in S_n$. However, σ does not play a role in their final result. We will discuss this further on.

they construct a projector to map the operators to a subspace where such redundancies have been factored out.

- Finally they construct a complete basis of operators using branching coefficients and Clebsch-Gordan coefficients.

As mentioned in the footnote in [7] they actually study operators of the form

$$\sigma X_{a_1} \otimes X_{a_2} \otimes \ldots \otimes X_{a_n} \sigma^{-1} = X_{a_{\sigma^{-1}(1)}} \otimes X_{a_{\sigma^{-1}(2)}} \otimes \ldots \otimes X_{a_{\sigma^{-1}(n)}}, \quad (5.0.11)$$

where σ is an element of S_n i.e. it swaps the indices of the fields\complex matrices or equivalently permutes the vectors in $V^{\otimes n}$ on which the matrices act. The point at which they use the above in [7] is to show that operators of the form

$$\sum_{\sigma \in S_n} [\Gamma_\Lambda(\sigma)]_{ij} \alpha \sigma X_1^{\mu_1} \otimes X_2^{\mu_2} \otimes \cdots X_M^{\mu_M} \sigma^{-1}, \quad (5.0.12)$$

now reside in $V_\Lambda^{U(M)} \otimes V_\Lambda^{S_n}$ instead of $V^{\otimes n}$ i.e. they found a connection to the Schur-Weyl duality. It is also important since Λ labels both a representation of the symmetric group and the unitary group. This is important in constructing the possible BHR operators, examples of which will be given in section 7.

In the above list we have just given a brief outline of the procedure they implemented to obtain an orthonormal basis for gauge invariant operators. In the subsequent subsections we will investigate their method in more detail.

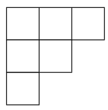
5.1 A Brief Aside on Branching Coefficients

In this subsection we will briefly discuss branching coefficients which we will require in order to understand the derivation of the BHR (Brown-Heslop-Ramgoolam) operators.

Suppose we have an arbitrary projector $P_{R \to R_\alpha}$ with matrix elements P_{ij} which projects a matrix from a representation R into some subduced representation R_α. Then we can write the projector as follows

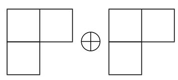
$$P_{ij} = \sum_\beta B_{i\beta} B_{\beta j}, \tag{5.1.1}$$

where the factors $B_{i\beta}$ are *branching coefficients.* The β index specifies which subduced representation the branching coefficient is mapping the index i (or j) to. For instance, consider the following S_6 representation labeled by the Young diagram



(5.1.2)

it turns out that this representation has two subductions (see section 6.5 for a more furbished discussion) of the form



(5.1.3)

so the β index of the branching coefficients would label each of these respective subductions. Suppose that the outer product $R \circ S$ appears n times as a subduction of the representation T then $g(R \circ S, T) = n$ is known as the Littlewood-Richardson coefficient. So β would run from $\beta = 1$ upto $\beta = g(R \circ S, T)$.

The branching coefficients project a given index in the representation R onto some subspace R_α. Suppose the projector $P_{R \to R_\alpha}$ maps vectors in some space V^D to some subspace V^d i.e. we have that

$$P_{R \to R_\alpha} : V^D \to V^d \text{ or } [P_{R \to R_\alpha}]_{ij} v_i = u_j, \tag{5.1.4}$$

where $v_i \in V^D$ and $u_j \in V^d$ and if we can write the projector in terms of branching coefficients as illustrated in equation (5.1.1) then the vectors

$$v_\beta = \sum_\beta P_{i\beta} v_i, \tag{5.1.5}$$

form a complete basis of the vector subspace V^d. This fact was used in [7] to develop their operators as we will shortly reveal.

Now we will define the branching coefficients in terms of bras and kets. Suppose $[\Gamma_R(P_{R\to R_\alpha})]_{ij}$ is a matrix representation of the projector $P_{R\to R_\alpha}$ which projects the representation R onto R_α then we have the following

$$\begin{aligned}[\Gamma_R(P_{R\to R_\alpha})]_{ij} &= \langle R, i | P_{R\to R_\alpha} | R, j\rangle \\ &= \sum_\beta \langle R, i | R \to R_\alpha, \beta\rangle \langle R \to R_\alpha, \beta | R, j\rangle \,. \end{aligned} \tag{5.1.6}$$

In the last line we inserted a complete set of operators. Thus the branching coefficients can be written as

$$B^{R\to R_\alpha}_{i\beta} = \langle R, i | R \to R_\alpha, \beta\rangle \,. \tag{5.1.7}$$

5.2 Deriving the BHR Operators

As mentioned already all gauge invariant operators can be built from linear combinations of operators of the form

$$\mathrm{Tr}(\alpha X_1^{\mu_1} \otimes X_2^{\mu_2} \otimes \cdots \otimes X_M^{\mu_M}),$$

where the trace is being taken over the $U(N)$ indices which have been omitted in the above. As mentioned before, they first transform the above operator to obtain the following

$$\mathcal{O}^{\mu R}_{pq} = \frac{1}{n!} \sum_{\alpha \in S_n} [\Gamma_R(\alpha)]_{pq} \mathrm{Tr}(\alpha X_1^{\mu_1} \otimes \ldots \otimes X_M^{\mu_M}) \tag{5.2.1}$$

The reason they sum over the α is primarily because the operators

$$\frac{1}{n!} \sum_{\alpha \in S_n} [\Gamma_R(\alpha)]_{pq} \alpha, \tag{5.2.2}$$

which transform the gauge invariant traced operators, are projection operators (see section 2.5 for a better discussion) so this is already creating a build-up to a diagonalised two-point function – since projection operators are orthogonal. We can also already see a hint of the burgeoning Schur polynomials in this operator.

Next they transform the operators to factor out any *redundancies*. Note that

$$\begin{aligned}\mathrm{Tr}(\alpha X_1^{\mu_1} \otimes X_2^{\mu_2} \otimes \cdots \otimes X_M^{\mu_M}) &= \mathrm{Tr}(\alpha\gamma X_1^{\mu_1} \otimes X_2^{\mu_2} \otimes \cdots \otimes X_M^{\mu_M}\gamma^{-1}) \\ &= \mathrm{Tr}(\gamma^{-1}\alpha\gamma X_1^{\mu_1} \otimes X_2^{\mu_2} \otimes \cdots \otimes X_M^{\mu_M}),\end{aligned}$$

where $\gamma \in H_\mu = S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_M}$. Note that in the last line we used the cyclicity of the trace. What this means is that it does not matter how we arrange the μ_1 copies of the X_1 fields amongst themselves (since they all represent the same physical field), or the μ_2 copies of the X_2 fields etc. To see mathematically that acting on α with γ in this manner does indeed leave things invariant let us consider their action on just two matrices labeled X and Y (we will assume there are m copies of X and n copies of Y) and we will use the fact that $\gamma^{-1}\alpha\gamma$ acts on the lower indices of the matries i.e.

$$\begin{aligned}&\gamma^{-1}\alpha\gamma X^{i_1}_{j_1} X^{i_2}_{j_2} \ldots X^{i_m}_{j_m} Y^{i_{m+1}}_{j_{m+1}} Y^{i_{m+2}}_{j_{m+2}} \ldots Y^{i_{m+n}}_{j_{m+n}} \\ &= X^{i_1}_{j_{\gamma^{-1}\alpha\gamma(1)}} \ldots X^{i_m}_{j_{\gamma^{-1}\alpha\gamma(m)}} Y^{i_{m+1}}_{j_{\gamma^{-1}\alpha\gamma(m+1)}} \ldots Y^{i_{m+n}}_{j_{\gamma^{-1}\alpha\gamma(m+n)}} \\ &= X^{i_{\gamma(1)}}_{j_{\alpha\gamma(1)}} \ldots X^{i_{\gamma(m)}}_{j_{\alpha\gamma(m)}} Y^{i_{\gamma(m+1)}}_{j_{\alpha\gamma(m+1)}} \ldots Y^{i_{\gamma(m+n)}}_{j_{\alpha\gamma(m+n)}} \\ &= X^{i_1}_{j_{\alpha(1)}} \ldots X^{i_m}_{j_{\alpha(m)}} Y^{i_{m+1}}_{j_{\alpha(m+1)}} \ldots Y^{i_{m+n}}_{j_{\alpha(m+n)}}.\end{aligned}$$

Note, in the second last line we have used theorem 1 in appendix B to turn the γ^{-1} acting on the lower indices into a γ acting on the upper indices. In the last line we have used the fact that $\gamma \in S_m \times S_n$ which means we are just swapping the X fields amongst themselves and the Y fields amongst themselves without mixing between the two different fields – thus leaving the system invariant.

So essentially what we want is a new operator which obeys $\gamma\tilde{\mathcal{O}}^{\mu R}_{pq}\gamma^{-1} = \tilde{\mathcal{O}}^{\mu R}_{pq}$ for all $\gamma \in H_\mu$. Clearly the factor involving the trace in (5.2.1) is invariant under the action of H_μ i.e.

$$\mathcal{O}^{\mu R}_{pq} = \frac{1}{n!}\sum_{\alpha\in S_n}[\Gamma_R(\alpha)]_{pq}\underbrace{\mathrm{Tr}(\alpha X_1^{\mu_1}\otimes\ldots\otimes X_M^{\mu_M})}_{\text{Invariant under } H_\mu \text{ action.}}.$$

To make the α representation (i.e. $\Gamma_R(\alpha)$) invariant under the action of H_μ we need to transform it in the following way.

$$\tilde{\mathcal{O}}^{\mu R}_{pq} = \frac{1}{|H_\mu|}\sum_{\gamma\in H_\mu}\gamma\mathcal{O}^{\mu R}_{pq}\gamma^{-1} = \frac{1}{|H_\mu|}\frac{1}{n!}\sum_{\alpha\in S_n}[\Gamma_R(\gamma)]_{ip}[\Gamma_R(\alpha)]_{pq}[\Gamma_R(\gamma^{-1})]_{qj} \times\mathrm{Tr}(\alpha X_1^{\mu_1}\otimes\ldots\otimes X_M^{\mu_M}).$$

The factor of $\frac{1}{|H_\mu|}$ is just to normalise it. Thus we are acting on $\mathcal{O}^{\mu R}_{pq}$ with the following projection operator

$$\begin{aligned} P_{ij;pq} &= \frac{1}{|H_\mu|}\sum_{\gamma\in H_\mu}[\Gamma_R(\gamma)]_{ip}[\Gamma_R(\gamma^{-1})]_{qj} \\ &= \frac{1}{|H_\mu|}\sum_{\gamma\in H_\mu}[\Gamma_R(\gamma)]_{ip}[\Gamma_R(\gamma)]_{jq}. \end{aligned}$$

This projector factors out any multiplicities. This is also where the branching and Clebsch-Gordan coefficients make their appearance. As mentioned in subsection 5.1, given a projector which maps onto a subspace, if we can decompose the projector into branching coefficients then we can use them to create a complete basis for the subspace.

To decompose this projector into branching coefficients we proceed as follows

$$\begin{aligned}
P_{ij;pq} &= \frac{1}{|H_\mu|}\sum_{\gamma\in H_\mu}[\Gamma_R(\gamma)]_{ip}[\Gamma_R(\gamma)]_{jq} \\
&= \langle R,i;R,j|P|R,p;R,q\rangle \\
&= \sum_{\Lambda,\tau}\sum_{\Lambda',\tau'}\langle R,i;R,j|\Lambda,\tau,s\rangle\langle\Lambda,\tau,s|P|\Lambda',\tau',t\rangle\langle\Lambda',\tau',t|R,p;R,q\rangle \\
&= \sum_{\Lambda,\tau}\langle R,i;R,j|\Lambda,\tau,s\rangle\langle\Lambda,\tau,s|P|\Lambda,\tau,t\rangle\langle\Lambda,\tau,t|R,p;R,q\rangle \\
&= \sum_{\Lambda,\tau} S^{\tau,\Lambda RR}_{sij}[\Gamma_\Lambda(P)]_{st}S^{\tau,\Lambda RR}_{tpq}. \qquad (5.2.3)
\end{aligned}$$

In the third line we have inserted two complete sets of states. In the following line we have used the fact that the states are orthonormal and that the projector is proportional to the identity matrix by Schur's lemma. As indicated the factors $\langle R,i;R,j|\Lambda,\tau,s\rangle$ just correspond to the Clebsch-Gordon coefficients $S^{\tau,\Lambda RR}_{sij}$ which in this particular instance allow us to write two operators in two representations as a single operator in a single representation. In general Clebsch-Gordon coefficients are used to rewrite $R\otimes S$ indices as a single T index say. Naturally they can be used for states, operators and more general tensors. For instance, suppose we have the two operators $\Gamma_R(\sigma)$ and $\Gamma_S(\sigma)$ which reside in the space $R\otimes S$ then we can use Clebsch-Gordon coefficients to write them in a single representation T as follows

$$\begin{aligned}
[\Gamma_R(\sigma)]_{ij}[\Gamma_S(\sigma)]_{kl} &= \langle R,i;S,k|\,\sigma\,|R,j;S,l\rangle \\
&= \sum_{T,\tau}\langle R,i;S,k|T,\tau,a\rangle\langle T,\tau,a|\,\sigma\,|T,\tau,b\rangle\langle T,\tau,b|R,j;S,l\rangle \\
&= \sum_{T,\tau}\langle R,i;S,k|T,\tau,a\rangle[\Gamma_T(\sigma)]_{ab}\langle T,\tau,b|R,j;S,l\rangle \\
&= \sum_{T,\tau} S^{\tau,RST}_{ika}[\Gamma_T(\sigma)]_{ab}S^{\tau,RST}_{jlb},
\end{aligned}$$

where τ is the multiplicity of T in the product space $R\otimes S$. If we recall that $[\Gamma_\Lambda(P)]$ in equation (5.2.3) is also a projector then we can split it into branching coefficients as explained in section 5.1

$$[\Gamma_\Lambda(P)]_{st} = \sum_\beta B_{s\beta} B_{\beta t},$$

and thus the projection operator $P_{ij;pq}$ can be written as

$$P_{ij;pq} = \sum_{\Lambda,\tau,\beta} S^{\tau,\Lambda RR}_{sij} B_{s\beta} B_{\beta t} S^{\tau,\Lambda RR}_{tpq}.$$

Since the projection operator $P_{ij;pq}$ can be split into branching coefficients like this we can use the results of section 5.1 to create a complete basis for the gauge invariant operators viz.

$$\begin{aligned} \mathcal{O}^{\Lambda\mu,R}_{\beta,\tau} &= B_{\beta t} S^{\tau,\Lambda RR}_{tpq} \mathcal{O}^{\mu R}_{pq} \\ &= \frac{1}{n!} \sum_{\alpha \in S_n} B_{\beta t} S^{\tau,\Lambda RR}_{tpq} \left[\Gamma_R(\alpha)\right]_{pq} \mathrm{Tr}(\alpha X_1^{\mu_1} \otimes X_2^{\mu_2} \otimes \ldots \otimes X_M^{\mu_M}). \end{aligned}$$

We have finally arrived at the operators which Brown, Heslop and Ramgoolam studied in [7]. From section 5.1 and assuming the projectors are complete it is patent that these operators do indeed form a complete basis for the gauge invariant operators $\mathrm{Tr}(\alpha X_1^{\mu_1} \otimes X_2^{\mu_2} \otimes \ldots \otimes X_M^{\mu_M})$.

5.3 Orthogonality of the BHR Operators

It is clear from the construction of the BHR operators that they are orthogonal. However we will briefly and explicitly verify their orthogonality in this section. Consider the two BHR operators

$$\mathcal{O}^{\Lambda\mu,R}_{\beta,\tau} = \frac{1}{n!} \sum_{\alpha \in S_n} B_{\beta s} S^{\tau,\Lambda RR}_{smn} \left[\Gamma_R(\alpha)\right]_{mn} \mathrm{Tr}(\alpha X_1^{\mu_1} \otimes X_2^{\mu_2} \otimes \ldots \otimes X_M^{\mu_M}),$$

and

$$\mathcal{O}^{\Lambda'\mu',R'}_{\beta',\tau'} = \frac{1}{n!}\sum_{\alpha'\in S_n} B_{\beta' t} S^{\tau',\Lambda' R' R'}_{tpq} \left[\Gamma_{R'}(\alpha')\right]_{pq} \mathrm{Tr}(\alpha' X_1^{\mu'_1} \otimes X_2^{\mu'_2} \otimes \ldots \otimes X_M^{\mu'_M}).$$

In [7] the two-point function was found to be

$$\langle \mathcal{O}^{\Lambda\mu,R}_{\beta,\tau} \mathcal{O}^{\dagger\Lambda'\mu',R'}_{\beta',\tau'} \rangle = \delta^{\Lambda\Lambda'}\delta^{RR'}\delta_{\beta\beta'}\delta_{\tau\tau'}\frac{|H_\mu|\mathrm{Dim}R}{d_R^2}.$$

Proceeding accordingly we have

$$\begin{aligned}
\langle \mathcal{O}^{\Lambda\mu,R}_{\beta,\tau} \mathcal{O}^{\dagger\Lambda'\mu',R'}_{\beta',\tau'} \rangle &= \left(\frac{1}{n!}\right)^2 \sum_{\alpha\in S_n}\sum_{\alpha'\in S_n} B_{\beta s} S^{\tau,\Lambda RR}_{smn} B_{\beta' t} S^{\dagger\tau',\Lambda' R' R'}_{tpq} \\
&\times \left[\Gamma_R(\alpha)\right]_{mn} \left[\Gamma_{R'}(\alpha')\right]^{\dagger}_{pq} \\
&\times \left\langle \mathrm{Tr}(\alpha X_1^{\mu_1}\otimes\ldots\otimes X_M^{\mu_M}) \left[\mathrm{Tr}(\alpha' X_1^{\mu'_1}\otimes\ldots\otimes X_M^{\mu'_M})\right]^{\dagger} \right\rangle \\
&= \left(\frac{1}{n!}\right)^2 \sum_{\alpha\in S_n}\sum_{\alpha'\in S_n} B_{\beta s} S^{\tau,\Lambda RR}_{smn} B_{\beta' t} S^{\tau',\Lambda' R' R'}_{tqp} \\
&\times \left[\Gamma_R(\alpha)\right]_{mn} \left[\Gamma_{R'}(\alpha')\right]_{qp} \sum_{\gamma\in H_\mu} \mathrm{Tr}(\alpha\gamma\alpha'\gamma^{-1}). \qquad (5.3.1)
\end{aligned}$$

The sum over γ is consistent with a sum over all possible Wick contractions. Note that the branching and Clebsch-Gordan coefficients are real so the adjoint operator † does not affect them save for swapping the Clebsch-Gordan indices in last line. Also the matrices $[\Gamma_R(\alpha)]$ can be chosen to be real so that the adjoint operator † just swaps the indices. The correlator of the traced fields is obtained by performing Wick contractions. See section 6.3 of this dissertation for a more in depth explanation. Next they employ the Schur-Weyl duality in [7] to evaluate the sum $\sum_{\gamma\in H_\mu}\mathrm{Tr}(\alpha\gamma\alpha'\gamma^{-1})$

$$V^{\otimes n} = \oplus_T V_T^{S_n} \otimes V_T^{U(N)}.$$

This is used to expand the trace as follows

$$\mathrm{Tr}(\sigma I) = \sum_T \chi_T(\sigma)\chi_T(I) = \sum_T \chi_T(\sigma)\mathrm{Dim}T,$$

where Dim$T$ is just the dimension of the U(N) representation T. Thus (5.3.1) becomes

$$\begin{aligned}
&\left(\frac{1}{n!}\right)^2 \sum_{\alpha\in S_n}\sum_{\alpha'\in S_n} B_{\beta s}S^{\tau,\Lambda RR}_{smn}B_{\beta' t}S^{\tau',\Lambda' R'R'}_{tqp}\left[\Gamma_R(\alpha)\right]_{mn}\left[\Gamma_{R'}(\alpha')\right]_{qp}\\
&\times\sum_{\gamma\in H_\mu}\sum_T \mathrm{Dim}T\left[\Gamma_T(\alpha)\right]_{ab}\left[\Gamma_T(\gamma)\right]_{bc}\left[\Gamma_T(\alpha')\right]_{cd}\left[\Gamma_T(\gamma^{-1})\right]_{da}\\
&=\frac{|H_\mu|}{d_R d_{R'}}\sum_T \mathrm{Dim}T P_{ab;cd}\delta^{RT}\delta^{R'T}\delta_{ma}\delta_{nb}\delta_{qc}\delta_{pd}B_{\beta s}S^{\tau,\Lambda RR}_{smn}B_{\beta' t}S^{\tau',\Lambda' R'R'}_{tqp}\\
&=\frac{|H_\mu|\mathrm{Dim}R}{d_R d_{R'}}\delta^{RR'}B_{\beta s}S^{\tau,\Lambda RR}_{sab}B_{\beta' t}P_{ab;cd}S^{\tau',\Lambda' R'R'}_{tcd}\\
&=\frac{|H_\mu|\mathrm{Dim}R}{d_R^2}B_{\beta s}S^{\tau,\Lambda RR}_{sab}B_{\beta' t}S^{\tau',\Lambda' R'R'}_{tab}\\
&=\delta^{\Lambda\Lambda'}\delta^{RR'}\delta_{\beta\beta'}\delta_{\tau\tau'}\frac{|H_\mu|\mathrm{Dim}R}{d_R^2}.
\end{aligned}$$

In the second line we have used the fundamental orthogonality relation (see [9]) for the sums over α and α' viz.

$$\sum_{\alpha\in S_n}\left[\Gamma_R(\alpha)\right]_{mn}\left[\Gamma_T(\alpha)\right]_{ab}=\frac{n!}{d_R}\delta^{RT}\delta_{ma}\delta_{nb}.$$

Note that the above is for a real orthogonal representation. We have also used the definition of the projector in the second line

$$P_{ab;cd}=\frac{1}{|H_\mu|}\sum_{\gamma\in H_\mu}\left[\Gamma_T(\gamma)\right]_{bc}\left[\Gamma_T(\gamma^{-1})\right]_{da}$$

In the second last line we have contracted all the delta functions and in the final line we have made use of the orthogonality relations of branching coefficients and Clebsch-Gordan coefficients viz.

$$\sum_{a,b}S^{\tau,\Lambda RR}_{sab}S^{\tau',\Lambda' R'R'}_{tab}=\delta^{\Lambda\Lambda'}\delta^{RR'}\delta^{\tau\tau'}\delta_{st},$$

and

$$\sum_s B_{\beta s} B_{\beta' s} = \delta_{\beta\beta'}.$$

5.4 Counting the BHR Operators

Here we will elaborate upon the counting of the BHR operators in order to show that they have the same counting as gauge invariant operators. This is done in order to show that the BHR basis is complete.

Let us recall the definition of the BHR operators

$$\mathcal{O}^{\Lambda\mu,R}_{\beta,\tau} = \frac{1}{n!} \sum_{\alpha \in S_n} B_{\beta t} S^{\tau,\Lambda RR}_{tpq} \left[\Gamma_R(\alpha)\right]_{pq} \mathrm{Tr}(\alpha X_1^{\mu_1} \otimes X_2^{\mu_2} \otimes \ldots \otimes X_M^{\mu_M}).$$

The β index runs over 1 to $g(\mu;\Lambda)$ and the τ index runs over 1 to $C(R,R,T)$. $g(\mu;\Lambda)$ is the Littlewood-Richardson coefficient and coefficient $C(R,S,T)$ enumerates the occurence of T in the inner product of R and S. Thus the total number of BHR operators is given by

$$\sum_R \sum_\Lambda C(R,R,\Lambda) g(\mu;\Lambda).$$

However this also corresponds to the number of gauge invariant operators given by the Polya counting formula

$$\prod_{k=1}^{\infty} \frac{1}{1-(x_1^k + \cdots + x_M^k)} = \sum_\mu N(\mu_1,\mu_2,\cdots \mu_M) x_1^{\mu_1} x_2^{\mu_2} \cdots x_M^{\mu_M},$$

where $N(\mu_1,\mu_2,\cdots \mu_M)$ is the number of gauge invariant operators i.e.

$$N(\mu_1,\mu_2,\cdots \mu_M) = \sum_R \sum_\Lambda C(R,R,\Lambda) g(\mu;\Lambda).$$

There are numerous methods for obtaining the Littlewood-Richardson coefficients [9] e.g. through the multiplication of Young diagrams. As a brief digression we will show how one explicitly computes the coefficients

$C(R,S,T)$. For example consider the following two direct products of representations of S_2.

$$\yng(2) \otimes \yng(2) = \yng(2) \qquad \text{and} \qquad \yng(1,1) \otimes \yng(1,1) = \yng(2).$$

Clearly $C(\yng(2),\yng(2),\yng(2)) = 1$ and $C(\yng(1,1),\yng(1,1),\yng(2)) = 1$ – with any other coefficients being zero. We obtain the coefficients as follows. We use the fact that

$$\chi_{\yng(2)\otimes\yng(2)} = \chi_{\yng(2)} \times \chi_{\yng(2)} \qquad \text{and} \qquad \chi_{\yng(1,1)\otimes\yng(1,1)} = \chi_{\yng(1,1)} \times \chi_{\yng(1,1)},$$

which are derived from simple properties of the tensor product of matrices. Using the orthogonality relation of characters

$$\sum_{g\in G} \chi_R(g)\chi_S(g^{-1}) = \delta_{RS} n!, \tag{5.4.1}$$

and the fact that $\chi_{\yng(2)\otimes\yng(2)}$ is a linear sum of the characters of the irreducible representations contained in $\yng(2)\otimes\yng(2)$ i.e.

$$\chi_{\yng(2)\otimes\yng(2)} = \sum_\alpha C_\alpha \chi_\alpha.$$

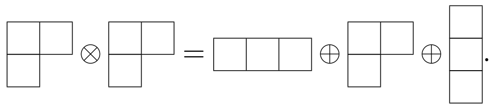
Substituting this into (5.4.1) we obtain the following

$$\sum_{g\in G}\sum_\alpha C_\alpha \chi_\alpha \chi_R(g^{-1}) = \delta_{R\alpha} n! C_\alpha = n! C_R.$$

Thus for instance to determine $\yng(2,1)\otimes\yng(2,1)$ we would need to calculate the coefficients $C_{\yng(3)}$, $C_{\yng(2,1)}$ and $C_{\yng(1,1,1)}$ e.g.

$$\begin{aligned} C_{\yng(2,1)} &= \frac{1}{3!}\sum_{g\in S_3}\left(\chi_{\yng(2,1)}(g)\right)^2 \chi_{\yng(2,1)}(g^{-1}) \\ &= \frac{1}{6}[8+0-2] \\ &= 1. \end{aligned}$$

The other coefficients are also found to be one. Thus we have that



The methods of [7] have recently been extended to non-compact groups [24]. This is needed if one wants to organize operators by exploiting the full $\mathrm{SU}(2,2|4)$ superconformal symmetry of $\mathcal{N}=4$ SYM. Further, one loop corrections to the BHR operators have been computed in [25]. For a recent review see [26].

6 The Restricted Schur Operators

6.1 Introduction

In this section we will be discussing the restricted Schur polynomials we studied in [8]. As with the BHR operators (Brown-Heslop-Ramgoolam see [7] and section 5 of this dissertation) the restricted Schur polynomials provide an orthonormal basis for operators in the Higgs sector of $\mathcal{N} = 4$ SYM.

Unlike the operators of Brown, Helsop and Ramgoolam in [7] and Kimura and Ramgoolam in [27], the restricted Schur polynomials allow us to deal with excited gravitons i.e. gravitons with strings attached [28], [16], [29], [30]. We will embelish upon this later.

Note that in this section, for the sake of readability, we will deviate from my previous notation. Where before we used

$$X_1,\ X_2,\ X_3,\ \cdots X_n, \tag{6.1.1}$$

to denote n different fields, we will be working predominantly with only two fields in this section. We will denote these fields by A and B respectively.

In general, we will be considering multi-matrix restricted Schur polynomials of the form

$$\chi_\alpha = \mathrm{Tr}(\mathcal{O}_\alpha A^{\otimes n} \otimes B^{\otimes m}). \tag{6.1.2}$$

In this case we are only concerned with two matrices but the extension to any number of matrices is trivial. The operator $\mathcal{O}_\alpha$ in (6.1.2) is given by

$$\mathcal{O}_\alpha = \frac{1}{n!m!} \sum_{\sigma \in S_n} \mathrm{Tr}_{R_\alpha} \left(\Gamma_R(\sigma)\right) \sigma. \tag{6.1.3}$$

This operator is what transforms the matrices/fields into a gauge invariant restricted Schur polynomial. As with all operators, hitherto, the permutation element σ acts on the matrices/fields as follows

$$\sigma A^{i_1}_{j_1} \otimes A^{i_2}_{j_2} \otimes \cdots \otimes A^{i_n}_{j_n} \otimes B^{i_{n+1}}_{j_{n+1}} \otimes B^{i_{n+2}}_{j_{n+2}} \otimes \cdots \otimes B^{i_{n+m}}_{j_{n+m}}$$
$$= A^{i_1}_{j_{\sigma(1)}} \otimes A^{i_2}_{j_{\sigma(2)}} \otimes \cdots \otimes A^{i_n}_{j_{\sigma(n)}} \otimes B^{i_{n+1}}_{j_{\sigma(n+1)}} \otimes B^{i_{n+2}}_{j_{\sigma(n+2)}} \otimes \cdots \otimes B^{i_{n+m}}_{j_{\sigma(n+m)}},$$

i.e. it permutes the lower U(N) indices of the fields.

As with the BHR operators in section 5 We will show in the subsequent subsections that the restricted Schur polynomials (or their associated projection operators $\mathcal{O}_\alpha$) indeed satisfy the following

1. The operators $\mathcal{O}_\alpha$ are in fact projection operators.
2. The restricted Schur polynomials are orthogonal.
3. The operators $\mathcal{O}_\alpha$ avoid overcounting of restricted Schur polynomials.
4. The restricted Schur polynomials have the same counting as the gauge invariant multi-trace operators.
5. We will show that restricted Schur polynomials can be trivially extended to multi-matrix models.

We will follow this with

- We will elucidate any relationships between our operators and those of Brown, Heslop and Ramgoolam [7]. (This is presented in section 7).
- Finally we elaborate upon the numerical computations we did to check the result for the correlation function.

6.2 The Restricted Schurs as Projectors

In this section we will show that the operators $\mathcal{O}_\alpha$ (defined in equation (6.1.3)) are indeed projection operators. That is, given the two operators

$$\mathcal{O}_\alpha = \frac{1}{n!m!} \sum_{\sigma \in S_{m+n}} \mathrm{Tr}_{R_\alpha} \left(\Gamma_R(\sigma) \right) \sigma,$$

and

$$\mathcal{O}_\beta = \frac{1}{n!m!} \sum_{\tau \in S_{m+n}} \mathrm{Tr}_{S_\beta} \left(\Gamma_S(\tau) \right) \tau,$$

we wish to show that

$$\mathcal{O}_\alpha \mathcal{O}_\beta \propto \delta_{\alpha\beta} \mathcal{O}_\beta.$$

Proceeding accordingly we have

$$\begin{aligned} \mathcal{O}_\alpha \mathcal{O}_\beta &= \left(\frac{1}{n!m!} \right)^2 \sum_{\sigma \in S_{n+m}} \sum_{\tau \in S_{n+m}} \mathrm{Tr}_{R_\alpha} \left(\Gamma_R(\sigma) \right) \mathrm{Tr}_{S_\beta} \left(\Gamma_S(\tau) \right) \sigma\tau \\ &= \left(\frac{1}{n!m!} \right)^2 \sum_{\psi \in S_{n+m}} \sum_{\tau \in S_{n+m}} \mathrm{Tr}_{R_\alpha} \left(\Gamma_R(\psi\tau^{-1}) \right) \mathrm{Tr}_{S_\beta} \left(\Gamma_S(\tau) \right) \psi. \end{aligned} \tag{6.2.1}$$

Let us now just perform the sum over τ

$$\begin{aligned} &\sum_{\tau \in S_{m+n}} \mathrm{Tr}_{R_\alpha} \left(\Gamma_R(\psi\tau^{-1}) \right) \mathrm{Tr}_{S_\beta} \left(\Gamma_S(\tau) \right) \\ &= \sum_{\tau \in S_{m+n}} \mathrm{Tr}_{R_\alpha} \left(\Gamma_R(\psi)\Gamma_R(\tau^{-1}) \right) \mathrm{Tr}_{S_\beta} \left(\Gamma_S(\tau) \right) \\ &= \sum_{\tau \in S_{m+n}} \mathrm{Tr}(P_{R \to R_\alpha}(\Gamma_R(\psi)\Gamma_R(\tau^{-1}))) \mathrm{Tr}(P_{S \to S_\beta}\Gamma_S(\tau)) \\ &= \sum_{\tau \in S_{m+n}} [P_{R \to R_\alpha}]_{ij} [\Gamma_R(\psi)]_{jk} [\Gamma_R(\tau^{-1})]_{ki} [P_{S \to S_\beta}]_{mn} [\Gamma_S(\tau)]_{nm} \\ &= \sum_{\tau \in S_{m+n}} [P_{R \to R_\alpha}\Gamma_R(\psi)]_{ik} [\Gamma_R(\tau^{-1})]_{ki} [P_{S \to S_\beta}]_{mn} [\Gamma_S(\tau)]_{nm}. \end{aligned} \tag{6.2.2}$$

Here we utilise the fundamental orthogonality relation (see [9]) in representation theory which states that

$$\sum_{\tau \in S_{m+n}} [\Gamma_R(\tau^{-1})]_{ki} [\Gamma_S(\tau)]_{mn} = \frac{(n+m)!}{d_R} \delta_{km} \delta_{in} \delta_{RS}. \tag{6.2.3}$$

Thus equation (6.2.2) becomes

$$\begin{aligned} &\frac{(n+m)!}{d_R} [P_{R\to R_\alpha} \Gamma_R(\psi)]_{ik} [P_{S\to S_\beta}]_{mn} \delta_{km} \delta_{in} \delta_{RS} \\ &= \frac{(n+m)!}{d_R} [P_{R\to R_\alpha} \Gamma_R(\psi)]_{im} [P_{S\to S_\beta}]_{mi} \delta_{RS} \\ &= \frac{(n+m)!}{d_R} \mathrm{Tr}_{R_\alpha}(\Gamma_R(\psi)) \delta_{RS} \delta_{R_\alpha S_\beta} \end{aligned} \tag{6.2.4}$$

If we substitute this back into the equation for $\mathcal{O}_\alpha \mathcal{O}_\beta$ (equation (6.2.1)) we obtain

$$\begin{aligned} O_\alpha O_\beta &= \left(\frac{1}{n!m!}\right)^2 \sum_{\psi \in S_{m+n}} \mathrm{Tr}_{R_\alpha}(\Gamma_R(\psi)) \frac{(n+m)!}{d_R} \delta_{RS} \delta_{R_\alpha S_\beta} \\ &= \frac{(m+n)!}{n!m!d_R} \underbrace{\delta_{RS} \delta_{R_\alpha S_\beta}}_{\delta_{\alpha\beta}} \underbrace{\frac{1}{n!m!} \sum_{\psi \in S_{m+n} \mathrm{Tr}_{R_\alpha}} (\Gamma_R(\psi))\, \psi}_{\mathcal{O}_\alpha}. \end{aligned} \tag{6.2.5}$$

Thus we have shown that the operator $\mathcal{O}_\alpha$ is indeed a projection operator i.e.

$$\mathcal{O}_\alpha \mathcal{O}_\beta \propto \delta_{\alpha\beta} \mathcal{O}_\beta.$$

6.3 The Orthogonality of the Restricted Schur Polynomials

In this section we will show that two restricted Schur polynomials are in fact orthonormal. Since the Schur polynomials map between operators and states – and one cannot have a brane in two different orthonormal states simultaneously – the Schur polynomials must be orthogonal.

Firstly, let us restate the following important identity we will be using.

$$\langle (A)^i_j (A^\dagger)^k_l \rangle = \delta^i_l \delta^k_j = \langle (B)^i_j (B^\dagger)^k_l \rangle\,. \tag{6.3.1}$$

A and B are in fact of the following form

$$A = \frac{1}{\sqrt{2}}(M_1 + iM_2)$$
$$B = \frac{1}{\sqrt{2}}(M_3 + iM_4),$$

where M_c for $c = 1, 2, 3, 4$ are all hermitian matrices.

We can see that (6.3.1) in fact comes from the two-point function of the free multi-matrix model which is given by

$$\begin{aligned}\langle (M_a)^i_j (M_b)^k_l \rangle &= \frac{d}{d(J_a)^j_i} \frac{d}{d(J_b)^l_k} I(J)\big|_{J_a=0} \\ &= \delta_{ab}\delta^i_l \delta^k_j,\end{aligned}$$

where a and b range from 1 to 4 and $I(J)$ is given by

$$I(J) = \int_{-\infty}^{\infty} [dM_a] e^{-\alpha \mathrm{Tr}(M_a^2) + \mathrm{Tr}(J_a M_a)}.$$

Given the two restricted Schur polynomials

$$\begin{aligned}\chi_\alpha &= \mathrm{Tr}(O_\alpha A \otimes A) \\ &= (O_\alpha)^{j_1 j_2}_{i_1 i_2} A^{i_1}_{j_1} A^{i_2}_{j_2},\end{aligned}$$

and

$$\begin{aligned}\chi_\beta &= \mathrm{Tr}(O_\beta A \otimes A) \\ &= (O_\beta)^{l_1 l_2}_{k_1 k_2} A^{k_1}_{k_1} A^{l_2}_{l_2},\end{aligned}$$

we will show that the two point function is diagonalised as follows

$$\langle \chi_\alpha \chi^\dagger_\beta \rangle = \delta_{\alpha\beta} \frac{(\mathrm{hooks})_R}{(\mathrm{hooks})_{R_\alpha}} f_R. \qquad (6.3.2)$$

To prove this relationship we will first need to give a derivation of the following identity

$$\langle \chi_\alpha \chi_\beta^\dagger \rangle = \sum_{\gamma \in S_n \times S_m} \mathrm{Tr}(O_\alpha \gamma O_\beta^\dagger \gamma^{-1}). \tag{6.3.3}$$

Let us first consider an illustrative, trivial example. Let us suppose there are only two copies of the field A in our operators and no B's i.e. our two operators are

$$\begin{aligned} \chi_\alpha &= \mathrm{Tr}(O_\alpha A \otimes A) \\ &= (O_\alpha)^{j_1 j_2}_{i_1 i_2} A^{i_1}_{j_1} A^{i_2}_{j_2}, \end{aligned}$$

and

$$\begin{aligned} \chi_\beta &= \mathrm{Tr}(O_\beta A \otimes A) \\ &= (O_\beta)^{l_1 l_2}_{k_1 k_2} A^{k_1}_{k_1} A^{l_2}_{l_2}. \end{aligned} \tag{6.3.4}$$

Thus our correlator becomes

$$\begin{aligned} \langle \chi_\alpha \chi_\beta^\dagger \rangle &= \langle (O_\alpha)^{j_1 j_2}_{i_1 i_2} A^{i_1}_{j_1} A^{i_2}_{j_2} (O_\beta^\dagger)^{l_1 l_2}_{k_1 k_2} (A^\dagger)^{k_1}_{l_1} (A^\dagger)^{k_2}_{l_2} \rangle \\ &= (O_\alpha)^{j_1 j_2}_{i_1 i_2} (O_\beta^\dagger)^{l_1 l_2}_{k_1 k_2} \langle A^{i_1}_{j_1} A^{i_2}_{j_2} (A^\dagger)^{k_1}_{l_1} (A^\dagger)^{k_2}_{l_2} \rangle \\ &= (O_\alpha)^{j_1 j_2}_{i_1 i_2} (O_\beta^\dagger)^{l_1 l_2}_{k_1 k_2} \left[\delta^{i_1}_{l_1} \delta^{i_2}_{l_2} \delta^{k_1}_{j_1} \delta^{k_2}_{j_2} + \delta^{i_1}_{l_2} \delta^{i_2}_{l_1} \delta^{k_2}_{j_1} \delta^{k_1}_{j_2} \right] \end{aligned}$$

The last line comes from the fact that there are two ways of combining the two different A's with the two different $A^\dagger$'s. The last line simplifies to

$$\begin{aligned} \langle \chi_\alpha \chi_\beta^\dagger \rangle &= (O_\alpha)^{j_1 j_2}_{i_1 i_2} (O_\beta^\dagger)^{l_1 l_2}_{k_1 k_2} \sum_{\sigma \in S_2} \delta^{i_1}_{l_{\sigma(1)}} \delta^{i_2}_{l_{\sigma(2)}} \delta^{k_{\sigma(1)}}_{j_1} \delta^{k_{\sigma(2)}}_{j_2} \\ &= (O_\alpha)^{j_1 j_2}_{i_1 i_2} (O_\beta^\dagger)^{l_1 l_2}_{k_1 k_2} \sum_{\sigma \in S_2} \delta^{i_1}_{l_{\sigma(1)}} \delta^{i_2}_{l_{\sigma(2)}} \delta^{k_1}_{j_{\sigma^{-1}(1)}} \delta^{k_2}_{j_{\sigma^{-1}(2)}} \\ &= \sum_{\sigma \in S_2} \mathrm{Tr}(O_\alpha \sigma O_\beta^\dagger \sigma^{-1}). \end{aligned} \tag{6.3.5}$$

The last two lines deserve some explanation. Obviously since we are contracting the indices in the second last line this gives rise to the trace in the last line. The fact that σ^{-1} acts on the lower indices of the operators (and not σ) can be deduced from theorem 1 in appendix B.

We can expand this to multiple A fields so that in general we have

$$\langle \chi_\alpha \chi_\beta^\dagger \rangle = \mathrm{Tr}(O_\alpha \sigma O_\beta^\dagger \sigma^{-1}) \tag{6.3.6}$$

where $\chi_\alpha = \mathrm{Tr}(O_\alpha A^{\otimes n})$ and $\chi_\beta = \mathrm{Tr}(O_\beta A^{\otimes n})$.

We proceed as follows

$$\begin{aligned}\langle \chi_\alpha \chi_\beta^\dagger \rangle &= \langle (O_\alpha)^{j_1 j_2 \cdots j_n}_{i_1 i_2 \cdots i_n} A^{i_1}_{j_1} \cdots A^{i_n}_{j_n} (O_\beta^\dagger)^{l_1 l_2 \cdots l_n}_{k_1 k_2 \cdots k_n} (A^\dagger)^{k_1}_{l_1} (A^\dagger)^{k_2}_{l_2} \cdots (A^\dagger)^{k_n}_{l_n} \rangle \\ &= (O_\alpha)^{j_1 j_2 \cdots j_n}_{i_1 i_2 \cdots i_n} (O_\beta^\dagger)^{l_1 l_2 \cdots l_n}_{k_1 k_2 \cdots k_n} \langle A^{i_1}_{j_1} A^{i_2}_{j_2} \cdots A^{i_n}_{j_n} (A^\dagger)^{k_1}_{l_1} (A^\dagger)^{k_2}_{l_2} \cdots (A^\dagger)^{k_n}_{l_n} \rangle\end{aligned} \tag{6.3.7}$$

To evaluate this we use the fact that there are $n!$ ways of combining the n A's with the n $A^\dagger$'s (since the first A can be combined with any of the n $A^\dagger$'s, the second A can be combined with any of the $n-1$ remaining $A^\dagger$'s, $\cdots$.) Thus these combinations are encapsulated by the permutation group S_n. Therefore (6.3.7) can be simplified as

$$\begin{aligned}\langle \chi_\alpha \chi_\beta^\dagger \rangle &= (\mathcal{O}_\alpha)^{j_1 \cdots j_n}_{i_1 \cdots i_n} (\mathcal{O}_\beta^\dagger)^{l_1 \cdots l_n}_{k_1 \cdots k_n} \sum_{\sigma \in S_n} \delta^{i_1}_{l_{\sigma(1)}} \cdots \delta^{i_n}_{l_{\sigma(n)}} \delta^{k_{\sigma(1)}}_{j_1} \cdots \delta^{k_{\sigma(n)}}_{j_n}, \\ &= (\mathcal{O}_\alpha)^{j_1 \cdots j_n}_{i_1 \cdots i_n} (\mathcal{O}_\beta^\dagger)^{l_1 \cdots l_n}_{k_1 \cdots k_n} \sum_{\sigma \in S_n} \delta^{i_1}_{l_{\sigma(1)}} \cdots \delta^{i_n}_{l_{\sigma(n)}} \delta^{k_1}_{j_{\sigma^{-1}(1)}} \cdots \delta^{k_n}_{j_{\sigma^{-1}(n)}}, \\ &= \sum_{\sigma \in S_n} \mathrm{Tr}(\mathcal{O}_\alpha \sigma \mathcal{O}_\beta^\dagger \sigma^{-1}).\end{aligned} \tag{6.3.8}$$

where we have again used theorem 1 in appendix B to turn the σ acting on the upper indices into a σ^{-1} acting on the lower indices. Thus what we have done is perform all possible Wick contractions.

To extend the result for operators containing B fields follows naturally. We know from equation (6.3.1) that the two point function of the A and

B fields is zero so the extension is quite trivial – we just perform the Wick contractions over the A and B fields separately. Thus we would have that

$$\langle \chi_\alpha \chi_\beta^\dagger \rangle = \sum_{\sigma \in S_m \times S_n} \mathrm{Tr}(\mathcal{O}_\alpha \sigma \mathcal{O}_\beta \sigma^{-1}).$$

Now we will explicitly evaluate

$$\begin{aligned} \langle \chi_\alpha \chi_\beta^\dagger \rangle &= \sum_{\gamma \in S_m \times S_n} \mathrm{Tr}(O_\alpha \gamma O_\beta^\dagger \gamma^{-1}) \\ &= \sum_{\gamma \in S_m \times S_n} \mathrm{Tr}(O_\alpha O_\beta^\dagger) \\ &= m!n! \mathrm{Tr}(O_\alpha O_\beta^\dagger). \end{aligned} \tag{6.3.9}$$

In the succeeding subsection we will show that $\sigma \mathcal{O}_\beta \sigma^{-1} = \mathcal{O}_\beta$ validating the above. Note that the factor of $m!n!$ comes from summing over $\gamma \in S_m \times S_n$ since this is the order of the group $S_m \times S_n$.

Evaluating (6.3.9) is very similar to how we proved that $\mathcal{O}_\alpha$ was a projector in subsection 6.2. We proceed as follows.

$$\begin{aligned} m!n! \mathrm{Tr}(\mathcal{O}_\alpha \mathcal{O}_\beta^\dagger) &= \frac{1}{m!n!} \mathrm{Tr} \left[\sum_{\sigma, \tau \in S_{m+n}} \mathrm{Tr}_{R_\alpha} \left(\Gamma_R(\sigma) \right) \mathrm{Tr}_{S_\beta} \left(\Gamma_S(\tau) \right)^* \sigma \tau^{-1} \right] \\ &= \frac{1}{m!n!} \sum_{\psi, \tau \in S_{m+n}} \mathrm{Tr}_{R_\alpha} \left(\Gamma_R(\psi\tau) \right) \mathrm{Tr}_{S_\beta} \left(\Gamma_S(\tau) \right)^* \mathrm{Tr}(\psi) \\ &= \frac{1}{m!n!} \sum_{\psi, \tau \in S_{m+n}} \mathrm{Tr}_{R_\alpha} \left(\Gamma_R(\psi\tau) \right) \mathrm{Tr}_{S_\beta} \left(\Gamma_S(\tau) \right)^* N^{C(\psi)}. \end{aligned} \tag{6.3.10}$$

In the second line of the above we used the substitution $\psi = \sigma \tau^{-1}$ and in the last line we used theorem 2 from appendix B. Next we just perform the sum over τ

$$
\begin{aligned}
&\sum_{\tau \in S_{m+n}} \mathrm{Tr}_{R_\alpha}\left(\Gamma_R(\psi\tau)\right) \mathrm{Tr}_{S_\beta}\left(\Gamma_S(\tau)\right)^* \\
&= \sum_{\tau \in S_{m+n}} \mathrm{Tr}_{R_\alpha}\left(\Gamma_R(\psi)\Gamma_R(\tau)\right) \mathrm{Tr}_{S_\beta}\left(\Gamma_S(\tau)\right)^* \\
&= \sum_{\tau \in S_{m+n}} \mathrm{Tr}(P_{R\to R_\alpha}(\Gamma_R(\psi)\Gamma_R(\tau)))\mathrm{Tr}(P_{S\to S_\beta}\Gamma_S(\tau))^* \\
&= \sum_{\tau \in S_{m+n}} [P_{R\to R_\alpha}]_{ij}[\Gamma_R(\psi)]_{jk}[\Gamma_R(\tau)]_{ki}[P_{S\to S_\beta}]^*_{mn}[\Gamma_S(\tau)]^*_{nm} \\
&= \sum_{\tau \in S_{m+n}} [P_{R\to R_\alpha}\Gamma_R(\psi)]_{ik}[\Gamma_R(\tau)]_{ki}[P_{S\to S_\beta}]_{nm}[\Gamma_S(\tau)]^*_{nm}. \qquad (6.3.11)
\end{aligned}
$$

In the above we have used the fact that $[P_{S\to S_\beta}]^*_{mn} = [P_{S\to S_\beta}]_{nm}$. Yet again we employ the fundamental orthogonality relation from representation theory (see [9]) which states that

$$
\sum_{\tau \in S_{m+n}} [\Gamma_R(\tau)]_{ki}\,[\Gamma_R(\tau)]^*_{nm} = \frac{(m+n)!}{d_R}\delta_{kn}\delta_{im}\delta_{RS}.
$$

Substituting this into (6.3.11) yields the following

$$
\begin{aligned}
&\delta_{RS}\frac{(m+n)!}{d_R}[P_{R\to R_\alpha}\Gamma_R(\psi)]_{in}[P_{S\to S_\beta}]_{ni} \\
&= \delta_{RS}\delta_{R_\alpha S_\beta}\frac{(m+n)!}{d_R}\mathrm{Tr}(P_{R\to R_\alpha}\Gamma_R(\psi)) \\
&= \delta_{RS}\delta_{R_\alpha S_\beta}\frac{(m+n)!}{d_R}\mathrm{Tr}_{R_\alpha}(\Gamma_R(\psi)). \qquad (6.3.12)
\end{aligned}
$$

Substituting the above into (6.3.10) yields the following

$$
\begin{aligned}
m!n!\mathrm{Tr}(\mathcal{O}_\alpha\mathcal{O}^\dagger_\beta) &= \delta_{RS}\delta_{R_\alpha S_\beta}\frac{1}{m!n!}\frac{(m+n)!}{d_R}\sum_{\psi \in S_{m+n}}\mathrm{Tr}_{R_\alpha}(\Gamma_R(\psi))N^{C(\psi)} \\
&= \delta_{RS}\delta_{R_\alpha S_\beta}\frac{d_{R_\alpha}}{m!n!}\frac{(m+n)!}{d_R}f_R \\
&= \delta_{RS}\delta_{R_\alpha S_\beta}\frac{(\mathrm{hooks})_R}{(\mathrm{hooks})_{R_\alpha}}f_R. \qquad (6.3.13)
\end{aligned}
$$

In the above we have made use of the fact that the dimension of a particular representation K of S_n is given by

$$d_K = \frac{n!}{(\text{hooks})_K}.$$

See appendix A for the definition of *hooks*. We have also made use of the identity

$$\sum_{\psi \in S_{m+n}} \text{Tr}_{R_\alpha}(\Gamma_R(\psi)) N^{C(\psi)} = f_R,$$

from appendix F of [16]. See appendix A for the definition of f_R in terms of the Young diagram labeling the representation R.

Thus we have shown that the two point function for two restricted Schur polynomials is in fact orthogonal i.e.

$$\langle \chi_\alpha \chi_\beta^\dagger \rangle = \delta_{\alpha\beta} \frac{(\text{hooks})_R}{(\text{hooks})_{R_\alpha}} f_R. \tag{6.3.14}$$

6.4 Elimination of Overcounting in Restricted Schur Polynomials

Here we will briefly show that the operators $\mathcal{O}_\alpha$ avoid any overcounting of the restricted Schur polynomials. When considering gauge invariant operators we have that

$$\text{Tr}(\sigma A^{\otimes n} \otimes B^{\otimes m}) = \text{Tr}(\gamma\sigma\gamma^{-1} A^{\otimes n} \otimes B^{\otimes m}), \tag{6.4.1}$$

where $\sigma \in S_{m+n}$ and $\gamma \in S_m \times S_n$. Acting with the γ and γ^{-1} in the indicated manner is equivalent to just swapping the A fields amongst themselves and the B fields amoungst themselves without mixing them thus not affecting the overall interpretation (see the section on the BHR operators for a more elaborate discussion of this). Thus we don't want the $\mathcal{O}_\alpha$ operators to count the above two operators as two different operators when the construct the restricted Schur polynomials i.e. we want

$$\gamma \mathcal{O}_\alpha \gamma^{-1} = \mathcal{O}_\alpha, \tag{6.4.2}$$

since the restricted Schur polynomials are given by

$$\chi_\alpha = \mathrm{Tr}(\mathcal{O}_\alpha A^{\otimes m} \otimes B^{\otimes n}).$$

By definition we have that

$$\begin{aligned}
\gamma\mathcal{O}_\alpha\gamma^{-1} &= \frac{1}{m!n!}\sum_{\sigma\in S_{m+n}} \mathrm{Tr}_{R_\alpha}\left(\Gamma_R(\sigma)\right)\gamma\sigma\gamma^{-1} \\
&= \frac{1}{m!n!}\sum_{\psi\in S_{m+n}} \mathrm{Tr}_{R_\alpha}\left(\Gamma_R(\gamma^{-1}\psi\gamma)\right)\psi \\
&= \frac{1}{m!n!}\sum_{\psi\in S_{m+n}} \mathrm{Tr}_{R_\alpha}\left(\Gamma_R(\gamma^{-1})\Gamma_R(\psi)\Gamma_R(\gamma)\right)\psi \\
&= \frac{1}{m!n!}\sum_{\psi\in S_{m+n}} \mathrm{Tr}_{R_\alpha}\left(\Gamma_{R_\alpha}(\gamma^{-1})\Gamma_R(\psi)\Gamma_{R_\alpha}(\gamma)\right)\psi \\
&= \frac{1}{m!n!}\sum_{\psi\in S_{m+n}} \mathrm{Tr}\left(\Gamma_{R_\alpha}(\gamma^{-1})[P_{R\to R_\alpha}\Gamma_R(\psi)]\Gamma_{R_\alpha}(\gamma)\right)\psi \\
&= \frac{1}{m!n!}\sum_{\psi\in S_{m+n}} \mathrm{Tr}_{R_\alpha}\left(\Gamma_R(\psi)\right)\psi.
\end{aligned}$$

In the third last line we have used the fact that $\gamma \in S_m \times S_n$ and since R_α is an irreducible representation of $S_m \times S_n$ we have that $\Gamma_R(\gamma) = \Gamma_{R_\alpha}(\gamma)$. In the last line we have used the fact that the trace is invariant under a similarity transformation.

Thus the projection operators $\mathcal{O}_\alpha$ do eliminate overcounting when used to construct the restricted Schur polynomials. Acting with $\gamma^{-1}\sigma\gamma$ on the fields is equivalent to swapping the fields amongst themselves.

6.5 Counting the Restricted Schur Polynomials

In this section we will discuss the $N = \infty$ counting of the restricted Schur operators. In the following section we will consider finite N counting of the restricted Schur operators.

The number of ways of building a gauge invariant operator out of n copies of some field X corresponds to the number of ways of partitioning the integer n. The generating function for the number of ways of partitioning n (or equivalently the number of possible gauge invariant fields built from n copies of X) is given by the following Polya counting function

$$P(x) = \prod_{n=1}^{\infty} \left(\frac{1}{1 - x^n} \right). \tag{6.5.1}$$

we can naturally extend this generating function to two fields/matrices as follows

$$N(x, y) = \prod_{n=1}^{\infty} \left(\frac{1}{1 - x^n - y^n} \right), \tag{6.5.2}$$

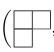
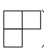
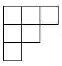
which counts the number of ways of building gauge invariant operators from p copies of x and q copies of y. The Polya counting function of multi-matrices is just a natural extension of this (see equation (6.5.4)). We have expanded (6.5.2) to order 6 in both x and y in equation (6.5.3) below.

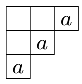
$$\begin{aligned}
&\left(1 + y + 2y^2 + 3y^3 + 5y^4 + 7y^5 + 11y^6\right) + \\
&\left(1 + 2y + 4y^2 + 7y^3 + 12y^4 + 19y^5 + 30y^6\right) x + \\
&\left(2 + 4y + 10y^2 + 18y^3 + 34y^4 + 56y^5 + 94y^6\right) x^2 + \\
&\left(3 + 7y + 18y^2 + 38y^3 + 74y^4 + 133y^5 + 233y^6\right) x^3 + \\
&\left(5 + 12y + 34y^2 + 74y^3 + 158y^4 + 297y^5 + 550y^6\right) x^4 + \\
&\left(7 + 19y + 56y^2 + 133y^3 + 297y^4 + 602y^5 + 1166y^6\right) x^5 + \\
&\left(11 + 30y + 94y^2 + 233y^3 + 550y^4 + 1160y^5 + 2382y^6\right) x^6.
\end{aligned} \tag{6.5.3}$$

So for example let us consider the coefficient of $x^3 y$ which is 7. Then this means there are seven ways of building gauge invariant operators from the fields $XXXY$. We listed these possibilities below.

$$\text{Tr}(XXXY) \quad \text{Tr}(XXX)\text{Tr}(Y) \quad \text{Tr}(XX)\text{Tr}(XY)$$
$$\text{Tr}(XX)\text{Tr}(X)\text{Tr}(Y) \quad \text{Tr}(X)\text{Tr}(XXY) \quad \text{Tr}(X)\text{Tr}(X)\text{Tr}(XY)$$
$$\text{Tr}(X)\text{Tr}(X)\text{Tr}(X)\text{Tr}(Y)$$

Now the problem that we face is checking whether the restricted Schur operators have the same required counting as these gauge invariant operators. In tables 6.1, 6.2 and 6.3 we have listed the all the possible subductions for $S_4 = S_2 \times S_2$, $S_5 = S_3 \times S_2$ and $S_6 = S_3 \times S_3$ respectively. As can be seen the number of subductions in table 6.1 is 10, the number of subductions in table 6.2 is 18 and in table 6.3 its 36. These match the coefficients of x^2y^2 and x^3y^2 in (6.5.3) respectively. Thus, in these instances, the number of subductions and thus the number of restricted Schur operators do in fact agree with the number of gauge invariant operators. However the coefficient for x^3y^3 in (6.5.3) is 38 and the number of subductions is 36 in table 6.3. The discrepancy comes from the fact that we can have 2 *twisted states* [16] accounting for these apparently missing operators. We will elaborate upon this shortly. We have checked that the counting does concur for all possible subductions for all permutation groups less than or equal to S_6.

In table 6.3 a subtlety emerges. Note that there are two copies of the subduction (,) for . We have labelled these distinct subductions as R_1 and R_2 respectively. These two different subductions are obtained as follows. If we remove the 3 boxes labelled with a's in the Young diagram below



then we can produce new, legal Young diagrams from the removed boxes as follows.

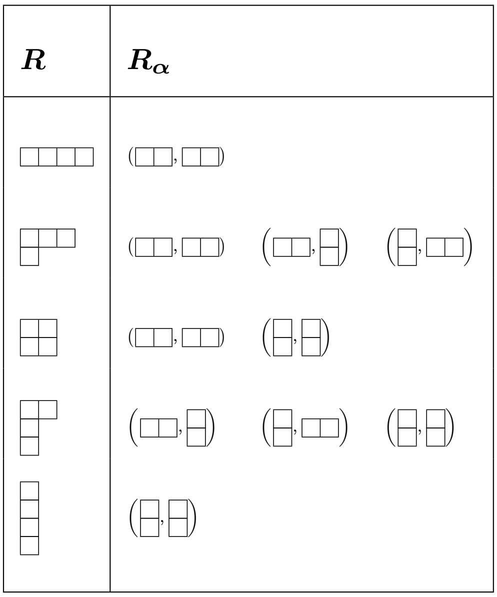


Table 6.1: A complete list of $S_4 = S_2 \times S_2$ subductions.

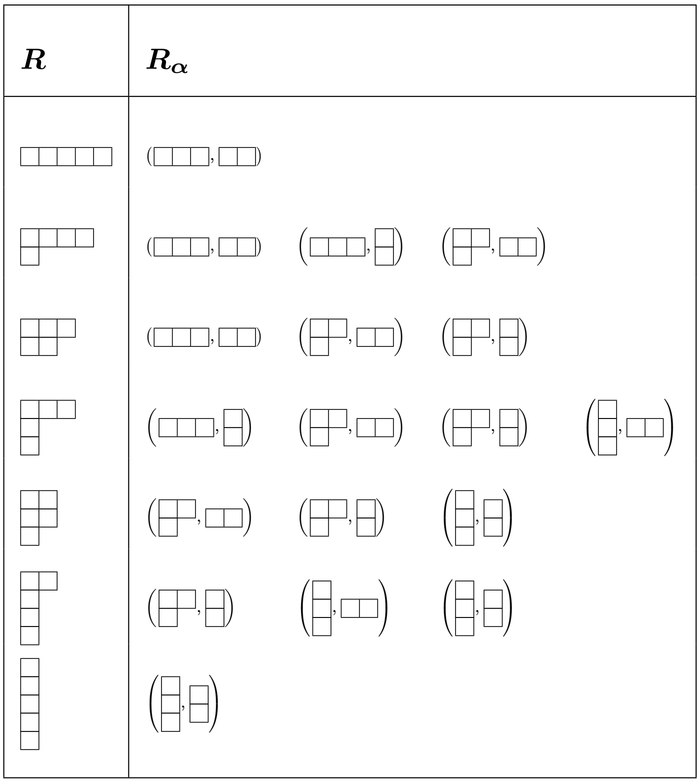
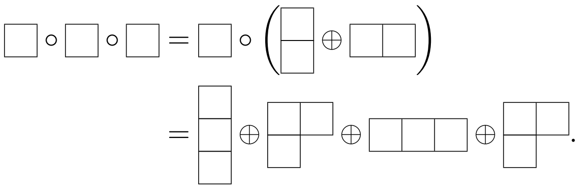


Table 6.2: A complete list of $S_5 = S_3 \times S_2$ subductions.

$\boldsymbol{R} \in \mathrm{Rep}(S_6)$	$\boldsymbol{R_\alpha} \in \mathrm{Rep}(S_3 \times S_3)$
$\yng(6)$	$\left(\yng(3),\yng(3)\right)$
$\yng(5,1)$	$\left(\yng(3),\yng(3)\right)$ $\left(\yng(3),\yng(2,1)\right)$ $\left(\yng(2,1),\yng(3)\right)$
$\yng(4,2)$	$\left(\yng(3),\yng(3)\right)$ $\left(\yng(3),\yng(2,1)\right)$ $\left(\yng(2,1),\yng(3)\right)$ $\left(\yng(2,1),\yng(2,1)\right)$
$\yng(4,1,1)$	$\left(\yng(3),\yng(1,1,1)\right)$ $\left(\yng(3),\yng(2,1)\right)$ $\left(\yng(2,1),\yng(3)\right)$ $\left(\yng(2,1),\yng(2,1)\right)$ $\left(\yng(1,1,1),\yng(3)\right)$
$\yng(3,3)$	$\left(\yng(3),\yng(3)\right)$ $\left(\yng(2,1),\yng(2,1)\right)$
$\yng(3,2,1)$	$\left(\yng(3),\yng(2,1)\right)$ $\left(\yng(2,1),\yng(3)\right)$ $\underbrace{\left(\yng(2,1),\yng(2,1)\right)}_{R_1}$ $\underbrace{\left(\yng(2,1),\yng(2,1)\right)}_{R_2}$ $\left(\yng(2,1),\yng(1,1,1)\right)$ $\left(\yng(1,1,1),\yng(2,1)\right)$
$\yng(2,2,2)$	$\left(\yng(2,1),\yng(2,1)\right)$ $\left(\yng(1,1,1),\yng(1,1,1)\right)$
$\yng(3,1,1,1)$	$\left(\yng(3),\yng(1,1,1)\right)$ $\left(\yng(2,1),\yng(2,1)\right)$ $\left(\yng(2,1),\yng(1,1,1)\right)$ $\left(\yng(1,1,1),\yng(2,1)\right)$ $\left(\yng(1,1,1),\yng(3)\right)$
$\yng(2,2,1,1)$	$\left(\yng(2,1),\yng(2,1)\right)$ $\left(\yng(2,1),\yng(1,1,1)\right)$ $\left(\yng(1,1,1),\yng(2,1)\right)$ $\left(\yng(1,1,1),\yng(1,1,1)\right)$
$\yng(2,1,1,1,1)$	$\left(\yng(2,1),\yng(1,1,1)\right)$ $\left(\yng(1,1,1),\yng(2,1)\right)$ $\left(\yng(1,1,1),\yng(1,1,1)\right)$
$\yng(1,1,1,1,1,1)$	$\left(\yng(1,1,1),\yng(1,1,1)\right)$

Table 6.3: A complete list of $S_6 = S_3 \times S_3$ subductions.



Note that this produces two distinct copies of $\yng(2,1)$. When we take the restricted trace of $\Gamma_{\yng(3,2,1)}$ we are in fact only tracing over a segment of the matrix. We have depicted some of the ways of taking the restricted trace of the S_6 representation in question in figure 3. As can be seen in figure 3, the two equivalent representations R_1 and R_2 span a matrix consisting of 2×2 blocks. We can take the trace of any one of these four blocks (labeled $R_1 \otimes R_1$, $R_1 \otimes R_2$, $R_2 \otimes R_1$ and $R_2 \otimes R_2$ respectively.) Thus there are four possible restricted Schur operators we can create. And thus, inductively, it is clear that if we have m copies of some subduction R_α then they will span $m \times m$ block matrices which would mean we could construct m^2 independent operators from them.

Note that the trace taken over the off-diagonal blocks (i.e. $R_1 \otimes R_2$ and $R_2 \otimes R_1$) correspond to the *twisted states* we mentioned earlier. We use *intertwiners* to describe them (see [16] and [30]). These twisted states correspond to two strings stretching between two different branes.There are two twisted states since there is an orientation associated with open strings.

Extending the counting of operators to more than two fields is trivial. For M distinct fields the generating function for the number of unique gauge invariant operators is given by

$$N(X_1, X_2, \cdots, X_M) = \prod_{n=1}^{\infty} \left(\frac{1}{1 - X_1^n - X_2^n - \cdots - X_M^n} \right). \tag{6.5.4}$$

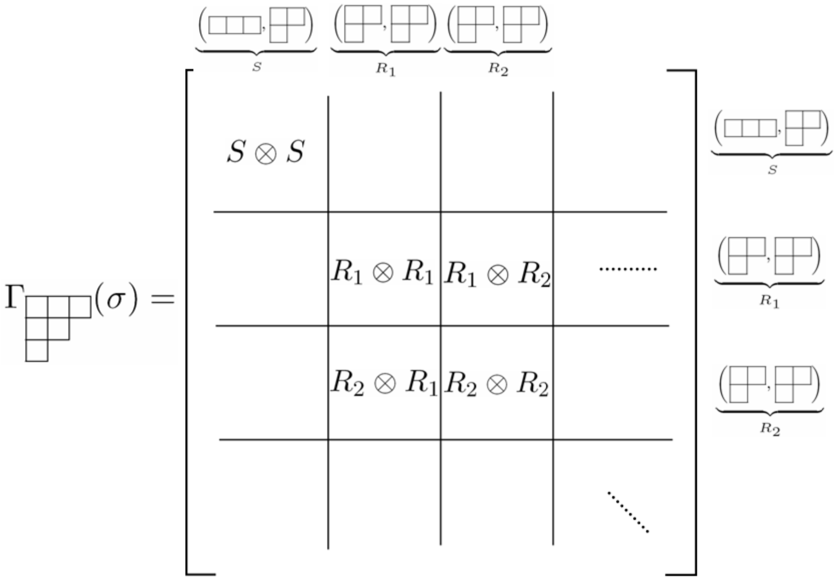


Figure 3: The above figure represents the indicated S_6 representation which has been diagonalised into its various possible subductions – note that we have only used 3 of the possible subductions, it should in fact be a 16×16 matrix subduced into 6 possible subductions (see table 6.3). When computing the restricted trace $\mathrm{Tr}_{R_\alpha}(\Gamma_R(\sigma))$ it corresponds to tracing over only the diagonal block of the matrix $\Gamma_R(\sigma)$ corresponding to given subduced representation. For instance if we consider $\mathrm{Tr}_S(\Gamma_R(\sigma))$ then we only trace over the diagonal block in the above matrix labeled $S \otimes S$. However in the case of the indicated R_1 and R_2 subductions we can trace over the diagonal block elements labeled $R_1 \otimes R_1$ and $R_2 \otimes R_2$ or we can also trace over the off-diagonal blocks labeled $R_1 \otimes R_2$ and $R_2 \otimes R_1$. These off diagonal traces would create operators known as *intertwiners* (see [16] and [30]). In general for m copies of some subduction R_α there are clearly m^2 possible operators to construct from them.

6.6 Finite N Counting of the Restricted Schur Polynomials

We are concerned with counting the number of *independent* gauge invariant operators. In the case of $N = 3$ the independent operators are

$$\mathrm{Tr}(Z) \quad \mathrm{Tr}(Z^2) \quad \mathrm{Tr}(Z^3),$$

where we have restricted ourselves to one matrix Z. In the case of $N = 3$ there are only 3 eigenvalues (Z is an $N \times N$ matrix) and they can be determined from the above 3 independent operators

$$\begin{aligned}\mathrm{Tr}(Z) &= \lambda_1 + \lambda_2 + \lambda_3,\\ \mathrm{Tr}(Z^2) &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2,\\ \mathrm{Tr}(Z^3) &= \lambda_1^3 + \lambda_2^3 + \lambda_3^3.\end{aligned}$$

This is related to the manifestation of the stringy exclusion principle discussed in section 3 i.e. a giant graviton expanding in the S^5 component of the $\mathrm{AdS}_5\times\mathrm{S}^5$ spacetime has a cutoff on its angular momentum. The operator dual to a graviton with p units of angular momentum is given by (see [31] et. al.)

$$\frac{\mathrm{Tr}(Z^p)}{\sqrt{pN^p}}. \tag{6.6.1}$$

Thus we can see that a cutoff on the number of independent operators naturally leads to a cutoff on the angular momentum and hence the size of a giant graviton in the spherical component of the $\mathrm{AdS}_5\times\mathrm{S}^5$ spacetime. If, however, we consider an operator of the form $\mathrm{Tr}(Z^4)$ when $N = 3$ it can be shown that this operator can be written as a linear combination of the above lower order operators. That is $\mathrm{Tr}(Z^4)$ can be written as

$$\mathrm{Tr}(Z^4) = a\mathrm{Tr}(Z)\mathrm{Tr}(Z^3) + b\mathrm{Tr}(Z^2)^2 + c\mathrm{Tr}(Z^2)\mathrm{Tr}(Z)^2 + \cdots, \tag{6.6.2}$$

where the coefficients a, b, c, $\cdots$ are subject to certain constraints

The number of single matrix finite N gauge invariant operators is given by the following Polya generating function (see [32])

$$N(Z) = \prod_{n=1}^{N} \left(\frac{1}{1 - Z^n} \right), \tag{6.6.3}$$

where n is limited to N in contrast to equation (6.5.1) which counts the number of $N = \infty$ operators. Unfortunately if we restrict n in equation (6.5.2) to at most N we do not get the generating function for the number of finite N gauge invariant multi-matrix operators i.e.

$$N(Z_1, Z_2, \cdots Z_M) \neq \prod_{n=1}^{N} \left(\frac{1}{1 - Z_1^n - Z_2^n - \cdots - Z_M^n} \right).$$

In [32] Dolan shows that for $N = \infty$ counting the number of multi-matrix operators, for M fields say, is given by

$$N(1, 2, \cdots, M) = \sum_R \sum_\Lambda C(R, R, \Lambda) g(\mu; \Lambda), \tag{6.6.4}$$

where $\mu = [\mu_1] \circ [\mu_2] \circ \cdots \circ [\mu_M]$ denotes the field content i.e. μ_1 is the number of times field 1 appears, μ_2 is the number of times field 2 appears etc.. $[m]$ denotes the Young diagram composed of a single row of length m. For finite N counting one restricts the above sum over T to Young diagrams with at most N rows.

Let us restrict our discussion to at most two fields. The number of finite N operators constructed from m X fields and n Y fields is given by

$$N(m, n) = \sum_R \sum_\Lambda C(R, R, \Lambda) g([m] \circ [n]; \Lambda), \tag{6.6.5}$$

where the the sum over T has implicitly been restricted to Young diagrams with at most N rows. As an example of the counting let us consider operators built from 2 X fields and 2 Y fields. Since we are considering only two fields the Young diagrams Λ in equation (6.6.5) is limited to at most two rows since it is a representation of U(2) [7]. Thus the relevant Littlewood-Richardson coefficients are

$$g\,(\yng(2) \circ \yng(2); \yng(4)) = 1,$$
$$g(\yng(2) \circ \yng(2); \yng(3,1)) = 1,$$
$$g(\yng(2) \circ \yng(2); \yng(2,2)) = 1.$$

And the relevant S_4 direct products are

$$\yng(4) \otimes \yng(4) = \yng(4)$$
$$\yng(3,1) \otimes \yng(3,1) = \yng(4) \oplus \yng(3,1) \oplus \yng(2,2) \oplus \yng(2,1,1)$$
$$\yng(2,2) \otimes \yng(2,2) = \yng(4) \oplus \yng(2,2) \oplus \yng(1,1,1,1)$$
$$\yng(2,1,1) \otimes \yng(2,1,1) = \yng(4) \oplus \yng(3,1) \oplus \yng(2,2) \oplus \yng(2,1,1)$$
$$\yng(1,1,1,1) \otimes \yng(1,1,1,1) = \yng(4)$$

For $N = \infty$ equation (6.6.5) becomes

$$\begin{aligned} N(2,2) &= \sum_R \sum_\Lambda C(R,R,\Lambda) g(\yng(2) \circ \yng(2); \Lambda) \\ &= \sum_R \Big[C(R,R,\yng(4)) + C(R,R,\yng(3,1)) + C(R,R,\yng(2,2)) \Big] \\ &= C(\yng(4),\yng(4),\yng(4)) + C(\yng(3,1),\yng(3,1),\yng(4)) \\ &+ C(\yng(2,2),\yng(2,2),\yng(4)) + C(\yng(2,1,1),\yng(2,1,1),\yng(4)) + C(\yng(1,1,1,1),\yng(1,1,1,1),\yng(4)) \\ &+ C(\yng(3,1),\yng(3,1),\yng(3,1)) + C(\yng(2,1,1),\yng(2,1,1),\yng(3,1)) + C(\yng(3,1),\yng(3,1),\yng(2,2)) \\ &+ C(\yng(2,2),\yng(2,2),\yng(2,2)) + C(\yng(2,1,1),\yng(2,1,1),\yng(2,2)) \\ &= 10. \end{aligned}$$

Comparing this with the Polya generating function in equation (6.5.3) we see that there are indeed 10 operators constructed from 2 X fields and 2

Y fields at $N = \infty$. Now let us consider and arbitrary finite N example e.g. $N = 2$ then equation (6.6.5) becomes

$$\begin{aligned}
N(2,2) &= \sum_R \sum_\Lambda C(R,R,\Lambda) g(\yng(2) \circ \yng(2); \Lambda) \\
&= \sum_R \Big[C(R,R,\yng(4)) + C(R,R,\yng(3,1)) + C(R,R,\yng(2,2)) \Big] \\
&= C(\yng(4),\yng(4),\yng(4)) + C(\yng(3,1),\yng(3,1),\yng(4)) \\
&+ C(\yng(2,2),\yng(2,2),\yng(4)) + C(\yng(3,1),\yng(3,1),\yng(3,1)) + C(\yng(3,1),\yng(3,1),\yng(2,2)) \\
&+ C(\yng(2,2),\yng(2,2),\yng(2,2)) \\
&= 6.
\end{aligned}$$

Let us compare this to the finite N counting of restricted Schur polynomials. At $N = \infty$ there is no restriction on the representations labeling the restricted Schurs. In this case the total number of restricted Schur polynomials is 10 as evidenced by table 6.1. For finite N counting, R, which labels the restricted Schur χ_{R,R_α}, is limited to at most N rows. Thus for $N = 2$ counting of restricted Schur polynomials, where R labels a representation of S_4 we can only have restricted Schurs of the form

$$\chi_{\yng(4),R_\alpha} \qquad \chi_{\yng(3,1),R_\alpha} \qquad \chi_{\yng(2,2),R_\alpha}.$$

Referring to table 6.1 we see that the above representations ($\yng(4)$, $\yng(3,1)$ and $\yng(2,2)$) have 6 subductions in total. This is in agreement with the finite N counting given by formula (6.6.5).

As explained in the previous section (see figure 3), if the Young diagram R can be subduced into $R_1 \circ R_2$ in $m = g(R_1, R_2; R)$ possible ways then there are $m^2 = g(R_1, R_2; R)^2$ possible restricted Schur polynomials which can be constructed from it. Thus, considering all the possible representations of S_n and all their possible subductions into $S_{n_1} \times S_{n-n_1}$ representations (where

n_1 can vary), the total number of restricted Schur polynomials which can be constructed is given by

$$N = \sum_{n_1=0}^{n} \sum_{R \vdash n} \sum_{R_1 \vdash n_1} \sum_{R_2 \vdash n-n_1} \left(g(R_1, R_2; R)\right)^2 , \tag{6.6.6}$$

where $g(R_1, R_2, R)$ again denotes the Littlewood-Richardson coefficient and $R \vdash n$ denotes that R is a partition of n. In other words the above function counts the number of ways of obtaining $N = \infty$ restricted Schur polynomials constructed from n_1 X fields and $n - n_1$ Y fields – where n_1 can vary.

Let us suppose n_1 is fixed i.e. the S_n representations are being subduced into representations of a particular $S_{n_1} \times S_{n-n_1}$. In other words we are considering restricted Schurs constructed from n_1 X fields and $n - n_1$ Y fields again but now n_1 is fixed i.e. the field content is fixed. In this case the total number of restricted Schur polynomials is given by

$$N = \sum_{R \vdash n} \sum_{R_1 \vdash n_1} \sum_{R_2 \vdash n_2} \left(g(R_1, R_2, R)\right)^2 , \tag{6.6.7}$$

where we have set $n_2 = n - n_1$. Using the following two identities

$$g(R_1, R_2; R) = \frac{1}{n_1! n_2!} \sum_{\sigma \in S_{n_1}} \sum_{\tau \in S_{n_2}} \chi_{R_1}(\sigma) \chi_{R_2}(\tau) \chi_R(\sigma \circ \tau) \tag{6.6.8}$$

and

$$\sum_R \chi_R(\sigma) \chi_R(\tau) = |\mathrm{sym}(\sigma)| \delta([\sigma] = [\tau]), \tag{6.6.9}$$

we can simplify equation (6.6.7). Note that $[\sigma]$ denotes the conjugacy class of σ and and thus $\delta([\sigma] = [\tau])$ will be nonzero for $|[\sigma]|$ values. By $\mathrm{sym}(\sigma)$ we mean

$$|\mathrm{sym}(\sigma)| = i_1! 1^{i_1} i_2! 2^{i_2} \cdots i_n! 1^{i_n}, \tag{6.6.10}$$

where i_j denotes the number of cycles of length j and we have assumed $\sigma \in S_n$. The size of the conjugacy class $[\sigma]$ is given by

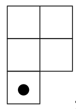
$$|[\sigma]| = \frac{n!}{|\mathrm{sym}(\sigma)|}. \tag{6.6.11}$$

Using the above identities in equation (6.6.7) we obtain

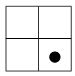
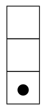
$$\begin{aligned}
N = &\sum_{R\vdash n}\sum_{R_1\vdash n_1}\sum_{R_2\vdash n_2}\left(\frac{1}{n_1!n_2!}\right)^2\left[\sum_{\sigma_1\in S_{n_1}}\sum_{\tau_1\in S_{n_2}}\chi_{R_1}(\sigma_1)\chi_{R_2}(\tau_1)\chi_R(\sigma_1\circ\tau_1)\right]\\
&\times\left[\sum_{\sigma_2\in S_{n_1}}\sum_{\tau_2\in S_{n_2}}\chi_{R_1}(\sigma_2)\chi_{R_2}(\tau_2)\chi_R(\sigma_2\circ\tau_2)\right]\\
&=\left(\frac{1}{n_1!n_2!}\right)^2\sum_{\sigma_1,\sigma_2\in S_{n_1}}\sum_{\tau_1,\tau_2\in S_{n_2}}|\mathrm{sym}(\sigma_1)|\delta([\sigma_1]=[\sigma_2])\\
&\times|\mathrm{sym}(\tau_1)|\delta([\tau_1]=[\tau_2])\sum_{R\vdash n}\chi_R(\sigma_1\circ\tau_1)\chi_R(\sigma_2\circ\tau_2)\\
&=\frac{1}{n_1!n_2!}\sum_{\sigma_1\in S_{n_1}}\sum_{\tau_1\in S_{n_2}}\sum_{R\vdash n}(\chi_R(\sigma_1\circ\tau_1))^2\\
&=\frac{1}{n_1!n_2!}\sum_{\sigma_1\in S_{n_1}}\sum_{\tau_1\in S_{n_2}}|\mathrm{sym}(\sigma_1\circ\tau_1)|.
\end{aligned}\tag{6.6.12}$$

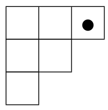
However going from the second last line to the last is only valid if we are not considering some finite cutoff on N. If we are considering a finite cutoff on N then we only sum over the Young diagrams R with at most N rows and equation (6.6.9) is unfortunately no longer valid.

We have shown that the finite N counting of the restricted Schur operators does indeed agree with the Polya counting for $n_1 = n - 1$ and $n_2 = 1$ i.e. when we subduce R by removing only one box. This becomes apparent when we consider the following injective mapping between subductions and the multi-trace gauge invariant operators. Let us consider gauge invariant operators built from any number of X fields but only one Y field. We can then translate the individual traces in the operator into columns of a Young diagram. A few examples will make this clear. Consider

$$\mathrm{Tr}(XXY)\mathrm{Tr}(XX) =$$  , (6.6.13)

here Tr(XXY) corresponds to the first column of the Young diagram and the box labeled with the bullet corresponds to the Y field. Here are a few more examples

$$\mathrm{Tr}(XX)\mathrm{Tr}(XY) =$$  $$\mathrm{Tr}(XXY) =$$  (6.6.14)

$$\mathrm{Tr}(XXX)\mathrm{Tr}(XX)\mathrm{Tr}(Y) =$$  (6.6.15)

We have explored several ways of trying to provide a simpler counting of finite N BPS operators. The following method we employed eventually breaks down but we have also considered ways of remedying this. We attempted to count the finite N BPS operators in terms of multi-traces. Our reasoning was as follows. In section 2.3 we discussed the fact that the six Higgs fields of the matrix model can be combined into 3 complex matrices

$$\begin{aligned} X &= \phi_1 + i\phi_2, \\ Y &= \phi_3 + i\phi_4, \\ Z &= \phi_5 + i\phi_6. \end{aligned}$$

The matrices can be rotated into each other through SO(6) transformations. Thus we thought it plausible that all finite N counting could be encapsulated by counting the number of independent multi-traces by making the restriction that at finite N there could be at most N matrices in any given trace contained in the multi-trace operator – we drew this conclusion from the limit on the angular momentum of the giant gravitons at finite N

(see the discussion above relating to equation (6.6.1)). For example consider $N = 3$. The possible independent multi-trace operators built from Z and X with at most 3 matrices in any given individual trace are

$$\begin{gathered}\mathrm{Tr}(X) \quad \mathrm{Tr}(X^2) \quad \mathrm{Tr}(X^3), \\ \mathrm{Tr}(Z) \quad \mathrm{Tr}(Z^2) \quad \mathrm{Tr}(Z^3), \\ \mathrm{Tr}(ZX) \quad \mathrm{Tr}(ZX^2) \quad \mathrm{Tr}(Z^2X).\end{gathered}$$

The first six traces involving only X or only Z would obviously allow us to determine the respective eigenvalues of X and Z. We assumed that the mixed traces (those involving both X and Z) would allow us to determine the unitary transformation which takes us from the basis where X is diagonal to the basis where Z is diagonal. However as mentioned above the 3 complex matrices X, Y and Z can be rotated into each other. Thus the traces involving only the X's say

$$\mathrm{Tr}(X) \quad \mathrm{Tr}(X^2) \quad \mathrm{Tr}(X^3),$$

can be rotated into all the other traces through some SO(6) action. Thus we suspected that they would form a complete basis for finite N BPS operators. However this approach has not been successful as we shall discuss below. We suspect that perhaps a group other than SO(6) is necessary to generate all the multi-trace operators sufficient as a basis for fintie N BPS operators.

As mentioned already, if we restrict the product over n to N in equation (6.5.2) it does not yield the number of finite N operators. However let us consider a variation of this. It is clear that

$$\frac{1}{1-Z} = 1 + Z + Z^2 + Z^3 + \cdots$$

counts the number of ways of building trace operators from several matrices where each trace is over only one matrix i.e. it counts $\mathrm{Tr}(Z)^n$. In other words if we consider the possible operators which can be built from n copies of some field Z and each trace contains only one matrix then its clear that there is only one way of doing this – $\mathrm{Tr}(Z)^n$. Similarly

$$\frac{1}{1-Z^2} = 1 + Z^2 + Z^4 + Z^6 + \cdots$$

and

$$\frac{1}{1-ZX} = 1 + ZX + (ZX)^2 + (ZX)^3 + \cdots$$

count operators of the form $\mathrm{Tr}(Z^2)^n$ and $\mathrm{Tr}(ZX)^n$ respectively. Thus suppose we want to count multi-trace operators where each trace contains at most 2 matrices then the generating function for the number of these operators is

$$\begin{aligned}
&\left(\frac{1}{1-X}\right)\left(\frac{1}{1-X^2}\right)\left(\frac{1}{1-Z}\right)\left(\frac{1}{1-Z^2}\right)\left(\frac{1}{1-ZX}\right)\\
&= \left(1 + Z + 2Z^2 + 2Z^3 + 3Z^4 + 3Z^5 + 4Z^6 + \cdots\right)\\
&+ \left(1 + 2Z + 3Z^2 + 4Z^3 + 5Z^4 + 6Z^5 + 7Z^6 + \cdots\right) X\\
&+ \left(2 + 3Z + 6Z^2 + 7Z^3 + 10Z^4 + 11Z^5 + 14Z^6 + \cdots\right) X^2\\
&+ \left(2 + 4Z + 7Z^2 + 10Z^3 + 13Z^4 + 16Z^5 + 19Z^6 + \cdots\right) X^3\\
&+ \left(3 + 5Z + 10Z^2 + 13Z^3 + 19Z^4 + 22Z^5 + 28Z^6 + \cdots\right) X^4\\
&+ \left(3 + 6Z + 11Z^2 + 16Z^3 + 22Z^4 + 28Z^5 + 34Z^6 + \cdots\right) X^5\\
&+ \left(4 + 7Z + 14Z^2 + 19Z^3 + 28Z^4 + 34Z^5 + 44Z^6 + \cdots\right) X^6\\
&+ \cdots
\end{aligned}$$

where we have expanded the generating function to order 6 in both X and Z. As an illustration of the generating function's validity let us consider all possible multi-trace operators built from 3 Xs and 2 Zs where there can be at most 2 matrices in any given trace

$$\mathrm{Tr}(Z)^3\mathrm{Tr}(X)^2 \quad \mathrm{Tr}(Z^2)\mathrm{Tr}(Z)\mathrm{Tr}(X)^2 \quad \mathrm{Tr}(Z)^3\mathrm{Tr}(Z)\mathrm{Tr}(X^2)$$
$$\mathrm{Tr}(Z^2)\mathrm{Tr}(Z)\mathrm{Tr}(X^2) \quad \mathrm{Tr}(Z)^2\mathrm{Tr}(X)\mathrm{Tr}(ZX) \quad \mathrm{Tr}(Z^2)\mathrm{Tr}(X)\mathrm{Tr}(ZX)$$
$$\mathrm{Tr}(Z)\mathrm{Tr}(ZX)^2.$$

There are 7 operators in total which corresponds to the coefficient of Z^2X^3. Using Matlab we checked that

$$N_2 = \left(\frac{1}{1-X}\right)\left(\frac{1}{1-X^2}\right)\left(\frac{1}{1-Z}\right)\left(\frac{1}{1-Z^2}\right)\left(\frac{1}{1-ZX}\right) \tag{6.6.16}$$

does indeed give the correct $N = 2$ counting up to order 15 i.e. where the total number of X fields and Z fields does not exceed 15. We can extend equation (6.6.16) to $N = 3$ as follows

$$\begin{aligned}
N_3 = & \left(\frac{1}{1-X}\right)\left(\frac{1}{1-X^2}\right)\left(\frac{1}{1-X^3}\right)\left(\frac{1}{1-Z}\right)\left(\frac{1}{1-Z^2}\right)\left(\frac{1}{1-Z^3}\right) \\
& \times \left(\frac{1}{1-ZX}\right)\left(\frac{1}{1-Z^2X}\right)\left(\frac{1}{1-Z^2X}\right) \\
& = \left(1+Z+2Z^2+3Z^3+4Z^4+5Z^5+7Z^6+\cdots\right) \\
& + \left(1+2Z+4Z^2+6Z^3+9Z^4+12Z^5+16Z^6+\cdots\right)X \\
& + \left(2+4Z+8Z^2+13Z^3+20Z^4+27Z^5+37Z^6+\cdots\right)X^2 \\
& + \left(3+6Z+13Z^2+22Z^3+34Z^4+48Z^5+67Z^6+\cdots\right)X^3 \\
& + \left(4+9Z+20Z^2+34Z^3+55Z^4+79Z^5+111Z^6+\cdots\right)X^4 \\
& + \left(5+12Z+27Z^2+48Z^3+79Z^4+116Z^5+166Z^6+\cdots\right)X^5 \\
& + \left(7+16Z+37Z^2+67Z^3+111Z^4+166Z^5+241Z^6+\cdots\right)X^6 \\
& + \cdots
\end{aligned} \tag{6.6.17}$$

This formula unfortunately breaks down for Z^2X^2 and for all other Z^mX^n where $m+n>4$. Above we see that the coefficient for Z^2X^2 is 8 when in fact it should be 9. Multiplying N_2 by

$$\frac{1}{1-Z^2X^2},$$

corrects many of the coefficients but also ultimately breaks down at Z^6X^6 and for all Z^mX^n where $m+n>12$. This may mean that there is no simple generating function for finite N BPS operators in terms of multi-traces.

We will now show that the number of finite N restricted Schur polynomials is in agreement with the finite N partition function of Dolan [32]. This partition function is given by

$$Z_{U(N)}(t)=\frac{1}{(2\pi i)^N N!}\oint\prod_{i=1}^{N}\frac{dz_i}{z_i}\Delta(z)\Delta(z^{-1})\prod_{j=1}^{k}\prod_{r,s=1}^{N}\frac{1}{1-t_j z_r z_s^{-1}}. \quad (6.6.18)$$

To evaluate this we use the Cauchy-Littlewood formula which is given by

$$\prod_{i=1}^{L}\prod_{j=1}^{M}\frac{1}{1-x_iy_i}=\sum_{\substack{\lambda\\ l(\lambda)\leq\min(L,M)}}\chi_\lambda(x_1,\cdots,x_L)\chi_\lambda(y_1,\cdots,y_M), \quad (6.6.19)$$

where χ_λ denotes a Schur polynomial. Thus equation (6.6.18) becomes

$$Z_{U(N)}(t)=\frac{1}{(2\pi i)^N N!}\oint\prod_{i=1}^{N}\frac{dz_i}{z_i}\Delta(z)\Delta(z^{-1})\prod_{j=1}^{k}\sum_{\substack{\lambda_j\\ l(\lambda_j)\leq N}}\chi_{\lambda_j}(t_jz)\chi_{\lambda_j}(z^{-1}). \quad (6.6.20)$$

The Schur polynomial can be written as

$$\chi_{\lambda_j}(t_jz)=t_j^{|\lambda_j|}\chi_{\lambda_j}(z). \quad (6.6.21)$$

Utilising this in equation (6.6.20) and swapping the product and the sum we have

$$Z_{U(N)}(t) = \frac{1}{(2\pi i)^N N!} \sum_{\substack{\lambda_1,\lambda_2,\cdots,\lambda_k \\ l(\lambda_i)\leq N}} (t_1)^{|\lambda_1|}(t_2)^{|\lambda_2|}\cdots(t_k)^{|\lambda_k|}$$
$$\times \oint \prod_{i=1}^{N} \frac{dz_i}{z_i} \Delta(z)\Delta(z^{-1})\chi_{\lambda_1}(z)\chi_{\lambda_1}(z^{-1})\chi_{\lambda_2}(z)\chi_{\lambda_2}(z^{-1})$$
$$\cdots\chi_{\lambda_k}(z)\chi_{\lambda_k}(z^{-1}). \quad (6.6.22)$$

Using the product rule for Schur polynomials we get

$$Z_{U(N)}(t) = \frac{1}{(2\pi i)^N N!} \sum_{\substack{\lambda_1,\cdots,\lambda_{k+2} \\ l(\lambda_i)\leq N}} (t_1)^{|\lambda_1|}(t_2)^{|\lambda_2|}\cdots(t_k)^{|\lambda_k|}$$
$$\times\, g_{\lambda_1\lambda_2\cdots\lambda_k\lambda_{k+1}} g_{\lambda_1\lambda_2\cdots\lambda_k\lambda_{k+2}}$$
$$\times \oint \prod_{i=1}^{N} \frac{dz_i}{z_i} \Delta(z)\Delta(z^{-1})\chi_{\lambda_{k+1}}(z)\chi_{\lambda_{k+2}}(z^{-1}). \quad (6.6.23)$$

In the above $g_{\lambda_1\lambda_2\cdots\lambda_k\lambda_{k+1}}$ counts how many times λ_{k+1} appears in the product $\lambda_1 \otimes \lambda_2 \otimes \cdots \otimes \lambda_k$. These can be expressed in terms of Littlewood-Richardson coefficients. For example if $k = 4$

$$g_{\lambda_1\lambda_2\lambda_3\lambda_4} = \sum_{\lambda} g_{\lambda_1\lambda_2\lambda} g_{\lambda\lambda_3\lambda_4}, \quad (6.6.24)$$

where $g_{\lambda_1\lambda_2\lambda}$ is the usual Littlewood-Richardson coefficient.Using the inner product employed by Dolan [32] we have

$$
\begin{aligned}
Z_{U(N)}(t) &= \sum_{\substack{\lambda_1,\cdots,\lambda_{k+2}\\ l(\lambda_i)\leq N}} (t_1)^{|\lambda_1|}\cdots(t_k)^{|\lambda_k|} g_{\lambda_1\lambda_2\cdots\lambda_k\lambda_{k+1}} \\
&\times g_{\lambda_1\lambda_2\cdots\lambda_k\lambda_{k+2}} \left\langle \chi_{\lambda_{k+1}}(z), \chi_{\lambda_{k+2}}(z^{-1}) \right\rangle \\
&= \sum_{\substack{\lambda_1,\cdots,\lambda_{k+2}\\ l(\lambda_i)\leq N}} (t_1)^{|\lambda_1|}\cdots(t_k)^{|\lambda_k|} g_{\lambda_1\lambda_2\cdots\lambda_k\lambda_{k+1}} \\
&\times g_{\lambda_1\lambda_2\cdots\lambda_k\lambda_{k+2}} \delta_{\lambda_{k+1},\lambda_{k+2}} \\
&= \sum_{\substack{\lambda_1,\cdots,\lambda_{k+1}\\ l(\lambda_i)\leq N}} (t_1)^{|\lambda_1|}\cdots(t_k)^{|\lambda_k|} \left(g_{\lambda_1\lambda_2\cdots\lambda_k\lambda_{k+1}}\right)^2 .
\end{aligned}
\tag{6.6.25}
$$

The last line is exactly the counting function for the number of finite N restricted Schur polynomials. Thus the number of finite N restricted Schur polynomials is in agreement with the finite N partition function.

As $N \to \infty$ we get

$$
\sum_{\lambda_1\cdots\lambda_{k+1}} g_{\lambda_1\cdots\lambda_{k+1}} t_1^{|\lambda_1|}\cdots t_1^{|\lambda_k|} = \prod_{k=1}^{\infty} \frac{1}{1-\left(t_1^k + t_2^k + \cdots + t_m^k\right)} \tag{6.6.26}
$$

This generalises theorem 4.1 of Willenbring [33].

6.7 Extension to the Multi-Matrix Model

Up until now we have focused only on two fields in our model. Extending this to multiple matrices is relatively straightfoward. Directly above in equation (6.5.4) we have given the Polya counting formula for enumerating the number of multi-matrix operators in general.

Suppose that we have M matrices labeled X_1, X_2 $\ldots X_M$ and that we have m_1, m_2, $\ldots m_M$ copies of each respective matrix. The restricted Schur polynomial then becomes

$$\chi_{R,R_\alpha} = \frac{1}{m_1!m_2!\dots m_M!} \sum_{\sigma\in S_{m_1+m_2+\dots+m_M}} \mathrm{Tr}_{R_\alpha}(\Gamma_R(\sigma)) \\ \times\mathrm{Tr}(\sigma X_1^{\otimes m_1}\otimes X_2^{\otimes m_2}\dots X_M^{\otimes m_M}),$$

where the subduction R_α would now consist of M Young diagrams with m_1, m_2, $\dots m_M$ boxes respectively.

To extend the exact formula for the two-point function is also trivial. The formula itself does not change, we still have

$$\langle\chi_{R,R_\alpha}\chi^\dagger_{S,S_\beta}\rangle = \delta_{RS}\delta_{R_\alpha S_\beta}\frac{(\mathrm{hooks})_R}{(\mathrm{hooks})_{R_\alpha}}f_R.$$

However we now have that

$$(\mathrm{hooks})_{R_\alpha} = \prod_{i=1}^{M}(\mathrm{hooks})_{R_i},$$

where R_i denotes the individual components/Young diagrams of the subduction R_α.

6.8 Numerical Checks

In this section we will offer some concrete examples of restricted Schur polynomials as well as a discussion of our numerical computations of the two-point function. We wrote Matlab code to check our analytical calculation for the correlation function viz.

$$\langle\chi_{R,R_\alpha}\chi^\dagger_{S,S_\beta}\rangle = \delta_{RS}\delta_{R_\alpha S_\beta}\frac{(\mathrm{hooks})_R}{(\mathrm{hooks})_{R_\alpha}}f_R, \tag{6.8.1}$$

for all representations R from S_1 through to S_6. We showed in subsection 6.3 (see equation (6.3.10)) that the two point-function of two restricted Schurs is given by the following

$$\langle \chi_{R,R_\alpha}, \chi^\dagger_{S,S_\beta} \rangle = \delta_{RS}\delta_{R_\alpha S_\beta}\frac{1}{m!n!}\sum_{\sigma,\tau\in S_{m+n}} \mathrm{Tr}_{R_\alpha}\left(\Gamma_R(\sigma)\right)\mathrm{Tr}^*_{R_\alpha}\left(\Gamma_R(\sigma)\right)N^{C(\sigma\tau)}, \tag{6.8.2}$$

where the $N^{C(\sigma\tau)}$ factor came from performing all the Wick contractions of the multi-point function

$$\begin{aligned}\Big\langle A^{i_1}_{i_{\sigma(1)}}\cdots A^{i_m}_{i_{\sigma(m)}} B^{i_{m+1}}_{i_{\sigma(m+1)}}\cdots (B^\dagger)^{i_{m+n}}_{i_{\sigma(m+n)}}\\ \times (A^\dagger)^{j_1}_{j_{\sigma(1)}}\cdots (A^\dagger)^{j_m}_{j_{\sigma(m)}}(B^\dagger)^{j_{m+1}}_{j_{\sigma(m+1)}}\cdots (B^\dagger)^{j_{m+n}}_{j_{\sigma(m+n)}}\Big\rangle.\end{aligned}$$

There are several steps to perform in the numerical computation of the two-point function

1. We have to determine the representation $\Gamma_R(\sigma)$ for each $\sigma \in S_n$.
2. We have to compute the restricted trace $\Gamma_{R_\alpha}(\sigma)$ by creating a projection operator $P_{R\to R_\alpha}$ to map the representation to the subduced space labeled by R_α.
3. We have to compute $N^{C(\sigma)}$ for each $\sigma \in S_n$.
4. We then have to sum $\mathrm{Tr}_{R_\alpha}\left(\Gamma_R(\sigma)\right)\mathrm{Tr}^*_{R_\alpha}\left(\Gamma_R(\tau)\right)N^{C(\sigma\tau)}$ over all $(m+n)!$ possible values for σ and τ.

The representation of $\sigma \in S_n$ is obtained using strand diagrams (see [30]). The restricted trace $\mathrm{Tr}_{R_\alpha}\left(\Gamma_R(\sigma)\right)$ is obtained by constructing a projection operator $P_{R\to R_\alpha}$ which maps the given representation R to the subduced representation R_α i.e.

$$\mathrm{Tr}_{R_\alpha}\left(\Gamma_R(\sigma)\right) = \mathrm{Tr}\left(P_{R\to R_\alpha}\Gamma_R(\sigma)\right).$$

We will briefly discuss how to construct a projection operator. Define the operator $\hat{O}$ (see [16]), which determines the symmetry relation between two boxes in a Young diagram, as follows

$$\hat{O}\left(\yng(2)\right) = 1 \quad \hat{O}\left(\yng(1,1)\right) = -1.$$

Since the Young diagrams are related to irreducible representations of SU(N) tensors they have a definite symmetry under the interchange of indices. Boxes in the same column have an antisymmetrical relationship whilst those in the same row have a symmetrical one. Thus $\hat{O}$ is given by $\hat{O} = \Gamma_R(12)$ i.e. the representation of $(12) \in S_2$ labelled by the Young diagram R. We can trivially extend this definition to any given number of boxes as follows

$$\hat{O} = \sum_{\substack{i,j \\ i \neq j}} (ij).$$

Thus for S_3 we would have $\hat{O} = \Gamma_R(12) + \Gamma_R(13) + \Gamma_R(23)$ as the symmetry operator and its affect on the 3 possible S_3 representations would be as follows

$$\hat{O}\left(\yng(3)\right) = 3 \quad \hat{O}\left(\yng(2,1)\right) = 0 \quad \hat{O}\left(\yng(3)\right) = -3.$$

For multiple fields (i.e. a multi-matrix restricted Schur polynomial) there would be multiple symmetry operators. For example suppose we have 5 fields viz. three A fields and two B fields then we would have the following symmetry operators

$$\hat{O}_A = \Gamma_R(12) + \Gamma_R(13) + \Gamma_R(23) \text{ and } \hat{O}_B = \Gamma_R(45).$$

Suppose our given representation is $\yng(3,2)$ and the subduction is $\yng(2,1) \oplus \yng(1,1)$ then our projection operator would be given by

$$P_{\yng(3,2) \to \yng(2,1) \oplus \yng(1,1)} = \frac{(\hat{O}_A - 3I)(\hat{O}_A + 3I)}{9} \frac{(\hat{O}_B - I)}{2},$$

where I denotes the identity matrix. If the above projection operator acts on any subduction other than $\yng(2,1) \oplus \yng(1,1)$ then it yields zero. Note that the factors of 9 and 2 in the denominator are to normalise the operator – if just the numerator acts on $\yng(2,1) \oplus \yng(1,1)$ we get 18.

Once we have computed the two-point function in equation (6.8.2) for $R = \yng(3,2)$ and $R_\alpha = \yng(2,1) \oplus \yng(1,1)$ we obtain the following

$$\langle \chi_{\yng(3,2),\, \yng(2,1) \oplus \yng(1,1)}, \chi^{\dagger}_{\yng(3,2),\, \yng(2,1) \oplus \yng(1,1)} \rangle = -8N^2 - 4N^3 + 8N^4 + 4N^5. \tag{6.8.3}$$

We proceeded to check the two-point function numerically for all the representations of S_n for $n \leq 6$ confirming in each instance the exact formula (equation (6.8.1)).

The numerical computation of the two-point function is computationally expensive. Since we have to sum over σ and τ in equation (6.8.2) the number of terms in the sum grows as $((m+n)!)^2$. Also explicitly computing the representation $\Gamma_R(\sigma)$ for $\sigma \in S_n$ imposes serious time penalties.

In [34] a product rule was recently obtained allowing us to decompose the product of restricted Schur polynomials into a sum of restricted Schurs. This product rule is similar to that of Schur polynomials (discussed in section 2.5) but uses restricted Littlewood-Richardson coefficients. Coupled with our method for determining the two-point function of restricted Schurs in [8] this now allows us to determine the multipoint function of any number of restricted Schur polynomials.

7 Relationship Between the BHR Operators and the Restricted Schur Polynomials

In this section we will elucidate the relationship between the BHR operators and our restricted Schur's for the case of just two fields. In [35] extra symmetries present in the free field theory limit are used to define Casimir's that organise the different multi-matrix operator bases. Let the two fields be labeled X and Y, with m copies of the prior and n of the latter. In review the restricted Schur polynomials are given by

$$\chi_{R,R_\alpha} = \frac{1}{m!n!} \sum_{\sigma \in S_{m+n}} \mathrm{Tr}_{R_\alpha} \left(\Gamma_R(\sigma)\right) \mathrm{Tr}(\sigma X^{\otimes m} \otimes Y^{\otimes n}), \tag{7.0.4}$$

while the BHR operators are given by

$$\mathcal{O}^{\Lambda\mu,R'}_{\beta,\tau} = \frac{1}{(m+n)!} \sum_{\sigma \in S_{m+n}} B_{j\beta} S^{\tau,\Lambda R' R'}_{jpq} \left[\Gamma_{R'}(\sigma)\right]_{pq} \mathrm{Tr}(\sigma X^{\otimes m} \otimes Y^{\otimes n}). \tag{7.0.5}$$

The trace $\mathrm{Tr}(\sigma X^{\otimes m} \otimes Y^{\otimes n})$ can be written in two possible ways – using either the definition of the restricted Schur polynomials or the BHR operators.

$$\begin{aligned}\mathrm{Tr}(\sigma X^{\otimes m} \otimes Y^{\otimes n}) = & \sum_{R;R_1,R_2,\beta_1,\beta_2} \frac{d_R}{d_{R_1} d_{R_2} (m+n)!} \\ & \times B^{R\to R_1 \circ R_2,\beta_1}_{acd} B^{R\to R_1 \circ R_2,\beta_2}_{bcd} \left[\Gamma_R(\sigma)\right]_{ab} \chi_{R,R_1\circ R_2},\end{aligned}$$

and

$$\mathrm{Tr}(\sigma X^{\otimes m} \otimes Y^{\otimes n}) = \sum_{R'} d_{R'} \left[\Gamma_{R'}(\sigma)\right]_{ab} B_{c\beta} S^{\tau,\Lambda R' R'}_{cab} \mathcal{O}^{\Lambda\mu,R'}_{\beta,\tau},$$

where the first instance is written in terms of restricted Schur polynomials and the second in terms of the BHR operators. Thus we can write the restricted Schur polynomials in terms of the BHR operators yielding

$$\chi_{R,R_\alpha} = \frac{1}{m!n!} \sum_{\sigma \in S_{m+n}} \mathrm{Tr}_{R_\alpha} \left(\Gamma_R(\sigma)\right) \sum_{R'} d_{R'} \left[\Gamma_{R'}(\sigma)\right]_{ab} B_{c\beta} S^{\tau,\Lambda R' R'}_{cab} \mathcal{O}^{\Lambda\mu,R'}_{\beta,\tau}.$$

Note that we have mixed notation here using both the restricted trace and the branching coefficients. We will rectify this shortly. First let us simplify the sum as follows

$$\begin{aligned}
\chi_{R,R_\alpha} &= \frac{1}{m!n!} \sum_{R'} \left[P_{R\to R_\alpha}\right]_{ij} \left(\sum_{\sigma \in S_{m+n}} \left[\Gamma_R(\sigma)\right]_{ji} \left[\Gamma_{R'}(\sigma)\right]_{ab} \right) \\
&\times d_{R'} B_{c\beta} S^{\tau,\Lambda R' R'}_{cab} \mathcal{O}^{\Lambda\mu,R'}_{\beta,\tau} \\
&= \frac{1}{m!n!} \sum_{R'} \left[P_{R\to R_\alpha}\right]_{ij} \frac{(m+n)!}{d_R} \delta_{RR'} \delta_{ja} \delta_{ib} \\
&\times d_{R'} B_{c\beta} S^{\tau,\Lambda R' R'}_{cab} \mathcal{O}^{\Lambda\mu,R'}_{\beta,\tau} \\
&= \frac{(m+n)!}{m!n!} \left[P_{R\to R_\alpha}\right]_{ba} B_{c\beta} S^{\tau,\Lambda RR}_{cab} \mathcal{O}^{\Lambda\mu,R}_{\beta,\tau}, \qquad (7.0.6)
\end{aligned}$$

where we have utilised the fundamental orthogonality relation in the sum over σ. Next we write the projector in bra-ket notation as follows

$$\left[P_{R\to R_1\oplus R_2;(\beta_1,\beta_2)}\right]_{ij} = \sum_{k,l} \langle R, i | R, R_1 \circ R_2, kl, \beta_1 \rangle \langle R, R_1 \circ R_2, kl, \beta_2 | R, j \rangle ,$$

where β_1 and β_2 label the multiplicity of R in the *outer* product $R_1 \circ R_2$. Similarly, converting the branching and the Clebsh-Gordon coefficients to the bra and ket form yields the following

$$\begin{aligned}
B_{c\beta} &= \langle \Lambda, c | [m] \circ [n], \beta \rangle \\
S^{\tau,\Lambda RR}_{cab} &= \langle R \otimes R, ab | \Lambda, \tau, c \rangle .
\end{aligned}$$

Note that by $[m]$ and $[n]$ we mean the Young diagram (and thus the corresponding representation) consisting of a single row of m boxes and n boxes respectively. Thus we can rewrite equation (7.0.6) as follows

$$\chi_{R,R_1\circ R_2;(\beta_1,\beta_2)} = \frac{(m+n)!}{m!n!} \langle R,b|R,R_1\circ R_2,kl,\beta_1\rangle \langle R,R_1\circ R_2,kl,\beta_2|R,a\rangle \times \langle \Lambda,c|[m]\circ[n],\beta\rangle \langle R\otimes R,ab|\Lambda,\tau,c\rangle \mathcal{O}^{\Lambda\mu,R}_{\beta,\tau}.$$

Proceeding accordingly and applying rules related to bras and kets we can simplify the expression

$$\begin{aligned}\chi_{R,R_1\oplus R_2;(\beta_1,\beta_2)} =& \frac{(m+n)!}{m!n!} \langle R,R_1\circ R_2,kl,\beta_1|R,b\rangle \langle R,R_1\circ R_2,kl,\beta_2|R,a\rangle \\ &\times \langle R,a| \langle R,b| |\Lambda,\tau,c\rangle \langle \Lambda,c|[m]\circ[n],\beta\rangle \mathcal{O}^{\Lambda\mu,R}_{\beta,\tau} \\ &\frac{(m+n)!}{m!n!} \langle R,R_1\circ R_2,kl,\beta_1| \langle R,R_1\circ R_2,kl,\beta_2| |\Lambda,\tau,c\rangle \\ &\times \langle \Lambda,c|[m]\circ[n],\beta\rangle \mathcal{O}^{\Lambda\mu,R}_{\beta,\tau} \\ =& \frac{(m+n)!}{m!n!} \langle R,R_1\circ R_2,kl,\beta_1;R,R_1\circ R_2,kl,\beta_2|\Lambda,\tau,c\rangle \\ &\times \langle \Lambda,c|[m]\circ[n],\beta\rangle \mathcal{O}^{\Lambda\mu,R}_{\beta,\tau}.\end{aligned} \tag{7.0.7}$$

Equation (7.0.7) is the central result of this section. It shows that we can rewrite restricted Schur polynomials as a linear combination of the BHR operators where

$$\frac{(m+n)!}{m!n!} \langle R,R_1\circ R_2,kl,\beta_1;R,R_1\circ R_2,kl,\beta_2|\Lambda,\tau,c\rangle \langle \Lambda,c|[m]\circ[n],\beta\rangle, \tag{7.0.8}$$

acts as a Clebsch-Gordon coefficient relating the two operators. An analogy of this can be drawn. Consider a 2 dimensional harmonic oscillator. It can be quantised using either cartesian coordinates, X and Y, or polar coordinates which are related to angular momentum. The restricted Schur operators correspond to the first type of quantisation, the BHR operators to the second.

The factor $\langle \Lambda,c|[m]\circ[n],\beta\rangle$ is just the branching coefficient $B_{c\beta}$ which is relatively straightforward to calculate for small representations as illustrated in appendix D of [7]. As can be seen in tables 7.1 and 7.2 the restricted Schur polynomials and BHR operators corresponding to just two distinct fields (labeled X and Y) are in fact identical. This speaks to the fact that there is a unique trivial space which can be subduced from S_2 i.e. $S_1 \times S_1$ and S_1 is one dimensional.

For the other operators (with no β and τ multiplicities) it is relatively straightforward to show their equivalence if the irreducible representation is small enough. We will first show by comparision and then by direct computation how to determine the coefficients relating the restricted Schurs to the BHR operators. Note that we have omitted the β and τ indices (and thus the branching coefficients) in the following – since they can only assume one value i.e. there are no multiplicities. Implementing equation (7.0.6) for $\chi_{\yng(3);\yng(2)\oplus\yng(1)}$ we have

$$\chi_{\yng(3);\yng(2)\oplus\yng(1)} = \frac{3!}{2!1!}\left\langle \yng(3), \yng(2)\circ\yng(1), kl; \yng(3), \yng(2)\circ\yng(1), kl \middle| \Lambda \right\rangle \mathcal{O}^{\Lambda,\yng(3)},$$

Since the inner tensor product is given by

$$\yng(3) \otimes \yng(3) = \yng(3),$$

the only representation Λ can assume is $\yng(3)$. Thus we have that

$$\begin{aligned}
\chi_{\yng(3);\yng(2)\oplus\yng(1)} &= \frac{3!}{2!1!}\left\langle \yng(3), \yng(2)\circ\yng(1), kl; \yng(3), \yng(2)\circ\yng(1), kl \middle| \yng(3) \right\rangle \mathcal{O}^{\yng(3),\yng(3)} \\
&= 3\left\langle \yng(3), \yng(2)\circ\yng(1), kl; \yng(3), \yng(2)\circ\yng(1), kl \middle| \yng(3) \right\rangle \frac{1}{6}[\mathrm{Tr}(X)\mathrm{Tr}(X)\mathrm{Tr}(Y) \\
&+ \mathrm{Tr}(XX)\mathrm{Tr}(Y) + 2\mathrm{Tr}(X)\mathrm{Tr}(XY) + 2\mathrm{Tr}(XXY)].
\end{aligned}$$

It is clear that if we compare this to $\chi_{\yng(3);\yng(2)\oplus\yng(1)}$ in table 7.1 that the coefficient $\left\langle \yng(3), \yng(2)\circ\yng(1), kl; \yng(3), \yng(2)\circ\yng(1), kl \middle| \yng(3) \right\rangle$ must be equal to 1. Next let us rewrite $\chi_{\yng(2,1);\yng(2)\oplus\yng(1)}$ in terms of the BHR operators.

$$
\begin{aligned}
\chi_{\yng(2,1);\yng(2)\oplus\yng(1)} &= \frac{3!}{2!1!}\left\langle \yng(2,1),\yng(2)\circ\yng(1),kl;\yng(2,1),\yng(2)\circ\yng(1),kl|\Lambda\right\rangle \mathcal{O}^{\Lambda,\yng(2,1)} \\
&= 3\left\langle \yng(2,1),\yng(2)\circ\yng(1),kl;\yng(2,1),\yng(2)\circ\yng(1),kl|\yng(3)\right\rangle \mathcal{O}^{\yng(3),\yng(2,1)} \\
&+ \left\langle \yng(2,1),\yng(2)\circ\yng(1),kl;\yng(2,1),\yng(2)\circ\yng(1),kl|\yng(2,1)\right\rangle \mathcal{O}^{\yng(2,1),\yng(2,1)} \\
&= 3\left\langle \yng(2,1),\yng(2)\circ\yng(1),kl;\yng(2,1),\yng(2)\circ\yng(1),kl|\yng(3)\right\rangle \frac{1}{3\sqrt{2}}[\mathrm{Tr}(X)\mathrm{Tr}(X)\mathrm{Tr}(Y) \\
&- \mathrm{Tr}(XXY)] \\
&+ \left\langle \yng(2,1),\yng(2)\circ\yng(1),kl;\yng(2,1),\yng(2)\circ\yng(1),kl|\yng(2,1)\right\rangle \frac{1}{3\sqrt{2}}[\mathrm{Tr}(XX)\mathrm{Tr}(Y) \\
&- \mathrm{Tr}(X)\mathrm{Tr}(XY)]. \qquad (7.0.9)
\end{aligned}
$$

Where in the second line we have used the fact that the inner tensor product is

$$\yng(2,1)\otimes\yng(2,1) = \yng(3)\oplus\yng(2,1)\oplus\yng(1,1,1)$$

but since Λ is a representation of $U(2)$ only the first two Young diagrams are allowed. Comparing equation (7.0.9) to the restricted Schur polynomials in table 7.1 we see that

$$\left\langle \yng(2,1),\yng(2)\circ\yng(1),kl;\yng(2)\circ\yng(1),kl|\yng(3)\right\rangle = \left\langle \yng(2,1),\yng(2)\circ\yng(1),kl;\yng(2)\circ\yng(1),kl|\yng(2,1)\right\rangle = \frac{1}{\sqrt{2}}.$$

Similarly the relationship can be shown to hold for other restricted Schur polynomials with representations consisting of 3 boxes.

Now let us directly compute the coefficients relating the restricted Schur polynomials to the BHR operators. The easiest way to do this is using equation (7.0.6). Then we have that

$$
\begin{aligned}
&\langle R, R_1\circ R_2, kl, \beta_1; R, R_1\circ R_2, kl, \beta_2|\Lambda,\tau,c\rangle\,\langle\Lambda,c|[m]\circ[n],\beta\rangle = \\
&[P_{R\to R_\alpha}]_{ba}\,B_{c\beta}S^{\tau,\Lambda RR}_{\ \ cab}. \qquad (7.0.10)
\end{aligned}
$$

$$\chi_{\yng(2);\yng(1)\oplus\yng(1)} = \frac{1}{2}\left[\mathrm{Tr}(X)\mathrm{Tr}(Y) + \mathrm{Tr}(XY)\right]$$

$$\chi_{\yng(1,1);\yng(1)\oplus\yng(1)} = \frac{1}{2}\left[\mathrm{Tr}(X)\mathrm{Tr}(Y) - \mathrm{Tr}(XY)\right]$$

$$\begin{aligned}\chi_{\yng(3);\yng(2)\oplus\yng(1)} = \frac{1}{2}[&\mathrm{Tr}(X)\mathrm{Tr}(X)\mathrm{Tr}(Y) + \mathrm{Tr}(XX)\mathrm{Tr}(Y)\\ &+2\mathrm{Tr}(X)\mathrm{Tr}(XY) + 2\mathrm{Tr}(XXY))]\end{aligned}$$

$$\begin{aligned}\chi_{\yng(2,1);\yng(2)\oplus\yng(1)} = \frac{1}{2}[&\mathrm{Tr}(X)\mathrm{Tr}(X)\mathrm{Tr}(Y) + \mathrm{Tr}(XX)\mathrm{Tr}(Y)\\ &-\mathrm{Tr}(X)\mathrm{Tr}(XY) - \mathrm{Tr}(XXY)]\end{aligned}$$

$$\begin{aligned}\chi_{\yng(2,1);\yng(1,1)\oplus\yng(1)} = \frac{1}{2}[&-\mathrm{Tr}(X)\mathrm{Tr}(X)\mathrm{Tr}(Y) + \mathrm{Tr}(XX)\mathrm{Tr}(Y)\\ &-\mathrm{Tr}(X)\mathrm{Tr}(XY) + \mathrm{Tr}(XXY)]\end{aligned}$$

$$\begin{aligned}\chi_{\yng(1,1,1);\yng(2)\oplus\yng(1)} = \frac{1}{2}[&-\mathrm{Tr}(X)\mathrm{Tr}(X)\mathrm{Tr}(Y) + \mathrm{Tr}(XX)\mathrm{Tr}(Y)\\ &+2\mathrm{Tr}(X)\mathrm{Tr}(XY) - 2\mathrm{Tr}(XXY))]\end{aligned}$$

Table 7.1: A list of the first few restricted Schur polynomials.

$$\mathcal{O}^{\yng(2),\yng(2)} = \frac{1}{2}[\mathrm{Tr}(X)\mathrm{Tr}(Y) + \mathrm{Tr}(XY)]$$

$$\mathcal{O}^{\yng(2),\yng(1,1)} = \frac{1}{2}[\mathrm{Tr}(X)\mathrm{Tr}(Y) - \mathrm{Tr}(XY)]$$

$$\begin{aligned}\mathcal{O}^{\yng(3),\yng(3)} = \frac{1}{6}[&\mathrm{Tr}(X)\mathrm{Tr}(X)\mathrm{Tr}(Y) + \mathrm{Tr}(XX)\mathrm{Tr}(Y)\\ &+2\mathrm{Tr}(X)\mathrm{Tr}(XY) + 2\mathrm{Tr}(XXY)]\end{aligned}$$

$$\mathcal{O}^{\yng(3),\yng(2,1)} = \frac{1}{3\sqrt{2}}[\mathrm{Tr}(X)\mathrm{Tr}(X)\mathrm{Tr}(Y) - \mathrm{Tr}(XXY)]$$

$$\begin{aligned}\mathcal{O}^{\yng(3),\yng(1,1,1)} = \frac{1}{6}[&\mathrm{Tr}(X)\mathrm{Tr}(X)\mathrm{Tr}(Y) - \mathrm{Tr}(XX)\mathrm{Tr}(Y)\\ &-2\mathrm{Tr}(X)\mathrm{Tr}(XY) + 2\mathrm{Tr}(XXY)]\end{aligned}$$

$$\mathcal{O}^{\yng(2,1),\yng(2,1)} = \frac{1}{3\sqrt{2}}[\mathrm{Tr}(XX)\mathrm{Tr}(Y) - \mathrm{Tr}(X)\mathrm{Tr}(XY)]$$

Table 7.2: A list of the first few BHR operators. In this case there are no multiplicities so the only labels are Λ and R in $\mathcal{O}^{\Lambda,R}$.

Here we list the pertinent Clebsch-Gordon coefficients and representation matrices to our ensuing computations – these were taken from appendix D of [7]. The Clebsch-Gordon coefficients are

$$\begin{aligned} S_{1ab}^{\yng(3)RR} &= \frac{1}{\sqrt{d_R}}\delta_{ab}, \\ S_{111}^{\yng(2,1)\,\yng(2,1)\,\yng(2,1)} &= \frac{1}{\sqrt{2}}, \\ S_{111}^{\yng(2,1)\,\yng(2,1)\,\yng(2,1)} &= -\frac{1}{\sqrt{2}} \\ S_{112}^{\yng(2,1)\,\yng(2,1)\,\yng(2,1)} = S_{121}^{\yng(2,1)\,\yng(2,1)\,\yng(2,1)} &= 0, \end{aligned}$$

and the representation matrices are

$$\Gamma_{\yng(2,1)}((1)(2)(3)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \Gamma_{\yng(2,1)}((12)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

First we have that

$$\begin{aligned} \langle \yng(3), \yng(2)\circ\yng(1), kl; \yng(3), \yng(2)\circ\yng(1), kl | \yng(3)\rangle &= \left[P_{\yng(3)\to\yng(2)\oplus\yng(1)}\right]_{ba} S_{1ab}^{\yng(3)\,\yng(3)\,\yng(3)} \\ &= 1\cdot\frac{1}{\sqrt{d_{\yng(3)}}} \\ &= 1\cdot\frac{1}{\sqrt{1}} \\ &= 1 \end{aligned} \tag{7.0.11}$$

Multiply the above by the factor of $\frac{3!}{2!1!} = 3$ and we get the correct result. Next we compute the coefficients for BHR operators comprising $\chi_{\yng(2,1);\yng(2)\oplus\yng(1)}$ – see equation (7.0.9). The first coefficient is given by

$$\begin{aligned} \frac{3!}{2!1!}\langle \yng(2,1), \yng(2)\circ\yng(1), kl; \yng(2,1), \yng(2)\circ\yng(1), kl | \yng(3)\rangle &= 3\left[P_{\yng(2,1)\to\yng(2)\oplus\yng(1)}\right]_{ba} S_{1ab}^{\yng(3)\,\yng(2,1)\,\yng(2,1)} \\ &= 3\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\frac{1}{\sqrt{d_{\yng(2,1)}}}\delta_{ab} \\ &= \frac{3}{\sqrt{2}}. \end{aligned} \tag{7.0.12}$$

Which is in agreement with our comparison above. The projector

$$P_{\yng(2,1)\to\yng(2)\oplus\yng(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

is obtained from

$$P_{\yng(2,1)\to\yng(2)\oplus\yng(1)} = \frac{\left(\Gamma_{\yng(2,1)}((12)) + 1\right)}{2}.$$

See section 6.8 for a more in depth discussion of projectors. We have recently posted the results of this section on the arXiv in [36].

Conclusion

Restricted Schur polynomials are a prominent basis for the change of variables from super Yang-Mills theory to Type IIB string theory. This prominence is due to several of their features. Their two-point functions are diagonal [8] and there exists a product rule for them [34] thus allowing us to compute their multi-point functions in general. Their counting agrees with the number of BPS operators for both $N \to \infty$ as well as N finite, where N is the rank of the gauge group $U(N)$. They are also interchangeable with other BPS operators such as the BHR operators thus strengthening their candidacy.

Restricted Schurs have been suggested as being dual to excited giant gravitons, and further their relation to the dynamics of open strings attached to giants has been made explicit in [16, 29, 30]. Ultimately they may perhaps be a key player in the resolution of the AdS/CFT correspondence by parametrising the gauge invariant variables of the multi-matrix model.

A A Super-Succint Review of Young Diagrams and Representations

In this section we will briefly discuss some basic results of Young diagrams and their relation to representations of S_n and $SU(N)$ groups. For more furbished discussions see [9] and [37] respectively.

The dimension of a given representation of a permutation group S_n is obtained by dividing the order of the group (i.e. $n!$) by the hooks of the Young diagram labelling the representation. As an example let us consider the Young diagram

$$\begin{array}{|c|c|c|}\hline 5 & 3 & 1 \\ \hline 3 & 1 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}$$

which labels a representation of S_6. In each box we have filled in the corresponding hook length for that box. The hook length of a box just corresponds to number of boxes an Γ-shaped curve intersects, with the corner of the Γ-shaped curve located in that particular box. The dimension of the representation is given obtained by dividing 6! by the product of the hooks i.e.

$$d_{\yng(3,2,1)} = \frac{6!}{5 \times 3 \times 3} = 16.$$

Also, the sum of the dimensions of the various subductions of a given Young diagram always adds up to the dimension of the original Young diagram. In table 6.3 we have listed the various $S_3 \times S_3$ subductions for S_6. Let us consider the listed subductions for the Young diagram whose dimension we computed above. The dimensions of the various S_3 components are given by (using the same formula as above)

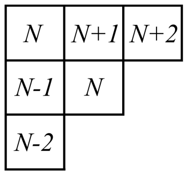


Figure 4: Weights for a Young diagram labelling an S_6 representation.

$$d_{\yng(1,1,1)} = d_{\yng(3)} = 1 \text{ and } d_{\yng(2,1)} = 2.$$

Thus the dimension of the subduction $\left(\yng(2,1),\yng(2,1)\right)$ is 4 (we just multiply the dimensions of the two components.) Thus if we add up the dimensions of all the subductions then we have

$$d_{\left(\yng(3),\yng(2,1)\right)} + d_{\left(\yng(2,1),\yng(3)\right)} + \underbrace{d_{\left(\yng(2,1),\yng(2,1)\right)}}_{R_1} + \underbrace{d_{\left(\yng(2,1),\yng(2,1)\right)}}_{R_2} + d_{\left(\yng(2,1),\yng(1,1,1)\right)} + d_{\left(\yng(1,1,1),\yng(2,1)\right)}$$
$$= 2 + 2 + 4 + 4 + 2 + 2$$
$$= 16,$$

which is the same as the dimension of $\yng(3,2,1)$.

Finally, the weights of a Young diagram, are best defined using an example. Consider figure 4. We start in the top right hand corner and label it with an N. Moving one block to the right we augment the label with 1. Moving one block down we subtract 1. We proceed in this fashion until all the boxes have been filled in. The product of the weights yields the algebraic term f_R. In this case we have that

$$f_R = N \times (N+1) \times (N+2) \times (N-1) \times N \times (N-2).$$

B Micellaneous Mathematical Results

Here we have collected various mathematical results that were not inserted into the main text of the report so as to avoid too many digressions.

Theorem 1 *Given some permutation $\sigma \in S_n$ acting on the upper indices of n copies of some field labelled Z we can rewrite it as σ^{-1} acting on the lower indices instead i.e.*

$$Z^{i_{\sigma(1)}}_{j_1} Z^{i_{\sigma(2)}}_{j_2} \cdots Z^{i_{\sigma(n)}}_{j_n} = Z^{i_1}_{j_{\sigma^{-1}(1)}} Z^{i_2}_{j_{\sigma^{-1}(2)}} \cdots Z^{i_n}_{j_{\sigma^{-1}(n)}}.$$

Proof. We use the fact that if we permute the upper indices in the same way as we permute the lower indices then nothing is changed. Suppose we apply some permutation $\gamma \in S_n$ to both the lower and upper indices of the term on the left in the above i.e.

$$Z^{i_{\sigma(1)}}_{j_1} Z^{i_{\sigma(2)}}_{j_2} \cdots Z^{i_{\sigma(n)}}_{j_n} = Z^{i_{\gamma\sigma(1)}}_{j_{\gamma(1)}} Z^{i_{\gamma\sigma(2)}}_{j_{\gamma(2)}} \cdots Z^{i_{\gamma\sigma(n)}}_{j_{\gamma(n)}}.$$

This is true for any given $\gamma \in S_n$ - thus setting $\gamma = \sigma^{-1}$ yields the desired result. ■

Theorem 2 *The trace of $\sigma \in S_n$ is given by $N^{C(\sigma)}$ (where $C(\sigma)$ denotes the number of cycles in σ) i.e.*

$$Tr(\sigma) = N^{C(\sigma)}.$$

Proof.

We can write σ as

$$\sigma \equiv \delta^{i_1}_{j_\sigma(1)} \delta^{i_2}_{j_\sigma(2)} \cdots \delta^{i_n}_{j_\sigma(n)}.$$

Therefore we have that

$$\mathrm{Tr}(\sigma) = \delta^{i_1}_{i_{\sigma}(1)} \delta^{i_2}_{i_{\sigma}(2)} \dots \delta^{i_n}_{i_{\sigma}(n)}.$$

Now we can group the δ functions into cycles. Each cycle only contributes a factor of N and we are done.

■

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