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Masuku, Mthokozisi

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RADIAL DYNAMICS OF THE LARGE N LIMIT OF MULTIMATRIX MODELS

Mthokozisi Masuku

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Declaration

I, the undersigned, hereby declare that the work contained in this thesis is my own original work. It is being submitted for the Degree of Doctor of Philosophy in the University of the Witwatersrand, Johannesburg. It has not previously in its entirety or in part been submitted for any degree or examination in any other University.

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Mthokozisi Masuku

_____ day of _____ 20 _____ .

Abstract

Matrix models, and their associated integrals, are encoded with a rich structure, especially when studied in the large N limit. In our project we study the dynamics of a Gaussian ensemble of m complex matrices or $2m$ hermitian matrices for $d = 0$ and $d = 1$ systems.

We first investigate the two hermitian matrix model parameterized in "matrix valued polar coordinates", and study the integral and the quantum mechanics of this system. In the Hamiltonian picture, the full Laplacian is derived, and in the process, the radial part of the Jacobian is identified. Loop variables which depend only on the eigenvalues of the radial matrix turn out to form a closed subsector of the theory. Using collective field theory methods and a density description, this Jacobian is independently verified.

For potentials that depend only on the eigenvalues of the radial matrix, the system is shown to be equivalent to a system of non-interacting (2+1)-dimensional "radial fermions" in a harmonic potential.

The matrix integral of the single complex matrix system, ($d = 0$ system), is studied in the large N semi-classical approximation. The solutions of the stationary condition are investigated on the complex plane, and the eigenvalue density function is obtained for both the single and symmetrically extended intervals of the complex plane.

The single complex matrix model is then generalized to a Gaussian ensemble of m complex matrices or $2m$ hermitian matrices. Similarly, for this generalized ensemble of matrices, we study both the integral of the system and the Hamiltonian of the system.

A closed sector of the system is again identified consisting of loop variables that only depend on the eigenvalues of a matrix that has a natural interpretation as that of a radial matrix. This closed subsector possess an enhanced $U(N)^{m+1}$ symmetry. Using the Schwinger-Dyson equations which close on this radial sector we derive the Jacobian of the change of variables to this radial sector.

The integral of the system of m complex matrices is evaluated in the large

N semi-classical approximation in a density description, where we observe the emergence of a new logarithmic term when $m \geq 2$. The solutions of the stationary condition of the system are investigated on the complex plane, and the eigenvalue density functions for $m \geq 2$ are obtained in the large N limit.

The "fermionic description" of the Gaussian ensemble of m complex matrices in radially invariant potentials is developed resulting in a sum of non-interacting Hamiltonians in $(2m + 1)$-dimensions with an induced singular term, that acts on radially anti-symmetric wavefunctions.

In the last chapter of our work, the Hamiltonian of the system of m complex matrices is formulated in the collective field theory formalism. In this density description we will study the large N background and obtain the eigenvalue density function.

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Chapter 1

Structure Of This Thesis

This thesis is organized as follows: in chapter two we provide a very schematic review of large N matrix systems and the AdS/CFT correspondence. We also provide an example of the AdS/CFT correspondence by briefly discussing the BMN limit.

In chapter three we provide a brief review of matrix models, and discuss some examples of theories that have successfully demonstrated the application of matrix models. Chapters two and three provide us insight into some of the powerful methods that can be used to solve a system of $N \times N$ matrices in the large N limit. These two chapters render us the motivation for the research work undertaken in this thesis.

In chapter four we review some of the properties of the single hermitian matrix model. For the integral, we also establish the form of the eigenvalue density function for a harmonic oscillator potential, and show that it satisfies the well known Wigner semi-circle distribution. The Laplacian of this matrix model is identified. Following this, the single hermitian matrix model is given a fermionic description. In this chapter we further present the generalized collective field theory framework and demonstrate its importance by applying it to the single hermitian matrix model.

The density description of the single hermitian matrix model, obtained through the collective field theory formalism, is obtained in the large N limit where we can observe the background geometry that arises.

Chapter five will see us investigate some of the complex properties of the two hermitian matrix model. A new parameterization will be defined, inducing "matrix valued polar coordinates" into the aforementioned hermitian system. After this new parameterization, the infinitesimal line element is used to derive the full Laplacian that mixes both radial and angular degrees of freedom, from which, a Vandermonde determinant with positive definite radial eigenvalues will be identified.

In chapter six, we introduce a (Gaussian) potential for the single complex matrix with radial symmetry, that will be generalized later for a larger system of Gaussian complex matrices. With this potential we identify the partition function of the system, whose stationary condition is obtained. The solutions for this stationary condition will be investigated along the single cut $[x_-, x_+]$ interval of the complex plane such that $x_+ > x_- > 0$ for the radial eigenvalues $x = \rho_i = r_i^2$. The single cut will be extended symmetrically to a double cut $[-x_-, -x_+]$ and $[x_-, x_+]$, whereupon the solution will be investigated.

For the Hamiltonian, the fermionic picture of the singlet sector of the $N \times N$ single hermitian matrix model, obtained in chapter four, is well known. In chapter seven we consider the radially restricted sector of the single complex matrix model and show that this system can also be described by N non-interacting "radial fermions" in $(2+1)$-dimensions.

In chapter eight, we will develop the density description for the single complex matrix model, restricted to the radially symmetric subsector. In this description, the Jacobian of the system will be derived by defining the collective field theory variables that close under "joining" and "splitting", first encountered in chapter four.

To extend the formalism from chapter eight for the single complex matrix model, in chapter nine, a Hamiltonian restricted to radial degrees of freedom with a Gaussian potential that possess enhanced radial symmetry will be presented using the collective field theory formalism. In this radial sector, we will obtain the eigenvalue density function of the system.

In chapter ten we will generalize the number of (the) Gaussian ensemble of

matrices from a single complex matrix to a general number m. This chapter will again see us identify gauge invariant correlators with radial symmetry that close in the radial sector and in which the Schwinger-Dyson equations also close. A more generalized Jacobian will be identified by establishing an identity using Schwinger-Dyson equations.

Following the complexities of chapter ten, in chapter eleven, the partition function with a Gaussian potential that possesses enhanced symmetry for a general Gaussian ensemble of m complex matrices will be investigated using the large N semi-classical approximation. The solutions of the system of complex matrices will be investigated along the single cut $[\rho_-, \rho_+]$ of the real line of the complex plane for $\rho_+ > \rho_- > 0$. Also, these solutions will be extended symmetrically to a double cut of the complex plane to $[-\rho_-, -\rho_+]$ and $[\rho_-, \rho_+]$. In both scenarios, the eigenvalue distribution function will be investigated for the radially restricted sector.

In chapter twelve we mention how the stationary condition and densities for the partition function for a system of a Gaussian ensemble of complex matrices identified in chapter eleven can be related to the zeros of Laguerre and Hermite polynomials.

In chapter thirteen the radially restricted Hamiltonian for a general number of complex matrices is interpreted using "radial fermions" in higher dimensions where a singular form is observed.

In the penultimate chapter of our work, we again work in the radially symmetric restricted closed subsector where the Hamiltonian of a general number of m complex matrices is represented in the density description using the collective field theory formalism. A Gaussian potential will be introduced and defined, following this, the eigenvalue density function of this radially restricted sector will be derived.

The final chapter, chapter fifteen, will be reserved for the discussion and conclusion regarding our work, identifying and addressing problems that we would wish to pursue for future research.

Chapter 2

From t' Hooft To BMN: An Overview

The goal of this chapter is to introduce, in a non-technical manner, important works whose ideas and results influence, to a certain extent, the purpose and objective of our project.

Some of these important concepts presented in this chapter provide a historical bridge to the more recent work that has been carried out in this project.

A map of key ideas, not necessarily following a specific chronological order, illustrating important earlier works that set the precedent for our project, will be presented in this chapter.

To be precise, the following are objectives we wish to carry out in chapter 2:

- Provide a general introduction of the t' Hooft model of large N QCD
- Present a general introduction of Random Matrix Theory
- Briefly review the Dyson Gas approach to solving an ensemble of random matrices
- Provide a non-technical introduction to the AdS/CFT correspondence
- Give an example of an AdS/CFT correspondence: the BMN correspondence

2.1 When QCD Is Non-Perturbative

Quantum Chromodynamics (QCD) has positioned itself as the theory of strong interactions, and an impressive building block that forms part of an edifice of physical theories whose objective is to:
(i) aid in providing a consistent model of Quantum Gravity and
(ii) contribute towards a formulation of physical theories whose aim is to unify all of nature's fundamental forces.

Quantum Chromodynamics is responsible for describing the dynamics surrounding the strong nuclear force. This theory of strong interactions belongs to the SU(3) non-abelian gauge symmetry group.

The theory of strong interactions is made up of quarks and gluons. The quarks are six spin half fermion particles that come in six flavours namely up, down, strange, charm, bottom and top.

Quarks also carry an inherent internal degree of freedom that enable them to have a "charge". This quark charge is referred to as the "colour charge". Colour is the quantum charge associated with QCD. Interactions amongst quarks are mediated by gauge gluon bosons. The colour charge of quarks come in three variations, these being red, blue and yellow charge.

The consistent success of QCD in its ability to make correct predictions that can be verified experimentally can be attributed, in part, to the application of perturbative methods that are applied when performing computations in QCD [8] [9]. These computations are carried out at different energy scales of the theory. QCD has a running coupling constant and this generally means that the coupling constant, g, has an explicit dependence on the energy scales of the theory [10].

In order for us to understand how the running coupling constant evolves with the varying energy scales of QCD, one has to study the so called "beta" function $\beta(g)$ given by the following equation

$$\beta(g) = \frac{\mathrm{d}g(\mu)}{\mathrm{d}\ln(\mu)}, \tag{2.1.1}$$

where $g(\mu)$ represents the running coupling constant of the gauge theory and

μ is associated with the energy/momentum scale of the theory [11].

This function, $\beta(g)$, is determined by the renormalization group of equations and relates the coupling constant of the theory to its momentum scale.

QCD can be probed and understood at both high and low energy regimes. QCD studied in the high energy regime renders the function $\beta(g)$ to be negative therefore the running coupling constant becomes very small [13].

At these high energy/short distance scales, perturbative methods are applicable to QCD since the running coupling constant of the theory becomes weakly coupled at this ultraviolet (UV) (high energy) regime. In this limit, the function $\beta(g)$ has a form that is determined by the renormalization group of equations of the ultraviolet (energy) regime. The function $\beta(g)$, in the ultraviolet limit has a Gaussian fixed point at which QCD becomes an "asymptotically free theory" [12].

Computations in QCD become easier to handle since a dictionary can be developed for the theory of strong interactions in the ultraviolet regime where the perturbative techniques are applicable. In addition, when QCD is probed at such short distance scales, the running coupling constant of the theory diminishes to zero, therefore rendering QCD a weakly coupled theory resulting in calculations that are much easier to compute.

Since the perturbative tools are applicable in studying QCD at high energies where the theory is weakly coupled and reveals asymptotic freedom, it would be interesting to explore general features of QCD at low energies.

As one descends from the ultraviolet regime and moves down to the physics of lower energy scales, the tools of perturbation theory breaks down. When QCD is studied at low energy scales, or more technically the infrared (IR) energy scales, the methods of perturbation theory no longer apply.

The renormalization group of equations that determine the perturbative expansion of the $\beta(g)$ function indicates that as one investigates QCD at large distance scales, the running coupling constant increases, resulting in a strongly coupled theory of strong interactions that is non-perturbative.

One can ask: How can a low energy, strongly coupled and non-perturbative

QCD be understood physically?

In the infrared regime, one can consider a pair of quarks that are strongly coupled. In such a scenario, the force between the pair of quarks is relatively strong and this force increases as the distance between the two quarks is increased. It becomes energetically more favourable to produce two quark anti-quark ($q\bar{q}$) pairs.

The force between the two quarks increases with increasing distance between the quarks. This unique phenomena leads one to deduce that the quarks could never be observed in an isolated state. This feature of low energy QCD is more commonly known as "quark confinement" [14].

From the $\beta(g)$ obtained from the renormalization group equations of the infrared regime, a infrared (Gaussian) fixed point can be observed when the first term of $\beta(g)$ becomes very large ($g \to \infty$) [14]. This represents the point at which quark confinement occurs in low energy QCD.

In essence, the low energy scheme of QCD is where infrared instabilities are prevalent and this is the energy domain of the theory of strong interactions where the tools of perturbation theory fail to provide a physically consistent model of QCD that would explain phenomena such as chiral symmetry breaking [15] [16] and quark confinement.

Hence, non-perturbative low energy QCD still needs to explain the observed phenomena and associated properties of sub-nucleic particles of strong interactions.

The preceding argument highlights, without any technicalities, that QCD, a theory of strong interactions with six quarks, three colour charges and non-abelian symmetry gauge group SU(3) in $(3+1)$-dimensions cannot be consistently solved using perturbative methods across all energy scales accessible to the theory.

The following section will discuss an alternative method to probe the low energy regime of QCD.

2.1.1 Low Energy QCD With A Large Number Of Colour Charges

In his famous work [17], t' Hooft proposed an approximation method to investigate the low energy regime of QCD.

It was suggested by t' Hooft that instead of studying QCD with SU(3) gauge symmetry with a fixed colour charge parameter $N_c = 3$, rather the low energy dynamics of QCD should be investigated for an infinitely large number of colour charges, $N_c \to \infty$, with non-abelian gauge group SU(N_c).

Strong interactions understood using gauge group SU(3) is a more realistic picture of nature. For a generalized case where one considers a model with an infinite number of colour charges, it would be natural to assume that the theory would be complex and completely non-tractable. In fact the opposite is true, QCD becomes perturbatively tractable when one adopts the t' Hooft approach to probe low energy dynamics.

The framework proposed by t' Hooft in order to understand QCD requires that the inverse of number of colour charges N_c be considered as the perturbative expansion parameter of the theory. Therefore the t' Hooft model will have an infinitely large amount of colour charges $N_c \to \infty$ and a non-abelian SU(N_c) symmetry gauge group. With such characteristic features, the t' Hooft model of QCD will provide a perturbative expansion of QCD in the low energy regime where $N_c = 3$ in reality.

The dynamics governing SU(N_c) gauge theory are studied perturbatively in the double scaling limit. In this double scaling limit, the t' Hooft model requires $N_c \to \infty$ and the (square of the) Yang-Mills coupling constant g_{YM} of the theory be taken to zero whilst keeping the product of two constants fixed. The product of N_c and g^2_{YM} provides a definition for the t' Hooft coupling constant λ defined by $\lambda = g^2_{YM} N_c$.

With the t' Hooft coupling constant held fixed in the low energy limit of large N_c QCD, the first term of the $\beta(g)$ function perturbative expansion continues to be negative [13]. In this case, it means that large N_c QCD proposed by t' Hooft

is consistent with an asymptotically free theory.

This ingenious proposal by t' Hooft allows perturbative techniques to be applied to investigate the low-energy physics of QCD since the theory of strong interactions becomes weakly coupled for large N_c and the t' Hooft coupling constant, $\lambda = g_{YM}^2 N_c$, held fixed.

Perturbative QCD studied in the large N_c limit, coupled to SU(N_c) gauge symmetry, can reveal interesting properties of the model such as simplified diagrams that can be associated with physical generic process that governs quark-gluon dynamics [18] .

Fermions of strong interactions are fields in the fundamental representation of SU(N_c) gauge invariance. The propagation of these quark fields can be represented using index notation and can be diagrammatically represented by an oriented line.

It can be elegantly shown that generic physical processes of strong interactions can be represented by planar Feynman diagrams using this index associated with the propagating quark fields [17].

On the other hand, the gauge bosons mediating the strong nuclear force, are represented and transform in the adjoint representation of the SU(N_c) symmetry gauge group. The gluon fields can be shown by a pair of oriented lines in opposite directions [17].

The gluon and quark fields can be coupled and together be diagrammatically shown in "double-line" notation or ribbon graphs as Feynman diagrams in the large N_c limit. The physics that govern large N_c QCD is considerably simplified in the perturbative expansion since the quark-gluon interactions can be understood through Feynman diagrams in double line notation. The double-line Feynman diagrams are coupled to the factors λ and N_c, which are both perturbative expansion parameters of large N_c QCD.

The generic physical processes of large N_c QCD that involve quark-gluon propagators have different double-line Feynman diagrams that appear with different factors of N_c (for propagation lines) and g_{YM} (associated with interaction vertices). Alternatively, double-line Feynman diagrams can be grouped according to

different powers of the t' Hooft coupling constant λ.

The double-line Feynman diagrams that are proportional to N_c^2 are grouped as leading order diagrams. These leading order double-line Feynman diagrams appear as being planar. Planar diagrams can loosely be understood to be diagrams that can be drawn on a plane or on the surface of a Riemann sphere with no lines crossing over each other or intersecting.

In the perturbative regime of large N_c QCD, planar diagrams become dominant as they grow exponentially large compared to non-planar diagrams. The non-planar diagrams tend to be sub-leading in N^2, and these group of diagrams grow factorally in the large N_c limit.

As mentioned previously, different factors of N_c and λ can be associated as coefficients to different double-line Feynman diagrams that could be either planar or non-planar. For the diagrammatic representation of physical processes, the vertex of each diagram can be associated with N_c/λ and propagators can be assigned factors of λ/N_c. The quark loops contribute a factor of N_c.

The double-line Feynman diagrams that are proportional to the $1/N_c$ perturbation expansion parameter have a topological interpretation [19]. The $1/N_c$ factor that is coupled to planar Feynman diagrams can be understood through the Riemann surface on which the planar diagram can be drawn. One can nest the relationship between planar Feynman diagrams and surfaces on which they are mapped [19] by the following equation

$$N_c^{V-E+F}\lambda^{E-V} = N_c^{\chi}\lambda^{E-V}. \tag{2.1.2}$$

In (2.1.2) above, the term V represents the number of vertices of the planar diagram, E are the propagators (or equivalently the number of edges) and the term F are the loops (or equivalently the number of faces). The term
$\chi = V - E + F$ denotes the Euler character of the surface upon which the corresponding Feynman diagram in double line notation will be drawn.

As a result of the topological invariance of the Euler character χ, we can represent it in terms of the number of handles and boundaries of the surface on

which the planar Feynman diagram can be drawn as:

$$\chi = 2 - 2g - b. \tag{2.1.3}$$

Equation (2.1.3) consists of the number of handles g of the surface and the number of boundaries b.

Through the applications of perturbative methods, the generic amplitudes $\mathcal{A}_{\mathcal{QCD}}$ that represent quark-gluon physical processes in terms of double line Feynman diagrams can be neatly represented as follows:

$$\mathcal{A}_{\mathcal{QCD}} = \sum_{h,b=0}^{\infty} (N_c)^{2-2g-b} f_{g_{YM}}(\lambda). \tag{2.1.4}$$

The polynomial $f_{g_{YM}}(\lambda)$, is related to the t' Hooft coupling constant λ. One can deduce from equation (2.1.4) above that the leading order diagrams occur in the absence of quark loops ($b = 0$) and are also planar ($g = 0$), therefore this represents the planar limit sector of large N_c QCD.

In general, surfaces with maximum possible Euler character $\chi = 2$ have the topologies of a sphere or a plane. Planar diagrams can be consistently and smoothly mapped to surfaces with maximum Euler character.

Higher order diagrams that have $g > 0$, will appear to be sub-leading in N_c. Each term in the perturbative expansion that is sub-leading in N_c will have a $1/N_c$ factor resulting in a multitude of non-planar diagrams for higher orders of perturbative expansions.

The low energy dynamics of strong interactions, when perturbatively studied in the large N_c limit of QCD yields rich and insightful features. The simplification of large N_c QCD which uses the number of colour charges N_c as a perturbation expansion parameter is useful because one is able to understand quark-gluon physical processes by studying double line Feynman diagrams.

In the large N_c perturbation regime of QCD, both planar and non-planar diagrams appear but it is the former that tend to dominate and therefore provide an extensively simplified formulation of QCD with an infinite number of colour charges N_c and SU($\mathrm{N_c}$) symmetry gauge group.

Though large N_c QCD is considerably simplified by enabling the applications of perturbative techniques to understand the planar limit, it should be emphasized that the t' Hooft model is an approximation to the real world non-perturbative low energy limit of QCD with $N_c = 3$ number of colour charges and SU(3) non-abelian symmetry gauge group.

Large N_c QCD in itself is approximative but at the moment lacks an exact solution. Therefore large N_c QCD provides an indicative solution to the low energy regime of QCD in $(3 + 1)$-dimensions.

2.1.2 What Of Random Matrix Theory?

In the preceding section, crucial fundamental ideas were introduced. The model proposed by t' Hooft represents an ensemble of $N_c \times N_c$ matrices which can be understood to be a multimatrix model of sorts. This statistical ensemble of matrices can be linked to the action that can be used to derive both planar and non-planar double line Feynman diagrams [21].

In addition, the inverse of the size N_c of the matrices plays the role of an expansion parameter in the perturbation of large N_c QCD, and this parameter is explicitly related to the topology of these double line Feynman diagrams.

The ingenious model provided by t' Hooft to understand the low energy dynamics of QCD can be further simplified when studied through the framework of Random Matrix Theory (RMT) [22].

Matrix models have contributed significantly in the pursuit to unlock and understand the physics that underlies the sub-nuclear nature of atomic particles. Even in the context of modern day physics, multimatrix models have evolved to become a critical analytical tool that can be used to provide a deeper comprehension and mathematical solutions to cutting edge science problems across a multitude of disciplines.

Through the application of matrix models in our project, insightful and unique analytical observations have been arrived at in the large N_c limit. Therefore, large N_c multimatrix models leave much to be desired.

The t' Hooft model of large N_c QCD becomes solvable in some instances of

Random Matrix Theory [21]. The first application of RMT to QCD was by t' Hooft where the author was studying a model of mesons in $(1 + 1)$-spacetime dimensions in the large N_c limit [23].

Random Matrix models were first used by E. P. Wigner to understand and provide a model for nucleic properties that occur during reactions [24]. Since then, Random Matrices have been put to good use in other disciplines of science like statistical physics, mathematics etc.

Wigner wished to provide a mathematical model that would describe the properties of energy spacings when heavy nuclei were undergoing excitations. It made sense to Wigner to let the varying distributions of energy spacings be represented by a statistical ensemble of real symmetric matrices.

The distribution of the energy spacings can be understood by considering an hermitian operator $\vec{H}$ which would represent the Hamiltonian of the system. Wigner proposed that the Hamiltonian operator $\vec{H}$ should be a very large random matrix that is an element of an even larger ensemble of random matrices. Wigner further proposed that this group of random matrices have the same general properties as that of the Hamiltonian operator $\vec{H}$.

From these preceding conditions, the following system can be set up

$$\vec{H}\, v(x_i) = \lambda_W(x_i)\, \vec{H}\ . \tag{2.1.5}$$

The $N \times N$ matrix $\lambda_W(x_i)$ is the eigenvalue matrix of the Hamiltonian operator $\vec{H}$ and $v(x_i)$ is the eigenvalue dependent wavefunction of the system. The intervals between successive $N \times N$ eigenvalue matrices equivalently represents the intervals between successive energy levels. The random $N \times N$ matrices studied by Wigner are taken to be large such that $N \to \infty$.

The distribution of eigenvalues of the system of an ensemble of $N \times N$ random matrices enabled Wigner to solve and show exactly how the nuclear energy spacings are distributed.

In this limit, large N_c QCD can also be understood and be solved using RMT [25]. What truly is exciting and indeed fascinating, is that the perturbative large N_c limit of QCD in the planar limit is exactly solvable [21], but not necessarily

integrable [26]. We now continue to motivate how the t' Hooft model can be understood through RMT.

Random Matrix Theory can be understood as a theory of a random ensemble of $N \times N$ matrices that are generated through a probability function of the theory. In this case we will require that the size of the matrix N be the same as the number of colour charges N_c. This random matrix will be discussed in the large N_c limit for some arbitrary generated potential $V(\psi)$ where ψ denotes randomly generated $N_c \times N_c$ hermitian matrices.

For this system of randomly generated $N_c \times N_c$ matrices ψ, one is able to determine the eigenvalue density function $\rho(E)$. To do this, the following partition function should be defined

$$Z = \int d\psi e^{-N_c \mathrm{Tr}(V(\psi))}. \tag{2.1.6}$$

The probability function that generates the random matrices ψ is given by

$$P(\psi) = \frac{1}{Z} e^{-N \mathrm{Tr}(V(\psi))}, \tag{2.1.7}$$

and has the normalization condition $\int D\psi P(\psi) = 1$.

The above probability function is invariant under the unitarity transformation $P(\psi) = P(U^\dagger \psi U)$.

For this system of randomly generated $N_c \times N_c$ matrices ψ, we would wish to solve an eigenvalue/eigenfunction equation $\psi \mathbf{v} = E\mathbf{v}$ in the planar limit. The eigenvalue system will be solved for a randomly generated $N_c \times N_c$ matrix ψ in the large N_c limit that is generated with probability $P(\psi)$.

Once the system is solved, the distribution of eigenvalues will be captured by the eigenvalue density function $\rho(E)$ and represented through a graph. This eigenvalue distribution graph can reveal interesting features about the system and one can learn about the underlying geometry of the system.

The system of randomly generated hermitian matrices will be solved for the action $S = -N\mathrm{Tr}\left(V(\psi)\right)$, through the path integral Z. The path integral will be integrated over $N_c \times N_c$ matrices using the measure of the theory $d\psi$. Once

solved, the path integral can be entirely represented in terms of the eigenvalues of ψ which simplifies the system and becomes more tractable.

Using RMT to understand large N_c QCD is beneficial. Solving large N_c QCD in this matrix model is similar, in principle, to solving a quantum field theory in $(0+0)$-dimensional spacetime.

In the same spirit as in RMT, one can consider, in particular, Gaussian matrix integrals that are expressed in terms of $N_c \times N_c$ hermitian matrices. The solutions to Gaussian matrix integrals can be associated with the double-line Feynman diagrams as a result of how the matrices ψ are defined. This approach of treating Gaussian matrix model integrals is very similar to the t' Hooft model of large N_c QCD.

The planar limit of RMT can be understood through double line Feynman diagrams using the index notation that is assigned to quark-gluon interactions. Just as in the t' Hooft model, planar diagrams dominate the large N_c limit of RMT.

For some arbitrary chosen potential $V(\psi)$, the real analytic function $G(z)$ that represents a series of quark propagators can be defined for quark-gluon generic processes [21]. From this analytic function, $G(z)$, planar Feynman diagrams can be generated which are topologically invariant. In this limit, under special conditions the density of eigenfunctions $\rho(E)$ can be defined from the solution of the real analytic function $G(z)$.

A famous example that elegantly shows the summation of planar Feynman diagrams and provides an analytic function is shown in the work of BIPZ [27] for eigenvalue distributions/densities. The authors of BIPZ [27] consider an $N_c \times N_c$ single hermitian matrix model.

In their work, the authors provide a solution for planar diagrams that are summed exactly. In addition, for this single hermitian matrix model, the analytic function that satisfies special conditions provides a solution to the density of eigenvalues. The eigenvalue distribution of the single hermitian matrix Gaussian model is shown to be the famous Wigner's semi-circle distribution.

Regarding the work presented in our current project, we will review the

methodology of arriving at the Wigner distribution used by the authors of BIPZ [27] to study the single hermitian matrix model. In the framework of our project, this methodology will be extended by applying it to a multimatrix model. The way in which this is done is unique for multimatrix models and we hope this will offer valuable insight into multimatrix models.

In the framework of RMT, for a particular potential $V(\psi)$ [1] , it can be shown that the Wigner type distribution characterizes the eigenvalues of the density function $\rho(E)$ in the large N_c limit.

2.1.3 The Dyson Gas Approach

There is an alternative method that can be used to understand and solve the real analytical function $G(z)$. This alternate method that is popularly applied in RMT is known as the "Dyson Gas" approach [28] [29] [30].

To heuristically motivate, without any technical details, and argue why the Dyson gas approach is preferable, we consider the following potential:

$V(\psi) = a\psi^2 + g\psi^4$, defined for a random $N_c \times N_c$ matrix ψ. The term a appearing in $V(\psi)$ is defined up to a constant and g can be thought of as the coupling constant of the RMT associated with the potential $V(\psi)$.

The random $N_c \times N_c$ matrix can be diagonalized using unitary $N \times N$ matrices U and $U^\dagger$. Naturally, the random matrix ψ is invariant under unitary transformation. To obtain an explicit representation of ψ, we carry out the transformation $\psi = U^\dagger \lambda_D U$. The matrix λ_D is an $N_c \times N_c$ diagonal matrix of the eigenvalues of the random matrix ψ.

The partition function of the system of random matrices in terms of eigenvalues takes the form

$$\begin{aligned} Z &= \int d\psi e^{-N_c \mathrm{Tr}(V(\psi))} \\ &= \int dU \int \prod_i d\lambda_{D_i} \mathcal{J}_{\mathcal{R}} e^{-N_c \sum_i V(\lambda_i)}. \end{aligned} \tag{2.1.8}$$

[1] The simplest example can be $V(\psi) = \frac{1}{2} m^2 \psi^2$.

The Jacobian $\mathcal{J}_{\mathcal{R}}$ arises when one changes the variables of integration of the measure from $d\psi$ to the new coordinates $d\lambda_{D_i}$. In the second line of equation (2.1.8), the partition function is purely in terms of the eigenvalues λ_i of the matrix ψ.

The integral that runs over the measure dU gives the volume of the gauge group $\mathrm{SU(N_c)}$. The degrees of freedom that are associated with the unitary matrix U of the system decouple. This simplification lends itself as a necessity since the decoupling of the unitary matrix model from the partition function leaves a system that is purely in terms of eigenvalues.

When our model is purely in terms of eigenvalues, the distribution of these eigenvalues when plotted on a graph can reveal much about the system. This eigenvalue distribution can also indicate whether this system defined with a large number of randomly distributed matrices is in fact a system of non-interacting gas molecules.

At this point, without presenting a derivation, the partition function Z, can be shown to be

$$Z = \prod_i \int d\lambda_{D_i} e^{-NE(\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_{N_c})}, \tag{2.1.9}$$

where we have that

$$E(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{N_c}) = \sum_i V(\lambda_i) - 1/N_c \sum_{i \neq j} \log\left(\lambda_i - \lambda_j\right)^2. \tag{2.1.10}$$

The partition function in equation (2.1.9) is in fact equivalent to the partition function of a classical one-dimensional gas with N_c molecules.

One of the advantages of the Dyson approach is that it simplifies the system of $N_c \times N_c$ matrix models ψ with N_c^2 degrees of freedom to a more accessible and simplified model with N_c degrees of freedom.

This system of eigenvalues appearing in the partition function (2.1.9) and the effective action (2.1.10) has a very simple and yet extraordinary physical interpretation. In the semi-classical limit, equation (2.1.9) coupled to (2.1.10)

represents an interacting one-dimensional classical gas with N_c degrees of freedom where the i'th eigenvalue λ_i denotes the position of the i'th molecule.

The gas itself is confined to a potential well $V(\lambda_i)$. Any two molecules confined in the potential well $V(x)$ will repel each other according to $-1/N_c \log{(x-y)^2}$.

Further physical properties can be revealed for this one-dimensional classical gas by studying this system in the semi-classical approximation. Using this approximation method, the classical gaseous system is defined in the continuum limit where the variables of the system become continuous. Previously, the partition function appearing in equation (2.1.9) was in terms of eigenvalues which are discretized. Therefore, according to the semi-classical approximation method we find that

$$V'(\lambda_i) = \frac{2}{N_c} \sum_{i \neq j} \frac{1}{\lambda_i - \lambda_j}. \tag{2.1.11}$$

To obtain a solution to equation (2.1.11) above we need to solve it for continuous variables in the continuum limit. In this limit, an analytical function $G(z)$ can be introduced, and its solution will indicate the form of the eigenvalue density function of the system.

The analytic function is solved in the complex plane z on an interval that is cut along the real axis. A very elegant method has been shown by BIPZ [27] on how this analytical can be solved. We adopt a similar method for our argument in our work. The function $G(z)$ will be subject to special conditions. For instance, on the complex plane, we will require that when $z \to \infty$ then the function $G(z)$ must converge such that $G(z) \to 1/z$.

The function $G(z)$, by satisfying boundary conditions along the cut of the real axis on the complex plane and meeting strict restrictions, does provide a solution that can be related to the eigenvalue density function of the system in the potential well $V(x)$. This will be reviewed in detail in chapter 4.

For quadratic potentials (Gaussian), the one-dimensional classical gas with N_c degrees of freedom can be shown to have an eigenvalue distribution that obeys the Wigner semi-circle distribution.

In our motivation of the Dyson gas approach we originally considered the

system with potential $V(\lambda) = a\lambda^2 + \frac{g}{N_c}\lambda^4$ represented in terms of eigenvalues. The $1/N_c$ term can be understood to represent the temperature of the system [21]. This potential can be shown to have a system of eigenvalues that obeys the Wigner semi-circle distribution.

The work of BIPZ [27] provides an in depth and elegant example of the Dyson approach for a single hermitian $N_c \times N_c$ matrix. The RMT model of BIPZ [27] is a framework that will be understood through our project by considering a parameterized multimatrix model.

The aim of discussing RMT was to highlight some of the special features of the framework that will be of great importance for our project. We also hoped to motivate and illustrate the richness and the usefulness of the methodology considered by Dyson in studying a large system of random matrices.

In general we indicated, through the Dyson gas approach, how a system of $N_c \times N_c$ matrices with N^2 degrees of freedom can be reduced to N_c degrees of freedom by studying a RMT framework.

RMT can be used to study systems with a large number of degrees of freedom to make these more calculable. The big advantage of RMT is that it is exactly solvable, whereas the same cannot be said about the t' Hooft model of large N_c QCD. This is due to the complexity of the space-time valued multimatrix systems.

A solution of large N_c QCD would have to be an exact summation of all the planar (and non-planar diagrams) that emerge from the perturbation theory of the t' Hooft model. This method of solving the low energy regime would be crude and laborious. Alternatively a more elegant approach similar to the Dyson gas approach would be needed to solve the t' Hooft model of QCD where the (planar) large N limit corresponds to a semi-classical limit. In turn, this solution will provide a deeper understanding on how low energy QCD can be understood and solved.

In the context of our project, the issue/challenge becomes that of parametrization for a large system of matrices. For instance, for two matrices M_1 and M_2, it is easy to find a parametrization that yields the eigenvalues of the matrices under

the trace. We observe that:

$$Tr(M_1^2) = \sum_i \lambda_i^2 \qquad Tr(M_2^2) = \sum_i \lambda_j^2, \tag{2.1.12}$$

which both yield N degrees of freedom.

Things become complicated when the product of matrices mix under the trace, for instance: $Tr(M_1 M_2)$, which no longer yields a parameterization that will give us terms that strictly depend on eigenvalues of the matrices M_1 and M_2 since the angular degrees of freedom under the trace don't get eliminated. In such an instance, the number of degrees of freedom no longer grows like N.

In terms of parameterization, for three matrices, for instance, one may try to parametrize invariant loops as:

$$Tr(\prod_{p_i}^{\infty}\prod_{q_i}^{\infty}\prod_{k_i}^{\infty} M_1^{p_i} M_2^{q_i} M_3^{k_i}). \tag{2.1.13}$$

Clearly such an explicit parametrization becomes complicated and unyieldy.

2.1.4 A String At The End Of The Gauge Theory Tunnel

The theory of strong interactions has, within its framework, string-like objects which are flux tubes or Wilson lines [31] [32] [33]. When you have a pair of quark and anti-quark particles and you try to separate the them, a flux tube forms between the two. This flux tube is a result of the force between the quark and anti-quark particle increasing with the separation distance between the quark pair (quark confinement). The pair of quarks transmit a gluon gauge boson between themselves even at large distances.

This behavior of quarks is contrary to what classical physics suggests (Newton's laws) that the force of attraction between two bodies diminishes with increasing distance between the two bodies

In general, upon close observation, these flux tubes of QCD seem to behave like "strings". Hence, motivated by this observation, there have been many attempts to formulate a theory of strong interactions whose fundamental objects are these "string" like flux tubes.

The description of QCD in terms of these string like objects unveiled many surprisingly interesting phenomenological attributes of QCD, as a consequence, a "theory of strings" was born.

2.1.5 Could This Be A Hint For A Gauge/String Theory Duality?

String theory was discovered during a time when particle accelerators were unveiling a zoo of particles such as mesons and hadrons that exist on a sub-atomic scale. In an attempt to formulate a theoretical model that would consistently describe these particles, string theory came about [34] [35].

The motivation behind string theory was that these particles could be understood as different oscillation modes of strings. The idea was great in that it described very well some of the features of the hadron spectrum. Unfortunately, string theory could not provide a complete picture for the hadron spectrum as it suffered from inconsistencies. As a result, Quantum Chromodynamics proved itself as a successful theory of strong interactions and succeeded where string theory could not.

Quantum Chromodynamics, a non-abelian SU(3) gauge theory with three colour charges, $N_c = 3$, successfully described the physics of strongly interacting particles. The t' Hooft model, provided a more simplified model of QCD when the number of colour charges N_c of the theory were made infinitely large bringing back the idea of a "QCD" string underlying the topological expansion. Also, the diagrammatic perturbative expansion of large N_c QCD points towards a free string theory in the large N_c limit.

In the planar limit, the coupling constant of the free string theory g_s can be identified with the double line Feynman diagram expansion parameter $1/N_c$.

If QCD with $N_c = 3$ colour charges possesses similar features as QCD with an infinite number of colour charges $N_c \to \infty$, then this similarity would help provide the correct string model of strong interactions. Considering the preceding relation and using string theory to describe strong interactions, the string model

provides a consistent relation between the mass (m) and angular momentum (J) for the lightest hadron with a given spin. The relation for the lightest hadron produced by the string model was $m^2 \approx TJ^2 + const$ [36].

The preceding relation, is based on a rotating, relativistic string with tension T. The above mentioned relation is an example of how a model of strings can be related to QCD in the large N_c limit. It is evident that the large N_c limit of QCD connects gauge theory with string theory.

The argument presented in this section seems to suggest that a large N_c gauge theory can be related to string theory. This argument is very general in its nature and serves as an indication and not an explicit rigorous derivation.

2.2 Lo and Behold: Gauge Theory Is "Stringy"!

Without presenting pedantic and exhaustive details, we will motivate how gauge theory could have a possible dual string description.

String theory, just like large N_c QCD is a mathematical model that is based on perturbation theory. It is not exactly soluble. In this perturbation framework of string theory, a topological expansion that is associated with generic string processes can be identified.

The Feynman diagrams that represent how strings interact can be likened to the planar diagrams of large N_c QCD [13] [19] [37]. In string theory, the amplitude $\mathcal{A}_{\mathcal{ST}}$ associated with the generic processes of string interactions is given by the following equation

$$\mathcal{A}_{\mathcal{ST}} = \sum_{g,h} g_s^{2g+h-2} k_{g,h}. \tag{2.2.1}$$

In equation (2.2.1) above, the constants g and h have the same definition as in the amplitude $\mathcal{A}_{\mathcal{QCD}}$ (in equation (2.1.4)) of large N_c QCD. Also, g_s in the above equation represents the coupling constant of string theory.

When one compares $\mathcal{A}_{\mathcal{QCD}}$ in equation (2.1.4), and $\mathcal{A}_{\mathcal{ST}}$ in equation (2.2.1), the following identification can be made: $g_s = 1/N_c$. Essentially, by matching

equation (2.1.4) with equation (2.2.1), we observe that the string theory coupling constant g_s plays the role of the $1/N_c$ expansion parameter in large N_c QCD.

The matching of the expansion parameters of string theory and $1/N_c$ of gauge theory presents an important though non-rigorous and non-lucid guide towards the existence of gauge theories having a string theory description.

The equivalence of equation (2.1.4) and equation (2.2.1) seems to suggests that large N_c QCD could be reformulated using string theory. The double line Feynman diagrams in the planar limit of QCD have a topological expansion encoded in the $1/N_c$ expansion parameter. It is this $1/N_c$ parameter that may be mapped onto the world sheet of a propagating string with the same topology as the double line Feynman diagram in the large N_c limit.

Clearly, both equations (2.1.4) and (2.2.1) represent amplitudes and are both based on perturbative expansions. Therefore, the equivalence of $\mathcal{A}_{\mathcal{QCD}}$ and $\mathcal{A}_{\mathcal{ST}}$ should be treated as an allusion that beyond the terrain of gauge theories, string theory resides. In other words, gauge theories and string theories might be related by dualities.

More recently, many examples of dualities are well known. For example, the different dualities that relate the different string theories [38] [39] [40] [41]. When a single theory can be described using (at least) two descriptions to describe the same theory, then these two descriptions are related through a duality.

For example, one description of the theory could be strongly coupled while the other description of the same theory is weakly coupled and vice-versa, then these two theories are dual to each other.

In order to for us to gain a deeper understanding of the low energy regime of QCD, one would hope that there could possibly exist a dual description of QCD. In the low energy domain, QCD is strongly coupled, therefore a dual theory, one might hope, could possibly provide deeper understanding and elucidate on the nature of the strongly coupled gauge theory.

Several indications have been suggested that the dual description of the strongly coupled theory of strong interactions might possibly be string theory. The most well known example of a gauge theory/string theory duality is the AdS/CFT

correspondence.

2.3 The AdS/CFT Correspondence: A General Motivation

The AdS/CFT (Anti-de Sitter/Conformal Field Theory) correspondence is a conjecture that acts as a bridge between theories described by different frameworks, which upon a deeper inspection, are in fact related to each other. On the one end of the AdS/CFT correspondence, one can identify gauge theories that are formulated through the prescriptions of Quantum Field Theory. In addition, on the gauge theory side of the correspondence, a gravitational prescription is absent.

On the other end of the AdS/CFT correspondence, resides superstring theory in ten-dimensions. On this side of the correspondence, gravity is present and the superstring model includes a massless spin two gauge boson naturally.

According to the previous description of the AdS/CFT correspondence, we can essentially deduce that what the correspondence does is to relate a theory that incorporates gravity into its framework to a theory that does not. The theory whose framework incorporates gravity is in D-dimensions and this framework is related to a local field theory in $(D-1)$-dimensions whose model lacks a gravitational description.

The idea of relating theories from different dimensions separated by a single dimension was suggested by t' Hooft in attempting to reconcile the physics of general relativity and the premises of quantum mechanics into a single framework [42] [43].

This idea proposed by t' Hooft can further be understood as an extension of the Holographic principle [44] [45]. The holographic principle was critical in understanding the macroscopic variables of black holes [46] [47] [48] [49] [50] [51].

Therefore, according to the holographic principle, the information of a body in D-dimensions can be uniformly mapped to and stored on its boundary which is a region of lower dimension $(D-1)$. From this boundary region, one can infer properties of the body in higher dimensions.

Classically, the holographic principle is based on the idea of the hologram. A hologram is a devise that is able to contain all the information of a body in $(3+1)$-dimensions on a surface in $(2+1)$-dimensions. t' Hooft extended the principles of holography to understand and attempt to unravel insightful ideas regarding Quantum Gravity.

Therefore the AdS/CFT correspondence provides a dictionary that is based on the ideas of the holographic principle [47]. This dictionary offers a translation between a higher dimensional string theory and a lower dimensional local field theory.

Specifically, the AdS/CFT duality provides an equivalence between conformal field theories and string theories.

On the conformal theory side of the correspondence, one can identify Quantum field theories that are conformal and gauge invariant. These gauge invariant models on the conformal side of the correspondence describe particles, both fermions and bosons, up to spin one.

On the string theory side of the correspondence, the theory is defined on the Anti-de Sitter background and its framework includes a massless spin two particle that acts as a gauge boson for the gravitational force.

The most common conjecture that provides a demonstration of the AdS/CFT correspondence is the one proposed by Juan Maldacena [52] [53] [54] [55]. The Maldacena conjecture suggests an equivalence between a $(9+1)$-dimensional type IIB string theory compactified on the $AdS_5 \times S^5$ and $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory on $(3+1)$-dimensional spacetime.

In this part of the introduction we will present a non-technical approach to demonstrate and understand the AdS/CFT correspondence.

2.3.1 A Tale Of Two Theories

A Brane New World

The Maldacena conjecture is largely understood through Dp-branes. Dp-branes are a group of extended objects on which open strings can attach their end points

[56] [57] [58]. These extended objects are teeming with physical properties that allow for elegant physical solutions.

Dp-branes can be viewed differently in different p-dimensions. For instance, a D0-brane can be identified with a point particle, a D1-brane can be viewed as a one-dimensional extended object such as a string, a D2-brane can be viewed as a two-dimensional surface like the world sheet swept out by a propagating string and so forth. Naturally, the number of dimensions p can be generalized to higher dimensions.

The Dp-branes have inherent properties such as mass, tension and energy. In addition, Dp-branes have a tension that is proportional to the inverse of the coupling constant g_s of a string.

Both open and closed strings display some sort of interaction with Dp-branes. The end points of an open string attached onto the surface of a Dp-brane satisfy particular boundary conditions. These boundary conditions can either be Dirichilet boundary conditions or Neumann boundary conditions. Our argument regarding the Maldacena conjecture will be based on D(irichilet)- branes i.e. Dp-branes defined according to Dirichilet boundary conditions of open strings.

The physics of open and closed strings coupled to Dp-branes can offer insight into the gauge theory side of the AdS/CFT correspondence. The spectrum of open strings can be associated with gauge fields and from the vibrational modes of the closed string the graviton arises naturally.

Parameters of the t' Hooft Model

The t' Hooft model was presented as a large N_c gauge theory where $N_c \to \infty$ and with gauge symmetry group SU(N_c). The coupling constant $\lambda = g_{YM}^2 N_c$ of the t' Hooft model was defined in the double scaling limit, $N_c \to \infty$ and $g_{YM} \to 0$, where it was held fixed.

The parameter N_c represents the number of colour charges of QCD and g_{YM} is a dimensionless coupling constant that controls the interactions of the gauge bosons, also the parameter N_c can be understood as the degrees of freedom of SU(N_c).

In sections that follow we will elucidate the AdS/CFT correspondence in the limit where the degrees of freedom of SU(N_c) are large. To accomplish this we will provide a heuristic non-technical demonstration of the Maldacena conjecture.

The t' Hooft Coupling Constant Related To Strings

The objective for this section is to provide a general motivation as to how, in the $N_c \to \infty$ limit, the 't Hooft coupling constant λ arises from considering interacting open strings attached to Dp-branes [59].

As we mentioned previously, coupled to the open strings are massless gauge fields, and these fields live on a stack of N coincident Dp-branes. Under these conditions it is possible to show that the string coupling constant g_o is related to the Yang-Mills dimensionless coupling constant g_{YM} associated with strong interactions.

The end points of open strings attached to the stack of N Dp-branes can be labeled from 1 to N i.e. we can assign an index i to each of the end points of the open strings attached to any one of the N Dp-branes such that $i = 1, 2, 3 \ldots, N$.

Since g_o governs the interactions of open strings, and the massless fields are gauge bosons living on the open strings, then one can associate the gauge bosons with open strings. Therefore, the coupling constant g^2_{YM} of gauge theory will coincide with the coupling constant of open strings g_o.

We remind ourselves that, for large N_c QCD, whenever the gauge bosons of the theory interact, the process of interaction involves a factor of g^2_{YM}. In general, we also demonstrated that for the SU(N_c) gauge theory, the amplitude that represents the generic processes of interacting gauge bosons involved the g^2_{YM} coupling constant.

Naturally, we can generalize the number of interacting open strings on the stack of N coincident Dp-branes to be very large. In this instance, the amplitude that represents the generic process of interacting massless open strings will be

$$\begin{aligned}\mathcal{A}_n &= \sum_{n=0}^{\infty} c_n \left(g_{YM}^2 N\right)^n \\ &= \sum_{n=0}^{\infty} c_n \lambda^n, \end{aligned} \tag{2.3.1}$$

where c_n is some constant of order n.

The above equation elucidates how, for a system of interacting open strings whose end points are attached to a stack of N coincident Dp-branes, the 't Hooft coupling constant emerges naturally (compare with equation (2.1.4)).

In general, the amplitude that governs interacting open strings on N Dp-branes can be shown to be equal to the amplitude that describes the interactions of gauge bosons in $\mathrm{SU(N_c)}$ gauge theory.

It is also possible to associate both planar and non-planar diagrams to the amplitude that governs interacting open strings on the N coincident Dp-branes.

If we take a number of N Dp-branes to be equivalent to the number of colour charges N_c (which we assumed implicitly without stating it), interesting indicative results can be observed between the parameters of the 't Hooft model and the framework of interacting open strings.

The emergence of the 't Hooft coupling constant λ in the framework of open strings attached to N coincident Dp-branes essentially demonstrates the following:

(i) a relationship between the dimensionless coupling constant g_{YM} of strong interactions and the coupling constants that regulates the interaction of the massless open strings and,

(ii) how the 't Hooft coupling constant is involved in the amplitude $\mathcal{A}_n$ of open string generic processes.

The above argument is a non-technical indicative demonstration that large N_c gauge theories with $\mathrm{SU(N_c)}$ symmetry might be related to string theories. This is shown through the relationship between the parameters of both gauge and string theory.

Gravitational Effects In The Company Brane-iacs

In this section, we provide some insight as to how Dp-branes induce gravitational effects in the background geometry [59] [60]. This insight plays an important role in understanding the Maldacena conjecture.

We mentioned earlier that Dp-branes have properties such as mass/energy and tension. Therefore according to General Relativity, it means that Dp-branes can induce curvature into the background geometry. Curvature introduced into background geometry can loosely translate to mean that there is a gravitational force as a result of Dp-branes introduced into the spacetime geometry.

To get a clearer understanding on how the presence of Dp-branes can induce gravitational effects, first we consider a stack of N Dp-branes. The parameter N can be taken to be large [2].

These N coincident Dp-branes with mass M will be wrapped around a compact space of volume V_p and will be a distance r away from some arbitrary body with mass m. These N coincident Dp-branes will have tension T_p given by the following equation

$$T_p = \frac{1}{g_c} \frac{1}{\left(\sqrt{\alpha'}\right)^{p+1}}. \tag{2.3.2}$$

In equation (2.3.2) above, the tension of the N coincident Dp-branes T_p is proportional to the inverse of the string coupling constant g_c and $\alpha' \approx l_s^2$ is the characteristic string length.

We introduce a scaling factor R in D' spacetime dimensions such that $R^{D'-3} \equiv G^{(D')}M$. This scaling factor, can be related to the parameter N and the string coupling constant g_c. The constant D' represents the spacetime dimensions in which the N Dp-branes are defined and $G^{(D')}$ is the Newton gravitational constant in D'-dimensions [3].

The scale factor R can be identified with the radius of a Schwarzschild black hole of mass M, up to terms of order one.

[2] We will follow here the discussion in [59].

[3] The Newton gravitational constant in D'-dimensions: $G^{(D')} \approx g_c^2 \left(\sqrt{\alpha'}\right)^{D'-2}$.

A stack of N Dp-branes compactified on a p-dimensional space with volume V_p will have the following mass M

$$M = NV_p \frac{1}{g_c} \frac{1}{\left(\sqrt{\alpha'}\right)^{p+1}}. \tag{2.3.3}$$

In the dimensionally reduced $(D' - p)$ spacetime the stack of N Dp-branes with total mass M will appear as a point source for an observer a large distance away.

At this point, we can obtain a characteristic equation for the N coincident Dp-branes that involves the string length α', g_c and N. This system of Dp-branes can be understood through the following equation

$$\left(\frac{R}{\sqrt{\alpha'}}\right)^{D'-p-3} \cong g_c N. \tag{2.3.4}$$

Equation (2.3.4) can be taken in two limits. In the first limit, we can study equation (2.3.4) for weakly coupled string theory where $g_c N \to 0$. In this weak coupling limit of string theory, $R \to 0$. This behavior for R means that we can neglect gravitational effects in the background geometry.

Curvature of the spacetime region occupied by N coincident Dp-branes with total mass M will be minimal, resulting in negligible gravitational effects [59].

For the second limit, we consider a strongly coupled string theory specified by the condition $g_c N \to \infty$. We see in equation (2.3.4) that in the limit $g_c N \to \infty$ we find that $R >> 1$. In the limit where the string theory is strongly coupled, the gravitational effects cannot be neglected, this will induce a back reaction into calculations performed in this spacetime region occupied by the N coincident Dp-branes.

Maldacena Conjecture

The preceding section provided an idea on how a stack of N Dp-branes can introduce gravitational effects into the spacetime geometry they occupy. These N Dp-branes have properties such as total mass M, volume V_p and tension T_p.

Understanding the effects due to the N Dp-branes on the spacetime region they occupy is important in understanding the AdS/CFT correspondence.

Maldacena formulated the AdS/CFT background by considering a stack of N D3-branes in $D' = (9+1)$-dimensional Minkowski spacetime.

On the N D3-branes one finds the low energy limit of type IIB string theory. Maldacena conjectured that the low energy limit of type IIB string theory must be equivalent to a conformally invariant maximally supersymmetric $\mathcal{N} = 4$ Supersymmetric Yang-Mills (SYM) theory in $(3+1)$-spacetime dimensions.

The type IIB string theory lives in the $AdS_5 \times S^5$ space, and its low energy limit is a supergravity theory. On the string theory side of the Maldacena conjecture, are the closed strings of type IIB (string) theory whose spectrum contains the graviton. The spectrum of massless open strings is coupled to the gauge theory degrees of freedom.

On the field theory side, the maximum amount of supercharges the $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory can have is four i.e. four spinor charges ($\mathcal{N} = 4$).

To show the equivalence between the conformal field theory and string theory, as conjectured by the AdS/CFT correspondence, one must consider the symmetries of both theories. To make manifest the beautiful structure of the Maldacena conjecture, it is of utmost importance that

(i) we consider the low energy regime of type IIB string theory in
$D' = (9+1)$-dimensional spacetime approximated by type IIB supergravity on N coincident D3-branes and,

(ii) we show that the symmetries of $\mathcal{N} = 4$ SYM match with those of low energy type IIB string theory.

The AdS/CFT correspondence will be examined starting in the limit where the spacetime geometry is a flat $D' = (9+1)$-dimensional Minkowski spacetime and there is no curvature in the background geometry such that $g_c N << 1$. We will also comment about the AdS/CFT correspondence in the limit where

gravitational effects due to the N D3-branes cannot be ignored, in the limit $g_c N >> 1$.

In Weakness And In Strength: Gravitational Effects Due To D3-branes

Below, we will review the role of type IIB string theory whose low energy limit is approximated by type IIB supergravity theory.

Previously we mentioned that the coupling constant g_o of open strings restricted to the surfaces of N coincident Dp-branes is related to the dimensionless Yang-Mills coupling constant g_{YM} that governs the interactions of $\mathrm{SU(N_c)}$ gauge theory for a large number of degrees of freedom N_c. The two coupling constants were shown to be related as follows: $g_o^2 \approx g_{YM}^2$.

The coupling constant g_c of closed strings permeating in $D' = (9+1)$-dimensional Minkowski spacetime is related to g_o by the relation $g_c \approx g_o^2$. In total we can relate all coupling constants across the spectrum for gauge theory, open and closed string theory through $g_o^2 \approx g_{YM}^2 \approx g_c$.

The above relations of the gauge and string theory coupling constants are merely approximations and not exact, they just give us an inclination of how they are related.

We will consider a stack of N coincident D3 branes in flat $(9+1)$-dimensional spacetime. Since the N Dp-branes have properties such as mass and tension, we can investigate the gravitational effects of this system in flat $(9+1)$-dimensional spacetime.

We will first start off by considering the system of N coincident D3-branes taken in the limit $g_c N <<< 1$.

In the case where $g_c \to 0$, the closed strings propagate freely and are non-interacting in $(9+1)$-dimensional spacetime. Similarly the open strings on the N coincident D3-branes are non-interacting due to the relation of the coupling constants.

In the limit $g_c N << 1$, the gravitational effects due to the N coincident D3-branes on the background geometry is minimal. This means that the N D3-branes can be treated as though they are embedded in flat $(9+1)$-dimensional

spacetime.

The system we consider will be studied in the low energy limit E of type IIB string theory, this implies the following condition

$$E << \frac{1}{l_s}, \tag{2.3.5}$$

where l_s is the characteristic string length. We will require that the energy E be very small when compared to the string energy scale $1/\ l_s$. When $l_s \to 0$, the relation (2.3.5) will be required to hold.

In the limit where the string length approaches zero, the massive open string states restricted onto the stack of N D3-branes are mostly intractable and inaccessible. As a result, only massless gauge degrees of freedom are confined to the D3-branes.

Also, when $l_s \to 0$, the Newton gravitational constant $G^{(10)} \to 0$. Since $G^{(10)}$ vanishes, this naturally implies that the closed string coupling constant g_c also goes to zero. Therefore in the limit $G^{(10)} \to 0$, closed strings propagate freely in background geometry with no interactions.

For string length scales that are very small, the remaining gauge degrees of freedom on the N coincident D3-branes are massless U(N) Yang-Mills fields. In the limit where $l_s \to 0$, the U(1) gauge field decouples from the system, leaving supersymmetric SU(N) non-abelian gauge fields in four-dimensions.

In summary, by considering the low energy regime of N coincident D3-branes in the limit where $g_c N << 1$, we can make the following inferences regarding the physics of this system,

- Freely non-interacting closed strings propagating in $(9+1)$-dimensional flat Minkowski spacetime are decoupled from the physics of the bulk

- The physics of the N coincident D3-branes is described by SU(N) $\mathcal{N} = 4$ Supersymmetric Yang-Mills massless gauge fields confined to the D3-branes.

The second case that will be considered in the argument that follows is for a system of N coincident D3-branes that induce curvature into the background geometry. Simply put, we now consider the system of D3-branes in the limit $g_c N >>> 1$, but still for low energy.

In the limit $g_c N >> 1$, the N coincident D3-branes are defined on a background geometry of $(9+1)$-dimensional spacetime that is no longer flat. In addition to having total mass M and other physical properties, the stack of N D3-branes also carry charge[4]. Complex and intractable field equations of massless modes of type IIB supergravity string theory describe the physics that governs the stack N D3-branes system.

The configuration of the system of Dp-branes has a solution that could be understood through the background geometry with induced curvature. This geometrical solution that describes the configuration of D3-branes is a plane with an infinite well at the center (the throat).

An infinite distance away down the throat, lies the solution whose geometry describes the physics of the N D3-branes where the horizon is found.

The throat asymptotically converges into a cylinder as one moves towards the origin of the throat. The circles that surround the throat converge into circles of constant radius. These circles are known as the "circumference of the throat". The circle an infinite distance down the throat is the horizon. Any point that lies on the plane will naturally be an infinite distance away from the location of the horizon.

We should remember that we are discussing the geometry of N coincident D3-branes that induce curvature into the geometry of the spacetime fabric in $(9+1)$-dimensions.

In this spacetime, when the N D3-branes are extended along the coordinates (x^1, x^2, x^3), an observer who assumes a position specified by the coordinates $(x^4, x^5, x^6, x^7, x^8, x^9)$ will see the configuration of D3-branes as a point mass. The coordinates that specify the framework of the observer are transverse to the coordinates of the system of D3-branes.

[4]The N coincident D3-branes carry a Ramond-Ramond charge see [19] [61] [62].

The configuration of D3-branes along the six-dimensional transverse coordinates, are surrounded by five-dimensional spheres S^5. The geometry of the horizon arises in this transverse space an infinite distance down the throat.

The horizon, a circle that surrounds the origin of the throat an infinite distance away, is surrounded by S^5 spheres with constant radius R. The volume of the S^5 spheres with radius R that surround the horizon converge to a constant value.

We can now see the emergence of a throat geometry with a horizon arising from the system of N coincident D3-branes. This is the same configuration of Dp-branes with mass M considered in the limit $g_c N >> 1$ yielding non-negligible gravitational effects on the background geometry.

When one provides a mathematical formulation to describe the gravitational solution of the throat geometry, the metric of this geometry is encoded with the mathematical structure of the near horizon geometry. Within the metric that provides a gravitational solution, one finds the metric that describes the near-horizon geometry which is identical to the metric of the $AdS_5 \times S^5$ geometry.

The metric that describes the $AdS_5 \times S^5$ is

$$\mathrm{ds}^2 = \underbrace{R^2\left[d\rho^2 - dt^2\cosh^2\rho + \sinh^2\rho d\Omega_3^2\right]}_{AdS_5} + \overbrace{R^2\left[d\theta^2 + d\varphi^2\cos^2 + \sin^2\theta\left(d\Omega_3'\right)^2\right]}^{S^5}. \tag{2.3.6}$$

In the above metric of $AdS_5 \times S^5$, the global coordinates that describe the anti-de Sitter geometry AdS_5 are (t, ρ, Ω_i) and the global coordinates that describe the geometry of the five sphere are $(\varphi, \theta, \Omega_i')$.

The four spacetime dimensions that make up the AdS_5 geometry are coordinates parallel to the system of N coincident D3-branes plus a radial dimension that extends transverse to the configuration. In total, there are five-dimensions that describe the AdS_5 space.

The S^5 spheres are made up by five-dimensions in the transverse direction. In total, the sum of the AdS_5 and S^5 spatial extensions make up the total coordinates of the $(9+1)$-dimensional spacetime.

Having discussed the geometry underlying the system of N coincident D3-branes which consequently gave rise to the $AdS_5 \times S^5$ background, we now wish to understand the physics of the system of D3-branes coupled to the near horizon throat geometry.

An observer whose frame of observation is positioned an infinite distance away from the near horizon geometry, sees redshifted energy excitations that come from the horizon. This simply means that discrete finite energy excitations produced near the horizon will appear as being redshifted for an observer far away from the throat.

The discrete finite amount of energy excitations from the horizon appear with an even lesser amount of energy for an observer an infinite distance away since this energy radiating from the near horizon is redshifted. Therefore, the near horizon energy excitations will reduce the amount of energy for an excitation or equivalently result in a longer wavelength for the radiation when observed from far away.

An observer positioned far away who sees low energy excitation from the throat geometry, he/she either observes fixed amounts of energy excitations produced from the horizon or the observer sees long wavelength (low energy) excitations from the system of D3-branes.

Interestingly, the two observations made by the observer decouple. This means that for an observer located an infinite distance away, the horizon of the throat geometry looks much smaller than the long wavelength of the energy excitations. Therefore the energy excitations from the horizon are never captured by the observer.

In addition, the original energy excitations do not have enough energy to overcome the energy barrier near the horizon geometry to escape to infinity. These primary excitations are never captured.

The above system of the near horizon geometry can be well approximated by a system of two decoupled configurations. These decoupled systems are

- A flat $(9 + 1)$-dimensional Minkowski spacetime with low energy closed strings propagating freely a distance far away from the near horizon geom-

etry

- A configuration of type IIB superstrings that are permeating the near horizon $AdS_5 \times S^5$ geometry.

We have presented an argument wherein we consider a system of N coincident D3-branes in the low energy limit E defined by equation (2.3.5). The system of N coincident D3-branes was characterized by the two limits of g_cN. In both limits the parameter N was assumed to be very large yet finite and the closed string coupling constant g_c adjusted to define the limits reached by the product g_cN.

In the first limit, where $g_cN << 1$, the gravitational affects on the $(9+1)$-dimensional Minkowski spacetime, due to the presence of the system of D3-branes, are negligible and close to zero. In this limit, the background is approximated to be flat.

The second limit, where $g_cN >> 1$, causes curvature of the Minkowski spacetime and in this instance the gravitational effects due to the presence of the D3-branes cannot be ignored. Calculations performed under these conditions have to take the backreaction due to the D3-branes into account.

In both limits of g_cN, the system of N coincident D3-branes resulted in two decoupled systems.

When $g_cN << 1$, we established that one of the decoupled systems is a local SU(N) gauge symmetry field theory of $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory on the N coincident D3-branes. In the limit where $g_cN >> 1$, the system reduced to a theory of type IIB closed strings propagating in $AdS_5 \times S^5$ geometry.

In the limit where the product g_cN is generalized, we would still observe the same decoupled systems with closed strings.

Therefore, for any value g_cN, the same physics can be described consistently by the theories that arise from the decoupled systems. One can use the prescription of $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory or one can use closed strings near the horizon geometry of type IIB superstring theory whose low energy limit is type IIB supergravity in the $AdS_5 \times S^5$ to describe the physics (of the D3-brane

configuration). Both theories will provide a consistent description.

The argument provided above provides an indicative motivation for the Maldacena conjecture and demonstrates that $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory is equivalent to type IIB string theory on the $AdS_5 \times S^5$ spacetime geometry.

Matching Global Symmetries Of The Maldacena Conjecture

Preceding this current section, we provided an argument that demonstrated how the AdS/CFT correspondence can be deduced by considering a system of N coincident D3-branes in $(9 + 1)$ spacetime dimensions.

Further proof that supports and elucidates the AdS/CFT correspondence are the matching global symmetries associated with the theories on either side of the duality [52] [63] [64]. Below, in what follows, we will explicate the matching global symmetries between $\mathcal{N} = 4$ SYM and type IIB string theory on the $AdS_5 \times S^5$ geometry.

On the gauge theory side of the AdS/CFT correspondence, one finds conformal field theory that preserves conformal invariance in $(3+1)$-dimensions. These conformal symmetries consist of operators that define the Lie algebra of the respective field theories and are made up of a combination of Lorentz generators and spacetime translation generators. In addition, there are other operators of the conformal symmetry group that generate scale transformations and special conformal translations.

In total, there are 15 operators that generate the symmetries of conformal field theories in $(3 + 1)$-dimensional Minkowski spacetime.

We will now consider the symmetry dynamics on the string theory side of the AdS/CFT correspondence.

The closed strings of type IIB string theory propagate freely in the bulk of the spacetime geometry due to the gravitational excitations when $g_c N >> 1$. The geometry where the closed string propagate is described by five-dimensional anti-de Sitter space AdS_5.

The anti-de Sitter spacetime is described by a Lorentz manifold that possess

maximal symmetry. The Lorentz manifold is associated with a constant negative scalar curvature. This five-dimensional anti-de Sitter space represents a vacuum solution with maximal symmetry of the Einstein equations with a negative cosmological constant.

The isometries of AdS_5 spacetime are generated by 15 operators and satisfy an algebraic structure similar to $\mathcal{N} = 4$ SYM conformal field theory in $(3 + 1)$-dimensions.

In order for the AdS/CFT to be consistent, the symmetries of (4+1)-dimensional AdS_5 spacetime must match those of conformal field theory in (3+1)-dimensional spacetime. The isometries of S^5, the five-dimensional sphere, have a symmetry algebra that matches the superconformal symmetry of conformal field theories in $(3 + 1)$-dimensions. Within Supersymmetric Yang-Mills, the symmetries are preserved when a set of scalar fields and a set of fermionic fields are rotated amongst each other and these symmetries match the ones found in S^5.

The conformal field theory in $(3 + 1)$-dimensions possesses a superconformal symmetry group $\mathrm{SU}(2, 2|4)$. This superconformal symmetry has bosonic subgroups which are the conformal groups: $\mathrm{SU}(2, 2) \times \mathrm{SU}(4)_{\mathcal{R}} \approx \mathrm{SO}(2, 4) \times \mathrm{SO}(6)_{\mathcal{R}}$.

The bosonic subgroup $\mathrm{SU}(2, 2) \approx \mathrm{SO}(2, 4)$ represents the conformal symmetries in $(3 + 1)$-dimensions. The symmetry $\mathrm{SU}(4)_{\mathcal{R}} \approx \mathrm{SO}(6)_{\mathcal{R}}$ represents the $\mathcal{R}$-symmetry group that rotates the $\mathcal{N} = 4$ supercharges into one another. The $\mathcal{R}$-symmetry group can be understood to be a symmetry that does not commute with other supersymmetries and acts as a rotation group of the space transverse to the N D3-branes.

The $\mathcal{N} = 4$ SYM is conformally invariant, and this means that it is invariant under the global superconformal transformations that we showed above.

We will now take a look at the global symmetries of type IIB string theory on $AdS_5 \times S^5$ spacetime.

The AdS_5 geometry has the isometry group $\mathrm{SO}(4, 2)$. Also, the S^5 sphere has the rotation symmetry group $\mathrm{SO}(6) \equiv \mathrm{SU}(4)$. In total, the $AdS_5 \times S^5$ spacetime will have the full symmetry $\mathrm{SU}(2, 2) \times \mathrm{SU}(4)$ which is a product of the symmetries of AdS_5 and S^5. Therefore the full symmetry group of the string theory on the

$(9+1)$-dimensional $AdS_5 \times S^5$ is the supergroup SU$(2,2|4)$.

The supergroup SU$(2,2|4)$ emerges on the string theory side because of all the 32 Poincaré symmetries, 16 of these are preserved by the N coincident D3-branes and the maximally supersymmetric AdS spacetime. The 16 Poincaré supersymmetries will be increased to the full 32 by the addition of 16 conformal symmetries. This will give the total global symmetry of SU$(2,2|4)$ on the string theory side.

In conclusion we see that on the conformal theory side, the $\mathcal{N}=4$ Supersymmetric Yang-Mills consists of 32 supercharges characterized by the superconformal symmetry SU$(2,2|4)$ which is the same as that of the supergroup SU$(2,2|4)$ on the string theory side on the $AdS_5 \times S^5$ geometry.

Matching The Fundamental Parameters Of The Maldacena Conjecture

By considering global symmetries and the physics of N coincident D3-branes in a special limit, the preceding sections illustrated the correspondence between an SU(N) $\mathcal{N}=4$ Supersymmetric Yang-Mills local gauge field theory and type IIB superstring theory whose low energy limit is type IIB supergravity on the $AdS_5 \times S^5$ spacetime.

Essentially the correspondence provides a dictionary for a gauge theory and a string theory.

The gauge theory, an $\mathcal{N}=4$ SYM, is defined in $(3+1)$ spacetime dimensions and type IIB supergravity theory exists in $(9+1)$ spacetime dimensions of the $AdS_5 \times S^5$ geometry. To crystallize the dictionary of the AdS/CFT correspondence that relates gauge and string theories, we can relate the fundamental parameters of two theories on either side of the duality.

For the gauge theory side of the correspondence, we find two-dimensionless parameters,

- the QCD dimensionless coupling constant g_{YM}

- the number of colour charges $N_c \equiv N$ of the quarks.

We also defined the t' Hooft coupling constant $\lambda = g_{YM}^2 N$ in the large N limit, $N \to \infty$, that acts as a parameter of the theory and is associated with topological Feynman diagrams.

On the string theory side of the Maldacena conjecture we find type IIB superstring theory on $AdS_5 \times S^5$ spacetime. Associated with the string theory are the following parameters

- the coupling constant g_c for closed strings, related to the coupling constant of open string g_o through $g_c \approx g_o^2$

- the radius of S^5, $\left(R/\sqrt{\alpha'}\right)$, expressed in natural units of string length,which are identical to that of AdS_5.

Massless open strings propagating on the surfaces of N coincident D3-branes give rise to massless gauge bosons of SU(N) gauge theory, we can relate the coupling constants of the two theories by $g_{YM} \approx g_o$. The coupling constants of both open and closed strings can be related to the gauge coupling constant by $g_{YM}^2 \approx g_o^2 \approx g_c$. To be precise, we have

$$g_{YM}^2 = 4\pi g_c. \tag{2.3.7}$$

Previously we mentioned that the spacetimes of AdS_5 and S^5 have the same effective radius $\left(R/\sqrt{\alpha'}\right)$, where R is also the radius of the horizon of the throat geometry.

In terms of the parameters R, l_s, g_c and N, we can define the following relations between the parameters of gauge and string theories

$$g_c = \frac{g_{YM}^2}{4\pi} \qquad \Longleftrightarrow \qquad g_c = \frac{\lambda}{4\pi N}, \tag{2.3.8}$$

$$\frac{R^4}{l_s^2} = g_{YM}^2 N \qquad \Longleftrightarrow \qquad \sqrt{\lambda} = \frac{R^2}{l_s}. \tag{2.3.9}$$

When N is taken to be very large, $N \to \infty$, with the t' Hooft coupling constant held fixed, the closed string coupling constant g_c in equation (2.3.8) becomes very small, and in turn λ will be small.

In this regard, equation (2.3.8) seems to suggest that when N is large, both type IIB supergravity string theory and $\mathcal{N} = 4$ SYM seem to be weakly coupled. If we extend this inference to equation (2.3.9), we observe that when λ is small, then the radius R of $AdS_5 \times S^5$ geometry is small which implies that the radius of S^5 will be small.

With R being small, it presents a challenge because type IIB supergravity becomes intractable and very difficult to handle. It is also difficult to produce meaningful results when the radius R is small.

To perform calculations with meaningful results on the string theory side, we need R to be large for a weakly coupled, $g_c \to 0$, string theory. In the limit that the radius R is large, this means that S^5 will have a large radius and the curvature of the background geometry will be minimal therefore making it much easier to perform calculations of type IIB superstring theory.

Upon close inspection in equation (2.3.9), in the limit where R is large, this means that λ will also be large. If the t' Hooft coupling constant is large, it means that large N QCD becomes a strongly coupled SU(N) gauge theory therefore making it nearly impossible to do calculations that will yield meaningful results on the gauge theory side.

It is very difficult to work with a strongly coupled gauge theory, therefore a special ingredient will be needed to make things easier. This special ingredient is supersymmetry. What supersymmetry introduces is the ability for some gauge theory observables not to depend on λ when $g^2_{YM} = 0$. These observables are said to be "protected observables".

The quantities computed on the gauge theory side with supersymmetry can now be compared to the observables appearing on the string theory side. Also, in the limit that $\lambda \to \infty$, the string modes decouple and type IIB supergravity string theory holds and it becomes easier to perform calculations. Hence, we see a weak-strong coupling duality.

In the limit that $\mathcal{N} = 4$ SYM is defined in the large λ limit, it acquires observables that are hard to calculate, but these observables can easily be calculated on the string theory side of the duality.

This demonstrates the weak/strong coupling nature between gauge theories and string theories of the AdS/CFT correspondence.

2.4 The Maldacena Berenstein Nastase Limit

The Maldacena conjecture can be extended beyond the supergravity states of type IIB string theory.

In the spirit of the AdS/CFT correspondence, there is a holographic principle that relates the spectrum of closed strings permeating the pp-wave geometry and $\mathcal{N} = 4$ Supersymmetric Yang-Mills defined in $(3 + 1)$-dimensions.

This holographic relationship between strings on the pp-wave geometry and a gauge theory in $(3 + 1)$-dimensions is famously known as the "BMN limit", or Berenstein-Maldacena-Nastase limit [65] [66] [67] [68] [69].

In the BMN limit, there is a restricted class of operators on the gauge theory (SYM) side that correspond to strings in pp-wave geometry. This restricted class of BMN operators, are coupled to an anomalous dimension that is a function of the effective 't Hooft coupling constant λ'. These BMN operators can be shown to be equivalent to string states in the BMN limit.

In general, the restricted class of BMN operators on the SYM side of the correspondence come with an anomalous dimension that can be equivalently mapped to the string theory side. On the string theory side of the BMN correspondence, the gauge operators correspond to a spectrum of closed string states with large angular momentum J in a flat spacetime geometry

On the gauge theory side of the BMN correspondence one finds the single trace operator, namely chiral primary operator $\mathrm{Tr}\left(Z^J\right)$. These chiral primary operators on the Yang-Mills side can be shown to be equivalent to the spectrum of a perturbative closed string theory in a space with no curvature in the pp-wave geometry.

The gauge theory operators, $\mathrm{Tr}\left(Z^J\right)$, are defined with large angular momentum J and are equivalent, in the BMN correspondence, to a massless gauge particle with spin two, the graviton.

The BMN correspondence seems to extend the original dictionary of the Maldacena conjecture by providing us with a more concrete example of an AdS/CFT correspondence. The emergence of the graviton on the string theory side that can be related to operators on the gauge theory side is strongly indicative of a gauge/gravity duality.

2.4.1 BMN Parameters Of The Theory

Having hinted in the preceding section that the BMN correspondence could be a possible holographic principle between a gauge theory and a theory of gravity, we provide further details of this notion in this section. Our objective for this part of the introduction is to show how parameters of the gauge theory relate with those on the string theory side in the BMN correspondence.

The background geometry in which the BMN correspondence is defined is special, it is a pp-wave geometry. To obtain the pp-wave background geometry, the $AdS_5 \times S^5$ geometry of type IIB supergravity is considered in the Penrose limit. In this limit, closed strings move along the circumference of S^5 with a large angular momentum J.

The metric that encodes the plane wave geometry is maximally supersymmetric, this means that, the closed strings that propagate in this spacetime geometry can be described by a type IIB maximally supersymmetric string theory. Since the action of these type IIB closed strings propagating in the plane wave geometry is easily defined and tractable in the light cone gauge, this means that the closed string spectrum is easily calculable in this gauge.

Now, on the gauge theory side of the correspondence we find the chiral primary operators $\mathrm{Tr}\left(Z^J\right)$. These chiral primary operators are defined with large R-charge J and are also required to be gauge invariant.

The gauge theory side of the BMN correspondence is described by $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory. On the gauge theory of the correspondence, the framework is defined in the double scaling limit where the number of colour charges N and the R-charge J are taken to be infinitely large.

In this double scaling limit, the perturbative closed string spectrum can be

obtained in the pp-wave geometry. The parameters of the gauge theory side of the correspondence are given by

$$\lambda' = \frac{g_{YM}^2}{J^2}, \qquad g_f = \frac{J^2}{N}. \tag{2.4.1}$$

In the BMN limit, the effective 't Hooft coupling constant λ' and g_f are kept fixed when $J, N \to \infty$. The constant g_f is the effective genus of the double line Feynman diagrams [70].

The effective t' Hooft coupling constant λ' and the effective genus counting parameter g_f can be related to the energy/mass scale μ, the square of the characteristic string length or the Regge slope α' and the momentum of the vibrating closed string mode p^+.

The relationship is

$$\lambda' = \frac{g_{YM}^2 N}{J^2} = \frac{1}{\left(\mu p^+ \alpha'\right)^2}, \tag{2.4.2}$$

and

$$g_f = \frac{J^2}{N} = 4\pi g_c \left(\mu p^+ \alpha'\right)^2. \tag{2.4.3}$$

Equations $(2.4.1) - (2.4.3)$ reconcile the fundamental parameters that are important in describing the physics of $\mathcal{N} = 4$ SYM gauge theory and the parameters that govern the physics that occurs on the string theory side of the correspondence on a pp-wave background geometry.

The parameter μ, when taken in the limit approaching zero, approximates the deviation of the pp-wave geometry from flat space Minkowski. The limit $\mu \to 0$ can be viewed in the same spirit as with the N coincident D3-branes of the Maldacena conjecture in the limit where $g_s N >>> 1$, resulting in curvature being induced into flat $(9+1)$-dimensional Minkowski spacetime.

When the parameter μ is taken to be small, $\mu \to 0$, this introduces a strong coupling limit into the gauge theory side of the correspondence according to equation (2.4.2). In addition, we see how the effective genus parameter g_f is proportional to the closed string coupling constant g_c according to equation (2.4.3).

2.4.2 More Parameters More Relations

We will further extend on the key equations that are important in illustrating the BMN correspondence.

To start off, the Lagrangian of $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory consists of six real (Higgs) scalar fields ϕ^i where $i = 1, 2, 3 \ldots, 6$. These scalar fields ϕ^i transform in the adjoint representation of the gauge group SU(N).

The symmetry of the $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory is the conformal symmetry group $\mathrm{SO}(2,4) \times \mathrm{SO}(6)$. The conformal group SO(6) is an R-symmetry group. The R-symmetry group operates on the six Higgs scalar fields, in addition, the angular momentum J can be understood as an operator that acts on two of the six Higgs scalars. Therefore J generates rotations on the (1 2) plane defined by the $\phi^1 - \phi^2$ Higgs of R^6.

The BMN framework is considered in the limit where $N \to \infty$ and $g_{YM} << 1$ whilst g_{YM}, kept fixed. With these limits, it will mean that the 't Hooft coupling constant will be very large. In addition, on the field theory side, we consider operator states that carry large R-charge J, where J is the SO(2) generator of rotations that is responsible for rotating two of the six Higgs scalar fields (ϕ^1, ϕ^2).

Symmetry breaking takes place in the plane wave geometry of the BMN correspondence due to the dynamics of the closed string.

In the Penrose limit of the $AdS_5 \times S^5$ geometry, we considered a closed string propagating on the periphery of S^5. The scenario of a closed string propagating on the equator of S^5 is equivalent to considering the neighbourhood of the trajectory of a particle that is positioned at the center of AdS_5 that traces out a path along the circumference of S^5 and moving close to the speed of light.

As we mentioned previously, the metric of S^5 has an inherent SO(6) isometry. In the Penrose limit, by restricting ourselves to the neighbourhood trajectory of a particle traveling along the equator of S^5, this restriction breaks the SO(6) R-symmetry of S^5. The symmetry that remains on the gauge theory side is $\mathrm{SO}(4) \times \mathrm{U}(1)$.

It should be emphasized that the symmetry is broken on both the gauge theory side and string theory side of the BMN correspondence in the instance where we

restrict ourselves to the geometry seen by the particle centered on AdS_5 and traveling close to the speed of light along the equator of S^5.

The symmetry subgroup U(1) of SO(6) can be identified with the angle between the $\phi^1 - \phi^2$ plane of R^6, generated by J.

We can further couple an R-charge to a particular state that is associated with the U(1) symmetry group whose rotations are generated by J. Equivalently, the R-charge of a state can be understood to mean that U(1) $\approx$ SO(2) is a subgroup of SO(6).

A particle or a closed string propagating in the pp-wave geometry has properties that are associated with the variables p^+ and p^-. The variable p^- describes the energy of closed string propagating in the pp-wave geometry, and p^+ is its momentum. Both variables are defined in the light cone gauge.

In this light cone framework, we can understand p^+ and p^- to be the light cone energy and the light cone momentum that are associated with a closed string propagating in the pp-wave geometry.

The variables p^+ and p^- can be related to variables that appear on the gauge theory side through the coordinate transformations of $AdS_5 \times S^5$ taken in the Penrose limit.

Without presenting the details, it can be shown that the correspondence between operators of $\mathcal{N} = 4$ SYM theory and type IIB string theory on the pp-wave background adhere to the following relations

$$\begin{aligned} 2p^- &= -p_+ = \Delta - J, \\ 2p^+ &= -p_- = \frac{\Delta + J}{R^2}. \end{aligned} \tag{2.4.4}$$

In equation (2.4.4), Δ is the conformal dimension and J is the R-charge and both are BMN operators. The above equations must satisfy the BPS condition which requires that $\Delta \geq |J|$, and naturally this implies that $p^{\pm} > 0$ be non-negative.

We remember that on the string theory side we consider the neighbourhood of a particle traveling close to the speed of light along the equator of S^5 therefore

the fluctuations of the closed string of type IIB (string) theory are localized in the neighbourhood of the particle.

On the gauge side of the BMN correspondence, the operators $\Delta + J$ and $\Delta - J$ will be required to be finite due to the picture of the closed strings in the pp-wave background. Equation (2.4.4) is a realization of the BMN correspondence.

The light cone energy p^- represents the light cone Hamiltonian of the closed string action that has been quantized in the light cone gauge.

The arguments presented so far demonstrate a generic dictionary constructed in the BMN limit. This dictionary provides a relationship regarding the BMN correspondence between the spectrum of closed strings with a Hamiltonian p^- in the light cone gauge.

2.4.3 Chiral Primary Operators Of BMN

In this section we will delve further into the BMN correspondence. We dedicate this section to considering the BMN correspondence in terms of chiral primary operators $\mathrm{Tr}\left(Z^J\right)$ and the closed string spectrum.

On the field theory side of the BMN correspondence, we have six Higgs scalars ϕ^i for $i = 1, 2, \ldots, 6$ that make up R^6. A pair from the six Higgs scalars, ϕ^1 and ϕ^2, are used to define the complex matrix Z such that $Z = \phi^1 + i\phi^2$.

The matrix Z is used to construct the chiral primary operator $\mathrm{Tr}\left(Z^J\right)$. The generator J is the $\mathrm{U}(1) \approx \mathrm{SO}(2)$ generator that rotates the $\phi^1 - \phi^2$ plane.

These chiral primary operators are the 1/2 BPS states that carry large R-charge J. Also, the 1/2 BPS states have a scaling dimension that is equivalent to J for all orders of the effective t' Hooft coupling constant λ'.

The gauge operator $\mathrm{Tr}\left(Z^J\right)$ is a single trace state of Yang-Mills theory on $R \times S^3$ and the trace runs over the N colour indices where N is taken to be very large. Also, these chiral primary operators can be identified with a spectrum of dimensions that are made up of single trace operators of the theory on R^4 [65], the two descriptions of these gauge theory operators are interchangeable.

The objective is to match operators appearing on the field theory side of the correspondence with those of string theory defined on pp-wave geometry.

What makes the BMN correspondence so important is the property that the non-interacting type IIB string theory is exactly solvable in the BMN limit where the action of the type IIB string theory is quantized in the light cone gauge in the plane wave geometry.

The Hamiltonian in the light cone gauge of the quantized string theory is given by

$$H_{lc} = 2p^- = -p_+ = \sum_{n=-\infty}^{\infty} \sum_{I=1}^{8} \left(a_n^I\right)^\dagger a_n^I \sqrt{\mu^2 + \frac{n^2}{\left(\alpha' p^+\right)^2}}. \tag{2.4.5}$$

Equation (2.4.5) clearly shows how the light cone energy p^- is equal to the light cone Hamiltonian H_{lc} of the closed strings of type IIB string theory. The creation operator a_n^I and annihilation operators $\left(a_n^I\right)^\dagger$ generate bosonic excitations when acting on the ground state or the vacuum state $|0; p^+>$ in the light cone gauge.

The $|0; p^+>$ vacuum state in the light cone gauge has zero light cone energy, therefore according to equation (2.4.4) it should correspond to a gauge operator with $\Delta - J = 0$. This correspondence will be expanded upon later.

A general state of quantized type IIB string theory, is described by 8 massive bosons and 8 massive fermions. We will restrict ourselves to the general state that generates the bosonic excitations, and this state is represented as follows

$$a_{n_1}^{I_1} a_{n_2}^{I_2} a_{n_3}^{I_2} \dots a_{n_m}^{I_m} |0; p^+> . \tag{2.4.6}$$

The index I counts the number of bosons, that is $I = 1, 2 \dots .8$.

The vibrational modes n associated with a propagating closed string can be decomposed into Fourier modes. On the closed string, the right moving modes of vibration are denoted by $n < 0$ and the left moving modes of vibration are denoted by $n > 0$. The zeroth mode of vibration is represented by $n = 0$.

The light cone gauge framework in which we consider closed strings with modes of vibration will require that we impose a physical restriction on the closed strings. This physical condition is such that the total momentum of the vibrational modes along the closed string conserve the total momentum. The mo-

mentum of both the left and right moving modes of vibration cancel each other, therefore rendering the total momentum of the modes of vibration conserved.

This physical condition will be represented as follows

$$P = \sum_{n=-\infty}^{\infty} \sum_{I=1}^{8} n N_n^I = 0. \tag{2.4.7}$$

The number N_n^I represents the total occupation of the modes of vibration or equivalently it represents the eigenvalue number of the pair of creation/annihilation operators $\left(a_n^I\right)^\dagger a_n^I$ acting on the vacuum energy state $|0; p^+>$.

The physics of the $n = 0$ mode of vibration for a closed string necessitate a comment. The excitation of the $n = 0$ mode of vibration generates the spectrum of massless supergravity closed string states that propagate in the plane wave geometry. If we consider a particle with fixed p^+ that traces out a trajectory in the plane wave geometry, it would feel like it was trapped in a potential well and therefore cannot escape to infinity because its energy p^- is fixed.

The light cone Hamiltonian H_{lc} in equation (2.4.5) can be expressed in terms of the occupation number N_n as follows

$$H_{lc} = 2p^- = -p_+ = \sum_{n=-\infty}^{\infty} \sum_{I=1}^{8} N_n \sqrt{\mu^2 + \frac{n^2}{\left(\alpha' p^+\right)^2}}. \tag{2.4.8}$$

We can also use the dictionary of parameters that we developed to rewrite H_{lc} in terms of the gauge theory parameters. Therefore the spectrum of H_{lc} in terms of gauge theory operators is given by the following

$$\Delta - J = \sum_{n=-\infty}^{\infty} \sum_{I=1}^{8} N_n^I \left(1 + \lambda' n^2\right)^{1/2}. \tag{2.4.9}$$

In the pp-wave geometry, we generate the spectrum of closed string states by acting with the creation and annihilation operators, a_n^I and $\left(a_n^I\right)^\dagger$, acting on the unique ground string state $|0; p^+>$ that has zero light cone energy $p^- = 0$.

The vacuum state $|0; p^+>$ with zero light cone energy should correspond to a gauge operator with $\Delta - J = 0$, which is the lowest value that can be assigned to the chiral primary operator $\mathrm{Tr}\left(Z^J\right)$ with large R-charge J.

To demonstrate the dictionary of the BMN correspondence between the single trace operator with the lowest value and the vacuum state of a closed string we present the following equivalence

$$\frac{1}{\sqrt{JN^J}} Tr\left(Z^J\right) \longleftrightarrow |0; p^+\rangle. \tag{2.4.10}$$

The vacuum string state in the light cone gauge appearing in equation (2.4.10), is defined for the $n = 0$ mode, and is shown to be equivalent to the normalized chiral primary operator with normalization factor $1/\sqrt{JN^J}$. Naturally, as we mentioned before, the above correspondence appearing in equation (2.4.10) describes a spectrum of massless supergravity mode, $n = 0$, of closed strings propagating in the pp-wave geometry.

The string states of the $n = 0$ mode in the pp-wave geometry generate the flat space spectrum.

2.4.4 Impurity States Of BMN

Now that we have shown the correspondence of the gauge operator with the lowest possible value $\Delta - J = 0$ and the ground string state in the light cone gauge, we now want to look at excited string states.

The excited massless supergravity modes will be generated by creation/annihilation operators acting on the vacuum string state $|0; p^+\rangle$. These creation/annihilation operators will be restricted to zero momentum modes of the closed string.

The zero momentum oscillators are a_0^i for $i = 1, 2 \ldots 8$ (and S_0^b for $b = 1, 2 \ldots 8$ for their fermionic counterparts), and these oscillators act on the string vacuum state in order to generate excitations for the closed string spectrum.

It should be noted that all these oscillators have the same light cone energy and this property leads to the general condition that the "total light cone energy is equivalent to the total number of oscillators that are acting on the light cone ground state" [65].

In order for us to generate the excited states, we first have to obtain the gauge theory operators. These gauge theory operators that correspond to excited string

states are represented by generalized chiral primary operators that are coupled to scalar fields inside the single trace of the operator. These scalar fields can be shown to generate a string spectrum in flat space.

On the gauge theory side, to generate the first excited state, the scalar fields ϕ^r, where $r = 3, 4, 5, 6$ are treated as impurities and are added into the single trace operator $\mathrm{Tr}\left(Z^J\right)$. The position of the impurity inside the single trace gauge operator is important since the physics is carried out in the large N limit, that is, the planar limit. The positions of the impurity inside the trace operator will only shift in the presence of interactions.

The impurity states ϕ^r inside the chiral primary operators are summed over all possible positions they can occupy inside the trace. The lowest lying mode with $\Delta - J = 0$ has already been shown in the previous section, and there is a single mode with this condition.

The next lying mode, that is associated with the first excited states has $\Delta - J = 1$, and this will correspond to supergravity states with zero momentum on the string theory side. Therefore, we can generate the first excited state in the BMN limit

$$\frac{1}{\sqrt{N^{J+1}}}\mathrm{Tr}\left(\phi^r Z^J\right) \longleftrightarrow a_0^i|0;p^+\rangle. \tag{2.4.11}$$

The gauge theory operator has been normalized in the large N limit.

The number of excitations can be generalized by inserting k impurities into the single trace operator. Every time an impurity ϕ^r is inserted into the trace, it is summed over all positions it could occupy inside the trace.

To obtain higher excited states in the BMN limit, we simply add more impurities into the single trace gauge operator. On the string theory side, we still have supergravity modes that are associated with the $n = 0$ mode.

In general, higher excited states are given by

$$\sum \frac{1}{\sqrt{N^{J+k}}}\mathrm{Tr}\left(\ldots Z\phi_{i_1} Z\phi_{i_2} \ldots Z\phi_{i_k}\right) \longleftrightarrow a_0^{i_1} a_0^{i_2} \ldots a_0^{i_k}|0;p^+\rangle. \tag{2.4.12}$$

The above string states in equation (2.4.12) will correspond to a gauge oper-

ator state with $\Delta - J = k$ for a general number k of impurities ϕ^r.

The sum ($\sum$) in equation (2.4.12), represents the summation over all the positions that could be occupied by the impurities inside the single trace operator.

The summation that runs over the position of the impurities inside the trace in equation (2.4.12) neglects the situation where we have two impurities occupying the same position since these have a coefficient that is subleading in J, such that $1/J$ is the leading coefficient. Gauge operators subleading in J with $1/J$ are neglected in the large J limit.

We are working in the limit where we assume that the number of impurities appearing inside the trace will always be smaller than the number of the Z matrices. This scenario suggests that we are working in a system that is approximated by a "dilute gas".

We have illustrated that supergravity modes with $n = 0$ are in precise correspondence with chiral primary operators $\mathrm{Tr}\left(Z^J\right)$. Operators appearing on the gauge theory side of the correspondence are 1/2 BPS states and these states have dimension Δ that does not depend on the coupling parameters of the theory.

The oscillators that act on the light cone vacuum string state $|0; p^+\rangle$ to generate excitations were supergravity modes with zero momentum modes $n = 0$.

The BPS operators are gauge operators that have $(\Delta - J)$ finite and have $\Delta \approx J \approx \sqrt{N}$. The BPS condition $\Delta \geq |J|$ is implicit to BPS operators. For these BPS operators, the conformal dimension Δ remains unchanged by the interactions since Δ does not depend on g_{YM}. This unique property corresponds to supergravity states with zero momentum $n = 0$ modes.

2.4.5 Non-BPS States

In the previous section we discussed BPS operators that were identified with supergravity states of the zeroth mode i.e. $n = 0$.

In what follows we extend the dictionary of the BMN correspondence to include stringy non-BPS states, which are essentially the non-supergravity modes that have $n \neq 0$.

These non-supergravity states will correspond to gauge operators on the Yang-

Mills side coupled to impurity states. The action of string oscillators with non-zero modes of vibration of the closed string on the vacuum string state will be identified with the gauge theory operators.

In the pp-wave geometry, these non-BPS string states can be understood to represent "massive" string states that have total momentum along the closed string conserved. Therefore the constraint that the total momentum for string states be conserved means that more than a single oscillator $\left(a_n^\dagger\right)^i$ and $(a_{-n})^i$ should act on the vacuum string state $|0;p^+\rangle$ such that $\left(a_n^\dagger\right)^i (a_{-n})^i |0;p^+\rangle$.

Equivalent to these string states on the gauge theory side we find gauge operators that have more than one impurity ϕ that is position dependent inside the single trace operator. The action of more than one oscillator acting on the vacuum state and more than one position dependent impurity inserted on the gauge theory side ensures that there is zero total momentum along the closed string.

An interesting new feature appears on the Yang-Mills side of the BMN correspondence for non-supergravity states. The chiral primary operators will have more than one impurity inserted into the trace. The trace operator sums the impurity over all possible positions it could occupy inside the trace along the Z fields. A phase term that is related to the positions of the impurities inside the trace appears to be multiplied by the single trace operator.

The position dependent phase term that is related to the position of the impurity ϕ inside the single trace operator is given by $exp\left((2\pi inl)/J\right)$. The term l appearing inside the phase term represents the position of the impurity ϕ inside the single trace operator.

Below we present the simplest demonstration of the BMN correspondence for non-supergravity states and phase dependent chiral primary operators

$$\left(a_n^\dagger\right)^i (a_{-n})^i |0;p^+\rangle \longleftrightarrow \frac{1}{\sqrt{J}} \frac{1}{N^{J/2+1}} \mathrm{Tr}\left(\phi_i Z^l \phi_j Z^{J-l}\right) e^{\frac{2\pi inl}{J}}. \tag{2.4.13}$$

The gauge theory operator appearing on the right hand side of equation (2.4.13) is a non-BPS operator when the phase term $exp\left((2\pi inl)/J\right) \neq 0$.

Equation (2.4.13) demonstrates how the two oscillator operators acting on

the vacuum string state equivalently correspond to a gauge theory state with two impurities inserted inside the chiral primary operator. The two impurities appearing inside the gauge theory operator are summed over all possible positions that they could occupy inside the trace along the string of Z fields.

The correspondence appearing in equation (2.4.13) can be generalized for many excitations of the non-supergravity states. Therefore, for a general number of excitations we have

$$(a_{n_1})^\dagger \dots (a_{n_m})^\dagger |0; p^+\rangle \longleftrightarrow \frac{1}{J} \sum_{l_1 \dots l_m = 1}^{J} \frac{1}{N^{\frac{J+m}{2}}} \mathrm{Tr}\left(\phi \dots Z\phi \dots Z\phi Z \dots\right) \times e^{(2\pi i[n_1 l_1 + n_2 l_2 \dots n_m l_m]/J)}. \tag{2.4.14}$$

The pattern that governs the correspondence for non-supergravity states and non-BPS states becomes easy to follow. An oscillator mode for each closed string corresponds to $\Delta - J = 1$ on the gauge theory side for each single field.

The single (impurity) field appearing inside the gauge operator is summed at all possible positions inside the trace according to a phase that depends on the momentum. In the case where there is a state whose total momentum does not disappear, that is, the total momentum along the string is not zero, this results in operators being zero automatically due to the cyclicity of the trace. The constraint, equation (2.4.7), that the total momentum should vanish along the closed string will always be enforced on the string spectrum.

The constraint that total momentum be zero for a closed string with $n \neq 0$ vibration modes when a single creation/annihilation operator acts on a vacuum string state corresponds to a single impurity appearing in the trace operator $Tr\left(Z^J\right)$ for large R-charge J.

However, for the $n \neq 0$ modes of vibration, the impurity inside the trace is position dependent and is coupled to a phase dependent term multiplying the single trace operator.

Single oscillator states acting on the string vacuum states will vanish as a result of the conservation of momentum constraint. Explicitly, the BMN non-supergravity states with $n \neq 0$ modes that vanish due to the conservation of

momentum have the following form

$$a_n^i|0;p^+\rangle \longleftrightarrow \frac{1}{J}\sum_{l=1}^{J}\frac{1}{N^{(2J+1)/2}}\mathrm{Tr}\left(Z^l\phi^i Z^{J-l}\right)e^{\frac{2\pi inl}{J}}. \tag{2.4.15}$$

The right hand side of equation (2.4.15) above vanishes due to the cyclicity of the trace for the $n \neq 0$ modes of vibration. The left hand side of equation (2.4.15) also vanishes due to the salient constraint that the total momentum along the closed string should vanish. The conservation of total momentum can be identified with a physical state.

In the double scaling limit of the BMN correspondence of the gauge parameters, since $J \approx \sqrt{N}$, then the difference between the conformal dimension Δ (dilatation operator) and the R-charge J coupled to an impurity with phase n, gives the following relation [71]

$$(\Delta - J)_n = \sqrt{1+\lambda' n^2}. \tag{2.4.16}$$

Equation (2.4.16) is exactly the same as equation (2.4.9).

In conclusion, the argument provided for this section demonstrated how the spectrum of type IIB string theory defined on the pp-wave geometry is equivalent to a special class of BMN operators in $\mathcal{N}=4$ Supersymmetric Yang-Mills theory.

This introduction, provided to motivate the BMN correspondence which represents a special limit of the AdS/CFT correspondence, and serves only as an heuristic demonstration of the BMN correspondence and not a rigorous derivation.

We hope that this introduction provided a general idea behind the equivalence between a string spectrum of type IIB string theory defined on the plane wave geometry and a restricted class of BPS states or BMN operators in the $\mathcal{N}=4$ SU(N) Supersymmetric Yang-Mills Theory.

Chapter 3

Matrix Models

In the previous chapter we presented an introduction that provided highlights of important works that serve as a motivation for the work carried in this project.

This chapter will (formally) introduce matrix models. We will carry out the following objectives:

- Overall, provide a general introduction on matrix models whose characteristic features are aligned with our project
- Present a general introduction on Plane Wave Matrix Theory
- Highlight the significance of matrix models through examples
- Briefly review 1/2 BPS states and LLM
- Provide examples of different treatments of the two matrix model

We remember that, in the previous chapter, we classified the BMN limit as a special limit of the AdS/CFT correspondence that possessed the special property of being exactly solvable for a theory of closed strings restricted to the plane wave background geometry.

The gauge theory operators, $\mathrm{Tr}\left(Z^J\right)$, of the BMN correspondence were shown to mix with scalar fields ϕ appearing inside the single trace. These chiral primary gauge operators appear at the boundary of the plane wave spacetime with a large angular momentum J.

BMN introduces the idea of an $(1+1)$-dimensional theory as a result of an impurity ϕ hopping in the background of Z fields, as described in the previous chapter.

The physics of the scalar fields amongst the Z fields inside the chiral primary operator can be correctly described by a $(1+1)$-dimensional string field theory in the plane wave geometry. In the limit where we describe the plane wave string theory in terms of the light cone gauge degrees of freedom, the plane wave string theory is reduced to a free and massive two-dimensional matrix model [72].

The perimeter of the plane wave geometry where the chiral primary gauge operators are defined is in fact a one-dimensional region and the string theory lives in the plane wave $(1+1)$-dimensional region. The correspondence between the two regions where the string theory lives and the gauge theory lives, seem to satisfy the prescriptions of the Maldacena conjecture.

Since the idea of BMN, the authors of [73] provided a derivation of a quantum mechanical Hamiltonian by considering the $\mathcal{N}=4$ SYM theory in the BMN limit for the two impurity case. This Hamiltonian in [73] is derived from the one loop part of the dilatation operator matrix element.

In the spirit of BMN, [73] makes it more precise from $\mathcal{N}=4$ SYM that string field theory results in a quantum mechanical system of matrices [74].

It turns out that this quantum mechanical system can also be obtained as a dimensional reduction on S^3, which is referred to as plane wave matrix theory [75]. In the section that follows we discuss this "Plane Wave Matrix Model".

3.1 Plane Waves Coupled To Matrix Degrees Of Freedom

The theory living on the boundary of the plane wave geometry is a one-dimensional $\mathcal{N}=4$ Supersymmetric Yang-Mills theory. This theory is a quantum mechanical system with matrix degrees of freedom [75].

The authors of [75] suggested that this matrix model can be obtained by performing a dimensional reduction through studying a Kaluza-Klein reduction

of $D = (3+1)$, $\mathcal{N} = 4$ SYM on a round three sphere S^3.

3.2 A Flavour For Plane Wave Matrix Theory

The authors of [75] start by considering an $\mathcal{N} = 4$ Supersymmetric Yang-Mills in $(3+1)$-dimensions compactified on a three sphere. The action that describes the physics of $\mathcal{N} = 4$ Supersymmetric Yang-Mills in $(3+1)$-dimensional Minkowski spacetime is well known to result from a reduction on $\mathcal{N} = 1$ Supersymmetric Yang-Mills in (9+1)-dimensions on a six torus formulated on a curved background that is invariant under supersymmetry.

The $\mathcal{N} = 4$, $D = 4$ Supersymmetric Yang-Mills defined on $R \times S^3$ is described by the following metric

$$ds^2 = g_{\alpha\nu} dx^\alpha dx^\nu = -dt^2 + R^2 \left(d\theta^2 + \sin^2\theta d\psi^2 + \sin^2\theta \sin^2\psi d\chi \right), \qquad (3.2.1)$$

where R is the radius of S^3 and (θ, ψ, χ) are the global coordinates.

The dimensional reduction from $D = (3+1)$-dimensions of $\mathcal{N} = 4$ SYM is carried out by performing an expansion of the $D = (3+1)$ fields in terms of the spherical harmonics of S^3. These spherical harmonics are expressed in irreducible representation $(m_L; m_R)$ of the isometry group $\mathrm{SO}(4) \equiv \mathrm{SU}(2)_\mathrm{L} \otimes \mathrm{SU}(2)_\mathrm{R}$. Also, these spherical harmonics depend on the spin of the $D = (3+1)$ fields.

Upon inserting the spherical harmonics into the $\mathcal{N} = 4$, $D = (3+1)$ SYM theory and integrating out the action over S^3, what remains is a one-dimensional theory that is made up of an infinite number of fields. These infinite number of fields have an associated mass spectrum.

To determine the mass spectrum of the excited fields, the spherical harmonics must be orthonormalized. The mode expansions (of the orthonormal spherical harmonics) are substituted into the action of the theory and a new action is obtained that is proportional to the term $\left(4\pi^2 R^3\right)/g_{YM}^2$ appearing in equation (13) in [75].

In general, without discussing details, the masses of the vector modes are calculated using vector spherical harmonics.

As expected these masses, presented in an infinite Kaluza-Klein mass tower, depend on the radius R of S^3. Since the masses appearing in the Kaluza-Klein mass tower come in different states, one needs to act with the two supercharges Q_L and Q_R to move from one mass state to the next.

The entire Kaluza-Klein tower with mass states is a uniform irreducible representation that is not decomposed into finite portions of irreducible representations. One finds masses of the Kaluza-Klein tower that come in different states, for example, the lowest lying states of the mass tower are 1/2 BPS states, the following level of mass states are 1/4 BPS states etc.

The infinite Kaluza-Klein mass tower spectrum of $\mathcal{N} = 4$ SYM on $R \times S^3$ is reduced to lower dimensions to derive the plane wave matrix theory. This is done by showing that the infinite tower of the Kaluza-Klein mass spectrum can be consistently truncated to the lowest lying modes and that the action that governs the lowest lying modes is the plane wave matrix theory model.

In [75], the one-dimensional Lagrangian with quantum mechanical degrees of freedom is given by the following Lagrangian

$$\begin{aligned} L &= \mathrm{Tr}\left(\frac{1}{2}\left(D_t X_I\right)^2 - \frac{1}{2}\left(\frac{m}{3}\right)^2 X_a^2 - \frac{1}{2}\left(\frac{m}{6}\right)^2 X_i^2\right) \\ &+ \mathrm{Tr}\left(\frac{m}{3} i\varepsilon_{abc} X_a X_b X_c + \frac{1}{4}\left[X_I, X_J\right]^2 - 2i\theta^\dagger D_t \theta + \frac{m}{2}\theta^\dagger \theta\right) \\ &- \mathrm{Tr}\left(2\theta^\dagger \sigma^a [X_a, \theta] - \theta^\dagger i\sigma^2 \rho_i \left[X_i, \theta^*\right] + \theta^T i\sigma^2 \rho_i^\dagger \left[X_i, \theta\right]\right). \end{aligned} \tag{3.2.2}$$

The variables appearing in equation (3.2.2) above are defined as follows: the X_I's are one-dimensional bosonic hermitian $N \times N$ matrices, θ is the one-dimensional fermionic hermitian $N \times N$ matrix, D_t represents the covariant derivative with respect to time t, σ^a is identified with Pauli matrices, ρ_i represents Clebsch-Gordon co-efficients and m is the mass parameter of the plane-wave matrix model $= 6/R$. For the Lagrangian above, $I = 1, \ldots\ldots, 9$, $a = 1, 2, 3$, $i = 1, 2, \ldots, 6$.

It should be mentioned that the action with fields in one-dimension of plane wave matrix theory can be used to retrieve the equations of motion of the $D = (3 + 1)$-dimensions.

At this point, we can further introduce the coupling constant parameter g_{YM} of $\mathcal{N}=4$, $D=(3+1)$ SYM theory and relate it to m of the plane wave matrix model in $(1+0)$-dimensions. By requiring that the prefactor of the plane wave background action of $\mathcal{N}=4$ SYM in $(9+1)$-dimensions match the prefactor of the quadratic action of $\mathcal{N}=4$ SYM in $D=(3+1)$-dimensions compactified on S^3, the following is shown

$$\frac{m^3}{9}=\frac{32\pi^2}{g_{YM}^2}. \tag{3.2.3}$$

The above relation appearing in equation (3.2.3), relates the parameters of $\mathcal{N}=4$, $D=(3+1)$ Super Yang-Mills theory and the $(1+0)$-dimensional plane wave matrix model.

There is a remarkable agreement between the plane wave matrix model Hamiltonian and the dilatation operator calculated to higher loop order.

The work of [75] established precisely how a quantum mechanical model with matrix degrees of freedom in one-dimension arises as the Kaluza-Klein reduction of $\mathcal{N}=4$, $D=(3+1)$ Supersymmetric Yang-Mills theory on $R\times S^3$.

This result obtained in [75] of matrix degrees of freedom in one-dimension has been motivated by [78] through the holographic principle, by showing that the BMN gauge theory dual of type IIB plane wave superstring theory should be given by a one-dimensional (plane wave matrix) model.

3.3 Examples: Application Of Matrix Models

In general, in this thesis, (path) integrals of matrices or the quantum mechanics of matrices will be referred to as matrix models.

Matrix models or multimatrix models have generated great interest due to their extensive applicability. The more fascinating framework is the study of multimatrix models of hermitian matrices of size $N\times N$ studied in the large N limit. Large N matrix models are truly fascinating because of their richness in structure and the physical results they yield.

In what follows, and in addition to the example discussed in the previous

subsection, we give further examples of where matrix models have successfully found applications.

M(atrix)-Theory Conjecture

String theory was shown to be consistent and contain a gravitational description only in $(9+1)$-dimensions. It was further demonstrated that string theory is made of five supersymmetric perturbative theories in $(9+1)$-dimensions. These five superstring theories (where some have been mentioned in earlier chapters), described in terms of both open and closed strings, are Type IA, Type IIA, Type IIB, Heterotic $E_8 \times E_8$ and Heterotic SO(32).

What's fascinating is that all these five superstring theories are not distinct, that is, these superstring theories are related by dualities namely the T-duality, S-duality and U-duality. The U-duality is a combination of the T-duality and the S-duality [79] [80] [81].

Witten conjectured that these five superstring theories are all a special limit of a $(10+1)$-dimensional supergravity theory known as M-theory [40] [82] [83] [84]. The M-theory in $(10+1)$-dimensions provides a single framework for all the five superstring theories in $(9+1)$-dimensions.

It was further conjectured that $(10+1)$-dimensional M-theory is in fact a theory of matrices. The conjecture that proposes that M-theory is a theory of matrices is more commonly known as the BFSS conjecture [85] named after its authors and is known as M(atrix)-theory [85] [86] [87] [88] [89].

The BFSS conjecture considers an "infinite momentum frame" from where an observer of this frame is moving at light speed. In this model, BFSS conjectures that a simple Supersymmetric Yang-Mills theory of D0-branes is equivalent to $(10+1)$-dimensional supergravity in the infinite momentum reference frame.

The D0-branes are the fundamental objects upon which the BFSS conjecture is based. These D0-branes can be treated as point particles with no spatial extension. The spacetime coordinates of D0-branes are $N \times N$ matrices that act as the degrees of freedom of the theory.

Computations done in M(atrix)-theory, based on matrix models can give a

better understanding about (10+1)-dimensional supergravity and equivalently M-theory. Also, M(atrix)-theory is a quantum mechanical model that could provide a non-perturbative formulation of M-theory and also help better understand the five variations of the perturbative superstring theories.

Other Examples

We give further examples of theories that have descriptions related to quantum mechanical or zero dimensional matrix degrees of freedom.

- In the plane wave limit, operators are constructed from a complex matrix Z made up of two Higgs fields. These group of operators can be considered as operators acting on an N dimensional vector space $\mathbf{V}$ and dual to 1/2 BPS giant gravitons living in the $AdS_5 \times S^5$ geometry. The gauge theory operators are defined in the zero coupling limit, $g_{YM} = 0$, and large N of the Yang-Mills theory. The operators living on the gauge theory side which are equivalent to these giant gravitons are in fact, by their definition and construction, multi-trace matrix operators. On the $\mathcal{N} = 4$, U(N), Yang-Mills side, the gauge theory operators are represented by a special class of 1/2 BPS operators. Their two point functions can be diagonalized, and in this diagonalized space one can find correlators of 1/2 BPS operators that are identified with 1/2 BPS states of giant gravitons in the $AdS_5 \times S^5$ via the AdS/CFT correspondence [90] [91].

- The three point correlation functions of chiral primary operators of $\mathcal{N} = 4$, $D = (3 + 1)$ Supersymmetric Yang-Mills theory of supergravity states and 1/2 BPS states are known to be independent from the t' Hooft coupling constant $\lambda = g_{YM}^2 N$ [93]. These three-point functions are calculated for the strong coupling limit $\lambda >> 1$ and the weak coupling limit $\lambda << 1$. The weak coupling limit represents a weak field theory in which exact non-perturbative solutions are obtained, whereas in the strong coupling limit, operators are studied perturbatively through type IIB supergravity. The calculation of three point functions is equivalent to calculating free matrix

model overlaps and reinforces the Maldacena conjecture.

- In [94], the large N limit of gauge quantum mechanics for a single hermitian matrix model in a harmonic oscillator potential is studied as a toy model for the AdS/CFT conjecture. This single matrix model is argued, by [94] and in agreement with [90], to be dual to giant gravitons, which are identified with 1/2 BPS states in terms of D-branes. The propagation of these giant gravitons is understood in the $AdS_5 \times S^5$ geometry are argued to be associated with the eigenvalues of a matrix.

- The authors of [95] manage to demonstrate how a canonical string field theory is derived by considering large N matrix models taken in the BMN double scaling limit of Supersymmetric Yang-Mills theory. The parameters N, J were previously defined to be the colour charges and the SO(2) generator of rotations. In [95], to derive the string field theory, the authors use collective field theory [96].

- The $c = 1$ string theory has been shown to correspond to a large N matrix model [97] [98] [99]. This matrix model can be decomposed into eigenvalues that become fermions of the quantum theory (BIPZ) [27]. The string theory whose dual theory is a matrix model has a two-dimensional spacetime. The two-dimensional string theory possesses a linear dilaton background.

- It has been shown that a large N theory with SU(N) gauge symmetry defined on a d-dimensional spacetime is equivalent to an SU(N) symmetry gauge theory also defined in the large N limit but living on a single point. This equivalence demonstrates a reduction in degrees of freedom from d-dimensional fields to a single point for a large N theory with SU(N) symmetry. This reduction of degrees of freedom was shown by Euguchi and Kawai and is popularly referred as the Euguchi-Kawai model [100]. In the weak coupling limit, the Euguchi-Kawai model is troubled by anomalies and inconsistencies as a result of the symmetry breaking that takes place at $g_{YM} \to 0$. To overcome these troubles and recover large N QCD

with SU(N) symmetry, a "quenched" Euguchi-Kawai model was proposed to overcome the challenge of spontaneous symmetry breaking that occurs at weak coupling [101]. In the "quenched" Euguchi-Kawai model, the eigenvalues of the hermitian matrices are quenched (arranged over) so that the original large N SU(N) gauge theory on the d-dimensional spacetime can be recovered [102] [103].

To list all of the research work that makes use of the richness of matrix models would indeed truly be a project on its own. Our goal of giving examples of work that make use of matrix models was to motivate and hopefully generate an interest into the wide and extensive field of matrix models and also indicate their importance since our entire project is based on matrix models.

In our project, we start by considering a complexified two hermitian matrix model that we parameterize through "matrix polar coordinates" into a single complex matrix. We then generalize the model from a single complex matrix to an arbitrary number of complex matrices, (or equivalently to an even number of hermitian matrices).

As it will be shown, there is a closed subsector of these systems that can be thought of as being associated with a matrix valued radial coordinate. We will study many aspects of the dynamics of this radial subsector.

In studying the dynamics of this radial degree of freedom, we will make use and occasionally generalize the collective field approach developed by Jevicki and Sakita [96] [104]. This framework of collective field theory will be reviewed later in this thesis.

The idea of singling out one of the hermitian matrices of a complex matrix system is not new, and it is briefly described in the next subsection.

3.4 Review: 1/2 BPS States And LLM

We follow the approach of [108] in showing that the dynamics of 1/2 BPS states associated with giant gravitons corresponds to the dynamics of a single hermitian matrix, obtained by truncation of a single complex matrix.

The two hermitian matrix model is made up of two of the six Higgs scalar fields Φ_a, $a = 1, 2 \ldots, 6$. The Hamiltonian of the system will be a dilatation operator and the Higgs degrees of freedom will be identified with the coordinates of the system. The action of the two matrix system is given in terms of the Higgs scalar fields by the following

$$\mathrm{S} = \frac{1}{g_{YM}^2} \int \mathrm{dtTr} \left(\left(\dot{\Phi}_1 \right)^2 + \left(\dot{\Phi}_2 \right)^2 - \Phi_1^2 - \Phi_2^2 - \frac{1}{2} \left[\Phi_1, \Phi_2 \right]^2 \right). \tag{3.4.1}$$

The commutator term appearing in the equation (3.4.1) above plays no role in establishing a correspondence between matrix theory and supergravity states, and it can therefore be neglected. Therefore, of relevance to the 1/2 BPS sector will be the following quadratic harmonic oscillator Hamiltonian for the two matrix model

$$H = \frac{1}{2} \mathrm{Tr} \left(P_1^2 + P_2^2 + \Phi_1^2 + \Phi_2^2 \right). \tag{3.4.2}$$

A complex matrix, along with its complex conjugate, is constructed from two of the Higgs scalar fields as follows

$$Z = \frac{1}{\sqrt{2}} \left(\Phi_1 + i \Phi_2 \right), \qquad Z^\dagger = \frac{1}{\sqrt{2}} \left(\Phi_1 - i \Phi_2 \right). \tag{3.4.3}$$

Naturally, the conjugates to Z and $Z^\dagger$ given by $\partial / \partial Z$ and $\partial / \partial Z^\dagger$ can also be defined.

The matrix model has an $SL(2, R)$ or SU(2) symmetry, and the generator of rotations J amongst the scalar fields is a U(1) charge symmetry.

A special class of states that are defined on the matrix theory side will be defined using the complex matrix in equation (3.4.3). These states will represent a restricted class of chiral primary operators and are associated with 1/2 BPS states, and are of the general form

$$\mathrm{Tr} \left(Z^{k_1} \right) \mathrm{Tr} \left(Z^{k_2} \right) \ldots \mathrm{Tr} \left(Z^{k_n} \right). \tag{3.4.4}$$

Creation and annihilation operators A and $B^\dagger$ are related to the field Z and the conjugate $\Pi = -i\frac{\partial}{\partial Z^\dagger}$ as follows

$$A = \frac{1}{2}\left(Z + i\Pi\right) \qquad B^\dagger = \frac{1}{2}\left(Z - i\Pi\right). \tag{3.4.5}$$

Using these operators, the Hamiltonian of the system in the reduced Hilbert space and the U(1) charge J can also expressed in a new form such that

$$H = \mathrm{Tr}\left(A^\dagger A + B^\dagger B\right) \qquad J = \mathrm{Tr}\left(A^\dagger A - B^\dagger B\right). \tag{3.4.6}$$

The 1/2 BPS sector of the theory is defined by $H = J$, in other words, no B excitations.

The eigenvalue configuration of the system is obtained by diagonalizing the system of A oscillators from which these eigenvalues can be treated as fermionic degrees of freedom. The diagonalized oscillators [90] [108] are of the form

$$A_{ij} = \lambda_i \delta_{ij} \qquad A^\dagger_{ij} = \lambda^\dagger_i \delta_{ij}. \tag{3.4.7}$$

In this new representation of the sub-sector of the Hilbert space in terms of the eigenvalues of the oscillators, the Hamiltonian of the system takes the form

$$H = \sum_i \lambda^\dagger_i \lambda_i. \tag{3.4.8}$$

The fermionic wavefunctions of the system are shown by [90] [108], and these wavefunctions are represented as states of the restricted Hilbert space. This is obtained by multiplication by the Vandermonde determinant.

The different states of the fermionic wavefunctions can be represented by Schur polynomials. In particular, Schur polynomials with l boxes in a single row and l boxes in a single column correspond to particles/holes respectively and to giant gravitons in the AdS geometry and S sphere respectively.

The 1/2 BPS sector, $(J = \Delta)$, corresponds to the A, $A^\dagger$ system, with no B excitations, which can be associated with the single hermitian matrix
$M = 1/\sqrt{2}\left(A + A^\dagger\right)$, and it can be given a bosonized density description.

As it will be shown later, the bosonized density description of a single hermitian matrix in an harmonic potential is given by the following

$$\mathrm{H}^0_{eff} = \frac{1}{2N^2}\int dx \partial_x \Pi(x)\phi(x,0)\partial_x\Pi(x) + N^2\int dx\left(\frac{\pi^2}{6}\phi^3(x,0) + \phi(x,0)\left(\frac{\omega^2x^2}{2}-\mu\right)\right), \tag{3.4.9}$$

where ϕ is the density of eigenvalues M.

As $N \to \infty$, the background configuration ϕ_0, is given by the well known Wigner distribution

$$\begin{aligned}\pi\phi(x,0) &= \pi\phi_0 \\ &= \sqrt{2\mu-\omega^2x^2} \\ &= \sqrt{2\omega-\omega^2x^2}.\end{aligned} \tag{3.4.10}$$

One can see the emergence of the droplet picture proposed by Lin, Lunin and Maldacena (LLM) [107], if the following is defined: $p_\pm \equiv \partial_x\Pi(x)/N^2 \pm \pi\phi(x,0)$. Using this definition, equation (3.4.9) can be given the following phase space representation

$$\mathrm{H}^0_{eff} = \frac{N^2}{2\pi}\int dp\int dx\left(\frac{p^2}{2}+\frac{x^2}{2}-\mu\right). \tag{3.4.11}$$

In the large N limit, $N\to\infty$, we observe $p_\pm \to \pm\pi\phi(x,0) = \pm\pi\phi_0$, and the boundary of the droplet is given by the following: $p_\pm^2 = 2\mu - x^2$.

Referring now to LLM [107], their final expression for the flux and energy of the bosonized free fermion droplet takes the following form

$$N = \frac{1}{4\pi l_p^2}\int dx_1\int dx_2\left(u(t,x_1,x_2)+\frac{1}{2}\right), \tag{3.4.12}$$

$$\begin{aligned}\Delta &= \frac{1}{4\pi\hbar^2}\int dx_1\int dx_2\left(x_1^2+x_2^2\right)\left(u(t,x_1,x_2)+\frac{1}{2}\right) \\ &- \frac{1}{8\pi\hbar^2}\int dx_1\int dx_2\left(x_1^2+x_2^2\right)\left(u(t,x_1,x_2)+\frac{1}{2}\right)^2.\end{aligned} \tag{3.4.13}$$

Equation (3.4.13), is nothing but equation (3.4.11), where the coordinates (x_1, x_2) correspond to the phase space coordinates (x, p). In addition, equation (3.4.12) is nothing but the condition

$$\int dx \phi_0(x) = 1. \tag{3.4.14}$$

Up to this point, we were considering a system whose dynamics was restricted to 1/2 BPS configurations, corresponding to a single hermitian matrix model subsector defined by A, $A^\dagger$ oscillators. This restriction can be removed to a sector with a larger number of matrices, that is, a matrix model whose states now include B excitations.

First define the two matrix model by matrices M and N with the Hamiltonian H:

$$H = -\frac{1}{2}\mathrm{Tr}\left(\frac{\partial}{\partial M}\frac{\partial}{\partial M}\right) - \frac{1}{2}\mathrm{Tr}\left(\frac{\partial}{\partial N}\frac{\partial}{\partial N}\right) + \frac{1}{2}\mathrm{Tr}\left(M^2 + N^2\right) \tag{3.4.15}$$

The third term in equation (3.4.15) represents the quadratic potential of the system of two matrices.

The Hamiltonian in equation (3.4.15) is treated asymmetrically in the approach followed in [108], in that the matrix $N = \frac{1}{\sqrt{2}}\left(B + B^\dagger\right)$ is treated in a coherent state basis. The oscillators B are the impurities in the system.

The Hamiltonian (3.4.15) will act on functionals of invariant loop variables: $\Phi(\psi(k, s = 0, 1, 2 \ldots, s))$, where $\psi(k, s = 0, 1, 2, \ldots s)$ represents the states with a general number of s "B" impurities. The spectrum of these states about the background generated by the matrix M is studied.

The sector with a single impurity in a harmonic oscillator potential has eigenfunctions which are identical to Tchebychev polynomials of the second kind [108]. In general, for s impurities subject to a harmonic oscillator potential, a shifted Marchesini-Onofri operator. On the matrix theory side, [108] finds wavefunctions that take the form of plane waves. On the SUGRA side, a kernel operator is constructed that establishes a relation of AdS and S geometry wavefunctions to the wavefunctions of the matrix model.

3.5 More Examples Of The Two Matrix Model

In this section, we provide other (brief) examples of models of two matrices, where the matrices are treated asymmetrically.

The author of [109] adopts a similar approach to [108], in that the two matrix model considered by [109], where one of the matrices is treated exactly in the large N limit and the second matrix is treated in the large N background of the first matrix, in a creation/annihilation operator basis. The second matrix is treated as an impurity in the harmonic background of the first matrix and shown to agree with the Fock space spectrum.

A g_{YM} dependent interaction Hamiltonian is considered and first order perturbation carried out in the two cases: in the first case, the impurity is associated with an anti-holomorphic matrix, and the second case, this matrix represents one of the transverse Higgs scalars. In the second case, a BMN type Hamiltonian is identified.

This BMN type Hamiltonian is further studied in [110], where, following a Bogoliubov transformation, the action of this Hamiltonian is considered when acting on multi-impurity states. The two impurity states are shown to diagonalize this Hamiltonian, and the spectrum is obtained, which depends on the 2 momenta. A relationship to the magnon spectrum is pointed out.

Reference [111] attempts to incorporate the g_{YM} dependence on the background itself, again in an approach where one of the matrices is treated exactly, where the other is restricted to the ground state in a creation/annihilation basis. In this sector, the background configurations obtained in closed form and shown to be well defined in the limit when $\lambda = g_{YM}^2 N \to \infty$.

Now that we have provided a motivation, an explanation and gave examples regarding the relevance of multimatrix models and their applications in general, from this point onwards we proceed with our project.

The following chapter will serve to strengthen our understanding of some of the underlying features of the single hermitian matrix model as presented by BIPZ [27]. The results obtained by BIPZ will be generalized to decode the intricate

structure of the two hermitian matrix model or single complex matrix model that has been parameterized through "matrix valued polar coordinates".

Having gained confidence from studying the two hermitian matrix model we will pursue our ambition to investigate some of the unique properties of a larger and generalized Gaussian ensemble of m $N \times N$ complex matrices in the large N limit where $m \geq 2$, which will require us to identify a closed subsector, where Schwinger-Dyson equations close and there is an underlying enhanced radial symmetry.

Chapter 4

Large N Limit Of The Single Hermitian Model

In this chapter we will review certain key properties that arise from studying the single hermitian matrix model. Some of the methods used to obtain solutions in the single hermitian matrix model will be extended and applied to a system that is larger, encompassing greater degrees of freedom.

The objectives for this chapter are as follows:

- Introduce the single hermitian matrix model and derive its integral equation in the $d = 0$ case
- Solve for the eigenvalue density function and observe the geometry due to distribution of eigenvalues (Dyson Gas approach)
- For $d = 1$, show how the single hermitian matrix model can be mapped to a system of fermionic degrees of freedom
- Introduce the collective field theory formalism in its generality
- Apply the methods of collective field theory to the single hermitian matrix model and as a result, study the background geometry that arises from the distribution of eigenvalues in the leading large N limit.

4.1 Single Hermitian Matrix Model

We consider a matrix model described by a single $N \times N$ hermitian matrix M. By performing an angular parametrization on the matrix M we are able to diagonalize it and obtain its eigenvalue representation as follows:

$$M = V^{\dagger} \lambda V. \tag{4.1.1}$$

In the above, V is a $N \times N$ unitary matrix that represents angular degrees of freedom of the matrix M. This unitary matrix belongs to the U(N) gauge group. The matrix λ is a diagonal $N \times N$ matrix with real entries on the main diagonal which are the eigenvalues of the matrix M.

With the matrix M, we can define a single hermitian matrix model partition function

$$Z_{SM} = \int dM \exp -(W(M)), \tag{4.1.2}$$

whose potential $W(M)$ is also invariant under the angular similarity transformations of the unitary matrices $V^{\dagger}$ and V

$$W(V^{\dagger} M V) = W(M). \tag{4.1.3}$$

If the above invariance for the potential is true then it means that the potential can be represented in terms of the eigenvalues of the matrix M or in terms of the moments

$$\mathrm{Tr}\left(\lambda^n\right), \tag{4.1.4}$$

for some integer n.

As an example, we choose the potential for the single hermitian matrix model as

$$W(M) = \mathrm{Tr}\left(\frac{1}{2} M^2 + \frac{g}{N} M^4\right), \tag{4.1.5}$$

then for our partition function we have

$$Z_{SM} = \int dM \exp\left(-\frac{1}{2}\mathrm{Tr}\left(M^2\right) - \frac{g}{N}\mathrm{Tr}\left(M^4\right)\right). \tag{4.1.6}$$

We need to introduce a change of coordinates from the N^2 degrees of freedom of the matrix M to the N degrees of freedom of the diagonal matrix and the $N^2 - N$ degrees of freedom of the unitary matrix that diagonalizes the matrix M.

We write the line element dM of the partition function in equation (4.1.6) in terms of the eigenvalues of the hermitian matrix M and the angular degrees of freedom V and $V^\dagger$ using equation (4.1.1). Having done this, we will need to compute the square of the line element $\mathrm{Tr}\left(dM^2\right)$. We draw the following analogy

$$\mathrm{d}s^2 = g_{\mu\nu}dx^\mu dx^\nu \equiv \mathrm{Tr}\left(dM^2\right) = \eta_{\mu\nu}dX^\mu dX^\nu. \tag{4.1.7}$$

The detailed calculation that shows the measure dM expressed in terms of the eigenvalues $d\lambda$ and dV is shown in the appendix A.1.

Using this result, the partition function Z_{SM} can now be written in terms of the eigenvalues of the diagonal matrix λ such that

$$\begin{aligned}
Z_{SM} &= \int dM \exp\left(-\frac{1}{2}\mathrm{Tr}\left(M^2\right) - \frac{g}{N}\mathrm{Tr}\left(M^4\right)\right) \\
&= \int dM \exp\left(-\frac{1}{2}\mathrm{Tr}\left(\lambda^2\right) - \frac{g}{N}\mathrm{Tr}\left(\lambda^4\right)\right) \\
&= \int d\lambda_i \prod_{i<j}\left(\lambda_i - \lambda_j\right)^2 \exp\left(-\frac{1}{2}\sum_i \lambda_i^2 - \frac{g}{N}\sum_i \lambda_i^4\right) \\
&= \int d\lambda_i \prod_{i\neq j}\left(\lambda_i - \lambda_j\right) \exp\left(-\frac{1}{2}\sum_i \lambda_i^2 - \frac{g}{N}\sum_i \lambda_i^4\right) \\
&= \int d\lambda_i \exp\left(-\frac{1}{2}\sum_i \lambda_i^2 - \frac{g}{N}\sum_i \lambda_i^4 + \ln\left(\prod_{i\neq j}\left(\lambda_i - \lambda_j\right)\right)\right) \\
\Rightarrow Z_{SM} &= \int d\lambda_i \exp\left(-\frac{1}{2}\sum_i \lambda_i^2 - \frac{g}{N}\sum_i \lambda_i^4 + \sum_{i\neq j}\ln\left(\lambda_i - \lambda_j\right)\right).
\end{aligned} \tag{4.1.8}$$

In equation (4.1.8) we see the well known Vandermonde determinant for the

single hermitian matrix model:

$$\Delta(\lambda) = \prod_{i<j} (\lambda_i - \lambda_j) \,. \tag{4.1.9}$$

We note that the problem has been reduced from N^2 degrees of freedom of the $N \times N$ hermitian matrix M in equation (4.1.6) to N degrees of freedom in terms of the the eigenvalues of M.

Equation (4.1.8) describes N particles in the common potential $\left(\frac{1}{2}\lambda_i^2 + \frac{g}{N}\lambda_i^4\right)$ subject to an interacting repulsive two-dimensional Coulomb law potential.

In the large N limit, $N \to \infty$, we solve for our saddle point equations of motion for the partition function Z_{SM} in equation (4.1.8) above. We compute our saddle point equation

$$\begin{aligned}
& \frac{\partial}{\partial \lambda_k}\left(-\frac{1}{2}\sum_i \lambda_i^2 - \frac{g}{N}\sum_i \lambda_i^4 + \sum_{i\neq j} \ln(\lambda_i - \lambda_j)\right) = 0 \\
\Rightarrow \quad & -\lambda_k - \frac{4g}{N}\lambda_k^3 + \left[\sum_{j(j\neq k)} \frac{1}{\lambda_k - \lambda_j} - \sum_{i(i\neq k)} \frac{1}{\lambda_i - \lambda_k}\right] = 0 \\
\Rightarrow \quad & -\lambda_k - \frac{4g}{N}\lambda_k^3 + \sum_{j(j\neq k)} \frac{2}{\lambda_k - \lambda_j} = 0 \\
\Rightarrow \quad & \sum_{j(j\neq k)} \frac{1}{\lambda_k - \lambda_j} = \frac{1}{2}\lambda_k + \frac{2g}{N}\lambda_k^3.
\end{aligned} \tag{4.1.10}$$

Hence, the eigenvalues λ_i, satisfy the following stationary condition

$$\frac{1}{2}\lambda_i + \frac{2g}{N}\lambda_i^3 = \sum_{i\neq j} \frac{1}{\lambda_i - \lambda_j}. \tag{4.1.11}$$

Equation (4.1.11) is the large N single hermitian matrix model saddle point equation.

The Vandermonde determinant appearing in the measure of equation (4.1.8) is a result of diagonalizing the hermitian matrix M and integrating out the angular degrees V. When we compute our saddle point equations in the large N limit, we find that the mutually repulsive Coulomb interaction balances the force derived for the common potential, N particles will be evenly distributed around the minimum of the potential.

One-cut solution

Equation (4.1.11) is an eigenvalue problem that we wish to solve in the large N limit. To do this we need to move from a framework whose variables are discretized i.e. $\lambda_{i,j}$ for $i, j = 1.2.3.....N$ to a framework where the variables are no longer discrete but are now continuous.

In equation (4.1.11) we rescale the discrete variables to introduce the continuous variables

$$\lambda_i \to \sqrt{N}\lambda(x), \tag{4.1.12}$$

then following this we define an eigenvalue density function $\rho(\lambda)$ that satisfies the following condition

$$\frac{\partial x}{\partial \lambda} = \rho(\lambda). \tag{4.1.13}$$

The potential $W(M)$ in equation (4.1.5) of the single hermitian matrix model can have one global minima but several local minima that are all occupied by eigenvalues. Due to these multiple minima, it means that the eigenvalue density function $\rho(\lambda)$ can have multiple discontinuities, thus we need to define a support $(-\alpha, \alpha)$ on the real line for $\rho(\lambda)$ and require that in the support the eigenvalue density function be even [5], positive and normalized as follows

$$\int_{-\alpha}^{\alpha} d\lambda \rho(\lambda) = 1. \tag{4.1.14}$$

Therefore in the continuum limit, we have the following equation

$$\frac{1}{2}\lambda + 2g\lambda^3 = \fint_{-\alpha}^{\alpha} d\zeta \frac{\rho(\zeta)}{\lambda - \zeta} \qquad |\lambda| \leq \alpha, \tag{4.1.15}$$

where the variables described above are continuous.

For $N \to \infty$ we require that $\rho(\lambda)$ be continuous and positive in the compact support $(-\alpha, \alpha)$, but we will also require that the density function vanish outside this support.

[5] This is a consequence of the fact that the potential is even.

To obtain the solution of the equation of the semi-classical approximation we introduce an analytic function $G(z)$ for complex z, with a single cut along the interval defined by the support $(-\alpha, \alpha)$ [27], defined as follows

$$G(z) = \int_{-\alpha}^{\alpha} d\zeta \frac{\rho(\zeta)}{z - \zeta}. \tag{4.1.16}$$

The analytic function $G(z)$ has a unique solution that satisfies particular conditions. This unique solution is analytic on the complex plane outside the cut of the support $(-\alpha, \alpha)$ and when the continuous variable z is real outside the support $(-\alpha, \alpha)$ then the $G(z)$ must also be real.

From equation (4.1.16) we also observe that $z \to \pm\infty$ requires that $G(z)$ linearly decay as $\frac{1}{z}$ as a consequence of the normalization condition (4.1.15). On the complex plane when z approaches the support, $z \to (-\alpha, \alpha)$, the analytic function $G(\lambda)$ will have the following behaviour

$$G(z \pm i\epsilon) = \frac{1}{2}z + 2gz^3 \mp i\pi\rho(z). \tag{4.1.17}$$

The unique solution which satisfies the previously mentioned conditions for the analytic function $G(z)$ on the complex plane was constructed by Brézin et.al [27]. It is given by the following

$$G(z) = \frac{1}{2}z + 2gz^3 - \left(\frac{1}{2} + 2gz^3 + \alpha^2 g\right)\sqrt{z^2 - \alpha^2}. \tag{4.1.18}$$

The derivation of equation (4.1.18) is shown in appendix A.2

The above equation is encoded with the structure of the eigenvalue density function $\rho(\lambda)$. From equation (4.1.18) above, it follows that

$$\rho(\lambda) = \frac{1}{\pi}\left(\frac{1}{2} + 2g\lambda^2 + \alpha^2 g\right)\sqrt{\alpha^2 - \lambda^2}, \quad |\lambda| \leq \alpha^2. \tag{4.1.19}$$

If we switch off the coupling constant, that is, we consider a system with no interactions defined by $g = 0$, we find that equation (4.1.19) yields the Wigner semi-circle law for the dispersion of eigenvalues for a system of a single hermitian

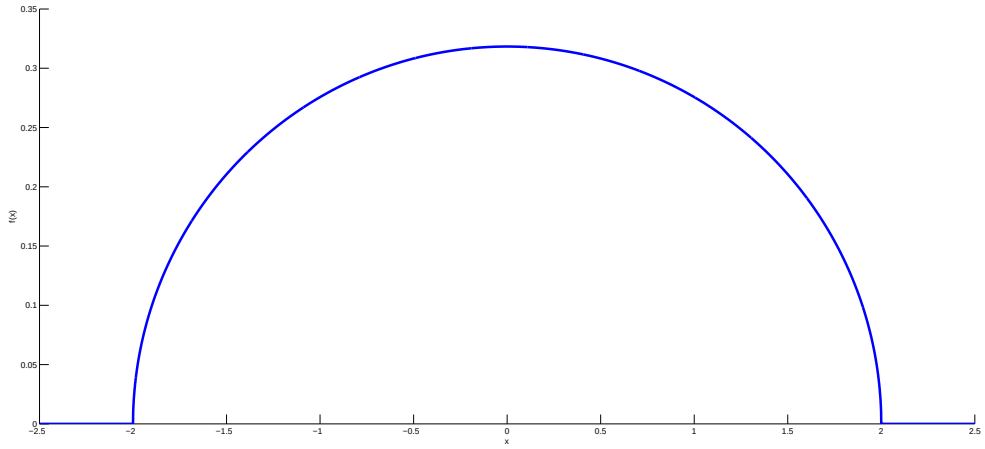

Figure 4.1: Single hermitian matrix model: Wigner semi-circle eigenvalue distribution.

matrix M i.e.

$$\rho(\lambda) = \frac{1}{\pi}\sqrt{1 - \frac{\lambda^2}{4}}. \tag{4.1.20}$$

For the $g = 0$ system, the support of the function is on the real line where $\alpha = 2$. Equation (4.1.20) represents an equation obeying the semi-circle law of the Wigner distribution. The distribution of eigenvalues is shown in Figure (4.1).

4.2 One-Dimensional Fermionic Picture

In this section we continue with the single hermitian matrix model but in an Hamiltonian setting, and show that it is equivalent to a Fermi gas of N non-interacting particles [27] subject to the potential in equation (4.1.5)

$$W(M) = W(\lambda_i) = \left(\frac{1}{2}\lambda_i^2 + \frac{g}{N}\lambda_i^4\right). \tag{4.2.1}$$

We start off by introducing the Hamiltonian

$$
\begin{aligned}
H_{SM} &= -\frac{1}{2}\nabla^2 + W(M) \qquad (4.2.2)\\
&= -\frac{1}{2}\mathrm{Tr}\left(\frac{\partial^2}{\partial M^2}\right) + W(M)\\
&= -\frac{1}{2}\left(\frac{1}{\Delta^2}\frac{\partial}{\partial\lambda_i}\Delta^2\frac{\partial}{\partial\lambda_i}\right) + W(\lambda_i) + \ldots(\text{angular degrees of freedom}),
\end{aligned}
$$

for the $N \times N$ hermitian matrix M and the ϕ^4 interaction $W(M)$ as defined in equation (4.2.2). In equation (4.2.2) above, the Laplacian operator appearing on the right hand side has been derived in appendix A.1 and Δ defined previously in equation (4.1.9).

In the ground state of the system, we see no appearance of any angular degrees of freedom. Therefore all singlet wavefunctions disappear when operated upon by operators that represent the angular degrees of freedom in the kinetic piece of the Laplacian.

Using the above Hamiltonian operator, we set up the following eigenvalue equation

$$
H_{SM}\phi(\lambda_i) = E_G\phi(\lambda_i), \qquad (4.2.3)
$$

for the ground state energy E_G, and identify $\phi(\lambda_i)$ as the ground state symmetric wavefunction, that is, it is symmetric under the exchange of any two eigenvalues.

Consider the kinetic piece of the Hamiltonian that acts as an operator in the Hamiltonian of equation (4.2.2). This Laplacian operator is the one that we use to arrive at the fermionic description of the single hermitian matrix model. We consider

$$
\left(-\frac{1}{2}\frac{1}{\Delta^2}\frac{\partial}{\partial\lambda_i}\Delta^2\frac{\partial}{\partial\lambda_i}\right)\phi(\lambda_i). \qquad (4.2.4)
$$

Let us introduce a ground state anti-symmetric wavefunction $\Omega(\lambda_i)$ defined by the following equation

$$
\Omega(\lambda_i) = \Delta\phi(\lambda_i), \qquad (4.2.5)
$$

and consider the action of the Laplacian on $\Omega(\lambda_i)$.

The anti-symmetric function $\Omega(\lambda_i)$ is a product of the symmetric wavefunction $\phi(\lambda_i)$ and the anti-symmetric Vandermonde determinant.

The kinetic operator of equation (4.2.4) ((4.2.2)) is now purely dependent on the eigenvalues λ_i of the system, allowing us to define an eigenvalue equation as follows

$$\begin{aligned} -\frac{1}{2}\left(\frac{1}{\Delta^2}\frac{\partial}{\partial\lambda_i}\Delta^2\frac{\partial}{\partial\lambda_i}\right)\phi(\lambda_i) &= E_G\phi(\lambda_i) \\ -\frac{1}{2}\left(\frac{1}{\Delta^2}\frac{\partial}{\partial\lambda_i}\Delta^2\frac{\partial}{\partial\lambda_i}\right)\frac{\Omega(\lambda_i)}{\Delta} &= E_G\frac{\Omega(\lambda_i)}{\Delta} \\ -\left(\frac{1}{\Delta}\frac{\partial}{\partial\lambda_i}\Delta\right)\left(\Delta\frac{\partial}{\partial\lambda_i}\frac{1}{\Delta}\right)\Omega(\lambda_i) &= 2E_G\Omega(\lambda_i). \end{aligned} \tag{4.2.6}$$

In total, it turns out that

$$H_{SM}\Omega(\lambda_i) = \sum_i^N \Lambda_i\Omega(\lambda_i) = E_G\Omega(\lambda_i), \tag{4.2.7}$$

for the discretized Hamiltonian operator

$$\sum_i^N \Lambda_i = -\frac{1}{2}\sum_i \frac{\partial^2}{\partial\lambda_i^2} + \frac{1}{2}\sum_i \lambda_i^2 + \frac{g}{N}\sum_i \lambda_i^4. \tag{4.2.8}$$

The derivation of equation (4.2.8) from equation (4.2.6) is shown in the appendix A.3.

Equation (4.2.8) has been reduced to a problem with just N non-interacting degrees of freedom.

By introducing the anti-symmetric wavefunction $\Omega(\lambda_i)$ into our one dimensional system, our eigenvalue problem has become that of N single particle fermions in the potential defined by $W(\lambda_i)$,

$$W(\lambda_i) = \frac{1}{2}\sum_i \lambda_i^2 + \frac{g}{N}\sum_i \lambda_i^4. \tag{4.2.9}$$

These N particles can be interpreted as a Fermi gas in the potential $W(\lambda_i)$ in equation (4.2.9) above.

4.3 Collective Field Theory Formalism

In this section we review the collective field theory formalism. In the next section, we will use this method to obtain a density description of the single hermitian matrix Hamiltonian.

The reason for this is that the collective field theory formalism simplifies a problem posed in a quantum field theoretical framework defined in the large N limit. In this large N limit the variables of collective field theory become independent and the original solution of quantum field theory can be interpreted in a quantum mechanical framework, via collective field theory.

Our first step of transitioning into the collective field theory frame work is to reformulate the single hermitian matrix Hamiltonian description in terms of our new invariant trace variables. So in essence this means that we are rewriting the Hamiltonian for the single hermitian matrix model defined in equation (4.2.2) using independent (in the large N limit) collective field theory invariant variables.

As a first step, we wish to demonstrate how to derive a generalized Hamiltonian using collective field theory variables.

We start of by defining a Hamiltonian using general coordinates q_i with some arbitrary potential $V(q_i)$

$$\begin{aligned} H &= H_K + H_V \\ &= -\frac{1}{2}\sum_i \frac{\partial}{\partial q_i}\frac{\partial}{\partial q_i} + \sum_i V(q_i). \end{aligned} \tag{4.3.1}$$

The notation H_K and H_V denotes the kinetic and potential parts of the Hamiltonian for general coordinates q_i. Also in the first line of equation (4.3.1) the term $-i\partial/\partial q_i$ is the conjugate momentum of the coordinate q_i.

From equation (4.3.1) we perform a change of variables, that is, we are now performing a transition into the framework of collective field theory from a theory defined using general coordinates q_i to the collective field theory variable(s) ϕ_α.

Acting with the Hamiltonian on wavefucntions that only depend on the invariant collective field variables, one has

$$
\begin{aligned}
2H &= -\sum_i \frac{\partial}{\partial q_i}\frac{\partial}{\partial q_i} + V(q_i) \\
&= -\sum_{i,\alpha} \frac{\partial^2 \phi_\alpha}{\partial q_i^2}\frac{\partial}{\partial \phi_\alpha} - \sum_i \sum_{\alpha,\beta} \frac{\partial \phi_\alpha}{\partial q_i}\frac{\partial \phi_\beta}{\partial q_i}\frac{\partial}{\partial \phi_\alpha}\frac{\partial}{\partial \phi_\beta} + V(\phi_\alpha) \\
&= -\sum_{i,\alpha} \omega_\alpha \frac{\partial}{\partial \phi_\alpha} - \sum_{\alpha,\beta} \Omega_{\alpha,\beta}\frac{\partial}{\partial \phi_\alpha}\frac{\partial}{\partial \phi_\beta} + V(\phi_\alpha). \qquad (4.3.2)
\end{aligned}
$$

In equation (4.3.2) above we have defined the following terms

$$
\omega_\alpha = \sum_i \frac{\partial^2 \phi_\alpha}{\partial q_i^2}, \qquad \Omega_{\alpha,\beta} = \sum_i \frac{\partial \phi_\alpha}{\partial q_i}\frac{\partial \phi_\beta}{\partial q_i}, \qquad (4.3.3)
$$

where ω_α is the "splitting" operator and $\Omega_{\alpha,\beta}$ is the "joining" operator.

The relevance of these terms will be seen when we do an explicit example using the single hermitian matrix model.

After performing the change of variables, the kinetic and potential energy components of the Hamiltonian in equation (4.3.1) take the form

$$
2H_K = -\sum_{i,\alpha} \omega_\alpha \frac{\partial}{\partial \phi_\alpha} - \sum_{\alpha,\beta} \Omega_{\alpha,\beta}\frac{\partial}{\partial \phi_\alpha}\frac{\partial}{\partial \phi_\beta}, \qquad H_V = V(\phi_\alpha). \qquad (4.3.4)
$$

As a result of the change of variables, naturally we would expect a Jacobian. In the coordinate space encoded by the general coordinates q_i, the wavefunctions $\psi(q)$ carry information regarding the energy states of the Hamiltonian.

Similarly, the invariant operator wavefunctions $\Psi(\phi)$ carry information about the space in which the coordinates ϕ_α live. Thus, requiring a change of coordinates from the old variables q_i to the new variables ϕ_α requires that the inner products of the wavefunctions from the two respective coordinate spaces be preserved, hence

$$
\int dq \psi^\dagger(q)\psi(q) = \int d\phi J \Psi^\dagger(\phi)\Psi(\phi), \qquad (4.3.5)
$$

where J is the Jacobian of the transformation.

The explicit hermiticity of the kinetic piece H_K should be apparent after performing the following similarity transformation

$$\begin{aligned}\frac{\partial}{\partial\phi_\alpha} &\rightarrow J^{1/2}\frac{\partial}{\partial\phi_\alpha}J^{-1/2} \\ &= \left(\frac{\partial}{\partial\phi_\alpha}-\frac{1}{2}\frac{\partial \ln J(\phi)}{\partial\phi_\alpha}\right). \end{aligned} \tag{4.3.6}$$

In equation (4.3.6) above we note the subtlety of the dependence of the Jacobian J on the invariant variable ϕ. Our attention is now solely on the kinetic piece H_K of the Hamiltonian (4.3.2).

The equation stemming from the similarity transformation in the last line of equation (4.3.6) is substituted into the kinetic piece H_K of the collective field theory Hamiltonian:

$$\begin{aligned}-2H_K &= \sum_\alpha \omega_\alpha\frac{\partial}{\partial\phi_\alpha}+\sum_{\alpha,\beta}\Omega_{\alpha,\beta}\frac{\partial}{\partial\phi_\alpha}\frac{\partial}{\partial\phi_\beta} \\ &\longrightarrow \sum_\alpha \omega_\alpha\left(\frac{\partial}{\partial\phi_\alpha}-\frac{1}{2}\frac{\partial \ln J(\phi)}{\partial\phi_\alpha}\right)+\sum_{\alpha,\beta}\Omega_{\alpha,\beta}\left(\frac{\partial}{\partial\phi_\alpha}-\frac{1}{2}\frac{\partial \ln J(\phi)}{\partial\phi_\alpha}\right)\left(\frac{\partial}{\partial\phi_\beta}-\frac{1}{2}\frac{\partial \ln J(\phi)}{\partial\phi_\beta}\right) \\ &= \sum_\alpha \omega_\alpha\frac{\partial}{\partial\phi_\alpha}-\frac{1}{2}\sum_\alpha\omega_\alpha\frac{\partial \ln J(\phi)}{\partial\phi_\alpha} \\ &+\sum_{\alpha,\beta}\left(\frac{\partial}{\partial\phi_\alpha}\Omega_{\alpha,\beta}\frac{\partial}{\partial\phi_\beta}-\left(\frac{\partial}{\partial\phi_\alpha}\Omega_{\alpha,\beta}\right)\frac{\partial}{\partial\phi_\beta}\right)-\frac{1}{2}\sum_{\alpha,\beta}\Omega_{\alpha,\beta}\left(\frac{\partial}{\partial\phi_\alpha}\frac{\partial}{\partial\phi_\beta}\ln J(\phi)\right) \\ &-\sum_{\alpha,\beta}\Omega_{\alpha,\beta}\left(\frac{\partial}{\partial\phi_\alpha}\ln J(\phi)\right)\frac{\partial}{\partial\phi_\beta}+\frac{1}{4}\sum_{\alpha,\beta}\Omega_{\alpha,\beta}\left(\frac{\partial \ln J(\phi)}{\partial\phi_\alpha}\frac{\partial \ln J(\phi)}{\partial\phi_\beta}\right). \end{aligned} \tag{4.3.7}$$

We have assumed $\Omega_{\alpha,\beta}=\Omega_{\beta,\alpha}$. Substituting the similarity transformation equation (4.3.6), H_K should be explicitly hermitian and any non-hermitian terms must be set equal to zero. Therefore the real function $J(\phi)$ can be uniquely determined from this requirement. This gives

$$\begin{aligned} 0 &= \sum_\alpha\omega_\alpha\frac{\partial}{\partial\phi_\alpha}-\left(\frac{\partial}{\partial\phi_\alpha}\Omega_{\alpha,\beta}\right)\frac{\partial}{\partial\phi_\beta}-\Omega_{\alpha,\beta}\left(\frac{\partial \ln J(\phi)}{\partial\phi_\beta}\frac{\partial}{\partial\phi_\alpha}\right) \\ \Rightarrow 0 &= \omega_\alpha-\left(\frac{\partial}{\partial\phi_\beta}\Omega_{\alpha,\beta}\right)-\Omega_{\alpha,\beta}\left(\frac{\partial \ln J(\phi)}{\partial\phi_\beta}\right). \end{aligned} \tag{4.3.8}$$

From equation (4.3.8) above it follows that

$$\frac{\partial \ln J(\phi)}{\partial\phi_\gamma}=\Omega^{-1}_{\gamma,\alpha}\omega_\alpha-\Omega^{-1}_{\gamma,\alpha}\left(\frac{\partial}{\partial\phi_\beta}\Omega_{\alpha,\beta}\right), \tag{4.3.9}$$

with the definition $\sum_\alpha \Omega^{-1}_{\gamma,\alpha}\Omega_{\alpha,\beta} = \delta_{\gamma,\beta}$.

Equation (4.3.9) may not be solvable for $J(\phi)$ in closed form, but we make use of it by substituting it back into H_K which is purely hermitian. By performing this substitution H_K becomes

$$\begin{aligned} H_K &= -\frac{1}{2}\sum_\alpha \omega_\alpha \frac{\partial \ln J(\phi)}{\partial \phi_\alpha} + \sum_{\alpha,\beta} \frac{\partial}{\partial \phi_\alpha}\Omega_{\alpha,\beta}\frac{\partial}{\partial \phi_\beta} \\ &- \sum_{\alpha,\beta}\Omega_{\alpha,\beta}\left((\frac{\partial}{\partial \phi_\alpha}\frac{\partial \ln J(\phi)}{\partial \phi_\beta}) + \frac{1}{4}\frac{\partial \ln J(\phi)}{\partial \phi_\alpha}\frac{\partial \ln J(\phi)}{\partial \phi_\beta}\right) \\ &= -\frac{1}{2}\omega_\alpha\left(\Omega^{-1}_{\alpha,\beta}\omega_\beta - \Omega^{-1}_{\alpha,\beta}\left(\frac{\partial}{\partial \phi_\gamma}\Omega_{\beta,\gamma}\right)\right) + \frac{\partial}{\partial \phi_\alpha}\Omega_{\alpha,\beta}\frac{\partial}{\partial \phi_\beta} \\ &- \Omega_{\alpha,\beta}\left(\frac{\partial}{\partial \phi_\alpha}\left[\left(\Omega^{-1}_{\beta,\alpha}\omega_\alpha - \Omega^{-1}_{\beta,\alpha}\left(\frac{\partial}{\partial \phi_\gamma}\Omega_{\alpha,\gamma}\right)\right)\right]\right) \\ &+ \frac{1}{4}\Omega_{\alpha,\beta}\left(\Omega^{-1}_{\alpha,\beta}\omega_\beta - \Omega^{-1}_{\alpha,\beta}\left(\frac{\partial}{\partial \phi_\gamma}\Omega_{\beta,\gamma}\right)\right)\left(\Omega^{-1}_{\beta,\alpha}\omega_\alpha - \Omega^{-1}_{\beta,\alpha}\left(\frac{\partial}{\partial \phi_\gamma}\Omega_{\alpha,\gamma}\right)\right). \end{aligned} \tag{4.3.10}$$

We multiply the $1/2$ factor that appeared in front of the kinetic piece and set equal to zero any non-hermitian terms that emerge in equation (4.3.10) above. So now, we see our complete hermitian collective field theory Hamiltonian

$$\begin{aligned} H &= -\frac{1}{2}\frac{\partial}{\partial \phi_\alpha}\Omega_{\alpha,\beta}\frac{\partial}{\partial \phi_\beta} + \frac{1}{8}\omega_\alpha\Omega^{-1}_{\alpha,\beta}\omega_\beta - \frac{1}{8}\omega_\alpha\Omega^{-1}_{\alpha,\beta}\left(\frac{\partial}{\partial \phi_\gamma}\Omega_{\beta,\gamma}\right) \\ &- \frac{1}{8}\left(\frac{\partial}{\partial \phi_\gamma}\Omega_{\beta,\gamma}\right)\Omega^{-1}_{\beta,\alpha}\omega_\alpha - \frac{1}{8}\left(\frac{\partial}{\partial \phi_\gamma}\Omega_{\beta,\gamma}\right)\Omega^{-1}_{\beta,\alpha}\left(\frac{\partial}{\partial \phi_\gamma}\Omega_{\alpha,\gamma}\right) \\ &+ \frac{1}{2}\left(\frac{\partial}{\partial \phi_\alpha}\omega_\alpha\right) + \frac{1}{2}\left(\frac{\partial}{\partial \phi_\alpha}\frac{\partial}{\partial \phi_\gamma}\Omega_{\alpha,\gamma}\right) + V[\phi]. \end{aligned} \tag{4.3.11}$$

In our case, we will be investigating the dynamics of matrix models in the large N limit where not all of the terms appearing in equation (4.3.11) will survive. The only terms that appear in the Hamiltonian in the large N limit for the kinetic piece will be the first two terms appearing in the first line of equation (4.3.11). The rest of the terms are sub-leading, do not contribute to the leading large N limit and fluctuations.

In total, our effective collective field theory Hamiltonian will be as follows

$$H_{eff} = \frac{1}{2}\sum_{\alpha,\beta}\frac{\partial}{\partial \phi_\alpha}\Omega_{\alpha,\beta}\frac{\partial}{\partial \phi_\beta} + \sum_{\alpha,\beta}\frac{1}{8}\omega_\alpha\Omega^{-1}_{\alpha,\beta}\omega_\beta + V[\phi] \tag{4.3.12}$$

In equation (4.3.12) above, we re-introduced the summations over α and β to express the effective Hamiltonian formally. If the invariant variable labels are continuous we can also express the above equation in the coordinate representation (x, y) using the fact that $\sum_i f(x_i) \to \int dx f(x)$ to give us the following effective Hamiltonian

$$\begin{aligned} H_{eff} &= -\frac{1}{2}\int dx \int dy \frac{\partial}{\partial\phi(x)}\Omega(x,y,[\phi])\frac{\partial}{\partial\phi(y)} \\ &+ \frac{1}{8}\int dx \int dy \omega(x,[\phi])\Omega^{-1}(x,y,[\phi])\omega(y,[\phi]) + V[\phi]. \end{aligned} \quad (4.3.13)$$

Equation (4.3.13) will be used more effectively by applying it to the single hermitian matrix model. This is what we will do in the following section.

4.4 Single Hermitian Matrix Model Density Description

Now that we have formally presented the framework of collective field theory for a general set of variables, the goal is now to apply the results of the previous section to the single hermitian matrix model.

Our starting point is the single hermitian matrix model Hamiltonian

$$\begin{aligned} H_{SM} &= -\frac{1}{2}\nabla^2 + W \\ &= -\frac{1}{2}\mathrm{Tr}\left(\frac{\partial}{\partial M}\frac{\partial}{\partial M}\right) + W(M). \end{aligned} \quad (4.4.1)$$

We emphasize again that the Hamiltonian (4.4.1) for the $N \times N$ hermitian matrix M is invariant under U(N) gauge transformations.

We introduce our collective field variables which constitute the invariant set of operators

$$\phi_k = \mathrm{Tr}\left(e^{ikM}\right) = \sum_i e^{ik\lambda_i}, \quad (4.4.2)$$

for some real number k.

The degrees of freedom in this framework are the eigenvalues λ_i of the matrix M with the index i specified by the size of the matrix such that $i = 1, 2, ...N$. The variable ϕ_k represents the exponentiated sum of the eigenvalues λ_i lying on the main diagonal of the hermitian matrix M. The invariant set of operators ϕ_k are specified in the coordinate space with the following Fourier transformation

$$\phi(x) = \int \frac{dk}{2\pi} e^{-ik} \phi_k = \sum_i \delta(x - \lambda_i). \tag{4.4.3}$$

Using our new set of variables that we introduced in equations (4.4.2) and (4.4.3), and following the steps described in the previous section, we can write down our newly formulated Hamiltonian in equation (4.4.1), taking equations (4.3.12) and (4.3.13) as a starting point

$$\begin{aligned} H_{SM} &= -\frac{1}{2}\mathrm{Tr}\left(\frac{\partial}{\partial M}\frac{\partial}{\partial M}\right) + W(M) \\ &= -\frac{1}{2}\frac{\partial}{\partial M_{ij}}\frac{\partial}{\partial M_{ji}} + W(M) \\ &= -\frac{1}{2}\left[\left(\frac{\partial^2 \phi_k}{\partial M_{ij}\partial M_{ji}}\right)\frac{\partial}{\partial \phi_k} + \left(\frac{\partial \phi_k}{\partial M_{ji}}\frac{\partial \phi_{k'}}{\partial M_{ij}}\right)\frac{\partial}{\partial \phi_k}\frac{\partial}{\partial \phi_{k'}}\right] + W[\phi] \\ &= -\frac{1}{2}\omega(k;[\phi])\frac{\partial}{\partial \phi_k} - \frac{1}{2}\Omega(k,k';[\phi])\frac{\partial}{\partial \phi_k}\frac{\partial}{\partial \phi_{k'}} + W[\phi], \end{aligned} \tag{4.4.4}$$

where the "joining" and "splitting" operators in our new variables take the form

$$\Omega(k,k';[\phi]) = \frac{\partial \phi_k}{\partial M_{ji}}\frac{\partial \phi_{k'}}{\partial M_{ij}} \qquad \omega(k;[\phi]) = \frac{\partial^2 \phi_k}{\partial M_{ij}\partial M_{ji}}. \tag{4.4.5}$$

Explicitly, the joining operator in the collective field theory language takes the form

$$\begin{aligned}
\Omega(k,k';[\phi]) &= \sum_{ij}\left(\frac{\partial[\phi_k]}{\partial M_{ij}}\right)\left(\frac{\partial[\phi_{k'}]}{\partial M_{ji}}\right) \\
&= \sum_{ij}\left(\frac{\partial}{\partial M_{ij}}\mathrm{Tr}\left(e^{ikM}\right)\right)\left(\frac{\partial}{\partial M_{ji}}\mathrm{Tr}\left(e^{ik'M}\right)\right) \\
&= \sum_{ij} ik\left(e^{ikM}\right)_{ij} ik'\left(e^{ik'M}\right)_{ji} \\
&= -kk'\sum_{i}\left(e^{i(k+k')M}\right)_{ii} \\
&= -kk'\mathrm{Tr}\left(e^{i(k+k')M}\right) \\
&= -kk'\phi_{k+k'}. \qquad (4.4.6)
\end{aligned}$$

The joining operator performs the following $(\phi_k, \phi_{k'}) \to (\phi_{k+k'})$.

As for the splitting operator, we operate on ϕ_k twice, this is shown below:

$$\begin{aligned}
\omega(k;[\phi]) &= \sum_{ij}\frac{\partial}{\partial M_{ij}}\frac{\partial}{\partial M_{ji}}[\phi_k] \\
&= \frac{\partial}{\partial M_{ij}}\left(ik\left(e^{ikM}\right)_{ij}\right) \\
&= ik\int_0^1 d\alpha\left(e^{i\alpha kM}\right)_{ik}\left(ik\frac{\partial M_{kl}}{\partial M_{ij}}\right)\left(e^{i(1-\alpha)kM}\right)_{lj} \\
&= (ik)^2\int_0^1 d\alpha\left(e^{i\alpha kM}\right)_{ii}\left(e^{i(1-\alpha)kM}\right)_{jj} \qquad \text{let} \quad k' = k\alpha \\
&= -k\int_0^k dk'\mathrm{Tr}\left(e^{ik'M}\right)\mathrm{Tr}\left(e^{i(k-k')M}\right) \\
&= -k\int_0^k dk'\phi_{k'}\phi_{k-k'}. \qquad (4.4.7)
\end{aligned}$$

In third line of equation (4.4.7) above, we made use of the following critical identity

$$\frac{\partial}{\partial M_{ij}}\left(e^{ikM}\right)_{qq'} = (ik)\int_0^1 d\alpha\left(e^{ikM}\right)_{qi}\left(e^{i(1-\alpha)kM}\right)_{jq'}, \qquad (4.4.8)$$

which allowed us to see the splitting process on $\phi_{k'}$: $(\phi_k, \phi_{k-k'}; k, k' \in \Re)$.

To demonstrate the usefulness of collective field theory for the single hermitian matrix model, we need to make use of equation (4.4.3). It means that we need

to define all our variables so that they can be substituted into equation (4.4.4) in the coordinate space (x, y).

We proceed with the Fourier transform of the splitting operator ω_k, defined in k-space. We first resume our calculation from the last line of equation (4.4.7)

$$\begin{aligned}
\omega_k &= -k \int_0^k dk' \left(e^{ikM}\right)_{ii} \left(e^{i(k-k')M}\right)_{jj} \\
&= -k \sum_{ij} \int_0^k dk' \left(e^{ik'(\lambda_i - \lambda_j)}\right) \left(e^{ik\lambda_j}\right) \\
&= \sum_{i \neq j} \frac{ik}{\lambda_i - \lambda_j} \left(e^{ik\lambda_i} - e^{ik\lambda_j}\right) \\
&= 2 \sum_{i \neq j} \frac{ike^{ik\lambda_i}}{\lambda_i - \lambda_j} \qquad (4.4.9)
\end{aligned}$$

Equation (4.4.9) is the version of equation (4.4.7) in the eigenvalue representation. From this point onwards, we will define all our variables that are necessary to be substituted into the Hamiltonian in equation (4.4.4) from the k-space representation to the coordinate representation (x, y) by Fourier transform.

We proceed as follows

$$\begin{aligned}
\omega_k &= 2 \int dx \int dy \delta(x - \lambda_i)\delta(y - \lambda_j) \sum_{i \neq j} \frac{ike^{ikx}}{x - y} \\
&= 2ik \int dx \fint dy \sum_i \delta(x - \lambda_i) \sum_j \delta(y - \lambda_j) \frac{e^{ikx}}{(x - y)}. \qquad (4.4.10)
\end{aligned}$$

In equation (4.4.10) (above) we made and will make use (below) of the following definitions

$$\int dx \delta(x - \lambda_i) f(x) = f(\lambda_i); \qquad \int \frac{dk}{2\pi} e^{-ikx} \phi_k = \phi(x) = \sum_i \delta(x - \lambda_i), \quad (4.4.11)$$

and we obtain the following for ω_k

$$\omega_k = 2ik \int dx \fint dy \frac{\phi(x)\phi(y)}{(x - y)} \left(e^{ikx}\right). \qquad (4.4.12)$$

From the equation (4.4.12) above, to obtain the Fourier transform $\omega(x)$ of ω_k we perform

$$\begin{aligned}
\omega(x) &= 2\int \frac{dk}{2\pi} e^{-ikx}\omega_k \\
&= 2\int \frac{dk}{2\pi} e^{-ikx}\left(ik\int dz \fint dy \frac{\phi(z)\phi(y)}{(z-y)} e^{ikz}\right) \\
&= -2\int \frac{dk}{2\pi}\left(\frac{\partial}{\partial x} e^{-ikx}\right)\left(\int dz \fint dy \frac{\phi(z)\phi(y)}{(z-y)} e^{ikz}\right) \\
&= -2\partial_x \int \frac{dk}{2\pi}\left(\int dz \fint dy \frac{\phi(z)\phi(y)}{(z-y)} e^{ik(z-x)}\right) \\
&= -2\partial_x \int dz\phi(z) \fint dy \frac{\phi(y)}{(z-y)} \int \frac{dk}{2\pi} e^{ik(z-x)} \\
&= -2\partial_x \left(\int dz\delta(z-x)\right) \fint dy\phi(z)\phi(y)\frac{1}{z-y} \\
&= -2\partial_x \fint dy\phi(x)\phi(y)\frac{1}{x-y} \\
\Rightarrow \omega(x) &= -2\partial_x\left(\phi(x) \fint dy \frac{\phi(y)}{x-y}\right) \\
\Rightarrow \omega(x) &= -2\partial_x\phi(x)G(x). \qquad (4.4.13)
\end{aligned}$$

In the fifth line of equation (4.4.13) we define $\partial_x = \partial/\partial x$, also we see the density description of the splitting operator $\omega(x)$.

We will now continue by presenting the density description of the joining operator, using the definition of equation (4.4.6):

$$
\begin{aligned}
\Omega(x,y,[\phi]) &= \int \frac{dk}{2\pi}\int \frac{dk'}{2\pi} e^{-ikx}e^{-ik'y}\Omega(k,k',[\phi]) \\
&= \int \frac{dk}{2\pi}\int \frac{dk'}{2\pi} e^{-ikx}e^{-ik'y}\left(-kk'\phi_{k+k'}\right) \\
&= \int \frac{dk}{2\pi}\int \frac{dk'}{2\pi} \left(\partial_x e^{-ikx}\right)\left(\partial_y e^{-ik'y}\right)\left(\sum_i e^{i(k+k')\lambda_i}\right) \\
&= \partial_x\partial_y \int \frac{dk}{2\pi}\int \frac{dk'}{2\pi} \left(e^{-ikx}\right)\left(e^{-ik'y}\right)\left(\sum_i e^{i(k+k')\lambda_i}\right) \\
&= \partial_x\partial_y \int \frac{dk}{2\pi}\int \frac{dk'}{2\pi} e^{-ikx}e^{-ik'y}\left(\int dz\delta(z-\lambda_i)\sum_i e^{i(k+k')z}\right) \\
&= \partial_x\partial_y \int \frac{dk}{2\pi} e^{-ik(x-z)}\int \frac{dk'}{2\pi} e^{-ik'(y-z)}\left(\int dz\sum_i \delta(z-\lambda_i)\right) \\
&= \partial_x\partial_y \left(\delta(x-z)\delta(y-z)\int dz\phi(z)\right) \\
&= \partial_x\partial_y \left(\int dz\delta(x-z)\delta(y-z)\phi(z)\right) \\
\Rightarrow \Omega(x,y,[\phi]) &= \partial_x\partial_y\left(\delta(x-y)\phi(x)\right). \qquad (4.4.14)
\end{aligned}
$$

We have computed all the necessary terms to give us the form/structure of the collective field theory effective Hamiltonian for the single hermitian matrix model.

Since we wish to rewrite the effective Hamiltonian H_{eff} in equation (4.3.13) in terms of the invariant set of variables ϕ_k and $\phi(x)$ of the single hermitian matrix model, we need to calculate the term that expresses the contribution of the repulsion amongst the eigenvalues of the system. The term, in equation (4.3.13), associated with the eigenvalue repulsion is $\omega(x,[\phi])\Omega^{-1}(x,y,[\phi])\omega(y,[\phi])$.

Below, we explicitly calculate this repulsion directly from equation (4.3.13) as follows

$$
\begin{aligned}
& \frac{1}{8}\int dx \int dy \omega(x,[\phi])\Omega^{-1}(x,y,[\phi])\omega(y,[\phi]) \\
= & \frac{1}{8}\int dx \int dy \left(-2\partial_x\left(\phi(x)G(x)\right)\right)\Omega^{-1}(x,y,[\phi])\left(-2\partial_y\left(\phi(y)G(y)\right)\right) \\
= & \frac{1}{2}\int dx \int dy \left(\partial_x\left(\phi(x)G(x)\right)\right)\Omega^{-1}(x,y,[\phi])\left(\partial_y\left(\phi(y)G(y)\right)\right) \\
= & \frac{1}{2}\int dx\phi(x)G(x)\int dy \left(\partial_y\partial_x\Omega^{-1}_{x.y}\right)\phi(y)G(y) \\
= & \frac{1}{2}\int dx \int dy \left(\phi(x)G(x)\right)\left(\partial_x\partial_y\Omega^{-1}_{x,y}\right)\left(\phi(y)G(y)\right). \qquad (4.4.15)
\end{aligned}
$$

In equation (4.4.15) above, we made use of the following definition (see (4.4.13))

$$
G(x) = \oint dy \frac{\phi(y)}{x-y}, \qquad (4.4.16)
$$

which is a term that captures the repulsion amongst the eigenvalues of the single hermitian matrix model.

In the integration by parts above, all surface terms do not contribute as $\phi(x)$ has finite support.

Equation (4.4.15) can be simplified further. The inverse of the joining operator $\Omega^{-1}_{x,y}$, partially differentiated with respect to the density variables ∂_x and ∂_y can be shown explicitly to simplify as follows

$$
\begin{aligned}
& \int dy \Omega^{-1}_{x,y}\Omega_{y,z} = \delta(x-z) \\
\Rightarrow \quad & \int dy \Omega^{-1}_{x,y}\partial_y\partial_z\left(\delta(y-z)\phi(y)\right) = \delta(x-z) \\
\Rightarrow \quad & \partial_z \int dy \Omega^{-1}_{x,y}\partial_y\left(\delta(y-z)\phi(y)\right) = \delta(x-z) \\
\Rightarrow \quad & -\left[\partial_z \int dy \left(\partial_y\Omega^{-1}_{x,y}\right)\delta(y-z)\phi(y)\right] = \delta(x-z) \\
\Rightarrow \quad & -\partial_z\left[\left(\partial_z\Omega^{-1}_{x,z}\right)\phi(z)\right] = \delta(x-z) \\
\Rightarrow \quad & -\partial_x\partial_z\left[\left(\partial_z\Omega^{-1}_{x,z}\right)\phi(z)\right] = \partial_x\delta(x-z) \\
\Rightarrow \quad & -\partial_z\left[\left(\partial_z\partial_x\Omega^{-1}_{x,z}\right)\phi(z)\right] = -\partial_z\delta(x-z) \\
\Rightarrow \quad & \left(\partial_x\partial_z\Omega^{-1}_{x,z}\right) = \frac{\delta(x-z)}{\phi(x)}. \qquad (4.4.17)
\end{aligned}
$$

The identity derived in equation (4.4.17) is substituted into equation (4.4.15) to give us the following

$$\begin{aligned}
& \frac{1}{8}\int dx \int dy \omega(x,[\phi])\Omega^{-1}(x,y,[\phi])\omega(y,[\phi]) \\
= & \frac{1}{2}\int dx \int dy \left(\phi(x)G(x)\right)\left(\partial_x\partial_y\Omega^{-1}_{x,y}\right)\left(\phi(y)G(y)\right) \\
= & \frac{1}{2}\int dx \int dy \left(\phi(x)G(x)\right)\left(\frac{\delta(x-y)}{\phi(y)}\right)\left(\phi(y)G(y)\right) \\
= & \frac{1}{2}\int dx \int dy \phi(x)G(x)\delta(x-y)G(y) \\
= & \frac{1}{2}\int dx \phi(x)G^2(x) \\
= & \frac{1}{2}\int dx \phi(x)\left(⨍ dy \frac{\phi(y)}{x-y}\right)^2. \qquad (4.4.18)
\end{aligned}$$

The last line of equation (4.4.18) shows a repulsion amongst eigenvalues that contribute to the potential in the denominator. There is an identity that allows this term to simplify. First note that

$$\begin{aligned}
G(x) &= ⨍ dy \frac{\phi(y)}{x-y} \\
&= ⨍ dy \frac{1}{x-y}\int \frac{dk}{2\pi} e^{-iky}\phi_k \\
&= \int \frac{dk}{2\pi}\phi_k \left\{⨍ dy \frac{e^{-iky}}{x-y}\right\}. \qquad (4.4.19)
\end{aligned}$$

The last line of equation (4.4.19) requires that we perform contour integration, and can be shown to be

$$⨍ dy \frac{e^{-iky}}{x-y} = i\pi\varsigma(k)e^{-ikx}, \qquad (4.4.20)$$

for $\varsigma(\pm k) = \pm 1$.

Substituting into (4.4.18)

$$
\begin{aligned}
& \frac{1}{2}\int dx\phi(x)G^2(x) \\
= \; & \frac{1}{2}\int dx\phi(x)\left(⨍ dy\frac{\phi(y)}{x-y}\right)^2 \\
= \; & \int dx\int\frac{dk_1}{2\pi}\int\frac{dk_2}{2\pi}\int\frac{dk_3}{2\pi}\phi_{k_1}\phi_{k_2}\phi_{k_3}e^{-ikx_1}\times \\
& \left(⨍ dy\frac{e^{-ik_2}}{x-y}⨍ dy'\frac{e^{-ik_3}}{x-y'}\right) \\
= \; & -\pi^2\int\frac{dk_1}{2\pi}\int\frac{dk_2}{2\pi}\int\frac{dk_3}{2\pi}\phi_{k_1}\phi_{k_2}\phi_{k_3}\varsigma(k_2)\varsigma(k_3)\times \\
& \int dxe^{-i(k_1+k_2+k_3)x} \\
= \; & \frac{\pi^2}{3}\int dx\phi^3(x). \qquad (4.4.21)
\end{aligned}
$$

In arriving at the result above, the different signs of k_1, k_2, k_3 are considered.

So what we have proved is that the term that represents the repulsion amongst eigenvalues for the single hermitian matrix model contributing to the common potential is cubic and local, that is

$$
\frac{1}{8}\int dx\int dy\omega(x,[\phi])\Omega^{-1}(x,y,[\phi])\omega(y,[\phi]) = \frac{\pi^2}{6}\int dx\phi^3(x). \quad (4.4.22)
$$

For the kinetic piece we have,

$$
\begin{aligned}
& \frac{1}{2}\int dx\int dy\Pi(x)\Omega(x,y,[\phi])\Pi(y) = \\
& \frac{1}{2}\int dx\int dy\Pi(x)\left(\left(\partial_x\partial_y\delta(x-y)\right)\phi(x)\right)\Pi(y) = \\
& \frac{1}{2}\int dx\left(\partial_x\Pi(x)\right)\phi(x)\int dy\delta(x-y)\left(\partial_y\Pi(y)\right) = \\
& \frac{1}{2}\int dx\left(\partial_x\Pi(x)\right)\phi(x)\left(\partial_x\Pi(x)\right). \qquad (4.4.23)
\end{aligned}
$$

We arrive at the form of the (effective) collective field Hamiltonian,

$$
\begin{aligned}
H_{SMeff} &= \frac{1}{2}\int dx \int dy \Pi(x)\Omega(x,y,[\phi])\Pi(y) \\
&+ \frac{\pi^2}{6}\int dx\phi^3(x) + W[\phi] \\
&= \frac{1}{2}\int dx \left(\partial_x \Pi(x)\right)\phi(x)\left(\partial_x \Pi(x)\right) + \mathcal{W}[\phi], \qquad (4.4.24)
\end{aligned}
$$

subject to the constraint

$$
\int dx\phi(x) = N. \qquad (4.4.25)
$$

The "effective" potential $\mathcal{W}[\phi]$, is defined as follows

$$
\mathcal{W}[\phi] = \frac{\pi^2}{6}\int dx\phi^3(x) + W[\phi]. \qquad (4.4.26)
$$

Below we will give an example that demonstrates the application of the collective field theory formalism using the single hermitian matrix model.

We consider a system coupled to the harmonic oscillator potential, described by the following Lagrangian

$$
\mathcal{L} = \frac{1}{2}\mathrm{Tr}\left(\frac{\partial}{\partial M}\frac{\partial}{\partial M}\right) - \frac{1}{2}\bar{\omega}^2\mathrm{Tr}\left(M^2\right), \qquad (4.4.27)
$$

where $\bar{\omega}$ is the angular frequency of the system.

Using our new set of invariant variables, the harmonic potential becomes the following

$$
\begin{aligned}
\frac{1}{2}\bar{\omega}^2\mathrm{Tr}\left(M^2\right) &= \frac{1}{2}\bar{\omega}^2 \sum_i \lambda_i^2 \\
&= \frac{1}{2}\bar{\omega}^2 \int dx \sum_i \delta(x-\lambda_i)x^2 \\
&= \frac{1}{2}\bar{\omega}^2 \int dx x^2\phi(x). \qquad (4.4.28)
\end{aligned}
$$

So now our system can be described by the following effective Hamiltonian

$$
H_{SMeff} = \frac{1}{2}\int dx \left(\partial_x \Pi(x)\right)\phi(x)\left(\partial_x \Pi(x)\right) + \mathcal{W}(\phi), \qquad (4.4.29)
$$

with the effective potential defined by

$$\mathcal{W}[\phi] = \frac{\pi^2}{6}\int dx\phi^3(x) + \frac{1}{2}\bar{\omega}^2\int dx x^2\phi(x). \tag{4.4.30}$$

To obtain a solution, we set up the following functional

$$\begin{aligned}\mathcal{K}(\mu,\phi) &= \frac{\pi^2}{6}\int dx\phi^3(x) + \frac{1}{2}\bar{\omega}^2\int dx x^2\phi(x) \\ &+ \mu\left(N - \int dx\phi(x)\right),\end{aligned} \tag{4.4.31}$$

in order to enforce the constraint in equation (4.4.25).

To make visible the N dependence for the rest of the terms appearing in equation (4.4.31), we rescale according to the following

$$x \to \sqrt{N}x, \qquad \phi(x) \to \sqrt{N}\phi(x), \qquad \mu \to N\mu \quad \text{and} \quad \Pi \to \frac{\Pi}{N}, \tag{4.4.32}$$

which will give us

$$\begin{aligned}H_{SMeff} &= \frac{1}{2N^2}\int dx\left(\partial_x\Pi(x)\right)\phi(x)\left(\partial_x\Pi(x)\right) \\ &+ N^2\left[\frac{\pi^2}{6}\int dx\phi^3(x) + \frac{1}{2}\bar{\omega}^2\int dx x^2\phi(x) + \mu\left(1 - \int dx\phi(x)\right)\right].\end{aligned} \tag{4.4.33}$$

We see that the large N limit corresponds to the minimum of the effective potential.

Differentiating the functional $\mathcal{K}(\mu,\phi)$ with respect to ϕ, we obtain the following

$$\begin{aligned}&\frac{\partial}{\partial\phi(x)}\mathcal{K}(\mu,\phi) = 0 \\ \Rightarrow\ &\left(\frac{\pi^2}{2}\phi^2(x) + \frac{1}{2}\bar{\omega}^2x^2 - \mu\right) = 0.\end{aligned} \tag{4.4.34}$$

Solving for $\phi_0(x)$, we obtain

$$\phi_0(x) = \frac{\sqrt{2}}{\pi}\left(\mu - \frac{1}{2}\bar{\omega}^2x^2\right)^{1/2} \qquad |x| \leq \frac{\sqrt{2\mu}}{\bar{\omega}}. \tag{4.4.35}$$

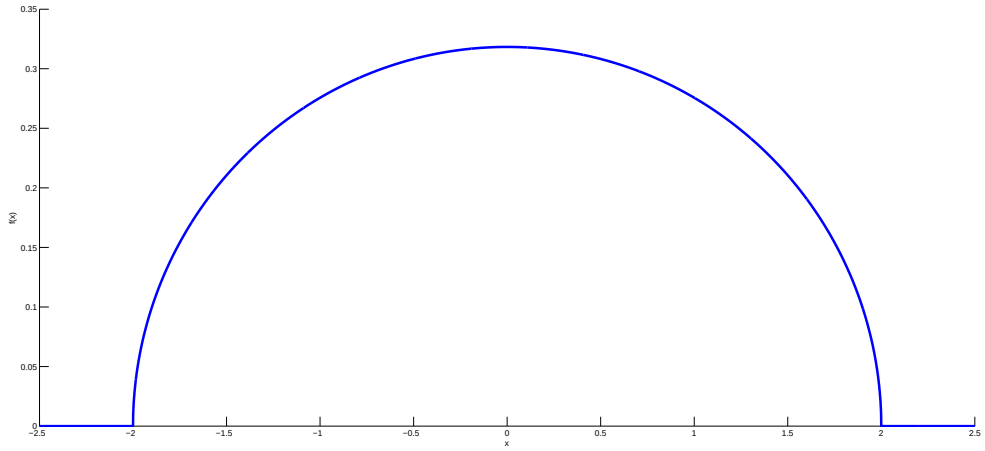

Figure 4.2: Single hermitian matrix model collective field theory solution: Wigner semi-circle distribution.

Equation (4.4.35) is the Wigner semi-circle distribution for the single hermitian matrix model. We denoted by $\phi_0(x)$ the background large N configuration. It represents the ground state wavefunction of the system which is the classical stationary point. This is the same equation that was obtained by Itzykson et.al [27] for $\mu = 1$ and $\bar{\omega} = 1/2$.

The Lagrange multiplier μ is fixed by the constraint $\int dx\phi(x) = 1$, and in our case is $\mu = \bar{\omega}$, hence the normalized density is

$$\phi_0(x) = \frac{\sqrt{2\bar{\omega}}}{\pi}\sqrt{1 - \frac{1}{2}\bar{\omega}x^2}. \tag{4.4.36}$$

The large N background geometry represented by equation (4.4.36) is shown in Figure 4.2 for $\bar{\omega} = 1/2$.

Chapter 5

Two Hermitian Matrices

The preceding chapter focused on general special properties of the single hermitian matrix model and in particular the emergence of the Wigner distribution that arose from studying the single hermitian matrix model in an harmonic potential.

In this chapter, we consider the two hermitian matrix model, in the hope that we can learn more about the matrix system. We will study this multimatrix model parameterized in terms of "matrix valued polar coordinates".

This parameterization yields a more symmetric approach when studying the multimatrix model of two hermitian matrices. In the framework that we propose, we treat the quantum mechanics of a system of two hermitian matrices in a non-supersymmetric framework.

In this chapter, we will carry out the following objectives:

- Introduce the parameterization of "matrix valued polar coordinates" for the two hermitian matrix system
- Obtain the Jacobian due to the two hermitian matrix system that has been parameterized
- Obtain the Laplacian in terms of the polar matrix coordinates

5.1 Polar Matrix Coordinates

To start off, we consider the system of two hermitian $N \times N$ matrices X_1 and X_2. Using these two hermitian matrices, we study the system whose quantum mechanics is described by the following Hamiltonian

$$\begin{aligned} \hat{H} &= -\frac{1}{2}\nabla^2 + V(X_1, X_2) \\ &= -\frac{1}{2}\left(\frac{\partial}{\partial (X_1)_{ij}} \frac{\partial}{\partial (X_1)_{ji}} + \frac{\partial}{\partial (X_2)_{ij}} \frac{\partial}{\partial (X_2)_{ji}} \right) + V(X_1, X_2)\,. \end{aligned} \tag{5.1.1}$$

These two matrices are projected onto the complex plane where they are represented in a real-imaginary basis. On this complex plane, the matrix Z is defined as follows

$$Z = X_1 + iX_2. \tag{5.1.2}$$

The two $N \times N$ hermitian matrices X_1 and X_2 can be viewed as two of the six scalar fields X_i where $i = 1, 2 \ldots, 6$ appearing in the bosonic sector of $\mathcal{N} = 4$ SYM. The linear combination of X_1 and X_2 on the complex plane that makes up the matrix Z can be parameterized into a product of a radial component and an angular component as per usual treatment of coordinates defined on the complex plane.

While we imply the Euler treatment of X_1 and X_2, we should remember that we are working in a matrix model and therefore the treatment of the hermitian matrices X_1 and X_2 should follow the principles of matrices and not real numbers.

The symmetrical treatment of the quantum mechanics of two hermitian matrices X_1 and X_2 generally means that both matrices are treated exactly in the coordinate basis and there is no holomorphic projection of either matrix.

We write the single complex matrix Z as follows

$$Z = X_1 + iX_2 \equiv RU \qquad Z^\dagger = X_1 - iX_2 \equiv U^\dagger R. \tag{5.1.3}$$

The complex matrix Z is a product of the $N \times N$ hermitian matrix R and the $N \times N$ unitary matrix U. The radial hermitian matrix R constitutes the radial

degrees of freedom of Z and the unitary matrix U makes up the angular degrees of freedom of Z. Note that

$$\mathrm{Tr}\left(Z^\dagger Z\right) = \mathrm{Tr}\left(ZZ^\dagger\right) = \mathrm{Tr}\left(X_1^2 + X_2^2\right). \tag{5.1.4}$$

The new parametrization of the complex matrix Z does in fact preserve the number of degrees of freedom of the system of matrices. The number of independent degrees of freedom of R and U is N^2 for each of them, so therefore, in total Z will have $2N^2$ degrees of freedom.

One should be able to express the Hamiltonian operator appearing in equation (5.1.1) in terms of the new "matrix valued polar coordinates" shown in equation (5.1.3). Specifically, we would like to write the Laplacian appearing in equation (5.1.1) in the new parameterization of matrix polar coordinates in terms of the eigenvalues of the radial matrix R and angular variables.

For this, we need to compute the polar matrix line element. This line element in matrix polar coordinates will be encoded with the Jacobian of our system. We will therefore make the following identification

$$\mathrm{Tr}\left(dZdZ^\dagger\right) = \mathrm{Tr}\left(dZ^\dagger dZ\right) \equiv g_{AB}dX^A dX^B. \tag{5.1.5}$$

The standard procedure to change from standard matrices (X_1, X_2) to radial and angular matrix variables (R, U) will require the diagonalization and reconfiguration of the complex matrix Z. To accomplish this, the radial matrix R is diagonalized such that $R \to \left(V^\dagger rV\right)$ where the unitary matrices V and $V^\dagger$ are $N \times N$ matrices belonging to the U(N) gauge group and r is an $N \times N$ diagonal matrix of the eigenvalues of R.

Therefore, the following diagonalization of Z follows naturally

$$Z = RU = V^\dagger r(VU) \qquad Z^\dagger = \left(U^\dagger V^\dagger\right) rV. \tag{5.1.6}$$

With the above expressions for Z and $Z^\dagger$, the following differentials are obtained

$$
\begin{aligned}
dZ &= V^\dagger \left(dr + r dV V^\dagger - dV V^\dagger r + r V dU U^\dagger V^\dagger \right) VU \qquad (5.1.7)\\
&= V^\dagger \left(dr + \left[r, dV V^\dagger \right] + r V dU U^\dagger V^\dagger \right) VU.
\end{aligned}
$$

and

$$
dZ^\dagger = U^\dagger V^\dagger \left(dr + \left[r, dV V^\dagger \right] - VU dU^\dagger V^\dagger r \right) V. \qquad (5.1.8)
$$

Following the diagonalization of the complex matrix Z, anti-hermitian Lie algebra differentials are defined in terms of the variables of Z. These differentials are given by

$$
dX = V dU U^\dagger V^\dagger \quad dS = dV V^\dagger. \qquad (5.1.9)
$$

We substitute the Lie algebra differentials into dZ and $dZ^\dagger$, we obtain

$$
dZ = V^\dagger \left(dr + [r, dS] + r dX \right) VU \qquad (5.1.10)
$$

and

$$
dZ^\dagger = U^\dagger V^\dagger \left(dr + [r, dS] - dX r \right) V. \qquad (5.1.11)
$$

One way to compute the Jacobian that arises as a result of the change in coordinates $(X_1, X_2) \to (r, V, U)$ is to obtain the line element $dZ dZ^\dagger \equiv g_{AB} dX^A dX^B$.

From the two expressions of dZ and $dZ^\dagger$, we can obtain the full expression for the square of the infinitesimal length

$$
\begin{aligned}
\mathrm{Tr}\left(dZ^\dagger dZ \right) &= \sum_i \left\{ (dr_i)^2 + (r_i)^2 \, dX_{ii} dX_{ii}^* \right\} \qquad (5.1.12)\\
&+ \sum_{i<j} \left\{ 2 (r_i - r_j)^2 \, dS_{ij} dS_{ij}^* + (r_i - r_j)^2 \left[dS_{ij} dX_{ij}^* + dX_{ij} dS_{ij}^* \right] \right\}\\
&+ \sum_{i<j} \left\{ \left(r_i^2 + r_j^2 \right) dX_{ij} dX_{ij}^* \right\}.
\end{aligned}
$$

The full expression of equation (5.1.12) is derived in appendix B.1, as shown in equation (B.1.20).

From the expression in equation (5.1.12), one is able to specify the matrix of the metric g_{AB} that will allow us to write the line element in a short hand version as it appears in equation (5.1.5). The full expression and complete derivation of the metric g_{AB} is shown in the appendix B.1, in equation (B.1.21).

With some standard algebra techniques and using equation (B.1.21), we compute the following

$$\begin{aligned} \det|g_{AB}| &= \prod_i r_i^2 \left(\frac{1}{4} \prod_{i<j} \left(r_i^2 - r_j^2\right)^2 \right)^2 \\ &= \prod_i r_i^2 \Delta_{TMM}^2, \end{aligned} \tag{5.1.13}$$

which is the determinant of the metric g_{AB}.

We define the Jacobian J_{TMM}, which emerges from the change in coordinates $\mathrm{Tr}\left(dZ^\dagger dZ\right) \to J_{TMM}\, dr\, dS\, dX$, as follows

$$J_{TMM} = \sqrt{\det|g_{AB}|} = \prod_i r_i \Delta_{TMM}^2, \tag{5.1.14}$$

where

$$\Delta_{TMM}^2 = \prod_{i<j} \frac{1}{4} \left(r_i^2 - r_j^2\right)^2. \tag{5.1.15}$$

Equation (5.1.15) is the Vandermonde determinant, for positive definite variables r_i^2.

The Jacobian J_{TMM} is the Jacobian of the two hermitian matrix system in matrix polar coordinates represented purely in terms of the eigenvalues of the radial matrix R and is decoupled from any angular degrees of freedom.

The Laplacian of the Hamiltonian operator (5.1.1), defined in the standard way, can be represented in terms of the new coordinates as follows

$$
\begin{aligned}
\nabla^2 &= \frac{1}{\prod_k r_k}\frac{1}{\Delta^2_{TMM}}\frac{\partial}{\partial r_i}\left[\prod_k r_k \Delta^2_{TMM}\right]\frac{\partial}{\partial r_i} + \left\{\frac{1}{r_i^2}\frac{\partial}{\partial X_{ii}}\frac{\partial}{\partial X^*_{ii}}\right\} \\
&+ \sum_{i\neq j}\frac{2(r_i^2+r_j^2)}{(r_i^2-r_j^2)^2}\frac{\partial}{\partial S_{ij}}\frac{\partial}{\partial S^*_{ij}} - \sum_{i\neq j}\frac{2}{(r_i+r_j)^2}\left\{\frac{\partial}{\partial S_{ij}}\frac{\partial}{\partial X^*_{ij}} + \frac{\partial}{\partial X_{ij}}\frac{\partial}{\partial S^*_{ij}}\right\} \\
&+ \sum_{i\neq j}\frac{4}{(r_i+r_j)^2}\frac{\partial}{\partial X_{ij}}\frac{\partial}{\partial X^*_{ij}} \\
&= \frac{1}{\prod_k r_k}\left\{\frac{\partial}{\partial r_i}\prod_k r_k\right\}\frac{\partial}{\partial r_i} + \frac{1}{\Delta^2_{TMM}}\left\{\frac{\partial}{\partial r_i}\Delta^2_{TMM}\right\}\frac{\partial}{\partial r_i} + \frac{\partial}{\partial r_i}\frac{\partial}{\partial r_i} \\
&+ \left\{\frac{1}{r_i^2}\frac{\partial}{\partial X_{ii}}\frac{\partial}{\partial X^*_{ii}}\right\} + \sum_{i\neq j}\frac{2(r_i^2+r_j^2)}{(r_i^2-r_j^2)^2}\frac{\partial}{\partial S_{ij}}\frac{\partial}{\partial S^*_{ij}} \qquad (5.1.16) \\
&- \sum_{i\neq j}\frac{2}{(r_i+r_j)^2}\left\{\frac{\partial}{\partial S_{ij}}\frac{\partial}{\partial X^*_{ij}} + \frac{\partial}{\partial X_{ij}}\frac{\partial}{\partial S^*_{ij}}\right\} + \sum_{i\neq j}\frac{4}{(r_i+r_j)^2}\frac{\partial}{\partial X_{ij}}\frac{\partial}{\partial X^*_{ij}}.
\end{aligned}
$$

The full derivation of equation (5.1.16) is shown in appendix (B.1) and is given by equation (B.1.29).

Equation (5.1.16) above is the Laplacian of the Hamiltonian operator in equation (5.1.1) represented in terms of our new coordinate system defined on the complex plane.

It should be noted that the computation of the Laplacian operator in equation (5.1.16) was performed for one of the possible parameterizations [112]. We only felt it necessary to present one of the two parameterizations because it carries a significant result that appears for both parameterizations. This common result is the Jacobian J_{TMM}.

The potential of the new system of two matrices in polar matrix coordinates has a rich $U(N)\times U(N)$ symmetry which simplifies our system even more when the potential depends on the eigenvalues r_i of the radial matrix R. Therefore, in general, the form of the potential considered, will be of the following form

$$
V(X_1,X_2) = \mathrm{Tr}\left(F\left(ZZ^\dagger\right)\right) = \mathrm{Tr}\left(F\left(Z^\dagger Z\right)\right) = \mathrm{Tr}\left(F(r)\right) = F(r_i) = V(r_i). \qquad (5.1.17)
$$

In chapters $6-9$, we will continue with the discussion of the properties of two hermitian matrices. In chapters $10-14$ we will show how many of the

results obtained for two hermitian matrices can be generalized to an arbitrary even number of hermitian matrices.

Chapter 6

Wigner Distribution And Harmonic Potential For The Two Matrix Model

In the previous section we managed to derive the Jacobian J_{TMM} and the Laplacian operator for the two hermitian matrix model parameterized in "matrix valued polar coordinates". This Jacobian emerges as a result of a change in coordinates from matrices to a system expressed purely in terms of the eigenvalues of the radial matrix R and angular matrices V and $V^{\dagger}$.

This chapter will be dedicated to investigating whether a Wigner semi-circle distribution does arise for the polarized two matrix system with an harmonic potential as we saw for the single hermitian matrix model.

In this chapter, we aim to accomplish the following objectives:

- Introduce an integral system for the two hermitian matrix model parameterized in "matrix valued polar coordinates" (partition function) for a central harmonic potential
- Obtain the radial eigenvalue density defined on the single interval of the complex plane
- Extend the large N radial eigenvalue density to a double cut interval

- Present an eigenvalue density function associated with the double cut interval of the complex plane showing agreement with the restriction to the single cut solution

6.1 Polar Matrix Model Integral

We will now establish the matrix integral in our new polar matrix coordinates. We will first consider the Gaussian potential

$$G = \frac{\omega^2}{2}\mathrm{Tr}\left(Z^\dagger Z\right) = \frac{\omega^2}{2}\mathrm{Tr}\left(R^2\right) = \frac{\omega^2}{2}\sum_i r_i^2 = \frac{\omega^2}{2}\sum_i \rho_i \tag{6.1.1}$$

In equation (6.1.1) we let $\rho_i = r_i^2$ for the index $i = 1, 2 \ldots, N$. For the Gaussian partition function, one obtains

$$\begin{aligned} Z_{TMM} &= \int dZ dZ^\dagger e^{-G} \\ &\approx \int dr \prod_i r_i \prod_{i<j}(r_i^2 - r_j^2)^2 e^{-\frac{\omega^2}{2}\sum_i r_i^2} \\ &= \int \prod_i d\rho_i \prod_{i<j}(\rho_i - \rho_j)^2 e^{-\frac{\omega^2}{2}\sum_i \rho_i} \\ &= \int \prod_i d\rho_i e^{\sum_{i\neq j}\ln(\rho_i-\rho_j) - \frac{1}{2}\omega^2\sum_i \rho_i} \\ \Rightarrow Z_{TMM} &= \int \prod_i d\rho_i e^{-G_{eff}(\rho_i)}. \end{aligned} \tag{6.1.2}$$

In the last line of equation (6.1.2) we defined the effective action $G_{eff}(\rho_i)$ of the system as follows

$$G_{eff}(\rho_i) = \frac{\omega^2}{2}\sum_i \rho_i - \sum_{i\neq j}\ln|\rho_i - \rho_j|. \tag{6.1.3}$$

Equations (6.1.2) and (6.1.3) are used to obtain the stationary condition below

$$\begin{aligned}
\frac{\partial}{\partial\rho_k}\left(\sum_{i\neq j}\ln(\rho_i-\rho_j)^2-\frac{1}{2}\omega^2\sum_i\rho_i\right) &= 0 \\
\Rightarrow\sum_{i\neq j}\left(\frac{1}{\rho_i-\rho_j}(\delta_{ik}-\delta_{jk})\right)-\frac{1}{2}\omega^2\sum_i\delta_{ik} &= 0 \\
\sum_{k\neq j}\frac{1}{\rho_k-\rho_j}+\sum_{k\neq j}\frac{1}{\rho_k-\rho_j}-\frac{1}{2}\omega^2 &= 0 \\
\Rightarrow 2\sum_{k\neq j}\frac{1}{\rho_i-\rho_j}=\frac{1}{2}\omega^2. &
\end{aligned} \tag{6.1.4}$$

Similar to the single hermitian matrix model, the stationary condition seen in the last line of equation (6.1.4) will be defined in the continuum limit. In this limit, the variables of our system are continuous, therefore we find that

$$⨍\frac{dx'\Phi(x')}{x-x'}=\frac{\omega^2}{4}, \tag{6.1.5}$$

where $x=\rho$ and $x'=\rho'$.

A short comment is in order. The two hermitian matrix model that was parameterized through angular and radial degrees of freedom into the single complex matrix Z represents the $m=1$ model whose parameterization is treated more symmetrically instead of the "impurity" state basis where the matrices are represented as creation/annihilation operators [108] [109] [110] [111].

By treating the two matrix model in the creation/annihilation basis, where one matrix generates the large N limit background and the other matrix is treated in the "impurity" sector of the first matrix, the matrix that generates the background becomes the holomorphic component of a complex matrix. This is different from the approach followed in this thesis.

Also, in the parameterization considered in [113] used to study the eigenvalue distribution of matrix ensembles (complex, quaternion and real matrices), the single complex matrix is parameterized in terms of it's eigenvalues and the upper diagonal matrix. This type of parameterization is appropriate for holomorphic projections.

6.2 Positive Single Cut Ansatz

Equation (6.1.5) is a stationary condition taken in the continuum limit. To provide a standard solution for equation (6.1.5), we will adopt the methods of BIPZ [27]. This will require that we introduce an analytical function $\mathcal{F}(z)$ as already described in chapter 4,

$$\mathcal{F}(z) = \int_{x_-}^{x_+} \mathrm{d}x' \frac{\Phi(x')}{z - x'}. \tag{6.2.1}$$

It is important to note that the function $\mathcal{F}(z)$ defined in the complex plane, can only have a cut on the positive real axis, i.e. on the interval $[x_-, x_+]$ with $x_+ > x_- > 0$.

We recall that when z approaches the support $[x_-, x_+]$ from either side

$$\begin{aligned} \mathcal{F}(x \pm i\epsilon) &= \int_{x_-}^{x_+} \mathrm{d}x' \frac{\Phi(x')}{x - x'} \mp i\pi\Phi(x) \\ &= \frac{\omega^2}{4} \mp i\pi\Phi(x). \end{aligned} \tag{6.2.2}$$

The analytic function that satisfies the above mentioned conditions is unique in its construction and is given by

$$\mathcal{F}(z) = \frac{\omega^2}{4} - \frac{\omega^2}{4z}\sqrt{(z - x_+)(z - x_-)}, \tag{6.2.3}$$

with

$$x_- = 0 \qquad\qquad x_+ = \frac{8}{\omega^2}. \tag{6.2.4}$$

Note that the condition $x_- = 0$ removes the apparent pole in equation (6.2.3). The derivation of this ansatz and the conditions for the end points are described in appendix C.1.

From equations (6.2.2) and (6.2.3), we obtain the eigenvalue density function

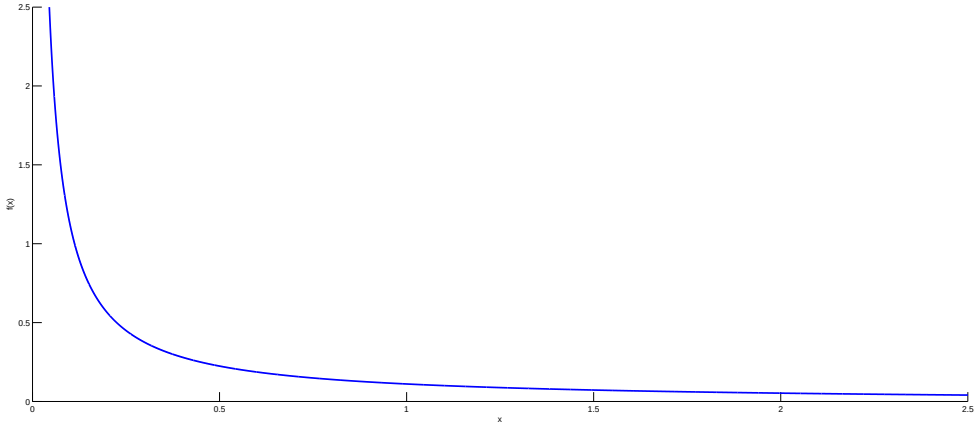

Figure 6.1: Eigenvalue distribution $\Phi(\rho)$ of the two matrix model parameterized in "matrix valued polar coordinates" Z.

$$\begin{aligned}
\Phi(\rho) &= \frac{\omega^2}{4\pi\rho}\sqrt{(x_+ - \rho)(\rho - x_-)} \\
&= \frac{\omega^2}{4\pi\rho}\sqrt{(\frac{8}{\omega^2} - \rho)\rho} \\
\Rightarrow \Phi(\rho) &= \frac{\omega^2}{4\pi}\sqrt{\frac{8}{\omega^2\rho} - 1} \\
&= \frac{\omega^2}{4\pi\sqrt{\rho}}\sqrt{\frac{8}{\omega^2} - \rho} \qquad 0 \leq \rho \leq \frac{8}{\omega^2}. \qquad (6.2.5)
\end{aligned}$$

Equation (6.2.5) does not resemble the Wigner semi-circle law as can be observed in Figure (6.1) plotted for $\omega = 1/2$.

6.3 Symmetric Extension Of The Real Line

In order to confirm the solution (6.2.5), in particular the handling of the divergence as $\rho \to 0$, in this section we will treat the stationary condition appearing in equation (6.1.5) symmetrically in terms of the radial coordinate r ($\rho = r^2$).

We first start with the stationary condition in equation (6.1.4), and express it in terms of the eigenvalues r_i of the radial matrix R such that

$$2\sum_{j(i\neq j)}\frac{1}{\rho_i-\rho_j}=\frac{1}{2}\omega^2$$
$$\Rightarrow\quad\sum_{i\neq j}\frac{r_i}{r_i^2-r_j^2}=\frac{1}{4}\omega^2 r_i. \tag{6.3.1}$$

We will now introduce the following identity into the left hand side of equation (6.3.1)

$$\frac{2r_i}{r_i^2-r_j^2}=\frac{1}{r_i-r_j}+\frac{1}{r_i+r_j}. \tag{6.3.2}$$

Once the identity (6.3.2) is substituted into equation (6.3.1) we obtain

$$2\sum_{i\neq j}\frac{r_i}{r_i^2-r_j^2}=\sum_{i\neq j}\left(\frac{1}{r_i-r_j}+\frac{1}{r_i+r_j}\right)=\frac{1}{2}\omega^2 r_i. \tag{6.3.3}$$

In this equation, both r_i and r_j are positive. The second term however suggests that if we allow r_j to be both positive and negative, then equation (6.3.5) can be written as

$$\sum_{-\infty<r_j<\infty(r_i\neq r_j)}\frac{1}{r_i-r_j}=\frac{1}{2}\omega^2 r_i, \tag{6.3.4}$$

and strongly resembles the solution of the single hermitian matrix model that yielded the Wigner distribution of eigenvalues appearing in equation (4.1.11) (for the free case $g=0$).

It is therefore natural to extend the density of the eigenvalues to the whole real line, by defining a density function $\phi(r')$

$$\phi(r')\equiv 2r'\Phi(r'^2)\equiv\phi(-r'),\qquad r>0, \tag{6.3.5}$$

under the change of coordinates $x=\rho=r^2$. By construction $\phi(r')$ is symmetric.

It follows that the integral (6.1.5) takes the following form

$$\int_0^\infty\frac{dx'\Phi(x')}{x-x'}=\int_0^\infty\frac{dr'2r'\Phi(r'^2)}{r^2-r'^2}=\int_0^\infty\frac{dr'\phi(r')}{r^2-r'^2}. \tag{6.3.6}$$

So in total we will have the following equation

$$\fint_0^\infty \frac{dr'\phi(r')}{r^2 - r'^2} = \frac{\omega^2}{4}. \tag{6.3.7}$$

Then:

$$\begin{aligned}
\fint_0^\infty \frac{dr'\phi(r')}{r^2 - r'^2} &= \frac{1}{2r}\fint_0^\infty dr'\phi(r')\left(\frac{1}{r+r'} + \frac{1}{r-r'}\right) \\
&= \frac{1}{2r}\left[\fint_{-\infty}^0 \frac{dr'\phi(-r')}{r-r'} + \fint_0^\infty \frac{dr'\phi(r')}{r-r'}\right] \\
&= \frac{1}{2r}\fint_{-\infty}^\infty \frac{dr'\phi(r')}{r-r'},
\end{aligned} \tag{6.3.8}$$

so that (6.1.5) takes the form

$$\fint_{-\infty}^\infty \frac{dr'\phi(r')}{r-r'} = \frac{\omega^2 r}{2}. \tag{6.3.9}$$

Note the constraint

$$\int_{-\infty}^\infty dr'\phi(r') = 2, \tag{6.3.10}$$

which is a normalization condition for the integral of the radial function $\phi(r')$ as a result of integrating over the entire real axis for both positive and negative values of r.

Equation (6.3.9) is similar to the single Hamiltonian with a harmonic potential. The ansatz for the extended real line $\mathcal{F}'(z)$ follows from (4.1.18) from which one gets immediately

$$\mathcal{F}'(z) = \frac{\omega^2}{2}z - \frac{\omega^2}{2}\sqrt{z^2 - \alpha^2}, \tag{6.3.11}$$

where the asymptotic expansion $\mathcal{F}'(z) = 2/z + \ldots$ is in accordance with the constraint (6.3.10) when z is large.

From equation (6.3.11) one can follow the methods of the single hermitian matrix model to obtain the density function, given by

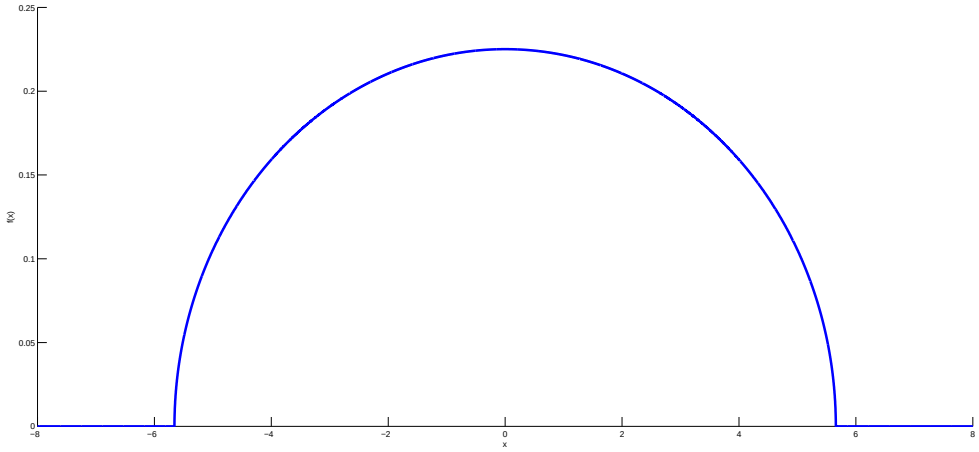

Figure 6.2: Wigner semi-circle eigenvalue distribution of the two matrix model parameterized in "matrix valued polar coordinates" Z.

$$\phi(r) = \frac{\omega^2}{2\pi}\left[\frac{8}{\omega^2} - r^2\right]^{1/2} \qquad -\frac{\sqrt{8}}{\omega} \leq r^2 \leq \frac{\sqrt{8}}{\omega}. \tag{6.3.12}$$

The eigenvalue distribution (6.3.12) is a Wigner semi-circle distribution as can be observed in Figure (6.2) above for $\omega = 1/2$. However, one needs to remember that the physical region is the restriction to $r \geq 0$.

After obtaining the eigenvalue density function appearing in equation (6.3.12), we now return to our original variables where we had $r^2 = \rho > 0$ and $2r\Phi(r^2) = \phi(r) = 2\sqrt{\rho}\Phi(\rho)$ we re-write equation (6.3.12) as follows

$$\begin{aligned} \phi(r) &= \frac{\omega^2}{2\pi}\left[\frac{8}{\omega^2} - r^2\right]^{1/2} \\ 2\sqrt{\rho}\Phi(\rho) &= \frac{\omega^2}{2\pi}\left[\frac{8}{\omega^2} - \rho\right]^{1/2} \\ \Rightarrow \Phi(\rho) &= \frac{\omega^2}{4\pi}\left(\frac{8}{\rho\omega^2} - 1\right)^{1/2} \qquad 0 \leq \rho \leq \frac{8}{\omega^2}. \end{aligned} \tag{6.3.13}$$

In agreement with with equation (6.2.5).

Chapter 7

Fermionization In The Radial Sector Of Two Hermitian Matrices

It is well known that the singlet sector of the $N \times N$ single hermitian matrix model Hamiltonian with a potential that depends strictly on the eigenvalues of the system is equivalent to a configuration of N non-interacting fermions [27] as reviewed in chapter 4.

As shown, this is a result of the fact that the Jacobian (4.1.9) is anti-symmetric under the exchange of any two matrix eigenvalues.

In this chapter, we will carry out the following objectives:

- Consider the radial sector of the two matrix Laplacian in matrix valued polar coordinates when acting on symmetric wavefunctions
- Show that the system has an equivalent description in terms of decoupled two dimensional "radial fermions".

For the sake of convenience, we will denote this Jacobian from equation (4.1.9) (the Vandermonde determinant) by J_{SM}.

As a reminder, the single hermitian matrix model Jacobian J_{SM} is of the

following form

$$J_{SM} = \prod_{i<j} (x_i - x_j) = \Delta(x_k). \tag{7.0.1}$$

We consider the Laplacian operator appearing in equation (5.1.16) restricted to act on wavefunctions which depend on radial eigenvalues only. We refer to these as "s-states", decoupled from any angular degrees of freedom.

Recalling that $\rho_i = r_i^2$ are the eigenvalues of R, the Laplacian in equation (5.1.16) acting on the "s-states" takes the form:

$$\begin{aligned} -\frac{1}{2}\nabla^2 &= -\frac{1}{2}\frac{1}{\Delta^2_{TMM}} \sum_i \frac{\partial}{\partial r_i} \left(r_i \Delta^2_{TMM} \right) \frac{\partial}{\partial r_i} \\ &= -\frac{2}{\Delta^2(\rho)} \sum_i \frac{\partial}{\partial \rho_i} \left(\rho_i \Delta^2(\rho) \right) \frac{\partial}{\partial \rho_i}. \end{aligned} \tag{7.0.2}$$

The second line in equation (7.0.2) will act on symmetric wavefunctions Φ. Similar to the single hermitian matrix case, this suggests that we introduce an anti-symmetric wavefunction Ψ defined as

$$\Psi = \Delta\Phi. \tag{7.0.3}$$

The following Schrödinger equation is set-up

$$\begin{aligned} &-\left(\frac{2}{\Delta^2(\rho_i)} \sum_i \frac{\partial}{\partial \rho_i} \rho_i \Delta^2(\rho_i) \frac{\partial}{\partial \rho_i} \right) \frac{\Psi}{\Delta(\rho_i)} = E \frac{\Psi}{\Delta(\rho_i)} \\ &-\left[\frac{2}{\Delta(\rho_i)} \sum_i \frac{\partial}{\partial \rho_i} \Delta(\rho_i) \rho_i \Delta(\rho_i) \frac{\partial}{\partial \rho_i} \frac{1}{\Delta(\rho_i)} \right] \Psi = E\Psi. \end{aligned} \tag{7.0.4}$$

In appendix D we verify that the left hand side of the equation (7.0.4) above simplifies to

$$-\left[\frac{2}{\Delta(\rho_i)} \sum_i \frac{\partial}{\partial \rho_i} \Delta(\rho_i) \rho_i \Delta(\rho_i) \frac{\partial}{\partial \rho_i} \frac{1}{\Delta(\rho_i)} \right] = -2 \sum_i \frac{\partial}{\partial \rho_i} \rho_i \frac{\partial}{\partial \rho_i}. \tag{7.0.5}$$

Recalling that $r_i = \sqrt{\rho_i}$, we obtain the system of eigenvalue equations of the form

$$\left(-2\sum_i \frac{\partial}{\partial\rho_i}\rho_i\frac{\partial}{\partial\rho_i} + V(\rho_i)\right)\Psi = \left(-\frac{1}{2}\frac{1}{r_i}\frac{\partial}{\partial r_i}r_i\frac{\partial}{\partial r_i} + V(r_i)\right)\Psi. \tag{7.0.6}$$

For some index i, the discretized system in equation (7.0.6) corresponds to one single particle Hamiltonian describing the energy of a $(1+1)$-dimensional non-interacting "radial fermion" subjected to a potential $V(r_i)$. We refer to these particles as "fermions" or "radial fermions" because the anti-symmetrization has only taken place for the radial coordinate.

In conclusion, equation (7.0.6) represents an "s-state" Schrödinger equation for N non-interacting and non-relativistic 2-dimensional "radial fermions" subjected to the potential $V(r_i)$. The appearance of $(1+1)$-dimensional particle coordinates from two matrices is a new result [112].

Chapter 8

Closed Subsector And Matrix Radial Coordinates

The radial sector discussed in the previous chapter turns out to be a closed subsector of the two matrix model [112] [114]. In this chapter we will apply the collective field theory method to obtain a density description of this subsector.

The objectives of this chapter are:

- Develop a density description for the radial sector of (the) two hermitian matrix model parameterized in polar matrix coordinates
- Rederive the Jacobian of the two matrix polar system in the density description. It will be shown later that this approach can be generalized to an arbitrary even number of hermitian matrices

We first start off by defining the following set of invariant states in terms of the complex matrix Z and $Z^\dagger$

$$\Phi_k = \mathrm{Tr}\left(e^{ikZ^\dagger Z}\right) = \sum_i e^{ikr_i^2}, \tag{8.0.1}$$

and the coordinate x representation given by the Fourier transform

$$\begin{aligned}\Phi(x) &= \int \frac{dk}{2\pi} e^{-ikx}\phi_k \\ &= \sum_i \int \frac{dk}{2\pi} e^{-ik(x-r_i^2)} \\ &= \sum_i \delta(x-r_i^2). \end{aligned} \tag{8.0.2}$$

We note that

$$\begin{aligned}\frac{\partial \Phi_k}{\partial Z_{ij}} &= \frac{\partial}{\partial Z_{ij}} \mathrm{Tr}\left(e^{ikZ^\dagger Z}\right) \\ &= ik\left(e^{Z^\dagger Z} Z^\dagger\right)_{ji}, \end{aligned} \tag{8.0.3}$$

and that

$$\begin{aligned}\frac{\partial \Phi_k}{\partial Z^\dagger_{ij}} &= \frac{\partial}{\partial Z^\dagger_{ij}} \mathrm{Tr}\left(e^{ikZ^\dagger Z}\right) \\ &= ik\left(Z e^{ikZ^\dagger Z}\right)_{ji}. \end{aligned} \tag{8.0.4}$$

We find that the joining operator is given by

$$\begin{aligned}\Omega_{kk'} &= \left[\frac{\partial \Phi_k}{\partial Z^\dagger_{ij}} \frac{\partial \Phi_{k'}}{\partial Z_{ji}}\right] \\ &= ik\left(Z e^{ikZ^\dagger Z}\right)_{ji} ik'\left(e^{ik'Z^\dagger Z} Z^\dagger\right)_{ij} \\ &= -kk' \mathrm{Tr}\left(Z^\dagger Z e^{i(k+k')Z^\dagger Z}\right). \end{aligned} \tag{8.0.5}$$

The joining operator has a Fourier transform in coordinate space x given by the following, $(x = r_i^2)$,

$$
\begin{aligned}
\Omega_{xx'} &= \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} e^{-ikx} e^{-ik'x'} \Omega_{kk'} \\
&= -kk' \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} e^{-ikx} e^{-ik'x'} \mathrm{Tr}\left(Z^\dagger Z e^{i(k+k')Z^\dagger Z} \right) \\
&= \partial_x \partial_{x'} \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} e^{-ikx} e^{-ik'x'} \mathrm{Tr}\left(Z^\dagger Z e^{i(k+k')Z^\dagger Z} \right) \\
&= \partial_x \partial_{x'} \left(\sum_i x_i \delta(x - x_i) \delta(x' - x_i) \right) \\
\Rightarrow \Omega_{xx'} &= \partial_x \partial_{x'} \left[x \Phi(x) \delta(x' - x) \right]. \qquad (8.0.6)
\end{aligned}
$$

We will now take a close look at the splitting operator ω_k defined by

$$
\begin{aligned}
\omega_k &= \left[\frac{\partial^2 \phi_k}{\partial Z^\dagger_{ij} \partial Z_{ji}} \right] \\
&= (ik)^2 \int_0^1 d\alpha \left(e^{ik\alpha Z^\dagger Z} \right)_{ii} \left(Z e^{ik(1-\alpha) Z^\dagger Z} Z^\dagger \right)_{jj} \\
&+ ik \left(e^{ikZ^\dagger Z} \right)_{ii} \underbrace{\delta_{jj}}_{N} \qquad (8.0.7) \\
&= -k^2 \int_0^1 d\alpha \left(e^{ik\alpha Z^\dagger Z} \right)_{ii} \left(Z^\dagger Z e^{ik(1-\alpha) Z^\dagger Z} \right)_{jj} + ikN \left(e^{ikZ^\dagger Z} \right)_{ii}.
\end{aligned}
$$

We now introduce a change of variables into equation (8.0.7):

$$
k' = \alpha k : \alpha = 1 \Rightarrow k' = k; \alpha = 0 \Rightarrow k' = 0, \qquad (8.0.8)
$$

and obtain

$$
\omega_k = -k \sum_{ij} \int_0^k dk' \left(e^{ik' r_i^2} \right) \left(r_j^2 e^{i(k-k') r_j^2} \right) + ikN \sum_i e^{ikr_i^2}. \qquad (8.0.9)
$$

The above equation is integrated over the k'-space and is represented purely in terms of eigenvalues. We will proceed and break up the integral above into terms that have $i = j$ and $i \neq j$. We obtain

$$
\omega_k = -k^2 \sum_i r_i^2 e^{ikr_i^2} - k \sum_{i \neq j} \int_0^k dk' e^{ik' r_i^2} r_j^2 e^{i(k-k') r_j^2} + ikN \sum_i e^{ikr_i^2}. \qquad (8.0.10)
$$

The second term appearing in equation (8.0.10) will be dealt with separately below so as to simplify it even further. We write:

$$\begin{aligned} - \quad & k\sum_{i\neq j}\int_0^k dk' e^{ik'r_i^2} r_j^2 e^{i(k-k')r_j^2} = \sum_{i\neq j}\frac{ikr_j^2}{(r_i^2-r_j^2)}\left(e^{ikr_i^2}-e^{ikr_j^2}\right) \\ = \quad & \sum_{i\neq j}\frac{ikr_j^2 e^{ikr_i^2}}{(r_i^2-r_j^2)} - \sum_{i\neq j}\frac{ikr_j^2 e^{ikr_j^2}}{(r_i^2-r_j^2)} \\ = \quad & -ik\underbrace{\sum_{i\neq j}}_{N-1}\sum_i e^{ikr_i^2} + \sum_{i\neq j}\frac{ikr_j^2 e^{ikr_j^2}}{(r_j^2-r_i^2)} - \sum_{i\neq j}\frac{ikr_j^2 e^{ikr_j^2}}{(r_i^2-r_j^2)} \\ = \quad & -ik(N-1)\sum_i e^{ikr_i^2} - 2\sum_{i\neq j}\frac{ikr_j^2 e^{ikr_j^2}}{(r_i^2-r_j^2)}. \end{aligned} \tag{8.0.11}$$

We substitute equation (8.0.11) into equation (8.0.10) and continue to solve for ω_k to obtain the following

$$\Rightarrow \omega_k \quad = \quad -k^2\sum_i r_i^2 e^{ikr_i^2} - 2ik\sum_{i\neq j}\frac{r_j^2 e^{ikr_j^2}}{(r_i^2-r_j^2)} + ik\sum_i e^{ikr_i^2}. \tag{8.0.12}$$

Having solved for the splitting operator in k-space, we can naturally consider its Fourier transform in x-coordinate space denoted by ω_x:

$$\begin{aligned} \omega_x \quad = \quad & \int\frac{dk}{2\pi}e^{-ikx}\left(-k^2\sum_i r_i^2 e^{ikr_i^2}\right) + \int\frac{dk}{2\pi}e^{-ikx}\left(-2ik\sum_{i\neq j}\frac{r_j^2 e^{ikr_j^2}}{(r_i^2-r_j^2)}\right) \\ + \quad & \int\frac{dk}{2\pi}e^{-ikx}\left(ik\sum_i e^{ikr_i^2}\right) \end{aligned} \tag{8.0.13}$$

For the sake of simplicity and clarity, each of the three terms appearing in equation (8.0.13) will be calculated individually below. We start with the second term appearing in the first line of equation (8.0.13)

$$\int \frac{dk}{2\pi} e^{-ikx} \left(-2ik \sum_{i \neq j} \frac{r_j^2 e^{ikr_j^2}}{(r_i^2 - r_j^2)} \right) = 2\partial_x \sum_{i \neq j} \int \frac{dk}{2\pi} e^{ik(r_j^2 - x)} \frac{r_j^2}{(r_i^2 - r_j^2)}$$

$$= -2\partial_x \int dx' \Phi(x') \fint dy' \Phi(y') \delta(y' - x) \frac{y'}{y' - x'}$$

$$= -2\partial_x \fint dx' \Phi(x') \Phi(x) \frac{x}{x - x'} \tag{8.0.14}$$

$$\Rightarrow \int \frac{dk}{2\pi} e^{-ikx} \left(-2ik \sum_{i \neq j} \frac{r_j^2 e^{ikr_j^2}}{(r_i^2 - r_j^2)} \right) = -2\partial_x \left(x\Phi(x) \fint dy \frac{\Phi(y)}{x - y} \right)$$

Below, we now solve for the rest of the terms appearing in equation (8.0.13)

$$\int \frac{dk}{2\pi} e^{-ikx} \left(-k^2 \sum_i r_i^2 e^{ikr_i^2} \right) = \partial_x^2 \int dx' \delta(x' - x) \Phi(x') x'$$

$$\Rightarrow \int \frac{dk}{2\pi} e^{-ikx} \left(-k^2 \sum_i r_i^2 e^{ikr_i^2} \right) = \partial_x^2 \left(x\Phi(x) \right), \tag{8.0.15}$$

and also

$$\int \frac{dk}{2\pi} e^{-ikx} \left(ik \sum_i e^{ikr_i^2} \right) = -\partial_x \int dx' \Phi(x') \delta(x' - x)$$

$$\Rightarrow \int \frac{dk}{2\pi} e^{-ikx} \left(ik \sum_i e^{ikr_i^2} \right) = -\partial_x \left(\Phi(x) \right). \tag{8.0.16}$$

Equations (8.0.14), (8.0.15) and (8.0.16) will be substituted back into equation (8.0.13) to obtain the full expression for ω_x:

$$\begin{aligned} \omega_x &= \partial_x^2 \left(x\Phi(x) \right) - 2\partial_x \left(x\Phi(x) \fint \frac{dy \Phi(y)}{(x - y)} \right) - \partial_x \Phi(x) \\ &= -\partial_x \left[-\partial_x (x\Phi(x)) + 2x\Phi(x) \fint \frac{dy \Phi(y)}{(x - y)} + \Phi(x) \right] \\ \Rightarrow \omega_x &= -\partial_x \left(x\Phi(x) \left[2 \fint \frac{dy \Phi(y)}{(x - y)} - \frac{\partial_x \Phi(x)}{\Phi(x)} \right] \right). \end{aligned} \tag{8.0.17}$$

As has been observed in the collective field theory for the single hermitian matrix model, the term $\partial_x \Phi(x)/\Phi(x)$ can be neglected [115], leaving us with

$$\Rightarrow \omega_x = -\partial_x \left[x\Phi(x) \left(2 \fint \frac{dy \Phi(x)}{(x - y)} \right) \right]. \tag{8.0.18}$$

In the collective field density description, the Jacobian J that arises from a change of variables from matrices to invariant state operators will satisfy the following equation (refer to equation (4.3.9))

$$\int dx' \Omega_{xx'} \frac{\partial \ln J}{\partial \Phi(x')} + \int dx' \frac{\partial \Omega_{xx'}}{\partial \Phi(x')} = \omega_x. \tag{8.0.19}$$

The second term appearing on the left hand side of equation (8.0.19) vanishes as previously observed in the density description of the single hermitian matrix model. With this condition, it necessarily follows that

$$\int dx' \Omega_{xx'} \frac{\partial \ln J}{\partial \Phi(x')} - \omega_x = 0. \tag{8.0.20}$$

In equation (8.0.20) above, we will make use of $\Omega_{xx'}$ and ω_x that have already been defined, it then follows that

$$\begin{aligned}
& \int dx' \Omega_{xx'} \frac{\partial \ln J}{\partial \Phi(x')} - \omega_x = 0 \\
\Rightarrow \quad & \int dx' \partial_x \partial_{x'} \left(x\Phi(x) \delta(x-x') \right) \frac{\partial \ln J}{\partial \Phi(x')} + \partial_x \left(x\Phi(x) \left[2 \fint \frac{dx' \Phi(x')}{x-x'} \right] \right) = 0 \\
\Rightarrow \quad & -\left(x\Phi(x) \right) \partial_x \frac{\partial \ln J}{\partial \Phi(x)} + \left(x\Phi(x) \right) \left[2 \fint \frac{dx' \Phi(x')}{x-x'} \right] = 0 \\
\Rightarrow \quad & \partial_x \frac{\partial \ln J}{\partial \Phi(x)} = 2 \fint \frac{dx' \Phi(x')}{x-x'}.
\end{aligned} \tag{8.0.21}$$

We now show that the Jacobian $J = \Delta_{TMM}^2$ that we had previously obtained in (5.1.15) namely

$$J = \prod_{i<j} \frac{1}{4} \left(r_i^2 - r_j^2 \right)^2, \tag{8.0.22}$$

satisfies equation (8.0.21).

One has

$$\begin{aligned}
\ln J &= \sum_{i \neq j} \ln \left| r_i^2 - r_j^2 \right| + const \\
&= \fint dx \fint dy \Phi(x) \Phi(y) \ln |x-y| + const.
\end{aligned} \tag{8.0.23}$$

Substituting into the left hand side,

$$\partial_x \frac{\partial \ln J}{\partial \Phi(x)} = 2 \fint \frac{dx' \Phi(x')}{x - x'}. \tag{8.0.24}$$

This collective field theory treatment of the radial sector of two hermitian matrices shows that this set of invariant operators close under the process of joining and splitting. This is equivalent to the closure that characterizes Schwinger-Dyson equations as will be observed in later chapters.

Chapter 9

Radial Sector Density Description Of The Two Matrix Hamiltonian

We will continue with the collective field theory formalism, elegantly demonstrated by Jevicki and Sakita [96], that we first applied to the single hermitian matrix model and the previous chapter, by presenting the Hamiltonian description of the restricted radial sector made up of two hermitian matrices

In this chapter we will carry out the following objectives:

- Consider a radially restricted Hamiltonian formalism for the two hermitian matrices coupled to a Gaussian potential
- Obtain the eigenvalue density function and couple this to the geometrical representation due to the distribution of eigenvalues of the system

We recall equation (4.3.13)

$$\begin{aligned} H_{eff}[x,x';[\phi]] &= -\frac{1}{2}\int \mathrm{dx}\int dx'\Pi(x)\Omega[x,x',[\phi]]\Pi(x') \\ &+ \frac{1}{8}\int \mathrm{dx}\int \mathrm{dx}'\omega(x;[\phi])\Omega^{-1}(x,x';[\phi])\omega(x';[\phi]) + \mathcal{P}[x,[\phi]]. \end{aligned} \tag{9.0.1}$$

We (also) recall that previously we defined the following invariant states:

$$\begin{aligned}
\phi_k &= \mathrm{Tr}\left(e^{ikZ^\dagger Z}\right) = \sum_i e^{ikr_i^2} = \sum_i e^{ikx_i}, \\
\phi(x) &= \int dx e^{-ikx}\phi_k = \sum_i \delta(x - r_i^2) = \sum_i \delta(x - x_i). \qquad (9.0.2)
\end{aligned}$$

We should emphasize that in equation (9.0.2), $x_i = r_i^2 > 0$.

In the previous chapter, we have derived the expressions of both the joining and the splitting operators. For the joining operator

$$\begin{aligned}
\Omega(k, k'; [x]) &= \frac{\partial \phi_k}{\partial Z_{ij}^\dagger}\frac{\partial \phi_{k'}}{\partial Z_{ji}} = -kk'\mathrm{Tr}\left(Z^\dagger Z e^{i(k+k')Z^\dagger Z}\right), \qquad (9.0.3) \\
\Omega(x, x'; [\phi]) &= \int \frac{dk'}{2\pi}\int \frac{dk}{2\pi} e^{-ikx}e^{-ik'x'}\Omega(k, k'; [\phi]) = \partial_x\partial_{x'}\left[x\phi(x)\delta(x - x')\right],
\end{aligned}$$

and the splitting operator with its associated Fourier transform is

$$\begin{aligned}
\omega(k, k'; [\phi]) &= \frac{\partial^2 \phi_k}{\partial Z_{ij}^\dagger Z_{ji}} = -k^2\sum_i x_i e^{ikx_i} - 2ik\sum_{i\neq j}\frac{x_j e^{ikx_i}}{(x_i - x_j)} \\
&+ ik\sum_i e^{ikx_i}, \\
\omega(x; [\phi]) &= \int \frac{dk}{2\pi} e^{-ikx}\omega(k; [\phi]) \\
&= \partial_x\left[(x\phi(x))\left[2\fint \frac{dx'\phi(x')}{(x - x')}\right]\right] \\
&= \partial_x\left[(x\phi(x))\,\mathcal{L}(x)\right]. \qquad (9.0.4)
\end{aligned}$$

We will first study the effective potential term which represents the repulsion amongst the eigenvalues coupled to each other:

$$\begin{aligned}
&\int \mathrm{dx}\int \mathrm{dx}'\omega(x; [\phi])\Omega^{-1}(x, x'; [\phi])\omega(x'; [\phi]) = \qquad (9.0.5) \\
&\int \mathrm{dx}\int \mathrm{dx}'(\partial_x(x\phi(x))\mathcal{L}(x))\Omega^{-1}(x, x'; [\phi])(\partial_{x'}(x'\phi(x'))\mathcal{L}(x')) = \\
= \ &\frac{1}{8}\int \mathrm{dx}\int \mathrm{dx}'\left(x\phi(x)\mathcal{L}(x)\right)\left(\partial_x\partial_{x'}\Omega^{-1}(x, x'; [\phi])\right)x'\phi(x')\mathcal{L}(x').
\end{aligned}$$

Using exactly the same argument used in obtaining equation (4.4.17), we obtain

$$\left(\partial_x \partial_{x'} \Omega^{-1}_{x,x'}\right) \;=\; \frac{\delta(x-x')}{x'\phi(x')}. \tag{9.0.6}$$

We substitute the definition appearing in equation (9.0.6) into the last line of equation (9.0.5) to obtain the following expression

$$\begin{aligned} & \frac{1}{8}\int \mathrm{dx} \int \mathrm{dx}' \left(x\phi(x)\mathcal{L}(x)\right)\left(\partial_x \partial_{x'}\Omega^{-1}(x,x';[\phi])\right) x'\phi(x')\mathcal{L}(x) \\ = \;& \frac{1}{8}\int \mathrm{dx}\left(x\phi(x)\right)\left[2\fint \frac{dx'\phi(x')}{(x-x')}\right]^2. \end{aligned} \tag{9.0.7}$$

Using equation (9.0.7), and (9.0.3) we obtain a new expression for the effective Hamiltonian operator

$$\begin{aligned} H_{eff}[x,x';[\phi]] \;=\; & \frac{1}{2}\int \mathrm{dx}\left(\partial_x \Pi(x)\right)\left[x\phi(x)\right]\left(\partial_x \Pi(x)\right) \\ + \; & \frac{1}{8}\int \mathrm{dx}\left(x\phi(x)\right)\left[2\fint \frac{dx'\phi(x')}{(x-x')}\right]^2 + \mathcal{P}[x,[\phi]]. \end{aligned} \tag{9.0.8}$$

The first term appearing in the last line of equation (9.0.8), encodes the repulsive behavior of the eigenvalues of the physical system. The denominator of this term seems to suggest that the eigenvalues repel each other when trapped in a potential.

We saw that in the single hermitian (matrix) system, this term was shown to be equivalent to a local cubic interaction. We wish to investigate if this is also the case for the radial sector of the two (hermitian) matrix system.

To do so, and as was the case for the single matrix integral, we extend the range and the definition of the radial eigenvalue density to the whole real line.

From our original definition of the eigenvalues $x = r^2 > 0$, we will extend the definition of our eigenvalues to include both positive values and negative values of r. We will define: x and x' such that $x = r^2$, $x' = s^2$ for both $s > 0$ and $r > 0$. Also, by definition, we induce a change of variables for the wavefunctions of the system as follows: $\Phi(r) = 2r\phi(r^2)$. As previously, we extend the domain of the definition of the density to the whole real line by requiring $\Phi(-r) \equiv \Phi(r)$, $r > 0$.

We will proceed as follows

$$\begin{aligned}
\int \mathrm{dx}\left(x\phi(x)\mathcal{L}^2(x)\right) &= \int_0^\infty \mathrm{dx}\left(x\phi(x)\right)\left[2\fint_0^\infty \frac{\mathrm{dx}'\phi(x')}{(x-x')}\right]^2 \\
&= 4\int_0^\infty (dr^2)\left(r^2\phi(r^2)\right)\left(\fint_0^\infty \frac{d(s^2)\phi(s^2)}{(r^2-s^2)}\right)^2 \\
&= 4\int_0^\infty dr\Phi(r)r^2\left(\fint_0^\infty \frac{ds\Phi(s)}{(r^2-s^2)}\right)^2. \qquad (9.0.9)
\end{aligned}$$

Equation (9.0.9) carries an identity that can be used to simplify the entire expression, but first we extend the range of integration

$$\begin{aligned}
& 4\int_0^\infty dr\Phi(r)r^2\left(\fint_0^\infty \frac{ds\Phi(s)}{(r^2-s^2)}\right)^2 \\
&= \int_0^\infty dr\Phi(r)\left(\fint_0^\infty ds\Phi(s)\frac{2r}{(r^2-s^2)}\right)^2 \\
&= \int_0^\infty dr\Phi(r)\left(\fint_0^\infty ds\Phi(s)\left(\frac{1}{(r+s)}+\frac{1}{(r-s)}\right)\right)^2 \\
&= \int_0^\infty dr\Phi(r)\left(\left(\fint_{-\infty}^0 \frac{ds\Phi(-s)}{(r-s)}+\fint_0^\infty \frac{ds\Phi(s)}{(r-s)}\right)\right)^2 \\
&= \int_0^\infty dr\Phi(r)\left(\fint_{-\infty}^\infty \frac{ds\Phi(s)}{(r-s)}\right)^2 \\
&= \frac{1}{2}\int_{-\infty}^\infty dr\Phi(r)\left(\fint_{-\infty}^\infty \frac{ds\Phi(s)}{(r-s)}\right)^2. \qquad (9.0.10)
\end{aligned}$$

The squared term appearing in the brackets is in fact an identity that first appeared in the collective field theory framework of the single hermitian matrix model. This is remarkable because it will allow us to simplify the above expression. As we previously saw, we can write

$$\left(\fint_{-\infty}^\infty \frac{ds\Phi(s)}{(r-s)}\right)^2 = \frac{\pi^2}{3}\Phi^2(r). \qquad (9.0.11)$$

We make use of equation (9.0.11) in equation (9.0.10) to obtain the following

$$
\begin{aligned}
& 4\int_0^\infty dr\Phi(r)r^2\left(\fint_0^\infty \frac{ds\Phi(s)}{(r^2-s^2)}\right)^2 \\
= \;& \frac{1}{2}\int_{-\infty}^\infty dr\Phi(r)\left(\fint_{-\infty}^\infty \frac{ds\Phi(s)}{(r-s)}\right)^2 \\
= \;& \frac{1}{2}\int_{-\infty}^\infty dr\Phi(r)\left[\frac{\pi^2}{3}\Phi^2(r)\right] \\
= \;& \frac{\pi^2}{6}\int_{-\infty}^\infty dr\Phi^3(r). \qquad (9.0.12)
\end{aligned}
$$

We note that in equation (9.0.12) above, the integral over the radial eigenvalues r integrating the radial function $\Phi(r)$ runs over the entire real line domain. Also, as we saw for the single hermitian matrix case, we get a cubic potential that's generated in the radially symmetric bosonic sector of $(2+1)$-dimensional fermions [112].

We substitute equation (9.0.12) into equation (9.0.8), to obtain a more simplified expression for the effective Hamiltonian

$$
\begin{aligned}
H_{eff}[x,x';[\phi]] &= \frac{1}{2}\int \mathrm{dx}\left(\partial_x\Pi(x)\right)\left[x\phi(x)\right]\left(\partial_x\Pi(x)\right) \\
&+ \frac{1}{8}\left(\frac{\pi^2}{6}\int_{-\infty}^\infty dr\Phi^3(r)\right)+\mathcal{P}[x,[\phi]]. \qquad (9.0.13)
\end{aligned}
$$

We will define the effective potential

$$
\mathcal{P}'(x;[\phi]) = \left[\frac{\pi^2}{48}\int_{-\infty}^\infty dr\Phi^3(r)\right]+\mathcal{P}[x,[\phi]]. \qquad (9.0.14)
$$

The eigenvalues for our system of two matrices will be subjected to a Gaussian potential that has $U(N)\times U(N)$ symmetry. At this point, we will use, in the calculations that follow, a potential for our matrix system that is expressed in terms of collective field theory variables.

We introduce the following Gaussian potential for our system

$$\begin{aligned}
\mathcal{P}[x,[\phi]] &= \frac{\omega^2}{2}\mathrm{Tr}\left(Z^\dagger Z\right) = \frac{\omega^2}{2}\sum_i r_i^2 \\
&= \frac{\omega^2}{2}\int \mathrm{dx}\phi(x)x \\
&= \frac{\omega^2}{2}\int_0^\infty dr(2r\phi(r^2))r^2 \qquad [x=r^2] \\
&= \frac{\omega^2}{2}\int_0^\infty dr\Phi(r)r^2 \qquad [\Phi(r)=2r\phi(r^2)] \\
\Rightarrow \mathcal{P}[x,[\phi]] &= \frac{\omega^2}{4}\int_{-\infty}^\infty dr\Phi(r)r^2. \qquad (9.0.15)
\end{aligned}$$

Using equation (9.0.15), we assemble the effective potential to obtain the following

$$\begin{aligned}
\mathcal{P}'(x;[\phi]) &= \frac{\pi^2}{48}\int_{-\infty}^\infty dr\Phi^3(r) + \mathcal{P}[x,[\phi]] \\
&= \left[\frac{\pi^2}{48}\int_{-\infty}^\infty dr\Phi^3(r)\right] + \frac{\omega^2}{4}\int_{-\infty}^\infty dr\Phi(r)r^2 \\
&= \mathcal{P}'[r,[\phi]]. \qquad (9.0.16)
\end{aligned}$$

The density function $\Phi(r)$ is subject to the constraint

$$\int_{-\infty}^\infty dr\Phi(r) = 2N. \qquad (9.0.17)$$

The functional is given by the following expression

$$\begin{aligned}
\mathcal{M}(\vartheta,[\phi]) &= \mathcal{P}'(r;[\phi]) + \vartheta\left(2N - \int_{-\infty}^\infty dr\Phi(r)\right) \\
&= \frac{\pi^2}{48}\int_{-\infty}^\infty dr\Phi^3(r) + \frac{\omega^2}{4}\int_{-\infty}^\infty dr\Phi(r)r^2 \\
&+ \vartheta\left(2N - \int_{-\infty}^\infty dr\Phi(r)\right). \qquad (9.0.18)
\end{aligned}$$

The term ϑ represents the Lagrange multiplier.

The functional appearing above will be solved in the large N limit, but before we do this we will need to rescale our collective field theory variables to explicitly show how the N dependence emerges. One has:

$$r \to \sqrt{N}r, \quad \Phi(r) \to \sqrt{N}\Phi(r), \quad \vartheta \to N\vartheta. \qquad (9.0.19)$$

We substitute the rescaled variables appearing in equation (9.0.19) into equation (9.0.18), to obtain the following equation

$$\begin{aligned}\mathcal{M}(\vartheta,[\Phi]) &= \mathcal{P}'_A(r;[\phi]) + \vartheta\left(2N - \int_{-\infty}^{\infty} dr\Phi(r)\right) \\ &= N^2\left[\frac{\pi^2}{48}\int_{-\infty}^{\infty} dr\Phi^3(r) + \frac{\omega^2}{4}\int_{-\infty}^{\infty} dr\Phi(r)r^2\right] \\ &+ \vartheta N^2\left(2 - \int_{-\infty}^{\infty} dr\Phi(r)\right). \end{aligned} \tag{9.0.20}$$

In order to obtain the large N background solution, we can vary equation (9.0.20) with respect to the radial density (Φ) to obtain the following

$$\begin{aligned}&\frac{\partial}{\partial\Phi(r)}\mathcal{M}(\vartheta,[\phi]) = 0 \\ \Rightarrow\ & N^2\left(\frac{\pi^2}{16}\Phi^2(r) + \frac{\omega^2}{4}r^2 - \vartheta\right) = 0 \\ \Rightarrow \Phi(r) &= \frac{2}{\pi}\left(4\vartheta - \omega^2 r^2\right)^{1/2}. \end{aligned} \tag{9.0.21}$$

Proceeding from equation (9.0.21), we will use the constraint (9.0.17) to fix the Lagrange multiplier ϑ such that $\vartheta = \omega/2$ and to obtain the normalized ground state eigenvalue density function

$$\Phi_0(r) = \frac{2\omega}{\pi}\left(\frac{2}{\omega} - r^2\right)^{1/2} \qquad |r| \leq \pm\sqrt{\frac{2}{\omega}}. \tag{9.0.22}$$

Equation (9.0.22) is the normalized eigenvalue density function whose density description yields the semi-circle law that defines the Wigner distribution.

Figure (9.1) shows the Wigner semi-circle distribution obeyed by equation (9.0.22) for $\omega = 1/2$ and is perfectly symmetric about the zero point.

Note that in terms of the original variables $\rho = r^2$, one has

$$\phi_0(\rho) = \frac{\sqrt{2\omega}}{\pi}\sqrt{\frac{1}{\rho} - \frac{\omega}{2}} \qquad 0 < \rho < \frac{2}{\omega}, \tag{9.0.23}$$

which no longer obeys the Wigner semi-circle distribution law.

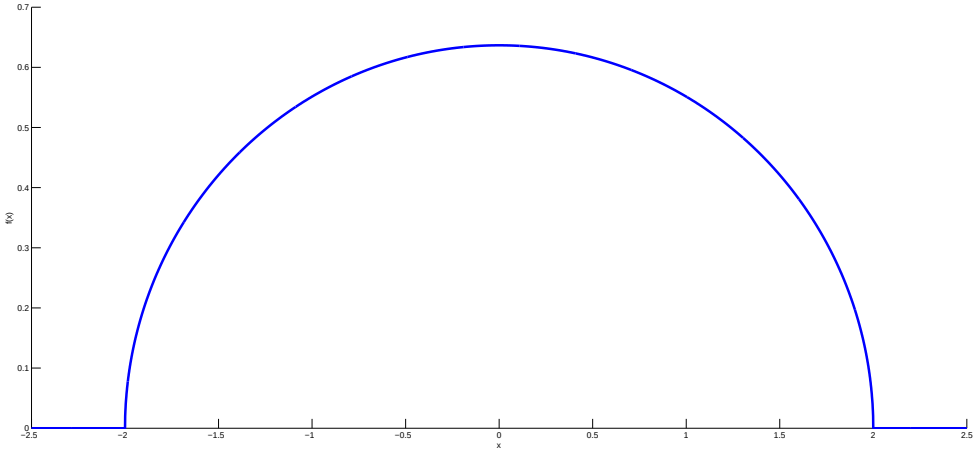

Figure 9.1: Wigner semi-circle eigenvalue distribution for the two hermitian matrix model in collective field theory.

Chapter 10

Radial Sector Of Systems With An Even Number Of Hermitian Matrices

Earlier chapters in the main body of the work discussed the single hermitian matrix model and the single complex model by treating the partition function and the collective field theory Hamiltonian formalism of both matrix models in the large N limit.

Both the single and two hermitian matrix model yielded remarkable physical results like the manifestation of the Wigner semi-circle law, though the latter required special treatment by restricting it to a sector strictly dependent on radial degrees of freedom.

Also, reducing these matrix models, and decoupling any angular degrees of freedom resulted in both systems of matrices in their respective frameworks yielding a fermionic type description.

So far, we have shown remarkable parallels between the radial sector of the single complex matrix model and the single hermitian matrix model with respect to the results obtained and how uniquely the matrix systems were treated observing the appearance of $(1+1)$-dimensional quasi-particles.

The natural question that arises is the following: Can the properties that we observed and learnt in earlier chapters related to the single hermitian matrix

model and the two hermitian matrix model be generalized for a larger system of matrices. It turns out that many of these properties can indeed be generalized to systems of an arbitrary number of complex matrices.

The chapters that follow will be dedicated to answering the preceding questions and also develop and learn more about the richness and physical properties of matrix models.

The main goals of this chapter are as follows:

- Provide a definition of the generalized radial sector constituted by a large ensemble of complex matrices
- Construct an integral for the generalized radial sector (partition function)
- Develop a density description formalism for the generalized radial sector
- Obtain an expression for the Jacobian for the general ensemble of matrices that make up the radial sector

10.1 Jacobian In The Radial Sector

In this chapter we consider a generalized model of m $N \times N$ complex matrices

$$Z_A \qquad A = 1, 2, ...m, \tag{10.1.1}$$

or equivalently a matrix model of $2m$ $N \times N$ hermitian matrices

$$Z_A = X_{(2A-1)} + iX_{(2A)}. \tag{10.1.2}$$

We would like to look at the large N description of the system mentioned above. Consider a Gaussian partition function

$$\mathcal{Z} = \int \prod_A \prod_{ij} \left(dZ_A^\dagger\right)_{ij} (dZ_A)_{ij}\, e^{-S_g(Z_A, Z_A^\dagger)}, \tag{10.1.3}$$

where $S_g(Z_A, Z_A^\dagger)$ is the Gaussian potential:

$$S(Z_A, Z_A^\dagger) = \frac{\omega^2}{2} \mathrm{Tr} \left(\sum_A Z_A^\dagger Z_A \right). \tag{10.1.4}$$

This potential has a $U(N)^{m+1}$ symmetry as it is invariant under the following transformation

$$Z_A \to V_A Z_A V^\dagger. \tag{10.1.5}$$

The potential above depends on the positive definite hermitian matrix

$$\sum_A Z_A^\dagger Z_A, \tag{10.1.6}$$

whose eigenvalues can be written as follows

$$\rho_i = r_i^2 \qquad i = 1, 2, 3, ..., N. \quad \rho_i \geq 0. \tag{10.1.7}$$

Equation (10.1.7) represents the eigenvalues of our system whose dynamics will be investigated in the large N limit.

We will consider potentials that are dependent only on ρ_i:

$$S(Z_A, Z_A^\dagger) = S(\rho_i). \tag{10.1.8}$$

We are interested in the radial dynamics first. We expect many of these properties not to change when angular degrees of freedom are included, as in this construction they are dimensionless.

In addition the (radial) Gaussian potential which is the subject of more detailed analysis in this thesis is relevant to LLM backgrounds [107] possibly in a more symmetric description of the geometry.

Our main objective is to study the distribution of the eigenvalues of the following partition function

$$\begin{aligned} \mathcal{Z} &= \int \prod_A \prod_{ij} \left(dZ_A^\dagger \right)_{ij} (dZ_A)_{ij} \, e^{-S(Z_A, Z_A^\dagger)} \\ &= \int \prod_i d\rho_i \mathcal{J}(\rho_i) e^{-S(\rho_i)}, \end{aligned} \tag{10.1.9}$$

in the large N framework.

In order for us to obtain the Jacobian $\mathcal{J}(\rho_i)$, we need to perform a change of variables from the original degrees of freedom of the theory $(Z_A^\dagger, Z_A)$, to the $U(N)^{m+1}$ invariant operators $(\Phi(\rho), \Phi_k)$.

To derive the complete expression of the Jacobian $\mathcal{J}(\rho_i)$ we will use the density description.

As a starting point, we turn to Quantum Field Theory, making use of a familiar identity which can be utilized to obtain Schwinger-Dyson equations. These Schwinger-Dyson equations occur as a result of deducing conservation laws from the symmetries of functional integrals i.e. integrals over fields. We use an identity from these Schwinger-Dyson equations:

$$\int \prod_A \prod_{ij} (dZ_A)^\dagger_{ij} (dZ_A)_{ij} \frac{\partial}{\partial (Z_A)_{ji}} \left(\frac{\partial \Phi_k}{\partial (Z_A)^\dagger_{ij}} F[\Phi] e^{-S_g} \right) = 0 \tag{10.1.10}$$

In equation (10.1.10), the function $F[\Phi]$ represents the product of invariant operators Φ. These operators, defined in the density description, are in fact invariant under the symmetry of equation (10.1.5) and depend on the eigenvalues ρ_i of the system. When constructing these invariant operators we use the hermitian matrix of equation (10.1.6), defined under a trace. When referring to the operators in our system, we in fact mean operators that depend on the eigenvalues of the system i.e.

$$\Phi \equiv \Phi(\rho). \tag{10.1.11}$$

We construct the definition of our operators as follows

$$\begin{aligned} \Phi_k &= \mathrm{Tr}\left(e^{ik\sum_A Z_A^\dagger Z_A} \right) \\ &= \sum_i e^{ik\rho_i} \\ &= \sum_i e^{ikr_i^2}. \end{aligned} \tag{10.1.12}$$

The Fourier transform of the operator above defined in radial coordinate space is as follows

$$\begin{aligned}\Phi(\rho) &= \int \frac{dk}{2\pi} e^{-ik\rho}\Phi_k \\ &= \sum_i \delta(\rho - r_i^2). \end{aligned} \tag{10.1.13}$$

Equations (10.1.12) and (10.1.13) are variables that are defined in the density description which will later be used to develop a collective field theory model for this generalized system of $2m$ matrices.

In the first line of equation (10.1.10) we use the standard product rule by differentiating with respect to $\partial/\partial(Z_A)_{ji}$, we deduce the following

$$\begin{aligned} & \int \prod_A \prod_{ij} (dZ_A)^\dagger_{ij} (dZ_A)_{ij} \frac{\partial}{\partial (Z_A)_{ji}} \left(\frac{\partial \Phi_k}{\partial (Z_A)^\dagger_{ij}} F[\Phi] e^{-S_g(Z_A, Z_A^\dagger)} \right) = 0 \\ \Rightarrow & \int \prod_A \prod_{ij} (dZ_A)^\dagger_{ij} (dZ_A)_{ij} \left(\frac{\partial^2 \Phi_k}{\partial (Z_A)_{ji} \partial (Z_A)^\dagger_{ij}} F[\Phi] \right) e^{-S_g(Z_A, Z_A^\dagger)} \\ + & \int \prod_A \prod_{ij} (dZ_A)^\dagger_{ij} (dZ_A)_{ij} \left(\frac{\partial \Phi_k}{\partial (Z_A)^\dagger_{ij}} \frac{\partial F[\Phi]}{\partial (Z_A)_{ji}} \right) e^{-S_g(Z_A, Z_A^\dagger)} \\ - & \int \prod_A \prod_{ij} (dZ_A)^\dagger_{ij} (dZ_A)_{ij} \left(\frac{\partial \Phi_k}{\partial (Z_A)^\dagger_{ij}} F[\Phi] \frac{\partial S_g(Z_A, Z_A^\dagger)}{\partial (Z_A)_{ji}} \right) e^{-S_g(Z_A, Z_A^\dagger)} \\ = & \ 0. \end{aligned} \tag{10.1.14}$$

The above equation will lead us to an equation whose time-ordered correlation functions depend on the operators Φ,

$$\begin{aligned} & \left\langle \frac{\partial^2 \Phi_k}{\partial (Z_A)_{ji} \partial (Z_A)^\dagger_{ij}} F[\Phi] \right\rangle + \left\langle \frac{\partial \Phi_k}{\partial (Z_A)^\dagger_{ij}} \frac{\partial F[\Phi]}{\partial (Z_A)_{ji}} \right\rangle \\ - & \left\langle \frac{\partial \Phi_k}{\partial (Z_A)^\dagger_{ij}} F[\Phi] \frac{\partial S_g(Z_A, Z_A^\dagger)}{\partial (Z_A)_{ji}} \right\rangle = 0. \end{aligned} \tag{10.1.15}$$

From the functional integrals that are dependent on the invariant operators Φ, one can obtain all the time ordered product expectation values. Therefore we define a product expectation value for the $U(N)^{m+1}$ invariant operators

$$\langle G[\Phi] \rangle \equiv \int \prod_A \prod_{ij} (dZ_A^\dagger)_{ij} (dZ_A)_{ij} G[\Phi] e^{-S_g}, \tag{10.1.16}$$

where we still maintain $S_g \equiv S_g(Z_A, Z_A^\dagger)$.

With the change of variables we will require the following equation to hold true

$$\int \prod_A \prod_{ij} (dZ_A^\dagger)_{ij} (dZ_A)_{ij} G[\Phi] e^{-S_g} = \int [d\Phi] J(\Phi) G[\Phi] e^{-S_g}. \tag{10.1.17}$$

As a result of changing variables, the above two integrals should be the same. We will again consider an identity that will generate Schwinger-Dyson equations under a new reformulation of the theory in terms of the invariant variables Φ using the methods based on [116] [117] [118]. The identity is as follows

$$\int [d\Phi] \int dk' \frac{\partial}{\partial \Phi'_k} \left(\sum_A \left[\frac{\partial \Phi_k}{\partial (Z_A^\dagger)_{ij}} \frac{\partial \Phi_{k'}}{\partial (Z_A)_{ji}} \right] J(\Phi) F[\Phi] e^{-S_g} \right) = 0. \tag{10.1.18}$$

Operating with $\partial / \partial \Phi'_k$ we obtain,

$$\begin{aligned}
& \int [d\Phi] \int dk' \left(\frac{\partial}{\partial \Phi'_k} \left[\frac{\partial \Phi_k}{\partial (Z_A^\dagger)_{ij}} \frac{\partial \Phi_{k'}}{\partial (Z_A)_{ji}} \right] F[\Phi] \right) J(\Phi) e^{-S_g} \\
+ & \int [d\Phi] \int dk' \left(\left[\frac{\partial \Phi_k}{\partial (Z_A^\dagger)_{ij}} \frac{\partial \Phi_{k'}}{\partial (Z_A)_{ji}} \right] \frac{\partial J(\Phi)}{\partial \Phi'_k} \frac{1}{J(\Phi)} F[\Phi] \right) J(\Phi) e^{-S_g} \\
+ & \int [d\Phi] \int dk' \left(\left[\frac{\partial \Phi_k}{\partial (Z_A^\dagger)_{ij}} \frac{\partial \Phi_{k'}}{\partial (Z_A)_{ji}} \right] \frac{\partial F[\Phi]}{\partial \Phi'_k} \right) J(\Phi) e^{-S_g} \\
- & \int [d\Phi] \int dk' \left(\left[\frac{\partial \Phi_k}{\partial (Z_A^\dagger)_{ij}} \frac{\partial \Phi_{k'}}{\partial (Z_A)_{ji}} \right] F[\Phi] \frac{\partial S_g}{\partial \Phi'_k} \right) J(\Phi) e^{-S_g} \\
= & \ 0.
\end{aligned} \tag{10.1.19}$$

We obtain time-ordered correlation functions integrated over the variable k'

$$\int dk' \left\langle \frac{\partial}{\partial \Phi'_k} \left[\frac{\partial \Phi_k}{\partial (Z_A)^\dagger_{ij}} \frac{\partial \Phi_{k'}}{\partial (Z_A)_{ji}} \right] F[\Phi] \right\rangle +$$
$$\int dk' \left\langle \left[\frac{\partial \Phi_k}{\partial (Z_A)^\dagger_{ij}} \frac{\partial \Phi_{k'}}{\partial (Z_A)_{ji}} \right] \frac{\partial \ln J(\Phi)}{\partial \Phi'_k} F[\Phi] \right\rangle +$$
$$\int dk' \left\langle \left[\frac{\partial \Phi_k}{\partial (Z_A)^\dagger_{ij}} \frac{\partial \Phi_{k'}}{\partial (Z_A)_{ji}} \right] \frac{\partial F[\Phi]}{\partial \Phi'_k} \right\rangle -$$
$$\int dk' \left\langle \left[\frac{\partial \Phi_k}{\partial (Z_A)^\dagger_{ij}} \frac{\partial \Phi_{k'}}{\partial (Z_A)_{ji}} \right] F[\Phi] \frac{\partial S_g}{\partial \Phi'_k} \right\rangle = 0. \tag{10.1.20}$$

In the last two terms of equation (10.1.20), we integrate the variable k' using the chain rule. The third term becomes

$$\int dk' \left\langle \left[\frac{\partial \Phi_k}{\partial (Z_A)^\dagger_{ij}} \frac{\partial \Phi_{k'}}{\partial (Z_A)_{ji}} \right] \frac{\partial F[\Phi]}{\partial \Phi'_k} \right\rangle = \left\langle \left[\frac{\partial \Phi_k}{\partial (Z_A)^\dagger_{ij}} \frac{\partial F[\Phi]}{\partial (Z_A)_{ji}} \right] \right\rangle. \tag{10.1.21}$$

We treat the last term appearing in equation (10.1.20) in a similar manner, and we obtain

$$\int dk' \left\langle \frac{\partial}{\partial \Phi'_k} \left[\frac{\partial \Phi_k}{\partial (Z_A)^\dagger_{ij}} \frac{\partial \Phi_{k'}}{\partial (Z_A)_{ji}} \right] F[\Phi] \right\rangle +$$
$$\int dk' \left\langle \left[\frac{\partial \Phi_k}{\partial (Z_A)^\dagger_{ij}} \frac{\partial \Phi_{k'}}{\partial (Z_A)_{ji}} \right] \frac{\partial \ln J(\Phi)}{\partial \Phi'_k} F[\Phi] \right\rangle +$$
$$\left\langle \left[\frac{\partial \Phi_k}{\partial (Z_A)^\dagger_{ij}} \frac{\partial F[\Phi]}{\partial (Z_A)_{ji}} \right] \right\rangle - \left\langle \left[\frac{\partial \Phi_k}{\partial (Z_A)^\dagger_{ij}} \frac{\partial S_g}{\partial (Z_A)_{ji}} \right] F[\Phi] \right\rangle = 0. \tag{10.1.22}$$

Equation (10.1.15) and equation (10.1.22) are the same equations represented in different coordinates. Naturally both equations should generate the same Schwinger-Dyson equations for the arbitrary function $F[\Phi]$.

Therefore when comparing equations (10.1.15) and (10.1.22) the following identity for the Jacobian must be true

$$\int dk' \frac{\partial}{\partial \Phi'_k}\left[\frac{\partial \Phi_k}{\partial (Z_A)^\dagger_{ij}}\frac{\partial \Phi_{k'}}{\partial (Z_A)_{ji}}\right] + \int dk'\left[\frac{\partial \Phi_k}{\partial (Z_A)^\dagger_{ij}}\frac{\partial \Phi_{k'}}{\partial (Z_A)_{ij}}\right]\frac{\partial \ln J(\Phi)}{\partial \Phi'_k}$$
$$= \frac{\partial^2 \Phi_k}{\partial (Z_A)^\dagger_{ij}\partial (Z_A)_{ij}}. \tag{10.1.23}$$

Equation (10.1.23) above is the equation for the Jacobian $\mathcal{J}(\rho_i) = J(\Phi)$ [5].

At this point we make use of the definitions of our invariant variables in equation (10.1.12) and equation (10.1.13). We will apply some of the principles we used in the collective field theory framework. We first start off by introducing the following definitions

$$\Omega_{kk'} = \sum_A \left[\frac{\partial \Phi_k}{\partial (Z_A)^\dagger_{ij}}\frac{\partial \Phi_{k'}}{\partial (Z_A)_{ji}}\right], \qquad \omega_k = \sum_A \frac{\partial^2 \Phi_k}{\partial (Z_A)^\dagger_{ij}\partial (Z_A)_{ij}}. \tag{10.1.24}$$

In the equation (10.1.24) above, $\Omega_{kk'}$ is the joining operator and ω_k is the splitting operator of collective field theory. We define the Fourier transforms of equation (10.1.24) ,

$$\Omega_{\rho\rho'} = \int \frac{dk}{2\pi}\int \frac{dk'}{2\pi} e^{-ik\rho}e^{-ik'\rho'}\Omega_{kk'}, \quad \omega_\rho = \int \frac{dk}{2\pi} e^{-ik\rho}\omega_k. \tag{10.1.25}$$

With the use of equations (10.1.25) and (10.1.24), we can rewrite equation (10.1.23) in a more compact form. We start by substituting the definitions of equation (10.1.24) into equation (10.1.23)

$$\int dk' \frac{\partial}{\partial \Phi'_k}\left[\frac{\partial \Phi_k}{\partial (Z_A)^\dagger_{ij}}\frac{\partial \Phi_{k'}}{\partial (Z_A)_{ji}}\right] + \int dk'\left[\frac{\partial \Phi_k}{\partial (Z_A)^\dagger_{ij}}\frac{\partial \Phi_{k'}}{\partial (Z_A)_{ij}}\right]\frac{\partial \ln J(\Phi)}{\partial \Phi'_k}$$
$$= \frac{\partial^2 \Phi_k}{\partial (Z_A)^\dagger_{ij}\partial (Z_A)_{ij}}$$
$$\Rightarrow \int dk' \frac{\partial \Omega_{kk'}}{\partial \Phi'_k} + \int dk' \Omega_{kk'}\frac{\partial \ln J(\Phi)}{\partial \Phi'_k} = \omega_k \tag{10.1.26}$$

[5] We deem this change of notation necessary to preserve consistency from earlier chapters. Later we will return to the initial definition of the Jacobian ($\mathcal{J}(\rho_i)$).

Proceeding from equation (10.1.26), we substitute the Fourier transform definitions of equation (10.1.25) to obtain a more compact form of the equation of the Jacobian

$$\Rightarrow \quad \int d\rho' \Omega_{\rho\rho'} \frac{\partial \ln J(\Phi)}{\partial \Phi(\rho')} + \int d\rho' \frac{\partial \Omega_{\rho\rho'}}{\partial \Phi(\rho')} = \omega_\rho \tag{10.1.27}$$

Equation (10.1.27) is the equation of the Jacobian $J(\Phi)$, this equation is defined in terms of the collective field theory Fourier space variables $\Omega_{\rho\rho'}$ and ω_ρ. We recognize the equation for the Jacobian that results from collective field theory. This approach shows that this equation can also be obtained from Schwinger-Dyson equations [116].

As observed for the single and the two hermitian matrix model, the second term on the left hand side of equation (10.1.27) vanishes, leaving us with the following

$$\int d\rho' \Omega_{\rho\rho'} \frac{\partial \ln J(\Phi)}{\partial \Phi(\rho')} = \omega_\rho. \tag{10.1.28}$$

To solve for $J(\Phi)$, it would be fundamental that we explicitly express the collective field theory operators $\Omega_{\rho\rho'}$ and ω_ρ of equations (10.1.24) and (10.1.25). Thus, from equation (10.1.12) we have

$$\begin{aligned} \frac{\partial \Phi_k}{\partial (Z_A)_{ij}} &= \frac{\partial}{\partial (Z_A)_{ij}} \mathrm{Tr}\left(e^{ik\left(\sum_B Z_B^\dagger Z_B\right)} \right) \\ &= ik \left(e^{\sum_B (Z_B^\dagger)(Z_B)} Z_A^\dagger \right)_{ji}. \end{aligned} \tag{10.1.29}$$

Naturally, using a similar process to equation (10.1.29) above, we can deduce the following to be true

$$\begin{aligned} \frac{\partial \Phi_k}{\partial (Z_A^\dagger)_{ij}} &= \frac{\partial}{\partial (Z_A^\dagger)_{ij}} \mathrm{Tr}\left(e^{ik \sum_B Z_B^\dagger Z_B} \right) \\ &= ik \left(Z_A e^{ik \sum_B (Z_B^\dagger)(Z_B)} \right)_{ji}. \end{aligned} \tag{10.1.30}$$

On our way to defining $\Omega_{\rho\rho'}$ and ω_ρ, we use the above two equations (10.1.29) and (10.1.30). We first start with the joining operator

$$\begin{aligned}\Omega_{kk'} &= \sum_A \left[\frac{\partial\Phi_k}{\partial(Z_A)^\dagger_{ij}}\frac{\partial\Phi_{k'}}{\partial(Z_A)_{ji}}\right] \\ &= ik\sum_A \left(Z_A e^{ik\sum_B (Z_B^\dagger)(Z_B)}\right)_{ji} ik' \left(Z_A e^{ik'\sum_B (Z_B^\dagger)(Z_B)} Z_A^\dagger\right)_{ij} \\ &= -kk'\mathrm{Tr}\left(\sum_A Z_A^\dagger Z_A e^{i(k+k')\sum_B Z_B^\dagger Z_B}\right), \qquad (10.1.31)\end{aligned}$$

and then its Fourier transform becomes

$$\begin{aligned}\Omega_{\rho\rho'} &= \int\frac{dk}{2\pi}\int\frac{dk'}{2\pi}e^{-ik\rho}e^{-ik'\rho'}\Omega_{kk'} \\ \Rightarrow \Omega_{\rho\rho'} &= \partial\rho\partial\rho'\left[\rho\Phi(\rho)\delta(\rho'-\rho)\right]. \qquad (10.1.32)\end{aligned}$$

We now wish to move on and define the splitting operator $\omega(\rho)$, for this we start off with the definition of the splitting operator in the real k-space. This implies that

$$\begin{aligned}\omega_k &= \sum_A \frac{\partial^2\phi_k}{(\partial Z_A^\dagger)_{ij}(\partial Z_A)_{ji}} \\ &= \sum_A (ik)^2\int_0^1 d\alpha\left(e^{ik\alpha\sum_A Z_A^\dagger Z_A}\right)_{ii}\left(Z_A e^{ik(1-\alpha)\sum_B Z_B^\dagger Z_B} Z_A^\dagger\right)_{jj} \\ &+ ik\left(e^{ik\sum_A Z_A^\dagger Z_A}\right)_{ii}\underbrace{\delta_{jj}}_{N}\ \underbrace{\sum_A \delta_{A,A}}_{m} \qquad (10.1.33) \\ &= -k^2\int_0^1 d\alpha\left(e^{ik\alpha\sum_A Z_A^\dagger Z_A}\right)_{ii}\left(Z_A e^{ik(1-\alpha)\sum_A Z_A^\dagger Z_A} Z_A^\dagger\right)_{jj} + ik\left(e^{ik\sum_A Z_A^\dagger Z_A}\right)_{ii} Nm.\end{aligned}$$

The treatment of equation (10.1.33) above will be no different from earlier chapters, the only significant difference is the emergence of a constant m. We rewrite the above equation as

$$\omega_k = -k\sum_{ij}\int_0^k dk'\left(e^{ik'r_i^2}\right)\left(r_j^2 e^{i(k-k')r_j^2}\right) + ikNm\sum_i e^{ikr_i^2}. \qquad (10.1.34)$$

Equation (10.1.34) will be further simplified by extending the summation over i and j such that $\sum_{ij} = \sum_i + \sum_{i\neq j}$. Therefore we obtain the following

$$\omega_k = -k^2 \sum_i r_i^2 e^{ikr_i^2} - k \sum_{i \neq j} \int_0^k dk' e^{ik' r_i^2} r_j^2 e^{i(k-k')r_j^2} + ikNm \sum_i e^{ikr_i^2}. \tag{10.1.35}$$

As we mentioned earlier that the last term in equation (10.1.35) is non-trivial and unique from the single complex matrix. With a bit of algebra, but similar to the manipulations carried in (8.0.10), equation (10.1.35) is shown to be

$$\begin{aligned} \omega_k &= -k^2 \sum_i r_i^2 e^{ikr_i^2} - 2ik \sum_{i \neq j} \frac{r_j^2 e^{ikr_j^2}}{(r_i^2 - r_j^2)} + ikN(m-1) \sum_i e^{ikr_i^2} \\ &+ ik \sum_i e^{ikr_i^2}. \end{aligned} \tag{10.1.36}$$

We now proceed to define the Fourier transform of ω_k

$$\begin{aligned} \Rightarrow \omega_x &= \int \frac{dk}{2\pi} e^{-ikx} \left(-k^2 \sum_i r_i^2 e^{ikr_i^2} \right) + \int \frac{dk}{2\pi} e^{-ikx} \left(-2ik \sum_{i \neq j} \frac{r_j^2 e^{ikr_j^2}}{(r_i^2 - r_j^2)} \right) \\ &+ \int \frac{dk}{2\pi} e^{-ikx} \left(ikN(m-1) \sum_i e^{ikr_i^2} \right) + \int \frac{dk}{2\pi} e^{-ikx} \left(ik \sum_i e^{ikr_i^2} \right). \end{aligned} \tag{10.1.37}$$

Equation (10.1.37) bears a striking resemblance to the splitting operator of equation (8.0.13) except for the $(m-1)$ factor in the third term. When $m = 1$, we recover equation (8.0.13). The above equation will be simplified as in (8.0.13) to get

$$\omega_x = -\partial_x \left(x\Phi(x) \right) \left[2 \fint \frac{dy \Phi(x)}{(x-y)} + \frac{N(m-1)}{x} - \frac{\partial_x \Phi(x)}{\Phi(x)} \right]. \tag{10.1.38}$$

As we argued previously, in the large N limit we can neglect the third term of equation (10.1.38). Also, to return to a more common notation, we had previously defined $\rho = r^2$. We reset variables as follows: $x = \rho, y = \rho'$. In total, the density description of the splitting operator in coordinate representation will take the following form

$$\omega_\rho = -\partial_\rho \left(\rho\Phi(\rho) \right) \left[2 \fint \frac{d\rho' \Phi(\rho')}{(\rho - \rho')} + \frac{N(m-1)}{\rho} \right]. \tag{10.1.39}$$

The definitions obtained for $\Omega_{\rho\rho'}$ and ω_ρ, will now be put into effect in equation (10.1.38) which will give us the following

$$\begin{aligned}
& \int d\rho' \Omega_{\rho\rho'} \frac{\partial \ln J(\Phi)}{\partial \Phi(\rho')} - \omega_\rho = 0 \\
\Rightarrow \quad & \int d\rho' \partial_\rho \partial_{\rho'} \left(\rho\Phi(\rho)\delta(\rho-\rho')\right) \frac{\partial \ln J(\Phi)}{\partial \Phi(\rho')} + \partial_\rho \left(\rho\Phi(\rho) \left[2 \fint \frac{d\rho' \Phi(\rho')}{\rho - \rho'} \right] \right) \\
+ \quad & \partial_\rho \left(\rho\Phi(\rho) \left[\frac{N(m-1)}{\rho} \right] \right) = 0 \\
\Rightarrow \quad & -\left(\rho\Phi(\rho)\right) \partial_\rho \frac{\partial \ln J(\Phi)}{\partial \Phi(\rho)} + \left(\rho\Phi(\rho)\right) \left[2 \fint \frac{d\rho' \Phi(\rho')}{\rho - \rho'} \right] + \left(\rho\Phi(\rho)\right) \left[\frac{N(m-1)}{\rho} \right] = 0 \\
\Rightarrow \quad & \partial_\rho \frac{\partial \ln J(\Phi)}{\partial \Phi(\rho)} = \left[2 \fint \frac{d\rho' \Phi(\rho')}{\rho - \rho'} \right] + \frac{N(m-1)}{\rho}. \qquad (10.1.40)
\end{aligned}$$

The last line of equation (10.1.40) is the equation of the Jacobian.

The Jacobian is

$$\ln J(\Phi) = \fint d\rho \fint d\rho' \Phi(\rho)\Phi(\rho') \ln|\rho - \rho'| + N(m-1) \fint d\rho \Phi(\rho) \ln|\rho|, \qquad (10.1.41)$$

as it can be shown to straight forwardly satisfy (10.1.40).

To shorten the notation, we will write $a = m - 1$. In terms of the eigenvalues

$$\begin{aligned}
\ln J(\Phi) &= \fint d\rho \fint d\rho' \Phi(\rho)\Phi(\rho') \ln|\rho - \rho'| + N(m-1) \fint d\rho \Phi(\rho) \ln|\rho| \\
&= \sum_{i \neq j} \ln\left|r_i^2 - r_j^2\right| + Na \sum_i \ln\left|r_i^2\right| \\
&= \sum_{i \neq j} \ln|\rho_i - \rho_j| + Na \sum_i \ln|\rho_i|. \qquad (10.1.42)
\end{aligned}$$

Having expressed the equation of the Jacobian in terms of radial eigenvalues in equation (10.1.42), we now proceed to write the Jacobian for a general even number of $2m$ matrices

$$
\begin{aligned}
\ln J &= \sum_{i\neq j} \ln|\rho_i-\rho_j| + Na\sum_i \ln|\rho_i| \\
&= a\sum_i \ln(\rho_i) + a(N-1)\sum_i \ln(\rho_i) + \sum_{i\neq j}\ln|\rho_i-\rho_j| \\
&= \sum_i \ln(\rho_i^a) + \sum_{i\neq j}\sum_i \ln(\rho_i^{a/2}) + \sum_{i\neq j}\sum_j \ln(\rho_j^{a/2}) + \sum_{i\neq j}\ln|\rho_i-\rho_j| \\
&= \ln\left(\prod_{i=1}^N \rho_i^a\right) + \sum_{i\neq j}\ln\left(\rho_i^{a/2}\rho_j^{a/2}|\rho_i-\rho_j|\right) \\
\Rightarrow \ln J &= \ln\left[\prod_{i=1}^N \rho_i^a\left(\prod_{i\neq j}\rho_i^{a/2}\rho_j^{a/2}|\rho_i-\rho_j|\right)\right] \\
\Rightarrow J &= \prod_{i=1}^N \rho_i^{m-1}\left(\prod_{i\neq j}\rho_i^{(m-1)/2}\rho_j^{(m-1)/2}|\rho_i-\rho_j|\right) \\
\Leftrightarrow J &= \prod_{i=1}^N \rho_i^{m-1}\left(\prod_{i>j}\rho_i^{(m-1)/2}\rho_j^{(m-1)/2}|\rho_i-\rho_j|\right)^2 \\
\Rightarrow J &= \prod_{i=1}^N \rho_i^{m-1}\prod_{i>j}\rho_i^{m-1}\rho_j^{m-1}(\rho_i-\rho_j)^2 = \mathcal{J}(\rho_i). \qquad (10.1.43)
\end{aligned}
$$

For the Jacobian J above we introduce shorthand notation such that

$$
\mathcal{J}(\rho_i) = J = \prod_{i=1}^N \rho_i^{m-1}\mathcal{V}_R^2(\rho_i), \qquad (10.1.44)
$$

where the antisymmetric term

$$
\begin{aligned}
\mathcal{V}_R(\rho_i) &= \prod_{i>j}\rho_i^{\frac{(m-1)}{2}}\rho_j^{\frac{(m-1)}{2}}(\rho_i-\rho_j) \\
&= \prod_{i>j} r_i^{(m-1)} r_j^{(m-1)}\left(r_i^2-r_j^2\right) \\
&= \mathcal{V}_R(r_i^2), \qquad (10.1.45)
\end{aligned}
$$

generalizes the commonly known Vandermonde determinant $\Delta = \prod_{i>j}(\rho_i-\rho_j)$. Equation (10.1.45) represents the Jacobian for a general $2m$ number of complex matrices in the radial sector.

Earlier on we had defined a Gaussian ensemble of m $N \times N$ complex matrices $(Z_A)_{ij}$ for $A = 1, 2, ..., m$, or equivalently $2m$ hermitian $N \times N$ matrices. Fundamentally we were motivated by the desire to obtain the large N description of the Gaussian partition function Z in equation (10.1.3) in terms of radial eigenvalues of the system $\rho_i = r_i^2$.

In summary:

$$\begin{aligned}
&\int \prod_A \prod_{ij} (dZ_A^\dagger)_{ij} (dZ_A)_{ij} e^{-S} = \int \prod_i d\rho_i \mathcal{J}(\rho_i) e^{-S_g(\rho_i)} \\
&= G_m \int \prod_i d\rho_i \prod_{i=1}^N \rho_i^{m-1} \left[\prod_{i>j} \rho_i^{m-1} \rho_j^{m-1} |\rho_i - \rho_j|^2 \right] e^{-S_g(\rho_i)} \\
&= G_m \int \prod_i d\rho_i \prod_{i=1}^N \rho_i^{m-1} \mathcal{V}_R^2(\rho_i) e^{-S_g(\rho_i)}, \qquad (10.1.46)
\end{aligned}$$

where in the last line of equation (10.1.46) G_m is some numerical constant.

On the left hand side of equation (10.1.46) is the probability density of the Gaussian partition function in equation (10.1.3), and on the right hand side we have defined our new measure for the Gaussian partition function using the generalized Jacobian J, for a general ensemble of $2m$ Gaussian $N \times N$ complex matrices, expressed purely in terms of radial eigenvalues.

In equation (10.1.46), when we set $m = 1$, we recover the partition function in equation (6.1.2) of the single complex matrix model Z.

We should remember that our generalized sector is a Gaussian ensemble of m complex $N \times N$ matrices. The rectangular $M \times N$ Gaussian ensemble of matrices considered in [119] [120] [121] [122], is a system of matrices that can be related to our generalized radial sector matrix model by letting: $M = mN$.

The approach developed in our project treats our matrix model in a gauge invariant manner.

Chapter 11

Gaussian Potential For More Complex Matrices

The generalized Jacobian J in equation (10.1.44), for a system of an even number of $2m$ hermitian matrices, has been derived using variables from the density description. We will now proceed to investigate the geometric properties of the eigenvalue distribution associated with equation (10.1.44).

In this chapter, we aim to demonstrate the following objectives:

- Using the partition function for a general number of m complex matrices (generalized radial sector), establish a stationary condition
- Study the solutions of the above mentioned stationary condition along a positive single interval of the complex plane, and obtain an eigenvalue density distribution
- Investigate the solutions of the stationary condition along an extended symmetric interval on the complex plane, and obtain an eigenvalue density distribution and its graphic description

We recall that previously we defined our Gaussian partition function as follows

$$\mathcal{Z} = \prod_{A} \prod_{ij} \left(dZ_A^{\dagger} \right)_{ij} (dZ_A)_{ij} \, e^{-S_g(Z_A^{\dagger}, Z_A)}, \tag{11.0.1}$$

where the Gaussian potential $S_g(Z_A^\dagger, Z_A)$ has been defined in equation (10.1.4) for $\rho_i = r_i^2$.

The the Gaussian partition will be

$$\begin{aligned}
\mathcal{Z} &= \int \prod_A \prod_{ij} \left(dZ_A^\dagger\right)_{ij} (dZ_A)_{ij}\, e^{-S_g(Z_A^\dagger, Z_A)} \\
&= \int \prod_i d\rho_i \mathcal{J}(\rho_i) e^{-S_g(\rho_i)} \\
&= \int \prod_i d\rho_i e^{\ln J - S_g(\rho_i)} \qquad [\mathcal{J}(\rho_i) = J] \\
&= \int \prod_i d\rho_i e^{-S_{eff}}. \qquad (11.0.2)
\end{aligned}$$

Even though in the Gaussian case it is true that the partition function factorizes into the partition functions of two hermitian matrices, this is not sufficient to fully characterize mixed matrix correlators.

In particular we are interested in the radial density which, to higher powers, will mix the two hermitian matrices.

In the last line of equation (11.0.2) we have defined the effective action S_{eff} of the system

$$S_{eff} = S_g(\rho_i) - \ln J. \qquad (11.0.3)$$

We consider the methods of the density description which we will use to define the Gaussian potential originally introduced in equation (10.1.4)

$$\begin{aligned}
S_g(Z_A^\dagger, Z_A) &= \frac{\omega^2}{2} \mathrm{Tr} \left(\sum_A Z_A^\dagger Z_A \right) = \frac{\omega^2}{2} \sum_i r_i^2 \\
&= \frac{\omega^2}{2} \int d\rho \Phi(\rho) \rho \\
&= S_g(\rho_i). \qquad (11.0.4)
\end{aligned}$$

First, we introduce the following constraint

$$\int_0^\infty d\rho \Phi(\rho) = N, \qquad (11.0.5)$$

and we rescale the variables of our theory such that $\rho \to N\rho$, $\rho' \to N\rho'$ and $\Phi(\rho) \to \Phi(\rho)$, applying these conditions, we can proceed to compute our effective action

$$\begin{aligned} S_{eff} &= \frac{\omega^2}{2}\int d\rho\Phi(\rho)\rho - \int d\rho\Phi(\rho)\fint d\rho'\Phi(\rho')\ln|\rho-\rho'| \\ &- N(m-1)\int d\rho\Phi(\rho)\ln|\rho| \\ \Rightarrow S_{eff} &= N^2\left[\frac{\omega^2}{2}\int d\rho\Phi(\rho)\rho - \int d\rho\Phi(\rho)\fint d\rho'\Phi(\rho')\ln|\rho-\rho'|\right] \\ &- N^2\left[(m-1)\int d\rho\Phi(\rho)\ln|\rho|\right]. \end{aligned} \tag{11.0.6}$$

The large N eigenvalue density configuration of our system is determined by the following stationary condition

$$\begin{aligned} &\partial_\xi\frac{\partial}{\partial\Phi(\xi)}S_{eff} = 0 \\ \Rightarrow\ &\partial_\xi\frac{\partial}{\partial\Phi(\xi)}\left(N^2\left[\frac{\omega^2}{2}\int d\rho\Phi(\rho)\rho - \int d\rho\Phi(\rho)\fint d\rho'\Phi(\rho')\ln|\rho-\rho'|\right]\right) - \\ &\partial_\xi\frac{\partial}{\partial\Phi(\xi)}\left(N^2\left[(m-1)\int d\rho\Phi(\rho)\ln|\rho|\right]\right) = 0 \\ \Rightarrow\ &\partial_\xi\left(\frac{\omega^2}{2}\xi - 2\fint d\rho'\Phi(\rho')\ln|\xi-\rho'| - (m-1)\ln|\xi|\right) = 0 \\ \Rightarrow\ &\frac{\omega^2}{2} - 2\fint\frac{d\partial'\Phi(\rho')}{\xi-\rho'} - \frac{m-1}{\xi} = 0. \end{aligned} \tag{11.0.7}$$

If we let $\xi = \rho$, then the semi-classical radial eigenvalue distribution in the large N limit satisfies

$$2\fint\frac{d\rho'\Phi(\rho')}{\rho-\rho'} = \frac{\omega^2}{2} - \frac{m-1}{\rho}, \tag{11.0.8}$$

which is the stationary condition of the system.

On the left hand side of equation (11.0.8), we see the standard Coulomb potential $(\rho-\rho')^{-1}$ representing the repulsion amongst eigenvalues centered at ρ. On the right hand side of equation (11.0.8), the second term represents a logarithmic potential whose strength varies with the number $(m-1)$ of matrices in our system. The radial eigenvalues will experience a repulsion centered at

$\rho = 0$ for the logarithmic potential. This is a new feature of $m \geq 2$ complex matrices.

The case for the $m = 1$ complex matrix Z has already been discussed in earlier chapters, and it can be recovered from equation (11.0.8). To obtain the solution to equation (11.0.8), which is a generalization of solutions associated with Penner potentials [123], we use the techniques applied by BIPZ [27] to study the single hermitian matrix model.

Due to the pole that appears on the right hand side of equation (11.0.8), the $\rho \to 0$ limit has to be treated cautiously according to the methods of Tan [124].

11.1 Positive Single Cut Solution

Below we find a solution for equation (11.0.8), that is, we obtain the density of eigenvalues for our system of m complex matrices.

As before, we first start off by introducing the analytic function

$$\mathcal{G}(z) = \int_{\rho_-}^{\rho_+} \frac{d\rho' \Phi(\rho')}{z - \rho'}, \tag{11.1.1}$$

along some support $[\rho_-, \rho_+]$ for $\rho_+ > \rho_- > 0$ on the complex plain z.

The above function $\mathcal{G}(z)$ is defined on the complex plane of z and is analytic along the cut $[\rho_-, \rho_+]$ also defined on complex plane z. For large $|z|$, that is $z \to \pm\infty$, the analytic function behaves as $\mathcal{G}(z) \approx 1/z$. For small z, $G(z)$ cannot have a pole [124]. In addition to this, we require that the analytic function $\mathcal{G}(z)$ be real for real z outside the support $[\rho_-, \rho_+]$, and that when z approaches the support $[\rho_-, \rho_+]$, then the analytic function will be given by the following solution

$$\begin{aligned} \mathcal{G}(\rho \pm i\epsilon) &= \fint_{\rho_-}^{\rho_+} \frac{d\rho' \Phi(\rho')}{\rho - \rho'} \mp i\pi\Phi(\rho) \\ &= \frac{\omega^2}{4} - \frac{m-1}{2\rho} \mp i\pi\Phi(\rho). \end{aligned} \tag{11.1.2}$$

The solution to equation (11.1.2) is a unique analytic function with a single cut along the complex plane z on which all of the previously mentioned conditions

hold for the density of eigenvalues $\Phi(\rho)$ on the support $[\rho_-, \rho_+]$, and this solution is given by

$$\mathcal{G}(z) = \frac{\omega^2}{4} - \frac{m-1}{2\rho} - \frac{\omega^2}{4z}\sqrt{(z-\rho_-)(z-\rho_+)}. \tag{11.1.3}$$

The solution appearing in equation (11.1.3) for the analytic function $\mathcal{G}(z)$ will implicitly enforce the requirement that in the limit $z \to 0$ the function $\mathcal{G}(z)$ holds and is defined. Equation (11.1.3) is the single cut ansatz for the $m \geq 2$ complex matrix model.

The derivation of equation (11.1.3) and the details of calculations that follow are shown in Appendix E.

From the unique function in equation (11.1.3), we can obtain the equation of the roots:

$$\begin{aligned} &\rho_\pm^2 - \frac{8}{\omega^2}\left(\frac{a}{2}+1\right)\rho_\pm + \frac{4a^2}{\omega^4} = 0 \\ &\Rightarrow \rho_\pm = \frac{2}{\omega^2}(m+1) \pm \frac{4}{\omega^2}\sqrt{m}. \end{aligned} \tag{11.1.4}$$

Equation (11.1.4) fixes the boundaries of the single cut on the complex plane z. Using equations (11.1.3) and (11.1.4), we can obtain the density of eigenvalues and this is shown to be

$$\begin{aligned} \Phi(\rho) &= \frac{\omega^2}{4\pi\rho}\sqrt{(\rho_+ - \rho)(\rho - \rho_-)} \\ &= \frac{\omega^2}{4\pi\rho}\left[\frac{16m}{\omega^4} - \left(\rho - \frac{2(m+1)}{\omega^2}\right)^2\right]^{1/2} \\ \Rightarrow \Phi(\rho) &= \frac{1}{\pi\rho}\sqrt{m - \frac{\omega^4}{16}\left(\rho - \frac{2(m+1)}{\omega^2}\right)^2}. \end{aligned} \tag{11.1.5}$$

As can be seen above that equation (11.1.5) is not a Wigner distribution due to the appearance of a pole.

11.1.1 Symmetric Solutions: Extended Domain

In this section, we will again extend the domain of integration along the real line of the complex plane z and evaluate the stationary condition in equation (11.0.8).

This will confirm the solutions (11.1.5) for $m \geq 2$. Recall that

$$\phi(r) = \phi(-r) = 2r\Phi(r^2) \qquad \int_{-\infty}^{\infty} dr\phi(r) = 2, \tag{11.1.6}$$

where $\phi(r)$ is an even/symmetric function of the radial eigenvalues r.

Previously we have obtained

$$\fint_0^{\infty} \frac{d\rho'\Phi(\rho')}{\rho - \rho'} = \frac{1}{2r}\fint_{-\infty}^{\infty} \frac{dr'\phi(r')}{r - r'}. \tag{11.1.7}$$

Using equation (11.1.7) in equation (11.0.8), we observe the following

$$\begin{aligned} \fint_0^{\infty} \frac{d\rho'\Phi(\rho')}{\rho - \rho'} &= \frac{\omega^2}{4} - \frac{(m-1)}{2\rho} \\ &= \frac{1}{2r}\fint_{-\infty}^{\infty} \frac{dr'\phi(r')}{r - r'} \\ \Rightarrow \fint_{-\infty}^{\infty} \frac{dr'\phi(r')}{r - r'} &= \frac{\omega^2 r}{2} - \frac{(m-1)}{r}. \end{aligned} \tag{11.1.8}$$

In the case of the extended real domain, we consider a symmetric double cut along the complex plane of z such that $[-r_-, -r_+]$ and $[r_-, r_+]$ where $r_+ > r_- > 0$.

The analytic function that we introduce on the complex plane z for the last line in equation (11.1.8) is as follows

$$\begin{aligned} \mathcal{G}'(r \pm i\epsilon) &= \fint_{-\infty}^{\infty} \frac{dr'\phi(r')}{r - r'} \mp i\pi\phi(r) \\ &= \frac{\omega^2}{2}r - \frac{m-1}{r} \mp i\pi\phi(r). \end{aligned} \tag{11.1.9}$$

The analytic function $\mathcal{G}'(z)$ will adhere to certain special conditions, for example, for large z we require that $\mathcal{G}'(z) \approx 2/z$ and that the function have no

poles when $z \to 0$. The unique function that satisfies the properties of $\mathcal{G}'(r \pm i\epsilon)$ is given by

$$\mathcal{G}'(z) = \frac{\omega^2}{2} z - \frac{a}{z} - \frac{\omega^2}{2z}\sqrt{(z^2 - r_+^2)(z^2 - r_-^2)}, \tag{11.1.10}$$

where in the equation (11.1.10) we denoted $a = m - 1$.

Equation (11.1.10) above satisfies the properties of equation (11.1.9), ergo it is the solution. From the properties of the analytic function $\mathcal{G}'(z)$, we can obtain the equations that fix the boundary conditions of the support

$$\begin{aligned} & r_\pm^4 - \frac{4(2+a)}{\omega^2} r_\pm^2 + \frac{4a^2}{\omega^4} = 0 \\ \Rightarrow \quad & r_\pm^2 = 2\frac{(m+1)}{\omega^2} \pm \frac{4}{\omega^2}\sqrt{m}. \end{aligned} \tag{11.1.11}$$

The derivation of equation (11.1.10) and (11.1.11) are shown in the appendix E.

The boundary limits of the double cut ansatz intervals where our eigenvalue density function is defined is given by

$$\begin{aligned} & r_-^2 \leq r^2 \leq r_+^2 \\ \Rightarrow \quad & 2\frac{(m+1)}{\omega^2} - \frac{4}{\omega^2}\sqrt{m} \leq r^2 \leq 2\frac{(m+1)}{\omega^2} + \frac{4}{\omega^2}\sqrt{m}. \end{aligned} \tag{11.1.12}$$

Having fixed the boundaries of our double cut interval on the complex plane and using equations (11.1.11) and (11.1.12), we can show our eigenvalue density function to be

$$\begin{aligned} \phi(r) &= \frac{\omega^2}{2\pi r}\sqrt{(r_+^2 - r^2)(r^2 - r_-^2)} \qquad r_-^2 \leq r^2 \leq r_+^2 \\ &= \frac{\omega^2}{2\pi}\left[4\frac{(2+a)}{\omega^2} - r^2 - \frac{1}{r^2}\frac{4a^2}{\omega^4}\right]^{1/2} \\ \Rightarrow \phi(r) &= \frac{\omega^2}{2\pi}\left[4\frac{(m+1)}{\omega^2} - r^2 - \frac{1}{r^2}\frac{4(m-1)^2}{\omega^4}\right]^{1/2}. \end{aligned} \tag{11.1.13}$$

Equation (11.1.13) is the generalized eigenvalue density function over the entire extended real number domain for a general number of m complex matrices.

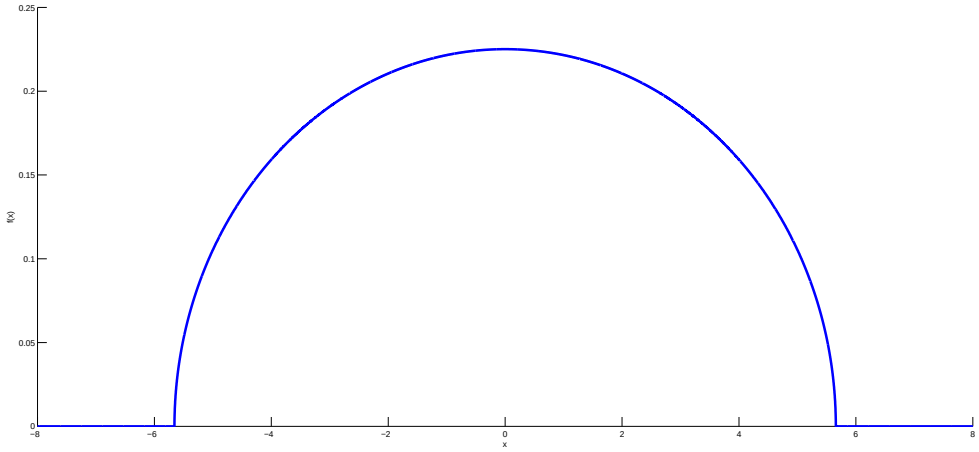

Figure 11.1: Generalized radial sector eigenvalue distribution for $m = 1$.

We consider the $m = 1$ model for complex matrices, this implies that the constant of the previously defined density $a = m - 1$ becomes zero, and we recover

$$\phi(r) = \frac{\omega^2}{2\pi}\left[\frac{8}{\omega^2} - r^2\right]^{1/2} \qquad -\frac{\sqrt{8}}{\omega} \leq r \leq \frac{\sqrt{8}}{\omega}, \qquad (11.1.14)$$

graphically represented in Fig (11.1) for $\omega = 1/2$.

Equation (11.1.13) will be plotted for different values of m as shown in figures 11.2, 11.3 and 11.4 all for $\omega = 1/2$.

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We return to our original variables and obtain

$$\begin{aligned} \Rightarrow 2\sqrt{\rho}\Phi(\rho) &= \frac{\omega^2}{2\pi}\left[\frac{4(m+1)}{\omega^2} - r^2 - \frac{1}{r^2}\frac{4}{\omega^4}(m-1)^2\right]^{1/2} \\ \Rightarrow \Phi(\rho) &= \frac{\omega^2}{4\pi}\left[\frac{1}{\rho}\frac{4}{\omega^2}(m+1) - \frac{1}{\rho^2}\frac{4}{\omega^2}(m-1)^2 - 1\right]^{1/2}. \end{aligned} \qquad (11.1.15)$$

It is clearly evident that equation (11.1.15) is an eigenvalue density function of an ensemble of random $N \times N$ Gaussian matrices whose eigenvalue distribution is no longer governed by the semi-circle law of the Wigner description.

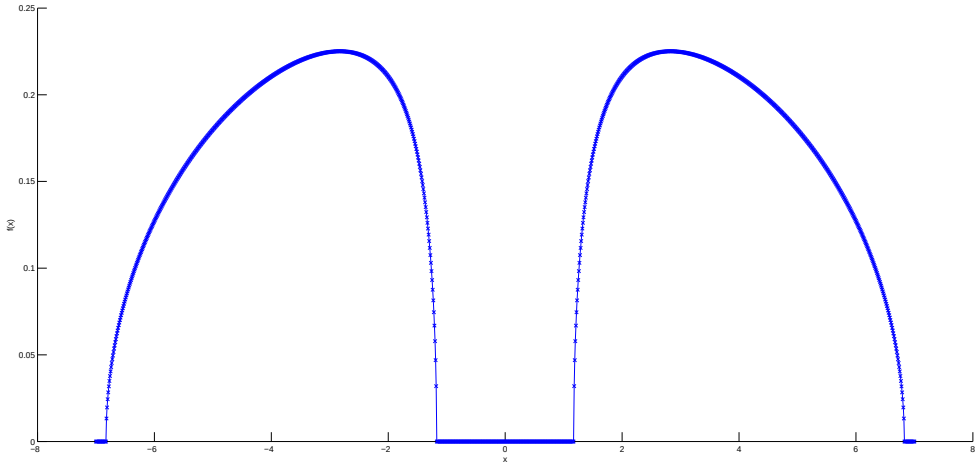

Figure 11.2: Generalized radial sector eigenvalue distribution for $m = 2$.

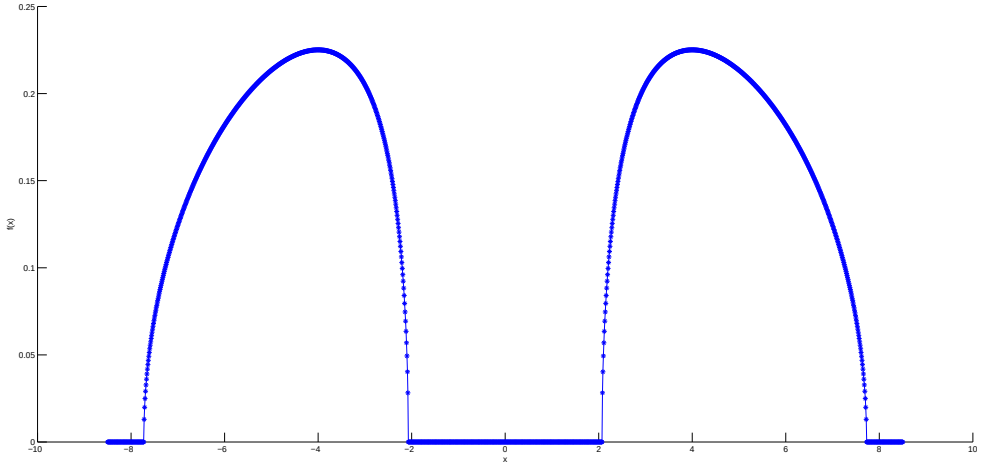

Figure 11.3: Generalized radial sector eigenvalue distribution for $m = 3$.

Chapter 12

Laguerre vs. Hermite

Equations that characterize the generalized saddle point approximations (equation (11.0.8)) for the generalized radially restricted sector coupled to a Gaussian potential can be related to the density of zeros of either Hermite or Laguerre polynomials.

In this chapter we identify a relationship between the stationary condition of the generalized radial sector and Laguerre and Hermite polynomials.

This relationship was previously demonstrated in the collective field description of the matrix models and supersymmetric matrix models in [125] [126].

To establish a relation to the density of zeros of our work and polynomials, we will draw inspiration from [127] [128], whose work is largely based on the results of [129].

The Laguerre polynomial $L_N^{\alpha}(x)$ satisfies a discretized equation, whose zero modes are given by the following equation

$$\sum_{j=1,i\neq j}^{N} \frac{1}{x_i - x_j} = \frac{1}{2}\left(1 - \frac{1+\alpha}{x_i}\right). \tag{12.0.1}$$

Earlier on, we defined the stationary condition in terms of the density variables, given by equation (11.0.8) which we showed to be the following

$$\sum_{j=1,i\neq j}^{N} \frac{1}{\rho_i - \rho_j} = \frac{\omega^2}{4} - \frac{(m-1)}{2\rho_i}. \tag{12.0.2}$$

One can compare equations (12.0.1) and (12.0.2) and establish a correspondence between the two equations when

$$x_i = \frac{\omega^2}{2}\rho_i, \qquad \omega^2 = 2, \qquad \text{and} \qquad \alpha = m - 2. \tag{12.0.3}$$

It follows that the radial density corresponds to the density of zeros of the Laguerre polynomial $L_N^{m-2}(\frac{\omega^2}{2}\rho)$ as $N \to \infty$.

It is also well known [127] [128] [129] that there exists a class of Hermite polynomials $H_{2N}(r)$ whose equation for their zero's is given by the following:

$$\sum_{j=-N, i\neq j}^{N} \frac{1}{x_i - x_j} = x_i. \tag{12.0.4}$$

Using the techniques of semiclassical approximation, we derived the classical condition in equation (6.3.4) for the symmetric treatment of the two hermitian matrix model in polar coordinates which is given by

$$\sum_{i\neq j} \frac{1}{r_i - r_j} = \frac{\omega^2}{2} r_i. \tag{12.0.5}$$

Recall that the above equation is the $m = 1$ case of the more general equation (11.1.8)

$$\sum_{i\neq j} \frac{1}{r_i - r_j} = \frac{\omega^2}{2} r_i - \frac{(m-1)}{r_j}. \tag{12.0.6}$$

Comparing (12.0.4) with (12.0.5) it follows that for $m = 1$, (12.0.5) is satisfied by the zeros of $H_{2N}(\frac{\omega}{\sqrt{2}}r)$ [126].

The Hermite and Laguerre polynomials as they appear in equations (12.0.1) and (12.0.4) are represented in discretized variables such that $x_i > 0$ and the condition: $x_{-j} = -x_j$ is required.

With these requirements in place, the left hand side of equation (12.0.4) can be expressed as

$$\sum_{j=-N,i\neq j}^{N}\frac{1}{r_i-r_j} = \sum_{j=1}^{N}\frac{1}{r_i+r_j}+\sum_{j=1,i\neq j}^{N}\frac{1}{r_i-r_j}$$
$$= \frac{1}{2r_i}+\sum_{j=1,i\neq j}^{N}\frac{2r_i}{r_i^2-r_j^2}, \qquad (12.0.7)$$

this can be used to establish the usual relationship between Hermite and Laguerre polynomials.

Chapter 13

Radial Fermionisation

So far, as we observed in chapter 10, we were able to generalize the Jacobian J for an even number of m $N \times N$ complex matrices, or $2m$ $N \times N$ hermitian matrices. This, in its own right, is a remarkable result.

In this chapter, we wish to construct a fermionic picture for our multimatrix model with a general number of m complex matrices Z_A. The following objectives are in order:

- Introduce the Laplacian operator independent from angular degrees of freedom for the generalized radial sector
- Set up an eigenvalue equation for the system of m complex matrices in the radial sector
- Develop a fermionic description for the generalized radial sector

In the paper by E. Brézin et.al [27], it was shown that summing the planar diagrams in the large N limit in one dimension corresponds to a problem of determining the ground state energy of a Hamiltonian which is also equivalent to solving a fermionic problem with N degrees of freedom coupled to some common potential of the single hermitian matrix model.

Previously, for the single hermitian matrix model, we showed that the kinetic

piece of the Laplacian when acting on a symmetric wavefunction $\Phi(\lambda_i)$

$$\nabla_{S.M} = -\frac{1}{2}\frac{1}{\Delta^2}\frac{\partial}{\partial r_i}\Delta^2\frac{\partial}{\partial r_i}\Phi, \tag{13.0.1}$$

where $\Delta = \prod_{i>j}(r_i - r_j)$ is the Vandermonde determinant, became the sum of single particle Laplacians acting on the anti-symmetric wavefunction $\Psi = \Delta\Phi$ i.e.

$$-\frac{1}{2}\sum_i \frac{\partial^2}{\partial\lambda_i^2}\Psi. \tag{13.0.2}$$

A similar result was accomplished for the single complex matrix $Z = RU$ or equivalently two hermitian matrix model, where the Laplacian of the matrix was defined as

$$\begin{aligned}
\nabla^2_{radial} &= \frac{1}{[\Delta^2_{MR}(r_i^2)\prod_k r_k]}\sum_i \frac{\partial}{\partial r_i}[\Delta^2_{MR}(r_i^2)\prod_k r_k]\frac{\partial}{\partial r_i} \\
&= \frac{1}{\Delta^2_{MR}(r_i^2)}\sum_i \frac{1}{r_i}\frac{\partial}{\partial r_i} r_i \Delta^2_{MR}(r_i^2)\frac{\partial}{\partial r_i} \\
&= \frac{4}{\Delta^2(\rho_i)}\sum_i \frac{\partial}{\partial\rho_i}\rho_i\Delta^2(\rho_i)\frac{\partial}{\partial\rho_i}.
\end{aligned} \tag{13.0.3}$$

In equation (13.0.3) above, we defined $\rho_i = r_i^2$ and the term

$$\begin{aligned}
\Delta^2_{MR}(r_i^2) &= \frac{1}{2}\prod_{i<j}\left(r_i^2 - r_j^2\right)^2 \\
&= \frac{1}{2}\prod_{i<j}\left(\rho_i - \rho_j\right)^2 \\
&= \Delta^2(\rho_i),
\end{aligned} \tag{13.0.4}$$

is the term that generalizes the Vandermonde determinant of the single hermitian matrix model theory.

It was shown that when acting on the anti-symmetric wavefunction $\Psi(\rho_i) = \Delta(\rho_i)\Phi(\rho_i)$, the Laplacian became the sum of 2-dimensional single particle Laplacians.

We now wish to see how our generalized model of m complex matrices would look like if we attempt to rewrite it in a fermionic framework. With this being said, we consider the generalized radial part of the Laplacian in $2m$ dimensions which takes the following form

$$\begin{aligned}
\nabla^2_{Radial} &= \sum_i \frac{1}{\mathcal{V}_R^2(r_i^2)} \frac{1}{r_i^{2m-1}} \frac{\partial}{\partial r_i} r_i^{2m-1} \mathcal{V}_R^2(r_i^2) \frac{\partial}{\partial r_i} \\
&= \frac{1}{\mathcal{V}_R^2(\rho_i)} \sum_i \frac{1}{\rho_i^{1/2(2m-1)}} 2\sqrt{\rho_i} \frac{\partial}{\partial \rho_i} \rho^{1/2(2m-1)} 2\sqrt{\rho_i} \mathcal{V}_R^2(\rho_i) \frac{\partial}{\partial \rho_i} \\
&= \sum_i \frac{4}{\mathcal{V}_R^2(\rho_i)} \frac{1}{\rho_i^{m-1}} \frac{\partial}{\partial \rho_i} \rho_i^m \mathcal{V}_R^2(\rho_i) \frac{\partial}{\partial \rho_i}. \qquad (13.0.5)
\end{aligned}$$

Recall that

$$\begin{aligned}
\mathcal{V}_R(\rho_i) &= \prod_{i>j} \rho_i^{\frac{(m-1)}{2}} \rho_j^{\frac{(m-1)}{2}} (\rho_i - \rho_j) \\
&= \prod_{i>j} r_i^{(m-1)} r_j^{(m-1)} \left(r_i^2 - r_j^2\right) \\
&= \mathcal{V}_R(r_i^2). \qquad (13.0.6)
\end{aligned}$$

Equation (13.0.5) above generalizes the radial part of the Laplacian that we derived earlier for the single complex matrix model Z in equation (5.1.16).

We proceed to set up the Schrödinger equation:

$$\begin{aligned}
\hat{H}\Phi(\rho_i) &= E\Phi(\rho_i) \\
\left(-\frac{1}{2}\nabla^2_{Radial} + K(\rho_i)\right) \Phi(\rho_i) &= E\Phi(\rho_i), \qquad (13.0.7)
\end{aligned}$$

where $K(\rho_i)$ is some common potential of the system that is a function only of the radial eigenvalues of the system.

We introduce the following wavefunctions

$$\begin{aligned}
\Psi(\rho_i) &= \mathcal{V}_R(\rho_i)\Phi(\rho_i) \\
\Rightarrow \Phi(\rho_i) &= \frac{\Psi(\rho_i)}{\mathcal{V}_R(\rho_i)}. \qquad (13.0.8)
\end{aligned}$$

The wave function $\Psi(\rho_i)$ is an anti-symmetric wave function that depends on the radial eigenvalues ρ_i of the system. Using the wavefunction $\Psi(\rho_i)$ we can proceed to obtain a fermionic description of our model therefore we re-write the Laplacian operator of equation (13.0.5) as follows

$$\begin{aligned}
\left(\sum_i \frac{4}{\mathcal{V}_R^2(\rho_i)}\frac{1}{\rho_i^{m-1}}\frac{\partial}{\partial\rho_i}\rho_i^m\mathcal{V}_R^2(\rho_i)\frac{\partial}{\partial\rho_i}\right)\Phi(\rho_i) &= E\Phi(\rho_i) \\
\sum_i \frac{4}{\mathcal{V}_R^2(\rho_i)}\frac{1}{\rho_i^{m-1}}\frac{\partial}{\partial\rho_i}\rho_i^m\mathcal{V}_R^2(\rho_i)\frac{\partial}{\partial\rho_i}\frac{\Psi(\rho_i)}{\mathcal{V}_R(\rho_i)} &= E\frac{\Psi(\rho_i)}{\mathcal{V}_R(\rho_i)} \\
4\sum_i\left(\frac{1}{\rho_i^{m-1}}\frac{1}{\mathcal{V}_{\mathcal{R}}(\rho_i)}\frac{\partial}{\partial\rho_i}\mathcal{V}_R(\rho_i)\right)\rho_i^m\left(\mathcal{V}_R(\rho_i)\frac{\partial}{\partial\rho_i}\frac{1}{\mathcal{V}_R(\rho_i)}\right)\Psi(\rho_i) &= E\Psi(\rho_i).
\end{aligned} \tag{13.0.9}$$

Our attention will now be mainly focused on the left hand side of equation (13.0.9). We first consider the following

$$\begin{aligned}
\left(\mathcal{V}_R(\rho_i)\frac{\partial}{\partial\rho_i}\frac{1}{\mathcal{V}_R(\rho_i)}\right) &= \left(\mathcal{V}_R(\rho_i)\left(\frac{-1}{\mathcal{V}_R^2(\rho_i)}\frac{\partial\mathcal{V}_R(\rho_i)}{\partial\rho_i}\right)\right) \\
&= -\frac{\partial}{\partial\rho_i}\ln\left(\mathcal{V}_R(\rho_i)\right) \\
&= -\frac{\partial}{\partial\rho_i}\ln\left(\prod_{i>j}\rho_i^{(m-1)/2}\rho_j^{(m-1)/2}(\rho_i-\rho_j)\right) \\
&= -\sum_{i>j}\frac{\partial}{\partial\rho_i}\ln\left(\rho_i^{(m-1)/2}\rho_j^{(m-1)/2}(\rho_i-\rho_j)\right) \\
&= -\frac{\partial}{\partial\rho_i}\sum_{k>j}\ln\left(\rho_k^a\rho_j^a(\rho_k-\rho_j)\right) \qquad \text{let} \quad a=\frac{m-1}{2} \\
&= -\sum_{k>j}\frac{1}{\rho_k^a\rho_j^a(\rho_k-\rho_j)}\left[\frac{\partial\rho_k^a}{\partial\rho_i}\rho_j^a(\rho_k-\rho_j)+\rho_k^a\frac{\partial\rho_j^a}{\partial\rho_i}(\rho_k-\rho_j)\right] \\
&- \sum_{k>j}\frac{1}{\rho_k^a\rho_j^a(\rho_k-\rho_j)}\left[\rho_k^a\rho_j^a\frac{\partial}{\partial\rho_i}(\rho_k-\rho_j)\right] \\
&= -\left[\frac{2a}{\rho_i}+\sum_{k>j}\frac{\delta_{ki}-\delta_{ji}}{(\rho_k-\rho_j)}\right] \\
&= -\left[\frac{2a}{\rho_i}+\sum_{i>j}\frac{1}{\rho_i-\rho_j}-\sum_{k>i}\frac{1}{\rho_k-\rho_i}\right] \\
\Rightarrow\left(\mathcal{V}_R(\rho_i)\frac{\partial}{\partial\rho_i}\frac{1}{\mathcal{V}_R(\rho_i)}\right) &= -\left[\frac{2a}{\rho_i}+\sum_{i\neq k}\frac{1}{\rho_i-\rho_k}\right],
\end{aligned} \tag{13.0.10}$$

where in the second last line of equation (13.0.10) we used the following identity

$$\sum_{i>j}\frac{1}{\rho_i-\rho_j}-\sum_{k>i}\frac{1}{\rho_k-\rho_i}=\sum_{i\neq k}\frac{1}{\rho_i-\rho_k}. \tag{13.0.11}$$

It follows that

$$\begin{aligned}\left(\frac{1}{\mathcal{V}_R(\rho_i)}\frac{\partial}{\partial\rho_i}\mathcal{V}_R(\rho_i)\right) &= \frac{1}{\mathcal{V}_R(\rho_i)}\frac{\partial\mathcal{V}_R(\rho_i)}{\partial\rho_i}\\ &= \frac{\partial\ln\left(\mathcal{V}_R(\rho_i)\right)}{\partial\rho_i}\\ \Rightarrow\left(\frac{1}{\mathcal{V}_R(\rho_i)}\frac{\partial}{\partial\rho_i}\mathcal{V}_R(\rho_i)\right) &= \left(\frac{2a}{\rho_i}+\sum_{i\neq j}\frac{1}{\rho_i-\rho_j}\right).\end{aligned} \tag{13.0.12}$$

The two identities from equations (13.0.10) and (13.0.12) are substituted into equation (13.0.9) to simplify the equation [6]. We proceed as follows

$$\begin{aligned}&4\sum_i\left(\frac{1}{\mathcal{V}_{\mathcal{R}}(\rho_i)}\frac{1}{\rho_i^{m-1}}\frac{\partial}{\partial\rho_i}\mathcal{V}_R(\rho_i)\right)\rho_i^m\left(\mathcal{V}_R(\rho_i)\frac{\partial}{\partial\rho_i}\frac{1}{\mathcal{V}_R(\rho_i)}\right)=\\ &\sum_i\frac{4}{\rho_i^b}\left(\frac{\partial}{\partial\rho_i}+\frac{2a}{\rho_i}+\sum_{i\neq k}\frac{1}{\rho_i-\rho_k}\right)\rho_i^m\left(\frac{\partial}{\partial\rho_i}-\frac{2a}{\rho_i}-\sum_{i\neq j}\frac{1}{\rho_i-\rho_j}\right)=\\ &\sum_i\frac{4}{\rho_i^b}\left(\frac{\partial}{\partial\rho_i}\rho_i^m\frac{\partial}{\partial\rho_i}-2a\frac{\partial}{\partial\rho_i}\rho_i^b-\frac{\partial}{\partial\rho_i}\rho_i^m\sum_{i\neq j}\frac{1}{\rho_i-\rho_j}+2a\rho_i^b\frac{\partial}{\partial\rho_i}\right)+\\ &\sum_i\frac{4}{\rho_i^b}\left(-4a^2\rho_i^{b-1}-2a\sum_{i\neq j}\frac{\rho_i^b}{\rho_i-\rho_j}+\sum_{i\neq k}\frac{\rho_i^m}{\rho_i-\rho_k}\frac{\partial}{\partial\rho_i}-\sum_{i\neq k}\frac{2a\rho_i^b}{\rho_i-\rho_k}\right)-\\ &\sum_i\frac{4}{\rho_i^b}\left(\sum_{i\neq k}\sum_{i\neq j}\frac{\rho_i^m}{\rho_i-\rho_k}\frac{1}{\rho_i-\rho_j}\right).\end{aligned} \tag{13.0.13}$$

Certain terms appear in our calculation of equation (13.0.13) which can be simplified, for instance the second term in the third last line can be simplified as follows

[6] We still maintain that $a=(m-1)/2$ and we introduce the notation $b=m-1$.

$$\begin{aligned}\frac{\partial}{\partial\rho_i}\rho_i^m\frac{2a}{\rho_i} &= m\rho_i^{m-1}\frac{\partial\rho_i}{\partial\rho_i}\frac{2a}{\rho_i}-\rho_i^m\frac{1}{\rho_i^2}\frac{\partial\rho_i}{\partial\rho_i}2a+\rho_i^{m-1}2a\frac{\partial}{\partial\rho_i}\\ &= 2ab\rho_i^{b-1}+2a\rho_i^b\frac{\partial}{\partial\rho_i}. \end{aligned} \tag{13.0.14}$$

Similarly, we also simplify the third term in the third last line of equation (13.0.13)

$$\begin{aligned}\frac{\partial}{\partial\rho_i}\rho_i^m\sum_{i\neq j}\frac{1}{\rho_i-\rho_j} &= m\rho_i^{m-1}\frac{\partial\rho_i}{\partial\rho_i}\sum_{i\neq j}\frac{1}{\rho_i-\rho_j}-\rho_i^m\sum_{i\neq j}\frac{1}{(\rho_i-\rho_j)^2}\frac{\partial}{\partial\rho_i}(\rho_i-\rho_j)\\ &+ \sum_{i\neq j}\frac{\rho_i^m}{\rho_i-\rho_j}\frac{\partial}{\partial\rho_i}\\ &= \sum_{i\neq j}\frac{m\rho_i^b}{\rho_i-\rho_j}-\sum_{i\neq j}\frac{\rho_i^m}{(\rho_i-\rho_j)^2}+\sum_{i\neq j}\frac{\rho_i^m}{\rho_i-\rho_j}\frac{\partial}{\partial\rho_i}. \end{aligned} \tag{13.0.15}$$

We therefore substitute the two simplified terms from equation (13.0.14) and equation (13.0.15) into the respective terms that appear in the third last line of equation (13.0.13) to obtain the following

$$\begin{aligned}&\sum_i\frac{4}{\rho_i^b}\left(\frac{\partial}{\partial\rho_i}\rho_i^m\frac{\partial}{\partial\rho_i}-2am\rho_i^{b-1}+2a\rho_i^{b-1}-2a\rho_i^b\frac{\partial}{\partial\rho_i}-m\rho_i^b\sum_{i\neq j}\frac{1}{\rho_i-\rho_j}\right)+\\ &\sum_i\frac{4}{\rho_i^b}\left(\sum_{i\neq j}\frac{\rho_i^m}{(\rho_i-\rho_j)^2}-\sum_{i\neq j}\frac{\rho_i^m}{\rho_i-\rho_j}\frac{\partial}{\partial\rho_i}+2a\rho_i^b\frac{\partial}{\partial\rho_i}-4a^2\rho_i^{b-1}-2a\sum_{i\neq j}\frac{\rho_i^b}{\rho_i-\rho_j}\right)+\\ &\sum_i\frac{4}{\rho_i^b}\left(\sum_{i\neq k}\frac{\rho_i^m}{\rho_i-\rho_k}\frac{\partial}{\partial\rho_i}-2a\sum_{i\neq k}\frac{\rho_i^b}{\rho_i-\rho_k}-\sum_{i\neq k}\sum_{i\neq j}\frac{\rho_i^m}{\rho_i-\rho_k}\frac{1}{\rho_i-\rho_j}\right)\\ =\;& 4\sum_i\left(\frac{1}{\rho_i^b}\frac{\partial}{\partial\rho_i}\rho_i^m\frac{\partial}{\partial\rho_i}-\frac{2am}{\rho_i}+\frac{2a}{\rho_i}-m\sum_{i\neq j}\frac{1}{\rho_i-\rho_j}+\sum_{i\neq j}\frac{\rho_i^{m-b}}{(\rho_i-\rho_j)^2}\right)\\ +\;& 4\sum_i\left(-\frac{4a^2}{\rho_i}-2a\sum_{i\neq j}\frac{1}{\rho_i-\rho_j}-2a\sum_{i\neq k}\frac{1}{\rho_i-\rho_k}-\sum_{i\neq k}\sum_{i\neq j}\frac{\rho_i^{m-b}}{\rho_i-\rho_k}\frac{1}{\rho_i-\rho_j}\right)\\ =\;& 4\sum_i\left(\frac{1}{\rho_i^b}\frac{\partial}{\partial\rho_i}\rho_i^m\frac{\partial}{\partial\rho_i}-\frac{2am}{\rho_i}+\frac{2a}{\rho_i}-\frac{4a^2}{\rho_i}+\sum_{i\neq j}\frac{\rho_i^{m-b}}{(\rho_i-\rho_j)^2}-\sum_{i\neq k}\sum_{i\neq j}\frac{\rho_i^{m-b}}{\rho_i-\rho_k}\frac{1}{\rho_i-\rho_j}\right). \end{aligned} \tag{13.0.16}$$

In the fourth and fifth line of equation (13.0.16), we have made use of the

following identity

$$\sum_{i\neq j}\frac{1}{(\rho_i-\rho_j)}=0, \tag{13.0.17}$$

which allowed us to simplify our problem even further.

Another identity that appears lies on the last line of equation (13.0.16) which was already used in the context of the two hermitian matrices, and will be proved later in appendix D, is the following

$$\begin{aligned}\left(\sum_{i\neq j}\frac{\rho_i^{m-b}}{(\rho_i-\rho_j)^2}-\sum_{i\neq k}\sum_{i\neq j}\frac{\rho_i^{m-b}}{\rho_i-\rho_k}\frac{1}{\rho_i-\rho_j}\right) &= \left(\sum_{i\neq j}\frac{\rho_i}{(\rho_i-\rho_j)^2}-\sum_{i\neq k}\sum_{i\neq j}\frac{\rho_i}{\rho_i-\rho_k}\frac{1}{\rho_i-\rho_j}\right)\\ &= 0. \end{aligned}\tag{13.0.18}$$

We resume our calculation from the last line of equation (13.0.16), putting our identities together we obtain the following

$$\begin{aligned}&4\sum_i\left(\frac{1}{\rho_i^b}\frac{\partial}{\partial\rho_i}\rho_i^m\frac{\partial}{\partial\rho_i}-\frac{2am}{\rho_i}+\frac{2a}{\rho_i}-\frac{4a^2}{\rho_i}+\sum_{i\neq j}\frac{\rho_i^{m-b}}{(\rho_i-\rho_j)^2}-\sum_{i\neq k}\sum_{i\neq j}\frac{\rho_i^{m-b}}{\rho_i-\rho_k}\frac{1}{\rho_i-\rho_j}\right)\\ =\;& 4\sum_i\left(\frac{1}{\rho_i^b}\frac{\partial}{\partial\rho_i}\rho_i^m\frac{\partial}{\partial\rho_i}-\frac{2am}{\rho_i}+\frac{2a}{\rho_i}-\frac{4a^2}{\rho_i}\right)\\ =\;& 4\sum_i\left(\frac{1}{\rho_i^b}\frac{\partial}{\partial\rho_i}\rho_i^m\frac{\partial}{\partial\rho_i}-\frac{8a^2}{\rho_i}\right).\end{aligned}\tag{13.0.19}$$

In the last line of equation (13.0.19) we can substitute our constants $a=(m-1)/2$ and $b=(m-1)$ and in total this Laplacian operator of equation (13.0.9) is now

$$\begin{aligned}&4\sum_i\left(\frac{1}{\mathcal{V}_{\mathcal{R}}(\rho_i)}\frac{1}{\rho_i^{m-1}}\frac{\partial}{\partial\rho_i}\mathcal{V}_R(\rho_i)\right)\rho_i^m\left(\mathcal{V}_R(\rho_i)\frac{\partial}{\partial\rho_i}\frac{1}{\mathcal{V}_R(\rho_i)}\right)=\\ &4\sum_i\left(\frac{1}{\rho_i^b}\frac{\partial}{\partial\rho_i}\rho_i^m\frac{\partial}{\partial\rho_i}-\frac{8a^2}{\rho_i}\right)\\ =\;& 4\sum_i\left(\frac{1}{\rho_i^{m-1}}\frac{\partial}{\partial\rho_i}\rho_i^m\frac{\partial}{\partial\rho_i}-\frac{2}{\rho_i}(m-1)^2\right).\end{aligned}\tag{13.0.20}$$

Equation (13.0.20) is the Laplacian in terms of the radial eigenvalues, we now surmise our calculation by substituting our equation (13.0.20) into equation (13.0.9):

$$
\begin{aligned}
4\sum_i \left(\frac{1}{\mathcal{V}_{\mathcal{R}}(\rho_i)}\frac{1}{\rho_i^{m-1}}\frac{\partial}{\partial\rho_i}\mathcal{V}_R(\rho_i)\right)\rho_i^m\left(\mathcal{V}_R(\rho_i)\frac{\partial}{\partial\rho_i}\frac{1}{\mathcal{V}_R(\rho_i)}\right)\Psi(\rho_i) &= E\Psi(\rho_i) \\
4\sum_i \left(\frac{1}{\rho_i^{m-1}}\frac{\partial}{\partial\rho_i}\rho_i^m\frac{\partial}{\partial\rho_i}-\frac{2}{\rho_i}(m-1)^2\right)\Psi(\rho_i) &= E\Psi(\rho_i)
\end{aligned}
\tag{13.0.21}
$$

The equation (13.0.21) exhibits the Laplacian as a sum of single particle Laplacians in $2m$ dimensions for our matrix model of a general number of m complex matrices, including a potential $\mathcal{K}(\rho_i)$ depending only on the radial eigenvalues, the new Hamiltonian is

$$
\begin{aligned}
\hat{\mathcal{H}}\Psi(\rho_i) &= E\Psi(\rho_i) \\
\left(-\frac{1}{2}\nabla^2_{Radial}+\mathcal{K}(\rho_i)\right)\Psi(\rho_i) &= E\Psi(\rho_i) \\
\left[-\left(2\sum_i\frac{1}{\rho_i^{m-1}}\frac{\partial}{\partial\rho_i}\rho_i^m\frac{\partial}{\partial\rho_i}-\frac{4(m-1)^2}{\rho_i}\right)+\mathcal{K}(\rho_i)\right]\Psi(\rho_i) &= E\Psi(\rho_i)
\end{aligned}
\tag{13.0.22}
$$

The form of the radial Laplacian

$$
\nabla^2_{Radial}=\left(4\sum_i\left(\frac{1}{\rho_i^{m-1}}\frac{\partial}{\partial\rho_i}\rho_i^m\frac{\partial}{\partial\rho_i}-\frac{2}{\rho_i}(m-1)^2\right)\right), \tag{13.0.23}
$$

is new.

One finds that in addition to the radial $2m$-dimensional Laplacian, a new $(1/\rho_i)$ potential is induced which is absent when $m=1$.

Chapter 14

Hamiltonian Density Description: Radial Sector Of An Arbitrary Number Of Complex Matrices

In this chapter we will obtain the collective field theory description for the radial sector of a system of an arbitrary number of m complex matrices Z_A.

This chapter will be structured according to the following objectives:

- Develop the density description picture of the sector with m complex matrices through the definition of density variables associated with the effective Hamiltonian of the collective field theory formalism restricted to the radial degrees of freedom
- Investigate the eigenvalue density function of the generalized radial sector in the collective field theory framework and demonstrate its eigenvalue distribution on a graph for a Gaussian potential

This chapter cannot avoid some redundancy as some equations and respective variables have already been explained in the previous chapters, but reappear as a consequence of their importance in formulating a comprehensive theoretical model of the radially restricted complex matrix model in the collective field theory framework.

For the complex, positive definite hermitian matrix model

$$\sum_A Z_A^\dagger Z_A, \tag{14.0.1}$$

where $A = 1, 2, 3, ..., m$ and m is a positive integer, $m > 0$, we define our collective field theory variables as follows

$$\begin{aligned} \phi_k &= \mathrm{Tr}\left(e^{ik\sum_A Z_A^\dagger Z_A}\right) = \sum_i e^{ikr_i^2} = \sum_i e^{ik\rho_i}; \\ \phi(x) &= \int dx e^{-ikx}\phi_k = \sum_i \delta(x - r_i^2) = \sum_i \delta(\rho - \rho_i) = \phi(\rho), \end{aligned} \tag{14.0.2}$$

where k is an arbitrary real number and the loop variables ϕ_k and $\phi(\rho)$ are Fourier transforms of each other. Above we retained the definition that requires that $x = \rho = r^2$.

We remind ourselves of the effective Hamiltonian that generalizes the description of our complex matrix model which is

$$\begin{aligned} H_{eff}[\rho;[\phi]] &= \frac{1}{2}\int d\rho \int d\rho' \Pi(\rho)\Omega(\rho,\rho';[\phi])\Pi(\rho') \\ &+ \frac{1}{8}\int d\rho \int d\rho' \omega(\rho;[\phi])\Omega^{-1}(\rho,\rho';[\phi])\omega(\rho';[\phi]) + V_A[\rho,[\phi]], \\ &= \hat{H}_K + \frac{1}{8}\int d\rho \int d\rho' \omega(\rho;[\phi])\Omega^{-1}(\rho,\rho';[\phi])\omega(\rho';[\phi]) + V_A[\rho,[\phi]], \end{aligned} \tag{14.0.3}$$

where we have introduced the conjugate momentum of the system:
$\Pi(\rho) = \partial/i\partial\phi(\rho)$.

The following variables have been defined in preceding chapters: the "joining operator" with its Fourier transform is

$$\begin{aligned} \Omega(k,k';[\phi]) &= \sum_A \frac{\partial\phi_k}{\partial Z_A^\dagger}\frac{\partial\phi_{k'}}{\partial Z_A} = -kk'\mathrm{Tr}\left(Z_A^\dagger Z_A e^{i(k+k')\sum_A Z_A^\dagger Z_A}\right), \\ \Omega(\rho,\rho';[\phi]) &= \int \frac{dk'}{2\pi}\int \frac{dk}{2\pi} e^{-ik\rho}e^{-ik'\rho'}\Omega(k,k';[\phi]) = \partial_\rho\partial_{\rho'}\left[\rho\phi(\rho)\delta(\rho-\rho')\right], \end{aligned} \tag{14.0.4}$$

and the splitting operator with its respective Fourier transform

$$\begin{aligned}
\omega(k,k';[\phi]) &= \frac{\partial^2\phi_k}{\partial Z_A^\dagger \partial Z_A} = -k^2\sum_i \rho_i e^{ik\rho_i} - 2ik\sum_{i\neq j}\frac{\rho_j e^{ik\rho_i}}{(\rho_i-\rho_j)} \\
&+ ikN(m-1)\sum_i e^{ik\rho_i} + ik\sum_i e^{ik\rho_i}, \\
\omega(\rho;[\phi]) &= \int \frac{dk}{2\pi} e^{-ik\rho}\omega(k;[\phi]) \\
&= \partial_\rho\left((\rho\phi(\rho))\left[2\fint \frac{d\rho'\phi(\rho')}{(\rho-\rho')} + \frac{N(m-1)}{\rho}\right]\right). \qquad (14.0.5)
\end{aligned}$$

The $\hat{H}_K$ term appearing in equation (14.0.3) is the kinetic piece of the effective Hamiltonian $H_{eff}[\rho;[\phi]]$ whose form can be simplified to obtain the following

$$\begin{aligned}
\hat{H_K} &= \frac{1}{2}\int d\rho \int d\rho' \Pi(\rho)\Omega(\rho,\rho';[\phi])\Pi(\rho') \\
\Rightarrow \hat{H_K} &= \frac{1}{2}\int d\rho\,(\partial_\rho\Pi(\rho))\,[\rho\phi(\rho)]\,(\partial_\rho\Pi(\rho)). \qquad (14.0.6)
\end{aligned}$$

Equation (14.0.6) is substituted back into equation (14.0.3) to obtain the following

$$\begin{aligned}
H_{eff}[\rho;[\phi]] &= \frac{1}{2}\int d\rho\,(\partial_\rho\Pi(\rho))\,[\rho\phi(\rho)]\,(\partial_\rho\Pi(\rho)) \qquad (14.0.7) \\
&+ \frac{1}{8}\int d\rho\int d\rho'\omega(\rho;[\phi])\Omega^{-1}(\rho,\rho';[\phi])\omega(\rho';[\phi]) + V_A[\rho,[\phi]].
\end{aligned}$$

We will now compute the first term appearing in the last line of equation (14.0.7), which represents the repulsion amongst the radial eigenvalues of our system. To accomplish this we consider equations (14.0.4) and (14.0.5) and define

$$F(\rho) = \left[2\fint \frac{d\rho'\phi(\rho')}{(\rho-\rho')} + \frac{N(m-1)}{\rho}\right], \qquad (14.0.8)$$

from equation (14.0.5).

As we observed before, the repulsion term can be shown to be

$$\begin{aligned}
&\frac{1}{8}\int d\rho\int d\rho'\omega(\rho;[\phi])\Omega^{-1}(\rho,\rho';[\phi])\omega(\rho';[\phi]) = \qquad (14.0.9) \\
&= \frac{1}{8}\int d\rho\int d\rho'\,(\rho\phi(\rho)F(\rho))\left(\partial_\rho\partial_{\rho'}\Omega^{-1}(\rho,\rho';[\phi])\right)\rho'\phi(\rho')F(\rho').
\end{aligned}$$

In equation (14.0.9) above we will make use of the identity that was derived in earlier chapters as observed in equation (4.4.17) and equation (9.0.6):

$$\left(\partial_\rho \partial_{\rho'} \Omega^{-1}_{\rho,\rho'}\right) = \frac{\delta(\rho-\rho')}{\rho'\phi(\rho')}. \tag{14.0.10}$$

We substitute equation (14.0.10) above into equation (14.0.9), and this in turn simplifies our equation to become

$$\begin{aligned} & \frac{1}{8}\int d\rho \int d\rho' \left(\rho\phi(\rho)F(\rho)\right)\left(\frac{\delta(\rho-\rho')}{\rho'\phi(\rho')}\right)\rho'\phi(\rho')F(\rho') \\ = \; & \frac{1}{8}\int d\rho \left(\rho\phi(\rho)F^2(\rho)\right) \\ = \; & \frac{1}{8}\int d\rho \left(\rho\phi(\rho)\right)\left[2\not{\int}\frac{d\rho'\phi(\rho')}{(\rho-\rho')}+\frac{N(m-1)}{\rho}\right]^2. \end{aligned} \tag{14.0.11}$$

In the last line we see the explicit N dependence being retained.

We let $\rho = x = r^2$ and $\rho' = y$, using this we can simplify equation (14.0.11) further (for now we ignore the 1/8 factor, we shall multiply this factor back into equation (14.0.11) at the end of the computation):

$$\begin{aligned} & \int d\rho \int d\rho' \left(\rho\phi(\rho)F(\rho)\right)\left(\partial_\rho\partial_{\rho'}\Omega^{-1}(\rho,\rho';[\phi])\right)\rho'\phi(\rho')F(\rho') \\ = \; & \int d\rho \left(\rho\phi(\rho)F^2(\rho)\right) = \int dx \left(x\phi(x)F^2(x)\right) \\ = \; & \int_0^\infty dx \left(x\phi(x)\right)\left[2\not{\int}_0^\infty \frac{dy\phi(y)}{(x-y)}+\frac{N(m-1)}{x}\right]^2 \\ = \; & 4\int_0^\infty dx \left(x\phi(x)\right)\left(\not{\int}_0^\infty \frac{dy\phi(y)}{(x-y)}\right)^2 + 4N(m-1)\int_0^\infty dx\phi(x)\not{\int}_0^\infty \frac{dy\phi(y)}{(x-y)} \\ + \; & N^2(m-1)^2\int_0^\infty dx\left[\frac{\phi(x)}{x}\right]. \end{aligned} \tag{14.0.12}$$

The domain of the radial eigenvalues is now extended over the entire real line, that is, we define the radial eigenvalues r for $r \to \pm\infty$. Therefore we again induce the following definitions: $x = r^2$, $y = s^2$ for $r > 0, s > 0$ and $\Phi(r) \equiv 2r\phi(r^2) = \Phi(-r)$ for $d(r^2) = 2rdr$. Using these definitions (14.0.12) takes the form

$$\begin{aligned}\int d\rho\left(\rho\phi(\rho)F^2(\rho)\right) &= 4\int_0^\infty dr\Phi(r)r^2\left(\fint_0^\infty \frac{ds\Phi(s)}{(r^2-s^2)}\right)^2 \\ &+ 4N(m-1)\int_0^\infty dr\Phi(r)\fint_0^\infty \frac{ds\Phi(s)}{(r^2-s^2)} + N^2(m-1)^2\int_0^\infty dr\left[\frac{\Phi(r)}{r^2}\right].\end{aligned} \tag{14.0.13}$$

Each of the terms appearing in last two lines of equation (14.0.13) are computed individually. We start with the first term of equation (14.0.13)

$$\begin{aligned}&4\int_0^\infty dr\Phi(r)r^2\left(\fint_0^\infty \frac{ds\Phi(s)}{(r^2-s^2)}\right)^2 = 4\int_0^\infty dr\Phi(r)\frac{1}{4}\left(\fint_0^\infty \frac{ds2r\Phi(s)}{(r^2-s^2)}\right)^2 \\ =& \int_0^\infty dr\Phi(r)\left(\fint_0^\infty ds\Phi(s)\frac{2r}{(r^2-s^2)}\right)^2 \\ =& \int_0^\infty dr\Phi(r)\left(\left(\fint_0^\infty \frac{ds\Phi(s)}{(r+s)}+\fint_0^\infty \frac{ds\Phi(s)}{(r-s)}\right)\right)^2 \\ =& \int_0^\infty dr\Phi(r)\left(\left(\fint_{-\infty}^0 \frac{ds\Phi(-s)}{(r-s)}+\fint_0^\infty \frac{ds\Phi(s)}{(r-s)}\right)\right)^2 \\ =& \int_0^\infty dr\Phi(r)\left(\fint_{-\infty}^\infty \frac{ds\Phi(s)}{(r-s)}\right)^2 \\ =& \frac{1}{2}\int_{-\infty}^\infty dr\Phi(r)\left(\fint_{-\infty}^\infty \frac{ds\Phi(s)}{(r-s)}\right)^2\end{aligned} \tag{14.0.14}$$

In the last line of equation (14.0.14) we now see that we are integrating over the entire real line, $r \in (-\infty, \infty)$, not just positive of r. We again make use of the identity that was first seen in the calculation of the single hermitian matrix model, that is,

$$\left(\fint_{-\infty}^\infty \frac{ds\Phi(s)}{(r-s)}\right)^2 = \frac{\pi^2}{3}\Phi^2(r), \tag{14.0.15}$$

substituting this above identity into the last line of equation (14.0.14), it naturally follows that

$$4\int_0^\infty dr\Phi(r)r^2\left(\fint_0^\infty \frac{ds\Phi(s)}{(r^2-s^2)}\right)^2 = \frac{\pi^2}{6}\int_{-\infty}^\infty dr\Phi^3(r). \tag{14.0.16}$$

We now shift our attention to the first term appearing in the second last line of equation (14.0.13), that is

$$
\begin{aligned}
& 4N(m-1)\int_0^\infty dr\Phi(r)\left(\not\int_0^\infty \frac{ds\Phi(s)}{(r^2-s^2)}\right) \\
= \;& 2N(m-1)\int_{-\infty}^\infty dr\Phi(r)\left(\frac{1}{2r}\not\int_0^\infty \frac{ds2r\Phi(s)}{(r^2-s^2)}\right) \\
= \;& N(m-1)\int_{-\infty}^\infty dr\Phi(r)\frac{1}{r}\left[\not\int_{-\infty}^0 \frac{ds\Phi(s)}{r-s}+\not\int_0^\infty \frac{ds\Phi(s)}{r-s}\right] \\
= \;& N(m-1)\int_{-\infty}^\infty dr\Phi(r)\frac{1}{r}\left[\not\int_{-\infty}^\infty \frac{ds\Phi(s)}{r-s}\right] \\
= \;& \frac{1}{2}N(m-1)\int_{-\infty}^\infty dr\Phi(r)\left[\not\int_{-\infty}^\infty ds\Phi(s)\left(\frac{2}{r(r-s)}\right)\right].
\end{aligned}
\tag{14.0.17}
$$

The last line of equation (14.0.17) will be understood to be under the double integral, we simplify the following term

$$
\begin{aligned}
\frac{2}{r(r-s)} &= \frac{1}{r(r-s)}+\frac{1}{r(r-s)}=\frac{1}{r(r-s)}+\frac{1}{s(s-r)} \qquad \text{swop}: r\to s \\
\Rightarrow \frac{2}{r(r-s)} &= -\frac{1}{rs}.
\end{aligned}
\tag{14.0.18}
$$

We will use the result of equation (14.0.18) in the last line of equation (14.0.17), and this gives us the following expression

$$
\begin{aligned}
& \frac{1}{2}N(m-1)\int_{-\infty}^\infty dr\Phi(r)\left[\not\int_{-\infty}^\infty ds\Phi(s)\left(\frac{2}{r(r-s)}\right)\right] \\
= \;& \frac{1}{2}N(m-1)\int_{-\infty}^\infty dr\Phi(r)\left[\not\int_{-\infty}^\infty ds\Phi(s)\left[\frac{-1}{rs}\right]\right] \\
= \;& -\frac{1}{2}N(m-1)\left(\int_{-\infty}^\infty dr\frac{\Phi(r)}{r}\right)\left(\not\int_{-\infty}^\infty ds\frac{\Phi(s)}{s}\right) \\
= \;& 0,
\end{aligned}
\tag{14.0.19}
$$

because by construction $\Phi(r)$ even.

What remains now is the last term that appears in the last line of equation (14.0.13). This term in fact works out to be simpler, such that

$$
N^2(m-1)^2\int_0^\infty dr\left[\frac{\Phi(r)}{r^2}\right] = \frac{1}{2}N^2(m-1)^2\int_{-\infty}^\infty dr\left[\frac{\Phi(r)}{r^2}\right]. \tag{14.0.20}
$$

Now that we have simplified the terms that appear in the last two lines of equation (14.0.13), we can reassemble everything and continue with our calculation of the repulsion amongst eigenvalues,

$$\begin{aligned}
&\int d\rho \int d\rho' \omega(\rho;[\phi])\Omega^{-1}(\rho,\rho';[\phi])\omega(\rho';[\phi]) \\
&= \int d\rho\,(\rho\phi(\rho))\left[2\fint \frac{d\rho'\phi(\rho')}{(\rho-\rho')} + \frac{N(m-1)}{\rho}\right]^2 \\
&= \frac{\pi^2}{6}\int_{-\infty}^{\infty} dr\Phi^3(r) + \frac{1}{2}N^2(m-1)^2\int_{-\infty}^{\infty} dr\left[\frac{\Phi(r)}{r^2}\right]. \qquad (14.0.21)
\end{aligned}$$

Therefore we write the effective Hamiltonian as

$$H_{eff}[\rho;[\phi]] = \frac{1}{2}\int d\rho\,(\partial_\rho\Pi(\rho))\,[\rho\phi(\rho)]\,(\partial_\rho\Pi(\rho)) + \mathcal{V}_A(\rho;[\phi]), \qquad (14.0.22)$$

where

$$\begin{aligned}
\mathcal{V}_A(\rho;[\phi]) &= \frac{1}{8}\left[\int d\rho\int d\rho'\omega(\rho;[\phi])\Omega^{-1}(\rho,\rho';[\phi])\omega(\rho';[\phi])\right] + V_A[\rho,[\phi]] \\
&= \frac{1}{8}\left[\frac{\pi^2}{6}\int_{-\infty}^{\infty} dr\Phi^3(r) + \frac{1}{2}N^2(m-1)^2\int_{-\infty}^{\infty} dr\left[\frac{\Phi(r)}{r^2}\right]\right] + V_A[\rho,[\phi]]. \qquad (14.0.23)
\end{aligned}$$

We consider a Gaussian ensemble of complex matrices with a Gaussian potential $V_A[\rho,[\phi]]$ given by

$$\begin{aligned}
V_A[\rho,[\phi]] &= \frac{\omega^2}{2}\mathrm{Tr}\left(\sum_A Z_A^\dagger Z_A\right) = \frac{\omega^2}{2}\sum_i r_i^2 \\
&= \frac{\omega^2}{2}\int d\rho\phi(\rho)\rho \\
&= \frac{\omega^2}{4}\int_{-\infty}^{\infty} dr\Phi(r)r^2. \qquad (14.0.24)
\end{aligned}$$

As we observed in earlier chapters for partition functions of the single complex matrix and the more general system of m complex matrices, the collective field theory framework is also restricted to potentials that strictly depend on radial

coordinates as has been emphasized. This restriction has been briefly discussed in chapter 10 below equation (10.1.8).

Having defined our Gaussian potential, we can use this in the effective potential defined in equation (14.0.23)

$$\begin{aligned}\mathcal{V}_A(\rho;[\phi]) &= \frac{1}{8}\left[\frac{\pi^2}{6}\int_{-\infty}^{\infty} dr\Phi^3(r) + \frac{1}{2}N^2(m-1)^2\int_{-\infty}^{\infty} dr\left[\frac{\Phi(r)}{r^2}\right]\right] + V_A[\rho,[\phi]] \\ &= \frac{1}{8}\left[\frac{\pi^2}{6}\int_{-\infty}^{\infty} dr\Phi^3(r) + \frac{1}{2}N^2(m-1)^2\int_{-\infty}^{\infty} dr\left[\frac{\Phi(r)}{r^2}\right]\right] \\ &+ \frac{\omega^2}{4}\int_{-\infty}^{\infty} dr\Phi(r)r^2. \\ &= \mathcal{V}_A(r;[\phi]). \end{aligned} \tag{14.0.25}$$

We introduce the Lagrange functional $\mathcal{D}(\eta,[\phi])$, with the Lagrange multiplier η:

$$\begin{aligned}\mathcal{D}(\eta,[\phi]) &= \mathcal{V}_A(r;[\phi]) + \eta\left(2N - \int_{-\infty}^{\infty} dr\Phi(r)\right) \\ &= \frac{\pi^2}{48}\int_{-\infty}^{\infty} dr\Phi^3(r) + \frac{1}{16}N^2(m-1)^2\int_{-\infty}^{\infty} dr\left[\frac{\Phi(r)}{r^2}\right] \\ &+ \frac{\omega^2}{4}\int_{-\infty}^{\infty} dr\Phi(r)r^2 + \eta\left(2N - \int_{-\infty}^{\infty} dr\Phi(r)\right). \end{aligned} \tag{14.0.26}$$

In the large N limit where our calculation is performed, to show the N dependence explicitly in the functional $\mathcal{D}(\eta,[\phi])$, we rescale the variables as follows

$$r \to \sqrt{N}r \quad \Phi(r) \to \sqrt{N}\Phi(r) \quad \eta \to N\eta.$$

As a result of extending the domain of our radial eigenvalues, the following constraint holds true in the large N limit

$$\int_{-\infty}^{\infty} dr\Phi(r) = 2. \tag{14.0.27}$$

We set up the functional $\mathcal{D}(\eta,[\phi])$ which is rescaled and minimized with respect to $(\eta,\Phi(r))$

$$\begin{aligned}\mathcal{D}(\eta,[\phi]) &= N^2\left[\frac{\pi^2}{48}\int_{-\infty}^{\infty}dr\Phi^3(r)+\frac{1}{16}(m-1)^2\int_{-\infty}^{\infty}dr\left[\frac{\Phi(r)}{r^2}\right]\right]\\ &+ N^2\left[\frac{\omega^2}{4}\int_{-\infty}^{\infty}dr\Phi(r)r^2+\eta\left(2-\int_{-\infty}^{\infty}dr\Phi(r)\right)\right]. \quad (14.0.28)\end{aligned}$$

We now vary our functional $\mathcal{D}(\eta,[\phi])$ in equation (14.0.28) with respect to $\Phi(r)$, and this gives us the following minima

$$\begin{aligned}&\frac{\partial}{\partial\Phi(r)}\mathcal{D}(\eta,[\phi])=0\\ \Rightarrow\ &\frac{\partial}{\partial\Phi(r)}\left(\frac{\pi^2}{48}\int_{-\infty}^{\infty}dr\Phi^3(r)+\frac{1}{16}(m-1)^2\int_{-\infty}^{\infty}dr\left[\frac{\Phi(r)}{r^2}\right]\right)\\ +\ &\frac{\partial}{\partial\Phi(r)}\left(\frac{\omega^2}{4}\int_{-\infty}^{\infty}dr\Phi(r)r^2+\eta\left(2-\int_{-\infty}^{\infty}dr\Phi(r)\right)\right)=0\\ \Rightarrow\Phi^2(r) &= \frac{4}{\pi^2}\left(4\eta-\omega^2r^2-\frac{(m-1)^2}{4}\frac{1}{r^2}\right)\\ \Rightarrow\Phi(r) &= \frac{2}{\pi}\left(4\eta-\omega^2r^2-\frac{(m-1)^2}{4}\frac{1}{r^2}\right)^{1/2}. \quad (14.0.29)\end{aligned}$$

In equation (14.0.29) above, when we set $m=1$, we recover the equation of the Wigner semi-circle distribution of eigenvalues.

Using the techniques adopted by [125], we can simplify equation (14.0.29) by solving for the Lagrange multiplier η to obtain the following (normalized) eigenvalue density function

$$\Phi(r)=\frac{2}{\pi}\left(\frac{\omega}{2}(2m-1)-\omega^2r^2-\frac{(m-1)^2}{4}\frac{1}{r^2}\right)^{1/2}\qquad r_-^2\le r^2\le r_+^2, \quad (14.0.30)$$

where

$$r_\pm^2=\frac{(2m-1)}{4\omega}\pm\sqrt{\frac{(2m-1)^2}{16\omega^2}-\frac{(m-1)^2}{4\omega^2}}. \quad (14.0.31)$$

Below, we have shown the graphical representation using equation (14.0.30) for the cases $m=1$, $m=2$ and $m=3$ using $\omega=1/2$. Figure 14.1 clearly shows a distribution that follows the Wigner semi-circle law. In Figure 14.2, the eigenvalues are split symmetrically about the interval where the density function collapses.

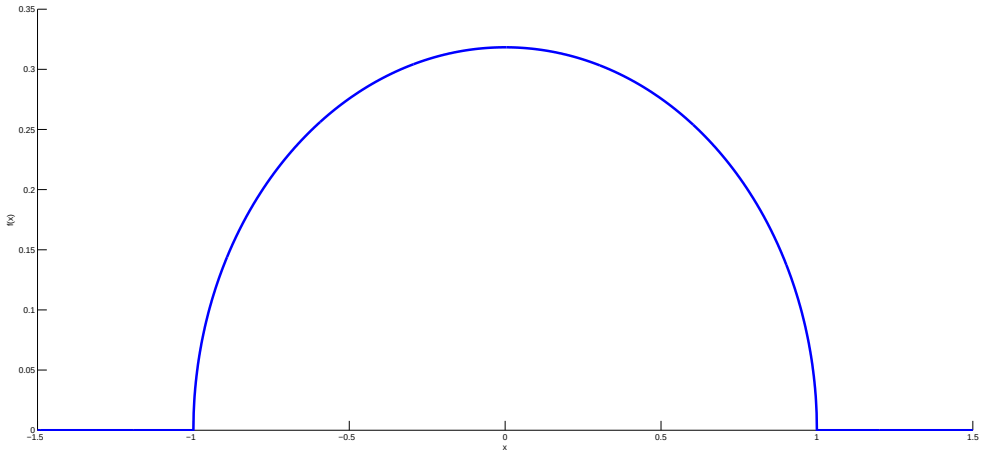

Figure 14.1: Generalized radial sector eigenvalue distribution in collective field theory for $m = 1$.

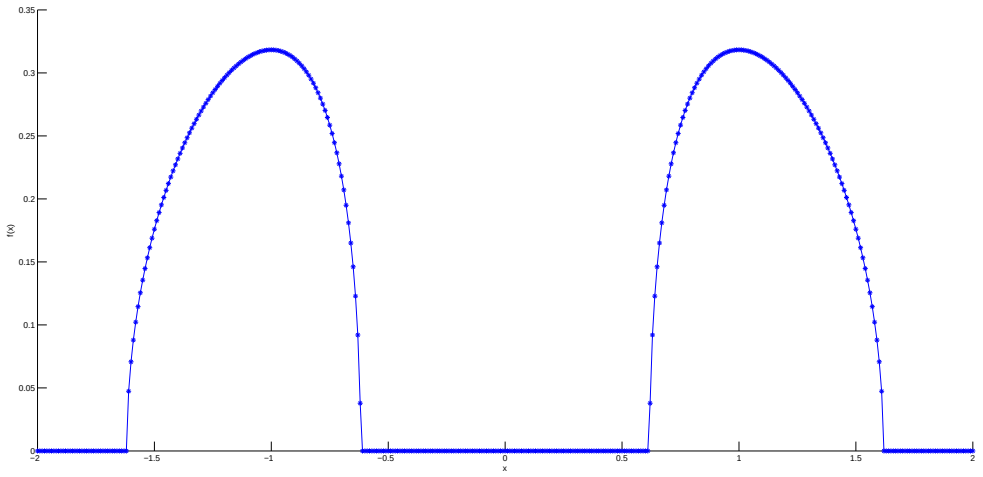

Figure 14.2: Generalized radial sector eigenvalue distribution in collective field theory for $m = 2$.

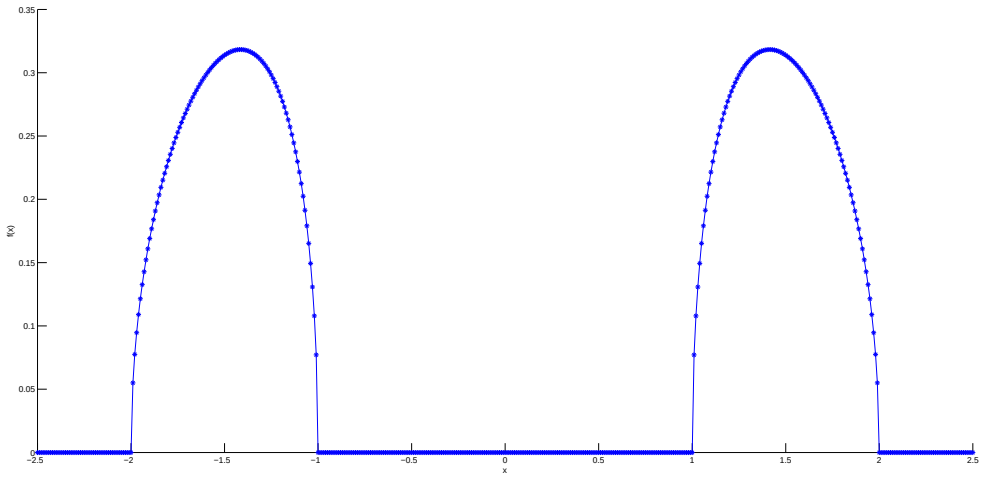

Figure 14.3: Generalized radial sector eigenvalue distribution in collective field theory for $m = 3$.

Chapter 15

Conclusion

The large N dynamics of matrix models studied in our work provided us with unique, non-trivial and remarkable results. The work carried out in this project gave us the opportunity to ask exciting and challenging questions which will hopefully be addressed in future research work. Below we provide conclusions drawn form the work carried out in this research project.

The single hermitian matrix model reviewed in chapter four, provided a blueprint from which we could compare our results and observe whether new unique results can be extrapolated for multimatrix models.

The reader would also recognize that we introduced the Hamiltonian treatment of the single hermitian matrix model by providing a density description through the collective field theory framework. Close inspection reveals that the system of the single hermitian matrix model was treated using two methods:

- through the treatment of the partition function, Z_{SM}, with a potential that is invariant under angular similarity transformations and,

- through the application of the collective field theory formalism (the density description), where gauge invariant state operators that close are identified in the Hamiltonian formalism.

Both treatments resulted in us being able to define a background geometry in the large N limit for the single hermitian matrix model.

In our work, we also investigated the one-dimensional fermionic picture of the single hermitian matrix model. We showed that this fermionic picture becomes a one dimensional system with the Hamiltonian being reduced to a sum of N independent Hamiltonian operators. This system can be identified with N single particle fermions subject to some potential or equivalently as the N degrees of freedom of a Fermi gas.

The challenge we seek(ed) to address in our work was to generalize the template developed for the single hermitian matrix model (study the partition function, develop a fermionic picture, formulate a collective field theory formalism) and use this template to study a large Gaussian ensemble of $2m$ hermitian matrices or m complex matrices in a subsector of the theory which has a natural interpretation as a radially invariant sector.

In chapter five we started off by considering the quantum mechanics of a two hermitian matrix system X_1 and X_2. For this system, a "matrix valued polar coordinate" parameterization was introduced:

$$Z = X_1 + iX_2 \equiv RU.$$

With this parameterization defined in the radial sector of the single complex matrix, we obtained/derived the Jacobian J_{TMM}, defined in positive definite eigenvalues of the radial matrix R, decoupled from any angular degrees of freedom. This Jacobian generalized the Jacobian of the single hermitian matrix model. In both the single hermitian and complex matrix systems, the Vandermonde determinants preserved their anti-symmetric properties, though different eigenvalues.

The Laplacian operator associated with the single complex matrix sector would act on gauge invariant states

$$\mathrm{Tr}\left(\ldots Z^{n_p} Z^{\dagger m_p} \ldots Z^{n_q} Z^{\dagger m_q} \ldots\right).$$

The above gauge invariant state operators depend strictly on the eigenvalues of the radial matrix R and angular degrees of freedom. There is still a need to develop our understanding on how we can construct explicit wavefunctions made

up of these gauge invariant states and observe what type of spectrum would result when acted upon by the Laplacian operator of the single complex matrix model.

It should be noted that the complexity of the number of degrees of freedom increases when one compares the Laplacian of the single hermitian matrix model (equation (4.2.2)/ (A.1.18)) and the single complex matrix model (equation (5.1.16)/(B.1.29)).

For the single complex matrix model, we presented a Gaussian partition function in chapter six, associated with a Gaussian potential with enhanced symmetry. This partition function was investigated in the semi-classical large N limit and the solutions were obtained by studying the analytical functions that satisfy special constraints on the complex plane. This single cut ansatz resulted in a large N background geometry whose eigenvalue distribution function did not satisfy the Wigner semi-circle law.

It was only when the single cut was symmetrically extended to the double cut, and redefining the eigenvalue density function, did we observe a Wigner semi-circle distribution in the large N limit. We should point out that the type of background distribution obtained in the large N limit depends on the potential of the system that one chooses to specify.

The radial fermionic description was investigated in chapter seven for the single complex matrix. In this chapter we made use of the higher dimensional radially restricted Laplacian that we derived in chapter five that depends on the radial eigenvalues $\rho_i = r_i^2$. This system would naturally depend on a potential that has an associated $U(N) \times U(N)$ enhanced symmetry due to the imposed radially restricted sector.

In the fermionic description of the single complex matrix, we identified an "s-state" Schrödinger equation describing N non-interacting (2+1)-dimensional non-relativistic fermions, a new result. This result generalized the single hermitian matrix model fermionic description uniquely.

The density description of the single complex matrix model was developed in chapter eight, where we introduced and defined a set of invariant states that are restricted to the closed subsector and depend strictly on radial eigenvalues. In

this density description, the state operators close under "joining" and "splitting", and were generalized from the collective field theory formalism that we presented in chapter four for the single hermitian matrix model.

We managed to derive the Jacobian in the density description (equation (8.0.22)) and showed that it was the same as the Jacobian derived in equation (5.1.15) in the single complex matrix sector. This was remarkable since we applied two different treatments of the single complex matrix model, in addition, this agreement highlights, demonstrates and emphasizes the usefulness and importance of the collective field theory formalism.

The remarkable dynamics of the single complex matrix model in the density description were further investigated where the Hamiltonian of the system was expressed using the collective field theory formalism. By choosing a Gaussian potential in this radially closed subsector, we managed to derive a Wigner type semi-circle distribution for the eigenavlue density function in the large N limit.

It should be noted with keen interest that the background distribution in the density description of the radial sector of the single complex matrix model depends on how one chooses to specify the radially symmetric wavefunction as can be compared in equations (6.3.12) and (6.3.13). In the radially restricted closed subsector, one only obtains the Wigner type distribution when the wavefunction $\Phi(r)$ is dependent on r and not ρ, and defined on an symmetrically extended double cut interval on the real line of the complex plane.

The biggest motivation for this project was to investigate the dynamics surrounding a large general ensemble of m complex matrices or $2m$ hermitian matrices, using the template developed in chapter four. The treatment of this complex system was carried out in chapter ten where we considered m complex matrices Z_A, $A = 1, \ldots, m$.

The large N limit was investigated for the positive definite, hermitian matrix system

$$\sum_A Z_A^\dagger Z_A,$$

which was identified as a radial coordinate.

Of importance in chapter ten was the identification of correlators in the radially restricted sector, that close in the subsector and this equivalently translated to the closure of the underlying Schwinger-Dyson equation. This identification of correlators mirrors the density description, collective field theory formalism, treatment of this generalized radial sector and restricts us to working in a closed subsector with an enhanced $U(N)^{m+1}$ radial symmetry in higher dimensions depending on the number of Gaussian ensemble of complex matrices we are working with.

The invariant operators $(\Omega_{kk'}, \Omega_{\rho\rho'}, \omega_{kk'}, \omega_{\rho\rho'})$, restricted to the closed sector, generalize the invariant state operators observed for the single complex matrix sector.

Remarkably, for a system of m complex matrices, the Jacobian that describes the change of coordinates from $(Z_A^\dagger; Z_A) \rightarrow (\Phi(\rho); \Phi_k)$, is shown in equation (10.1.43)/(10.1.44). This result generalizes the Jacobian derived for the single complex matrix model, and was derived using Schwinger-Dyson equations in the closed subsector.

In chapter eleven, the stationary condition, equation (11.0.8), for the partition function integrated over a general number of m complex matrices in the radial sector was investigated. It was refreshing to observe that this stationary condition generalized that of the single hermitian and complex matrix model.

The stationary condition, for a general system of m complex matrices, possessed the standard Coulomb potential, in addition, we also identified a new term that represents a logarithmic potential whose strength varies according to $(m-1)$. One can deduce that this new feature is true strictly for $m > 1$, but for $m = 1$, we recover the stationary condition of the single complex matrix.

When the solutions of the stationary condition, equation (11.0.8), were investigated in the large N limit, along the single cut interval (on the real line of the complex plane), we observed an eigenvalue density function that did not satisfy a Wigner type semi-circle distribution for the closed radially restricted subscetor.

In the double cut ansatz, the only time we observe a Wigner type eigenvalue distribution is when $m = 1$, and not for $m > 1$.

Chapter twelve saw us identify the density of zeros of the closed subsector, restricted to radial degrees of freedom, related to the density of zeros of both Laguerre and Hermite polynomials. This was a remarkable observation.

In chapter thirteen, we introduced a radially restricted Laplacian operator in $2m$-dimensions associated with the Jacobian derived in chapter ten for a general gaussian ensemble of m complex matrices. The goal to develop a fermionic description for this system of matrices was accomplished with remarkable, unique and non-trivial results.

This fermionic description of m complex matrices yielded a radially restricted discretized Hamiltonian operator, and we observed a singular potential centered at zero whose strength varies according to $(m-1)^2$. This discretized operator acts on radially anti-symmetric wavefunctions and generalizes the results obtained for the fermionic description of the single complex matrix model.

The penultimate chapter of our work saw us rigorously develop the Hamiltonian description for a Gaussian ensemble of m complex matrices. In this chapter, we introduced a density description, making use of the collective field theory formalism, identifying a closed radially restricted subsector.

As we observed earlier, the invariant state operators identified in this chapter for the system of m complex matrices associated with an enhanced radial symmetry generalize the radial invariant state operators of the single complex matrix density description.

Remarkably, after specializing to a Gaussian potential for system of m complex matrices, we managed to obtain th large N background eigenvalue density for arbitrary m complex matrices.

Although the results derived in our work are non-trivial in their own right, the work carried out in our project opened up a myriad of questions that should be considered for future research. Some of the questions read as follows

- In our work we restricted ourselves (to the radial sector) to study a Gaussian ensemble of m complex matrices or $2m$ hermitian matrices. How do we define and what can we learn from studying a Gaussian ensemble of $(2m+1)$ hermitian matrices, a system that is important in QCD?

- If one pays close attention, the work carried out in our project was in the closed subsector of the radially restricted "free case" with no interactions, that is, $g_{YM} = 0$. What dynamics would be involved if we had to introduce a coupling constant g_{YM} associated with the potential of our system?

- What features can be extrapolated in the limit where supersymmetry could be considered when working with a Gaussian ensemble of complex matrices? Can we obtain exact solutions the way we did in our current project?

- We managed to successfully apply the "matrix valued polar coordinate" parameterization into our matrix model, can we extend this application of the radial degrees of freedom to other sectors like the AdS/CFT and hopefully observe a particular emergence of spacetime geometry.

The work carried out in this project demonstrated the richness of matrix models and their extensive applications. We hope that the methods carried out in our project can be extended and applied further to investigate complex problems associated with string theory.

Appendix A

Single Matrix Model

A.1 Defining The Laplacian Of Single Hermitian Matrix

The objective for this section of appendix A is to determine the Jacobian $\Delta(\lambda)$ appearing in equation (4.1.9) and the Laplacian of the single hermitian matrix model which will be used later in the fermionic description of the system.

We start off by computing the integration measure dM of the $N \times N$ single hermitian matrix model M by diagonalizing the matrix M such that

$$M \to U^{\dagger} \lambda U, \tag{A.1.1}$$

where $U^{\dagger}$ and U are $N \times N$ unitary matrices of the gauge group $U(N)$.

The measure dM will be written in terms of the angular matrix U and the eigenvalue matrix λ, and will satisfy the following

$$\mathrm{Tr}\left(dM^2\right) = \eta_{\mu\nu} dX^{\mu} dX^{\nu}. \tag{A.1.2}$$

The matrix differential element dM is given by

$$
\begin{aligned}
dM &= d(U^\dagger \lambda U) \\
&= dU^\dagger \lambda U + U^\dagger d\lambda U + U^\dagger \lambda dU \\
&= U^\dagger \left(d\lambda + U(dU^\dagger)\lambda + \lambda(dU)U^\dagger \right) U \\
&= U^\dagger \left(d\lambda + \lambda(dU)U^\dagger - (dU)U^\dagger\lambda \right) U \\
&= U^\dagger \left(d\lambda + \left[\lambda, (dU)U^\dagger\right] \right) U. \qquad \text{(A.1.3)}
\end{aligned}
$$

In equation (A.1.3) above, we used the properties of the unitarity of the angular matrix U such that

$$
\begin{aligned}
& UU^\dagger = 1 \\
\Rightarrow \quad & (dU)U^\dagger + U(dU^\dagger) = 0 \\
\Rightarrow \quad & (dU)U^\dagger = -U(dU^\dagger). \qquad \text{(A.1.4)}
\end{aligned}
$$

We now compute the square of the line element, $\mathrm{Tr}\,(dM^2) = \mathrm{Tr}\left(dMdM^\dagger\right)$, therefore we have

$$
\begin{aligned}
\mathrm{Tr}\left(dMdM^\dagger\right) &= \mathrm{Tr}\left(U^\dagger \left(d\lambda + \left[\lambda, (dU)U^\dagger\right] \right) UU^\dagger \left(d\lambda + \left[\lambda, (dU)U^\dagger\right] \right) U \right) \\
&= \mathrm{Tr}\left(d\lambda^2 + d\lambda \left[\lambda, (dU)U^\dagger\right] + \left[\lambda, (dU)U^\dagger\right] d\lambda + \left[\lambda, (dU)U^\dagger\right]^2 \right) \\
&= \mathrm{Tr}\left(d\lambda + (dU)U^\dagger \left[d\lambda, \lambda\right] + \left[d\lambda, \lambda\right](dU)U^\dagger + \left[\lambda, (dU)U^\dagger\right]^2 \right) \\
&= \mathrm{Tr}\left(d\lambda^2 + \left[\lambda, (dU)U^\dagger\right]^2 \right) \\
&= \mathrm{Tr}\left(dM^2\right). \qquad \text{(A.1.5)}
\end{aligned}
$$

In the third last line of equation (A.1.5) we used the property of the cyclicity of the trace for the real elements of the diagonal matrix λ as follows

$$
\begin{aligned}
& \mathrm{Tr}\left(A\left[B, C\right]\right) = \mathrm{Tr}\left(B\left[C, A\right]\right) = \mathrm{Tr}\left(C\left[A, B\right]\right) \\
\Rightarrow \quad & \mathrm{Tr}\left(d\lambda \left[\lambda, (dU)U^\dagger\right]\right) = \mathrm{Tr}\left((dU)U^\dagger \left[d\lambda, \lambda\right]\right) = 0 \\
& \text{since} \qquad \left[d\lambda, \lambda\right] = 0. \qquad \text{(A.1.6)}
\end{aligned}
$$

We introduce indices $(i,j) = 1, 2, ..., N$ to specify the entries in our matrices. Therefore we express the square of the line element dM^2 in terms of the indices (i,j) as follows

$$\begin{aligned}
\mathrm{Tr}\left(dM^2\right) &= \mathrm{Tr}\left(d\lambda^2 + \left[\lambda, (dU)U^\dagger\right]^2\right) \\
&= (d\lambda_i)^2 + \left[\lambda, (dU)U^\dagger\right]_{ij}\left[\lambda, (dU)U^\dagger\right]_{ji} \\
&= (d\lambda_i)^2 \\
&+ \left(\lambda_i\left((dU)U^\dagger\right)_{ij} - \left((dU)U^\dagger\right)_{ij}\lambda_j\right) \times \left(\lambda_j\left((dU)U^\dagger\right)_{ji} - \left((dU)U^\dagger\right)_{ji}\lambda_i\right) \\
&= (d\lambda_i)^2 + (\lambda_i - \lambda_j)\left((dU)U^\dagger\right)_{ij}(\lambda_j - \lambda_i)\left((dU)U^\dagger\right)_{ji} \\
\Rightarrow \mathrm{Tr}\left(dM^2\right) &= \sum_i (d\lambda_i)^2 - \sum_{ij}(\lambda_i - \lambda_j)^2\left((dU)U^\dagger\right)_{ij}\left((dU)U^\dagger\right)_{ji}. \qquad \text{(A.1.7)}
\end{aligned}$$

Below we introduce and define Lie algebra valued matrices dP which are anti-hermitian. These are expressed as follows

$$\begin{aligned}
(dP)_{ij} &= \left((dU)U^\dagger\right)_{ij} \\
\text{where} \qquad dP &= -dP^\dagger. \qquad \text{(A.1.8)}
\end{aligned}$$

Using (A.1.8) above, we can rewrite $\mathrm{Tr}\left(dM^2\right)$ as follows

$$\begin{aligned}
\mathrm{Tr}\left(dM^2\right) &= \sum_i (d\lambda_i)^2 - \sum_{ij}(\lambda_i - \lambda_j)^2\left((dU)U^\dagger\right)_{ij}\left((dU)U^\dagger\right)_{ji} \qquad \text{(A.1.9)} \\
&= \sum_i (d\lambda_i)^2 - \sum_{ij}(\lambda_i - \lambda_j)^2 (dP)_{ij}(dP)_{ji} \\
&= \sum_i (d\lambda_i)^2 + \sum_{i\neq j}(\lambda_i - \lambda_j)^2 (dP)_{ij}(dP^*)_{ij} \\
&= \sum_i (d\lambda_i)^2 + \frac{1}{2}\sum_{i\neq j}(\lambda_i - \lambda_j)^2\left[(dP)_{ij}(dP^*)_{ij} + (dP^*)_{ij}(dP)_{ij}\right] \\
&= \sum_i (d\lambda_i)^2 + \sum_{i>j}(\lambda_i - \lambda_j)^2\left[(dP)_{ij}(dP^*)_{ij} + (dP^*)_{ij}(dP)_{ij}\right].
\end{aligned}$$

From equation (A.1.9), we pull out the metric, which is given by

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (\lambda_i - \lambda_j)^2 & 0 \\ 0 & 0 & (\lambda_i - \lambda_j)^2 \end{pmatrix}. \tag{A.1.10}$$

We define the Jacobian as

$$\begin{aligned} J &= \sqrt{\det \eta_{\mu\nu}} = \sqrt{\prod_{i>j} (\lambda_i - \lambda_j)^4} \\ &= \prod_{i<j} (\lambda_i - \lambda_j)^2 . \end{aligned} \tag{A.1.11}$$

We see the very well known Vandermonde determinant

$$\Delta(\lambda) = \prod_{i<j} (\lambda_i - \lambda_j) , \tag{A.1.12}$$

that appears in equation (4.1.9).

Under integration, the measure of dM translates into the following

$$\begin{aligned} \int dM &\rightarrow \int d\lambda \int dP J \\ &= \int d\lambda \int (dU) U^\dagger J. \end{aligned} \tag{A.1.13}$$

In the second line of equation (A.1.13), we see the unitary measure dU which represents the normalized Haar measure of the unitary gauge group $U(N)$. The unitary measure dU decouples from the rest of the integral and we can integrate it out since it is invariant under the similarity transformation

$$\begin{aligned} & (dU) U^\dagger \rightarrow (dU) \\ \Rightarrow & \int dU = 1. \end{aligned} \tag{A.1.14}$$

In total our measure becomes

$$\begin{aligned}\int dM &= \int d\lambda \int (dU) U^\dagger J \\ &= \int d\lambda \int (dU) J \\ &= \int d\lambda J \\ \Rightarrow \int dM &= \int d\lambda \prod_{i>j} (\lambda_i - \lambda_j)^2. \end{aligned} \tag{A.1.15}$$

We will now derive the Laplacian operator ∇^2 of the Hamiltonian

$$H_{SM} = -\frac{1}{2}\nabla^2 + W(M), \tag{A.1.16}$$

as defined in equation (4.2.2) for some eigenvalue dependent potential $W(M)$. We first start off with the definition of the Laplacian

$$\nabla^2 = \frac{1}{\sqrt{\det \eta_{\mu\nu}}} \frac{\partial}{\partial X^\nu} \eta^{\nu\mu} \sqrt{\det \eta_{\mu\nu}} \frac{\partial}{\partial X^\mu}. \tag{A.1.17}$$

In equation (A.1.17), we substitute in the Jacobian (A.1.11) to obtain the following

$$\begin{aligned}\nabla^2 &= \frac{1}{\sqrt{\det \eta_{\mu\nu}}} \frac{\partial}{\partial X^\mu} \sqrt{\det \eta_{\mu\nu}} \eta^{\mu\nu} \frac{\partial}{\partial X^\nu} \\ &= \frac{1}{\prod_{i<j}(\lambda_i - \lambda_j)^2} \frac{\partial}{\partial \lambda_i} \left[\prod_{i<j} (\lambda_i - \lambda_j)^2 \right] \frac{\partial}{\partial \lambda_i} \\ &+ \frac{1}{\prod_{i<j}(\lambda_i - \lambda_j)^2} \frac{\partial}{\partial P_{ij}} \left[\prod_{i<j} (\lambda_i - \lambda_j)^2 \sum_{i>j} \frac{1}{(\lambda_i - \lambda_j)^2} \right] \frac{\partial}{\partial P^*_{ij}} \\ &+ \frac{1}{\prod_{i<j}(\lambda_i - \lambda_j)^2} \frac{\partial}{\partial P^*_{ij}} \left[\prod_{i<j} (\lambda_i - \lambda_j)^2 \sum_{i>j} \frac{1}{(\lambda_i - \lambda_j)^2} \right] \frac{\partial}{\partial P_{ij}} \\ &= \frac{1}{\prod_{i<j}(\lambda_i - \lambda_j)^2} \frac{\partial}{\partial \lambda_i} \left[\prod_{i<j} (\lambda_i - \lambda_j)^2 \right] \frac{\partial}{\partial \lambda_i} \\ &- \frac{2}{\prod_{i<j}(\lambda_i - \lambda_j)^2} \frac{\partial}{\partial P_{ij}} \left[\prod_{i<j} (\lambda_i - \lambda_j)^2 \sum_{i>j} \frac{1}{(\lambda_i - \lambda_j)^2} \right] \frac{\partial}{\partial P_{ji}} \\ &= \frac{1}{\Delta^2} \frac{\partial}{\partial \lambda_i} \left[\Delta^2 \right] \frac{\partial}{\partial \lambda_i} - \sum_{i \neq j} \frac{1}{(\lambda_i - \lambda_j)^2} \frac{\partial}{\partial P_{ij}} \frac{\partial}{\partial P_{ji}}. \end{aligned} \tag{A.1.18}$$

The Laplacian in equation (A.1.18) is substituted into our Hamiltonian operator H_{SM} to obtain the following

$$\begin{aligned} \hat{H}_{SM} &= -\frac{1}{2}\nabla^2 + W(M) \qquad \text{(A.1.19)} \\ &= -\frac{1}{2}\frac{1}{\Delta^2}\frac{\partial}{\partial r_i}\left[\Delta^2\right]\frac{\partial}{\partial r_i} + \frac{1}{2}\sum_{i\neq j}\frac{1}{(r_i - r_j)^2}\frac{\partial}{\partial P_{ij}}\frac{\partial}{\partial P_{ji}} + W(M). \end{aligned}$$

In equation (A.1.19), the first term appearing in last line is the kinetic piece of the Laplacian that is strictly dependent on the eigenvalues λ_i of the hermitian matrix M. The second term in equation (A.1.19) represents the "angular" component of the Laplacian that preserves the angular degrees of freedom [130]. This angular component is the non-singlet $SU(N)$ angular momentum degrees of freedom of the Hamiltonian operator.

We will restrict ourseles to potentials that only depend on the eigenvalues λ_i. In addition we note that when the angular component of the kinetic piece acts on ground state wavefunctions in the singlet sector of $SU(N)$, we must get zero. The singlet wavefunctions ϕ will be independent of the angular variables V and $V^\dagger$ and should be symmetric wavefunctions of the eigenvalues λ_i of M.

A.2 Single Cut Solution Of The Single Matrix Model

In this section of the appendix, we will derive the solution appearing in equation (4.1.19). From the saddle point equation taken in the continuum limit in equation (4.1.11) we introduce the following analytic function

$$G(\lambda) = \oint_{-\alpha}^{\alpha} d\xi \frac{\rho(\xi)}{\lambda - \xi}, \qquad \text{(A.2.1)}$$

defined on the support $(-\alpha, \alpha)$ on the complex plane of λ.

Due to the constraints that must be satisfied by the analytic function appearing in equation (A.2.1), there exists a unique function that provides a solution

to equation (A.2.1) on the single cut along the real axis of the complex plane for the specific potential $W(M)$ defined in equation (4.1.5). This analytic function is given by

$$G(z) = \frac{1}{2}z + 2gz^3 - \left(qz^2 + d\right)\sqrt{z^2 - \alpha^2}, \tag{A.2.2}$$

where the variables q and d are constants that we are required to solve for.

For both equation (A.2.1) and (A.2.2), we will require that as $z \to \pm\infty$ then $G(z) \approx 1/z$. By taking z to be very large, we expand the right hand side of equation (A.2.1) as follows

$$\begin{aligned} G(z) &= \oint_{-\alpha}^{\alpha} d\xi \frac{\rho(\xi)}{z - \xi} \\ &= \frac{1}{z}\oint_{-\alpha}^{\alpha} d\xi\rho(\xi)\left(1 - \frac{\xi}{z}\right)^{-1} \\ &= \frac{1}{z} + \frac{1}{z^2}\int_{-\alpha}^{\alpha} d\xi\rho(\xi)\xi + \frac{1}{z^3}\oint_{-\alpha}^{\alpha} d\xi\rho(\xi)\xi^2 + \ldots\ldots \end{aligned} \tag{A.2.3}$$

In equation (A.2.3), we used the fact that the density function $\rho(\xi)$ is normalized as follows

$$\int_{-\alpha}^{\alpha} d\xi\rho(\xi) = 1. \tag{A.2.4}$$

We now consider equation (A.2.2) for large z

$$\begin{aligned} G(z) &= \frac{1}{2}z + 2gz^3 - \left(qz^2 + d\right)\sqrt{z^2 - \alpha^2} \\ &= z\left(\frac{1}{2} + \frac{q\alpha^2}{2} - d\right) + \frac{1}{z}\left(\frac{q\alpha^4}{8} + \frac{d\alpha^2}{2}\right) + z^3\left(2g - q\right) \\ &+ \frac{1}{z^3}\left(\frac{1}{16}q\alpha^6 + \frac{1}{8}d\alpha^4\right) + \ldots\ldots \end{aligned} \tag{A.2.5}$$

Using equations (A.2.3) and (A.2.5), we compare the coefficients of the variable z, and we can solve the follwing equations

$$\text{z}: \qquad \left(\frac{1}{2} + \frac{q\alpha^2}{2} - d\right) = 0, \tag{A.2.6}$$

$$\mathrm{z}^3 : \qquad (2g - q) = 0, \tag{A.2.7}$$

and

$$\frac{1}{\mathrm{z}} : \qquad \left(\frac{1}{8}q\alpha^4 + \frac{1}{2}d\alpha^2\right) = 1. \tag{A.2.8}$$

Solving for q and d in terms of α, we obtain the following result

$$G(z) = \frac{1}{z} + 2gz^3 - \left(2gz^2 + \frac{1}{2} + g\alpha^2\right)\sqrt{z^2 - \alpha^2}. \tag{A.2.9}$$

From (A.2.8)

$$\begin{aligned} &\left(\frac{1}{8}q\alpha^4 + \frac{1}{2}d\alpha^2\right) = 1 \\ \Rightarrow \qquad &3g\alpha^4 + \alpha^2 - 4 = 0. \end{aligned} \tag{A.2.10}$$

In equation (A.2.10), the physically acceptable roots of the equation are given by

$$\alpha^2 = \frac{1}{6}\left(-\frac{1}{g} + \sqrt{\frac{1}{g^2} + \frac{48}{g}}\right). \tag{A.2.11}$$

In equation (A.2.11) when $g = 0$, we can obtain the solution for the boundary values of the single cut interval $(-\alpha, \alpha)$ along the real axis of the complex plane. The boundary values are

$$\alpha = \pm 2. \tag{A.2.12}$$

A.3 Discretized Hamiltonian Operator

For this section of appendix A, we derive the Hamiltonian operator H_{SM} that appears in equation (4.2.8).

We will consider the Hamiltonian $\hat{H}$ operator that only depends on the eigenvalues r_i of the single hermitian matrix M, defined as follows

$$
\begin{aligned}
H_{SM} &= -\frac{1}{2}\nabla^2 + W(M) \\
&= -\frac{1}{2}\mathrm{Tr}\left(\frac{\partial^2}{\partial M^2}\right) + W(M) \\
&= -\frac{1}{2}\left(\frac{1}{\Delta^2}\frac{\partial}{\partial r_i}\Delta^2\frac{\partial}{\partial r_i}\right) + W(r_i), \qquad \text{(A.3.1)}
\end{aligned}
$$

defined for the Vandermonde determinant

$$
\Delta = \prod_{i>j}(r_i - r_j)\,. \qquad \text{(A.3.2)}
$$

The Laplacian operator, earlier derived in equation (A.1.18), is decoupled from any angular degrees of freedom as it appears in equation (A.3.1).

Our Hamiltonian operator acts on symmetric singlet sector ground state wavefunctions $\phi(r_i)$, giving us the ground state energy E_G, this is defined by the eigenvalue equation

$$
H_{SM}\phi(r_i) = E_G\phi(r_i). \qquad \text{(A.3.3)}
$$

The ground state wavefunctions $\phi(r_i)$ are functions of the eigenvalues r_i of the matrix M and preserve the symmetry of the $U(N)$ gauge group.

We introduce an anti-symmetric wavefunction $\Omega(r_i)$ defined as follows

$$
\Omega(r_i) = \Delta\phi(r_i). \qquad \text{(A.3.4)}
$$

By introducing the anti-symmetric wavefunction $\Omega(r_i)$, we are implicitly redefining our problem to be that of N fermions in the common potential $W(r_i)$.

We consider the eigenvalue equation (A.3.3) above and consider the kinetic operator

$$
\begin{aligned}
-\frac{1}{2}\left(\frac{1}{\Delta^2}\frac{\partial}{\partial r_i}\Delta^2\frac{\partial}{\partial r_i}\right)\phi(r_i) &= E_G\phi(r_i) \\
-\frac{1}{2}\left(\frac{1}{\Delta^2}\frac{\partial}{\partial r_i}\Delta^2\frac{\partial}{\partial r_i}\right)\frac{\Omega(r_i)}{\Delta} &= E_G\frac{\Omega(r_i)}{\Delta} \\
-\left(\frac{1}{\Delta}\frac{\partial}{\partial r_i}\Delta\right)\left(\Delta\frac{\partial}{\partial r_i}\frac{1}{\Delta}\right)\Omega(r_i) &= 2E_G\Omega(r_i), \qquad \text{(A.3.5)}
\end{aligned}
$$

which now acts on the anti-symmetric wavefunction $\Omega(r_i)$.

To solve the eigenvalue equation appearing in equation (A.3.5), we start with the left hand side of the equation where we consider the following

$$\begin{aligned}
\left(\frac{1}{\Delta}\frac{\partial\Delta}{\partial r_i}\right) &= \frac{\partial}{\partial r_i}\ln\Delta \\
&= \frac{\partial}{\partial r_i}\sum_{k<j}\ln\left(r_k - r_j\right) \\
&= \sum_{k<j}\frac{1}{(r_k - r_j)}\left(\frac{\partial r_k}{\partial r_i} - \frac{\partial r_k}{\partial r_i}\right) \\
&= \sum_{i<j}\frac{1}{(r_i - r_j)} - \sum_{k<i}\frac{1}{(r_k - r_i)} \\
\Rightarrow \left(\frac{1}{\Delta}\frac{\partial\Delta}{\partial r_i}\right) &= \sum_{k\neq i}\frac{1}{(r_i - r_k)}. \qquad \text{(A.3.6)}
\end{aligned}$$

Similarly, we can define the following identity from the left side of equation (A.3.5)

$$\begin{aligned}
\Delta\left(\frac{\partial}{\partial r_i}\frac{1}{\Delta}\right) &= -\Delta\frac{1}{\Delta^2}\frac{\partial\Delta}{\partial r_i} = -\frac{1}{\Delta}\frac{\partial\Delta}{\partial r_i} = -\frac{\partial}{\partial r_i}\ln\Delta \\
\Rightarrow \Delta\left(\frac{\partial}{\partial r_i}\frac{1}{\Delta}\right) &= -\sum_{j\neq i}\frac{1}{(r_i - r_j)}. \qquad \text{(A.3.7)}
\end{aligned}$$

Using the two preceding equations we can combine them in order to simplify the operator on the left hand side of equation (A.3.5)

$$
\begin{aligned}
& \left(\frac{1}{\Delta}\frac{\partial \Delta}{\partial r_i}\right)\left(\Delta\frac{\partial}{\partial r_i}\frac{1}{\Delta}\right) \\
= & \left(\frac{\partial}{\partial r_i}+\sum_{k\neq i}\frac{1}{(r_i-r_k)}\right)\left(\frac{\partial}{\partial r_i}-\sum_{j\neq i}\frac{1}{(r_i-r_j)}\right) \\
= & \frac{\partial^2}{\partial r_i^2}-\sum_{i\neq j}\frac{\partial}{\partial r_i}\sum_{j\neq i}\frac{1}{(r_i-r_j)}+\sum_{k\neq i}\frac{1}{(r_i-r_k)}\frac{\partial}{\partial r_i} \\
- & \left(\sum_{k\neq i}\sum_{j\neq i}\frac{1}{(r_i-r_k)}\frac{1}{(r_i-r_j)}\right) \\
= & \frac{\partial^2}{\partial r_i^2}-\sum_{i\neq j}\frac{\partial}{\partial r_i}\sum_{j\neq i}\frac{1}{(r_i-r_j)}+\sum_{k\neq i}\frac{1}{(r_i-r_k)}\frac{\partial}{\partial r_i} \\
- & \left(\sum_{k\neq i\neq j}\frac{1}{(r_i-r_k)}\frac{1}{(r_i-r_j)}+\sum_{i\neq j}\frac{1}{(r_i-r_j)^2}\right) \\
= & \left(\frac{\partial^2}{\partial r_i^2}-\sum_{k\neq i\neq j}\frac{1}{(r_i-r_k)}\frac{1}{(r_i-r_j)}-\sum_{i\neq j}\frac{1}{(r_i-r_j)^2}\right) \\
+ & \left(\sum_{k\neq i}\frac{1}{(r_i-r_k)}\frac{\partial}{\partial r_i}-\sum_{i\neq j}\frac{\partial}{\partial r_i}\sum_{j\neq i}\frac{1}{(r_i-r_j)}\right). \qquad \text{(A.3.8)}
\end{aligned}
$$

We simplify the last two terms that appear in equation (A.3.8). We first start with the two terms appearing in the last line of equation (A.3.8)

$$
\begin{aligned}
& \left(\sum_{k\neq i}\frac{1}{(r_i-r_k)}\frac{\partial}{\partial r_i}-\frac{\partial}{\partial r_i}\sum_{j\neq i}\frac{1}{(r_i-r_j)}\right) \\
= & \sum_{k\neq i}\frac{1}{(r_i-r_k)}\frac{\partial}{\partial r_i}-\left(-\sum_{j\neq i}\frac{1}{(r_i-r_j)^2}\frac{\partial}{\partial r_i}(r_i-r_j)+\sum_{j\neq i}\frac{1}{(r_i-r_j)}\frac{\partial}{\partial r_i}\right) \\
= & \sum_{k\neq i}\frac{1}{(r_i-r_k)}\frac{\partial}{\partial r_i}-\sum_{j\neq i}\frac{1}{(r_i-r_j)}\frac{\partial}{\partial r_i}+\sum_{j\neq i}\frac{1}{(r_i-r_j)^2} \\
& \Rightarrow \left(\sum_{k\neq i}\frac{1}{(r_i-r_k)}\frac{\partial}{\partial r_i}-\frac{\partial}{\partial r_i}\sum_{j\neq i}\frac{1}{(r_i-r_j)}\right)=\sum_{j\neq i}\frac{1}{(r_i-r_j)^2}. \qquad \text{(A.3.9)}
\end{aligned}
$$

If we substitute equation (A.3.9) into (A.3.8), we can establish the following result

$$\begin{aligned}
&\left(\frac{1}{\Delta}\frac{\partial\Delta}{\partial r_i}\right)\left(\Delta\frac{\partial}{\partial r_i}\frac{1}{\Delta}\right)\\
= &\left(\frac{\partial^2}{\partial r_i^2}-\sum_{k\neq i\neq j}\frac{1}{(r_i-r_k)}\frac{1}{(r_i-r_j)}\right). \qquad \text{(A.3.10)}
\end{aligned}$$

The second term appearing in the second line of equation (A.3.10) can be shown to be zero. To demonstrate this, we will assign values to the indices such that $i, j, k = 1, 2, 3$ and use these to show how this identity becomes zero. For these 3 eigenvalues, their contribution to equation (A.3.10) is:

$$\begin{aligned}
&\frac{1}{(r_1-r_2)(r_1-r_3)}+\frac{1}{(r_2-r_1)(r_2-r_3)}+\frac{1}{(r_3-r_1)(r_3-r_2)}\\
= &\frac{1}{(r_1-r_2)}\left(\frac{1}{(r_1-r_3)}-\frac{1}{(r_2-r_3)}\right)+\frac{1}{(r_1-r_2)(r_1-r_3)}\\
= &\frac{1}{(r_1-r_2)}\left(\frac{r_2-r_3-r_1+r_3}{(r_1-r_3)(r_3-r_3)}\right)+\frac{1}{(r_3-r_1)(r_3-r_2)}\\
= &\frac{1}{(r_1-r_2)}\frac{(r_1-r_2)}{(r_1-r_3)(r_2-r_3)}+\frac{1}{(r_3-r_1)(r_3-r_2)}\\
= &-\frac{1}{(r_1-r_2)}\frac{(r_1-r_2)}{(r_1-r_3)(r_2-r_3)}+\frac{1}{(r_3-r_1)(r_3-r_2)}\\
= &-\frac{1}{(r_1-r_3)(r_2-r_3)}+\frac{1}{(r_3-r_1)(r_3-r_2)}=0.
\end{aligned}$$

Having shown that in equation (A.3.10) the second term in the last line is reduced zero, the kinetic operator is simply

$$-\sum_i\left(\frac{1}{\Delta}\frac{\partial\Delta}{\partial r_i}\right)\left(\Delta\left(\frac{\partial}{\partial r_i}\frac{1}{\Delta}\right)\right)=-\sum_i\frac{\partial^2}{\partial r_i^2}. \qquad \text{(A.3.11)}$$

The kinetic part of the Hamiltonian H_{SM} in equation (A.3.1) has been reduced to an operator with N degrees of freedom. The potential $W(r_i)$ explicitly represented in terms of eigenvalues r_i is given by

$$W(r_i)=\frac{1}{2}\sum_i r_i^2+\frac{g}{N}\sum_i r_i^4, \qquad \text{(A.3.12)}$$

our Hamiltonian H_{SM} acting on the anti-symmetric wavefunction $\Omega(r_i)$ can be expressed using eigenvalue representation r_i:

$$
\begin{aligned}
H_{SM} &= -\frac{1}{2}\sum_i \frac{\partial^2}{\partial r_i^2} + \frac{1}{2}\sum_i r_i^2 + \frac{g}{N}\sum_i r_i^4 \\
&= \sum_i \Lambda_i. \qquad \text{(A.3.13)}
\end{aligned}
$$

Therefore using the eigenvalue of equation (A.3.5), we can have our operator acting on the anti-symmetric wavefunction $\Omega(r_i)$ i.e. a sum of decoupled single particle Hamiltonians.

Appendix B

Radial Sector: Polar Matrix Model

B.1 Jacobian In Radial Sector: Parameterized Two Matrix Model

In this appendix we derive the Jacobian, J_{TMM}, seen in equation (5.1.14) and the anti-symmetric term in equation (5.1.15), also the Laplacian appearing in equation (5.1.16), all equations resulting from the parameterization of the two hermitian matrix model. To accomplish this we first consider the two $N \times N$ hermitian matrices X_1 and X_2 used to construct the complex matrix Z, expressed as a product of a radial part and an angular part [112].

We will consider

$$\begin{aligned} Z &= X_1 + iX_2 \\ &\equiv \Gamma\Omega. \end{aligned} \tag{B.1.1}$$

The matrix Γ is an $N \times N$ radial matrix representing the radial degrees of freedom of the matrix Z and the $N \times N$ unitary matrix Ω represents the angular degrees of freedom of the matrix Z.

We first diagonalize the radial matrix Γ to obtain the eigenvalue representa-

tion, this we carry out as follows

$$\Gamma = U^{\dagger}\gamma_D U. \tag{B.1.2}$$

In equation (B.1.2) above we have used the matrices U and $U^{\dagger}$ which are unitary matrices of the gauge group $U(N)$. The diagonal matrix γ_D is an $N \times N$ matrix consisting of the eigenvalues of Γ. Using equation (B.1.2) we have that

$$\begin{aligned} Z &= \Gamma\Omega \\ &= U^{\dagger}\gamma_D U\Omega \\ &= U^{\dagger}\gamma_D \left(U\Omega\right), \end{aligned} \tag{B.1.3}$$

and similarly

$$\begin{aligned} Z^{\dagger} &= \Omega^{\dagger}\Gamma^{\dagger} \\ &= \left(\Omega^{\dagger}U^{\dagger}\right)\gamma_D U. \end{aligned} \tag{B.1.4}$$

In order for us to obtain the Jacobian for the two matrix model, we first need to compute the following

$$\mathrm{Tr}\left(dZdZ^{\dagger}\right) = \eta_{\mu\nu}dX^{\mu}dX^{\nu}. \tag{B.1.5}$$

Equation (B.1.5) is the equivalent of the squared line element in $(3+1)$-dimensional Minkowski space, above we have presented it using the matrix dictionary.

We first start by computing the matrix differential element dZ, which can be viewed as an arbitrary distance along the matrix Z. This we carry out as follows

$$\begin{aligned} dZ &= d\left(\Gamma\Omega\right) = d\left(U^{\dagger}\gamma U\Omega\right) \\ &= dU^{\dagger}\gamma U\Omega + U^{\dagger}d\gamma U\Omega + U^{\dagger}\gamma dU\Omega + U^{\dagger}\gamma Ud\Omega \\ &= U^{\dagger}\left(UdU^{\dagger}\gamma + d\gamma + \gamma dUU^{\dagger} + \gamma Ud\Omega\Omega^{\dagger}U^{\dagger}\right)U\Omega \\ &= U^{\dagger}\left(d\gamma + \gamma dUU^{\dagger} - dUU^{\dagger}\gamma + \gamma Ud\Omega\Omega^{\dagger}U^{\dagger}\right)U\Omega \\ \Rightarrow dZ &= U^{\dagger}\left(d\gamma + \left[\gamma, dUU^{\dagger}\right] + \gamma Ud\Omega\Omega^{\dagger}U^{\dagger}\right)U\Omega. \end{aligned} \tag{B.1.6}$$

The same is done for $Z^\dagger$, that is, we compute its matrix differential element

$$dZ^\dagger = \Omega^\dagger U^\dagger \left(d\gamma + \left[\gamma, dUU^\dagger \right] - Ud\Omega\Omega^\dagger U^\dagger \gamma \right) U. \tag{B.1.7}$$

In both equations (B.1.6) and (B.1.7) we used the property of the unitarity of the matrices U and $U^\dagger$ such that

$$U^\dagger_{\alpha\beta} U_{\beta\varsigma} = \delta_{\alpha\varsigma} \quad \text{and} \quad dUU^\dagger = -UdU^\dagger. \tag{B.1.8}$$

In equation (B.1.6) and (B.1.7) we introduce anti-hermitian Lie algebra valued differential matrices defined as follows

$$\begin{aligned} dS &= dUU^\dagger \\ \Rightarrow dS &= -dS^\dagger, \end{aligned} \tag{B.1.9}$$

and

$$\begin{aligned} dN &= Ud\Omega\Omega^\dagger U^\dagger \\ \Rightarrow dN &= -dN^\dagger. \end{aligned} \tag{B.1.10}$$

Using the Lie valued anti-hermitian differentials in equations (B.1.9) and (B.1.10), these are substituted into equations (B.1.6) and (B.1.7) for the definitions of the matrix differential dZ and $dZ^\dagger$, and the following is obtained

$$dZ = U^\dagger \left(d\gamma + [\gamma, dS] + \gamma dN \right) U\Omega, \tag{B.1.11}$$

and

$$dZ^\dagger = \Omega^\dagger U^\dagger \left(d\gamma + [\gamma, dS] - dN\gamma \right) U. \tag{B.1.12}$$

Using equations (B.1.11) and (B.1.12), we can now obtain $\mathrm{Tr}\left(dZdZ^\dagger \right)$, this we carry out as follows

$$
\begin{aligned}
\mathrm{Tr}\left(dZdZ^{\dagger}\right) &= \mathrm{Tr}\left(dZ^{\dagger}dZ\right) \\
&= \mathrm{Tr}\left(U^{\dagger}\left(d\gamma+[\gamma,dS]+\gamma dN\right)U\Omega\Omega^{\dagger}U^{\dagger}\left(d\gamma+[\gamma,dS]-dN\gamma\right)U\right) \\
&= \mathrm{Tr}\left(\left(d\gamma+[\gamma,dS]+\gamma dN\right)\left(d\gamma+[\gamma,dS]-dN\gamma\right)\right) \\
&= \mathrm{Tr}\left(d\gamma^{2}+d\gamma\left[\gamma,dS\right]-d\gamma dN\gamma+\left[\gamma,dS\right]d\gamma+\left[\gamma,dS\right]^{2}\right) \\
&- \mathrm{Tr}\left(\left[\gamma,dS\right]dN\gamma+\gamma dNd\gamma+\gamma dN\left[\gamma,dS\right]-\gamma dN^{2}\gamma\right) \\
&= \mathrm{Tr}\left(d\gamma^{2}+\left[\gamma,dS\right]\left[\gamma,dS\right]-\gamma^{2}dN^{2}+2d\gamma\left[\gamma,dS\right]\right) \\
&+ \mathrm{Tr}\left(dN\left[d\gamma,\gamma\right]+\left[\gamma,dS\right]\left[\gamma,dN\right]\right). \qquad \text{(B.1.13)}
\end{aligned}
$$

As pointed out before:

$$
\mathrm{Tr}\left(d\gamma\left[d\gamma,\gamma\right]\right)=0. \qquad \text{(B.1.14)}
$$

Taking into consideration the constraints observed in equation (B.1.14), we express equation (B.1.13) as follows

$$
\begin{aligned}
\mathrm{Tr}\left(dZdZ^{\dagger}\right) &= \mathrm{Tr}\left(d\gamma^{2}+\left[\gamma,dS\right]\left[\gamma,dS\right]-\gamma^{2}dN^{2}\right) \\
&+ \mathrm{Tr}\left(\left[\gamma,dS\right]\left[\gamma,dN\right]\right). \qquad \text{(B.1.15)}
\end{aligned}
$$

Equation (B.1.15) will be expressed in terms of indices. Doing this will allow us to obtain the entries to matrix of the metric $\eta_{\mu\nu}$ appearing in equation (B.1.5).

Equation (B.1.15) is expressed in index notation as follows

$$
\begin{aligned}
\mathrm{Tr}\left(dZdZ^{\dagger}\right) &= \mathrm{Tr}\left(d\gamma^{2}\right)+\mathrm{Tr}\left(\left[\gamma,dS\right]\left[\gamma,dS\right]\right)-\mathrm{Tr}\left(\gamma^{2}dN^{2}\right) \\
&+ \mathrm{Tr}\left(\left[\gamma,dS\right]\left[\gamma,dN\right]\right) \\
&= \sum_{i}\left(d\gamma_{i}^{2}\right)+\sum_{ij}\left[\gamma,dS\right]_{ij}\left[\gamma,dS\right]_{ji}-\sum_{ij}\gamma_{i}^{2}dN_{ij}dN_{ji} \\
&+ \sum_{ij}\left[\gamma,dS\right]_{ij}\left[\gamma,dN\right]_{ji}. \qquad \text{(B.1.16)}
\end{aligned}
$$

The terms associated with angular degrees of freedom appearing in the commutators can be reduced to their eigenvalue representation, therefore we focus our attention on the commutators appearing in equation (B.1.16)

$$\begin{aligned}\sum_{ij}\left[\gamma, dS\right]_{ij}\left[\gamma, dS\right]_{ji} &= \sum_{ij}\left(\gamma_i dS_{ij} - dS_{ij}\gamma_j\right)\left(\gamma_j dS_{ji} - dS_{ji}\gamma_i\right) \\ &= \sum_{ij}\left(\gamma_i - \gamma_j\right) dS_{ij}\left(\gamma_j - \gamma_i\right) dS_{ji} \\ &= \sum_{ij}\left(\gamma_i - \gamma_j\right)\left(\gamma_j - \gamma_i\right) dS_{ij} dS_{ji} \\ &= -\sum_{ij}\left(\gamma_i - \gamma_j\right)^2 dS_{ij} dS_{ji}, \end{aligned} \tag{B.1.17}$$

and in a similar fashion we treat the following commutator as the latter

$$\begin{aligned}\sum_{ij}\left[\gamma, dS\right]_{ij}\left[\gamma, dN\right]_{ji} &= \sum_{ij}\left(\gamma_i dS_{ij} - dS_{ij}\gamma_j\right)\left(\gamma_j dN_{ji} - dN_{ji}\gamma_i\right) \\ &= \sum_{ij}\left(\gamma_i - \gamma_j\right) dS_{ij}\left(\gamma_j - \gamma_i\right) dN_{ji} \\ &= \sum_{ij}\left(\gamma_i - \gamma_j\right)\left(\gamma_j - \gamma_i\right) dS_{ij} dN_{ji} \\ &= -\sum_{ij}\left(\gamma_i - \gamma_j\right)^2 dS_{ij} dN_{ji}. \end{aligned} \tag{B.1.18}$$

We substitute equation (B.1.17) and (B.1.18) into equation (B.1.16) to introduce the eigenvalue representation into the expression of $\mathrm{Tr}\left(dZdZ^\dagger\right)$, this will give the following

$$\begin{aligned}\mathrm{Tr}\left(dZdZ^\dagger\right) &= \sum_i\left(d\gamma_i^2\right) + \sum_{ij}\left[\gamma, dS\right]_{ij}\left[\gamma, dS\right]_{ji} - \sum_{ij}\gamma_i^2 dN_{ij} dN_{ji} \\ &+ \sum_{ij}\left[\gamma, dS\right]_{ij}\left[\gamma, dN\right]_{ji} \\ &= \sum_i\left(d\gamma_i^2\right) - \sum_{ij}\left(\gamma_i - \gamma_j\right)^2 dS_{ij} dS_{ji} \\ &- \frac{1}{2}\sum_{ij}\left(\gamma_i - \gamma_j\right)^2\left[dS_{ij} dN_{ji} + dN_{ij} dS_{ji}\right] \\ &- \frac{1}{2}\sum_{ij}\left[\gamma_i^2 + \gamma_j^2\right] dN_{ij} dN_{ji}. \end{aligned} \tag{B.1.19}$$

In order for us to take into account all of the components that are being summed over, we will separate the terms that appear in equation (B.1.19) into

summations that require $\sum_{i=j}$ and those that are defined by the summation $\sum_{i\neq j}$, this will leave us with the following

$$\begin{aligned}
\mathrm{Tr}\left(dZdZ^{\dagger}\right) &= \sum_{i}\left(d\gamma_i^2\right)+\sum_{i}\gamma_i^2 dN_{ii}dN_{ii}^{*}+\sum_{i\neq j}\left(\gamma_i-\gamma_j\right)^2 dS_{ij}dS_{ij}^{*} \\
&+ \frac{1}{2}\sum_{i\neq j}\left(\gamma_i-\gamma_j\right)^2\left[dS_{ij}dN_{ij}^{*}+dN_{ij}dS_{ij}^{*}\right] \\
&+ \frac{1}{2}\sum_{i\neq j}\left[\gamma_i^2+\gamma_j^2\right]dN_{ij}dN_{ij}^{*} \\
&= \sum_{i}\left(d\gamma_i^2\right)+\sum_{i}\gamma_i^2 dN_{ii}dN_{ii}^{*}+2\sum_{i>j}\left(\gamma_i-\gamma_j\right)^2 dS_{ij}dS_{ij}^{*} \\
&+ \sum_{i>j}\left(\gamma_i-\gamma_j\right)^2\left[dS_{ij}dN_{ij}^{*}+dN_{ij}dS_{ij}^{*}\right] \\
&+ \sum_{i>j}\left[\gamma_i^2+\gamma_j^2\right]dN_{ij}dN_{ij}^{*}. \qquad \text{(B.1.20)}
\end{aligned}$$

From the last three lines of equation (B.1.20) we can pull out the coefficients of the anti-hermitian matrix differential that will be used to define the matrix of the metric $\eta_{\mu\nu}$ as it appears in equation (B.1.5). The respective variables that will define the rows and columns of entries to the matrix defining metric will be $\left(d\gamma_i, dN_{ii}, dS_{ij(i<j)}, dS^{*}_{ij(i<j)}, dN_{ij(i<j)}, dN^{*}_{ij(i<j)}\right)$.

Using equation (B.1.5) and the coefficients of the matrix differentials appearing in equation (B.1.20) we can obtain the matrix of the metric $\eta_{\mu\nu}$ which defines $\mathrm{Tr}\left(dZdZ^{\dagger}\right)$ and this is given by the following

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \gamma_i^2 & 0 & 0 & 0 & 0 \\
0 & 0 & (\gamma_i-\gamma_j)^2 & \frac{1}{2}(\gamma_i-\gamma_j)^2 & 0 & 0 \\
0 & 0 & \frac{1}{2}(\gamma_i-\gamma_j)^2 & \frac{1}{2}(\gamma_i^2+\gamma_j^2) & 0 & 0 \\
0 & 0 & 0 & 0 & (\gamma_i-\gamma_j)^2 & \frac{1}{2}(\gamma_i-\gamma_j)^2 \\
0 & 0 & 0 & 0 & \frac{1}{2}(\gamma_i-\gamma_j)^2 & \frac{1}{2}(\gamma_i^2+\gamma_j^2)
\end{pmatrix}. \qquad \text{(B.1.21)}$$

Using the matrix in equation (B.1.21), we compute the determinant of the matrix above which we deduce as follows

$$
\begin{aligned}
\det \eta_{\mu\nu} &= \prod_i \gamma_i^2 \prod_{i>j} \left[\frac{1}{2} \left(\gamma_i^2 + \gamma_j^2 \right) \left(\gamma_i - \gamma_j \right)^2 - \frac{1}{4} \left(\gamma_i - \gamma_j \right)^2 \left(\gamma_i - \gamma_j \right)^2 \right] \\
&\times \left[\frac{1}{2} \left(\gamma_i^2 + \gamma_j^2 \right) \left(\gamma_i - \gamma_j \right)^2 - \frac{1}{4} \left(\gamma_i - \gamma_j \right)^2 \left(\gamma_i - \gamma_j \right)^2 \right] \\
&= \prod_i \gamma_i^2 \prod_{i>j} \left[\frac{1}{4} \left(\gamma_i^2 + \gamma_j^2 \right)^2 \left(\gamma_i - \gamma_j \right)^4 - \frac{1}{4} \left(\gamma_i^2 + \gamma_j^2 \right) \left(\gamma_i - \gamma_j \right)^6 + \frac{1}{16} \left(\gamma_i - \gamma_j \right)^8 \right] \\
&= \prod_i \gamma_i^2 \prod_{i>j} \frac{1}{16} \left(\gamma_i - \gamma_j \right)^4 \left[4 \left(\gamma_i^2 + \gamma_j^2 \right)^2 - 4 \left(\gamma_i^2 + \gamma_j^2 \right) \left(\gamma_i - \gamma_j \right)^2 + \left(\gamma_i - \gamma_j \right)^4 \right] \\
&= \prod_i \gamma_i^2 \prod_{i>j} \frac{1}{16} \left(\gamma_i - \gamma_j \right)^4 \left[2 \left(\gamma_i^2 + \gamma_j^2 \right) - \left(\gamma_i - \gamma_j \right)^2 \right]^2 \\
&= \prod_i \gamma_i^2 \prod_{i>j} \frac{1}{16} \left(\gamma_i - \gamma_j \right)^4 \left[2\gamma_i^2 + 2\gamma_j^2 - \left(\gamma_i^2 - \gamma_i\gamma_j - \gamma_j^2 \right) \right]^2 \\
&= \prod_i \gamma_i^2 \prod_{i>j} \frac{1}{16} \left(\gamma_i - \gamma_j \right)^4 \left[\gamma_i^2 + \gamma_i\gamma_j + \gamma_j^2 \right]^2 \\
&= \prod_i \gamma_i^2 \prod_{i>j} \frac{1}{16} \left(\gamma_i - \gamma_j \right)^4 \left[\gamma_i + \gamma_j \right]^4 \\
&= \prod_i \gamma_i^2 \prod_{i>j} \frac{1}{16} \left[\left(\gamma_i - \gamma_j \right) \left(\gamma_i + \gamma_j \right) \right]^4 \\
&= \prod_i \gamma_i^2 \prod_{i>j} \frac{1}{16} \left[\gamma_i^2 - \gamma_i\gamma_j + \gamma_i\gamma_j - \gamma_j^2 \right]^4 \\
\Rightarrow \det \eta_{\mu\nu} &= \prod_i \gamma_i^2 \left[\prod_{i>j} \frac{1}{4} \left(\gamma_i^2 - \gamma_j^2 \right)^2 \right]^2 . \qquad \text{(B.1.22)}
\end{aligned}
$$

The determinant of the metric $\eta_{\mu\nu}$ appearing in equation (B.1.22) is then obtained from a two hermitian matrix model that was parameterized using polar coordinates. In equation (B.1.22) we introduce a short hand notation

$$
\Delta_{TMM}^2 = \prod_{i>j} \frac{1}{4} \left(\gamma_i^2 - \gamma_j^2 \right)^2 , \qquad \text{(B.1.23)}
$$

therefore we can rewrite equation (B.1.22) using equation (B.1.23) as follows

$$
\begin{aligned}
\det \eta_{\mu\nu} &= \prod_i \gamma_i^2 \left[\prod_{i>j} \frac{1}{4} \left(\gamma_i^2 - \gamma_j^2 \right)^2 \right]^2 \\
&= \prod_i \gamma_i^2 \left[\Delta_{TMM}^2 \right]^2 . \qquad \text{(B.1.24)}
\end{aligned}
$$

To obtain the Jacobian J_{TMM} for two hermitian matrix model in polar coordinates, we will need to take the square root of equation (B.1.24), this will be given by

$$\begin{aligned} J_{TMM} &= \sqrt{\det \eta_{\mu\nu}} \\ &= \sqrt{\prod_i \gamma_i^2 \left[\Delta_{TMM}^2\right]^2} \\ &= \prod_i \gamma_i \Delta_{TMM}^2 \\ \Rightarrow J_{TMM} &= \prod_i \gamma_i \prod_{i>j} \frac{1}{4}\left(\gamma_i^2 - \gamma_j^2\right)^2. \end{aligned} \tag{B.1.25}$$

The Jacobian J_{TMM} is the Jacobian obtained from a measure defined by the polar matrices $\mathrm{Tr}\left(dZdZ^\dagger\right)$ to a set of new variables whose measure is defined by the Lie algebra anti-hermitian matrix differentials (dr, dS, dN) such that

$$dZdZ^\dagger \;\rightarrow\; J_{TMM} dr dM dN, \tag{B.1.26}$$

where J_{TMM} is given by equation (B.1.25).

We will now proceed to compute the Laplacian of the two matrix model in polar coordinates. We start off with the definition of the Laplacian

$$\nabla^2 = \frac{1}{\sqrt{\det g_{\mu\nu}}} \frac{\partial}{\partial X^\beta} g^{\nu\mu} \sqrt{\det g_{\mu\nu}} \frac{\partial}{\partial X^\alpha}. \tag{B.1.27}$$

Evidently, for us to apply equation (B.1.27) above, the inverse of the metric g_{AB} in equation (B.1.21) is necessary, and is given by the following

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\gamma_i^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4(\gamma_i^2+\gamma_j^2)}{(\gamma_i-\gamma_j)^2(\gamma_i+\gamma_j)^2} & \frac{-2}{(\gamma_i+\gamma_j)^2} & 0 & 0 \\ 0 & 0 & \frac{-2}{(\gamma_i+\gamma_j)^2} & \frac{4}{(\gamma_i+\gamma_j)^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{4(\gamma_i^2+\gamma_j^2)}{(\gamma_i-\gamma_j)^2(\gamma_i+\gamma_j)^2} & \frac{-2}{(\gamma_i+\gamma_j)^2} \\ 0 & 0 & 0 & 0 & \frac{-2}{(\gamma_i+\gamma_j)^2} & \frac{4}{(\gamma_i+\gamma_j)^2} \end{pmatrix}. \tag{B.1.28}$$

The Laplacian is obtained by using equation (B.1.28) in combination with equation (B.1.27) to obtain the following

$$\begin{aligned}
\nabla^2 &= \frac{1}{\prod_k \gamma_k}\frac{1}{\Delta^2_{TMM}}\frac{\partial}{\partial\gamma_i}\left[\prod_k \gamma_k \Delta^2_{TMM}\right]\frac{\partial}{\partial\gamma_i} + \left\{\frac{1}{\gamma_i^2}\frac{\partial}{\partial N_{ii}}\frac{\partial}{\partial N^*_{ii}}\right\} \\
&+ \sum_{i\neq j}\frac{2(\gamma_i^2+\gamma_j^2)}{(\gamma_i^2-\gamma_j^2)^2}\frac{\partial}{\partial S_{ij}}\frac{\partial}{\partial S^*_{ij}} - \sum_{i\neq j}\frac{2}{(\gamma_i+\gamma_j)^2}\left\{\frac{\partial}{\partial S_{ij}}\frac{\partial}{\partial N^*_{ij}} + \frac{\partial}{\partial N_{ij}}\frac{\partial}{\partial S^*_{ij}}\right\} \\
&+ \sum_{i\neq j}\frac{4}{(\gamma_i+\gamma_j)^2}\frac{\partial}{\partial N_{ij}}\frac{\partial}{\partial N^*_{ij}} \\
&= \frac{1}{\prod_k \gamma_k}\left\{\frac{\partial}{\partial\gamma_i}\prod_k \gamma_k\right\}\frac{\partial}{\partial\gamma_i} + \frac{1}{\Delta^2_{TMM}}\left\{\frac{\partial}{\partial\gamma_i}\Delta^2_{TMM}\right\}\frac{\partial}{\partial\gamma_i} + \frac{\partial}{\partial\gamma_i}\frac{\partial}{\partial\gamma_i} \\
&+ \left\{\frac{1}{\gamma_i^2}\frac{\partial}{\partial N_{ii}}\frac{\partial}{\partial N^*_{ii}}\right\} + \sum_{i\neq j}\frac{2(\gamma_i^2+\gamma_j^2)}{(\gamma_i^2-\gamma_j^2)^2}\frac{\partial}{\partial S_{ij}}\frac{\partial}{\partial S^*_{ij}} \qquad \text{(B.1.29)} \\
&- \sum_{i\neq j}\frac{2}{(\gamma_i+\gamma_j)^2}\left\{\frac{\partial}{\partial S_{ij}}\frac{\partial}{\partial N^*_{ij}} + \frac{\partial}{\partial N_{ij}}\frac{\partial}{\partial S^*_{ij}}\right\} + \sum_{i\neq j}\frac{4}{(\gamma_i+\gamma_j)^2}\frac{\partial}{\partial N_{ij}}\frac{\partial}{\partial N^*_{ij}}.
\end{aligned}$$

This is the result shown in equation (5.1.16).

Appendix C

The Two Matrix Model

C.1 The Single Cut Ansatz

In this appendix, we will demonstrate the complete derivation of the solution (6.2.3) and the boundary limits in equation (6.2.4) of the single cut ansatz for the two hermitian matrix model in polar coordinates.

We first start off with the function appearing in equation (6.2.2)

$$\begin{aligned} \mathcal{F}(x \pm i\epsilon) &= \oint_{x_-}^{x_+} \frac{dx' \Phi(x')}{x - x'} \mp i\pi\Phi(x) \\ &= \frac{\omega^2}{4} \mp i\pi\Phi(x). \end{aligned} \tag{C.1.1}$$

Equation (C.1.1) appearing above satisfies special conditions that have already been mentioned and is also defined along the interval $[x_-, x_+]$ which represents a single cut along the real axis of the complex plane z. The endpoints of the cut satisfy $x_+ > x_- > 0$.

We first define the analytic function $\mathcal{F}(z)$ along the complex plane of z

$$\mathcal{F}(z) = \int_{x_-}^{x_+} \frac{dx' \Phi(x')}{z - x'}. \tag{C.1.2}$$

To start off we require the condition $\mathcal{F}(z) \approx 1/z$ for large z. With this condition, an expansion is performed for large z, i.e. $(z \to \pm\infty)$. Therefore $\mathcal{F}(z)$ becomes

$$\begin{aligned}
\mathcal{F}(z) &= \oint_{x_-}^{x_+} \frac{dx'\Phi(x')}{z-x'} \\
&= \frac{1}{z}\oint_{x_-}^{x_+} dx'\Phi(x')\left(1-\frac{x'}{z}\right)^{-1} \\
&= \frac{1}{z}+\frac{1}{z^2}\int_{x_-}^{x_+} dx'\Phi(x')x' + \frac{1}{z^3}\int_{x_-}^{x_+} dx'\Phi(x')x'^2 + \frac{1}{z^4}\int_{x_-}^{x_+} dx'\Phi(x')x'^3 + \ldots\ldots
\end{aligned} \tag{C.1.3}$$

The eigenvalue density function $\Phi(x')$ in equation (C.1.3) when integrated along the single cut $[x_-, x_+]$ will be required to be even, positive definite and normalized such that

$$\int_{x_-}^{x_+} dx'\Phi(x') = 1. \tag{C.1.4}$$

The complete solution to the function in equation (C.1.1) is perturbatively derived by introducing the following ansatz

$$\mathcal{F}(z) = \frac{\omega^2}{4} - \frac{f}{z}\sqrt{(z-x_+)(z-x_-)}, \tag{C.1.5}$$

In the ansatz appearing in equation (C.1.5), we will solve for the function f using perturbative methods that were developed for the single hermitian matrix model. To start off, we will require that $\mathcal{F}(z)$ have no poles as $z \to 0$ and that for very large z, we must have $\mathcal{F}(z) \approx 1/z$. With these two conditions, we expand the ansatz appearing in equation (C.1.5) as follows

$$\begin{aligned}
\mathcal{F}(z) &= \frac{\omega^2}{4} - \frac{f}{z}\sqrt{(z-x_+)(z-x_-)} \\
&= \frac{\omega^2}{4} - \frac{f}{z}\times z\sqrt{(1-\frac{x_+}{z})(1-\frac{x_-}{z})} \\
&= \frac{\omega^4}{4} - f\left(1-\frac{1}{2z}\left(x_+ + x_-\right) - \frac{1}{8z^2}\left(x_+^2 - 2x_+x_- - x_-^2\right)\right) \\
&- f\left(\frac{1}{16z^3}\left(x_+^3 - x_-x_+^2 - x_-^2x_+ + x_-^3\right) - \frac{1}{64z^4}\left(2x_-x_+^3 + x_-^2x_+^2 + 2x_-^3x_+\right)\right) \\
&+ f\left(\frac{1}{128z^5}\left(x_-^2x_+^3 + x_-^3x_+^2\right) + \frac{1}{256z}x_-^3x_+^3\right) + \ldots.
\end{aligned} \tag{C.1.6}$$

We will use equations (C.1.3) and (C.1.6) to equate the coefficients of expansion of the function $\mathcal{F}(z)$ whose expansion parameters are $z^0, z^{-1} \ldots$ for large z. We will start off by comparing the z^0 coefficients for both equations (C.1.3) and (C.1.6), to obtain the following

$$\begin{aligned} & \frac{\omega^2}{4} - f = 0 \\ \Rightarrow \quad & f = \frac{\omega^2}{4}. \end{aligned} \tag{C.1.7}$$

Equation (C.1.7) above assigns a definitive value to the function f, which we substitute back into the ansatz to obtain the following

$$\mathcal{F}(z) = \frac{\omega^2}{4} - \frac{\omega^2}{4z}\sqrt{(z - x_+)(z - x_-)}. \tag{C.1.8}$$

We will now proceed to obtain the end points of the single cut ansatz on the complex plane z along the real axis.

We start off by equating the coefficients of $1/z$ for the function $\mathcal{F}(z)$ whose expansion we performed in equations (C.1.3) and (C.1.6). We obtain

$$\begin{aligned} & 1 = \frac{f}{2}\,(x_- + x_+) \\ \Rightarrow \quad & 1 = \frac{\omega^2}{8}\,(x_- + x_+) \\ \Rightarrow \quad & (x_- + x_+) = \frac{8}{\omega^2}. \end{aligned} \tag{C.1.9}$$

Earlier we required $\mathcal{F}(z)$ have no pole when $z \to 0$, we will now expand our analytic function appearing in equation (C.1.8) for the preceding condition to obtain the following

$$\begin{aligned} \mathcal{F}(z \to 0) \quad &= \quad \frac{\omega^2}{4} - \frac{f(z)}{z}\sqrt{(z - x_+)(z - x_-)} = 0 \\ &= \quad -\frac{\omega^2}{4z}\sqrt{x_+ x_-} = 0 \\ &\Rightarrow \quad x_+ x_- = 0 \\ &\Rightarrow \quad x_+ = 0 \quad \text{or} \quad x_- = 0. \end{aligned} \tag{C.1.10}$$

Since $x_+ > x_-$, $x_- = 0$, we substitute the results of equation (C.1.10) into equation (C.1.9) to obtain the following

$$x_+ \;=\; \frac{8}{\omega^2}. \tag{C.1.11}$$

These are the results quoted in equation (6.2.4).

Appendix D

Two Matrix Model Radial Fermionic Picture

This appendix will be dedicated to showing how the system of two hermitian matrices in polar coordinates can be restricted to a configuration of purely radial coordinates, therefore replicating a system of "radial fermions" with N degrees of freedom as seen in equation (7.0.6).

We notice that the Laplacian in equation (7.0.2) possesses a strictly radial part that only depends on the eigenvalues of R. This is the same radial piece of the Laplacian that we derived earlier in equation (5.1.16) and (B.1.29).

The radial part of the Laplacian is represented by the following term

$$\begin{aligned} -\frac{1}{2}\nabla^2 &= -\frac{1}{2}\frac{1}{\Delta^2_{TMM}(r_i^2)}\frac{1}{\prod_k r_k}\sum_i \frac{\partial}{\partial r_i}(\prod_k r_k)\Delta^2_{TMM}(r_i^2)\frac{\partial}{\partial r_i} \quad \text{(D.0.1)} \\ &= -\frac{1}{2}\frac{1}{\Delta^2_{TMM}}\sum_i \frac{1}{r_i}\frac{\partial}{\partial r_i} r_i \Delta^2_{TMM}(r_i^2)\frac{\partial}{\partial r_i}. \end{aligned}$$

In the second line of equation (D.0.1), we introduce the variable $\rho_i = r_i^2$, resulting in the following

$$-\frac{1}{2}\nabla^2 = -\frac{2}{\Delta^2(\rho_i)}\sum_i \frac{\partial}{\partial \rho_i}\rho_i \Delta^2(\rho_i)\frac{\partial}{\partial \rho_i}, \quad \text{(D.0.2)}$$

since

$$\begin{aligned}\Delta^2_{TMM}(r_i^2) &= \prod_{i<j}\frac{1}{4}\left(r_i^2 - r_j^2\right)^2 \\ &= \prod_{i<j}\frac{1}{4}\left(\rho_i - \rho_j\right)^2 \\ &= \Delta^2(\rho_i). \end{aligned} \tag{D.0.3}$$

Equation (D.0.3) was generally understood to represent the modified Vandermonde determinant when compared to the determinant of the single hermitian matrix model.

The radial operator (D.0.2) will act on "s-state" symmetric wavefunctions Φ that are independent of any angular degrees of freedom. We define the anti-symmetric wavefunctions with strict radial dependence ρ_i as follows

$$\begin{aligned}\Psi &= \Delta(\rho_i)\Phi \\ &= \prod_{i<j}\frac{1}{4}\left(\rho_i - \rho_j\right)\Phi. \end{aligned} \tag{D.0.4}$$

Therefore, the eigenvalue/eigenfunction equation in terms of the operator (D.0.2) and the wavefunction Ψ becomes

$$\begin{aligned}&-\left(\frac{2}{\Delta^2(\rho_i)}\sum_i \frac{\partial}{\partial\rho_i}\rho_i\Delta^2(\rho_i)\frac{\partial}{\partial\rho_i}\right)\frac{\Psi}{\Delta(\rho_i)} = E\frac{\Psi}{\Delta(\rho_i)} \\ &-\left[\frac{2}{\Delta(\rho_i)}\sum_i \frac{\partial}{\partial\rho_i}\Delta(\rho_i)\rho_i\Delta(\rho_i)\frac{\partial}{\partial\rho_i}\frac{1}{\Delta(\rho_i)}\right]\Psi = E\Psi. \end{aligned} \tag{D.0.5}$$

On the left hand side of equation (D.0.5), simple algebraic methods will be used to help simplify the radial operator term. To achieve this, we expand equation (D.0.5) as follows

$$
\begin{aligned}
& -\frac{2}{\Delta(\rho_i)}\sum_i \frac{\partial}{\partial\rho_i}\Delta(\rho_i)\rho_i\Delta(\rho_i)\frac{\partial}{\partial\rho_i}\frac{1}{\Delta(\rho_i)} = \qquad (D.0.6)\\
& -2\sum_i\left(\frac{1}{\Delta(\rho_i)}\frac{\partial}{\partial\rho_i}\Delta(\rho_i)\right)\rho_i\left(\Delta(\rho_i)\frac{\partial}{\partial\rho_i}\frac{1}{\Delta(\rho_i)}\right)\\
= \; & -2\sum_i\left(\frac{\partial}{\partial\rho_i}+\sum_{k\neq i}\frac{1}{\rho_i-\rho_k}\right)\rho_i\left(\frac{\partial}{\partial\rho_i}-\sum_{j\neq i}\frac{1}{\rho_i-\rho_j}\right)\\
= \; & -2\left\{\sum_i\left(\frac{\partial}{\partial\rho_i}\rho_i\frac{\partial}{\partial\rho_i}-\sum_{j\neq i}\frac{1}{\rho_i-\rho_j}+\sum_{j\neq i}\frac{\rho_i}{(\rho_i-\rho_j)^2}\right)\right\}\\
- \; & 2\sum_i\left(\sum_{k\neq i}\frac{1}{\rho_i-\rho_k}\rho_i\frac{\partial}{\partial\rho_i}-\rho_i\sum_{j\neq i}\frac{1}{\rho_i-\rho_j}\rho_i\frac{\partial}{\partial\rho_i}\right)\\
+ \; & 2\sum_i\left(\sum_{j\neq i,k\neq i}\frac{\rho_i}{\rho_i-\rho_k}\frac{1}{\rho_i-\rho_j}\right).
\end{aligned}
$$

From equation (D.0.6) above, we will make use of identity terms that will help us simplify the above expression clearly:

$$\sum_i\sum_{i\neq j}\frac{1}{\rho_i-\rho_j}=0. \qquad (D.0.7)$$

With the use of equation (D.0.7) in equation (D.0.6), we obtain

$$
\begin{aligned}
& -\frac{2}{\Delta(\rho_i)}\sum_i \frac{\partial}{\partial\rho_i}\Delta(\rho_i)\rho_i\Delta(\rho_i)\frac{\partial}{\partial\rho_i}\frac{1}{\Delta(\rho_i)} = \qquad (D.0.8)\\
& -2\left\{\sum_i\frac{\partial}{\partial\rho_i}\rho_i\frac{\partial}{\partial\rho_i}+\sum_{i\neq j}\frac{\rho_i}{(\rho_i-\rho_j)^2}-\sum_{j\neq i\neq k}\frac{\rho_i}{(\rho_i-\rho_j)(\rho_i-\rho_k)}\right\}.
\end{aligned}
$$

Equation (D.0.8) above can further be simplified by observing that the following identity is true:

$$\sum_{i\neq j}\frac{\rho_i}{(\rho_i-\rho_j)^2}-\sum_{i\neq j\neq k}\frac{\rho_i}{(\rho_i-\rho_k)(\rho_i-\rho_j)}=\sum_{i\neq j\neq k}\frac{\rho_i}{(\rho_i-\rho_k)(\rho_i-\rho_j)}=0. \qquad (D.0.9)$$

The right hand side of equation (D.0.9), can be explicitly shown to vanish by considering any three eigenvalues ρ_1,ρ_2,ρ_3:

$$\begin{aligned}
\frac{\rho_1}{(\rho_1-\rho_2)(\rho_1-\rho_3)}+\frac{\rho_2}{(\rho_2-\rho_1)(\rho_2-\rho_3)}+\frac{\rho_3}{(\rho_3-\rho_1)(\rho_3-\rho_2)} &= \quad \text{(D.0.10)}\\
\frac{1}{(\rho_1-\rho_2)}\left(\frac{\rho_1}{(\rho_1-\rho_3)}-\frac{\rho_2}{(\rho_2-\rho_3)}\right)+\frac{\rho_3}{(\rho_3-\rho_1)(\rho_3-\rho_2)} &= \\
\frac{1}{(\rho_1-\rho_2)}\left(\frac{\rho_1\rho_2-\rho_1\rho_3-\rho_2\rho_1+\rho_2\rho_3}{(\rho_3-\rho_1)(\rho_3-\rho_2)}\right)+\frac{\rho_3}{(\rho_3-\rho_1)(\rho_3-\rho_2)} &= \\
\frac{1}{(\rho_1-\rho_2)}\frac{\rho_3(\rho_2-\rho_1)}{(\rho_1-\rho_3)(\rho_2-\rho_3)}+\frac{\rho_3}{(\rho_1-\rho_3)(\rho_2-\rho_3)} &= \\
-\frac{\rho_3}{(\rho_1-\rho_3)(\rho_2-\rho_3)}+\frac{\rho_3}{(\rho_1-\rho_3)(\rho_2-\rho_3)} &= 0.
\end{aligned}$$

We make use of equations (D.0.7), (D.0.9) and (D.0.10), what remains in total is the following

$$-\frac{2}{\Delta(\rho_i)}\sum_i\frac{\partial}{\partial\rho_i}\Delta(\rho_i)\rho_i\Delta(\rho_i)\frac{\partial}{\partial\rho_i}\frac{1}{\Delta(\rho_i)}=-2\sum_i\frac{\partial}{\partial\rho_i}\rho_i\frac{\partial}{\partial\rho_i}. \qquad \text{(D.0.11)}$$

The right hand side of equation (D.0.11) acts as an operator on the anti-symmetric wavefunctions defined in equation (D.0.4), giving us the following energy system

$$\begin{aligned}
-\frac{2}{\Delta^2(\rho_i)}\sum_i\frac{\partial}{\partial\rho_i}\rho_i\Delta^2(\rho_i)\frac{\partial}{\partial\rho_i}\Psi &= E\Psi \qquad \text{(D.0.12)}\\
\Rightarrow -2\left(\sum_i\xi_i\right)\Psi &= E\Psi,
\end{aligned}$$

where

$$\begin{aligned}
\xi_i &= \frac{\partial}{\partial\rho_i}\rho_i\frac{\partial}{\partial\rho_i}-\frac{1}{2}V(\rho_i)\\
&= \frac{1}{4}\left(\sum_i\frac{1}{r_i}\frac{\partial}{\partial r_i}r_i\frac{\partial}{\partial r_i}-4V'(r_i)\right). \qquad \text{(D.0.13)}
\end{aligned}$$

The methodology used to derive equations (D.0.12), (D.0.13) and (7.0.6) was adopted from the procedure used to demonstrate that the single hermitian matrix model is a model of free fermions with N degrees of freedom. The fermion picture

was made famous by BIPZ [27] in their calculations that are based on the single hermitian matrix model. In our work, the emergence of higher dimensionality is new.

Appendix E

Radial Sector: System With An Even General Number Of Matrices

E.1 Eigenvalue Density Function: Single Cut Solution

In this appendix we show the calculations for the solution obtained appearing in equation (11.1.5) which is a solution for equations (11.1.3) and (11.1.4). The semi classical approximation in equation (11.0.8) will be solved in the large N limit. In this limit, we will define an analytic function whose behavior along the cut $[\rho_-, \rho_+]$ is

$$\begin{aligned} \mathcal{G}(\rho \pm i\epsilon) &= \oint_{\rho_-}^{\rho_+} \frac{d\rho' \Phi(\rho')}{\rho - \rho'} \mp i\pi\Phi(\rho) \\ &= \frac{\omega^2}{4} - \frac{m-1}{2\rho} \mp i\pi\Phi(\rho), \end{aligned} \tag{E.1.1}$$

as seen in equation (11.1.2).

The above analytic function satisfies the conditions discussed before, assumed to be true along the interval $[\rho_-, \rho_+]$, which is defined as a cut on the complex

plane z where we require $\rho_+ > \rho_- > 0$. The analytic function is given by:

$$\mathcal{G}(z) = \int_{\rho_-}^{\rho_+} \frac{d\rho' \Phi(\rho')}{z - \rho'}, \tag{E.1.2}$$

which is used to obtain the solution for equation (E.1.1) and also obtain an explicit representation of the eigenvalue density function $\Phi(\rho)$. We will require that $\mathcal{G}(z) \approx 1/z$ for large z and that it has no poles at $z = 0$. Using equation (E.1.2) we perform an expansion for large z, $(z \to \pm\infty)$, which will give us the following

$$\begin{aligned}
\mathcal{G}(z) &= \fint_{\rho_-}^{\rho_+} \frac{d\rho' \Phi(\rho')}{z - \rho'} \\
&= \frac{1}{z} \fint_{\rho_-}^{\rho_+} d\rho' \Phi(\rho') \left(1 - \frac{\rho'}{z}\right)^{-1} \\
&= \frac{1}{z} + \frac{1}{z^2} \int_{\rho_-}^{\rho_+} d\rho' \Phi(\rho') \rho' + \frac{1}{z^3} \int_{\rho_-}^{\rho_+} d\rho' \Phi(\rho') \rho'^2 + \frac{1}{z^4} \int_{\rho_-}^{\rho_+} d\rho' \Phi(\rho') \rho'^3 + \ldots .
\end{aligned} \tag{E.1.3}$$

In equation (E.1.3) above we have required that the eigenvalue density function $\Phi(\rho)$ be positive, normalized and even such that

$$\int_{\rho_-}^{\rho_+} d\rho' \Phi(\rho') = 1. \tag{E.1.4}$$

As z approaches the support, $z \to [\rho_+, \rho_-]$, we will require equation (E.1.1) to be true, therefore in order for us to find the density of eigenvalues, we will define the following analytic function

$$\mathcal{G}(z) = \frac{\omega^2}{4} - \frac{m-1}{2z} - \frac{q}{z} \sqrt{(z - \rho_-)(z - \rho_+)}, \tag{E.1.5}$$

which must satisfy all the conditions required for equation (E.1.1).

In equation (E.1.5) we will solve for the constant q, ρ_+ and ρ_-. We expand $\mathcal{G}(z)$ perturbatively

$$
\begin{aligned}
\mathcal{G}(z) &= \frac{\omega^2}{4} - \frac{m-1}{2\rho} - \frac{q}{z}\sqrt{(z-\rho_-)(z-\rho_+)} \\
&= \mathcal{G}(z)\frac{\omega^2}{4} - \frac{m-1}{2\rho} - \frac{q}{z} \times z\sqrt{(1-\frac{\rho_-}{z})(1-\frac{\rho_+}{z})} \\
&= \frac{\omega^4}{4} - \frac{a}{2z} - q\left(1 - \frac{1}{2z}\left(\rho_+ + \rho_-\right) - \frac{1}{8z^2}\left(\rho_+^2 - 2\rho_+\rho_- - \rho_-^2\right)\right) \\
&- q\left(\frac{1}{16z^3}\left(\rho_+^3 - \rho_-\rho_+^2 - \rho_-^2\rho_+ + \rho_-^3\right) - \frac{1}{64z^4}\left(2\rho_-\rho_+^3 + \rho_-^2\rho_+^2 + 2\rho_-^3\rho_+\right)\right) \\
&+ q\left(\frac{1}{128z^5}\left(\rho_-^2\rho_+^3 + \rho_-^3\rho_+^2\right) + \frac{1}{256z}\rho_-^3\rho_+^3\right) + \ldots . \qquad \text{(E.1.6)}
\end{aligned}
$$

In equation (E.1.6) above we denoted: $a = m - 1$.

Equations (E.1.3) and (E.1.6) are from the same analytic function. These two equations represent a large z expansion on both sides of the analytic function $\mathcal{G}(z)$, so as a result of this, we can equate the coefficients of the z terms that appear on both sides of the expansion for $\mathcal{G}(z)$. We start with the z^0 coefficients in both equations (E.1.3) and (E.1.6), and this gives us

$$
\begin{aligned}
&\frac{\omega^2}{4} - q = 0 \\
\Rightarrow \quad & q = \frac{\omega^2}{4}. \qquad \text{(E.1.7)}
\end{aligned}
$$

Using equation (E.1.7), we can rewrite equation (E.1.5) as follows

$$
\mathcal{G}(z) = \frac{\omega^2}{4} - \frac{m-1}{2z} - \frac{\omega^2}{4z}\sqrt{(z-\rho_-)(z-\rho_+)}. \qquad \text{(E.1.8)}
$$

In order for us to obtain the boundary limits that define the support $[\rho_-, \rho_+]$, we will again equate the coefficients of equations (E.1.3) and (E.1.6) for the analytic function $\mathcal{G}(z)$. We equate the coefficients for the $1/z$ term

$$
\begin{aligned}
&1 = -\frac{a}{2} + \frac{q}{2}\left(\rho_- + \rho_+\right) \\
\Rightarrow \quad & 1 + \frac{a}{2} = \frac{\omega^2}{8}\left(\rho_- + \rho_+\right) \\
\Rightarrow \quad & \left(\rho_- + \rho_+\right) = \frac{8}{\omega^2}\left(\frac{a}{2} + 1\right). \qquad \text{(E.1.9)}
\end{aligned}
$$

We refer back to the previous condition that requires that our analytic function $\mathcal{G}(z)$ have no pole when $z \to 0$, this condition will present us with the following constraint for the boundary limits of the support

$$\begin{aligned}
\mathcal{G}(z \to 0) &= \frac{\omega^2}{4} - \frac{m-1}{2z} - \frac{q}{z}\sqrt{(z-\rho_-)(z-\rho_+)} = 0 \\
&\approx -\frac{a}{2z} - \frac{\omega^2}{4z}\sqrt{\rho_-\rho_+} = 0 \\
&\Rightarrow \sqrt{\rho_-\rho_+} = -2\frac{a}{\omega^2} \\
&\Rightarrow \rho_-\rho_+ = \frac{4a^2}{\omega^4}. \qquad \text{(E.1.10)}
\end{aligned}$$

Therefore, using equations (E.1.9) and (E.1.10), we can define an equation of roots for the boundary limits of the support $[\rho_-, \rho_+]$. We substitute equation (E.1.10) into equation (E.1.9) for either ρ_+ or ρ_- to obtain the following

$$\rho_+^2 - \frac{8}{\omega^2}\left(1 + \frac{a^2}{2}\right)\rho_+ + \frac{4a^2}{\omega^4} = 0. \qquad \text{(E.1.11)}$$

The same methods that were used to obtain the quadratic equation for ρ_+ can be used for ρ_-, whose equation of roots will be

$$\rho_-^2 - \frac{8}{\omega^2}\left(1 + \frac{a^2}{2}\right)\rho_- + \frac{4a^2}{\omega^4} = 0. \qquad \text{(E.1.12)}$$

Finding the roots of both equations (E.1.11) and (E.1.12) is straight forward in both equations, therefore the values of the boundary limits of the support are shown to be

$$\rho_\pm = \frac{2}{\omega^2}(m+1) \pm \frac{4}{\omega^2}\sqrt{m}. \qquad \text{(E.1.13)}$$

Equation (E.1.13) represents the boundary limits of the support $[\rho_-, \rho_+]$, and we see that these limits are related to the number of complex matrices m.

E.2 Eigenvalue Density Function: Symmetric Solutions

For this section of the appendix we will derive equations (11.1.10) and (11.1.11) to obtain or define the symmetric solutions of the eigenvalue density function. We start with equation (11.1.9), whose analytic function is defined as follows

$$\begin{aligned}\mathcal{G}'(r \pm i\epsilon) &= \fint_{-\infty}^{\infty} \frac{dr'\phi(r')}{r - r'} \mp i\pi\phi(r) \\ &= \frac{\omega^2}{2} r - \frac{m-1}{r} \mp i\pi\phi(r), \end{aligned} \tag{E.2.1}$$

and has to satisfy constraints similar to those of the analytic function defined for the single cut ansatz.

These properties will now be extended. These symmetric solutions will be defined over the intervals: $[-r_+, -r_-]$ and $[r_+, r_-]$ where $r_+ > r_- > 0$, on the complex plane z. Define the analytic function $\mathcal{G}'(z)$ extended over the real domain as follows

$$\mathcal{G}'(z) = \int_{-\infty}^{\infty} \frac{dr'\phi(r')}{z - r'}. \tag{E.2.2}$$

Equation (E.2.2) will be expanded for large z as follows

$$\begin{aligned}\mathcal{G}'(z) &= \fint_{-\infty}^{\infty} \frac{dr'\phi(r')}{z - r'} \\ &= \frac{1}{z} \fint_{-\infty}^{\infty} dr'\phi(r') \left(1 - \frac{r'}{z}\right)^{-1} \\ &= \frac{2}{z} + \frac{1}{z^2}\int_{-\infty}^{\infty} dr'\phi(r')r' + \frac{1}{z^3}\int_{-\infty}^{\infty} dr'\phi(r')r'^2 + \frac{1}{z^4}\int_{\infty}^{\infty} dr'\phi(r')r'^3 + \ldots. \end{aligned} \tag{E.2.3}$$

In the last line of equation (E.2.3) we used the definition of the normalization of the eigenvalue density function

$$\int_{-\infty}^{\infty} dr'\phi(r') = 2, \tag{E.2.4}$$

and required that it must be an integral that is positive, even and normalized.

To obtain a solution for the eigenvalue density function in equation (E.2.1), we propose the following ansatz for the analytic function

$$\mathcal{G}'(z) = \frac{\omega^2}{2}z - \frac{a}{z} - \left(\frac{d}{z} + c\right)\sqrt{(z^2 - r_+^2)\,(z^2 - r_-^2)}, \tag{E.2.5}$$

where we have set $a = m - 1$. The prerequisite for the analytic function in equation (E.2.5) is that $\mathcal{G}'(z) \approx 2/z$ for large z, using this condition we will expand equation (E.2.5) as follows

$$\begin{aligned}
\mathcal{G}'(z) &= \frac{\omega^2}{2}z - \frac{a}{z} - \left(\frac{d}{z} + c\right)\sqrt{(z^2 - r_+^2)\,(z^2 - r_-^2)} \qquad \text{(E.2.6)}\\
&= \frac{\omega^2}{2}z - \frac{a}{z} - \left(\frac{d}{z} + c\right)\sqrt{z^4}\left(1 - \frac{r_+^2}{z^2}\right)^{1/2}\left(1 - \frac{r_+^2}{z^2}\right)^{1/2}\\
&= \frac{c}{2}\left(r_+^2 + r_-^2\right) + z\left(\frac{1}{2}\omega^2 - d\right) - cz^2 + \frac{1}{z}\left(\frac{d}{2}\left(r_+^2 + r_-^2\right) - a\right)\\
&+ \frac{d}{8z^3}\left(r_+^4 - 2r_+^2r_-^2 + r_-^4\right) + \frac{d}{16z^5}\left(r_+^6 - r_+^2r_-^4 - r_+^4r_-^2 + r_-^6\right)\\
&+ \frac{d}{64z^7}\left(2r_+^2r_-^6 + r_+^4r_-^4 - 2r_+^6r_-^2\right) - \frac{d}{128z^9}\left(r_+^4r_-^6 + r_+^6r_-^4\right) - \frac{d}{256z^{11}}r_+^6r_-^6\\
&+ \frac{c}{8z^2}\left(r_+^4 - 2r_+^2r_-^2 + r_-^4\right) + \frac{c}{16z^4}\left(r_+^6 - r_+^2r_-^4 - r_+^4r_-^2 + r_-^6\right)\\
&- \frac{c}{64z^6}\left(2r_+^2r_-^6 + r_+^4r_-^4 - 2r_+^6r_-^2\right) - \frac{c}{128z^8}\left(r_+^4r_-^6 + r_+^6r_-^4\right) - \frac{c}{256z^{10}}r_+^6r_-^6.
\end{aligned}$$

Using equations (E.2.6) and (E.2.3), we equate the coefficients of the factor z, and we obtain the following equations

$$\begin{aligned}
\text{coefficient} : z \qquad & \left(\frac{1}{2}\omega^2 - d\right) = 0\\
\Rightarrow \quad & d = \frac{1}{2}\omega^2, \qquad \text{(E.2.7)}
\end{aligned}$$

$$\text{coefficient} : z^2 \qquad c = 0, \qquad \text{(E.2.8)}$$

and

$$\begin{aligned}
\text{coefficient} : 1/z \qquad & \left(\frac{d}{2}\left(r_+^2 + r_-^2\right) - a\right) = 2\\
\Rightarrow \quad & \frac{\omega^2}{4}\left(r_+^2 + r_-^2\right) - a - 2 = 0. \qquad \text{(E.2.9)}
\end{aligned}$$

The definitions of equations (E.2.7) and (E.2.8) are substituted into the analytic function ansatz in equation (E.2.5) and we obtain the following

$$\mathcal{G}'(z) = \frac{\omega^2}{2}z - \frac{a}{z} - \frac{\omega^2}{2z}\sqrt{(z^2 - r_+^2)\,(z^2 - r_-^2)}. \tag{E.2.10}$$

We have to consider the behavior of $\mathcal{G}'(z)$ when $z \to 0$. This will require that we consider another prerequisite that has to be satisfied by our analytical function in equation (E.2.10) above. Therefore, we require that $\mathcal{G}'(z \to 0) = 0$, this will give us the following

$$\begin{aligned}
& \mathcal{G}'(z \to 0) = \frac{\omega^2}{2}z - \frac{a}{z} - \frac{\omega^2}{2z}\sqrt{(z^2 - r_+^2)\,(z^2 - r_-^2)} = 0 \\
\Rightarrow \quad & -\frac{a}{z} - \frac{\omega^2}{2z}\sqrt{r_+^2 r_-^2} = 0 \\
\Rightarrow \quad & \sqrt{r_+^2 r_-^2} = 2\frac{a}{\omega^2} \\
\Rightarrow \quad & r_+^2 r_-^2 = 4\frac{a^2}{\omega^4}.
\end{aligned} \tag{E.2.11}$$

At this point we are presented with two equations that will be solved simultaneously, that is equations (E.2.9) and the last line of equation (E.2.11). These two equations will be used to solve for the boundary limits of the support $[-r_+, -r_-]$ and $[r_+, r_-]$.

Using the two previously mentioned equations, we can substitute equation (E.2.11) into (E.2.9) by eliminating either r_-^2 or r_+^2 to obtain the following equation of roots

$$r_+^4 - \frac{4\,(a+2)}{\omega^2}r_+^2 + \frac{4a^2}{\omega^4} = 0. \tag{E.2.12}$$

Using the same method to derive the quadratic equation (E.1.12) for r_+^2, the same can be done for r_-^2, we can show that

$$r_-^4 - \frac{4\,(a+2)}{\omega^2}r_-^2 + \frac{4a^2}{\omega^4} = 0. \tag{E.2.13}$$

Both equation (E.2.12) and (E.2.13) are fundamentally the same, we will therefore introduce a short hand notation

$$r_{\pm}^4 - \frac{4\,(a+2)}{\omega^2} r_{\pm}^2 + \frac{4a^2}{\omega^4} = 0. \tag{E.2.14}$$

If we treat equation (E.2.14) as a quadratic equation by defining $x = r_{\pm}^2$, the above equation can be shown to have the following roots

$$\begin{aligned} r_{\pm}^2 &= \frac{2\,(2+a)}{\omega^2} \pm \frac{4}{\omega^2}\,(a+1)^{1/2} \\ &= \frac{2\,(m+1)}{\omega^2} \pm \frac{4}{\omega^2}\,(m)^{1/2}\,. \end{aligned} \tag{E.2.15}$$

Equation (E.2.15) is the same equations as (11.1.11). For the $m = 1$ matrix model, the constant a defined earlier becomes zero: $a = m - 1 = 0$, therefore our boundary limits are given by

$$r_- = 0 \qquad r_+^2 = \frac{8}{\omega^2}. \tag{E.2.16}$$

as we saw earlier for the radially restricted two matrix model.

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