A SEARCH FOR CONFIGURATIONAL INVARIANTS

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ABSTRACT

In this dissertation a search is conducted for systems of the form

$$H = \frac{1}{2}(p_x^2 + p_y^2) + A(x,y)p_x + B(x,y)p_y + V(x,y),$$

which possess configurational invariants polynomial or rational in the momenta as well as for axially symmetric systems of the form

$$H = \frac{1}{2}(p_r^2 + \frac{p_\theta^2}{r^2}) + V(r,t),$$

which possess configurational invariants polynomial in the momenta. Several new systems are found.

First integrals are discussed and a survey made of systems of the first form above which possess polynomial first integrals. A survey of the literature pertaining to configurational invariants is also given.

DECLARATION

I declare that this thesis is my own, unaided work. It is being submitted for the degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

K Cyrus

(Name of candidate)

12th day of March, 1991.

To my Family

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PREFACE

The purpose of this search is to find more systems possessing configurational invariants giving us a better idea of what types of system possess them and also to establish how abundent they are. The concept of a configurational invariant is relatively new and was introduced by L. S. Hall in 1982, who then also conducted the first search for them. Here his ideas are extended to include more general systems and different ansätze for the configurational invariants. The existence of configurational invariants with respect to angular momentum (instead of the energy as has previously been done) is also investigated. As in his study this search is restricted to two-dimensional systems.

Configurational invariants arise in the study of dynamical systems in which one is usually engaged in the solution of equations of motion, which constitute the mathematical statement of Newton's second law of motion. These equations form a system of differential equations of order $2n$, where n is the number of degrees-of-freedom. Just as first integrals (constants of the motion) may be used to reduce the order of the system so configurational invariants may be used to do the same when the motion is restricted to a surface in phase space defined by $I(\mathbf{q}, \mathbf{p}, t) = C$, where I is a first integral of the system (e.g. energy) and C is a specific number.

The existence of configurational invariants is of importance in some physical problems. In some physical applications one is only interested in the system at a particular value of the energy, as in the context of Fokker-Plank equations and their use in nonequilibrium statistical mechanics (Graham 1985). It was the study of a physical phenomenon, namely axially symmetric magnetohydrodynamic reverse-field equilibria, which led L. S. Hall to introduce configurational invariants.

I wish to draw the readers' attention to the following notation used in this dissertation. The abreviation *wrt* denotes "with respect to", *ito* denotes "in terms of" and *rhs* and *lhs* denote "right-hand side" and "left-hand side" respectively. A dot above a variable or function denotes total differentiation *wrt* time. Vectors or vector functions are written in boldface.

Throughout the my research of this topic I have received valuable assistance and guidance from Prof. P. G. L. Leach, to whom I wish to express my sincere gratitude.

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CHAPTER I: FIRST INTEGRALS AND THEIR APPLICATIONS

1.1 Introduction

Consider a system of differential equations of order k in l dependent variables and one independent variable. Such a system can always be rewritten in the form

$$\frac{\mathrm{d}x_r}{\mathrm{d}t} = X_r(x_1, x_2, ..., x_l, t), \qquad r = 1, k, \tag{1.1.1}$$

where the X_r's are known functions of their arguments in which all the derivatives except the highest derivatives of the dependent variables $(x_1, x_2, ..., x_l)$, $l < k$, are defined as new variables $(x_{l+1}, ..., x_k)$. For example let

$$\begin{aligned} \frac{\mathrm{d}^3 x}{\mathrm{d}t^3} &= \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + X(x, y), \\ \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} &= \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}y}{\mathrm{d}t} + Y(x, y), \end{aligned} \tag{1.1.2}$$

be a system of differential equations in two variables x and y, then the system may be rewritten as

$$\begin{aligned} \frac{\mathrm{d}x_2}{\mathrm{d}t} &= x_1^2 + x_2 + X(x, y), \\ \frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_2, \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= x_1, \\ \frac{\mathrm{d}y_1}{\mathrm{d}t} &= x_1 y_1 + Y(x, y), \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= y_1, \end{aligned} \tag{1.1.3}$$

where x, x_1, x_2, y and y_1 are all treated as variables.

Definition 1:

Let $I(x_1, x_2, ..., x_k, t)$ be a functional such that $dI/dt = 0$ by virtue of the system of differential equations (1.1.1), ie.,

$$\frac{\partial I}{\partial x_1}\dot{x}_1 + \frac{\partial I}{\partial x_2}\dot{x}_2 + ... + \frac{\partial I}{\partial x_k}\dot{x}_k + \frac{\partial I}{\partial t} = 0 \tag{1.1.4}$$

when $(x_1, x_2, ..., x_k)$ are functions satisfying the system of differential equations. Then I is called a first integral of the system and

$$I(x_1, x_2, ..., x_k, t) = c, \tag{1.1.5}$$

where c is a constant, the value of which is obtained from the initial conditions. From this equation it is clear that any function of I will also be a first integral.

Expressions for the derivatives in (1.1.4) may be obtained from (1.1.1). Equation (1.1.4) becomes

$$\frac{\partial I}{\partial x_1}X_1 + \frac{\partial I}{\partial x_2}X_2 + ... + \frac{\partial I}{\partial x_k}X_k + \frac{\partial I}{\partial t} = 0. \tag{1.1.6}$$

This is the defining equation for a first integral of a system of the form of equation (1.1.1).

If k functionally independent first integrals

$$I_r(x_1, x_2, ..., x_k, t) = a_r, \qquad r = 1, k, \tag{1.1.7}$$

are known, it is possible, at least locally, to invert these equations to obtain

$$x_r = \phi(a_1, a_2, ..., a_k, t), \qquad r = 1, k, \tag{1.1.8}$$

and hence solve the problem. It follows that the process of finding k independent first integrals and that of solving the system completely are equivalent and this is the importance of first integrals in the study of dynamical systems. Even if fewer than k first integrals are known, they still give valuable information about the system.

The one-dimensional harmonic oscillator with the equation of motion

$$\ddot{q} + q = 0, \tag{1.1.9}$$

is a system for which sufficient first integrals are known to solve the problem. The two first integrals are:

$$\begin{aligned} H &= \frac{1}{2}(p^2 + q^2), \\ I &= q\sin(t) + p\cos(t), \end{aligned} \tag{1.1.10}$$

where $p = \mathrm{d}q/\mathrm{d}t$ is treated as another dependent variable. The solution for q follows easily from these first integrals:

$$q = I\sin(t) \pm \sqrt{H - I^2}\cos(t). \tag{1.1.11}$$

Since the beginnings of the subject of classical mechanics, researchers have been searching for first integrals of dynamical systems. Early attempts to find systematic methods of finding first integrals of a second order system of differential equations were made by Bertrand in 1852 (Whittaker 1944) and Ermakov in 1880 (Ray and Reid 1979). More recent methods used to find first integrals include the direct method (which is the generalization of Bertrand's method to time-dependent systems), Noether's theorem and the Lax-pair method. The method that is used throughout this study is the direct method since it is well suited to a search for configurational invariants.

In this chapter the direct method will be discussed and known results for polynomial first integrals of two-dimensional Hamiltonian systems of the form

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y), \tag{1.1.12}$$

will be given. The other methods mentioned above will be discussed briefly.

Further methods for analysing dynamical systems include Painlevé analysis and Ziglin analysis which do not involve finding first integrals, but, rather, determine whether the system is integrable and therefore whether it will have first integrals. The Hamilton-Jacobi method involves obtaining a solution to the Hamilton-Jacobi equation from which the full set of first integrals may easily be derived. The Lie theory of extended groups makes use of the point symmetries of the equations of motion to find a solution.

1.2 The Hamiltonian formulation

In the Hamiltonian formulation a system in n-dimensional space is characterised by a functional of the phase space variables and time, i.e. (q,p,t), called the Hamiltonian of the system and denoted by H. The Hamiltonian is derived in such a way that the equations of motion are given by

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q_i}, \quad \frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i} \quad i = 1, n. \tag{1.2.1}$$

(These equations are of the form (1.1.1).)

Suppose that $I(q,p,t)$ is a first integral of a Hamiltonian system then $\mathrm{d}I/\mathrm{d}t = 0$, i.e.,

$$\dot{I} = \frac{\partial I}{\partial q_i}\dot{q}_i + \frac{\partial I}{\partial p_i}\dot{p}_i + \frac{\partial I}{\partial t} = 0 \tag{1.2.2}$$

(summation over repeated indices is assumed) or, in the form of (1.1.6),

$$\frac{\partial I}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial I}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial I}{\partial t} = 0. \tag{1.2.3}$$

In Poisson Bracket notation this is written as

$$\frac{\mathrm{d}I}{\mathrm{d}t} = [I,H]_{PB} + \frac{\partial I}{\partial t} = 0. \tag{1.2.4}$$

The Poisson Bracket of the function $A = A(q,p,t)$ with the function $B = B(q,p,t)$ is defined as

$$[A,B]_{PB} = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}. \tag{1.2.5}$$

If I does not depend on time explicitly, then (1.2.4) reduces to

$$\frac{\mathrm{d}I}{\mathrm{d}t} = [I,H]_{PB} = 0. \tag{1.2.6}$$

Definition 2:
A function is called an autonomous function if it does not depend on time explicitly.

The first integrals considered in this chapter are autonomous and therefore their time derivative will be given by (1.2.6). If H is autonomous, then

$$\frac{\mathrm{d}H}{\mathrm{d}t} = [H,H]_{PB} = 0 \tag{1.2.7}$$

and H is a first integral. The types of system considered in this chapter are autonomous and therefore these systems will already have one first integral.

Definition 3:
If the Poisson Bracket of two functions is zero, then they are said to commute.

Definition 4:
Several functions, say, $F_i(q_i, p_i, t)$ $i = 1, n$ are said to be in involution if

$$[F_i, F_j]_{PB} = 0, \text{ for } i, j = 1, n. \tag{1.2.8}$$

In 1855 J. M. Liouville showed that for a n-dimensional Hamiltonian system (with $2n$ phase space variables) one only needs n first integrals, which are in involution, to solve the system (Liouville 1855, Whittaker 1944).

Theorem 1 (Liouville's theorem):

Suppose n functionally independent first integrals

$$\phi_r(q_1, q_2, ..., q_n, p_1, p_2, ..., p_n, t) = a_r \quad r = 1, n, \tag{1.2.9}$$

where $(a_1, a_2, ..., a_n)$ are arbitrary constants within one or more continuous intervals of nonzero measure so that differentiation *wrt* $a_i = i, n$ is meaningful, are known for the dynamical system

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q_i} \qquad i = 1, n. \tag{1.2.10}$$

If the functions $(\phi_1, \phi_2, ..., \phi_n)$ are in involution, it is possible to find the remaining n integrals and solve the differential equation.

Proof:

Using the integrals $(\phi_1, \phi_2, ..., \phi_n)$ the variables $(p_1, p_2, ..., p_n)$ can be determined in terms of $(q_1, q_2, ..., q_n, a_1, a_2, ..., a_n, t)$, say,

$$p_r = f_r(q_1, q_2, ..., q_n, a_1, a_2, ..., a_n, t), \qquad r = 1, n. \tag{1.2.11}$$

Since $(\phi_1 - a_1, \phi_2 - a_2, ..., \phi_n - a_n)$ are in involution, i.e.

$$[\phi_s - a_s, \phi_r - a_r]_{PB} =$$

$$\sum_{i=1}^{n} \left(\frac{\partial(\phi_r - a_r)}{\partial q_i} \frac{\partial(\phi_s - a_s)}{\partial p_i} - \frac{\partial(\phi_r - a_r)}{\partial p_i} \frac{\partial(\phi_s - a_s)}{\partial q_i} \right) = 0, \qquad r, s = 1, n, \tag{1.2.12}$$

the function $\phi_r - a_r$ is invariant *wrt* the following infinitesimal transformation

$$\bar{q}_i = q_i + \epsilon\frac{\partial \phi_s}{\partial p_i}, \qquad \bar{p}_i = p_i - \epsilon\frac{\partial \phi_s}{\partial q_i}, \qquad s, r = 1, n. \tag{1.2.13}$$

Since $(p_1 - f_1, p_2 - f_2, ..., p_n - f_n)$ are derivable from $(\phi_1 - a_1, \phi_2 - a_2, ..., \phi_n - f_n)$, they must also be invariant *wrt* these transformations and consequently

$$[\phi_s, p_r - f_r]_{PB} = 0 \qquad s, r = 1, n. \tag{1.2.14}$$

Similarly for $r = 1, ..., n,\ \phi_r - a_r$ is invariant *wrt* to the transformations

$$\bar{q}_i = q_i + \epsilon\frac{\partial(p_s - f_s,)}{\partial p_i}, \quad \bar{p}_i = p_i - \epsilon\frac{\partial(p_s - f_s,)}{\partial q_i} \quad s = 1, n. \tag{1.2.15}$$

Therefore, since $(p_1 - f_1, p_2 - f_2, ..., p_n - f_n)$ are derivable from $(\phi_1 - a_1, \phi_2 - a_2, ..., \phi_n - a_n)$, it follows that $(p_1 - f_1, p_2 - f_2, ..., p_n - f_n)$ are also in involution. As a result of this

$$\frac{\partial f_s}{\partial q_r} - \frac{\partial f_r}{\partial q_s} = 0 \qquad s, r = 1, n. \tag{1.2.16}$$

From Hamilton's equations of motion

$$\begin{aligned} -\frac{\partial H}{\partial q_r} &= \frac{\mathrm{d}p_r}{\mathrm{d}t} = \frac{\mathrm{d}f_r}{\mathrm{d}t} \\ &= \frac{\partial f_r}{\partial t} + \sum_{s=1}^{n} \frac{\partial f_r}{\partial q_s}\frac{\mathrm{d}q_s}{\mathrm{d}t} \\ &= \frac{\partial f_r}{\partial t} + \sum_{s=1}^{n} \frac{\partial f_s}{\partial q_r}\frac{\partial H}{\partial p_s}. \end{aligned} \tag{1.2.17}$$

Thus

$$\frac{\partial f_r}{\partial t} = -\frac{\partial H}{\partial q_r} - \sum_{s=1}^{n} \frac{\partial H}{\partial p_s}\frac{\partial f_s}{\partial q_r}$$

$$\frac{\partial f_r}{\partial t} = \frac{\partial H_1}{\partial q_r}, \tag{1.2.18}$$

where H_1 is H in the variables $(q_1, q_2, ..., q_n, a_1, a_2, ..., a_n, t)$.

Define the function W as follows:

$$\frac{\partial W}{\partial t} = -H_1, \quad \frac{\partial W}{\partial q_i} = f_i, \qquad i = 1, n. \tag{1.2.19}$$

This definition is consistent with equations (1.2.16) and (1.2.18). The remaining first integrals necessary to solve the problem are

$$I_r = \frac{\partial W}{\partial a_r} \qquad r = 1, n. \tag{1.2.20}$$

That these are first integrals can easily be seen by considering their time derivatives:

$$\begin{aligned}
\frac{\mathrm{d}I_r}{\mathrm{d}t} &= \frac{\partial I_r}{\partial t} + \sum_{k=1}^{n} \frac{\partial I_r}{\partial q_k}\frac{\partial H}{\partial p_k}(q_1, q_2, ..., q_n, f_1, f_2, ..., f_n, t) \\
&= \frac{\partial^2 W}{\partial t \partial a_r} + \sum_{k=1}^{n} \frac{\partial^2 W}{\partial q_k \partial a_r}\frac{\partial H}{\partial p_k}\left(q_1, q_2, ..., q_n, \frac{\partial W}{\partial q_1}, \frac{\partial W}{\partial q_2}, ..., \frac{\partial W}{\partial q_n}, t\right) \\
&= \frac{\partial}{\partial a_r}\left(\frac{\partial W}{\partial t} + H_1\right) \\
&= 0 \qquad\qquad \text{(in view of equation (1.2.19)).}
\end{aligned} \tag{1.2.21}$$

Although this seems to be a very useful theorem, it is very difficult to construct the remaining first integrals and is probably easier to find them by other methods. According to Poisson's theorem (Goldstein 1980), if the second derivatives of two first integrals of a system are continuous, their Poisson Bracket will also be a first integral and may be new. Thus, if some first integrals of a system are known, it may be possible to find more by calculating their Poisson Bracket expressions.

Definition 5:

An n-dimensional autonomous Hamiltonian system is said to be Liouville integrable if it possesses n functionally independent functions, including the Hamiltonian, which are in involution.

Since in this thesis only two-dimensional systems will be considered and, for the systems under consideration, the Hamiltonian is already a first integral, it is only necessary to find one extra first integral for the system to be Liouville integrable.

Note that an n-dimensional Hamiltonian system can have no more than n distinct first integrals in involution. It cannot have more than $2n - 1$ autonomous first integrals.

Definition 6:

A Hamiltonian system which possesses this maximum number of autonomous first integrals which are globally defined and single-valued is called superintegrable.

The Kepler problem is an example of such a system. For this system $n = 2$, therefore three first integrals are required for the definition to be satisfied. Two first integrals are given by the Hamiltonian

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{k}{r} \tag{1.2.22}$$

and the angular momentum

$$L = r \times \dot{r}. \tag{1.2.23}$$

Either of the two components of conserved vector

$$K = \dot{r} - \frac{k\hat{\theta}}{L}, \tag{1.2.24}$$

called Hamilton's vector, give the third first integral. The other component is a first integral which is functionally dependent on the first three.

The orbit equation can be determined from L and K. Denote the cross product of K and L by

$$J = K \times L, \tag{1.2.25}$$

called the Laplace-Runge-Lenz vector. Measuring θ from J, taking the scalar product of r with J and rearranging the result gives

$$r = \frac{L^2}{k - J\cos(\theta)}. \tag{1.2.26}$$

1.3 Canonical transformations

Transformations are usually made to simplify the system to a form for which the solution is more obvious. If a generalized canonical transformation is applied to two commuting functions, the two transformed functions will also commute. Therefore generalized canonical transformations transform integrable systems into integrable systems, i.e., they preserve integrability.

Definition 7:
A generalized canonical transformation is a transformation of the form

$$Q_i = f_i(q,p), \qquad P_i = g_i(q,p), \qquad i = 1, n \tag{1.3.0.1}$$

for which $[f_j, f_k]_{PB} = [g_j, g_k]_{PB} = 0$ and $[f_j, g_k]_{PB} = \delta_{jk} Z(q,p)$ as defined in (Hietarinta 1987). At present use of the term Generalized Canonical transformation is not standard. An alternative definition to that given here allows the transformation to depend on time (Burgan 1978). When $Z = 1$, the transformation is called a canonical transformation. That a generalized canonical transformation preserves involution is easily proven.

Let $K(Q,P)$ and $L(Q,P)$ be two functions in involution. Under a generalized canonical transformation they transform to $k(q,p) = K(f(q,p), g(q,p))$ and $l(q,p) = L(f(q,p), g(q,p))$ respectively. Making use of the chain rule and the commutation properties of the f's and g's one obtains

$$[k, l]_{PB(q,p)} = \left(\frac{\partial K}{\partial Q_i}\frac{\partial L}{\partial P_j} - \frac{\partial K}{\partial P_j}\frac{\partial L}{\partial Q_i}\right)[f_i, g_j]_{PB(p,q)}[K, L]_{PB(Q,P)} Z. \tag{1.3.0.2}$$

Therefore k and l are also in involution.

1.3.1 Time reflection symmetry

Consider the transformation $p \to -p$. If a function $K(q,p)$ has the property $K(-q,p) = cK(q,p)$, K is said to have good time reflection parity. If $c = 1$, K is even and, if $c = -1$, K is odd.

Theorem 2:
Suppose that two non-trivial functions K and L commute (*definition* 3). If K has good time reflection parity, then there exists a non-trivial function L_* that has good time reflection parity and $[K, L_*]_{PB} = 0$.

Proof:
The transformation $\mathbf{p} \to -\mathbf{p}$ is a generalized canonical transformation. Therefore $K(\mathbf{q}, -\mathbf{p}) = cK(\mathbf{q}, \mathbf{p})$ and $L(\mathbf{q}, -\mathbf{p})$ will also commute. K will now also commute with $L_+(\mathbf{q}, \mathbf{p}) = \frac{1}{2}[L(\mathbf{q}, \mathbf{p}) + L(\mathbf{q}, -\mathbf{p})]$ and $L_-(\mathbf{q}, \mathbf{p}) = \frac{1}{2}[L(\mathbf{q}, \mathbf{p}) - L(\mathbf{q}, -\mathbf{p})]$, both of which have good time reflection parity. The first is even and the second is odd. At least one of these must be nonzero and the required function has been obtained.

Corollary:
The function $H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y)$ has good time reflection parity and therefore all first integrals of this system will be either odd or even.

The following transformations which are discussed here are generalized canonical transformations and enable one to eliminate unnecessary freedom in a system thereby reducing the number of degrees of freedom.

1.3.2 Scaling

A scaling is a transformation of the form

$$\mathbf{q} \to c\mathbf{q}, \quad \mathbf{p} \to d\mathbf{p}, \qquad (1.3.2.1)$$

where c and d are constants. It preserves integrability and if $d = 1/c$ it is canonical. Here scaling will mainly be applied to homogeneous polynomial functions.

Definition 8:
A polynomial $P(q)$ is said to be homogeneous of degree n if $P(cq) = c^n P(q)$.

Definition 9:
A function $F(\mathbf{q}, \mathbf{p})$ is said to be weighted homogeneous if there exist integers n, m and k so that $F(c^m\mathbf{q}, c^n\mathbf{p}) = c^k F(\mathbf{q}, \mathbf{p})$.

As an example of how scaling may be used, consider the Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + ax^4 + bx^2y^2 + cy^4,$$

where a, b and c are constants. If $b \neq 0$ then the scaling $(p_x, p_y) \rightarrow (p_x, p_y)$, $(x, y) \rightarrow (b^{-1/4}x, b^{-1/4}y)$ will scale b to 1 and leave the system a little simpler. If there are two constants in the Hamiltonian with different homogeneous degrees, it is possible to scale both of them to 1 by performing the transformation $(x, y) \rightarrow (cx, cy)$, $(p_x, p_y) \rightarrow (dp_x, dp_y)$ and then dividing by d^2. The constants c and d may then be chosen to obtain the desired values of the coefficients.

For weighted homogeneous functions a theorem similar to the previous one exists.

Theorem 3:

If $K(\mathbf{q}, \mathbf{p})$ is a weighted homogeneous function which commutes with a polynomial L, i.e. $[K, L]_{PB} = 0$, then L may be written as $L = \sum_{j=1}^{M} L_j(\mathbf{q}, \mathbf{p})$, where each L_j is weighted homogeneous of a different degree and commutes with K.

Proof:

Since K is weighted homogeneous there exist m, n and k such that under the transformation $\mathbf{q} \rightarrow c^n\mathbf{q}$, $\mathbf{p} \rightarrow c^m\mathbf{p}$ K has the property $K(c^n\mathbf{q}, c^m\mathbf{p}) = c^k K(\mathbf{q}, \mathbf{p})$. Each term of L will gain a factor of the form c^l under this transformation. The sum of all the terms with the same value of l will be weighted homogeneous. Denote such a sum by L_j. Now $L(\mathbf{q}, \mathbf{p}) = \sum_{j=1}^{M} L_j(\mathbf{q}, \mathbf{p})$, where $L_j(c^n\mathbf{q}, c^m\mathbf{p}) = c^{l_j} L_j(\mathbf{q}, \mathbf{p})$ and $l_i \neq l_j$ for $i \neq j$.

Now it will be shown that each L_j commutes with $K(\mathbf{q}, \mathbf{p})$. The above scaling preserves integrability therefore $K(c^n\mathbf{q}, c^m\mathbf{p})$ and $L(c^n\mathbf{q}, c^m\mathbf{p})$ still commute. Since $c^k K(\mathbf{q}, \mathbf{p}) = K(c^n\mathbf{q}, c^m\mathbf{p})$, $K(\mathbf{q}, \mathbf{p})$ will also commute with $L(c^n\mathbf{q}, c^m\mathbf{p})$ for any value of c. Therefore K also commutes with the sum

$$\sum_{i=1}^{M} d_i L(c_i\mathbf{q}, c_i\mathbf{p}) = \sum_{i=1}^{M} d_i \sum_{j=1}^{M} c_i^{l_j} L_j(\mathbf{q}, \mathbf{p}) = \sum_{j=1}^{M} \left(\sum_{i=1}^{M} d_i c_i^{l_j} \right) L_j. \tag{1.3.2.2}$$

Now the function L_m may be isolated by choosing the d_i's so that

$$\sum_{i=1}^{M} d_i c_i^{l_j} = \delta_{jm}, \qquad j = 1, M. \tag{1.3.2.3}$$

This set of equations has a non-trivial solution since the c_i's can be chosen such that $\det(c_i^{l_j})_{ji} \neq 0$.

Corollary:
From this theorem it follows that weighted homogeneous Hamiltonians may be assumed to have weighted homogeneous first integrals.

The following theorem is a generalization of the previous theorem.

Theorem 4:
Let K be a weighted homogeneous polynomial and L a polynomial which commute. Suppose that a scaling

$$\mathbf{q} \to c^n \mathbf{q}, \qquad \mathbf{p} \to c^m \mathbf{p}$$

exists such that

$$K'(\mathbf{q}, \mathbf{p}) = K(c^n \mathbf{q}, c^m \mathbf{p}) c^k = K_0(\mathbf{q}, \mathbf{p}) + \sum_{l=1}^{M} c^{m_l} K_l(\mathbf{q}, \mathbf{p}), \text{ where } 0 < m_l, \tag{1.3.2.4}$$

then K_0 will commute with a polynomial contained in L.

Proof:
As in the last proof L may be written as

$$L(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^{M} L_i(\mathbf{q}, \mathbf{p}), \text{ where } L_i(c^n \mathbf{q}, c^m \mathbf{p}) = c^{k_i} L_i(\mathbf{q}, \mathbf{p}) \tag{1.3.2.5}$$

and $k_i \neq k_j$ for $i \neq j$. If k_r is the smallest of the k's, then

$$L'(\mathbf{q}, \mathbf{p}) = L(c^n \mathbf{q}, c^m \mathbf{p}) c^{-k_r} = L_r + \sum_{j=1}^{M} c^{k_j - k_r} L_j. \tag{1.3.2.6}$$

Since integrability is preserved under this scaling, K' and L' will commute. Now K_0 and L_r will also commute since they are the only terms which will contribute to the c^0 term in the expression for the Poisson Bracket. If it is found that $L_r = f(K_0)$, then one could construct a new L namely $L^* = L - f(K_0)$ which will commute with K. If L is functionally independent of K, then this procedure will eventually yield a functionally independent function which commutes with K_0.

Corollary :

Suppose $H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y)$ where V is polynomial and consists of weighted homogeneous parts of degrees $n_1 < n_2 < ... < n_N$. Then $H_* = \frac{1}{2}(p_x^2 + p_y^2) + V_1$ and $H^* = \frac{1}{2}(p_x^2 + p_y^2) + V_N$ are both integrable.

Proof :

Let $K = H_*$, $m = n_1$, $n = 2$ and $k = -2n_1$ for the first case. For the second case let $K = H^*$, $m = -n_N$, $n = -2$ and $k = 2n_N$.

1.3.3 Rotation

A rotation is a transformation of the form

$$x = \cos(\theta)X + \sin(\theta)Y \qquad p_x = \cos(\theta)P_X + \sin(\theta)P_Y,$$
$$y = -\sin(\theta)X + \cos(\theta)Y, \qquad p_y = -\sin(\theta)P_X + \cos(\theta)P_Y \tag{1.3.3.1}$$

and is canonical. The rotation in the co-ordinate space (x, y) induces the rotation in the momentum space (p_x, p_y).

In the present context the rotational degree of freedom may be used to fix a constant in the potential or in the second invariant. Consider a homogeneous polynomial in x and y (or p_x and p_y)

$$P(x, y) = \sum_{n=0}^{N} a_n x^{N-n} y^n. \tag{1.3.3.2}$$

Under the above rotation it becomes

$$\begin{aligned} P(x, y) = P'(X, Y) = & X^N \left\{ \sum_{n=0}^{N} a_n \cos(\theta)^{N-n} (-\sin(\theta))^n \right\} \\ & + X^{N-1} Y \left\{ \sum_{n=0}^{N} \cos(\theta)^{N-n} (-\sin(\theta))^n [(n+1)a_{n+1} \right. \\ & \left. - (N-n+1)a_{n-1}] \right\} + \ldots\ldots . \end{aligned} \tag{1.3.3.3}$$

If $a_{N-1} = 0$, then the coefficient of $X^{N-1}Y$ will vanish if $\cos(\theta) = 0$. For all other cases the equation

$$\sum_{n=0}^{N} (-\tan(\theta))^n [(n+1)a_{n+1} - (N-n+1)a_{n-1}] = 0, \tag{1.3.3.4}$$

where a_0, a_1 and a_2 are constants. The above rotation will take it to

$$\begin{aligned}
&\left[a_2\cos^2(\theta) - a_1\cos(\theta)\sin(\theta) + a_0\sin^2(\theta)\right] P_X^2 \\
&+ \left[2(a_2 - a_0)\cos(\theta)\sin(\theta) + a_1(\cos^2(\theta) - \sin^2(\theta))\right] P_X P_Y \\
&+ \left[a_2\sin^2(\theta) + a_1\sin(\theta)\cos(\theta) + a_0\cos^2(\theta)\right] P_Y^2.
\end{aligned} \tag{1.3.3.6}$$

It will only be possible to eliminate the $P_X P_Y$ term if

$$\tan(\theta) = \frac{(a_2 - a_0)}{a_1} \pm \frac{1}{a_1}((a_2 - a_0)^2 + a_1^2)^{1/2}. \tag{1.3.3.7}$$

Unless $(a_2 - a_0) = \pm i a_1$ in which case $\tan^2(\theta) = -1$, a rotation may be found to satisfy this equation. In the former case a rotation may be found to rotate the quadratic to $p_x(p_x \pm i p_y)$, or $(p_x \pm i p_y)^2$.

If the polynomial is written in the form

$$P(x, y) = \Pi_{n=1}[a_n x + b_n y], \tag{1.3.3.8}$$

a different method of fixing a constant may be applied. Under the transformation it becomes

$$P(X,Y) = \Pi_{n=1}[(a_n\cos(\theta) - b_n\sin(\theta))X + (a_n\sin(\theta) + b_n\cos(\theta))Y]. \tag{1.3.3.9}$$

Now by choosing θ so that

$$a_n\sin(\theta) + b_n\cos(\theta) = 0, \tag{1.3.3.10}$$

one or more factors of X may be extracted from the product so that

$$P(X,Y) = X^k Q(X,Y).$$

This may always be done provided that P is not of the form

$$P(x,y) = (x + iy)^K (x - iy)^L. \tag{1.3.3.11}$$

1.3.4 Translation

A translation takes the form

$$q_i \to q_i + u_i, \qquad i = 1, n \tag{1.3.4.1}$$

where the u_i's are constants.

The leading terms of a first integral often contain angular momentum terms. Translations may be used to remove some of the free constants. For example consider the first integral $(ax+b)p_y+(-ay+c)p_x$. If $a \neq 0$, then the translation $x \to x-a/b$, $y \to y + c/a$ will change I to $I = a(xp_y - yp_x)$. If $a = 0$, then a rotation and scaling will change I to $I = p_x$ or $I = p_x \pm ip_y$.

An example of how a translation may be used to simplify a potential is the tranformation of the potential

$$V = x^4 + 6x^2y^2 + 8y^4 + a(8y^3 + 3yx^2) + k(x^2 + 4y^2) + \frac{3a^2}{8}(x^2 + 8y^2), \tag{1.3.4.2}$$

to

$$V = x^4 + 6x^2y^2 + 8y^4 + k(x^2 + 4y^2) + ey, \tag{1.3.4.3}$$

using the transformation $y \to y - a/4$ with $e = -2ak - a^3/2$. Since the latter potential is integrable and the transformation is canonical, the former is also integrable.

1.3.5 Gauge transformations

The gauge transformation is only of use when dealing with Hamiltonians which contain terms linear in the momenta e.g.

$$H = \frac{1}{2}(p_x^2 + p_y^2) + A(x,y)p_x + B(x,y)p_y + V(x,y). \tag{1.3.5.1}$$

The transformation takes the form

$$p_x \to p_x + F_x(x,y), \qquad p_y \to p_y + F_y(x,y), \tag{1.3.5.2}$$

where $F(x,y)$ is arbitrary. The functions A, B and C change to

$$\begin{gathered} A \to A + F_x, \qquad B \to B + F_y, \\ C \to C + AF_x + BF_y + \frac{1}{2}(F_x^2 + F_y^2). \end{gathered} \tag{1.3.5.3}$$

The following quantities remain invariant with respect to a gauge transformation

$$U = A_y - B_x,$$
$$W = C - \frac{1}{2}(A^2 + B^2). \tag{1.3.5.4}$$

These quantities are of great assistance in identifying potentials which may be in one of many transformed forms. If for a system $U = 0$, one may transform to the gauge where $A = B = 0$ and considerably simplify the system.

The inverse problem of finding A, B and C given U and W is quite difficult. In later chapters it is necessary to do this in order to obtain particular potentials.

1.3.6 Coupling constant metamorphosis

The transformation discussed here is very different from those that have been discussed up till now. The transformation of variables is not stated explicitly, but the transformation takes a Hamiltonian of the form $H = H_0(\mathbf{q},\mathbf{p}) + gH_1(\mathbf{q},\mathbf{p})$ to one of the form $K = H_0(\mathbf{q},\mathbf{p})/H_1(\mathbf{q},\mathbf{p}) - e/H_1(\mathbf{q},\mathbf{p})$. The interesting property of this tranformation is contained in the following theorem.

Theorem 5:
Assume the Hamiltonian $H = H_0 + gH_1$ is integrable at any energy and value of the coupling constant g, with a first integral $I = I(\mathbf{q},\mathbf{p};g)$. Then the Hamiltonian $K = H_0/H_1 - e/H_1$ is integrable for any value of the energy or coupling constant e and has a second invariant $J = I(\mathbf{q},\mathbf{p};(e - H_0)/H_1)$.

Proof:
Since J is I with g replaced by $-K$ the Poisson Bracket of K and J may be written as follows is

$$[K,J]_{PB} = \frac{[H_0,J]_{PB}}{H_1} - (H_0 - e)\frac{[H_1,J]_{PB}}{H_1^2} = \frac{[H_0(\mathbf{q},\mathbf{p}) - KH_1(\mathbf{q},\mathbf{p}); I(\mathbf{q},\mathbf{p};-K)]_{PB}}{H_1(\mathbf{q},\mathbf{p})}.$$

Since K may be treated as a constant and H is integrable for all values of g the Poisson Bracket vanishes and K is integrable.

The constant g is related to the new energy K in that K has the same expression as the solution for g from the equation $H = H_0 + gH_1$ and hence the name coupling constant metamorphosis. If H is only integrable at specific value of the energy, then the integrability of K can only be garanteed at one value of the energy. This transformation has been used in (Hietarinta 1984c *et al*) to obtain some interesting results.

1.4 The direct method and polynomial first

1.4.1 The general problem

The method is quite straightforward and needs little
first integral and the Hamiltonian must be made.
to be a polynomial in the momenta, for example i
choose

$$I = a(x, y)p_x^2 + b(x, y)p_x p_y + c($$

where a, b c and d are unknown functions of x and

$$H = \frac{1}{2}(p_x^2 + p_y^2) + A(x, y)p_x + B($$

where A, B and V are unknown. The unknowns in
the requirement

$$\dot{I} = [I, H]_{PB} = ($$

In this section the existence of polynomial fi
Hamiltonian systems of the form

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V($$

will be discussed. According to the corollary to T
a Hamiltonian of this form must be either odd or e
first integral may be written as

$$I = \sum_{n=0}^{[N/2]} \sum_{m=0}^{N-2n} p_x^m p_y^{N-2n-m} d^{n}$$

Manipulation of the indices so that the exponents of the p_x's are equal gives

$$\sum_{n=0}^{[N/2]} \sum_{m=1}^{N-2n+1} p_x^m p_y^{N-2n-m+1} \frac{\partial d^{m-1,N-2n}}{\partial x}$$

$$+ \sum_{n=0}^{[N/2]} \sum_{m=0}^{N-2n} p_x^m p_y^{N-2n-m+1} \frac{\partial d^{m,N-2n}}{\partial y}$$

$$- \frac{\partial V}{\partial x} \sum_{n=0}^{[N/2]} \sum_{m=-1}^{N-2n-1} (m+1) p_x^m p_y^{N-2n-m-1} d^{m+1,N-2n}$$

$$- \frac{\partial V}{\partial y} \sum_{n=0}^{[N/2]} \sum_{m=0}^{N-2n} (N-2n-m) p_x^m p_y^{N-2n-m-1} d^{m,N-2n} = 0, \qquad (1.4.1.4)$$

where $[N/2]$ means the integer value of $N/2$.

The index n may now be manipulated so that the p_y's have the same exponents:

$$\sum_{n=0}^{[N/2]} \sum_{m=1}^{N-2n+1} p_x^m p_y^{N-2n-m+1} \frac{\partial d^{m-1,N-2n}}{\partial x}$$

$$+ \sum_{n=0}^{[N/2]} \sum_{m=0}^{N-2n} p_x^m p_y^{N-2n-m+1} \frac{\partial d^{m,N-2n}}{\partial y}$$

$$- \frac{\partial V}{\partial x} \sum_{n=1}^{[N/2+1]} \sum_{m=0}^{N-2n+1} (m+1) p_x^m p_y^{N-2n-m+1} d^{m+1,N-2n+2}$$

$$- \frac{\partial V}{\partial y} \sum_{n=0}^{[N/2+1]} \sum_{m=0}^{N-2n+2} (N-2n-m+2) p_x^m p_y^{N-2n-m+1} d^{m,N-2n+2} = 0, \qquad (1.4.1.5)$$

where the summation in the third term now begins at 0 since for $m = -1$ this term is zero anyway. The summation in the last term here may terminate at $N - 2n + 1$ since the next term in the summation is zero. Rearrangement of the previous result gives

$$\sum_{n=0}^{[N/2+1]} \sum_{m=0}^{N-2n+1} p_x^m p_y^{N-2n-m+1} \left\{ \frac{\partial d^{m-1,N-2n}}{\partial x} + \frac{\partial d^{m,N-2n}}{\partial y} - \frac{\partial V}{\partial x}(m+1) d^{m+1,N-2n+2} - \frac{\partial V}{\partial y}(N-2n-m+2) d^{m,N-2n+2} \right\} = 0, \qquad (1.4.1.6)$$

where $d^{s,t} = 0$ if $s < 0$ or $s > t$ or $t < 0$ or $t > N$. For a polynomial in p_x and p_y to be identically zero the coefficients of combinations of different powers of p_x and p_y must be zero. For the last equation this implies that

$$(m+1)\frac{\partial V}{\partial x}d^{m+1,k+1} + (k+1-m)\frac{\partial V}{\partial y}d^{m,k+1} = \frac{\partial d^{m-1,k-1}}{\partial x} + \frac{\partial d^{m,k-1}}{\partial y}, \qquad (1.4.1.7)$$

where $m = 0, k$, $k = N+1$, 0 or 1(2) and $d^{s,t}$ is as above. Henceforth this process will simply be refered to as separation of coefficients. There are now $(N+2)(N+4)/4$ differential equations to solve for $(N+2)^2/4$ unknowns if N is even and if N is odd there are $(N+3)^2/4$ equations to solve for $(N+3)(N+1)/4$ unknowns.

If $k = N+1$, then (1.4.1.7) takes a simpler form:

$$\frac{\partial d^{m-1,N}}{\partial x} + \frac{\partial d^{m,N}}{\partial y} = 0, \qquad m = 0, N+1. \qquad (1.4.1.8)$$

These equations may be solved to give

$$d^{m,N} = \sum_{v=0}^{m}\sum_{u=0}^{N-m}(-1)^v\binom{u+v}{v}x^u y^v C^{u+v,m-v,N}, \quad m = 0, N, \qquad (1.4.1.9)$$

where the C's are constants. If the Hamiltonian is a homogeneous polynomial of degree n_v, the first integral must also be homogeneous according to the corollary to *Theorem* 3. As a result all $d^{m,k}$ must have homogeneous degree $n_c + n_v(N-k)/2$, where n_c is the homogeneous degree of the $d^{m,N}$ above.

For $k = N-1$ equations (1.4.1.7) become

$$(m+1)\frac{\partial V}{\partial x}d^{m+1,N} + (N-m)\frac{\partial V}{\partial y}d^{m,N} = \frac{\partial d^{m-1,N-2}}{\partial x} + \frac{\partial d^{m,N-2}}{\partial y}, \qquad (1.4.1.10)$$

for $m = 0, N-1$. Here there are N equations for $N-1$ unknowns $d^{m,N-2}$. The coefficient $d^{0,N-2}$ may be eliminated between the equation for $m = 0$ and $m = 1$ by use of the integrability condition. The resultant equation has a term $\partial^2 d^{1,N-2}/\partial y^2$. Application of the integrability condition to this function enables one to eliminate it from the equation for $m = 2$. Continuing in this manner one gets the final equation

$$\sum_{n=0}^{N-1}(-1)^n\frac{\partial}{\partial x}^{N-n-1}\frac{\partial}{\partial y}^{n}\left[(n+1)\frac{\partial V}{\partial x}d^{n+1,N} + (N-n)\frac{\partial V}{\partial y}d^{n,N}\right] = 0. \qquad (1.4.1.11)$$

In this equation the $d^{n,N}$'s may be obtained from equation (1.4.1.9). If for a given V no C's can be found to satisfy this equation, then V does not have a first integral of order N. For any polynomial potential their exist constants C and a value N such that (1.4.1.11) will be satisfied. For some potentials the following result may be formulated.

Theorem 6:
A Hamiltonian of the form (1.4.1.1) has no non-trivial polynomial first integrals if all the partial derivatives of V are linearly independent over the ring of polynomials in x and y.

Proof:
Consider (1.4.1.11) with V satisfying the requirements of the theorem. As a result of the assumptions made, the coefficient of $(\partial/\partial x)^{N-n-1}(\partial/\partial y)^n V$ must be zero:

$$(n+1)d^{n+1,N} - (N-n+1)d^{n-1,N} = 0, \qquad n = 0, N-1. \tag{1.4.1.12}$$

Solving these equations, starting at $n = 0$, it soon becomes clear that, if n or N is odd, then $d^{n,N} = 0$ and, if n and N are both even, then

$$d^{2m,2M} = \binom{M}{m} d^{0,2M}. \tag{1.4.1.13}$$

The leading part of I is then given by $d^{0,2M}[p_x^2 + p_y^2]^M$, which is trivial since it contains the kinetic term of the Hamiltonian which may now be replaced by $2(H - V)$.

The equations (1.4.1.7) will now be solved for certain values of N.

1.4.2 The case N=1

According to the corollary to *Theorem* 2 a linear first integral of a system of the form (1.4.1.1) must be odd. Therefore it must have the form

$$I = a(x, y)p_x + b(x, y)p_y. \tag{1.4.2.1}$$

Taking the Poisson Bracket of I with H and separating by coefficients of p_x and p_y gives

$$\frac{\partial a}{\partial x} = 0, \tag{1.4.2.2a}$$

$$\frac{\partial a}{\partial y} - \frac{\partial b}{\partial x} = 0, \tag{1.4.2.2b}$$

$$\frac{\partial b}{\partial y} = 0, \tag{1.4.2.2c}$$

$$a\frac{\partial V}{\partial x} + b\frac{\partial V}{\partial y} = 0. \tag{1.4.2.2d}$$

The first three equations are easily integrated and yield the solution

$$a = cy + d,$$
$$b = -cx + e. \tag{1.4.2.3}$$

Equation (1.4.2.2d) now becomes

$$(cy + d)V_x + (-cx + e)V_y = 0,$$

which may be solved by the method of characteristics: The equation for the characteristic is

$$\frac{dx}{cy + d} = \frac{dy}{-cx + e},$$

with solution

$$u = \frac{c}{2}(x^2 + y^2) + ex - dy.$$

The general form of V is

$$V = V(u) = V(\frac{c}{2}(x^2 + y^2) + ex - dy). \tag{1.4.2.4}$$

The solution for I is

$$I = c(yp_x - xp_y) + dp_x + ep_y. \tag{1.4.2.5}$$

1.4.3 The case N=2

There has been interest in this problem for a long time. The first systematic attempt to solve this problem appeared in (Darboux 1901). His results are reproduced by (Whittaker 1944). However, these results were incomplete and more recent treatments have yielded the remaining cases. The cases 1), 2), 4), and 7) below may be found in (Winternitz *et al* 1966), other references in this regard are (Dorizzi *et al* 1983), (Ankiewicz and Pask 1983), (Thompson 1984a), and (Sen 1985). The cases 3), and 6) are discussed in (Dorizzi *et al* 1983). The case 5) was first given in (Hietarinta 1987).

For the given Hamiltonian the first integral must be either odd or even. Therefore for $N = 2$ it must be even. Let

$$I = Ap_x^2 + Bp_xp_y + Cp_y^2 + D, \tag{1.4.3.1}$$

where $A, ..., D$ are functions of x and y which still have to be determined. Taking the Poisson Bracket with H and separating by coefficients of p_x and p_y gives the following set of equations to solve

$$\begin{aligned}
\frac{\partial A}{\partial x} &= 0,\\
\frac{\partial A}{\partial y} + \frac{\partial B}{\partial x} &= 0,\\
\frac{\partial B}{\partial y} + \frac{\partial C}{\partial x} &= 0,\\
\frac{\partial C}{\partial y} &= 0,
\end{aligned} \tag{1.4.3.2}$$

$$\begin{aligned}
\frac{\partial D}{\partial x} &= 2A\frac{\partial V}{\partial x} + B\frac{\partial V}{\partial y},\\
\frac{\partial D}{\partial y} &= B\frac{\partial V}{\partial x} + 2C\frac{\partial V}{\partial y}.
\end{aligned} \tag{1.4.3.3}$$

Equations (1.4.3.2) are readily solved :

$$\begin{aligned}
A &= ay^2 + by + c,\\
B &= -2axy - bx - dy - e,\\
C &= ax^2 + dx + f,
\end{aligned} \tag{1.4.3.4}$$

where $a, ..., f$ are constants.

The integrability condition on D may be used to eliminate it from (1.4.3.3) giving

$$(2axy + bx + dy + e)\left(\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial y^2}\right) - 2[a(x^2 - y^2) + dx - by + f - c]\frac{\partial^2 V}{\partial x \partial y}$$
$$+3(2ay + b)\frac{\partial V}{\partial x} - 3(2ax + d)\frac{\partial V}{\partial y} = 0. \tag{1.4.3.5}$$

It is convenient to break the problem up into eight different subcases to solve it. The first case is that which Darboux solved.

<u>case 1)</u>

In this case $a \neq 0$ and for simplicity let $a = 1$. Translations may be used to eliminate b and d. By subtracting the correct amount of H from I, f may be eliminated. If

$e^2 + c^2 \neq 0$, then the system may be rotated to eliminate e from I and therefore also from (1.4.3.5). In this case it is convenient to replace c by c^2 and equation (1.4.3.5) may then be written as :

$$xy\left(\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial y^2}\right) + (y^2 - x^2 + c^2)\frac{\partial^2 V}{\partial x \partial y} + 3y\frac{\partial V}{\partial x} - 3x\frac{\partial V}{\partial y} = 0. \qquad (1.4.3.6)$$

The equation of the characteristic of this equation is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(y^2 - x^2 + c^2) \pm \sqrt{y^4 - x^2y^2 + x^4 + 2(y^2 - x^2)c^2 + c^4}}{xy},$$

which is equivalent to

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - \frac{(y^2 - x^2 + c^2)}{xy}\frac{\mathrm{d}y}{\mathrm{d}x} - 1 = 0. \qquad (1.4.3.7)$$

Let $v = y^2$ and $u = x^2$ then

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{y\mathrm{d}y}{x\mathrm{d}x}.$$

By multiplying (1.4.2.7) through by y^2/x^2 and using the latter result the following equation is obtained

$$\left(\frac{\mathrm{d}v}{\mathrm{d}u}\right)^2 - \frac{v - u + c^2}{u}\frac{\mathrm{d}v}{\mathrm{d}u} - \frac{v}{u} = 0. \qquad (1.4.3.8)$$

Rearrangement gives

$$v = u\frac{\mathrm{d}v}{\mathrm{d}u} - c^2\left(1 + \frac{\mathrm{d}v}{\mathrm{d}u}\right)^{-1}\frac{\mathrm{d}v}{\mathrm{d}u}. \qquad (1.4.3.9)$$

This is a Clairaut's equation the solution of which is given in (Ince 1956) which is repeated here for convenience. Differentiation of (1.4.3.9) *wrt* u gives

$$p = p + \left\{u + \frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{-c^2p}{1+p}\right)\right\}\frac{\mathrm{d}p}{\mathrm{d}u},$$

i.e.,

$$\left\{u + \frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{-c^2p}{1+p}\right)\right\}\frac{\mathrm{d}p}{\mathrm{d}u} = 0, \qquad (1.4.3.10)$$

where $p = \mathrm{d}v/\mathrm{d}u$. Either one of the factors in this equation must be zero. It is found that choosing $\mathrm{d}p/\mathrm{d}u = 0$ namely $p = m = constant$ yields the general solution. In this case (1.4.3.9) becomes, after some rearrangement,

$$(m+1)(um - v) - c^2m = 0, \qquad (1.4.3.11)$$

or *ito* x and y

$$(m+1)(x^2m - y^2) - c^2m = 0. \qquad (1.4.3.12)$$

This may be rewritten as

$$\frac{x^2}{\gamma^2} + \frac{y^2}{\gamma^2 - c^2} = 1, \tag{1.4.3.13}$$

where $\gamma^2 = c^2/(m+1)$.

From the above equation it is evident that the characteristic curves are confocal ellipses and hyperbolae. It is then natural to take as new variables the parameters of confocal ellipses and hyperbolae denoted by α and β:

$$x = \frac{\alpha\beta}{c}, \quad y = \frac{1}{c}\{(\alpha^2 - c^2)(c^2 - \beta^2)\}^{\frac{1}{2}}. \tag{1.4.3.14}$$

Under this transformation (1.4.3.6) becomes

$$(\beta^2 - \alpha^2)\frac{\partial^2 V}{\partial\alpha\partial\beta} + 2\beta\frac{\partial V}{\partial\alpha} - 2\alpha\frac{\partial V}{\partial\beta} = 0, \tag{1.4.3.15}$$

the solution of which is

$$V = \frac{f(\alpha) - \phi(\beta)}{(\alpha^2 - \beta^2)}, \tag{1.4.3.16}$$

where f and ϕ are arbitrary functions of their arguments and

$$2\alpha^2 = r^2 + c^2 + [(r^2 + c^2)^2 - 4c^2x^2]^{\frac{1}{2}},$$

$$2\beta^2 = r^2 + c^2 - [(r^2 + c^2)^2 - 4c^2x^2]^{\frac{1}{2}}, \tag{1.4.3.17}$$

where $r^2 = x^2 + y^2$.

The corresponding first integral is

$$I = (xp_y - yp_x)^2 + c^2p_x^2 + \frac{2[\beta^2 f(\alpha) - \alpha^2\phi(\alpha)]}{(\alpha^2 - \beta^2)}. \tag{1.4.3.18}$$

<u>case 2)</u>

If $a \neq 0$ but $c = 0$, then (1.4.3.6) may be written as

$$2B + x\frac{\partial B}{\partial x} + y\frac{\partial B}{\partial y} = 0, \tag{1.4.3.19}$$

where $B = y(\partial V/\partial x) - x(\partial V/\partial y)$. Using the method of characteristics to solve this equation gives

$$B = \frac{F(y/x)}{y^2}, \tag{1.4.3.20}$$

where F is arbitrary. This is equivalent to

$$yV_x - xV_y = F(y/x)\frac{1}{y^2}, \tag{1.4.3.21}$$

where B has been replaced by the expression it represents. The characteristics are determined from

$$\frac{dx}{y} = \frac{-dy}{x} = \frac{dV}{[F(y/x)/y^2]}. \tag{1.4.3.22}$$

From the first and second components of the characteristic $u = x^2+y^2$ is easily obtained. From the second and third components of (1.4.3.22), with the transformation $x = \cos(\theta)$, $y = \sin(\theta)$, the folllowing equation is obtained

$$-d\theta = \frac{dV}{F(\tan(\theta))\sec^2(\theta)}. \tag{1.4.3.23}$$

This equation is easily integrated and the final solution is

$$V = g(x^2+y^2) + \frac{f(y/x)}{x^2+y^2}, \tag{1.4.3.24}$$

where f and g are arbitrary functions of their arguments. The associated first integral is

$$I = (xp_y - yp_x)^2 + 2f(\frac{y}{x})/r^2. \tag{1.4.3.25}$$

<u>case 3)</u>

If $a \neq 0$ and $\mathbf{c} \neq 0$, but $e^2 + c^2 = 0$, e may not be removed by a rotation of (1.4.3.5), as was done to obtain (1.4.3.6). The solution for V has the the same form as in the first case except that now

$$\begin{aligned} 2\alpha^2 &= r^2 + [r^4 - 2c(x \pm iy)^2]^{\frac{1}{2}}, \\ 2\beta^2 &= r^2 - [r^4 - 2c(x \pm iy)^2]^{\frac{1}{2}}. \end{aligned} \tag{1.4.3.26}$$

The corresponding first integral is given by

$$I = (xp_y - yp_x)^2 + cp_x(p_x \pm ip_y) + \frac{2[\beta^2 f(\alpha) - \alpha^2\phi(\beta)]}{\alpha^2 - \beta^2}. \tag{1.4.3.27}$$

<u>case 4)</u>

If $a = 0$ and $b^2 + d^2 \neq 0$, a translation and rotation of the axis by $\tan^{-1}(d/b)$ gives new co-ordinates

$$\begin{aligned} X &= b(x-u) - d(y-w), \\ Y &= d(x-u) + b(y-w). \end{aligned} \tag{1.4.3.28}$$

If u and w are chosen as

$$u = \frac{be + dc}{b^2 + d^2}, \quad w = \frac{de - bc}{b^2 + d^2}, \tag{1.4.3.29}$$

equation (1.4.3.5) becomes

$$X\left(\frac{\partial^2 V}{\partial X^2} - \frac{\partial^2 V}{\partial Y^2}\right) + 2Y\frac{\partial^2 V}{\partial X \partial Y} + 3\frac{\partial V}{\partial X} = 0. \tag{1.4.3.30}$$

If U is defined by

$$V = \frac{U}{\xi^2 + \eta^2}, \tag{1.4.3.31}$$

where

$$X = \xi\eta \text{ and } Y = \frac{1}{2}(\xi^2 - \eta^2), \tag{1.4.3.32}$$

equation (1.4.3.30) becomes

$$\frac{\partial^2 U}{\partial \xi \partial \eta} = 0 \tag{1.4.3.33}$$

with solution $U = g_1(\xi) + g_2(\eta)$, where g_1 and g_2 are arbitrary functions. V is given by

$$V = \frac{g_1(\xi) + g_2(\eta)}{\xi^2 + \eta^2} \tag{1.4.3.34}$$

and the first integral is

$$I = (yp_x - xp_y)p_x + \frac{\xi^2 g_2(\eta) - \eta^2 g_1(\xi)}{\xi^2 + \eta^2}. \tag{1.4.3.35}$$

<u>Case 5)</u>

If $a = 0$ but $b \neq 0$ and $b^2 + d^2 = 0$, then by means of a scaling one can obtain $b = 1$, $d = \pm i$. If then $e \neq i(c - f)$, transformations can be found to take c, e and f to $c = i/8$, $e = -1/4$ and $f = -i/8$. The potential and corresponding first integral are

$$V = \frac{1}{\sqrt{\bar{z}}}[f(z + \sqrt{\bar{z}}) + g(z - \sqrt{\bar{z}})], \tag{1.4.3.36}$$

$$\begin{aligned} I =& (yp_x - xp_y)(p_x + ip_y) + \frac{i}{8}(p_x - ip_y)^2 \\ &+ i[1 + z\sqrt{\bar{z}}]f(z + \sqrt{\bar{z}}) + i[-1 + z\sqrt{\bar{z}}]g(z - \sqrt{\bar{z}}), \end{aligned} \tag{1.4.3.37}$$

where $z = x + iy$.

case 6)

If in the previous case $e = i(c-f)$, c, e and f may be eliminated and the singular limit is obtained :

$$V = \frac{F(x \pm iy)}{r} + G'(x \pm iy), \tag{1.4.3.38}$$

with first integral

$$\begin{aligned} I = &(yp_x - xp_y)(p_x \pm ip_y) - i(x \pm iy)[F(x \pm iy)/r + G'(x \pm iy)] \\ &+ iG(x \pm iy), \end{aligned} \tag{1.4.3.39}$$

where F and G are arbitrary functions of their arguments.

Case 7)

If $a = 0$, $b = 0$, $d = 0$, subtraction of the correct amount of H from I will eliminate f. Furthermore, unless $e^2 + c^2 = 0$, which will be discussed in the next case, the system may be rotated so that $c = 1$ and $e = 0$. The equation for V reduces to

$$V_{xy} = 0, \tag{1.4.3.40}$$

with solution

$$V = f(x) + g(y) \tag{1.4.3.41}$$

and first integral

$$I = p_x^2 + 2f(x) \tag{1.4.3.42}$$

(or alternatively $I = p_y^2 + 2g(y)$).

case 8)

If $a = 0$, $b = 0$, $d = 0$, $f = 0$, $c \neq 0$ but $c^2 + e^2 = 0$, then c and e may be chosen as $c = 1$, $e = \pm i/2$ in which case

$$V = r^2 F''(x \pm iy) + G(x \pm iy) \tag{1.4.3.43}$$

and

$$\begin{aligned} I = &p_x(p_x \pm ip_y) + r^2 F''(x \pm iy) + G(x \pm iy) + 2(x \pm iy)F'(x \pm iy) \\ &- 2F(x \pm iy). \end{aligned} \tag{1.4.3.44}$$

Particular examples

The Hénon-Heiles-type potential

$$V = \frac{1}{2}(x^2 + y^2) + \frac{y^3}{3} + x^2 y, \tag{1.4.3.45}$$

with first integral

$$I = p_x p_y + xy + \frac{x^3}{3} + xy^2, \tag{1.4.3.46}$$

given in (Aizawa and Saito 1972), may be transformed by a rotation to a potential of the type in case 7) above.

A special case of case 4) is

$$V = \frac{1}{r} \log\left(\frac{r+y}{r-y}\right), \tag{1.4.3.47}$$

found by (Dorizzi *et al* 1983).

By choosing the arbitrary functions in the results for V in the cases studied above particular examples are obtained. A special case of case 2) is $V = r^n$. $V = [a(r+y)^{n+1} + b(r-y)^{n+1}]/r$ is a particular case of case 4). $V = a(x \pm iy)^{n+1}/r + b(x \pm iy)^n$, $V = ax^n + by^n$ and $V = ar^2(x \pm iy)^{n-2} + b(x \pm iy)^n$ are special cases of 6), 7) and 8) respectively.

There are some potentials which belong to more than one of the eight cases : The potential (Fris *et al* 1965)

$$V = a(x^2 + y^2) + bx^{-2} + cy^{-2}, \tag{1.4.3.48}$$

belongs to cases 2) and 7). The potential

$$V = ax^2 + by^2 + cx^{-2} \tag{1.4.3.48}$$

belongs to the cases 4) and 7) and

$$V = \frac{a}{r} + \frac{1}{r}\left[\frac{b}{r+y} + \frac{c}{r-y}\right] \tag{1.4.3.50}$$

belongs to the cases 2) and 4).

1.4.4 The case N=3

In this case the integral is odd. Denote it by

$$I = Ap_x^3 + Bp_x^2p_y + Cp_xp_y^2 + Dp_y^3 + Fp_x + Gp_y, \tag{1.4.4.1}$$

where $A, ..., G$ are functions of x and y. The set of equations to solve for the existence of a first integral are :

$$\frac{\partial A}{\partial x} = 0, \tag{1.4.4.2}$$

$$\frac{\partial B}{\partial x} + \frac{\partial A}{\partial y} = 0, \tag{1.4.4.3}$$

$$\frac{\partial C}{\partial x} + \frac{\partial B}{\partial y} = 0, \tag{1.4.4.4}$$

$$\frac{\partial D}{\partial x} + \frac{\partial C}{\partial y} = 0, \tag{1.4.4.5}$$

$$\frac{\partial D}{\partial y} = 0, \tag{1.4.4.6}$$

$$\frac{\partial F}{\partial x} - 3A\frac{\partial V}{\partial x} - B\frac{\partial V}{\partial y} = 0, \tag{1.4.4.7}$$

$$\frac{\partial G}{\partial x} + \frac{\partial F}{\partial y} - 2B\frac{\partial V}{\partial x} - 2C\frac{\partial V}{\partial y} = 0, \tag{1.4.4.8}$$

$$\frac{\partial G}{\partial y} - C\frac{\partial V}{\partial x} - 3D\frac{\partial V}{\partial y} = 0, \tag{1.4.4.9}$$

$$F\frac{\partial V}{\partial x} + G\frac{\partial V}{\partial y} = 0. \tag{1.4.4.10}$$

Equations (1.4.4.2)-(1.4.4.6) are easily solved :

$$A = a_0 + a_1y + a_2y^2 + a_3y^3, \tag{1.4.4.11}$$

$$B = b_0 + b_1y + d_2y^2 - a_1x - 2a_2xy - 3a_3xy^2, \tag{1.4.4.12}$$

$$C = c_0 - b_1x + a_2x^2 - d_1y - 2d_2xy + 3a_3x^2y, \tag{1.4.4.13}$$

$$D = d_0 + d_1x + d_2x^2 - a_3x^3. \tag{1.4.4.14}$$

It follows from (1.4.4.10) that

$$F = \frac{\partial V}{\partial y}Z, \qquad G = -\frac{\partial V}{\partial x}Z, \tag{1.4.4.15}$$

where Z is an unknown function of x and y. Substitution of this into the remaining three equations, (1.4.4.7)-(1.4.4.9), yields

$$Z\frac{\partial^2 V}{\partial x\partial y}+\frac{\partial Z}{\partial x}\frac{\partial V}{\partial y}-3A\frac{\partial V}{\partial x}-B\frac{\partial V}{\partial y}=0, \quad (1.4.4.16)$$

$$Z\left(\frac{\partial^2 V}{\partial y^2}-\frac{\partial^2 V}{\partial x^2}\right)+\frac{\partial Z}{\partial y}\frac{\partial V}{\partial y}-\frac{\partial Z}{\partial x}\frac{\partial V}{\partial x}-2B\frac{\partial V}{\partial x}-2C\frac{\partial V}{\partial y}=0, \quad (1.4.4.17)$$

$$-Z\frac{\partial^2 V}{\partial x\partial y}-\frac{\partial Z}{\partial y}\frac{\partial V}{\partial x}-3D\frac{\partial V}{\partial y}-C\frac{\partial V}{\partial x}=0. \quad (1.4.4.18)$$

The two remaining unknown functions still to be determined are Z and V. The sum of (1.4.4.16) and (1.4.4.18) is

$$\left(3A+C+\frac{\partial Z}{\partial y}\right)\frac{\partial V}{\partial x}+\left(3D+B-\frac{\partial Z}{\partial x}\right)\frac{\partial V}{\partial y}=0, \quad (1.4.4.19)$$

which may be solved for Z :

$$Z=\Phi(V)+Y, \quad (1.4.4.20)$$

$$\begin{aligned}Y={}&z_0+(3d_0+b_0)x+\frac{1}{2}(3d_1-a_1)x^2+d_2x^3\\&-(3a_0+c_0)y-\frac{1}{2}(3a_1-d_1)y^2-a_2y^3\\&-a_2x^2y-d_2xy^2+b_1xy-\frac{3}{4}a_3(x^2+y^2)^2.\end{aligned} \quad (1.4.4.21)$$

With this expression for Z equations (1.4.4.16) and (1.4.4.17) become

$$Y\frac{\partial^2 V}{\partial x\partial y}+3D\frac{\partial V}{\partial y}-3A\frac{\partial V}{\partial x}=-\frac{\partial^2\Phi(V)}{\partial x\partial y}, \quad (1.4.4.22)$$

$$\begin{aligned}&Y\left(\frac{\partial^2 V}{\partial y^2}-\frac{\partial^2 V}{\partial x^2}\right)-3(A+C)\frac{\partial V}{\partial y}-3(D+B)\frac{\partial V}{\partial x}\\&=\frac{\partial^2\Phi(V)}{\partial x^2}-\frac{\partial^2\Phi(V)}{\partial y^2}.\end{aligned} \quad (1.4.4.23)$$

Depending on Φ these equations may be highly nonlinear in V. The only integrable solution known for these equations with $\Phi\neq 0$ was obtained by (Fokas 1980) with the ansatz $V=V(x^2+ny^2)$:

$$H=\frac{1}{2}(p_x^2+p_y^2)+(x^2-y^2)^{-2/3}, \quad (3.3.24)$$

$$I=(p_x^2-p_y^2)(xp_y-yp_x)-4(yp_x+xp_y)(x^2-y^2)^{-2/3}. \quad (3.3.25)$$

(Holt 1982) solved the equations (1.4.4.22) and (1.4.4.23) for $\Phi = 0$ and for particular values of some of the constants to obtain two new integrable systems

$$V = \frac{3}{4}y^{4/3} + x^2y^{-2/3} + \delta y^{-2/3} \tag{1.4.4.26}$$

with

$$I = 2p_x^3 + 3p_xp_y^2 + 3p_x(-3y^{4/3} + 2x^2y^{-2/3} + 2\delta y^{-2/3}) + 18p_yxy^{1/3}, \tag{1.4.4.27}$$

and

$$V = x^2 + 4y^2 + \delta x^{-2} \tag{1.4.4.28}$$

with

$$I = p_x^2p_y + 8xyp_x + (-x^2 + \delta x^{-2})p_y, \tag{1.4.4.29}$$

where δ is an arbitrary constant.

The latter potential also has a quadratic first integral (cf. (1.4.3.41)) and is therefore superintegrable.

The three particle system with Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + v(q_1 - q_2) + u(q_2 - q_3) + w(q_3 - q_1), \tag{1.4.4.30}$$

has a cubic first integral for some choices of v, u, and w. For this system the centre of mass moves at a constant velocity. Therefore after transformation to the co-ordinates

$$\begin{aligned} x &= \frac{(2q_3 - q_1 - q_2)}{\sqrt{3}}, \\ y &= q_1 - q_2, \\ z &= \frac{\sqrt{6}}{3}(q_1 + q_2 + q_3). \end{aligned} \tag{1.4.4.31}$$

one finds that z is an ignorable co-ordinate. A subsequent scaling reduces the Hamiltonian to

$$H = \frac{1}{2}(p_x^2 + p_y^2) + v(-2y) + u(\sqrt{3}x + y) + w(-\sqrt{3}x + y) \tag{1.4.4.32}$$

For the choice of parameters $A = 1$, $B = 0$, $C = -3$ and $D =$
from (1.4.4.7)-(1.4.4.9):

$$F = 3\big(-2v(-2y) + u(\sqrt{3}x + y) + w(-\sqrt{3}x$$

$$G = -3\sqrt{3}\big(u(\sqrt{3}x + y) - w(-\sqrt{3}x + y)\big).$$

The remaining equation to be solved is (1.4.4.10)

$$(u + w - 2v)(u' - w') - (u' + w' - 2v')(u - w) =$$

A solution to this is the Toda-type potential

$$V = e^{[\sqrt{3}x-y]/2} + e^{y} + e^{[-\sqrt{3}x-y]/2},$$

$$I = p_x^3 - 3p_xp_y^2 + 3\left(e^{[\sqrt{3}x-y]/2} - 2e^{y} + e^{[-\sqrt{3}}\right.$$

$$\left. + 3\sqrt{3}\left(e^{[\sqrt{3}x-y]/2} - e^{[-\sqrt{3}x-y]/2}\right)p_y.\right.$$

The case $v(z) = u(z) = w(z)$ has been studied using the Lax-pa
1975) and (Kulish 1976) who solved the problem when v is
$P_w(z)$. (Moser 1975) studied the cases $v = z^{-2}$ and $[\sin(az)]^{-}$
first integral is

$$I = \frac{1}{3\sqrt{6}}p_x^3 - \frac{1}{\sqrt{6}}p_xp_y^2 - \frac{3(3x^4 - 6x^2y^2 - y^4)}{2\sqrt{6}(3x^2 - y^2)^2y^2}p_x + \frac{2}{(3x^2}$$

This system is also superintegrable since it has a quadratic fir

Assume a cubic first integral with an angular momentum lea

$$I = (yp_x - xp_y)^3 + \ldots.$$

In this case it best to transform to polar co-ordinates r and
$y = r\sin(\theta)$. Then the Hamiltonian and first integral become

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + V(r,\theta),$$

$$I = p_\theta^3 + A(r,\theta)p_r + B(r,\theta)p_\theta.$$

For I to be a first integral A, B and V must satisfy the following relations.

$$A = a(\theta), \;\; B = \frac{a'(\theta)}{r} + b(\theta) \tag{1.4.4.41}$$

and

$$V = \frac{c}{r} + \frac{g(\theta)}{r^2} + \frac{h(\theta)}{r^3}, \tag{1.4.4.42}$$

where

$$\begin{aligned} a'h' - 3ah &= 0, \\ a'g' - 2ag + bh' &= 0, \\ a'' + a - 3h' &= 0, \\ 3g' - b' &= 0, \\ bg' - ac &= 0, \end{aligned} \tag{1.4.4.43}$$

and the prime denotes differentiation *wrt* θ. The general solution to these equations is not known, but a particular solution may be obtained for $b = g = c = 0$ as in (Thompson 1984b):

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{h(y/x)}{(x^2+y^2)^{\frac{3}{2}}},$$

$$I = -(yp_x - xp_y)^3 + \frac{1}{2(x^2+y^2)^{\frac{1}{2}}} \times \tag{1.4.4.44}$$

$$\left[Kx(xp_x + yp_y) - \left(\frac{K(x^2+y^2)^{\frac{1}{2}}}{x}\right)'(yp_x - xp_y)\right],$$

where x and y are the original co-ordinates and the prime denotes differentiation *wrt* $z := \arctan(y/x)$. K and h satisfy the following pair of differential equations.

$$K'' + 2K = 6h'\cos(z), \tag{1.4.4.45}$$

$$3hK = h'K' + h'K\tan(z). \tag{1.4.4.46}$$

1.4.5 The case N=4

The form for a quartic first integral is

$$\begin{aligned} I = {} & f_0p_x^4 + f_1p_x^3p_y + f_2p_x^2p_y^2 + f_3p_xp_y^3 + f_4p_y^4 \\ & + g_op_x^2 + g_1p_xp_y + g_2p_y^2 + h, \end{aligned} \tag{1.4.5.1}$$

where f_i, g_i and h are unknown functions of x and y. On taking the Poisson Bracket with H and separating by coefficients the following equations are obtained

$$
\begin{aligned}
\frac{\partial f_0}{\partial x} &= 0,\\
\frac{\partial f_1}{\partial x} + \frac{\partial f_0}{\partial y} &= 0,\\
\frac{\partial f_2}{\partial x} + \frac{\partial f_1}{\partial y} &= 0,\\
\frac{\partial f_3}{\partial x} + \frac{\partial f_2}{\partial y} &= 0,\\
\frac{\partial f_4}{\partial x} + \frac{\partial f_3}{\partial y} &= 0,\\
\frac{\partial f_4}{\partial y} &= 0,
\end{aligned}
\tag{1.4.5.2}
$$

$$
\begin{aligned}
\frac{\partial g_0}{\partial x} &= 4f_0\frac{\partial V}{\partial x} + f_1\frac{\partial V}{\partial y},\\
\frac{\partial g_1}{\partial x} + \frac{\partial g_0}{\partial y} &= 3f_1\frac{\partial V}{\partial x}V + 2f_2\frac{\partial V}{\partial y},\\
\frac{\partial g_2}{\partial x} + \frac{\partial g_1}{\partial y} &= 2f_2\frac{\partial V}{\partial x} + 3f_3\frac{\partial V}{\partial y},\\
\frac{\partial g_2}{\partial y} &= f_3\frac{\partial V}{\partial x} + 4f_4\frac{\partial V}{\partial y},
\end{aligned}
\tag{1.4.5.3}
$$

$$
\begin{aligned}
\frac{\partial h}{\partial x} &= 2g_0\frac{\partial V}{\partial x} + g_1\frac{\partial V}{\partial y},\\
\frac{\partial h}{\partial y} &= g_1\frac{\partial V}{\partial x} + 2g_2\frac{\partial V}{\partial y},
\end{aligned}
\tag{1.4.5.4}
$$

The 5 equations (1.4.5.3) and (1.4.5.6) remain to be solved for the three unknown functions g_i, V and the 15 constants in (1.4.5.5). The general solution to these equations is not known, but many particular solutions are known. Some examples are given below.

<u>Examples:</u>

1) The following cubic potential was found to have a quartic first integral in (Hall 1983) and (Grammaticos *et al* 1982):

$$V = \frac{1}{2}a(x^2 + 16y^2) + d(\frac{16}{3}y^3 + x^2y). \tag{1.4.5.7}$$

A generalized version of this is given in (Grammaticos *et al* 1984) and (Hietarinta 1984b):

$$V = \frac{16}{3}y^3 + x^2y + \frac{a}{2}(x^2 + 16y^2) + mx^{-2} + nx^{-6}, \tag{1.4.5.8}$$

which has the first integral

$$\begin{aligned} I = &p_x^4 + (2ax^2 + 4x^2y + 4mx^{-2} + 4mx^{-2} + 4nx^{-6})p_x^2 - \frac{4}{3}x^3p_xp_y - \frac{4}{3}ax^4y - \frac{4}{3}x^4y^2 \\ &+ \frac{8}{3}my + 8nyx^{-4} - \frac{2}{9}x^6 + a^2x^4 + 4(an + m^2)x^{-4} + 8mnx^{-8} + 4n^2x^{-12}. \end{aligned}$$

2) For $n = 0$ the quartic potential

$$V = a(x^4 + 12x^2y^2 + 16y^4) + b(x^2 + 4y^2) + cx^{-6}(x^2 + 4y^2) + mx^{-2} + ny^{-2}, \quad (1.4$$

corresponds to the potential (1.4.3.34) for $g_1(z) = g_2(z) = \frac{1}{2}(az^5 + bz^3 + mz^{-1} +$
which was shown to be integrable with a quadratic first integral. Let the first in
of the former system be denoted by I_2. Then the above system with nonzero n ha
quartic first integral

$$I = I_2^2 + n[2x^2y^{-2}p_x^2 + 16(ax^4 + cx^{-4})]. \quad (1.4$$

In (Ramani *et al* 1982) and (Grammaticos *et al* 1983) it is found that the potent

$$V = x^4 + 6x^2y^2 + 8y^4 \quad (1.4$$

is integrable with a quartic first integral. The following generalization of this pot
is integrable if $e = 0$ (Grammaticos 1984) or $n = 0$ and $l = 0$ (Hietarinta 1984b)

$$V = x^4 + 6x^2y^2 + 8y^4 + k(x^2 + 4y^2) + mx^{-2} + nx^{-6} + ly^{-2} + ey, \quad (1.4$$

$$\begin{aligned} I =& p_x^4 + 4p_x^2(x^4 + 6x^2y^2 + kx^2 + mx^{-2} + nx^{-6} + ey) - (16x^3y + e4x)p_xp_y + 4x \\ &+ 4m^2x^{-4} + 8mx^2 + 16my^2 + 4k^2x^4 + 8kx^6 + 16x^4y^2 + 4x^8 \\ &+ 16x^6y^2 + 16x^4y^4 + e(8mx^{-2}y - 2ex^2 - 8x^4y - 16x^2y^3 - 16kx^2y) \\ &+ 81x^4y^{-2} + n(8mx^{-8} + 4nx^{-12} + 8kx^{-4} + 8x^{-2} + 48x^{-4}y^2). \end{aligned} \quad (1.4$$

3) The Holt-type potential

$$V = 9x^{4/3} + y^2x^{-2/3}, \quad (1.4$$

was found to be integrable in (Hietarinta 1983) by the use of Painlevé analysis
following generalization thereof found in (Hietarinta 1984b)

$$V = \frac{9}{2}x^{4/3} + (y^2 + d)x^{-2/3} + mx^{2/3}ny^{-2} + a(9x^2 + 4y^2), \quad (1.4$$

has the quartic first integral

$$\begin{aligned} I =& p_y^4 + 2p_x^2p_y^2 + p_x^2(16ay^2 + 4ny^{-2}) + 24x^{1/3}yp_xp_y \\ &+ 4p_y^2(x^{-2/3}(y^2 + d) + mx^{2/3} + a(9x^2 + 4y^2) + ny^{-2}) + 16my^2 \\ &+ 32adx^{-2/3}y^2 + 8dnx^{-2/3}y^{-2} + 8nx^{-2/3} + 32amx^{2/3}y^2 + 8nmx^{2/3}y^{-2} \\ &+ 72x^{2/3}y^2 + 72anx^2y^{-2} + 4n^2y^{-4} + 32ax^{-2/3}y^2(9x^2 + y^2) + 32a^2y^2(9x^2 + \end{aligned} \quad (1.4$$

4) A Toda-type potential representing the basic "three particle free-end lattice" may be rotated to

$$H = \frac{1}{2}(p_x^2 + p_y^2) + e^y + e^{x-y},$$
$$I = p_x^2 p_y^2 + 2e^y p_x^2 - 2e^{x-y} p_x p_y + e^{2x-2y} + 2e^x. \tag{1.4.5.18}$$

Assume a first integral with an angular momentum leading part:

$$I = (yp_x - xp_y)^4 + g_0 p_x^2 + g_1 p_x p_y + g_2 p_y^2 + h. \tag{1.4.5.19}$$

To solve the remaining equations it is best to transform to polar co-ordinates ($x = r\cos(\theta)$, $y = r\sin(\theta)$) and then one finds that

$$V = v_1 r^{-1} + v_2(\theta) r^{-2} + v_3(\theta) r^{-3} + v_4(\theta) r^{-4}, \tag{1.4.5.20}$$

$$g_0 = a(\theta),$$

$$g_1 = a'(\theta) r^{-1} + b(\theta),$$

$$g_2 = [\frac{1}{2}a''(\theta) + a(\theta)] r^{-2} + b'(\theta) r^{-1} + c(\theta), \tag{1.4.5.21}$$

$$h = \int [2cv_2 - bv_1] d\theta + [2av_1 - bv_2] r^{-1} + [2av_2 - \frac{1}{2}bv_3' - \frac{1}{2}a'v_2'] r^{-2}$$
$$+ [2av_3 - \frac{1}{3}bv_4' - \frac{1}{3}a'v_3'] r^{-3} + [2av_4 - \frac{1}{4}a'v_4'] r^{-4}$$

where v_1 is a constant.

1.4.6 Higher order first integrals

No in-depth searches for first integrals of order higher than four have been made, but some interesting examples of systems possessing higher than fourth order first integrals have been found. All known systems possessing sixth order first integrals are obtained by generalizing systems possessing cubic first integrals with an additive term.

The application of Painlevé analysis in (Grammaticos 1984) and (Hietarinta 1983) suggests that the potential

$$V = 12x^{4/3} + y^2 x^{-2/3}, \tag{1.4.6.1}$$

is integrable. A search revealed that this system has a first integral of order six:

$$\begin{aligned} I =& p_y^6 + 3p_y^4 p_x^2 + (18x^{4/3} + 6x^{-2/3}y^2)p_y^4 \\ &+ 72x^{1/3} y p_y^3 p_x + 648x^{2/3} y^2 p_y^2 + 648y^4. \end{aligned} \tag{1.4.6.2}$$

A Toda-type potential with a sixth order first integral was found in (Dorizzi 1984). The result has been rotated here to make apparent its relationship with the Toda-type potential $V = e^{[\sqrt{3}x-y]/2} + e^{y} + e^{-[\sqrt{3}x+y]/2}$ (cf. (1.4.4.36)) obtained in the section on cubic first [illegible] If the cubic first integral is denoted by I_{3T}, the sixth order results

In § 1.4.3 it was shown that the nearest neighbour potential with $v(z) = z^{-2}$, is superintegrable. The generalization

$$
\begin{aligned}
V &= v(-2y) + v(\sqrt{3}+y) + v(-\sqrt{3}+y), \text{ with} \\
v(z) &= z^{-2} + \frac{1}{2}a^2 z^2,
\end{aligned}
\tag{1.4.6.7}
$$

is integrable with a sixth order first integral (Wojciechowski 1983, Adler 1977). Denote the cubic first integral for V when $a = 0$ by $I_3(p_x, p_y, x, y)$. Then the sixth order first integral for (1.4.6.7) is

$$
I = I_3(p_x + iax, p_y + iay, x, y) I_3(p_x - iax, p_y - iay, x, y). \tag{1.4.5.8}
$$

This system is superintegrable since it is of the form (1.4.3.24).

A class of superintegrable models was found in (Thompson 1984b). Systems of the form

$$
H = \frac{1}{2}(p_x^2 + p_y^2) + V(x - y), \tag{1.4.6.9}
$$

were considered. They already have the first integral $I = p_x + p_y$. Thompson found that a higher order invariant existed if V had the form

$$
V = (x - y)^{-2a}, \tag{1.4.6.10}
$$

where $a = 1/(2k+1)$, $k = 0, 1, 2, \ldots$. This result may be generalized for arbitrary a. If the co-ordinates are rotated so that

$$
V = x^{-2a}, \tag{1.4.6.11}
$$

a first integral is given by

$$
I = p_y \int^{p_x x^a} (u^2 + 2)^{(1-a)/2a} du - ay(p_x^2 + 2x^{-2a})^{(1+a)/2a}. \tag{1.4.6.12}
$$

1.5 Noether's theorem

Another method to determine first integrals of dynamical systems is to use Noether's theorem (Noether 1918). One of the versions of the theorem found in (Sarlet and Cantrijn 1981) is given here. Let

$$\begin{aligned} \bar{t} &= t + \epsilon\tau(t, q_i), \\ \bar{q}_i &= q_i + \epsilon\xi_i(t, q_i), \end{aligned} \tag{1.5.0.1}$$

be an infinitesimal point transformation. Here the subscript i refers to the different spatial co-ordinates. The transformed velocities are given by

$$\frac{d\bar{q}_i}{d\bar{t}} = \frac{\dot{q}_i + \epsilon\dot{\xi}_i}{1 + \epsilon\dot{\tau}} = \dot{q}_i + \epsilon(\dot{\xi}_i - \dot{q}_i\dot{\tau}). \tag{1.5.0.2}$$

For this transformation to leave the action integral of a system invariant up to a gauge term it must be such that

$$\int_{\bar{t}_1}^{\bar{t}_2} L\left(\bar{t}, \bar{q}(\bar{t}), \frac{d\bar{q}}{d\bar{t}}(\bar{t})\right) d\bar{t} = \int_{t_1}^{t_2} L(t, q(t), \dot{q}(t))dt$$
$$+\epsilon \int_{t_1}^{t_2} \frac{df}{dt}(t, q(t))dt + O(\epsilon^2),$$

where $[t_1, t_2]$ is any subinterval of the interval $[a, b]$ on which $q(t)$ is defined. This will only be the case if

$$L\left(\bar{t}(t), \bar{q}(t), \frac{d\bar{q}}{d\bar{t}}(t)\right) \frac{d\bar{t}}{dt}(t) = L(t, q(t), \dot{q}(t)) + \epsilon\frac{df}{dt}(t, q(t)) + O(\epsilon^2). \tag{1.5.0.3}$$

To first order in ϵ the *lhs* may be written as

$$L(t, q, \dot{q}) + \epsilon\xi_i\frac{\partial L}{\partial q} + \epsilon(\dot{\xi}_i - \dot{q}\dot{\tau})\frac{\partial L}{\partial \dot{q}} + \epsilon\tau\frac{\partial L}{\partial t}.$$

From the first order in ϵ terms in the previous equation it is easily seen that

$$\tau\frac{\partial L}{\partial t} + \frac{\partial L}{\partial q_i}\xi_i + \frac{\partial L}{\partial \dot{q}_i}(\dot{\xi}_i - \dot{q}_i\dot{\tau}) + L\dot{\tau} = \frac{df}{dt}. \tag{1.5.0.4}$$

A few simple manipulations of this equation and use of the Euler-Lagrange equations give

$$\frac{df}{dt} = (\xi_i - \dot{q}_i\tau)\left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right] + \frac{d}{dt}\left[L\tau + \frac{\partial L}{\partial \dot{q}_i}(\xi_i - \dot{q}_i\tau)\right], \tag{1.5.0.5}$$

from which it is easily deduced that

$$I = f - \left[L\tau + \frac{\partial L}{\partial \dot{q}_i}(\xi_i - \dot{q}_i\tau)\right] \tag{1.5.0.6}$$

is an integral.

Infinitesimal transformations of the form (1.5.0.1) are called Noether-transformations of L if they satisfy (1.5.0.4) and to each corresponds a first integral of the form (1.5.0.6).

Consider the one-dimensional harmonic oscillator with Lagrangian

$$L = \frac{1}{2}(\dot{q}^2 - q^2). \tag{1.5.0.7}$$

Equation (1.5.0.4) becomes

$$\begin{aligned} &-q\xi + \dot{q}\left(\frac{\partial \xi}{\partial q}\dot{q} + \frac{\partial \xi}{\partial t} - \dot{q}\left(\frac{\partial \tau}{\partial q}\dot{q} + \frac{\partial \tau}{\partial t}\right)\right) \\ &+\frac{1}{2}(\dot{q}^2 - q^2)\left(\frac{\partial \tau}{\partial q}\dot{q} + \frac{\partial \tau}{\partial t}\right) = \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial t}. \end{aligned} \tag{1.5.0.8}$$

Separation by coefficients of powers of $\dot{q}$ yields :

$$-\frac{\partial \tau}{\partial q} + \frac{1}{2}\frac{\partial \tau}{\partial q} = 0, \tag{1.5.0.9a}$$

$$\frac{\partial \xi}{\partial q} - \frac{\partial \tau}{\partial t} + \frac{1}{2}\frac{\partial \tau}{\partial t} = 0, \tag{1.5.0.9b}$$

$$\frac{\partial \xi}{\partial t} - \frac{1}{2}q^2\frac{\partial \tau}{\partial q} = \frac{\partial f}{\partial q}, \tag{1.5.0.9c}$$

$$-q\xi - \frac{1}{2}q^2\frac{\partial \tau}{\partial t} = \frac{\partial f}{\partial t} \tag{1.5.0.9d}$$

which yield the solutions

$$\tau = A + B\sin(2t) + C\cos(2t), \tag{1.5.0.10a}$$

$$\xi = [B\cos(2t) - C\sin(2t)]q + D\sin(t) + G\cos(t), \tag{1.5.0.10b}$$

where the constants are arbitrary. By choosing each constant equal to unity in turn, whilst the others are set at zero, gives the following Noether-transformations and the corresponding first integrals

$$\bar{t}_1 = t + \epsilon, \quad \bar{q}_1 = q, \tag{1.5.0.12.1.a}$$

$$I_1 = \frac{1}{2}(\dot{q}^2 + q^2), \tag{1.5.0.12.1.b}$$

$$\bar{t}_2 = t + \epsilon \sin(2t), \quad \bar{q}_2 = q + \epsilon q \cos(2t), \tag{1.5.0.12.2.a}$$

$$I_2 = \frac{1}{2}(\dot{q}^2 - q^2)\sin(2t) - q\dot{q}\cos(2t), \tag{1.5.0.12.2.b}$$

$$\bar{t}_3 = t + \epsilon \cos(2t), \quad \bar{q}_3 = q - \epsilon q \sin(2t), \tag{1.5.0.12.3.a}$$

$$I_3 = \frac{1}{2}(\dot{q}^2 - q^2)\cos(2t) + q\dot{q}\sin(2t), \tag{1.5.0.12.3.b}$$

$$\bar{t}_4 = t, \quad \bar{q}_4 = q + \epsilon \sin(t), \tag{1.5.0.12.4.a}$$

$$I_4 = q\cos(t) - \dot{q}\sin(t), \tag{1.5.0.12.4.b}$$

$$\bar{t}_5 = t, \quad \bar{q}_5 = q + \epsilon \cos(t), \tag{1.5.0.12.5.a}$$

$$I_5 = q\sin(t) + \dot{q}\cos(t). \tag{1.5.0.12.5.b}$$

If in (1.5.0.1) τ and ξ^i are allowed to depend on velocity, i.e.,

$$\begin{aligned} \bar{t} &= t + \epsilon\tau(t, q, \dot{q}), \\ \bar{q}^i &= q^i + \epsilon\xi^i(t, q, \dot{q}), \end{aligned} \tag{1.5.0.13}$$

the equations determining the Noether-transformations are (Sarlet and Cantrijn 1981)

$$L\frac{\partial\tau}{\partial\dot{q}_j} + \frac{\partial L}{\partial\dot{q}_i}\left(\frac{\partial\xi_i}{\partial\dot{q}_j} - \dot{q}_i\frac{\partial\tau}{\partial\dot{q}_j}\right) = \frac{\partial f}{\partial\dot{q}_j}, \tag{1.5.0.14.a}$$

$$\begin{aligned} &\tau\frac{\partial L}{\partial t} + \xi_i\frac{\partial L}{\partial q_i} + L\left(\frac{\partial\tau}{\partial t} + \dot{q}_i\frac{\partial\tau}{\partial q_i}\right) + \\ &\frac{\partial L}{\partial\dot{q}_i}\left[\frac{\partial\xi_i}{\partial t} + \frac{\partial\xi_i}{\partial q_j}\dot{q}_j - \dot{q}_i\left(\frac{\partial\tau}{\partial t} + \dot{q}_j\frac{\partial\tau}{\partial q_j}\right)\right] = \frac{\partial f}{\partial t} + \dot{q}_i\frac{\partial f}{\partial q_i}. \end{aligned} \tag{1.5.0.14.b}$$

The first integrals are again given by (1.5.0.6). If the complete solution to the equations (1.5.0.14) can be found, all the first integrals of a system can be determined. It is generally not possible to make progress with the solution of (1.5.0.14) and the precise nature of the velocity dependence of τ and the ξ's must be made.

The problem may also be solved by using the symmetries of the differential equations alone. This is the approach of the Lie theory of extended groups. This is somewhat further removed from the present method and is not reviewed here.

1.6 The Lax-pair method

This method consists of finding two $n \times n$ matrices L and B so that the equations of motion are given by

$$\frac{dL}{dt} = [L, B] = LB - BL. \tag{1.6.1}$$

The first integrals of the system are then given by the coeficients of γ^m in the expansion of $\det(L - \gamma I)$ or alternatively by $\mathrm{tr}(L^k)(k = 1, 2, \ldots)$ where tr stands for the trace. As an example consider Euler's equations of motion for a rigid body in the absence of torque (Steeb 1990)

$$\begin{aligned} \dot{q}_1 &= (\lambda_3 - \lambda_2) q_2 q_3, \\ \dot{q}_2 &= (\lambda_1 - \lambda_3) q_3 q_1, \\ \dot{q}_3 &= (\lambda_2 - \lambda_1) q_1 q_2. \end{aligned} \tag{1.6.2}$$

A Lax representation of this problem is given by

$$\frac{dL}{dt} = [L, \lambda L],$$

where

$$L = \begin{pmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{pmatrix}, \tag{1.6.3}$$

$$\lambda L = \begin{pmatrix} 0 & -\lambda_3 q_3 & \lambda_2 q_2 \\ \lambda_3 q_3 & 0 & -\lambda_1 q_1 \\ -\lambda_2 q_2 & \lambda_1 q_1 & 0 \end{pmatrix}. \tag{1.6.4}$$

From this Lax pair the first integral

$$I = q_1^2 + q_2^2 + q_3^2 \tag{1.6.5}$$

is obtained.

This method has been used with success (Calogero 1975, Moser 1975), but it has the drawback that there is no systematic way of finding L and B.

CHAPTER II: CONFIGURATIONAL INVARIANTS

2.1 Introduction

First integrals are useful in the solution of differential equations as was pointed out in the previous chapter. If, however, the system is non-integrable it will not possess sufficient first integrals to solve the problem. In such a case one may look at special cases of the problem to determine whether they are solvable.

A natural way of restricting or reducing the system is to consider it on a hypersurface of phase space defined by $I = C$, where I is a first integral of the system and C is a fixed value. Under these restrictions it may be possible to find further first integrals of the reduced system. These first integrals will, however, not be first integrals of the original system. They have been given the name configurational invariants.

Configurational invariants were first introduced by L. S. Hall in 1982 (Hall 1983). His attempts to explain the existence of a certain class of axially symmetric magnetohydrodynamic reversed-field equilibria led him to the idea.

2.2 Configurational invariants and a weak form of complete integrablity

The following formal definition of a configurational invariant was first given in (Sarlet et al. 1985).

Let

$$\begin{aligned} \ddot{x} &= f_1(x, y, \dot{x}, \dot{y}), \\ \ddot{y} &= f_2(x, y, \dot{x}, \dot{y}), \end{aligned} \tag{2.2.1}$$

be a dynamical system. Suppose that

$$F(x, y, \dot{x}, \dot{y}) = E \tag{2.2.2}$$

is a first integral of the system, e.g. energy, then its total time derivative must be equal to zero :

$$\frac{dF}{dt} = \dot{x}\frac{\partial F}{\partial x} + \dot{y}\frac{\partial F}{\partial y} + f_1\frac{\partial F}{\partial \dot{x}} + f_2\frac{\partial F}{\partial \dot{y}} = 0.$$

(2.2.1) may be rewritten as

$$\begin{aligned}\ddot{x} &= f_1(x, y, \dot{x}, \phi(x, y, \dot{x}, E)) \\ &= \bar{f}_1(x, y, \dot{x}, E), \\ \dot{y} &= \phi(x, y, \dot{x}, E),\end{aligned} \tag{2.2.4}$$

where the parameter E replaces $\dot{y}$ as a variable. If we choose the value of E, say $E = E_0$, then we are only looking at the class of systems with solutions confined to the hypersurface

$$F(x, y, \dot{x}, \dot{y}) = E_0$$

in phase space. The corresponding dynamical system is

$$\begin{aligned}\ddot{x} &= \bar{f}_1(x, y, \dot{x}, E_0), \\ \dot{y} &= \phi(x, y, \dot{x}, E_0).\end{aligned} \tag{2.2.5}$$

Definition 8:
A function $J = J(x, y, \dot{x}, \dot{y}, t)$ is said to be configurational invariant of the original system (2.2.1) *wrt* the first integral (2.2.2) if its restriction to the surface $E = E_0$ namely

$$\bar{J}(x, y, \dot{x}, t, E_0) = J((x, y, \dot{x}, \phi(x, y, \dot{x}, E_0), t), \tag{2.2.6}$$

is a first integral of the reduced system above.

Clearly a first integral is always a configurational invariant, but configurational invariants are not always first integrals. To illustrate the concept of a configurational invariant consider the following example considered by Sarlet et al..

Let

$$\begin{aligned}\ddot{x} &= \Omega\dot{y} - \frac{\partial V}{\partial x}, \\ \ddot{y} &= -\Omega\dot{x} - \frac{\partial V}{\partial y},\end{aligned} \tag{2.2.7}$$

where Ω and V are functions of x and y, be a dynamical system. To find a linear first integral of the form

$$J(x, y, \dot{x}, \dot{y}) = S(x, y)\dot{x} + T(x, y)\dot{y} + K(x, y), \tag{2.2.8}$$

we must solve $\dot{J} = 0$, i.e.,

$$\frac{\partial S}{\partial x}\dot{x}^2 + \left(\frac{\partial S}{\partial y} + \frac{\partial T}{\partial x}\right)\dot{x}\dot{y} + \frac{\partial T}{\partial y}\dot{y}^2 + \left(\frac{\partial K}{\partial x} - T\Omega\right)\dot{x}$$
$$+ \left(\frac{\partial K}{\partial y} + S\Omega\right)\dot{y} - \left(S\frac{\partial V}{\partial x} + T\frac{\partial V}{\partial y}\right) = 0. \tag{2.2.9}$$

Separating by coefficients of the different powers of $\dot{x}$ and $\dot{y}$ gives

$$\frac{\partial S}{\partial x} = 0, \quad \frac{\partial T}{\partial y} = 0, \quad \frac{\partial S}{\partial y} + \frac{\partial T}{\partial x} = 0,$$
$$\frac{\partial K}{\partial x} - T\Omega = 0, \quad \frac{\partial K}{\partial y} + S\Omega = 0, \quad S\frac{\partial V}{\partial x} + T\frac{\partial V}{\partial y} = 0. \tag{2.2.10}$$

The general solution to these equations is

$$V = V(c(x^2 + y^2) + ay + bx),$$
$$\Omega = \Omega(c(x^2 + y^2) + ay + bx),$$
$$S = -(2cy + a), \tag{2.2.11}$$
$$T = 2cx + b,$$
$$K = \int \Omega(u)\ du + d,$$

where a, b, c and d are arbitrary constants.

The energy of this system,

$$E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + V, \tag{2.2.12}$$

is a first integral. To find a configurational invariant $\bar{J}$ *wrt* E of the same form as J above let

$$\dot{y} = \sqrt{2(E_0 - V) - \dot{x}^2}. \tag{2.2.13}$$

The equation for the time derivative is

$$\dot{\bar{J}} = \dot{x}\left(-\Omega T + \frac{\partial K}{\partial x}\right) + \dot{x}\sqrt{2(E_0 - V) - \dot{x}^2}\left(\frac{\partial S}{\partial y} + \frac{\partial T}{\partial x}\right)$$
$$+ \dot{x}^2\left(-\frac{\partial T}{\partial y} + \frac{\partial S}{\partial x}\right) + \sqrt{2(E_0 - V) - \dot{x}^2}\left(S\Omega + \frac{\partial K}{\partial y}\right) \tag{2.2.14}$$
$$+ \left[-S\frac{\partial V}{\partial x} + 2(E_0 - V)\frac{\partial T}{\partial y} - T\frac{\partial V}{\partial y}\right] = 0.$$

Separation by coefficients of different powers of $\dot{x}$ and $\dot{y}$ yields

$$\frac{\partial S}{\partial y}+\frac{\partial T}{\partial x}=0, \qquad \frac{\partial S}{\partial x}-\frac{\partial T}{\partial y}=0,$$
$$\frac{\partial K}{\partial x}-\Omega T=0, \qquad \frac{\partial K}{\partial y}+S\Omega=0, \tag{2.2.15}$$
$$-S\frac{\partial V}{\partial x}+2(E_0-V)\frac{\partial T}{\partial y}-T\frac{\partial V}{\partial y}=0.$$

This is a smaller set of equations than (2.2.10).

The solution to (2.2.15) as obtained by L. S. Hall is :

$$E_0-V=\left(\frac{\partial\phi}{\partial\eta}\right)|\nabla\eta|^2, \qquad \Omega=\left(\frac{\partial K}{\partial\eta}\right)|\nabla\eta|^2,$$
$$S=\frac{\partial\eta}{\partial y}, \qquad T=-\frac{\partial\eta}{\partial x}, \tag{2.2.16}$$

where ϕ and K are arbitrary functions of η which is any solution of Laplace's equation. This is more general than the solution of (2.2.10).

Sarlet et al. found a particular solution to (2.2.15) which is given here. Let $\Omega=0$, $S=x$ and $T=y$. All that is then needed for the equations to be solved is for K to be equal to zero and for $W=(V-E_0)$ to satisfy

$$x\frac{\partial W}{\partial x}+y\frac{\partial W}{\partial y}+2W=0. \tag{2.2.17}$$

The general solution to this is

$$W=x^{-2}f(\frac{y}{x}), \tag{2.2.18}$$

where f is an arbitrary function. The corresponding system is

$$\ddot{x}=x^{-3}[2f+yx^{-1}f'],$$
$$\ddot{y}=-x^{-3}f', \tag{2.2.19}$$

where the prime denotes differentiation of f *wrt* its argument. This is a generalized Ermakov system (Ray and Reid 1982). The energy is given by

$$E=\frac{1}{2}(\dot{x}^2+\dot{y}^2)+x^{-2}f \tag{2.2.20}$$

and $\bar{J}$ is given by

$$\bar{J} = x\dot{x} + y\dot{y}. \tag{2.2.21}$$

Now

$$\dot{\bar{J}} = 2E_0,$$

which is zero only when $E_0 = 0$. Thus $\bar{J}$ is a configurational invariant of the system above on the hypersurface $E = 0$. Note that a first integral is always a configurational invariant since its time derivative is always zero. Here configurational invariants that are also first integrals are referred to as trivial configurational invariants.

A weak form of complete integrability

Consider a Hamiltonian system with n degrees of freedom with Hamiltonian

$$H = H(\mathbf{q}, \mathbf{p}, t). \tag{2.2.22}$$

Here boldface is used to indicate vectors. Suppose that the system has a first integral Φ_n and $n-1$ corresponding configurational invariants Φ_α, $\alpha = 1, n-1$. One may assume that the constant value of Φ_n is zero since by adding the appropriate constant to Φ_n one obtains another first integral the constant value of which is zero. Suppose that these configurational invariants together with the first integral are in weak involution, by which is meant

$$\dot{\Phi}_\alpha\big|_{\Phi_n=0} = 0, \quad \alpha = 1, n-1, \tag{2.2.23}$$

$$[\Phi_i, \Phi_j]\big|_{\Phi_n=0} = 0, \quad i, j = 1, n. \tag{2.2.24}$$

(Here and in the sequel the subscript PB has been omitted for reasons of clarity.)

In the following Greek indices run from 1 to $n-1$ and Roman indices run from 1 to n. The summation convention over repeated indices applies.

Introduce functions F_i defined by

$$F_\alpha = \Phi_\alpha - a_\alpha, \;\; F_n = \Phi_n, \tag{2.2.25}$$

where the a_α's are arbitrary constants. Assuming that the Jacobian $(\partial\Phi_i/\partial p_j) = (\partial F_i/\partial p_j)$ is nonsingular in a neighbourhood of the region of interest we can locally solve the relations $F_i = 0$ for the momenta p_j :

$$p_i = f_i(\mathbf{q}, t, a_\alpha). \tag{2.2.26}$$

Let

$$G_i(\mathbf{q}, \mathbf{p}, t, a_\alpha) = p_i - f_i(\mathbf{q}, t, a_\alpha). \tag{2.2.27}$$

Since the F_i's are configurational invariants, they satisfy the relation

$$\dot{F}_i\big|_{F_n=0} = 0 \tag{2.2.28}$$

and the following is also true

$$\dot{F}_i\big|_{F=0} = 0, \tag{2.2.29}$$

where $F = 0$ means $F_i = 0$ for $i = 1, n$. In view of (2.2.23)-(2.2.24) the F's must satisfy

$$[F_i, F_j]\big|_{F_n=0} = 0,$$

i.e.,

$$[F_i, F_j]\big|_{F=0} = 0, \;\; i = 1, n. \tag{2.2.30}$$

To obtain the final result two lemmas are necessary.

Lemma 1:

$$[F_i, F_j]\big|_{F=0} = 0 \quad \Longleftrightarrow \quad [G_i, G_j] = 0. \tag{2.2.31}$$

Proof:

From the definition of the G_i's it is easy to show that

$$[G_i, G_j] = 0 \quad \Longleftrightarrow \quad \frac{\partial f_i}{\partial q_j} - \frac{\partial f_j}{\partial q_i} = 0. \tag{2.2.32}$$

The *lhs* of (2.2.31) when expanded is:

$$[F_i, F_j]\big|_{F=0} = \frac{\partial F_i}{\partial q_l}(\mathbf{q}, f(\mathbf{q}, t), t)\frac{\partial F_j}{\partial p_l}(\mathbf{q}, f(\mathbf{q}, t), t) - \frac{\partial F_i}{\partial p_l}(\text{—})\frac{\partial F_j}{\partial q_l}(\text{—}). \tag{2.2.33}$$

Taking the derivatives of the identities $F_i(\mathbf{q}, f(\mathbf{q}, t), t) = 0$ *wrt* q_l yields :

$$\frac{\partial F_i}{\partial q_l}(\mathbf{q}, f(\mathbf{q}, t), t) + \frac{\partial F_i}{\partial p_k}(\text{—})\frac{\partial f_k}{\partial q_l} = 0. \tag{2.2.34}$$

Replacing $\partial F/\partial q$-terms in the expression above gives

$$[F_i, F_j]\big|_{F=0} = \frac{\partial F_i}{\partial p_k}(\mathbf{q}, f(\mathbf{q}, t), t)\frac{\partial F_j}{\partial p_l}(\text{—})\left(\frac{\partial f_l}{\partial q_k} - \frac{\partial f_k}{\partial q_l}\right). \tag{2.2.35}$$

The result (2.2.31) now follows from (2.2.32) and the regularity of $\partial F_i/\partial p_k$.

Lemma 2:

$$\dot{F}_i\big|_{F=0} = 0 \iff \dot{G}_i\big|_{G=0} = 0. \tag{2.2.36}$$

The expression

$$\dot{F}_i\big|_{F=0} = \left(\frac{\partial F_i}{\partial t} + \frac{\partial F_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial F_i}{\partial p_j}\frac{\partial H}{\partial q_j}\right)\bigg|_{p=f(q,t)} \tag{2.2.37}$$

may be rewritten using (2.2.34) and the analogous identity

$$\frac{\partial F_i}{\partial t}(q, f(q,t), t) + \frac{\partial F_i}{\partial p_j}(\text{—})\frac{\partial f_j}{\partial t} = 0. \tag{2.2.38}$$

It then becomes

$$\dot{F}\big|_{F=0} = -\left[\frac{\partial F_i}{\partial p_j}\left(\frac{\partial f_j}{\partial t} + \frac{\partial f_j}{\partial q_k}\frac{\partial H}{\partial p_k} + \frac{\partial H}{\partial q_j}\right)\right]\bigg|_{p=f(q,t)} \tag{2.2.39}$$

or

$$\dot{F}_i\big|_{F=0} = \frac{\partial F_i}{\partial p_j}(q, f(q,t), t)(\dot{G}_j\big|_{G=0}). \tag{2.2.40}$$

The lemma follows.

Since the F's satisfy (2.2.40), it follows from *Lemma* 1 that the f's satisfy (2.2.31). Equation (2.2.29) with *Lemma* 2 gives

$$\frac{\partial f_j}{\partial t} + \frac{\partial f_j}{\partial q_k}\frac{\partial H}{\partial p_k}(q, f(q,t), t) + \frac{\partial H}{\partial q_j}(q, f(q,t), t) = 0. \tag{2.2.41}$$

Using (2.2.31) in this equation yields

$$\frac{\partial f_j}{\partial t} + \frac{\partial H_1}{\partial q_j} = 0, \tag{2.2.42}$$

where $H_1(q,t,a) = H(q, f(q,t), t)$. Equations (2.2.31) and (2.2.42) indicate the existence of a function W such that

$$f_j = \frac{\partial W}{\partial q_j} \quad , H_1 = -\frac{\partial W}{\partial t}. \tag{2.2.43}$$

Define $n-1$ new functions $I_\alpha(q,t,a)$ by

$$I_\alpha(q,t,a) = \frac{\partial W}{\partial a_\alpha}. \tag{2.2.44}$$

Now $\dot{I}|_{F=0} = 0$ since

$$\begin{aligned}\dot{I}|_{F=0} &= \frac{\partial I_\alpha}{\partial t} + \frac{\partial I_\alpha}{\partial q_k}\frac{\partial H}{\partial p_k}(\mathrm{q}, f(\mathrm{q},t),t) \\ &= \frac{\partial^2 W}{\partial t \partial a_\alpha} + \frac{\partial^2 W}{\partial q_k \partial \alpha_\alpha}\frac{\partial H}{\partial p_k}(\mathrm{q}, \frac{\partial W}{\partial \mathrm{q}}, t) \\ &= \frac{\partial}{\partial a_\alpha}(\frac{\partial W}{\partial t} + H_1),\end{aligned}$$

which is zero in view of (2.2.43). Thus $n-1$ additional invariants have been constructed and the following "weak complete integrability" theorem has been proved.

Theorem 7:
Let $H(\mathrm{q},\mathrm{p},t)$ be a Hamiltonian of an n-dimensional Hamiltonian system. Assume that a first integral Φ_n and $n-1$ associated configurational invariants Φ_α are known. Suppose further that they are in weak involution in the sense of (2.2.24). Then there exist $n-1$ additional configurational invariants I_α so that the integration of the system along the hypersurface $\Phi_n = 0$ reduces to solving a single first order differential equation.

2.3 First search for configurational invariants

The first search for configurational invariants was made by (Hall 1983) and his work is now reviewed.

Dynamical systems of the following form were considered

$$\ddot{x} = \Omega\dot{y} - \frac{\partial V}{\partial x},$$
$$\ddot{y} = -\Omega\dot{x} - \frac{\partial V}{\partial y}, \tag{2.3.1}$$

where both $\Omega(x, y)$ and $V(x, y)$ depend upon position. The invariant was assumed to be a general polynomial

$$I(\dot{x}, \dot{y}, x, y) = \sum_{n=0}^{N} \sum_{m=0}^{n} A_{nm}(x, y)\dot{x}^m \dot{y}^{n-m}. \tag{2.3.2}$$

The energy integral is given by

$$E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + V,$$

which suggests the following change of variables

$$\dot{x} = u\cos(\gamma), \quad \dot{y} = u\sin(\gamma), \tag{2.3.3}$$

where $u^2 = 2(E_0 - V)$. Now I can be written as

$$I(\dot{x}, \dot{y}, x, y) = \frac{1}{2}C_0 + \sum_{n=1}^{N}[C_n\cos(n\gamma) + S_n\sin(n\gamma)], \tag{2.3.4}$$

where S_n and C_n are functions of x and y and

$$S_n = 0 \text{ for } n \leq 0 \text{ or for } n \geq N + 1,$$

$$C_n = 0 \text{ for } n \leq -1 \text{ or for } n \geq N + 1.$$

The total time derivative of I is

$$\dot{I}_N = \frac{1}{2}\sum_{n=0}^{N+1}\Bigg\{\bigg[u^n\frac{\partial}{\partial x}\left(u^{1-n}C_{n-1}\right) - u^n\frac{\partial}{\partial y}\left(u^{1-n}S_{n-1}\right) - 2n\Omega S_n$$

$$+u^{-n}\frac{\partial}{\partial x}\left(u^{1+n}C_{n+1}\right) + u^{-n}\frac{\partial}{\partial y}\left(u^{1+n}S_{n+1}\right)\bigg]\cos(n\gamma)$$

$$+\bigg[u^n\frac{\partial}{\partial y}\left(u^{1-n}C_{n-1}\right) + u^n\frac{\partial}{\partial x}\left(u^{1-n}S_{n-1}\right) + 2n\Omega C_n$$

$$-u^{-n}\frac{\partial}{\partial y}\left(u^{1+n}C_{n+1}\right) + u^{-n}\frac{\partial}{\partial x}\left(u^{1+n}S_{n+1}\right)\bigg]\sin(n\gamma)\Bigg\}, \qquad (2.3.5)$$

where V has been written *ito* u and $\dot{\gamma}$ has been replaced by

$$\dot{\gamma} = u^{-1}(\ddot{y}\cos(\gamma) - \ddot{x}\sin(\gamma))$$
$$= -\Omega - u^{-1}\left(\frac{\partial V}{\partial y}\cos(\gamma) - \frac{\partial V}{\partial x}\sin(\gamma)\right). \qquad (2.3.6)$$

For $\dot{I}_N$ to be zero the coefficients of $\cos(n\gamma)$ and $\sin(n\gamma)$ must vanish except for $\sin(0\gamma)$. This yields $2N+3$ equations in $2N+3$ unknowns; C_n for $0 \le n \le N$ S_n for $1 \le n \le N$, u and Ω.

By introducing the function

$$F_n = u^{-n}(C_n + iS_n) \qquad (2.3.7)$$

and the operators

$$\nabla = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}, \quad \nabla^* = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}, \qquad (2.3.8)$$

the equations obtained by separating (2.3.5) may be written as

$$\nabla^* F_{n-1} + 2ni\Omega F_n + u^{-2n}\nabla\left(u^{2n+2}F_{n+1}\right) = 0, \;\; 1 \le n \le N, \qquad (2.3.9.a)$$

$$\mathrm{Im}(F_0) = 0, \qquad (2.3.9.b)$$

$$\mathrm{Im}([\nabla(iu^2F_1)] = 0, \qquad (2.3.9.c)$$

$$\nabla^* F_N = 0. \qquad (2.3.9.d)$$

From the last condition it follows that F_N is an analytic function for any value of N. The necessary and sufficient condition for a configurational invariant to exist is that a solution to equations (2.3.9) exists.

There are more convenient variables to use than the present ones. They may be obtained as follows. Let G be an analytic function such that

$$F_N = \frac{G^N}{N}. \tag{2.3.10}$$

A conformal transformation may be defined from the complex variable $z = x + iy$ to the complex variable $\zeta = \xi + i\eta$ by

$$\mathrm{d}z = G\mathrm{d}\zeta. \tag{2.3.11}$$

The differential operators corresponding to ∇ and ∇^* defined previously are

$$\mathrm{D} = G\nabla = \frac{\partial}{\partial\xi} - i\frac{\partial}{\partial\eta} = 2\frac{\partial}{\partial\zeta}$$

and

(2.3.13)

$$\mathrm{D}^* = G^*\nabla^* = \frac{\partial}{\partial\xi} + i\frac{\partial}{\partial\eta} = 2\frac{\partial}{\partial\zeta^*}.$$

Define λ_n, β and T as follows

$$\lambda_n = G^{-n}F_n = u^{-n}G^{-n}(C_n + iS_n), \tag{2.3.14.a}$$

$$\beta = 2GG^*\Omega, \tag{2.3.14.b}$$

$$T = 2GG^*u^2 = 2GG^*(E_0 - V). \tag{2.3.14.c}$$

Equations (2.3.9) transform to

$$\mathrm{D}^*\lambda_{n-1} + ni\beta\lambda_n + T^{-n}\mathrm{D}(T^{n+1}\lambda_{n+1}) = 0, \quad 1 \le n \le N, \tag{2.3.15.a}$$

$$\mathrm{Im}(\lambda_0) = 0, \tag{2.3.15.b}$$

$$\mathrm{Im}[\mathrm{D}(iT\lambda_1)] = 0, \tag{2.3.15.c}$$

where $\lambda_N = 1/N$ and $\lambda_{N+1} = 0$.

Notice that $\lambda_N = 1/N$ has replaced F_N thereby greatly simplifying the calculations to solve (2.3.9.a). The configurational invariant takes the form

$$I_N = \frac{1}{2}\lambda_0 + \mathrm{Re}\left\{\sum_{n=0}^{N}(\lambda_n G^n u^n e^{-in\gamma})\right\}. \tag{2.3.16}$$

For nonzero Ω equations (2.3.15) are difficult to solve. Therefore only the case $\Omega = 0$ will be considered further. Firstly the conditions for the existence of a configurational invariant will be reduced to a single equation for the cases $N = 1,4$ and then a few examples obtained using these results.

When $\Omega = 0$ equations (2.3.15.a) decouple :

$$\mathrm{D}^*\lambda_{n-2} + T^{1-n}\mathrm{D}(T^n\lambda_n) = 0, \qquad 1 \leq n \leq N. \tag{2.3.17}$$

For $n = N$ this becomes

$$\mathrm{D}^*\lambda_{N-2} + \mathrm{D}T = 0.$$

Integrating this *wrt* ζ^* gives

$$\lambda_{N-2} = -\int \mathrm{D}(T)d\zeta^* + \Lambda_{N-2}(\zeta), \tag{2.3.18}$$

where $\Lambda_{N-2}(\zeta)$ is any analytic function of ζ.

Let

$$\psi = \mathrm{DD}^*(T), \tag{2.3.19}$$

then

$$\lambda_{N-2} = -\int \mathrm{D}^2\mathrm{D}^*\psi + \Lambda_{N-2}(\zeta),$$

i.e.

$$\lambda_{N-2} = -\mathrm{D}^2\psi + \Lambda_{N-2}(\zeta). \tag{2.3.20}$$

Since Λ_{N-2} is an analytic function of ζ, $\mathrm{D}^*\Lambda_{N-2} = 0$ and therefore $\mathrm{DD}^*\Lambda_{N-2} = 0$. Consequently $\Psi = \psi + \Lambda_{N-2}$ will also be a solution of (2.3.19) and Λ_{N-2} may be absorbed into ψ giving

$$\lambda_{N-2} = -\mathrm{D}^2\psi. \tag{2.3.21}$$

Then

$$\lambda_{N-2} = -X + iY, \tag{2.3.22}$$

where

$$X = \frac{\partial^2\psi}{\partial\xi^2} - \frac{\partial^2\psi}{\partial\eta^2} \text{ and } Y = 2\frac{\partial^2\psi}{\partial\xi\partial\eta}. \tag{2.3.23}$$

For the cases $N = 1,4$ the conditions for the existence of a configurational invariant will now be reduced to one equation for each case for the unknown function ψ.

Case N=1:
Since $\lambda_1 = 1$, (2.3.15.c) becomes

$$\begin{aligned} &\mathrm{Im}[\mathrm{D}(iT)] = 0, \\ &\mathrm{Im}[(\frac{\partial}{\partial\xi} - i\frac{\partial}{\partial\eta})T] = 0, \\ &\frac{\partial T}{\partial\xi} = 0, \end{aligned} \tag{(2.3.24}$$

or equivalently

$$\frac{\partial\psi}{\partial\xi} = 0. \tag{2.3.25}$$

The functions ψ or T must satisfy the respective constraints above for a configurational invariant to exist.

Case N=2:
In this case (2.3.21) is

$$\lambda_0 = -\mathrm{D}^2\psi. \tag{2.3.26}$$

According to (2,3,15.b) $\mathrm{Im}(\lambda_0) = 0$ therefore simplifying the previous equation. The equation to determine ψ is

$$\frac{\partial^2\psi}{\partial\xi\partial\eta} = 0. \tag{2.3.27}$$

Case N=3:
In this case λ_1 can be obtained from (2.3.22) above. Substituting this into (2.3.15.c) yields

$$\frac{\partial}{\partial\xi}(XT) - \frac{\partial}{\partial\eta}(YT) = 0. \tag{2.3.15.c}$$

Rewritten in terms of ψ this is

$$\frac{\partial^2\psi}{\partial\xi^2}\frac{\partial^3\psi}{\partial\xi^3}-\left(\frac{\partial^2\psi}{\partial\xi^2}+2\frac{\partial^2\psi}{\partial\eta^2}\right)\frac{\partial^3\psi}{\partial\xi\partial\eta^2}$$

$$-\frac{\partial^2\psi}{\partial\xi\partial\eta}\left(\frac{\partial^3\psi}{\partial\xi^3\partial\eta}+\frac{\partial^3\psi}{\partial\eta^3}\right)=0. \tag{2.3.29}$$

In general for odd N (2.3.15.c) is the determining equation for ψ. Since $\lambda_N = 1/N$ one can use equation (2.3.17) to determine λ_{n-2} *ito* λ_n and ψ and hence ultimately *ito* λ_N and ψ until $n-2=1$. At this point the condition (2.3.15.c) is applied and a equation only containing the unknown ψ obtained.

<u>Case N=4:</u>

By setting $n=2$ in equation (2.3.17), writing $\lambda_2=\lambda_2^R+i\lambda_2^I$ and setting both the real and imaginery parts equal to zero the following is obtained

$$\frac{\partial\lambda_0^R}{\partial\xi}-\frac{\partial\lambda_0^I}{\partial\eta}=-\left(2\frac{\partial T}{\partial\xi}\lambda_2^R+2\frac{\partial T}{\partial\eta}\lambda_2^I+T\frac{\partial\lambda_2^R}{\partial\xi}+T\frac{\partial\lambda_2^I}{\partial\eta}\right), \tag{2.3.30}$$

$$\frac{\partial\lambda_0^R}{\partial\eta}+\frac{\partial\lambda_0^I}{\partial\xi}=-2\frac{\partial T}{\partial\xi}\lambda_2^I+2\frac{\partial T}{\partial\eta}\lambda_2^R-T\frac{\partial\lambda_2^I}{\partial\xi}+T\frac{\partial\lambda_2^R}{\partial\eta}. \tag{2.3.31}$$

Differentiating the two equations so as to obtain $(\partial^2\lambda_0^R/\partial x\partial y)$ and then eliminating it from them gives

$$\frac{\partial}{\partial\xi}\left(-2\frac{\partial T}{\partial\xi}\lambda_2^I+2\frac{\partial T}{\partial\eta}\lambda_2^R-T\frac{\partial\lambda_2^I}{\partial\xi}+T\frac{\partial\lambda_2^R}{\partial\eta}\right)$$

$$+\frac{\partial}{\partial\eta}\left(2\frac{\partial T}{\partial\xi}\lambda_2^R+2\frac{\partial T}{\partial\eta}\lambda_2^I+T\frac{\partial\lambda_2^R}{\partial\xi}+T\frac{\partial\lambda_2^I}{\partial\eta}\right)=0, \tag{2.3.32}$$

where the condition $\lambda_0^I=0$ has been used.

This is the equation determining ψ when N is even since equation (2.3.17) may be repeatedly solved for λ_{n-2} *ito* λ_N and ψ until $n=4$ and the result substituted into the determining equation above. Thus for $N=4$, ψ must satisfy

$$\frac{\partial}{\partial\xi}\left(\frac{\partial^2\psi}{\partial\xi\partial\eta}\frac{\partial^3\psi}{\partial\xi^3}+\frac{\partial^2\psi}{\partial\xi^2}\frac{\partial^3\psi}{\partial\xi^2\partial\eta}\right)$$

$$-\frac{\partial}{\partial\eta}\left(\frac{\partial^2\psi}{\partial\xi\partial\eta}\frac{\partial^3\psi}{\partial\eta^3}+2\frac{\partial^2\psi}{\partial\eta^2}\frac{\partial^3\psi}{\partial\xi\partial\eta^2}\right)=0. \tag{2.3.33}$$

Examples:

i) Third order

Since the equation for ψ to exist for a third order configurational invariant is difficult to solve choose

$$-\psi = H(\eta) + f(\xi)g(\eta), \tag{2.3.34}$$

where H, f and g are still to be determined. The equation for ψ becomes

$$\begin{aligned} &\frac{f^{(1)}}{g^{(1)}}\frac{\partial}{\partial\eta}\left[H^{(2)}(g^{(1)})^2\right] + \left\{ff^{(1)}\left[g^{(1)}g^{(3)} + 2(g^{(2)})^2\right]\right. \\ &\left. + f^{(1)}f^{(2)}\left[(g^{(1)})^2 + gg^{(2)}\right] - f^{(2)}f^{(3)}g^2\right\} = 0, \end{aligned} \tag{2.3.35}$$

where the superscripts in parentheses denote the order of the derivative. Let $f = a + \frac{1}{2}(\xi - \xi_0)^2$ and g be the solution to

$$g^{(1)}g^{(3)} + 2(g^{(2)})^2 = 0. \tag{2.3.36}$$

Then $g = K + \alpha(\eta - \eta_0)^{\frac{4}{3}}$ and the equation for ψ becomes

$$\frac{\partial}{\partial\eta}\left[H^{(2)}(g^{(1)})^2\right] = -g^{(1)}\left[(g^{(1)})^2 + gg^{(2)}\right]. \tag{2.3.37}$$

It is easy to solve this equation, but the general solution is long and shall therefore be omitted. Choosing G (of the transformation) to be a constant gives rise to the following algebraic potential

$$V = K_0 + \frac{4}{9}a\alpha(y-y_0)^{-2/3} + \frac{\alpha}{6}(y-y_0)^{4/3} + \frac{2}{9}\alpha(y-y_0)^{-2/3}(x-x_0). \tag{2.3.38}$$

If f is chosen as before and g is chosen as $g = e^{\lambda\eta}$, where λ is a constant, then

$$T = A - Be^{-2\lambda\eta} - (C_1e^{\sqrt{3}\lambda\xi} + C_2e^{-\sqrt{3}\lambda\xi})e^{\lambda\eta}. \tag{2.3.39}$$

With $\lambda = 2$ and appropriate values for the other constants the corresponding potential is a Toda-type potential (Hall 1983):

$$V = \frac{1}{24}\left\{e^{(2y+2\sqrt{3}x)} + e^{(2y-2\sqrt{3}x)} + e^{-4y}\right\} - \frac{1}{8}. \tag{2.3.40}$$

The corresponding configurational invariant is

$$\begin{aligned} I_3 &= 8\dot{x}(\dot{x}^2 - 3\dot{y}^2) + (\dot{x} + \sqrt{3}\dot{y})e^{(2y-2\sqrt{3}x)} \\ &+ (\dot{x} - \sqrt{3}\dot{y})e^{(2y+2\sqrt{3}x)} - 2\dot{x}e^{-4y}. \end{aligned} \tag{2.3.41}$$

ii) Fourth order

Let

$$-\psi = \Xi(\xi) + H(\eta) + f(\xi)\eta. \tag{2.3.42}$$

The equation for determining ψ when $N = 4$ now requires that

$$\begin{aligned}
&(\frac{\partial f}{\partial \xi}\frac{\partial^4 f}{\partial \xi^4} + 5\frac{\partial^2 f}{\partial \xi^2}\frac{\partial^3 f}{\partial \xi^3})\eta - \frac{\partial f}{\partial \xi}\frac{\partial^4 H}{\partial \eta^4} \\
&+ \frac{\partial f}{\partial \xi}\frac{\partial^4 \Xi}{\partial \xi^4} + 3\frac{\partial^2 f}{\partial \xi^2}\frac{\partial^3 \Xi}{\partial \xi^3} + 2\frac{\partial^3 f}{\partial \xi^3}\frac{\partial^2 \Xi}{\partial \xi^2} = 0.
\end{aligned} \tag{2.3.43}$$

For $G = constant$ the potential has the form

$$V = \frac{1}{2}(x^2 + 16y^2) + 4x^2 y + \frac{64}{3}y^3 \tag{2.3.44}$$

and

$$\begin{aligned}
I_4 = &\frac{1}{4}\dot{x}^4 + (\frac{1}{4}x^2 + 4x^2 y)\dot{x}^2 - \frac{4}{3}x^3\dot{x}\dot{y} + \frac{1}{4}x^4 - \frac{4}{3}x^4 y \\
&- \frac{8}{9}x^6 - \frac{16}{3}x^4 y^2.
\end{aligned} \tag{2.3.45}$$

2.4 Further work on configurational invariants

In this section results obtained by (Hietarinta 1987) for configurational invariants are discussed. Transformations are introduced which transform integrable systems to systems of which the integrability can only be garanteed at zero energy. These results can be applied in an anologous way to find systems integrable at fixed values of other first integrals. Here the fixed value of the energy will be taken as zero for simplicity.

Consider a Hamiltonian of the form

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y). \tag{2.4.1}$$

The transformation

$$\begin{aligned} x &= u(X, Y), \;\; p_x = (v_Y P_X - v_X P_Y)/D, \\ y &= v(X, Y), \;\; p_y = (-u_Y P_X + u_X P_Y)/D, \end{aligned} \tag{2.4.2}$$

where $D = u_X v_Y - u_Y v_X$, is canonical if

$$\begin{aligned} u_X u_Y + v_X v_Y &= 0, \\ u_X^2 + v_X^2 = u_Y^2 + v_Y^2 &\neq 0 \end{aligned} \tag{2.4.3}$$

and then the form of the Hamiltonian will remain unchanged except for a term multiplying the kinetic energy term. The solution to these equations is

$$u_X = v_Y, \;\; u_Y = -v_X, \tag{2.4.4}$$

(the complex conjutate form of these is also a solution). These are the Cauchy-Riemann equations which characterize analycity. Therefore, if $u = \mathrm{Re}(f(Z))$ and $v = \mathrm{Im}(f(Z))$, where f is a analytic function of $Z = X + iY$, then u and v will satisfy (2.4.4). The transformed Hamiltonian is

$$H_1 = \frac{1}{2}|f'(Z)|^{-2}(P_X^2 + P_Y^2) + V(\mathrm{Re}(f(Z)), \mathrm{Im}(f(Z))). \tag{2.4.5}$$

Theorem 8:

The system with Hamiltonian

$$H_2 = \frac{1}{2}(P_X^2 + P_Y^2) + |f'(Z)|^2 V(\mathrm{Re}(f(Z)), \mathrm{Im}(f(Z))), \tag{2.4.6}$$

is integrable at zero energy.

Proof:

Note that the phase space variables of the system H_1 in (2.4.5) and H_2 are the same. Denote the rate of change of these variables as determined from H_1 and H_2 by a dot above the variable and a prime respectively. For H_1 Hamilton's equations of motion give

$$\dot{X} = P_X|f'|^{-2}, \qquad \dot{P}_X = (P_X^2 + P_Y^2)|f'|^{-3}\frac{\partial}{\partial X}|f'| - \frac{\partial V}{\partial X}. \tag{2.4.7}$$

Analogous expressions may be obtained for $\dot{Y}$ and $\dot{P}_Y$. For H_2 Hamilton's equations are

$$X' = P_X = |f'|^2\dot{X} \qquad P_X' = -2|f'|V\frac{\partial}{\partial X}|f'| - |f'|^2\frac{\partial V}{\partial X}. \tag{2.4.8}$$

Since energy is zero, V may be found from $H_2 = 0$ for, from (2.4.6),

$$\begin{aligned} P_X' &= -2|f'|\left[-\frac{1}{2}(P_X^2 + P_Y^2)|f'|^{-2}\right]\frac{\partial}{\partial X}|f'| - |f'|^2\frac{\partial V}{\partial X} \\ &= |f'|^2\left[(P_X^2 + P_Y^2)|f'|^{-3}\frac{\partial}{\partial X}|f'| - \frac{\partial V}{\partial X}\right]. \end{aligned}$$

In view of (2.4.7) this reduces to

$$P_X' = |f'|^2\dot{P}_X. \tag{2.4.9}$$

Y' and P_Y' have similar expressions.

Let $I(X, Y, P_X, P_Y)$ be a first integral of the integrable system H_1, then I is a configurational invariant of H_2:

$$\begin{aligned} I'|_{E=0} &= \left[\frac{\partial I}{\partial X}X' + \frac{\partial I}{\partial Y}Y' + \frac{\partial I}{\partial P_X}P_X' + \frac{\partial I}{\partial P_Y}P_Y'\right]_{E=0} \\ &= |f'|^2\left[\frac{\partial I}{\partial X}\dot{X} + \frac{\partial I}{\partial Y}\dot{Y} + \frac{\partial I}{\partial P_X}\dot{P}_X + \frac{\partial I}{\partial P_Y}\dot{P}_Y\right]_{E=0} \\ &= 0. \end{aligned}$$

Thus H_2 is integrable at zero energy.

As an illustration consider the potential $V = ax^2 + by^2$. Let $f = Z^2$. Then V becomes

$$V' = 4[ax^6 + (4b - a)x^4y^2 + (4b - a)x^2y^4 + ay^6], \tag{2.4.10}$$

which is only known to be integrable when $a = 4b$, $b = 4a$ or $a = b$. For all other values of a and b the system must at least be integrable at zero energy. For $a = 1$ and $b = -1$ it was found that the system possessed the following configurational invariant

$$I = \frac{\dot{x}^2(x^2 - y^2)}{(x^2 + y^2)^2} - \frac{2xy\dot{x}\dot{y}}{(x^2 + y^2)^2} + \frac{8(x^6 - 2x^4y^2 + 5x^2y^4)}{(x^2 + y^2)}. \tag{2.4.11}$$

Application of Ziglin's theorem (Yoshida 1987) to this system showed that it is non-integrable.

As another example consider the transformation represented by $f = Z^N$ applied to the integrable potential $V = a + b/r$. If $N = -1$ and $N = -2$ respectively, the following two integrable potentials are obtained

$$V = br^{-3} + ar^{-4}, \qquad V = br^{-4} + ar^{-6}. \tag{2.4.12}$$

A relationship between polynomial and Toda-type potentials can be established using the transformation (2.4.2). When $f = e^z$ the system

$$H = \frac{1}{2}(p_x^2 + p_y^2) + W(x, y), \tag{2.4.13}$$

where W is a polynomial in x and y, transforms to

$$H' = \frac{1}{2}(p_X^2 + p_Y^2) + e^{2X}W(\frac{1}{2}(e^{X+iY} + e^{X-iY}), \frac{1}{2}(e^{X+iY} - e^{X-iY})), \tag{2.4.14}$$

where W is now a Toda-type potential.

On the other hand starting from a two term Toda-type potential

$$V = e^{ax}(e^{by} + e^{cy}) \tag{2.4.15}$$

and applying the transformation defined by $f = \log(z)$ which in polar co-ordinates becomes $x = \log(r)$, $y = \theta$, gives the potential

$$V' = r^{a-2}(e^{b\theta} + e^{c\theta}). \tag{2.4.16}$$

If $b = im$ and $c = in$ where a, m and n are suitable integers, V is a polynomial in cartesian co-ordinates.

Complexification of the transformations

Under the transformation

$$\begin{aligned} z &= (x + iy)\sqrt{2}, \quad p_z = (p_x - ip_y)\sqrt{2}, \\ w &= (x - iy)\sqrt{2}, \quad p_w = (p_x + ip_y)\sqrt{2}, \end{aligned} \tag{2.4.17}$$

(2.4.1) becomes

$$K = p_z p_w + U(z, w). \tag{2.4.18}$$

When working at zero energy one uses the zero-energy condition to eliminate one of the momenta or some other variable in the energy function. Now solving for one of the momenta from the zero-energy condition

$$p_z p_w + U(z, w) = 0 \tag{2.4.19}$$

does not give square roots as would have been the case for a system of the form (2.4.1), making this a convenient tranformation. The transformation does not really complicate the potential since it is linear.

The analogue of (2.4.2) in these co-ordinates is

$$\begin{aligned} z &= R(Z), \quad p_z = P_Z/R'(Z), \\ w &= S(W), \quad P_W/S'(W). \end{aligned} \tag{2.4.20}$$

The tranformed Hamiltonian is

$$K' = [P_Z P_W + R'(Z)S'(W)U(R(Z), S(W))]/[R'(Z)S'(W)]. \tag{2.4.21}$$

Multiplying through by $R'(Z)S'(W)$ gives

$$K' = [P_Z P_W + R'(Z)S'(W)U(R(Z), S(W))], \tag{2.4.22}$$

which will be integrable at zero energy. To retain the reality of the potential it is necessary that $S'(W) = R'(Z)^*$.

<u>Some Applications of these Results</u>

Suppose that I is a configurational invariant even in the momenta. Then I may be written in the form

$$I = \sum_{n=1}^{N} p_z^{2n} e(2n; z, w) + e(0; z, w) + \sum_{n=1}^{M} p_w^{2n} e(-2n; z, w). \tag{2.4.23}$$

Setting the total time derivative of I equal to zero, applying the zero-energy condition to eliminate terms of the form $p_w p_z^{2n}$ and separating by coefficients of different powers of p_z and p_w yields

$$\frac{\partial}{\partial w} e(2n-1) = U \frac{\partial}{\partial z} e(2n) + 2n \frac{\partial U}{\partial z} e(2n), \qquad n = 1, N+1, \tag{2.4.24.a}$$

$$\frac{\partial}{\partial z} e(-2n+1) = U \frac{\partial}{\partial w} e(-2n) + 2n \frac{\partial U}{\partial w} e(-2n), \qquad n = 1, M+1. \tag{2.4.24.b}$$

Here it is assumed that $e(2n) = 0$ for $n > N$ or $n < -M$.

If the configurational invariant is odd, it may be written as

$$I = \sum_{n=0}^{N} p_z^{2n+1} e(2n+1; z, w) + \sum_{n=0}^{M} p_w^{2n+1} e(-2n-1; z, w) \tag{2.4.25}$$

and the equations for the functions e are

$$\frac{\partial}{\partial w} e(2n-1) = U \frac{\partial}{\partial z} e(2n+1) + (2n+1) \frac{\partial U}{\partial z} e(2n+1), \tag{2.4.26.a}$$

$$n = 1, N+1,$$

$$\frac{\partial}{\partial z} e(-2n+1) = U \frac{\partial}{\partial w} e(-2n-1) + (2n+1) \frac{\partial U}{\partial w} e(-2n-1), \tag{2.4.26.b}$$

$$n = 1, M+1,$$

$$\frac{\partial}{\partial z}(U e(1)) + \frac{\partial}{\partial w}(U e(-1)) = 0. \tag{2.4.26.c}$$

For small values of N and M equations (2.4.24) and (2.4.25) may easily be simplified. Consider a linear configurational invariant of the form (2.4.25). The solution to (2.4.26.a,b) is $e(1) = f(z)$, $e(-1) = g(w)$. The transformation (2.4.20) may be used, with a possible reflection, to give $e(1) = 1$ and $e(-1) = 1$ or 0. The remaining equation

$$\frac{\partial U}{\partial z} + e(-1)\frac{\partial U}{\partial w} = 0 \tag{2.4.27}$$

has the solution $U = U(w)$ if $e(-1) = 0$ or $U = U(z-w)$ if $e(-1) = 1$.

For a quadratic configurational invariant of the form (2.4.23) the leading coefficients may be chosen just as before, viz. $e(2) = 1$ and $e(-2) = 1$ or 0. The integrability condition for $e(0)$ is

$$\frac{\partial^2 U}{\partial z^2} - e(-2)\frac{\partial^2 U}{\partial w^2} = 0 \tag{2.4.28}$$

with the solution $U = zf(w)+g(w)$ if $e(-2)=0$ and $U = f(z+iw)+g(z-iw)$ if $e(-2) = 1$.

For a cubic configurational invariant of the form (2.4.25) the leading coefficients are chosen as previously. Equations (2.4.26) become

$$\frac{\partial}{\partial w}e(1) = 3\frac{\partial U}{\partial z}, \qquad \frac{\partial}{\partial z}e(-1) = 3\frac{\partial U}{\partial w}e(-3), \tag{2.4.29.a}$$

$$\frac{\partial}{\partial z}(Ue(1)) + \frac{\partial}{\partial w}(Ue(-1)) = 0. \tag{2.4.29.b}$$

To simplify these equations an auxilary function W, defined by

$$U = \frac{\partial^2 W}{\partial z \partial w}, \tag{2.4.30}$$

is introduced and equations (2.4.29.a) can now be integrated :

$$e(1) = 3\frac{\partial^2 W}{\partial z^2}, \qquad e(-1) = 3\frac{\partial^2 W}{\partial w^2}. \tag{2.4.31}$$

W is determined from (2.4.29.b):

$$\frac{\partial}{\partial z}\left(\frac{\partial^2 W}{\partial z \partial w}\frac{\partial^2 W}{\partial z^2}\right) + e(-3)\frac{\partial}{\partial w}\left(\frac{\partial^2 W}{\partial z \partial w}\frac{\partial^2 W}{\partial w^2}\right) = 0. \tag{2.4.32}$$

For quartic configurational invariants

$$e(4) = 1, \;\; e(-4) = 1 \text{ or } e(-4) = 0,$$

$$e(2) = 4\frac{\partial^2 W}{\partial z^2}, \;\; e(-2) = 4e(-4)\frac{\partial^2 W}{\partial z^2}, \tag{2.4.33}$$

and from the integrability condition for e(0)

$$\frac{\partial}{\partial z}\left(\frac{\partial^3 W}{\partial z^3}\frac{\partial^2 W}{\partial z\partial w}+2\frac{\partial^3 W}{\partial z^2\partial w}\frac{\partial^2 W}{\partial z^2}\right)-e(-4)\frac{\partial}{\partial w}\left(\frac{\partial^3 W}{\partial w^3}\frac{\partial^2 W}{\partial z\partial w}+2\frac{\partial^3 W}{\partial z\partial w^2}\frac{\partial^2 W}{\partial w^2}\right)=0. \tag{2.4.34}$$

These results are equivalent to those found by (Hall 1983), discussed in § **2.3** above.

The transformation introduced in this section to find systems integrable at zero energy is interesting. It is easier to implement than the direct method since there are no differential equations to solve. However, it is less systematic since it is up to the individual to decide what Hamiltonian to begin with and what transformation to use.

The complexification of the co-ordinates follows naturally from the two-dimensional problems being considered since they possess the "natural" kinetic energy which gives rise to Laplace's equation, when the zero-energy condition is imposed. The solution contains the complex co-ordinates μ and ν.

CHAPTER III: CONFIGURATIONAL INVARIANTS POLYNOMIAL IN THE MOMENTA FOR TWO-DIMENSIONAL POTENTIALS LINEAR IN THE MOMENTA

3.1 Introduction

In the previous chapter L. S. Hall's work on configurational invariants was discussed. He looked for polynomial configurational invariants for systems of the form

$$\begin{aligned} \ddot{x} &= \Omega\dot{y} - \frac{\partial V}{\partial x}, \\ \ddot{y} &= -\Omega\dot{x} - \frac{\partial V}{\partial y}. \end{aligned} \tag{3.1.1}$$

Here slightly more general systems will be considered, specifically systems of the form

$$H = \frac{1}{2}(p_x^2 + p_y^2) + A(x,y)p_x + B(x,y)p_y + V(x,y). \tag{3.1.2}$$

Hall (1983) found the general solution to the problem for configurational invariants linear in the momenta, but for quadratic configurational invariants the general solution is still unknown. For systems of the second type above the general solution has not been found even for linear configurational invariants as a result of the complexity of the problem. It was decided only to investigate the existence of configurational invariants which are most quadratic, in the momenta, in this chapter. Here the approach adopted in (Hietarinta 1987) is used. The constant value of the energy is taken to be zero since it is equivalent to consider the system H at $E = E_0$ and the system $H - E_0$ at $E = 0$.

3.2 Configurational invariants linear in the momenta.

For the simplification of later calculations the Hamiltonian is written as

$$H = \frac{1}{2}[(p_x + A)^2 + (p_y + B)^2] + W(x,y), \tag{3.2.1}$$

where

$$W = V - \frac{1}{2}[A^2 + B^2].$$

Similarly the linear configurational invariant is written as

$$I = (p_x + A)a(x,y) + (p_y + B)b(x,y) + c(x,y). \tag{3.2.2}$$

When separating $\dot{I}|_{E=0}$ by coefficients of powers of $(p_x + A)$ and $(p_y + B)$ the following set of equations is obtained.

$$\frac{\partial a}{\partial x} - \frac{\partial b}{\partial y} = 0, \tag{3.2.3.1}$$

$$\frac{\partial a}{\partial y} + \frac{\partial b}{\partial x} = 0, \tag{3.2.3.2}$$

$$\frac{\partial c}{\partial x} - bU = 0, \qquad U := \frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}, \tag{3.2.3.3}$$

$$\frac{\partial c}{\partial y} + aU = 0, \tag{3.2.3.4}$$

$$2\frac{\partial b}{\partial y}W + \frac{\partial W}{\partial x}a + \frac{\partial W}{\partial y}b = 0. \tag{3.2.3.5}$$

The solutions to (3.2.3.1) and (3.2.3.2) are

$$b = b_1(x + iy) + b_2(x - iy), \tag{3.2.4.1}$$

$$a = i(b_1 - b_2), \tag{3.2.4.2}$$

where b_1 and b_2 are arbitrary functions of their arguments.

Using equations (3.2.3.3) and (3.2.3.4) the integrability condition on c may be expressed as

$$\frac{\partial}{\partial x}(aU) + \frac{\partial}{\partial y}(bU) = 0. \tag{3.2.5}$$

Let $x + iy = \mu$ and $x - iy = \nu$. Then this equation becomes

$$\left(\frac{\mathrm{d}b_1}{\mathrm{d}\mu} - \frac{\mathrm{d}b_2}{\mathrm{d}\nu}\right)U + \left(b_1\frac{\partial U}{\partial \mu} - b_2\frac{\partial U}{\partial \nu}\right) = 0. \tag{3.2.6}$$

Since $\partial a/\partial x = \partial b/\partial y$, equation (3.2.3.5) may be rewritten as

$$\left(\frac{\partial a}{\partial x} + \frac{\partial b}{\partial y}\right)W + \frac{\partial W}{\partial x}a + \frac{\partial W}{\partial y}b = 0. \tag{3.2.7}$$

Thus U and W must satisfy the same equation. To solve (3.2.6) or equivalently (3.2.7) assume a separation of variables, i.e.,

$$U = f(\mu)g(\nu). \tag{3.2.8}$$

Then equation (3.2.6) becomes

$$\left(\frac{\mathrm{d}b_1}{\mathrm{d}\mu} - \frac{\mathrm{d}b_2}{\mathrm{d}\nu}\right) fg + b_1 g\frac{\mathrm{d}f}{\mathrm{d}\mu} - b_2 f\frac{\mathrm{d}g}{\mathrm{d}\nu} = 0.$$

If $f \neq 0$ and $g \neq 0$, division by them and rearrangement gives

$$\frac{1}{f}\left(\frac{\mathrm{d}b_1}{\mathrm{d}\mu} f + b_1\frac{\mathrm{d}f}{\mathrm{d}\mu}\right) = \frac{1}{g}\left(\frac{\mathrm{d}b_2}{\mathrm{d}\nu} g + b_2\frac{\mathrm{d}g}{\mathrm{d}\nu}\right). \tag{3.2.9}$$

The *lhs* is only a function of μ and the *rhs* only a function of ν. Therefore

$$\frac{1}{f}\left(\frac{\mathrm{d}b_1}{\mathrm{d}\mu} f + b_1\frac{\mathrm{d}f}{\mathrm{d}\mu}\right) = k, \tag{3.2.10.1}$$

$$\frac{1}{g}\left(\frac{\mathrm{d}b_2}{\mathrm{d}\nu} g + b_2\frac{\mathrm{d}g}{\mathrm{d}\nu}\right) = k, \tag{3.2.10.2}$$

where k is a constant. Division of (3.2.10.1) by b_1 and subsequent integration yields

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\ln(b_1) + \frac{\mathrm{d}}{\mathrm{d}\mu}\ln(f) = \frac{k}{b_1},$$

and hence

$$f = \frac{l_1}{b_1}\mathrm{e}^{\int k/b_1 \mathrm{d}\mu}, \tag{3.2.11.1}$$

similarly

$$g = \frac{l_2}{b_2}\mathrm{e}^{\int k/b_2 \mathrm{d}\nu}, \tag{3.2.11.2}$$

where l_1 and l_2 are constants of integration.

If W is assumed to have the same form as U, the solutions to U and W are

$$U = \frac{u}{b_1 b_2}\mathrm{e}^{k_1\left(\int 1/b_1 \mathrm{d}\mu + \int 1/b_2 \mathrm{d}\nu\right)}, \tag{3.2.12.1}$$

$$W = \frac{w}{b_1 b_2}\mathrm{e}^{k_2\left(\int 1/b_1 \mathrm{d}\mu + \int 1/b_2 \mathrm{d}\nu\right)}, \tag{3.2.12.2}$$

where u, w, k_1 and k_2 are constants.

For this particular U equations (3.2.3.3) and (3.2.3.4) give

$$c = \frac{u}{k_1}\mathrm{e}^{k_1\left(\int 1/b_1 \mathrm{d}\mu + \int 1/b_2 \mathrm{d}\nu\right)}, \qquad k_1 \neq 0, \tag{3.2.13.1}$$

$$c = u\left(\int \frac{1}{b_1}\mathrm{d}\mu + \int \frac{1}{b_2}\mathrm{d}\nu\right), \qquad k_1 = 0. \tag{3.2.13.2}$$

The only remaining equation to be solved is $\partial A/\partial y - \partial B/\partial x = U$. To solve it and find a few examples of potentials several functional forms for b_1 and b_2 have been chosen below and the corresponding Hamiltonian and configurational invariant determined. In the following examples the arbitrary constants in U and W above will be assigned the following values: $k_1 = 0$, $k_2 = 1$, $u = 1$ and $w = -1$.

Let $b_1 = (x + iy)^{-1}$ and $b_2 = (x - iy)^{-1}$. then

$$U = x^2 + y^2. \tag{3.2.14}$$

A solution to $U = \partial A/\partial y - \partial B/\partial x$ is

$$\begin{aligned} A &= \frac{1}{4}y(x^2 + y^2), \\ B &= -\frac{1}{4}x(x^2 + y^2). \end{aligned} \tag{3.2.15}$$

The corresponding Hamiltonian and configurational invariant are

$$\begin{aligned} H = \frac{1}{2}&\left[\left(p_x + \frac{1}{4}y(x^2 + y^2)\right)^2 + \left(p_y - \frac{1}{4}x(x^2 + y^2)\right)^2\right] \\ &- (x^2 + y^2)e^{x^2 - y^2} \end{aligned} \tag{3.2.16.1}$$

and

$$\begin{aligned} I = &\left(p_x + \frac{1}{4}y(x^2 + y^2)\right)\frac{2y}{x^2 + y^2} + \left(p_y - \frac{1}{4}x(x^2 + y^2)\right)\frac{2x}{x^2 + y^2} \\ &+ x^2 - y^2. \end{aligned} \tag{3.2.16.2}$$

The equations of motion of this system were numerically integrated using a Runge-Kutta-Merson method for various initial values. Two orbits obtained are given in figg 1 and 2.

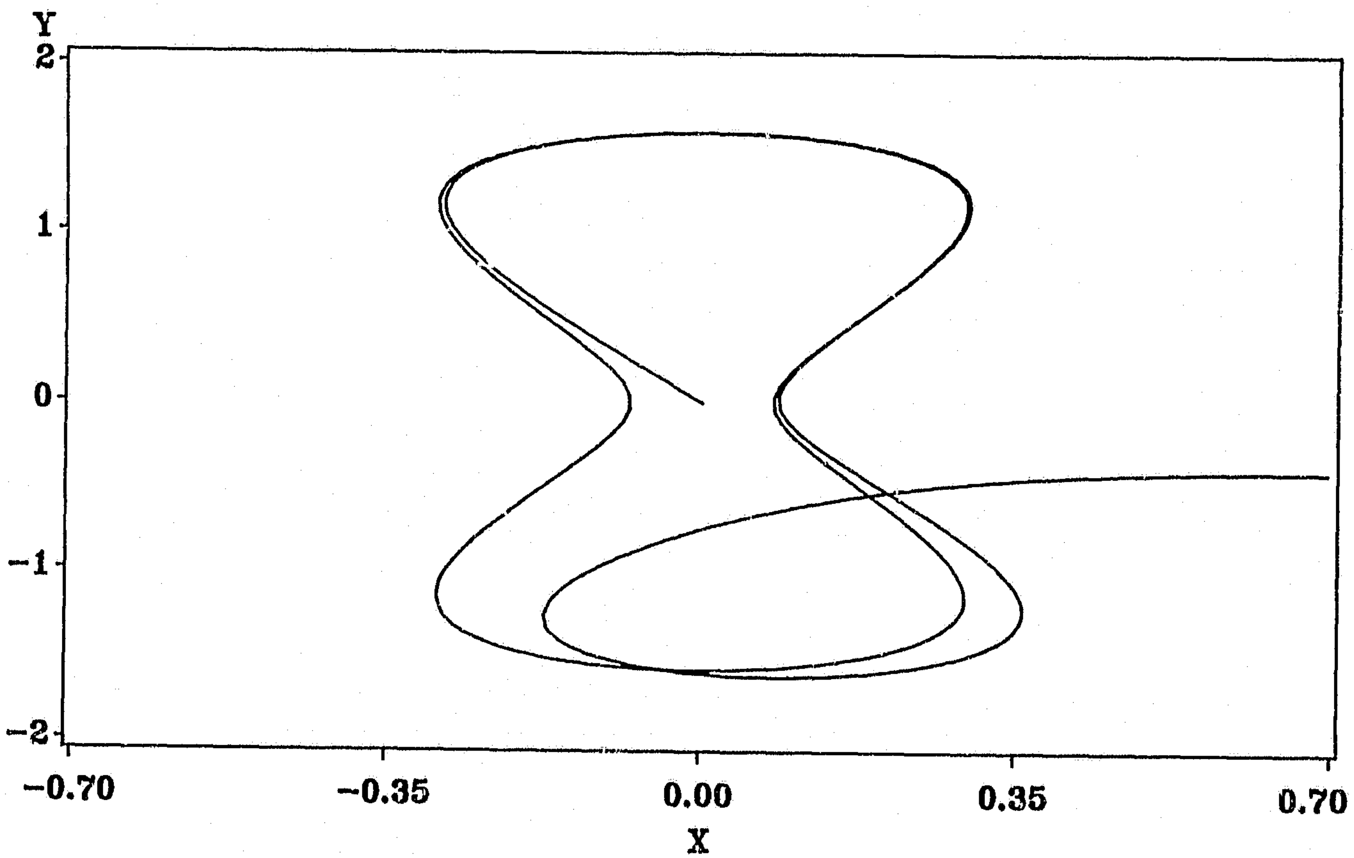

fig 1: An orbit of the system described by

$$H = \frac{1}{2}\left[\left(p_x + \frac{1}{4}y(x^2+y^2)\right)^2 + \left(p_y - \frac{1}{4}x(x^2+y^2)\right)^2\right] - (x^2+y^2)e^{x^2-y^2}$$

with initial conditions $x = 0.0$, $y = 0.0$, $\dot{x} = -0.1$ and $\dot{y} = 0.3088612$.

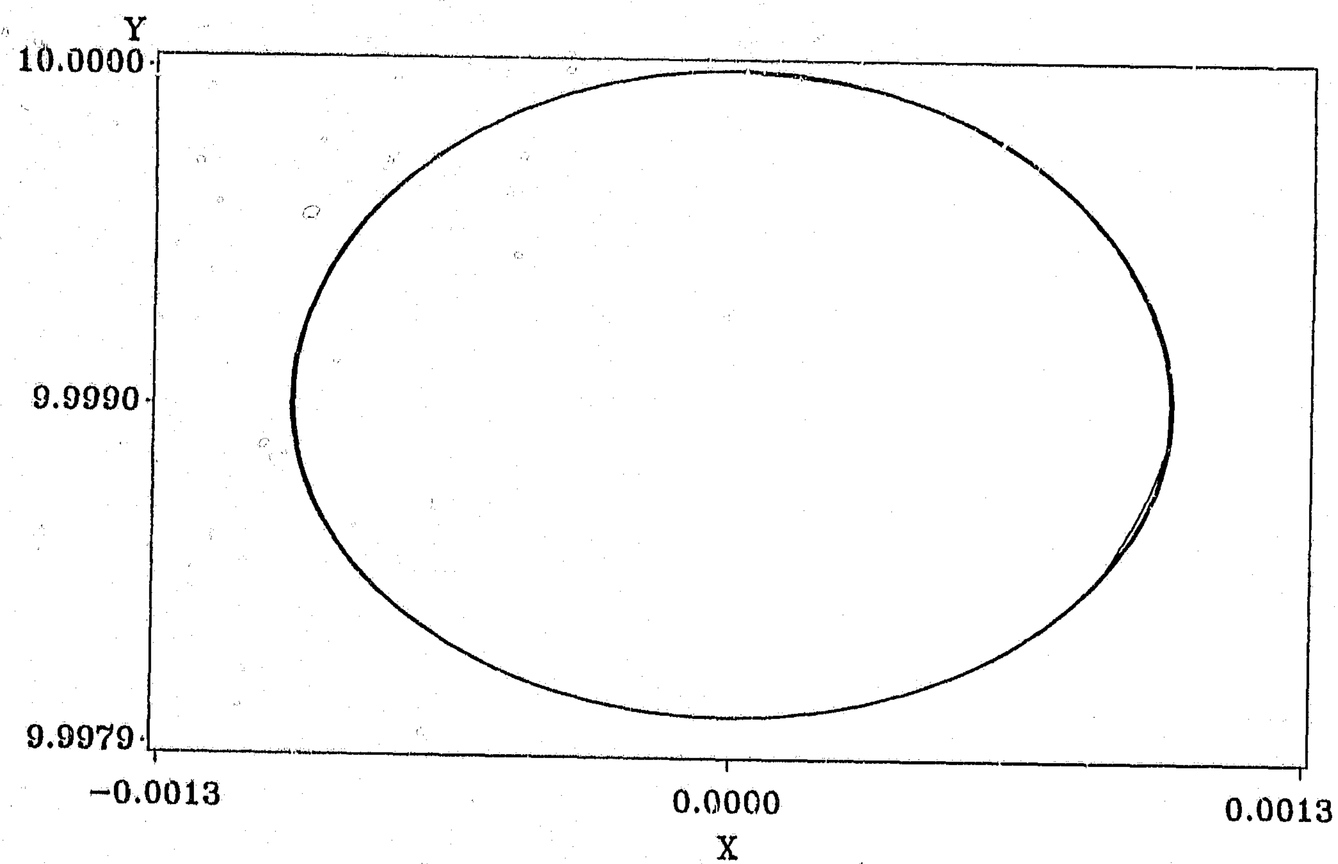

fig 2: An orbit of the system described by

$$H = \frac{1}{2}\left[\left(p_x + \frac{1}{4}y(x^2+y^2)\right)^2 + \left(p_y - \frac{1}{4}x(x^2+y^2)\right)^2\right] - (x^2+y^2)e^{x^2-y^2}$$

with initial conditions $x = 0.0$, $y = 10.0$, $\dot{x} = 0.1$ and $\dot{y} = 0.0$.

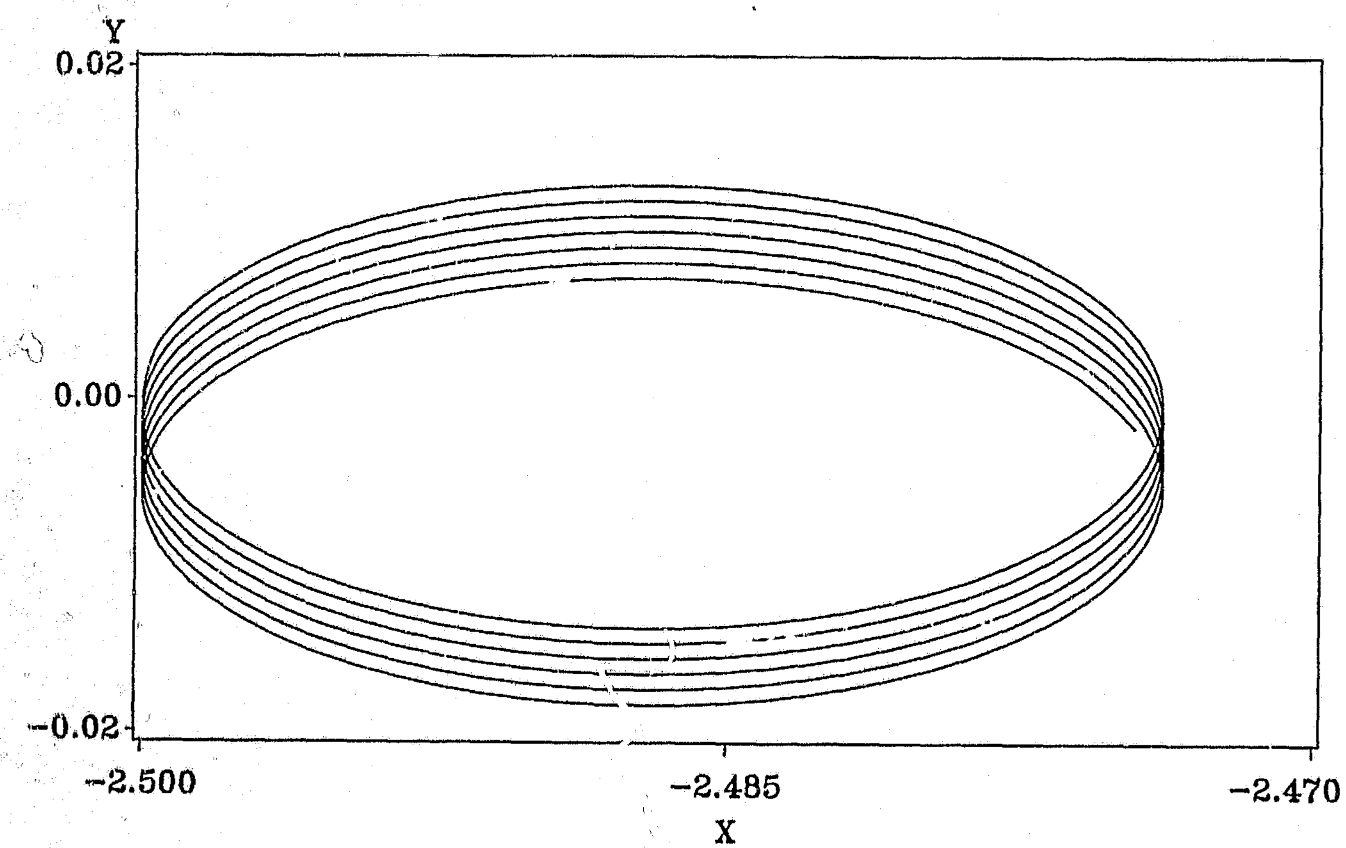

fig 3: An orbit of the system described by

$$H = \frac{1}{2}\left[\left(p_x + \frac{1}{6}y(x^2+y^2)^2\right)^2 + \left(p_y - \frac{1}{6}x(x^2+y^2)^2\right)^2\right] - (x^2+y^2)^2 e^{2x(x^2/3-y^2)}$$

with initial conditions $x = -2.5$, $y = 0.0$, $\dot{x} = 0.0$ and $\dot{y} = 0.5$.

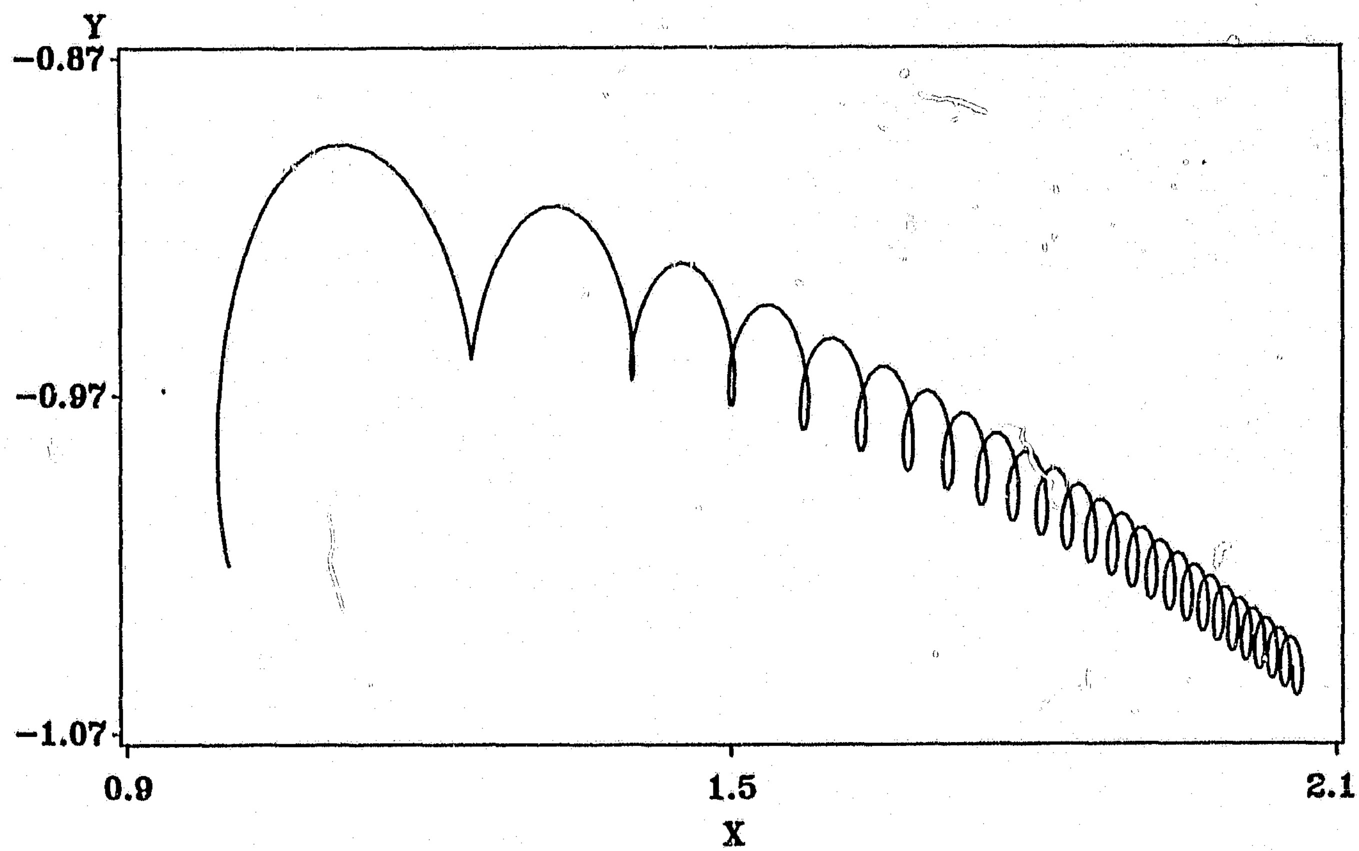

fig 4: An orbit for the system described by

$$H = \frac{1}{2}\left[\left(p_x + \frac{1}{8}y(x^2+y^2)^3\right)^2 + \left(p_y - \frac{1}{8}x(x^2+y^2)^3\right)^2\right] - (x^2+y^2)^3 e^{((x^4+y^4)/2-3x^2y^2)}$$

with initial conditions $x = 1.0$, $y = -1.02$, $\dot{x} = 0.0$ and $\dot{y} = 0.0$.

For $b_1 = (x+iy)^{-2}$ and $b_2 = (x-iy)^{-2}$ a possible solution for H and I is

$$H = \frac{1}{2}\left[\left(p_x + \frac{1}{6}y(x^2+y^2)^2\right)^2 + \left(p_y - \frac{1}{6}x(x^2+y^2)^2\right)^2\right] - (x^2+y^2)^2 e^{2x(x^2/3-y^2)} \tag{3.2.17.1}$$

and

$$I = \left(p_x + \frac{1}{6}y(x^2+y^2)^2\right)\frac{4xy}{(x^2+y^2)^2} + \left(p_y - \frac{1}{6}x(x^2+y^2)^2\right)\frac{2(x^2-y^2)}{(x^2+y^2)^2} + 2x\left(\frac{x^2}{3} - y^2\right) \tag{3.2.17.2}$$

An orbit of this system is given in fig 3.

For $b_1 = (x+iy)^{-3}$ and $b_2 = (x-iy)^{-3}$ a possible solution for H and I is

$$H = \frac{1}{2}\left[\left(p_x + \frac{1}{8}y(x^2+y^2)^3\right)^2 + \left(p_y - \frac{1}{8}x(x^2+y^2)^3\right)^2\right] - (x^2+y^2)^3 e^{((x^4+y^4)/2-3x^2y^2)}, \tag{3.2.18.1}$$

$$I = \left(p_x + \frac{1}{8}y(x^2+y^2)^3\right)\frac{2y(x^2-3y^2)}{(x^2+y^2)^3} + \left(p_y - \frac{1}{8}x(x^2+y^2)^3\right)\frac{2x(3x^2-y^2)}{(x^2+y^2)^3} + \frac{1}{2}(x^4+y^4) - 3x^2y^2. \tag{3.2.18.2}$$

An orbit of this system is given in fig 4.

For the choice $b_1 = \mu$, $b_2 = \nu$

$$U = \frac{1}{x^2+y^2} \tag{3.2.19}$$

and

$$W = -1. \tag{3.2.20}$$

A possible solution for A and B is

$$A = \frac{1}{2x}\arctan\left(\frac{y}{x}\right), \qquad B = -\frac{1}{2y}\arctan\left(\frac{x}{y}\right). \tag{3.2.21}$$

The Hamiltonian and corresponding configurational invariant are

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2x}\arctan\left(\frac{y}{x}\right)p_x - \frac{1}{2y}\arctan\left(\frac{x}{y}\right)p_y$$
$$+\frac{1}{8}\left\{\frac{1}{x^2}\arctan^2\left(\frac{y}{x}\right) + \frac{1}{y^2}\arctan^2\left(\frac{x}{y}\right)\right\} - 1. \tag{3.2.22.1}$$

$$I = -2yp_x + 2xp_y - \frac{y}{x}\arctan\left(\frac{y}{x}\right) - \frac{x}{y}\arctan\left(\frac{x}{y}\right)$$
$$+\ln(x^2+y^2). \tag{3.2.22.2}$$

A different approach to the solution of (3.2.6) is to look at the equations of its characteristics which are given by

$$\frac{-\mathrm{d}U}{U\left(\mathrm{d}b_1/\mathrm{d}\mu - \mathrm{d}b_2/\mathrm{d}\nu\right)} = \frac{\mathrm{d}\mu}{b_1} = -\frac{\mathrm{d}\nu}{b_2}. \tag{3.2.23.a, b}$$

These equations are difficult to solve so let

$$b_1 = \mu^p \quad \text{and} \quad b_2 = \nu^q. \tag{3.2.24}$$

Equations (3.2.23.a,b) then become

$$\frac{-\mathrm{d}U}{U\left(p\mu^{p-1} - q\nu^{q-1}\right)} = -\frac{\mathrm{d}\mu}{\mu^p} = \frac{\mathrm{d}\nu}{\nu^q}. \tag{3.2.25.a, b}$$

The equation (3.2.25.b) may be integrated to yield

$$k = \frac{1}{p-1}\mu^{-p+1} + \frac{1}{q-1}\nu^{-q+1}, \tag{3.2.26}$$

provided $p \neq 1$ and $q \neq 1$. From (3.2.25.a)

$$\frac{-\mathrm{d}U}{U} = \frac{\left(p\mu^{p-1} + q\nu^{q-1}\right)\mathrm{d}\mu}{\mu^p}. \tag{3.2.27}$$

For the method of characteristics it is necessary to eliminate ν from this equation before integrating it. ν^{q-1} may be obtained from (3.2.26):

$$\nu^{q-1} = \frac{1}{((q-1)k - (q-1)/(p-1)\mu^{1-p})}. \tag{3.2.28}$$

Equation (3.2.27) may now easily be integrated giving

$$L = U\mu^p\left[\frac{(p-1)\mu^{p-1}}{(p-1)k\mu^{p-1} - 1}\right]^{q/(q-1)}, \tag{3.2.29}$$

where I_1 is a constant. On replacing k by its expression in (3.2.26) L becomes

$$L = U\mu^p\nu^q, \tag{3.2.30}$$

where a constant factor has been absorbed into L. The solution for U is therefore of the form

$$U = \frac{F(k)}{\mu^p\nu^q}, \tag{3.2.31}$$

where F is a arbitrary function of k.

Now equations (3.2.3.3)-(3.2.3.5) are still to be solved. The solution to (3.3.3.5) is of the same form as that obtained for (3.2.6) above. To solve (3.2.3.3) substitute for U and change to the (μ, ν) variables:

$$\frac{\partial c}{\partial \mu} + \frac{\partial c}{\partial \nu} - (\nu^{-q} + \mu^{-p})F(k) = 0. \tag{3.2.32}$$

Similarly (3.2.3.4) becomes

$$\frac{\partial c}{\partial \mu} - \frac{\partial c}{\partial \nu} + (\nu^{-q} - \mu^{-p})F(k) = 0. \tag{3.2.33}$$

The sum and difference of the two equations give respectively

$$\frac{\partial c}{\partial \mu} - \mu^{-p}F = 0 \tag{3.2.34.1}$$

and

$$\frac{\partial c}{\partial \nu} - \nu^{-q}F = 0. \tag{3.2.34.2}$$

The second terms of (3.2.34.1) and (3.2.34.2) are $\partial F/\partial\mu$ and $\partial F/\partial\nu$ respectively therefore

$$c = -\int F\ dk\ +\ constant. \tag{3.2.35}$$

Now for $b_1 = \mu^p$, $b_2 = \nu^q$ with $p \neq 0$ and $q \neq 0$ the equations (3.2.3.1)-(3.2.3.5) are solved.

As an example consider the case $F = k$, $p = q = 2$. Then

$$U = \frac{2x}{(x^2+y^2)^3}. \tag{3.2.36}$$

From the definition of U

$$\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x} = \frac{2x}{(x^2+y^2)^3}.$$

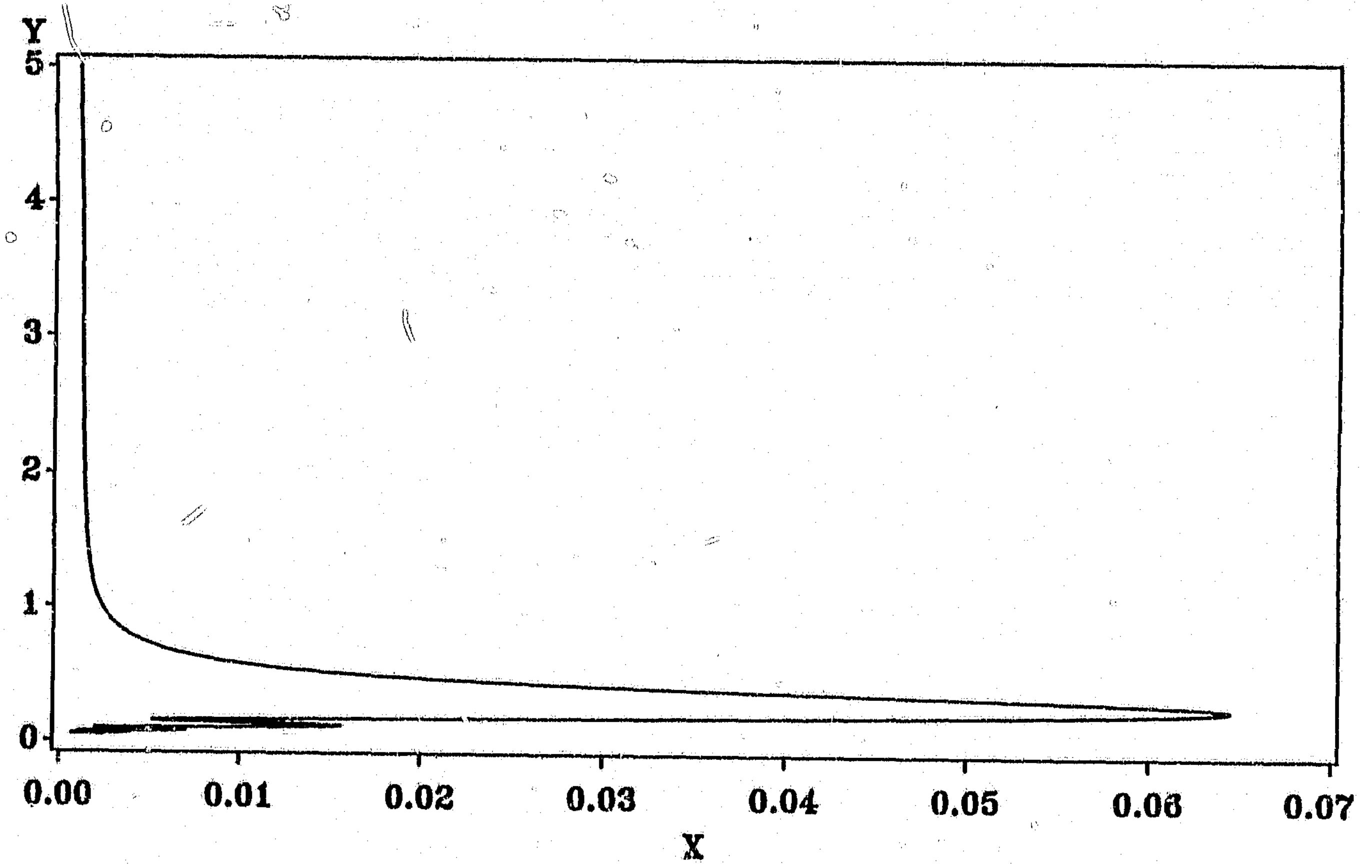

fig 5: An orbit of the system described by

$$H = \frac{1}{2}\left[\left(p_x + \frac{x}{(x^2+y^2)^2}\right)^2 + \left(p_y - \frac{1}{2}\frac{(1+2y)}{(x^2+y^2)^2}\right)^2\right] - \frac{2x}{(x^2+y^2)^3}$$

with initial conditions $x = 0.001$, $y = 5.0$, $\dot{x} = 0.001$ and $\dot{y} = -10.0$.

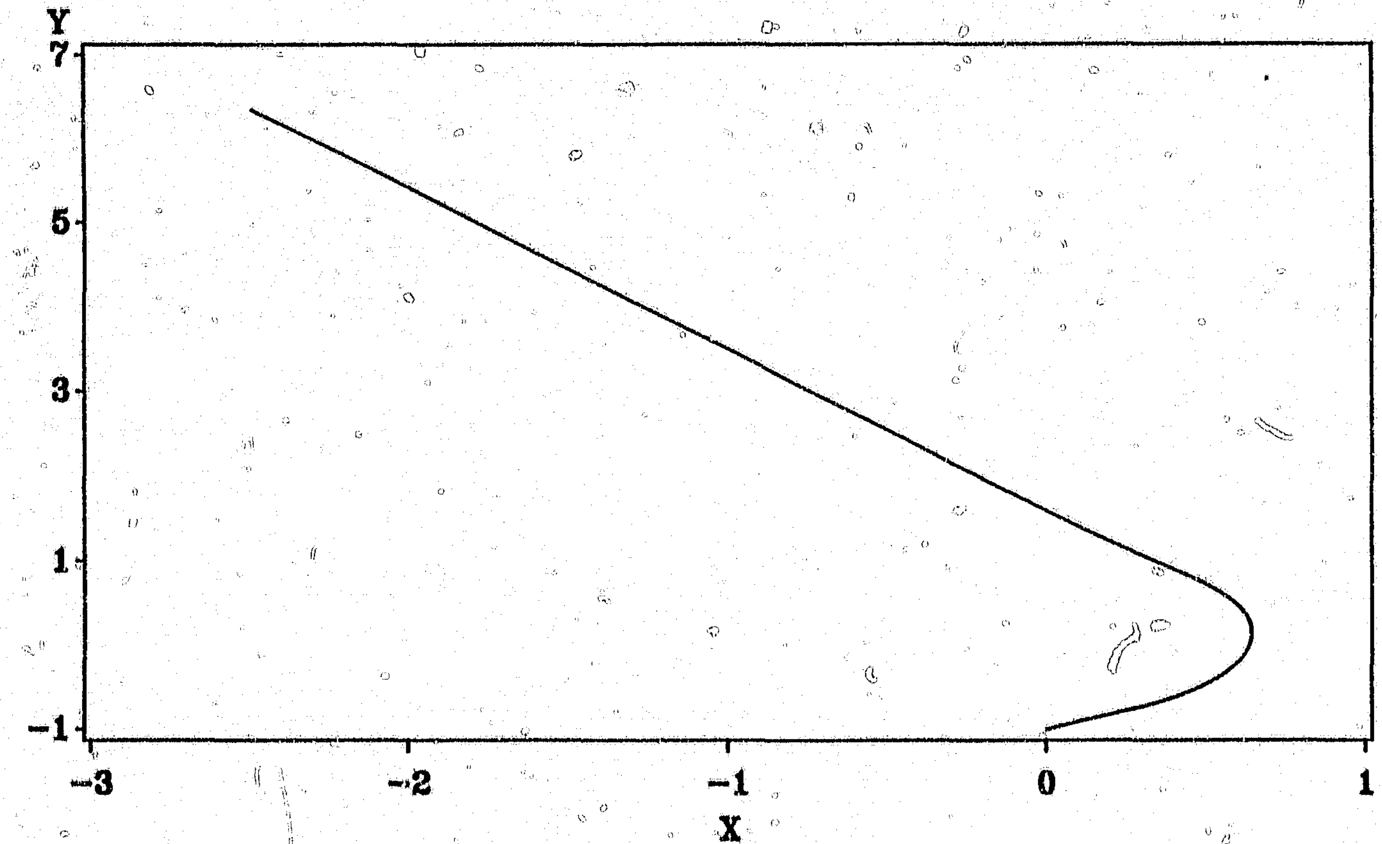

fig 6: An orbit of the system described by

$$H = \frac{1}{2}\left[\left(p_x + \frac{x}{(x^2+y^2)^2}\right)^2 + \left(p_y - \frac{1}{2}\frac{(1+2y)}{(x^2+y^2)^2}\right)^2\right] - \frac{2x}{(x^2+y^2)^3}$$

with initial conditions $x = 0.0$, $y = -1.0$, $\dot{x} = 1.0$ and $\dot{y} = 1.0$.

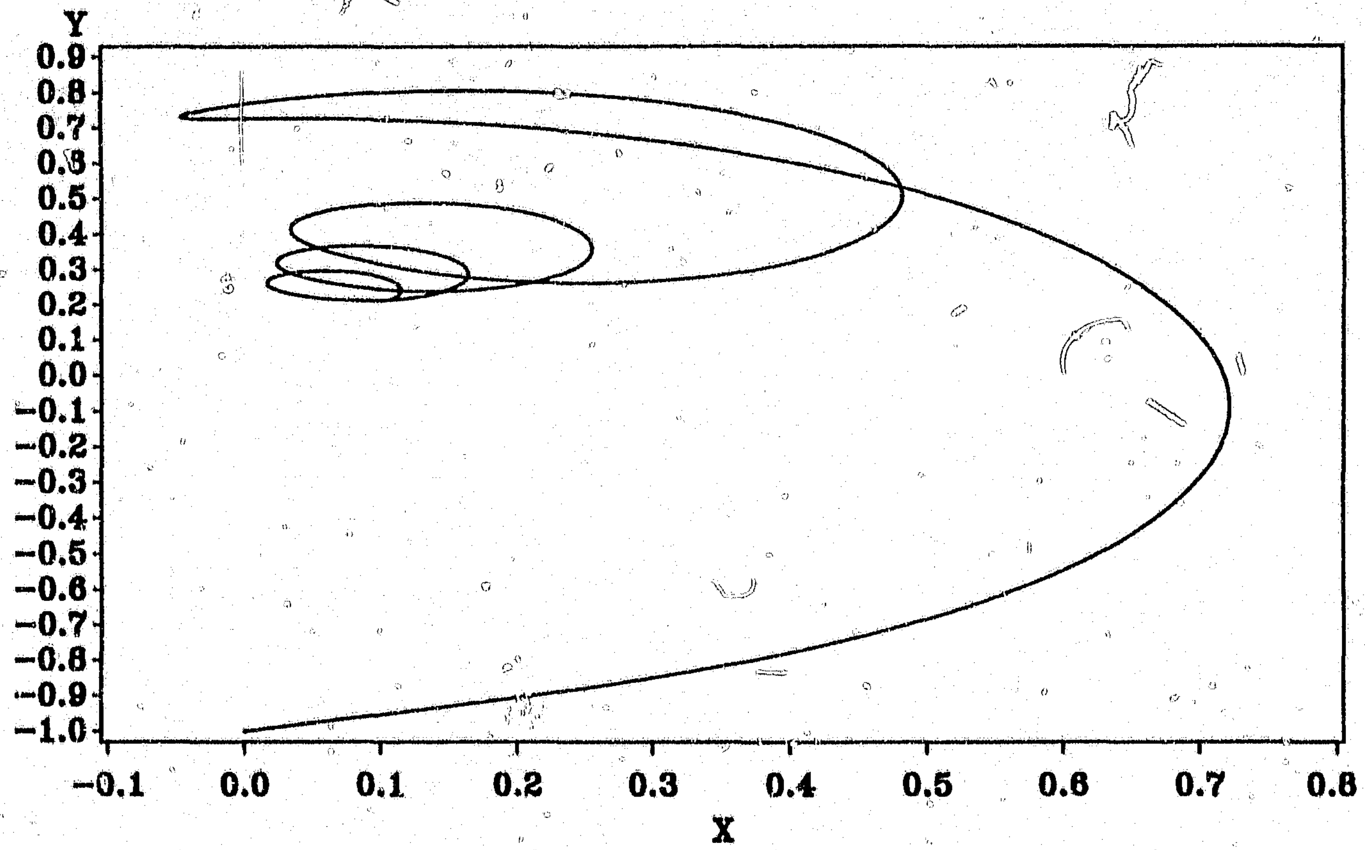

fig 7: An orbit of the system described by

$$H = \frac{1}{2}\left[\left(p_x + \frac{x}{(x^2+y^2)^2}\right)^2 + \left(p_y - \frac{1}{2}\frac{(1+2y)}{(x^2+y^2)^2}\right)^2\right] - \frac{2x}{(x^2+y^2)^3}$$

with initial conditions $x = 0.0$, $y = -1.0$, $\dot{x} = 1.0$ and $\dot{y} = 0.5$.

A and B will satisfy this equation if

$$A = \frac{x}{(x^2+y^2)^2} \tag{3.2.37}$$

and

$$B = \frac{1}{2(x^2+y^2)^2} + \frac{y}{(x^2+y^2)^2}. \tag{3.2.38}$$

When the same solution for W as for U is assumed, the Hamiltonian and configurational invariant are given by

$$H = \frac{1}{2}\left[\left(p_x + \frac{x}{(x^2+y^2)^2}\right)^2 + \left(p_y - \frac{1/2+y}{(x^2+y^2)^2}\right)^2\right] - \frac{2x}{(x^2+y^2)^3}, \tag{3.2.39.1}$$

$$I = -4xy\left(p_x + \frac{x}{(x^2+y^2)^2}\right) + 2(x^2-y^2)\left(p_y + \frac{1/2+y}{(x^2+y^2)^2}\right) + \frac{(-1-2y+2x^2y-x^2+y^2-2y^3)}{(x^2+y^2)^2}. \tag{3.2.39.2}$$

Numerical integration of this system yielded the orbits in figg 5-7.

If in equations (3.2.25.a,b) $p=1$ and $q \neq 1$, then from (3.2.25.b)

$$k = \mu e^{(\nu^{1-q}/(1-q))}, \tag{3.2.40}$$

Integration of (3.2.25.a) yields

$$L = U\nu^q e^{(\nu^{1-q}/(q-1))}. \tag{3.2.41}$$

The general solution of U is of the form

$$U = F(k)\nu^{-q}e^{(\nu^{1-q}/(1-q))}. \tag{3.2.42}$$

If $p=1$ and $q=1$, then

$$k = \nu\mu, \tag{3.2.43}$$

$$L = U \tag{3.2.44}$$

and

$$U = F(\mu\nu). \tag{3.2.45}$$

In all the examples given above W was chosen to have the same form as U, but forms different to that of U could have been chosen for W, giving many other systems with configurational invariants. Even more systems may be found by finding different solutions to $U = \partial A/\partial y - \partial B/\partial x$.

3.3 Configurational invariants quadratic in the momenta

Following the structure of the ansatz of (3.2.2) the ansatz for the quadratic configurational invariant is written as follows

$$
\begin{aligned}
I = a(x,y)(p_x + A)^2 + b(x,y)(p_x + A)(p_y + B) \\
+ c(x,y))(p_x + A) + d(x,y)(p_y + B) + f(x,y),
\end{aligned}
\tag{3.3.1}
$$

where the zero-energy condition has been used to eliminate the $(p_y + B)^2$ term.

The conditions for the existence of a quadratic configurational invariant are

$$2\frac{\partial d}{\partial y}W + \frac{\partial W}{\partial x}c + \frac{\partial W}{\partial y}d + 2bUW = 0, \tag{3.3.2.1}$$

$$+\frac{\partial f}{\partial y} - \frac{\partial W}{\partial x}b + cU = 0, \tag{3.3.2.2}$$

$$-2\frac{\partial b}{\partial y}W + \frac{\partial f}{\partial x} - 2\frac{\partial W}{\partial x}a - \frac{\partial W}{\partial y}b - dU = 0, \tag{3.3.2.3}$$

$$\frac{\partial c}{\partial y} + \frac{\partial d}{\partial x} + 2aU = 0, \tag{3.3.2.4}$$

$$\frac{\partial c}{\partial x} - \frac{\partial d}{\partial y} - 2bU = 0, \quad U = \frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}, \tag{3.3.2.5}$$

$$\frac{\partial a}{\partial y} + \frac{\partial b}{\partial x} = 0, \tag{3.3.2.6}$$

$$\frac{\partial a}{\partial x} - \frac{\partial b}{\partial y} = 0. \tag{3.3.2.7}$$

As usual the last two equations have the solution

$$a = i(b_1 - b_2), \quad b = b_1 + b_2, \tag{3.3.3}$$

where b_1 and b_2 are arbitrary functions of their arguments.

To make use of the fact that b_1 and b_2 are only functions of single arguments a transformation to these variables, namely $\mu = x + iy$ and $\nu = x - iy$, is made. It is also convenient to replace c and d by $h_1 = (c + id)/2$ and $h_2 = (c - id)/2$. The remaining equations then become

$$-\frac{\partial W}{\partial \nu}h_2 - \frac{\partial W}{\partial \mu}h_1 - \frac{\partial h_2}{\partial \nu}W + \frac{\partial h_2}{\partial \mu}W + \frac{\partial h_1}{\partial \nu}W - \frac{\partial h_1}{\partial \mu}W$$

$$-UWb_2 - UWb_1 = 0, \tag{3.3.4.1}$$

$$-\frac{\partial f}{\partial \nu}i + \frac{\partial f}{\partial \mu}i + 2\frac{\partial W}{\partial \nu}ci - \frac{\partial W}{\partial \nu}b_2 - \frac{\partial W}{\partial \nu}b_1$$

$$-\frac{\partial W}{\partial \mu}b_2 - \frac{\partial W}{\partial \mu}b_1 + Uh_2 + Uh_1 = 0, \tag{3.3.4.2}$$

$$+\frac{\partial f}{\partial \nu} + \frac{\partial f}{\partial \mu} + 3\frac{\partial W}{\partial \nu}ib_2 - \frac{\partial W}{\partial \nu}ib_1 + \frac{\partial W}{\partial \mu}ib_2$$

$$-3\frac{\partial W}{\partial \mu}ib_1 + 2\frac{\partial b_2}{\partial \nu}iW - 2\frac{\partial b_1}{\partial \mu}iW - iUh_2 + iUh_1 = 0, \tag{3.3.4.3}$$

$$\frac{\partial h_2}{\partial \mu} - \frac{\partial h_1}{\partial \nu} - Ub_2 + Ub_1 = 0, \tag{3.3..4.4}$$

$$\frac{\partial h_2}{\partial \mu} + \frac{\partial h_1}{\partial \nu} - Ub_2 - Ub_1 = 0. \tag{3.3.4.5}$$

From equations (3.3.4.4) and (3.3.4.5) it is easily deduced that

$$h_1 = b_1 \int U d\nu + r(\mu), \tag{3.3.5.1.}$$

$$h_2 = b_2 \int U d\mu + s(\nu), \tag{3.3.5.2}$$

where r and s are arbitrary functions of their arguments.

By eliminating U from (3.3.4.1) and simplifying, (3.3.4.1) may be written as

$$\frac{\partial}{\partial \nu}(Wh_2) + \frac{\partial}{\partial \mu}(Wh_1) = 0. \tag{3.3.6}$$

Application of the integrability condition on f in equations (3.3.4.2) and (3.3.4.3) yields (after some simplification) :

$$\begin{aligned} &\frac{\partial}{\partial \mu}\left(\frac{1}{b_1}\frac{\partial h_1}{\partial \nu}h_1 - \frac{\partial W}{\partial \mu}b_1 - \frac{\partial}{\partial \mu}(Wb_1)\right) \\ &+\frac{\partial}{\partial \nu}\left(\frac{1}{b_2}\frac{\partial h_2}{\partial \mu}h_2 - \frac{\partial W}{\partial \nu}b_2 - \frac{\partial}{\partial \nu}(Wb_2)\right) = 0. \end{aligned} \tag{3.3.7}$$

Equation (3.3.6) may be used to obtain an expression for $\partial W/\partial \nu$

$$\frac{\partial W}{\partial \nu} = -\frac{1}{h_2}\left(\frac{\partial W}{\partial \mu}h_1 + \frac{\partial h_2}{\partial \nu}W + \frac{\partial h_1}{\partial \mu}W\right). \tag{3.3.8}$$

Substitution of (3.3.8) into (3.3.7) yields

$$\begin{aligned}
&-2\frac{\partial^2 W}{\partial \mu^2}\frac{(b_2 h_1^2 + b_1 h_2^2)}{h_2^2} + \frac{\partial W}{\partial \mu}\left(3\frac{\partial b_2}{\partial \nu}h_2^2 h_1 - 3\frac{\partial b_1}{\partial \mu}h_2^3 - 6\frac{\partial h_2}{\partial \nu}b_2 h_2 h_1 + 2\frac{\partial h_2}{\partial \mu}b_2 h_1^2\right.\\
&\left.+2\frac{\partial h_1}{\partial \nu}b_2 h_2^2 - 6\frac{\partial h_1}{\partial \mu}b_2 h_2 h_1\right)\Big/h_2^3 + W\left(-\frac{\partial^2 b_2}{\partial \nu^2}h_2^3 + 3\frac{\partial b_2}{\partial \nu}\frac{\partial h_2}{\partial \nu}h_2^2 + 3\frac{\partial b_2}{\partial \nu}\frac{\partial h_1}{\partial \mu}h_2^2\right.\\
&-\frac{\partial^2 b_1}{\partial \mu^2}h_2^3 - 2\frac{\partial^2 h_2}{\partial \mu \partial \nu}b_2 h_2 h_1 + 2\frac{\partial^2 h_2}{\partial \nu^2}b_2 h_2^2 - 4\left(\frac{\partial h_2}{\partial \nu}\right)^2 b_2 h_2 + 2\frac{\partial h_2}{\partial \nu}\frac{\partial h_2}{\partial \mu}b_2 h_1\\
&-6\frac{\partial h_2}{\partial \nu}\frac{\partial h_1}{\partial \mu}b_2 h_2 + 2\frac{\partial h_2}{\partial \mu}\frac{\partial h_1}{\partial \mu}b_2 h_1 + 2\frac{\partial^2 h_1}{\partial \mu \partial \nu}b_2 h_2^2 - 2\frac{\partial^2 h_1}{\partial \mu^2}b_2 h_2 h_1\\
&\left.-2\left(\frac{\partial h_1}{\partial \mu}\right)^2 b_2 h_2\right)\Big/h_2^3 + \left(-\frac{\partial b_2}{\partial \nu}\frac{\partial h_2}{\partial \mu}b_1^2 h_2 - \frac{\partial b_1}{\partial \mu}\frac{\partial h_1}{\partial \nu}b_2^2 h_1 + \frac{\partial^2 h_2}{\partial \mu \partial \nu}b_2 b_1^2 h_2\right.\\
&\left.+\frac{\partial h_2}{\partial \nu}\frac{\partial h_2}{\partial \mu}b_2 b_1^2 + \frac{\partial^2 h_1}{\partial \mu \partial \nu}b_2^2 b_1 h_1 + \frac{\partial h_1}{\partial \nu}\frac{\partial h_1}{\partial \mu}b_2^2 b_1\right)\Big/(b_2^2 b_1^2) = 0,
\end{aligned} \tag{3.3.9}$$

which may be treated as an ordinary differential equation in μ with ν as a constant. To find a solution to ths equation the following particular case was considered. Let $b_1 = b_2 = U = 1$, $h_1 = \nu + r$, $h_2 = \mu + s$ and r and s be constants. Then (3.3.9), after some simplification, reduces to

$$\frac{1}{(\mu+s)^2}\frac{\partial^2 W}{\partial \mu^2} - \frac{1}{(\mu+s)^3}\frac{\partial W}{\partial \mu} = 0, \tag{3.3.10}$$

with solution

$$W = \frac{1}{2}(\mu+s)^2 W_1(\nu) + W_2(\nu), \tag{3.3.11}$$

where W_1 and W_2 are arbitrary functions of their arguments. From equation (3.3.8) it follows, after separation by coefficients of μ, that

$$W_2 = -\frac{1}{2}W_1(\nu+r)^2, \qquad W_1 = constant. \tag{3.3.12}$$

An additive constant has been omitted from W_2 since in the final solution the constant is only added to the potential. Of the original set, the remaining two equations to be solved are equations (3.3.2.2) and (3.3.2.3). When solved these equations yield the solution for f. The final solution is given by

$$H = \frac{1}{2}(p_x + A)^2 + \frac{1}{2}(p_y + B)^2 + \frac{W_1}{2}\left[(\mu + s)^2 - (\nu + r)^2\right], \tag{3.3.13.1}$$

$$\begin{aligned} I &= 2(p_x + A)(p_y + B) + (2x + r + s)(p_x + A) - (2y + i(r - s))(p_y + B) \\ &\quad - 2xy + 2iW_1(x^2 + y^2) + (2iW_1(r + s) + i(s - r))x \\ &\quad + (2W_1(s - r) - (s + r))y, \end{aligned} \tag{3.3.13.2}$$

where $\partial A/\partial y - \partial B/\partial x = 1$. I is also a first integral of the system.

CHAPTER IV: CONFIGURATIONAL INVARIANTS RATIONAL IN THE MOMENTA

4.1 General results

Usually one assumes a polynomial form for the first integral when looking for first integrals of a system. What about different forms ? First 'ntegrals of rational and transcendental forms in the momenta are found in (Hietarinta 1984a, Lewis and Leach 1985 and Taylor and Leach 1990). In (Hietarinta 1984a) it is proved that, for the systems under discussion, it is not possible to construct polynomial first integrals out of the rational and transcendental first integrals that the system has. (Hall 1985) points out that these first integrals may be rewritten as trivial polynomials. In a subsequently article (Hietarinta 1985) argues that these trivial first integrals are of little value when trying gain information about a system; it is better to use the original first integrals. Here the existence of configurational invariants of forms other than polynomials will be assumed and a search for configurational invariants of certain rational forms will be conducted.

In (Hietarinta 1987) an interesting theorem on rational first integrals is derived. Consider a first integral of the form

$$I = \frac{R}{S}, \tag{4.1}$$

where r and s do not contain t. The total time derivative set equal to zero gives

$$S[H,R] = R[H,S]. \tag{4.2}$$

This is equivalent to the pair

$$[H,R] = GR, \quad [H,S] = GS, \tag{4.3.a, b}$$

where G is a rational function.

The function G is not only rational but polynomial in the momenta. Returning to (4.2) one sees that this is an equation for polynomials. Now, since S divides the lhs, it must also divide the rhs, but it may be assumed that it does not divide R for then I would have been a polynomial. Therefore S divides $[H,S]$. Similarly R divides $[H,R]$. Consequently G is a polynomial.

The theorem states the following:
The leading parts of R and S are of the form $r = f_1 w$, $s = f_2 w$ where the f_i's commute with the kinetic energy term and w is a particular solution of (4.4) below. The function w is a constant *iff* g is independent of the p's.

Proof:
Here it is assumed that the Hamiltonian has the usual leading part, i.e., $\frac{1}{2}(p_x^2 + p_y^2)$. Denote the leading parts of R and S by r and s respectively. The highest order term in (4.3.a) is

$$p_x r_x + p_y r_y = gr, \tag{4.4}$$

where g is the linear (in the momenta) part of G. G can be at most of linear order since the order of $[H, R]$ is one greater than the order of r. The same equation holds for s. The solution of the equation may be written as $r = fw$, where f is the general solution when $g = 0$ and w is a particular solution. Since r/s is a solution to (4.4) when $g = 0$, the particular solutions of r and s may be assumed to be the same.

This theorem does not carry over for configurational invariants. To see this suppose firstly that I is a first integral. Then (4.3) is obtained from (4.2) by first rewriting (4.2) as

$$S[H, R] = GSR = R[H, S] \tag{4.5.a, b}$$

and then simply dividing the first equation by S and the second by R. If, however, I is a configurational invariant, which is not also a first integral, then the zero-energy condition must be imposed which may result in S no longer being a factor of the *lhs* of (4.2) or R no longer being a factor of the *rhs*. As an example consider a rational configurational invariant with

$$R = a(x, y)p_y + b(x, y) \text{ and } S = c(x, y)p_x + d(x, y). \tag{4.6}$$

Let

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y). \tag{4.7}$$

As in the previous chapter the zero-energy condition is imposed by replacing p_y^2 by

$p_y^2 = -p_x^2 - 2V$. The lhs of (4.2) becomes

$$
\begin{aligned}
&-a\frac{\partial c}{\partial y}p_x^3 + (a\frac{\partial c}{\partial x}p_y + b\frac{\partial c}{\partial x} - a\frac{\partial d}{\partial y})p_x^2 \\
&+(-2aV\frac{\partial c}{\partial y} + b\frac{\partial c}{\partial y}p_y + a\frac{\partial d}{\partial x}p_y + b\frac{\partial d}{\partial x})p_x \\
&-2aV\frac{\partial d}{\partial y} + b\frac{\partial d}{\partial y}p_y - ac\frac{\partial V}{\partial x}p_y - bc\frac{\partial V}{\partial x}.
\end{aligned}
\tag{4.8}
$$

of which S is not a factor. G cannot be a polynomial since then division of the of (4.5.a) wrt S will yield a rational on the lhs and a polynomial on the rhs, which is impossible.

4.2 Configurational invariants of the form linear / linear

The three forms derived in (Hietarinta 1987) for $w = 1$ will be used here as ansätze for configurational invariants. They are

$$I_1 = \frac{[p_x + h(x,y)]}{[p_y + k(x,y)]}, \tag{4.9.1}$$

$$I_2 = \frac{[yp_x - xp_y + h(x,y)]}{[p_y + k(x,y)]} \tag{4.9.2}$$

and

$$I_3 = \frac{[yp_x - xp_y + h(x,y)]}{[p_x + ip_y + k(x,y)]}. \tag{4.9.3}$$

Consider the form (4.9.1). A gauge transformation

$$P_x = p_x + f_x, \quad P_y = p_y + f_y,$$

may be used to eliminate V and k. The transformed invariant and Hamiltonian are

$$I = \frac{(h^* + P_x)}{P_y} \tag{4.10}$$

and

$$H = \frac{1}{2}(P_x^2 + P_y^2) + A^* P_x + B^* P_y, \tag{4.11}$$

where

$$A^* = A - f_x,$$

$$B^* = B - f_y,$$

$$h^* = h - f_x,$$

$$V = Af_x + Bf_y - \frac{1}{2}(f_x^2 + f_y^2),$$

$$k = f_y.$$

The set of equations which the unknowns must satisfy for a configurational invariant to exist are

$$A^*\frac{\partial h^*}{\partial x} + B^*\frac{\partial h^*}{\partial y} + h^*\frac{\partial B^*}{\partial y} - 2B^*\frac{\partial A^*}{\partial y} = 0, \tag{4.12.1}$$

$$-\frac{\partial B^*}{\partial x} + \frac{\partial h^*}{\partial y} - \frac{\partial A^*}{\partial y} = 0, \tag{4.12.2}$$

$$\frac{\partial A^*}{\partial y}(h^* - 2A^*) = 0, \tag{4.12.3}$$

$$\frac{\partial h^*}{\partial x} - \frac{\partial A^*}{\partial x} + \frac{\partial B^*}{\partial y} = 0. \tag{4.12.4}$$

From (4.12.3) either

$$h^* = 2A^* \tag{4.13.a}$$

or

$$\frac{\partial A^*}{\partial y} = 0. \tag{4.13.b}$$

In the case of (4.13.a), (4.12.2) and (4.12.4) respectively become

$$\frac{\partial B^*}{\partial x} = \frac{\partial A^*}{\partial y}$$

and

$$\frac{\partial B^*}{\partial y} = -\frac{\partial A^*}{\partial x},$$

with the solution

$$A^* = A_1^*(x+iy) + A_2^*(x-iy), \tag{4.14}$$

$$B^* = i(A_1^* - A_2^*), \tag{4.15}$$

where A_1^* and A_2^* are arbitrary functions of $x+iy$ and $x-iy$ respectively. With (4.13a), (4.14) and (4.15) (4.12.1) is satisfied identically.

All the equations are now solved. The configurational invariant and corresponding Hamiltonian are

$$I_1 = \frac{[2(A_1^* + A_2^*) + P_x]}{P_y}, \tag{4.16}$$

$$H_1 = \frac{1}{2}(P_x^2 + P_y^2) + (A_1^* + A_2^*)P_x + i(A_1^* - A_2^*)P_y. \tag{4.17}$$

Consider the case (4.13.b). The integrability condition on h^* and A may be invoked to eliminate them from $\partial/\partial y$ (4.12.4) $-\partial/\partial x$ (4.12.2) yielding

$$\frac{\partial^2 B^*}{\partial x^2} + \frac{\partial^2 B^*}{\partial y^2} = 0, \tag{4.18}$$

with solution

$$B^* = B_3^*(x+iy) + B_4^*(x-iy). \tag{4.19}$$

It now follows from the final equations (4.12.2) and (4.12.4) that

$$h^* = i(B_4^* - B_3^*) + A^* + h_0^*, \tag{4.20}$$

where h_0^* is a arbitrary constant.

Differentiation of (4.12.1) *wrt* y eliminates A from that equation and gives the result

$$\begin{aligned} &2i\left(\left(\frac{\partial B_4^*}{\partial y}\right)^2 - \left(\frac{\partial B_3^*}{\partial y}\right)^2\right) + 2i\left(B_4^*\frac{\partial^2 B_4^*}{\partial y^2} - B_3^*\frac{\partial^2 B_3^*}{\partial y^2}\right) \\ &+ h_0^*\left(\frac{\partial^2 B_3^*}{\partial y^2} + \frac{\partial^2 B_4^*}{\partial y^2}\right) = 0. \end{aligned} \tag{4.21}$$

Since B_3^* and B_4^* depend on $\mu = x + iy$ and $\nu = x - iy$ respectively, it is convenient to transform to these variables. Equation (4.21) may be separated giving

$$(-h_0^* + 2iB_3^*)\frac{\mathrm{d}^2 B_3^*}{\mathrm{d}\mu^2} + 2i\left(\frac{\mathrm{d}B_3^*}{\mathrm{d}\mu}\right)^2 = c_2, \tag{4.22}$$

$$(+h_0^* + 2iB_4^*)\frac{\mathrm{d}^2 B_4^*}{\mathrm{d}\nu^2} + 2i\left(\frac{\mathrm{d}B_4^*}{\mathrm{d}\nu}\right)^2 = c_2, \tag{4.23}$$

where c_2 is an arbitrary constant. Integration of these equations twice yields equations for B_3^* and B_4^* :

$$B_3^*(-h_0^* + iB_3^*) = \frac{1}{2}c_2\mu^2 + c_1\mu + c_0, \tag{4.24}$$

and

$$B_4^*(+h_0^* + iB_4^*) = \frac{1}{2}c_2\nu^2 + d_1\nu + d_0. \tag{4.25}$$

The final equation, viz. (4.12.1) is

$$\frac{1}{2}\left(\frac{\partial A^*}{\partial \mu} + \frac{\partial A^*}{\partial \nu}\right) = ic_2(\mu + \nu) + i(c_1 + d_1). \tag{4.26}$$

This equation is easily solved by the method of characteristics to give the solution

$$A^{*2} = \frac{i}{2}c_2(\mu + \nu)^2 + i(c_1 + d_1)(\mu + \nu). \tag{4.27}$$

In this case the Hamiltonian contains the square roots of complex numbers unless c_2 and c_1 are zero in which case the linear in the momenta part of H has constant coefficients.

For the ansatz I_2, k and V are also eliminated by means of a gauge transformation. The equations to solve are

$$2\frac{\partial B^*}{\partial x}yB^* + \frac{\partial B^*}{\partial y}h^* + \frac{\partial h^*}{\partial x}A^* - \frac{\partial h^*}{\partial y}B^* + 2A^*B^* = 0, \tag{4.28.1}$$

$$2\frac{\partial B^*}{\partial x}yA^* + \frac{\partial A^*}{\partial y}h^* - 2\frac{\partial h^*}{\partial y}A^* + 2A^{*2} = 0, \tag{4.28.2}$$

$$\frac{\partial B^*}{\partial y}y - \frac{\partial A^*}{\partial x}y + \frac{\partial h^*}{\partial x} + B^* = 0, \tag{4.28.3}$$

$$\frac{\partial B^*}{\partial x}y + \frac{\partial A^*}{\partial y}y - \frac{\partial h^*}{\partial y} + A^* = 0. \tag{4.28.4}$$

Application of the integrability condition on A^* and h^* in (4.28.3) and (4.28.4) gives

$$\left(\frac{\partial^2 B^*}{\partial y^2}+\frac{\partial^2 B^*}{\partial x^2}\right)y+2\frac{\partial B^*}{\partial y}=0. \tag{4.29}$$

By transforming to the variables $\mu = x+iy$, $\nu = x-iy$ this equation is simplified and the solution follows easily:

$$B^*=\frac{1}{y}[b_1(x+iy)+b_2(x-iy)], \tag{4.30}$$

where b_1 and b_2 are arbitrary functions of their arguments. On substituting B^* into (4.28.4) one finds that

$$h^*=yA^*-i(b_1-b_2)+h_0. \tag{4.31}$$

In the variables $\mu = x+iy$ and $\nu = x-iy$ (4.28.2) is

$$\frac{\partial A^*}{\partial y}[-A^*y+i(b_2-b_1)+h_0]=0. \tag{4.32}$$

Now either

$$A^*=\frac{1}{y}[i(b_2-b_1)+h_0] \tag{4.33.a}$$

or

$$\frac{\partial A^*}{\partial y}=0. \tag{4.33.b}$$

If the first answer for A^* is taken, (4.28.1) vanishes and the system is solved. The Hamiltonian and corresponding configurational invariant are

$$H_2=\frac{1}{2}(P_x^2+P_y^2)+\frac{i}{y}(b_2-b_1)P_x+\frac{1}{y}(b_1+b_2)P_y, \tag{4.34.a}$$

$$I_2=\frac{1}{P_y}[yP_x-xP_y+2i(b_2-b_1)+2h_0]. \tag{4.34.b}$$

If instead $\partial A^*/\partial y=0$, i.e. $A^*=A^*(x)$, then $\partial^4/\partial\mu^2\partial\nu^2$ (4.28.1) is an equation in A^* only with solution

$$A^{*2}=\frac{a_2}{2}x^2+a_1x+a_0. \tag{4.35}$$

This result simplifies (4.28.1) which may now be solved for b_1 and b_2:

$$b_1^2(w)+ih_0b_1(w)=b_2^2-ih_0b_2=\frac{a_2}{32}w^4+\frac{a_1}{8}w^3+c_2w^2+c_1w+c_0. \tag{4.36}$$

In this case, in addition to being a configurational invariant, I is also a first integral and for $h_0 = 0$ it agrees with the result obtained in (Hietarinta 1987).

For the third ansatz the gauge transformation is chosen so that the results can be compared with those for first integrals in (Hietarinta 1987). The gauge transformation is firstly used to remove k. As in the above mentioned paper the co-ordinates $\mu = (x+iy)/\sqrt{2}$, $\nu = (x-iy)/\sqrt{2}$ are used and A and B replaced by $A^* = (A+iB)/\sqrt{2}$ and $B^* = (A-iB)/\sqrt{2}$. The remaining gauge freedom is used to eliminate V's dependence on ν so that $V = V(\mu)$. The set of equations to solve is

$$\left(\frac{\partial h}{\partial \mu} - \frac{\partial h}{\partial \nu}\right) V + \left(\frac{\partial B^*}{\partial \nu} + \frac{\partial B^*}{\partial \mu} - \frac{\partial A^*}{\partial \nu} - \frac{\partial A^*}{\partial \mu}\right) iV\mu$$
$$+i(A^* + B^*)V = 0, \tag{4.37.1}$$

$$\frac{\partial h}{\partial \nu} iA^* + \frac{\partial h}{\partial \mu} iB^* - \frac{\partial V}{\partial \mu}\mu - \left(\frac{\partial B^*}{\partial \nu} + \frac{\partial B^*}{\partial \mu}\right)\mu B^*$$
$$+ \left(\frac{\partial B^*}{\partial \nu} + \frac{\partial B^*}{\partial \mu}\right)\mu A^* + \left(\frac{\partial B^*}{\partial \nu} - \frac{\partial A^*}{\partial \nu}\right) ih + \left(\frac{\partial A^*}{\partial \nu} + \frac{\partial A^*}{\partial \mu}\right)\mu B^*$$
$$- \left(\frac{\partial A^*}{\partial \nu} + \frac{\partial A^*}{\partial \mu}\right)\mu A^* + (A^*)^2 - (B^*)^2 = 0, \tag{4.37.2}$$

$$i\frac{\partial h}{\partial \mu} - \frac{\partial B^*}{\partial \mu}\mu - \frac{\partial A^*}{\partial \nu}\mu - B^* = 0, \tag{4.37.3}$$

$$\frac{\partial h}{\partial \mu} + \frac{\partial B^*}{\partial \mu} i\mu - \frac{\partial A^*}{\partial \nu} i\mu + iB^* = 0. \tag{4.37.4}$$

The solution to the equation (4.37.3)+i(4.37.4) is

$$h = -i\mu B^* - id(\mu), \tag{4.38}$$

where d is an arbitrary function of its argument. From (4.37.3) it follows that $A^* = A^*(\mu)$. B^* may be obtained from (4.37.1):

$$B^* = \frac{1}{2\mu}\left(-A^*\nu + \frac{\partial A^*}{\partial \mu}\mu\nu - d\right) + g(\mu). \tag{4.39}$$

The solution to the remaining equation (4.37.2) is

$$(A^*)^2 = a_2\mu^2 + a_1\mu + \frac{d_2}{2}, \tag{4.40.1}$$

$$d^2 = \frac{d_2}{2}\nu^2 + d_1\nu + d_0, \tag{4.40.2}$$

$$V = A^*g + \frac{d_1}{4\mu}. \tag{4.40.3}$$

Differentiating the resultant invariant *wrt* time shows that I is a first integral. The result differs from that in (Hietarinta 1987) in that there $V = 1/2\mu$ and here the remaining gauge freedom was used to remove the μ dependence of V.

CHAPTER V: CONFIGURATIONAL INVARIANTS POLYNOMIAL IN THE MOMENTA FOR TWO DIMENSIONAL AXIALLY SYMMETRIC TIME-DEPENDENT SYSTEMS

5.1 General results

For two-dimensional axially symmetric time-dependent systems with a velocity-independent potential the Hamiltonian is given by :

$$H = \frac{1}{2}p_r^2 + \frac{1}{2}\frac{p_\theta^2}{r^2} + V(r,t). \tag{5.1.1}$$

Here r and θ are the radial and angular co-ordinates respectively. The angular momentum p_θ is a first integral for a system of this form since

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0$$

and it is the first integral with respect to which configurational invariants will be sought. Its constant value is denoted by L.

Assume a configurational invariant polynomial in the momenta :

$$I = \sum_{i=0}^{N} a_i(r,\theta,t)p_r^i. \tag{5.1.2}$$

It is not necessary for I be a polynomial in p_θ, since p_θ is a constant.

Now

$$\dot{I} = \sum_{i=0}^{N}\left[\frac{\partial a_i}{\partial r}p_r^i\dot{r} + \frac{\partial a_i}{\partial \theta}p_r^i\dot{\theta} + \frac{\partial a_i}{\partial t}p_r^i + ia_ip_r^{i-1}\dot{p}_r\right]. \tag{5.1.3}$$

Hamilton's equations of motion are

$$\begin{aligned}\dot{r} &= \frac{\partial H}{\partial p_r} = p_r, \qquad \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{r^2} = \frac{L}{r^2},\\ \dot{p}_r &= -\frac{\partial H}{\partial r} = -\frac{\partial V}{\partial r} + \frac{p_\theta^2}{r^3} = \frac{L^2}{r^3} - \frac{\partial V}{\partial r}.\end{aligned} \tag{5.1.4}$$

Setting $\dot{I}$ equal to zero, using the above results and separating by coefficients of p_r yields

$$a_1\left(\frac{L^2}{r^3}-\frac{\partial V}{\partial r}\right)+\frac{\partial a_0}{\partial \theta}\frac{L}{r^2}+\frac{\partial a_0}{\partial t}=0, \tag{5.1.5.1}$$

$$\frac{\partial a_{i-1}}{\partial r}+\frac{\partial a_i}{\partial \theta}\frac{L}{r^2}+\frac{\partial a_i}{\partial t}+(i+1)a_{i+1}\left(\frac{L^2}{r^3}-\frac{\partial V}{\partial r}\right)=0, \quad i=1,N-1, \tag{5.1.5.2}$$

$$\frac{\partial a_{N-1}}{\partial r}+\frac{\partial a_N}{\partial \theta}\frac{L}{r^2}+\frac{\partial a_N}{\partial t}=0, \tag{5.1.5.3}$$

$$\frac{\partial a_N}{\partial r}=0. \tag{5.1.5.4}$$

The solutions to (5.1.5.4) and (5.1.5.3) are

$$a_N=a_N(\theta,t), \tag{5.1.6}$$

and

$$a_{N-1}=\frac{\partial a_N}{\partial \theta}\frac{L}{r}-\frac{\partial a_N}{\partial t}r+b_{N-1}(\theta,t), \tag{5.1.7}$$

where b_{N-1} is an arbitrary function of its arguments. By substituting these results into (5.1.5.2) with $i=N-1$ one can find an expression for a_{N-2} :

$$\begin{aligned}a_{N-2}=&\frac{\partial^2 a_N}{\partial \theta^2}\frac{L^2}{2r^2}+\frac{\partial b_{N-1}}{\partial \theta}\frac{L}{r}+\frac{1}{2}\frac{\partial^2 a_N}{\partial t^2}r^2-\frac{\partial b_{N-1}}{\partial t}r\\&+a_N\frac{NL^2}{2r^2}+Na_NV+b_{N-2}(\theta,t).\end{aligned} \tag{5.1.8}$$

Similarly expressions for $a_{N-3},\dots,a_0$ are obtained from (5.1.5.2) with $i=N-2,\ 1$. The functions $b_i(\theta,t)$ are determined from (5.1.5.1).

These equations are now solved for $N=1$ and $N=2$.

5.2 Case N=1

From (5.1.6) and (5.1.7)

$$a_1=a_1(\theta,t), \tag{5.2.1}$$

and

$$a_0=\frac{\partial a_1}{\partial \theta}\frac{L}{r}-\frac{\partial a_1}{\partial t}r+b_0(\theta,t). \tag{5.2.2}$$

Equation (5.1.5.1) becomes

$$a_1\left(\frac{L^2}{r^3}-\frac{\partial V}{\partial r}\right)+\left(\frac{\partial^2 a_1}{\partial\theta^2}\frac{L}{r}+\frac{\partial b_0}{\partial\theta}\right)\frac{L}{r^2}-\frac{\partial^2 a_1}{\partial t^2}r+\frac{\partial b_0}{\partial t}=0. \tag{5.2.3}$$

Integration and rearrangement gives

$$V=\frac{1}{a_1}\left(-a_1\frac{L^2}{2r^2}-\frac{\partial^2 a_1}{\partial\theta^2}\frac{L^2}{2r^2}-\frac{\partial b_0}{\partial\theta}\frac{L}{r}-\frac{1}{2}\frac{\partial^2 a_1}{\partial t^2}r^2+\frac{\partial b_0}{\partial t}r+V_1(t)\right). \tag{5.2.4}$$

Here division by a_1 is permissible as it is assumed that $a_1 \neq 0$ since the degree of the invariant being sought is 1.

For V to depend on t and r only the coefficients of the different powers of r on the *rhs* must be functions t alone, i.e.,

$$1+\frac{1}{a_1}\frac{\partial^2 a_1}{\partial\theta^2}=-K(t), \tag{5.2.5.1}$$

$$\frac{1}{a_1}\frac{\partial b_0}{\partial\theta}=M(t), \tag{5.2.5.2}$$

$$\frac{1}{a_1}\frac{\partial^2 a_1}{\partial t^2}=F(t), \tag{5.2.5.3}$$

$$\frac{1}{a_1}\frac{\partial b_0}{\partial t}=Q(t), \tag{5.2.5.4}$$

$$\frac{1}{a_1}V_1(t)=T(t), \tag{5.2.5.5}$$

where the right-hand sides are arbitrary functions of t.

Suppose $V_1(t)\neq 0$. Then a_1 cannot be a function of θ. Therefore

$$a_1=a_1(t). \tag{5.2.6}$$

Equation (5.2.5.1) becomes redundant in this case.

From (5.2.5.2)

$$b_0 = Ma_1\theta + M_1(t), \tag{5.2.7}$$

where M_1 is an arbitrary function of t. Substitution of b_0 into (5.2.5.4) and separation by coefficients of θ gives:

$$M = \frac{C}{a_1}, \tag{5.2.8}$$

$$M_1 = \int Qa_1 dt + c_1, \tag{5.2.9}$$

where C and c_1 are constants.

Equation (5.2.5.3) is the equation of motion for the time-dependent harmonic oscillator with the spatial variable replaced by a_1. The time-dependent harmonic oscillator, given by

$$\ddot{x} + \Omega^2(t)x = 0 \tag{5.2.10}$$

has the Hamiltonian

$$H = \frac{1}{2}[p^2 + \Omega^2(t)x^2]. \tag{5.2.11}$$

The system possesses a first integral, viz.

$$I = \frac{1}{2}\left[\frac{x^2}{\rho^2} + (p\rho - x\dot{\rho})^2\right], \tag{5.2.12}$$

where the function $\rho = \rho(t)$ satisfies the Pinney-Lewis equation

$$\ddot{\rho} + \Omega^2(t)\rho = \frac{1}{\rho^3}. \tag{5.2.13}$$

Sometimes it is easier to find a solution to (5.2.13) than (5.2.10). Suppose that one has found a partcular solution to (5.2.13) for a specific value of Ω. Then for this Ω a general solution to (5.2.10) may be found by substituting $p = \dot{x}$ and $\Phi = x/\rho$ into I and solving. The solution is

$$x = \rho[A\cos(\phi) + B\sin(\phi)], \tag{5.2.14}$$

where $\dot{\phi}(t) = 1/\rho^2$ and A and B are arbitrary constants (Lutsky 1978) .

Particular solutions to (5.2.13) for various functional forms of Ω have been found in (Eliezer and Gray 1976):

$$\Omega = \frac{a}{t}, \qquad \rho^2 = \alpha t, \text{ where } \alpha = \frac{1}{(a^2 - 1/4)^{1/2}} \qquad \text{if } a \neq \frac{1}{2}, \tag{5.2.15.1}$$

$$\Omega = \frac{1}{2}t, \qquad \rho^2 = t\{A(\ln(t) + B)^2 + \frac{1}{A}\}, \tag{5.2.15.2}$$

$$\Omega = \frac{a}{t^2}, \qquad \rho = \beta t, \text{ where } \beta = \left(\frac{1}{a}\right)^{1/2}, \tag{5.2.15.3}$$

$$\Omega = \frac{1}{t^4}, \qquad \rho = t. \tag{5.2.15.4}$$

The corresponding general solutions of (5.2.10) for the Ω's in (5.2.15.1), (5.2.15.3) and (5.2.15.4) above, obtained using (5.2.14), are respsectively

$$x = (\alpha t)^{\frac{1}{2}}[A\cos(\alpha^{-1}\ln(t)) + B\sin(\alpha^{-1}ln(t))], \tag{5.2.16.1}$$

$$x = \beta t[A\cos(\beta^{-2}t^{-1}) - B\sin(\beta^{-2}t^{-1})], \tag{5.2.16.2}$$

$$x = t[A\cos(t^{-1}) + B\sin(t^{-1})], \tag{5.2.16.3}$$

where A and B are arbitrary constants.

When Ω is a constant (5.2.10) is the equation for an autonomous harmonic oscillator with solution

$$x = A\cos(\Omega t) + B\sin(\Omega t). \tag{5.2.16.4}$$

If a_1 takes on any of these solutions, M will be undefined for an infinite number of values of t, except if $\Omega = 0$ in (5.2.16.4). In this case $a_1 = A + Bt$ and M is undefined only for $t = A/B$. The potential and configurational invariant are given by

$$V = -\frac{L^2}{2r^2} - \frac{CL}{(A+Bt)r} + Qr + T, \tag{5.2.17.1}$$

$$I = p_r + \frac{C}{(A+Bt)}\theta + \int Q. \tag{5.2.17.2}$$

The function T in (5.2.17.1) is redundant since it is only a function of time and does not enter into Hamilton's equations of motion.

If $V_1 = 0$, (5.2.5.1) does not disappear and it may be solved for a_1:

$$a_1 = A(t)\cos(\gamma\theta + p(t)),$$

where $\gamma = \pm\sqrt{1+K}$ and A and p are arbitrary functions of t.

Substitution of a_1 into (5.2.5.3) yields

$$\begin{aligned}&\ddot{A}\cos(\gamma\theta+p) - 2\dot{A}\sin(\gamma\theta+p)(\dot{\gamma}\theta+\dot{p})\\&- A\cos(\gamma\theta+p)(\dot{\gamma}\theta+\dot{p})^2 - A\sin(\gamma\theta+p)(\ddot{\gamma}\theta+\ddot{p})\\=&FA\cos(\gamma\theta+p).\end{aligned} \tag{5.2.18}$$

Separation by coefficients of different powers of $\cos(\gamma\theta+p)$, $\sin(\gamma\theta+p)$ and θ yields

$$FA - \ddot{A} + A\dot{p}^2 = 0, \tag{5.2.19.1}$$

$$A\dot{\gamma} = 0, \tag{5.2.19.2}$$

$$2\dot{A}\dot{p} + A\ddot{p} = 0, \tag{5.2.19.3}$$

$$2\dot{A}\dot{\gamma} + A\ddot{\gamma} = 0. \tag{5.2.19.4}$$

From equation (5.2.19.2) either $A = 0$ or γ is a constant. Since $a_1 \neq 0$, the latter possibility must be true. Equation (5.2.19.4) then disappears. Assuming that $\dot{p} \neq 0$, (5.2.19.3) may be divided by $(A\dot{p})$:

$$2\frac{\dot{A}}{A} + \frac{\ddot{p}}{\dot{p}} = 0. \tag{5.2.20}$$

Integration and subsequent rearrangement yields

$$\dot{p} = \frac{W}{A^2}, \tag{5.2.21}$$

where W is a constant.

Equation (5.2.19.1) becomes

$$FA - \ddot{A} + \frac{W^2}{A^3} = 0, \tag{5.2.22}$$

which is the Pinney-Lewis equation discussed earlier. This equation will not be solved, but is simply used to obtain an expression for F :

$$F = \frac{\ddot{A}}{A} - \frac{W^2}{A^4}. \tag{5.2.23}$$

From (5.2.5.2)

$$b_0 = \frac{MA}{\gamma}\sin(\gamma\theta + p) + M_1(t). \tag{5.2.24}$$

Substitution of this result into (5.2.5.4) gives

$$\frac{(\dot{M}A + M\dot{A})}{\gamma}\sin(\gamma\theta + p) + \frac{MA}{\gamma}\cos(\gamma\theta + p)\dot{p} + \dot{M}_1 = QA\cos(\gamma\theta + p).$$

Separating by coefficients of different powers of $\cos(\gamma\theta + p)$, $\sin(\gamma\theta + p)$ and θ yields

$$\frac{MA\dot{p}}{\gamma} - QA = 0, \tag{5.2.25.1}$$

$$\frac{\dot{M}A + M\dot{A}}{\gamma} = 0, \tag{5.2.25.2}$$

$$\dot{M}_1 = 0. \tag{5.2.25.3}$$

From the last equation above it follows that M_1 is a constant. From (5..2.25.2) it follows that the product MA is a constant and therefore

$$M = \frac{G}{A}, \tag{5.2.26}$$

where $p = \int \frac{W}{A^2} dt$, (cf "new time"$T = \int^t \rho^{-2}(t')dt'$ (Burgan 1978)).

If in (5.2.19.3) above $\dot{p} = 0$, p is a constant and the equation vanishes. The solution to (5.2.19.1) is then

$$F = \frac{\ddot{A}}{A}. \tag{5.2.29}$$

In this case, in view of (5.2.27), Q is zero. The potential and configurational invariant are then given by

$$V = \frac{(\gamma^2 - 1)L^2}{2r^2} - \frac{GL}{Ar} - \frac{\ddot{A}}{2A} r^2, \tag{5.2.30.1}$$

$$\begin{aligned} I = & A\cos(\gamma\theta + p)p_r - A\gamma\sin(\gamma\theta + p)\frac{L}{r} \\ & - \dot{A}\cos(\gamma\theta + p)r + \frac{G}{\gamma}\sin(\gamma\theta + p). \end{aligned} \tag{5.2.30.2}$$

5.3 Case N=2

Expressions for a_2, a_1 and a_0 follow directly from (5.1.6), (5.1.7) and (5.1.8):

$$a_2 = a_2(\theta, t), \tag{5.3.1}$$

$$a_1 = \frac{\partial a_2}{\partial \theta}\frac{L}{r} - \frac{\partial a_2}{\partial t} r + b_1(\theta, t), \tag{5.3.2}$$

$$\begin{aligned} a_0 = & \frac{\partial^2 a_2}{\partial \theta^2}\frac{L^2}{2r^2} + \frac{\partial b_1}{\partial \theta}\frac{L}{r} + \frac{1}{2}\frac{\partial^2 a_2}{\partial t^2} r^2 - \frac{\partial b_1}{\partial t} r \\ & + a_2 \frac{L^2}{r^2} + 2a_2 V + b_0(\theta, t). \end{aligned} \tag{5.3.3}$$

The remaining unknowns are b_0, b_1 and V. The last equation to solve is (5.1.5.1) and substituting the above expressions for a_2, a_1 and a_0 into this equation gives

$$\begin{aligned} & 2a_2\frac{\partial V}{\partial t} + \left(\frac{\partial a_2}{\partial t} r - \frac{\partial a_2}{\partial \theta}\frac{L}{r} - b_1\right)\frac{\partial V}{\partial r} + 2\left(\frac{\partial a_2}{\partial t} + \frac{\partial a_2}{\partial \theta}\frac{L}{r^2}\right)V \\ & + \frac{\partial b_0}{\partial t} + \frac{L}{r^2}\frac{\partial b_0}{\partial \theta} - r\frac{\partial^2 b_1}{\partial t^2} + \frac{L^2}{r^3}\frac{\partial^2 b_1}{\partial \theta^2} + \frac{L^2}{r^3} b_1 + 2\frac{L^3}{r^4}\frac{\partial a_2}{\partial \theta} \\ & + \frac{r^2}{2}\frac{\partial^3 a_2}{\partial t^3} + \frac{L}{2}\frac{\partial^3 a_2}{\partial t^2 \partial \theta} + \frac{L^2}{2r^2}\frac{\partial^3 a_2}{\partial t \partial \theta^2} + \frac{L^3}{2r^4}\frac{\partial^3 a_2}{\partial \theta^3} = 0. \end{aligned} \tag{5.3.4}$$

Since V is independent of θ, the coefficients of V and its derivatives in the above equation must be independent of θ up to the same multiplier.

By dividing (5.3.4) by a_2 the θ dependence of the coefficient of $\partial V/\partial t$ will be removed. Requiring that the coefficient of $\partial V/\partial r$ in this new equation be independent of θ gives the equation

$$\frac{\partial}{\partial\theta}\left(\frac{r}{a_2}\frac{\partial a_2}{\partial t}-\frac{L}{a_2 r}\frac{\partial a_2}{\partial\theta}-\frac{b_1}{a_2}\right)=0. \tag{5.3.5}$$

For this to be true the coefficient of each power of r must be zero. From the r term

$$\frac{\partial}{\partial\theta}\left(\frac{1}{a_2}\frac{\partial a_2}{\partial t}\right)=0.$$

On integration this gives

$$a_2=\beta(\theta)e^{\alpha(t)}.$$

Substitution into the r^{-1} term yields

$$\beta=Ke^{\gamma\theta}$$

and therefore

$$a_2=Ke^{\gamma\theta+\alpha(t)}. \tag{5.3.6}$$

The r^0 term may be integrated to give

$$b_1=\delta(t)Ke^{\gamma\theta+\alpha}. \tag{5.3.7}$$

With these results (5.3.4) divided by a_2 becomes

$$\begin{aligned}
&2\frac{\partial V}{\partial t}+\left(r\dot\alpha-\frac{L\gamma}{r}-\delta\right)\frac{\partial V}{\partial r}+2\left(\dot\alpha+\frac{L\gamma}{r^2}\right)V\\
&+\frac{1}{K}\left(\frac{\partial b_0}{\partial t}+\frac{L}{r^2}\frac{\partial b_0}{\partial\theta}\right)e^{-\gamma\theta-\alpha}-r(\ddot\delta+2\dot\delta\dot\alpha+\delta\ddot\alpha+\delta\dot\alpha^2)\\
&+\frac{L^2\delta}{r^3}(\gamma^2+1)+\frac{2L^3\gamma}{r^4}+\frac{1}{2}r^2e^{-\alpha}\frac{d^3}{dt^3}(e^\alpha)\\
&+\frac{L\gamma}{2}e^{-\alpha}\frac{d^2}{dt^2}(e^\alpha)+\frac{L^2\gamma^2}{2r^2}e^{-\alpha}\frac{d}{dt}(e^\alpha)+\frac{L^3\gamma^3}{2r^4}=0.
\end{aligned} \tag{5.3.8}$$

To render this equation free of θ, the terms containing b_0 need to be independent of θ for all powers of r, i.e.,

$$\frac{\partial}{\partial\theta}\left(e^{-\gamma\theta}\frac{\partial b_0}{\partial t}\right)=0,\text{ and }\frac{\partial}{\partial\theta}\left(e^{-\gamma\theta}\frac{\partial b_0}{\partial\theta}\right)=0.$$

From the first of these

$$b_0 = \nu(t)e^{\gamma\theta} + \omega(\theta).$$

The second equation then determines ω:

$$\omega = constant \times e^{\gamma\theta}$$

and then

$$b_0 = K\nu(t)e^{\gamma\theta}, \tag{5.3.9}$$

where the constant has been incorporated into ν and K has been introduced to eliminate $1/K$ in (5.3.8).

Now that the θ dependence has been removed, (5.3.8) may be integrated. The first characteristic of (5.3.8) is obtained from

$$\frac{\mathrm{d}t}{2} = \frac{\mathrm{d}r}{r\dot{\alpha} - L\gamma/r - \delta}.$$

This is Abel's equation of the second kind for which a solution in closed form is not known for nonzero values of δ.

Let $\delta = 0$. Then the characteristic is

$$\omega_1 = r^2 e^{-\alpha} + L\gamma \int e^{-\alpha}, \tag{5.3.10}$$

which may be inverted to obtain r^2 :

$$r^2 = e^{\alpha}\omega_1 - L\gamma e^{\alpha} \int e^{-\alpha}. \tag{5.3.11}$$

With $\delta = 0$ equation (5.3.8) becomes

$$\begin{aligned} &2\frac{\partial V}{\partial t} + \left(\dot{\alpha}r - \frac{L\gamma}{r}\right)\frac{\partial V}{\partial r} + 2\left(\dot{\alpha} + \frac{L\gamma}{r^2}\right)V + e^{-\alpha}\left(\dot{\nu} + \frac{L\gamma\nu}{r^2}\right) \\ &+ 2\frac{L^3\gamma}{r^4} + \frac{1}{2}r^2 e^{-\alpha}\frac{\mathrm{d}^3}{\mathrm{d}t^3}(e^{\alpha}) + \frac{1}{2}L\gamma e^{-\alpha}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(e^{\alpha}) + \frac{L^2\gamma^2}{2r^2}e^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}(e^{\alpha}) \\ &+ \frac{L^3\gamma^3}{2r^4} = 0. \end{aligned} \tag{5.3.12}$$

The expression for r^2 above may be used to remove all occurences of r^2 in equation (5.3.12) and hence r will not appear in the equation of the second charateristic:

$$
\begin{aligned}
\frac{dt}{2} = -\,\mathrm{d}V \Big/ \Big[& 2\left(\dot{\alpha} + \frac{L\gamma}{e^{\alpha}(\omega_1 - L\gamma \int e^{-\alpha})}\right) V + e^{-\alpha}\left(\dot{\nu} + \frac{L\gamma\nu}{e^{\alpha}(\omega_1 - L\gamma \int e^{-\alpha})}\right) \\
& + \frac{2L^3\gamma}{e^{2\alpha}(\omega_1 - L\gamma \int e^{-\alpha})^2} + \frac{1}{2}(\omega_1 - L\gamma \int e^{-\alpha})\frac{\mathrm{d}^3}{\mathrm{d}t^3}(e^{\alpha}) + \frac{1}{2}L\gamma e^{-\alpha}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(e^{\alpha}) \\
& + \frac{L^2\gamma^2}{2e^{\alpha}(\omega_1 - L\gamma \int e^{-\alpha})} e^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}(e^{\alpha}) + \frac{L^3\gamma^3}{2e^{2\alpha}(\omega_1 - L\gamma \int e^{-\alpha})^2}\Big].
\end{aligned}
\tag{5.3.13}
$$

The integrating factor for this equation is

$$
e^{\alpha}(\omega_1 - L\gamma \int e^{-\alpha})^{-1}
$$

and the equation may then be rewritten as

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}t}(Ve^{\alpha}(\omega_1 - L\gamma \int e^{-\alpha})^{-1}) + \frac{\dot{\nu}}{2(\omega_1 - L\gamma \int e^{-\alpha})} \\
& + \frac{L^3\gamma}{e^{\alpha}(\omega_1 - L\gamma \int e^{-\alpha})^3} + \frac{L\gamma\nu e^{-\alpha}}{2(\omega_1 - L\gamma \int e^{-\alpha})^2} + \frac{e^{-\alpha}}{4}\frac{\mathrm{d}^3}{\mathrm{d}t^3}(e^{\alpha}) \\
& + \frac{L\gamma(\mathrm{d}^2 e^{\alpha}/\mathrm{d}t^2)}{4(\omega_1 - L\gamma \int e^{-\alpha})} + \frac{L^2\gamma^2 e^{-\alpha}(\mathrm{d}e^{\alpha}/\mathrm{d}t)}{4(\omega_1 - L\gamma \int e^{-\alpha})^2} + \frac{L^3\gamma^3}{4e^{\alpha}(\omega_1 - L\gamma \int e^{-\alpha})^3} = 0.
\end{aligned}
\tag{5.3.14}
$$

Integration now gives the second characteristic :

$$
\begin{aligned}
\omega_2 = & Ve^{\alpha}(\omega_1 - L\gamma \int e^{-\alpha})^{-1} + \frac{\nu}{2(\omega_1 - L\gamma \int e^{-\alpha})} \\
& + \frac{e^{\alpha}}{4}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(e^{\alpha}) - \frac{1}{8}(\frac{\mathrm{d}}{\mathrm{d}t}(e^{\alpha}))^2 + \frac{L\gamma(\mathrm{d}e^{\alpha}/\mathrm{d}t)}{4(\omega_1 - L\gamma \int e^{-\alpha})} \\
& + \frac{L^2}{2(\omega_1 - L\gamma \int e^{-\alpha})^2} + \frac{L^2\gamma^2}{8(\omega_1 - L\gamma \int e^{-\alpha})^2}.
\end{aligned}
\tag{5.3.15}
$$

Finally for $\delta = 0$, V and I take the form

$$
\begin{aligned}
V = & e^{-2\alpha} r^2 f(\omega_1) - \frac{1}{2}e^{-\alpha}\nu - \frac{1}{4}r^2(\ddot{\alpha} + \frac{1}{2}\dot{\alpha}^2) \\
& - \frac{L\gamma\dot{\alpha}}{4} - \frac{4L^2 + L^2\gamma^2}{8r^2},
\end{aligned}
\tag{5.3.16.1}
$$

$$I = \left[p_r^2 + \left(\frac{L\gamma}{r} - \dot{\alpha} r \right) \dot{p}_r + \frac{\alpha^2}{4} r^2 - \frac{1}{2} \alpha L \gamma + \frac{L^2\gamma^2}{4r^2} \right] e^{\gamma\theta + \alpha} + 2 f r^2 e^{\gamma\theta - \alpha}, \tag{5.3.16.2}$$

where a constant factor of K has been removed from I and f is an arbitrary function of ω_1.

CHAPTER 6: CONCLUSION

A few interesting potentials, in particular those in Chapter III have arisen out of this study. It is not yet known whether these systems are integrable or not, but it was found that they do not possess first integrals quadratic in the momenta other than the Hamiltonian. If the systems are non-integrable, then the configurational invariants are non-trivial. If a system is integrable, then on the zero energy surface one of the first integrals simplifies to the configurational invariant found, making the system easier to solve on that surface. Only one system possessing a non-trivial configurational invariant, given in § **2.4** of chapter II, is so far known.

Only certain types of Hamiltonians have been considered here. The search could be extended to systems of the form

$$H = a(q_1, q_2)p_1^2 + b(q_1, q_2)p_1p_2 + c(q_1, q_2)p_2^2 + \epsilon V(q_1, q_2)$$

considered in (Havas 1975). The search may also be extended to a higher dimensional space.

In Chapter II it was shown that Liouville's theorem could be adapted to give a weak complete integrability theorem for systems possessing configurational invariants. It may be possible to derive further results for configurational invariants from other theorems and methods valid for integrable systems and first integrals e.g. the Painlevé test and the Lie theory of extended groups.

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