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Tidal effects on pulsation modes in close binary star systems

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Tidal effects on oscillation modes in close binary systems

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Dissertation submitted in partial fulfilment of the requirements for the degree Master of Science in the School of Physics at the University of the Witwatersrand

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October, 2018



Declaration of authenticity:

I declare that the research project, Tidal effects on oscillation modes in close binary systems, is my own work, and that each source of information used has been acknowledged by means of a complete reference. This dissertation has not been submitted before for any other research project, degree or examination at any university.



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This thesis is dedicated to my future wife Natasha, who is always by my side, walking with me through life, providing everything I need for the journey.

Acknowledgements

I would like to thank first and foremost my supervisors Prof. Fabio Frescura and Prof. Chris Engelbrecht for their support, patience and guidance for the duration of this thesis. I am very grateful for all the effort you generously put in to helping me complete this thesis and appreciate all the knowledge you shared.

I would also like to thank Prof. Daniel Joubert, who kindly took up the role of administrative supervisor to help me complete the submission of this thesis.

I would also like to thank Sheldon Herbst, for assisting me with a very interesting discussion on techniques for solving differential equations.

I would also like to thank the Univeristy of the Witwatersrand for giving me the opportunity to complete this work.

I would like to express my wholehearted thanks to Natasha and my family for their ongoing support during the process of acquiring this degree. Because of their unconditional love and sacrifice, I had the strength to complete this thesis.

Abstract

This thesis investigates theoretically the effect of tides on pulsation modes in close binary star systems. It demonstrates by construction of a rudimentary model the ability of tidal forces to excite pulsations in the component stars provided that the orbit of the binary is either eccentric and /or asynchronous. An asynchronous system with perfectly circularised orbit is able to drive pulsations at a single frequency. An eccentric orbit drives pulsations at a variety of frequencies. The magnitude of the response of the driven system depends on how close the driving frequencies are to its natural pulsation frequencies.

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Chapter 1

Introduction

Current models of stellar structure do not incorporate in a realistic way five key features of real stars. These are: convection, the non-linearity of real stellar pulsations, the effects of rapid rotation, the effects of magnetic fields, and the effects of dynamic tides on stellar structure and evolution. Convection is treated by mixing length theory, which is known to underestimate severely the rate of energy transport through stellar materials [1]; Stellar pulsation is treated by a linear approximation of the equations of motion that is valid only in the limit of infinitesimally small pulsation amplitudes [2]; No existent model has adequately taken into account all of the effects of rapid rotation [3], magnetohydrodynamics and dynamic tides on stellar structure, convective layers, pulsation, and on observational features such as the shape of spectral lines. These shortcomings in the existent models are due principally to two factors: past shortage of computational power, and poorly developed theories of the underlying physical phenomena. Recent advances in both of these fields make it possible to address these shortcomings.

Until recently, it was thought that binary systems formed the majority of all stellar configurations constituting our galaxy: "Single stars which move alone in space are now known to represent only a minority of objects constituting our Galaxy" [4]. By the middle to late twentieth century studies had shown a high fraction of main sequence F and G spectral type star systems consisted of binary or multiple stars. It was assumed that this trend was imitated by stars of all spectral type, however a recent discovery showing that M stars, which are actually the most numerous, comprising 80% of the stars in our galaxy, exist as single stars and that the fraction of stars per spectral type that exist as binaries decreases as we move towards late type stellar configurations: "... studies suggest that the binary frequency declines from the G star value, being only around 30% for M stars" [5]. The fraction of binary or multiple star systems in early and middle-aged type stars still seem to follow the trend of being large, to our advantage, since without binary system configurations the amount of information we can obtain about individual stars is almost non-existent: "... why binaries are important is that they are the primary source of our knowledge of the fundamental properties of stars" [6]. Zdeněk Kopal states "As long as a star is single, no empirical way exists for gauging (or analysing) the 'anatomy' of its external gravitational field; or learning anything about the distribution

of light on its surface. Place, however, another star in its proximity: and various properties of the combined gravitational field of both components can be deduced from observable characteristics of their motions" [4]. For binary systems at known distances (up to a limit), making use of the directly determined gravitational field allows for the indirect calculation of the absolute masses of their components. Beyond this limit the absolute masses as well as the dimensions of these components can only be determined for 'spectroscopic' binaries, which happen to be eclipsing binaries by virtue of their favourable orientation [4]. Information about the gravitational field of the binary system also allows us, under specific conditions, to probe the internal structure of the components: "A gravitational field emanates from the interior, which the opacity of the overlying layers cannot appreciably modify" [4].

An eclipsing close binary system is a star system consisting of two stars orbiting a common center of mass in a plane along our line of sight, which are sufficiently close together that their gravitational fields can disturb their respective outer atmospheres. If the two components do not fill their Roche lobes (region of space around the star where orbiting matter is gravitationally bound), the system is termed a detached close binary. If one star or both stars fill their Roche lobes, the system is termed a semi-detached or contact close binary respectively. Mass transfer can occur in both detached and contact binaries, however, through different processes. Naturally, one can expect mass transfer to effect pulsations [7], however, since the focus of this thesis involves the effect of tidal forces as small quantities (to be used as perturbations), for the simplification of this study, mass transfer between stellar components is considered negligible and the component masses remain constant throughout the study. The components in a binary system are assigned the terms '*primary*' and '*secondary*' for the most and least luminous component respectively. This designation has no relation to the masses of the components, however in most cases (not always) the brightest component is also the most massive of the two in the binary system. For the sake of this study, this is assumed and therefore reference to the components by means of primary and secondary is also an indication of the mass of that component. A tide arises in a binary system as a result of the gravitational potential generated by the star of smaller mass (secondary) and is experienced by the star of larger mass (primary). Currently, there are two principal investigations leading the front on the effects of tides on stellar structure and evolution. One being the effects of tides through tidal dissipation and structural deformation on apsidal motion, the decay rate of orbital eccentricity from elliptical to circular orbits, and internal stellar evolution through the modification of dissipative processes [8, 9, 10, 11, 12, 13]. The other being the effects of tides on stellar oscillation modes and frequencies [12, 14, 15].

Most eclipsing close binary systems that are currently being studied have already evolved into circular, synchronised orbits: "a state of minimum kinetic energy" [9] whereas for the few remaining, they continue for the time being to display elliptical orbits. Through time, ignoring kinetic energy losses through interstellar winds and gravitational waves, these binary systems will also have their orbits decay to circular orbits through the process of tidal dissipation: "Through tidal interaction, kinetic energy and angular momentum are exchanged between the rotation of the components and their orbital motion" [16]. The rate at which this decay occurs is dependent on 2 factors, the strength of the tidal interaction

(governed by the separation distance between binary components) and the physical processes that constitute tidal dissipation. These mechanisms commonly include "viscous and radiative dissipation" [9] with many more mechanisms under review [8, 13, 17, 18, 19]. The presence of a tide induces a structural deformation of the primary to the shape of a bulge. This distorts the gravitational field generated by the primary and in turn affects the apsidal motions and orbital eccentricity decay rates. It needs to be noted that these effects are applied to both the secondary and the primary components, occurring simultaneously for both. Our interest lies in one of the stars, traditionally the most massive (primary) as the gravitational forces responsible for these effects are considered a fraction of those generated by the primary. This is important mathematically and conceptually.

The effects of tides on oscillation modes and frequencies have been investigated in a way largely similar to the effects of axial rotation [14]. A perturbation of the equations of motion representing the stellar medium, with the addition of an external driving force term, is performed. In the case of rotation this term includes a centrifugal and coriolis component, whereas for tides this extra term is a modified version of the gravitational potential generated by the companion. There are two principal formalisms in use when discussing tides: the '*equilibrium tide*' model and the '*dynamic tide*' model [13, 20, 21]. The equilibrium (static) tide is characteristic of the tidal interaction generated by a companion star in synchronous and circular orbit with its primary, where the mean motion of the companion is the same as the rotational angular velocity of the primary. As a result a tidal bulge forms on the surface of the primary that is fixed in the co-rotating frame of reference as the companion orbits. The dynamic tide is characteristic of the opposite, a tidal interaction generated by a companion star in an asynchronous orbit. This results in the tidal bulge being dragged across the surface of the primary in the co-rotating frame of reference as the companion orbits. For convenience, slow stellar rotation is considered when solving for these systems. The reason being that the coriolis and centrifugal terms, originally present in the equations, can be neglected as these terms contain twofold or products of velocity terms. The tidal term is expressed as a multipole expansion and is restricted, for convenience, to second order. The first order component of the tidal expansion is the equilibrium part and the second order component the dynamic part.

From its launch in March 2009 until the failure of the second of four reaction wheels in May 2013 the *Kepler* satellite has been continuously collecting data from stars in a fixed field of view with only brief occasional interruptions in which it rolls to face the sun while recharging its batteries. The data obtained from the *Kepler* satellite has ushered in a new era in asteroseismology which will enable us to probe and analyse the internal structure of distant stars by observation of their pulsational behaviour. Until its failure in 2013 the principal objective of the *Kepler* mission was the identification of exoplanets, now the K2 'Second Light' mission plan is in place which utilises the disabled *Kepler* in a way that could detect habitable planets around smaller, dimmer red dwarf stars. An important side product of the data it collected was the continuous monitoring of the light curves of stars in its field of view. A few hundred close eclipsing binary systems with δ Scuti type primaries have emerged from the *Kepler* database and many of these have been studied intensively since its launch.

Due to the precision (noise levels around a few micromagnitudes in white light) of the data, weak pulsation signatures have been detected in many close binary systems with both eccentric [22] and circularised orbits [23, 24]. These pulsation signatures frequently point to a resonant coupling of pulsation frequencies with orbital motion in both eccentric [22, 25, 26, 27] and circular [23, 24, 28] orbits. This is known to be the case for systems with eccentric orbits [22, 27], however initial expectations are that resonance should not occur in systems with circular orbits, since there is no substantial difference between the periastron and apastron separations of the binary components that could drive such a resonance. However, given that these are close systems, tidal effects might be able to produce the apparent observed resonances [29, 30, 31]. Further investigations also predict that tidally forced oscillations lead to a splitting of the frequency spectrum into three bands [28, 32, 33]. Hiromoto Shibahashi predicts, among others: "A triplet fine structure with an equal spacing of twice the orbital frequency appears in the frequency spectrum" [32].

A long term objective of this project is to test this hypothesis by comparing results obtained computationally with those reduced from observations. For this to be done one needs first to develop theoretically and computationally the theory of tides and, more specifically, study their effects on the oscillations of primary components in eclipsing close binary systems through computational models. By incorporating stellar parameters, obtainable from observational data, into these tidal models one is able to computationally determine the modified oscillation frequencies and by comparing these results to data determine whether tides are truly responsible for the observed phenomena.

There has also been a dramatic increase in computational power over the last few years. This allows us to perform calculations to unprecedented precision and to explore parameter space far more densely than has previously been possible. As a result we can develop highly detailed models of tides, which can then be incorporated into current stellar evolution models such as MESA and ASTEC. Currently, MESA and ASTEC do not consider the effect of tides on stellar evolution [34, 35]. Due to the accurate results obtained by *Kepler*, a reasonably new computational model called CAFein was introduced in 2013, which fully investigates the effects of dynamic tides on the eigenfunctions of massive main sequence stars in eccentric binaries [36]. This provides a window of opportunity for entering the field of stellar modelling at a time when it is due to make exciting new advances.

For a Masters project the above objectives are ambitious and so the immediate objective of this thesis is first and foremost the development of the theory, with a few elementary computational models. This thesis focuses on the study of tides and pulsations separately, where an entry level simulation for determining pulsation frequencies is investigated. In the absence of a realistic computational tidal model I develop a numerical simulation for a polytropic star. This is not completely unrealistic as white dwarf stars and fully convective stars are well described by polytropes. There are other studies where the effect of tides on the pulsations of polytropes are investigated using different techniques, aside from perturbation theory. An example of this can be found in [37, 38] where averaging techniques are used. In the final parts of the thesis, I briefly look at combining the two subjects to give a unified theory of pulsations with tides

and investigate the outcome. The main body of this thesis includes six chapters, with this introduction being the first and the results chapter being the last.

In Chapter 2 I introduce the stellar configuration to be used for the remainder of this thesis. By setting up the correct reference frames and coordinate systems it is easy to manoeuvre between the equations and parameters evaluated within them. This chapter also focuses on deriving a definition for the second order tide generating potential that describes the effects of both the static and dynamic tide. This is done by treating the secondary star as a point mass enabling me to expand its gravitational potential in terms of the Legendre polynomials. Further manipulation and assumption reveals Fourier coefficients, which when evaluated to second order result in the second order tide generating potential. By evaluating the second order tide generating potential in the frequency domain, I show that for a circularised orbit the static tide has no effect on stellar pulsations with only the dynamic tide driving pulsations for a single frequency $2\,(\Omega - \mu)$.

In Chapter 3 I introduce the standard pulsation theory. A major component of the theory is the study of perturbations. I briefly introduce its concepts and derive equations for the unperturbed and perturbed configurations for the problem. I then investigate the radial and non-radial components of the perturbation equations and show that to derive a characteristic equation for an eigenvalue problem, the Cowling approximation is assumed.

In Chapter 4 I introduce the numerics involved in solving the unperturbed and perturbed equations. Under the assumption of the Cowling approximation I first de-dimensionalise the equations and then discretise them by applying a central difference approximation to all first and second order derivatives. By discretising the perturbation equations I reveal a characteristic equation with eigenvalues proportional to second order pulsation frequencies of the system. I then calculate the eigenvalues by running a two part simulation. The first part determines values for the unperturbed density ρ_0 and is done in C++. Using these values for density, the second part calculates the eigenvalues using a built in Octave function called *eig()*. The results for the simulation of a ploytrope of index $n = 3$ and physical parameters similar to that of the sun are presented in chapter 6.

In Chapter 5 I briefly investigate the combination of theories described in chapters 2 and 3. I introduced the effect of tides to the pulsation equations as a perturbation of the gravitational potential of the primary. Under the initial assumption that I follow the same strategy of finding pulsation frequencies as defined in chapter 3, I also showed that the best choice of configuration for the new problem is to introduce a third configuration called the tidal configuration. In deriving the tidal and perturbed equations I discovered that, for the dynamic tide, it is in fact not possible to use an eigenvalue problem to determine pulsation frequencies without simplifying the physics with assumptions. To simplify the problem, I isolate the static tide. I also investigate the implications for a numerical simulation of the merged theories.

Chapter 2

The Tide Generating Potential

2.1 Introduction

Viewed locally, terrestrial tides can be defined as the periodic rising and falling of the level of the sea at a fixed position on the surface of the Earth. The rising and falling motion is produced by variations in the gravitational field of the moon and of the sun, as their positions with respect to the Earth change.

A similar effect is produced in the primary star of a binary system by variations in the gravitational field of the secondary as it changes its position relative to the primary. A stellar tide is thus the movement of stellar material relative to the position that it might have occupied had it been in spherically symmetric equilibrium.

The magnitude of this effect depends on the configuration of the binary system. Later in this chapter, we introduce a parameter ε_T, the tidal parameter, to quantify the size of the forces responsible for tides in relation to the other forces that drive the system. Generally, tide producing forces are sufficiently small to be regarded as perturbations of the equilibrium configuration of the body of interest. In this thesis, I consider only systems where this is the case.

In both the terrestrial and stellar cases, we assume that the point of observation is a fixed point in a frame of reference that co-rotates with the body of interest. In the terrestrial case, where we are dealing with an object that has a solid crust, this hypothesis is easily justified. Stars, however, are known to undergo differential rotation, so this hypothesis may not be altogether satisfactory. However, if the average rotation rate of the primary is significantly smaller than the binary period, this assumption may be satisfactory.

We model the stellar material as a fluid continuum. Its equilibrium and motions are therefore described by the equations of hydrodynamics. This is discussed in more detail in Appendix A.

The response of the fluid to applied forces can be modelled using Euler's equation. This equation is essentially Newton's second law applied to a fluid continuum whose viscosity is zero or negligibly small. The assumption that the viscosity of the stellar material is negligibly small may be justified as follows. In general, fluid viscosity has a significant effect only near boundaries or in regions with significant velocity gradients. This might occur, for example, at the interface of high velocity wind belts moving in opposite directions, such as are seen in the atmospheres of gas giants such as Jupiter or Saturn. Unlike rocky planets, stars have no boundaries or basins to interact with flows. It is also reasonable to assume that they do not support wind belts or regions with significant tangential velocity gradients. This means that we can therefore neglect viscosity and expect Euler's equation to provide a relatively good model for the description of stellar tides.

Euler's equation describes the motion of a fluid relative to an inertial frame. This is inconvenient for our problem. The fluid equations are solved most easily when fluid velocities are small. Furthermore, small perturbations to a flow are easily masked in situations where the unperturbed parameters are large. Also, from a computational point of view, in a model where the relevant parameters have large values, small perturbations might be lost altogether in numerical calculations because of the way that large numbers are stored in computer memory. Both problems are easily avoided by moving to a frame of reference relative to which fluid velocities are small. This means that we ought to move to a frame of reference that moves with the primary star and which also co-rotates with it. Such a frame is necessarily non-inertial, so Euler's equation will need to be modified to take into account the acceleration of the frame.

Several frames of reference are naturally associated with a binary system. If the centre of mass of the binary moves inertially, we can construct an inertial frame by placing its origin at the centre of mass of the system and choosing axes whose directions are fixed with respect to the distant quasars. The orbits of the centres of mass of the binary stars around their common centre of mass lie in a plane. It makes sense therefore to choose this plane as one of the coordinate planes of the inertial frame, with the z-axis perpendicular to it.

If the binary system is isolated, it will not precess. If, on the other hand, it is influenced by the gravitational fields of its nearest neighbours, its precession will generally be slow, with a period that is much longer than the orbital period of the binary. This means that we can safely ignore the effects of precession.

If the orbits of the component stars are perfectly or nearly perfectly circularised, the system will not prefer any direction in the orbital plane above any other. In this case, there is no natural choice for the direction of the x-axis, and we can choose it arbitrarily. The right-hand rule then determines the direction of the y-axis. However, the case of very nearly circularised and synchronous orbits is only of limited interest, since a binary system of this kind does not support significant dynamic tides.

A more interesting case is that in which the binary orbits are sufficiently eccentric and asynchronous for the component stars to support dynamic tides of

non-negligible magnitude. In this case, the binary orbits themselves define preferred directions in the orbital plane. We choose the x-axis on the line that joins the centres of mass of the components and directed toward of the periastron of the orbit of the secondary. The direction of the y-axis is then determined by the right hand rule.

The reference frame defined by these prescriptions is inertial and is the most natural inertial frame to associate with the binary system. Denote it by I.

Since Newton's laws are valid only in inertial frames, our first instinct may be work in the frame I. However, as noted above, using this frame is problematic. The velocity of the stellar fluid relative to this frame is large in general, and the trajectory of a fluid element is unnecessarily complex. This frame unnecessarily complicates computations and masks the much smaller perturbations of the motion produced by tidal effects. We must thus search for a more convenient frame of reference.

A second frame of reference that is naturally associated with the primary is one whose origin C_1 is at its centre of mass and whose axes have the same directions as those of the inertial frame I. Denote this frame by $\overline{Q}$. Its advantage is that it is non-rotating, so the effects of its acceleration are easily taken into account. Its disadvantage is that the velocity of the fluid of the primary is unlikely to be small.

Suppose that the primary displays no differential rotation but rotates as if it were a solid body. Then we can choose a frame of reference whose origin is at the centre of mass of the primary and whose axes rotate with it. Relative to this frame, in the absence of tidal effects, the stellar material has zero velocity. If, on the other hand, the star has differential rotation, we can choose a frame whose axes rotate with the average rotation rate of the stellar material. This way, the velocities of the various parts of the stellar material is maximally reduced. Denote this frame by $\widetilde{Q}$ and call it the *co-rotating frame*, even though this term is not altogether appropriate.

A more detailed description of the frames I, $\overline{Q}$ and $\widetilde{Q}$, together with the notation for their respective axes and coordinates, is given in Appendix B. The configurations and the notation described there will be used throughout this Chapter.

2.2 Euler's Equation in the Inertial Frame I

The signature feature of Euler's approach to fluids, in contrast with the Lagrangian formalism, is the description of the motion of the fluid by means of a velocity field $\vec{u}\,(\vec{x},t)$ rather than by following the trajectories of individual fluid points. The field $\vec{u}\,(\vec{x},t)$ gives the value of the velocity of the fluid that occupies position $\vec{x}$ at time t.

The concept of a velocity field provides a completely different description of velocity from that familiar to Newtonian mechanics, where we follow the mo-

tion of a particle as it moves along a trajectory. The trajectory is a vector valued function of the single variable t that gives the position vector of the particle. The velocity of the particle is then a simple derivative of the position vector with respect to time, and its acceleration is the second derivative of this function with respect to time.

By using a velocity field to describe the fluid, we focus on the global behaviour of the fluid and not on the motion of individual fluid points. Since we no longer explicitly track individual fluid points, their acceleration must be calculated by a different technique.

Consider an infinitesimal fluid element p situated anywhere within the volume or on the surface of the primary. This element is modelled as a point particle that occupies position $\vec{x}$ at time t. The acceleration of this point is defined as the limit of the ratio of the increase of its velocity to the time taken for this increase to occur, in the limit as the time taken tends to zero. The problem here is that, in time Δt, the fluid point moves to a new position $\vec{x} + \Delta\vec{x}$, where

$$\Delta\vec{x} = \vec{u}\left(\vec{x}, t\right)\, \Delta t.$$

Its velocity at this position is not given by $\vec{u}\left(\vec{x}, t+\Delta t\right)$, but by $\vec{u}\left(\vec{x}+\Delta\vec{x}, t+\Delta t\right)$. The acceleration of the fluid point at time t is therefore given by

$$\vec{A}\left(\vec{x}, t\right) = \lim_{\Delta t \to 0} \frac{\vec{u}\left(\vec{x} + \Delta\vec{x}, t + \Delta t\right) - \vec{u}\left(\vec{x}, t\right)}{\Delta t},$$

which is not the time derivative of the velocity field, as we might have expected naively, but something more complicated. We can evaluate the limit on the right hand side using Cartesian coordinates, index notation and the multivariable Taylor's expansion to get

$$A^i\left(\vec{x}, t\right) = u^k\left(\vec{x}, t\right)\, \frac{\partial u^i}{\partial x^k}(\vec{x}, t) + \frac{\partial u^i}{\partial t}(\vec{x}, t).$$

All fields are now evaluated at the same arguments, so these may safely be omitted.

If we now restore the vector notation, we obtain a coordinate-free formula that is valid in all systems of coordinates for the inertial frame I,

$$\vec{A} = \vec{u} \cdot \nabla\vec{u} + \frac{\partial\vec{u}}{\partial t}.$$

The acceleration in this formula is an *acceleration field*. The quantity $\vec{A}\left(\vec{x}, t\right)$ is the acceleration of the fluid at position $\vec{x}$ at time t, and represents the acceleration at time t of any sufficiently small fluid element that occupies position $\vec{x}$.

According to Newton's second law, the acceleration of a fluid element is equal to the total force per unit mass that acts on the fluid at the position of the element at time t. I shall assume in this thesis that the stellar fluid is subject to two forces only: internal stress, and gravity. We have assumed that the fluid is

inviscid, so the only internal stress it experiences is hydrostatic. Per unit mass, it is given by

$$\frac{1}{\rho}\,\vec{\nabla}P,$$

where P is the pressure field of the fluid, and ρ is its density.

The gravitational force on the element consists of two parts: that exerted on the fluid element by the gravitational attraction of the primary, and that exerted on the fluid element by the gravitational attraction of the secondary. Each can be expressed in terms of a potential, so the total gravitational force on the element can be expressed as

$$-\vec{\nabla}(\Phi_P + \Phi_S),$$

where Φ_P and Φ_S are respectively the potentials for the field of the primary and that for the field of the secondary.

Euler's equation is just Newton's second law for the fluid element, but with this difference: the acceleration is replaced by Euler's acceleration field. Cancelling the mass of the fluid element from both sides of the equation leaves

$$\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = -\frac{1}{\rho}\,\vec{\nabla}P - \vec{\nabla}\Phi_P - \vec{\nabla}\Phi_S, \tag{2.1}$$

where the right hand side is the total force per unit mass acting on the element. This is an equation that relates the fields $\vec{u}$, ρ, P, Φ_P and Φ_S.

2.3 Euler's Equation in an Accelerated Frame

We arrive at Euler's equation for a general accelerated frame Q by first considering the equation of motion of a point particle in this frame, and then replacing the particle velocity and acceleration by the velocity and acceleration fields of the fluid discussed in the previous section.

The equation of motion for a particle in an accelerated frame is

$$\vec{a} = \frac{\vec{F}}{m} - \ddot{\vec{x}}_1 - 2\,\vec{\omega}\times\dot{\vec{r}} - \dot{\vec{\omega}}\times\vec{r} - \vec{\omega}\times(\vec{\omega}\times\vec{r}). \tag{2.2}$$

This equation is derived in Appendix B. Note that the vector $\vec{\tilde{x}}$ in that derivation is replaced in the above formula by the symbol $\vec{r}$. This change is purely cosmetic.

The force $\vec{F}$ is the total physical force that acts on the particle in the inertial frame I, $\vec{a}$ is the acceleration of the particle relative to the accelerated frame Q, $\vec{\omega}$ is the angular velocity of the frame Q, $\vec{r}$ is the position vector of the particle relative to the frame Q, and $\ddot{\vec{x}}_1$ is the acceleration of the origin C_1 of the frame Q relative to I.

To obtain Euler's equation of motion for the accelerated frame, we note that

the fluid point p at position $\vec{r}$ at time t in the accelerated frame Q has velocity $\dot{\vec{r}}$ and acceleration $\vec{a}$ given by the Eulerian velocity and acceleration fields $\vec{v}\,(\vec{r},t)$ and $\vec{a}\,(\vec{r},t)$ of the fluid *relative to the frame* Q,

$$\dot{\vec{r}} = \frac{d\vec{r}}{dt}(t) = \vec{v}\,(\vec{r}\,(t),t),$$

and

$$\ddot{\vec{r}}\,(t) = \vec{a}\,(\vec{r},t) = \frac{\partial \vec{v}}{\partial t}\,(\vec{r}\,(t),t) + \vec{v}\,(\vec{r}\,(t),t)\cdot\nabla\vec{v}\,(\vec{r}\,(t),t).$$

The equation of motion of a fluid relative to $\widetilde{Q}$ is thus

$$\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} = \vec{f} - \ddot{\vec{x}}_1 - 2\,\vec{\omega}\times\vec{v} - \dot{\vec{\omega}}\times\vec{r} - \vec{\omega}\times(\vec{\omega}\times\vec{r}). \tag{2.3}$$

where $\vec{f}$ is the force per unit mass that acts on the fluid in the inertial frame I. In our model, therefore, the equation of motion becomes

$$\begin{aligned}\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} = -\frac{1}{\rho}\,\vec{\nabla}P - \vec{\nabla}\Phi_P - \vec{\nabla}\Phi_S - \ddot{\vec{x}}_1 \\ - 2\,\vec{\omega}\times\vec{v} - \dot{\vec{\omega}}\times\vec{r} - \vec{\omega}\times(\vec{\omega}\times\vec{r}).\end{aligned} \tag{2.4}$$

Euler's Equation for a Binary System

The angular velocity $\vec{\omega}$ in equation (2.4) is the angular velocity of the accelerated frame Q. In the case of a binary system, we choose this frame to be the frame whose origin is at the centre of mass of the primary and whose axes co-rotate with the primary, as described in Section 2.1. I denote this frame by $\widetilde{Q}$. The angular velocity $\vec{\omega}$ of this frame is the angular velocity of rotation of the primary, which I denote by Ω. This angular velocity is distinct from, and should not be confused with, the angular velocity of the binary system, called the mean motion of the orbit, denoted by μ.

If the binary system is isolated, the external torque on it is zero. This means that there is no mechanism for producing angular acceleration of the frame $\widetilde{Q}$. We thus have

$$\dot{\vec{\Omega}} = 0.$$

In this thesis, I assume that the angular velocity of the primary relative to the inertial frame I is sufficiently small for the centrifugal term $\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$ to be neglected.

Furthermore, since the frame $\widetilde{Q}$ co-rotates with the primary, the stellar fluid of the primary will have zero velocity in the frame $\widetilde{Q}$ if the star moves as a rigid body, or low velocity if it is in differential rotation. In either case, I assume that this makes the Coriolis term $\vec{\Omega}\times\vec{v}$ negligible.

Incorporating all of these assumptions into the equation of motion of the fluid of the primary, its equation of motion then becomes

$$\vec{a} = \frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} = -\frac{1}{\rho}\,\vec{\nabla}P - \vec{\nabla}\Phi_P - \vec{\nabla}\Phi_S - \ddot{\vec{x}}_1. \tag{2.5}$$

2.4 The Gravitational Potential Φ_S

In this thesis, I model the secondary as a point mass. Its gravitational potential Φ_S at the fluid point p shown in Figure 2.1 is then given by

$$\Phi_S(p) = -\frac{GM_2}{d}, \tag{2.6}$$

where d is the distance between fluid element p and the secondary.

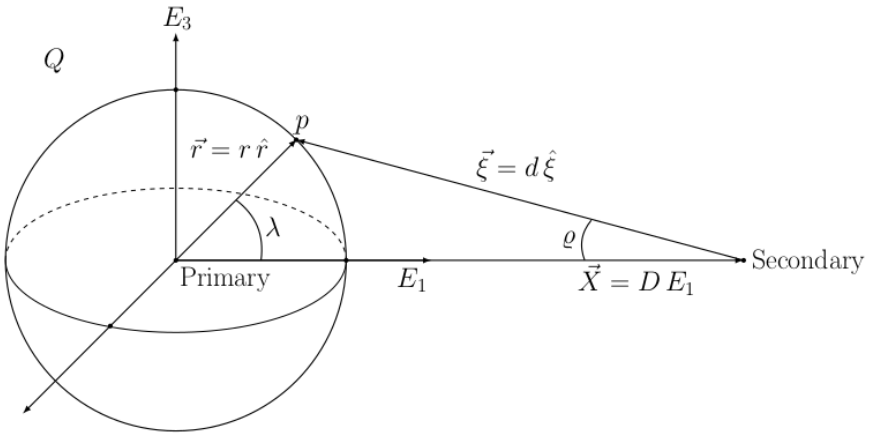


Figure 2.1: A simplified vector diagram of the binary system relating the positions of its components.

We now express this potential in terms of the known quantities D and λ. Denote by $\vec{X}$ the position of the secondary relative to the primary, as shown in Figure 2.1. Then

$$\vec{\xi} = \vec{r} - \vec{X}, \tag{2.7}$$

and so

$$\vec{\xi}^{\,2} = \left(\vec{r} - \vec{X}\right)^2 = r^2 + D^2 - 2\,r\,D\,\left(\widehat{r}\cdot\widehat{X}\right).$$

Since $r \ll D$ always, we can write

$$\left|\vec{\xi}\right| = d = D\sqrt{\frac{r^2}{D^2} + 1 - 2\,\frac{r}{D}\,\left(\widehat{r}\cdot\widehat{X}\right)}, \tag{2.8}$$

where $\widehat{r}\cdot\widehat{X} = \cos\lambda$, and λ is the angle between the vectors $\vec{X}$ and $\vec{r}$, as shown in Figure 2.1. Let $t = r/D$ and $s = \cos\lambda$, equation (2.8) becomes

$$d = D\sqrt{1 + t^2 - 2\,t\,s}. \tag{2.9}$$

Substituting equation (2.9) into (2.6) then gives

$$\Phi_S\,(p) = -\frac{GM_2}{D}\,\frac{1}{\sqrt{1 + t^2 - 2\,t\,s}}. \tag{2.10}$$

The term $\left(\sqrt{1+t^2-2ts}\right)^{-1}$ is the generating function for the Legendre polynomials (see appendix C). Using equation (C.10), we expand (2.10) to give

$$\begin{aligned}\Phi_S(p) &= -\frac{GM_2}{D}\sum_{n=0}^{\infty} t^n P_n(s) \\ &= -\frac{GM_2}{D}\sum_{n=0}^{\infty}\left(\frac{r}{D}\right)^n P_n(\cos\lambda).\end{aligned} \tag{2.11}$$

Writing out the first two terms explicitly we have

$$\Phi_S(p) = -\frac{GM_2}{D} - \frac{GM_2}{D^2}(r\cos\lambda) - \frac{GM_2}{D}\sum_{n=2}^{\infty}\left(\frac{r}{D}\right)^n P_n(\cos\lambda).$$

Put

$$\Phi_0(p) = -\frac{GM_2}{D} \quad \text{and} \quad \Phi_1(p) = -\frac{GM_2}{D^2}(r\cos\lambda).$$

The first term, Φ_0, is a constant. It therefore does not contribute to the gravitational field. It is determined by the choice of zero for the gravitational potential and could be altered by choosing a different reference point for the zero of the potential. It therefore carries no real physical information. Since its gradient is zero, it can safely be omitted from the expression for Φ_S.

To understand physically the significance of the second term Φ_1, evaluate the physical force per unit mass exerted by the secondary at position p. The simplest way to calculate this force per unit mass is as follows: consider the position of the secondary relative to the primary *at some chosen instant* t; construct an orthonormal frame $\{E_1, E_2, E_3\}$ with origin is at the centre of mass of the primary and with Cartesian coordinates (x^1, x^2, x^3); calculate the quantity $-\vec{\nabla}\Phi_1$ in these coordinates; finally, write this result in a coordinate independent form.

For simplicity, choose the first axis of this frame to point in the direction of the secondary. Then $\widehat{X} = E_1$, and $\widehat{r}\cdot\widehat{X} = \widehat{r}\cdot E_1 = x^1$, where x^1 is the x-coordinate of p in this system of coordinates. We thus have, in terms of these coordinates,

$$\Phi_1(p) = -\frac{GM_2}{D^2}\,x^1\,E_1,$$

and

$$-\vec{\nabla}\Phi_1(p) = -\bar{\nabla}\left(-\frac{GM_2}{D^2}\,x^1\right) = \frac{GM_2}{D^2}\,E_1.$$

This result is easily converted into coordinate free form by noting that $E_1 = \widehat{X}$. We thus get a coordinate free formula for the force per unit mass exerted by the secondary at the point p,

$$-\vec{\nabla}\Phi_1(p) = \frac{GM_2}{D^2}\,\widehat{X}.$$

The force per unit mass produced by Φ_1 thus has magnitude GM_2/D^2 in the direction of the unit vector $\widehat{X}$. Both quantities are constant. The term Φ_1 is thus responsible for accelerating each point of the fluid uniformly in the fixed direction Φ_1. Each fluid element in the primary thus undergoes the same constant acceleration. Stated differently, the entire primary is accelerated as if it were a rigid body falling toward the secondary under the action of the gravitational attraction of the secondary.

The gravitational field $\Phi_S(p)$ at p due to the secondary can thus be written in the form

$$-\vec{\nabla}\Phi_S(p) = \frac{GM_2}{D^2}\,\widehat{X} + \vec{\nabla}\left[\frac{GM_2}{D}\sum_{n=2}^{\infty}\left(\frac{r}{D}\right)^n P_n(\cos\lambda)\right]. \tag{2.12}$$

The negative gradient of the terms in square brackets in equation (2.12) represent that part of the gravitational force that is responsible for tides.

Equation of Motion in the Co-Rotating Frame $\widetilde{Q}$

Substituting equation (2.12) into equation (2.5) gives

$$\vec{a} + \ddot{\vec{x}}_1 = -\frac{1}{\rho}\vec{\nabla}P - \vec{\nabla}\Phi_P + \frac{GM_2}{D^2}\,\widehat{X} + \vec{\nabla}\left[\frac{GM_2}{D}\sum_{n=2}^{\infty}\left(\frac{r}{D}\right)^n P_n(\cos\lambda)\right]. \tag{2.13}$$

If we assume that the orbits of the binary are nearly (but not completely) circularised, that is, that their eccentricities are small, then we can treat their orbits as circular in all terms except the tidal ones. In particular, the acceleration $\ddot{\vec{x}}_1$ will have a strongly dominant radial component, and a negligibly small tangential component which we ignore.

Now, $\ddot{\vec{x}}_1$ is the acceleration of the centre of mass of the primary relative to the centre of mass of the binary system. This acceleration is provided by the total gravitational force exerted by the secondary on the primary. By Newton's second law, we have

$$M_1\,\ddot{\vec{x}}_1 \approx \frac{GM_1M_2}{D^2}\,\widehat{X},$$

where we have ignored the small tangential components, and where D is the distance between centre of masses C_1 and C_2. Thus

$$\ddot{\vec{x}}_1 = \frac{GM_2}{D^2}\,\widehat{X}. \tag{2.14}$$

This term is identical to the third term on the right hand side of equation (2.13), so the equation of motion of the stellar fluid in the co-rotating frame becomes

$$\vec{a} = -\frac{1}{\rho}\vec{\nabla}P - \vec{\nabla}\Phi_P + \vec{\nabla}\left[\frac{GM_2}{D}\sum_{n=2}^{\infty}\left(\frac{r}{D}\right)^n P_n(\cos\lambda)\right]. \tag{2.15}$$

This result shows that the second term of the series in equation (2.11) cancels the acceleration of the centre of mass of the co-rotating frame $\widetilde{Q}$, leaving only the terms in $-\vec{\nabla}\Phi_S$ with $n \geq 2$. This means that, from the point of view of an observer at rest in the co-rotating frame $\widetilde{Q}$, the only force exerted on the fluid at p by the secondary is the force whose potential is given by

$$\Phi_T = -\frac{GM_2}{D}\sum_{n=2}^{\infty}\left(\frac{r}{D}\right)^n P_n\left(\cos\lambda\right). \tag{2.16}$$

The potential Φ_T is therefore responsible for the generation of tides on the primary.

2.5 The Tidal Parameter ε_T

The distance from the secondary to the fluid element p is large compared with the radius of the primary. We thus expect the tidal terms in equation (2.15) to be small when compared with the term $-\vec{\nabla}\Phi_P$.

In this section, we investigate the relative sizes of these two terms. For this, we need to compare the magnitudes of these two forces on a given fluid element. Equivalently, we can compare the magnitudes of the accelerations of the given fluid element produced by these two forces.

It is clear that the accelerations produced will be a function of the position of the fluid element in the primary. The fluid elements in which the secondary produces the greatest acceleration are those closest to it. All other elements will undergo an acceleration that is smaller by a factor smaller than 1. We will therefore consider a fluid element that lies at a given instant on the line joining the centres of mass of the two stars, and which is at the surface of the primary. This is shown in Figure 2.2.

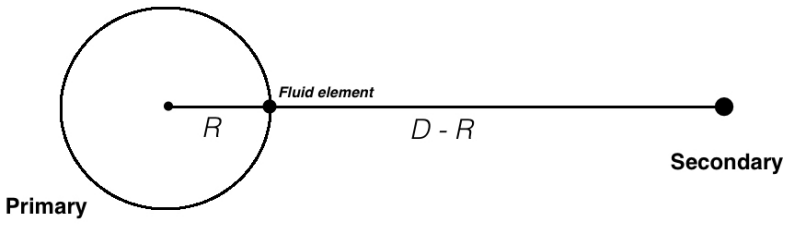


Figure 2.2: Fluid element with greatest acceleration due to action of the gravitational field of the secondary.

Using the vectors introduced in Figure 2.2, the magnitude of the acceleration of the fluid element produced by the gravitational field of the primary is given by

$$\frac{GM_1}{R^2}.$$

Note that this is an acceleration relative to the primary. That is, it is an acceleration relative to the frame $\widetilde{Q}$.

The magnitude of the acceleration produced by the gravitational field of the secondary is given by

$$\frac{GM_2}{(D-R)^2} = \frac{GM_2}{D^2}\left(1-\frac{R}{D}\right)^{-2} \approx \frac{GM_2}{D^2}\ \left(1+\frac{2R}{D}\right) = \frac{GM_2}{D^2} + \frac{2GM_2R}{D^3}.$$

The first term on the right hand side is the acceleration of the primary, considered as a point particle, produced by the field of the secondary. It is therefore an acceleration relative to the inertial frame I, and *not* relative to the frame $\widetilde{Q}$. To get the acceleration of the fluid element relative to $\widetilde{Q}$, we need to subtract the acceleration of the primary, giving the acceleration of the fluid element relative to the frame $\widetilde{Q}$ produced by the action of the gravitational field of the secondary as

$$\frac{2GM_2R}{D^3}.$$

The factor 2 in this result is irrelevant, since we are interested only in orders of magnitude. We thus write this acceleration as

$$\frac{GM_2R}{D^3}.$$

To compare the effects of these two gravitational fields, we find the ratio of the magnitudes of the accelerations of the fluid element produced by the secondary to that produced by the primary. The parameter obtained in this way is denoted by ε_T and is called the *tidal parameter*. It is given by

$$\varepsilon_T = \left(\frac{R}{D}\right)^3 \frac{M_2}{M_1}.$$

This parameter is a good measure of the strength of the tidal term in the equation of motion in situations where D is a constant. This occurs if the stellar orbits of the binary have circularised. If they have not, then D is a function of time, and ε_T given by the above expression is not a constant. This makes it unsuitable.

The orbit of the secondary relative to the primary is Keplerian, and is characterised by the length of its semi-major axis. The semi-major axis also represents an average distance between primary and secondary. We may therefore replace D by a in the above formula to obtain a constant that is a measure of the relative strength of the tidal force. We therefore define ε_T to be

$$\varepsilon_T = \left(\frac{R}{a}\right)^3 \frac{M_2}{M_1}. \tag{2.17}$$

Typical parameter ranges for δ Scuti Algol-type close binaries are $M_1 \approx 0.2 \to 2\ M_\odot$, $M_2 \approx 0.2 \to 2\ M_\odot$, $a \approx 0.02 \to 0.06\ Au$ and $R \approx 1 \to 3\ R_\odot$. The mass M_2 of the secondary must at all times be $\leq$ to mass M_1 of the primary. Using the maximum and minimum values for these parameters, we calculate values for ε_T for a range of possible binary configurations. They are displayed in Table 2.1). We see from table 2.1 that, for binary configurations in the range specified above, the value of ε_T is of the order 10^{-10} or smaller. This justified treating the effect of the tidal term as a perturbation of the equilibrium configuration of the primary.

M_1 ($M_\odot$)	M_2 ($M_\odot$)	a (Au)	R ($R_\odot$)	ε_T
0.2	0.2	0.02	1	1.25605×10^{-11}
0.2	0.2	0.02	3	3.39133×10^{-10}
0.2	0.2	0.06	1	4.65203×10^{-13}
0.2	0.2	0.06	3	1.25605×10^{-11}
2	0.2	0.02	1	1.25605×10^{-12}
2	0.2	0.02	3	3.39133×10^{-11}
2	0.2	0.06	1	4.65203×10^{-14}
2	0.2	0.06	3	1.25605×10^{-12}
2	2	0.02	1	1.25605×10^{-11}
2	2	0.02	3	3.39133×10^{-10}
2	2	0.06	1	4.65203×10^{-13}
2	2	0.06	3	1.25605×10^{-11}

Table 2.1: Table of ε_T values calculated for different binary system configurations.

2.6 The Tide Generating Potential

It is useful to express the potential Φ_T in terms of ε_T by writing

$$\Phi_T = \varepsilon_T W(r, \lambda, t).$$

Since ε_T is dimensionless, W has the units of a gravitational potential and is called the *tide generating potential* or, more briefly, the *tidal potential*.

Using the above definition, we get the following expression for the tide generating potential W,

$$\begin{aligned} W(r,\lambda,t) &= -\frac{M_1}{M_2}\left(\frac{a}{R}\right)^3 \frac{GM_2}{D} \sum_{n=2}^{\infty} \left(\frac{r}{D}\right)^n P_n(\cos\lambda) \\ &= -\frac{GM_1}{D}\left(\frac{a}{R}\right)^3 \sum_{n=2}^{\infty} \left(\frac{r}{D}\right)^n P_n(\cos\lambda)\,. \end{aligned}$$

In this expression, the first factor on the right hand side has the units of gravitational potential. The remaining factors in this expression are all dimensionless.

The leading factor contains D, which is time dependent. This is not useful. We therefore replace D by R, which is a constant, in the leading factor, and then reorganise the remaining terms into more useful dimensionless ratios. It is especially useful to include the dimensionless ratios R/a, r/R and R/D. We write W in terms of these as follows:

$$\begin{aligned} W(r,\lambda,t) &= -\frac{GM_1}{R} \sum_{n=2}^{\infty} \left(\frac{R}{D}\right)^{n+1} \left(\frac{a}{R}\right)^3 \left(\frac{r}{R}\right)^n P_n(\cos\lambda) \\ &= -\frac{GM_1}{R} \sum_{n=2}^{\infty} \left(\frac{R}{a}\right)^{n+1} \left(\frac{a}{D}\right)^{n+1} \left(\frac{a}{R}\right)^3 \left(\frac{a}{a}\right)^{n+1} \left(\frac{r}{R}\right)^n P_n(\cos\lambda) \\ &= -\frac{GM_1}{R} \sum_{n=2}^{\infty} \left(\frac{R}{a}\right)^{n-2} \left(\frac{a}{D}\right)^{n+1} \left(\frac{r}{R}\right)^n P_n(\cos\lambda)\,. \end{aligned} \tag{2.18}$$

$W(r, \lambda, t)$ is expressed in the form of (2.18) so that it is in terms of easily interpretable factors. GM_1/R is the gravitational potential generated by the primary at its surface. It is a scaling factor that carries the dimensions of gravitational potential. The factors in the terms of the summation that follow are all dimensionless. The function $W(r, \lambda, t)$ is thus measured in units of GM_1/r.

The ratio R/a is constant, dimensionless and small. The distance $D = D(t)$ is not a fixed value, but changes with time as the secondary orbits the primary. a is the semi-major axis of the secondary orbit and is the maximum value of $D(t)$. The ratio a/D is thus a time dependent, dimensionless factor of order of one.

r is the radial position of a fluid element p situated within the primary. The ratio r/R is then the radial distance to the fluid element p measured in units of the average radius of the primary and takes values between zero and one.

2.7 W in Spherical Coordinates $\vec{r}$

Equation (2.18) expresses W in terms of the parameters r, λ and t. r with its base unit $\hat{r}$ is the radial coordinate of the spherical coordinate system assigned to reference frame $\widetilde{Q}$ (see Appendix B.9). The angle λ however is not one of the standard spherical angles. We thus need to express the polynomials $P_n(\cos\lambda)$ as functions of the spherical angles ϕ', θ, $\widetilde{\theta} = \pi/2$ and $\widetilde{\phi}$ (see Figure B.3).

The angle λ is the angle between vectors $\vec{r}$ and $\vec{X}$. Therefore

$$\cos\lambda = \widehat{r} \cdot \widehat{X}. \tag{2.19}$$

In terms of the Cartesian coordinates $\bar{x}^{\,i}$, with the coordinates of vectors $\vec{r}$ and $\vec{X}$ as defined in appendix B.9, we have

$$\begin{aligned}
\widehat{r} &= \sin\frac{\pi}{2}\,\cos\widetilde{\phi}\,\vec{E}_1 + \sin\frac{\pi}{2}\,\sin\widetilde{\phi}\,\vec{E}_2 + \cos\frac{\pi}{2}\,\vec{E}_3, \\
\widehat{X} &= \sin\theta\,\cos\phi'\,\vec{E}_1 + \sin\theta\,\sin\phi'\,\vec{E}_2 + \cos\theta\,\vec{E}_3.
\end{aligned}$$

Substituting for $\hat{r}$ and $\hat{X}$ in equation (2.19) gives

$$\begin{aligned}
\cos\lambda &= \sin\frac{\pi}{2}\,\cos\widetilde{\phi}\,\sin\theta\,\cos\phi' \\
&\qquad + \sin\frac{\pi}{2}\,\sin\widetilde{\phi}\,\sin\theta\,\sin\phi' + \cos\frac{\pi}{2}\,\cos\theta \\
&= \sin\frac{\pi}{2}\,\sin\theta\left(\cos\widetilde{\phi}\,\cos\phi' + \sin\widetilde{\phi}\,\sin\phi'\right) \\
&\qquad + \cos\frac{\pi}{2}\,\cos\theta \\
&= \cos\frac{\pi}{2}\,\cos\theta + \sin\frac{\pi}{2}\,\sin\theta\cos\left(\phi' - \widetilde{\phi}\right).
\end{aligned}$$

With $\cos\lambda$ in this form we can now use the *Spherical Harmonic Addition Theorem* (see equation (E.18)) to write the Legendre polynomial with argument $\cos\lambda$ in the form of

$$P_n(\cos\lambda) = \frac{4\pi}{2n+1}\sum_{m=-n}^{n} Y_n^m(\theta,\phi')\,Y_n^{m*}\left(\frac{\pi}{2},\widetilde{\phi}\right). \tag{2.20}$$

The vector $\vec{X}$ connects the centre of mass of the primary to that of the secondary. The angle $\widetilde{\phi}$ is the azimuth for the co-rotating frame $\widetilde{Q}$ and is equivalent to the true anomaly ν of the orbit of the secondary (see appendix F). Equation (2.20) then becomes

$$P_n(\cos\lambda) = \frac{4\pi}{2n+1} \sum_{m=-n}^{n} Y_n^m(\theta,\phi')\, Y_n^{m*}\left(\frac{\pi}{2},\nu\right). \tag{2.21}$$

From equation (E.17) we have

$$Y_n^m(\theta,\phi') = \sqrt{\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}}\, P_n^{|m|}(\cos\theta)\, e^{im\phi'},$$
$$Y_n^{m*}(90,\nu) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}}\, P_n^{|m|}(0)\, e^{-im\nu}.$$

Substituting these definitions into equation (2.21) gives

$$P_n(\cos\lambda) = \sum_{m=-n}^{n} \frac{(n-|m|)!}{(n+|m|)!} \times P_n^{|m|}(\cos\theta)\, P_n^{|m|}(0)\, e^{im\phi'}\, e^{-im\nu}. \tag{2.22}$$

Substituting equation (2.22) into (2.18)[1] gives

$$W = -\left(\frac{GM_1}{R}\right)\sum_{n=2}^{\infty}\left(\frac{R}{a}\right)^{n-2}\left(\frac{a}{D}\right)^{n+1}\left(\frac{r}{R}\right)^{n} \times \sum_{m=-n}^{n}\frac{(n-|m|)!}{(n+|m|)!}\, P_n^{|m|}(\cos\theta)\, P_n^{|m|}(0)\, e^{im\phi'}\, e^{-im\nu}. \tag{2.23}$$

The second summation is nested within the first and therefore terms containing only the index n can be moved across the m-summation. Let

$$\widetilde{f}_n^m(t) = \frac{(n-|m|)!}{(n+|m|)!}\left(\frac{R}{a}\right)^{n-2}\left(\frac{a}{D}\right)^{n+1} P_n^{|m|}(0)\, e^{-im\nu}. \tag{2.24}$$

The time dependence arises on account of $D = D(t)$[2]. Equation (2.23) then becomes

$$W = -\left(\frac{GM_1}{R}\right)\sum_{n=2}^{\infty}\sum_{m=-n}^{n}\widetilde{f}_n^m(t)\left(\frac{r}{R}\right)^{n} P_n^{|m|}(\cos\theta)\, e^{im\phi'}. \tag{2.25}$$

Taking the Fourier transform of $\widetilde{f}_n^m(t)$ we obtain

$$\widetilde{f}_n^m(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F_n^m(\omega)\, e^{i\omega t}\, d\omega. \tag{2.26}$$

[1] Function arguments are omitted unless there is confusion

[2] In general, the orbit of the binary system is not assumed to be completely circularised

It is convenient, when comparing our results with observational data, to measure time from a reference epoch t_0. We thus replace t with $t - t_0$, equation (2.26) then becomes

$$f_n^m(t) = \widetilde{f}_n^m(t - t_0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_n^m(\omega)\, e^{i\omega(t-t_0)}\, d\omega. \tag{2.27}$$

If we assume that the orbit of the secondary is periodic with period T, not all of the components $e^{i\omega(t-t_0)}$ will feature in equation (2.27). We need only retain those components that satisfy

$$e^{i\omega([t+T]-t_0)} = e^{i\omega(t-t_0)}.$$

Therefore

$$e^{i\omega T} = 1,$$

which implies that

$$\omega T = 2\pi k,$$

where $k \in \mathbb{Z}$. This means that the Fourier integral should contain only components $e^{i\omega(t-t_0)}$ that have frequencies

$$\omega = \frac{2\pi}{T} k = k\mu, \tag{2.28}$$

where μ is the mean motion of the orbit of the secondary (see appendix F). This reduces the integral in equation (2.27) to a discrete sum

$$f_n^m(t) = \sum_{\omega=-\infty}^{\infty} F_n^m(\omega)\, e^{i\omega(t-t_0)},$$

or in terms of the mean motion

$$f_n^m(t) = \sum_{k=-\infty}^{\infty} F_n^m(k\mu)\; e^{ik\mu(t-t_0)}.$$

Put

$$C_n^m(k) = F_n^m(k\mu),$$

and then

$$f_n^m(t) = \sum_{k=-\infty}^{\infty} C_n^m(k)\, e^{ik\mu(t-t_0)}. \tag{2.29}$$

Substituting equation (2.29) into (2.25) gives

$$W = -\left(\frac{GM_1}{R}\right) \sum_{n=2}^{\infty} \sum_{m=-n}^{n} \sum_{k=-\infty}^{\infty} C_n^m(k) \times \left(\frac{r}{R}\right)^n P_n^{|m|}(\cos\theta)\; e^{ik\mu(t-t_0)}\; e^{im\phi'}. \tag{2.30}$$

Using equation (B.6) for ϕ', equation (2.30) becomes

$$W = \sum_{n=2}^{\infty} \sum_{m=-n}^{n} \sum_{k=-\infty}^{\infty} e^{it(k\mu+m\Omega)-it_0(k\mu+m\Omega)} \times -\left(\frac{GM_1}{R}\right) C_n^m(k) \left(\frac{r}{R}\right)^n P_n^{|m|}(\cos\theta)\, e^{im\phi}. \tag{2.31}$$

Let

$$\Gamma_{n,m,k}(\vec{r}) = -\left(\frac{GM_1}{R}\right) C_n^m(k) \left(\frac{r}{R}\right)^n P_n^{|m|}(\cos\theta)\, e^{im\phi}, \tag{2.32}$$

where $\vec{r} = (r, \theta, \phi)$. Equation (2.31) is then

$$W(\vec{r}, t) = \sum_{n=2}^{\infty} \sum_{m=-n}^{n} \sum_{k=-\infty}^{\infty} \Gamma_{n,m,k}(\vec{r})\, e^{it(k\mu+m\Omega)-it_0(k\mu+m\Omega)}. \tag{2.33}$$

Thus separating the spatial terms from the time dependent one.

Evaluating the Coefficients $C_n^m(k)$

To find the coefficients $C_n^m(k)$ we use the orthogonality property of the exponential functions. First we write equation (2.29) in terms of the mean anomaly M (see appendix F). We then have

$$f_n^m(t) = \sum_{k=-\infty}^{\infty} C_n^m(k)\, e^{ikM}, \tag{2.34}$$

where the mean anomaly M, for time period $t - t_0$, is defined as

$$\begin{aligned} M &= \mu(t - t_0) \\ &= \frac{2\pi}{T}(t - t_0), \end{aligned}$$

and T is the orbital period. The mean anomaly M increases uniformly with time, completing one orbit for every increment of 2π radians. However M is not to be interpreted as an angle, except in the special case where the orbit is circular. If the orbit is not circular, the angle made by the radius vector and the reference direction increases non-linearly with time and cannot be identified with M. At $t = t_0$[3] we have that

$$M = 0,$$

and at $t = t_0 + T$[4] we have

$$M = 2\pi.$$

By integrating from 0 to 2π, for a single period, over the mean anomaly M instead of time t we have essentially recalibrated the time axis in such a way

[3] t_0 is the time at which the orbiting body passes the reference point in a chosen set of observed cycles. t_0 is sometimes called the *epoch*.

[4] T is the time taken for the orbiting body to complete one orbit.

that the time taken for one complete orbit is 2π units. Equation (2.34) then becomes

$$\int_0^{2\pi} e^{-ik'M}\,\widetilde{g}_n^m\,(M)\,dM = \int_0^{2\pi} e^{-ik'M} \sum_{k=-\infty}^{\infty} C_n^m(k)\,e^{ikM}\,dM$$
$$= \sum_{k=-\infty}^{\infty} C_n^m(k) \int_0^{2\pi} e^{-ik'M}\,e^{ikM}\,dM, \qquad (2.35)$$

where $\widetilde{g}_n^m(M) = \widetilde{f}_n^m(M/\mu)$. Using equation (I.1), equation (2.35) is reduced to

$$\int_0^{2\pi} e^{-ik'M}\,\widetilde{g}_n^m\,(M)\,dM = \sum_{k=-\infty}^{\infty} C_n^m(k)\,2\pi\,\delta_{kk'}$$
$$= C_n^m(k')\,2\pi.$$

The coefficients $C_n^m(k')$ are then evaluated according to

$$C_n^m(k') = \frac{1}{2\pi}\int_0^{2\pi} e^{-ik'M}\,\widetilde{g}_n^m\,(M)\,dM. \qquad (2.36)$$

Substituting equation (2.24) into (2.36), where $D = D\,(M)$ and the dummy index k' is relabelled as k, gives

$$C_n^m(k) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikM}$$
$$\times \left(\frac{(n-|m|)!}{(n+|m|)!}\left(\frac{R}{a}\right)^{n-2}\left(\frac{a}{D\,(M)}\right)^{n+1} P_n^{|m|}\,(0)\,e^{-im\nu}\right) dM$$
$$= \left(\frac{(n-|m|)!}{(n+|m|)!}\left(\frac{R}{a}\right)^{n-2} P_n^{|m|}\,(0)\right)$$
$$\times \frac{1}{2\pi}\int_0^{2\pi} e^{-i(kM+m\nu)}\left(\frac{a}{D\,(M)}\right)^{n+1} dM. \qquad (2.37)$$

Since $e^{-i(kM+m\nu)}$ is complex and we are only interested in finding real solutions to these equations, only its real part is considered. Using Euler's formula we have

$$e^{-i(kM+m\nu)} = \cos\,(kM + m\nu)\,.$$

Substituting this result into equation (2.37) then gives

$$C_n^m(k) = \left(\frac{(n-|m|)!}{(n+|m|)!}\left(\frac{R}{a}\right)^{n-2} P_n^{|m|}\,(0)\right)$$
$$\times \frac{1}{2\pi}\int_0^{2\pi} \cos\,(kM + m\nu)\left(\frac{a}{D\,(M)}\right)^{n+1} dM. \qquad (2.38)$$

Since $D\,(M)$ is a distance, it will always be positive irrespective of whether M is negative or positive, therefore $D(M) = D(-M)$. It is possible to rewrite equation (2.38) as (see equation I.6)

$$C_n^m(k) = \left(\frac{(n-|m|)!}{(n+|m|)!}\left(\frac{R}{a}\right)^{n-2} P_n^{|m|}\,(0)\right)$$

$$\times \frac{1}{2\pi}\int_0^{2\pi} \cos\left(kM + m\nu\right)\left(\frac{a}{D\left(M\right)}\right)^{n+1} dM. \tag{2.39}$$

Let

$$\Lambda_{n,m,k} = \frac{1}{\pi}\int_0^{\pi} \cos\left(kM + m\nu\right)\left(\frac{a}{D\left(M\right)}\right)^{n+1} dM, \tag{2.40}$$

equation (2.39) then becomes

$$C_n^m(k) = \frac{(n-|m|)!}{(n+|m|)!}\left(\frac{R}{a}\right)^{n-2} P_n^{|m|}\left(0\right)\Lambda_{n,m,k}. \tag{2.41}$$

Up until now, we have assumed that the orbit of the secondary is not completely circularised. We now make the assumption that it is. However, we do not here assume that the primary is phase-locked with the binary orbital motion, that is, we do not assume that the uniform angular velocity Ω of the primary is synchronised with the orbital motion of the secondary. The orbit is thus asynchronous with the rotation of the primary.

A circularised orbit is a degenerate ellipse. This leads to some simplifications in the description of the orbit. It also requires some revision of previous definitions. For example, the orbit no longer has a periastron, so we can no longer assign a natural reference direction in the orbit. The direction of the axis $\bar{x}$ [1] can now be chosen to be any convenient fixed direction in the plane of the orbit. The epoch t_0 will thus be any conveniently chosen time at which the radius vector to the secondary coincides with this direction. Furthermore, since a circle is an ellipse of zero eccentricity, its radius will coincide with the semi-major axis a (and also with the semi-minor axis $b = a$). The distance D between the primary and the secondary is no longer a function of time and and has value $D = a$. Lastly, the mean anomaly M is the same as the true anomaly ν of the orbit. This happens because the origin of the barycentric coordinate system, where the true anomaly maps an ellipse, is located at a focus of the orbital ellipse, which now merges with the second focus of the orbital ellipse to become the centre of the circular orbit. We loose no generality if we set our clocks to zero at the epoch t_0. This means that we can take the reference time t_0 to be 0. With these changes, equation (2.40) becomes

$$\Lambda_{n,m,k} = \frac{1}{\pi}\int_0^{\pi} \cos\Big(M\left(k+m\right)\Big)\, dM. \tag{2.42}$$

If $k + m \neq 0$, then for $u = M(k + m)$ where $du/dM = k + m$, equation (2.42) becomes

$$\begin{aligned}
\Lambda_{n,m,k} &= \frac{1}{\pi\left(k+m\right)}\int_0^{\pi(k+m)} \cos(u)\, du \\
&= \frac{1}{\pi\left(k+m\right)}\ \sin(u)\Big|_0^{\pi(k+m)} \\
&= \frac{1}{\pi\left(k+m\right)}\ \sin(\pi k + \pi m)
\end{aligned}$$

$$= \frac{1}{\pi\,(k+m)}\Big(\sin(\pi k)\cos(\pi m) + \sin(\pi m)\cos(\pi k)\Big).$$

Since $k, m \in \mathbb{Z}$ we will always have

$$\begin{aligned}\cos(\pi k) &= \cos(\pi m) = 1,\\ \sin(\pi k) &= \sin(\pi m) = 0.\end{aligned}$$

Therefore

$$\Lambda_{n,m,k} = 0.$$

On the other hand, if $k + m = 0$, equation (2.42) becomes

$$\begin{aligned}\Lambda_{n,m,k} &= \frac{1}{\pi}\int_0^\pi \cos\Big(M\,(k+m)\Big)\,dM\\ &= \frac{1}{\pi}\int_0^\pi \cos(0)\,dM\\ &= \frac{1}{\pi}\int_0^\pi 1\,dM\\ &= \frac{1}{\pi}\,M\Big|_0^\pi\\ &= 1.\end{aligned}$$

The only non-zero coefficients $\Lambda_{n,m,k}$ are thus those for which $k + m = 0$. The coefficients $C_n^m(k) \neq 0$ are therefore those for which $k = -m$.

The sum that defines the tide generating potential begins at $n = 2$. The first term in the expansion is then the term with $n = 2$. According to equations (2.33) and (2.41), we get

$$W_2(\vec{r}, t) = \sum_{m=-2}^{2} \Gamma_{2,m,-m}(\vec{r})\; e^{imt(\Omega-\mu)}, \tag{2.43}$$

and

$$C_2^m(-k) = \frac{(2-|m|)!}{(2+|m|)!}\,P_2^{|m|}(0). \tag{2.44}$$

For $n = 2$, we have the following possibilities for the values of m and k

$$\begin{aligned}n &= 2\\ m &= -2,-1,0,1,2\\ k &= -m.\end{aligned}$$

For each of these possibilities, the coefficients $C_n^m(k)$ have the following values

$$\begin{aligned}C_2^{-2}(2) = C_2^2(-2) &= \frac{0!}{4!}\,P_2^2(0)\\ &= \frac{1}{8},\end{aligned}$$

$$C_2^{-1}(1) = C_2^1(-1) = \frac{1!}{3!}\,P_2^1(0)$$
$$= 0,$$

$$C_2^0(0) = \frac{2!}{2!}\,P_2^0(0)$$
$$= -\frac{1}{2}.$$

Substituting the above results for coefficients $C_n^m(k)$ into equation (2.32) we have

$$\Gamma_{2,2,-2}\,(\vec{r}) = -\left(\frac{GM_1}{R}\right) C_2^2(-2)\left(\frac{r}{R}\right)^2 P_2^2(\cos\theta)\,e^{2i\phi}$$
$$= -\frac{1}{8}\left(\frac{GM_1}{R}\right)\left(\frac{r}{R}\right)^2 P_2^2(\cos\theta)\,e^{2i\phi},$$

$$\Gamma_{2,-2,2}\,(\vec{r}) = -\left(\frac{GM_1}{R}\right) C_2^{-2}(2)\left(\frac{r}{R}\right)^2 P_2^2(\cos\theta)\,e^{-2i\phi}$$
$$= -\frac{1}{8}\left(\frac{GM_1}{R}\right)\left(\frac{r}{R}\right)^2 P_2^2(\cos\theta)\,e^{-2i\phi},$$

$$\Gamma_{2,1,-1}\,(\vec{r}) = -\left(\frac{GM_1}{R}\right) C_2^1(-1)\left(\frac{r}{R}\right)^2 P_2^1(\cos\theta)\,e^{i\phi}$$
$$= 0,$$

$$\Gamma_{2,-1,1}\,(\vec{r}) = -\left(\frac{GM_1}{R}\right) C_2^{-1}(1)\left(\frac{r}{R}\right)^2 P_2^1(\cos\theta)\,e^{-i\phi}$$
$$= 0,$$

$$\Gamma_{2,0,0}\,(\vec{r}) = -\left(\frac{GM_1}{R}\right) C_2^0(0)\left(\frac{r}{R}\right)^2 P_2^0(\cos\theta)$$
$$= \frac{1}{2}\left(\frac{GM_1}{R}\right)\left(\frac{r}{R}\right)^2 P_2^0(\cos\theta).$$

Equation (2.43) is then

$$W_2 = \Gamma_{2,-2,2}\,(\vec{r})\;e^{-2it(\Omega-\mu)} + \Gamma_{2,0,0}\,(\vec{r}) + \Gamma_{2,2,-2}\,(\vec{r})\;e^{2it(\Omega-\mu)}$$
$$= \left(\frac{GM_1}{R}\right)\left(\frac{r}{R}\right)^2\Bigg(-\frac{1}{8}\,P_2^2(\cos\theta)\,e^{-2i\phi}\,e^{-2it(\Omega-\mu)} + \frac{1}{2}\,P_2^0(\cos\theta)$$
$$-\frac{1}{8}\,P_2^2(\cos\theta)\,e^{2i\phi}\,e^{2it(\Omega-\mu)}\Bigg)$$
$$= \left(\frac{GM_1}{R}\right)\left(\frac{r}{R}\right)^2\left(\frac{1}{2}\,P_2^0(\cos\theta) - \frac{1}{8}\,P_2^2(\cos\theta)\,e^{-2i\left[\phi+t(\Omega-\mu)\right]}\right.$$

$$
\begin{aligned}
&\qquad - \frac{1}{8}\, P_2^2(\cos\theta)\, e^{2i\left[\phi + t(\Omega-\mu)\right]} \Bigg) \\
&= \left(\frac{GM_1}{R}\right)\left(\frac{r}{R}\right)^2 \Bigg(\frac{1}{2}\, P_2^0(\cos\theta) \\
&\qquad - \frac{1}{8}\, P_2^2(\cos\theta) \Big(e^{-2i\left[\phi + t(\Omega-\mu)\right]} + e^{2i\left[\phi + t(\Omega-\mu)\right]} \Big) \Bigg).
\end{aligned}
$$

Using the identity

$$
\cos\Big(2\big(\phi + t\,(\Omega-\mu)\,\big)\Big) = \frac{e^{2i\left[\phi + t(\Omega-\mu)\right]} + e^{-2i\left[\phi + t(\Omega-\mu)\right]}}{2},
$$

we get

$$
\begin{aligned}
W_2\,(\vec{r},t) &= \left(\frac{GM_1}{R}\right)\left(\frac{r}{R}\right)^2 \\
&\quad \times \left(\frac{1}{2}\, P_2(\cos\theta) - \frac{1}{4}\, P_2^2(\cos\theta)\, \cos\Big(2\big(\phi + t\,(\Omega-\mu)\,\big)\Big)\right), \qquad (2.45)
\end{aligned}
$$

where the associated Legendre polynomials of order $m = 0$, $P_n^0(\cos\theta)$, are just the Legendre polynomials of the first kind, $P_n(\cos\theta)$.

Equation (2.45) is the leading term of the tide generating potential. It is the dominant term, and may be taken as a good approximation to W. It is a sum of two terms. The first term is time-independent and is responsible for generating what is called the *equilibrium tide* or the *static tide*. The second term is time dependent. It is responsible for generating what is called the *dynamic tide*.

The static tide is characteristic of a synchronous orbit, in which both components are tidally locked and the mean motion of the secondary is the same as the rotational angular velocity of the primary. The static tide distorts the primary by creating a tidal bulge that is fixed on its surface.

The dynamic tide is characteristic of an asynchronous orbit. It too distorts the shape of the primary by creating a tidal bulge on the surface of the primary. Viewed from the co-rotating reference frame $\widetilde{Q}$, this bulge is dragged across the surface of the primary with the movement of the orbit of the secondary. A person standing at a fixed position on the surface of the primary (not recommended!) would see the "surface" of the star rising and falling as the secondary orbits the primary.

2.8 W in the Frequency Domain

In chapter 3 the oscillation frequencies of a pulsating star will be determined by Fourier transformation of the perturbation equations. In preparation for incorporating the effect of the tidal potential on pulsations, in this section I calculate the Fourier transform of the tide generating potential (2.45).

Denote the Fourier transform of the tide generating potential by $W(\vec{r},\omega)$. It can be calculated by looking at the inverse Fourier transform of $W(\vec{r},t)$ such that

$$W(\vec{r},\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} W(\vec{r},t)\, e^{-i\omega t}\, \mathrm{d}t. \tag{2.46}$$

Using equation (2.45), (2.46) becomes

$$\begin{aligned} W(\vec{r},\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} & \left(\frac{GM_1}{R}\right)\left(\frac{r}{R}\right)^2 \\ & \times \left(\frac{1}{2}\, P_2(\cos\theta) - \frac{1}{8}\, P_2^2(\cos\theta)\, e^{-2i[\phi+t(\Omega-\mu)]}\right. \\ & \left. - \frac{1}{8}\, P_2^2(\cos\theta)\, e^{2i[\phi+t(\Omega-\mu)]}\right) e^{-i\omega t}\, \mathrm{d}t \\ = \frac{r^2}{\sqrt{2\pi}} & \left(\frac{GM_1}{R^3}\right)\left(\frac{1}{2}\, P_2(\cos\theta) \int_{-\infty}^{\infty} e^{-i\omega t}\, \mathrm{d}t\right. \\ & - \frac{1}{8}\, P_2^2(\cos\theta) \int_{-\infty}^{\infty} e^{-2i[\phi+t(\Omega-\mu)]}\, e^{-i\omega t}\, \mathrm{d}t \\ & \left. - \frac{1}{8}\, P_2^2(\cos\theta) \int_{-\infty}^{\infty} e^{2i[\phi+t(\Omega-\mu)]} e^{-i\omega t}\, \mathrm{d}t\right) \\ = \frac{r^2}{\sqrt{2\pi}} & \left(\frac{GM_1}{R^3}\right)\left(\frac{1}{2}\, P_2(\cos\theta) \int_{-\infty}^{\infty} e^{-i\omega t}\, \mathrm{d}t\right. \\ & - \frac{1}{8}\, P_2^2(\cos\theta)\, e^{-2i\phi} \int_{-\infty}^{\infty} e^{-it[\omega+2(\Omega-\mu)]}\, \mathrm{d}t \\ & \left. - \frac{1}{8}\, P_2^2(\cos\theta)\, e^{2i\phi} \int_{-\infty}^{\infty} e^{-it[\omega-2(\Omega-\mu)]}\, \mathrm{d}t\right). \end{aligned} \tag{2.47}$$

Using the integral identity for the Dirac delta function

$$\int_{-\infty}^{\infty} e^{\pm ixt} \mathrm{d}t = 2\pi\, \delta(x),$$

we have

$$\begin{aligned} W(\vec{r},\omega) = \frac{r^2}{\sqrt{2\pi}} & \left(\frac{GM_1}{R^3}\right)\Big(\pi\, P_2(\cos\theta)\, \delta(\omega) \\ & - \frac{\pi}{4}\, P_2^2(\cos\theta)\, e^{-2i\phi}\, \delta\left(\omega + 2\left[\Omega-\mu\right]\right) \\ & - \frac{\pi}{4}\, P_2^2(\cos\theta)\, e^{2i\phi}\, \delta\left(\omega - 2\left[\Omega-\mu\right]\right)\Big), \end{aligned} \tag{2.48}$$

or

$$\begin{aligned} W(\vec{r},\omega) = \frac{r^2}{\sqrt{2\pi}} & \left(\frac{GM_1}{R^3}\right)\Big(\pi\, P_2(\cos\theta)\, \delta(\omega) \\ & - \frac{\pi}{4}\, P_2^2(\cos\theta)\, e^{-2i\phi}\, \delta\left(\omega + 2\left[\Omega-\mu\right]\right) \\ & - \frac{\pi}{4}\, P_2^2(\cos\theta)\, e^{2i\phi}\, \delta\left(\omega - 2\left[\Omega-\mu\right]\right)\Big) \end{aligned}$$

$$= r^2 \left(\frac{GM_1}{R^3}\right) \left(\sqrt{\frac{\pi}{2}}\, P_2(\cos\theta)\, \delta(\omega) \right.$$
$$- \sqrt{\frac{\pi}{32}}\, P_2^2(\cos\theta) \left\{ e^{-2i\phi}\, \delta\left(\omega + 2\left[\Omega - \mu\right]\right) \right.$$
$$\left. \left. + e^{2i\phi}\, \delta\left(\omega - 2\left[\Omega - \mu\right]\right) \right\} \right). \qquad (2.49)$$

By definition $\delta(x) = 0$ if $x \neq 0$, so we see from equation (2.49) that the tidal term can only drive pulsations at values of ω given by

$$\omega = 0, \pm 2\left(\Omega - \mu\right).$$

The value $\omega = 0$ arises from the static component of equation (2.49). A frequency of $\omega = 0$ means no oscillation at all and corresponds to a permanent static deformation of the system. The static component of the tide generating term is thus incapable of driving pulsations in the primary.

The values $\omega = \pm 2\left(\Omega - \mu\right)$ arise from the dynamic tidal terms of the tidal potential and corresponds to a to a single time domain frequency of $2\left(\Omega - \mu\right)$. Provided that $\Omega \neq \mu$, the tidal term drives steady state pulsations at frequency $2\left(\Omega - \mu\right)$. The effect of this driving force on the primary depends on whether it coincides with, or approximates, a natural pulsation frequency of the primary.

If the frequency of the driving force coincides with a natural pulsation frequency of the primary, the driver would produce a strong resonant response in the primary. The primary would thus pulsate strongly at this natural frequency. However, to estimate the amplitude of this oscillation, we would need to include a viscosity term in the fluid equations to dissipate the energy input into the oscillation by the driver. In the absence of viscosity, the oscillation amplitude would increase without limit, which is clearly absurd. With viscosity, the pulsations would reach a finite steady state amplitude.

On the other hand, if the frequency is close to, but not coincident with, a natural pulsation frequency, the dynamic tide would drive pulsations at the forcing frequency $2\left(\Omega - \mu\right)$. The amplitude of the induced pulsations would depend on how close the driving frequency is to the natural pulsation frequency, with the amplitude decreasing as the difference in these frequencies increases. If the driving frequency is too far from a natural frequency, the induced pulsations would be weak or non-existent.

Equation (2.49) is valid only for circularised orbits. In the case of an elliptical orbit, the Fourier transform would contain a spectrum of frequencies. If the orbit is periodic with a constant period, its spectrum would be discrete and would contain frequencies that are overtones of a fundamental. The tidal force would thus drive pulsations at a variety of frequencies. Comments similar to the above would now apply to each frequency present in the Fourier transform. The strength of the driver at a given frequency would be proportional to the amplitude of the corresponding frequency component in the transform, with an induced response that depends on how closely that frequency matches the natural pulsation frequencies of the primary.

Interestingly, only the dynamic tide has any effect on the pulsation of the primary. The static component only produces a distortion of the initial shape of a star and will not have any effect on the pulsation frequencies of a star.

Chapter 3

Pulsation Theory

3.1 Introduction

This chapter introduces standard pulsation theory without the effect of tides. As we saw in Chapter 2, the stellar material is modelled as a fluid continuum. Its equilibrium and motions about the equilibrium configuration are described by the hydrodynamic equations (see Appendix A).

In this thesis, I model the star as a polytrope. Accordingly, a polytropic equation of state has been used to close the set of fluid equations. For a detailed description of polytropes and polytropic processes, see Appendix G.

The goal of pulsation theory is the study of pulsation modes in stars, their stability, and their frequencies. The main tool in this analysis is perturbation theory.

Perturbation theory is a mathematical technique that looks for approximate solutions to a problem in a situation where an exact solution to a simpler similar problem is known. The parameters of the system are assumed to differ by only a small amount from their values in the simpler problem. We thus replace the system parameters by a sum consisting of the parameter value of the known solution, say q, called the *unperturbed parameter*, and a small deviation δq from that value, called the *perturbation*. The sum $q + \delta q$ is then substituted into the governing equations. Many terms in the equations then cancel by virtue of the fact that the unperturbed values q satisfy the governing equations for the unperturbed system, leaving us with equations for the perturbations δq.

The precise definition of a perturbation depends on which formalism one chooses for the description of the fluid. Two formalisms are available, called Eulerian and Lagrangian. Each of these formalisms requires the concept of perturbation to be conceptualised differently.

The Two Formalisms for Fluid Dynamics

To describe the motion of a fluid continuum, we need to state the position of each of its points at each time t. To do this, we must first identify each of the

fluid points by assigning to it a label, which is a unique and permanent identifier. Commonly, the identifier of a given fluid point is chosen to be its initial position $\vec{a}$, with coordinates $\vec{a} = (a^1, a^2, a^3)$.

One way to think of this unique label is by analogy with a field of football players. Each football player carries a unique label on his shirt. This label enables viewers to state unambiguously which player is in view, irrespective of his position on the field. Similarly, imagine "painting" the label $\vec{a}$ onto a fluid point. Whatever actual position this point occupies in its motion, the label it carries uniquely identifies that fluid point. In this way, we can track the motion of each point in the flow.

The identifier label $\vec{a}$ must not be confused with the coordinates $\vec{x} = (x^1, x^2, x^3)$ that specify the actual position in space occupied by that fluid point at time t.

Consider first a single point in the fluid. Its motion is fully described by its trajectory through space. A point trajectory has equations of the form

$$\vec{x} = \vec{\gamma}(t).$$

The fluid element with initial position $\vec{a}$ will then have trajectory

$$\vec{x} = \vec{\gamma}_{\vec{a}}(t).$$

In the continuum model, a fluid does not contain a discrete set of fluid points, but a continuous distribution of them. Therefore, the values of the labels $\vec{a}$ range continuously over some domain $\mathcal{D}$ in $\mathbb{R}^3$. The trajectory $\vec{x}$ is thus a continuous function of the parameters $\vec{a}$, as well as of the time t. It is customary to express this by writing the trajectory equations in the form

$$\vec{x} = \vec{F}(\vec{a}, t). \tag{3.1}$$

The function $\vec{F}$ is called the *flow function* of the fluid. The flow function fully encapsulates all information about the flow of the fluid.

In this description, the velocity of a fluid element is given by

$$\vec{v}_\ell(\vec{a}, t) = \frac{d\vec{\gamma}_{\vec{a}}}{dt}(t) = \frac{\partial \vec{F}}{\partial t}(\vec{a}, t). \tag{3.2}$$

The variable $\vec{v}_\ell$ delivers the velocity of the fluid element as a function of time t and of initial position $\vec{a}$ of the fluid point. The velocity calculated in this way can be called the *Lagrangian velocity field*.

This is not the most convenient way to express the fluid velocity. It is more convenient, when working with the equations of motion of the fluid, to express the velocity in terms of the current position $\vec{x}$ of the fluid point. The reason for this is that the force fields that determine the motion of the fluid through Newton's second law must be evaluated at the *present* position of a fluid element in order to determine its acceleration at this position. It is therefore useful to express the fluid velocity as a function of its present position, and then also its acceleration.

To express $\vec{v}_\ell$ as a function of $\vec{x}$ and t we must eliminate $\vec{a}$ in favour of $\vec{x}$. The current position of the fluid element is given by (3.1). Solving this equation to obtain $\vec{a}$ as a function of $\vec{x}$ we get[1]

$$\vec{a} = \vec{f}\,(\vec{x}, t)\,.$$

This expresses the initial position $\vec{a}$ of the fluid point as a function of its present position $\vec{x}$. The velocity of the fluid at position $\vec{x}$ at time t is thus given by

$$\vec{v}\,(\vec{x},\, t) = \vec{v}_\ell\left(\vec{f}\,(\vec{x},\, t)\,, t\right). \tag{3.3}$$

The vector field $\vec{v}\,(\vec{x}, t)$ is called the *Eulerian velocity field*.

The Eulerian velocity field is a vector field in the usual sense of the word. It delivers the velocity of the fluid point that occupies position $\vec{x}$ at time t.

The Eulerian and Lagrangian velocity fields differ conceptually in a very important respect. The Eulerian velocity does not track the motion of individual fluid points. It views the flow as a global entity, and provides information about the flow pattern at each instant t. In particular, the fluid point that occupies a given position $\vec{x}$ at time t will in general be different from the fluid point that occupies that same position at a later time.

It is characteristic of the Eulerian description that it treats all physical quantities that describe the fluid as fields. It has no interest in the individual experiences of a given fluid element or point. In particular, given a particular fluid point $\vec{a}$, The Eulerian description is agnostic about where this point is at time t, nor does it keep track of what forces it experiences, or its pressure or temperature. The Eulerian description is concerned only with the values of the relevant parameters at fixed positions in space.

Consider a fixed region in space. Construct a lattice of fixed grid points in this region. In the Eulerian formalism, the grid points are the points at which we measure the physical parameters of the fluid at any given time t. As the fluid flows, the values of these parameters change with time, but the fluid that occupies those points is not the fluid that occupied them at some previous time. The fluid flows continuously through the reference grid points at which we measure the properties of the fluid. The means that we do not track the evolution of the fluid parameters in a given parcel of fluid, but only their evolution at fixed points in space.

On the other hand, the Lagrangian description follows individual fluid elements as they move, and measures the values of the fluid parameters in each fluid parcel. It tracks the evolution of the parameters in any given fluid parcel.

[1]Note that $\vec{a}$ here is *not* an acceleration, but the identifying label of the fluid point.

3.2 Perturbation Theory

Perturbation theory is useful for two things. Firstly, it is useful in the study of the stability of a known exact solution, and secondly, for the investigation of solutions to a problem that differ only slightly from a problem with known exact solution. Pulsation theory is an application of the latter kind of study.

In pulsation theory, exact solutions to the problem of interest are not known, but an exact solution to a simpler, directly related problem is available. We then search for a solution to the problem of interest, usually only approximate, by expressing the unknown solution as a power series expansion in terms of some suitably defined small, dimensionless parameter. The series is constructed in such a way that, when the expansion parameter is set to zero, the problem reduces to the one with known solution and the power series collapses to the known solution. The problem of interest can then be approximated to any desired degree of accuracy by evaluating a sufficient number of terms in the power series.

Problems with known solutions that are considered in pulsation theory are most commonly suitably chosen equilibrium stellar configurations. The known solution is called *the unperturbed solution*, and the system it describes is called the *unperturbed system*. This known solution is then "disturbed" by applying a small disturbing influence to it. The disturbance is called the *perturbation*, and the force that produces the perturbation is called *the perturbing force*. When the unperturbed system is a flow rather than an equilibrium configuration, the perturbation need not be produced by an applied force, but could be due to a change in the initial or boundary condition of the problem. Whatever the case may be, the perturbing influence is almost always assumed to be small. The solution to the perturbed system is called the *perturbed solution*, and the parameters that describe the difference between the unperturbed and perturbed solutions are called *the variations* or *the perturbations*.

Perturbing influences lead to equations for the perturbations that are generally non-linear. In general, however, they can be linearised if the perturbing influences are sufficiently small. In the process of linearisation, all terms which are quadratic in the perturbed quantities, or of higher order, are neglected.

Linearised equations enable us to search for normal modes of oscillation in the perturbed quantities and then to construct the general solution for the perturbed quantities by the principle of linear superposition. Without linearisation, the principle of superposition fails and the search for solutions is considerably complicated.

Corresponding to the two formalisms available for the formulation of fluid dynamics, there are two formalisms available for the treatment of perturbations. These are called respectively Eulerian and Lagrangian. Both are used in pulsation theory, often together, with authors moving from one to the other as convenient. Nevertheless, it is possible to develop the theory with strict adherence to one formalism only. There may, however, be disadvantages in adhering to one formalism only. In the Lagrangian formalism, the final perturbation

equations can become difficult to solve analytically (see Appendix J). On the other hand, adhering only to the Eulerian formalism may be undesirable because we are generally interested in the motion of a particular fluid element and the values of the fluid parameters associated with it. This information is not directly provided by the Eulerian formalism. It is best therefore to develop the ability to move freely from one formalism to the other. This is done by means of formulas that we will develop below.

Lagrangian and Eulerian perturbation theory essentially represent different approaches to the solution of one and the same problem. They describe the perturbation of the physical quantities that describe a given system. They differ only in the way that the system is viewed, and accordingly lead to different definitions of the perturbation, which we shall now discuss.

Lagrangian and Eulerian Perturbations

Let Q be a physical quantity. The Eulerian formalism describes Q as a field $Q(\vec{r},t)$. Denote the field that describes Q in the unperturbed system by $Q_0(\vec{r},t)$. The difference in the value of Q between the perturbed and the unperturbed flows is thus

$$Q'(\vec{r},t) = Q(\vec{r},t) - Q_0(\vec{r},t).$$

This difference is called the *Eulerian perturbation* in the variable Q.

In the Lagrangian formalism, we follow the motion of a given fluid point $\vec{a}$, and track the value of Q at this point with time. We can thus write the value of Q as a function of $\vec{a}$ and t. To distinguish the functions used in the Lagrangian formalism from their counterparts in the Eulerian formalism, we label them with a subscript ℓ. In the Lagrangian formalism, therefore, the value of Q for the point $\vec{a}$ at time t is given by $Q_\ell(\vec{a},t)$. Denote the value of this function in the unperturbed flow by $Q_{\ell,0}(\vec{a},t)$. The difference between the value of Q at time t in the perturbed and unperturbed flows is then given by

$$\delta Q_\ell(\vec{a},t) = Q_\ell(\vec{a},t) - Q_{\ell,0}(\vec{a},t).$$

This difference is called the *Lagrangian perturbation* in the variable Q.

Note that we have adopted the convention that differences in the value of Q are denoted by a prime in the Eulerian formalism, and by the symbol δ in the Lagrangian formalism.

In the systems of interest in this thesis, the unperturbed system is an equilibrium configuration. This means that in both Eulerian and Lagrangian formalisms, the value of Q in the unperturbed flow will not depend on time. Accordingly, the expressions for the Eulerian and Lagrangian perturbations of Q will be, respectively,

$$\left.\begin{array}{rcl} Q'(\vec{r},t) & = & Q(\vec{r},t) - Q_0(\vec{r}) \\ \delta Q_\ell(\vec{a},t) & = & Q_\ell(\vec{a},t) - Q_{\ell,0}(\vec{a}) \end{array}\right\} \tag{3.4}$$

The points of view embodied in the Eulerian and Lagrangian definitions of perturbations are precisely the same as those that characterise their respective

theories of fluid flow. Thus Q' is the difference in values of the property Q *at the same point in space* at time t in the perturbed and unperturbed flows. Q' is thus viewed as a property of the fluid at a fixed point in space, and is not related to any individual fluid element. On the other hand, δQ_ℓ is the difference in the value of Q *in the same fluid element* at time t as it takes part in the two flows. Lyndon-Bell and Ostriker [39] ask us to imagine two flows simultaneously: one is the perturbed flow, which is the real flow; the other is the unperturbed flow, which does not really take place and which they call the "ghostly flow". The same fluid element moves along different paths in space in these flows, and with different values of the property Q at any given time. δQ_ℓ is the difference in these values at a given time.

Of particular interest in perturbation theory is the vector *displacement* of a given fluid point. This is the vector that joins the position that it would have had at time t in the unperturbed flow to the position that it actually has at time t in the perturbed flow. In the Lagrangian formalism, the position of a fluid point in the unperturbed flow is given by the function $\vec{r}_0\,(\vec{a},t)$, while in the perturbed flow it is given by the function $\vec{r}\,(\vec{a},t)$. This displacement, denoted by $\vec{\xi}$, is given by

$$\vec{\xi}_\ell\,(\vec{a},t) = \vec{r}\,(\vec{a},t) - \vec{r}_0\,(\vec{a},t).$$

The quantity $\vec{\xi}_\ell$ is called the *Lagrangian displacement* of the perturbed flow. It measures the displacement of a given fluid point in the perturbed flow from the position that it would have had in the unperturbed flow. As represented above, it is expressed as a function of $\vec{a}$ and t, and so is in Lagrangian representation.

The velocity of a fluid point in the perturbed flow can now be expressed, in the Lagrangian description, in terms of the Lagrangian displacement,

$$\frac{\partial\vec{\xi}_\ell}{\partial t}(\vec{a},t) = \frac{\partial\vec{r}}{\partial t}(\vec{a},t) - \frac{\partial\vec{r}_0}{\partial t}(\vec{a},t) = \vec{v}_\ell\,(\vec{a},t) - \vec{v}_{\ell,0}\,(\vec{a},t),$$

or

$$\vec{v}_\ell\,(\vec{a},t) = \vec{v}_{\ell,0}\,(\vec{a},t) + \frac{\partial\vec{\xi}_\ell}{\partial t}(\vec{a},t).$$

This equation states the intuitively obvious fact that $\partial\vec{\xi}_\ell/\partial t$ is the relative velocity of a fluid point in the perturbed flow measured with respect to its velocity the unperturbed flow.

By eliminating $\vec{a}$ in favour of $\vec{r}_0$, or of $\vec{r}$, we get an Eulerian representation of the Lagrangian displacement. Evidently, two Eulerian representations are possible, depending on whether we choose to eliminate $\vec{a}$ in favour of position $\vec{r}_0$ in the unperturbed flow, or of position $\vec{r}$ in the perturbed flow.

Notation for the Eulerian representation of the Lagrangian displacement can be complicated. Some authors, such as Cox [40] or Lyndon-Bell and Ostriker [39], write $\vec{r}_0$ for the position variables used in the unperturbed case, and $\vec{r}$ for those used in the perturbed case. This convention is unnecessarily clumsy.

In this thesis, I adopt a different convention in which the position of a given fluid point in the unperturbed flow is denoted by $\vec{r}$, while its position in the perturbed flow is given by $\vec{r}+\vec{\xi}(\vec{r},t)$ or, more simply, by $\vec{r}+\vec{\xi}$, where the Lagrangian displacement $\vec{\xi}$ is assumed to have been expressed as a function of the position of the point $\vec{r}$ in the unperturbed flow, and of t.

The unperturbed flows considered in this thesis are equilibrium configurations. This means that a given fluid point does not change its position with time, and hence

$$\vec{r}_0\left(\vec{a},t\right)=\vec{a},$$

and

$$\frac{\partial\vec{r}_0}{\partial t}(\vec{a},t)=\frac{\partial\vec{a}}{\partial t}=0,$$

since, in Lagrangian formalism, the variables $\vec{a},t$ are independent. The Lagrangian displacement then becomes, in Lagrangian representation,

$$\vec{\xi}\left(\vec{a},\,t\right)=\vec{r}\left(\vec{a},\,t\right)-\vec{a}, \tag{3.5}$$

and the Lagrangian velocity field becomes

$$\vec{v}_\ell\left(\vec{a},t\right)=\frac{\partial\vec{\xi}_\ell}{\partial t}(\vec{a},t).$$

Relation between Lagrangian and Eulerian Perturbations

Lagrangian and Eulerian perturbations are very different from one another as concepts. The Eulerian difference is a simple difference in values of two fields, perturbed and unperturbed, at the same position $\vec{r}$ in space at the same time t. This difference does not care that the fluid points that occupy a given position at different times are different. The Lagrangian description, however, is the difference in values of the parameter for the *same fluid point*, which occupies different positions in space at the same instant t. These two quantities can nevertheless be related to one another, provided that we convert both to the same representation. In what follows, I convert the Lagrangian perturbation into an Eulerian representation where the position $\vec{r}$ represents the position of the fluid point in the *unperturbed flow*.

Expressing the Lagrangian displacement as a function of position in the unperturbed flow, we can write the position of a given fluid point in the unperturbed flow as $\vec{r}$, and its position in the perturbed flow as $\vec{r}+\vec{\xi}(\vec{r},t)$. The Lagrangian perturbation of the fluid parameter Q, *expressed in this Eulerian representation*, then becomes

$$\delta Q\left(\vec{r},t\right)=Q\left(\vec{r}+\vec{\xi}(\vec{r},t),t\right)-Q_0\left(\vec{r},t\right).$$

This relation is exact. No approximation has yet been made. In the case where the Lagrangian displacement $\vec{\xi}$ is small, that is $|\vec{\xi}|\ll|\vec{r}|$, we get an approximate relation by expanding the right hand side of the above equation by a Taylor's series such that

$$\delta Q\left(\vec{r},t\right)=\left[Q\left(\vec{r},t\right)+\vec{\xi}\left(\vec{r},t\right)\cdot\vec{\nabla}Q\left(\vec{r},t\right)+\cdots\right]-Q_0\left(\vec{r},t\right)$$

$$= Q\left(\vec{r},t\right) - Q_0\left(\vec{r},t\right) + \vec{\xi}\left(\vec{r},t\right)\cdot\nabla Q\left(\vec{r},t\right) + \cdots .$$

Note that $Q\left(\vec{r},t\right) - Q_0\left(\vec{r},t\right) = Q'(\vec{r},t)$ is the Eulerian perturbation in Q, so

$$\delta Q\left(\vec{r},t\right) = Q'\left(\vec{r},t\right) + \vec{\xi}\left(\vec{r},t\right)\cdot\vec{\nabla} Q\left(\vec{r},t\right) + \cdots .$$

All functions in this relation are now evaluated at $(\vec{r},t)$, so we may safely omit all arguments. We thus get, correct to first order,

$$\delta Q \approx Q' + \vec{\xi}\cdot\vec{\nabla} Q. \tag{3.6}$$

This equation relates the Lagrangian to the Eulerian perturbation at a point $\vec{r}$ in the *unperturbed flow* at time t. Equation (3.6) allows us to switch between Lagrangian and Eulerian perturbations in a calculation.

Note that all quantities in this equation are in Eulerian representation, that is, they are functions of $\vec{r}$, t and not of $\vec{a}$, t.

3.3 Equations for the Unperturbed Configuration

Perturbation theory requires us to know an exact solution that plays the role of the unperturbed system. Pulsation theory requires the unperturbed system to be the equilibrium configuration of a star. We thus consider a spherically symmetric star in hydrostatic equilibrium. Choose the origin of coordinates at its centre of mass, with axes arbitrarily oriented, and use spherical coordinates in this frame of reference.

Physically, the assumption of spherical symmetry means that the star must be non-rotating or, if it is rotating, that the rotation rate must be sufficiently slow for the star not to display any significant departures from spherical symmetry.

In hydrostatic equilibrium, all physical parameters that characterise the stellar fluid are time independent, and its velocity field $\vec{v}_0$ is identically 0. In the case of slow rotation, the velocity field must be negligibly small. The requirement of spherical symmetry is met by requiring all physical parameters to be functions of r alone, independent of the angles θ and ϕ.

For a hydrostatic equilibrium, the first of the hydrodynamic equations (A.1) is satisfied identically, since $\vec{v} = 0$ and ρ is time independent. This equation thus yields no information. The remaining equations then become

$$\frac{dP_0}{dr} = -\rho_0\,\frac{d\Phi_0}{dr}$$

$$\frac{1}{r^2}\,\frac{d}{dr}\left(r^2\,\frac{d\Phi_0}{dr}\right) = 4\pi G\rho_0$$

where all fluid parameters relating to the unperturbed equilibrium configuration carry subscript 0, and all fields are functions of r alone.

These are two equations for three unknown fields P_0, ρ_0 and Φ_0, and are insufficient for a complete solution. One more equation is needed. As discussed in Appendix A, there is only one more general physical principle available to provide another equation to close the system, namely the principle of the conservation of energy for the fluid. However, this too is insufficient for closing the system of equations since it introduces one more variable, the temperature T of the system. To achieve closure, we need to identify some particular physical process for the stellar material that provides an additional relation between the variables. This is equivalent to choosing a stellar model.

Stellar models are highly complex and beyond the scope of the aims of this thesis. Their complexity requires that their equations be implemented in computer code. Implementations of this kind are lifetime pursuits that typically require teams of astrophysicists and numerical analysts. Setting up such a model is therefore beyond the scope of this thesis. Some open source codes are available, but at the time when the work for this thesis was carried out, no open source implementation was available that contained working sub-modules for stellar pulsation or for the effect of tides. For these reasons, it was decided to work in this thesis with the simplest of all the stellar models, namely that of the polytropic star. Even though this model is clearly simplistic, it is not altogether unrealistic: degenerate stars like white dwarfs and neutron stars are known to be polytropes. Moreover, Eddington's "standard model" of a star, the first fairly successful model of a star ever proposed, is also a polytrope.

The polytropic law is a power law that relates the pressure and density of a material. It has the form

$$P_0 = k\,\rho_0^{\gamma'} \tag{3.7}$$

where k is a constant and γ' is a constant exponent, sometimes called the *polytropic exponent*, not to be confused with the polytropic index which we shall introduce later. The polytropic law has the advantage both of closing the above system of equations for hydrostatic equilibrium and of providing a relatively simple model to illustrate the workings of pulsation theory and the effect of tides. The polytropic equation of state is discussed more fully in Appendix G.2.

The equations for the static equilibrium of a polytropic star thus become

$$\left.\begin{aligned} \frac{dP_0}{dr} &= -\rho_0 \frac{d\Phi_0}{dr} \\ \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\Phi_0}{dr}\right) &= 4\pi G\rho_0 \\ P_0 &= k\,\rho_0^{\gamma'} \end{aligned}\right\} \tag{3.8}$$

These equations can be decoupled. Substituting equation three into equation one and then that result into equation two of (3.8) we get

$$-4\pi G\rho_0 = \frac{k}{r^2}\frac{d}{dr}r^2\left(\frac{1}{\rho_0}\frac{d\rho_0^{\gamma'}}{dr}\right)$$

$$= \frac{k}{r^2}\frac{d}{dr}r^2\left(\frac{\gamma'\,\rho_0^{\gamma'-1}}{\rho_0}\frac{d\rho_0}{dr}\right)$$
$$= \frac{k\,\gamma'}{r^2}\frac{d}{dr}r^2\left(\rho_0^{\gamma'-2}\,\frac{d\rho_0}{dr}\right) \tag{3.9}$$

Using the identity

$$\rho_0^{\gamma'-2}\,\frac{d\rho_0}{dr} = \frac{d}{dr}\left(\frac{\rho_0^{\gamma'-1}}{\gamma'-1}\right)$$

equation (3.9) becomes

$$-4\pi G\rho_0 = k\left(\frac{\gamma'}{\gamma'-1}\right)\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\,\rho_0^{\gamma'-1} \tag{3.10}$$

Decoupling the equations in this way, and de-dimensionalising them in such a way that $\gamma'-1$ is removed from the derivative of ρ, leads to the Lane-Emden equation (see Appendix G.3). The Lane-Emden equation can be solved analytically only for a very limited number of values of γ' [41, 42]. For other values, these equations need to be solved numerically.

Numerical solutions to equation (3.10) can be found by reducing it to a finite difference equation (see Appendix H), with r discretised in a chosen number of steps. The total mass and radius of the polytropic model can be chosen to match a star of interest. However, note that, for a given value of $\gamma\,'$, a polytrope is subject to a *mass-radius relation* [42]. We can therefore not match the mass and radius of a star to our model if $\gamma\,'$ has been chosen. Nevertheless, we may succeed in matching the model to a star of given mass and radius by adjustment of the value of $\gamma\,'$. In the case of the Lane-Emden equation this relationship is defined by equation (G.21). We can then calculate the unperturbed parameters to any desired degree of accuracy, and tabulate them in look-up tables that can be used in conjunction with perturbation theory when modelling a perturbed system.

In this thesis I adopt a non-standard de-dimensionalising formula for the density. This choice is a matter of convenience and has no effect on the results of the simulation. It is convenient to use the same de-dimensionalsing technique in both the unperturbed and the perturbed equations. The choice made here is better suited to the form of the perturbed radial equation (3.50) (see Chapter 4).

The fact that our unperturbed configuration is a hydrostatic equilibrium simplifies the relation between the Eulerian and Lagrangian velocity perturbations: they become one and the same quantity. From equation (3.6) we have, in Eulerian representation,

$$\delta\vec{v}\,(\vec{r},t) = \vec{v}\,'\,(\vec{r},t) + \vec{\xi}\cdot\nabla\vec{v}_0\,(\vec{r}\,)\,, \tag{3.11}$$

which becomes, since $\vec{v}_0 = 0$,

$$\delta\vec{v}\,(\vec{r},t) = \vec{v}\,'\,(\vec{r},t)\,. \tag{3.12}$$

If, in Eulerian representation, we denote the position of a *given fluid point* in the perturbed flow by $\vec{r}\,'$, then this position is related to the position $\vec{r}$ of the same fluid point in the unperturbed flow by

$$\vec{r}\,' = \vec{r} + \vec{\xi}.$$

In this equation, $\vec{r}$ and $\vec{r}\,'$ are a shorthand for the unperturbed and perturbed trajectories of the given fluid point, and $\vec{\xi}$ represents the corresponding Lagrangian displacement of the given point, so we get

$$\frac{d\vec{r}'}{dt} = \frac{d\vec{r}}{dt} + \frac{d\vec{\xi}}{dt},$$

and hence the velocity of the fluid point in the perturbed flow relative to the same point in the unperturbed flow is

$$\delta v = \frac{d\vec{r}'}{dt} - \frac{d\vec{r}}{dt} = \frac{d\vec{\xi}}{dt}.$$

This relation holds for each fluid point at time t, not only the given one, so in terms of the Eulerian fields δv and $\vec{\xi}$, this equation then becomes

$$\delta v\left(\vec{r},t\right) = \frac{\partial\vec{\xi}}{\partial t}\left(\vec{r},t\right).$$

3.4 Equations for the Perturbed Configuration

Using the definition for the Eulerian perturbation given in equation (3.4), we write

$$\left.\begin{array}{rcl} \rho\left(\vec{r},t\right) & = & \rho_0\left(r\right) + \rho'\left(\vec{r},t\right), \\ P\left(\vec{r},t\right) & = & P_0\left(r\right) + P'\left(\vec{r},t\right), \\ \Phi\left(\vec{r},t\right) & = & \Phi_0\left(r\right) + \Phi'\left(\vec{r},t\right), \\ \vec{v}\left(\vec{r},t\right) & = & \vec{v}_0\left(r\right) + \vec{v}\,'\left(\vec{r},t\right) = \vec{v}\,'\left(\vec{r},t\right), \end{array}\right\} \tag{3.13}$$

where quantities with subscript 0 are the functions calculated in the unperturbed problem, and so are data for this problem. The unknowns are therefore the perturbations ρ', P', Φ' and $\vec{v}\,'$.

We obtain equations for these unknowns by substituting ρ, P, Φ and $\vec{v}$ into the hydrodynamic equations. The resulting equations can be simplified by using the governing equations of the unperturbed problem. This leads to a set of non-linear equations in the unknowns.

Generally, a perturbation of a system leads to small departures from the unperturbed solutions. We shall therefore assume that all the perturbed quantities are small. Consequently, all products of the perturbations that are of second or higher order may be neglected. This process of simplification leads to a set of approximate linear equations for the perturbations. The process by which we get these approximate linear equations is called *linearisation*. We shall now linearise each of the governing equations in turn. The arguments of all functions will be suppressed, except when this leads to confusion.

The Continuity Equation

The first equation of the set (A.1) becomes

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\Big(\rho_0 + \rho'\Big) + \vec{\nabla}\cdot\Big(\,(\rho_0 + \rho'\,)\,\vec{v}\,'\Big) \\ &= \frac{\partial \rho'}{\partial t} + \vec{\nabla}\cdot\Big(\rho_0\,\vec{v}\,' + \rho'\,\vec{v}\,'\Big). \end{aligned} \tag{3.14}$$

Linearisation of (3.14) then gives

$$\frac{\partial \rho'}{\partial t} + \vec{\nabla}\cdot\Big(\rho_0\,\vec{v}\,'\Big) = 0.$$

Using equations (3.12) and (3.13) for $\vec{v}\,'$ we get

$$\frac{\partial \rho'}{\partial t} + \vec{\nabla}\cdot\left(\rho_0 \frac{\partial \vec{\xi}}{\partial t}\right) = 0.$$

Since ρ_0 is not dependent on time, it and the time derivative commute and we have

$$\frac{\partial}{\partial t}\left(\rho' + \vec{\nabla}\cdot\Big(\rho_0\,\vec{\xi}\Big)\right) = 0. \tag{3.15}$$

Integrating both sides gives

$$\rho' + \vec{\nabla}\cdot\Big(\rho_0\,\vec{\xi}\Big) = f\,(t)$$

where $f\,(t)$ is an arbitrary function of time. I shall assume for this thesis that $f\,(t) = 0$. Then the equation reduces to

$$\rho' + \vec{\nabla}\cdot\Big(\rho_0\,\vec{\xi}\Big) = 0 \tag{3.16}$$

In fact, it can be proven by using a completely different and more fundamental approach to perturbation theory that this equation follows with no further assumptions [43]. We shall therefore accept this as the correct form of the continuity equation for perturbation theory.

Euler's Eequation

Since $\vec{v}_0 = 0$, the second equation of system (A.1) becomes

$$\frac{\partial \vec{v}\,'}{\partial t} + \Big(\vec{v}\,'\cdot\vec{\nabla}\Big)\vec{v}\,' = -\frac{1}{\rho_0 + \rho'}\vec{\nabla}\Big(P_0 + P'\Big) - \vec{\nabla}\Big(\Phi_0 + \Phi'\Big).$$

The second term on the left hand side is quadratic in $\vec{v}\,'$ and so may be neglected, leaving

$$\frac{\partial \vec{v}\,'}{\partial t}\Big(\rho_0 + \rho'\Big) = -\vec{\nabla}\Big(P_0 + P'\Big) - (\rho_0 + \rho'\,)\,\vec{\nabla}\Big(\Phi_0 + \Phi'\Big),$$

or

$$\rho_0\frac{\partial \vec{v}\,'}{\partial t} + \rho'\frac{\partial \vec{v}\,'}{\partial t} = -\vec{\nabla}P_0 - \vec{\nabla}P' - \rho_0\vec{\nabla}\Phi_0 - \rho'\vec{\nabla}\Phi_0$$

$$- \rho_0 \vec{\nabla}\Phi' - \rho'\vec{\nabla}\Phi'. \tag{3.17}$$

Linearising equation (3.17) gives

$$\rho_0 \frac{\partial \vec{v}'}{\partial t} = -\vec{\nabla}P_0 - \vec{\nabla}P' - \rho_0\vec{\nabla}\Phi_0 - \rho'\vec{\nabla}\Phi_0 - \rho_0\vec{\nabla}\Phi'. \tag{3.18}$$

Using equation one of (3.8), where $\vec{\nabla}P_0 = \partial P/\partial r$ owing to radial symmetry in the unperturbed configuration, equation (3.18) becomes

$$\rho_0 \frac{\partial \vec{v}'}{\partial t} = -\vec{\nabla}P' - \rho'\vec{\nabla}\Phi_0 - \rho_0\vec{\nabla}\Phi'.$$

Using equations (3.12) and (3.13) for $\vec{v}'$ then gives

$$\rho_0 \frac{\partial^2 \vec{\xi}}{\partial t^2} = -\vec{\nabla}P' - \rho'\vec{\nabla}\Phi_0 - \rho_0\vec{\nabla}\Phi'. \tag{3.19}$$

This is Euler's equation for the perturbed system.

Poisson's Equation

The third equation of the set (A.1) becomes

$$\nabla^2\Big(\Phi_0 + \Phi'\Big) = 4\pi G\Big(\rho_0 + \rho'\Big),$$

and since

$$\nabla^2\Phi_0 = 4\pi G\rho_0$$

this equation becomes

$$\nabla^2\Phi' = 4\pi G\rho'. \tag{3.20}$$

This is Poisson's equation for the perturbed system.

The Polytropic Equation of State

The polytropic equation (3.7) becomes

$$P_0 + P' = k\left(\rho_0 + \rho'\right)^{\gamma'} = k\,\rho_0^{\gamma'}\left(1 + \frac{\rho'}{\rho_0}\right)^{\gamma'}.$$

According to our assumptions, we have $\rho'/\rho_0 \ll 1$, so expanding the bracket on the right hand side in a Taylor series we get

$$P_0 + P' \approx k\,\rho_0{}^{\gamma'}\left(1 + \gamma'\frac{\rho'}{\rho_0}\right) = k\,\rho_0{}^{\gamma'} + k\,\gamma'\,\rho_0{}^{\gamma'}\frac{\rho'}{\rho_0}.$$

In the unperturbed flow, we have

$$P_0 = k\,\rho_0{}^{\gamma'},$$

so the linearised equation of state becomes

$$P' = k\,\gamma'\,\rho_0{}^{\gamma'}\,\frac{\rho'}{\rho_0} = k\,\gamma'\,(\rho_0)^{\gamma'-1}\,\rho'\,.$$

For convenience, define

$$\tau\,(r) = k\,\gamma'\,(\rho_0)^{\gamma'-1}\,. \tag{3.21}$$

The function $\tau\,(r)$ can be calculated from the known equilibrium parameter ρ_0, and the polytropic exponent γ', and is therefore a known quantity. Then equation (3.21) becomes

$$P' = \tau\,\rho'\,. \tag{3.22}$$

This is the equation of state for the perturbed system.

Summary

The system of linearised equations that determine the perturbed quantities v', ρ', P' and Φ' are:

$$\boxed{\begin{aligned} 0 &= \rho' + \vec{\nabla}\cdot\left(\rho_0\,\vec{\xi}\right), \\ \rho_0\frac{\partial^2\vec{\xi}}{\partial t^2} &= -\vec{\nabla}P' - \rho'\vec{\nabla}\Phi_0 - \rho_0\vec{\nabla}\Phi', \\ \nabla^{\,2}\Phi' &= 4\pi G\rho', \\ P' &= \tau\,\rho'\,. \end{aligned}} \tag{3.23}$$

3.5 The Frequency Domain

To determine the pulsation frequencies of a star, we search for normal mode solutions to the perturbation equations. The general solution to the pulsation equations can then be found from the principle of superposition which guarantees that a general pulsation will be a linear superposition of the normal mode solutions.

The perturbation equations are functions of time, not of frequency. They therefore need to be transformed to the frequency domain. In this thesis, I adopt the convention that the Fourier transform and the inverse Fourier transform of an arbitrary physical parameter $Q\,(\vec{r},\,t)$ are given respectively by the formulas

$$Q\,(\vec{r},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} Q\,(\vec{r},\omega)\,e^{i\omega t}d\omega,$$
$$Q\,(\vec{r},\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} Q\,(\vec{r},t)\,e^{-i\omega t}dt.$$

If we Fourier transform an equation with independent variables r and t only with respect to t, only terms containing derivatives with respect to t are affected. Equations without time derivatives are left essentially unchanged. In particular,

of all the perturbation equations, only (3.19) is affected. For example, replacing $\rho'(\vec{r},t)$ and $\vec{\xi}(\vec{r},t)$ in equation (3.16) by their corresponding Fourier integrals gives

$$0 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \rho'\left(\vec{r},\omega\right) e^{i\omega t}\, d\omega + \vec{\nabla} \cdot \left(\rho_0\left(r\right) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \vec{\xi}\left(\vec{r},\omega\right) e^{i\omega t}\, d\omega \right).$$

Removing the constant factors and noting that the operations of multiplying by $\rho_0\left(r\right)$ and taking the divergence commute with integration with respect to ω, this becomes

$$0 = \int_{-\infty}^{\infty} \rho'\left(\vec{r},\omega\right) e^{i\omega t}\, d\omega + \int_{-\infty}^{\infty} \vec{\nabla} \cdot \left(\rho_0\left(r\right) \vec{\xi}\left(\vec{r},\omega\right) \right) e^{i\omega t} d\omega,$$

which means that

$$\rho'\left(\vec{r},\omega\right) + \vec{\nabla} \cdot \left(\rho_0\left(r\right) \vec{\xi}\left(\vec{r},\omega\right) \right) = 0.$$

This equation has the same form as equation (3.16), but with $\rho'\left(\vec{r},t\right)$ and $\vec{\xi}(\vec{r},t)$ replaced respectively by $\rho'\left(\vec{r},\omega\right)$ and $\vec{\xi}(\vec{r},\omega)$. Equations (3.20) and (3.22) thus become

$$\begin{aligned} \nabla^2 \Phi'(\vec{r},\omega) &= 4\pi G \rho'(\vec{r},\omega), \\ P'(\vec{r},\omega) &= \tau(r)\, \rho'(\vec{r},\omega)\,. \end{aligned}$$

The transformation of equation (3.19) is evaluated as follows. After cancelling the numerical factor from both sides, the left hand side becomes

$$\begin{aligned} \rho_0(r) \frac{\partial^2}{\partial t^2} \int_{-\infty}^{\infty} \vec{\xi}\left(\vec{r},\omega\right) e^{i\omega t}\, d\omega &= \int_{-\infty}^{\infty} \rho_0\left(r\right) \vec{\xi}\left(\vec{r},\omega\right) \left(\frac{\partial^2}{\partial t^2}\, e^{i\omega t} \right) d\omega \\ &= -\int_{-\infty}^{\infty} \rho_0\left(r\right) \vec{\xi}\left(\vec{r},\omega\right) \omega^2 e^{i\omega t}\, d\omega \end{aligned}$$

so that the equation becomes

$$\begin{aligned} &-\int_{-\infty}^{\infty} \rho_0\left(r\right) \vec{\xi}\left(\vec{r},\omega\right) \omega^2 e^{i\omega t}\, d\omega \\ &\qquad = \int_{-\infty}^{\infty} \left(-\vec{\nabla} P'\left(\vec{r},\omega\right) - \rho'\left(\vec{r},\omega\right) \vec{\nabla} \Phi_0\left(r\right) - \rho_0\left(r\right) \vec{\nabla} \Phi'\left(\vec{r},\omega\right) \right) e^{i\omega t}\, d\omega. \end{aligned}$$

After again transforming, and dividing by $\rho_0\left(r\right)$, this equation becomes

$$-\omega^2\, \vec{\xi}\left(\vec{r},\omega\right) = -\frac{1}{\rho_0\left(r\right)} \vec{\nabla} P'\left(\vec{r},\omega\right) - \frac{\rho'\left(\vec{r},\omega\right)}{\rho_0\left(r\right)} \vec{\nabla} \Phi_0\left(r\right) - \vec{\nabla} \Phi'\left(\vec{r},\omega\right).$$

In the above equation, ω is a potential frequency of oscillation for the star. In principle, this equation could be solved for any choice of value for ω. However, the solutions obtained in this way will not, in general, satisfy the boundary conditions for the oscillation problem. The boundary conditions thus select from among potential values for ω a discrete subset of values. The values admitted by the boundary conditions are called the *eigenvalues* or *characteristic values* for the problem. The eigenvalues represent the frequencies of the normal mode

oscillations of the system.

The eigenfrequencies ω are unknowns for which we have to solve. Once the eigenfrequencies are known, the corresponding solutions for the remaining perturbation variables can be found. The solutions for the perturbation variables can then be used to construct the corresponding variables for the perturbed system by adding them to their corresponding equilibrium values. These then represent the parameters of the perturbed system as functions of $\vec{r}$ and t.

In practise, it is necessary to find only the eigenfrequencies ω of the system. These can then be compared with frequencies contained in the light curves of observed stars. The data can also be made to yield the relative strength of these frequencies, which in turn could be used to reconstruct the oscillational modes of the observed star up to an unknown common scaling factor. Reconstructions of this kind, however, are not interesting.

Summary

The equations that, together with the boundary conditions, determine the oscillation frequencies of the star are:

$$\left.\begin{aligned}
\rho'(\vec{r},\omega) &= -\vec{\nabla}\cdot\left(\rho_0(r)\ \vec{\xi}(\vec{r},\omega)\right) \\
\omega^2\,\vec{\xi}(\vec{r},\omega) &= \frac{1}{\rho_0(r)}\,\vec{\nabla}P'(\vec{r},\omega) + \vec{\nabla}\Phi'(\vec{r},\omega) + \frac{\rho'(\vec{r},\omega)}{\rho_0(r)}\,\vec{\nabla}\Phi_0(r) \\
\nabla^2\Phi'(\vec{r},\omega) &= 4\pi G\rho'(\vec{r},\omega) \\
P'(\vec{r},\omega) &= \tau(r)\,\rho'(\vec{r},\omega)
\end{aligned}\right\} \quad (3.24)$$

Separation into Radial and Non-Radial Terms

The second of equations (3.24) is a vector equation that can be separated into radial and non-radial components. Let $\vec{\xi}_r$ and $\vec{\xi}_\perp$ be respectively the radial and non-radial components of Lagrangian displacement $\vec{\xi}$. Then $\vec{\xi}_r = \xi_r\,\hat{r}$, and

$$\vec{\xi} = \xi_r\,\hat{r} + \vec{\xi}_\perp,$$

where

$$\vec{\xi}_\perp = \xi_\theta\,\hat{\theta} + \xi_\phi\,\hat{\phi}.$$

Note that, in general, all three components ξ_r, ξ_θ and ξ_ϕ will be functions of all three of the variables r, θ, ϕ.

It is useful to introduce the following notation for $\vec{\nabla}$. In spherical coordinates, we have

$$\vec{\nabla}Q = \frac{\partial Q}{\partial r}\,\hat{r} + \frac{1}{r}\,\frac{\partial Q}{\partial \theta}\,\hat{\theta} + \frac{1}{r\sin\theta}\,\frac{\partial Q}{\partial \phi}\,\hat{\phi}.$$

The gradient thus separates nicely into a radial and a tangential component at each point. It is useful to introduce a separate symbol for its tangential part. Denote it by $\vec{\nabla}_\perp Q$. Thus

$$\vec{\nabla}_\perp Q = \frac{1}{r}\left(\frac{\partial Q}{\partial \theta}\,\hat{\theta} + \frac{1}{\sin\theta}\frac{\partial Q}{\partial \phi}\,\hat{\phi}\right).$$

Note that if Q is a function of r alone, then $\vec{\nabla} Q$ has only a radial component,

$$\vec{\nabla} Q = \frac{\partial Q}{\partial r}\,\hat{r}.$$

The formula for $\vec{\nabla}_\perp Q$ shows that we can define an associated tangential gradient operator $\vec{\nabla}_\perp$, given by

$$\vec{\nabla}_\perp = \frac{1}{r}\left(\frac{\partial}{\partial \theta}\,\hat{\theta} + \frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\,\hat{\phi}\right).$$

Using these notations, the second equation of system (3.24) becomes

$$\omega^2\,\xi_r\,\hat{r} + \omega^2\,\vec{\xi}_\perp = \left(\frac{1}{\rho_0}\frac{\partial P'}{\partial r} + \frac{\partial \Phi'}{\partial r} + \frac{\rho'}{\rho_0}\frac{d\Phi_0}{dr}\right)\hat{r} + \left(\frac{1}{\rho_0}\,\vec{\nabla}_\perp P' + \vec{\nabla}_\perp \Phi'\right),$$

so that, separating radial and tangential parts, we get

$$\omega^2 \xi_r = \frac{1}{\rho_0}\frac{\partial P'}{\partial r} + \frac{\partial \Phi'}{\partial r} + \frac{\rho'}{\rho_0}\frac{d\Phi_0}{dr}, \tag{3.25}$$

$$\omega^2 \vec{\xi}_\perp = \frac{1}{\rho_0}\vec{\nabla}_\perp P' + \vec{\nabla}_\perp \Phi'. \tag{3.26}$$

3.6 Radial Oscillations

We say that a star oscillates radially when the Lagrangian displacement $\vec{\xi}$ has no tangential component, $\vec{\xi}_\perp = 0$. Since an oscillation of this type is spherically symmetric, it is not unreasonable to assume that all other parameters of the star will also be spherically symmetric. Accordingly, all perturbations will be assumed to depend only on the variables r and t. This means that their Fourier transforms will be functions of r and ω alone.

With these assumptions, the equations (3.24) become

$$\left.\begin{aligned} \rho' &= -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho_0 \xi_r\right) \\ \rho_0\,\omega^2\,\xi_r &= \frac{\partial P'}{\partial r} + \rho_0 \frac{\partial \Phi'}{\partial r} + \rho'\,\frac{d\Phi_0}{dr} \\ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi'}{\partial r}\right) &= 4\pi G \rho' \\ P' &= \tau \rho' \end{aligned}\right\} \tag{3.27}$$

where ξ_r, ρ', P' and Φ' are all functions of r and ω, while ρ_0, Φ_0 and τ are all functions of r only.

Equations (3.27) are a set of coupled differential equations. They can be decoupled as follows. First, eliminate ξ_r from the first and second equations, to get

$$0 = \omega^2 \rho' + \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \left(\frac{\partial P'}{\partial r} + \rho_0 \frac{\partial \Phi'}{\partial r} + \rho' \frac{d\Phi_0}{dr} \right). \tag{3.28}$$

Expanding the derivative on the right hand side of equation (3.28) then gives, after some tedious calculation

$$0 = \omega^2 \rho' + \frac{2}{r}\frac{\partial P'}{\partial r} + \frac{\partial^2 P'}{\partial r^2} + \frac{d\rho_0}{dr}\frac{\partial \Phi'}{\partial r} + \frac{\partial \rho'}{\partial r}\frac{d\Phi_0}{dr} + 8\pi G\, \rho_0\, \rho' \tag{3.29}$$

The first and second derivative for P' in equation (3.29) can be calculated by using the fourth of equations (3.27), and gives

$$\frac{\partial P'}{\partial r} = \frac{d\tau}{dr}\rho' + \tau \frac{\partial \rho'}{\partial r} \tag{3.30}$$

and

$$\frac{\partial^2 P'}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{d\tau}{dr}\rho' + \tau \frac{\partial \rho'}{\partial r} \right) = \frac{d^2\tau}{dr^2}\rho' + 2\frac{d\tau}{dr}\frac{\partial \rho'}{\partial r} + \tau \frac{\partial^2 \rho'}{\partial r^2} \tag{3.31}$$

Substituting these equations into equation (3.29) then gives, after some tidying up,

$$\begin{aligned} 0 = \omega^2\, \rho' + \frac{d\rho_0}{dr}\frac{\partial \Phi'}{\partial r} + \left(\frac{2}{r}\frac{d\tau}{dr} + \frac{d^2\tau}{\partial r^2} + 8\pi G \rho_0 \right) \rho' \\ + \left(\frac{2\,\tau}{r} + 2\,\frac{d\tau}{dr} + \frac{d\Phi_0}{dr} \right) \frac{\partial \rho'}{\partial r} + \tau \frac{\partial^2 \rho'}{\partial r^2}. \end{aligned} \tag{3.32}$$

The coefficients of all terms except that containing ω are known from the unperturbed case.

We make the equations more manageable by defining the following quantities:

$$\begin{aligned} \eta\ (r) &= \frac{2}{r}\frac{d\tau}{dr} + \frac{d^2\tau}{dr^2} + 8\pi G \rho_0 \\ \epsilon\ (r) &= \frac{2\,\tau}{r} + 2\,\frac{d\tau}{dr} + \frac{d\Phi_0}{dr} \\ \zeta\ (r) &= \frac{d\rho_0}{dr}. \end{aligned}$$

Equation (3.32) then becomes

$$0 = \omega^2\, \rho' + \zeta \frac{\partial \Phi'}{\partial r} + \eta\, \rho' + \epsilon \frac{\partial \rho'}{\partial r} + \tau \frac{\partial^2 \rho'}{\partial r^2}.$$

Next, we use this result to eliminate Φ' from the third of equations (3.27). Write this last result in the form

$$\frac{\partial \Phi'}{\partial r} = -\left(\frac{\omega^2}{\zeta}\, \rho' + \frac{\eta}{\zeta}\, \rho' + \frac{\epsilon}{\zeta}\frac{\partial \rho'}{\partial r} + \frac{\tau}{\zeta}\frac{\partial^2 \rho'}{\partial r^2} \right),$$

and define

$$\eta/\zeta = \widetilde{\eta}, \quad \epsilon/\zeta = \widetilde{\epsilon}, \quad \tau/\zeta = \widetilde{\tau}, \quad 1/\zeta = \widetilde{\zeta}.$$

Then

$$\frac{\partial \Phi'}{\partial r} = -\left(\widetilde{\zeta}\,\omega^2\,\rho' + \widetilde{\eta}\,\rho' + \widetilde{\epsilon}\,\frac{\partial \rho'}{\partial r} + \widetilde{\tau}\,\frac{\partial^2 \rho'}{\partial r^2}\right). \tag{3.33}$$

Write the third equation of system (3.27) in the form

$$0 = \frac{2}{r}\frac{\partial \Phi'}{\partial r} + \frac{\partial^2 \Phi'}{\partial r^2} - 4\pi G \rho',$$

and use result (3.33) to eliminate Φ'. This gives

$$\begin{aligned}
0 &= \frac{2}{r}\frac{\partial \Phi'}{\partial r} + \frac{\partial^2 \Phi'}{\partial r^2} - 4\pi G\rho' \\
&= \frac{2}{r}\left(\widetilde{\zeta}\,\omega^2\,\rho' + \widetilde{\eta}\,\rho' + \widetilde{\epsilon}\,\frac{\partial \rho'}{\partial r} + \widetilde{\tau}\,\frac{\partial^2 \rho'}{\partial r^2}\right) \\
&\qquad + \frac{\partial}{\partial r}\left(\widetilde{\zeta}\,\omega^2\,\rho' + \widetilde{\eta}\,\rho' + \widetilde{\epsilon}\,\frac{\partial \rho'}{\partial r} + \widetilde{\tau}\,\frac{\partial^2 \rho'}{\partial r^2}\right) + 4\pi G\rho' \\
&= \omega^2\,\frac{2\,\widetilde{\zeta}}{r}\,\rho' + \frac{2\,\widetilde{\eta}}{r}\,\rho' + \frac{2\,\widetilde{\epsilon}}{r}\,\frac{\partial \rho'}{\partial r} + \frac{2\,\widetilde{\tau}}{r}\,\frac{\partial^2 \rho'}{\partial r^2} \\
&\qquad + \omega^2\,\frac{\partial \widetilde{\zeta}}{\partial r}\,\rho' + \omega^2\,\widetilde{\zeta}\,\frac{\partial \rho'}{\partial r} + \frac{\partial \widetilde{\eta}}{\partial r}\,\rho' + \frac{\partial \rho'}{\partial r}\,\widetilde{\eta} \\
&\qquad\qquad + \frac{\partial \widetilde{\epsilon}}{\partial r}\frac{\partial \rho'}{\partial r} + \widetilde{\epsilon}\,\frac{\partial^2 \rho'}{\partial r^2} + \frac{\partial \widetilde{\tau}}{\partial r}\frac{\partial^2 \rho'}{\partial r^2} + \widetilde{\tau}\,\frac{\partial^3 \rho'}{\partial r^3} + 4\pi G\rho'.
\end{aligned}$$

After cleaning up, this becomes

$$\begin{aligned}
0 = &\left[\omega^2\left(\frac{2\,\widetilde{\zeta}}{r} + \frac{\partial \widetilde{\zeta}}{\partial r}\right) + \frac{2\,\widetilde{\eta}}{r} + \frac{\partial \widetilde{\eta}}{\partial r} + 4\pi G\right]\rho' \\
&+ \left(\frac{2\,\widetilde{\epsilon}}{r} + \omega^2\widetilde{\mu} + \widetilde{\eta} + \frac{\partial \widetilde{\epsilon}}{\partial r}\right)\frac{\partial \rho'}{\partial r} \\
&+ \left(\frac{2\,\widetilde{\tau}}{r} + \widetilde{\epsilon} + \frac{\partial \widetilde{\tau}}{\partial r}\right)\frac{\partial^2 \rho'}{\partial r^2} + \widetilde{\tau}\,\frac{\partial^3 \rho'}{\partial r^3}.
\end{aligned} \tag{3.34}$$

There are only two unknowns in equation (3.34). These are the function ρ' and the constant ω. All other quantities in this equation can be calculated from the known solutions of the unperturbed configuration.

Before finding the unknown function ρ', we need first the value of the constant ω. It should be noted that solutions can be constructed for any given value of ω. In general, however, these solutions do not satisfy the boundary conditions of the problem. Solutions that satisfy the boundary conditions exist only for a reduced set of values of ω. Unfortunately, this set of values cannot be found by analytical methods for the above equation. We therefore need to resort to numerical techniques.

Problems of this kind are common in theoretical physics, and there exist a variety of techniques for solving them. Amongst others, some methods commonly

used for equations like (3.34) are "shooting" and "relaxation" techniques. Introductions to these techniques, as well as a few others, can be found in "Stellar Oscillations" by Christensen-Dalsgaard [44]. A more detailed description of shooting techniques applicable to this problem can be found in "The internal constitution of the stars" by Eddington [41].

In this thesis, I use a finite difference technique that reduces equation (3.34) by discretisation to a matrix eigenvalue problem (see Section 4.4). This makes it possible to calculate values for ω irrespective of whether values for ρ' are known.

To solve equation (3.34) as a discretised eigenvalue problem, it cannot contain ω as a coefficient of either ρ' or $\partial\rho'/\partial r$. We get an equation of the required form by adopting the Cowling approximation, described in Section 3.8. The importance of this is discussed in Section 4.4

3.7 Non-Radial Oscillations

Since ρ_0 has no non-radial dependence, equation (3.26) can be further simplified to give

$$\vec{\xi}_\perp = \frac{1}{\omega^2}\vec{\nabla}_\perp\left(\frac{P'}{\rho_0} + \Phi'\right). \tag{3.35}$$

Using the identity

$$\vec{\nabla}\cdot\left(\rho_0\,\vec{\xi}\right) = \vec{\xi}\cdot\left(\vec{\nabla}\rho_0\right) + \rho_0\left(\vec{\nabla}\cdot\vec{\xi}\right),$$

equation one of (3.24) becomes

$$0 = \rho' + \vec{\xi}\cdot\left(\vec{\nabla}\rho_0\right) + \rho_0\left(\vec{\nabla}\cdot\vec{\xi}\right).$$

Using equation (3.6) with $Q = \rho$ gives

$$\begin{aligned} 0 &= \delta\rho_\ell + \rho_0\left(\vec{\nabla}\cdot\vec{\xi}\right) \\ &= \frac{\delta\rho_\ell}{\rho_0} + \vec{\nabla}\cdot\vec{\xi}, \end{aligned} \tag{3.36}$$

where $\delta\rho_\ell$ is the Lagrangian perturbation of ρ. Using equation (3.35) for $\vec{\xi}_\perp$, the divergence in equation (3.36) can be separated into radial and non-radial components giving

$$\begin{aligned} 0 &= \frac{\delta\rho_\ell}{\rho_0} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\xi_r\right) + \vec{\nabla}_\perp\cdot\vec{\xi}_\perp \\ &= \frac{\delta\rho_\ell}{\rho_0} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\xi_r\right) + \frac{1}{\omega^2}\nabla_\perp^2\left(\frac{P'}{\rho_0} + \Phi'\right), \end{aligned} \tag{3.37}$$

where $\nabla_\perp^2$ is the non-radial component of the Laplace operator and is defined by

$$\nabla_\perp^2 = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}\right]. \tag{3.38}$$

Similarly, equation three of (3.24) can be separated into radial and non-radial components giving

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi'}{\partial r}\right) + \nabla_\perp^2 \Phi' = 4\pi G\rho' \tag{3.39}$$

Equation (3.6) shows clearly the difference between Lagrangian and Eulerian perturbations of a physical quantity. For equations of state in which the pressure and density related as simply as they are in the polytropic model, this difference becomes most pronounced when applying the proceeding equations to the system.

For the polytropic model, the Lagrangian variant of the fourth equation of the set (3.24) can be shown to have the same form as its Eulerian counterpart. This is an instance of where combining the two formalisms is advantageous for simplifying the equations. From equation (3.6) with $Q = \rho$ we have[2]

$$\rho' = \delta\rho - \xi_r \frac{d\rho_0}{dr}.$$

Using this result with equation four of (3.24) in equation (3.6) with $Q = P$ we get[2]

$$\begin{aligned}
\delta P &= \xi_r \frac{dP_0}{dr} + P' \\
&= \xi_r \frac{dP_0}{dr} + \tau\,\rho' \\
&= \xi_r \frac{dP_0}{dr} + \tau\left(\delta\rho - \xi_r \frac{d\rho_0}{dr}\right) \\
&= \tau\,\delta\rho - \tau\,\xi_r \frac{d\rho_0}{dr} + \xi_r \frac{d}{dr}\left(k\,\rho_0^{\gamma'}\right) \\
&= \tau\,\delta\rho - \tau\,\xi_r \frac{d\rho_0}{dr} + \xi_r\,k\,\gamma'\rho_0^{\gamma'-1}\frac{d\rho_0}{dr} \\
&= \tau\,\delta\rho - \tau\,\xi_r \frac{d\rho_0}{dr} + \xi_r \frac{P_0}{\rho_0}\frac{d\rho_0}{dr} \\
&= \tau\,\delta\rho - \tau\,\xi_r \frac{d\rho_0}{dr} + \tau\,\xi_r \frac{d\rho_0}{dr} \\
&= \tau\,\delta\rho.
\end{aligned} \tag{3.40}$$

Using equation four of (3.24) and equation (3.40) with $\delta\rho$ as the subject of the formula we can eliminate $\delta\rho$ and ρ' from equations (3.26), (3.37) and (3.39). This gives

$$0 = -\omega^2\xi_r + \frac{1}{\rho_0}\frac{\partial P'}{\partial r} + \frac{\partial \Phi'}{\partial r} + \frac{P'}{P_0}\frac{d\Phi_0}{dr} \tag{3.41}$$

and

$$0 = \frac{1}{\tau\,\rho_0}\left(P' + \xi_r \frac{dP_0}{dr}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\xi_r\right) + \frac{1}{\omega^2}\nabla_\perp^2\left(\frac{P'}{\rho_0} + \Phi'\right)$$

[2] where $\vec{\xi}\cdot\vec{\nabla}Q_0(r) = \xi_r\,\frac{dQ_0}{dr}$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\xi_r\right) + \left(\frac{1}{P_0} + \frac{1}{\omega^2}\nabla_\perp^2\right)P' + \frac{\xi_r}{P_0}\frac{dP_0}{dr} + \frac{1}{\omega^2}\nabla_\perp^2\Phi'$$

or

$$0 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\left(\omega^2\,\xi_r\right) + \left(\frac{\omega^2}{P_0} + \nabla_\perp^2\right)P' + \omega^2\,\xi_r\,\frac{d\ln P_0}{dr} + \nabla_\perp^2\Phi' \tag{3.42}$$

and

$$0 = \left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \nabla_\perp^2\right)\Phi' - \frac{4\pi G}{\tau}P'. \tag{3.43}$$

Equations (3.41) - (3.43) are the governing equations for the non-radial oscillations of a polytrope and determine the four unknown variables ξ_r, P', Φ' and ω.

Solution of the Non-Radial Equations

Recent advances in computing make the task of solving difficult differential equations a matter of routine. By advanced numerical techniques, such as Finite Element and Finite Volume schemes, equations (3.41) - (3.43) can be solved by radially marching the problem forward at fixed values of the angles θ and ϕ. Finding solutions in this way is not easy. The process of reducing the governing equations to corresponding numerical form is tedious and time consuming. One also encounters considerable difficulties in developing robust reliable codes that guarantee numerical stability and precision.

In Chapter 4, a central space finite difference scheme is used to solve the problem of radial oscillation. More details on the method will be presented in that Section. See also Appendix H for information on Finite Difference Schemes.

Analytically, there is little that can be done to determine the form of the solutions to equations (3.41) - (3.43). In conventional pulsation theory a common approach is to use the method of the separation of variables. This assumes that we can find solutions to equations (3.41) - (3.43) in the form

$$\left.\begin{array}{rcl} P'\left(r,\theta,\phi\right) & = & P'_R\left(r\right)\,P'_A\left(\theta,\phi\right) \\ \xi_r\left(r,\theta,\phi\right) & = & \xi_R\left(r\right)\,\xi_A\left(\theta,\phi\right) \\ \Phi'\left(r,\theta,\phi\right) & = & \Phi'_R\left(r\right)\,\Phi'_A\left(\theta,\phi\right) \end{array}\right\} \tag{3.44}$$

and that the general solution to equations (3.41) - (3.43), which is not of separable type, will be a linear superposition of separable solutions (3.44). Of course, this procedure works only if equations (3.41) - (3.43) are linear.

Applying the method of the separation of variables to equations (3.41) - (3.43) leads to a difficult set of coupled differential equations. There is, however, an approximation scheme that simplifies those equations considerably and renders them more manageable. The approximation scheme is called Cowling's approximation. I discuss it in the next section.

The method of the separation of variables can be used only when the system of

equations considered admits reduction to a set of ordinary differential equations each of which involves one independent variable only. There is no guarantee that this is possible in general. In fact, most partial differential equations are not reducible in this way and must satisfy stringent conditions if they are to be separable.

Using (3.44), equation (3.43) becomes

$$\begin{aligned} P' &= \frac{\tau}{4\pi G}\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial \Phi'}{\partial r} + \nabla_\perp^2 \Phi'\right) \\ &= \frac{\tau}{4\pi G}\left(\frac{1}{r^2}\frac{d}{dr}r^2\frac{d\Phi'_R}{dr}\right)\Phi'_A + \left(\frac{\tau\,\Phi'_R}{4\pi G}\right)\nabla_\perp^2 \Phi'_A \\ &= H(r)\,\Phi'_A + Q(r)\,\nabla_\perp^2 \Phi'_A, \end{aligned} \tag{3.45}$$

where H and Q are functions that depend only on r. Using (3.44), equation (3.41) becomes

$$\omega^2 \xi_r = \frac{1}{\rho_0}\frac{\partial P'}{\partial r} + \frac{P'}{P_0}\frac{d\Phi_0}{dr} + \frac{d\Phi'_R}{dr}\,\Phi'_A. \tag{3.46}$$

Substituting equation (3.45) into (3.46) then gives

$$\begin{aligned} \omega^2 \xi_r &= \frac{1}{\rho_0}\frac{d}{dr}\Big(H(r)\,\Phi'_A + Q(r)\,\nabla_\perp^2 \Phi'_A\Big) \\ &\qquad + \frac{1}{P_0}\frac{d\Phi_0}{dr}\Big(H(r)\,\Phi'_A + Q(r)\,\nabla_\perp^2 \Phi'_A\Big) \\ &\qquad + \frac{d\Phi'_R}{dr}\Phi'_A \\ &= \left(\frac{1}{\rho_0}\frac{dH}{dr}(r) + \frac{1}{P_0}\frac{d\Phi_0}{dr}H(r) + \frac{d\Phi'_R}{dr}\right)\Phi'_A \\ &\qquad + \left(\frac{1}{\rho_0}\frac{dQ}{dr}(r) + \frac{1}{P_0}\frac{d\Phi_0}{dr}Q(r)\right)\nabla_\perp^2 \Phi'_A \\ &= G(r)\,\Phi'_A + F(r)\,\nabla_\perp^2 \Phi'_A, \end{aligned} \tag{3.47}$$

where G and F are functions that depend only on r. Lastly, using (3.44), equation (3.42) becomes

$$0 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\Big(\omega^2\,\xi_r\Big) + \left(\frac{\omega^2}{P_0} + \nabla_\perp^2\right)P' + \omega^2\,\xi_r\,\frac{d\ln P_0}{dr} + \Phi'_R\nabla_\perp^2\Phi'_A. \tag{3.48}$$

Substituting equations (3.45) and (3.47) into (3.48) we have

$$\begin{aligned} 0 &= \frac{1}{r^2}\frac{\partial}{\partial r}r^2\Big(G(r)\,\Phi'_A + F(r)\,\nabla_\perp^2 \Phi'_A\Big) \\ &\qquad + \left(\frac{\omega^2}{P_0} + \nabla_\perp^2\right)\Big(H(r)\,\Phi'_A + Q(r)\,\nabla_\perp^2 \Phi'_A\Big) \\ &\qquad + \frac{d\ln P_0}{dr}\Big(G(r)\,\Phi'_A + F(r)\,\nabla_\perp^2 \Phi'_A\Big) + \Phi'_R\,\nabla_\perp^2 \Phi'_A \\ &= \left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\,G(r) + \frac{\omega^2}{P_0}H(r) + \frac{d\ln P_0}{dr}G(r)\right)\Phi'_A \end{aligned}$$

$$+\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\,F\left(r\right)+\frac{\omega^2}{P_0}\,Q\left(r\right)+H\left(r\right)+\frac{d\ln P_0}{dr}\,F\left(r\right)\right.$$
$$\left.+\,\Phi'_R\right)\nabla_\perp^2\Phi'_A+Q(r)\,\nabla_\perp^2\left(\nabla_\perp^2\Phi'_A\right)$$
$$=S(r)\,\Phi'_A+Z(r)\,\nabla_\perp^2\Phi'_A+Q(r)\,\nabla_\perp^2\left(\nabla_\perp^2\Phi'_A\right), \tag{3.49}$$

where S and Z are functions that depend only on r.

The complicated structure of equation (3.49) makes it difficult to determine solutions to equations (3.41) - (3.43).

Before numerical methods were well developed, a common approach to dealing with equations like these was to look for similar sets of equations for which the solutions were known, and to exploit the similarities to search for solutions. At the time of writing this thesis, no sets of equations similar to the above are known. Another common approach to solving problems of similar complexity is, by exploiting expert knowledge on the actual behaviour of these systems, to introduce reasonable physical assumptions that simplify the equations. Effectively, this procedure replaces the model developed thus far by another simpler model whose underlying assumptions become the subject of investigation and testing. A common simplifying assumption used in this context is the Cowling approximation, which reduces equation (3.49) to one that is more tractable.

3.8 The Cowling Approximation

Cowling introduced his approximation in 1941 [45]. He simplified the perturbation equations by ignoring the Eulerian perturbation Φ' of the gravitational potential. The reason for this approximation is almost obvious: if perturbations are small, the flow of mass in the perturbation process is minimal and cannot produce a large effect on the gravitational field.

Though this approximation is intuitively obvious, Cowling gave a more detailed and more persuasive argument for it. The gravitational potential at any point of the star is given by an integral over the entire star. In this sense, it is an averaging process. As an averaging process, its effect is to smooth out any local disturbances in the gravitational potential. This conjecture has been verified by subsequent work due to Cowling and others [40].

This approximation is almost self evident in the case of a non-rotating star undergoing small perturbations. In the case of rotating stars, the distortion of the shape of the star due to centrifugal effects may become significant. In this case, the distortion of the equilibrium configuration may become significant, but further distortion due to perturbation can only be slight for the same reasons as in the non-rotating case, allowing us again to regard Φ' as insignificant. However, this approximation may be too severe in the case of rapidly rotating stars.

Another factor to consider before making this approximation is the density profile of the star. Stars that are centrally condensed are better suited to Cowling's approximation than those that are not. Central condensation guarantees that

most of the stellar mass is contained within a relatively small radius, where the Lagrangian displacements are limited in size because of the boundary conditions that they must satisfy. Most of the mass of the star will thus be less mobile than in other types of stars. This means that the effects of perturbation on the gravitational potential will be smaller than otherwise might have been the case. Central condensation mitigates the effects of rotation, allowing us to use Cowling's approximation successfully even in cases where rotation is relatively large.

Cowling's approximation allows us to write

$$\Phi\left(\vec{r},\omega\right) = \Phi'\left(\vec{r},\omega\right) + \Phi_0\left(r\right) \approx \Phi_0\left(r\right).$$

Equation three of the system (3.24) and equation (3.43) may thus be ignored.

Radial Oscillations in the Cowling Approximation

By removing all terms that contain Φ' from equation (3.28) and expanding its derivative on the right hand side we get, after some tedious calculation,

$$0 = \omega^2\rho' + \frac{2}{r}\frac{\partial P'}{\partial r} + \frac{\partial^2 P'}{\partial r^2} - \frac{k}{\rho_0}\frac{d\rho_0^{\gamma'}}{dr}\frac{\partial \rho'}{\partial r} + 4\pi G\,\rho_0\,\rho' \ ,$$

where in the unperturbed flow, we have

$$\frac{d\Phi_0}{dr} = -\frac{k}{\rho_0}\frac{d\rho_0^{\gamma'}}{dr}.$$

Using equations (3.30) and (3.31) for the first and second derivatives of P' we have

$$\begin{aligned} 0 &= \omega^2\rho' + \frac{2}{r}\frac{d\tau}{dr}\,\rho' + \frac{d^2\tau}{dr^2}\,\rho' + 4\pi G\rho'\rho_0 \\ &\qquad + \frac{2\tau}{r}\frac{\partial\rho'}{\partial r} + 2\frac{d\tau}{dr}\frac{\partial\rho'}{\partial r} - \frac{k}{\rho_0}\frac{d\rho_0^{\gamma'}}{dr}\frac{\partial\rho'}{\partial r} + \tau\,\frac{\partial^2\rho'}{\partial r^2} \\ &= \left(\omega^2 + \frac{2}{r}\frac{d\tau}{dr} + \frac{d^2\tau}{dr^2} + 4\pi G\rho_0\right)\rho' \\ &\qquad + \left(\frac{2\tau}{r} + 2\,\frac{d\tau}{dr} - \frac{\tau}{\rho_0}\,\frac{d\rho_0}{dr}\right)\frac{\partial\rho'}{\partial r} + \tau\,\frac{\partial^2\rho'}{\partial r^2}, \end{aligned} \tag{3.50}$$

where we have used, by the chain rule

$$\frac{k}{\rho_0}\frac{d\rho_0^{\gamma'}}{dr} = \frac{k\,\gamma'\,\rho_0^{\gamma'-1}}{\rho_0}\,\frac{d\rho_0}{dr} = \frac{\tau}{\rho_0}\,\frac{d\rho_0}{dr}.$$

As was the case with equation (3.34), each coefficient in equation (3.50) contains constants or known functions. The only two unknowns in this equation are the function ρ' and the constant ω, but now ω appears only as a coefficient of the term containing ρ'. The importance of this is discussed in Section 4.4.

By specifying boundary conditions relevant to the system of interest, equations (3.50) and (3.10) can be discretised and de-dimensionalised to calculate the radial frequencies for a polytrope of polytropic exponent γ' or index n. This is described in more detail in chapter 4 where a two part simulation is constructed and initial and surface conditions introduced to calculate the eigenfrequencies for a polytropic model of index $n = 3$ that has parameters similar to that of the sun.

Non-Radial Oscillations in Cowling's Approximation

Removing all terms that contain Φ' in equations (3.41) and (3.42) we get

$$0 = -\omega^2 \xi_r + \frac{1}{\rho_0}\frac{\partial P'}{\partial r} + \frac{1}{P_0}\frac{d\Phi_0}{dr}P', \tag{3.51}$$

$$0 = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \Big(\omega^2 \xi_r\Big) + \left(\frac{\omega^2}{P_0} + \nabla_\perp^2\right) P' + \omega^2 \xi_r \frac{d\ln P_0}{dr}. \tag{3.52}$$

In the Cowling approximation equation (3.43) is satisfied identically, since $\Phi' = 0$. This equation thus yields no information.

Non-Radial Solutions in Cowling's Approximation

Following the same procedure used in Section 3.7, where separable solutions were assumed, equation (3.51) becomes

$$\begin{aligned}
\omega^2 \xi_r &= \frac{1}{\rho_0}\frac{dP'_R}{dr}P'_A + \frac{P'_R}{P_0}\frac{d\Phi_0}{dr} \\
&= \left(\frac{1}{\rho_0}\frac{dP'_R}{dr} + \frac{P'_R}{P_0}\frac{d\Phi_0}{dr}\right) P'_A \\
&= A(r)\, P'_A,
\end{aligned} \tag{3.53}$$

where A is a function of r alone. Using solutions (3.44), equation (3.52) becomes

$$0 = \frac{2}{r}\Big(\omega^2 \xi_r\Big) + \frac{\partial}{\partial r}\left(\omega^2 \xi_r\right) + \omega^2 \frac{P'_R}{P_0} P'_A + \omega^2 \xi_r \frac{d\ln P_0}{dr} + P'_R \nabla_\perp^2 P'_A. \tag{3.54}$$

Substituting equation (3.53) into (3.54) then gives

$$\begin{aligned}
0 &= \frac{2A(r)}{r}P'_A + \frac{dA(r)}{dr}P'_A + \left(\omega^2 \frac{P'_R}{P_0}\right) P'_A + A(r)\frac{d\ln P_0}{dr}P'_A + P'_R \nabla_\perp^2 P'_A \\
&= \left(\frac{2}{r}A(r) + \frac{dA(r)}{dr} + \omega^2 \frac{P'_R}{P_0} + A(r)\frac{d\ln P_0}{dr}\right) P'_A + P'_R \nabla_\perp^2 P'_A \\
&= B(r)\, P'_A + P'_R \nabla_\perp^2 P'_A,
\end{aligned} \tag{3.55}$$

where B is the coefficient of P'_A in equation (3.55) and is a function of r alone. Substituting the definition (3.38) for $\nabla_\perp^2$ into (3.55) we get

$$0 = B(r)\, P'_A + \frac{P'_R}{r^2 \sin\theta}\left(\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}\right) P'_A. \tag{3.56}$$

By comparing equation (3.56) with equation (E.2) in Appendix E we see that their form is identical if

$$P'_R \equiv R\,(r),$$
$$B(r) \equiv \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR\,(r)}{dr}\right)$$

and

$$P'_A \equiv \Phi\,(\phi)\;\Theta\,(\theta).$$

Since N is a constant, it can be concluded that

$$P'_A(\theta,\phi) = Y_\ell^m(\theta,\phi).$$

The complete solution, namely $P'(\vec{r})$ is then

$$P'(\vec{r}) = P'_R\,(r)\,Y_\ell^m(\theta,\phi). \tag{3.57}$$

We can use equation (3.57) to get an equation for the components of $\vec{\xi}$. Using equations (3.25) for ξ_r and (3.35) for $\vec{\xi}_\perp$, with Φ' removed, we have

$$\vec{\xi} = \xi_r\,\hat{r} + \vec{\xi}_\perp = \left(\frac{1}{\omega^2\,\rho_0}\frac{\partial P'}{\partial r} + \frac{\rho\,'}{\omega^2\,\rho_0}\frac{d\Phi_0}{dr}\right)\hat{r} + \left(\frac{1}{\omega^2\,\rho_0}\vec{\nabla}_\perp P'\right).$$

Using equation four of (3.24) to replace ρ' with P', where $P'(\vec{r}) = P'_R\,(r)\,P'_A\,(\theta,\phi)$, we have

$$\vec{\xi} = \left[\left(\frac{1}{\omega^2\,\rho_0}\frac{dP'_R}{dr} + \frac{P'_R}{\omega^2\,P_0}\frac{d\Phi_0}{dr}\right)\hat{r} + \left(\frac{P'_R}{\omega^2\,\rho_0}\vec{\nabla}_\perp\right)\right]P'_A,$$

which can be rewritten to give

$$\vec{\xi} = \left[\xi_r(r)\,,\;\xi_\perp(r)\frac{\partial}{\partial\theta}\,,\;\xi_\perp(r)\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right]Y_\ell^m(\theta,\phi), \tag{3.58}$$

where

$$\xi_r\,(r) = \left(\frac{1}{\omega^2\,\rho_0}\frac{dP'_R}{dr} + \frac{P'_R}{\omega^2\,P_0}\frac{d\Phi_0}{dr}\right),$$
$$\xi_\perp(r) = \frac{P'_R}{\omega^2\,P_0}\frac{1}{r}.$$

Solutions in the form of (3.57) and (3.58) are important results. They demonstrate that in the Cowling approximation the only values we need to calculate in the perturbed configuration are those for P'_R and ω. From these values we can then determine solutions for P', $\vec{\xi}$ and by extension for $\rho\,'$ by using the spherical harmonics to calculate the angular components. This means that we can solve the angular problem analytically rather than numerically as we do for the radial equation (see equation (3.50)). The cost of this advantage is the adoption of Cowling's approximation. The fact that we can find solutions to the angular problem analytically also has the added advantage that we can import all the

familiar physical insights associated with these well known solutions. Gaining general insights from numerical solutions is a much more difficult task.

As we will see in Chapter 5, where we consider the effect of tides, the above simplifications are no longer possible. This is especially true for the case for dynamic tides, where the distortion of the shape of the star is a function of time.

Chapter 4

Radial Oscillation Frequencies in the Cowling Approximation

4.1 Introduction

This chapter focuses principally on the discretisation scheme adopted for the calculation of the equilibrium configuration of a polytrope, and for the calculation of the pulsation frequencies that the polytrope admits. The discussion of the discretisation schemes is preceded by a general discussion of how to reduce a physical differential equation to dimensionless form, why it is necessary to do so, and the interpretation of the parameters introduced in this process as natural scales for the physical problem considered.

Boundary conditions play an important role in the search for solutions to differential equations that involve derivatives with respect to space variables. In the absence of boundary conditions, a differential equation may admit a very large number of solutions. In the case of partial differential equations, this number may be so large that no single formula exists that encapsulates all of them; in other words, a general solution might not exist. It is essential in all cases to supply sufficient information to select an unique solution from among the huge number available.

The spherically symmetric polytrope is described by a single ordinary differential equation. In principle, such an equation admits a general solution. In practice, no one has yet found a formula for the general solution corresponding to values of the polytropic index other than $0, 1$ and 5. We thus have no alternative but to resort to numerical solutions. These require boundary conditions to be specified before the computation can begin. I will therefore discuss not only the discretisation technique that I have used for the computation, but will also establish what boundary conditions are needed to calculate the density of the unperturbed system by marching forward in r the discretised, de-dimensionalised unperturbed equation (3.10). These are presented in Section 4.3.

A different type of calculation is needed to find the pulsation frequencies of a polytrope. This calculation requires us to use the unperturbed density values found in the first calculation. These are fed into an eigenvalue equation. One way to solve for the eigenvalues is by discretising it. This converts it into a matrix eigenvalue problem that can be solved by standard matrix techniques. The boundary conditions from the the first calculation are used to infer boundary conditions for the density perturbations.

In the absence of boundary conditions, a differential equation eigenvalue problem admits solutions for every possible value of the eigenvalue. The equation admits a continuum of eigenvalues. Not all of these solutions, however, satisfy the required boundary conditions. The boundary values are only satisfied by solutions that correspond to a subset of the full range of possible eigenvalues. Those solutions that violate the boundary conditions are thus discarded as not suitable for describing the physical problem.

If we were to choose arbitrarily a value for the eigenvalue, we can always construct a corresponding solution, but this solution will not in general satisfy the boundary conditions that define the physical system. This observation can be used to construct an alternative method for searching for solutions: adjust the eigenvalue systematically until the solution matches the required boundary conditions to a specified degree of accuracy. This method of solution of the eigenvalue problem belongs to a class of solution methods called *shooting methods*.

I have calculated both the equilibrium configuration of a polytrope and its lowest oscillation frequencies as described above. I chose the mass of the polytrope equal to that of the Sun. There is a mass-radius relation for a polytrope [42], so once its mass and polytropic index have been chosen, the radius is determined by the physics of the polytrope. Eddington's Standard Model of a star is a polytrope of index $n = 3$, so I chose this value for the calculations for a solar mass polytrope. The Sun's central density is $\rho_{\text{central}} = 1.622 \times 10^6$ Kg/m^3.

4.2 De-dimensionalisation and Scaling

De-dimensionalisation of the governing equations is an useful first step to the construction of numerical models. The reasons for this are several. The most important is that both mathematics and numerical analysis assume that variables are pure numbers. In contrast, most variables in physics are dimensioned quantities. In general, these cannot be substituted into mathematical functions in a meaningful way.

De-dimensionalisation can be achieved by forming ratios of physical variables with physical constants that carry the same units. This introduces a second reason for de-dimensionalisation. Physical parameters encountered in astrophysical systems are generally very large values when measured in SI units. Large values are not well handled by computers. It is better to work with smaller numbers where possible. By choosing numerical constants with values that are typical of

the magnitudes encountered in a given system, we can form dimensionless ratios that are of more manageable size computationally. For example, the sun has a radius of the order of 10^8 m. Typical distances that need to be considered in a discussion of the structure of the sun's interior will be of the same order, or less. It is thus useful to introduce the dimensionless ratio $r/R_\odot$ in place of the radial distance r. All radial distances considered will then have values between 0 and 1.

A third reason for de-dimensionalisation is that, by judicious choice of a set of independent dimensional constants, solutions to equations can be written in a form that is independent of the size of the system. This means that the solution is "universal", in the sense that it solves all problems that involve the same equations irrespective of the size of the system. Stated differently, by choosing appropriate values of the numerical constants in the solution, we can generate the solution for the system of interest. In engineering, this is called scaling [46].

For these reasons, it is useful first to de-dimensionalise both the unperturbed and perturbed equations by means of characteristic values of selected physical parameters for the system of interest. For example, the unperturbed equation (3.10) contains two dimensioned variables, namely the independent variable r and the dependent variable ρ_0. To de-dimensionalise equation (3.10) we must introduce corresponding constants r_c and ρ_c that have the same dimensions as r and ρ_0 respectively, from which we can form the dimensionless ratios

$$\mu = \frac{r}{r_c}, \qquad \eta = \frac{\rho_0}{\rho_c}. \tag{4.1}$$

Introducing these ratios into equation (3.10) produces an equation for the new dependent variable η as a function of the new independent variable μ. Both variables are now dimensionless, so if we collect the constants in the equation into a single new constant C, this new constant will also be dimensionless. The solution of this equation is expected to be in the form

$$\eta = f(\mu, C)$$

where f is a function to be determined. The physical dimensions can be restored to this equation by replacing η and μ by their expressions above. This gives

$$\rho_0 = \rho_c \, f\left(\frac{r}{r_c}, C\right).$$

It is clear that the dimensionless solution solves all physical problems governed by the same de-dimensionalised differential equation. The constants ρ_c and r_c set the scales for the dimensionalised physical problem.

We can add a small refinement to the above procedure. In the case of equation (3.10), only one dimensionless constant C arises. Rather than carry this constant around, we can choose the value of just one of the two de-dimensionalising values ρ_c, r_c. The value of the other can then be set by simplifying the resultant equation as much as possible. This simplification is achieved by choosing the value of C to be 1. The differential equation then contains no constants, and the value of the second de-dimensionalising parameter is determined by the equation

$$C = 1.$$

By determining the scale of one variable, this equation then determines a natural scale for the other. The dimensionless solution to the differential equation now takes the form

$$\eta = f(\mu),$$

where f is a function to be determined. Restoring the physical dimensions for the problem then gives

$$\rho_0 = \rho_c \, f\left(\frac{r}{r_c}\right).$$

This procedure allows physical problems to be solved by constructing models of manageable size, which can the be scaled to the required size.

In the case of the polytrope, the density of the polytrope can be expected to range from zero at its surface to a maximum value at its centre. We can therefore choose ρ_c to be its central density. Choosing $C = 1$ then fixes the value r_c which sets the scale in a natural way for the radial variable r. The polytrope in equilibrium is expected to have a well defined radius. The constant r_c can then be expected to have a magnitude that is of the same order of magnitude as the radius of the star.

Similarly, the decoupled perturbation equation (3.50) is essentially a differential equation for determining the independent variable ρ' as a function of the dependent variable r. Now, though ω appears as an argument of ρ', it is not a second independent variable, for two reasons: equation (3.50) features only the partial derivative with respect to r, indicating that ω is being held constant; and, by taking the Fourier transform of equations (3.23) and decoupling them, we have essentially decomposed the original equation into an infinity of equations, one for each given value of ω. De-dimensionalisation of this equation thus requires the introduction of two constants ρ_c and r_c as before, giving rise to the dimensionless quantities

$$\eta' = \frac{\rho'}{\rho_c}, \qquad \mu = \frac{r}{r_c}.$$

To simplify the algebra, I have chosen to introduce also the following dimensionless quantities,

$$\widetilde{\omega} = \frac{\omega}{\omega_c}, \qquad \widetilde{\tau} = \frac{\tau}{\tau_c}. \tag{4.2}$$

The four non-dimensionalising quantities ρ_c, r_c, ω_c and τ_c are clearly not independent, as can be seen by inspecting their physical dimensions,

$$[\rho_c] = \frac{M}{L^3}, \qquad [r_c] = L, \qquad [\omega_c] = \frac{1}{T}, \qquad [\tau_c] = \frac{L^2}{T^2}, \tag{4.3}$$

where M, L and T are the physical dimensions of mass, length and time respectively. Since only three independent physical dimensions are involved in this problem, we will only be able to choose three scale values out of four for the

quantities ρ_c, r_c, ω_c and τ_c. Furthermore, if we follow the same procedure as was followed above in which we chose the value of one dimensionless constant in the resultant de-dimensionalised equation, we will be free to chose only values for two of these four quantities.

Since we have already established the scale values r_c and ρ_c for the polytrope, we will use these as scale values for the perturbed problem also. Fixing the value of a further dimensionless constant in the de-dimensionalised form of this equation will then fix also the scale values ω_c and τ_c.

4.3 Unperturbed Polytrope

The system of equations governing the spherically symmetric polytrope in hydrostatic equilibrium are discussed in Section 3.3. As shown in Section 3.3, the system can be decoupled to yield the following equation for the variable ρ_0:

$$-4\pi G\rho_0 = k\left(\frac{\gamma'}{\gamma'-1}\right)\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\,\rho_0^{\gamma'-1}. \tag{4.4}$$

The dependent variable in this equation is ρ_0, and the independent variable is r. Both are dimensioned. Following the method described in the previous section, we can de-dimensionalise this equation by introducing a reference density ρ_c and a reference radius r_c and defining the dimensionless variables η and μ using equations (4.1). Eliminating ρ_0 and r in favour of η and μ in (4.4) and collecting the constants in this equation, we get

$$\eta_0 = -\left(\frac{\rho_c^{\gamma'-2}}{r_c^2}\;\frac{k}{4\pi G}\;\frac{\gamma'}{\gamma'-1}\right)\frac{1}{\mu^2}\frac{d}{d\mu}\mu^2\frac{d}{d\mu}\,\eta_0^{\gamma'-1}. \tag{4.5}$$

Since all variables in equation (4.5) are dimensionless, so also is the bracketed coefficient on the right hand side.

Next, choose r_c in such a way that produces maximal simplification of equation (4.5). This occurs if we choose the value of the dimensionless coefficient to be 1. Therefore put

$$\frac{\rho_c^{\gamma'-2}}{r_c^2}\;\frac{k}{4\pi G}\;\frac{\gamma'}{\gamma'-1} = 1.$$

All quantities other than ρ_c and r_c in this relation are data for this problem[1], so the requirement of maximal simplification relates the reference values ρ_c and r_c. The price paid for simplification is thus that the distance scale r_c in the problem of the polytrope is set by the density scale ρ_c,

$$r_c = \sqrt{\rho_c^{\gamma'-2}\,\frac{k}{4\pi G}\left(\frac{\gamma'}{\gamma'-1}\right)},$$

[1]The equation of state for the polytrope is a datum. Equivalently, the values of γ' and k are given.

or, expressed in terms of the polytropic index n, defined implicitly by $\gamma' = 1/n + 1$,

$$r_c = \sqrt{\rho_c^{n-1} \frac{k}{4\pi G} (n+1)}, \tag{4.6}$$

This relationship is identical to that determined for the Lane-Emden equation (see Appendix G.3).

With these choices, equation (4.5) becomes

$$\eta_0 = -\frac{1}{\mu^2} \frac{d}{d\mu} \left(\mu^2 \frac{d}{d\mu} \eta_0^{\gamma'-1} \right). \tag{4.7}$$

Discretisation

To discretise equation (4.7), first rewrite it in the form

$$\eta_0 = -\frac{2}{\mu} \frac{d}{d\mu} \eta_0^{\gamma'-1} - \frac{d^2}{d\mu^2} \eta_0^{\gamma'-1}. \tag{4.8}$$

To reduce this equation to a numerical calculation, I used Finite Difference Methods (see appendix H). These require that we replace the first derivative in equation (4.8) by the finite difference formula (H.9) and the second derivative by (H.12). In equation (4.8), the independent variable is $\eta_0^{\gamma'-1}$, with μ in the role of the dependent variable. The discretised version of equation (4.8) thus becomes

$$\begin{aligned} 0 = \eta_0(\mu_i) + \frac{2}{\mu_i} & \left(\frac{\eta_0^{\gamma'-1}(\mu_{i+1}) - \eta_0^{\gamma'-1}(\mu_{i-1})}{2\,\Delta\mu} \right) \\ & + \left(\frac{\eta_0^{\gamma'-1}(\mu_{i+1}) - 2\eta_0^{\gamma'-1}(\mu_i) + \eta_0^{\gamma'-1}(\mu_{i-1})}{(\Delta\mu)^2} \right) \end{aligned}$$

or

$$\begin{aligned} 0 &= (\Delta\mu)^2 \eta_0(\mu_i) + \frac{\Delta\mu}{\mu_i} \left(\eta_0^{\gamma'-1}(\mu_{i+1}) - \eta_0^{\gamma'-1}(\mu_{i-1}) \right) \\ &\quad + \eta_0^{\gamma'-1}(\mu_{i+1}) - 2\,\eta_0^{\gamma'-1}(\mu_i) + \eta_0^{\gamma'-1}(\mu_{i-1}) \\ &= (\Delta\mu)^2 \eta_0(\mu_i) - 2\,\eta_0^{\gamma'-1}(\mu_i) + \left(\frac{\Delta\mu}{\mu_i} + 1 \right) \eta_0^{\gamma'-1}(\mu_{i+1}) \\ &\quad - \left(\frac{\Delta\mu}{\mu_i} - 1 \right) \eta_0^{\gamma'-1}(\mu_{i-1}), \end{aligned} \tag{4.9}$$

where $\Delta\mu$ is the step size for marching forward the value of μ, and μ_i the current value of μ at the i^{th} step.

Equation (4.9) allows us to calculate the value μ_{i+1} from the values of μ_{i-1} and μ_i. Making $\eta_0(\mu_{i+1})$ the subject of the formula, we get

$$\eta_0(\mu_{i+1}) = \left(\frac{\left(\frac{\Delta\mu}{\mu_i} - 1\right) \eta_0^{\gamma'-1}(\mu_{i-1}) + 2\,\eta_0^{\gamma'-1}(\mu_i) - (\Delta\mu)^2\,\eta_0(\mu_i)}{\frac{\Delta\mu}{\mu_i} + 1} \right)^{1/(\gamma'-1)} . \quad (4.10)$$

To use this equation to calculate $\eta_0(\mu_{i+1})$, we need to supply the values $\eta_0(\mu_{i-1})$ and $\eta_0(\mu_i)$. To initiate the calculation, we therefore need the values $\eta_0(\mu_0)$ and $\eta_0(\mu_1)$ at $\mu_0 = 0$ and $\mu_1 = \Delta\mu$ respectively. To find these, we use the boundary conditions for this equation, which we now discuss.

Equation (4.10) is a second order ordinary differential equation. It therefore requires two boundary conditions to define a solution uniquely. The standard boundary conditions for an equation of this type consist of the value of the dependent variable and of its first derivative at a chosen value of the independent variable. There are other possible boundary conditions, but these need not be considered here.

In the case of equation (4.7), the dependent variable is $\eta_0^{\gamma'-1}$. From the physics of the polytrope problem, it is clear that the greatest density of the polytrope will occur at its centre. We may therefore use this value of the density to define a density scale for the polytrope by choosing the de-dimensionalising constant ρ_c to be its central density. This means that, at $\mu_0 = 0$, we will have $\eta_0(\mu_0) = 1$, and hence $\eta_0^{\gamma'-1}(\mu_0) = 1$. This is the first boundary condition.

The second boundary condition follows from the continuity of the density of the polytrope. The variable μ is essentially a radial coordinate for a system of spherical coordinates with origin at the centre of the star. The properties of the stellar material change continuously throughout the volume of the polytrope. As we pass through the origin, therefore, ρ_0 must change continuously. For this to be true, its first derivative must be zero. It then follows that $d\eta_0^{\gamma'-1}/d\mu\,(\mu_0) = 0$ at $\mu_0 = 0$.

The second boundary condition allows us to calculate the value of $\eta_0^{\gamma'-1}(\mu_1)$. Using the forward difference formula

$$\frac{d\eta_0^{\gamma'-1}}{d\mu}(\mu_0) = \frac{\eta_0^{\gamma'-1}(\mu_1) - \eta_0^{\gamma'-1}(\mu_0)}{\Delta\mu},$$

we get

$$\eta_0^{\gamma'-1}(\mu_1) = \Delta\mu\,\frac{d\eta_0^{\gamma'-1}}{d\mu}(\mu_0) + \eta_0^{\gamma'-1}(\mu_0) = 0 + 1$$

or

$$\eta_0^{\gamma'-1}(\mu_1) = 1.$$

With these two values, we can march the solution forward using formula (4.10). The boundary of the polytrope is defined as that point at which the density of the polytrope reaches the value 0. When this happens, the calculation can be terminated.

In Phase 1 of the calculation (the unperturbed system), the step size $\Delta\mu$ is a matter of indifference. Too few steps will give a poorly resolved solution with relatively high errors developing as the solution is marched forward. On the other hand, too many steps may yield well resolved solutions with lower errors, but making increasingly heavy demands on computer memory since the calculated values need to be saved for use in Phase 2 of the calculation.

Theoretically, the calculation should be stopped when it returns a value of zero for $\eta_0^{\gamma'-1}(\mu_N) = 0$. Computationally, however, it is unlikely that $\eta_0^{\gamma'-1}$ will ever become exactly zero. We thus need to devise a stopping criterion that avoids this difficulty. I chose to terminate the calculation as soon as $\eta_0 < 0.00001$. I took the value of μ at the step when η_0 first drops below this value to be the dimensionless radius of the polytrope.

There is an alternative method for calculating the values of $\eta_0^{\gamma'-1}(\mu_0)$ and $\eta_0^{\gamma'-1}(\mu_1)$ that are used to start the marching forward of the solution. This method uses Chandrasekhar's power series solution to equation (4.7) given by equation (G.35). This power series is accurate for small values of μ and can be safely used to calculate $\eta_0^{\gamma'-1}(\mu_1)$. Note that in Appendix G.3 we dedimensionalised density using $\eta = \rho^{\gamma'-1}/\rho_c^{\gamma'-1}$. Therefore, to use the Chandrasekhar power series, the definition for η in (4.1) would need to be adapted to the conventions used here. It is easy to see that at small values of μ, the second and third terms in expansion (G.35) are considerably smaller than the first term, making the difference in the value calculated for $\eta_0^{\gamma'-1}(\mu_1)$ using this formula and the value that we have calculated negligible.

This phase of the computation was performed using C^{++}. The results reported in Chapter 6, and the code used for the calculation is found in Appendix L.1.1.

4.4 Perturbed System

Substituting quantities (4.1) and (4.2) into the perturbed equation (3.50) we get

$$\begin{aligned} 0 = &\left((\omega_c^2\,\rho_c)\,\widetilde{\omega}^2 + \left(\frac{\tau_c\,\rho_c}{r_c^2}\right)\frac{2}{\mu}\frac{d\widetilde{\tau}}{d\mu} + \left(\frac{\tau_c\,\rho_c}{r_c^2}\right)\frac{d^2\widetilde{\tau}}{d\mu^2} + 4\pi G\,\rho_c^2\,\eta_0\right)\eta' \\ &+ \left(\left(\frac{\tau_c\,\rho_c}{r_c^2}\right)\frac{2\,\widetilde{\tau}}{\mu} + \left(\frac{\tau_c\,\rho_c}{r_c^2}\right) 2\,\frac{d\widetilde{\tau}}{d\mu} - \left(\frac{\tau_c\,\rho_c}{r_c^2}\right)\frac{\widetilde{\tau}}{\eta_0}\,\frac{d\eta_0}{d\mu}\right)\frac{\partial\eta'}{\partial\mu} \\ &+ \left(\frac{\tau_c\,\rho_c}{r_c^2}\right)\widetilde{\tau}\,\frac{\partial^2\eta'}{\partial\mu^2}. \end{aligned} \tag{4.11}$$

Equation (4.11) contains three distinct constant coefficients. It is easy to check, using (4.3), that all three have the same physical dimensions, given by

$$\left[\omega_c^2\,\rho_c\right] = \left[\frac{\tau_c\,\rho_c}{r_c^2}\right] = \left[4\pi G\,\rho_c^2\right] = \frac{M}{L^3\,T^2},$$

where we have used the fact that $[G] = L^3/M\,T^2$. Dividing equation (4.11) by the coefficient of the leading term of equation (4.11) gives

$$\begin{aligned} 0 = &\left(\frac{r_c^2\,\omega_c^2}{\tau_c}\,\widetilde{\omega}^2 + \frac{2}{\mu}\frac{d\widetilde{\tau}}{d\mu} + \frac{d^2\widetilde{\tau}}{d\mu^2} + \frac{4\pi G\,\rho_c\,r_c^2}{\tau_c}\,\eta_0\right)\eta\,' \\ &+ \left(\frac{2\,\widetilde{\tau}}{\mu} + 2\,\frac{d\widetilde{\tau}}{d\mu} - \frac{\widetilde{\tau}}{\eta_0}\,\frac{d\eta_0}{d\mu}\right)\frac{\partial\eta\,'}{\partial\mu} + \widetilde{\tau}\,\frac{\partial^2\eta\,'}{\partial\mu^2}. \end{aligned} \tag{4.12}$$

This reduces the number of constants in the equation to two. Now, however, both constants are dimensionless. They are given by

$$\frac{r_c^2\,\omega_c^2}{\tau_c} \quad \text{and} \quad \frac{4\pi G\,\rho_c\,r_c^2}{\tau_c}.$$

In the unperturbed problem, we chose the value of ρ_c to be the central density of the star. We then fixed the value of r_c by demanding maximum simplification of the equation governing the unperturbed polytrope. This requirement was found to be equivalent to setting

$$r_c^2 = \rho_c^{\gamma\,'-2}\,\frac{k}{4\pi G}\,\frac{\gamma\,'}{\gamma\,'-1}.$$

We then defined τ_c to be

$$\tau_c = k\gamma\,'\rho_c^{\gamma\,'-1}.$$

This fixes the scale τ_c for the parameter τ in terms of ρ_c, the datum k in the polytropic equation of state, and the universal constant G. The second dimensionless constant listed above, $4\pi G\,\rho_c r_c^2/\tau_c$ thus has a fixed numerical value determined by ρ_c. This leaves only the first dimensionless constant listed above,

$$\frac{r_c^2\,\omega_c^2}{\tau_c}.$$

The scale ω_c for the pulsation frequencies of the polytrope is not determined by our previous scale choices, so we can determine it by the principle of maximal simplification of equation (4.12). We therefore determine ω_c by requiring that

$$\frac{r_c^2\,\omega_c^2}{\tau_c} = 1,$$

or

$$\omega_c^2 = \frac{\tau_c}{r_c^2}.$$

The maximal simplification requirement has forced a relation that effectively means that the pulsation frequency scale for the polytrope is determined by ρ_c.

Denote the second dimensionless constant by Λ. Thus,

$$\Lambda = \frac{4\pi G\,\rho_c\,r_c^2}{\tau_c}.$$

Since both r_c and τ_c are determined by ρ_c, we evaluate Λ in terms of ρ_c. This gives,

$$\Lambda = 4\pi G\,\rho_c\;\rho_c^{\gamma\,\prime-2}\;\frac{k}{4\pi G}\;\frac{\gamma\,\prime}{\gamma\,\prime-1}\;\frac{1}{k\gamma\,\prime\rho_c^{\gamma\,\prime-1}},$$

which simplifies to

$$\Lambda = \frac{1}{\gamma\,\prime-1}.$$

It will be noted that the definition of Λ contains a factor r_c^2/τ_c, which occurs also in the definition of ω_c. This enables us to relate ω_c directly to Λ. This relation is

$$\Lambda = \frac{4\pi G\,\rho_c}{\omega_c^2},$$

or

$$\omega_c^2 = \frac{4\pi G\,\rho_c}{\Lambda} = (\gamma'-1)4\pi G\,\rho_c.$$

This gives the very useful relation

$$\omega_c = \sqrt{(\gamma'-1)\,4\pi G\,\rho_c}. \tag{4.13}$$

The quantity ω_c determines a natural pulsation frequency scale for the polytrope. Formula (4.13) expresses this frequency scale in terms of the polytrope central density. We can use this formula to get a rough estimate of the lowest pulsation frequency of the polytrope.

Equation (4.11) becomes

$$\begin{aligned} 0 = \widetilde{\omega}^2\eta\,' &+ \left(\frac{2}{\mu}\,\frac{d\widetilde{\tau}}{d\mu} + \frac{d^2\widetilde{\tau}}{d\mu^2} + \frac{\eta_0}{\gamma'-1}\right)\eta\,' \\ &+ \left(\frac{2\,\widetilde{\tau}}{\mu} + 2\,\frac{d\widetilde{\tau}}{d\mu} + \frac{\widetilde{\tau}}{\eta_0}\,\frac{d\eta_0}{d\mu}\right)\frac{\partial\eta\,'}{\partial\mu} + \widetilde{\tau}\,\frac{\partial^2\eta\,'}{\partial\mu^2}. \end{aligned} \tag{4.14}$$

Other than the first term on the right hand side in equation (4.14), all coefficients in this equation are quantities that can be calculated from the solutions of the unperturbed polytrope. In principle, they are known values. The only unknowns in equation (4.14) are then η' and $\widetilde{\omega}$.

Equation (4.14) is an eigenvalue equation for the unknown function η' of μ,

where the eigenvalue is $\widetilde{\omega}$. The equation admits solutions for all possible values of $\widetilde{\omega}$. However, not all of these solutions satisfy the boundary conditions. Imposing the boundary conditions restricts the values that $\widetilde{\omega}$ can take to a discrete subset of the real line.

This equation cannot be solved by analytic methods, even in those few cases where we have solutions in closed form for the unperturbed problem. In the next section I will show that by discretising equation (4.14), we can convert it into a matrix eigenvalue problem that enables us to identify those values for $\widetilde{\omega}$ that lead to solutions that satisfy the boundary conditions. The eigenvalues can then be found without needing to solve for the values of η'.

The Eigenvalue Problem

To discretise equation (4.14) we replace the derivatives in this equation by their finite difference counterparts given in (H.9) and (H.12). Equation (4.14) then becomes

$$
\begin{aligned}
0 = & \\
&\left(\widetilde{\omega}^2 + \frac{\widetilde{\tau}\left(\mu_{i+1}\right) - \widetilde{\tau}\left(\mu_{i-1}\right)}{\mu\,\Delta\mu} + \frac{\eta_0\left(\mu_i\right)}{\gamma' - 1} + \frac{\widetilde{\tau}\left(\mu_{i+1}\right) - 2\,\widetilde{\tau}\left(\mu_i\right) + \widetilde{\tau}\left(\mu_{i-1}\right)}{(\Delta\mu)^2}\right)\eta'(\mu_i) \\
&\quad + \left[\frac{2\,\widetilde{\tau}\left(\mu_i\right)}{\mu} + \frac{\widetilde{\tau}\left(\mu_{i+1}\right) - \widetilde{\tau}\left(\mu_{i-1}\right)}{\Delta\mu} + \frac{\widetilde{\tau}\left(\mu_i\right)}{\eta_0\left(\mu_i\right)}\left(\frac{\eta_0\left(\mu_{i+1}\right) - \eta_0\left(\mu_{i-1}\right)}{2\,\Delta\mu}\right)\right] \\
&\qquad \times \left(\frac{\eta'\left(\mu_{i+1}\right) - \eta'\left(\mu_{i-1}\right)}{2\,\Delta\mu}\right) \\
&\qquad\quad + \widetilde{\tau}\left(\mu_i\right)\left(\frac{\eta'\left(\mu_{i+1}\right) - 2\,\eta'(\mu_i) + \eta'\left(\mu_{i-1}\right)}{(\Delta\mu)^2}\right)
\end{aligned}
$$

Collecting like terms gives

$$
\begin{aligned}
0 = {} & (\Delta\mu)^2\,\widetilde{\omega}^2\eta'\left(\mu_i\right) \\
& + \left[\left(1 + \frac{\Delta\mu}{\mu}\right)\widetilde{\tau}\left(\mu_{i+1}\right) + \left(1 - \frac{\Delta\mu}{\mu}\right)\widetilde{\tau}\left(\mu_{i-1}\right) - 4\,\widetilde{\tau}\left(\mu_i\right) + \frac{(\Delta\mu)^2\,\eta_0\left(\mu_i\right)}{\gamma' - 1}\right]\eta'\left(\mu_i\right) \\
& + \left[\left(\frac{\Delta\mu}{\mu} + \frac{\eta_0\left(\mu_{i+1}\right) - \eta_0\left(\mu_{i-1}\right)}{4\,\eta_0\left(\mu_i\right)} + 1\right)\widetilde{\tau}\left(\mu_i\right) + \frac{\widetilde{\tau}\left(\mu_{i+1}\right) - \widetilde{\tau}\left(\mu_{i-1}\right)}{2}\right]\eta'\left(\mu_{i+1}\right) \\
& - \left[\left(\frac{\Delta\mu}{\mu} + \frac{\eta_0\left(\mu_{i+1}\right) - \eta_0\left(\mu_{i-1}\right)}{4\,\eta_0\left(\mu_i\right)} - 1\right)\widetilde{\tau}\left(\mu_i\right) + \frac{\widetilde{\tau}\left(\mu_{i+1}\right) - \widetilde{\tau}\left(\mu_{i-1}\right)}{2}\right]\eta'\left(\mu_{i-1}\right)
\end{aligned}
$$

For simplicity, replace the bracketed coefficients by the letters A, B and C. We then have

$$
0 = \left((\Delta\mu)^2\,\widetilde{\omega}^2 + A_i\right)\eta'\left(\mu_i\right) + B_i\,\eta'\left(\mu_{i+1}\right) - C_i\,\eta'(\mu_{i-1}) \tag{4.15}
$$

where

$$A_i = A(\mu_i) = \left(1 + \frac{\Delta\mu}{\mu}\right)\widetilde{\tau}(\mu_{i+1}) + \left(1 - \frac{\Delta\mu}{\mu}\right)\widetilde{\tau}(\mu_{i-1}) - 4\,\widetilde{\tau}(\mu_i) + \frac{(\Delta\mu)^2\,\eta_0(\mu_i)}{\gamma' - 1}$$

$$B_i = B(\mu_i) = \left(\frac{\Delta\mu}{\mu} + \frac{\eta_0(\mu_{i+1}) - \eta_0(\mu_{i-1})}{4\,\eta_0(\mu_i)} + 1\right)\widetilde{\tau}(\mu_i) + \frac{\widetilde{\tau}(\mu_{i+1}) - \widetilde{\tau}(\mu_{i-1})}{2}$$

$$C_i = C(\mu_i) = \left(\frac{\Delta\mu}{\mu} + \frac{\eta_0(\mu_{i+1}) - \eta_0(\mu_{i-1})}{4\,\eta_0(\mu_i)} - 1\right)\widetilde{\tau}(\mu_i) + \frac{\widetilde{\tau}(\mu_{i+1}) - \widetilde{\tau}(\mu_{i-1})}{2}$$

Let

$$\lambda = -(\Delta\mu\,\widetilde{\omega})^2 \tag{4.16}$$

Equation (4.15) then becomes

$$0 = \Big(A_i - \lambda\Big)\,\eta'(\mu_i) + B_i\,\eta'(\mu_{i+1}) - C_i\,\eta'(\mu_{i-1}) \tag{4.17}$$

In matrix notation equation (4.17) has the following form

$$\begin{pmatrix} A_1 - \lambda & B_1 & 0 & . & . \\ -C_2 & A_2 - \lambda & B_2 & . & . \\ 0 & -C_3 & A_3 - \lambda & . & . \\ . & . & . & . & B_{N-1} \\ . & . & . & -C_N & A_N - \lambda \end{pmatrix} \begin{pmatrix} \eta'_1 \\ \eta'_2 \\ \eta'_3 \\ . \\ \eta'_N \end{pmatrix} = 0 \tag{4.18}$$

To simplify further, let

$$M = \begin{pmatrix} A_1 & B_1 & 0 & . & . \\ -C_2 & A_2 & B_2 & . & . \\ 0 & -C_3 & A_3 & . & . \\ . & . & . & . & B_{N-1} \\ . & . & . & -C_N & A_N \end{pmatrix} \tag{4.19}$$

then

$$\Big(M - \lambda * I\Big)\,\eta = 0 \tag{4.20}$$

where I is the identity matrix. Matrix M is a tridiagonal square matrix of order N with eigenvalues λ.

We now see more clearly than previously why it was necessary to implement Cowling's approximation in Section 3.8. Without Cowling's approximation, equation (4.14) would have had a term in $\widetilde{\omega}^2$ among the coefficients of $\partial\eta'/\partial\mu$. Discretisation of that equation would then have produced in place of the matrix in equation (4.18) a matrix with λ subtracted also from the upper and lower diagonals, and not just from the main diagonal. The problem would thus not have given rise to a matrix eigenvalue problem.

Equation (4.20) is linear and homogeneous in the unknowns η'. It therefore admits non-trivial solutions if and only if the determinant of the matrix $M - \lambda * I$ are zero,

$$|M - \lambda * I| = 0. \tag{4.21}$$

The determinant on the left hand side is a polynomial of order N in the unknown λ. It therefore always admits N complex roots. We are interested only in real solutions, which represent steady state oscillations. The coefficients of this polynomial are all constructed from the values of A_i, B_i and C_i, which can all be calculated from the solution to the unperturbed problem. This eigenvalue problem is therefore solvable[2].

The number of eigenvalues that can be calculated by discretisation is determined by the size of the matrix obtained. This, in turn is determined by the step size used when solving the unperturbed case and the number of steps needed to reach the surface of the star. Decreasing the step size in that calculation will thus increase the number of eigenfrequencies we can calculate. It should be noted that this procedure can only deliver the lowest eigenvalue of the problem. The higher the eigenvalue calculated, the closer to each other will be the nodes of the corresponding pulsations and the more evident will be the graininess of the discretisation. The highest frequencies obtained will thus fail to be accurate solutions to the problem.

To solve for λ we must calculate the determinant of matrix (4.18). The general formula for the determinant of a matrix is defined as follows[3]:

$$\det(\, M\,) = \sum_{p=1}^{N} m_{p\,n}\; M_{p\,n} \tag{4.22}$$

where M is a square matrix of order N, $M_{p\,n}$ the entry at row number p and column number n and $m_{p\,n}$ the cofactor of the entry $M_{p\,n}$. The cofactor $m_{p\,n}$ is the determinant of matrix obtained from M by removing its p^{th} row and its n^{th} column, multiplied by the factor $(-1)^{p+n}$.

Solutions for λ were found using the Octave function called *eig()*. With an equation in the form of (4.20), eigenvalues λ can be calculated by executing the function eig() on the matrix M defined in (4.19). From equation (4.16) we then have

$$\widetilde{\omega} = \frac{\sqrt{-\lambda}}{\Delta\mu}$$

where values of λ can be used to calculate values for $\widetilde{\omega}$. Finally, using equation (4.13), the values of the pulsation frequency ω can be calculated from $\widetilde{\omega}$.

[2]To calculate A_1 and B_1, values for $\widetilde{\tau}$ and η_0 at $i = -1$ must be known. Since there are no values for $\widetilde{\tau}$ and η_0 at this index value, the computation can proceed either by setting these values to zero, or by commencing at $i = 2$. A similar problem is encountered at index value $N - 1$. We thus terminate the computation at index value $i = N - 2$

[3]The matrix M used in equation (4.22) should not be confused with the matrix M defined in (4.19). In the following discussion, M represents a general matrix.

The exact algorithm used by *eig()* to calculate eigenvalues is a closely guarded secret in the MATLAB community and is not distributed or shared. However, according to source code documentation, based on smart but automatic assumptions about the nature of matrix M, *eig()* will select a suitable and optimised method from a list of built in routines to find the eigenvalues.

The results of the calculation can be found in Chapter 6. The code used for this phase of the calculation can be found in Appendix L.1.2.

Chapter 5

Stellar Pulsations with Tides

5.1 Introduction

In this chapter different ways to combine the effect of tides introduced in chapter 2 with the standard pulsation theory presented in chapter 3 are considered. In chapter 3 we considered a spherically symmetric system in hydrostatic equilibrium whose configuration was perturbed to introduce oscillations. The frequency of these oscillations were then calculated.

To take into account tidal effects, we need to introduce the gravitational field of the secondary into the fluid equations. This affects only the second equation of system (A.1). On the other hand, equations one, three and four of system (3.23) remain unchanged and will be used in the form in which they were presented in Chapter 3.

Adding the tide to Euler's equation as an additional force gives

$$\frac{\partial \vec{v}}{\partial t}\left(\vec{r},t\right)+\left(\vec{v}\left(\vec{r},t\right)\cdot\vec{\nabla}\right)\vec{v}\left(\vec{r},t\right)=-\frac{1}{\rho\left(\vec{r},t\right)}\vec{\nabla}P\left(\vec{r},t\right)-\vec{\nabla}\Phi\left(\vec{r},t\right) \\ -\varepsilon_T\,\vec{\nabla}W\left(\vec{r},t\right), \tag{5.1}$$

where the symbol Φ represents the gravitational potential of the primary. In this chapter, the symbol Φ_S no longer occurs in the equations, so we simplify the notation by omitting the subscript P on the gravitational potential of the primary.

Equation (5.1), together with equations one and three of system (A.1) and the polytropic equation of state (3.7), is the system equations that I will use to model stellar pulsations in a tide-supporting polytrope.

The final term on the right hand side of equation (5.1) is the tidal term. It contains the very small tidal parameter ε_T. The smallness of this parameter enables us, if desired, to regard the tidal term as a perturbing force.

Two approaches are therefore open to us. In the first, the unperturbed system is taken to be the polytrope in spherically symmetric hydrostatic equilibrium, as was done in Chapter 3, and we perturb this system by simultaneous application of both the tide generating gravitational force and the oscillation-inducing perturbations. This means that equation (3.19) will now include also the tidal perturbing force. In the second, we look for a solution to the tidal problem and regard this solution as the unperturbed system. Oscillations may then be introduced into the model by perturbing the tide-carrying polytrope.

Considering these two approaches, the first appears to be more problematic than the second for the following reasons. To calculate the oscillational frequencies of the system, we need an eigenvalue equation. When considering the perturbation equations in the non-tidal case, we arrived at a single eigenvalue equation by manipulating the system of perturbation equations to eliminate all perturbed quantities other than ρ'. In this way, (5.1) was reduced, using the other equations in the system, to an equation containing only ρ' and its derivatives up to some maximum order. The resultant equation for ρ' was in the form of an eigenvalue equation which could then be solved numerically (see equation (4.20)).

Manipulating equation (5.1) in the same way leads to an "eigenvalue equation" containing a term that does not involve ρ' in any way. The culprit term is the one involving $\varepsilon_T \vec{\nabla} W(\vec{r}, \omega)$, which leads to a time dependent term containing none of the perturbed quantities of the problem, and so leads to a term that does not involve ρ' or any of its derivatives. The "eigenvalue equation" obtained is thus of non-standard form. Solving it might therefore require new techniques which are beyond the scope of this thesis to investigate. Methods of solution of this new type of equation, if currently unknown, might be the subject to an independent research project.

In this thesis, I will therefore follow the second approach described above, in which we choose the unperturbed configuration to be a tide sustaining polytrope. Given the smallness of the coefficient ε_T, the solution to the tidal problem can be sought in two steps in which a spherically symmetric polytrope in hydrostatic equilibrium is subjected to the perturbation by the tidal term. The tide generating potential is thus introduced as a perturbation of the system as discussed in Chapter 2. The solution obtained in this way, which can be called the *tidal configuration*, or *the tide carrying polytrope*, can then be subjected independently to a second perturbation to produce oscillations of the system.

This procedure introduces its own difficulties. The unperturbed system, now given by the tidal configuration, is no longer spherically symmetric but is distorted by the tidal influences. Furthermore, the unperturbed system is, in general, no longer hydrostatic but is time dependent. This time dependence might arise from two physically independent causes: first, the binary orbit may not be circularised, in which case the distance D from primary to secondary is a function of time; second, the primary may not be phase locked with respect to the secondary. This means that the secondary rotates around the primary when viewed from the co-rotating frame of the primary. In both cases, the unperturbed configuration goes from being hydrostatic to hydrodynamic. As a

result of these two effects, the unperturbed velocity $\vec{v}_0$ will no longer be 0, and the values of all unperturbed parameters Q_0 will in general depend on all three spherical coordinates and on time.

If we follow this second approach, the effect of the tidal term is absorbed into the coefficients of the equations for the perturbed quantities and no longer appear as a solitary time dependent term in the eigenvalue equation. This method therefore leads to an eigenvalue equation in conventional form that can be solved by standard methods. The application of these standard methods, however, is now more complicated since the coefficients calculated from the tidal configuration are now time dependent and cannot be removed from the integrals of the Fourier transforms as if they were constants. Were this problem soluble by hand, this difficulty might not be insuperable. However, in a numerical calculation, the time dependence of the coefficients presents new complications, making it difficult to reduce the eigenvalue equation to a matrix eigenvalue problem by discretisation. We must thus use other methods to search for oscillation frequencies of this system. I comment further on these other methods in Section 5.2.

The second approach described above effectively introduces a third configuration, intermediate between the spherically symmetric hydrostatic polytrope and the tide-carrying oscillating polytrope, namely the tide-carrying shape-distorted polytrope. This third configuration is called the *surface perturbation*, which I discuss in the next section.

5.2 The Surface Perturbation

In the presence of the tidal force (with both its static and dynamic components, identified in Chapter 2), the unperturbed equations (3.8) along with the polytropic equation of state (3.7) become

$$\left.\begin{aligned} \frac{\partial \rho_0}{\partial t}\left(\vec{r},t\right) &= -\vec{\nabla}\cdot\left(\rho_0\left(\vec{r},t\right)\,\vec{v}_0\left(\vec{r},t\right)\right) \\ \frac{\partial \vec{v}_0}{\partial t}\left(\vec{r},t\right)+\left(\vec{v}_0\left(\vec{r},t\right)\cdot\vec{\nabla}\right)\vec{v}_0\left(\vec{r},t\right) &= -\frac{1}{\rho_0\left(\vec{r},t\right)}\,\vec{\nabla}P_0\left(\vec{r},t\right)-\vec{\nabla}\Phi_0\left(\vec{r},t\right) \\ &\qquad -\varepsilon_T\,\vec{\nabla}W\left(\vec{r},t\right) \\ \nabla^2\Phi_0\left(\vec{r},t\right) &= 4\pi G\rho_0\left(\vec{r},t\right) \\ P_0\left(\vec{r},t\right) &= k\left(\rho_0\left(\vec{r},t\right)\right)^{\gamma'}, \end{aligned}\right\} \quad (5.2)$$

where all functions now depend on all four variables $\vec{r}$ and t, and not on r and t alone, as previously had been the case. I have emphasised this fact by including the arguments in all equations.

System (5.2) are identical to the hydrodynamic equations (A.1), but with the tidal term included. These are called the *surface perturbation equations*. This name originates in the fact that the tidal term is essentially a perturbation of the

gravitational potential of the primary which distorts its gravitational equipotential surfaces.

System (5.2) consists of six equations, one vector and three scalar equations, for the six unknowns ρ_0, $\vec{v}_0$, P_0 and $\vec{v}_0$. This initial unperturbed system is no longer hydrostatic, but has become hydrodynamic. Calculating values for these variables thus becomes a much more difficult process, since they are no longer time independent or spherically symmetric. Numerical solution of these equations now requires a full three dimensional Direct Numerical Simulation (DNS) in which the entire range of spatial and temporal scales characteristic of both tidal and pulsation effects must be resolved.

5.3 The Perturbed Configuration

Taking the changes to the unperturbed configuration into account, the Eulerian perturbations (3.13) are now defined by

$$\left.\begin{array}{rcl} \rho\left(\vec{r},t\right) & = & \rho_0\left(\vec{r},t\right)+\rho'\left(\vec{r},t\right) \\ P\left(\vec{r},t\right) & = & P_0\left(\vec{r},t\right)+P'\left(\vec{r},t\right) \\ \Phi\left(\vec{r},t\right) & = & \Phi_0\left(\vec{r},t\right)+\Phi'\left(\vec{r},t\right) \\ \vec{v}\left(\vec{r},t\right) & = & \vec{v}_0\left(\vec{r},t\right)+\vec{v}'\left(\vec{r},t\right). \end{array}\right\} \tag{5.3}$$

The relationship between the Eulerian and Lagrangian velocity perturbation no longer reduces to equation (3.12) but is given by the more general equation (3.11). Since the unperturbed position of the fluid element, namely $\vec{r}_0=\vec{r}_0\big(\vec{a},t\big)$, is now dependent on time, equation (3.13) is no longer valid and becomes

$$\delta\vec{v}_\ell=\frac{d\vec{\xi}}{dt}+\frac{d\vec{r}_0}{dt}. \tag{5.4}$$

Combining equations (3.11) and (5.4) then gives

$$\begin{aligned} \vec{v}\,' &= \delta\vec{v}_\ell-\vec{\xi}\cdot\nabla\vec{v}_0 \\ &= \frac{d\vec{\xi}}{dt}+\frac{d\vec{r}_0}{dt}-\vec{\xi}\cdot\nabla\vec{v}_0. \end{aligned} \tag{5.5}$$

Using the newly defined perturbations (5.3), the first equation of system (A.1) becomes

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\Big(\rho_0+\rho'\Big)+\vec{\nabla}\cdot\Big(\left(\rho_0+\rho'\right)\left(\vec{v}_0+\vec{v}\,'\right)\Big) \\ &= \frac{\partial\rho_0}{\partial t}+\frac{\partial\rho'}{\partial t}+\vec{\nabla}\cdot\Big(\rho_0\,\vec{v}_0+\rho_0\,\vec{v}\,'+\rho'\,\vec{v}_0+\rho'\,\vec{v}\,'\Big). \end{aligned} \tag{5.6}$$

Linearising equation (5.6) then gives

$$0=\frac{\partial\rho_0}{\partial t}+\frac{\partial\rho'}{\partial t}+\vec{\nabla}\cdot\Big(\rho_0\,\vec{v}_0+\rho_0\,\vec{v}\,'+\rho'\,\vec{v}_0\Big). \tag{5.7}$$

Substituting equation (5.5) for $\vec{v}\,'$ into equation (5.7) we get

$$0=\frac{\partial\rho_0}{\partial t}+\frac{\partial\rho'}{\partial t}+\vec{\nabla}\cdot\left(\rho_0\,\vec{v}_0+\rho'\,\vec{v}_0+\rho_0\Big(\frac{d\vec{\xi}}{dt}+\frac{d\vec{r}_0}{dt}-\vec{\xi}\cdot\nabla\vec{v}_0\Big)\right)$$

$$= \frac{\partial \rho_0}{\partial t} + \frac{\partial \rho\,'}{\partial t} + \vec{\nabla} \cdot \left(\rho_0\, \vec{v}_0 + \rho\,'\, \vec{v}_0 + \rho_0\, \frac{d\vec{\xi}}{dt} + \rho_0\, \frac{d\vec{r}_0}{dt} - \rho_0\, \vec{\xi} \cdot \nabla \vec{v}_0 \right).$$

Using equation one of system (5.2) then gives

$$0 = \frac{\partial \rho\,'}{\partial t} + \vec{\nabla} \cdot \left(\rho\,'\, \vec{v}_0 + \rho_0\, \frac{d\vec{\xi}}{dt} + \rho_0\, \frac{d\vec{r}_0}{dt} - \rho_0\, \vec{\xi} \cdot \nabla \vec{v}_0 \right). \tag{5.8}$$

Equation (5.1) becomes

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} \left(\vec{v}\,' + \vec{v}_0 \right) + \Big(\left(\vec{v}_0 + \vec{v}\,' \right) \cdot \vec{\nabla} \Big) \left(\vec{v}_0 + \vec{v}\,' \right) \\ &\qquad + \frac{1}{\rho_0 + \rho\,'} \vec{\nabla} \Big(P_0 + P' \Big) + \vec{\nabla} \Big(\Phi_0 + \Phi' \Big) + \varepsilon_T\, \vec{\nabla} W \\ &= \frac{\partial \vec{v}\,'}{\partial t} + \frac{\partial \vec{v}_0}{\partial t} + \Big(\vec{v}_0 \cdot \vec{\nabla} \Big) \vec{v}_0 + \Big(\vec{v}_0 \cdot \vec{\nabla} \Big) \vec{v}\,' \\ &\qquad + \Big(\vec{v}\,' \cdot \vec{\nabla} \Big) \vec{v}_0 + \Big(\vec{v}\,' \cdot \vec{\nabla} \Big) \vec{v}\,' + \frac{1}{\rho_0 + \rho\,'} \vec{\nabla} \Big(P_0 + P' \Big) \\ &\qquad\qquad + \vec{\nabla} \Big(\Phi_0 + \Phi' \Big) + \varepsilon_T\, \vec{\nabla} W. \end{aligned} \tag{5.9}$$

The tidal term in this equation is a known function of $\vec{r}$ and t, and should not be confused with the perturbation procedure. Physically, it is a small external force of known form. Multiplying through by $(\rho_0 + \rho\,')$ and linearising equation (5.9) we get

$$\begin{aligned} 0 = \rho_0 \frac{\partial \vec{v}\,'}{\partial t} &+ \rho_0 \frac{\partial \vec{v}_0}{\partial t} + \rho\,' \frac{\partial \vec{v}_0}{\partial t} + \Big(\vec{v}_0 \cdot \vec{\nabla} \Big) \vec{v}_0\, \rho_0 \\ &+ \Big(\vec{v}_0 \cdot \vec{\nabla} \Big) \vec{v}_0\, \rho\,' + \Big(\vec{v}_0 \cdot \vec{\nabla} \Big) \vec{v}\,' \rho_0 + \Big(\vec{v}\,' \cdot \vec{\nabla} \Big) \vec{v}_0\, \rho_0 + \vec{\nabla} P_0 \\ &+ \vec{\nabla} P' + \rho_0 \vec{\nabla} \Phi_0 + \rho\,' \vec{\nabla} \Phi_0 + \rho_0 \vec{\nabla} \Phi' + \rho_0\, \varepsilon_T\, \vec{\nabla} W, \end{aligned}$$

where the term $\rho\,' \varepsilon_T\, \vec{\nabla} W$ is removed in the linearisation procedure since ε_T and $\rho\,'$ are both assumed to be small values of the same order of magnitude. Using the second equation of system (5.2), we then have

$$\begin{aligned} 0 = \rho_0 \frac{\partial \vec{v}\,'}{\partial t} &+ \rho\,' \frac{\partial \vec{v}_0}{\partial t} + \Big(\vec{v}_0 \cdot \vec{\nabla} \Big) \vec{v}_0\, \rho\,' \\ &+ \Big(\vec{v}_0 \cdot \vec{\nabla} \Big) \vec{v}\,' \rho_0 + \Big(\vec{v}\,' \cdot \vec{\nabla} \Big) \vec{v}_0\, \rho_0 + \vec{\nabla} P' \\ &+ \rho\,' \vec{\nabla} \Phi_0 + \rho_0 \vec{\nabla} \Phi', \end{aligned} \tag{5.10}$$

where equation (5.5) can be used to determine values for $\vec{\xi}$ using values for $\vec{v}\,'$ and a grid of initial values $\vec{a}$. The Poisson equation and polytropic equation of state for the perturbed system are the same as those found in section 3.4, except that all unperturbed quantities are now dependent on three spatial dimensions and on time. In the case of the perturbed polytropic equation of state we now have

$$\tau \left(\vec{r}, t \right) = k\, \gamma' \Big(\rho_0 \left(\vec{r}, t \right) \Big)^{\gamma' - 1}.$$

Equations (5.10) and (5.8) with equations (3.20) and (3.22) make up a full set of equations which can be used to solve for the four unknowns ρ', P', Φ' and $\vec{v}\,'$. Note that there is no tidal term in equation (5.10). Introducing the effect of tides into the unperturbed configuration leads to its cancellation in the equations for the perturbations.

The unknowns Φ', P' and $\vec{v}\,'$ can be eliminated from equations (5.10) using the remaining equations for the perturbed system.

A significant new complication arises in the resultant decoupled equations. This problem is already evident in equations (5.10), in which all terms are products of known functions of $\vec{r}$ and t with the unknown perturbations. When forming the Fourier transform of these terms, we can no longer remove the coefficients from the corresponding Fourier integrals since these too are functions of t. The transform of a product of functions of t is the convolution of the transforms of the factors. The Fourier transform of the decoupled equations will thus not be a linear equation in the Fourier coefficients of the perturbation variables, but an integral transform of them. The resultant problem is thus not a standard eigenvalue problem, and so it cannot be solved by standard eigenvalue techniques.

An alternative to this procedure is to solve the perturbation equations computationally by DNS. This second DNS needs the values of the unperturbed parameters obtained by the first simulation stored in memory in the form of a look-up table. Such a procedure is memory intensive and would require the use of a very substantial computer cluster.

5.4 The Static Tide

The difficulties described above arise from incorporating bot the static and dynamic tidal terms into the model. One should perhaps begin by solving a simpler problem that involves only the static tide, identified in Chapter 2.

Physically, we get a static tide in a binary system in which the components in the binary system are tidally locked. As a consequence of tidal locking, the angular velocity of the primary Ω is equal to the mean motion of the secondary μ. Equation (2.45) then becomes

$$W(\vec{r}) = \left(\frac{GM_1}{R}\right)\left(\frac{r}{R}\right)^2 \times \left(\frac{1}{2}\,P_2(\cos\theta) - \frac{1}{4}\,P_2^2(\cos\theta)\,\cos(2\,\phi)\right), \tag{5.11}$$

so that the time dependence of the tide generating potential falls away. Equation (5.11) thus describes the form of the tide generating potential that governs a system which supports a static tide with no dynamic component.

Since the tidal term is no longer time dependent, we may once again break up the problem into two parts. In the first, we consider an unperturbed system that is spherically symmetric and in hydrostatic equilibrium, as we did in Chapter 3. The effect of the small tidal term can be calculated by treating it as a

small perturbation. The new configuration is a new distorted hydrostatic equilibrium in which the system is in equilibrium under the action of the combined gravitational fields of both the primary and the secondary. This first solution solves the surface perturbation problem. The values of the fluid parameters in this new equilibrium can be stored in look-up tables in computer memory for use in the second phase of the solution.

In the second part of the static tide problem, we take as the unperturbed system the distorted hydrostatic equilibrium obtained from the surface perturbation problem, and we look for its oscillational modes by applying perturbation theory to it. Equations (5.2) now become

$$\left.\begin{aligned} \frac{\partial \rho_0}{\partial t}(\vec{r}) &= 0 \\ \vec{\nabla}\Phi_0(\vec{r}) &= -\frac{1}{\rho_0(\vec{r})}\vec{\nabla}P_0(\vec{r}) - \varepsilon_T \vec{\nabla}W(\vec{r}) \\ \nabla^2\Phi_0(\vec{r}) &= 4\pi G\rho_0(\vec{r}) \\ P_0(\vec{r}) &= k\left(\rho_0(\vec{r})\right)^{\gamma'}. \end{aligned}\right\} \tag{5.12}$$

The Eulerian perturbations (5.3) are given by

$$\left.\begin{aligned} \rho(\vec{r},t) &= \rho_0(\vec{r}) + \rho'(\vec{r},t) \\ P(\vec{r},t) &= P_0(\vec{r}) + P'(\vec{r},t) \\ \Phi(\vec{r},t) &= \Phi_0(\vec{r}) + \Phi'(\vec{r},t) \\ \vec{v}(\vec{r},t) &= \vec{v}'(\vec{r},t), \end{aligned}\right\} \tag{5.13}$$

where the unperturbed quantities are now not functions of r alone, but of $\vec{r}$.

The relationship between the Eulerian and Lagrangian velocity perturbations is again defined by equation (3.12), where the Lagrangian velocity is defined by equation (3.13), since $\vec{v}_0 = 0$ and $\vec{a} \neq \vec{a}(t)$.

Using equation (5.13), the first equation of system (A.1), system (5.13) again reduces to equation (3.16), as before, but now the unperturbed parameters are functions of $\vec{r}$.

Similarly, the Poisson and polytropic perturbation equations remain the same as those found in section 3.4.

Omitting the arguments of all functions, equation (5.1) becomes

$$\begin{aligned} 0 = \frac{\partial \vec{v}\,'}{\partial t} + \left(\vec{v}\,' \cdot \vec{\nabla}\right)\vec{v}\,' + \frac{1}{\rho_0 + \rho'}\vec{\nabla}\left(P_0 + P'\right) \\ + \vec{\nabla}\left(\Phi_0 + \Phi'\right) + \varepsilon_T \vec{\nabla}W. \end{aligned} \tag{5.14}$$

Multiplying through by $\rho_0 + \rho'$ and linearising equation (5.14), we get

$$0 = \rho_0 \frac{\partial \vec{v}\,'}{\partial t} + \vec{\nabla} P_0 + \vec{\nabla} P' \\ + \rho_0 \vec{\nabla} \Phi_0 + \rho\,' \vec{\nabla} \Phi_0 + \rho_0 \vec{\nabla} \Phi' + \rho_0\, \varepsilon_T\, \vec{\nabla} W.$$

Using equation two of (5.12) then gives

$$0 = \rho_0 \frac{\partial \vec{v}\,'}{\partial t} + \vec{\nabla} P' + \rho\,' \vec{\nabla} \Phi_0 + \rho_0 \vec{\nabla} \Phi'$$

with the tidal term falling away. Using equations (3.12) and (3.13) for $\vec{v}\,'$ we get

$$\rho_0 \frac{\partial^2 \vec{\xi}}{\partial t^2} (\vec{r}, t) = \vec{\nabla} P' (\vec{r}, t) + \rho\,' (\vec{r}\,)\, \vec{\nabla} \Phi_0 (\vec{r}\,) + \rho_0 (\vec{r}\,)\, \vec{\nabla} \Phi' (\vec{r}, t). \tag{5.15}$$

Equation (5.15) is identical in form to equation (3.19), but with the important difference that the unperturbed parameters are now functions of $\vec{r}$ rather than of r alone.

Since the equations developed in this section are identical in form to those in section 3.4, following the same procedure as was followed there yields an eigenvalue equation.

To solve the eigenvalue problem for the case of a static tide requires us to know the values of the unperturbed parameters in three dimensions, namely r, θ and ϕ. The process of solving for pulsation frequencies is considerably more complicated than it was in the case without the static tide and requires considerably more computational resources.

Chapter 6

Results

The results presented in this chapter were calculated by running the two part calculation presented in chapter 4. To de-dimensionalise the equations, I used the parameter values of the sun. The sun's central density is $\rho_c = 1.622 \times 10^6$ Kg/m^3 and its mean radius is $R_\odot = 6.957 \times 10^8$ m. The radius of the polytrope should not be expected to coincide with the solar radius value $R_\odot$. It is well known that the governing equations of a polytrope lead to a mass-radius relation [42]. This means that if we choose ρ_c to be the central density of the sun, the radius of the polytrope will be determined by ρ_c through equation (4.6), and there is no *a priori* reason to expect it to coincide with the solar radius. Choosing ρ_c to be central density of the sun, the radius of the polytrope will then be one of the outputs of the model.

I have also chosen to construct the model with polytropic index $n = 3$. This is the value used by Eddington to construct his Standard Model of a star, which appears to have been the first successful stellar model to have been constructed. The simulation was run for 100 steps. For the step size chosen, this number of steps exceeded the number needed to reach the cut-off condition. The code for the simulation can be found in Appendix L.

The values of ρ_0 and its non-dimensional counterpart η_0, calculated in the first part of the simulation, are presented as density vs radius plots in section 6.1. Tables of the calculated values of ρ_0 and η_0 are not included in this thesis due to their unwieldy size. These data can be found in the directory called *Results/100 Steps* that comes with the electronic copy of the thesis. The full list of frequencies calculated in the second part of the simulation are listed in ascending order in Table 6.1. I comment on these values in Section 6.2.

6.1 Density-Radius Plots for the Unperturbed System

The calculations were carried out using the de-dimensionalised form of the equation for the polytrope with $\eta_0 = \rho_0/\rho_c$, where ρ_c was taken to the be central density of the sun, and with $\mu = r/r_c$ where r_c was taken to be $R_\odot$. Initially the computation was run for a large number of steps, with fixed step size, in

order to locate the point where η_0 satisfied the condition $\eta_0 < 0.00001$. The value where this occurred was taken to define the radius R of the polytrope. This was found to occur at a radius of $r = 6.42 \times R_\odot$. Together with the chosen value $\gamma' = 3$, this value determines the value of k in the polytropic equation of state.

The calculation was then repeated with a step size that was calculated in such a way that 100 steps more than covered the region of interest. This number of steps is more than sufficient for resolving the polytrope density as a function of radius, as is confirmed by the plots displayed in this chapter and by similar plots generated for a 1000 steps covering the same region. These can be found in directory *Results/1000 Steps* on the accompanying CD. The choice of 100 grid points for the calculation is also sufficient for resolving the density perturbations up to about the 50^{th} harmonic, where the Nyquist limit is reached.

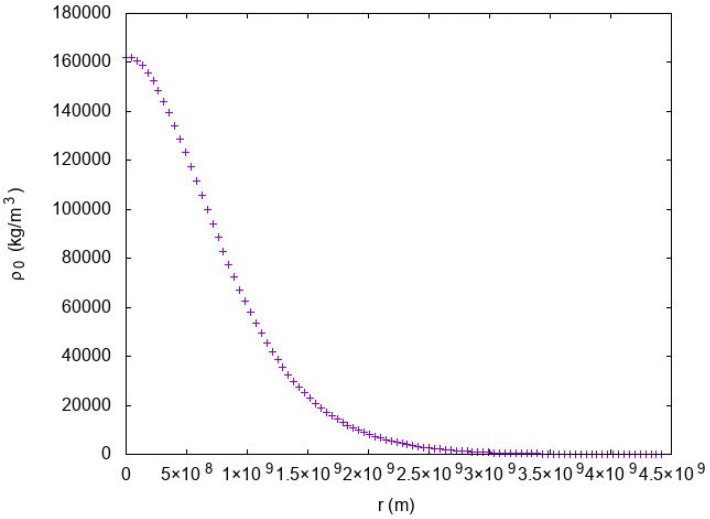


Figure 6.1: A Graph of ρ_0 vs r up to $r = 6.42 * r_c$ calculated in the unperturbed system.

Graphs 6.1 and 6.2 are plots of the values of ρ_0 vs r and of η_0 vs μ calculated by running the first part of the calculation from $\mu = 0$ to $\mu = 6.42$. These graphs show that the density ρ_0 of the unperturbed polytrope decreases slowly at first from its central value, consistent with the boundary condition that $d\rho_0/dr = 0$ at $r = 0$. However, the density begins to decrease very rapidly before μ has reached value $\mu = 0.1$, and decreases to 1/10 of its central density at about $\mu = 0.4$. The greater part of the mass of the polytrope is thus contained close to its core, making it a centrally condensed star.

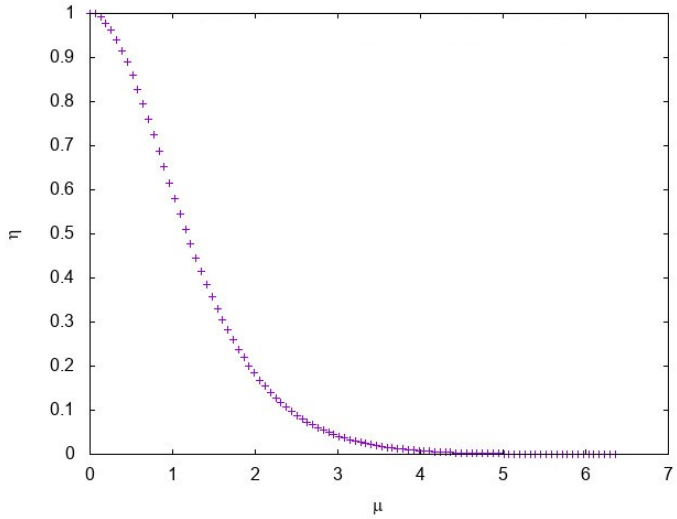


Figure 6.2: A Graph of η_0 vs μ up to $\mu = 6.42$ calculated in the unperturbed system.

6.2 Oscillation Frequencies ω in Hz

Table 6.1 lists in ascending order the oscillation frequencies of a polytrope with index $n = 3$ and central density equal to that of the sun. The polytrope has maximum radius $r_{max} = 6.42 \times R_\odot$ where $R_\odot$ is the mean radius of the sun. The characteristic equation for the eigenvalue problem, presented in Section 4.4, was solved for λ by calculating the determinant of the matrix $M - \lambda * I$ using a built in Octave function called *eig()*.

By running the second part of the calculation for 100 steps, 98 frequency values were generated. In section 4.4 we determined that the characteristic equation of matrix $M - \lambda * I$, namely (4.21), is a polynomial of order N which, in principle, admits N roots for the eigenvalue λ. We also showed that were we to calculate the eigenvalues of the matrix $M - \lambda * I$, we would need values for $\widetilde{\tau}$ and η_0 at index value $i = -1$. Since these values do not exist, the calculation must begin at index value $i = 2$ and terminate at $i = N - 1$. The number of roots obtained in this way is then 98 rather than 100. In general, if the number of steps used in the calculation is N, the calculation will produce $N-2$ eigenvalues.

Using equation (4.6), we can get an estimate of the lowest pulsation frequency for the polytrope. Using values $\rho_c = \rho_\odot$ and $\gamma' = 1.333...$ we get an estimate of $w_c = 0.00673$ Hz. Therefore, the lowest frequencies in Table 6.1 are expected to be close to this value. For a few of the lowest frequency values in Table 6.1, this is definitely the case and it is evident that the simulation is functioning as it should.

The list of frequencies is more easily explained if we consider that a radially pulsating star is essentially a continuous string that vibrates with a single degree

0	**0.040779**	**0.079458**	**0.138443**
0.004231	**0.042137**	**0.081295**	**0.141606**
0.005476	**0.043504**	**0.083163**	**0.144857**
0.006932	**0.044883**	**0.085062**	**0.148200**
0.008507	**0.046274**	**0.086996**	**0.151641**
0.010137	**0.047677**	**0.088964**	**0.155182**
0.011790	**0.049094**	**0.090969**	**0.158830**
0.013451	**0.050526**	**0.093011**	**0.162589**
0.015111	**0.051971**	**0.095093**	**0.166465**
0.016766	**0.053432**	**0.097216**	**0.170463**
0.018412	**0.054909**	**0.099381**	**0.174589**
0.020047	**0.056402**	**0.101592**	**0.178851**
0.021669	**0.057912**	**0.103848**	**0.183253**
0.023276	**0.059440**	**0.106153**	**0.187805**
0.024865	**0.060986**	**0.108508**	**0.192515**
0.026436	**0.062552**	**0.110916**	**0.197390**
0.027985	**0.064138**	**0.113379**	**0.202440**
0.029510	**0.065745**	**0.115899**	**0.207677**
0.031010	**0.067373**	**0.118479**	**0.213113**
0.032480	**0.069025**	**0.121121**	**0.218762**
0.033920	**0.070699**	**0.123828**	**0.224645**
0.035330	**0.072398**	**0.126603**	**0.230792**
0.036712	**0.074122**	**0.129448**	**0.237264**
0.038075	**0.075873**	**0.132368**	
0.039427	**0.077651**	**0.135365**	

Table 6.1: Calculated frequencies (Hz) in decreasing order for 100 steps

of freedom in the direction of $\hat{r}$. Equation (4.14) then describes the deviations of density η', from its known norm η_0, caused by the vibrational frequencies ω along the length of the string. When you discretise the continuous string, you are replacing it with a beady string that is made up of discrete masses, called oscillators, tied together by elastic bands. This approximates the system by reducing the function eigenvalue problem to an $N \times N$ matrix eigenvalue problem, which admits only N eigenvalues. This approximation is only good when low frequencies that can be distinctly measured are considered. By calculating eigenvalues for the frequency, we are sampling a continuous system by measuring discrete frequencies at discrete locations in the string. If the sampling rate[1] is high and the grid spacing between oscillators small, the measure of larger frequencies, which are approximately equal to the grid spacing, overlap with those of their neighbouring oscillators. This is known as the *Nyquist*

[1] number of oscillators per unit length

limit, which states that the minimum sampling rate must be twice the maximum frequency calculated to avoid the discreteness of the string dominating the calculated results, causing the approximation to fail. For frequencies much smaller than the grid spacing we still fall within this limit and these values are still considered useful. The largest of the frequencies in Table 6.1 are therefore physically meaningless as approximations to the actual eigenvalues.

Chapter 7

Discussion and Conclusions

7.1 Assumptions of the Model

This thesis investigates the effect of tides on pulsations. The forces responsible for tides are generally small compared with the forces driving fluid flows. This enables us to analyse their effect by means of perturbation theory.

The theory developed in this thesis is relevant to binary systems that are sufficiently well separated. The degree of separation of the components of the binary is quantified by the parameter ε_T, introduced in Section 2.5. This parameter is determined by the ratio of the masses of the binary components, and by the ratio of the radius of the primary to the average radius of the binary orbit. The masses of the binary components are in general of the same order of magnitude, so their ratio will be of order of 1. The critical factor is therefore the ratio of the radius of the primary to the average radius of the binary orbit. For the theory developed here to be applicable, therefore, this ratio must be significantly smaller than 1.

In this thesis, I have chosen values of the masses of the binary components that are realistic for δ Scuti Algol-type binaries, together with separation distances that ensure that ε_T will always be small, as illustrated in table 2.1.

In some binaries for which ε_T is small, mass transfer is known to occur. The theory developed in this thesis does not cover this eventuality, even though significant tides may characterise the system. It may nevertheless provide a good first approximation to the effect of the tidal forces.

Stars are essentially giant gas spheres. The stellar material is thus well modelled as a fluid continuum. Adopting the continuum model obviates the need to track the motion of individual particles. The assumption of a continuum is good provided that the mean free path of the fluid particles is significantly smaller than the size of the system considered. The smallness of the mean free path confines the constituent particles in such a way that clusters of particles may be considered to move collectively as a single unit whose mass is the total mass of the particles contained and whose internal motions can then be quantified by

thermodynamic variables. This could not be done were the overall identity of the unit not preserved. In the case of the sun, for example, the mean free path at its surface is ≈ 1 mm. The continuum model is thus more than adequate to model its properties.

Treating the stellar medium as a continuum permits us to describe its motions using the equations of hydrodynamics. These may be used to model the effect of applied forces on the fluid, and can be adjusted to represent particular stellar configurations where effects such as stellar rotation, fluid viscosity and magnetic forces are considered. In the case of stellar pulsation, other effects such as radiative transfer and ionisation need to be taken into account alongside gravitational forces and internal stresses. Tide producing forces are not internal to the stellar system and are generated by external time dependent gravitational fields.

Delta Scuti stars have convective cores and radiative envelopes. Their dominant pulsations originate in the radiative envelopes, in which the effects of viscosity may safely be ignored. I have therefore used Euler's equation when examining the effects of tides on pulsations.

Euler's equation is Newton's second law applied to fluid flow. It is therefore valid only in an inertial frame. Were an observer to measure the velocity of the stellar fluid relative to an inertial frame, the fluid velocity would in general be very large and the fluid trajectories would be unnecessarily complicated. Such a procedure would present severe computational problems. It is better therefore to move to a reference frame in which the fluid velocities are as close to zero as possible. In the case of the primary, this means moving to a frame with origin at its centre of mass and that co-rotates with it. Such a frame is not inertial, and Euler's equation needs to be modified to take the acceleration of the frame into consideration. As a result, two additional terms appear in the equation, namely the centrifugal and Coriolis "forces". If we assume that the angular velocity Ω of the primary is small, the centrifugal term, which is second order in Ω, is negligible and may be omitted. Moreover, the fluid velocity in the co-rotating frame is small, so the Coriolis term may also be neglected since $\Omega \times \vec{v}$ is quadratic in small quantities.

Dynamic tides arise when the components of the binary system have eccentric or asynchronous orbits. Eccentricity drives the tide through the time dependence of the distance of separation $D(t)$ of the components. Asynchronicity drives the tide through the time dependence of the phase in the dynamic term in equation (2.45).

To calculate the tidal coefficients $C_n^m(k)$, I assumed for simplicity that the orbit of the secondary is fully circularised. This means that the eccentricity is no longer a driver of the dynamic tide, since D is now independent of time. However the asynchronicity remains, and in this model becomes the principal driver of dynamic tides. We have assumed furthermore that the orbit of the secondary is periodic with period T. This converts the Fourier transform of the tide generating potential into a discrete Fourier series. These two assumptions might not be valid since binaries generally evolve into circular, synchronised orbits, which is "a state of minimum kinetic energy" [9]. This evolution is a result of

kinetic energy losses through multiple mechanisms of which tidal interactions is one of the most prominent. Older binaries might thus be both circularised and synchronous.

Applying perturbation theory to the hydrodynamic equations results in equations that are non-linear. This makes the search for solutions more difficult. However, if we assume the perturbations to be small, the equations can be linearised. In general, a perturbation of a system leads to small departures from the unperturbed solutions, so the assumption of small perturbations is not entirely unrealistic.

The model that I have developed relies strongly for its success on the Cowling approximation. The rationale for this approximation is that small perturbations can have no significant perturbative effect on the gravitational potential of the primary. Cowling argued that the perturbed potential at a point is given by an integral over the entire star. The process of integration averages and smooths out any departure from the unperturbed potential. This is a good approximation provided that the angular velocity of the primary is small, so that its shape is approximately spherical and that the greater part of its mass is centrally concentrated. The systems studied in this thesis meet all of these requirements.

Two principal reasons make the Cowling approximation desirable. Firstly, the non-radial equations are reduced to a workable mathematical form involving spherical harmonics which can then be combined with the radial solutions to construct a complete set of solutions. This simplifies the numerics considerably. Secondly, the approximation removes the pulsation frequency from multiple terms in the radial equation that contain the derivatives of ρ' leaving us with a standard eigenvalue problem. It should be noted, however, that if the Cowling approximation is not used, this does not leaves us in a dead end since numerical techniques exist which are capable of solving the associated equations, but with high computational overheads.

7.2 Conclusions from the Model

The Fourier transform of equation (2.45) contains three Dirac delta functions, $\delta(\omega)$, $\delta(\omega + 2[\Omega - \mu])$ and $\delta(\omega - 2[\Omega - \mu])$, where Ω is the rotation rate of the corotating frame and μ is the the mean motion of the binary. By definition of the Dirac delta function, this shows that ω must be either be 0 or $\pm 2(\Omega - \mu)$. Every other value for ω gives a Fourier component of the tide generating potential equal to zero. Because we have taken the complex Fourier transform of $W(\vec{r}, t)$, these two values of ω represent a single frequency in the time domain and correspond to a tidal driving force of frequency $2(\Omega - \mu)$.

This means that a tidal system is effectively a driven oscillator and can thus be expected to display the typical behaviour of such systems. In the case of a circularised orbit, the tidal force drives the system at a single frequency. In the steady state, it will drive pulsations at frequency $2(\Omega - \mu)$ provided that $\Omega \neq \mu$. If this frequency coincides with a natural frequency of the primary, the driver will produce a strong resonant response in the primary at this frequency. If the

driving frequency is close to a natural frequency of the primary, the response will be at the driving frequency, but with a reduced amplitude. The amplitude of the response will decreases the further away it is from this natural frequency.

I have considered only the effects of tides on radial pulsations. Shibahashi has shown that, if the pulsations are non-radial, the a single pulsation frequency in the spectrum will split into a "triplet fine structure with an equal spacing of twice the orbital frequency" [32].

The frequency $\omega = 0$ corresponds to a second contribution to the static tide. The static tide thus produces only a permanent deformation of the system and is therefore incapable of driving pulsations in the primary. The dynamic tide is therefore the only component of the tidal force that has any effect on the pulsation frequencies of a star

In this thesis, I have only considered the case of a fully circularised orbit. If the orbit is elliptical, the tide generating potential would contain a further time dependence in the form of a changing distance $D\,(t)$ between the binary components. Its Fourier transform would therefore contain a spectrum of driving frequencies rather than the two discussed above. Each frequency component would result in the same behaviour as described above. In the case of a periodic orbit, this spectrum would be discrete and would thus contain frequencies that are overtones of the natural frequencies. If we were instead to phase lock the components in a system with an eccentric orbit, the function $D(t)$ would still give rise to a transform with a spectrum of frequencies. In either event, the presence of driving terms at multiple frequencies is expected to produce a spectrum of pulsation frequencies.

7.3 Recommendations

In this thesis I have investigated only the effects of the tidal potential on radial pulsations. The action of a dynamic tide also produces non-radial motions in the stellar fluid. It is important to investigate whether the tidal force could produce non-radial pulsations. If so, these non-radial pulsations will support oscillatory quadrupole moments which have been shown by Shibahashi to result in the splitting of lines in the frequency spectrum of the pulsation spectrum [32]. The possibility of this type of spectral splitting could drive observational campaigns to identify this phenomenon in binary systems.

In this thesis I have investigated only the asynchronous contribution to the dynamic tide. The eccentricity as a driving component was removed by assuming a circular orbit for binary. It is important to investigate the effect of eccentricity as a driving force. The definition for the second order tide generating potential will need to be recalculated to include the eccentricity of the orbit.

At the time of writing this thesis, MESA had no functioning modules for the effects of tides on pulsations. Investigating this further would prove useful if, at its latest version, MESA could be used to model the star more realistically by taking these effects into consideration.

7.4 Conclusion

The investigation carried out in this thesis shows conclusively that tidal forces are capable of driving pulsations in a star. In fully circularised and synchronous orbits, there is no mechanism by which the tidal forces can produce pulsations, so no effect is expected in such systems. For tide driven pulsations to occur, the orbit of the binary must be asynchronous and / or eccentric. An asynchronous system with circularised orbit has a single driving frequency which could excite pulsations provided that this frequency is not too distant from a natural pulsation frequency of the primary. An eccentric periodic orbit has a discrete spectrum of driving frequencies, increasing the probability of resonant responses. Non periodic orbits have a continuous spectrum of driving frequencies, resulting in the excitation of multiple pulsation modes, but with response dependent on the strength of the corresponding driving Fourier component.

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Glossary

Binary Star System A binary system is a star system consisting of two stars orbiting around a common centre of mass. 1

Centrifugal Acceleration Inertial acceleration that is directed away from the rotational axis of a rotating reference frame. 111

Coriolis Acceleration Inertial acceleration that acts relative to a rotating reference frame. An object moving in a reference frame rotating clockwise will experience a Coriolis force to the left of its motion and vice versa. 111

Discretised To convert a continuous space into an equivalent discrete space for the purposes of easier calculation. 61

Dynamic Tide A tidal interaction generated by a companion star in an asynchronous and eccentric orbit. Forms the component of the tide-generating potential that results in a time dependent tidal effect. 3

Equilibrium/Static Tide A tidal interaction generated by a companion star in synchronous and circular orbit with its primary. Forms the component of the tide-generating potential that results in a time independent tidal effect. 3

Eulerian Formalism Formalism that treats the physical quantities of a fluid as fields. The Physical quantities of different fluid points as they pass the point of observation are measured. 12

Fluid Continuum A fluid continuum is a body that can be continually subdivided into infinitesimal fluid elements with properties being those of the bulk material. In the case of matter this is no longer an accepted theory since the discovery of the atom, however is a good assumption under certain conditions. 31

Keplerian Orbit The motion of one body relative to another that forms either an ellipse, parabola, or hyperbola. The trajectory of the orbit must traverse a two-dimensional orbital plane in three-dimensional space. 103

Lagrangian Formalism Formalism that follows individual fluid particles as they move. Physical quantities are then measured relative to an individual fluid point. 31

Mean Motion A measure of how fast an orbiting body progresses about its elliptical orbit. 21

Octave Open source alternative for the commonly used programming platform MATLAB. 5

Periapsis Same as periastron except references orbiting bodies as opposed to stars. 125

Periastron Nearest point from the centre of mass of the primary star in the secondary star's orbit. 9

Primary Star The brighter of the two stars in a binary system. Also commonly referred to as the smaller in size or larger in mass star in comparison to the other in the system. 2

Ratio Test A test to determine whether a series converges or diverges by calculating a ratio ρ. If $\rho < 1$ the series converges and if $\rho > 1$ the series diverges. A $\rho = 1$ is undetermined and another test must then be used. 113

Secondary Star The darker of the two stars in a binary system. Also commonly referred to as the larger in size or smaller in mass star in comparison to the other in the system. Synonymous with the name "companion". 2

Surface Perturbation Application of the effect of tides, in the absence of pulsations, to the unperturbed configuration resulting in it losing its spherical symmetry and becoming hydrodynamic . 70

Tidal Locking (Synchronous Orbit) Occurrence when the rotation rate of an orbiting body becomes the same as its orbital period, causing its one side to continuously face the body it is orbiting, also called a synchronous orbit. 83

True Anomaly The true anomaly is the angle between the direction of periapsis and the current position of an orbiting body, as seen from the origin of the barycentric coordinate system. 20

Nomenclature

Acronyms

ℓ	As a subscript - Lagrangian
ASTEC	Aarhus STellar Evolution Code
DNS	Direct numerical simulation
MESA	Modules for Experiments in Stellar Astrophysics
P	As a subscript - Primary
S	As a subscript - Secondary
T	Tidal

Symbols

γ'	Polytropic Exponent - Function of the Polytropic Index $\frac{n+1}{n}$
λ	Symbol for the Eigenvalue
$\mathcal{X}$	Rotational angle of co-rotating frame $\widetilde{Q}$
Ω	Angular velocity of the primary about axis $\widetilde{x}^3/\bar{x}^3$
ω	Oscillation frequency of the system
$\overline{Q}$	Co-moving reference frame with intertial axes
Φ	Gravitational Potential of a fluid element evaluated at its current position
ρ	Density of a fluid element evaluated at its current position
τ	$k\,\gamma'\,\rho_0^{\gamma'-1}$
ε_T	Tidal parameter
$\vec{A}$	Acceleration of fluid element p measured by an observer in reference frame I
$\vec{a}$	Acceleration of fluid element p measured by an observer in reference frame $\widetilde{Q}$

$\vec{u}$	Velocity of fluid element p measured by an observer in reference frame $\widetilde{Q}$
$\vec{v}$	Velocity of fluid element p measured by an observer in reference frame I
$\widetilde{Q}$	Co-rotating reference frame relative to $\overline{Q}$
C_1	Centre of mass of the primary star
C_2	Centre of mass of the secondary star
C_B	Centre of mass of the binary system
$d\Omega$	Solid Angle
D	Distance between the centre of the primary and the secondary
d	Distance between the secondary and an arbitrary fluid element p within the primary
I	Inertial reference frame
k	Proportionality constant in polytropic equation of state
M_1	Mass of primary star
M_2	Mass of secondary star
n	Polytropic Index
P	Pressure of a fluid element evaluated at its current position
p	Symbol given to an arbitrary fluid element of interest
Q_c	The charactersitic value of an arbitrary physical quantity Q
W	Tide generating potential
Y_n^m	Spherical harmonics
Y_n^{m*}	Complex conjugate of the spherical harmonics

Appendix A

Fluid Equations

The stellar material, like all matter, is atomic. This means that the mass of the material is concentrated in the form of particles, which are separated from each other by empty space. This is difficult to model mathematically and requires a statistical theory of fluids. We can avoid this by adopting a continuum model for the stellar material.

In the continuum model mass is modelled as continuously distributed through space by grouping an infinite number of fluid points as an infinitesimally small fluid element. The discontinuity between fluid points is then concealed preventing gaps in the distribution of mass. A fluid point cannot have a finite mass attributed to it. If its mass were finite, a fluid element with an infinite number of fluid points would have infinite mass. The mass of the fluid is therefore a property of a fluid element and not of the fluid points. This means that the basic building block of the stellar material in this model is the fluid element. The modelling of a fluid element is synonymous with particle kinematics and dynamics except that the measured properties are an average of those of the individual particles it contains.

This continuum model is a good conjecture only in cases where the mean free paths of the fluid points are significantly smaller than the size of the system, and can be comfortably confined to the size of the fluid element. For problems pertaining to stellar media this model is more than adequate.

To model the motion of stellar media the hydrodynamic equations of motion are used. These equations can be adjusted to represent explicit stellar configurations, allowing us the ability to consider particular effects and remove others such as stellar rotation, fluid viscosity and magnetic forces. The hydrodynamic equations used in conventional pulsation theory retain terms that represent the gravitational force and pressure force of the primary star only. In spherical

coordinates, these are listed below as

$$\left.\begin{aligned} \frac{\partial \rho}{\partial t}\left(\vec{r},t\right) &= -\vec{\nabla}\cdot\Big(\rho\left(\vec{r},t\right)\ \vec{v}\left(\vec{r},t\right)\Big), \\ \frac{\partial \vec{v}}{\partial t}\left(\vec{r},t\right)+\Big(\vec{v}\left(\vec{r},t\right)\cdot\vec{\nabla}\Big)\,\vec{v}\left(\vec{r},t\right) &= -\tfrac{1}{\rho(\vec{r},t)}\ \vec{\nabla}P\left(\vec{r},t\right)-\vec{\nabla}\Phi\left(\vec{r},t\right), \\ \nabla^2\Phi\left(\vec{r},t\right) &= 4\pi G\rho\left(\vec{r},t\right),\Big)^{\gamma'}, \end{aligned}\right\} \quad \text{(A.1)}$$

where ρ, P and Φ are the density, pressure and gravitational potential evaluated at the current position of the fluid element. $\vec{v}$ is the velocity of the fluid element itself. The first two equations are conservation equations, namely the continuity equation[1] and Euler's equation[2] and the third is Poisson's gravitational equation.

These are three equations for four unknown fields P, ρ, Φ and $\vec{v}$ and are insufficient for a complete solution. To close this set of equations we must either remove an unknown or add another equation. One way to remove density as an unknown is to assume that the fluid is incompressible making the density constant. This assumption is not suitable for stellar astrophysics. Another approach is to assume the fluid is barotropic in that there is a special relationship between pressure P and density ρ. This is a suitable assumption however cannot be generalised to include other thermodynamic processes. A better approach is to assume that the fluid is polytropic in that there is a power law relationship between P and ρ.

To find this relationship there is only one more general physical principle available, namely the principle of the conservation of energy. For a closed thermodynamic system in which only thermal and mechanical processes are considered, the first law of thermodynamics is stated as

$$\delta Q = dU + \delta W, \quad \text{(A.2)}$$

where δQ is the quantity of energy added to the system by heat, dU is the change in internal energy of the system and δW the quantity of energy lost to the system due to work done by the system on its surroundings. This is a general equation for the conservation of energy. The heat capacity of the system at a constant arbitrary physical quantity X is defined as

$$C_X = \frac{dQ_X}{dT}.$$

A polytropic process or change is defined as a quasi-static change of state carried out in such a way that the heat capacity C_X remains constant during the entire process so that

$$C_X = \frac{dQ_X}{dT} = Constant.$$

[1] mass conservation
[2] momentum conservation

Using this condition with the equation of state of an ideal gas the polytropic equation of state can be found. This is described in more detail in Appendix G where both polytropic processes as well as the equation of state are discussed.

The polytropic equation of state is defined as (see equation (G.13))

$$P\left(\vec{r},t\right)=k\Big(\rho\left(\vec{r},t\right)\Big)^{\gamma'},$$

where k is a proportionality constant, specifically determined for the star of interest and γ' a function of the polytropic index termed the polytropic exponent. This equation with equations (A.1) then form a complete set of equations.

Not only are power laws in general very effective in physics, another reason to consider a polytrope, as opposed to just a barotrope, is that all familiar thermodynamic processes are special cases of it. An adiabatic at one extreme and an isothermal at another extreme are then polytropes of zero and infinite heat capacities respectively. The same applies for all processes whose heat capacity ranges between 0 and ∞.

It is also important to note that some real systems, such as Fermi condensates like white dwarfs and neutron stars, obey polytropic law for particular values of the power γ'. Therefore, it is not an entirely unrealistic assumption.

Appendix B

Accelerated Frames of Reference

B.1 Introduction

Newton's second law of motion is valid only for an inertial frame of reference. It is often convenient to use a frame of reference that is not inertial.

For example, the most natural frame of reference for an earthbound observer to use is one whose origin is anchored to his fixed position on the surface of the earth. Similarly, a passenger on board a moving vehicle or craft would naturally use a frame of reference that is anchored to his craft. Neither of these frames is inertial. Each undergoes a complicated motion in which the origin of the frame is accelerated with the acceleration of the body to which it is attached, and in which the axes rotate.

An observer at rest in such a frame is said to be an accelerated observer, and the frame is called an accelerated frame. It is customary mentally to conflate the observer and the frame, and to regard them as one and the same item. Accordingly, we will use these two terms interchangeably.

The kinematic parameters used by an observer to describe the motion of a particle are defined with respect to his own frame. For example, the position vector of the particle is the vector from the origin of the frame to the position of the particle. Also, if we equip the frame with a set of Cartesian coordinates, x^1, x^2, x^3 say, the components of velocity and acceleration of the particle are respectively the first and second rates of change of the particle coordinates,

$$v^i = \frac{dx^i}{dt}, \qquad a^i = \frac{d^2x^i}{dt^2}.$$

Every frame of reference defines a set of three vectors. These are the vectors that run from the frames origin to the unit points on the axes, called the *frame vectors*. Denote them by e_1, e_2, e_3. All vectors used by the observer are constructed as linear combinations of the frame vectors. Thus the position, velocity

and acceleration vectors of a particle are given respectively by

$$\vec{x} = x^i \vec{e}_i, \qquad \vec{v} = v^i \ \vec{e}_i = \frac{dx^i}{dt} \ \vec{e}_i, \qquad \vec{a} = a^i \ \vec{e}_i = \frac{d^2x^i}{dt^2} \ \vec{e}_i,$$

where we have used the Einstein convention for summations.[1]

To calculate how a particle, viewed by an accelerated observer, moves under the action of given forces, we need to take into account not only the forces that act on the particle, but also the acceleration and rotation of the frame of reference and the definitions of position, velocity and acceleration used by the observer.

The purpose of this appendix is to find the equation of motion that must be used by an accelerated observer. This requires us first to relate the position, velocity and acceleration vectors used by the accelerated observer to those used by an inertial observer, and then to use these relations to set up the equation of motion for the accelerated frame.

B.2 The Binary System and its Reference Frames

Consider a binary system consisting of a primary star of mass M_1 and a secondary star of mass M_2. Introduce three reference frames, denoted I, $\overline{Q}$ and $\widetilde{Q}$. Please see Figure B.1 as an illustrative aid.

Frame I is inertial, with origin at the centre of mass of the binary system C_B. Denote its frame vectors by $E_i = E_1, E_2, E_3$, and the Cartesian coordinates defined with respect to its frame vectors by $x^i = (x^1, x^2, x^3)$. The direction of the x^3 axis is chosen to be perpendicular to the orbital plane of the binary. The direction of the x^1 axis is chosen so that it points towards the periastron of the orbit of the secondary.

Frame $\overline{Q}$ has its origin at the centre of mass of the primary C_1 and is co-moving with frame I. Since its axes are fixed and parallel to those of frame I, its frame vectors are also E_i, with the Cartesian coordinates defined with respect to its frame vectors denoted by $\bar{x}^i = (\bar{x}^{\,1}, \bar{x}^{\,2}, \bar{x}^{\,3})$. Its axes are fixed and parallel to those of frame I. Even though its axes are inertial, frame $\overline{Q}$ is not an inertial frame since its motion about I is non-linear.

Frame $\widetilde{Q}$ also has its origin at the centre of mass of the primary C_1, however co-rotates with frame $\overline{Q}$. Denote its frame vectors by $e_\mu = e_1, e_2, e_3$, and the Cartesian coordinates defined with respect to its frame vectors by $\widetilde{x}^\mu = (\widetilde{x}^{\,1}, \widetilde{x}^{\,2}, \widetilde{x}^{\,3})$.

The model considered in this thesis assumes that the primary rotates about

[1] Einstein summation convention: if in an index based expression an index appears twice, once in the superscript position and once in the subscript position, it is understood that this expression is implicitly preceded by a summation over that index that extends over its stated range. If no summation is to be performed, then the expression is to be followed by the statement $\not\Sigma i$. If an index is repeated more than twice, or twice with both in the same index positions, no summation is intended.

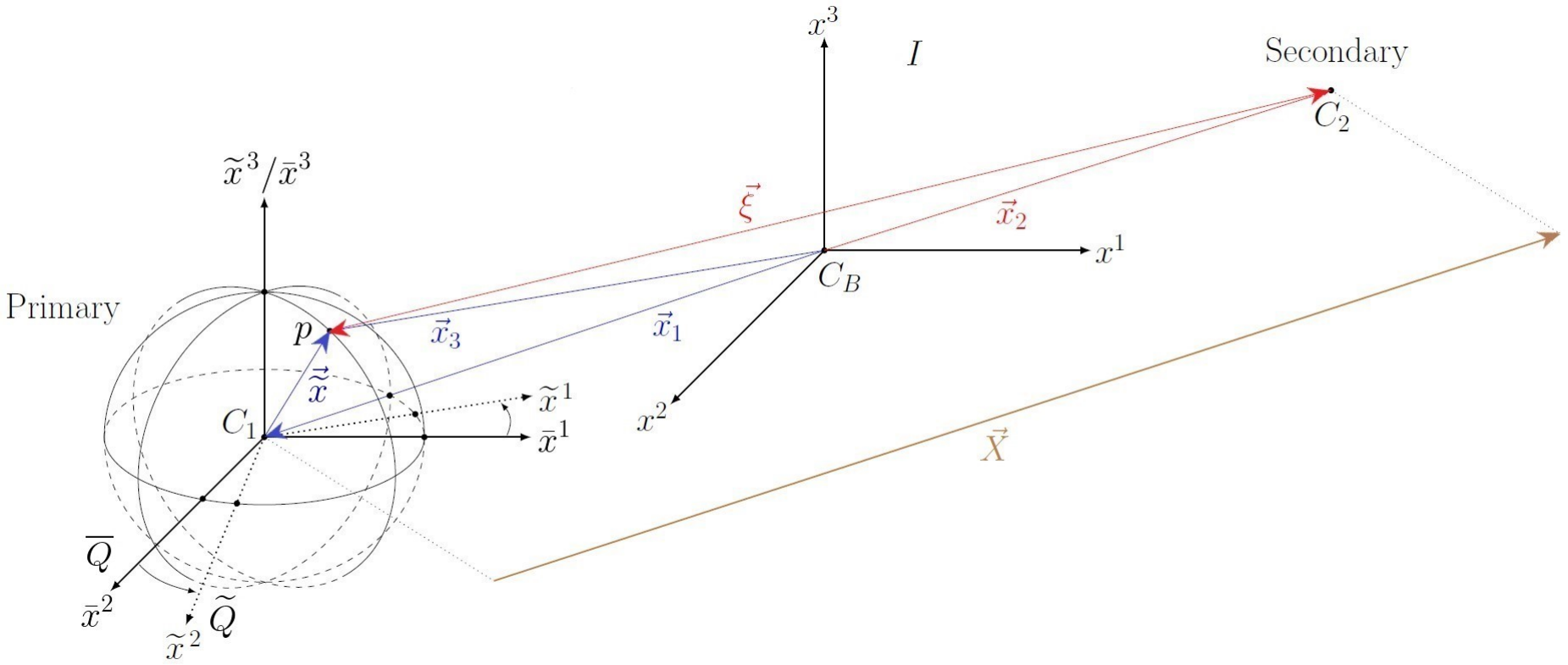


Figure B.1: A detailed vector diagram of the binary system and its components as well as the applied reference frames.

the $\widetilde{x}^3$ axis with uniform angular velocity Ω and that this rotational axis always remains perpendicular to the orbital plane of the binary. Therefore, axis $\widetilde{x}^3 = \bar{x}^3$ and remains fixed.

We are not interested in the detailed behaviour of the secondary, its only role is to produce a gravitational field which induces tidal distortions on the primary. We may therefore assume that the secondary be considered as a mass point, with its position denoted by C_2, that moves in a Keplerian orbit around the binary's centre of mass. This means that the orbit of the secondary around the centre of mass of the primary is also a Keplerian orbit. Let vector $\vec{\xi} = d\,\hat{\xi}$ represent the position of an arbitrary fluid element p relative to the mass point C_2.

B.3 Relation Between the Position Vectors

Consider an infinitesimal fluid element p situated anywhere within the volume or on the surface of the primary. This element is modelled as a point particle. Suppose there are two observers, one at rest at the origin C_B of the inertial frame, the other at rest at the origin C_1 of the accelerated frame. We will refer to these two observers respectively as "the observer I" and "the observer $\widetilde{Q}$".

Suppose that both observers observe the same particle p in motion. Observer I assigns coordinates x_3^i to the position of the particle and constructs its position vector $\vec{x}_3$ with respect to his origin C_B according to the formula

$$\vec{x}_3 = x_3^1\vec{E}_1 + x_3^2\vec{E}_2 + x_3^3\vec{E}_3 = x_3^i\vec{E}_i.$$

Using the classical definition of velocity as the rate of change of particle coordinates, observer I measures velocity components

$$u^i = \frac{dx_3^i}{dt}$$

for the particle, and constructs its velocity vector using the formula

$$\vec{u} = u^1\vec{E}_1 + u^2\vec{E}_2 + u^3\vec{E}_3 = u^i\vec{E}_i = \frac{dx_3^i}{dt}\vec{E}_i.$$

Similarly, he measures the components of the acceleration of the particle to be

$$A^i = \frac{du^i}{dt} = \frac{d^2x_3^i}{dt^2},$$

and constructs its acceleration vector from the formula

$$\vec{A} = A^1\vec{E}_1 + A^2\vec{E}_2 + A^3\vec{E}_3 = A^i\vec{E}_i = \frac{d^2x_3^i}{dt^2}\vec{E}_i.$$

Observer $\widetilde{Q}$ observes the same particle at the same instant t, but he assigns to it a different set of coordinates, $\widetilde{x}^{\mu}$[2]. He constructs its position vector $\vec{\widetilde{x}}$ with

[2]We could use Latin indices for both observers, but the calculation becomes more transparent if we use different types of indices for components taken relative to different reference frames. We will use Latin indices for coordinates and components relative to the inertial frame I, and Greek indices for those relative to the accelerated frame $\widetilde{Q}$

respect to C_1 from his coordinates using the formula

$$\vec{\widetilde{x}} = \widetilde{x}^1\vec{e}_1 + \widetilde{x}^2\vec{e}_2 + \widetilde{x}^3\vec{e}_3 = \widetilde{x}^\mu\vec{e}_\mu,$$

measures its velocity to have components

$$v^\mu = \frac{d\widetilde{x}^\mu}{dt},$$

constructs its velocity vector according to the formula

$$\vec{v} = v^1\vec{e}_1 + v^2\vec{e}_2 + v^3\vec{e}_3 = v^\mu\vec{e}_\mu = \frac{d\widetilde{x}^\mu}{dt}\vec{e}_\mu.$$

Similarly, he measures the components of its acceleration to be

$$a^\mu = \frac{dv^\mu}{dt} = \frac{d^2\widetilde{x}^\mu}{dt^2},$$

and constructs its acceleration vector according to the formula

$$\vec{a} = a^1\vec{e}_1 + a^2\vec{e}_2 + a^3\vec{e}_3 = a^\mu\vec{e}_\mu = \frac{d^2\widetilde{x}^\mu}{dt^2}\vec{e}_\mu.$$

The relation between the kinematic parameters measured by these two observers follows from the equation that relates their respective position vectors

$$\vec{x}_3 = \vec{x}_1 + \vec{\widetilde{x}}, \tag{B.1}$$

where the vector $\vec{x}_1 = \overrightarrow{C_BC_1}$ is the position of the origin C_1 of the accelerated frame relative to the origin C_B of the inertial frame. Expressing it in terms of the frame I, it is given by

$$\vec{x}_1 = x_1^i\vec{E}_i,$$

where the x_1^i are the coordinates assigned to C_1 by the observer I.

Relation (B.1) is easily deduced from Figure B.1.

Using the expressions given above for the respective position vectors, relation (B.1) becomes

$$x_3^i\vec{E}_i = x_1^i\vec{E}_i + \widetilde{x}^\mu\vec{e}_\mu. \tag{B.2}$$

B.4 Relation Between the Velocities

To deduce the relation between the velocities assigned by the two observers to the particle p, we differentiate equation (B.2) with respect to time. This gives,

$$\frac{dx_3^i}{dt}\vec{E}_i + x_3^i\frac{d\vec{E}_i}{dt} = \frac{dx_1^i}{dt}\vec{E}_i + x_1^i\frac{d\vec{E}_i}{dt} + \frac{d\widetilde{x}^\mu}{dt}\vec{e}_\mu + \widetilde{x}^\mu\frac{d\vec{e}_\mu}{dt}. \tag{B.3}$$

Now, the vectors E_i are the frame vectors of an inertial frame. They therefore change neither length nor direction in time and are therefore constant vectors, so we have

$$\frac{d\vec{E}_i}{dt} = 0.$$

The vectors e_μ, on the other hand, are the frame vectors of an accelerating frame. In general, therefore, they will change with time and their derivatives are not zero in general.

Now, the rate of change of a vector with respect to time is also a vector, and so can be expressed as a linear combination of a basis. The coefficients of the expansion measure the rate of change as seen by an observer whose frame has the given basis as its frame vectors. We are interested in the motion of the particle as seen by $\widetilde{Q}$. We therefore expand the $d\vec{e}_\mu/dt$ in terms of the frame vectors e_μ of $\widetilde{Q}$. This gives

$$\frac{d\vec{e}_\mu}{dt} = \omega^\nu{}_\mu \, \vec{e}_\nu.$$

We call these the *frame equations*.

What is the physical meaning of the frame equations? Imagine an old TV screen whose relaxation time is longer than the image refresh time. The old image thus persists for a short while after the new image has appeared. If we view the frame $\widetilde{Q}$ on such a screen, after time Δt the original positions of the axes would be visible at the same time as the new. We could then calculate the components of the new frame vectors relative to the old. Dividing these by Δt and taking the limit $\Delta t \to 0$ produces the quantities $\omega^\nu{}_\mu$ of the above equation. In this sense, the above equations are "self referential".

There is a second way to think about this. At time t, use the frame vectors $e_\mu(t)$ to construct an inertial frame with origin at the position of the origin C_B and axes in the directions defined by the $e_\mu(t)$. This inertial frame does not rotate, and is called the *instantaneous rest frame* of the system at instant t. As the vectors e_μ continue in their motion, we can measure their rotation rate relative to the instantaneous rest frame. At every instant, we repeat this construction. This is the physical content of the frame equations.

The frame $\widetilde{Q}$ is orthonormal. Its frame vectors e_μ must therefore always satisfy the orthonormality conditions

$$\vec{e}_\mu \cdot \vec{e}_\nu = \delta_{\mu\nu}.$$

Now, the quantities $\delta_{\mu\nu}$ are the components of the metric tensor in an orthonormal frame. For clarity, denote them by $g_{\mu\nu}$. Here, the $g_{\mu\nu}$ have constant values $\delta_{\mu\nu}$. The orthonormality condition can thus be written as

$$\vec{e}_\mu \cdot \vec{e}_\nu = g_{\mu\nu}.$$

Differentiating with respect to time, we get

$$\begin{aligned} 0 &= \dot{\vec{e}}_\mu \cdot \vec{e}_\nu + \vec{e}_\mu \cdot \dot{\vec{e}}_\nu \\ &= \omega^\alpha{}_\mu \, (\vec{e}_\alpha \cdot \vec{e}_\nu) + \vec{e}_\mu \cdot (\omega^\beta{}_\nu \, \vec{e}_\beta) \\ &= \omega^\alpha{}_\mu \, g_{\alpha\nu} + \omega^\beta{}_\nu \, g_{\mu\beta}. \end{aligned}$$

Using the metric tensor to lower indices, this relation becomes

$$\omega_{\nu\mu} + \omega_{\mu\nu} = 0.$$

The fully covariant tensors $\omega_{\mu\nu}$ are thus antisymmetric. This means that there are only three independent components out of the nine.

The orthonormality of the frame vectors means that the only change that they can undergo is rotation. The coefficients $\omega^{\nu}{}_{\mu}$ therefore measure the rotation rate of the frame $\widetilde{Q}$. Now, a rotation requires three parameters to specify it. A rotation rate is thus also specified by three parameters. This is consistent with the above result that only three of the nine components of $\omega_{\mu\nu}$ are independent.

It is customary to choose the quantities $\omega_{23}, \omega_{31}, \omega_{12}$ as the independent parameters. However, it is better at this stage to continue to work with all nine quantities $\omega_{\mu\nu}$ and refrain from making a choice until later.

Return now to equation (B.3). Incorporating the above information, this equation becomes

$$\dot{x}^i_3 \, \vec{E}_i = \dot{x}^i_1 \, \vec{E}_i + \dot{\widetilde{x}}^{\mu} \, \vec{e}_{\mu} + \widetilde{x}^{\mu} \, \omega^{\alpha}{}_{\mu} \, \vec{e}_{\alpha}. \tag{B.4}$$

B.5 Angular Velocity

The quantities $\omega_{\mu\nu}$ are rotation rates. They must therefore be related to the angular velocity of the frame $\widetilde{Q}$. To find this relationship, we first consider rotation of vectors about a fixed axis $\vec{n}$.

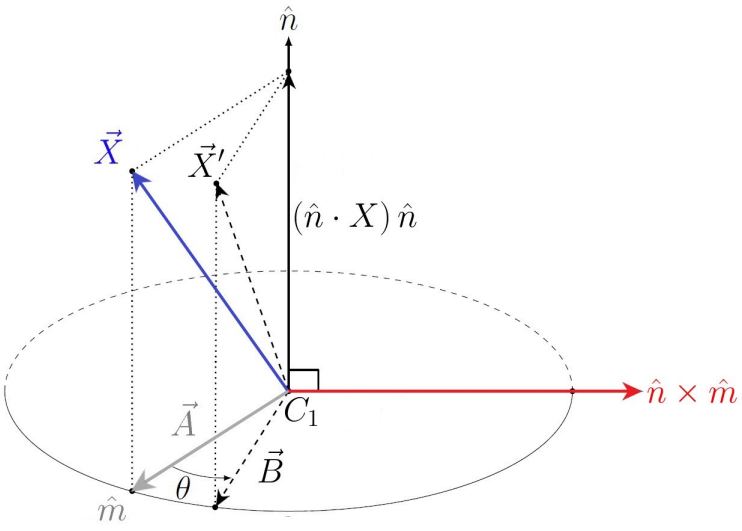


Figure B.2: Translation of the position of particle p to the origin of frame $\widetilde{Q}$.

Figure B.2 shows a vector $\vec{X}$ being rotated by angle θ around an axis defined by an unit vector $\hat{n}$. The angle θ is measured in the plane perpendicular to the axis of rotation.

Write $\vec{X}$ as a sum of components parallel to the axis $\vec{n}$ and perpendicular to it,

$$\vec{X} = (\hat{n} \cdot \vec{X})\,\hat{n} + \vec{A},$$

where $\vec{A}$ is the component of $\vec{X}$ perpendicular to $\hat{n}$.

Define a frame of reference with unit vectors $\hat{n}$, $\hat{m}$ and $\hat{n} \times \hat{m}$, where $\hat{m}$ is the unit vector in the direction of $\vec{A}$. Rotating $\vec{X}$ about $\hat{n}$ by angle θ is then equivalent to rotating $\vec{A}$ in the plane spanned by $\hat{m}$ and $\hat{n} \times \hat{m}$ into position $\vec{B}$ while leaving the axial component of $\vec{X}$ unchanged. Vector $\vec{B}$ is thus given by

$$\begin{aligned}\vec{B} &= A\,\cos\theta\;\hat{m} + A \sin\theta\;\hat{n} \times \hat{m} \\ &= \cos\theta\;\vec{A} + \sin\theta\;\hat{n} \times \vec{A}.\end{aligned}$$

We further eliminate $\vec{A}$ to express $\vec{B}$ in terms of the data for this problem,

$$\begin{aligned}\vec{B}\cos\theta\left[\vec{X} - (\hat{n} \cdot \vec{X})\,\hat{n}\right] + \sin\theta\;\hat{n} \times \left[\vec{X} - (\hat{n} \cdot \vec{X})\,\hat{n}\right] \\ = \cos\theta\left[\vec{X} - (\hat{n} \cdot \vec{X})\,\hat{n}\right] + \sin\theta\;\hat{n} \times \vec{X}.\end{aligned}$$

The rotated vector $\vec{X}'$ is given by

$$\vec{X}' = (\hat{n} \cdot \vec{X})\,\hat{n} + \vec{B}$$

or

$$\vec{X}' = \vec{X} + (1 - \cos\theta)\,(\hat{n} \cdot \vec{X})\,\hat{n} + \sin\theta\;\hat{n} \times \vec{X}.$$

To calculate the rate at which rotation occurs, consider a rotation by infinitesimal angle $\delta\theta$ and suppose that this rotation occurs in time δt. Then, to first order,

$$\vec{X}' \;\;=\;\; \vec{X} + \delta\theta\;\hat{n} \times \vec{X}.$$

Note that any change of direction of the axis or rotation would change $\hat{n}$ into $\hat{n} + \delta\hat{n}$, which gives rise to a second order term, and so vanishes in the limit as $\delta t \to 0$. We thus get

$$\frac{d\vec{X}}{dt} = \lim_{\delta t \to 0} \frac{\vec{X}' - \vec{X}}{\delta t} = \frac{d\theta}{dt}\;\hat{n} \times \vec{X}.$$

The rate of rotation of the vector $\vec{X}$ is thus given by $d\theta/dt$, which we denote by ω and call the angular velocity of rotation about the axis $\hat{n}$.

It is convenient to combine the scalar ω with the unit vector $\hat{n}$ into a single vector quantity which is called the *angular velocity vector* $\vec{\omega}$,

$$\vec{\omega} \;\;=\;\; \omega\,\hat{n}.$$

The formula for the rotation rate of the vector $\vec{X}$ then becomes

$$\frac{d\vec{X}}{dt} = \vec{\omega} \times \vec{X}$$

Expressed in terms of index notation in any given frame of reference, this equation becomes

$$\frac{dX^i}{dt} = \varepsilon^{ijk}\,\omega_j\,X_k.$$

Defining

$$\omega^{ik} = \varepsilon^{ijk}\,\omega_j = -\varepsilon^{ikj}\,\omega_j,$$

the rotation rate formula can be written in the form

$$\frac{dX^i}{dt} = -\,\omega^{ik}\,X_k,$$

or

$$\frac{dX^i}{dt} = -\,\omega^i{}_k\,X^k.$$

The ω^{ik}, or their mixed version $\omega^i{}_k$, are called the components of the angular velocity tensor.

Note that the formula relating the ω^{ik} to the ω^i can be inverted to give

$$\omega^i \;\;=\;\; -\frac{1}{2}\,\varepsilon^{ijk}\,\omega_{jk}.$$

These results enable us to interpret the rotation rate parameters $\omega^{\mu\nu}$ defined in previous sections in terms of the familiar angular velocity of elementary mechanics. Each frame vector e_μ of the accelerating frame Q is rotating relative to any instantaneous position that the frame assumes. Thus each frame vector e_μ is rotating around an instantaneous axis of rotation $\hat{n}$.

The axis $\hat{n}$ of rotation of the frame can be found by noting that

$$\omega^{\mu\nu} = \varepsilon^{\mu\nu\sigma}\,\omega_\sigma = \omega\,\varepsilon^{\mu\nu\sigma}\,n_\sigma,$$

so that

$$\omega^{\mu\nu}\,n_\nu = \omega\,\varepsilon^{\mu\nu\sigma}\,n_\sigma\,n_\nu = 0.$$

It is easy to show that the equation

$$\omega^{\mu\nu}\,n_\nu = 0$$

admits a solution that is unique up to a sign for an arbitrary non-zero factor and $\hat{n}$. This choice of sign is equivalent to choosing the direction of rotation that we regard as positive.

The above results show that the frame equations can be re-expressed in the form

$$\frac{d\vec{e}_\mu}{dt} = \vec{\omega} \times \vec{e}_\mu.$$

Note that the frame equations are a more natural way to express rotation rates than is the concept of angular velocity. This is shown by the fact that considering the rate of rotation of a rotating frame led to them directly. More importantly, the frame equations apply to rotating frames in a space of any dimension. The concept of angular velocity, on the other hand, does not generalise to dimensions higher than three.

B.6 Physical Interpretation of Equation (B.4)

We return now to equation (B.4),

$$\dot{x}_3^i\,\vec{E}_i = \dot{x}_1^i\,\vec{E}_i + \dot{\widetilde{x}}^\mu\,\vec{e}_\mu + \widetilde{x}^\mu\,\omega^\alpha{}_\mu\,\vec{e}_\alpha.$$

With the result of the previous section, we can write this equation in terms of vector notation,

$$\vec{u} = \vec{V} + \vec{v} + \vec{\omega}\times\vec{\widetilde{x}}, \tag{B.5}$$

where we have written

$$\vec{v} = \frac{dx_3^i}{dt}\vec{E}_i, \qquad \vec{V} = \frac{dx_1^i}{dt}\vec{E}_i, \qquad \vec{u} = \frac{d\widetilde{x}^\mu}{dt}\vec{e}_\mu, \qquad \vec{\omega} = \omega^\mu\,\vec{e}_\mu, \qquad \vec{\widetilde{x}} = \widetilde{x}^\mu\,\vec{e}_\mu.$$

Physically, $\vec{u}$ is the velocity of particle p as measured by the inertial observer, $\vec{V}$ is the velocity of the origin C_1 of the accelerated frame $\widetilde{Q}$ as measured by the inertial observer, $\vec{v}$ is the velocity of the particle as measured by the accelerated observer $\widetilde{Q}$, $\vec{\omega}$ is the angular velocity of the frame $\widetilde{Q}$, and $\vec{\widetilde{x}}$ is the position vector of the particle relative to the accelerated frame $\widetilde{Q}$. Both observers use the same vector space to describe vector quantities, but use different bases to work out components. They both agree therefore on vector quantities. In particular, they agree on the angular velocity vector ω, even though they disagree on its components.

B.7 Relation Between Accelerations

To obtain the equation that relates the accelerations, differentiate equation (B.4) that relates the velocities. Taking into account the previously deduced results, this gives

$$\ddot{x}_3^i\,\vec{E}_i = \ddot{x}_1^i\,\vec{E}_i + \ddot{\widetilde{x}}^\mu\,\vec{e}_\mu + \dot{\widetilde{x}}^\mu\dot{\vec{e}}_\mu + \dot{\widetilde{x}}^\mu\,\omega^\alpha{}_\mu\,\vec{e}_\alpha + \widetilde{x}^\mu\,\dot{\omega}^\alpha{}_\mu\,\vec{e}_\alpha + \widetilde{x}^\mu\,\omega^\alpha{}_\mu\dot{\vec{e}}_\alpha,$$

or

$$\ddot{x}_3^i\,\vec{E}_i = \ddot{x}_1^i\,\vec{E}_i + \ddot{\widetilde{x}}^\mu\,\vec{e}_\mu + 2\,\dot{\widetilde{x}}^\mu\,\omega^\alpha{}_\mu\,\vec{e}_\alpha + \widetilde{x}^\mu\,\dot{\omega}^\alpha{}_\mu\,\vec{e}_\alpha + \widetilde{x}^\mu\,\omega^\alpha{}_\mu\,\omega^\beta{}_\alpha\,\vec{e}_\beta.$$

The term on the left hand side is the vector acceleration $\vec{A}$ of particle p as measured by an observer at rest in the inertial frame I. The first and second terms on the right hand side are respectively the vector acceleration $\ddot{\vec{x}}_1$ of the origin C_1 of the accelerated frame $\widetilde{Q}$ as measured by an observer at rest in the inertial frame I, and the vector acceleration $\vec{a}$ of the particle as measured by an observer at rest in the accelerated frame $\widetilde{Q}$. We have already shown above that the third term on the right hand side can be written in coordinate-free vector form as $2\,\vec{\omega}\times\vec{\widetilde{x}}$. The fourth term can be expressed in a similar way as $\dot{\vec{\omega}}\times\vec{\widetilde{x}}$, while the last term can be expressed in vector form as

$$\begin{aligned}(\omega^{\beta\alpha}\,\omega_{\alpha\mu}\,\widetilde{x}^\mu)\,\vec{e}_\beta &= (\varepsilon^{\beta\alpha\sigma}\,\omega_\sigma\,\varepsilon_{\alpha\mu\tau}\,\omega^\tau\,\widetilde{x}^\mu\,)\,\vec{e}_\beta\\ &= (-\delta^{\beta\sigma}_{\mu\tau}\,\omega_\sigma\,\omega^\tau\,\widetilde{x}^\mu\,)\,\vec{e}_\beta\\ &= -\omega_\sigma\,(\omega^\beta\widetilde{x}^\sigma - \omega^\sigma\,\widetilde{x}^\beta\,)\,\vec{e}_\beta\end{aligned}$$

$$= -(\vec{\omega}\cdot\vec{\tilde{x}}\,)\,\vec{\omega} + (\vec{\omega}\cdot\vec{\omega}\,)\,\vec{\tilde{x}}$$

which is recognised through the BAC-CAB formula as the vector triple product

$$\omega^{\beta\alpha}\,\omega_{\alpha\mu}\,\tilde{x}^{\mu}\,\vec{e}_{\beta} = \vec{\omega}\times(\vec{\omega}\times\vec{\tilde{x}}\,).$$

In coordinate-free notation therefore, the accelerations of the particle measured in frames I and Q are related according to

$$\vec{A} = \ddot{\vec{x}}_1 + \vec{a} + 2\,\vec{\omega}\times\dot{\vec{\tilde{x}}} + \dot{\vec{\omega}}\times\vec{\tilde{x}} + \vec{\omega}\times(\vec{\omega}\times\vec{\tilde{x}}\,).$$

This is the equation that relates the particle accelerations as measured by observers I and $\widetilde{Q}$: the inertial observer measures the acceleration of the particle to be $\vec{A}$; the accelerated observer measures it to be $\vec{a}$; $\ddot{\vec{x}}_1$ is the acceleration of the origin C_1 of the frame $\widetilde{Q}$ as measured by I; $\vec{\omega}$ is the angular velocity of the accelerated frame; $\dot{\vec{\omega}}$ is the angular acceleration of the accelerated frame; and $\vec{x}$ is the position vector of the particle with respect to the accelerated frame.

B.8 Equation of Motion in the Accelerated Frame

Denote the mass of the particle p by m. Then Newton's second law gives

$$\vec{F} = m\,\vec{A},$$

where $\vec{F}$ is the total force that acts on the particle. This total force produces an acceleration $\vec{A}$ in the inertial frame I.

The acceleration of the particle observed in the accelerated frame $\widetilde{Q}$ is $\vec{a}$, and is related to the applied force by the equation

$$\vec{F} = m\,\ddot{\vec{x}}_1 + m\,\vec{a} + 2\,m\,\vec{\omega}\times\dot{\vec{\tilde{x}}} + m\,\dot{\vec{\omega}}\times\vec{\tilde{x}} + m\,\vec{\omega}\times(\vec{\omega}\times\vec{\tilde{x}}\,).$$

If an observer in the accelerated frame is unaware of the fact that he is accelerating, he would write this equation in the form

$$m\,\vec{a} = \vec{F} - m\,\ddot{\vec{x}}_1 - 2\,m\,\vec{\omega}\times\dot{\vec{\tilde{x}}} - m\,\dot{\vec{\omega}}\times\vec{\tilde{x}} - m\,\vec{\omega}\times(\vec{\omega}\times\vec{\tilde{x}}\,),$$

and he would interpret the terms on the right hand side of this equation as forces that act on the particle. This means that he will misinterpret the additional terms produced by the acceleration of his frame as physical forces that act on the particle. The true forces are those measured by an observer in the inertial frame, here represented by $\vec{F}$. The additional terms are not forces at all, but correction terms to take into account the acceleration of the frame $\widetilde{Q}$.

The additional terms are often (incorrectly) called *forces*. A better name for them might be *pseudo forces*, because they are not forces at all so that to call them "forces" is to lie (the meaning of the word *pseudo*). They might also be called *inertial forces*, since the effects that they describe arise not because of the action of actual forces on the particle, but because the inertia of the particle drives its motion relative to the accelerated frame.

The inertial forces $-m\,\ddot{\vec{x}}_1$ and $-m\,\dot{\vec{\omega}} \times \vec{\tilde{x}}$ have no name. They arise respectively because of the acceleration of the origin C_1 of the accelerated frame, and because of its angular acceleration as it rotates.

The terms $-2\,m\,\vec{\omega} \times \dot{\vec{\tilde{x}}}$ and $-m\,\vec{\omega} \times (\vec{\omega} \times \vec{\tilde{x}})$, on the other hand, are called respectively the Coriolis "force" and the centrifugal "force" respectively.

B.9 Frame $\widetilde{Q}$ in Spherical Coordinates

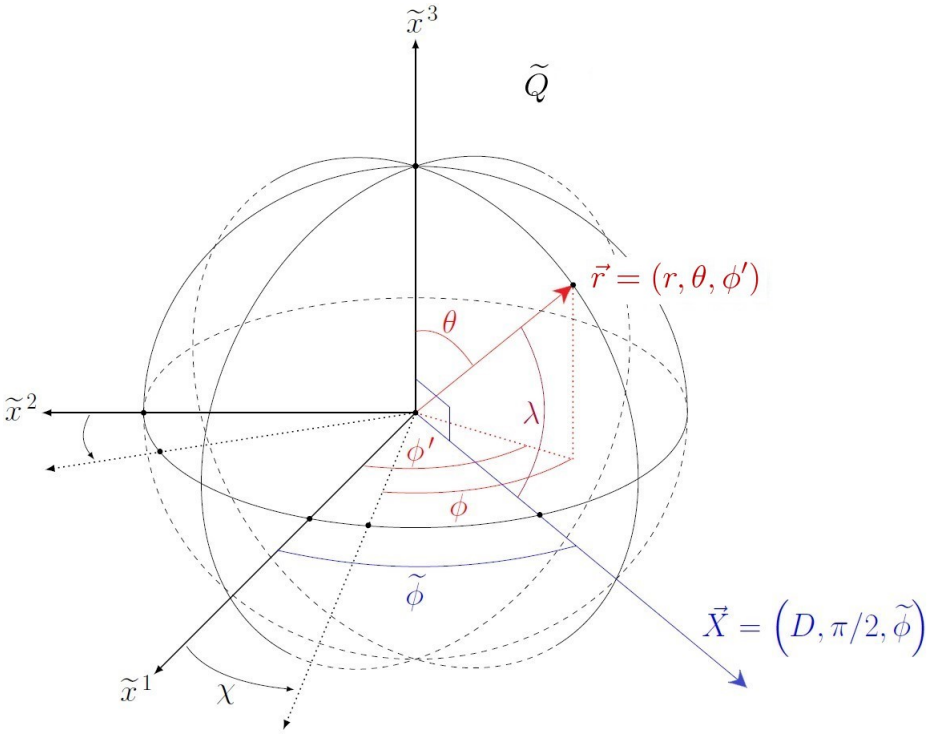


Figure B.3: A detailed vector diagram describing the positions of particle p and the secondary evaluated in the co-rotating reference frame $\widetilde{Q}$.

We can define spherical coordinates for the co-rotating reference frame $\widetilde{Q}$, with respect to its frame vectors e_μ, as (r, θ, ϕ) with centre C_1, polar axis $\widetilde{x}^3$ and x-axis $\widetilde{x}^1$. An observer in frame $\widetilde{Q}$ assigns coordinates $\widetilde{r}^\mu$ to the position of the particle p and constructs its position vector $\vec{r}$ with respect to his origin C_1 according to the formula

$$\vec{r} = r^\mu\,\vec{e}_\mu = r\,\vec{e}_1 + \theta\,\vec{e}_2 + \phi'\,\vec{e}_3.$$

He also assigns coordinates X^μ to the position of the secondary and constructs its position vector $\vec{X}$ with respect to his origin C_1 according to the formula

$$\vec{X} = X^\mu\,\vec{e}_\mu = D\,\vec{e}_1 + \widetilde{\theta}\,\vec{e}_2 + \widetilde{\phi}\,\vec{e}_3.$$

Since vector $\vec{X}$ will always lie on the $\widetilde{x}^1$ - $\widetilde{x}^2$ plane, its polar angle $\widetilde{\theta}$ is fixed with a value of $\pi/2$ (see Figure B.3). Therefore

$$\vec{X} = D\,\vec{e}_1 + \frac{\pi}{2}\,\vec{e}_2 + \widetilde{\phi}\,\vec{e}_3.$$

Let time t_0 be the time measured at the instance when vector $\vec{X}$ points towards the periastron of the orbit of the secondary, as well as the instance when the axes $\widetilde{x}^{\,i}$ of frame $\widetilde{Q}$ are parallel to the inertial axes $\bar{x}^i$ of frame $\overline{Q}$. Let time t_f be a later time measured after axis $\widetilde{x}^{\,1}$ has rotated by an angle of $\mathcal{X}$ and vector $\vec{X}$ has displaced itself, being no longer oriented in the direction of the periastron. ϕ is then the azimuthal angle of particle p measured from axis $\widetilde{x}^{\,1}$ after this time has elapsed and so $\mathcal{X} = \phi' - \phi$. By definition, from Figure B.3, the rotational angular velocity of the primary is

$$\omega = \frac{\mathcal{X}}{\delta t},$$

where $\delta t = t_f - t_0$. Therefore

$$\phi' = \phi + \omega\,(t_f - t_0)\,. \tag{B.6}$$

Appendix C

The Legendre Polynomials

An equation commonly encountered in mathematical physics when dealing with partial differential equations in spherical coordinates is Legendre's differential equation. It is defined as

$$\left(1-x^2\right)\frac{d^2y}{dx^2}(x) - 2x\frac{dy}{dx}(x) + c\,y\,(x) = 0, \tag{C.1}$$

where c is an arbitrary constant. The most useful solutions to the above equation are the Legendre Polynomials and can easily be found by assuming a power series solution of the following form

$$y\,(x) = \sum_{n=0}^{\infty} a_n\,x^n.$$

Substituting this series into equation (C.1) yields a recurrence relation for the coefficients a_n. The recurrence relation can be shown to be

$$a_{n+2} = \frac{n\,(n+1) - c}{(n+2)\,(n+1)}\,a_n. \tag{C.2}$$

Using the ratio test it is easily seen that the series is convergent for $-1 < x < 1$. In general the series is divergent at $x = \pm 1$. We are interested in the case where $x = \cos\theta$. This range includes the values $x = \pm 1$. To ensure that the solution is defined at these values, we must impose a condition that forces the series to terminate. The condition requires that all a_n be zero for all values of $n > n_{max}$, thus restricting the values of c to

$$n_{max}\,(n_{max}+1) - c = 0.$$

Conventionally $n_{max} = \ell$ and so

$$c = \ell\,(\ell+1).$$

This forces c to have an integer value. It can be shown that if ℓ is an even integer, then $y\,(x)$ is a polynomial of degree ℓ containing only even powers of x, and if ℓ is an odd integer, $y\,(x)$ is a polynomial of degree ℓ containing only odd powers. Thus equation (C.1) admits polynomial solutions that are defined in

the range $[-1, 1]$. From this set of solutions, by substituting $n = \ell - 2p$ where $p = 1, 2, 3, ..., \frac{\ell}{2}$, a general solution for $y(x)$ can be found to be

$$y(x) = c_\ell \frac{(\ell!)^2}{(2\ell)!} \sum_{p=0}^{\ell/2} \frac{(-1)^p}{p!} \frac{(2\ell - 2p)!}{(\ell - 2p)!\,(\ell - p)!} x^{\ell-2p}, \tag{C.3}$$

where c_ℓ is the normalisation coefficient. The Legendre polynomials can now be determined by implementing the normalisation condition $y(1) = 1$ giving

$$c_\ell = \frac{(2\ell)!}{2^\ell\,(\ell!)^2}.$$

Substituting this result into equation (C.3) gives

$$P_\ell(x) = \frac{1}{2^\ell} \sum_{p=0}^{\ell/2} \frac{(-1)^p}{p!} \frac{(2\ell - 2p)!}{(\ell - 2p)!\,(\ell - p)!} x^{\ell-2p}, \tag{C.4}$$

which is the standard definition for the Legendre polynomials.

Rodrigues' Formula

The Legendre polynomials can also be generated using Rodrigues' formula. It is obtained by using the following identity

$$\frac{d^s}{dx^s}(x^w) = \frac{w!}{(w-s)!} x^{w-s}.$$

Equation (C.4) can then be rewritten as

$$\begin{aligned} P_\ell(x) &= \frac{1}{2^\ell} \sum_{p=0}^{\ell/2} \frac{(-1)^p}{p!} \frac{1}{(\ell - p)!} \frac{d^\ell}{dx^\ell} x^{2\ell-2p} \\ &= \frac{1}{2^\ell\,\ell!} \frac{d^\ell}{dx^\ell} \sum_{p=0}^{[\ell/2]} \frac{(-1)^p}{p!} \frac{\ell!}{(\ell - p)!} (x^2)^{\ell-p}. \end{aligned}$$

We can continue the summation to ℓ since $\ell - p < \ell$ for terms above $\frac{\ell}{2}$. As a result when applying the derivative $\frac{d^\ell}{dx^\ell}$ to these terms they will all return zero. We then have

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} \sum_{p=0}^{\ell} \frac{(-1)^p}{p!} \frac{\ell!}{(\ell - p)!} (x^2)^{\ell-p}.$$

The form of the summation is similar to the definition of the binomial expansion defined in equation

$$(x^2 - 1)^\ell = \sum_{p=0}^{\ell} \frac{(-1)^p}{p!} \frac{\ell!}{(\ell - p)!} (x^2)^{\ell-p},$$

and so we have

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2 - 1)^\ell, \tag{C.5}$$

which is Rodrigues' formula.

Properties of the Legendre Polynomials

The most important properties of the Legendre polynomials, namely orthogonality and normalisation, are collectively described by the equation

$$\int_{-1}^{1} P_\ell\left(x\right) P_m\left(x\right) dx = \frac{2}{2\ell+1}\,\delta_{\ell m},$$

where both the orthogonality (if $\ell \neq m$)

$$\int_{-1}^{1} P_\ell\left(x\right) P_m\left(x\right) dx = 0$$

and normalisation (if $\ell = m$)

$$\int_{-1}^{1} \Big(P_\ell\left(x\right)\Big)^2 dx = \frac{2}{2\ell+1}$$

conditions are accounted for.

C.1 The Legendre Polynomial Generating Function

The Legendre polynomials P_ℓ can also be defined as the coefficients in a Taylor series expansion. To show this we begin by expanding the term $\left(1-y\right)^{-1/2}$ in a Taylor series such that

$$\begin{aligned}\left(1-y\right)^{-1/2} &= 1 + \left(\frac{1}{2}\right) y + \left(\frac{1}{2}\right)\left(\frac{3}{4}\right) y^2 + \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right) y^3 \\ &\qquad + \,...\, + \frac{1}{2^{2n}}\frac{(2n)!}{\left(n!\right)^2}\, y^n \\ &= \sum_{n=0}^{\infty} \frac{1}{2^{2n}}\frac{(2n)!}{\left(n!\right)^2}\, y^n.\end{aligned}$$

Let $y = t\left(2x-t\right)$, using the above expansion we then have

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} \frac{t^n}{2^{2n}}\frac{(2n)!}{\left(n!\right)^2}\,\left(2x-t\right)^n. \tag{C.6}$$

Let c be an arbitrary integer, equation (C.6) can then be expressed as

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty}\sum_{c=0}^{n} \frac{t^{n-c}}{2^{2n-2c}}\,\frac{(2n-2c)!}{\Big(\left(n-c\right)!\Big)^2}\,\left(2x-t\right)^{n-c}. \tag{C.7}$$

Expanding $\left(2x-t\right)^{n-c}$ using the binomial theorem gives

$$\left(2x-t\right)^{n-c} = \sum_{c=0}^{n-c} \frac{\left(-t\right)^c}{c!}\frac{\left(n-c\right)!}{\left(n-2c\right)!}\left(2x\right)^{n-2c},$$

which for a specific c-th term is then

$$(2x-t)^{n-c} = \frac{(-t)^c}{c!}\frac{(n-c)!}{(n-2c)!}(2x)^{n-2c}. \tag{C.8}$$

Substituting equation (C.8) into (C.7) gives

$$\begin{aligned}\frac{1}{\sqrt{1-2xt+t^2}} &= \sum_{n=0}^{\infty}\sum_{c=0}^{n}\frac{t^{n-c}}{2^{2n-2c}}\frac{(2n-2c)!}{\Big((n-c)!\Big)^2}\frac{(-t)^c}{c!} \\ &\qquad\times\frac{(n-c)!}{(n-2c)!}(2x)^{n-2c} \\ &= \sum_{n=0}^{\infty}\sum_{c=0}^{n} t^n \\ &\quad\times\frac{(-1)^c}{c!}\frac{2^{n-2c}}{2^{2n-2c}}\frac{(2n-2c)!}{(n-c)!\,(n-2c)!}x^{n-2k} \\ &= \sum_{n=0}^{\infty} t^n \\ &\quad\times\left[\frac{1}{2^n}\sum_{c=0}^{n}\frac{(-1)^c}{c!}\frac{(2n-2c)!}{(n-c)!\,(n-2c)!}x^{n-2c}\right].\end{aligned} \tag{C.9}$$

Since t^n and $1/2^n$ have indices which contain only n they can be moved outside of the c-summation.

The term in the square brackets of equation (C.9) looks very similar to equation (C.4) for the Legendre polynomials with c and n in place of p and ℓ. All that we are missing is the condition that equation (C.9) cannot contain powers of x in its first term at $c = n$. We therefore require that at $c = n$, x^{n-2k} becomes $x^0 = 1$ implying that $c = n/2$. Equation (C.9) is then

$$\begin{aligned}\frac{1}{\sqrt{1-2xt+t^2}} &= \sum_{n=0}^{\infty} t^n \\ &\quad\times\left[\frac{1}{2^n}\sum_{c=0}^{n/2}\frac{(-1)^c}{c!}\frac{(2n-2c)!}{(n-c)!\,(n-2c)!}(x)^{n-2c}\right]\end{aligned}$$

and by substituting equation (C.4) for the Legendre polynomials $P_n(x)$ we have

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty}(t)^n P_n(x). \tag{C.10}$$

The term on the LHS of equation (C.10) is commonly known as the generating function for the Legendre Polynomials.

Appendix D

The Associated Legendre Polynomials

Another differential equation[1] commonly encountered in mathematical physics is

$$\left(1-x^2\right)\frac{d^2y}{dx^2}(x)-2x\frac{dy}{dx}(x)+\left(c-\frac{m^2}{1-x^2}\right)y\left(x\right)=0, \tag{D.1}$$

with $m^2 \leq \ell^2$. There are many ways to find solutions to this equation, however the most useful solutions are the ones related to the Legendre polynomials. By substituting

$$y\left(x\right)=\left(1-x^2\right)^{m/2}u\left(x\right),$$

into equation (D.1) it can be shown that we get

$$\left(1-x^2\right)\frac{d^2u}{dx^2}(x)-2x\left(m+1\right)\frac{du}{dx}(x)+\Big(c-m(m+1)\Big)u\left(x\right)=0. \tag{D.2}$$

If $m=0$ it can easily be seen that, for $c=\ell(\ell+1)$, the above equation reduces to Legendre's differential equation which admits solutions $P_\ell\left(x\right)$.

By differentiating (D.2) we get

$$\begin{aligned}0=\left(1-x^2\right)\frac{d^2}{dx^2}u'\left(x\right)-2x\Big((m+1)+1\Big)\frac{d}{dx}u'\left(x\right)\\+\Big(c-(m+1)(m+2)\Big)\frac{du}{dx}(x),\end{aligned}$$

which is just equation (D.2) with $m+1$ in place of m and $u'(x)$ in place of $u\left(x\right)$.

Therefore if equation (D.2) admits solutions $P_\ell(x)$ for $m=0$ it must also admit solutions

$$\frac{dP_\ell}{dx}\left(x\right)\ for\ m=1,$$

[1] closely related to the Legendre differential equation

$$\frac{d^2 P_\ell}{dx^2}(x) \; for \; m = 2,$$

$$\vdots$$

$$\frac{d^m P_\ell}{dx^m}(x) \; for \; 0 \leq m \leq \ell. \tag{D.3}$$

The limit $0 \leq m \leq \ell$ is important because the degree ℓ of the Legendre polynomial is directly related to the degree of the polynomial it represents, if $m > \ell$ then

$$\frac{d^m P_\ell}{dx^m}(x) = 0.$$

Therefore the general derivative of the Legendre polynomials of degree ℓ, namely (D.3) is a solution to equation (D.2) and we have

$$\begin{aligned} y(x) &= \left(1 - x^2\right)^{m/2} u(x) \\ &= \left(1 - x^2\right)^{m/2} \frac{d^m P_\ell}{dx^m}(x) \\ &= P_\ell^m(x). \end{aligned}$$

These solutions are termed the Associated Legendre Polynomials.

Using Rodrigues' formula[2] the associated Legendre polynomials can be rewritten to give

$$\begin{aligned} P_\ell^m(x) &= \left(1 - x^2\right)^{m/2} \frac{d^m}{dx^m}\left(\frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}\left(x^2 - 1\right)^\ell\right) \\ &= \frac{\left(1 - x^2\right)^{m/2}}{2^\ell \ell!}\frac{d^{\ell+m}}{dx^{\ell+m}}\left(x^2 - 1\right)^\ell, \end{aligned}$$

which extends the limits of m to $-\ell \leq m \leq \ell$. Some authors include a factor of $(-1)^m$ in the definition of $P_\ell^m(x)$ to account for the parity of m.

Properties of the Associated Legendre Polynomials

The most important properties of the associated Legendre polynomials, namely orthogonality and normalisation, are collectively described by

$$\int_{-1}^{1} P_\ell^m(x)\, P_w^m(x)\, dx = \frac{2}{2\ell + 1}\frac{(\ell + m)!}{(\ell - m)!}\delta_{\ell w}, \tag{D.4}$$

where both the orthogonality (if $\ell \neq w$)

$$\int_{-1}^{1} P_\ell^m(x)\, P_w^m(x)\, dx = 0$$

and normalisation (if $\ell = w$)

$$\int_{-1}^{1} \left(P_\ell^m(x)\right)^2 dx = \frac{2}{2\ell + 1}\frac{(\ell + m)!}{(\ell - m)!}$$

conditions are accounted for.

[2] see equation (C.5)

Appendix E

Spherical Harmonics

Spherical Harmonics is the designation given to the angular portion of a set of solutions to Laplace's equation. In spherical coordinates, for a given scalar function $U(\vec{r}) = U(r, \theta, \phi)$, Laplace's equation is defined as

$$0 = \nabla^2 U = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial U}{\partial r}(\vec{r})\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial U}{\partial \theta}(\vec{r})\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 U}{\partial \phi^2}(\vec{r}). \tag{E.1}$$

One way of finding solutions to equation (E.1) is by using separation of variables. Let us assume a solution of the form $U = R(r)\,\Theta(\theta)\,\Phi(\phi)$[1]. Substituting it into equation (E.1) gives

$$\begin{aligned} 0 &= \frac{\Theta(\theta)\,\Phi(\phi)}{r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}(r)\right) + \frac{R(r)\,\Phi(\phi)}{r^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}(\theta)\right) \\ &\quad + \frac{R(r)\,\Theta(\theta)}{r^2 \sin^2\theta}\frac{d^2\Phi}{d\phi^2}(\phi) \qquad \text{(E.2)} \\ &= \frac{1}{R(r)}\frac{d}{dr}\left(r^2 \frac{dR}{dr}(r)\right) + \frac{1}{\Theta(\theta)\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}(\theta)\right) \\ &\quad + \frac{1}{\Phi(\phi)\sin^2\theta}\frac{d^2\Phi}{d\phi^2}(\phi) \\ &= \frac{\sin^2\theta}{R(r)}\frac{d}{dr}\left(r^2 \frac{dR}{dr}(r)\right) + \frac{\sin\theta}{\Theta(\theta)}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}(\theta)\right) \\ &\quad + \frac{1}{\Phi(\phi)}\frac{d^2\Phi}{d\phi^2}(\phi). \qquad \text{(E.3)} \end{aligned}$$

The third term in the above equation is the only term dependent on the variable ϕ, and the function Φ is dependent only on one variable. Therefore the third term in equation (E.3) is a constant such that

$$\frac{1}{\Phi(\phi)}\frac{d^2\Phi}{d\phi^2}(\phi) = c. \tag{E.4}$$

[1] This form does not represent the general solution to equation (E.1) but a specific set of solutions that make up part of the general solution. Because of the linearity of equation (E.1) a general solution can be constructed by combining this subset of solutions

The constant c is termed the separation constant and its form is subject to specific conditions.

In the case of a spherical coordinate system, with the variable ϕ used to describe the azimuthal angle, the periodicity of ϕ needs to be considered. For every chosen azimuthal angle ϕ, the azimuthal location of the exact same physical point can also be described by $\phi+2m\pi$ regardless of m, with $m \in \mathbb{Z}$. For this condition to be preserved the Φ solutions have to be sines and cosines instead of exponentials and must also be an integer to give a period of 2π. Therefore the constant c must have the following form

$$c = -m^2. \tag{E.5}$$

Where the negative ensures a complex exponential form and the square ensures m remains an integer. using (E.5) for c, equation (E.4) becomes

$$\frac{1}{\Phi(\phi)} \frac{d^2\Phi}{d\phi^2}(\phi) = -m^2,$$

which can very easily be shown to admit the following solutions

$$\Phi(\phi) = \begin{cases} e^{im\phi} \\ e^{-im\phi} \end{cases}. \tag{E.6}$$

Substituting equation (E.5) for c into (E.3) gives

$$\begin{aligned} 0 &= \frac{\sin^2\theta}{R(r)} \frac{d}{dr}\left(r^2 \frac{dR}{dr}(r)\right) + \frac{\sin\theta}{\Theta(\theta)} \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}(\theta)\right) - m^2 \\ &= \frac{1}{R(r)} \frac{d}{dr}\left(r^2 \frac{dR}{dr}(r)\right) + \frac{1}{\Theta(\theta)\sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}(\theta)\right) \\ &\qquad - \frac{m^2}{\sin^2\theta}. \end{aligned} \tag{E.7}$$

The first term in equation (E.7) is also a constant since it is the only term dependent on the variable r, and the function $R(r)$ is dependent only on one variable. Therefore, we have

$$\frac{1}{R(r)} \frac{d}{dr}\left(r^2 \frac{dR}{dr}(r)\right) = k \tag{E.8}$$

and equation (E.7) becomes

$$\begin{aligned} 0 &= k + \frac{1}{\Theta(\theta)\sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}(\theta)\right) - \frac{m^2}{\sin^2\theta} \\ &= \frac{1}{\sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}(\theta)\right) + \left(k - \frac{m^2}{\sin^2\theta}\right)\Theta(\theta). \end{aligned} \tag{E.9}$$

Let $x = \cos\theta$, we then have

$$d\theta = -\frac{1}{\sin\theta} dx.$$

Substituting this result into equation (E.9) we get

$$\begin{aligned}
0 &= \frac{d}{dx}\left(\sin^2\theta \frac{d\Theta}{dx}(x)\right) + \left(k - \frac{m^2}{\sin^2\theta}\right)\Theta(x) \\
&= \frac{d}{dx}\left(\left(1-\cos^2\theta\right)\frac{d\Theta}{dx}(x)\right) + \left(k - \frac{m^2}{\sin^2\theta}\right)\Theta(x) \\
&= \frac{d}{dx}\left(\frac{d\Theta}{dx}(x) - x^2\frac{d\Theta}{dx}(x)\right) + \left(k - \frac{m^2}{\sin^2\theta}\right)\Theta(x) \\
&= \frac{d^2\Theta}{dx^2}(x) - 2x\frac{d\Theta}{dx}(x) - x^2\frac{d^2\Theta}{dx^2}(x) + \left(k - \frac{m^2}{\sin^2\theta}\right)\Theta(x) \\
&= \left(1-x^2\right)\frac{d^2\Theta}{dx^2}(x) - 2x\frac{d\Theta}{dx}(x) + \left(k - \frac{m^2}{\sin^2\theta}\right)\Theta(x). \qquad \text{(E.10)}
\end{aligned}$$

When compared to equation (D.1), with $k = \ell(\ell+1)$, it is easy to see that equation (E.10) admits the associated Legendre polynomials as solutions. Therefore, the solutions to equation (E.9) are the associated Legendre polynomials with $x = \cos\theta$ such that

$$\begin{aligned}
\Theta(\theta) &= P_\ell^m(\cos\theta) \\
&= \left(1-\cos^2\theta\right)^{m/2}\frac{d^m}{dx^m}P_\ell(\cos\theta) \\
&= \left(\sin^2\theta\right)^{m/2}\frac{d^m}{dx^m}P_\ell(\cos\theta). \qquad \text{(E.11)}
\end{aligned}$$

As mentioned before, the spherical harmonics are defined as the angular part of $U(\vec{r})$ such that

$$\begin{aligned}
Y_\ell^m(\theta,\phi) &= N\,\Theta(\theta)\,\Phi(\phi) \\
&= \begin{cases} N\,P_\ell^m(\cos\theta)\,e^{im\phi} \\ N\,P_\ell^m(\cos\theta)\,e^{-im\phi} \end{cases}, \qquad \text{(E.12)}
\end{aligned}$$

where N is the normalisation constant and $0 \leq m \leq \ell$. To accommodate the extended limits $-\ell \leq m \leq \ell$, equation (E.12) is redefined as

$$Y_\ell^m(\theta,\phi) = N\,P_\ell^{|m|}(\cos\theta)\,e^{im\phi}, \qquad \text{(E.13)}$$

where $e^{-im\phi}$ is no longer considered since it is covered in the limits. The normalisation constant can be determined by using the orthonormal/normalisation property of the spherical harmonics

$$\int_0^{2\pi}\int_0^{\pi} Y_\ell^m(\theta,\phi)\,Y_{\ell'}^{m'*}(\theta,\phi)\,d\Omega = \delta_{\ell\ell'}\,\delta_{mm'}, \qquad \text{(E.14)}$$

where $d\Omega = \sin\theta\,d\theta\,d\phi$ is the solid angle and $Y_{\ell'}^{m'*}(\theta,\phi)$ the complex conjugate of $Y_\ell^m(\theta,\phi)$. Substituting equation (E.13) into (E.14) for both the spherical harmonics and its complex conjugate we get

$$\begin{aligned}
\delta_{\ell\ell'}\,\delta_{mm'} = N^2 \int_0^{2\pi}\int_0^{\pi} & e^{-im'\phi}\,e^{im\phi} \\
& \times P_{\ell'}^{|m'|}(\cos\theta)\,P_\ell^{|m|}(\cos\theta)\sin\theta\,d\theta\,d\phi
\end{aligned}$$

$$
\begin{aligned}
&= -N^2 \int_0^{2\pi} \int_1^{-1} e^{-im'\phi}\, e^{im\phi} \\
&\qquad \times P_{\ell'}^{|m'|}(\cos\theta)\; P_{\ell}^{|m|}(\cos\theta)\, d\cos\theta\, d\phi \\
&= N^2 \int_0^{2\pi} e^{-im'\phi}\, e^{im\phi}\, d\phi \\
&\qquad \times \int_{-1}^{1} P_{\ell'}^{|m'|}(\cos\theta)\; P_{\ell}^{|m|}(\cos\theta)\, d\cos\theta, \qquad \text{(E.15)}
\end{aligned}
$$

where

$$d\cos\theta = -\sin\theta d\,\theta.$$

Using equation (D.4) with $m = m'$ we have

$$N^2 \left(\int_0^{2\pi} 1\; d\phi \right) \frac{2}{2\ell+1} \frac{(\ell+|m|)!}{(\ell-|m|)!} = 1,$$

which is further simplified to

$$N = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-|m|)!}{(\ell+|m|)!}}. \qquad \text{(E.16)}$$

Substituting equation (E.16) for N in (E.13) gives the normalised spherical harmonics

$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-|m|)!}{(\ell+|m|)!}}\; P_\ell^{|m|}(\cos\theta)\; e^{im\phi}. \qquad \text{(E.17)}$$

The Spherical Harmonics are functions that are used to describe the surface of a sphere. Some of these configurations can be visualised in figure E.1 where the positive regions represent an outward expansion and the negative regions an inward contraction.

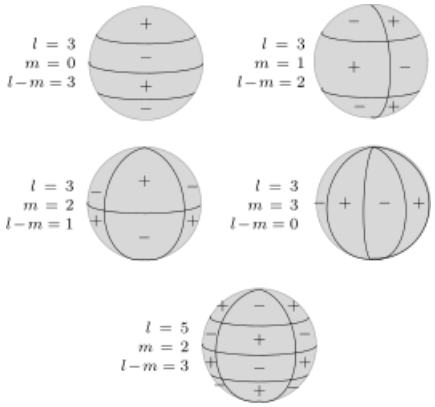


Figure E.1: Visual representation of Y_ℓ^m on a sphere with its nodal lines[47]

E.1 The Spherical Harmonic Addition Theorem

Given an angle χ between two vectors $\vec{r} = (r, \theta, \phi)$ and $\vec{r'} = (r', \theta', \phi')$ defined as

$$\cos\chi = \cos\theta\,\cos\theta' + \sin\theta\,\sin\theta'\cos\left(\phi - \phi'\right).$$

The Legendre polynomial of argument $\cos\chi$ can be written as

$$P_\ell\left(\cos\chi\right) = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y_\ell^m\left(\theta,\phi\right) Y_\ell^{m*}\left(\theta',\phi'\right), \tag{E.18}$$

where $Y_\ell^m\left(\theta,\phi\right)$ and $Y_\ell^{m*}\left(\theta',\phi'\right)$ are the spherical harmonics with arguments (θ,ϕ) and its complex conjugate with arguments (θ',ϕ') respectively. Equation (E.18) is commonly known as the Spherical Harmonic Addition Theorem or Legendre Addition Theorem.

Appendix F

The True, Eccentric and Mean Anomaly

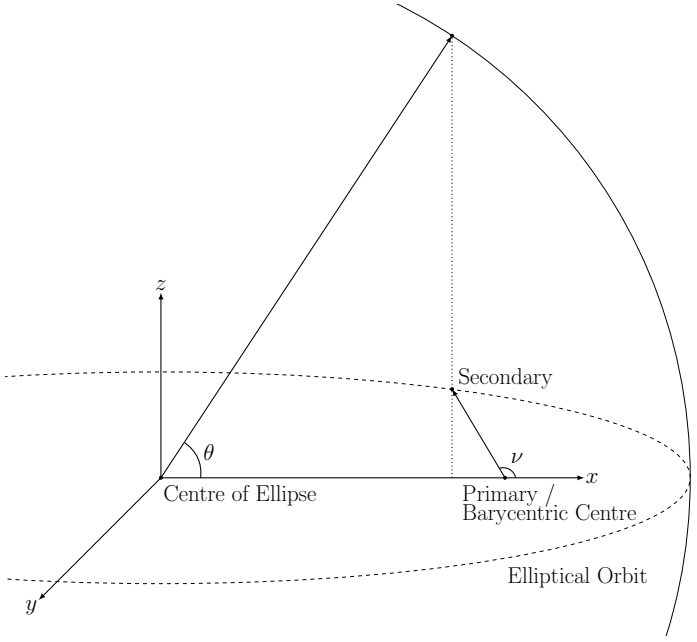


Figure F.1: A diagram illustrating the relation between the true and eccentric anomaly.

In celestial mechanics the angular position of an body orbiting in a Keplerian orbit is referred to as the anomaly. In the case of a binary system the angular position of the secondary with respect to the primary having been measured from a point of reference is termed the anomaly. There are three different anomalies used to define this position: The true, eccentric and mean anomaly.

F.1 The True Anomaly

A type of coordinate system commonly used in astronomy is the barycentric coordinate system. Its origin is commonly situated at the focus of the ellipse traversed by the orbiting body. The true anomaly is then the angle between the direction of periapsis and the current position of the body, as seen from the origin of the barycentric coordinate system. In figure F.1 it is the angular parameter ν.

F.2 The Eccentric Anomaly

The eccentric anomaly is a geographically defined angle measured from the centre of the ellipse to a point on the circumferential circle with radius equal to the semi-major axis of the ellipse traversed by the orbiting body. The point on the circle is geometrically located by drawing a perpendicular line from the semi-major axis of the ellipse through the location of the orbiting body. In Figure F.1 it is the angular parameter θ.

F.3 The Mean Anomaly

The mean anomaly is a parameter similar to the true anomaly except it relates the position of an orbiting body to the time since it last passed a reference point (usually the periastron of the orbit or when $\nu = 0$, see figure F.1). It is defined as

$$M = n\,t,$$

where n is the mean motion of the orbit. The time t denotes the duration since passing the reference point, therefore if the time measured at the reference point is t_0 and the time measured at some later point is t_f, the time t is then

$$t = t_f - t_0.$$

The mean motion n is a measure of how fast an orbiting body progresses about its elliptical orbit and is defined as

$$n = \frac{2\pi}{T}, \tag{F.1}$$

where T is the orbital period. The mean anomaly is then

$$M = \frac{2\pi}{T}\left(t_f - t_0\right). \tag{F.2}$$

If the orbit of the body is circular, then its mean anomaly and true anomaly are equivalent properties.

Appendix G

Polytropes

G.1 Polytropic Processes

Consider a system whose heat intake is δQ, which undergoes a corresponding rise in temperature dT. The heat capacity of the system is defined to be

$$C = \frac{\delta Q}{dT}. \tag{G.1}$$

According to the first law of thermodynamics[1], for any infinitesimal change in the equilibrium state of the system we have

$$\delta Q = dU + \delta W, \tag{G.2}$$

where dU is the change in internal energy that occurs when heat energy δQ is added to the system and energy due to work done by the system on its environment is lost δW. The internal energy dU is an exact differential of the system, whereas the heat transfer δQ and work δW are inexact differentials that denote small energy transfers in the process.

For a system undergoing a quasi-static change[2], the change in work is related to the change in volume by

$$\delta W = PdV, \tag{G.3}$$

where P denotes the internal pressure of the system and dV an infinitesimal change in volume. Substituting equation (G.3) for δW into equation (G.2) we get

$$\delta Q = dU + PdV. \tag{G.4}$$

The δQ in equation (G.1) implies that heat capacity C is a process function and is not well defined unless the change in heat energy δQ is quasi-static. However, expressing δQ in terms of the state functions dU and PdV eliminates the need for the change to be quasi-static and it is revealed that C is in actual fact a state

[1] considering only thermal and mechanical processes

[2] where the effect of work on the system results in a volume change

function. Since U, P and V are state variables, they may each be expressed as functions of two independent state variables, say temperature T and another arbitrary physical quantity X. The heat capacity at constant X is then

$$C_X = \frac{\delta Q_X}{dT} = \left(\frac{\partial U}{\partial T}\right)_X + P\left(\frac{\partial V}{\partial T}\right)_X. \tag{G.5}$$

Since the right hand side of equation (G.5) is a function of the two independent state variables T and X, so is the heat capacity C_X. The value of C_X for any given system is therefore, in general, a function of state and must change during any given process.

A polytropic process or change is defined as a quasi-static change of state carried out in such a way that the heat capacity C_X remains constant during the entire process so that

$$C_X = \frac{dQ_X}{dT} = Constant.$$

G.2 The Polytropic Equation of State

The equation of state of a perfect gas is

$$PV = nRT, \tag{G.6}$$

where nR is a constant. Using the result $(C_P - C_V) = nR$[3] equation (G.6) becomes

$$P = \frac{(C_P - C_V)}{V}T, \tag{G.7}$$

where C_P and C_V are the heat capacities at constant pressure and volume respectively. Using equation (G.5) it is easily shown that the specific heat capacity at constant volume is defined as

$$C_V = \frac{\delta Q_V}{dT} = \left(\frac{\partial U}{\partial T}\right)_V, \tag{G.8}$$

where $\delta Q_V/dT$ is the intake of heat into the system at constant volume. Substituting equations (G.7) and (G.8) into equation (G.5) we get

$$0 = (C_V - C)\frac{dT}{T} + (C_P - C_V)\frac{dV}{V}.$$

In the case of a perfect gas C_V and C_P are constant, therefore by integrating the above equation we have

$$\begin{aligned} Constant &= ln\, T^{(C_V - C)} + ln\, V^{(C_P - C_V)} \\ &= T^{(C_V - C)}\, V^{(C_P - C_V)} \\ &= T\, V^{\frac{C_P - C_V}{C_V - C}}. \end{aligned} \tag{G.9}$$

[3] a standard result for a perfect gas

Introducing a new parameter γ', termed the polytropic exponent, as

$$\gamma' = \frac{C_P - C}{C_V - C},$$

we have

$$\begin{aligned}\gamma' - 1 &= \left(\frac{C_P - C}{C_V - C}\right) - \left(\frac{C_V - C}{C_V - C}\right) \\ &= \frac{C_P - C_V}{C_V - C}.\end{aligned}$$

Using this result, equation (G.9) becomes

$$TV^{\gamma'-1} = Constant. \tag{G.10}$$

Using equation (G.9), we can easily deduce the following representations of equation (G.10)

$$P^{1-\gamma'}\,T^{\gamma'} = Constant, \tag{G.11}$$

$$P\,V^{\gamma'} = Constant. \tag{G.12}$$

The vapour density is defined as

$$\rho = \frac{M}{V},$$

where M is the mass of the gas and stays constant throughout the process. substituting this equation for V into equation (G.12) we get

$$P = k\,\rho^{\gamma'}, \tag{G.13}$$

where the constant has been labelled k arbitrarily.

Since γ' is constant it is easy to see that in the (P, ρ) plane, all polytropes with exponent γ' form a one-parameter family of curves with the constant k being the relating parameter. Equation (G.13) is typically known as the polytropic equation of state where the name polytropic implies that the formula encompasses all processes familiar to thermodynamics.

A polytropic fluid is the term given to a fluid that is described by an equation of state represented by (G.13).

An adiabatic ($C_X = 0$) as one extreme and an isothermal ($C_X = \infty$) as another extreme are then polytropics of zero and infinite heat capacities respectively. The same applies for all processes whose heat capacity ranges between 0 and ∞.

G.3 The Lane-Emden Equation

The Hydrostatic equations for a given polytropic system in spherical coordinates can be deduced from the hydrodynamic equations of motion (see appendix A)

with the polytropic equation of state (G.13) by applying the conditions that the system is spherically symmetric and that its fluid has a velocity $\vec{v} = 0$. Equations (A.1) with (G.13) then become

$$\left.\begin{aligned} \tfrac{\partial \rho}{\partial t}(r) &= 0, \\ \tfrac{d\Phi}{dr}(r) &= -\tfrac{1}{\rho(r)}\,\tfrac{dP}{dr}(r), \\ \tfrac{1}{r^2}\tfrac{d}{dr}r^2\tfrac{d\Phi}{dr}(r) &= 4\pi G\rho(r), \\ P(r) &= k\Big(\rho(r)\Big)^{\gamma'}. \end{aligned}\right\} \tag{G.14}$$

Substituting equation four into equation two and then that result into equation three of (G.14) we get[4]

$$\begin{aligned} -4\pi G\rho &= \frac{k}{r^2}\frac{d}{dr}r^2\left(\frac{1}{\rho}\frac{d\rho^{\gamma'}}{dr}\right) \\ &= \frac{k}{r^2}\frac{d}{dr}r^2\left(\frac{\rho^{\gamma'-1}\gamma'}{\rho}\frac{d\rho}{dr}\right) \\ &= \frac{k\,\gamma'}{r^2}\frac{d}{dr}r^2\Big(\rho^{\gamma'-2}\,\frac{d\rho}{dr}\Big). \end{aligned} \tag{G.15}$$

Using the identity that

$$\rho^{\gamma'-2}\,\frac{d\rho}{dr} = \frac{d}{dr}\left(\frac{\rho^{\gamma'-1}}{\gamma'-1}\right),$$

equation (G.15) becomes

$$-4\pi G\rho = k\left(\frac{\gamma'}{\gamma'-1}\right)\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\,\rho^{\gamma'-1}. \tag{G.16}$$

Now that the equations are decoupled, let

$$\eta = \frac{\rho^{\gamma'-1}}{\rho_c^{\gamma'-1}}$$

be a dimensionless variable where the characteristic value ρ_c is chosen to be the central density of the star of interest ρ_{cent}. Substituting this variable for $\rho^{\gamma'-1}$ in equation (G.16) we get

$$\begin{aligned} -4\pi G\,\rho_c\,\eta^{\frac{1}{\gamma'-1}} &= k\,\rho_c^{\gamma'-1}\left(\frac{\gamma'}{\gamma'-1}\right)\frac{1}{r^2}\frac{d}{dr}r^2\frac{d\eta}{dr} \\ -\eta^{\frac{1}{\gamma'-1}} &= \rho_c^{\gamma'-2}\left(\frac{k}{4\pi G}\right)\left(\frac{\gamma'}{\gamma'-1}\right)\frac{1}{r^2}\frac{d}{dr}r^2\frac{d\eta}{dr}. \end{aligned} \tag{G.17}$$

Eddington defines the polytropic index n as [41, 42]

$$n = \frac{1}{\gamma'-1}.$$

[4]Function arguments are omitted unless there is confusion

In terms of n, the polytropic exponent is then

$$\gamma' = \frac{1}{n} + 1. \tag{G.18}$$

Substituting equation (G.18) for γ' into equation (G.17) we get

$$-\eta^n = \rho_c^{\frac{1}{n}-1} \left(\frac{k}{4\pi G}\right)(n+1)\,\frac{1}{r^2}\frac{d}{dr}r^2\frac{d\eta}{dr}. \tag{G.19}$$

We now introduce the dimensionless variable ξ such that

$$\xi = \frac{r}{r_c},$$

where r_c is some characteristic value for r[5]. Substituting this definition for r in equation (G.19) we get

$$-\eta^n = \frac{\rho_c^{\frac{1}{n}-1}}{r_c^2}\left(\frac{k}{4\pi G}\right)(n+1)\,\frac{1}{\xi^2}\frac{d}{d\xi}\xi^2\frac{d\eta}{d\xi}. \tag{G.20}$$

We choose r_c so as to achieve maximal simplification by setting the constant term

$$\frac{\rho_c^{\frac{1}{n}-1}}{r_c^2}\left(\frac{k}{4\pi G}\right)(n+1) = 1.$$

Therefore

$$r_c = \sqrt{\rho_c^{\frac{1}{n}-1}\left(\frac{k}{4\pi G}\right)(n+1)}. \tag{G.21}$$

Equation (G.20) is then

$$-\eta^n = \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\eta}{d\xi}\right). \tag{G.22}$$

Equation (G.22) is referred to as the *Lane-Emden equation of index n* and is valid in any region where equation (G.15) can be expected to hold.

The Lane-Emden Function

By applying a set of initial conditions to the system, a set of solutions η_n can be found for equation (G.22). These solutions are called the *Lane-Emden function of index n*. Let ρ_c be the density at the centre of the star. At $\xi = 0$ we then have

$$\eta = 1. \tag{G.23}$$

Since the gravitational potential Φ at $\xi = 0$ is zero, we have that

$$\frac{d\eta}{d\xi} = 0 \tag{G.24}$$

[5]commonly set to the mean radius of the star of interest

implying that ρ is continuous at $\xi = 0$.

The left hand side of equation (G.22) is singular at $\xi = 0$. This means that any numerical solution would have to begin at a non-zero value for ξ. To find the form of the solution close to the origin let us assume a power series for η such that

$$\begin{aligned}\eta &= \sum_{p=0}^{\infty} a_p \, \xi^{\, p} \\ &= a_0 + a_1 \, \xi + a_2 \, \xi^{\, 2} + a_3 \, \xi^{\, 3} + ... \end{aligned} \tag{G.25}$$

The first and second derivative with respect to ξ is then

$$\begin{aligned}\frac{d\eta}{d\xi} &= \sum_{p=0}^{\infty} p \, a_p \, \xi^{\, p-1} \\ &= a_1 + 2 \, a_2 \, \xi + 3 \, a_3 \, \xi^{\, 2} + 4 \, a_4 \, \xi^{\, 3} + ..., \end{aligned} \tag{G.26}$$

$$\begin{aligned}\frac{d^2\eta}{d\xi^2} &= \sum_{p=0}^{\infty} p \, (p-1) \, a_p \, \xi^{\, p-2} \\ &= 2 \, a_2 + 6 \, a_3 \, \xi + 12 \, a_4 \, \xi^{\, 2} + 20 \, a_5 \, \xi^{\, 3} + ... \end{aligned} \tag{G.27}$$

Using the initial value for η (G.23) at $\xi = 0$ it is clear from series (G.25) that

$$a_0 = 1.$$

We also have from condition (G.24) at $\xi = 0$ and equation (G.26) that

$$a_1 = 0.$$

Since we are only interested in values for η very close to the centre of the star, we only consider values for ξ that are small. Therefore, higher order terms of ξ can be neglected. The more terms this assumption removes the lower the accuracy of the resulting numerics. In this situation this is an acceptable loss since we only require at most two values of η close to the centre of the star. These initial values will then be used to start a numerical simulation that calculates the rest. By removing terms of order higher than $\xi^{\, 3}$, equations (G.25) - (G.27) are reduced to

$$\begin{aligned}\eta &= a_0 + a_1 \, \xi + a_2 \, \xi^{\, 2} + a_3 \, \xi^{\, 3}, \\ \frac{d\eta}{d\xi} &= a_1 + 2 \, a_2 \, \xi + 3 \, a_3 \, \xi^{\, 2} + 4 \, a_4 \, \xi^{\, 3}, \\ \frac{d^2\eta}{d\xi^2} &= 2 \, a_2 + 6 \, a_3 \, \xi + 12 \, a_4 \, \xi^{\, 2} + 20 \, a_5 \, \xi^{\, 3}. \end{aligned}$$

With these results, the left and right hand side of equation (G.22) is evaluated separately to give

$$-\eta^n = -\left(1 + a_2\,\xi^{\,2} + a_3\,\xi^{\,3}\right)^n, \tag{G.28}$$

$$\frac{2}{\xi}\frac{d\eta}{d\xi} + \frac{d^2\eta}{d\xi^2} = 6\,a_2 + 12\,a_3\,\xi + 20\,a_4\,\xi^{\,2} + 30\,a_5\,\xi^{\,3}. \tag{G.29}$$

Let $x = a_2\,\xi^{\,2} + a_3\,\xi^{\,3}$, equation (G.28) is then

$$-\eta^n = -\left(1 + x\right)^n. \tag{G.30}$$

Using the standard definition for the binomial expansion, which is defined as

$$(1+x)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) x^k,$$

with

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!\,(n-k)!},$$

equation (G.30) becomes

$$\begin{aligned} -\eta^n &= -\sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!}\, x^k \\ &= -1 - \frac{n!}{(n-1)!}\, x - \frac{n!}{2\,(n-2)!}\, x^2 - ... \\ &= -1 - n\,x - \frac{n(n-1)}{2}\, x^2 - .... \end{aligned} \tag{G.31}$$

It is easy to see, by calculating x^2 to be

$$x^2 = a_2^2\,\xi^{\,4} + a_2\,a_3\,\xi^{\,5} + a_3^2\,\xi^{\,6}, \tag{G.32}$$

that since four is the smallest degree across all terms in equation (G.32), by choosing to keep a degree of three and lower for ξ, terms with x^2 and higher degrees are negligible. Equation (G.31) is then

$$\begin{aligned} -\eta^n &= -1 - n\,x \\ &= -1 - n\,a_2\,\xi^{\,2} - n\,a_3\,\xi^{\,3}. \end{aligned} \tag{G.33}$$

Therefore, equating this with the result given by equation (G.29), we have

$$-1 - n\,a_2\,\xi^{\,2} - n\,a_3\,\xi^{\,3} = 6\,a_2 + 12\,a_3\,\xi + 20\,a_4\,\xi^{\,2} + 30\,a_5\,\xi^{\,3}. \tag{G.34}$$

Since the power series of a function is unique[6], the left hand side and right hand side of equation (G.34) must be identical and the coefficients of corresponding powers of ξ must be equal. From this we can conclude that

[6]that is, there is only one power series of the form $\sum_p^\infty a_p\,\xi^{\,p} = 0$ which converges to a given function

$$
\begin{aligned}
a_2 &= -\frac{1}{6}, \\
a_3 &= 0, \\
a_4 &= \frac{n}{120}, \\
a_5 &= 0.
\end{aligned}
$$

The first three terms of η_n are then

$$\eta_n = 1 - \frac{1}{6}\,\xi^{\,2} + \frac{n}{120}\,\xi^{\,4}. \tag{G.35}$$

The Lane-Emden function η_n in its above form is only valid, since it is only accurate, at positions very close to the centre of the star. It can therefore only be used to calculate values for η very close to the centre of the star. More solutions, further away from the centre of the star, can be found by continuing the calculation using standard numerical methods.

Appendix H

Finite Difference Methods

Most differential equations are rarely solved analytically, with those having analytical solutions few in number. As a result the need for accurate numerical schemes is essential. Most basic numerical schemes for partial differential equations can be collectively identified as either finite difference methods, finite element methods or finite volume methods. Finite difference methods will be covered briefly in this section.

Finite difference methods are approximations, that when applied to a partial differential equation convert it into an algebraic equation. This is done by discretising the derivatives with linear combinations of function values at distinct grid points. This section serves as a very brief introduction to first and second order forward, central and backward finite difference approximations.

For an arbitrary function $f(x)$, that is infinitely differentiable at an arbitrary value for x named a, the Taylor expansion of $f(x)$ is defined as

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} \left(\frac{\partial^n f}{\partial x}(a) \right) \\ &= f(a) + (x-a) \frac{\partial f}{\partial x}(a) + \frac{1}{2!} (x-a)^2 \frac{\partial^2 f}{\partial x^2}(a) \\ &\quad + \frac{1}{3!} (x-a)^3 \frac{\partial^3 f}{\partial x^3}(a) + ... \end{aligned} \tag{H.1}$$

The first step in developing a finite difference approximation is to define a set of grid points in which a block size Δx is chosen. In the case of a forward difference approximation the block size is chosen to be

$$\Delta x = x_{i+1} - x_i, \tag{H.2}$$

and in the case of a backward finite difference approximation

$$\Delta x = x_i - x_{i-1}, \tag{H.3}$$

where if x_i is the value of x at a location on the grid, x_{i+1} is its next and x_{i-1} its previous immediate value. Using equations (H.2) and (H.3) with (H.1) we

get for the forward difference approximation

$$\begin{aligned} f(x_{i+1}) &= f(x_i) + (x_{i+1} - x_i)\,\frac{\partial f}{\partial x}(x_i) + \frac{1}{2!}\,(x_{i+1} - x_i)^2\,\frac{\partial^2 f}{\partial x^2}(x_i) \\ &\quad + \frac{1}{3!}\,(x_{i+1} - x_i)^3\,\frac{\partial^3 f}{\partial x^3}(x_i) + ... \\ &= f(x_i) + \Delta x\,\frac{\partial f}{\partial x}(x_i) + \frac{1}{2!}\,(\Delta x)^2\,\frac{\partial^2 f}{\partial x^2}(x_i) \\ &\quad + \frac{1}{3!}\,(\Delta x)^3\,\frac{\partial^3 f}{\partial x^3}(x_i) + ..., \end{aligned} \tag{H.4}$$

and backward difference approximation

$$\begin{aligned} f(x_{i-1}) &= f(x_i) + (x_{i-1} - x_i)\,\frac{\partial f}{\partial x}(x_i) + \frac{1}{2!}\,(x_{i-1} - x_i)^2\,\frac{\partial^2 f}{\partial x^2}(x_i) \\ &\quad + \frac{1}{3!}\,(x_{i-1} - x_i)^3\,\frac{\partial^3 f}{\partial x^3}(x_i) + ... \\ &= f(x_i) - \Delta x\,\frac{\partial f}{\partial x}(x_i) + \frac{1}{2!}\,(\Delta x)^2\,\frac{\partial^2 f}{\partial x^2}(x_i) \\ &\quad - \frac{1}{3!}\,(\Delta x)^3\,\frac{\partial^3 f}{\partial x^3}(x_i) + ... \end{aligned} \tag{H.5}$$

At this moment it is a matter of choice exactly how accurate the approximation needs to be. The accuracy of the approximation is related to the number of terms that are kept in equations (H.4) and (H.5), however the more we keep the higher the complexity of the resulting equations as well as the overhead cost during computation. Therefore it is matter of finding the right balance between accuracy and performance. The terms that are ignored are called the truncation error and are represented by the symbol O.

For first order finite difference approximations the first two terms on the right hand side of equations (H.4) and (H.5) are kept and the rest ignored. These equations are then

$$\left.\begin{aligned} f(x_{i+1}) &= f(x_i) + \Delta x\,\tfrac{\partial f}{\partial x}(x_i) + O\,(\Delta x^2), \\ f(x_{i-1}) &= f(x_i) - \Delta x\,\tfrac{\partial f}{\partial x}(x_i) + O\,(\Delta x^2). \end{aligned}\right\} \tag{H.6}$$

Rearranging equations (H.6) we get

$$\left.\begin{aligned} \tfrac{\partial f}{\partial x}(x_i) &= \tfrac{f(x_{i+1}) - f(x_i)}{\Delta x}, \\ \tfrac{\partial f}{\partial x}(x_i) &= \tfrac{f(x_i) - f(x_{i-1})}{\Delta x}, \end{aligned}\right\} \tag{H.7}$$

which represent the first order forward and backward difference approximations for function $f(x)$. To obtain the first order central difference approximation we subtract equation two from equation one in (H.6) so that

$$f(x_{i+1}) - f(x_{i-1}) = 2\,\Delta x\,\frac{\partial f}{\partial x}(x_i). \tag{H.8}$$

Rearranging equation (H.8) we get

$$\frac{\partial f}{\partial x}(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2\,\Delta x}, \tag{H.9}$$

which represents the first order central difference approximation for function $f(x)$. Another commonly used approximation[1] is the second order central difference approximation. It is derived by keeping terms up to $O(\Delta x^3)$ for equations (H.4) and (H.5) and adding the results. We then have

$$\left.\begin{aligned} f(x_{i+1}) &= f(x_i) + \Delta x\,\tfrac{\partial f}{\partial x}(x_i) + \tfrac{1}{2!}\,(\Delta x)^2\,\tfrac{\partial^2 f}{\partial x^2}(x_i) + O\,(\Delta x^3), \\ f(x_{i-1}) &= f(x_i) - \Delta x\,\tfrac{\partial f}{\partial x}(x_i) + \tfrac{1}{2!}\,(\Delta x)^2\,\tfrac{\partial^2 f}{\partial x^2}(x_i) + O\,(\Delta x^3). \end{aligned}\right\} \tag{H.10}$$

Adding equation one to two of (H.10) we get

$$f(x_{i+1}) + f(x_{i-1}) = 2\,f(x_i) + (\Delta x)^2\,\frac{\partial^2 f}{\partial x^2}(x_i). \tag{H.11}$$

Rearranging equation (H.11) we get

$$\frac{\partial^2 f}{\partial x^2}(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{(\Delta x)^2}, \tag{H.12}$$

which represents the second order central difference approximation for function $f(x)$.

The choice of whether to use a forward, central or backward approximation comes down to the desired level of accuracy and the availability of data. For example, a central difference approximation gives a much more accurate approximation of the derivatives than the forward and backward approximation. However, requires a previous and central data point to be known to initialise the simulation.

[1] and one that is used in this thesis

Appendix I

Additional Calculations

I.1 $\int_0^{2\pi} e^{ik'x}\, e^{-ikx} dx = \delta_{kk'}$

This equation has two results, if $k' = k$ then

$$\begin{aligned}\int_0^{2\pi} e^{ik'x}\, e^{-ikx} dx &= \int_0^{2\pi} 1\, dx \\ &= 2\pi. \end{aligned} \tag{I.1}$$

However if $k' \neq k$ then

$$\int_0^{2\pi} e^{ik'x}\, e^{-ikx} dx = \int_0^{2\pi} e^{ix\left(k'-k\right)} dx$$

and by substituting for $u = ix\,(k'-k)$, with $dx/du = i\,(k'-k)$, we have

$$\frac{1}{i\,(k'-k)} \int_0^{2i\pi\left(k'-k\right)} e^u du = \frac{1}{i\,(k'-k)} \left(e^{2i\pi\left(k'-k\right)} - 1\right). \tag{I.2}$$

Using Euler's formula we have

$$\begin{aligned} e^{2i\pi\left(k'-k\right)} &= \cos\left(2\pi k' - 2\pi k\right) + i \sin\left(2\pi k' - 2\pi k\right) \\ &= \cos\left(2\pi k'\right)\, \cos\left(2\pi k\right) + \sin\left(2\pi k'\right)\, \sin\left(2\pi k\right) \\ &\qquad + i \sin\left(2\pi k'\right) \cos\,\left(2\pi k\right) - i \sin\left(2\pi k'\right)\, \cos\left(2\pi k\right). \end{aligned}$$

Since $k,\, k' \in \mathbb{Z}$ for every $k,\, k'$

$$\begin{aligned} \cos\left(2\pi k'\right) &= \cos\left(2\pi k\right) = 1, \\ \sin\left(2\pi k'\right) &= \sin\left(2\pi k\right) = 0. \end{aligned}$$

Therefore

$$e^{2\pi\, i\left(k'-k\right)} = 1.$$

Using this result in equation (I.2) we get

$$\frac{1}{i\,(k'-k)} \int_0^{2\pi\, i\left(k'-k\right)} e^u du = \frac{1}{i\,(k'-k)}\,(1-1) = 0. \tag{I.3}$$

Collectively, equation (I.1) and (I.3) is written as

$$\frac{1}{2\pi}\int_0^{2\pi} e^{ik'x}\, e^{-ikx} dx = \delta_{kk'}. \tag{I.4}$$

I.2 $\int_0^{2\pi} \cos x dx = 2\int_0^{\pi} \cos x dx$

Splitting the integral we have

$$\begin{aligned}\int_0^{2\pi} \cos x\, dx &= \int_0^{\pi} \cos x\, dx + \int_{\pi}^{2\pi} \cos x\, dx \\ &= \int_0^{\pi} \cos x\, dx - \int_{2\pi}^{\pi} \cos x\, dx.\end{aligned} \tag{I.5}$$

Let $x = -y$, with $dx/dy = -1$, then

$$-\int_{2\pi}^{\pi} \cos x\, dx = \int_{-2\pi}^{-\pi} \cos(-y)\, dy.$$

Considering the cosine function is even, we have $\cos(y) = \cos(-y)$, therefore

$$-\int_{2\pi}^{\pi} \cos x\, dx = \int_{-2\pi}^{-\pi} \cos(y)\, dy$$

and by relabelling the variable y as x, can be rewritten as

$$-\int_{2\pi}^{\pi} \cos x\, dx = \int_{-2\pi}^{-\pi} \cos(x)\, dx.$$

Due to the periodicity of the cosine function

$$\int_{-2\pi}^{-\pi} \cos(x)\, dx = \int_0^{\pi} \cos(x)\, dx.$$

Substituting this result into equation (I.5) then gives

$$\begin{aligned}\int_0^{2\pi} \cos x\, dx &= \int_0^{\pi} \cos x\, dx + \int_0^{\pi} \cos x\, dx \\ &= 2\int_0^{\pi} \cos x\, dx.\end{aligned} \tag{I.6}$$

Appendix J

Introduction to Lagrangian Pulsation Theory

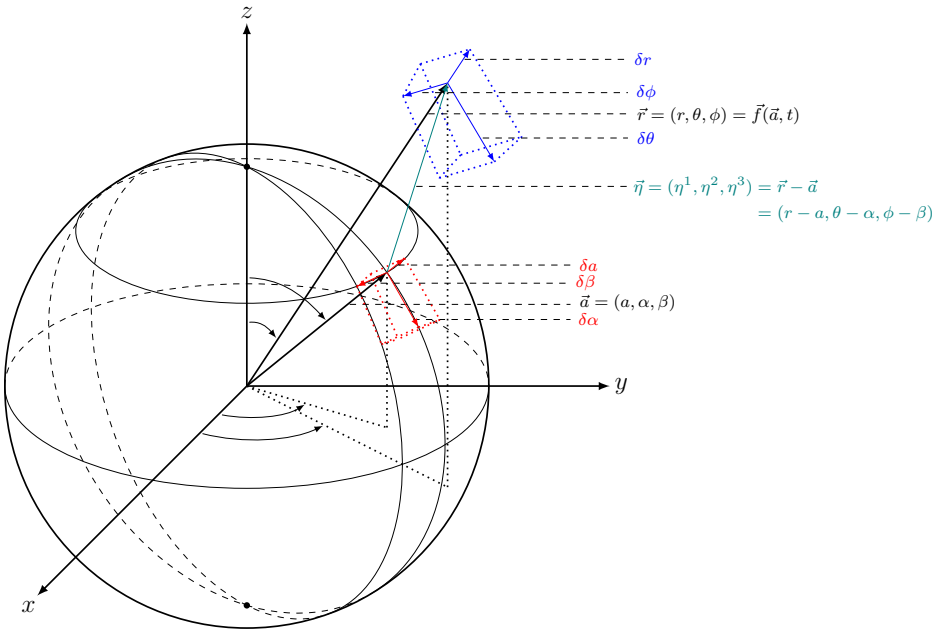


Figure J.1: A diagram illustrating the creation of the Lagrangian Displacement Vector $\vec{\eta}$ by connecting the initial and final positions of particle p.

In chapter 3 it was shown that conventional pulsation theory uses a combination of Eulerian and Lagrangian perturbation specifications to derive the perturbation equations. This section serves to demonstrate the difficulties that come with using a purely Lagrangian perturbation treatment to find these equations. From the definitions of both specifications in section 3.2 we see that, in the case of a purely Lagrangian perturbation, two additional concerns need to be

addressed. The first being that the variables $\vec{r} = \vec{r}(\vec{a}, t)$ are functions of time since the position of the fluid element changes with time. The second being that the resulting perturbation equations contain physical parameters that are evaluated in two different coordinate systems. To work around this we then need to find relationships between these parameters in both coordinate systems, determined through coordinate transformations, to solve for these equations. As we will see, these features add to the complication of determining the perturbation equations.

J.1 The Continuity Equation

Let $\vec{a} = (a, \alpha, \beta)$ represent the position of an arbitrary fluid point p in the reference frame of the unperturbed star. The standard assumption of the unperturbed configuration being hydrostatic and therefore spherically symmetric is still considered. A fluid element of fixed mass is then constructed with sides δa, $\delta\alpha$, $\delta\beta$ (see red cube in Figure J.1). The volume of this fluid element, with infinitesimally small sides, is then

$$\delta V_0 = a^2 \, sin\alpha \; da \; d\alpha \; d\beta.$$

We also have that

$$\delta V_0 = \frac{\delta m_0}{\rho_0},$$

where δm_0 and $\rho_0 = \rho_0(a)$ represent the mass and density contained within the fluid element, defined in the unperturbed configuration. As a result of spherical symmetry there is no dependence of the physical parameters on the angular variables α and β. A perturbation is then applied to the unperturbed configuration resulting in the fluid element moving to a new location. Let $\vec{r} = (r, \theta, \phi)$ represent the trajectory of fluid element p in the reference frame of the new perturbed configuration such that

$$\left.\begin{aligned} r &= f^1(\vec{a}, t) = f^1(a, \alpha, \beta, t), \\ \theta &= f^2(\vec{a}, t) = f^2(a, \alpha, \beta, t), \\ \phi &= f^3(\vec{a}, t) = f^3(a, \alpha, \beta, t). \end{aligned}\right\} \tag{J.1}$$

Even though we consider spherical symmetry in the unperturbed configuration, at this moment we are not looking specifically for radial oscillations and so every initial position of a fluid element[1], through perturbation, will have a different final position[2] that is a function of a, α and β. Fluid element p now has a new set of sides, namely δr, $\delta\theta$, $\delta\phi$ (see blue cube in Figure J.1) such that

$$\left.\begin{aligned} r + \delta r &= f^1(a + \delta a, \alpha, \beta, t), \\ \theta + \delta\theta &= f^2(a, \alpha + \delta\alpha, \beta, t), \\ \phi + \delta\phi &= f^3(a, \alpha, \beta + \delta\beta, t). \end{aligned}\right\}$$

Similarly, the volume of fluid element p in the perturbed configuration, with infinitesimally small sides, is now

[1] labelled as $\vec{a}$
[2] labelled as $\vec{r}$

$$\begin{aligned}\delta V &= r^2\, sin\theta\; dr\; d\theta\; d\phi \\ &= \frac{\delta m}{\rho},\end{aligned}$$

where δm and $\rho = \rho\left(\vec{r}, t\right)$ represent the mass and density contained within fluid element p, defined in the new perturbed configuration. Since the mass contained within the fluid element is fixed we have

$$\begin{aligned}\delta m &= \delta m_0 \\ \delta V\, \rho\, \left(\vec{r}, t\right) &= \delta V_0\, \rho_0\left(a\right) \\ r^2\, sin\theta\; dr\; d\theta\; d\phi\; \rho\, \left(\vec{r}, t\right) &= a^2\, sin\alpha\; da\; d\alpha\; d\beta\; \rho_0\left(a\right). \end{aligned} \tag{J.2}$$

Using equations (J.1) and (J.2) we have

$$\begin{aligned}dr &= (r + dr) - r = f^1\left(a + \delta a, \alpha, \beta, t\right) - f^1\left(a, \alpha, \beta, t\right), \\ d\theta &= (\theta + d\theta) - \theta = f^2\left(a, \alpha + \delta\alpha, \beta, t\right) - f^2\left(a, \alpha, \beta, t\right), \\ d\phi &= (\phi + d\phi) - \phi = f^3\left(a, \alpha, \beta + \delta\beta, t\right) - f^3\left(a, \alpha, \beta, t\right).\end{aligned}$$

Expanding each equation using a Taylor expansion to first order[3] we get for dr

$$\begin{aligned}dr &= f^1\left(a, \alpha, \beta, t\right) + da\; \frac{\partial f^1}{\partial a}\; \left(a, \alpha, \beta, t\right) - f^1\left(a, \alpha, \beta, t\right) \\ &= da\; \frac{\partial f^1}{\partial a}\; \left(\vec{a}, t\right)\end{aligned}$$

and similarly for $d\theta$ and $d\phi$

$$\begin{aligned}d\theta &= d\alpha\; \frac{\partial f^2}{\partial \alpha}\; \left(\vec{a}, t\right), \\ d\phi &= d\beta\; \frac{\partial f^3}{\partial \beta}\; \left(\vec{a}, t\right).\end{aligned}$$

Substituting these results into equation (J.2) then gives

$$\begin{aligned}r^2\, sin\theta\; da\; d\alpha\; d\beta\; \frac{\partial f^1}{\partial a}\left(\vec{a}, t\right)\; \frac{\partial f^2}{\partial \alpha}\left(\vec{a}, t\right)\; \frac{\partial f^3}{\partial \beta}\left(\vec{a}, t\right)\; \rho\left(\vec{r}, t\right) \\ = a^2\, sin\alpha\; da\; d\alpha\; d\beta\; \rho_0\left(a\right).\end{aligned}$$

Rearranging the above equation with ρ_0 as the subject of the formula we get

$$\begin{aligned}\rho_0\; \left(a\right) = \frac{\left(f^1\left(\vec{a}, t\right)\right)^2}{a^2}\; \frac{\sin\left(f^2\left(\vec{a}, t\right)\right)}{sin\,\alpha} \\ \times \frac{\partial f^1}{\partial a}\left(\vec{a}, t\right)\; \frac{\partial f^2}{\partial \alpha}\left(\vec{a}, t\right)\; \frac{\partial f^3}{\partial \beta}\left(\vec{a}, t\right)\; \rho\left(\vec{r}, t\right).\end{aligned} \tag{J.3}$$

Next, we introduce the non-dimensional Lagrangian displacement vector $\vec{\eta} = \eta^i\left(\vec{a}, t\right)$, which is defined so that it is infinitesimally small and has components J.1. These are defined as

[3] since the fluid element is considered infinitesimally small

$$\eta^1(\vec{a},t) = \frac{f^1(\vec{a},t) - a}{a},$$
$$\eta^2(\vec{a},t) = f^2(\vec{a},t) - \alpha,$$
$$\eta^3(\vec{a},t) = f^3(\vec{a},t) - \beta.$$

If we treat them as radians, the angles θ and ϕ are already considered dimensionless and do not need to be de-dimensionalised. Rearranging these equations we get[4]

$$\left.\begin{aligned} f^1 &= a\,\eta^1 + a, \\ f^2 &= \eta^2 + \alpha, \\ f^3 &= \eta^3 + \beta \end{aligned}\right\} \tag{J.4}$$

and taking their derivatives with respect to a, α and β we get

$$\left.\begin{aligned} \tfrac{\partial f^1}{\partial a} &= 1 + \eta^1 + a\,\tfrac{\partial \eta^1}{\partial a}, \\ \tfrac{\partial f^2}{\partial \alpha} &= 1 + \tfrac{\partial \eta^2}{\partial \alpha}, \\ \tfrac{\partial f^3}{\partial \beta} &= 1 + \tfrac{\partial \eta^3}{\partial \beta}. \end{aligned}\right\} \tag{J.5}$$

Substituting equations (J.4) and (J.5) into equation (J.3) then gives[4]

$$\begin{aligned}
\rho_0 = {} & \frac{\left(a\,\eta^1 + a\right)^2}{a^2}\,\frac{\sin\left(\eta^2 + \alpha\right)}{\sin\alpha}\left(1 + \eta^1 + a\,\frac{\partial \eta^1}{\partial a}\right) \\
& \left(1 + \frac{\partial \eta^2}{\partial \alpha}\right)\left(1 + \frac{\partial \eta^3}{\partial \beta}\right)\rho \\
= {} & \left(\frac{a^2\left(\eta^1\right)^2 + 2\,a^2\,\eta^1 + a^2}{a^2}\right)\left(\frac{\sin\left(\eta^2\right)\,\cos\alpha + \sin\alpha\,\cos\eta^2}{\sin\alpha}\right) \\
& \times \left(1 + \eta^1 + a\,\frac{\partial \eta^1}{\partial a}\right)\left(1 + \frac{\partial \eta^2}{\partial \alpha} + \frac{\partial \eta^3}{\partial \beta} + \frac{\partial \eta^2}{\partial \alpha}\,\frac{\partial \eta^3}{\partial \beta}\right)\rho \\
= {} & \left(\left(\eta^1\right)^2 + 2\,\eta^1 + 1\right)\left(\frac{\sin\eta^2\,\cos\alpha + \sin\alpha\,\cos\eta^2}{\sin\alpha}\right) \\
& \times \left(1 + \frac{\partial \eta^2}{\partial \alpha} + \frac{\partial \eta^3}{\partial \beta} + \frac{\partial \eta^2}{\partial \alpha}\,\frac{\partial \eta^3}{\partial \beta} + \eta^1 + \eta^1\,\frac{\partial \eta^2}{\partial \alpha} + \eta^1\,\frac{\partial \eta^3}{\partial \beta}\right. \\
& \quad + \eta^1\,\frac{\partial \eta^2}{\partial \alpha}\,\frac{\partial \eta^3}{\partial \beta} + a\,\frac{\partial \eta^1}{\partial a} + a\,\frac{\partial \eta^1}{\partial a}\,\frac{\partial \eta^2}{\partial \alpha} + a\,\frac{\partial \eta^1}{\partial a}\,\frac{\partial \eta^3}{\partial \beta} \\
& \quad \left. + a\,\frac{\partial \eta^1}{\partial a}\,\frac{\partial \eta^2}{\partial \alpha}\,\frac{\partial \eta^3}{\partial \beta}\right)\rho.
\end{aligned}$$

Since η^i is a small quantity we can neglect all two-fold and products of the perturbation through linearisation, as was done in chapter 3. The above equation

[4]Function arguments are omitted unless there is confusion

is then reduced to

$$\rho_0 = \left(2\,\eta^1 + 1\right)\,\left(\frac{\sin\eta^2\,\cos\alpha + \sin\alpha\,\cos\eta^2}{\sin\alpha}\right) \times \left(1 + \eta^1 + a\,\frac{\partial\eta^1}{\partial a} + \frac{\partial\eta^2}{\partial\alpha} + \frac{\partial\eta^3}{\partial\beta}\right)\rho. \tag{J.6}$$

For small values of x, $\sin x \approx x$ and $\cos x \approx 1$. Therefore, linearising terms along the way, equation (J.6) becomes

$$\begin{aligned}
\rho_0 &= \left(2\,\eta^1 + 1\right)\,\left(\frac{\eta^2\,\cos\alpha + \sin\alpha}{\sin\alpha}\right) \\
&\qquad \times \left(1 + \eta^1 + a\,\frac{\partial\eta^1}{\partial a} + \frac{\partial\eta^2}{\partial\alpha} + \frac{\partial\eta^3}{\partial\beta}\right)\rho \\
&= \left(2\,\eta^1 + 1\right)\,\left(\eta^2\,\cot\alpha + 1\right) \\
&\qquad \times \left(1 + \eta^1 + a\,\frac{\partial\eta^1}{\partial a} + \frac{\partial\eta^2}{\partial\alpha} + \frac{\partial\eta^3}{\partial\beta}\right)\rho \\
&= \left(2\,\eta^1\,\eta^2\,\cot\alpha + \eta^2\,\cot\alpha + 2\,\eta^1 + 1\right) \\
&\qquad \times \left(1 + \eta^1 + a\,\frac{\partial\eta^1}{\partial a} + \frac{\partial\eta^2}{\partial\alpha} + \frac{\partial\eta^3}{\partial\beta}\right)\rho \\
&= \left(\eta^2\,\cot\alpha + 2\,\eta^1 + 1\right) \\
&\qquad \times \left(1 + \eta^1 + a\,\frac{\partial\eta^1}{\partial a} + \frac{\partial\eta^2}{\partial\alpha} + \frac{\partial\eta^3}{\partial\beta}\right)\rho \\
&= \Big(\eta^2\,\cot\alpha + 2\,\eta^1 + 1 + \eta^1\,\eta^2\,\cot\alpha \\
&\qquad + a\,\eta^2\,\frac{\partial\eta^1}{\partial a}\,\cot\alpha + \eta^2\,\frac{\partial\eta^2}{\partial\alpha}\,\cot\alpha + \eta^2\,\frac{\partial\eta^3}{\partial\beta}\,\cot\alpha \\
&\qquad + 2\,\left(\eta^1\right)^2 + 2\,a\,\eta^1\,\frac{\partial\eta^1}{\partial a} + 2\,\eta^1\,\frac{\partial\eta^2}{\partial\alpha} + 2\,\eta^1\,\frac{\partial\eta^3}{\partial\beta} \\
&\qquad + \eta^1 + a\,\frac{\partial\eta^1}{\partial a} + \frac{\partial\eta^2}{\partial\alpha} + \frac{\partial\eta^3}{\partial\beta}\Big)\rho \\
&= \left(1 + 3\,\eta^1 + \eta^2\,cot\alpha + a\,\frac{\partial\eta^1}{\partial a} + \frac{\partial\eta^2}{\partial\alpha} + \frac{\partial\eta^3}{\partial\beta}\right)\rho.
\end{aligned} \tag{J.7}$$

Equation (J.7) defines a relationship between the density $\rho_0\,(a)$, evaluated in the unperturbed configuration and the density $\rho\,(\vec{r},t)$ evaluated in the perturbed configuration. The Lagrangian perturbation of an arbitrary physical parameter Q was defined in chapter 3 as being

$$\delta Q\,(\vec{a},t) = Q\,(\vec{r},t) - Q_0\,(a)\,. \tag{J.8}$$

Let Q_1 be a non-dimensional and small parameter defined as

$$Q_1\,(\vec{a},t) = \frac{\delta Q\,(\vec{a},t)}{Q_0\,(a)}. \tag{J.9}$$

Substituting the definition J.9 for Q_1 into equation (J.8) we get

$$Q\,(\vec{r},t) = Q_1\,(\vec{a},t)\,Q_0\,(a) + Q_0\,(a)\,. \tag{J.10}$$

Using equation (J.10) with $Q = \rho$, equation (J.7) becomes

$$\begin{aligned} 1 &= \left(1 + 3\,\eta^1 + \eta^2\,cot\alpha + a\,\frac{\partial \eta^1}{\partial a} + \frac{\partial \eta^2}{\partial \alpha} + \frac{\partial \eta^3}{\partial \beta}\right) \\ &\qquad \times (\rho_1 + 1)\,, \\ &= \rho_1 + 3\,\eta^1\,\rho_1 + \eta^2\,cot\alpha\,\rho_1 + a\,\frac{\partial \eta^1}{\partial a}\,\rho_1 \\ &\qquad + \frac{\partial \eta^2}{\partial \alpha}\,\rho_1 + \frac{\partial \eta^3}{\partial \beta}\,\rho_1 + 1 + 3\,\eta^1 + \eta^2\,cot\alpha \\ &\qquad\qquad + a\,\frac{\partial \eta^1}{\partial a} + \frac{\partial \eta^2}{\partial \alpha} + \frac{\partial \eta^3}{\partial \beta}. \end{aligned} \tag{J.11}$$

Linearising equation (J.11) we have

$$\rho_1 = -3\,\eta^1 - \eta^2\,cot\,\alpha - a\,\frac{\partial \eta^1}{\partial a} - \frac{\partial \eta^2}{\partial \alpha} - \frac{\partial \eta^3}{\partial \beta}. \tag{J.12}$$

This equation is the perturbed continuity equation, evaluated in the unperturbed reference frame, determined using a purely Lagrangian perturbation treatment. For the rest of equations (A.1) we won't be able to use the convenience of mass conservation and will have to apply Lagrangian perturbations to the equations directly.

J.2 Euler's Equation

Evaluated in the perturbed configuration, equation two of (A.1) is defined as

$$\frac{\partial \vec{v}}{\partial t}\,(\vec{r},t) + (\vec{v}\,(\vec{r},t)\cdot\nabla)\,\vec{v}\,(\vec{r},t) = -\frac{1}{\rho\,(\vec{r},t)}\,\nabla P\,(\vec{r},t) - \nabla\Phi\,(\vec{r},t)\,. \tag{J.13}$$

For an arbitrary fluid element p, the Lagrangian velocity vector $\vec{v}_\ell\,(\vec{a},t)$ is related to its trajectory by equation (3.2) such that

$$\begin{aligned} \vec{v}_\ell\,(\vec{a},t) &= \frac{d\vec{r}}{dt}(t) \\ &= \frac{d\vec{f}}{dt}\,(\vec{a},t)\,, \end{aligned} \tag{J.14}$$

where $\vec{f} = (f^1, f^2, f^3)$ and is the flow function of the fluid. The Lagrangian velocity $\vec{v}_\ell$ is related to the Eulerian velocity $\vec{v}$ by

$$\frac{d\vec{v}_\ell}{dt}\,(\vec{a},t) = \frac{\partial \vec{v}}{\partial t}\,(\vec{r},t) + (\vec{v}\,(\vec{r},t)\cdot\nabla)\,\vec{v}\,(\vec{r},t)\,.$$

From this result equation (J.13) can be written to give

$$\frac{d\vec{v}_\ell}{dt}\,(\vec{a},t) = -\frac{1}{\rho\,(\vec{r},t)}\,\nabla P\,(\vec{r},t) - \nabla\Phi\,(\vec{r},t)\,. \tag{J.15}$$

By substituting the Lagrangian velocity in place of the Eulerian we no longer need to consider the advective term $(\vec{v}\cdot\nabla)\,\vec{v}$. Using equation (J.14) we then have

$$\frac{d^2\vec{f}}{dt^2}\,(\vec{a},t) = -\frac{1}{\rho\,(\vec{r},t)}\,\nabla P\,(\vec{r},t) - \nabla\Phi\,(\vec{r},t)\,. \tag{J.16}$$

Since equation (J.16) is a vector equation with radial, polar and azimuthal components, it represents three equations, which can each be written separately. The radial equation is written as

$$\rho\left(\vec{r},t\right)\frac{\partial^2 f^1}{\partial t^2}(\vec{a},t) = -\frac{\partial P}{\partial r}\left(\vec{r},t\right) - \rho\left(\vec{r},t\right)\frac{\partial \Phi}{\partial r}\left(\vec{r},t\right). \tag{J.17}$$

Using equation (J.10) with $Q = \rho$ and the definition for f^1 in (J.4) the left hand side of equation (J.17) becomes

$$\begin{aligned}\left(\rho_1\rho_0 + \rho_0\right)\ \frac{\partial^2}{\partial t^2}(a\eta^1 + a) &= a\left(\rho_1\rho_0 + \rho_0\right)\frac{\partial^2\eta^1}{\partial t^2} \\ &= a\,\rho_1\,\rho_0\frac{\partial^2\eta^1}{\partial t^2} + a\,\rho_0\,\frac{\partial^2\eta^1}{\partial t^2}\end{aligned} \tag{J.18}$$

since $\partial a/\partial t = 0$. Since ρ_1 and η^1 are small quantities, by linearising equation (J.18) we have

$$\left(\rho_1\rho_0 + \rho_0\right)\ \frac{\partial^2}{\partial t^2}(a\eta^1 + a) = a\,\rho_0\,\frac{\partial^2\eta^1}{\partial t^2}. \tag{J.19}$$

Similarly for the angular components we have

$$\left.\begin{aligned}&\rho\left(\vec{r},t\right)f^1(\vec{a},t)\tfrac{\partial^2 f^2}{\partial t^2}(\vec{a},t) \\ &\qquad = -\tfrac{\partial P}{\partial\theta}\left(\vec{r},t\right) - \rho\left(\vec{r},t\right)\tfrac{\partial\Phi}{\partial\theta}\left(\vec{r},t\right), \\ &\rho\left(\vec{r},t\right)f^1(\vec{a},t)\Big(\sin f^2(\vec{a},t)\Big)\tfrac{\partial^2 f^3}{\partial t^2}(\vec{a},t) \\ &\qquad = -\tfrac{\partial P}{\partial\phi}\left(\vec{r},t\right) - \rho\left(\vec{r},t\right)\tfrac{\partial\Phi}{\partial\phi}\left(\vec{r},t\right),\end{aligned}\right\} \tag{J.20}$$

where $r = f^1$ and $\theta = f^2$. Using equation (J.10) with $Q = \rho$ and the definition for f^i in (J.4) the left hand side of the equations in (J.20) become

$$\left.\begin{aligned}&\left(\rho_1\rho_0 + \rho_0\right)\left(a\eta^1 + a\right)\tfrac{\partial^2}{\partial t^2}(\eta^2 + \alpha) \\ &\quad = \left(a\,\rho_1\,\rho_0\,\eta^1 + a\,\rho_1\,\rho_0 + a\,\eta^1\,\rho_0 + a\,\rho_0\right)\tfrac{\partial^2\eta^2}{\partial t^2} \\ &\quad = a\,\rho_1\,\rho_0\,\eta^1\tfrac{\partial^2\eta^2}{\partial t^2} + a\,\rho_1\,\rho_0\tfrac{\partial^2\eta^2}{\partial t^2} + a\,\eta^1\,\rho_0\tfrac{\partial^2\eta^2}{\partial t^2} \\ &\qquad\qquad + a\,\rho_0\tfrac{\partial^2\eta^2}{\partial t^2}, \\ &\left(\rho_1\rho_0 + \rho_0\right)\left(a\eta^1 + a\right)\Big[\sin\left(\eta^2 + \alpha\right)\Big]\tfrac{\partial^2}{\partial t^2}(\eta^3 + \beta) \\ &\quad = \left(a\,\rho_1\,\rho_0\,\eta^1 + a\,\rho_1\,\rho_0 + a\,\eta^1\,\rho_0 + a\,\rho_0\right)\tfrac{\partial^2\eta^3}{\partial t^2} \\ &\qquad\qquad \times\Big[\sin\left(\eta^2 + \alpha\right)\Big] \\ &\quad = \Big(a\,\rho_1\,\rho_0\,\eta^1\tfrac{\partial^2\eta^2}{\partial t^2} + a\,\rho_1\,\rho_0\tfrac{\partial^2\eta^3}{\partial t^2} + a\,\eta^1\,\rho_0\tfrac{\partial^2\eta^3}{\partial t^2} \\ &\qquad\qquad + a\,\rho_0\tfrac{\partial^2\eta^3}{\partial t^2}\Big)\Big[\sin\left(\eta^2 + \alpha\right)\Big].\end{aligned}\right\} \tag{J.21}$$

Since ρ_1 and η^i are small quantities, $\sin\eta^2 \approx \eta^2$ and $\cos\eta^2 \approx 1$. By linearising equations (J.21) we get

$$\left.\begin{aligned}
&(\rho_1\rho_0+\rho_0)\left(a\eta^1+a\right)\tfrac{\partial^2}{\partial t^2}(\eta^2+\alpha) = a\rho_0\tfrac{\partial^2\eta^2}{\partial t^2},\\
&(\rho_1\rho_0+\rho_0)\left(a\eta^1+a\right)\Big[\sin\left(\eta^2+\alpha\right)\Big]\tfrac{\partial^2}{\partial t^2}(\eta^3+\beta)\\
&\qquad = a\,\rho_0\,\tfrac{\partial^2\eta^3}{\partial t^2}\left(\sin\eta^2\sin\alpha+\cos\eta^2\cos\alpha\right)\\
&\qquad = a\,\rho_0\,\eta^2\,\sin\alpha\tfrac{\partial^2\eta^3}{\partial t^2}+a\,\rho_0\,\cos\alpha\,\tfrac{\partial^2\eta^3}{\partial t^2}\\
&\qquad = a\rho_0\cos\alpha\tfrac{\partial^2\eta^3}{\partial t^2}.
\end{aligned}\right\} \tag{J.22}$$

Using equations (J.19) and (J.22), (J.17) and (J.20) become

$$\left.\begin{aligned}
&a\,\rho_0(a)\tfrac{\partial^2\eta^1}{\partial t^2}(\vec{a},t)\\
&\qquad = -\tfrac{\partial P}{\partial r}(\vec{r},t)-\Big(\rho_1(\vec{a},t)\,\rho_0(a)+\rho_0(a)\Big)\tfrac{\partial\Phi}{\partial r}(\vec{r},t),\\
&a\,\rho_0(a)\,\tfrac{\partial^2\eta^2}{\partial t^2}(\vec{a},t)\\
&\qquad = -\tfrac{\partial P}{\partial\theta}(\vec{r},t)-\Big(\rho_1(\vec{a},t)\,\rho_0(a)+\rho_0(a)\Big)\tfrac{\partial\Phi}{\partial\theta}(\vec{r},t),\\
&a\,\cos\alpha\,\rho_0(a)\,\tfrac{\partial^2\eta^3}{\partial t^2}(\vec{a},t)\\
&\qquad = -\tfrac{\partial P}{\partial\phi}(\vec{r},t)-\Big(\rho_1(\vec{a},t)\,\rho_0(a)+\rho_0(a)\Big)\tfrac{\partial\Phi}{\partial\phi}(\vec{r},t).
\end{aligned}\right\} \tag{J.23}$$

Equations (J.23) contain terms that are evaluated in both the perturbed and unperturbed configuration. This is undesirable since we now need to determine the coordinate transformations that link the derivatives of $\vec{r}$ with those of $\vec{a}$. These equations are derived in Appendix K in the form of equations (K.9) - (K.11). Instead of moving from the perturbed frame to the unperturbed frame, we can also move the other way around by using coordinate transformations from $\vec{a}$ to $\vec{r}$. This is a matter of convenience. Because we need the transformations (K.9) - (K.11) to move forward, the calculations now become increasingly complex and difficult to manage. Since the intention of this appendix is to illustrate the complexity associated with using a pure Lagrangian perturbation in pulsation theory, it is adequate to focus on the radial equation. Using the form of equation (K.9) twice for $Q = P$ and $Q = \Phi$, equation one of (J.23) becomes[5]

$$\begin{aligned}
a\,\rho_0\,\frac{\partial^2\eta^1}{\partial t^2} =\\
&-\frac{1}{1+\eta^1+a\,\frac{\partial\eta^1}{\partial a}+\frac{\partial\eta^2}{\partial\alpha}+\frac{\partial\eta^3}{\partial\beta}}\\
&\qquad\times\left[\left(\left(1+\frac{\partial\eta^2}{\partial\alpha}+\frac{\partial\eta^3}{\partial\beta}\right)\frac{\partial\widetilde{P}}{\partial a}-\frac{\partial\eta^2}{\partial a}\frac{\partial\widetilde{P}}{\partial\alpha}-\frac{\partial\eta^3}{\partial a}\frac{\partial\widetilde{P}}{\partial\beta}\right)+(\rho_1\,\rho_0+\rho_0)\right.\\
&\qquad\left.\times\left(\left(1+\frac{\partial\eta^2}{\partial\alpha}+\frac{\partial\eta^3}{\partial\beta}\right)\frac{\partial\widetilde{\Phi}}{\partial a}-\frac{\partial\eta^2}{\partial a}\frac{\partial\widetilde{\Phi}}{\partial\alpha}-\frac{\partial\eta^3}{\partial a}\frac{\partial\widetilde{\Phi}}{\partial\beta}\right)\right].
\end{aligned} \tag{J.24}$$

After moving the fraction across the equals sign, it is easy to see through linearisation, the left hand side of equation (J.24) remains unchanged. The rest of

[5]Function arguments are omitted unless there is confusion

equation (J.24) becomes

$$
\begin{aligned}
a\,\rho_0\,\frac{\partial^2\eta^1}{\partial t^2} = &-\left(1+\frac{\partial\eta^2}{\partial\alpha}+\frac{\partial\eta^3}{\partial\beta}\right)\frac{\partial\widetilde{P}}{\partial a}+\frac{\partial\eta^2}{\partial a}\frac{\partial\widetilde{P}}{\partial\alpha}+\frac{\partial\eta^3}{\partial a}\frac{\partial\widetilde{P}}{\partial\beta}\\
&-\left[\left(1+\frac{\partial\eta^2}{\partial\alpha}+\frac{\partial\eta^3}{\partial\beta}\right)\frac{\partial\widetilde{\Phi}}{\partial a}-\frac{\partial\eta^2}{\partial a}\frac{\partial\widetilde{\Phi}}{\partial\alpha}-\frac{\partial\eta^3}{\partial a}\frac{\partial\widetilde{\Phi}}{\partial\beta}\right]\\
&\times\left(\rho_1\rho_0+\rho_0\right)\\
= &-\left(1+\frac{\partial\eta^2}{\partial\alpha}+\frac{\partial\eta^3}{\partial\beta}\right)\frac{\partial\widetilde{P}}{\partial a}+\frac{\partial\eta^2}{\partial a}\frac{\partial\widetilde{P}}{\partial\alpha}+\frac{\partial\eta^3}{\partial a}\frac{\partial\widetilde{P}}{\partial\beta}-\rho_0\,\rho_1\frac{\partial\widetilde{\Phi}}{\partial a}\\
&-\rho_0\left[\left(1+\frac{\partial\eta^2}{\partial\alpha}+\frac{\partial\eta^3}{\partial\beta}\right)\frac{\partial\widetilde{\Phi}}{\partial a}-\frac{\partial\eta^2}{\partial a}\frac{\partial\widetilde{\Phi}}{\partial\alpha}-\frac{\partial\eta^3}{\partial a}\frac{\partial\widetilde{\Phi}}{\partial\beta}\right]. \qquad \text{(J.25)}
\end{aligned}
$$

Since $Q\left(\vec{r},t\right)=\widetilde{Q}\left(\vec{a},t\right)$[6], using the form of equation (J.1) for both $Q=P$ and $Q=\Phi$ we have

$$
\left.\begin{aligned}
\widetilde{P}\left(\vec{a},t\right) &= P_1\left(\vec{a},t\right)P_0\left(a\right)+P_0\left(a\right),\\
\widetilde{\Phi}\left(\vec{a},t\right) &= \Phi_1\left(\vec{a},t\right)\Phi_0\left(a\right)+\Phi_0\left(a\right)
\end{aligned}\right\}. \qquad \text{(J.26)}
$$

Substituting equations (J.26) into equation (J.25) and linearising we get

$$
\begin{aligned}
a\rho_0\frac{\partial^2\eta^1}{\partial t^2} = &-\left(1+\frac{\partial\eta^2}{\partial\alpha}+\frac{\partial\eta^3}{\partial\beta}\right)\frac{\partial}{\partial a}\left(P_1P_0+P_0\right)\\
&+\frac{\partial\eta^2}{\partial a}\frac{dP_0}{d\alpha}+\frac{\partial\eta^3}{\partial a}\frac{dP_0}{d\beta}-\rho_0\,\rho_1\frac{d\Phi_0}{da}\\
&-\rho_0\left(1+\frac{\partial\eta^2}{\partial\alpha}+\frac{\partial\eta^3}{\partial\beta}\right)\frac{\partial}{\partial a}\left(\Phi_1\Phi_0+\Phi_0\right)\\
&+\rho_0\frac{\partial\eta^2}{\partial a}\frac{d\Phi_0}{d\alpha}+\rho_0\frac{\partial\eta^3}{\partial a}\frac{d\Phi_0}{d\beta}\\
= &-\frac{\partial}{\partial a}\left(P_1P_0\right)-\frac{dP_0}{da}-\frac{\partial\eta^2}{\partial\alpha}\frac{dP_0}{da}-\frac{\partial\eta^3}{\partial\beta}\frac{dP_0}{da}\\
&-\rho_0\,\rho_1\frac{d\Phi_0}{da}-\rho_0\frac{\partial}{\partial a}\left(\Phi_1\Phi_0\right)-\rho_0\frac{d\Phi_0}{da}\\
&-\rho_0\frac{\partial\eta^2}{\partial\alpha}\frac{d\Phi_0}{da}-\rho_0\frac{\partial\eta^3}{\partial\beta}\frac{d\Phi_0}{da}\\
= &-P_0\frac{\partial P_1}{\partial a}-P_1\frac{dP_0}{da}-\frac{\partial\eta^2}{\partial\alpha}\frac{dP_0}{da}-\frac{\partial\eta^3}{\partial\beta}\frac{dP_0}{da}\\
&-\rho_0\,\rho_1\frac{d\Phi_0}{da}-\rho_0\Phi_0\frac{\partial\Phi_1}{\partial a}-\rho_0\,\Phi_1\frac{d\Phi_0}{da}-\rho_0\frac{\partial\eta^2}{\partial\alpha}\frac{d\Phi_0}{da}\\
&-\rho_0\frac{\partial\eta^3}{\partial\beta}\frac{d\Phi_0}{da}-\left(\frac{dP_0}{da}+\rho_0\frac{d\Phi_0}{da}\right)\\
= &-P_0\frac{\partial P_1}{\partial a}-P_1\frac{dP_0}{da}-\frac{\partial\eta^2}{\partial\alpha}\left(\frac{dP_0}{da}+\rho_0\frac{d\Phi_0}{da}\right)
\end{aligned}
$$

[6] see Appendix K

$$
\begin{aligned}
&- \frac{\partial \eta^3}{\partial \beta}\left(\frac{dP_0}{da} + \rho_0 \frac{d\Phi_0}{da}\right) - \rho_0\,\rho_1 \frac{d\Phi_0}{da} \\
&\qquad - \rho_0\,\Phi_0 \frac{\partial \Phi_1}{\partial a} - \rho_0\,\Phi_1 \frac{d\Phi_0}{da} \\
&= -P_0 \frac{\partial P_1}{\partial a} - P_1 \frac{dP_0}{da} - \rho_0\,\rho_1 \frac{d\Phi_0}{da} - \rho_0\,\Phi_0 \frac{\partial \Phi_1}{\partial a} - \rho_0\,\Phi_1 \frac{d\Phi_0}{da}. \qquad \text{(J.27)}
\end{aligned}
$$

Since all equilibrium values are only functions of a, any derivative with respect to α and β of P_0 and Φ_0 is zero and has been removed. Euler's equation evaluated in the unperturbed configuration, see equation three of (3.8) with $r = a$, has also been used to remove some terms.

Equation (J.27) is the radial component of the perturbed Euler equation, evaluated in the unperturbed reference frame. The Lagrangian specification of fluids follows a single fluid element p from its original position $\vec{a}$ until its final position $\vec{r}$. Since the definition of the Lagrangian perturbation[7] relates quantities evaluated in both the unperturbed and perturbed configuration, the resulting equations include terms that are evaluated in both these frames. We then have to move the equation to be universally evaluated in either frame, which is done using coordinate transformations. It is these coordinate transformations that overly complicate the calculations, making this approach unfavourable. The Eulerian perturbation[8], with all its components, is evaluated entirely in the perturbed configuration and does not require a transformation of reference frames.

J.3 Polytropic Equation of State

In the perturbed configuration, equation (G.13) is defined as

$$
P\left(\vec{r}, t\right) = k\Big(\rho\left(\vec{r}, t\right)\Big)^{\gamma'} \qquad \text{(J.28)}
$$

Substituting the form of equation (J.1) for both $Q = P$ and $Q = \rho$ into equation (J.28)[9] we get

$$
\begin{aligned}
P_1 P_0 + P_0 &= \kappa\left(\rho_1\,\rho_0 + \rho_0\right)^{\gamma'} \\
&= k\,\rho_0^{\gamma'}\left(\rho_1 + 1\right)^{\gamma'} \\
&= P_0\left(\rho_1 + 1\right)^{\gamma'}
\end{aligned}
$$

$$
P_1 + 1 = \left(\rho_1 + 1\right)^{\gamma'}, \qquad \text{(J.29)}
$$

where equation four of (3.8) has been used to replace ρ_0 with P_0. Since ρ_1 is a small quantity we have that $\left(\rho_1 + 1\right)^{\gamma'} \approx \gamma' \rho_1 + 1$. Equation (J.29) is then

[7] equation two of (3.4)
[8] equation one of (3.4)
[9] Function arguments are omitted unless there is confusion

$$P_1 + 1 = \gamma' \rho_1 + 1$$

$$P_1 = \gamma' \rho_1 \tag{J.30}$$

Equation (J.30) is the equation for the perturbed polytropic equation of state, evaluated in the unperturbed reference frame.

J.4 Poisson's Equation

In the perturbed configuration, equation three of (A.1) is defined as

$$\begin{aligned} 4\pi G\rho\left(\vec{r},t\right) &= \nabla^2\Phi\left(\vec{r},t\right) \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)\left(\vec{r},t\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)\left(\vec{r},t\right) \\ &\qquad + \frac{1}{r^2\sin\theta^2}\frac{\partial^2\Phi}{\partial\phi^2}\left(\vec{r},t\right) \\ &= \frac{2}{r}\frac{\partial\Phi}{\partial r}\left(\vec{r},t\right) + \frac{\partial^2\Phi}{\partial r^2}\left(\vec{r},t\right) + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial\Phi}{\partial\theta}\left(\vec{r},t\right) \\ &\qquad + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial^2\theta}\left(\vec{r},t\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}\left(\vec{r},t\right) \end{aligned} \tag{J.31}$$

This is the most difficult of all Lagrangian perturbation equations to compute since we now have second order derivatives for $\vec{r}$. In appendix K we determined coordinate transformations for the first order derivatives of $\vec{r}$, which were lengthy and tedious. To determine the second order coordinate transformations from $\vec{r}$ to $\vec{a}$ are even more difficult and will require calculating Christoffel symbols of the second kind $\Gamma^i_{k\,l}$.

Appendix K

Lagrangian Coordinate Transformations

We are interested in finding the coordinate transformation of the derivatives $\frac{\partial}{\partial \vec{r}}$ of an arbitrary function $Q(\vec{r},t)$ from coordinates $\vec{r}$, defined in the perturbed reference frame to coordinates $\vec{a}$, defined in the unperturbed reference frame. Since $\vec{r}=\vec{f}(\vec{a},t)$ we can use the following relationship[1]

$$\begin{aligned} Q(\vec{r},t) &= Q(\vec{f},t) \\ &= \widetilde{Q}(\vec{a},t). \end{aligned}$$

Taking the derivative of $\widetilde{Q}$ with respect to a, α and β we get

$$\begin{aligned} \frac{\partial \widetilde{Q}}{\partial a}(\vec{a},t) = \frac{\partial Q}{\partial r}(\vec{r},t)\frac{\partial r}{\partial a}(\vec{a},t) + \frac{\partial Q}{\partial \theta}(\vec{r},t)\frac{\partial \theta}{\partial a}(\vec{a},t) + \\ \frac{\partial Q}{\partial \phi}(\vec{r},t)\frac{\partial \phi}{\partial a}(\vec{a},t), \end{aligned}$$

$$\begin{aligned} \frac{\partial \widetilde{Q}}{\partial \alpha}(\vec{a},t) = \frac{\partial Q}{\partial r}(\vec{r},t)\frac{\partial r}{\partial \alpha}(\vec{a},t) + \frac{\partial Q}{\partial \theta}(\vec{r},t)\frac{\partial \theta}{\partial \alpha}(\vec{a},t) + \\ \frac{\partial Q}{\partial \phi}(\vec{r},t)\frac{\partial \phi}{\partial \alpha}(\vec{a},t), \end{aligned}$$

$$\begin{aligned} \frac{\partial \widetilde{Q}}{\partial \beta}(\vec{a},t) = \frac{\partial Q}{\partial r}(\vec{r},t)\frac{\partial r}{\partial \beta}(\vec{a},t) + \frac{\partial Q}{\partial \theta}(\vec{r},t)\frac{\partial \theta}{\partial \alpha}(\vec{a},t) + \\ \frac{\partial Q}{\partial \phi}(\vec{r},t)\frac{\partial \phi}{\partial \beta}(\vec{a},t). \end{aligned}$$

[1]The function $\widetilde{Q}$ is not Q evaluated in the unperturbed reference frame but the symbol given to another arbitrary function

This result can be rewritten in matrix notation such that[2]

$$\begin{pmatrix} \frac{\partial \widetilde{Q}}{\partial a} \\ \\ \frac{\partial \widetilde{Q}}{\partial \alpha} \\ \\ \frac{\partial \widetilde{Q}}{\partial \beta} \end{pmatrix} = \begin{pmatrix} \frac{\partial r}{\partial a} & \frac{\partial \theta}{\partial a} & \frac{\partial \phi}{\partial a} \\ \\ \frac{\partial r}{\partial \alpha} & \frac{\partial \theta}{\partial \alpha} & \frac{\partial \phi}{\partial \alpha} \\ \\ \frac{\partial r}{\partial \beta} & \frac{\partial \theta}{\partial \alpha} & \frac{\partial \phi}{\partial \beta} \end{pmatrix} \begin{pmatrix} \frac{\partial Q}{\partial r} \\ \\ \frac{\partial Q}{\partial \theta} \\ \\ \frac{\partial Q}{\partial \phi} \end{pmatrix}$$

or

$$\frac{\partial \widetilde{Q}}{\partial a^\sigma} = A^\sigma_\gamma \frac{\partial Q}{\partial r^\gamma},$$

where A^σ_γ is the transformation matrix for the coordinate transformation from $\vec{a}$ to $\vec{r}$. Using equations (J.1) with their definitions (J.4), matrix A^σ_γ becomes

$$A^\sigma_\gamma = \begin{pmatrix} 1 + \eta^1 + a\,\frac{\partial \eta^1}{\partial a} & \frac{\partial \eta^2}{\partial a} & \frac{\partial \eta^3}{\partial a} \\ \\ a\,\frac{\partial \eta^1}{\partial \alpha} & 1 + \frac{\partial \eta^2}{\partial \alpha} & \frac{\partial \eta^3}{\partial \alpha} \\ \\ a\,\frac{\partial \eta^1}{\partial \beta} & \frac{\partial \eta^2}{\partial \beta} & 1 + \frac{\partial \eta^3}{\partial \beta} \end{pmatrix}. \tag{K.1}$$

To find the transformation from coordinates $\vec{r}$ to $\vec{a}$ we need to find the inverse of the matrix A^σ_γ, or A^γ_σ such that

$$\frac{\partial Q}{\partial r^\gamma} = A^\gamma_\sigma \frac{\partial \widetilde{Q}}{\partial a^\sigma}.$$

To do this we use the Adjoint method, defined as

$$A^\sigma_\gamma \; adj\left(A^\sigma_\gamma\right) = det(A^\sigma_\gamma)\; I, \tag{K.2}$$

where $adj(A^\sigma_\gamma)$ and $det(A^\sigma_\gamma)$ are the adjoint and determinant of matrix A^σ_γ respectively and I the identity matrix. Equation (K.2) can be rearranged to give

$$\begin{aligned} \left(A^\sigma_\gamma\right)^{-1} &= A^\gamma_\sigma \\ &= \frac{adj(A^\sigma_\gamma)}{det(A^\sigma_\gamma)}. \end{aligned} \tag{K.3}$$

Using the form of matrix A^σ_γ defined by (K.1), the determinant $det(A^\sigma_\gamma)$ is then

$$\begin{aligned} det(A^\sigma_\gamma) = \left(1 + \eta^1 + a\,\frac{\partial \eta^1}{\partial a}\right) &\left[\left(1 + \frac{\partial \eta^2}{\partial \alpha}\right)\left(1 + \frac{\partial \eta^3}{\partial \beta}\right) - \frac{\partial \eta^3}{\partial \alpha}\frac{\partial \eta^2}{\partial \beta}\right] \\ - \frac{\partial \eta^2}{\partial a} &\left[a\,\frac{\partial \eta^1}{\partial \alpha}\left(1 + \frac{\partial \eta^3}{\partial \beta}\right) - a\,\frac{\partial \eta^1}{\partial \beta}\frac{\partial \eta^3}{\partial \alpha}\right] \\ + \frac{\partial \eta^3}{\partial a} &\left[a\,\frac{\partial \eta^1}{\partial \alpha}\frac{\partial \eta^2}{\partial \beta} - a\left(1 + \frac{\partial \eta^2}{\partial \alpha}\right)\frac{\partial \eta^1}{\partial \beta}\right]. \end{aligned} \tag{K.4}$$

[2]Function arguments are omitted unless there is confusion

Since $\vec{\eta}$ is a small quantity, linearising equation (K.4) we get

$$\begin{aligned} det(A^{\sigma}_{\gamma}) &= \left(1+\eta^1+a\,\frac{\partial\eta^1}{\partial a}\right)\left(1+\frac{\partial\eta^2}{\partial\alpha}+\frac{\partial\eta^3}{\partial\beta}\right) \\ &= 1+\eta^1+a\,\frac{\partial\eta^1}{\partial a}+\frac{\partial\eta^2}{\partial\alpha}+\eta^1\frac{\partial\eta^2}{\partial\alpha}+a\,\frac{\partial\eta^1}{\partial a}\frac{\partial\eta^2}{\partial\alpha}+\frac{\partial\eta^3}{\partial\beta} \\ &\quad +\eta^1\frac{\partial\eta^3}{\partial\beta}+a\,\frac{\partial\eta^1}{\partial a}\frac{\partial\eta^3}{\partial\beta} \\ &= 1+\eta^1+a\,\frac{\partial\eta^1}{\partial a}+\frac{\partial\eta^2}{\partial\alpha}+\frac{\partial\eta^3}{\partial\beta}. \end{aligned} \tag{K.5}$$

The adjoint $adj(A^{\sigma}_{\gamma})$ of matrix A^{σ}_{γ} can be determined by calculating the transpose of the cofactor $\left(c^{\sigma}_{\gamma}\right)$ of A^{σ}_{γ} such that

$$\begin{aligned} adj(A^{\sigma}_{\gamma}) &= c^{\gamma}_{\sigma} \\ &= \left(c^{\sigma}_{\gamma}\right)^T \\ &= \left[(-1)^{\,\sigma+\gamma}\,B^{\sigma}_{\gamma}\right]^T, \end{aligned}$$

where B^{σ}_{γ} is the determinant for each entry in matrix A^{σ}_{γ} such that

$$B^{\sigma}_{\gamma} = \begin{pmatrix} \begin{vmatrix} 1+\frac{\partial\eta^2}{\partial\alpha} & \frac{\partial\eta^3}{\partial\alpha} \\ \frac{\partial\eta^2}{\partial\beta} & 1+\frac{\partial\eta^3}{\partial\beta} \end{vmatrix} & \begin{vmatrix} a\,\frac{\partial\eta^1}{\partial\alpha} & \frac{\partial\eta^3}{\partial\alpha} \\ a\,\frac{\partial\eta^1}{\partial\beta} & 1+\frac{\partial\eta^3}{\partial\beta} \end{vmatrix} & \begin{vmatrix} a\,\frac{\partial\eta^1}{\partial\alpha} & 1+\frac{\partial\eta^2}{\partial\alpha} \\ a\,\frac{\partial\eta^1}{\partial\beta} & \frac{\partial\eta^2}{\partial\beta} \end{vmatrix} \\ \begin{vmatrix} \frac{\partial\eta^2}{\partial a} & \frac{\partial\eta^3}{\partial a} \\ \frac{\partial\eta^2}{\partial\beta} & 1+\frac{\partial\eta^3}{\partial\beta} \end{vmatrix} & \begin{vmatrix} 1+\eta^1+a\,\frac{\partial\eta^1}{\partial a} & \frac{\partial\eta^3}{\partial a} \\ a\,\frac{\partial\eta^1}{\partial\beta} & 1+\frac{\partial\eta^3}{\partial\beta} \end{vmatrix} & \begin{vmatrix} 1+\eta^1+a\,\frac{\partial\eta^1}{\partial a} & \frac{\partial\eta^2}{\partial a} \\ a\,\frac{\partial\eta^1}{\partial\beta} & \frac{\partial\eta^2}{\partial\beta} \end{vmatrix} \\ \begin{vmatrix} \frac{\partial\eta^2}{\partial a} & \frac{\partial\eta^3}{\partial a} \\ 1+\frac{\partial\eta^2}{\partial\alpha} & \frac{\partial\eta^3}{\partial\alpha} \end{vmatrix} & \begin{vmatrix} 1+\eta^1+a\,\frac{\partial\eta^1}{\partial a} & \frac{\partial\eta^3}{\partial a} \\ a\,\frac{\partial\eta^1}{\partial\alpha} & \frac{\partial\eta^3}{\partial\alpha} \end{vmatrix} & \begin{vmatrix} 1+\eta^1+a\,\frac{\partial\eta^1}{\partial a} & \frac{\partial\eta^2}{\partial a} \\ a\,\frac{\partial\eta^1}{\partial\alpha} & 1+\frac{\partial\eta^2}{\partial\alpha} \end{vmatrix} \end{pmatrix}.$$

Using the first three entries B^1_1, B^1_2 and B^1_3 as an example, the determinants can be calculated as follow

$$\begin{aligned} B^1_1 &= \begin{vmatrix} 1+\frac{\partial\eta^2}{\partial\alpha} & \frac{\partial\eta^3}{\partial\alpha} \\ \frac{\partial\eta^2}{\partial\beta} & 1+\frac{\partial\eta^3}{\partial\beta} \end{vmatrix} \\ &= \left(1+\frac{\partial\eta^2}{\partial\alpha}\right)\left(1+\frac{\partial\eta^3}{\partial\beta}\right)-\frac{\partial\eta^3}{\partial\alpha}\frac{\partial\eta^2}{\partial\beta} \\ &= 1+\frac{\partial\eta^2}{\partial\alpha}+\frac{\partial\eta^3}{\partial\beta}+\frac{\partial\eta^2}{\partial\alpha}\frac{\partial\eta^3}{\partial\beta}-\frac{\partial\eta^3}{\partial\alpha}\frac{\partial\eta^2}{\partial\beta}, \end{aligned}$$

$$B^1_2 = \begin{vmatrix} a\,\frac{\partial\eta^1}{\partial\alpha} & \frac{\partial\eta^3}{\partial\alpha} \\ a\,\frac{\partial\eta^1}{\partial\beta} & 1+\frac{\partial\eta^3}{\partial\beta} \end{vmatrix}$$

$$
\begin{aligned}
&= a\,\frac{\partial\eta^1}{\partial\alpha}\left(1+\frac{\partial\eta^3}{\partial\beta}\right) - a\,\frac{\partial\eta^1}{\partial\beta}\frac{\partial\eta^3}{\partial\alpha}\\
&= a\,\frac{\partial\eta^1}{\partial\alpha} + a\,\frac{\partial\eta^1}{\partial\alpha}\frac{\partial\eta^3}{\partial\beta} - a\,\frac{\partial\eta^1}{\partial\beta}\frac{\partial\eta^3}{\partial\alpha},
\end{aligned}
$$

$$
\begin{aligned}
B_3^1 &= \begin{vmatrix} a\,\frac{\partial\eta^1}{\partial\alpha} & 1+\frac{\partial\eta^2}{\partial\alpha} \\ a\,\frac{\partial\eta^1}{\partial\beta} & \frac{\partial\eta^2}{\partial\beta} \end{vmatrix}\\
&= a\,\frac{\partial\eta^1}{\partial\alpha}\frac{\partial\eta^2}{\partial\beta} - a\,\frac{\partial\eta^1}{\partial\beta}\left(1+\frac{\partial\eta^3}{\partial\beta}\right)\\
&= a\,\frac{\partial\eta^1}{\partial\alpha}\frac{\partial\eta^2}{\partial\beta} - a\,\frac{\partial\eta^1}{\partial\beta} + a\,\frac{\partial\eta^1}{\partial\beta}\frac{\partial\eta^3}{\partial\beta}.
\end{aligned}
$$

By linearising these equations we get

$$
\begin{aligned}
B_1^1 &= 1+\frac{\partial\eta^2}{\partial\alpha}+\frac{\partial\eta^3}{\partial\beta},\\
B_2^1 &= a\,\frac{\partial\eta^1}{\partial\alpha},\\
B_3^1 &= -a\,\frac{\partial\eta^1}{\partial\beta}.
\end{aligned}
$$

In a similar way, we can calculate the remaining determinants in B_γ^σ to give

$$
B_\gamma^\sigma = \begin{pmatrix} 1+\frac{\partial\eta^2}{\partial\alpha}+\frac{\partial\eta^3}{\partial\beta} & a\,\frac{\partial\eta^1}{\partial\alpha} & -a\,\frac{\partial\eta^1}{\partial\beta} \\ \frac{\partial\eta^2}{\partial a} & 1+\eta^1+a\,\frac{\partial\eta^1}{\partial a}+\frac{\partial\eta^3}{\partial\beta} & \frac{\partial\eta^2}{\partial\beta} \\ -\frac{\partial\eta^3}{\partial a} & \frac{\partial\eta^3}{\partial\alpha} & 1+\eta^1+a\,\frac{\partial\eta^1}{\partial a}+\frac{\partial\eta^2}{\partial\alpha} \end{pmatrix}. \quad \text{(K.6)}
$$

Using matrix (K.6), c_γ^σ is then

$$
\begin{aligned}
c_\gamma^\sigma &= (-1)^{\sigma+\gamma}\\
&\quad\times\begin{pmatrix} 1+\frac{\partial\eta^2}{\partial\alpha}+\frac{\partial\eta^3}{\partial\beta} & a\,\frac{\partial\eta^1}{\partial\alpha} & -a\,\frac{\partial\eta^1}{\partial\beta} \\ \frac{\partial\eta^2}{\partial a} & 1+\eta^1+a\,\frac{\partial\eta^1}{\partial a}+\frac{\partial\eta^3}{\partial\beta} & \frac{\partial\eta^2}{\partial\beta} \\ -\frac{\partial\eta^3}{\partial a} & \frac{\partial\eta^3}{\partial\alpha} & 1+\eta^1+a\,\frac{\partial\eta^1}{\partial a}+\frac{\partial\eta^2}{\partial\alpha} \end{pmatrix}\\
&= \begin{pmatrix} 1+\frac{\partial\eta^2}{\partial\alpha}+\frac{\partial\eta^3}{\partial\beta} & -a\,\frac{\partial\eta^1}{\partial\alpha} & -a\,\frac{\partial\eta^1}{\partial\beta} \\ -\frac{\partial\eta^2}{\partial a} & 1+\eta^1+a\,\frac{\partial\eta^1}{\partial a}+\frac{\partial\eta^3}{\partial\beta} & -\frac{\partial\eta^2}{\partial\beta} \\ -\frac{\partial\eta^3}{\partial a} & -\frac{\partial\eta^3}{\partial\alpha} & 1+\eta^1+a\,\frac{\partial\eta^1}{\partial a}+\frac{\partial\eta^2}{\partial\alpha} \end{pmatrix}.
\end{aligned}
$$

The adjoint of matrix A^{σ}_{γ} is then

$$
\begin{aligned}
&adj(A^{\sigma}_{\gamma}) \\
&= \begin{pmatrix}
1+\frac{\partial\eta^2}{\partial\alpha}+\frac{\partial\eta^3}{\partial\beta} & -a\,\frac{\partial\eta^1}{\partial\alpha} & -a\,\frac{\partial\eta^1}{\partial\beta} \\
-\frac{\partial\eta^2}{\partial a} & 1+\eta^1+a\,\frac{\partial\eta^1}{\partial a}+\frac{\partial\eta^3}{\partial\beta} & -\frac{\partial\eta^2}{\partial\beta} \\
-\frac{\partial\eta^3}{\partial a} & -\frac{\partial\eta^3}{\partial\alpha} & 1+\eta^1+a\,\frac{\partial\eta^1}{\partial a}+\frac{\partial\eta^2}{\partial\alpha}
\end{pmatrix}^T \\
&= \begin{pmatrix}
1+\frac{\partial\eta^2}{\partial\alpha}+\frac{\partial\eta^3}{\partial\beta} & -\frac{\partial\eta^2}{\partial a} & -\frac{\partial\eta^3}{\partial a} \\
-a\,\frac{\partial\eta^1}{\partial\alpha} & 1+\eta^1+a\,\frac{\partial\eta^1}{\partial a}+\frac{\partial\eta^3}{\partial\beta} & -\frac{\partial\eta^3}{\partial\alpha} \\
-a\,\frac{\partial\eta^1}{\partial\beta} & -\frac{\partial\eta^2}{\partial\beta} & 1+\eta^1+a\,\frac{\partial\eta^1}{\partial a}+\frac{\partial\eta^2}{\partial\alpha}
\end{pmatrix}. \qquad \text{(K.7)}
\end{aligned}
$$

Using equation (K.3), with equation (K.5) for $det(A^{\sigma}_{\gamma})$ and matrix (K.7) for $adj(A^{\sigma}_{\gamma})$, the inverse matrix A^{γ}_{σ} is calculated to be

$$
\begin{aligned}
A^{\gamma}_{\sigma} = &\begin{pmatrix}
1+\frac{\partial\eta^2}{\partial\alpha}+\frac{\partial\eta^3}{\partial\beta} & -\frac{\partial\eta^2}{\partial a} & -\frac{\partial\eta^3}{\partial a} \\
-a\,\frac{\partial\eta^1}{\partial\alpha} & 1+\eta^1+a\,\frac{\partial\eta^1}{\partial a}+\frac{\partial\eta^3}{\partial\beta} & -\frac{\partial\eta^3}{\partial\alpha} \\
-a\,\frac{\partial\eta^1}{\partial\beta} & -\frac{\partial\eta^2}{\partial\beta} & 1+\eta^1+a\,\frac{\partial\eta^1}{\partial a}+\frac{\partial\eta^2}{\partial\alpha}
\end{pmatrix} \\
&\times \frac{1}{1+\eta^1+a\,\frac{\partial\eta^1}{\partial a}+\frac{\partial\eta^2}{\partial\alpha}+\frac{\partial\eta^3}{\partial\beta}}. \qquad \text{(K.8)}
\end{aligned}
$$

Using equation (K.8), the transformation equations from coordinates $\vec{r}$ to $\vec{a}$ are then

$$
\begin{aligned}
\frac{\partial Q}{\partial r} &= A^1_1\frac{\partial\widetilde{Q}}{\partial a}+A^1_2\frac{\partial\widetilde{Q}}{\partial\alpha}+A^1_3\frac{\partial\widetilde{Q}}{\partial\phi} \\
&= \frac{1}{1+\eta^1+a\,\frac{\partial\eta^1}{\partial a}+\frac{\partial\eta^2}{\partial\alpha}+\frac{\partial\eta^3}{\partial\beta}} \\
&\qquad \times\left[\left(1+\frac{\partial\eta^2}{\partial\alpha}+\frac{\partial\eta^3}{\partial\beta}\right)\frac{\partial\widetilde{Q}}{\partial a}-\frac{\partial\eta^2}{\partial a}\frac{\partial\widetilde{Q}}{\partial\alpha}-\frac{\partial\eta^3}{\partial a}\frac{\partial\widetilde{Q}}{\partial\beta}\right], \qquad \text{(K.9)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial Q}{\partial\theta} &= A^2_1\frac{\partial\widetilde{Q}}{\partial a}+A^2_2\frac{\partial\widetilde{Q}}{\partial\alpha}+A^2_3\frac{\partial\widetilde{Q}}{\partial\beta} \\
&= \frac{1}{1+\eta^1+a\,\frac{\partial\eta^1}{\partial a}+\frac{\partial\eta^2}{\partial\alpha}+\frac{\partial\eta^3}{\partial\beta}} \\
&\qquad \times\left[-a\,\frac{\partial\eta^1}{\partial\alpha}\frac{\partial\widetilde{Q}}{\partial a}+\left(1+\eta^1+a\,\frac{\partial\eta^1}{\partial a}+\frac{\partial\eta^3}{\partial\beta}\right)\frac{\partial\widetilde{Q}}{\partial\alpha}\right. \\
&\qquad\qquad \left.-\frac{\partial\eta^3}{\partial\alpha}\frac{\partial\widetilde{Q}}{\partial\beta}\right], \qquad \text{(K.10)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial Q}{\partial \phi} &= A_1^3 \frac{\partial \widetilde{Q}}{\partial a} + A_2^3 \frac{\partial \widetilde{Q}}{\partial \alpha} + A_3^3 \frac{\partial \widetilde{Q}}{\partial \beta} \\
&= \frac{1}{1 + \eta^1 + a \frac{\partial \eta^1}{\partial a} + \frac{\partial \eta^2}{\partial \alpha} + \frac{\partial \eta^3}{\partial \beta}} \\
&\qquad \times \left[-a \frac{\partial \eta^1}{\partial \beta} \frac{\partial \widetilde{Q}}{\partial a} + \left(1 + \eta^1 + a \frac{\partial \eta^1}{\partial a} + \frac{\partial \eta^2}{\partial \alpha} \right) \frac{\partial \widetilde{Q}}{\partial \beta} \right. \\
&\qquad\qquad \left. - \frac{\partial \eta^2}{\partial \beta} \frac{\partial \widetilde{Q}}{\partial \alpha} \right] . \qquad \text{(K.11)}
\end{aligned}
$$

Appendix L

Computation

L.1 Introduction

The code found within this chapter is available with the electronic copy of the thesis in a directory called Computation. It uses a combination of C++, Unix Make and Gnuplot for the unperturbed density calculation and Octave for the pulsation frequency calculation.

The files are listed in no particular order with their contents briefly explained. Most of the code contains comments and definitions, which should help the reader follow it more easily.

L.1.1 Unperturbed Simulation - C++ Code

Definitions

The text file below, called Definitions.txt, contains definitions and brief explanations of all variables, parameters and functions used in this simulation.

File L.1: Definitions.txt

```
#####################################
#####        Definitions        #####
#####################################

########################
###    Variables    ###
########################

gam                  -  1 + 1/n (n - Polytropic index)
delta_mu             -  1 / Number of steps       "Step-Size"
cur_step             -  Current step in the simulation
Parameters           -  File variable
Eta_Values           -  File variable
Eta_Values_Graph     -  File variable
mu_i                 -  i can be any int and represents the value for mu
                        at the cur_step i
mu                   -  Current value of mu used in while loop
eta_i                -  i can be any int and represents the value for eta
                        at the cur_step i
eta_prev             -  Previous value for eta at current step
eta_cur              -  Current value for eta at current step
eta_next             -  Next value for eta at current step
trigger              -  Variable used to break out of while loop if a
                        condition is met
```

```
top                  -  Variable assigned to an equation to simplify the
                        reading of code
bottom               -  Variable assigned to an equation to simplify the
                        reading of code

########################
###    Functions    ###
########################

eta_m(mu)  -  Function used to calculate the first value of eta at
              mu = delta_mu
              Uses --> mu
gamma_prime()  -  Function used to calculate the value gamma for use in the
                  polytropic equation of state
                  Uses --> pol_in

###################################
###     Constants\Parameters     ###
###################################

"Parameters"

cent_rho   -  Central density of the star used in the model
mean_rad   -  Mean radius of the star used in the model
pol_in     -  Polytropic index
num_steps  -  Number of steps we wish to take in the simulation (prompted for
                  input during simulation)
```

Makefile

The Make file below, called Makefile, is used to compile and build the code as well as provide some easy commands for plotting the results and cleaning directories.

File L.2: Makefile

```
main_dir = ./Main/
data_dir = ./Results/
graph_dir = ./Graphs/
gnu_instr_dir = ./Main/Plots/

Unpert_Calc: $(main_dir)Unpert_Calc.o
	g++ -o Unpert_Calc $(main_dir)Unpert_Calc.o

Unpert_Calc.o: $(main_dir)Unpert_Calc.cpp $(main_dir)Unpert_Parameters.h
	g++ -c $(main_dir)Unpert_Calc.cpp

plot:
	./Unpert_Calc
	@ gnuplot $(gnu_instr_dir)eta_vs_mu.p
	@ gnuplot $(gnu_instr_dir)rho_vs_r.p

clean:
	@ echo Cleaning up...
	@ rm $(main_dir)Unpert_Calc.o
	@ rm -r $(graph_dir)*
	@ rm -r $(data_dir)*
	@ rm Unpert_Calc
	@ echo "Done!"
```

Gnuplot Configuration Files

The gnuplot file below, called Eta_VS_Mu.p, is used to hold the plot parameters for the η_0 vs μ plot. The labels and title of each graph can be changed, however the below template is identical for both the non-dimensional and dimensional density plots. This set of rules can be called from the Makefile.

File L.3: Eta_VS_Mu.p

```
set autoscale                           # scale axes automatically
unset log                               # remove any log-scaling
unset label                             # remove any previous labels
set xtic auto                           # set xtics automatically
set ytic auto                           # set ytics automatically
set xlabel "{/Symbol m}"
set ylabel "{/Symbol h}"
set term jpeg
set key bottom
set output "./Graphs/Eta_VS_Mu.jpeg"
plot "./Results/Eta_Values_Graph.dat" using 1:2 title '{/Symbol h} vs {/
     Symbol m} up to {/Symbol m} = 1 (r = r_c) for 100 steps'
```

Header files

The header file below, called Unpert_Parameters.h, is used to hold all parameters for the simulation.

File L.4: Unpert_Parameters.h

```
#ifndef PARA1_H
#define PARA1_H

/*-------------------------------------------------------------------------
See "Definitions.txt" file for explanations of what each parameter means

These parameters represent those of the sun. Therefore a polytropic index
of 3 is used (This index value is commonly used to model main sequence
stars).
-------------------------------------------------------------------------*/

// Model Parameters

static const double cent_rho = 1.622e5;
static const double mean_rad = 6.957e8;
static const double pol_in = 3.0;

#endif
```

Unperturbed Density (η_0) Values for $0 < \mu < 1$ and $0 < \mu < 5$

The C++ file below, called Unpert_Calc.cpp, is the main program that calculates values for η_0 using equation (4.10). It is used to calculate values for the unperturbed density for two scenarios. Until $\mu = 1$ or $r = r_{max} = r_{mean}$ where r_{mean} is the mean radius of the sun, and until $\mu = 5$ or $r = r_{max} = 5 * r_{mean}$.

File L.5: Unpert_Calc.cpp

```
#include <cmath>
#include <iostream>
#include <fstream>
#include "Unpert_Parameters.h"

/*-------------------------------------------------------------------------
See "Definitions.txt" file for explanations of what each parameter means
and each function does
-------------------------------------------------------------------------*/

using namespace std;

double eta_m(double mu)
{
        return 1.0 + ((1.0/6) * pow(mu, 2)) + ((pol_in/120) * pow(mu, 4));
}
```

```
double gamma_prime()
{
        return (1.0 + pol_in) / pol_in;
}

int main()
{
        // Declare ofstream variables
        ofstream Parameters,
                 Eta_Values,
                 Eta_Values_Graph,
                 Rho_Values,
                 Rho_Values_Graph;

        int num_steps;

        // Prompt for num_steps on run
        cout << "Please specify number of steps: " << endl;
        cin >> num_steps;

        /*
        delta_r = r_max / num_steps, however: mu = r / r_c
        --> delta_mu = delta_r / r_c
        --> delta_mu = r_max / (r_c * num_steps)
        --> if r_max = r_mean where r_c = r_mean then delta_mu = 1 /
            num_steps
        --> if r_max = 5 * r_mean where r_c = r_mean then delta_mu = 5 /
            num_steps
        */
        double delta_mu = 1.0 / num_steps, // 5.0 / num_steps
               gam = gamma_prime();

        // Open file Parameters.dat
        Parameters.open("./Results/Parameters.dat");

        /*
        Write simulation parameters for use by the perturbation
        simulation to file
        */
        Parameters << "Step Size:"
                   << endl
                   << delta_mu
                   << endl << endl
                   << "Number of Steps:"
                   << endl
                   << num_steps
                   << endl << endl
                   << "Gamma:"
                   << endl
                   << gam
                   << endl << endl
                   << "r_c:"
                   << endl
                   << mean_rad
                   << endl << endl
                   << "rho_c:"
                   << endl
                   << cent_rho
                   << endl;

        Parameters.close();

        /*
        Shell specific radius at centre of star is mu = 0, at next step
        is delta_mu
        */
        double mu_0 = 0.0;
        double mu_1 = delta_mu;

        // Initial values for eta at mu = 0, mu_1 = delta_mu
        double eta_0 = 1.0;
        double eta_1 = eta_m(mu_1);

    // Can use eta_m to calculate first two density values, instead i used
```

```
 initial conditions.
// Kept for reference
// eta_0 = eta_m(mu_0);
// eta_1 = eta_m(mu_1);

/*
 * Open files Eta_Values.dat, Eta_Values_Graph.dat, Rho_Values.dat
 * and Rho_Values_Graph.dat
 */
Eta_Values.open("./Results/Eta_Values.dat");
Eta_Values_Graph.open("./Results/Eta_Values_Graph.dat");
Rho_Values.open("./Results/Rho_Values.dat");
Rho_Values_Graph.open("./Results/Rho_Values_Graph.dat");

// Print initial values to files
Eta_Values << eta_0
           << endl
           << eta_1
           << endl;

Eta_Values_Graph << mu_0 << "        " << eta_0
                 << endl
                 << mu_1 << "        " << eta_1
                 << endl;

// Print initial rho values to files
Rho_Values << eta_0 * cent_rho
           << endl
           << eta_1 * cent_rho
           << endl;

Rho_Values_Graph << mu_0 * mean_rad << "        " << eta_0
                     * cent_rho
                 << endl
                 << mu_1 * mean_rad << "        " << eta_1
                     * cent_rho
                 << endl;

////////////////////////////////////
// Numerical Simulation           //
// set to run till mu = 1 or mu = 5 //
////////////////////////////////////

double mu,
       top,
       bottom,
       eta_prev = eta_0,
       eta_cur = eta_1,
       eta_next;

/*
We will already know values for cur_step = 0 and cur_step = 1,
so we begin at step 2
*/
for (cur_step = 2; cur_step < num_steps - 1; cur_step++)
{
        // Calculate current mu depending on place in simulation
        mu = cur_step * delta_mu;

        // Break calculation up to make it more readable
        top = ((delta_mu / mu) - 1) * pow(eta_prev, (gam - 1))
                + 2 * pow(eta_cur, (gam - 1))
                  - (pow(delta_mu, 2) * eta_cur)

        bottom = ((delta_mu / mu) + 1)

        /*
        Calculate next value for eta using previous and current
        values
        */
        eta_next = pow((top / bottom), (1 / (gam-1)));

        /*
        Save eta values to file to be used in perturbation
```

```
                simulation later
                */
                Eta_Values << eta_next
                           << endl;

                /*
                Save eta values to file to be plotted later
                in gnuplot
                */
                Eta_Values_Graph << mu << "        " << eta_next
                                 << endl;

                /*
                Since eta = rho / cent_rho we have rho = cent_rho * eta
                */
                double rho = eta_next * cent_rho,
                       radius_r = mu * mean_rad;

                /*
                Save rho values to file
                */
                Rho_Values << rho
                           << endl;

                /*
                Save rho values to file to be plotted later
                in gnuplot
                */
                Rho_Values_Graph << radius_r << "        " << rho
                                 << endl;

                // Proceed to next step
                eta_prev = eta_cur;
                eta_cur = eta_next;

        }

        // Close files
        Eta_Values.close();
        Eta_Values_Graph.close();
        Rho_Values.close();
        Rho_Values_Graph.close();

        return 0;
}
```

Unperturbed Density (η_0) Values Until $\eta \approx 0$

The simulation below is similar to simulation L.1.1 except that it runs until $\eta_0 \approx 0$. The for loop is replaced by a while loop that is triggered to break below an arbitrary small value for η. This is chosen to be at $\eta = 0.00001$.

File L.6: Unpert_Calc.cpp

```
#include <cmath>
#include <iostream>
#include <fstream>
#include "Unpert_Parameters.h"

/*-------------------------------------------------------------------------
See "Definitions.txt" file for explanations of what each parameter means
and each function does
-------------------------------------------------------------------------*/

using namespace std;

double eta_m(double mu)
{
        return 1.0 + ((1.0/6) * pow(mu, 2)) + ((pol_in/120) * pow(mu, 4));
```

```
}

double gamma_prime()
{
        return (1.0 + pol_in) / pol_in;
}

int main()
{
        // Declare ofstream variables
        ofstream Parameters,
                 Eta_Values,
                 Eta_Values_Graph,
                 Rho_Values,
                 Rho_Values_Graph;

        int num_steps;

        //  Prompt for num_steps input on run
        cout << "Please specify number of steps: " << endl;
        cin >> num_steps;

        /*
        delta_r = r_max / num_steps, however: mu = r / r_c
        --> delta_mu = delta_r / r_c
        --> delta_mu = r_max / (r_c * num_steps)
        --> if r_max = r_mean where r_c = r_mean then delta_mu = 1 /
            num_steps
        */
        double delta_mu = 1.0 / num_steps,
               gam = gamma_prime();

        // Open file Parameters.dat
        Parameters.open("./Results/Parameters.dat");

        /*
        Write simulation parameters for use by the perturbation
        simulation to file
        */
        Parameters << "Step Size:"
                   << endl
                   << delta_mu
                   << endl << endl
                   << "Number of Steps:"
                   << endl
                   << num_steps
                   << endl << endl
                   << "Gamma:"
                   << endl
                   << gam
                   << endl << endl
                   << "r_c:"
                   << endl
                   << mean_rad
                   << endl << endl
                   << "rho_c:"
                   << endl
                   << cent_rho
                   << endl;

        Parameters.close();

        /*
        Shell specific radius at centre of star is mu = 0, at next step
        is delta_mu
        */
        double mu_0 = 0.0;
        double mu_1 = delta_mu;

        // Initial values for eta at mu = 0, mu_1 = delta_mu
        double eta_0 = 1.0;
        double eta_1 = eta_m(mu_1);

        // Can use eta_m to calculate first two density values, instead i
```

```
    used initial conditions
// Kept for reference
// eta_0 = eta_m(mu_0);
// eta_1 = eta_m(mu_1);

/*
 * Open files Eta_Values.dat, Eta_Values_Graph.dat, Rho_Values.dat
 * and Rho_Values_Graph.dat
 */
Eta_Values.open("./Results/Eta_Values.dat");
Eta_Values_Graph.open("./Results/Eta_Values_Graph.dat");
Rho_Values.open("./Results/Rho_Values.dat");
Rho_Values_Graph.open("./Results/Rho_Values_Graph.dat");

// Print initial values to files
Eta_Values << mu_0
           << endl
           << mu_1
           << endl;

Eta_Values_Graph << mu_0 << "        " << eta_0
                 << endl
                 << mu_1 << "        " << eta_1
                 << endl;

// Print initial rho values to files
Rho_Values << eta_0 * cent_rho
           << endl
           << eta_1 * cent_rho
           << endl;

Rho_Values_Graph << mu_0 * mean_rad << "        " << eta_0
                     * cent_rho
                 << endl
                 << mu_1 * mean_rad << "        " << eta_1
                     * cent_rho
                 << endl;

////////////////////////////////////
// Numerical Simulation           //
// set to run till eta < 0.00001 //
////////////////////////////////////

bool trigger = true;

double mu,
       top,
       bottom,
       eta_prev = eta_0,
       eta_cur = eta_1,
       eta_next;

/*
We will already know values for cur_step = 0 and cur_step = 1,
so we begin at step 2
*/
int cur_step = 2;


while (trigger)
{
        // Calculate current mu depending on place in simulation
        mu = cur_step * delta_mu;

        // Break calculation up to make it more readable
        top = ((delta_mu / mu) - 1) * pow(eta_prev, (gam - 1))
                + 2 * pow(eta_cur, (gam - 1))
                  - (pow(delta_mu, 2) * eta_cur)

        bottom = ((delta_mu / mu) + 1)

        /*
        Calculate next value for eta using previous and current
        values
```

```
                */
                eta_next = pow((top / bottom), (1 / (gam-1)));

                /*
                Save values to file to be used in perturbation simulation
                later
                */
                Eta_Values << eta_next
                           << endl;

                //Save values to file to be plotted later in gnuplot
                Eta_Values_Graph << mu << "         " << eta_next
                                 << endl;

                /*
                Since eta = rho / cent_rho we have rho = cent_rho * eta
                */
                double rho = eta_next * cent_rho,
                       radius_r = mu * mean_rad;

                /*
                Save rho values to file
                */
                Rho_Values << rho
                           << endl;

                /*
                Save rho values to file to be plotted later
                in gnuplot
                */
                Rho_Values_Graph << radius_r << "         " << rho
                                 << endl;

                // Proceed to next step
                cur_step += 1;
                eta_prev = eta_cur;
                eta_cur = eta_next;


                // Check if eta is close to zero for loop to end
                if (eta_cur < 0.00001)
                {
                        trigger = false;
                }
        }

        // Close files
        Eta_Values.close();
        Eta_Values_Graph.close();
        Rho_Values.close();
        Rho_Values_Graph.close();

        return 0;
}
```

L.1.2 Perturbed Simulation - Octave Code

Oscillation Frequencies (ω)

The Octave file below, called Eig_Val.m, is the main program that calculates values for ω using the eigenvalue problem described in Section 4.4. It is used to find the eigenvalues λ by running the popular Octave function *eig()* against matrix M, see equation (4.21).

File L.7: Eig_Val.m

```
clear;
clc;
```

```
% Assign result paths for both perturbed and unperturbed directories
% to variables (might change depending on your directory structure)
equil_dir = '../Unperturbed Values/R To R_c Or 5 R_c/Results';
results_dir = './Results';

% Open Parameters and Eta_Values files for reading and assign to variables
File1 = fopen(fullfile(equil_dir, 'Eta_Values.dat'), 'r');
File2 = fopen(fullfile(equil_dir, 'Parameters.dat'), 'r');

% Read line by line eta values and assign to array
eta = fscanf(File1, '%f');

% Read Step Size, Step No. and Gamma value from file and assign to variables
delta_mu = textscan(File2, '%f', 1, 'HeaderLines', 1){1};
num_steps = textscan(File2, '%f', 1, 'HeaderLines', 2){1};
gam = textscan(File2, '%f', 1, 'HeaderLines', 2){1};
r_c = textscan(File2, '%f', 1, 'HeaderLines', 2){1};
rho_c = textscan(File2, '%f', 1, 'HeaderLines', 2){1};

% Calculate non-dimensional frequency w_c from rho_c
w_c = sqrt( (gam - 1) * 4 * pi * 6.67408e-11 * rho_c);

% Instantiate a zero matrix of order num_steps to hold primary matrix
M = zeros(num_steps, num_steps);

% Instantiate a zero array of length num_steps to hold mu, tau, A, B and C
% information (See Section 4.3)
mu = zeros(num_steps, 1);
A = zeros(num_steps, 1);
B = zeros(num_steps, 1);
C = zeros(num_steps, 1);
tau = zeros(num_steps, 1);

% Build mu and tau arrays prior to main calculation
for j = 1:num_steps
    tau(j) = gam * (eta(j)**(gam - 1));
    mu(j) = j * delta_mu;
end

% Construct elements of matrix M
for k = 2:num_steps - 1
    A(k) = (1 + (delta_mu / mu(k))) * tau(k+1) + (1 - (delta_mu / mu(k))) *
        tau(k-1) - (4 * tau(k)) + (((delta_mu)**2 * eta(k)) / (gam - 1));
    B(k) = ((delta_mu / mu(k)) + ((eta(k+1) - eta(k-1)) / (4 * eta(k))) + 1)
        * tau(k) + ((tau(k+1) - tau(k-1)) / 2);
    C(k) = ((delta_mu / mu(k)) + ((eta(k+1) - eta(k-1)) / (4 * eta(k))) - 1)
        * tau(k) + ((tau(k+1) - tau(k-1)) / 2);
end

% Build matrix M for Eigenvalue problem
for i = 2:num_steps - 1
    M(i,i+1) = B(i);
    M(i,i) = A(i);
    M(i,i-1) = -C(i);
end

% Remove the first and last rows and columns
% This squares the matrix
rows = [1 num_steps];
cols = [1 num_steps];

M(rows, :) = [];
M(:, cols) = [];

% Calculate the eigenvalues for matrix M
lambda = eig(M);

% Calculate the non-dimensional frequency values
w_tilde = delta_mu * sqrt(-1 * lambda);

% Calculate the dimensional frequency values using w_c
w = w_tilde * w_c;

% Open Wide_Frequency and Frequency files for writing results
```

```
File3 = fopen(fullfile(results_dir, 'Tilde_Frequency.dat'), 'w');
File4 = fopen(fullfile(results_dir, 'Frequency.dat'), 'w');

% Define the print format for the results
format = repmat('%f \r\n', 1, num_steps);
format(end) = 'n';

% Print non-dimensional frequency values
fprintf(File3, format, w_tilde);

% Print dimensional frequency values
fprintf(File4, format, w);

% Close files
fclose(File1);
fclose(File2);
fclose(File3);
fclose(File4);
```