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Exploring two Foundation Phase teachers' selection and use of examples and representations in number-related tasks

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A research report submitted to the WITS SCHOOL OF EDUCATION, University of the Witwatersrand, Johannesburg, in fulfilment of the requirements for the degree of Master of Education (Primary Mathematics).

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Declaration

I, Samantha Sarah Morrison, hereby declare that this research report is my own work. It is being submitted for the Degree of Master of Education (Primary Mathematics) at the University of the Witwatersrand, Johannesburg. This report has not been submitted for any other degree or examination at any other university.

_____ _____

Samantha Sarah Morrison Date

Dedication

To Benjamin – my 'best boy'. You are a pure light that brightens my life even when dark clouds are gathering.

To Caitlin – my 'princess'. The high standards that you set for yourself, and your determination to achieve your goals has been a constant encouragement to me during this M. Ed process.

To Matthew – my 'gorgeous thing'. You are a gift to me and I treasure all the special ways that you make me feel loved and appreciated.

Acknowledgements

I would firstly like to acknowledge my *Heavenly Father*. Thank You for being faithful to complete the good work that You have started in me (Phil 1:6).

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Finally, I would also like to acknowledge my supervisor, *Prof. Hamsa Venkat*. Thank you for the kind, gracious manner in which you guided me through the past twelve months. You have patiently shared your wealth of knowledge and understanding with me, and for this I am very grateful. I appreciate all the time and effort that you took to teach me. My prayer is that the seeds that you have planted in my life will bear much fruit that will enrich the lives of teachers and children in South Africa and around the world. I could not have done this without you – thank you!

Abstract

National and international studies show that the standard of mathematics teaching and learning in South Africa is very low compared to other countries. These statistics are worrying because mathematics is one of the 'gatekeeper' subjects that determine learners' access to higher learning and a better future.

My study, aimed at exploring two Foundation Phase teachers' selection and use of examples and representations when teaching number, forms part of a longitudinal study currently underway within the Wits Maths Connect Primary (WMC-P) Project. One of the broad aims of the WMC-P Project is to improve primary teachers' mathematics content knowledge and also to see this translated into improved pedagogy on the ground. This qualitative study was carried out within the WMC-P Project's 20-Day in-service training course and one of the ten government schools participating in the broader study.

My study aimed to build on research that has been carried out on teachers' use of examples and representations with a focus on the South African terrain. The dataset comprised of two Foundation Phase teacher's pre-tests, course-work tasks, field notes, and transcripts of observed lessons. Data was analysed using an analytical framework based on current literature related to examples and representations within mathematics teaching. Findings from my study show possible associations between a higher content knowledge score and the extent of a teacher's example space and more coherent connections between different representational forms. More studies around this topic are needed because research shows that teachers' examples and representations in mathematics teaching are important for good teaching and conceptual understanding.

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CHAPTER ONE: INTRODUCTION

1.1 Background to the study

Performance in mathematics is often used as an indicator of a country's standard of education, as a benchmark of an individual's academic performance and as the 'gatekeeper' to higher education and a better chance at success in life. The standard of South Africa's science and mathematics education was recently ranked the worst out of 62 countries in the fifth Financial Development Report of the World Economic Forum (WEF, 2012) – putting poorer African countries like Kenya and Ghana ahead of us. Grave concerns continue to be expressed by government, institutes of higher learning, teachers and parents with regard to the country's performance in mathematics.

South African learners' poor performance in mathematics is well documented in both national and international studies, viz. the Annual National Assessments (ANA), the Trends in International Mathematics and Science Study (TIMSS), and the Southern and Eastern Africa Consortium for Managing Education Quality (SACMEQ). Statistics from the latest SACMEQ III study (Moloi & Chetty, 2010), as well as results from the comparative study conducted by Carnoy and Chisholm (2008), show that not only do South African learners need to improve their mathematical content knowledge, but the teachers do too. Thus the need for a content-knowledge focused in-service teacher training course like the 20-Day course that was piloted as part of the Wits Maths Connect Primary (WMC-P) Project. One of the aims of the 20-Day course was to see primary teachers improving in their mathematical content knowledge and also connecting what they have learned in the course to their teaching of mathematics in the classroom. After having been a tutor on the WMC-P Project's 20-Day course for a few months, my interest in understanding the ways in which teachers on the course selected and worked with examples and representations on course tasks and in their teaching practice was piqued. This followed evidence of teachers' poor selections of examples in the literature (Venkat & Naidoo, 2012) and limited ranges of examples in the low rates of task completion within and across lessons (Venkat, 2013) and more general slow pacing (Ensor et al., 2009; Reeves & Muller, 2005).

Examples are used in the classroom by both teachers and learners to communicate and explain mathematical ideas (Bills et al., 2006). Here an *example* includes anything that is used as 'raw material' in the classroom setting for the purpose of generalization, illustration of concepts, demonstration of possible variation, practising of a technique, or for conceptualization (Bills et al., 2006). Literature points to the importance of how examples are selected and used as this has bearing on what features the learners take note of, and consequently, on learners' mathematical understanding (Bills et al., 2006; Watson & Mason, 2005). Literature also indicates that how examples are represented in the classroom – in the form of concrete objects, spoken words, pictures, written words and symbols (Askew & Brown, 2003) – are important for both concept understanding in mathematics as well as for good mathematics teaching (Ball & Bass, 2003; Heize, Star, & Verschaffel, 2009). One way of further understanding and potentially mitigating the problems currently faced in the field of mathematics in South Africa is by re-considering how examples and their representations are used in the teaching and learning of mathematics, specifically in relation to the teaching and learning of number. The focus on early number in this study was motivated by literature which points to the importance of developing learners' number sense in the primary years (Anghileri, 2006; Askew, Brown, Rhodes, Wiliam, & Johnson, 1997; Haylock & Cockburn, 2008; McIntosh, Reys, & Reys, 1992; Shumway, 2011).

1.2 Literature-based Rationale

Number sense

Number sense is regarded as an essential outcome of the school curriculum in many countries and as such has received increased attention within mathematics research over the past three decades (The Cockcroft Report, 1982; NCTM, 1989; Reys, Reys, McIntosh, Emanuelsson, Johansson, & Yang, 1999; Anghileri, 2006; Shumway, 2011; and DBE, 2011a). Although it is an important outcome of most school curricula there is no single definition for number sense agreed upon by researchers and mathematics teachers because it is a complex concept that is difficult to define (Reys et al., 1999; Shumway, 2011). In the Cockcroft Report the term 'number sense' is used to describe someone who displays 'at-homeness with numbers' (1982, ch2, p. 8) while Anghileri describes a person who displays number sense as having a

'facility with numbers' and a 'feel for numbers' (Anghileri, 2006, p. 1). Based on this literature, and for the purpose of this report, I will use the term 'number sense' to refer to someone who: is aware of the relationships between numbers, can work flexibly with numbers and number operations when solving problems, can recognise patterns and make connections between number patterns and operations, and who is able to calculate efficiently using mental and written strategies.

Learners' flexibility in using number operations and procedures and their understanding of the relative effect of operating on numbers are considered as some of the essential components of number sense (Anghileri, 2006; McIntosh et al., 1992; Shumway, 2011). The four basic operations that constituted the syllabus for arithmetic in the past are still important for mathematics learning, but the emphasis within research findings suggest that they should not simply be taught as paper-and-pencil skills. Within addition and subtraction there is commentary that restricting conceptions of addition and subtraction to 'join' or 'take away' conceptions reduces the range of problems that learners are able to tackle (Carpenter, Fennema, Franke, Levi, & Empson, 1999). The body of work done by Carpenter et al. (1999) shows that there are different types of addition and subtraction problems that children must be exposed to, viz. *join*, *separate*, *part-part-whole*, and *compare problems*. These problem types are distinct in that children are initially likely to perform different actions to try to solve them (Carpenter et al., 1999). Similarly, these authors believe that learners use different initial direct modelling strategies to solve multiplication and division problems which they have grouped into *multiplication*, *measurement division* and *partitive division* problems based on the different information set types given in the problem and how learners solve these (Carpenter et al., 1999). These different conceptions of mathematical problems represent different levels of complexity, and Carpenter et al. have argued the need for learners to get exposure to the range of problem types to help them recognise the type of problems presented, and to solve them successfully (Carpenter et al., 1999).

Within the South African Foundation Phase curriculum the weighting for Numbers, Operations and Relationships (Learning Outcome 1) has been increased within recent curricula reform from 50% to 65%, 60% and 58% for Grades 1, 2 and 3 respectively (DBE, 2011a). This increase in notional time allocated to the teaching

and learning of number in South Africa's early primary curriculum, and the increased attention number sense has received within the international mathematics community, explains why the development of learners' number sense forms the broad context that undergirds my exploration of teachers' selection and use of examples and representations.

Examples

Examples form an integral part of the discipline of mathematics and have played an important role in the teaching and learning of mathematics throughout history (Bills et al., 2006; Watson & Mason, 2005). Examples offer insight into the nature of mathematics through their use in demonstrating methods, in explanations, in proofs and in concept development; and are one of the main tools used to illustrate and communicate concepts between teachers and learners (Bills et al., 2006).

The work done by Watson and Mason (2005) highlight some reasons why examples are considered important from the learning perspective. These authors argue that mathematics is learned by becoming familiar with examples that demonstrate or illustrate mathematical ideas and by constructing generalizations from examples (Watson & Mason, 2005). Therefore, the mathematical examples presented to learners directly influence what they learn. Examples are also important from the teaching perspective. The work done by Bills et al. (2006) show that examples in the form of worked solutions to problems (presented either by the teacher or in a textbook) are key features in virtually any instructional explanation. Leinhardt et al. (1990) as cited in Bills et al. (2006, p. 9) argue that examples are communicative devices that are fundamental to the mathematical explanations offered by teachers, and further note that:

> "Explanations consist of the orchestrations of demonstrations, analogical representations and examples. […]. A primary feature of explanations is the use of well-constructed examples, examples that make the point but limit the generalization, examples that are balanced by non- or counter-cases".

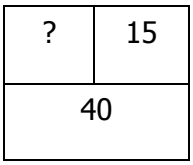
A person's *accessible example space* refers to the collection of examples which that person has access to at any moment and the richness of interaction between those examples (Bills et al., 2006). This example space plays a major role in the sense one can make of the task set, the activity engaged in, and how one construes what

a text says (Bills et al., 2006). Tall and Vinner (1981) use the term *concept image* to describe the total cognitive structure that one associates with a concept, including all the mental pictures, associated properties and processes. One's concept image of multiplication, for example, is built over time through different experiences. The concept of multiplication is usually first encountered as a process involving repeated addition of positive whole numbers in object or symbolic form that 'makes numbers bigger'. One's concept image of multiplication will change as one matures and meets new stimuli, like multiplying a whole number by a fraction (Tall & Vinner, 1981). Thus a person's concept image about any concept is not static but can be adapted and changed from a simple to a more sophisticated concept image as other examples thereof are encountered and assimilated (Tall & Vinner, 1981). The same can be said of someone's example space. Example spaces can be extended as new examples and counter-examples are encountered (Bills et al., 2006). Cognitive and contextual factors influence and inform both the example space and the concept images to which someone has access to at any moment (Bills et al., 2006).

Watson and Mason (2005) formulated the notion of *a personal example space* as a tool for helping teachers and learners become more aware of the potential and limitations of experience with examples. One's personal example space is what is accessible in response to a particular problem, in a particular context, in relation to one's disposition at that time (Bills et al., 2006). The contents and structure of example spaces are individual and situated (Bills et al., 2006). The 'individualness' of one's example space means that the extent to which any example can be considered useful is subjective. So, what the teacher may think is an excellent example to demonstrate a mathematical concept or procedure may not necessarily be an excellent example to the learners because their dispositions, prior knowledge of, and experiences with that concept differ. The nature and sequencing of examples, non-examples and counter-examples affords learners a unique opportunity to learn, but even more critical are the practices into which learners are inducted when working with and on examples (Bills et al., 2006). Rowland (2008) examines features of pre-service teachers' selection and use of examples in relation to taking account of *variables*, *sequencing*, *representations* and the *learning objectives* specified by the teacher. In this study I examine two Foundation Phase teachers' selection and use of examples in relation to the first three of Rowland's categories.

A teacher's selection and construction of examples, whether predetermined or spontaneous, reveals a good deal about that teacher's accessible example space in that situation, and hence the scope of her awareness and the focus of her attention (Goldenberg & Mason, 2008; Rowland, 2008). Thus for my study I will be looking at the examples selected, as well as the representational form thereof, to comment on teachers' example spaces in the context of their teaching.

A mathematical example used in the classroom can be represented in different ways – in the form of concrete objects, spoken and/or written words, pictures and symbols (Askew & Brown, 2003). Different representations of a particular example are transparent about some features and more opaque with regard to others (Bills et al., 2006). For example, in the problem represented with symbols as $\square + 15 = 40$, finding the missing addend seems more difficult to young learners to solve than if it were represented using this picture representing a part-part-whole conception of the same problem:



In the symbolic representation of the problem the possibility of using a subtraction operation to find the missing addend is not immediately noticeable to young learners (who frequently add the two numbers) whereas in the part-part-whole representation the possible use of the subtraction operation to find the solution to such an addition problem is more transparent. The 'variables' in this problem are the addend and total values, and varying these can make the problem easier or harder, as well as amenable to efficient solving through alternate models and strategies.

Representations

The importance of representations in mathematics teaching and learning has been widely acknowledged (Ball & Bass, 2003; Cobb, Yackel, & Wood, 1992; Heize et al., 2009; Pape & Tchoshanov, 2001; Terwel, van Oers, van Dijk, & van den Eeden, 2009). A study where children were asked to describe 'what was in their head' when they calculated answers to number problems (Bills, 1999, as cited in Askew & Brown, 2003) showed the extent to which their mental images were influenced by the physical representations used by their teachers. Reporting on this research,

Askew and Brown (2003, p. 7) noted that: 'Children need to be encouraged to develop efficient mental images and such a range will be influenced by physical representations offered by the teacher'. Thus the type of representations teachers use (progressing from concrete to abstract) and the way they use them (flexibly and in connected ways) in order to develop learners' number sense is critical, because this shapes the mental images learners bring to mind as they try to gain understanding of the given concept or as they try to use what they have learnt in a different situation.

Flexibility

The body of work done by Heize et al. (2009) suggests that classroom environments wherein learners are exposed to multiple representations of the mathematical concept being taught (including graphical, tabular, algebraic and verbal); and wherein they learn to flexibly shift between these representations are more effective in helping students understand and develop an appreciation for mathematics than classroom environments that do not emphasize multiple representations. Literature shows that teachers need to develop a representational repertoire and flexibly move from one representational form to another when teaching, based on the nature of the content to be taught, the context, and the characteristics of the students who will learn that content (Ball, 1993; Perkins & Unger, 1994).

Connections

While the use of multiple representations of examples is important, literature also shows that these representations should be used in connected ways. The body of work done by Askew et al. (1997) suggest that teachers who use approaches that 'connect different areas of mathematics and different ideas in the same area of mathematics' using various representations are more effective teachers of numeracy than teachers who are not considered 'connectionist'. Similarly, Haylock and Cockburn (2008) highlight the importance of establishing connections when teaching and learning number in the early primary years. These authors believe that when children build connections between actions, words, pictures and symbols as they learn number and number operations, their understanding thereof is enhanced and 'more secure' (Haylock & Cockburn, 2008, p. 9)

Progression

Ensor et al. (2009) report that South African teachers' persistent use of concrete representations of number has a negative effect on children's conceptual understanding thereof. Drawing on the work done by Dowling, Ensor et al. (2009) categorised a range of representations used by teachers, and ordered them from concrete to more abstract as follows: *concrete* (includes fingers, counters and beads); *iconic* (like drawings and cartoons); *indexical* (dots and tallies); *symbolic number-based* (includes number charts and number lines); and *symbolic-syntactical* (using mathematical statements). These authors believe that South African teachers can improve learners understanding of number in the Foundation Phase by progressing from concrete to more abstract representations when teaching number, (Ensor et al., 2009).

Literature within the mathematical landscape has shown that representations in a mathematical learning environment that are used in flexible, connected and progressively more abstract ways provide learners with good opportunities to develop mathematical thinking (Pape & Tchoshanov, 2001; Stylianou, 2010; Suh & Moyer, 2007). In light of the importance of examples and their representations in both concept development and good mathematics teaching (Ball & Bass, 2003; Heize et al., 2009) I think the present focus on teachers' selection and use of examples and their different representational forms when teaching number and number operations is warranted in the context of the reported poor standard of mathematics education in South Africa (WEF, 2012). Taken together, these findings from the literature lead to my research questions.

1.3 Research Questions

1. What comparisons can be made between two teachers' selection and use of representations in their course-work that relate to teaching number?
2. What comparisons can be drawn between these two teachers' selection and use of examples when teaching number-related tasks, taking account of variables, sequencing and representations?

1.4 Methodology

To answer these research questions I invited two Foundation Phase teachers – Zelda[1] and Deborah[2] who teach Grades 1 and 2 respectively, from one of the schools participating in the WMC-P Project's 20-Day course to participate in my study. My specific focus on Foundation Phase teachers was motivated by literature that shows the importance of teachers' choice of examples and representations when teaching early number skills (Anghileri, 2006; Askew & Brown, 2003; Haylock & Cockburn, 2008). The two teachers who make up my sample were selected based on their differing content knowledge as per their 20-Day course pre-test scores (one was stronger and the other weaker) – this feature being of interest in a context where gaps in the mathematics content knowledge (CK) and pedagogic content knowledge (PCK) of primary teachers have been widely reported (Carnoy & Chisholm, 2008; Rowland, Huckstep, & Thwaites, 2005); their willingness to participate in the study; and the language of learning and teaching (LoLT) of their school - which is English. Whilst comparison per se on the basis of two teachers is not the focus of this study, my inclusion of a comparative focus in the research questions is due to the ways in which looking across their differences allows me to see each individual teacher more deeply, through analysing relative presences in data drawn from the other teacher. The data set that I worked with in this study consisted of teachers' course-work tasks related to the teaching of early number, field notes made during the 20-Day course and lesson transcripts of two lessons presented at different points by each teacher – all focused on number work. As mentioned earlier, within the data analysis I used three of Rowland's (2008) categories of exemplification, viz. *variables, sequencing* and *representations*, as my broad analytical framework. Further details on the analytical and methodological approaches are given in Chapters 3 and 4 respectively.

1.5 Structure of research report

This current chapter is an introduction to the research report, which covers the background, the rationale for this study and the research questions that have guided this study.

[1] Anonymised
[2] Anonymised

Chapter 2 presents a survey of the literature I reviewed regarding number sense, and teachers' selection of examples and representations when teaching number.

Chapter 3 explains the analytical framework of this study and includes a table summarising the relationships between concepts used.

Chapter 4 explains the methodology used in my research. The research design and sampling procedure is described and my reasons for using these are explained. Thereafter the data collection techniques and methods of data analysis used are explained. After discussing the steps I took to attain rigour in my study I concluded this chapter by discussing ethical issues related to my study.

Chapter 5 presents the findings of this study and a discussion of the findings in light of the research questions and literature reviewed.

Chapter 6 presents the conclusion and limitations of my study. This is followed by a brief discussion of the areas highlighted for future research.

CHAPTER TWO: LITERATURE REVIEW

2.1 Introduction

I approached my literature review with a broad focus on issues relating to the teaching of number, with a particular interest in the development of learners' number-sense in the early years or the Foundation Phase. Here my focus was on what number-sense is, why developing learners' number-sense is considered important in the South African context, and how learners' number-sense can be developed. I also reviewed literature on the local and international mathematical landscape pertaining to primary teachers' selection and use of examples and representations when teaching number skills because of the body of evidence that points to the impact this can have on learners' conceptual understanding thereof (Ball & Bass, 2003; Heize et al., 2009). With regard to examples, my focus was on: what constitutes an example; why examples are important in teaching and learning mathematics; and how teachers' selection and use of examples can provide insight into their example spaces. With regard to how teachers' representations can offer their learners the opportunity to learn number, I focused on the notions of flexibility, connections and progression as these attributes of representations were highlighted in the literature reviewed. Although teachers' use of examples and representations impacts all areas of mathematics, my particular interest for this study was in exploring teachers' selection and use of examples and representations when teaching number.

2.2 Number Sense

What is number sense?

According to McIntosh et al. (1992) the exact origin of the term 'number sense' is not clear. These authors believe that this term was born out of the desire to replace the term 'numeracy' (coined in 1959) with its' associated conservative view of mathematics; while Anghileri (2006) believes that this term springs out of a reaction to the overemphasis on computational procedures that were devoid of understanding. The term 'number sense' gained general acceptance as one that encompasses the changes in mathematics education over recent years (McIntosh et al., 1992).

Number sense is described as an elusive, amorphous concept that exhibits itself in various ways within all strands of mathematics (McIntosh et al., 1992; Reys et al., 1999; Shumway, 2011). What makes this concept even more 'slippery' to define is the fact that it is referred to by different names within mathematics education literature. The Cockcroft Report (1982) describes a numerate adult (i.e. one who has number sense) as one who has a 'feeling for number which permits sensible estimation and approximation' and one who displays an 'an at-homeness with numbers' (The Cockcroft Report, 1982, ch2, p7, 8). A few years later Howden (1989, p. 11) described children with number sense as possessing 'good intuition about numbers and their relationships'. More recently, Anghileri (2006, p. 1) used the terms 'facility with numbers' and a 'feel for numbers' to describe children who have number sense.

Various essential components of number sense have been hypothesized in mathematics education literature. These include: understanding relationships between numbers, understanding the magnitude of numbers, understanding the relative effect of operating on numbers (e.g. that multiplication does not always 'make bigger'), counting strategies, making connections, the ability to recognise and use patterns and relationships between numbers, estimation, flexibility in using operations and procedures, mental calculation, multiple representations of numbers, fluent computation and using visual models (NCTM, 1989; Howden, 1982; McIntosh et al., 1992; Fennell & Landis, 1994; Anghileri, 2006; Shumway, 2011).

Why is number sense described as important?

Learners who lack number sense think of mathematics as a disconnected set of rules and algorithms that they must learn to use in order to solve problems successfully. This leads to rigidity in their use of number operations that resembles the procedural paper-and-pencil methods advocated in the past with no focus on conceptual understanding. For example, my 13 year old daughter recently explained that she was taught to calculate a sum like $\frac{2}{5} \div \frac{4}{9}$ by remembering the rule for dividing fractions, i.e. 'tip and times'. So when she sees this type of sum in an exercise or exam she 'tips' the second fraction (making it $\frac{9}{4}$) and then she multiplies the numerators, she does the same for the denominators, and the final step is to simplify. While this catchy phrase 'tip and times' certainly helps her remember the

procedure for getting the correct answer, it leads to a rigid use of an operation with no conceptual understanding of dividing with fractions. For children and adults living in the 21st century, having number sense is very important because this is a technological age in which they will encounter a greater range of numbers (e.g. when South Africa's Chad le Clos beat American, Michael Phelps, in the men's 200m butterfly final at the London Olympic Games by *five-hundredths* of a second); in a variety of different contexts like sport and politics; while utilizing new tools like iPads and GPS systems (McIntosh et al., 1992).

According to Shumway (2011) number sense facilitates learners': recognition of patterns and relationships between numbers, efficient computation, reasoning ability, and problem solving abilities. Some researchers believe that number sense is the foundation upon which all understandings of mathematics are built (Fennell & Landis, 1994; Reys et al., 1999; Shumway, 2011). Children who lack number sense face enormous barriers to learning mathematics because they see mathematics as a set of isolated, disconnected facts and algorithms which must be memorised and practised (Reys et al., 1999; Shumway, 2011). Although the development of learners' number sense is not a panacea for all the problems facing the teaching and learning of mathematics in South Africa (Schollar, 2008), it could lessen some of the barriers to learning mathematics that learners face.

McIntosh et al. (1992) argue that mathematics education literature contains lists of components of number sense and descriptions of children who possess it, but there is no description of how these components fit together. In an attempt to organise and interrelate some of the generally agreed upon components of number sense, these authors developed a framework for basic number sense with three broad categories: viz. *knowledge of and facility with numbers; knowledge of and facility with operations;* and, *applying knowledge of and facility with numbers and operations to computational settings* (McIntosh et al., 1992). Whilst they describe key features of number sense, McIntosh et al. (1992) do not attend to or deal with the nature of progression within number sense – which has been described as a critical issue in South Africa. This leads to a focus on literature detailing the nature of progression for the development of number sense.

How is number sense developed?

Wright, Martland, and Stafford (2006) developed the Learning Framework In Number (LFIN) as a guide to teaching and assessment in early number. This comprehensive framework sets out stages and levels of children's knowledge of number, facilitates profiling of children's knowledge, and indicates the likely progression in their learning of number (Wright et al., 2006). The LFIN details how aspects of 'children's counting strategies, their strategies for adding and subtracting, their knowledge of number word sequences and numerals, their ability to reason in terms of tens as well as ones, and their developing strategies for multiplication and division' are interrelated (Wright et al., 2006, p. 8). These authors also delineate other important aspects of children's early numerical knowledge such as the use of finger patterns, combining and partitioning small numbers, the role of spatial and temporal patterns, and the use of base-5 and base-10 strategies. The LFIN consists of eleven interrelated aspects of early number which are organised into four parts – of which the Stages of Early Arithmetic Learning (SEAL) is the most important. The SEAL focuses on the relative sophistication of children's strategies for counting, addition and subtraction – with strategies like 'flexible grouped counting' and 'working with 5 and 10 as benchmarks' which form the foundation of multiplication and division visible at the upper end (i.e. stage 5). This part of the framework also highlights the importance of using the base-ten structure and shows how the learning thereof develops which can be helpful to teachers who want to assist their learners along this trajectory (Wright et al., 2006).

Within the literature on number sense, the idea of developing more sophisticated counting strategies is seen as fundamental to the development of number sense. Alongside this is literature noting that development in counting strategies, whilst necessary, is not sufficient for building number sense. Learners' experience with a wide range of images that illustrate the logical structure of numbers is also needed to build good mental strategies that develop learners' flexibility and efficiency with the four basic number operations in different contextual situations.

Counting Strategies

Counting is seen as one of the most important foundations for the development of early number skills (Askew & Brown, 2003; Anghileri, 2006; Shumway, 2011) and as

the fundamental mathematical process that needs to be addressed from the child's earliest experiences in school (Thompson, 2008). Counting routines help children gain insight into the relationships between numbers and to recognize patterns in the number system – which in turn provides opportunities for children to create the benchmarks they need for solving problems efficiently and for doing mental calculations (Anghileri, 2006; McIntosh et al., 1992; Shumway, 2011; Thompson, 2008).

The general agreement in the literature base on the trajectory for developing counting skills is based on the work done by Carpenter and Moser, i.e. 'counting-all', 'counting from the first number', 'counting from the larger number', 'using known number facts' and 'using derived number facts' – which also figures within the SEAL progression (Askew & Brown, 2003; Carpenter & Moser, 1984; Clements and Samara as cited in Shumway, 2011; Wright et al., 2006).

When learning to count, very young children first develop skills such as: reciting number words in sequence, pattern recognition, subitising (naming spatial arrays like the dots on a domino card), one-to-one correspondence, and counting with different finger patterns (Thompson, 2008; Wright et al., 2006). In the counting trajectory the first strategy, *counting all*, is the most basic as this follows on from a direct modelling of the action described in the problem. The next strategy, *counting on*, is considered to be a 'big next step' from the former as it requires children to be able to recite number names in an oral sequence by starting from points other than the first number in that sequence (Thompson, 2008); and also be able to use the 'cardinality principle' – i.e. being able to assign a number name to a whole set, the 'abstraction principle' – i.e. knowing that concrete and abstract things can be counted, and the 'order-irrelevance principle' – i.e. knowing that they can assign any number name (tag) to any of the objects as long as there is a name for all the objects (Gelman & Gallistel, 1978). The *counting on from larger* strategy is the next level in the counting progression. Here children need to understand and use the commutative law of addition to know that the answer they will get from 4+9 will be the same as from 9+4. Thompson (2008) argues that at this stage many children will instinctively use this law without actually being aware of the name. Anghileri (2006, p. 54) maintains that children progress through the stages of 'counting all,

counting on and counting from the larger' before they come to recognize a number fact, like '3 and 5 together *always* makes 8'. The next strategy in the counting trajectory, *using known number facts*, sees children using number facts like bonds of ten and doubles. For the strategy called *derived number facts*, children work with near doubles, bridging-up-through-ten, partitioning of numbers, and step or skip counting. Counting strategies at this level are underpinned by knowledge of bonds of ten or 'complements of ten' (Thompson, 2008) – which also form the building blocks for mental strategies which are discussed next.

Mental Strategies

Although counting is acknowledged to be a key component in developing number sense, Shumway (2011) argues that it is not enough to build children's number sense. The conceptualisation of quantity is also important to the development of number sense in children; therefore Shumway (2011) believes that the visual, perceptual and conceptual understanding of quantity should be emphasised in mathematics teaching. Visualising is commonly overlooked in mathematics teaching, yet visual understandings help children think about the quantities conceptually, symbolically and abstractly (Shumway, 2011).

The work done by Wright et al. (2006) reports on children's mental strategies, like combining and partitioning, that develop alongside their counting strategies which do not necessarily rely on counting. Similarly, Askew and Brown (2003) maintain that children's effective mental strategies can be built on the partitioning of numbers in various ways. What is important here is that children need to understand the structure of numbers to use mental computation strategies, e.g. that 7 can be made of a 5 and a 2, or a 3 and a 4; they also need a good understanding of place value, e.g. that 253 is made of a 200 and a 50 and a 3; children also need to understand the relationships among numbers, e.g. that 16+14=30 because 15+15=30 (Askew & Brown, 2003; Shumway, 2011); and by using base-5 and base-10 mental strategies children will have the numbers 5 and 10 as reference points when combining and partitioning numbers (Wright et al., 2006). Number bonds (especially bonds of ten) are an important part of mental computation and should be memorized by children as they provide 'benchmarks' wherewith children can compute easily, and also derive other number facts from (Anghileri, 2006; Askew, 1998).

Estimation is an example of a mental computational skill often used by children and adults in everyday life as a means of determining the reasonableness of a calculation (Sowder & Schappelle, 1989; Shumway, 2011). Sowder and Schappelle (1989) believe that 'intentionality' of estimation should be the focus when teaching and learning this skill (i.e. the context it is to be used in) rather than 'not getting the exact answer' as in the case of estimation exercises devoid of context. Estimation is an important skill because it helps children develop 'conceptual structures for number', i.e. children learn about number size, about compensation, and about place value and the decimal system simultaneously (Sowder and Schappelle, 1989).

Classroom materials are powerful images for representing numbers and for illustrating the way numbers are related in a logical structure (Anghileri, 2006). In traditional classrooms the place value aspect of numbers has been emphasised more than counting, with images relating to 'tens sticks' 'unit cubes' and Dienes' blocks which are closely associated with column arithmetic that was central in the past (Anghileri, 2006). In contrast, research advocates for a focus on mental strategies that often relate, initially at least, to more intuitive ideas based on counting (Anghileri, 2006). Representations of number that relate closely to the counting sequence would include a 'bead frame', 'bead string' and number line (Anghileri, 2006). Therefore the teaching of algorithms that focus on a digit column value should be delayed in light of evidence pointing to children's mental strategies and sense of number being disrupted by early algorithm introduction (Askew & Brown, 2003). Apparatus like children's fingers, an arithmetic rack, a ten frame, and a bead string are some of the tools that teachers can use to initially help develop learners' mental strategies (Anghileri, 2006; Wright et al., 2006). Anghileri (2006) proposes that learners be taught how to use the empty number line (ENL) to support their mental calculations and later, also to produce effective written calculations.

Addition and Subtraction

The model presented by Carpenter and Moser (1984), which outlines the progression of children's counting strategies for addition, was referred to earlier in my discussion of counting strategies. I now briefly discuss Carpenter and Moser's (1984) progression in stages of children's subtraction strategies which are outlined by Anghileri (2006, p. 56) as: 'count out, counting back *from*, counting down *to*, count

up from, and using known facts and derived facts'. For calculating 8–3, these strategies are illustrated as follows: the *count out strategy* may involve the child counting out 8 counters, removing 3 of the 8 counters, and then counting the remaining counters to find the answer; the *counting back from strategy* – also referred to as the 'take away' perception of subtraction (Baroody as cited in Thompson, 2008) – will involve the child counting back from the bigger number, 8, as many counts as the smaller number, i.e. 8 … 7, 6, **5**; in the *counting down to* strategy – also known as the 'difference' perception of subtraction – the backward count goes from the bigger number to the smaller number: 8 … 7, 6, 5, 4, 3, using tallies to get the answer **5**; in *counting up from* the child will start at the smaller number, 3, and count up to the bigger number, 8, using tallies to get the answer **5**; and in *using known and derived facts* the child will use bonds of ten, partitioning of numbers, and their understanding of addition and subtraction as inverse operations (Thompson, 2008). Anghileri (2008) argues that at this stage children's development of number sense involves 'metacognition', because they are reflecting on their answers to problems and finding connections between different number facts.

Carpenter et al. (1999) have developed a classification scheme for different conceptions of word problems which I think are useful when considering abstract, symbolic problems as well. According to these authors, addition and subtraction problems can be classified based on the type of action or relationship described in the problem as *join*, *separate*, *part-part-whole*, and *compare* problems (Carpenter et al., 1999, p. 7). Join and separate problems are alike in that both involve an action over time, except in join problems the action results in the initial set being increased while in separate problems the initial set is decreased; and both problem types can be varied by changing the quantity that is unknown. Part-part-whole problems involve a constant relationship between a whole and its two separate parts. Unlike join and separate problems there is no action or change that takes place over time. There are only two part-part-whole problem types: either the two parts are given and the solver is asked to find the whole or one of the parts and the whole is given and the solver is asked to find the other part. Compare problems involve the comparison of two separate quantities or sets. Because one set is compared to another, one set is labelled the 'referent set', the other the 'compared set' and the third entity is the 'difference' (Carpenter et al., 1999, p. 9). Figure 1

shows the different types of addition and subtraction word problems classified by Carpenter et al. (1999).

Problem Type			
Join	*Result Unknown*: Ron had 3 toy cars. His parents gave him 2 more toy cars for his birthday. How many toy cars did he have then?	*Change Unknown*: Ron had 5 toy cars. His parents gave him some more toy cars for his birthday. Then he had 7 toy cars. How many toy cars did Ron's parents give him for his birthday?	*Start Unknown*: Ron had some toy cars. His parents gave him 2 more toy cars for his birthday. Then he had 7 toy cars. How many toy cars did Ron have before his birthday?
Separate	*Result Unknown*: Sam had 9 tops. He gave 4 tops to Pat. How many tops does Sam have left?	*Change Unknown*: Sam had 9 tops. He gave some tops to Pat. Then he had 5 tops left. How many tops did Sam give Pat?	*Start Unknown*: Sam had some tops. He gave 4 tops to Pat. Then he had 5 tops left. How many tops did Sam have to start with?
Part-Part-Whole	*Whole Unknown*: Ned has 6 green marbles and 9 orange marbles. How many marbles does Ned Have?	*Part Unknown*: Ned has 15 marbles. 6 are green and the rest are orange. How many orange marbles does Ned have?	
Compare	*Difference Unknown*: Tim has 16 stars. John has 9 stars. How many more stars does Tim have than John?	*Compare Quantity Unknown*: John has 9 stars. Tim has 7 more than John. How many stars does Tim have?	*Referent Unknown*: Tim has 16 stars. He has 9 more stars than John. How many stars does John have?

Figure 1.

Multiplication and Division

When children start learning about multiplication and division they link what they already know about addition and subtraction to these new operations (Anghileri, 2006). Introducing these new operations to young learners with practical experiences, rather than mathematical symbols, will help them establish meanings for the vocabulary associated with multiplication and division. Drawing on the work of Greer (1992), Anghileri (2006, p. 84) states that the multiplication of whole numbers can be represented in: '*repeated sets*, *many-to-one* correspondence, an *array* of rows and columns, and a *many-to-many* correspondence'. In all these situations the number of objects in each group, the number of groups, and the total number are involved in an interrelated mathematical relationship. When the total number is the unknown – that would be a multiplication problem, and when either the number of objects or number of groups is unknown – this would introduce a division problem. When the number of groups is missing in a division example this is also called a division-as-grouping or *partitive* problem; and when the number of

objects in each group is missing this is also called a division-as-sharing or *quotitive* problem (Anghileri, 2006). Anghileri (2006) believes that children need to think about multiplication and division in all these different ways to understand the interrelated mathematical relationships between them.

Carpenter et al. (1999) extended their framework to include multiplication and division – which they call 'grouping and partitioning' problems. According to these authors basic multiplication and division problems are grouped or partitioned into equivalent sets without remainders. These problems involve three quantities. So in the following example: *Tom has 5 bags of marbles. There are 3 marbles in each bag. All together Tom has 15 marbles*, the three quantities are: the number of bags, the number of marbles in each bag, and the total number of marbles. In grouping and partitioning problems any one of the three quantities can be unknown - representing a certain problem type (Carpenter et al., 1999) as shown in figure 2.

Unknown	**Problem Type**	**Example**
Total number of marbles	Multiplication problem	Tom has 5 bags of marbles. There are 3 marbles in each bag. How many marbles does Tom have altogether?
Number of groups	Measurement-division problem	Tom has 15 marbles. He puts 3 marbles in a bag. How many bags can he fill?
Number of marbles in each group (bag)	Partitive-division problem	Tom has 15 marbles. He put the marbles into 5 bags with the same amount of marbles in each bag. How many marbles are in each bag?

Figure 2.

In which type of learning environment is number sense best developed?

The acquisition of number sense is described as a gradual, evolutionary process that is highly personalised and influenced by the context in which mathematics evolves (McIntosh et al., 1992; Howden, 1989; Anghileri, 2006; Shumway, 2011). Children's number sense is developed within a supportive learning environment in which they can share their thinking, talk about ideas (even those that are not fully formed), learn from one another, work through their misunderstandings in front of peers, reflect on their successes and challenges, and be supported on their individual learning paths to number sense (Moore, 1994; Shumway, 2011). For teachers to build such a strong mathematical community, Shumway (2011) believes that three essential components must be in place, viz. math talk, using mistakes as learning

opportunities, and reflection – where learners have time to think about what they did or learnt, or some idea that was new. A poignant example of the impact this can have on a child's learning of number (and in this case specifically related to the observing of patterns) comes from Shumway (2011, p. 132) where a fourth grader, Catie, shared this during a time of reflection: 'The even-odd pattern (when counting in threes). Like 233 is odd, 236 is even, 239 is odd … that was really cool. I didn't notice it until Iliass talked about it.' This 'new idea' prompted Catie to continue looking for patterns when completing counting sequences, which helped develop her number sense (Shumway, 2011).

Number sense does not develop when the teacher's focus is on learners getting the correct answers or using the correct algorithm, but on the students' solution strategies and ways of thinking about the problem (Moore, 1994). Before teaching young learners to use addition and subtraction in symbolic examples and with the use of algorithms, teachers should encourage learners to identify number triples and explain different ways to represent findings using diagrams like the part-part-whole diagram and the ENL (Anghileri, 2006). The same can be said for multiplication and division - by highlighting number triples teachers will help learners understand the abstract relationship between numbers which in turn will encourage efficient calculation (Anghileri, 2006). Teachers' selection of examples must also help children to link new work to what they already know and to discuss the different approaches suggested by learners in the class so that learners become flexible in their approaches to problem solving (Anghileri, 2006; Thompson, 2008).

The phrase 'teaching *for* number sense' used by Moore (1994) brings to mind the idea that number sense is not a topic that can be taught directly, in a once-off manner, and then ticked off as 'done'. Rather, it is an approach to teaching mathematics that should permeate all teaching and learning activities in the mathematics classroom (Moore, 1994).

2.3 Examples

Drawing on the work of Leinhardt (2001), Rowland (2012) describes an 'instructional explanation' as an integral part of the task of teaching. This type of explanation is a pedagogical action that is complete when its constituent parts – like examples,

analogies, and representations – fit together coherently (Rowland, 2012). In this study my particular interest lay in the examples and representations selected by teachers as they do the work of explaining mathematical concepts and procedures related to number. While reviewing literature about exemplification in mathematics I tried to answer the following questions: What is a mathematical example?, Why are examples important in teaching and learning number? and How can teachers' selection and use of examples provide insight into their example spaces? I intend building my discussion about examples around the *what*, *why* and *how* questions listed above.

What is a mathematical example?

Exemplification may be described as 'any situation in which something specific is being offered to represent a general class to which learners' attention is to be drawn' (Bills et al., 2006). According to Watson and Mason (2005) mathematical examples include: *worked examples* which are questions worked through by a teacher or textbook; *exercises* which are questions to be worked on by learners as a means of practicing a specific technique; *representations of classes* used as raw material for inductive mathematical reasoning; *specific contextual situations* that can be used to motivate mathematical th inking; and *illustrations of concepts and principles*.

There are several types of examples identified in mathematics literature. Michener (1978) places examples into four 'epistemological classes', viz. *start-up examples* – which are usually transparent and helps one get started on a new topic; *reference examples* – these are very familiar and are referred to over and over again; *model examples* – are common or general examples; and *counter-examples* – which sharpen the distinction between concepts by showing that a statement is not true (Michener, 1978). *Extreme examples* allow one to 'go the edge of what usually happens within the particular mathematical context', e.g. 6x0=0 shows that multiplication does not always make bigger (Watson & Mason, 2005); while *non-examples* clarify boundaries of mathematical concepts and procedures. How the teacher or learner perceives a mathematical example depends on the individual, the context, and the representation – this is what makes it a special case, a generic example, a counter example or a non-example, rather than the qualities of the example itself (Bills et al., 2006; Watson & Mason, 2005). *Prototypical* examples are

those that are identified quicker as an example of something than other examples (Smith, Shoben, & Rips, 1974). In the literature I reviewed, the importance of prototypical examples are explained primarily in the context of teaching and learning geometry (Hershkowitz, 1989). I believe that the principle behind the caveat in literature regarding overuse of this type of example, i.e. that overexposure to prototypes may impede growth of fuller concept acquisition (Kellogg, 1980; Wilson, 1986), should be heeded with regard to the use of other examples too. When teachers and/or textbooks repeatedly use any example this can take on the nature of a prototypical example in the mind of a learner because a child's acquisition of a mathematical concept is strongly influenced by the examples, non-examples, representations, and contexts in which they have previously experienced this concept. Thus the need for teachers to incorporate a range of examples that works across the situation types discussed by Carpenter et al. (1999)

Sometimes what teachers offer as an example of a process may be viewed by learners as an example of an object (Watson & Mason, 2005; Gray & Tall, 2006). For instance, the equation $y = 2x + 3$ may be offered by the teacher as an example of a representation of a linear function, but learners may see it as an example of a procedure for drawing a graph. What affects the instructional value of an example is whether learners and teachers perceive the intended generality in the examples presented or not (Bills et al., 2006)

Why are examples important in the teaching and learning of mathematics?

Teachers' use of examples to represent a mathematical procedure or an abstract mathematical concept is a common pedagogical practice (Rowland, 2008). The work done by Tall and Vinner (1981) elaborates on the importance of teachers' choices of examples in improving learners' understanding of such concepts and procedures. Vinner (1983, 1991) conceptualised learners' misunderstandings in mathematics as a gap between learners' concept image and the actual concept definition. For these authors, improving learners' understanding of a particular concept implies reducing the gap between their concept image and the concept definition, and examples provided by teachers are vital for closing that 'gap'.

The purpose of a teacher's use of examples in mathematics teaching varies (Rowland, 2008). A teacher may use an example of a procedure as a particular

instance of a generality, i.e. as *an example of* something (Mason & Pimm, 1984; Rowland, 2008; Watson & Mason, 2005). When teaching a concept, the teacher's example selection is aimed at learners abstracting the mathematical concept embodied in the particular example, which is also an inductive process. The examples selected by the teacher should ideally be the result of careful thinking. Here the teacher's choice of example and the variables used (i.e. specific elements or components, like numbers) shows her understanding of the nature of that concept and her awareness of possible variation – knowing which elements may vary within the context of that example and which have to remain the same (Rowland, 2008). Based on Marton and Booth's (1997) idea of 'dimensions of variation' an aspect only becomes available to learn through discerning variation, thus the role of variables in mathematical objects and how they are varied across the example space is important for learning.

Teachers may also use examples as *an example for* something – these are usually called 'exercises' and are not examples used in an inductive manner (like for abstracting or generalising) but for reasons of illustration and practice (Rowland, 2008; Watson & Mason, 2005). So, if a teacher has recently explained to her Grade 3 class how to do column subtraction with decomposition, she may then want to give her learners a few examples as an exercise to help them remember the procedure and to gain fluency with it. Here too the teacher's choice of examples and variables is not arbitrary. These examples must ideally be graded from relatively easy to more challenging (so that learners experience success and gain confidence with the procedure). These exercise examples should also expose learners to the range of problem types that they may encounter (Rowland, 2008).

Examples have been used in many ways with regard to the teaching and learning of mathematics, including the concept of number sense. In some approaches the sequence or succession of examples is the important feature of their use (Bills et al., 2006). When thinking in terms of variation, the succession of examples and the aspects which are varied in that succession are important in affording learners access to key features of a concept or technique (Watson & Mason, 2006; Bills et al., 2006). Within this, it is important for the example space to range across different conceptions of key ideas – for the examples of subtraction presented to provoke the

need for viewing subtraction as 'take away' (e.g. 11-2) and as 'difference' (e.g. 11-8). Ultimately, learners make sense of mathematics by generalising and abstracting from the examples made available to them. Therefore teachers' choice of examples is crucial in determining what mathematics their learners can learn (Watson & Mason, 2005)

How can teachers' selection and use of examples provide insight into their example spaces?

According to Goldenberg and Mason (2008), the notion of 'a space of examples' is not new but has been used within mathematics literature by researchers such as Michener (1978) and Zaslavsky and Peled (1996) – although the use of the term *example spaces* has been popularised by the work done by Watson and Mason (2002, 2005). Watson and Mason (2005) believe that examples do not exist in isolation in one's mind but are interconnected and can be seen as members of a structured 'space', which they call a *personal example space*. The work done by Watson and Mason with regard to personal example spaces primarily refers to learners, but in my study I have extended this to refer to teachers who themselves are lifelong learners.

Watson and Mason (2005) have summarised a number of features of what constitutes an example space. In this section, I draw primarily on their work, with additional references drawn in where useful. The content and structure of teachers' example spaces are individual and situational. Where teachers have similar example spaces, how they access these can be different and therefore what they are able to access at a particular time also differs. What teachers have access to within their example space depends on many things, including: their recent experiences – so for the participants in my study, attending the WMC-P 20-Day course was a recent experience; the wording of particular prompts – sometimes a particular word is seen as a cue for a certain procedure; their inclination towards something – like whether they enjoy teaching a certain topic in mathematics or not; their assumptions about the topic – some teachers think that 'data handling' is too challenging and thus omit it from their teaching plan; and the particular situation – like the age of the learners in the class.

Teachers' example spaces can be both beneficial and limiting with regard to teaching and learning mathematics. The examples a teacher offers her class regarding a particular topic will come from her own example space. For example, whenever I thought of the concept 'hexagon' the image that came to my mind was of a shape with six equal sides, one side being parallel to the base of the book (figure 3). As a teacher to Grade 4 learners, my limited example space and concept image of a hexagon at that time caused me to only present the same example of a regular hexagon to my learners without ever showing them an example of an irregular hexagon, or a hexagon in an 'off-kilter' orientation (figure 4). Would the learners I taught be able to recognise figure 4 as a hexagon if this concept image was not part of their example space? I wonder, because until recently I certainly did not.

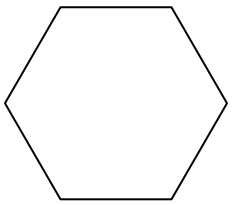


Figure3.

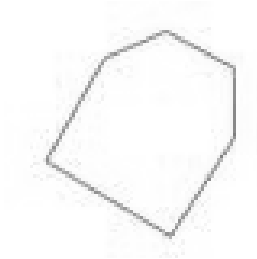


Figure4.

Similarly, a teacher's example space concerning number and number operations determines what she will make available to learn when teaching number topics (Watson & Mason, 2005). For example, in the context of teaching addition and subtraction, if a teacher only uses the 'join' and 'separate' conceptions of these operations this may suggest that only these two conceptions make up that teacher's example space and concept image for addition and subtraction. However, Zazkis and Leikin (2007) caution that the adage 'absence of evidence is not evidence of absence' applies to our understanding of teachers' example spaces too. Therefore, if a teacher only uses the 'join' and 'separate' conceptions of addition and subtraction it does not mean that the 'part-part-whole' or 'compare' conceptions of these operations are not part of her example space. It simply means that 'join' and 'separate' are the conceptions of addition and subtraction that she has access to from her example space in that situation, at that time. The collection of examples that a teacher has access to at any moment can be referred to as her *accessible example space* which is again a 'situated' space, i.e. dependant on many factors including the context, the trigger, and the state of the individual (Goldenberg &

Mason, 2008). Rowland (2008) maintains that a teacher's choice of example can show that she is aware of certain things, and I would like to discuss this next.

Rowland's (2008) four categories of exemplification – *variables*, *sequencing*, *representations*, and *learning objectives* – highlight particular aspects of teachers' awareness when selecting and using examples. These categories are not distinct – meaning that a teacher's use of an example in a particular situation can show more than one type of awareness. Teachers' selection of examples must *take account of variables* because examples of most mathematical objects consist of two or more parts, or variables, and by taking account of variables learners will be exposed to a range of possible problem types. What is important here is that the role of different variables must be clear, especially when teaching a new concept or procedure. Teachers' examples must also *take account of sequencing* and according to Van Patten, Chao, and Reigeluth (1986) there are two steps involved when a teacher wants to take account of sequencing: firstly, she must identify what elements are to be sequenced, and secondly, what organising principle will be used to sequence these. These authors delineate Merrill's (1983) five specific prescriptions for micro-sequencing which I found useful when analysing participants' sequencing of examples. These organising principles are discussed in greater detail in Chapter 3 of this report. Rowland (2008) also highlights the manner in which teachers' examples are *represented* because this can provide learners with more or less generality in relation to access to that concept or procedure being taught. (Representations are discussed in more detail in the following section). Finally, Rowland argues that teachers' examples should be tailored to the *learning objective* because this is the ultimate purpose for which they are used. Given that teachers' use of representations has been the focus of some discussion in the South African terrain (Askew, Venkat, & Mathews, 2012; Ensor et al., 2009), I break down my focus on representations into categories that have been highlighted within the literature.

2.4 Representations

My review of representations in the context of mathematics education is structured around representational flexibility, representational connections and representational progression.

Representational Flexibility

A key theme in the literature on mathematical representations is the idea of 'representational flexibility'. According to Graham, Pfannkuch, and Thomas (2009, p. 682) facets of the 'flexible use of representations, in particular establishing meaningful links between and amongst representational forms and translating from one representation to another, has been referred to by a number of terms, such as *representational fluency* (Lesh, 1999), *representational competence* (Shafrir, 1999) and *representational flexibility*'; while Nistal, Van Dooren, Clarebout, Elen, and Verschaffel (2009) refer to this as 'representational versatility'. In my research study I will use the term *representational flexibility* to refer to teachers' and learners' ability to seamlessly transition between different representations in their attempt to explain or construct mathematical understanding.

Teachers must develop a 'fruitful representational context' (Ball, 1993) to foster learners' mathematical thinking which (in the context of a mathematics class) means that the teacher should have more than one or two representations in her 'repertoire' when teaching any concept or procedure. Further, teachers need to consider the nature of the mathematical content to be taught and the characteristics of the learners who will learn that content (Perkins & Unger, 1994) when planning a lesson and deciding on appropriate examples and representations to use. Particular representations often foreground particular aspects of the given mathematical concept while obscuring other equally important aspects (Ball, 1993; Dreyfus & Eisenberg, 1996) while some representations might be more adequate as expressions of knowledge and as thinking tools than others for particular mathematical concepts (Cobb et al., 1992). Taken together this research points not only to the importance of the teacher being able to flexibly move from one representational form to another, but also to the teacher having an understanding of the different advantages and limitations a particular representation can have on the process of teaching and learning.

Cobb et al. (1992) suggest that teachers should not use representations in a rigid manner as this could lead to 'algorithmatization' of mathematics and could result in learners not applying their mathematical understanding gained in school in other out-of-school settings. Some researchers state that teachers should not impose their

expert representations on learners but should rather draw on students' prior knowledge and experiences to guide the negotiation of initial representations (Cobb et al., 1992; Terwel et al., 2009). Cobb et al. (1992) argue that representations used by teachers only have educational value to the extent that they facilitate learners' own construction of mathematical understanding; therefore they suggest that learners should co-construct representations to be used in problem solving.

The work done by Nistal et al. (2009) provides a critical look at literature concerning flexible representational choice in the teaching and learning of mathematics. These authors argue that traditional research on representational flexibility had a narrow focus, i.e. it only focused on the importance of matching the representation to the given task, while the personal characteristics of the learner and the characteristics of the context where learning was to take place, was over-looked (Nistal et al., 2009). These authors do not disagree with the importance of task characteristics in the selection of representation, but add that: *subject characteristics* such as learners' prior conceptual and procedural knowledge of representations, learners' domain-specific knowledge, and learners' representational preference; and *context characteristics* like an environment which provides active comparison and evaluation of representations and guidance in selection of representations also influence representational flexibility (Nistal et al., 2009).

When it comes to the number of representations that are considered appropriate for teachers to use, Askew and Brown provides some insight. These authors state that the trend among teachers in the UK is to use a wide range of representations when teaching a concept while teachers from other countries tend to use 'a few, well-researched and evidence-based representations such as the empty number line' (ENL) (Askew & Brown, 2003, p. 12). The suggestion here is that teachers should use a limited but effective set of representations in a structured and systematic way rather than a wide range of representations that are not coherently connected.

Representational Connections

The importance of establishing connections between different mathematical ideas when teaching numeracy using a range of representations is discussed in the work done by Askew et al. (1997) and Ball and Bass (2003). Similarly, the body of work done by Haylock and Cockburn (2008) speak of the importance of establishing

connections between language, pictures, symbols and concrete experiences when teaching and learning number in the early primary years. Haylock and Cockburn's (2008) framework of significant connections in the understanding of number as well as Askew et al.'s (1993) notion of effective teachers of numeracy being 'connectionist teachers' will form the basis of my discussion on representational connections.

Askew et al. (1993) argue that 'connectionist teachers', i.e. teachers who worked with learners' existing understandings and connected different mathematical ideas when teaching number (e.g. teaching fractions, decimals and percentages together rather than as separate topics), had classes with greater learner gains than teachers who were not considered 'connectionist'. On closer inspection one would discover that these 'connectionist' teachers believed that when teaching a learner to be numerate the teacher's duty was to make learners aware of a range of solution strategies to a mathematical problem (and thus different ways of representing the problem) so that the learner can identify a strategy and the accompanying representation that she is comfortable using (Askew et al., 1993). Connections between mathematical ideas were made possible as these 'connectionist' teachers worked actively with learners' explanations: drawing attention to similarities and differences between methods offered by learners, while always encouraging learners towards more efficient methods of calculation (Askew et al., 1993).

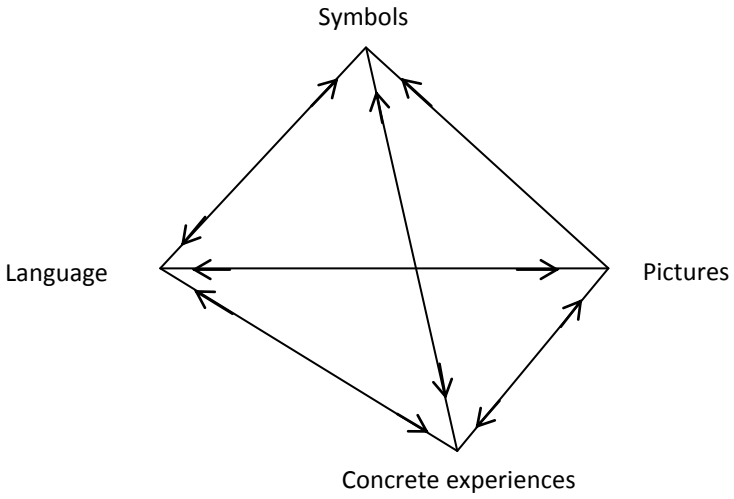


Figure 5: Significant connections in understanding numbers and number operations (Haylock & Cockburn, 2008)

Haylock and Cockburn (2008) have developed this simple model to demonstrate their understanding of the significant connections learners need to build in order for them to understand number and number operations.

These authors believe that for young learners to understand the concept of number they have to build a network of connections between the *symbol* (e.g. the numeral 3), the *language* (e.g. the word and number name: three), a *picture* or mental image (e.g. seeing 3 on a number line), and *concrete experiences* (like counting 3 hops or 3 counters). When learners build meaningful connections between these key mathematical experiences, and as they connect their new experiences in number to a web of previously connected experiences, their understanding of number and number operations is enhanced (Haylock & Cockburn, 2008). Numbers can be used in different ways: as labels for identification purposes, i.e. the nominal aspect; as labels for putting things in order, i.e. the ordinal aspect; and as indications of how many there are in a set of things, i.e. the cardinal aspect (Haylock & Cockburn, 2008). The work done by Haylock and Cockburn (2008) highlight the importance of young learners understanding the ordinal aspect of number, for example that 3 lies between 2 and 4. For this understanding to be developed in young learners these authors suggest that the image of the number line is one that strongly shows that 3 is a point on that number line that lies between 2 and 4. Linking back to the range in the example space mentioned earlier, Haylock and Cockburn (2008) warn primary teachers against familiarising young learners with only the cardinal aspect of number, e.g. that 3 represents a set of three things, as this is a limited view of number which can cause problems when learners have to operate on negative numbers later on.

In the South African context, representational connections are important to look at in the face of evidence of disconnections and limitations between teacher representations/explanations and the examples they are used with (Venkat & Adler, 2012), and limitations produced through ambiguity in teacher talk as they work with representations (Venkat & Naidoo, 2012).

Representational Progression

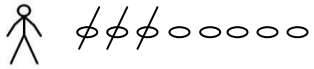
Carpenter et al. (1999) argue that young learners naturally try to solve problems by modelling the action or relationships in problems. For example, if learners are given

the problem: Ben has 3 sweets and his brother Matt has 4 sweets. How many sweets do they have altogether? Using counters or drawings, young learners will show Ben's 3 sweets and Matt's 4 sweets and after joining these, all the 'sweets' will be counted to find the answer.

If learners are given the problem: Sam has 8 marbles and gives 3 marbles to Jeff. How many marbles does Sam have now? Young learners will probably draw Sam with 8 marbles like this:



Thereafter, 3 of the 8 marbles will be scratched out like this to find the answer.



Carpenter et al. (1999) believe that this direct modelling, which usually requires physical or concrete representations of the problem, is an intuitive problem solving technique for very young learners, but argue that as children's mathematical thinking progresses they should move on to counting strategies in which the actions or relationships described in the problem become less explicit. When children learn to use counting strategies like 'counting on' or 'skip counting' their representation of the problem is more abstract and they are able to find solutions to problems involving larger numbers more efficiently.

The work done by Ensor et al. (2009) in the Cape Peninsula show that Foundation Phase learners continue using concrete representations when learning number and number operations from Grade 1 to Grade 3. As discussed previously, Carpenter et al. (1999) believe this to be a natural starting point in young learners' attempts at problem solving but should be replaced by more efficient strategies as learners' mathematical thinking matures. Having ordered the types of representations teachers' use from concrete (like counters, blocks, fingers, etc.) to more abstract representations (like symbols), Ensor et al. (2009) have found that teachers at these South African schools persistently used concrete apparatus when teaching number and number operations which negatively affected learners' conceptual understanding of number. These authors argue for teachers to 'specialise' pedagogic texts (i.e. oral

communication, written communication, bodily movements, and physical apparatus) when teaching number by progressing from using concrete to using more abstract representations when teaching (Ensor et al., 2009). So, if Grade 2 learners are given the sum 18+9= □, rather than making counters (concrete representation) available for learners to use, which would be time consuming and also encourage unit counting, teachers can, where appropriate, present models such as number lines (more abstract representation) which facilitates more efficient strategies like 'counting on' and working with 'friendly numbers' (i.e. multiples of 10). The complexities in selecting appropriate junctures in which more abstract models can be introduced is noted in Askew et al. (2012), where the teacher's jump to a more abstract symbolic manipulation of a missing addend task simply leaves learners confused.

2.5 Concluding comments

The literature reviewed points to the kinds of elements that would show good use of examples by teachers as: *taking account of variables* by presenting learners with different conceptions of number problems using an appropriate number range (Carpenter et al., 1999; Rowland, 2008; Watson & Mason, 2005); *taking account of sequencing* by presenting examples in a controlled or in a random sequence using any of Merrill's organising principles (Rowland, 2008; Van Patten et al., 1986); and, *taking account of representations* in a flexible, connected, and progressively more abstract way (Askew et al., 1997; Ball & Bass, 2003; Cobb et al., 1992; Haylock & Cockburn, 2008; Heize et al., 2009). In relation to developing learners' number sense more broadly, the literature reviewed show that teachers build number sense in learners through the use of counting strategies that help learners grasp the logical structure of numbers, the magnitude of numbers and the relationship between numbers whilst also moving them along the learning trajectory for counting (Anghileri, 2006; Shumway, 2011; Wright et al., 2006). Number sense is also developed when teachers' examples present number operations in a flexible way by using different conceptions of the operation as categorised by Carpenter et al. (1999) or by varying the 'unknown' therein so that learners are exposed to a range of problems. The body of work reviewed by Askew and Brown (2003) also points to teachers motivating and equipping learners to use mental strategies with the aid of visual representations like the number line and methods like the partitioning of

numbers so that they can develop efficiency in calculation which is a hallmark of number sense.

Literature reviewed also described the acquisition of number sense as a continual personalised process that is best developed in a supportive learning environment, therefore elements such as math talk, using mistakes as learning opportunities, and reflection are expected to be a part of the learning atmosphere in teachers' classes (Shumway, 2011).

CHAPTER THREE: ANALYTICAL FRAMEWORK

3.1 Introduction

The focus of my study, i.e. on teachers' selection and use of examples and representations when teaching number, is broadly driven by an interest in teachers' example spaces. Personal example spaces can be both beneficial and limiting when teaching and learning mathematics (Watson & Mason, 2005), therefore the examples that teachers use when teaching number are important as these directly influence learners' example spaces and thus the examples that learners have access to as they solve problems relating to number. Following on from the literature detailed in the previous chapter, and using the work done by Rowland (2008) in particular, the main concepts that form the analytical framework of this study are teachers' use of *variables, sequencing,* and *representations* in their choice of examples used when teaching *number-related topics*. Rowland's last category of exemplification, *learning objective*, is not included in my framework as I did not have access to teachers' lesson plans and thus did not know what their intended learning objective for each lesson or episode was.

3.2 Teachers' use of variables in examples

Developing Marton and Booth's (1997) idea of 'dimensions of variation' further, Watson and Mason (2005) argue that the 'dimensions of possible variation' and 'range of permissible change' within mathematical concepts being taught should be reflected in the examples provided by the teacher. These examples, in line with this theory, should also expose learners to variation in the range of types of problems that they may encounter.

Using the body of work done by Rowland (2012), I intend analysing teachers' examples by taking into account the variables or components described as relevant to the concept being worked with in the literature, and examine the ways in which the empirical example space makes these variables available to learn. I now link identifying dimensions of variation to the topics in focus in the empirical data, viz. counting strategies, mental strategies, and number operations.

Counting strategies

The variation of examples teachers use when teaching oral and written counting is expected to help children gain insight into: the logical structure of numbers, e.g. the base ten structure of our number system (Wright et al., 2006); the relationships between numbers, e.g. that 12 is *smaller than* 13 and *bigger than* 11 (Shumway, 2011) and recognition of patterns in our number system, e.g. the odd – even pattern of consecutive numbers (Anghileri, 2006). The recognition of patterns also helps young children establish benchmarks (these are initially multiples of 5 and 10) which assists them in efficient calculation – efficiency being one of the hallmarks of number sense (Kilpatrick, Swafford, & Findell, 2001). The variables within the examples teachers use in oral and written counting is also expected to help children grasp the magnitude of numbers and move children along the learning trajectory for counting which is broadly agreed upon by various researchers as: 'count all, count on from first, count on from larger, use known facts and use derived number facts' (Anghileri, 2006; Askew & Brown, 2003; Carpenter & Moser, 1984). The pace at which teachers are expected to move children along this counting trajectory, and the accompanying number range used by teachers, is expected to be guided by the specifications within the curriculum and by the learners' ability - because the development of number sense is a highly personalised process (Anghileri, 2006; DBE, 2011a). By varying how examples are represented when teaching counting, teachers can guard against overemphasising the cardinal aspect of number which is a very limited view of what numbers are (Haylock & Cockburn, 2008). Teachers are expected to use images that represent the logical structure of numbers, like the number line and bead string, as these relate closely to the counting sequence (Anghileri, 2006). As learners establish benchmarks during counting in variables of 5 and 10 this in turn also helps them to do mental calculations, which is discussed next.

Mental strategies

Mental images that children bring to mind as they visualise a solution to a number problem are significantly influenced by the physical representations their teachers use (Askew & Brown, 2003). Therefore I will expect to see teachers varying the type of examples and representational forms they use when teaching number, as these

build learners' visual, perceptual and conceptual understandings of quantity which is important for encouraging learners' mental strategies (Shumway, 2011).

The type of examples offered by teachers are not expected to encourage the early use of algorithms that focus on a digit column value because this detracts learners from using mental strategies (Askew & Brown, 2003). The use of images relating to 'tens sticks' 'unit cubes' and Dienes' blocks that are closely related to column arithmetic is thus anticipated to be delayed. Teachers are expected to rather offer examples that encourage learners to use number bonds (especially bonds of 10) as this is an important part of developing their ability to perform mental calculations successfully (Anghileri, 2006; Askew, 1998). The examples offered by teachers are also expected to encourage the partitioning of numbers in various ways (Askew & Brown, 2003). Learners are not expected to only be encouraged to partition numbers according to place value e.g. 358 = 300+50+8, but also be encouraged to partition numbers in ways that encourage fluent computation e.g. 75+76 can be partitioned as 75 + 75 + 1, which would make the problem easier to calculate.

Number operations

By taking account of variables when teaching number operations teachers will avoid rigidity in learners' use of algorithms to solve number problems. In my analysis of teachers' selection and use of examples when teaching number operations I will consider whether examples offered by teachers restricts learners' understanding of the operation or if the examples encourage flexible use of operations. For example, teachers' examples that only present subtraction problems as 'take away' problems will be considered restrictive, whereas examples that present subtraction problems using both the 'take away' and 'difference' conceptions would be considered flexible. Subtraction and addition examples presented using the 'join', 'separate', 'part-part-whole' and 'compare' conceptions detailed in the work done by Carpenter et al. (1999) will also be considered as encouraging flexible use of number operations. The same applies to multiplication and division problems as outlined in Chapter 2. As noted previously, I acknowledge that these authors developed their classification scheme in relation to word problems yet I have found it useful when analysing abstract/symbolic examples as well. Using the work done by Carpenter and Moser

(1984) I will also consider progression in the addition and subtraction strategies used by teachers.

3.3 Teachers' use of sequencing

Rowland (2008) maintains that teachers' examples must also *take account of sequencing* as examples are usually presented in a predetermined 'graded' sequence so that learners experience success with routine examples before trying more challenging ones. Although the sequence of most examples are controlled, examples can also be presented in a random sequence especially during interactive teaching (Rowland, 2008). Venkat and Naidoo (2012), as noted already, have pointed to 'random variation' which tends to obscure the connection needed for surfacing concepts in sequencing.

To elaborate on the notion of sequencing of examples I now draw on the work of Van Patten et al. (1986) who suggest that teachers need to take two basic steps when taking account of sequencing. Firstly, teachers will be expected to identify the elements to be sequenced by asking themselves 'What is to be sequenced?', and secondly, teachers are expected to select an organising principle by asking themselves 'How will it be sequenced?' (Van Patten et al., 1986). In the context of my study, teachers' worked examples and learner exercises will be the main elements considered for sequencing. According to Van Patten et al. (1986), Merrill's (1983) Component Display Theory provides a theoretical paradigm that identifies five specific prescriptions regarding micro-level sequencing – which concerns the organisation of generalities, examples, and practice exercises when teaching a particular content idea (Van Patten et al., 1986). These are:

1. presenting the 'worked example' or general rule before the learners' exercise for near transfer of the concept learned
2. presenting the learner exercise before the rule or generality for far transfer of the concept learned
3. arranging examples in a divergent sequence (i.e. make successive examples different from each other in some way)
4. arranging examples in an easy-to-difficult sequence
5. providing similar non-examples matched to examples

I now turn my attention to teachers' use of representations when teaching number.

3.4 Teachers' use of representations

In terms of my focus on representations, the literature reviewed identifies three broad categories of importance which form part of the analytical framework of my study: representational flexibility, representational connections, and representational progression. The detail of what constitutes these categories is discussed in Chapter 2 of this report. Here I additionally focus on what can vary within representational flexibility, connections and progression in relation to number sense.

I will expect to see teachers making use of a few researched-based representations in a flexible and coherent manner as they teach number (Askew & Brown, 2003; Ball & Bass, 2003; Heize et al., 2009). The connections that teachers establish between different representational forms like words, pictures and symbols will be analysed using the work done by Askew et al. (1997) and Haylock and Cockburn's (2008) model; while the progression within representations that literature suggests is needed to build good number sense will be analysed using the work done by Carpenter et al. (1999) and Ensor et al. (2009).

The table that follows provides an overview of the analytical framework developed for this study.

3.5 Table 1: Analytical Framework

<table>
<tr><th colspan="4">Rowland's (2008) Categories of Exemplification</th></tr>
<tr><th colspan="3">Variables</th><th>Sequencing</th><th>Representations</th></tr>
<tr><td>Counting strategies</td><td>Addition and Subtraction</td><td>Multiplication and Division</td><td rowspan="2">• usually graded
• controlled and/or random
• 2 steps regarding sequencing:
a) identify elements to be sequenced
b) select an organising principle

<u>Merrill's 5 specific prescriptions:</u>
1. rule before examples
2. examples before rule
3. divergent sequence
4. easy-to-difficult sequence
5. non-examples matched to examples</td><td><u>Flexibility</u>
• teacher to use multiple representations of:
+ & – (no. line, ENL, part-part-whole)
x & ÷ (repeated sets, arrays, many-to-one correspondence, many-to-many correspondence)
• use a few research-based representations
• flexibly shift between different representations
• extent of coherence of representations

<u>Connections</u>
• establish coherent connections between words, pictures, symbols and actions on objects
• establish connections between different ideas within and across mathematics topics
• connect representations to nature of task, characteristics of learners, and content
• connections between number operations (+ & –) and (x & ÷) as inverse operations

<u>Progression</u>
• direct modellers progressed to counting strategies and more abstract reps of number
• Ensor et al's (2009) categories of progression:
concrete – counters, beads
iconic – drawings, cartoons
indexical – dots, tallies
symbolic no. based – number line, 100 chart
symbolic syntactic – number sentences</td></tr>
<tr><td><u>Progression</u>
*count all
*count on
*count on from larger
*using known facts: doubles, no. bonds
*using derived facts: skip counting, near doubles, bridging thru 10, partitioning

**mental calculation strategies are included under 'using known facts' and 'using derived facts'</td><td>Problem Type
<u>Join</u>
*result unknown
*change unknown
*start unknown

<u>Separate</u>
*result unknown
*change unknown
*start unknown

<u>Part-part-whole</u>
*whole unknown
*part unknown

<u>Compare</u>
*difference unknown
*compare quantity unknown
*referent unknown

Progression
<u>Addition</u>
see counting strategies col.

<u>Subtraction</u>
*count out
*count back from – 'take away'
*count down to – difference
*count up from
*use known and derived facts</td><td><u>Multiplication</u>
*total number of objects unknown

<u>Measurement division</u>
*number of groups unknown
*division-as-grouping

<u>Partitive division</u>
*number of objects in each group unknown
*division-as-sharing</td></tr>
</table>

CHAPTER 4: METHODOLOGY

4.1 Introduction

Opie (2004, p. 33) describes educational research as 'the collection and analysis of information on the world of education so as to understand and explain it better'. Educational research can also be described as a systematic study of a topic of interest using data gathering techniques aimed at answering questions related to that topic (Leedy & Ormrod, 2010). The aim of this research study was to explore Foundation Phase teachers' selection and use of examples and representations when teaching number-related tasks. This was achieved by collecting and analysing data from teachers' course-work from early number pedagogy tasks in the 20-Day course, and two lesson presentations that related to teaching number in the Foundation Phase.

To answer my research questions I chose to employ a qualitative case study research approach which has been widely used in the field of education for approximately 30 years (Merriam, 2001). I gathered data in order to answer these research questions by: observing participants' interaction during the in-service training course and recorded these using field notes; document analysis of the participants' course-work tasks that related to the teaching of early number (2 tasks); lesson observations of 2 non-consecutive lessons presented towards the end of the 2012 academic year by each participant using video recording and field notes, which were later fully transcribed to capture all teacher talk and teacher-learner interaction, writing on the board and presentation or hand out of examples. In this chapter I discuss the design of my study, my sample selection, my data collection methods and the sources of information used, the way I organised the information and data collected, how I analysed the data collected, the steps I took to ensure the rigour of my study, and how I dealt with ethical issues related to my research.

4.2 Research design

Based on a constructivist paradigm, I believe that 'reality' is complex and multi-faceted, and that everyone (including teachers) constructs knowledge in a unique way using the experiences, tools and information available to them at that particular time. In my small-scale study I have attempted to illuminate the reality of two

teachers attending the WMC-P Project's 20-Day course in relation to their selection and use of examples and representations when teaching number.

I adopted a qualitative approach to my research – which involves studying phenomena that occur in natural settings in all their complexity (Leedy & Ormrod, 2010) – because I believed that this approach would best help me understand the experiences of the participants in my study 'as nearly as possible as (they) feel it or live it' (Sherman & Webb, 1988, p. 7). This naturalistic, in-depth inquiry was an interactive process wherein the participants shared information about themselves and their experiences with me (the researcher) in a context that was not contrived, but natural (Blaxter, Hughes, & Tight, 1996).

I chose to use the case study method of inquiry because this provided me with an opportunity for in-depth study of a bounded system, which in this case was two Foundation Phase teachers from a particular school who were attending the WMC-P projects' in-service training course (Merriam, 2001). A case study method of inquiry also suited my study because I was not interested in making broad generalizations about my topic of interest but rather wanted to generate 'thick description' (Geertz, 1973, as cited in Lincoln & Guba, 1990) which is needed to understand complex issues such as the topic of my study (Opie, 2004). For any case study to provide a rich account of the phenomenon studied it usually requires extensive data collection on the unit of study and on the context surrounding the case (Leedy & Ormrod, 2010). For this reason I had extended contact with research participants for the duration of the 20-Day course, and I used various data-gathering strategies like observations and document analysis to gather data.

One limitation of the method of inquiry that I chose to use for my study is the 'lack of rigor (which) is linked to the problem of bias … introduced by the subjectivity of the researcher' and others invested in the study (Hamel, 1993, as cited in Merriam, 2001, p. 43). This limitation is usually attributed to the fact that the researcher is the primary data collection instrument. To challenge the critique of reliability, validity, and generalizability issues related to my study, I draw on the work of Bassey (1981) who argues that the 'relatability' of a systematic and critically conducted case study is what makes it a valid form of educational research, rather than the generalizability thereof. Here 'relatability' refers to the extent to which a teacher in a similar

situation can relate her decision making to that described in the case study, thus making 'thick description' of the context and the unit of study imperative (Bassey, 1981). Other measures that I put in place to ensure the trustworthiness of the data collected during my study, and the subsequent analysis thereof, is discussed in more detail later in this report under the sub-heading 'Rigour'.

4.3 Sample

I used purposive sampling for my qualitative case study research as opposed to random sampling. Marshall (1996, p. 523) argues that through purposive sampling a researcher 'actively selects the most productive sample to answer the research question(s)'. These participants are those the researcher considers 'richer' than others and are the most likely to 'provide insight and understanding for the researcher' (Marshall, 1996, p. 523).

I invited 2 of the 38 teachers attending the WMC-P Projects' 20-day in-service training course to participate in my study. Zelda[1], who teaches a Grade 1 class (and is the Foundation Phase Head of Department), holds an Honours degree in Education and has English as her home language while Deborah[2], who teaches a Grade 2 class, holds a Higher Diploma in Education (Junior Primary) as her highest tertiary qualification and speaks Northern Sotho as her home language. Both teachers have been teaching in the Foundation Phase for more than 5 years. These teachers were selected as participants for my study because: their pre-test scores on a primary mathematics content knowledge test differed (one was high and the other low); they both taught in the Foundation Phase although not in the same grade; they worked at the same school; the LoLT of their school is English; and they were willing to participate in my study.

My focus on Foundation Phase teachers with a sample selection based on differing levels of content knowledge (CK) was motivated by literature which points to links between CK and pedagogical content knowledge (PCK) (Rowland et al., 2005) and findings of weak CK among South African teachers (SACMEQ III; Carnoy & Chisholm et al., 2008). The importance of developing early number skills in young learners is well documented in the field of mathematics education (Anghileri, 2006; Haylock &

[1] Anonymised
[2] Anonymised

Cockburn, 2008) – yet research evidence on the South African landscape show how pedagogic practices of Foundation Phase teachers often inhibits their learners' acquisition of number sense (Askew et al., 2012; Ensor et al., 2009). My sample selection was also pragmatic - by selecting participants who teach at the same school I only needed permission to gain access to one site – this made arranging for data collection easier – and with English as the LoLT of the school I could analyse data as I collected it without having to wait for translation.

4.4 Data collection

After receiving permission from the relevant authorities and participants, I collected and generated data using the aforementioned strategies. After reviewing different data-gathering instruments, I decided to use the following: field notes made during the 20-Day course and lesson observations, transcripts of recorded lessons, and document analysis of participants' course-work. A short discussion and motivation for the use of these data collection strategies and instruments used in my study follows.

Document analysis

Document analysis can be described as a systematic procedure in which printed and electronic material are reviewed and evaluated by a researcher in an attempt to answer specific research questions (Bowen, 2009). Documents that may be used for such systematic review and evaluation can take many forms such as: various public records; photographs; books and journals; agendas and minutes of meetings; newspaper and magazine clippings; maps and charts; individual and organisational reports; transcripts of events; and many more (Bowen, 2009; Merriam, 2002). Document analysis yields data such as description of events, explanation of processes, quotations, and excerpts that are subsequently organised into themes and categories (Labuschagne, 2003, as cited in Bowen, 2009). The analytic procedure involved in document analysis involves 'finding, selecting, appraising (making sense of), and synthesising' data contained in documents (Bowen, 2009, p. 28). In my study document analysis was used to complement the other forms of data collection although literature suggests that it can also be used by itself as a data-collection strategy (Bowen, 2009).

According to Bowen (2009) qualitative researchers ordinarily aim to draw on multiple sources of evidence through the use of different data-gathering strategies and techniques. I used observation together with document analysis to validate and corroborate data gathered in my qualitative study (Bowen, 2009) which allowed me to engage in 'triangulation' of data. According to McMillan and Schumacher (2003, p. 374) triangulation includes the 'cross-validation among data sources, data collection strategies, (and) time periods' which is often used by researchers to find regularities or recurring patterns in the data. Drawing on the work of Eisner and Patton, Bowen (2009) argues that qualitative researchers use the triangulation of data to ensure the credibility of their study – to guard against the accusation that their study's findings are the result of a single investigator's bias. How I dealt with the issue of bias in my study will be discussed in more detail later in this chapter under the sub-heading 'Rigour'.

The documents I examined for my study were: participants' 20-Day course pre-tests; participants' course-work tasks; field notes taken during the course and during lesson presentations; and transcripts of lessons presented by participants.

I chose to use document analysis as a data-gathering strategy in my study because it could yield rich descriptions of, and give me insight into, my participants' choice and use of examples and representations when teaching number (Merriam, 2001; Stake, 1995; Yin, 2003). Document analysis was also selected as a suitable data-gathering strategy for my study because: it could provide 'contextual richness' (Bowen, 2009); it could be used to highlight conditions that affect the participants in my study (e.g. the Foundation Phase CAPS documents' suggestions and/or prescriptions to teachers regarding the use of examples and representations when teaching number); and more importantly, because it could be used to corroborate evidence from other sources (Bowen, 2009). Given its efficiency and cost-effectiveness – the advantages of using document analysis in my study outweighed the limitations usually associated with it, viz. low retrievability and insufficient detail (Bowen, 2009).

Observation

Instead of relying on participants' self-report which is open to bias, I decided to observe and record participants' selection and use of examples and representations

when teaching number, first-hand. I asked participants if I could observe and record two lessons on any topic related to number and we selected dates and times that would fit into their current teaching schedule without disrupting their curricular duties. I observed participants during the 20-Day course and during lesson presentations at school and found this to be a very useful data collection strategy because it gave me direct access to participants' use of examples and representations when teaching number; and also because it helped me understand the setting wherein the participants worked (Wiersma & Jurs, 2004).

The role an observer plays in a research setting can vary: ranging on a continuum from complete participant to complete observer (Cohen & Manion, 1999; Fraenkel, Wallen, & Hyun, 2012). While observing participants' interaction during the 20-Day course I took on the role of a *participant observer* because I was a tutor on the course and as such I engaged in some of the activities I set out to observe. But, during my observation of lessons I was a *non-participant observer* because I did not take part in the lessons presented but only recorded my observations thereof (Cohen & Manion, 1999).

Any role a researcher assumes when she observes participants has advantages and disadvantages associated with it (Cohen & Manion, 1999; Fraenkel et al., 2012). Therefore I carefully considered the advantages and disadvantages associated with different roles before deciding on which role to assume for each observational setting. During my observation of participants in the 20-Day course I assumed the role of a participant observer because this helped me build a rapport with the teachers participating in my study - which proved invaluable during the latter parts of my data gathering process. During lesson observations I took on the role of a non-participant observer (or complete observer) because this was less likely to affect the actions of the participants being studied (Fraenkel et al., 2012).

Despite the advantages of the different observer roles I assumed, I was aware of the impact the 'observer effect' has on the behaviour of those being studied and consequently on the data gathered (Fraenkel et al., 2012) therefore I established a rapport with participants in the context of the course before using a more 'invasive' data-gathering technique like lesson observations. I was also aware of how 'observer bias' and 'observer expectations' could affect what I perceived (Fraenkel et al., 2012)

and I tried to alleviate this by using the video-recorded lesson transcripts to check the accuracy of my field notes made on site.

I recorded my observations using field notes taken during and immediately after the event observed in which I described participants' actions in as much detail as possible (Schumacher & McMillan, 1993). According to Cohen and Manion (1999) a researcher should never resume observations until the notes from the preceding observation are complete. In this way the danger of superimposing one set of events with a more recent set will be avoided. After initially thinking about recording my observations using an observation schedule, I decided against using this tool because it encourages an observer to only look for pre-selected observable behaviour which may result in overlooking significant unintended outcomes (Cohen, Manion, & Morrison, 2000).

A fellow researcher video-recorded the 2 lessons presented by each participant while I took field notes because this provided a permanent, comprehensive record of participants' use of examples and representations that I could preserve for subsequent analysis (Cohen & Manion, 1999). Video-recording these lessons also provided a snapshot of participants' instructional practice at that time as both their verbal and non-verbal interactions could be captured - which would have been difficult to do had I only taken field notes because so many things happen simultaneously in a classroom setting.

4.5 Data analysis

I analysed the data by firstly reading the written data and watching video-recorded lessons numerous times until I became very familiar with the details thereof. As mentioned earlier, this was an iterative process that included skimming (a superficial examination of material), reading (a more thorough examination of material), re-reading of selected material (more in-depth examination), summarising of lessons, transcription of video-recorded lessons, and then noting my preliminary interpretations of the data (Bowen, 2009).

Participants' course-work tasks were organised based on their relevance to early number topics. Tasks (or questions from tasks) that dealt with teaching an early number topic was included in the data set, and those tasks or questions that did not

deal with early number were excluded. For example, all the questions form participants' course-work tasks dated the 17th of March 2012 were not included in the data set because these dealt with decimals – a topic not related to teaching number in the Foundation Phase. My lesson observation data was organised by chunking lessons into episodes. An episode was determined by the task set and a change in the format of the lesson, e.g. from whole class to pair work. My data analysis was theory-driven - using the analytical framework that I developed mainly from Rowland's (2008) three categories of exemplification while also using interpretive indicators from the literature reviewed. The analytical framework used for analysis of data is detailed in Chapter 3 of this report.

4.6 Rigour

Criteria that are often used to judge the quality of any research include *validity* (also referred to as internal validity), *reliability*, and *generalizability* (or external validity). Drawing on the work of Marshall and Rossman (2010) these criteria can be broadly described as follows: *validity* refers to the truthfulness of a study; *reliability* speaks of the repeatability of a study; and *generalizability* speaks of how applicable the study is to other similar participants or settings. Issues related to the validity, reliability, and generalizability of any research speaks of the quality of that study. Thus all researchers should be concerned about these indicators as they help establish the readers' confidence in the outcome of the study. As a researcher I also considered using these indicators of a quality study to ensure the rigour and trustworthiness of my work but had to reconsider this after reading a few accounts of how these criteria for determining the quality of research applied mostly to quantitative research and not (as in the case of my study) as much to qualitative research (Feldman, 2003; Maxwell, 1992; Scaife, 2004; Sin, 2010). According to Guba and Lincoln (2003), although research rigor is a requisite in qualitative studies just as in quantitative studies, how this rigor is determined in qualitative studies differs simply because the two research paradigms are based on different ontological and epistemological assumptions.

Dependability of findings

In quantitative research reliability is described as: 'The extent to which a test or procedure produces similar results under constant conditions on all occasions' (Bell, 1999, p. 103). Scaife (2004) argues that it may be easy to judge scientific experiments according to the reliability of the dataset or data-gathering process because experimental conditions may be easy to control and thus easy to replicate. But, some researchers propose that this criterion may not be appropriate when judging research conducted in classrooms since we cannot expect 'constant conditions' in such an environment (Feldman, 2003; Merriam, 1995; Scaife, 2004).

Some suggestions for how the data-gathering process in educational research could be judged for dependability or consistency include: test – retest, equivalent forms, and split-half procedures (Scaife, 2004); and/or triangulation, peer examination and an audit trail (which involves the researcher providing a thick description of how data was collected and analysed) as proposed by Merriam (1995). To ensure the dependability of the findings emanating from my research I provided thick description of the data gathering and analysis processes I undertook in my study as well as triangulation of data sources. Whilst my study is exploratory, 'revisiting' teachers' example spaces across more than one data source over time allowed access to recurring regularities in teachers' examples and representations relating to them. This provides some dependability to the claims made in relation to teachers' selection and use examples and representations when teaching number-related tasks. A limitation of my data analysis is the relatively small classroom observation dataset, and the consequent need for care in relation to the breadth of ensuing claims.

Credibility of the findings

In traditional research the concept 'validity' is used to refer to 'the degree to which a method, a test or a research tool actually measures what it is supposed to measure' (Wellington, 2000, p. 201). Creswell (2012) believes that validity of any research study refers to the accuracy or credibility of the findings. While Merriam (1995) and Creswell (2012) make note of many strategies that could be used to validate the accuracy of findings in qualitative research, the following strategies were

used in my study, viz. 'methodological triangulation', i.e. looking for consistency in the data I gathered using different data-gathering strategies or methods (Cohen & Manion, 1999) to look for recurring patterns in the data ; submersion in the research situation; and transparency regarding the researcher's assumptions, experiences and biases. To deal with issues of bias I based interpretive accounts of participants' views using their own words as much as possible. I also provided a detailed description of: how data for my study was collected; what counted as data in my study; and how I transformed raw data for the purposes of analysis – which are strategies Feldman (2003) argues also ensures credibility in a qualitative study.

Relatability of the study

The generalizability or the external validity of any study refers to the extent to which findings from the study can be applied to other situations, and this is often noted as the greatest stumbling block to claims of value or 'goodness' in qualitative studies (Merriam, 1995). As mentioned previously, the aim of my study was not to make broad generalizations but rather to gain insight into how this sample of teachers selected and used examples and representations when teaching early number. Therefore I base the 'goodness' of my study on the relatability thereof (Bassey, 1981), i.e. the degree to which a teacher in a similar situation to that of the participants in my study can use the outcomes of this study to inform her own practice.

4.7 Ethical considerations

Before starting my research I obtained ethics clearance from the Human Research Ethics Committee (WITS SCHOOL OF EDUCATION – protocol number 2012ECE201) and informed consent from all participants. When I invited potential participants (i.e. teachers) to take part in my study I explained to them verbally and in writing that their participation would be voluntary, and that if they did choose to participate they had the right to withdraw from the study at any time without penalty. Potential participants were informed that taking part in my study would be neither an advantage nor disadvantage to them; that there were no foreseeable risks in participating; and that they would not be paid for their participation. Potential participants were assured that their names and identities would be kept confidential

at all times and in all academic writing emanating from the study. I informed potential participants that they would have an opportunity to verify information I gained through the use of different data-gathering strategies while the research report was still in draft form. Lastly, I assured potential participants of the safe keeping of confidential documents and that all data obtained during the course of my study would be destroyed between 3-5 years after completion of the study. All this information was given to potential participants to peruse at home with a consent form which they had to complete if they were indeed willing to participate in my study. Whilst the focus of my study was on teaching, information and informed consent was also sought from learners and parents in the two focal classes, and the school principal.

CHAPTER 5: FINDINGS AND DISCUSSION

5.1 Introduction

The aim of my study was to explore two Foundation Phase teachers' selection and use of examples and representations when teaching number-related tasks. This chapter presents the findings of my study and uses the research questions as the focal points of my discussion of the findings.

In order to answer my first research question, i.e. *What comparisons can be made between two teachers' selection and use of representations in their course-work that relate to teaching number?* I drew on two of the four course-work tasks completed by the participants during the WMC-P Projects' 20-Day course as my data source. These tasks make up this dataset because they were specifically related to teaching number in the Foundation Phase – which forms the nexus of my study. The representations participants chose to use when answering these tasks, and the explanations they offered for how they would use them in a teaching situation, were analysed and compared using the analytical framework formulated from key ideas that have emerged from literature, viz. *flexibility* (Ball, 1993; Cobb et al., 1992; Heize et al., 2009); *connections* (Askew et al., 1997; Ball & Bass, 2003; Haylock & Cockburn, 2008); and *progression* (Carpenter et al., 1999; Ensor et al., 2009) as outlined in Chapter 3.

I now present each participant's response to course-work tasks separately, followed by a short description thereof. Thereafter, I discuss participants' responses to the tasks set individually, before comparing them. I use the pseudonyms Zelda (higher content knowledge score on pre-test and course tasks) and Deborah (lower content knowledge) for the two focal teachers.

5.2 Course work – 12th May 2012

This task was aimed at exploring teacher's understanding of progression in counting strategies and representations that might support this progression. This task was presented as follows:

Learners worked out the following sums:

a) 18+3 b) 18+49 c) 202-8 d) 202-196

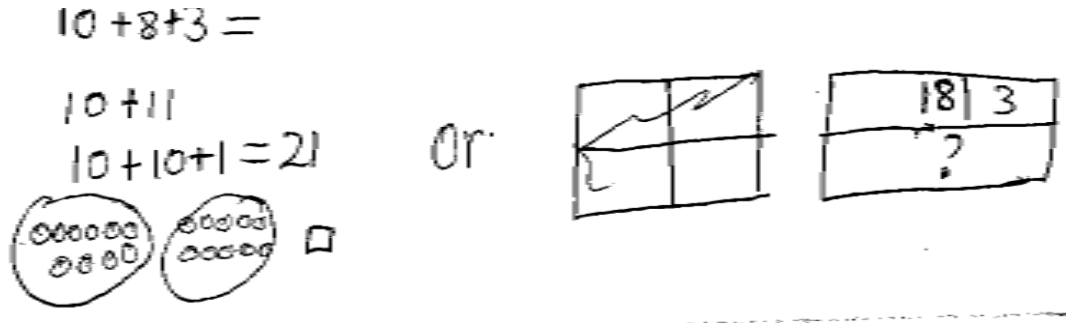
Here is how selected learners represented their thinking on each sum. In each case, suggest a strategy, including diagrams and explanation that might help to move the learner's thinking forward to a more efficient strategy. You can choose different representations to those shown below if you choose.

a) 18+3

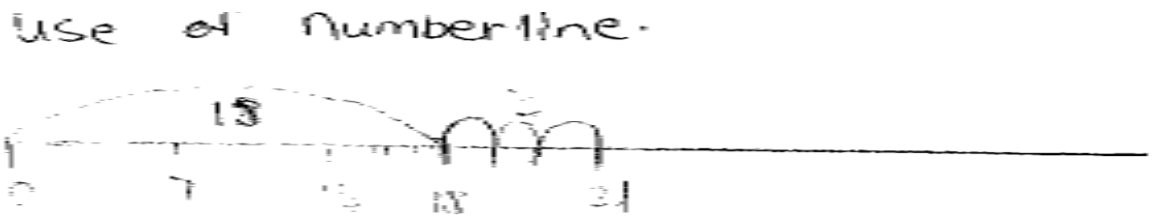
Learner's answer

000000000000000000 000 = 21

Deborah's answer



In the first solution Deborah partitions the first addend into tens and units. Thereafter the units are added using a recalled fact. Then the second addend is partitioned into tens and units. Next the three addends are added together to get the answer. In the final step a drawing is used to show either the answer or maybe the previous step. In the second solution Deborah uses the part-part-whole diagram showing the parts 18 and 3 (although not proportionally) and the unknown whole is indicated with a question mark.



In the third solution (previous page) Deborah drew an ENL with the following markings: 0, 7, 14, 18, and 21. A big 'jump' is drawn between 0 and 18 and the number 18 is written under the jump. Another 3 smaller jumps are shown between 18 and 21 with the number 3 written above the small jumps and no markings where the intermediate jumps land on the ENL.

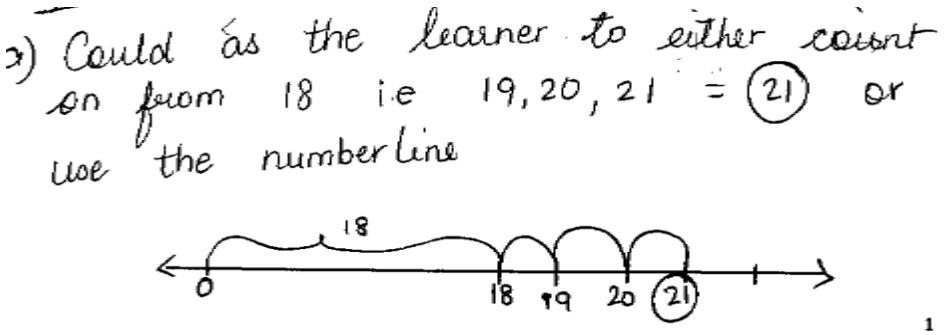
Commentary on **Deborah's** answer

In the first solution Deborah uses a symbolic-syntactical representation (Ensor et al., 2009) in her explanation which is a more abstract representation that lends itself to a more efficient strategy (partitioning) than that used by the learner. In the second step the 8+3 is added without any visible scaffolding or mediating representation shown. However, in the next step the 10+11 is not added in the same way. Instead, the 11 is partitioned into 10+1, and then in the third step 10+10+1 is added, again with no scaffolding or mediation visible. The last step is 'modelled' using an indexical representation that lends itself to a 'count all' strategy – the same strategy seen in the learner response presented in the task. In terms of the representational sequence, the teacher's thinking here seems somewhat circular and inconsistent in terms of sophistication of the strategies employed in the different steps. Thus, a lack of coherent expression of representational progression for developing more sophisticated counting strategies for addition is seen.

In Deborah's second solution she represents the problem using a part-part-whole diagram and in her third solution she represents the problem using an ENL which suggests that she has some awareness of multiple ways of representing the problem which points to representational flexibility (Heize et al., 2009).

In Deborah's first 2 solutions to this problem I see different conceptions of addition: the 'join' conception of addition in the first solution and a 'part-part-whole' conception in the second solution (Carpenter et al., 1999). In the third solution the teacher's use of the number line encourages a 'count on' action that is more efficient than the 'count all' shown in the learner's solution. This representation also encourages a focus on the ordinal aspect of number rather than the cardinal aspect, which Haylock and Cockburn (2008) maintain is important for building number sense in young children.

Zelda's answer



Zelda starts her explanation by stating that 'It would be fine for Grade 1 initially but later, as numbers become bigger, drawing pictures would not be practical'. Thereafter Zelda says she could either ask the learner to count on from 18, i.e. 19, 20, 21= 21 or use the number line. An ENL is drawn with hash marks 0, 18, 19, 20 and 21. A big jump is shown from the 0 to 18, and 3 smaller jumps from 18 to 21. The direction of the jumps is not shown on the number line. The number 21 is circled as was done in the count on explanation.

Commentary on **Zelda's** answer

The initial part of Zelda's explanation is synonymous with the argument made by Carpenter et al. (1999): that very young children naturally start off problem solving by directly modelling the action described in the problem and that this strategy should be replaced by a more efficient one later on. Zelda's suggestion to get learners to 'count on' is a more efficient strategy than the 'count all' used by the learner. There is a clear connection between Zelda's explanation of the 'count on' strategy (Carpenter & Moser, 1984), the action on the symbolic number-based representation - the ENL - (Ensor et al., 2009), and the symbolic notation used (Haylock & Cockburn, 2008).

Comparison

Both teachers offered some methods that were more efficient than that of the learner and both showed an ability to use multiple representations in their answers which suggests flexible use of representations – although Deborah was stronger in this aspect because she showed awareness of the part-part-whole diagram as well. Zelda established strong representational connections between the language she used in her explanation, the picture she drew, her actions on the representation, and

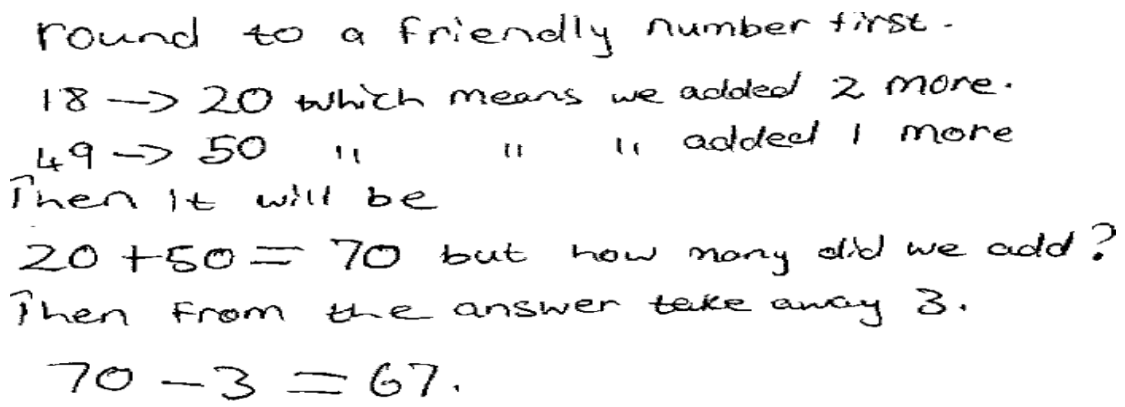
the symbolic notation used – which Haylock and Cockburn (2008) believe are the significant connections young learners need to build a good understanding of number – whereas Deborah's representational connections were not well established. Thus representational progression is much clearer in Zelda's presentation with coherent, consistent connections made between representations.

b) 18+49

Learner's answer

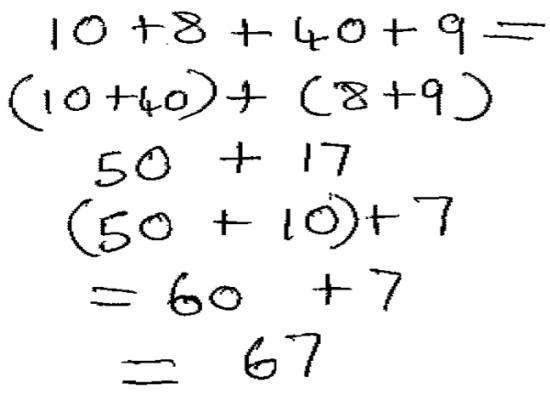


Deborah's answer



In Deborah's first solution (shown above) her explanation starts with an instruction to 'round [off] to friendly number first'. This instruction is followed by the first addend, 18, being rounded off to 20, shown with an arrow drawn from 18 to 20, with the following word explanation: 'which means we added 2 more'. Next the 49 is rounded to 50, also shown with an arrow drawn from 49 to 50 followed by this word explanation: 'which means we added 1 more'. The next step is introduced with the words 'Then it will be' followed by the insertion of 20+50=70. After this insertion:

'But how many did we add?' is asked. The next step in the explanation is: 'Then from the answer take away 3. The final step is the statement: 70-3=67.



In Deborah's second solution (above) no word explanation is given. Both addends are partitioned into tens and units with a transformed representation that reads as: 10+8+40+9. In the next step the tens are grouped together and the same is done for the units: (10+40)+(8+9). Next the numbers in brackets are added to give the representation 50+17. The 50 and 10 (from the 2nd addend) is now grouped with the unit '7' on its own. Next the tens are added producing a representation that now reads as 60+7. The final step is to add 60+7 to get the correct answer '67'.

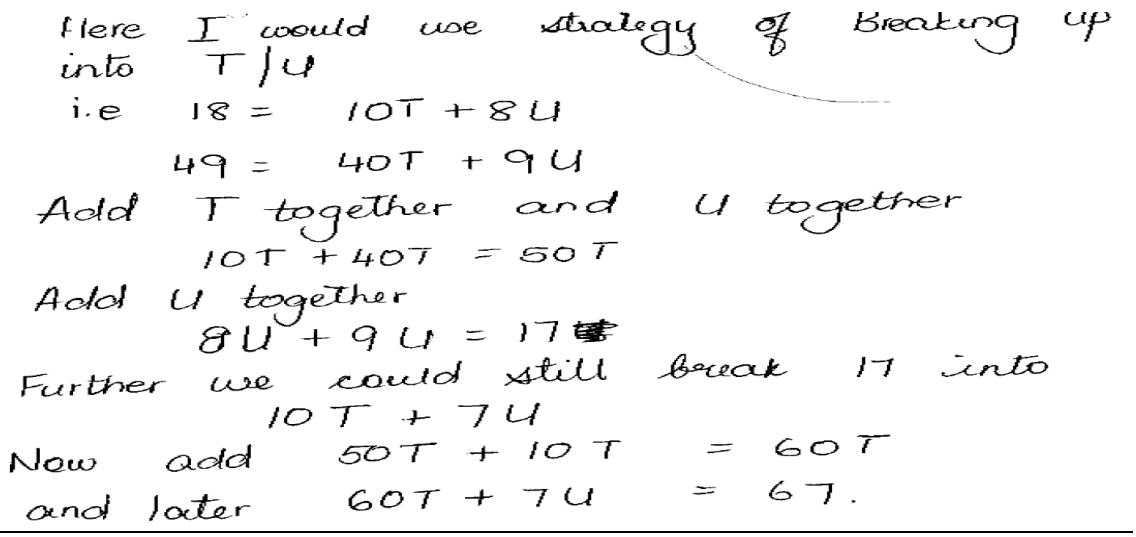
Commentary on **Deborah's** answer

In her first solution, Deborah's own number sense is apparent in terms of use of the nearest ten as a benchmark for rounding off both numbers (Wright et al., 2006), followed by an appropriate compensation step. Addition of 'rounded' addends to 70 is calculated as a known or recalled number fact with no elaboration on how this was done. The question of 'how much did we add?' does not unambiguously link to the previous explanation of what was added to the original addends in the process of rounding off. The next part of the explanation reads: 'Then from the answer take away 3'. Deborah does not explicitly establish the connection between adding 3 to the original addends and now doing the opposite, i.e. subtracting or taking away 3 from the answer, to get the final answer. There is an absence of explanation for this step linking the specific instance to the more general case so that the answer is not localised. The final solution is shown using symbolic notation: 70-3=67.

In Deborah's second solution she uses a symbolic-syntactical representation which according to Ensor et al. (2009) is more abstract than the symbolic-number based representation used by the learner – but the strategy is not more efficient than the

one shown. Partitioning of addends into tens and units connects to another idea in mathematics, viz. place value (Askew et al., 1997). The bracketed tens and units are added as known or recalled facts with no additional mediating representation shown. Of interest, is that the count-on-from-larger strategy presented as a 'near' next step in the literature is bypassed in this response (Carpenter & Moser, 1984).

Zelda's answer



Zelda's explanation states a strategy of 'breaking up' addends into 'tens and units'. Next Zelda shows how each addend is broken up (18=10T+8U and 49=40T+9U) and these are written below one another so that the tens and units of each partitioned addend lines up. The next part of the explanation 'Add tens together and units together' is followed by the tens from both addends being added (10T+40T=50T). Next 'Add units together' (8U+9U=17). Zelda goes on to explain that 'Further we could still break 17 into 10T+7U. 'Now add 50T+10T=60T, and later, 60T=7U=67.

Commentary on **Zelda's** answer

Zelda's explanation of the strategy she was going to use and the actions performed on addends are well connected (Askew et al., 1997; Haylock & Cockburn, 2008) but not clearly more efficient than the learner's method. The mathematical symbolic representations are potentially ambiguous in that 40T reads like '40 Tens'. Here too, the count on from larger step is bypassed. There is more overt mention here of the idea of 'exchange' between 10 units and 1 ten. In pedagogic terms then,

explanations are given that provide rationales for the steps taken (Leinhardt et al., 1990, as cited in Bills et al., 2006).

Comparison

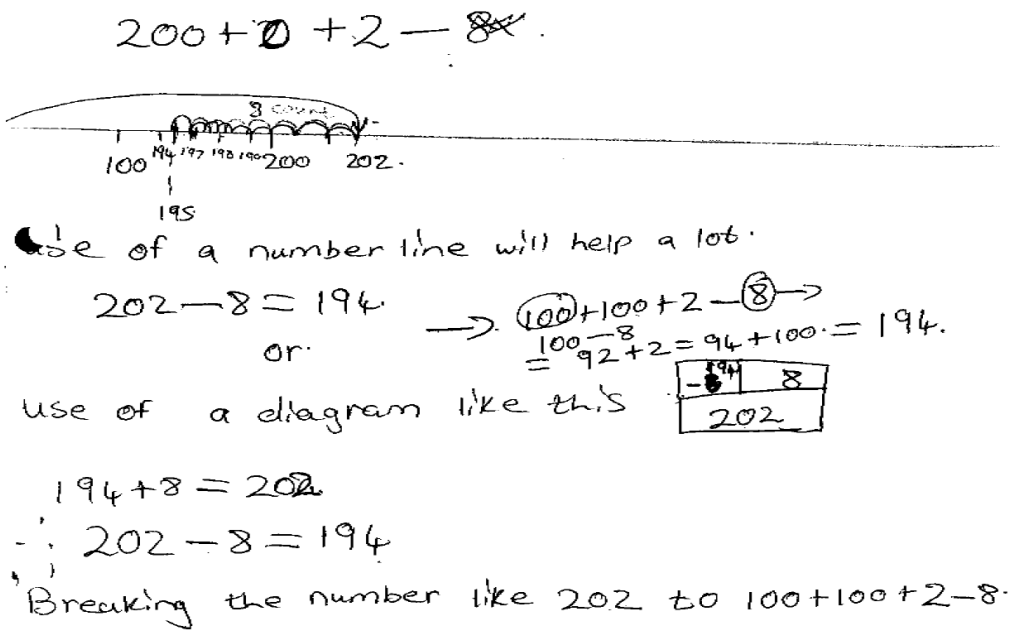
There was some ambiguity in the words that Deborah used which leaves gaps in the connection between her words, symbols and actions, whereas Zelda again showed strong representational connections between her word explanation and the action performed on addends, in spite of some ambiguity in the symbolic representation of place value.

c) 202-8

Learner's answer

'I begin at 202. [Opens eight fingers, then closes them one by one, Saying:] 201, 200, 199, 198, 197, 196, 195, 194. It's **194.**'

Deborah's answer



In Deborah's first solution the minuend is partitioned into hundreds, tens and units while the subtrahend (single digit) remains unchanged. So now the transformed problem reads: 200+0+2-8. Deborah then abandons this explanation here.

In the second solution Deborah uses an ENL with markings labelled: 100, 194, 195, 197, 198, 190, 200, and 202. A big 'jump' is drawn from the start of the number line to 202. The direction of this jump is indicated with the use of an arrow. Nine smaller jumps are drawn between 202 and 195 – the direction is not made clear and above these jumps Deborah wrote '8 counts'. From Deborah's action on the representation it seems as if the answer is 195, which is incorrect. Underneath the ENL the teacher wrote: 'use of a number line will help a lot'. Thereafter she wrote 202-8=194.

In Deborah's third solution the minuend is partitioned as 100+100+2-8. The next step involves only two of these numbers, the 100 and the 8, which are circled. The difference between 100 and 8, which is 92, is then added to 2 giving 94. The 94 is then added to the other 100 to get the final answer 194.

In her fourth solution Deborah uses a part-part-whole diagram to show the two parts, i.e. 194 and 8, and the whole, 202. Deborah then writes '194+8=202 therefore 202-8=194'. The sentence: 'Breaking the number like 202 to 100+100+2-8' follows.

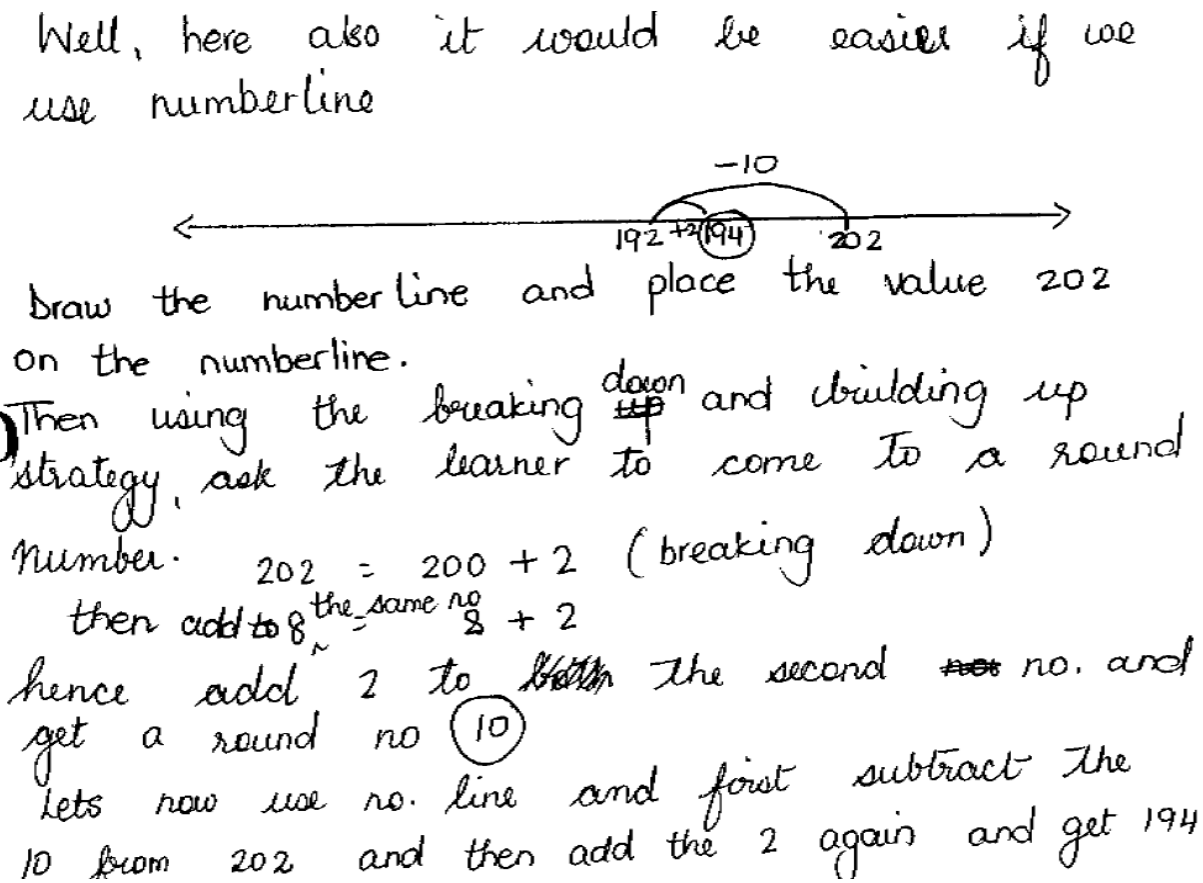
Commentary on **Deborah's** answer

After abandoning the first attempt, Deborah's use of the number line representation in the second solution is useful as it supports learners to see numerosity and develop more abstract number concept that incorporates ordinality and cardinality (Haylock & Cockburn, 2008). Even though Deborah chose a useful representation here, she did not use it judiciously. For example, the number line was not calibrated correctly and the incorrect number of jumps made resulted in an incorrect answer. Thus, whilst the correct symbolic problem and answer are written, these connect poorly with the number line representation as shown (Haylock & Cockburn, 2008). Breaking down 202−8 into 100+100+2−8 is mathematically accurate, but probably not more efficient than the 'count back from' method shown (Carpenter & Moser, 1984).

Deborah's representation of the solution - using symbolic notation, an ENL, and a part-part-whole diagram - shows flexible use of multiple representations (Heize et al., 2009). Here Deborah's explanation established a good connection between the part-part-whole diagram and the symbolic notation used (Haylock & Cockburn, 2008). A connection was also established (or strengthened) between two

mathematical operations: addition and subtraction – connecting different mathematical ideas is what Askew et al. (1997) believe helps learners to become numerate. Deborah also showed awareness of different conceptions of subtraction: 'separate' as well as a 'part-part-whole' conception (Carpenter et al., 1999). In her explanations Deborah showed partitioning in various ways, i.e. as 200+0+2-8 and as 100+100+2-8 which Askew and Brown (2003) argue is good for mental calculation.

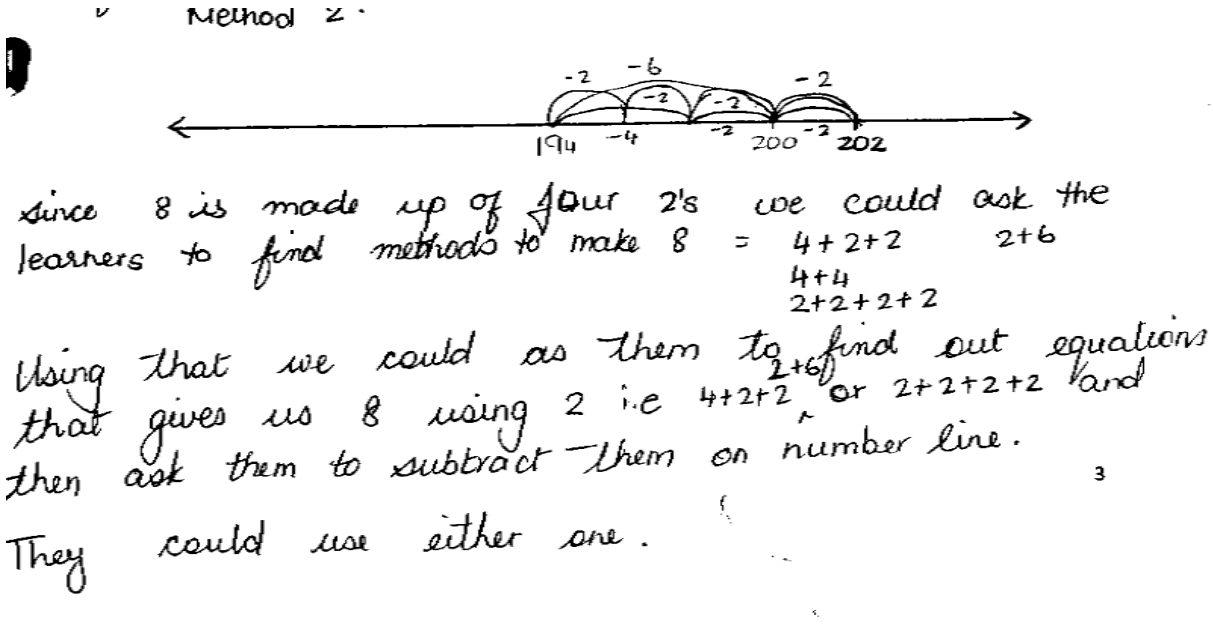
Zelda's answer



Description of **Zelda's** answer

Zelda starts her first solution off by saying that 'it would be easier if we use [a] number line'. This statement is followed by the drawing of an ENL with the markings 192, 194, and 202. A jump from 202 to 192 shown with -10 written above it and another jump from 192 to 194 is shown with +2 written below it. The number 194 is circled. Thereafter an explanation follows on how to do what is shown on the number line. First Zelda says 'Draw the number line and place the value 202 on the number line. Then using the breaking down and building up strategy, ask the learner

to come to a round number'. Symbolic notation is used next and the explanation 'add 2 to the second number and get a round number 10' follows. Lastly, Zelda writes 'Let's now use no. line and first subtract the 10 from the 202 and then add the 2 again and get 194'.



In the second solution (shown above) a number line is drawn with the markings 194, 200 and 202. 4 jumps are shown from 202 to 194 with -2 written above each. Other jumps are also shown: 2 jumps from 202 to 194 with -2 and -6 written above the respective jumps; and 3 jumps shown with -2, -2, and -4 written underneath these jumps. Zelda's word explanation follows: 'Since 8 is made up of four 2's we could ask the learners to find methods to make 8 = 4+2+2; 2+6; 4+4; 2+2+2+2'. Zelda then explains that learners could use the ways they've broken 8 down as ways of subtracting 8 from 202 on the number line, and learners could use either method that they felt comfortable with.

Commentary on **Zelda's** answer

Zelda's number line representation in both solutions is symbolic number-based which is more abstract than the fingers the learner used which is a concrete representation (Ensor et al., 2009). Zelda's first solution seems more efficient than that of the learner's because she got to the correct answer in 3 steps using the compensation strategy which is quicker than the 8 steps involved in the learner's 'count down from' method (Carpenter & Moser, 1984) – although there are some lines in the middle which seem superfluous and do not seem to connect coherently, e.g. 202=200+2.

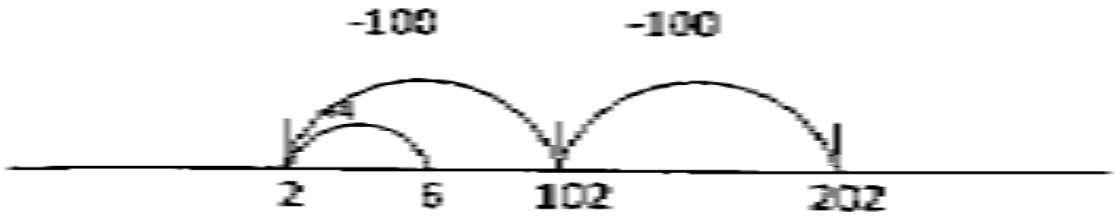
In the second solution the teacher's explanation connected well to the action shown on the diagrammatic representation (Askew et al., 1997; Haylock & Cockburn, 2008). Both methods presented by Zelda push for more efficient problem-solving than the learner's method. There are also indications of connections to prior learning in that ways of grouping 8 are referred to and used, with words coherently connected to the number line representation.

Comparison

Both teachers used a number line representation which shows representational progression when compared to the concrete representation used by the learner (Ensor et al., 2009). Other representations of the solution used by both teachers also point to representational flexibility. Instead of the 'count down from' method used by the learner, these teachers introduced other methods, e.g. compensation was used by Zelda, and the 'separate' and 'part-part-whole' conception of subtraction was used by Deborah. In her answer Deborah also highlighted the relationship between addition and subtraction as inverse operations. In this instance, more attempts at a rationale for the teacher's action is seen in Zelda's response.

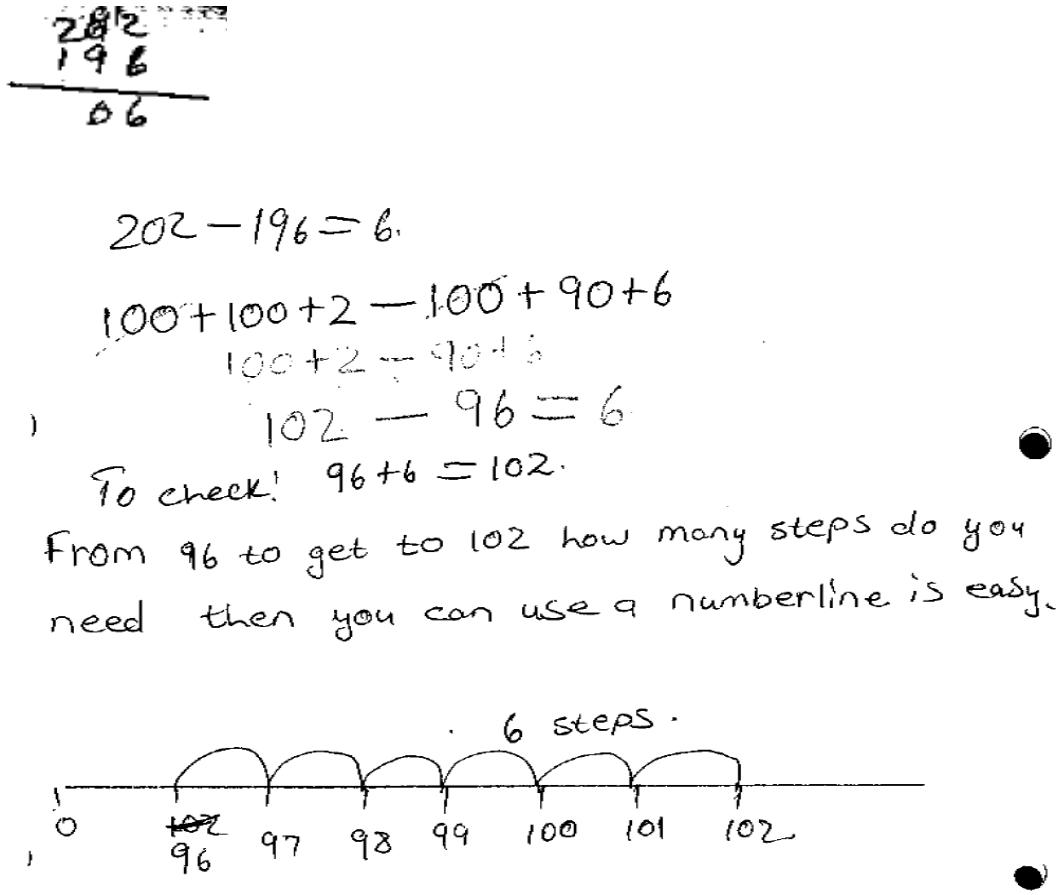
d) 202-196

Learner's answer



The learner uses an ENL – a symbolic number-based representation (Ensor et al., 2009) – and the compensation method (adding 4 to 196 to make it 200) to jump back 100 from 202 to 102, then another jump back by 100 to 2, and finally a jump forward 4 places to finish on 6. The learner's strategy shows a 'take away' conception of subtraction.

Deborah's answer



Description of **Deborah's** answer

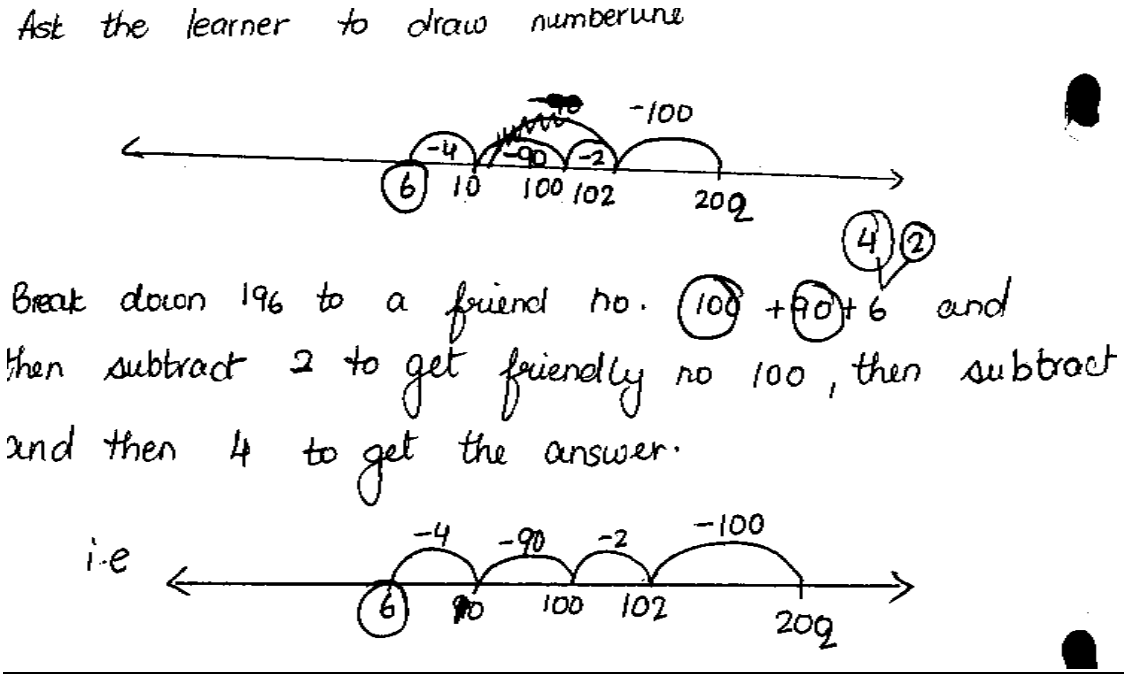
Deborah subtracts 196 from 202 (using column subtraction) in the top right hand corner of the page – this is much smaller than the rest of her answer. She then writes the problem and answer using symbolic notation with no word explanation. In the next step the minuend and subtrahend are partitioned as: 100+100+2 – 100+90+6. The 100's on either side of the 'equation' are cancelled out leaving 100+2 – 90+6. Both sides of the equation are then calculated leaving the problem: 102-96=6. Following this are the words 'To check' and the addition problem 96+6=102, although there is no accompanying explanation of the equivalent of the problem to the original. The next explanation 'From 96 to get to 102 how many steps do you need then you can use a number line is easy' is followed by an ENL showing the following markings: 0, 96, 97, 98, 99, 100, 101, 102 and 6 jumps between the 96 and 102 with no direction but '6 steps' written above all the jumps.

Commentary on **Deborah's** answer

The mathematical problem shown makes significant use of abstract notions of number and their relationship in this subtraction. The action of cancelling the two 100s is enacted but not explained. A connection is made between the operations of

addition and subtraction again – showing the relationship between them, as inverse operations (Askew et al., 1997). Another connection is made between the word explanation 'from 96 to get to 102 how many steps do you need …' and the ENL with the 6 steps shown between 96 and 102 (Haylock & Cockburn, 2008). The number line representation used to check that 102 – 96=6, with a difference model, could have been applied to the original problem: 202-196. Thus, once again, there is some circularity in Deborah's response which disrupts connections and efficiency.

Zelda's answer



Description of **Zelda's** answer

Zelda's explanation starts off with the instruction to ask learners to draw a number line. Thereafter an ENL is drawn with the markings 6, 10, 100, 102, and 202. 4 jumps are shown: from 202 to 102 with -100 written above it, then from 102 to 100 with -2 written below it, then from 100 to 10 with -90 below it, and finally from 10 to 6 with -4 written below the jump. The direction of the jumps is indicated by the symbols above or below it. Another 'neater' version of the same number line is drawn after the word representation. The word representation explains that the minuend is to be broken down to a friendly number: 100+90+6, and the 6 is further broken down into 4 and 2. The method of subtracting the partitioned minuend is explained by using the 'take away' conception of subtraction.

Commentary on **Zelda's** answer

The teacher used the same representation used by the learner, i.e. a symbolic number-based number line (Ensor et al., 2009). The learner's method shows a 'take away' conception of subtraction and so too does the teacher's method. Here the teacher's method (4 steps) does not seem more efficient than the learner's method (3 steps) – and thus representational progression is not incorporated here.

Comparison

Both teachers' solutions do not show more efficiency than that of the learner. One of Deborah's explanations made use of the 'difference' model whereas Zelda used the same 'take away' conception of subtraction as the learner. In this example Deborah shows awareness of two conceptions of subtraction as take away and difference – i.e. 'count down from' and 'count down to' in Wright et al. (2006) and Thompson's (2008) terms. So Deborah showed more flexibility than Zelda here, but the circularity and lack of rationale tends to limit the pedagogic potential of this awareness.

5.3 Course work – 15th June 2012

The course work for Day-8 followed two days of teaching around division-as-sharing and division-as-grouping. The following question (Q3), taken from the course work for Day-6, was selected for analysis because it corresponds with the topics covered in early number – while the other questions, i.e. Q1 - area, Q2 – multiplication of decimals, and Q3 – long division using the chunking method, are not included here.

Question 3 was presented as follows: **Write a grouping and a sharing problem to go with this sum: 35÷5**

Description of **Deborah's** grouping problem

Deborah wrote: 'We have 35 sweets and need to put them in packets equally. How many packets do we need?' The answer is given as: 'We need 7 packets for 5 children or 5 packets for 7 children'.

Description of **Deborah's** sharing problem

Deborah wrote: '35 sweets are shared among 5 children. How many sweets will each child get?' The solution is shown with a drawing of 5 children. Each child has the

number 7 above its head. The solution is finished with the words: 'each child will get 7 sweets'.

Commentary on **Deborah's** answers

The space allowed to answer the question in was tight so I am not sure if the teacher would have given more of an explanation if more space was given. Deborah's presentation of the sharing problem was coherent with the task. On the grouping task, her example fails to deal with 5 as the stated divisor and thus fails to connect coherently with the question.

Description of **Zelda's** grouping problem

Zelda wrote: 'Sam has 35 sweets. She wants to give 7 sweets to each child. How many children will get the sweets?'

Description of **Zelda's** sharing problem

Zelda wrote: 'Ben has 5 friends. He wants to share 35 pens with all his friends. How many pens will he give each friend?'

Commentary on **Zelda's** answers

Overall Zelda's presentation of the grouping and sharing problems were correct. However, in the sharing problem the notion of 'equal sharing' is not explicitly stated.

5.4 Overall commentary on course tasks

What seemed similar about the way in which these two teachers answered course-tasks was their ability to use more than one representational form when explaining the mathematical concept or procedure, which could suggest an ability to use representations flexibly (Ball, 1993; Cobb et al., 1992; Heize et al., 2009). Both teachers were mostly able to show a method and representation that was more sophisticated and more efficient than that used by the learner – pointing to some understanding of progression. Both teachers were also able to present the division-as-sharing example correctly, but only Zelda could present the division-as-grouping example correctly. It is worth mentioning here that during the follow-up session of the 20-Day course Deborah was then able to correctly present a grouping-as-sharing problem which seemed to be a thorny issue for many of the teachers on the course.

The most noteworthy difference between these teachers was where their strengths lay. Zelda's course-tasks show more coherent connections established between the different representational forms that she employed in her answers. In many of her answers I can see coherent, consistent connections between the words, diagrams, actions on diagrams, and symbols that she used, e.g. in her answers to 18+3, 18+49, and 202-8. In her answer to 202-8 Zelda also made connections to learners' prior learning by referring to different ways of grouping 8. Research shows that when teachers build connections between different representational forms, and between different aspects of mathematics, learners' understanding of number and number operations is more secure (Askew et al., 1997; Haylock & Cockburn, 2008). Although Deborah's connections between the language, actions and representations used were not always well connected, she did show an ability to do so in her answers to 202-8 and 202-196. Here Deborah made good connections between the diagrams and the symbolic notation used and the inverse operations of addition and subtraction.

Deborah's course-tasks showed a greater attempt at flexibility by her consistent use of more than one representation in her answers. By using multiple representations of the mathematical concept Deborah was able to show different conceptions of subtraction, i.e. 'separate' and 'part-part-whole', in her answer to 202-8. She was also able to show the problem 202-196 using a 'difference' conception of subtraction whereas Zelda used the same 'take away' conception as that of the learner. However, Deborah's use of these multiple representations was not always coherently connected with the language, actions, and the pictures she used. Here the admonition by Askew and Brown (2003) that teachers should rather use a few research-based representations instead of multiple representations that are not well connected is appropriate.

I now go onto examining classroom teaching aimed at answering my second research question: *What comparisons can be drawn between these two teachers' selection and use of examples when teaching number-related tasks, taking account of variables, sequencing and representations?*

I used field notes and transcripts of the two lessons presented by each teacher as my data source to answer my second research question. Data presentation in this

section begins with a lesson overview that details different episodes. Lessons are chunked into episodes based on the task given or when the teaching format changes – whole class (wh/cl), group work (gr/wk), or individual work (indivi/wk). Tasks were based on what was presented by the teacher as the focus of attention and in many instances these were broken down into a number of examples. Following Mason and Johnston-Wilder's (2006) distinction between task and activity, the activity outlined describes what happened in the enactment of the task. The representations states the symbolic, pictorial, and concrete apparatus used. Data from the lesson overviews is then analysed using the analytical framework formulated from three of Rowland's (2008) categories of exemplification, viz. *taking account of variables*, *sequencing*, and *representations* with additional literature-based indicators used under relevant categories – as detailed in Chapter 2 and summarised in Chapter 3 of this report.

5.5 Deborah - 1st Lesson Overview: Addition 7Aug 2012

Ep.	Task	Examples	Activity	Representations
1. wh /cl	Oral count in: 5s from 5 – 50 10s frm 10-100	a) multiples of 5 from 5 to 50 b) multiples of 10 from 10 to 100.	Whole class skip counts in 5s starting at 5 and ending at 50, and then in 10s from 10 to 100.	none
2. wh /cl	What is a number line?	What is a number line?	Lrs offer 'a line with no.s'. Pointing to an ENL running across top of the board tr says 'Here I have a line but it has no numbers. What do we call this?' Lrs answer 'an empty no. line' Tr says 'today we are going to add on the no. line' and writes heading: Addition on the number line.	ENL
3. wh /cl	What is another word for addition? Who knows the addition sign?	What is another word for addition? Who knows the addition sign?	Tr poses the question and lrs offer: minus, times, add. Tr accepts last offer. Lr comes to board and writes ÷, another writes =, another writes +. Tr accepts the last sign and says 'which means we put things together'.	see lr offerings of words in activity column symbols ÷, =, + written on board
4. wh /cl	How do we draw a number line? What do the arrows at both ends of the number line mean?		Tr poses the question and lrs offer: with a straight line. Tr draws a straight line on board with arrows at both ends. Tr asks the question and lrs call out answers at the same time – answers are not clear. Tr says 'because the no. line goes on and on and on. Do you understand?'. Lrs call out 'yes'.	ENL drawn on board with arrows at both ends (no demarcations on line)

Ep.	Task	Examples	Activity	Representations
5. wh /cl	Complete the number line	no. line	Tr places no.s 0-12 (in 2s) on ENL drawn in episode 4 saying 'We always start at zero. Then it depends how we are counting'. Lrs count from 0 to 12 in 2s as tr completes no. line.	number line symbolic no.s 0, 2, 4, 6, 8, 10, 12 filled in on no. line
	Fill in missing no.s on a number line	a) 4 and 10 b) 2 and 6	Tr erases 4 and 10 from no. line marked in multiples of 2 (0-12) and call a lr to board to fill in missing no.s. Lr is unable to do so and another lr is called to help him. The same activity is repeated with no.s 2 and 6 erased.	
6. wh /cl	Add 2+6 and 6+2 on a number line.	a) 2+6 b) 6+2	After writing the sums on the board the tr asks the class if the sums are the same and if the answers are the same. Lrs call out 'yes'. Tr asks the class what the answer is and lrs call out 'eight'. Then tr says 'It is eight, but today we must show it on a no. line'. Tr then demos how to add 2+6 on the no. line marked in multiples of 2 but her demo goes awry.	addition sums written in symbolic form on board no. line marked in 2s from 0 - 12 addition sum written in symbolic form on board
	Add 10+10 on a number line.	c) 10+10	Tr draws a no. line marked from 0 – 60 in 10s and writes the sum 10+10 on board. Tr asks for the answer and lrs call out 'twenty'. Deborah then demos how to add 10+10 on the no. line marked in multiples of 10, as above. Lrs call out their objections to what the teacher demonstrates more than once.	no. line marked in 10s from 0 – 60
7. gr/ wk	Copy and complete number lines using numbers cut from magazines.	a) no. line drawn on board from 0-9, marked in 1s	Tr draws a no. line (0-9) on board marked in ones with marking for no. 1 omitted by mistake and no.s 4, 7, 8 left out and replaced with empty boxes. Lrs work in groups to find no.s 0-9 in magazines, then cut no.s out, and paste them in order on the ENL.	number line
		b) no. line drawn on board from 0-50 marked in 5s	Tr presents a similar activity – now the no. line is marked in multiples of 5 from 0-50 with no.s 20, 25, 30, 35, 40, 45 erased and replaced with empty boxes. Lrs again work in groups to complete the activity.	number line
8. wh /cl	Verifying answers by completing no. lines as a wh/cl activity		Tr calls lrs to the board to fill in missing no.s on both no. lines. Tr then asks class to count in ones from 0-9 and in 5s from 0-50 to check if the no. lines (taken from groups and stuck on board) are completed correctly.	completed no. lines taken from groups

Ep.	Task	Examples	Activity	Representations
9. ind ivi/ wk	Copy and complete no. lines.	a) no. line b) no. line	Tr draws 2 no. lines on board and asks lrs to complete these individually in their books. The 1st no. line (a) is marked from 0-18 in 3s with no.s 9 and 15 erased and replaced with empty boxes. The 2nd no. line is marked from 0-100 in 10s with the no.s 30, 40, 50, 60, 70, 80 and 90 erased and replaced with empty boxes.	2 no. lines (formation as described under activity)
	Add using your number line.	a) 3+5 b) 10+6	Then tr writes the sub-heading 'Add using your number line' with two sums: 3+5 and 10+6 underneath it. Lrs complete the exercise in their books. Some lrs use recalled facts and others use their fingers to add.	a) 3+5 b) 10+6

5.6 Deborah - 2nd Lesson Overview: Sharing 31Oct 2012

Ep	Task	Examples	Activity	Representations
1. wh /cl	It is end of the month and your mummy is going to bring you nice things. Will you have it by yourself?		Deborah poses the question and learners answer 'No, we will share'. A short class discussion follows.	none
2 wh /cl	Share 1 cake between 2 people.	a) Share 1 cake between2 people.	Deborah uses a tub of play dough as a cake and calls a lr to share the cake with her. Deborah gives the lr a small piece of 'cake'. Class protests to this saying 'No Ma'am, she's going to cry'. Deborah then says that 'when we share it must be equal, equal'.	1 tub of play dough
	Share 2 cakes equally between 2 people.	b) Share 2 cakes equally between 2 people.	Deborah says 'some things are easy to share' and gives each lr 1 cake/tub of dough. Then Deborah says other things are difficult to share equally.	2 tubs of dough
	Share 3 cakes equally between 2 people.	c) Share 3 cakes equally between 2 people.	Deborah says 'I'm going to have a problem sharing 3 cakes' and calls a lr to share the cakes. One lr gets 1 cake and the other gets 2. Deborah asks the lr who received 1 cake if she is happy and she answers 'no'. Another lr calls out 'there's 1 remainder'. Deborah agrees and says 'some remainders can be shared – how do we write it?' A few lrs call out 'one and a half'.	3 tubs of dough no symbolic rep used for answer

Ep	Task	Examples	Activity	Representations
3. pr/ wk	Share 10 things equally between 2 people.	a) Share 10 things between 2 people.	Deborah hands out 1 tub of dough to each pair and a blank sheet of paper to every lr. Deborah gives the instruction 'divide your paper into 2. I'll call out a number and you make little balls, then you share the balls on your paper'. Deborah calls no. 10 and lrs start making small balls. Some lrs make 5 balls each, others make 10 each. A few lrs making 10 balls each are seen placing 5 on either half of page. Deborah writes 10 on the board and draws a line btwn the digits 1 and 0 (1/0) to 'share' it. Lrs object to this suggesting that Deborah rather buys 10 sweets and then share them. Deborah replies 'I can only share real things. I can't share the no. 10. So, if I have 10 sweeties and I must share it btwn 2 friends...' A lr calls out 'each will get 5'. Deborah then writes 10= 5 5 on the board.	tubs of dough 1/0 picture: 10= 5 5 10÷2
	Share 10 things between 5 people	b) Share 10 things between 5 people	Deborah instructs class to share their sweets btwn 5 people. A lr calls out 'each one will get 2'. Deborah says 'which sign can I use?' and lrs say 'division'. Then Deborah calls a lr to the board to write the division sign and the boy does so correctly. Deborah asks the class 'Is this a division sign, can I relate it to sharing?' Lrs call out 'yes'. Deborah then writes 10÷2 next to picture.Deborahwrites10= (giving each circle a name). Deborah then writes 2 in each circle, asks lrs to count in 2s, and writes 10÷5 next to picture. Deborah asks lrs if 2 and 5 are even or odd numbers. Lrs answer correctly.	picture: 10 = 2 2 2 2 2 10÷5
	Share 10 things between 3 people.	c) Share 10 things between 3 people.	Deborah gives class this instruction: Now share your 10 sweeties between 3 teachers. How many will each get?' Deborah dra 0= on board. A girl says 'Let's start with one each' and tr gives each circle 1 'sweet'. A few lrs call out 'No, we can give 3 each, one time' Deborah says 'Let's keep sharing' and gives each circle 1 sweet at a time. Lrs shout 'If you share one you'll take a looong time' but Deborah keeps giving 1 each till each circle has 3 sweets. Lrs	picture: 10 = ◯ ◯ ◯ (with 3 dots in each) symbols: $\frac{1}{3}$, 10÷3

			comment: 1 remainder; 3 is a odd number and 10 is not a odd no. Deborah asks the class what she should do with the remaining 1 sweet. One lr says 'then it's 3 over 1' Another says 'No, 1 over 3' Then Deborah draws a rectangle a divides it into 3 equal pieces, labelling each piece $\frac{1}{3}$, and says 'it's 1 over 3'	rectangle divided into 3 equal parts and labelled.
4. ind ivi/ wk	Word problems: Fundi and Yusuf want to share 3 chocolate bars equally. Show them how to do it. Jan, Sara and Ben want to share 4 chocolate bars equally. Show them how to do it. Yusuf, Ben, Jan and Fundi want to share 5 chocolate bars equally. Show them how to do it. Context free sharing: 36= _ _ _ _ 40= _ _ _ _ _ 24= _ _ _ _	Word problems: a) Fundi and Yusuf want to share 3 choc bars equally. Show them how to do it. b) Jan, Sara and Ben want to share 4 choc bars equally. Show them how to do it. c) Yusuf, Ben, Jan and Fundi want to share 5 choc bars equally. Show them how to do it. Context free sharing: a) 36= _ _ _ _ b) 40= _ _ _ _ _ c) 24= _ _ _ _	Deborah presents some lrs with the word problems and others with the context-free exercises. Lrs are allowed to work in pairs if needed. Deborah asks lrs doing the context-free exercises to use their dough when answering. Lrs are seen completing the word problems using pictures and words with varying levels of success. Other lrs are seen sharing the no. of apples (in context-free exercise) between the no. of groups shown using dough – also with varying levels of success.	2 worksheets: word problems and exercise using numbers and pictures

Deborah – Taking account of variables

Looking within and across both lessons presented by Deborah I notice that some aspects of the mathematical objects being taught are fixed while others are varied – the aspects which vary indicates the 'range of permissible change' (Watson & Mason, 2006a).

In Deborah's first lesson the most noteworthy aspect that remains fixed across all episodes is the 'join' conception of addition presented in worked examples (2+6 and 10+10, episode 6) and the learner exercise (3+5 and 10+6, episode 8). Carpenter et al. (1999) would argue that this is a limited view of addition that could cause problems for learners later on as they encounter other conceptions of addition like 'separate', 'part-part-whole' and 'compare' problems. By only presenting addition

examples as 'join' problems and reinforcing this with words like '[addition] means we put things together' (episode 3) Deborah may not only be restricting learners' understanding of the operation but also reducing the range of addition problems learners will be able to solve independently later on (Carpenter et al., 1999).

According to Carpenter et al. (1999) 'join' problems can be varied by changing the variable that is unknown. Deborah does not take account of this dimension of possible variation and presents all the 'join' problems with the 'result unknown', e.g. 3+5= □.

So, even though Deborah chose to 'fix' the type of addition problem to the 'join' conception, she could still have taken account of variables by presenting these using the 'change unknown' as 3+ □ = 8 or by presenting problems using the 'start unknown' as □+5=8. Thus the dimensions of possible variation and permissible change which Rowland (2008) maintains must be reflected within teachers' examples are not reflected in Deborah's selection of examples in her first lesson. Thus Deborah's examples do not expose learners to the range of types of addition problem that they may encounter (Carpenter et al., 1999).

What is also interesting to note is that when Deborah *did* take account of variables in her first lesson (e.g. using 2-digit numbers in successive examples) this was not done very successfully because the number range was low enough for the class to call out answers to examples as recalled facts - rendering the use of a number line to add redundant. Also, varying the calibration of the number lines (e.g. using a number line calibrated in 2s to add 2+6) resulted in Deborah obtaining an incorrect answer of 14 as seen from this transcript taken from episode 6:

A number line (drawn during the previous episode) calibrated in 2s from 0-12 is on the board.

Tr: *(Writes 2+6 and 6+2 on the board)* Is this the same? Will I get the same answer?

Cl: *(chorus)* Yes

Tr: What is the answer?

Cl: *(chorus)* Eight

Tr: It is eight, but today we must show it on the number line. First I must check if I have all the numbers. Do I have two? *(pointing to the number 2 on the no. line)*

Cl: *(chorus)* Yes

Tr: Do I have six? *(pointing to the number 6 on the no. line)*

Cl: *(chorus)* Yes

Tr: But I may not use the six. I don't know. I don't know the answer.

Now Deborah draws a single jump from 0 to 2 on the number line and then says: "We have to do six jumps". Then Deborah does 6 single jumps on the number line (calibrated in 2s) and lands on the number 14.

Tr: What do we get?

Cl: *(chorus)* Fourteen

Tr: Fourteen….Uhmmm…..

Here Deborah seems to realise her mistake.

Tr: But our number line is in twos. To count in ones we need a number line in ones.

Lr1: Ma'am you must count in twos because the number line is in twos.

Tr: No, I can count in ones *(Now Deborah erases the numbers 4, 6, 8, 10 and 12 and replaces these with 3, 4, 5, 6, 7, and 8 – leaving 0 and 2 unchanged. Deborah starts adding again on the number line with the first jump from 0 to 2 unchanged, followed by six single jumps to eight.*

So, even though Deborah did eventually get to the correct answer by adding on the number line here – *how* she used the representation in this worked example was not mathematically correct, suggesting problems in connecting grouped counting with addition.

In Deborah's second lesson she is seen taking account of variables in episodes 3, 4 and 5 by increasing the divisor and dividend in her worked examples and learner exercise. Deborah also takes account of variables in the type of examples presented in the learner exercise (i.e. word problems and context-free problems).

Deborah – Taking account of sequencing

Across both lessons presented by Deborah, Merrill's first prescription regarding micro sequencing (i.e. generality before example for near transfer) as cited in Van Patten et al. (1986) is evident as the organising principle because learners are required to use the concept and/or procedure just explained by the teacher to complete a similar exercise. For example, in Deborah's first observed lesson she demonstrates how to complete number lines and then how to add using a number line in episodes 5 and 6 respectively. Thereafter Deborah presents similar examples in the same order for learners to complete as an exercise, i.e. number lines to complete and then addition problems to calculate on a number line in episode 9. Here the sequencing of addition problems that can be solved using skip counting on the number line

represented is appropriate in episode 6. But the strategy disrupts this use. So the potential in the sequencing is disrupted in the enactment.

In Deborah's first observed lesson, the sequencing of worked examples (2+6 and 10+10) and examples used in the learner exercise (3+5 and 10+6) shows a 'divergent sequence' which relates to Merrill's third prescription regarding micro sequencing (Van Patten et al., 1986). The difference between these successive examples is that the initial problem in these pairs consists of single-digit numbers while the second problem in each pair has at least one two-digit number. I also see an attempt at graded sequencing because two-digit numbers are bigger in size than single-digit numbers and bigger numbers are generally more challenging to work with (Anghileri, 2006). However, if the sequencing here was indeed intended to be graded it is only so on a superficial level because the successive examples are actually less challenging than the initial examples because children often learn 'double' number facts like 10+10 relatively quickly and also find adding a single digit number to 10 (e.g. 10+6) relatively easy (Anghileri, 2006).

In both observed lessons Deborah used controlled sequencing and the 'easy-to-difficult' sequence – Merrill's fourth specific prescription for micro sequencing as cited in Van Patten et al. (1986) - in the worked examples and learner exercises. This is evident in the number range of successive worked examples, e.g. in the three worked examples presented in episode 3 of Deborah's second observed lesson there is an increase in the size of the divisor from example a) to example b), and an increase in the complexity of the example when the divisor changes again in example c) which necessitates the use of a unitary fraction in the answer. The word problems which form part of the learner exercises in the second observed lesson are also graded in this easy-to-difficult sequence – with an increase in dividend and divisor in each successive example, incorporating the possibility of fractions needed in each successive answer.

Deborah – Taking account of representations

One of the reasons that drive a teacher's selection and use of specific representations when teaching mathematics is to make an abstract concept more accessible to the learners (Rowland, 2008). In Deborah's first observed lesson – aimed at teaching learners how to add using a number line – Deborah's use of the

representation in both worked examples went awry. In the first example Deborah used a number line calibrated in two's from 0-12 to add 2+6 and got an incorrect answer due to ignoring the calibration and the unit counting enacted. Then after realising her mistake she changed the hash marks on the number line from multiples of 2 to ones, but did not do so for the whole number line, resulting in a calibrated number line with an irregular scale. In the second example Deborah tried to add 10+10 on a number line calibrated in tens from 0-60. This is an excerpt of the lesson transcript (episode 6) showing what took place:

Tr: Let's try another one *(draws number line marked in tens from 0-60 on the board).*

Now let's do 10+10. *(writes symbolic sum, 10+10, on board)*

The answer is …?

Cl: *(chorus)* Twenty.

Tr: *(Goes to tens number line on the board)*

Remember between 0 and 10 are numbers. Do you remember them?

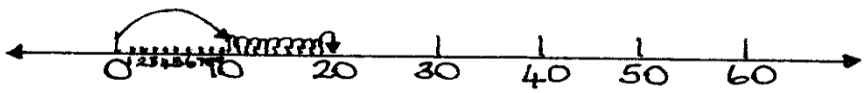
Cl: *(chorus)* Yes.

Tr: What are they?

Cl: *(chorus)* 1, 2, 3, 4, 5, 6, 7, 8, 9

Tr: *(writes 1-9 on no. line between hash marks for 0 and 10 and makes one big jump from 0-10 on the no. line)* After ten we have another numbers (*makes small, single jumps to 20)*

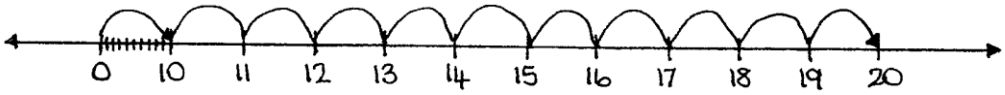
Now the representation looks like this:



Lr1: Haai, Ma'am. It's too small!

Tr: Okay. Let's make our number line big.

The teacher leaves her first jump from 0-10 on the number line unchanged and erases numbers 20, 30, 40, 50 and 60, putting numbers 11, 12, 13, 14, 15, 16, 17, 18, 19, and 20 in their place and extending the number line as needed. Now the teacher makes 10 big single jumps from 10 to 20 on the number line. These changes and the teacher's actions on the number line leave the representation looking like this:



Lr1: Haai, Ma'am. That doesn't fit – it's too big!

Research shows that number lines, which model the counting sequence, can provide learners with the mental imagery needed for calculation strategies (Beishuizen, 1999, as cited in Anghileri, 2006). Deborah's choice of a number line representation for learners to use as an a thinking tool (Cobb et al., 1992) for addition was thus a good choice here. However, *how* Deborah used the representation in these worked examples did not provide learners with greater access to the mathematical concept or procedure being taught (Rowland, 2008) for many reasons. Firstly, the number range of the examples was so limited that learners did not need to use the number line to add – they simply used a recalled fact; the choice of a well-known double (10+10) also negated the need for adding on a number line as doubles are some of the easiest number facts for children to remember (Anghileri, 2006); and finally, the way in which Deborah demonstrated how to add on the number line resulted in inconsistent calibrations in her number line representation, making it procedurally confusing. Learners' disagreement with the teacher's actions indicates at least some awareness of these inconsistencies. Across Deborah's second lesson the representations that she used to demonstrate worked examples were used flexibly and were also suited to the activity of sharing. Here Deborah showed awareness of representational progression by using representations that ranged from concrete to more abstract (Ensor et al., 2009)

5.7 Summary of Deborah's teaching

In both worked examples (2+6 and 10+10) presented in the first lesson there is a reversion to the unit counting pointed out by Schollar (2008) and Ensor et al. (2009). Deborah appears to have difficulty in working flexibly with the number line in increments of 2 in the context of addition – where she works in unit counting only. This occurs in the context of a representation which, initially at least, provides scaffolds into grouped counting – a phenomenon noted by Askew and Venkat (forthcoming). This suggests a fragmented and disconnected way of working across tasks.

Looking across both observed lessons there is evidence that Deborah has taken account of variables in a few aspects of the examples she selected (e.g. the calibrations on the number lines used in the first lesson) and the type of examples used (word problems and context-free problems) in the learner exercise in the

second lesson) although these were not always used successfully. Deborah should have focused on using greater variation in the number range of all examples and in the types of problems presented to learners, i.e. not only 'join' type of addition problems. Deborah took account of sequencing by: first presenting a general or worked example before a similar learner exercise; presenting all examples in an 'easy-to-difficult' sequence; and making successive examples different in terms of the number of digits used – all pointing to controlled rather than random sequencing of examples. In episode 2, example a) of the second lesson Deborah presents a non-example of equal sharing which shows awareness of Merrill's fifth organising principle regarding sequencing, i.e. matching non-examples to examples. In terms of exemplification in representations, Deborah moved easily between multiple representations of the mathematical concept in both lessons which showed representational flexibility, and her use of concrete (e.g. play dough), iconic (drawings) and symbolic-syntactic representations in the second lesson (Ensor et al., 2009) shows that Deborah took account of progression, i.e. using progressively more abstract representations.

5.8 Zelda - 1st Lesson Overview: Addition 29 Aug 2012

Ep	Task	Examples	Activity	Representations
1. wh /cl	Count in: 3s from 3 to 30, 2s from 6 to 30, 2s from 27 to 1	a) multiples of 3 from 3 – 30 b) multiples of 2 from 6 – 30 c) count backwards in 2s from 29 – 1	Each lr states the next no. in the sequence as tr points to him/her	No reps used
2. wh /cl	Match no.s 1-15 to no. names	a) 15 b) 6 c) 4 d) 11 e) 10 f) 1 g) 1 h) 9 I) 8 j) 12 k) 5 l) 2 m) 3 n) 13 o) 7 p) 14	Number names for 1-15 are written in sequence in 2 columns on board. Tr hands lrs a no. in symbolic form on card one at a time and asks them to match no.s to words	no. names one to fifteen symbolic no.s 1- 15 on cards

Ep	Task	Examples	Activity	Representations
3. wh /cl	Correctly place no.s 0-15 on number line.	a) 8 b) 7 c) 3 d) 11 e) 0 f) 13 g) 15 h) 1 i) 5 j) 6 k) 4 l) 12 m) 9 n) 14 o) 10 p) 2	Tr 'fishes' a fish-shaped card with no. 8 from a bowl and places it on no. line, counting from first hash on no. line from 0 in ones with the wh/class joining in the count. Lrs come to front to fish. The 1st lr gets 7 and correctly places it on the no. line without any overt counting. The 2nd lr places 3 – counting from 0 in ones. The 3rd lr gets 11 and starts counting from 0. The tr intervenes and shifts the lr to 'count on' from the 8 already on no. line to find 11. Lrs now continue placing no.s on no. line using 'count on' instead of 'count all'. Tr asks lr9 – lr15 to explain where they'll place their no.s using words like: before, after, between.	No. line on board Huge bowl (pond) with fish-shaped numbered cards (0-15) in it and 3 magnetic fishing rods
4. wh /cl	Adding on a no. line	a) 12+3 b) 5+7	Tr uses a rabbit to demonstrate 'jumps' on no. line. Tr explains that the 1st no. in the sum is where the rabbit starts and places the rabbit above no. 12 on the no. line. Then the tr explains that the 2nd no. in the sum is the no. of jumps the rabbit makes to the right on no. line. Tr explains that where the rabbit 'lands' is the answer. Tr completes number sentence by writing in answer. Once tr has demonstrated the first sum, she gets lrs to explain how to do the 2nd sum. Tr acts out lr instructions using the rabbit and no. line on board.	no. line with no.s 0-15 marked on sums given in symbolic form jump action on no. line
5. ind ivi/ wk	Filling in missing no.s on no. lines and use those no. lines to add.	5 no. lines – only 1st no. line with all markings 0-15 a) 10+4 b) 7+8 c) 9+2 d) 11+3 e) 6+6	Lrs are given a w/sheet with 5 no. lines spaced across an A4 page: one marked no. line (0-15) and 4 empty no. lines that have 0 marked at one end and space to go to 15. Lrs fill in missing no.s on no. lines and completes addition exercise while tr facilitates. Lrs are seen completing the exercise in diff ways: many lrs use their bead strings to add, others use their fingers, some use the no. lines on the w/sheet.	no. lines (0-15) bead strings available on each table a) 10+4 b) 7+8 c) 9+2 d) 11+3 e) 6+6

5.9 Zelda - 2nd Lesson Overview: Sharing 31Oct 2012

Ep	Task	Examples	Activity	Representations
1. wh /cl	Describe what 'estimate' means. Estimate the no. of chess pieces on the board. Count in 2s to find the actual no. of chess pieces. Which estimate was closest to the answer?	Learners offer: a) S - 4 b) G - 8 c) L - 30 d) M - 40 Looking at no.s 4, 8, 30, 40 from previous segment of episode 1	A few lrs call out 'guess' and tr affirms the answer. 4 lrs offer estimates: 4, 8, 30, 40. Tr writes estimates on board next to learners' initials. 2 lrs at the board move chess pieces in 2s while wh/class count in 2s out loud. They get 32. Lrs call out 'Levi was closest' and tr affirms their answer.	32 chess pieces. 4, 8, 30, 40 written in symbolic form Tr verbally repeats '32' and writes 32 on board in symbolic form. No rep given in acceptance of correct answer.
2. wh /cl	Share 32 chess pieces equally between 2 children	Share 32 chess pieces equally btwn 2 children	Tr introduces the task with a story about a mother with many children – two of whom want to play chess with the pieces counted in episode 1. When tr asks how these children can separate the pieces so that they can play, one lr calls out 'give the boy white and the girl black' and another says 'share the pieces'. Tr demonstrates sharing the corresponding black and white chess pieces 'one-by-one' between the 2 children.	32 magnetic chess pieces No rep given for final answer
3. wh /cl	Share 3 chocolates equally between 3 children Have chocs been shared equally, is it fair?	Share 3 chocolates between 3 children	Tr demo sharing 3 'chocolates' (magnetic cards in diff colours) between 3 children – each child gets 1 choc. Tr asks class the question and they call out 'yes'.	play dough to rep children and magnetic cardb/d to rep chocolates. Tr verbally repeats 'yes, it's fair because each child gets one'.
4. wh /cl	Share 5 chocolates equally between 4 children	Share 5 chocolates between 4 children	Tr calls lr to board to share chocs. Lr gives each child 1 choc and 'cuts' remaining choc into 2 equal pieces by drawing a line down the centre with a pen. Tr says 'the last choc was cut into 2 equal pieces but we need 4 pieces – what should we do next?' With help from the tr and another lr the 2 pieces are 'cut' again into 4 equal pieces.	play dough to rep children and magnetic cardb/d to rep chocolates. No representation (verbal/symbolic) given to show how much chocolate each child received.

Ep	Task	Examples	Activity	Representations
5. wh /cl	Share 14 fun fair tickets equally btwn 3 children	Share 14 tickets equally between 3 children. If 1 ticket equals 1 ride how many rides will each child get? How can we make the sharing fair? What can be done with the extra 2 tickets?	3 lrs are called to the front to share tickets until all tickets have been shared. One of the 3 lrs shares the tickets one-by-one – 2 lrs get 5 tickets each and 1 lr gets 4 tickets. One of the 3 lrs answers 'I'll get 4 and they will each get 5 rides'. Tr asks class if this sharing was fair and they call out 'no!' A lr on the mat says 'cut the tickets' and others shout 'No!' When teacher asks why tickets can't be cut a few lrs say 'then you won't get a ride'. Lrs offer 'buy another ticket and give each child 5 rides' and 'the mother can use it to go on 2 rides'.	Tickets
6. pr/ wk	Exercise: * R & K wants to share 3 choc bars equally. Show them how to do it. * S, K, & T want to share 4 choc bars equally. Show them how to do it. * S, K, P, & T want to share 5 choc bars equally. Show them how to do it.	* R & K wants to share 3 choc bars equally. Show them how to do it. * S, K, & T want to share 4 choc bars equally. Show them how to do it. * S, K, P, & T want to share 5 choc bars equally. Show them how to do it.	Tr explains the exercise to lrs before lrs go to their desks. Lrs use manipulatives already on their desks (play dough, colour paper strips, scissors) to solve the sharing word problems handed out by tr. Each pair receives 1 of the word problems listed.	sharing word problems presented in written form. No symbolic reps used.

Zelda – Taking account of variables

When looking for evidence of taking account of variables within and across lessons presented by Zelda, I asked myself these mathematical questions: What changes? and, What stays the same? (Watson & Mason, 2006a).

In episode 1 of Zelda's first observed lesson the aspects of the examples (i.e. counting routine) that could vary were: the interval or size of the count, the direction of the count, and the start and end points of the counting sequences. Zelda used variation in all these aspects, i.e. she varied the size or interval of the count using 3 and 2; she varied the direction of the counting routine: forward and backward; and

she varied the start and end points, i.e. starting the counting in 3s at 3; starting the counting in 2s at 6; and starting the backward counting in 2s at 29 – an odd number. By using variation in this way Zelda also varied the level of difficulty in this mental starter activity because counting from the first number in a counting sequence, e.g. starting at 3 when counting in threes, is easier than starting at a number that is further along in the counting sequence, e.g. starting at 6 when counting in twos. Anghileri (2006, p. 33) maintains that the latter is more demanding because she likens it to 'trying to complete the lines of a song or poem but starting in the middle'.

In episodes 4 and 5 of Zelda's first lesson the 'join-result unknown' conception of addition remained invariant in the worked examples and learner exercises (Carpenter et al., 1999). While there is a chance that if learners are presented with examples for addition in which all examples have a certain property 'then in the absence of counter-examples, the mind assumes the known properties to be implicit in other contexts' (Tall, 1991, p. 10). Given that data is drawn from a single lesson here it may be that other conceptions are dealt with on other occasions.

One aspect of Zelda's examples that did vary in these episodes was the position of the bigger addend. For example, in the first worked example the bigger addend was first (12+5) which meant that the number of jumps needed to complete the calculation on the number line was relatively limited compared to the second worked example in which the bigger addend was positioned second (5+7). Adding with the bigger addend positioned first is relatively easier for the learner because less jumps have to be made on the number line. Thus the variation here (i.e. using a simpler example first) was potentially suited to the introduction of the idea of commutativity, but this was not taken up in the enactment. The invariance of the 'rule' given further tends to limit flexibility, while there is no example with a larger 2nd addend given in episode 5.

Across episodes 3 and 4 in Zelda's second observed lesson the aspects of the examples that could vary from episode 3 to 4 were: the type of objects to be shared, the number of objects to be shared, the representation of those objects, the number of children it had to be shared amongst, and the action that would be needed to share it. Here Zelda chose to keep the type of objects to be shared (i.e. chocolates)

and the representation of those objects (i.e. magnetic cardboard pieces) invariant while the other aspects were varied. Watson and Mason (2006b) maintain that if too many aspects of successive examples change at once, the learner is likely to overlook possible variation and attempt successive problems individually without seeing the underlying mathematical structure in the set of examples. The aspects that Zelda did vary necessitated a slightly different action on the part of the learner (i.e. 'cutting' the remaining chocolate) which also worked well because attention was drawn to this aspect which learners had to appropriate in their pair-work later on.

Zelda took account of variables in many other examples and one of the most salient of these instances occurred in episode 5 of the second observed lesson. In the preceding episode of this lesson (as discussed above) the objects to be shared were 'chocolates' and the action that was needed to share the chocolates equally was to 'cut' the chocolates. In this successive example we see three learners sharing 14 fun fair tickets equally and getting 4 fun fair tickets each with 2 tickets left over. When asked what should be done with the tickets that were left over some learners offered: 'Cut the tickets'. Other learners objected to this suggestion and by way of a quick class discussion led by the teacher, all the learners agreed that the left over tickets could not be cut because this would render them useless. Here Zelda's use of variables in the objects to be shared was well thought out because sharing tickets was a counter-example that highlighted the difference between things that can be 'cut' when sharing and those that cannot. Zelda's gaze appeared to be on getting learners to practice sharing while working in pairs, which was what took place in the next episode. Zelda's choice of fun fair tickets in this example also shows her taking account of representations within context – because, depending on what is being shared, some things can be cut while others cannot.

Zelda – Taking account of sequencing

According to Rowland (2008) examples presented in class can be *controlled* (i.e. the teacher controls the sequence of numbers) or *randomly* generated (i.e. where the teacher does not have control over the sequence, e.g. obtaining a sequence of numbers by throwing a dice). Zelda used both random and controlled sequencing in examples presented in the lessons I observed. Worth noting here is the multiplicity of examples used in both lessons within tasks.

The manner in which Zelda used the sequencing of examples in episode 3 of the first lesson is an instance where a teacher used random sequencing successfully to achieve the intended outcome of the activity which is in contrast to how a pre-service teacher used random sequencing reported on by Rowland (2008). In the start of this episode Zelda told the class a story about a beautiful pond with many special fish – special, because each fish was numbered with a number from zero to fifteen. The activity involved 'fishing' a fish-shaped card (using a magnetic fishing rod) out of the pond (a huge bowl) and placing these numbered fish in numerical order on a number line drawn on the board (with hash marks for 16 numbers). Here is an extract from the transcript of this episode:

Tr: There are many fishes in the pond. These fish have numbers on them. When we fish we must place the fish on the number line.

(The teacher 'fishes' number 8. Starting at the first hash mark, the whole class helps her to count in ones from 0 to the 8th hash mark on the number line. The teacher correctly places fish no. 8. Now she calls learners to the front to 'fish'. Once a learner has 'fished' a number s/he must say the number out loud to the class and then place it on the number line.)

Lr1: I've got seven.

Tr: Where will number seven go on the number line?

(Lr1 points to the hash mark before fish number 8 previously placed on the number line.)

Tr: Very good.

(Lr1 places fish no. 7 correctly on the no. line without any overt counting)

Lr2: I have fish number three.

(Starting at the first hash mark, Lr2 counts incorrectly in ones from 0 and is helped by the teacher and class to recount. Lr2 correctly places fish number 3.)

Lr3: I have number eleven.

(Lr3 starts counting from the first hash mark in ones from 0 on the number line and then the teacher intervenes…)

Tr: Wait. Why are you counting from zero? Is eleven bigger than eight? *(pointing to 8 on no. line)*

Lr3: Yes

Tr: So you can count on from eight to find the place for number eleven.

Lr3: Eight, nine, ten, eleven. *(Lr3 counts on from fish no. 8, pointing to each hash mark with his finger, and places fish no. 11 correctly on the number line)*

Lr4: I've got thirteen.

Tr: Now look nicely at the number line.

Lr4: Eleven, twelve, thirteen. *(Lr4 counts on from no. 11 and correctly places fish no. 13 on the number line)*

By asking learners to 'fish' numbers out of the 'pond', the following random sequence of numbers was generated: 8, 7, 3, 11, 0, 13, 15, 1, 5, 6, 4, 12, 9, 14, 10 and 2. This random sequence worked well for this task because learners had to identify the symbolic number they 'fished' as numbers were not fished in numerical order. Learners who were struggling with identifying symbolic numbers (like Lr6 who struggled to identify number 15) were noticed and immediately helped. This random sequencing of numbers also helped the teacher to shift learners (e.g. Lr3) from using the 'count all' to the more efficient 'count on' strategy. After observing the teacher's interaction with Lr3 other learners also used 'count on' rather than 'count all' when ordering numbers on the number line. During this episode Zelda also used the random sequencing of numbers generated by the 'fishing' activity to help Lrs 9 - 15 to use correct mathematical terms like 'before', 'after' and 'between' to describe the placement of their numbered fish.

In episode 1 of the second lesson Zelda called on four learners to give an estimate of the number of magnetic chess pieces on the board and she wrote this random sequence of numbers on the board (i.e. 4, 8, 30, 40) which was then compared to the actual number of chess pieces counted by the class (i.e. 32). Here Zelda had no control over the numbers that learners would offer or the sequence of the numbers. The first offer (4) was very small and the second (8) was double the first but still far from the actual number. The third offer was a better estimate (30) and I believe the teacher's verbal affirmation of this offer (good), prompted the last learner to also give a relatively bigger number than the first two learners, viz. 40.

In episode 2 of the second lesson Zelda used controlled sequencing of symbolic numbers given to learners which they had to match to the number names written previously on the board. The number names from one to fifteen were written in sequence on the board but the order of the symbolic numbers was controlled by Zelda (who wrote a number on a blank card as learners come to fetch a card, while keeping her eye on the board to see which numbers have not yet been used). The order in which Zelda wrote the symbolic numbers were 15, 6, 4, 11, 10, 1, 1, 9, 8 12, 5, 2, 3, 13, 7 and 14. I am sure Zelda had a reason for asking learners to match the symbolic numbers in this sequence to the number names written in order from one to fifteen. By looking at the sequence of the symbolic numbers chosen by Zelda

it seems as if Merrill's third prescription regarding micro sequencing, i.e. arranging examples in a divergent sequence, as cited in Van Patten et al. (1986) was used an as organising principle because she alternated between bigger and smaller numbers which could have made the task of identifying symbolic numbers and matching them to their number names more challenging for her Grade 1 learners. Because the categories of exemplification used for my study are not distinct, Zelda's selection and use of examples here could also be discussed under 'taking account of variables' because she kept the order of the number names fixed (from one to fifteen) and varied the sequence of the symbolic numbers.

Zelda – Taking account of representations

Literature suggests that learners' understanding of number is enhanced when teachers use a few research-based representations rather than many different representations (Askew & Brown, 2003). Across both observed lessons Zelda seemed to be very aware of the benefit different types of representations could have on her young learners' understanding of mathematical concepts. Zelda used a few well thought out representations in a planned (and sometimes spontaneous) manner that were suited to the tasks, e.g. number line for adding using a 'count on' strategy; and suited to the learners' interests, e.g. using stories to set the contexts of worked examples.

Episode 1 from the first lesson presented by Zelda was an oral starter activity that involved learners counting forward and backward in intervals of 3 and 2 as directed by the teacher. No representation was used for most of this task but towards the end of the last counting routine (counting backwards in 2s from 29) learners got stuck and Zelda referred them to the 100 wall chart which was hanging on a window towards the back of the class. The use of the 100 wall chart seemed like a spontaneous decision (because all other representations that were used in the lesson were either stuck on the board in the front of the class or kept close-by on Zelda's table). Here the spontaneous use of a representation afforded the learners greater access to skip counting backwards from an odd number which they struggled to do mentally. I believe that in this episode Zelda's PCK was evident because she knew almost instinctively which representation (of those available in the class) could be used successfully as a mediating tool.

Literature within the educational landscape points to the importance of connections being established between different representational forms (e.g. words, pictures, symbols, and concrete experiences) and between different mathematical ideas and different facets of the mathematics curriculum in the teaching and learning of early number (Askew et al., 1997; Haylock & Cockburn, 2008). Bearing the importance of connections in mind as I observed Zelda's lessons, I became increasingly aware of the ample empirical evidence that shows how Zelda has taken account of representational connections. For example, during episodes 2, 3, 4 and 5 of the first lesson a clear connection is established between the words (written and spoken), symbols, pictures, and actions that Zelda used to explain adding on a number line. The number line that was constructed in episode 3 was also used in episode 4 (when Zelda demonstrated how to add on the number line) and replicas thereof was used on learners' worksheets in episode 5 for individual work. Across these four episodes the same number range, and thus also the same number symbols and spoken and written words were used in successive inter-linking tasks. However, in episode 4 of this lesson, the teacher's worked example (5+7) showed evidence of lack of flexibility of strategy. Following the use of 'count all' which was appropriate for 12+3, Zelda used the 'count all' strategy for 5+7 as well, whilst she shifted learners from 'count all' on the number line to 'count on' during the previous episode.

Zelda started episode 3 of the same lesson with a short story about a beautiful pond with numbered fish. The activity for this episode involved learners placing the fish in numerical order on the number line. In episode 4 the story continued with the introduction of a rabbit (wearing a tube because he couldn't swim) who troubled these numbered fish because he needed them to add. Zelda's word explanation of how the rabbit jumped from one numbered fish to the next to add (e.g. 'The first number told Rabbit where to start and the second number told him how many jumps to make'); her action of moving the rabbit (made of cardboard) on the number line; the picture of the rabbit's 'jumps' drawn with chalk on the number line; and the symbolic number sentence written on the board to show the calculation – all connected well to the idea of making 'jumps' on the number line to add. Here Zelda used multiple representations of the mathematical object and also displayed an ability to flexibly shift between these different representational forms. Meaningful connections were also established between the language used, the pictures, the

action on the number line, and the symbols used in the number sentence (Haylock & Cockburn, 2008). Even though this lesson consisted of different tasks that were progressively more challenging, these tasks flowed easily into each other – each one building on the former by using the same representations and number range – so that shifting from one task to the next was seamless.

5.10 Summary of Zelda's teaching

Zelda took account of variables in different ways. In episode 1 of the first lesson Zelda used variables *within* the episode by varying the size of the count, the direction of the count, and the start and end points of the counting sequence. In the second lesson Zelda takes account of variables *across* episodes 3, 4 and 5 by varying the objects to be shared, i.e. 'chocolates' in episodes 3 and 4 and then fun fair tickets in episode 5, which made learners realise that they have to consider the context of the problem when choosing a problem solving technique. Zelda did not take account of variables by presenting all the addition examples in episodes 4 and 5 of the first lesson as 'join-result unknown' problems which Carpenter et al. (1999) argue limits the range of addition problems learners will be able to solve when they encounter other conceptions of addition like 'separate', 'part-part-whole' and 'compare' problems. Zelda's use of controlled sequencing in the second lesson and random sequencing during episode 3 of the first lesson was used successfully.

With regard to her representation of examples, Zelda used a few well thought out representations in a pre-planned manner – but also showed an ability to use representations in a 'contingency' situation to make the mathematical concept more accessible to the Grade 1 learners in her class. Zelda's use of representations during the first observed lesson showed her ability to use progressively more abstract representations of the mathematical object, and during episode 3 of the first lesson she shifts the focus from encouraging learners to 'count on' using the number line to describing the order of numbers using mathematical terms like 'before', 'after' and 'between'. Here Zelda used the same representation during a single activity in a flexible manner by broadening the pedagogic intent to stimulate mathematical thinking in her learners. What stood out about Zelda's use of representations across her course-work tasks and observed lessons was the coherently connected way in which she used the selected representations.

5.11 What comparisons can be drawn between the two teachers?

To answer this question I draw on all data collected during the course of this study. My comparison of the two teachers' choice and use of examples in the context of their teaching is structured around three of Rowland's categories of exemplification and the essential components of number sense as discussed in Chapters 2 and 3 of this report.

With regard to taking account of variables – both teachers' addition examples (worked examples and learner exercises) only used the 'join - result unknown' type of problem. These teachers could have used either the 'compare', 'separate' or 'part-part-whole' conceptions of addition in some of the problems, or vary the 'unknown' in each problem, to bring in some variation and expose learners to a broader range of addition problems. Although no variation was used in the type of addition problem used by both teachers, variation was evident in other aspects of teachers' lessons with differing levels of success. Deborah varied the calibrations of the number lines she used for worked examples during the first lesson but got the wrong answer and evoked passionate disagreement from an outspoken learner regarding her use of number lines when adding. In contrast, when Zelda took account of variables, e.g. during episodes 3 and 4 of the second lesson, she did not vary too many aspects thereby encouraging learners to see tasks as 'conceptually related' instead of separate, individual tasks (Zawijewski and Silver, 1998, as cited in Watson & Mason, 2006b). What is also interesting to note is that both teachers selected the same sharing examples for the learners' exercise in their second lessons even though one class is Grade 1 learners and the other Grade 2. This raises a question with regard to the number range of the examples Deborah is exposing her Grade 2 learners to.

Both teachers took account of sequencing in their selection and use of examples by making use of Merrill's second (generality before example), third (divergent sequence) and fourth (easy-to-difficult sequence) specific prescriptions regarding micro sequencing (Merrill, 1983, as cited in Van Patten et al., 1986). Both teachers presented a general worked example to the class before expecting learners to attempt similar examples in their learner exercise either as pair-work or as individual tasks. Both teachers also used different successive examples, e.g. Deborah varied the number of digits in examples during her first lesson and Zelda used tickets as

the objects that should be shared which could not be 'cut' like the chocolates shared in the previous example in her second lesson. Both teachers used controlled sequencing of examples so that examples were more or less graded in an easy-to-difficult sequence, but only Zelda used random sequencing in her lessons (e.g. lesson 1 episode 3; lesson 2 episode 1).

Both teachers introduced the number line as a 'thinking tool' that learners could use when adding (Cobb et al., 1992) and this was a judicious choice of representation because literature points to number lines providing learners with good mental imagery for calculation strategies (Anghileri, 2006). However, as noted already, Deborah struggled to connect her non-unit calibrated number line to her worked examples on addition, while Zelda's explanation and demonstration regarding the use of a number line was always coherently connected. The incidence of disrupted connections in Deborah's use of representations is higher than Zelda's. Deborah's choice of examples negated the usefulness of her representations because most learners appeared able to solve the problems mentally.

During Deborah's second lesson, representations were used in progressively more abstract ways, i.e. from the concrete sharing of play dough, to iconic pictures of sharing drawn on the board, and eventually symbolic-syntactic number sentences (Ensor et al., 2009). Similar representational progression was evident in Zelda's first observed lesson, i.e. acting on concrete objects ('fishing' numbers out of a 'pond' using magnetic fishing rods), using iconic cartoon-like drawings of fish and a rabbit, using a number line to add (symbolic number-based), and using number sentences to show each calculation (symbolic-syntactic). After observation of Zelda's second lesson (on sharing) I asked her what the next step in the lesson would be because the lesson was not finished when the bell rang to signal the start of recess. Through her explanation of what would take place after recess (i.e. that the learners would paste the strips they had shared/cut into their books and also write their sharing sum using symbols) I believe that progression to a more abstract representation of the problem was to follow.

A notable difference across teachers' lessons was that the number of examples presented by Zelda as *examples of* something, as well as the number of *examples for* practicing the concept or procedure learnt, was higher than that presented by

Deborah. Further, the number range in the examples presented by Deborah was generally lower than that of the examples presented by Zelda, while the learner exercise given for practicing sharing was the same. These differences in the number range used and in the multiplicity of the two teachers' examples are significant because Zelda is teaching a Grade 1 class while Deborah is teaching Grade 2.

My interest in teachers' selection and use of examples and representations was ultimately driven by an interest in how this relates to their example spaces. As mentioned previously, a teacher's example space concerning a particular subject or topic within that subject determines what they will make available to learners when they teach that subject or topic (Watson & Mason, 2005). Therefore a teacher's example space related to number skills and number operations has bearing on what opportunities to learn number she creates for the learners in her class. Data from my study provides empirical evidence of how these two teachers provided their learners with opportunities to learn number in different ways.

Literature shows that teaching for number sense involves (amongst others) developing an awareness of the relationship between numbers. Zelda's course-work tasks and observed lessons provide evidence of this while Deborah shows some elements of this in her course-work tasks, and less evidence thereof in her teaching (see lesson one episode 6). Calculating efficiently using mental strategies is another element of teaching for number sense that is more evident in Zelda's teaching than in Deborah's. Looking more broadly than examples and representations, Shumway (2011) states that when teachers use learners' mistakes as 'learning opportunities' this helps to create a supportive learning environment wherein learners' number sense can develop. During Deborah's first lesson (episode 3) she did not use learners' incorrect responses as 'learning opportunities' but rather continued asking the same question until someone gave the response she was looking for. In contrast, when Zelda introduced a non-example of 'cutting when sharing' in episode 5 of lesson 2, the learner's incorrect response was used as a *learning opportunity* where learners could *reflect* on answers using *math talk* – all three elements that Shumway (2011) argues is needed for a supportive learning environment wherein number sense can be developed.

By looking across all the data gathered and analysed in this study, with regard to opening opportunities for learners to develop number skills an number sense, there is evidence pointing to more connected and more extended handling of ideas in Zelda's lessons.

CHAPTER SIX – CONCLUSION

In this chapter I present my concluding remarks by summarising some of the interesting findings of my study, noting the limitations of the study, and offering recommendations for future research that may emanate from this study.

6.1 Concluding remarks

Despite the importance of teachers' examples and representations in mathematics teaching and learning evident in literature, South African-based research in this area is surprisingly sparse. My study aimed at making a small contribution to closing this gap in the South African literature base on teachers' examples and representations within the context of teaching number skills or number sense. My particular focus on number was motivated by literature which points to the importance of building children's early number skills as a foundation to increased mathematics learning, as well as the reported poor quality of mathematics teaching and learning in South Africa (WEF, 2012).

This opportunistic study was sparked through my involvement in the WMC-P Project's 20-Day course as this was where my interest in teachers' selection and use of examples and representations was ignited. The 20-Day course also provided me with the opportunity to explore the use of examples and representations within the course-tasks and teaching of two of the Foundation Phase teachers. The two teachers who participated in my study displayed differences in their range of mathematics content knowledge (CK) as per the 20-Day pre-test score: Zelda was in the higher range and Deborah in the lower. However, my interest was not in their CK because the thrust of my study was not in discovering whether these teachers could themselves do the mathematics, but rather in their understandings of which examples and representations to select (and how to use these optimally) when teaching number-related tasks so that the mathematics was more accessible to their learners. Thus my focus was more related to pedagogic content knowledge (PCK) issues. However, because the participants represented a range of outcomes in the CK focused pre-test, this provided me with an avenue for a better nuanced approach to my analysis based on literature that reports on gaps in mathematics CK and PCK amongst primary teachers (Carnoy & Chisholm, 2008; Rowland et al., 2005). Data

gathered and analysed in the course of my study showed some similarities and differences in the way participants selected and used examples and how these examples were represented.

One salient aspect of participants' selection and use of representations when explaining concepts or procedures related to addition and subtraction was their frequent use of calibrated and empty number lines. This was salient, because field notes taken during the 20-Day course show that most teachers attending the course reported that they had not used number lines as a tool when teaching *before* attending the course. This is interesting to me because the work done by Watson and Mason (2005) around example spaces state that a person's recent experience can be a trigger for accessing a particular example at a particular time. The teachers attending the 20-Day course may have had experience with number lines before coming to the course, so it is possible that these representations already existed in their example spaces. My data suggests that after participants' experience on the course, where the number line was often used as a model that supported a visual representation of ordinality and cardinality (Anghileri, 2006; Haylock & Cockburn, 2008) this representation was fore-grounded in their example spaces. Because the number line was fore-grounded in participants' example spaces, it may have provided openings for these two teachers to incorporate it into their teaching. Thus, this overlap between the two teachers' frequent use of number lines in their explanations evident in their course-work and classroom teaching may corroborate Watson and Mason's notion that example spaces can be extended and explored (Watson & Mason, 2005).

Literature within the South African primary mathematics landscape has pointed over time to a lack of representational *progression* within mathematics teaching (Ensor et al., 2009; Pietersen, 2006; Reeves & Muller, 2005; Schollar, 2008) while international literature within mathematics education has pointed to the need for representational *flexibility* (Heize et al., 2009; Nistal et al., 2009). However, there is a greater preponderance in international literature regarding the importance of establishing *connections* between different representational forms and different ideas in the teaching and learning of primary mathematics, as this is essential for the conceptual learning of mathematics (Askew et al., 1997; Ball & Bass, 2003; Cobb et

al., 1992; Graham et al., 2009; Haylock & Cockburn, 2008; Perkins & Unger, 1994; Terwel et al., 2009). So, although my study focused on these three aspects of representations, it seems like establishing coherent representational connections is the most fundamental of the three. In my study the teacher with the higher CK score (Zelda) was also the teacher who made coherent connections more consistently between different representational forms and between different ideas in mathematics. Findings from my study point to possible associations between a higher CK score and the extent of a teacher's example space and more coherent connections - which could prove interesting to explore further using a bigger sample. However, my findings are less clear on differences in terms of representational flexibility and progression. My analysis shows that the gains of flexibility and progression can be disrupted by circularity and disconnections between examples and representations used to solve some problems. The analysis of data collected during this study and the preponderance of literature regarding the importance of establishing coherent connections when teaching and learning mathematics suggests that improving representational connections may be more immediately important in order to support the work of primary mathematics teaching in the South African landscape.

6.2 Limitations

There are many limitations attached to a qualitative study such as mine related to issues of rigour – all of which have been discussed in detail in Chapter 4 of this study. Other limitations of my study, that were not directly related to qualitative studies in general, is worth mentioning here.

One of the main limitations related to my study was the relatively small data set used for analysing teachers' selection and use of examples and representations. However, for the purpose of my explorative study the small data set proved sufficient as this allowed me to do a finer-grained analysis of the data collected. Also, the findings emanating from my study were never intended to be generalised.

Another limitation of my study was that inferences drawn about participants' examples and representations used in the context of teaching number were not corroborated as initially planned (through the use of interviews) due to a time constraint. Nonetheless, trustworthiness of inferences drawn was ensured through

triangulation of data sources and thick description of how data was gathered and analysed.

I also did not take other factors such as the participants' prior teaching experience, their level of tertiary qualification, or their home languages into account, which could also have affected their selection and use of examples and representations when teaching number. This was indeed a limitation of this study.

6.3 Recommendations for future research

Through analysing teachers' treatment of examples and representations one is able to get a glimpse of the complex web of understandings and considerations underlying teachers' choices. Untangling all the relevant issues that come into play simultaneously during a teacher's example-selection process is something that researchers have begun to look at, but there are still many strands of this web of understanding that have not yet been tugged on, e.g. affective factors affecting teachers' choices. I think that more studies looking at units of teaching, rather than single lessons, regarding the considerations that teachers have to take into account when selecting and using examples and representations for teaching particular topics, should be carried out and fed back into teacher training programmes. This could assist teachers in making judicious choices with regard to selecting and using examples and representations that will ultimately enhance the teaching and learning of mathematics in their classes.

References

Anghileri, J. (2006). *Teaching number sense*. London: Continuum.

Askew, M. (1998). *Teaching primary mathematics: A guide for newly qualified and student teachers* London: Hodder and Stoughton.

Askew, M., & Brown, M. (2003). How do we teach children to be numerate? *A BERA Professional User Review* (pp. 1-17). Kings College London: BERA.

Askew, M., Brown, M., Rhodes, V., Wiliam, D., & Johnson, D. (1997). Effective teachers of numeracy: A report of a study carried out for the Teacher Training Agency. London: Kings College, University of London.

Askew, M., & Venkat, H. (forthcoming). Mediating early number learning: Specialising across teacher talk and tools? *Journal of Education, 56*.

Askew, M., Venkat, H., & Mathews, C. (2012). *Coherence and consistency in South African primary mathematics lessons.* Paper presented at the 36th Conference of the International Group for the Psychology of Mathematics Education: Opportunities to learn in mathematics education, Taipei, Taiwan.

Ball, D. L. (1993). Halves, pieces and twoths: Constructing and using representational contexts in teaching fractions. In T. P. Carpenter, E. Fennema & T. A. Romberg (Eds.), *Rational numbers: An investigation of research.* (pp. 328-375). Hillsdale, NJ: Erlbaum.

Ball, D. L., & Bass, H. (2003). *Toward a practice-based theory of mathematical knowledge for teaching.* Paper presented at the Annual meeting of the Canadian mathematics education study group.

Bassey, M. (1981). Pedagogic research: On the relative merits of the search for generaisation and study of single events. *Oxford Review of Education, 7*(1), 73-93.

Bell, J. (1999). *Doing your research project* (3rd ed.). Buckingham: Open University Press.

Bills, L., Dreyfus, T., Mason, J., Tsamir, P., Watson, A., & Zaslavsky, O. (2006). *Exemplification in mathematics education.* Paper presented at the 30th Conference of the International Group for the Psychology of Mathematics Education, Prague, Czech Republic.

Blaxter, L., Hughes, C., & Tight, M. (1996). *How to research*. Buckingham: Open University Press.

Bowen, G. A. (2009). Document analysis as a qualitative research method. *Qualitative Reseach Journal, 9*(2), 27-40.

Carnoy, M., & Chisholm, L., et al. (2008). Towards understanding student academic performance in South Africa: A pilot study of Grade 6 mathematics lessons in South Africa. Pretoria: HSRC.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). *Children's Mathematics: Cognitively Guided Instruction*. Portsmouth, NH: Heinemann.

Carpenter, T. P., & Moser, J. M. (1984). The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education, 15*, 179-202.

Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to a representational view of mind in mathematics education. *Journal for Research in Mathematics Education, 23*(1), 2-33.

Cohen, L., & Manion, L. (1999). *Research Methods in education* (4th ed.). London: Routledge.

Cohen, L., Manion, L., & Morrison, K. (2000). *Research methods in educaton* (5th ed.). London: Routledge Falmer.

DBE. (2011a). *Curriculum and Assessment Policy Statement (CAPS): Foundation Phase Mathematics, Grade R - 3*. Pretoria: DBE.

DBE. (2011b). *Report on the Annual National Assessments of 2011*. Pretoria: DBE.

Dreyfus, T., & Eisenberg, T. (1996). On different facets of mathematical thinking. In R. J. Sternberg & T. Ben - Zeev (Eds.), *The nature of mathematical thinking.* (pp. 253-284). Hillsdale, NJ: Erlbaum.

Ensor, P., Hoadley, U., Jacklin, H., Kuhne, C., Schmitt, E., Lombard, A., & van den Huewel-Panhuizen, M. (2009). Specialising pedagogic text and time in Foundation Phase numeracy classrooms. *Journal of Education, 47*, 5-29.

Feldman, A. (2003). Validity and quality in self-study. *Educational Researcher, 32*(3), 26-28.

Fennell, F., & Landis, T. E. (1994). Number sense and operation sense. In C. A. Thornton & N. S. Bley (Eds.), *Window of opportunity: Mathematics for students with special needs* (pp. 187-203). Reston, VA: National Council of Teachers of Mathematics.

Fraenkel, J. R., Wallen, N. E., & Hyun, H. H. (2012). *How to design and evaluate research in education* (8th ed.). New York: McGraw-Hill.

Gelman, R., & Gallistel, C. (1978). *The child's understanding of number*. Cambridge: Harvard University Press.

Goldenberg, P., & Mason, J. (2008). Shedding light on and with example spaces. *Educational Studies in Mathematics, 69*, 183-194.

Graham, A. T., Pfannkuch, M., & Thomas, M. O. J. (2009). Versatile thinking and the learning of statistical concepts. *ZDM Mathematics Education, 41*, 681-695.

Haylock, D., & Cockburn, A. (2008). *Understanding mathematics for young children*. London: Sage.

Heize, A., Star, J. R., & Verschaffel, L. (2009). Flexible and adaptive use of strategies and representations in mathematics education. *ZDM Mathematics Education, 41*, 535-540.

Hershkowitz, R. (1989). Visualization in geometry - two sides of the coin. *Focus on Learning Problems in Mathematics, 11*(1), 61-76.

Howden, H. (1989). Teaching number sense. *The Arithmetic Teacher, 36*(6), 6-11.

Kellogg, R. (1980). Feature frequency and hypothesis testing in the acquisition of rule-governed concepts. *Memory and Cognition, 8*, 297-303.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Leedy, P. D., & Ormrod, J. E. (2010). *Practical research: Planning and design* (9th ed.). Upper saddle River, N. J.: Pearson Education.

Lincoln, Y. S., & Guba, E. G. (1990). Judging the quality of case study reports. *Qualitative Studies in Education, 3*(1), 53-59.

Marshall, C., & Rossman, G. B. (2010). *Designing qualitative research*. London: Sage.

Marshall, M. N. (1996). Sampling for qualitative research. *Family Practice, 13*(6), 522-525.

Mason, J., & Johnston-Wilder, S. (2006). *Designing and using mathematical tasks*. St Albans, England: Tarquin Publications.

Mason, J., & Pimm, D. (1984). Generic examples: Seeing the general in the particular. *Educational Studies in Mathematics, 15*(3), 277-290.

Maxwell, J. A. (1992). Understanding and validity in qualitative research. *Harvard Educational Review, 62*(3), 279-300.

McIntosh, A., Reys, B. J., & Reys, R. E. (1992). A proposed framework for examining number sense. *For the Learning of Mathematics, 12*(3), 2-8.

Merriam, S. B. (1995). Theory to Practice. What can you tell from an N of 1?: Issues of validity and reliability in qualitative research. *PAACE Journal of Lifelong Learning, 4*, 51-60.

Merriam, S. B. (2001). *Qualitative Research and Case Study Applications in Education*. San Francisco: Jossey - Bass.

Merriam, S. B. (2002). *Qualitative research in practice: Examples for discussion and analysis* (1st ed.). San Francisco, CA: Jossey-Bass.

Michener, E. (1978). Understanding understanding mathematics. *Cognitive Science, 2*, 361-383.

Moloi, M. Q., & Chetty, M. (2010). *The SACMEQ III Project in South Africa: A study of the conditions of schooling and the quality of education*. Pretoria: Ministry of Basic Education.

Moore, R. C. (1994). Teaching number sense in the elementary school. *Adventist Education*, 12-17.

Mullis, I. V. S., Martin, M. O., & Fay, P. (2008). TIMSS 2007 International Mathematics Report: Findings from IEA's Trends in International Mathematics and Science Study at the Fourth and Eighth Grades. Chestnut Hill, MA: TIMSS & PIRLS International Study Centre, Boston College.

Nistal, A. A., Van Dooren, W., Clarebout, G., Elen, J., & Verschaffel, L. (2009). Conceptualising, investigating, and stimulating representational flexibility in mathematical problem solving and learning: A critical review. *ZDM Mathematics Education, 41*, 627-636.

Opie, C. (2004). *Doing educational research - a guide to first time researchers*. London: SAGE Publications.

Pape, S. J., & Tchoshanov, M. A. (2001). The role of representation(s) in developing mathematical understanding. *Theory into Practice., 40*(2), 118-127.

Perkins, D. N., & Unger, C. (1994). A new look in representations for Mathematics and Science. *Instructional Science, 2*, 1-37.

Pietersen, C. (2006). Evaluation of a number skills development programme. *South African Journal of Education, 26*(3), 413-426.

Reeves, C., & Muller, J. (2005). Picking up the pace: Variation in the structure and organisation of learning school mathematics. *Journal of Education, 37*, 103-130.

Reys, R., Reys, B., McIntosh, A., Emanuelsson, G., Johansson, B., & Yang, D. (1999). Assessing number sense of students in Australia, Sweden, Taiwan, and the United States. *School Science and Mathematics, 99*(2), 61-70.

Rowland, T. (2008). The purpose, design and use of examples in the teaching of elementary mathematics. *Educational studies in mathematics, 69*, 149-163.

Rowland, T. (2012). *Explaining explaining.* Paper presented at the 18th Annual National Congress of the Association for Mathemaics Education of South Africa

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education., 8*, 255-281.

Scaife, J. (2004). Reliability, validity and credibility. In C. Opie (Ed.), *Doing educational research* (pp. 58-72). London: Sage.

Schollar, E. (2008). Final Report: The primary mathematics research project 2004-2007 - Towards evidence-based educational development in South Africa. Johannesburg.

Schumacher, S., & McMillan, J. (1993). *Research in education* (3rd ed.). New York: Harper Collins College Publishers.

Sherman, R. R., & Webb, R. B. (1988). *Qualitative research in education: Focus and methods*. London: Routledge Falmer.

Shumway, J. (2011). *Number sense routines: Building numerical literacy every day in K - 3*. USA: Stenhouse Publishers.

Sin, S. (2010). Considerations of quality in phenomenographic research. *International Journal of Qualiative Methods, 9*(4), 305-318.

Smith, E., Shoben, E., & Rips, L. (1974). Structure and process in semantic memory: A featural model for semantic decisions. *Psychological Review, 81*, 214-241.

Sowder, J. T., & Schappelle, B. P. (1989). Establishing Foundations for Research on Number Sense and Related Topics: Report of a Conference. San Diego, California: National Science Foundation.

Stake, R. E. (1995). *The art of case study research*. Thousand Oaks, California: Sage.

Stylianou, D. A. (2010). Teachers' conceptions of representation in middle school mathematics. *Journal of Mathematics Teacher Education., 13*, 325-343.

Suh, J., & Moyer, P. S. (2007). Developing students' representational fluency using virtual and physical algebra balances. *The Journal of Computers in Mathematics and Science Teaching, 26*(2), 155-173.

Tall, D. (1991). The psychology of advanced mathematical thinking. In D. Tall (Ed.), *Advanced Mathematical Thinking*. Dordrecht: Kluwer Academic Publishers.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in Mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*, 151-169.

Terwel, J., van Oers, B., van Dijk, I., & van den Eeden, P. (2009). Are representations to be provided or generated in primary mathematics education? Effects on transfer. *Educational Research and Evaluation, 15*, 25-44.

The Cockcroft Report. (1982). Mathematics Counts. London: Her Majesty's Stationery Office.

Thompson, I. (2008). From counting strategies to deriving number facts. In I. Thompson (Ed.), *Teaching and learning early number* (2nd ed., pp. 97-109). London: Open University Press.

Van Patten, J., Chao, C., & Reigeluth, C. M. (1986). A review of strategies for sequencing and synthesizing instruction. *Review of Educational Research, 56*(4), 437-471.

Venkat, H. (2013). Reading between the lines: Examining 'opportunity to learn' in a sample of Eastern Cape workbooks. In S. Motala, V. Dieltiens & Y. Sayed (Eds.), *Finding place and keeping pace - Exploring meaningful and equitable learning in South African schools*. South Africa: HSRC Press.

Venkat, H., & Naidoo, D. (2012). Analyzing coherence for conceptual learning in a Grade 2 numeracy lesson. *Education as Change, 16*(1), 21-23.

Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples*. Mahwah, NJ, USA: Erlbaum.

Watson, A., & Mason, J. (2006a). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. *Mathematics Thinking and Learning, 8*(2), 91-111.

Watson, A., & Mason, J. (2006b). Variation and mathematical structure. *Mathematics Teaching (incorporating Micromath), 194*, 3-5.

WEF. (2012). The Financial Development Report 2012. New York, USA: World Economic Forum.

Wellington, J. (2000). *Educational research: Contemporary issues and practical approaches*. London: Continuum.

Wiersma, W., & Jurs, S. G. (2004). *Research methods in education: An introduction* (8th ed.). London, England: Allyn and Bacon.

Wilson, S. (1986). Feature frequency and the use of negative instances in a geometric task. *Journal for Research in Mathematics Education, 17*, 130-139.

Wright, R. J., Martland, J., & Stafford, A. K. (2006). Early numeracy: Assessment for teaching and intervention. London: Sage.

Yin, R. (2003). *Case study research: Design and methods* (3rd ed.). Thousand Oaks, California: Sage.

Zazkis, R., & Leikin, R. (2007). Generating examples: From pedagogical tool to research tool. *For the Learning of Mathematics, 27*(2), 15-21.