The role of invariants in obtaining exact solutions of differential equations

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DECLARATION

I declare that the contents of this are original except where due references have been made. It is being submitted for the degree of PhD at University of the-Witwatersrand in Johannesburg. It has not been submitted before for any degree to any other institution.

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Abstract

We show here that variational and gauge symmetries have additional applications to the integrability of differential equations. We present a general method to construct first integrals for some classes. In particular, we present a broad class of diffusion type equations, viz., the Fisher Kolmorov and Fitzhugh Nagumo equations, which satisfy the Painlevé properties of their respective travelling wave forms and solitons. It is then shown how a study of invariance properties and conservation laws is used to 'twice' reduce the equations to solutions. We further constructing the first integrals of a large class of the well-known second-order Painlevé equations. In some cases, variational and gauge symmetries have additional applications following a known Lagrangian in which case the first integral is obtained by Noether's theorem. Generally, it is more convenient to adopt the 'multiplier' approach to find the first integrals.

The main chapters of this thesis have either been published or submitted for publication in accredited journals. The contents of Chapters 2, 3 and 5 has been published ([54], [55]).
All computations were done either by hand or Maple.

Contents

Introduction

A large part of Lie's work on symmetry groups of partial differential equations (pdes) concerned linear systems of first order equations, which it can be shown are equivalent to ordinary differential equation-systems. Lie did, however, study the symmetries of higher order. For example in two independent variables including the heat equation. Lie's work on higher order was never developed by other researchers. One possible explanation being that, unlike ordinary differential equations, knowing the symmetry group does not aid in determining its general solution. Wave equations are the only other early work on the symmetries. Despite the availability of Noether's theorem, work on symmetry groups came to a halt [52]. In the 1950s, this area of research was rekindled by Ovsiannikov [53].

This dissertation is structured as follows.

In the first chapter, we state the definitions and the fundamental concepts that will be required.
In the second chapter, we will discusses a novel approach to dealing with the integrability of certain classes of some diffusion type equations. The method utilises knowledge of a Lagrangian of the reduced (ode) to obtain conditions under which gauge symmetries exist. It proposes an alternative way of obtaining values of param-

eters for which equations may possess Painlevé properties and, hence, integrability leading to soliton solutions.

In chapters three and four, we propose a general method to construct first integrals for some classes of some well-known second-order ordinary differential equations, viz., the Emden and Liénard classes of equations. The separate chapters differentiate the linearizable case from the non linearizable case. The approach does not require knowledge of a Lagrangian and, instead, utilises the 'multiplier approach'. Then, we showed how a study of invariance properties and conservation laws is used to 'twice' reduce the equations to solutions.

In chapter five, we construct the first integrals of a large class of well-known second-order Painlevé equations. In some cases, variational or gauge symmetries have additional applications following a known Lagrangian in which case the first integral is obtained by Noether's theorem. We, almost always, adopt the 'multiplier' approach to find the first integrals.

Chapter 1

Preliminaries

1.1 Introduction

In this chapter, we state the necessary definitions of the fundamental concepts that will be used throughout this dissertation.

1.2 Fundamental Concepts

A fundamental operation on vector fields is the Lie bracket or commutator. This is defined in terms of their actions as derivations on functions [52]. Specifically, if α and β are vector fields, their Lie bracket $[\alpha, \beta]$ is a unique vector field satisfying

$$[\alpha, \beta](f) = \alpha(\beta(f)) - \beta(\alpha(f)) \tag{1.1}$$

for all smooth functions f. It is easy to verify that $[\alpha, \beta]$ is a vector field. In local coordinates, if

$$\alpha = \sum_{i=1}^{m} \xi^i(x) \frac{\partial}{\partial x^i} , \qquad \beta = \sum_{i=1}^{m} \eta^i(x) \frac{\partial}{\partial x^i},$$

then

$$[\alpha, \beta] = \sum_{i=1}^{m} \left[\alpha(\eta^i) - \beta(\xi^i)\right] \frac{\partial}{\partial x^i} = \sum_{i=1}^{m} \sum_{i=1}^{m} \left[\xi^i \frac{\partial \eta^i}{\partial x^i}\right] \frac{\partial}{\partial x^i} \tag{1.2}$$

Proposition. The Lie bracket has the following properties:

(i) Bilinearity

$$\begin{aligned} [c_1\alpha + c_2\rho, \beta] &= c_1[\alpha, \beta] + c_2[\rho, \beta], \\ [\alpha, c_1\beta + c_2\gamma] &= c_1[\alpha, \beta] + c_2[\alpha, \gamma], \end{aligned} \tag{1.3}$$

where c_1, c_2 are constants.

(ii) Skew-Symmetry

$$[\alpha, \beta] = -[\beta, \alpha]. \tag{1.4}$$

(iii) Jacobi Identity

$$[\gamma, [\alpha, \beta]] + [\beta, [\gamma, \alpha]] + [\alpha, [\beta, \gamma]] = 0. \tag{1.5}$$

If

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y},$$

is vector field in (x, y), then in the equivalent a vector field is given by The mapping X to translation $\bar{X} = \frac{\partial}{\partial s}$, we need to solve the system of first-order linear

$$X(s) = 1 \qquad \text{and} \qquad X(t) = 0.$$

Suppose an rth-order system of pdes of n independent and m dependent variables is given by,

$$E^{\beta}(x, u, u_{(1)}, \ldots, u_{(r)}) = 0\ , \qquad \beta = 1, \ldots, m\,. \tag{1.6}$$

A conservation law of (1.6) is the equation $D_i T^i = 0$ on the solutions of (1.6). Here, called a *conserved vector (flow)* of (1.6).
A vector field

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^{\alpha} \frac{\partial}{\partial u^{\alpha}} + \zeta_i^{\alpha} \frac{\partial}{\partial u_i^{\alpha}} + \zeta_{i_1 i_2}^{\alpha} \frac{\partial}{\partial u_{i_1 i_2}^{\alpha}} + \cdots\ , \tag{1.7}$$

is a Lie point symmetry generator, if it leaves invariant (1.6) X represents a one parameter Lie group of transformations.
The Euler-Lagrange equations, if they exist, associated with (1.6) are the system $\delta L/\delta u^{\alpha} = 0$, $\alpha = 1, \ldots, m$, where $\delta/\delta u^{\alpha}$ is the Euler-Lagrange operator given by

$$\frac{\delta}{\delta u^{\alpha}} = \frac{\partial}{\partial u^{\alpha}} + \sum_{s \geq 1} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1 \cdots i_s}^{\alpha}}\ , \quad \alpha = 1, \ldots, m. \tag{1.8}$$

L is referred to as a Lagrangian and a Noether symmetry operator X of L arises from a study of the invariance properties of the associated functional

$$\mathcal{L} = \int_{\Omega} L(x, u, u_{(1)}, \ldots, u_{(r)}) \mathrm{d}x \tag{1.9}$$

defined over Ω. If we include point dependent gauge terms $f_1, \ldots, f_n$, the Noether symmetries or gauge symmetries, X, are given by

$$XL + LD_i\xi^i = D_i f_i. \tag{1.10}$$

Corresponding to each X.
For the gauge terms equal to zero, we refer to X as 'variational symmetries'. Corresponding to each X, a conserved flow is obtained via Noether's Theorem.

Moreover, for the scalar scalar case, if there exists a nontrivial differential function Q, called a 'multiplier', such that

$$\frac{\delta}{\delta u}(QE) = 0,$$

then QE is a total divergence, i.e.,

$$QE = D_t T^t + D_x T^x,$$

for some (conserved) vector (T^t, T^x). Thus, a knowledge of each multiplier Q leads to a conserved vector determined by, inter alia, a Homotopy operator. See details and references in [46, 56, 49].

Chapter 2

Painlevé properties, gauge invariance and solitons of some classes of reaction-diffusion equations

2.1 Introduction and Background

It is well known that there is strong interplay between invariance and conservation laws of differential equations (des) and that this relationship can be physical, for e.g., in variational systems, energy conservation is associated with time invariance and linear momentum conservation is linked to space invariance. Often, though, the relationship may be purely mathematical and this has its merits in the analysis and integrability of the system (see [52] or [45], [41] and [42], for e.g.). In the latter two, the authors use the underlying invariance with specific algorithms to deal with diffusion systems such as the KdV, Fizhugh-Nagumo and Fisher equations. These equations are non variational so that Noether's Theorem cannot be appealed to for first integrals, among other things. Nevertheless, a particular reduction of such systems may be variational. We will show that, often, a travelling wave reduction leads to (odes) that generate a Lagrangian and first integrals via Noether's theorem. Moreover, what is novel here is that this Noether or variational symmetry approach leads to Painlevé properties and integrability. We demonstrate this on broad class of Fisher-Kolomorov and Nagumo type equations that have been models that represent reaction-diffusion models.

The contents of this chapter appeared in [54].

2.2 Symmetries, invariants and double reduction

2.2.1 The Fisher equation

In [9], we considered, in detail, the Fisher equation

$$u_t = u_{xx} + \lambda u(1-u) \tag{2.1}$$

which describes reaction-diffusion waves in biology, inter alia; for e.g., in population dynamics, to describe the spatial spread of an advantageous gene [10, 11]. A more general example is one in which the inhomogeneous term is given by $\lambda u(1-u^q)$, $q>0$, see [12]. In [13], the authors discussed the example with $q=2$ in the context of population genetics. It was shown in [9] that (2.1) only admits point symmetries involving time and space translations and does not admit wave reduction ($y = x - ct$

and $w = u$) from a combination of these yields

$$w'' + cw' + \lambda w(1 - w) = 0. \tag{2.2}$$

and that (2.2) has a *Painleve property* for $\lambda = \pm 6c^2/25$ (see [42]). Via a Lagrangian of (2.2)

$$L = e^{cy}[\frac{1}{2}w'^2 - \lambda(\frac{1}{2}w^2 - \frac{1}{3}w^3)]. \tag{2.3}$$

a variational/Noether symmetry $G = \xi(y, w)\partial/\partial y + \eta(y, w)\partial/\partial w$, approach obtained through

$$GL + L\frac{\mathrm{d}\xi}{\mathrm{d}y} = \frac{\mathrm{d}f}{\mathrm{d}y}, \tag{2.4}$$

where $f = f(y, w)$ is some gauge term, the only solution was obtained for $\lambda = \pm 6c^2/25$. Then, a complete solution by first integrals and double reduction (see [52]) is obtainable. This confirms a significant correlation between Noether symmetries and the Painleve property in the analysis of ordinary differential equations and integrability.

Notes.

1. If $\lambda = 1$, the wave speed is $c = \frac{5}{\sqrt{6}}$ for a Painleve property.

2. The axisymmetric case [14]

$$u_t = u_{rr} + \frac{1}{r}u_r + \lambda r(1 - r) \tag{2.5}$$

A generalisation of the Fisher equation

We now consider the first of generalisation of (2.1), viz.,

$$u_t = u_{xx} + \lambda u(1 - u^q) \tag{2.6}$$

discussed, for e.g., in [14]. With $y = x - ct$ and $w = u$ (2.6) reduces to the ode. In [14], solutions of this type are studied in detail and that with increasing q the wavespeed c, it is shown, increases and the steepness decreases as would have been the case with the FisherKolmogoroff wavefront solutions.

In the spirit of a Painleve analysis, admits a Lagrangian

$$L = e^{cy}[\frac{1}{2}w'^2 - \lambda(\frac{1}{2}w^2 - \frac{1}{q+2}w^{q+2})], \qquad q \neq -2. \tag{2.7}$$

In the expansion of (4.5a), it can be shown that $\xi_w = 0$, so that $\xi = a(y)$, say. By equating terms of w'^2, it turns out that $\eta = \frac{1}{2}(\dot{a} - ca)w$ and, similarly, from w', we obtain a nonzero gauge term $f = \frac{1}{4}e^{cy}(\ddot{a} - \dot{a})w^2$. From the remaining terms in (2.4), i.e., those independent of w', a further split based on functions of w is performed. From w^{q+2}, after some details, we get $a(y) = e^{\frac{q\,c}{q+4}y}$. Compatibility with w^2 terms lead to the condition

$$\lambda = \frac{c^2}{4}(1 - (\frac{q}{q+4})^2) = 2\,\frac{c^2\,(q+2)}{(q+4)^2}, \tag{2.8}$$

(if $q = 1$, $\lambda = \frac{6c^2}{25}$, as above). Thus, (2.6) has a Painleve property for λ as in (2.8). Furthermore, (2.8) admits a single gauge symmetry based on λ for which we can construct, by Noether's theorem, a first integral invariant under the same symmetry lead to, by double reduction, a general solution of (2.6) for λ as in (2.8). The gauge/Noether symmetry is

$$G = e^{\frac{q\,c}{q+4}y}[\frac{\partial}{\partial y} - \frac{2c}{q+4}w\frac{\partial}{\partial w}] \tag{2.9}$$

with gauge

$$f = -(\frac{c}{q+4})^2 q e^{\frac{2(q+2)}{q+4}cy} w^2 \tag{2.10}$$

so that the first integral is

$$
\begin{aligned}
I &= L\xi + (\eta - \xi w')\tfrac{\partial L}{\partial w'} - f \\
&= e^{cy}[\tfrac{1}{2}w'^2 - \tfrac{c^2}{4}(1 - (\tfrac{q}{q+4})^2)(\tfrac{1}{2}w^2 - \tfrac{1}{q+2}w^{q+2})]e^{\frac{q\,c}{q+4}y} \\
&-e^{\frac{q\,c}{q+4}y}(\tfrac{2c}{q+4}w + w')e^{cy}w' + (\tfrac{c}{q+4})^2 q e^{\frac{2(q+2)}{q+4}cy}w^2 \\
&= -\tfrac{1}{(q+4)^2}e^{\frac{2(q+2)}{q+4}cy}[(q+4)^2w'^2 + 4c(q+4)ww' + 4c^2w^2(1-w^q)]
\end{aligned}
\tag{2.11}
$$

The equation under the Painleve property for (**??**) is

$$
w'' + cw' + (\frac{c^2}{4}(1 - (\frac{q}{q+4})^2))w(1-w^q) = 0 \tag{2.12}
$$

and its reduced form is

$$
-\tfrac{1}{(q+4)^2}e^{\frac{2(q+2)}{q+4}cy}[(q+4)^2w'^2 + 4c(q+4)ww' + 4c^2w^2(1-w^q)] = k, \tag{2.13}
$$

for some arbitrary constant k and (2.13) inherits the symetry G above (see [52] for details). To reduce the first order ode, we map G to the vector field $\frac{\partial}{\partial r}$ in the transformed coordinates (r, s) via the equations

$$
G(s) = 0, \qquad G(r) = 1 \tag{2.14}
$$

to obtain, after some details,

$$
s = we^{(\frac{2c}{q+4})y}, \qquad r = -(\tfrac{q+4}{qc})e^{-(\frac{qc}{q+4})y}. \tag{2.15}
$$

Thus, it can be shown that

$$
\tfrac{\mathrm{d}s}{\mathrm{d}r} = e^{(\frac{2+q}{q+4})y}[w' + (\tfrac{2c}{q+4})w] \tag{2.16}
$$

and after dividing $w' + (\frac{2c}{q+4})w$ into $(q+4)^2w'^2 + 4c(q+4)ww' + 4c^2w^2(1-w^q)$, we get

$$
-\tfrac{1}{(q+4)^2}e^{\frac{2(q+2)}{q+4}cy}[(q+4)^2(w' + \tfrac{2c}{q+4}w)^2 + w^{2+q}] = k \tag{2.17}
$$

and after some manipulation with (2.16)

$$
(\tfrac{\mathrm{d}s}{\mathrm{d}r})^2 - (\tfrac{2c}{q+4})^2 s^{2+q} = -k \tag{2.18}
$$

whose leads to a solution of (**??**) and (2.6) for λ as given in (2.8).

Notes.

1. If $\lambda = 1$ above, then $c = \frac{q+4}{\sqrt{4+2q}}$ for a Painleve property.

2. For the density independent general case of the above, the travelling wave ode

$$w'' + cw' + \lambda w^p(1 - w^q) = 0, \qquad p, q \neq 1 \tag{2.19}$$

has the Lagrangian

$$L = e^{cy}[\frac{1}{2}w'^2 - \lambda(\frac{1}{p+1}w^{p+1} - \frac{1}{q+p+1}w^{p+q+1})], \qquad p, q \neq -1. \tag{2.20}$$

The equivalent calculations lead to

$$c = \frac{(p+q-1)(p+3)}{(p+q+3)(p-1)} \tag{2.21}$$

with further conditions leading to

$$c = \frac{(p+q-1)}{(p+q+3)}, \qquad c = -\frac{(p+q-1)}{(p+q+3)} \tag{2.22}$$

so that $p = -1$.

2. The case

$$w'' + cw' + \lambda w^{-1}(1 - w^q) = 0, \tag{2.23}$$

leads to $c = 0$ so that a travelling wave solution is in effect a 'steady state' solution (with $y = x$) from

$$w'' + \lambda w^{-1}(1 - w^q) = 0 \tag{2.24}$$

with Lagrangian

$$L = \frac{1}{2}w'^2 - \lambda(\ln w - \frac{1}{q}w^q). \tag{2.25}$$

2.2.2 The Density-Dependent Diffusion-Reaction Models equation

To study long time scales then we ought to take into consideration such growth terms in the model. An extension to incorporate density-dependent diffusion is thus, in the one-dimensional situation, [14]

$$u_t = D_x(f(u)u_x) + \lambda u(1-u). \tag{2.26}$$

Firstly, we consider the first of generalisation, viz.,

$$u_t = D_x(uu_x) + \lambda u(1-u). \tag{2.27}$$

'Physically this model implies that the population disperses to regions of lower density more rapidly as the population gets more crowded' [14]. A travelling wave reduction leads to

$$ww'' + cw' + u'^2 + \lambda w(1-w) = 0 \tag{2.28}$$

Solvability of equation (2.28) and a Painleve approach cannot be performed via the Lagrangian approach as done above. The conservation law study necessitates an alternative approach, the 'multiplier approach' [56]. Here, we suppose that

$$\frac{\delta}{\delta w}[Q(y,w,\ldots)(ww'' + cw' + u'^2 + \lambda w(1-w)] = 0 \tag{2.29}$$

so that $Q(y,w,w')(ww'' + cw' + u'^2 + \lambda w(1-w)$ is a total divergence, i.e., there exists a function $T(y,w,w')$ such that

$$Q(y,w,\ldots)(ww'' + cw' + u'^2 + \lambda w(1-w) = D_y T \tag{2.30}$$

so that T is a first integral of (2.28) and $T = k^*$ is a reduced form of (2.28). It turns out that $Q = e^{2cy}$ and using a Homotopy integral we get

$$T = -\frac{1}{2}e^{2cy}(-2w' + 2cw + c)w \tag{2.31}$$

so that

$$ww' - cw(w-1) = k^* e^{-2cy}, \qquad k^* arbitrary. \tag{2.32}$$

2.2.3 The Convection dependent Fisher-Kolmogoroff equation

If the convection arises as a natural extension of a conservation law, the model is given by

$$u_t + D_x h(u) = u_{xx} + f(u). \tag{2.33}$$

With $h'(u) = ku$ with k a positive or negative constant and $f(u)$ logistic, (2.33) becomes

$$u_t + kuu_x = u_{xx} + \lambda u(1-u) \tag{2.34}$$

with the travelling wave reduction

$$w'' + w'(c - kw) + \lambda w(1-w) = 0. \tag{2.35}$$

The multiplier approach yields $Q = e^{\frac{1}{2}(2c-k)y}$ for $\lambda = \frac{k}{4}(2c-k)$ with the reduced ode

$$e^{\frac{1}{2}(2c-k)y}\left(-2\,w' - kw + kw^2\right) = k^*. \tag{2.36}$$

2.2.4 The Fitzhugh-Nagumo equation

We carry out a similar study of a generalized version of the Fitzhugh-Nagumo equation

$$u_t = u_{xx} + \lambda u(1-u)(u-a), \quad a \neq 1. \tag{2.37}$$

The calculations, as before, are long and tedious but we briefly present the results. A Painleve analysis has been done on (2.37) for $\lambda = 1$ in [5]. A travelling wave reduction $y = x - ct$ and $w = u$ yields the o.d.e

$$w'' + cw' + \lambda w(1-w)(w-a) = 0. \tag{2.38}$$

A Lagrangian of (2.38) is $L = e^{cy}[\frac{1}{2}w'^2 - \lambda(-\frac{1}{4}w^4 - \frac{a}{2}w^2 + \frac{a+1}{3}w^3)]$ which has a single Noether symmetry $G = e^{cy/3}[\partial/\partial y + c/3(\frac{a+1}{3} - w)\partial/\partial w]$ for simultaneous forms of λ given by

$$\begin{aligned} \lambda &= \tfrac{2c^2}{3(a^2-a+1)}, \\ \lambda &= \tfrac{4c^2}{9a} \end{aligned} \tag{2.39}$$

from which we obtain $a = 1/2$ and $a = 2$ ($\lambda = 2\frac{4c^2}{9}$ and $\lambda = (1/2)\frac{4c^2}{9}$, respectively). Following from results regarding the Fisher equation, we propose that the equation (2.37) possesses Painleve properties for these combinations of a and λ. Furthermore, the exact solutions for the equation are obtainable from a double reduction using the single point symmetry generator G.
For $a = 2$, $\lambda = \frac{2c^2}{9}$, we get the first integral,

$$I = -\tfrac{1}{6}\,\mathrm{e}^{4/3\,cy}\left(-6c\,u'u + 6\,cu' - 9\,{u'}^2 + 5\,c^2u^2 - 2\,c^2u - 4\,c^2u^3 + u^4c^2\right). \tag{2.40}$$

Invariants of G are

$$s = (w-1)e^{\frac{1}{3}cy}, \qquad r = \tfrac{1}{c}e^{-\frac{1}{3}cy} \tag{2.41}$$

so that, by $I = k$, we get the integrable first-order ode

$$(\tfrac{\mathrm{d}s}{\mathrm{d}r})^2 - c^2s^4 = k \tag{2.42}$$

being a 'reduction'/transformation of (2.38) for this combination of a and λ.

2.3 Discussion and Conclusion

We have presented a novel approach in dealing with the integrability of certain classes of reductions of some diffusion type equations, viz., the Fisher-Kolmogorov and Nagumo equations. The method utilises a knowledge of a Lagrangian of the reduced ode to obtain conditions under which gauge symmetries exists. It proposes an alternative way of obtaining values of parameters for which equations may possess Painleve properties and, hence, integrability leading to soliton solutions.

Chapter 3

On first integrals, conservation laws and reduction of classes of Emden and Liénard equations-linearizable case

3.1 Introduction and Background

The general version of the modified version of the Emden type equations

$$y'' + \alpha y y' + \beta y^3 = 0 \tag{3.1}$$

has been the subject of many studies; in a detailed study by Mahomed and Leach [30], the linearizability conditions for (3.1) are presented along with the transformations that lead it to the canonical equation $Y'' = 0$, for $\alpha^2 = 9\beta$, that generate the maximal eight-dimensional Lie point symmetry generators of the algebra $sl(3, R)$. As a consequence. Nucci and Tamizhmani [51] discussed a procedure to determine Lagrangians using the method of the 'Jacobi last multiplier'. For example, for $\alpha = 3, \beta = 1$, they show a Lagrangian is $L = \frac{1}{(y^2+\dot{y})^2}$ generating five first integrals. Another important work in this regard is that of Tiwari et al [32] in which Lie point symmetry classifications of similar equations are performed. Lagrangians for differential equations are generally pursued to construct first integrals (conservation laws) via Noether's theorem [33] and, for (odes), one can use the variational symmetry and first integral association to perform double reductions of the odes (using the single symmetry) which for second-order odes lead to solutions [34]. For (pdes), a number of works dealing with conservation laws, associated symmetries and double reductions have been done recently, see [35], [36] and very recently, extended the approach to include the role of 'multipliers'.

The role of multipliers essential in the construction of conserved flows (conservation laws) of pdes as they surpass the need of a variational principle. Moreover, the existence of higher-order multipliers lead to a larger class of conservation laws and the computation depends on the power of one's software. Much work has been done in this regard, most notably [28], [29], [49]. For odes, however, the multiplier approach to construct first integrals has not been popular largely due to the position of Noether's theorem which relies on the existence and knowledge of a Lagrangian. In what follows, we will show how the multiplier approach can be exploited to construct first integrals of (3.1).

The contents of this chapter has been submitted to Afrika Matematika.

A differential function Q is a **multiplier** that leads to a conservation law of (3.1) if there exists a differential function T that satisfies the total divergence

$$QG = DT, \qquad G = y'' + \alpha yy' + \beta y^3 \tag{3.2}$$

Then, since the Euler operator annihilates total divergences,

$$\tfrac{\delta}{\delta y}(QG) = 0, \tag{3.3}$$

where D is the 'total derivative operator' and $\frac{\delta}{\delta y}$ is the 'Euler (-Lagrange) operator'.

In what follows, we study the first integrals and reduction of

$$y'' + 3yy' + y^3 = 0 \tag{3.4}$$

and other classes of similar equations using the multiplier approach. We show how the first integrals are 'associated' with Lie point symmetry generators which provides a 'double reduction' and, in this case, solutions of the ode.

Note. A well known procedure for constructing conservation laws involve Noether's theorem for variational problems. Some cases of (3.1) are variational and the non variational cases require alternative approaches. An approach adopted here requires the determination of **multipliers** followed by certain procedures to then construct the conserved flows/vectors, like the homotopy formula. For a detailed account, see [28, 29, 46], inter alia. We note that the method adopted here recovers all the results where the cases of the equation are variational; Noether's theorem is not utilised here.

3.2 The Emden equations

The equation (3.3) becomes

$$\frac{\delta}{\delta y}[Q(x,y,y')(y''+3yy'+y^3)]) = 0, \tag{3.5}$$

After some lengthy calculations and observations, it turns out that the multiplier functions take the form

$$Q = \frac{f(x,y)-y'g(x,y)}{(y^2+y')^3} \tag{3.6}$$

which we enumerate below

$$\begin{aligned} Q_1 &= \tfrac{y'}{(y^2+y')^3} \\ Q_2 &= \tfrac{-y^3-yy'}{(y^2+y')^3} \\ Q_3 &= \tfrac{-xy^3+y^2-xyy'}{(y^2+y')^3} \\ Q_4 &= \tfrac{-\frac{1}{2}x^2y^3+xy^2-\frac{2}{3}y-y'(\frac{1}{2}x^2y-\frac{1}{3}x)}{(y^2+y')^3} \\ Q_5 &= \tfrac{-\frac{1}{6}x^3y^3+\frac{1}{2}x^2y^2-\frac{2}{3}xy+\frac{1}{3}-y'(\frac{1}{6}x^3y-\frac{1}{6}x^2)}{(y^2+y')^3} \end{aligned} \tag{3.7}$$

Each multiplier Q leads to a first integral that satisfy equation (3.7). Regarding Q_1, we get the following.

$$D_x I = \frac{y'}{(y^2+y')^3}\left(y''+3yy'+y^3\right), \tag{3.8}$$

where $I = I(x,y,y')$ is the first integral and $D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots$ is the total derivative with respect to x. After applying the total derivative in equation (3.8), we obtain

$$I_x + y'I_y + y''I_{y'} = \frac{y'y''+3y'^2y+y'y^3}{y^6+3y^4y'+3y^2y'^2+y'^3}. \tag{3.9}$$

Splitting on y'' yields

$$y'' : I_{y'} = \frac{y'}{y^6+3y^4y'+3y^2y'^2+y'^3}, \tag{3.10}$$

$$: I_x + y'I_y = \frac{3y'^2y+y'y^3}{y^6+3y^4y'+3y^2y'^2+y'^3}. \tag{3.11}$$

Integrating equation (3.10) with respect to y' gives

$$I(x,y,y') = -\frac{y^2+2y'}{2(y^2+y')^2} + A(x,y), \tag{3.12}$$

where $A(x,y)$ is an arbitrary function of x and y. Substituting the value of $I(x,y,y')$ into equation (3.11) leads into the following

$$A_x - \frac{yy'}{(y^2+y')^2} + \frac{2y^3y'}{(y^2+y')^3} + \frac{4yy'^2}{(y^2+y')^3} + y'A_y = \frac{3yy'^2}{(y^2+y')^3} + \frac{y^3y'}{(y^2+y')^3}. \tag{3.13}$$

After simplifying equation (3.13), we get that

$$A_x + y'A_y = 0. \tag{3.14}$$

Splitting equation (3.13) on y' and integrating the equation with respect to y yields

$$A(x,y) = B(x), \tag{3.15}$$

the first integral is given by

$$I(x,y,y') = -\frac{y^2+2y'}{2(y^2+y')^2} + C_1. \tag{3.16}$$

Thus, we may let the first integral corresponding to Q_1 be

$$I_1(x,y,y') = -\frac{y^2+2y'}{2(y^2+y')^2}.$$

Nonlinearizable case

In this section, we consider another class (3.1) which also generates nontrivial multipliers and, hence, first integrals of the form (4.54), viz.,

$$y'' + \tfrac{7}{2}yy' + \tfrac{3}{2}y^3 = 0. \tag{3.17}$$

Equation (3.17) admits just two such multipliers, i.e.,

$$\begin{aligned} P_1 &= \frac{-\frac{2}{3}y^3 - yy'}{(y^2+y')^3} \\ P_2 &= \frac{-\frac{2}{3}xy^3 + \frac{5}{9}y^2 - (xy - \frac{8}{9})y'}{(y^2+y')^3}. \end{aligned} \tag{3.18}$$

The first integrals are, respectively,

$$\begin{aligned} J_1 &= \tfrac{1}{3}\, y \left(-\tfrac{1}{2}\, \frac{y^2}{(y^2+y')^2} + 3\, \frac{1}{y^2+y'} \right) - x \\ J_2 &= -\tfrac{1}{9}\, \frac{-9\, xy + 8}{y^2+y'} - \tfrac{1}{6}\, \frac{y^2(xy-1)}{(y^2+y')^2} - \tfrac{1}{2}\, x^2. \end{aligned} \tag{3.19}$$

We show the first integrals can be used to perform a double reduction of (3.17). Firstly, (3.17) admit the Lie symmetry generators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

A reduced first order ode reduction of (3.17) is given by $J_1 = k$, k constant. This ode, viz.,

$$\tfrac{1}{3}\, y \left(-\tfrac{1}{2}\, \frac{y^2}{(y^2+y')^2} + 3\, \frac{1}{y^2+y'} \right) - x = k \tag{3.20}$$

inherits X_2 if $k = 0$, since $X_2^{[1]} J_1 =$, where $X_2^{[1]}$ is the first prolongation of X_2. It is, in general, not easy to construct symmetry generators of first-order odes. To reduce (3.20) to a solution using X_2, we need to map X_2 to $X^* = 1\frac{\partial}{\partial t} + 0\frac{\partial}{\partial s}$, where $t = t(x, y)$ and $s = s(x, y)$ are transformed variables obtained by solving

$$x\frac{\partial t}{\partial x} - y\frac{\partial t}{\partial y} = 1, \qquad x\frac{\partial s}{\partial x} - y\frac{\partial s}{\partial y} = 0.$$

We get

$$t = \ln x, \qquad s = xy.$$

Therefore,

$$\begin{aligned} y' &= \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}s} \Big/ \frac{\mathrm{d}x}{\mathrm{d}s} \\ &= \frac{1 - s\frac{\mathrm{d}t}{\mathrm{d}s}}{x^2\, \frac{\mathrm{d}t}{\mathrm{d}s}} \end{aligned}$$

so that

$$\tfrac{1}{3}\, y \left(-\tfrac{1}{2}\, \tfrac{y^2}{(y^2+y')^2} + 3\, \tfrac{1}{y^2+y'} \right) - x = k \tag{3.21}$$

becomes the variable separable ode

$$6\, s^4 t'^2 - 17\, s^3 t'^2 + 12\, s^2 t'^2 + 12\, s^2 t' - 18\, s t' + 6 = 0, \tag{3.22}$$

where $t = t(s)$. We get

$$t' = \frac{-6\, s + 9 + \sqrt{-6\, s + 9}}{(6\, s^2 - 17\, s + 12)\, s}$$

3.3 Classes of the quadratic Liénard equations

In [32], a detailed Lie point symmetry analysis of the Liénard equations, $y'' + f(y) y'^2 + g(y) = 0$, is done. In particular, they showed two classes are linearizable as they admit an 8-dimensional Lie algebra of point symmetries. The complete set of five independent first integrals are constructed below.

3.3.1 $f = \alpha$ constant

We first consider the class

$$y'' + \alpha y'^2 + \tfrac{1}{\alpha} \beta^2 \left(1 - e^{-\alpha y}\right) = 0. \tag{3.23}$$

The multipliers that lead to nontrivial and independent first integrals are

$$
\begin{aligned}
Q_1 &= -e^{2\alpha y}y', \\
Q_2 &= \mathrm{e}^{\alpha\,y}\cos(\beta\,x)\,, \\
Q_3 &= \mathrm{e}^{\alpha\,y}\sin(\beta\,x)\,, \\
Q_4 &= -2\,\left(\mathrm{e}^{\alpha\,y}\right)^2\sin(\beta\,x)\cos(\beta\,x)\,y' + \tfrac{2\,(\mathrm{e}^{\alpha\,y})^2(\cos(\beta\,x))^2\beta - 2\,\mathrm{e}^{\alpha\,y}(\cos(\beta\,x))^2\beta - (\mathrm{e}^{\alpha\,y})^2\beta + \mathrm{e}^{\alpha\,y}\beta}{\alpha}, \\
Q_5 &= -\left(2\,\left(\mathrm{e}^{\alpha\,y}\right)^2(\cos(\beta\,x))^2 - \left(\mathrm{e}^{\alpha\,u}\right)^2\right)y' + \tfrac{-2\,(\mathrm{e}^{\alpha\,y})^2\sin(\beta\,x)\cos(\beta\,x)\beta + 2\,\mathrm{e}^{\alpha\,y}\sin(\beta\,x)\cos(\beta\,x)\beta}{\alpha}
\end{aligned}
\tag{3.24}
$$

with corresponding first integrals

$$
\begin{aligned}
I_1 &= -\tfrac{1}{\alpha^2}[\tfrac{1}{2}\,\alpha^2\mathrm{e}^{2\,\alpha\,y}y'^2 + \tfrac{1}{2}\,\mathrm{e}^{2\,\alpha\,y}\beta^2 - \mathrm{e}^{\alpha\,y}\beta^2 + \tfrac{1}{2}\,\beta^2], \\
I_2 &= -\tfrac{1}{\alpha}[-\mathrm{e}^{\alpha\,y}\cos(\beta\,x)\,\alpha\,y' - \mathrm{e}^{\alpha\,y}\sin(\beta\,x)\,\beta + \sin(\beta\,x)\,\beta], \\
I_3 &= -\tfrac{1}{\alpha}[-\mathrm{e}^{\alpha\,y}\sin(\beta\,x)\,\alpha\,y' + \mathrm{e}^{\alpha\,y}\cos(\beta\,x)\,\beta - \cos(\beta\,x)\,\beta], \\
I_4 &= -\tfrac{1}{\alpha^2}[\sin(\beta\,x)\cos(\beta\,x)\,\mathrm{e}^{2\,\alpha\,y}\alpha^2y'^2 - 2\,\left(\cos(\beta\,x)\right)^2\mathrm{e}^{2\,\alpha\,y}\alpha\,\beta\,y' \\
&\quad - \sin(\beta\,x)\cos(\beta\,x)\,\mathrm{e}^{2\,\alpha\,y}\beta^2 + 2\,\mathrm{e}^{\alpha\,y}\left(\cos(\beta\,x)\right)^2\beta\,y'\alpha + 2\,\mathrm{e}^{\alpha\,y}\cos(\beta\,x)\,\beta^2\sin(\beta\,x) \\
&\quad + \mathrm{e}^{2\,\alpha\,y}\alpha\,\beta\,y' - \cos(\beta\,x)\,\beta^2\sin(\beta\,x) - \mathrm{e}^{\alpha\,y}\alpha\,\beta\,y'], \\
I_5 &= -\tfrac{1}{\alpha^2}[\left(\cos(\beta\,x)\right)^2\mathrm{e}^{2\,\alpha\,y}\alpha^2y'^2 + 2\,\cos(\beta\,x)\sin(\beta\,x)\,\mathrm{e}^{2\,\alpha\,y}\alpha\,\beta\,y' \\
&\quad - \left(\cos(\beta\,x)\right)^2\mathrm{e}^{2\,\alpha\,y}\beta^2 - 2\,\mathrm{e}^{\alpha\,y}\cos(\beta\,x)\,\alpha\,y'\sin(\beta\,x)\,\beta - \tfrac{1}{2}\mathrm{e}^{2\,\alpha\,y}\alpha^2y'^2 \\
&\quad + 2\,\mathrm{e}^{\alpha\,y}\left(\cos(\beta\,x)\right)^2\beta^2 - \left(\cos(\beta\,x)\right)^2\beta^2 + \tfrac{1}{2}\mathrm{e}^{2\,\alpha\,y}\beta^2 - \mathrm{e}^{\alpha\,y}\beta^2 + \tfrac{1}{2}\beta^2]
\end{aligned}
\tag{3.25}
$$

3.3.2 $f = -\frac{2\alpha}{1+\alpha y}$

We first consider the class

$$y'' - \tfrac{2\alpha}{1+\alpha y}y'^2 + \beta^2\left(y + \alpha y^2\right) = 0. \tag{3.26}$$

The multipliers that lead to nontrivial and independent first integrals are

$$\begin{aligned}
Q_1 &= \tfrac{\cos(\beta\,x)}{\alpha^2 y^2+2\,\alpha\,y+1},\\
Q_2 &= \tfrac{\sin(\beta\,x)}{\alpha^2 y^2+2\,\alpha\,y+1},\\
Q_3 &= -\tfrac{y'}{(\alpha y+1)^4},\\
Q_4 &= \tfrac{2\,(\cos(\beta\,x))^2\beta\,y-\beta\,y}{\alpha^3 y^3+3\,\alpha^2 y^2+3\,\alpha\,y+1} - y'\big(\tfrac{2\sin(\beta\,x)\cos(\beta\,x)}{\alpha^4 y^4+4\,\alpha^3 y^3+6\,\alpha^2 y^2+4\,\alpha\,y+1}\big),\\
Q_5 &= -\tfrac{-2\beta y\sin(\beta\,x)\cos(\beta\,x)}{\alpha^3 y^3+3\,\alpha^2 y^2+3\,\alpha\,y+1} - y'\big(\tfrac{(2\,(\cos(\beta\,x))^2-1}{\alpha^4 y^4+4\,\alpha^3 y^3+6\,\alpha^2 y^2+4\,\alpha\,y+1}\big)
\end{aligned} \tag{3.27}$$

with corresponding first integrals

$$\begin{aligned}
I_1 &= \tfrac{\sin(\beta\,x)\alpha\,\beta\,y^2+\sin(\beta\,x)\beta\,y+\cos(\beta\,x)y'}{(\alpha\,y+1)^2},\\
I_2 &= \tfrac{\sin(\beta\,x)y'}{\alpha^2 y^2+2\,\alpha\,y+1} + \tfrac{\cos(\beta\,x)\beta}{\alpha\,(\alpha\,y+1)} - \tfrac{\cos(\beta\,x)\beta}{\alpha},\\
I_3 &= -\tfrac{1}{2}\,\tfrac{y'^2}{(\alpha y+1)^4} - \beta^2\left(-\tfrac{1}{\alpha^2(\alpha\,y+1)} + \tfrac{1}{2}\,\tfrac{1}{\alpha^2(\alpha\,y+1)^2}\right),\\
I_4 &= -\tfrac{1}{2}\,\tfrac{\sin(2\,\beta\,x)y'^2}{(\alpha\,y+1)^4} + \tfrac{\cos(2\,\beta\,x)\beta\,yy'}{(\alpha\,y+1)^3} + \sin(2\,\beta\,x)\,\beta^2\left(\tfrac{1}{2}\,\tfrac{1}{\alpha^2(\alpha\,y+1)^2} - \tfrac{1}{\alpha^2(\alpha\,y+1)}\right) + \tfrac{1}{2}\,\tfrac{\sin(2\,\beta\,x)\beta^2}{\alpha^2},\\
I_5 &= -\tfrac{1}{2}\,\tfrac{\cos(2\,\beta\,x)y'^2}{(\alpha\,y+1)^4} - \tfrac{\sin(2\,\beta\,x)\beta\,yy'}{(\alpha\,y+1)^3} + \cos(2\,\beta\,x)\,\beta^2\left(\tfrac{1}{2}\,\tfrac{1}{\alpha^2(\alpha\,y+1)^2} - \tfrac{1}{\alpha^2(\alpha\,y+1)}\right) + \tfrac{1}{2}\,\tfrac{\cos(2\,\beta\,x)\beta^2}{\alpha^2}.
\end{aligned} \tag{3.28}$$

3.4 Discussion and Conclusion

We presented a general method to construct first integrals for some classes of some well known second-order, viz., the Emden and Liénard classes of equations. The approach does not require a knowledge of a Lagrangian and, instead, utilises the 'multiplier approach'. Then, we showed how a study of the invariance properties and conservation laws are used to 'twice' reduce the equations to solutions.

Chapter 4

On first integrals, conservation laws and reduction of classes of Emden and Liénard equations

4.1 Introduction

In [32], Tiwari et. al. carry out a complete classification of the Lie point symmetry groups associated with the quadratic. For the non-maximal cases, the identified equations are integrable and contain some examples such as the 'Mathews-Lakshmanan oscillator particle on a rotating parabolic well'.
In [54], we computed the first integrals of the various classes of the linearizable cases without recourse to constructing the respective Lagrangians and Noether's Theorem ([30, 33]). In essence, the method generally assumes that there exists a relationship between symmetries and conservation laws ([34, 35]) and the computation adopts

the method expounded in the 'multiplier approach' ([28, 29]). In each case, a maximal five first integrals (conserved quantities) were obtained corresponding to the five independent multipliers. This matches with the equivalent five that are obtainable from the standard Lagrangian and variational symmetries of the linear free particle or oscillator equations via Noether's Theorem, we focus on the first integrals of the classes of nonlinearizable quadratic Liénard type equations. The method of multipliers is, vastly, more useful as there is no need to construct a Lagrangian for the underlying ode. To demonstrate a possible relationship between Lie point symmetries and first integrals, we need only check if the first integral is invariant under the symmetry. The primary application of this relationship lies in the double reduction of the ode ([34]).

4.2 The Emden equations

I. The three paramenter case

It has been shown in [32] that a class of equations generating a three dimensional algebra of Lie point symmetries is

$$y'' - \frac{2}{y}y'^2 + \lambda y^5 = 0$$

basis of symmetries being

$$X_1 = \tfrac{\partial}{\partial x}, \qquad X_2 = x^2\tfrac{\partial}{\partial x} - xy\tfrac{\partial}{\partial y} \qquad X_3 = 2x\tfrac{\partial}{\partial x} - y\tfrac{\partial}{\partial y} \tag{4.1}$$

so that

$$[X_1, X_2] = \tfrac{1}{2}X_3, \qquad [X_1, X_3] = 2X_1, \qquad [X_2, X_3] = 2X_2. \tag{4.2}$$

The multipliers $P(x,y,y')$, it turns out,

$$\tfrac{\delta}{\delta y}[P(x,y,y')(y''-\tfrac{2}{y}y'^2+\lambda y^5)]) = 0, \tag{4.3}$$

where $\frac{\delta}{\delta y}$ is the Euler operator. After some lengthy calculations we get that the multiplier functions take the form

$$P = \tfrac{f(x,y)-y'g(x,y)}{4y^7} \tag{4.4}$$

which we enumerate below to obtain

$$\begin{aligned} P_1 &= \tfrac{-x}{2y^3} - \tfrac{-x^2}{2y^4}y', \\ P_2 &= \tfrac{-1}{2y^3} - \tfrac{-x}{y^4}y,' \\ P_3 &= \tfrac{-y'}{y^4}. \end{aligned} \tag{4.5}$$

Each multiplier P_i leads to a first integral that satisfy equation (4.3).

(a) Regarding P_1, we get

$$D_x I = \left(\frac{-x}{2y^3} - \frac{-x^2}{2y^4}y'\right)\left(y'' - \frac{2}{y}y'^2 + \lambda y^5\right), \tag{4.6}$$

where $I = I(x,y,y')$ is the first integral and $D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots$ is the total derivative with respect to x. After Expanding and applying the total derivative in equation (4.6), we obtain

$$I_x + y'I_y + y''I_{y'} = \frac{y'y'' + 3y'^2y + y'y^3}{y^6 + 3y^4y' + 3y^2y'^2 + y'^3}. \tag{4.7}$$

Splitting on y'' yields

$$\begin{aligned} y'' \quad &: I_{y'} = -\tfrac{x^2y'}{2y^4} - \tfrac{x}{2y^3}, \\ &: I_x + y'I_y = -\tfrac{x(xy+y)\left(\lambda y^6 - 2y'^2\right)}{2y^5}. \end{aligned} \tag{4.8}$$

Integrating equation (4.8) with respect to y' gives

$$I(x,y,y') = -\frac{y^2\lambda\, x^2}{4} - \frac{1}{4y^2} + A(x,y), \tag{4.9}$$

where $A(x,y)$ is an arbitrary function of x and y. Substituting the value of $I(x,y,y')$ into equation (4.8) leads into the following

$$-A_x - \frac{xy^2}{2y^4} - \frac{y'}{2y^3} - \left(\frac{xy'\,(2xy'+3y)}{2y^5} + A_y\right) y' = \frac{-x(xy'+y)(\lambda y^6 - 2y'^2)}{2y^5}. \quad (4.10)$$

we get that

$$A_x + y' A_y = 0. \quad (4.11)$$

Splitting equation (4.10) on y' and integrating the equation with respect to y yields

$$A(x,y) = -\frac{y^2\lambda\, x^2}{4} - \frac{1}{4y^2} + B(x), \quad (4.12)$$

where $B(x)$ is an arbitrary function of x. After replacing A into equation (**??**) and integrating with respect to x, we obtain $B = C_1$, where C_1 is an arbitrary constant of integration. Therefore, the first integral is given by

$$I(x,y,y') = -\frac{x^2y^2}{4y^4} - \frac{xy'}{2y^3} - \frac{y^2\lambda\, x^2}{4} - \frac{1}{4y^2} + C_1. \quad (4.13)$$

Thus, we may let the first integral corresponding to P_1 be

$$I_1(x,y,y') = -\frac{x^2y^2}{4y^4} - \frac{xy'}{2y^3} - \frac{y^2\lambda\, x^2}{4} - \frac{1}{4y^2}.$$

It is easy to check that $D_x I_1 = \left(\frac{-x}{2y^3} - \frac{-x^2}{2y^4}y'\right)\left(y'' - \frac{2}{y}y'^2 + \lambda y^5\right)$.

(b) The first integral for the case P_2,viz.,

$$P_2 = \tfrac{-1}{2y^3} - \tfrac{-x}{y^4}y' \quad (4.14)$$

satisfy the divergent form

$$D_x I = -\frac{(2xy'+y)\,(\lambda y^6 + y''y - 2y'^2)}{2y^5}. \quad (4.15)$$

On Expanding and applying the total derivative in equation (4.15), we obtain

$$I_x + y'I_y + y''I_{y'} = -\frac{(2xy'+y)\left(\lambda y^6 + y''y - 2y'^2\right)}{2y^5}. \tag{4.16}$$

Splitting on y'' yields

$$\begin{aligned} y'' \quad &: I_{y'} = \tfrac{-xy'}{y^4} - \tfrac{1}{2y^3}, \\ &: I_x + y'I_y = -\tfrac{(2xy'+y)\left(\lambda y^6 - 2y'^2\right)}{2y^5}. \end{aligned} \tag{4.17}$$

Integrating equation (4.17) with respect to y' gives

$$I(x,y,y') = -\frac{xy'^2}{2y^4} - \frac{y'}{2y^3} + A(x,y), \tag{4.18}$$

where $A(x,y)$ is an arbitrary function of x and y. Substituting the value of $I(x,y,y')$ into equation (4.17) leads into the following

$$-A_x + \frac{y'^2}{2y^4} - \left(\frac{(4xy'^2+3y'y)}{2y^5} + A_y\right)y' = -\frac{(2xy'+y)\left(\lambda y^6 - 2y'^2\right)}{2y^5}. \tag{4.19}$$

$$y'A_y = 0. \tag{4.20}$$

Splitting equation (4.20) on y' gives

$$\begin{aligned} y' \quad &: A_x = -\tfrac{y^2\lambda}{2}, \\ &: A_y = -xy\lambda. \end{aligned} \tag{4.21}$$

Integrating equation (4.21) with respect to y yields

$$A(x,y) = -\frac{xy^2\lambda}{2} + B(x), \tag{4.22}$$

where $B(x)$ is an arbitrary function of x. After replacing the value of A into equation (4.22) and integrating with respect to x, we obtain $B = C_1$, where C_1 is an arbitrary constant of integration. Therefore, the first integral is given by

$$I(x,y,y') = -\frac{xy'^2}{2y^4} - \frac{y'}{2y^3} - \frac{y^2\lambda x}{2} + C_1 \tag{4.23}$$

from which we can choose the first integral corresponding to P_2 to be

$$I_2(x,y,y') = -\frac{xy'^2}{2y^4} - \frac{y'}{2y^3} - \frac{y^2\lambda x}{2}$$

so that that $D_x I_2 = \left(\frac{-1}{2y^3} - \frac{-x}{y^4}y'\right)\left(y'' - \frac{2}{y}y'^2 + \lambda y^5\right)$.

(c) For

$$P_3 = -\tfrac{y'}{y^4} \tag{4.24}$$

we get

$$D_x I = -\frac{y'\left(\lambda y^6 + y''y - 2y^2\right)}{y^5}. \tag{4.25}$$

and

$$I_x + y' I_y + y'' I_{y'} = -\frac{y'\left(\lambda y^6 + y''y - 2y^2\right)}{y^5}. \tag{4.26}$$

Splitting on y'' yields

$$\begin{aligned} y'' \quad &: I_{y'} = \tfrac{-y'}{y^4}, \\ &: I_x + y' I_y = -\tfrac{-y'\left(\lambda y^6 - 2y'^2\right)}{y^5}. \end{aligned} \tag{4.27}$$

Integrating equation (4.27) with respect to y' gives

$$I(x,y,y') = -\frac{y'^2}{y^4} + A(x,y), \tag{4.28}$$

where $A(x,y)$ is an arbitrary function of x and y. Substituting the value of $I(x,y,y')$ into equation (4.27) leads into the following

$$-\left(\frac{2y'^2}{y^5} + A_y\right)y' = -\frac{-y'\left(\lambda y^6 - 2y'^2\right)}{y^5}. \tag{4.29}$$

After integrating the equation with respect to y yields

$$A(x,y) = -\frac{\lambda y^2}{2} + B(x), \tag{4.30}$$

where $B(x)$ is an arbitrary function of x. After replacing the value of A into equation (**??**) and integrating with respect to x, we obtain $B = C_1$, where C_1 is an arbitrary constant of integration. Therefore, the first integral is given by

$$I(x, y, y') = -\frac{y'^2}{2y^4} - \frac{y^2\lambda}{2} + C_1. \tag{4.31}$$

Finally, a first integral corresponding to P_3 is

$$I_3(x, y, y') = -\frac{y'^2}{2y^4} - \frac{y^2\lambda}{2}$$

and $D_x I_3 = \left(\frac{-y'}{y^4}\right)\left(y'' - \frac{2}{y}y'^2 + \lambda y^5\right)$.

II. Two dimensional algebra case

The form of the second-order ode generating a Lie point symmetry algebra of two dimensions is

$$y'' + \frac{1}{y}y'^2 + \frac{1}{4}\lambda y^3 = 0$$

with algebra

$$X_1 = \tfrac{\partial}{\partial x}, \qquad X_2 = x\tfrac{\partial}{\partial x} - y\tfrac{\partial}{\partial y}. \tag{4.32}$$

The multiplier condition is

$$\tfrac{\delta}{\delta y}[P(x, y, y')(y'' + \tfrac{1}{y}y'^2 + \tfrac{1}{4}\lambda y^3)]) = 0 \tag{4.33}$$

and after some lengthy calculations, the multiplier function takes the form

$$P = -y^2 y' \tag{4.34}$$

so that

$$D_x I = -y^2 y' \left(y'' + \frac{1}{y} y'^2 + \frac{1}{4} \lambda y^3 \right), \tag{4.35}$$

where $I = I(x, y, y')$ is the first integral and $D_x = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + \cdots$ is the total derivative with respect to x. After Expanding and applying the total derivative in equation (4.35), we obtain

$$I_x + y' I_y + y'' I_{y'} = -\frac{yy' \left(y^4 + 4y'' y + 4y'^2\right)}{4}. \tag{4.36}$$

Splitting on y'' yields

$$\begin{aligned} y'' \quad &: I_{y'} = -y^2 y', \\ &: I_x + y' I_y = \tfrac{y'^2}{y} + \tfrac{\lambda y^3}{4}. \end{aligned} \tag{4.37}$$

Integrating equation (4.37) with respect to y' gives

$$I(x, y, y') = -\frac{y^2 y'^2}{2} + A(x, y), \tag{4.38}$$

where $A(x, y)$ is an arbitrary function of x and y. Substituting the value of $I(x, y, y')$ into equation (4.38) leads into the following

$$-A_x - \left(-yy'^2 + A_y\right) y' = -\frac{-yy' \left(\lambda y^4 + 4y'^2\right)}{4}. \tag{4.39}$$

After simplifying equation (4.39), we get that

$$A_x + y' A_y = 0. \tag{4.40}$$

Splitting equation (4.40) on y' gives

$$\begin{aligned} y' \quad &: A_x = 0, \\ &: A_y = -\tfrac{\lambda y^5}{4}. \end{aligned} \tag{4.41}$$

Integrating equation (4.41) with respect to y yields

$$A(x, y) = -\frac{\lambda y^6}{24} + B(x), \tag{4.42}$$

where $B(x)$ is an arbitrary function of x. After replacing the value of A into equation (4.41) and integrating with respect to x, we obtain $B = C_1$, where C_1 is an arbitrary constant of integration. Therefore, the first integral is given by

$$I(x, y, y') = -\frac{1}{2}y^2y'^2 - \frac{1}{24}\lambda y^6 + C_1. \tag{4.43}$$

Thus, we may let the first integral corresponding to P_1 be

$$I(x, y, y') = -\frac{1}{2}y^2y'^2 - \frac{1}{24}\lambda y^6.$$

It is easy to check that $D_xI = (-y^2y')\left(y'' + \frac{1}{y}y'^2 + \frac{1}{4}\lambda y^3\right)$.

Note. Even though the ode $y'' + \frac{1}{y}y'^2 + \frac{1}{4}\lambda y^3 = 0$ admits a two-dimensional algebra of Lie point symmetries, we obtain a single first integral. Nevertheless, the ode is completely integrable and, also, P is invariant under X_1 and conformally invariant under X_2 since $X_2P = -6P$.

III. One dimensional algebra case

(a) The first form of the second-order ode generating a Lie point symmetry algebra of dimension one is the **Mathews-Lakshmanan Oscillator** equation

$$y'' - \frac{\lambda yy'^2}{\lambda y^2 + 1} + \frac{w_0^2y}{\lambda y^2 + 1} = 0$$

with symmetry $X = \partial/\partial t$. The multiplier condition is

$$\frac{\delta}{\delta y}\left[P(x, y, y')\left(y'' - \frac{\lambda yy'^2}{\lambda y^2+1} + \frac{w_0^2y}{\lambda y^2+1}\right)\right] = 0, \tag{4.44}$$

so that

$$P = -\frac{y'}{\lambda y^2+1} \tag{4.45}$$

and

$$D_xI = \left(-\frac{y'}{\lambda y^2+1}\right)\left(y'' - \frac{\lambda y y'^2}{\lambda y^2+1} + \frac{w_0^2 y}{\lambda y^2+1}\right), \tag{4.46}$$

where $I = I(x, y, y')$ is the first integral. After expanding and applying the total derivative in equation (4.46), we obtain

$$I_x + y'I_y + y''I_{y'} = -\frac{y'\left(\lambda y^2 y'' + \left(-\lambda y'^2 + w_0^2\right) y + y''\right)}{\left(\lambda y^2+1\right)^2}. \tag{4.47}$$

Splitting on y'' yields

$$\begin{aligned} y'' \quad &: I_{y'} = -\tfrac{y'}{\lambda y^2+1}, \\ &: I_x + y'I_y = -\tfrac{\lambda y y'^2}{\lambda y^2+1} + \tfrac{w_0^2 y}{\lambda y^2+1}. \end{aligned} \tag{4.48}$$

Integrating equation (4.48) with respect to y' gives

$$I(x, y, y') = -\frac{y'^2}{2\lambda y^2+2} + A(x, y), \tag{4.49}$$

where $A(x, y)$ is an arbitrary function of x and y. Substituting the value of $I(x, y, y')$ into equation (4.48) leads into the following

$$-A_x - \left(\frac{y'^2\lambda y}{\left(\lambda y^2+1\right)^2} + A_y\right) y' = \frac{y'y\left(-\lambda y'^2 + w_0^2\right)}{\left(\lambda y^2+1\right)^2}. \tag{4.50}$$

After simplifying equation (4.50), we get that

$$A_x + y'A_y = 0. \tag{4.51}$$

Splitting equation (4.51) on y' gives

$$\begin{aligned} y' \quad &: A_x = 0, \\ &: A_y = -\tfrac{w_0^2 y}{\left(\lambda y^2+1\right)^2}. \end{aligned} \tag{4.52}$$

Integrating equation (4.52) with respect to y yields

$$A(x, y) = \frac{w_0^2}{2\lambda\left(\lambda y^2+1\right)} + B(x), \tag{4.53}$$

where $B(x)$ is an arbitrary function of x. After replacing the value of A into equation (4.52) and integrating with respect to x, we obtain $B = C_1$, where C_1 is an arbitrary constant of integration. Therefore, the first integral is, finally, given by

$$I(x, y, y') = -\frac{y'^2}{2(\lambda y^2 + 2)} + \frac{w_0^2}{2\lambda\left(\lambda y^2 + 1\right)}$$

and $D_x I = \left(-\frac{y'}{\lambda y^2+1}\right)\left(y'' - \frac{\lambda y y'^2}{\lambda y^2+1} + \frac{w_0^2 y}{\lambda y^2+1}\right)$.

(b) For **a particle in a rotating parabolic well**, the ode is

$$y'' + \frac{\lambda y y'^2}{\lambda y^2 + 1} + \frac{w_0^2 y}{\lambda y^2 + 1} = 0.$$

The multiplier functions take the form

$$P = f(x, y) - y' g(x, y) \tag{4.54}$$

which we showed to be

$$P = -(\lambda y^2 + 1)y' \tag{4.55}$$

leading to a first integral that satisfy equation (4.55)

$$D_x I = \left(-(\lambda y^2 + 1)y'\right)\left(y'' - \frac{\lambda y y'^2}{\lambda y^2 + 1} + \frac{w_0^2 y}{\lambda y^2 + 1}\right), \tag{4.56}$$

where $I = I(x, y, y')$ is the first integral and $D_x = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + \cdots$ is the total derivative with respect to x. After Expanding and applying the total derivative in equation (4.56), we obtain

$$I_x + y' I_y + y'' I_{y'} = -\left(\lambda y^2 y'' + \left(\lambda y'^2 + w_0^2\right) y + y''\right) u_1. \tag{4.57}$$

Splitting on y'' yields

$$\begin{aligned} y'' \quad &: I_{y'} = -\left(\lambda y^2 + 1\right) y', \\ &: I_x + y' I_y = \tfrac{\lambda y y'^2}{\lambda y^2+1} + \tfrac{w_0^2 y}{\lambda y^2+1}. \end{aligned} \tag{4.58}$$

Integrating equation (4.58), with respect to y' gives

$$I(x,y,y') = -\frac{1}{2}\lambda y^2 y'^2 - \frac{1}{2}y^2 + A(x,y), \tag{4.59}$$

where $A(x,y)$ is an arbitrary function of x and y. Substituting the value of $I(x,y,y')$ into equation (4.59). leads into the following

$$-A_x - \left(-\lambda y y'^2 + A_y\right) y' = -y\left(\lambda y'^2 + w_0^2\right) y'. \tag{4.60}$$

After simplifying equation (4.60), we get that

$$A_x + y' A_y = 0. \tag{4.61}$$

Splitting equation (4.61) on y' gives

$$\begin{aligned} y' \quad &: A_x = 0, \\ &: A_y = -w_0^2 y. \end{aligned} \tag{4.62}$$

Integrating equation (4.62) with respect to y yields

$$A(x,y) = -\frac{w_0^2 y^2}{2} + B(x), \tag{4.63}$$

where $B(x)$ is an arbitrary function of x. After replacing the value of A into equation (4.62) and integrating with respect to x, we obtain $B = C_1$, where C_1 is an arbitrary constant of integration. Therefore, the first integral is given by

$$I(x,y,y') = -\frac{1}{2}\lambda y^2 y'^2 - \frac{1}{2}y'^2 - \frac{1}{2}w_0^2 y^2 + C_1. \tag{4.64}$$

Thus, we may let the first integral corresponding to P_1 be

$$I(x,y,y') = -\frac{1}{2}\lambda y^2 y'^2 - \frac{1}{2}y'^2 - \frac{1}{2}w_0^2 y^2.$$

It is easy to check that $D_x I = (-(\lambda y^2+1)y')\left(y'' - \frac{\lambda y y'^2}{\lambda y^2+1} + \frac{w_0^2 y}{\lambda y^2+1}\right)$.

Discussion and Conclusion

We presented a general method to construct first integrals for some classes of some well known second-order ordinary differential equations, viz., the Emden and Liénard classes of equations. The approach does not require a knowledge of a Lagrangian and, instead, utilises the 'multiplier approach'. Then, we showed how a study of the invariance properties and conservation laws are used to 'twice' reduce the equations to solutions.

Chapter 5

On the first integrals of the Painlevé classes of equations

5.1 Introduction and Background

In this chapter, we will consider a well known class of equations known as the Painlevé type equations ([47], [50]) which are second-order odes. In [50], Lagrangians are constructed for the odes using the 'Jacobi Last Multiplier' (see also [51]). In principle, one would utilise the Lagrangians to determine the variational symmetries and Noether's Theorem to construct the first integrals. We will adopt this approach or the 'multiplier method' ([56], [46], [49]) to attain first integrals. In some cases, we use both approaches as the former, although more convenient, only delivers the zero-gauge symmetries. For second-order odes that are linearizable, there exists five independent first integrals (corresponding to five Noether symmetries of which at least two have non-zero gauge dependence) - the multiplier method would lead to all five.

The contents of this chapter appeared in [55].

We now present a method devised by *Jacobi* to derive Lagrangians of any second-order differential equation which involves finding the 'Last Multiplier'. This method provides a procedure [48] to determine the solutions of the pde,

$$Af = \sum_{i=1}^{n} a_i(x_1, x_2 ... x_n) \frac{\partial f}{\partial x_i} = 0, \tag{5.1}$$

or its equivalent associated Lagrange's system,

$$\frac{dx_1}{a_1} = \frac{dx_2}{a_2} = ... = \frac{dx_n}{a_n}. \tag{5.2}$$

There are several ways to construct the Multiplier. For eg., if we know all solutions less one, it is obtainable by

$$\frac{\partial(f, \omega_1, \omega_2 ... \omega_{n-1})}{\partial(x_1, x_2 ... x_n)} = MAf, \tag{5.3}$$

where

$$\frac{\partial(f, \omega_1, \omega_2, ..., \omega_{n-1})}{\partial(x_1, x_2, ..., x_n)} = \det \begin{pmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \\ \frac{\partial \omega_1}{\partial x_1} & \cdots & \frac{\partial \omega_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \omega_{n-1}}{\partial x_1} & \cdots & \frac{\partial \omega_{n-1}}{\partial x_n} \end{pmatrix} = 0 \tag{5.4}$$

and $\omega_1, \omega_2, ..., \omega_{n-1}$ are the solutions of (5.3). Thus, M is a function of the variables $(x_1, ..., x_n)$ and depends on the $n-1$ solutions chosen. The main properties of the Jacobi Last Multiplier are

(i) If one chooses a different set of $n-1$ independent solutions $\eta_1, \eta_2, ..., \eta_{n-1}$ for the equation $Af = 0$, then the corresponding Last Multiplier N is related to M as

$$N = M\frac{\partial(\eta_1, \eta_2...\eta_{n-1})}{\partial(\omega_1, \omega_2...\omega_{n-1})}. \tag{5.5}$$

(ii) Given a nonsingular transformation of variables,

$$\tau : (x_1, x_2...x_n) \to (x'_1, x'_2...x'_n), \tag{5.6}$$

the Multiplier M' of $A'f = 0$ is given by

$$M' = M\frac{\partial(x'_1, x'_2...x'_n)}{\partial(x_1, x_2...x_n)}, \tag{5.7}$$

where M is obtained from the $n-1$ solutions of $Af = 0$ corresponding to those chosen for $A'f = 0$ through the inverse transformation τ^{-1}.

(iii) One can prove that every Multiplier M is a solution of the linear partial differential equation,

$$\frac{\partial(Ma_1)}{\partial x_1} + \frac{\partial(Ma_2)}{\partial x_2} + ... + \frac{\partial(Ma_n)}{\partial x_n} = 0, \tag{5.8}$$

or of its equivalent,

$$\sum_{i=1}^{n} a_i \frac{\partial(\log M)}{\partial x_i} + \sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i} = 0; \tag{5.9}$$

vice versa every solution M of this equation is a Jacobi Last Multiplier.
(iv) If one knows two Jacobi Last Multipliers, M_1 and M_2, of (5.1) then their ratio is a solution ω of (5.1) or a first integral of (5.2). Since the existence of a solution is a result of the existence of a symmetry, an alternative formulation in terms of symmetries is provided in [44] . If we know $n-1$ symmetries of 5.1/5.2, suppose

$$\Gamma_i = \sum_{j=1}^{n} \xi_{ij}(x_1, x_2 \ldots x_n)\partial_{x_j}, \qquad i = 1 \ldots n-1, \tag{5.10}$$

then Jacobi's Last Multiplier is given by $M = \Delta^{-1}$, provided that $\Delta \neq 0$ where

$$\Delta = \det \begin{pmatrix} a_1 & \cdots & a_n \\ \xi_{1,1} & \cdots & \xi_{1,n} \\ \vdots & \ddots & \vdots \\ \xi_{n-1,1} & \cdots & \xi_{n-1,n} \end{pmatrix} \tag{5.11}$$

For second-order odes,

$$u'' = F(x, u, u') \tag{5.12}$$

the Jacobi Last Multiplier is as follows.
From property (iii) of the previous section we deduce that we can write

$$\frac{d}{dt}(\log M) + \frac{\partial F}{\partial u'} = 0 \Longrightarrow M = \exp\left[-\int \frac{\partial F}{\partial u'} dx\right] \tag{5.13}$$

that is to be satisfied by the Multiplier which we are going to find. Then, it can be shown that

$$M = \frac{\partial^2 L}{\partial u'^2}, \tag{5.14}$$

where $L = L(x, u, u')$ is the Lagrangian sought. This means that, if we know one Last Multiplier, then we can obtain L by two successive quadratures

$$L = \int \left(\int M du' \right) du' + f_1(x, u)u' + f_2(x, u) \tag{5.15}$$

where f_1 and f_2 are arbitrary functions. Since every pair of Lagrangians which differ only by a total derivative with respect to time of a differentiable function give rise to the same equations, we can define equivalence classes among Lagrangians up to total derivatives of an arbitrary function (gauge function). Then

$$f_1 = \frac{\partial g}{\partial u} \tag{5.16}$$

$$f_2 = \frac{\partial g}{\partial x} + f_3(x, u). \tag{5.17}$$

Hence, we get the Lagrangian

$$L = \int \left(\int M du' \right) du' + \frac{\partial g}{\partial x} + f_3, \tag{5.18}$$

where g is the arbitrary gauge function and using the Euler equation

$$-\frac{d}{dx}\left(\frac{\partial L}{\partial u'}\right) + \frac{\partial L}{\partial u} = 0, \tag{5.19}$$

it turns out that f_3 is not arbitrary. Using the above properties of the Last Multiplier, the Lagrangians of the second-order ordinary differential equations of Painlev'e type were performed [50]. Each of them can be written as a second-degree polynomial in u' with coefficients analytical in x and u

$$u'' + A(u)u'^2 + B(x,u)u' + C(x,u) = 0 \qquad (5.20)$$

and the equation defining the Last Multiplier (5.13) is

$$\frac{d}{dx}(\log M) - 2A(u)u' - B(x,u) = 0 \Longrightarrow M = \exp\left[\int (2A(u)u' + B(x,u))dx\right]. \qquad (5.21)$$

5.2 Lagrangians, multipliers and first integrals

In [51, 50], the authors establish a connection between the Jacobi Last Multiplier and the Lagrangians of second-order ordinary differential equations and, in the latter, the Lagrangians of some fifty second-order ordinary differential equations of Painlevé type are constructed. We consider a large class of these and show how we can obtain first integrals using either the Lagrangian formulation and the multiplier approach. Since the former sometimes require a knowledge of gauge terms, the latter produces a larger group of first integrals.

For $u = u(x)$, the conserved vector is reduced to a scalar differential function $T(x, u, u', \ldots)$.

5.2.1 Painleve classes and first integrals

For the ode

$$u'' - \frac{1}{2}\frac{u'^2}{u} - 4u^2 - \alpha u + \frac{1}{2u} = 0,$$

we obtain a single multiplier $Q = -\frac{u'}{u}$, have a lagrangian $L = \frac{1}{2}\frac{u'^2}{u} + \alpha\, u + 2\, u^2 + \frac{1}{2}\, u^{-1}$ with a variational symmetry $\frac{\partial}{\partial x}$ with corresponding first integral via Noether's theorem

$$T = \frac{1}{2}\frac{u'^2}{u} - \alpha u - 2u^2 - \frac{1}{2u}.$$

II. $u'' = 6u^2$, $L = \frac{1}{2}u'^2 + 2u^3$

The Lagrangian L leads to a single variational symmetry $\frac{\partial}{\partial x}$ with first integral

$$T = \frac{1}{2}u'^2 - 2u^3$$

and the multiplier is $Q = -u'$.

III. $u'' = 6u^2 + \frac{1}{2}$, $L = \frac{1}{2}u'^2 + 2u^3 + \frac{1}{2}u$

$$T = \frac{1}{2}u'^2 - 2u^3 - \frac{1}{2}u$$

from $\frac{\partial}{\partial x}$ with multiplier $Q = -u'$.

IV. $u'' = 6u^2 + x$, $L = xu + \frac{1}{2}u'^2 + 2u^3$, with no first integrals.

IV. $u'' = -3uu' - u^3 + q(x)(u' + u^2)$ has Lagrangian $L = \frac{1}{2u^2+2u'}e^{2\int q(x)\mathrm{d}x}$ which generates no first integrals.

VIII. $u'' = 2u^3$, $L = \frac{1}{2}u'^2 + \frac{1}{2}u^4$

$$T = \frac{1}{2}u'^2 - \frac{1}{2}u^4$$

with variational symmetry $\frac{\partial}{\partial x}$ and multiplier $Q = -u'$.

IX. $u'' = 2u^3 + xu + y$ is not conserved.

XI. $u'' = \frac{u'^2}{u}$ with Lagrangian $L = \frac{1}{2}\frac{u'^2}{u^2}$ leading to variational symmetries

$$u\partial_u, \qquad 2x\partial_x + u\ln u\partial_u, \qquad \partial_x$$

leading to first integrals, respectively,

$$-\frac{u'}{u}, \qquad -\frac{u'(u\ln u - 2xu')}{u^2}, \qquad \frac{1}{2}\frac{u'^2}{u^2}.$$

The multiplier approach yields the multipliers

$$Q_1 = \frac{1}{2}\frac{x\ln u}{u} - \frac{1}{2}u'\frac{x^2}{u^2}, \quad Q_2 = \frac{1}{2}\frac{\ln u}{u} - u'\frac{x}{u^2}, \quad Q_3 = \frac{u'}{u^2}, \quad Q_4 = \frac{x}{u}, \quad Q_5 = \frac{1}{u}$$

for which a couple of the first integrals are

$$T_3 = \frac{u'\left(uu'' - u'^2\right)x}{u^3}, \qquad T_5 = \frac{x\left(uu'' - u'^2\right)}{u'^2}.$$

Alternatively, if the ode is written as $\frac{uu''-u'^3}{u^3} = 0$, we obtain the multipliers to match the characteristics of the variational symmetries, viz.,

$$Q_1^c = \frac{1}{2}xu\ln u - \frac{1}{2}u'x^2, \quad Q_2^c = \frac{1}{2}u\ln u - xu', \quad Q_3^c = -u', \quad Q_4^c = xu, \quad Q_5^c = u$$

of which the last two correspond to the Noether symmetries which are gauge dependent and the first integrals are

$$T_4^c = \frac{1}{2}\frac{x^2\left(uu'' - u'^2\right)}{u^2}, \qquad T_5^c = \frac{x\left(uu'' - u'^2\right)}{u^2}$$

Clearly, the ode $u'' = \frac{u'^2}{u}$ is linearizable.

XII. $u'' = \frac{u'^2}{u} + \alpha u^3 + \beta u^2 + \gamma + \frac{\delta}{u}$ with Lagrangian $L = \frac{1}{2}\frac{u'^2}{u^2} + \frac{1}{2}\alpha u^2 + \beta u - \frac{1}{2}\frac{\delta}{u^2} - \frac{\gamma}{u}$ admits a single symmetry ∂_x with first integral

$$T = \frac{u'^2}{u^2} - \frac{\frac{1}{2}\alpha u^4 + \beta u^3 - \gamma u + \frac{1}{2}u'^2 - \frac{1}{2}\delta}{u^2}$$

and $u^2u'' = u^2[\frac{u'^2}{u} + \alpha u^3 + \beta u^2 + \gamma + \frac{\delta}{u}]$ admits a single multiplier $Q = -u'$ corresponding to the characteristic of the symmetry ∂_x.

XIII. $u'' = \frac{u'^2}{u} - \frac{u'}{u} + \alpha\frac{u^2}{x} + \beta\frac{1}{x} + \gamma u^3 + \delta\frac{1}{u}$ admits no symmetry and no first integrals.

XXXVII. $u'' = (\frac{1}{2u} + \frac{1}{u-1})\frac{u'^2}{u}$ has Lagrangian $L = \frac{1}{2}\frac{u'^2}{u(u-1)^2}$ which generate the zero gauge (variational) symmetries

$$(u-1)\sqrt{u}\partial_u, \qquad -x\partial_x + (u-1)\sqrt{u}\arctan\left(\sqrt{u}\right)\partial_u, \qquad \partial_x$$

with corresponding first integrals

$$-\frac{u'}{\sqrt{u}\,(u-1)}, \qquad -\frac{u'[u^{\frac{3}{2}}\mathrm{arctan}h(\sqrt{u}) - \sqrt{u}\mathrm{arctan}h(\sqrt{u}) + xu']}{u\,(u-1)^2} + \frac{1}{2}\frac{xu'^2}{u\,(u-1)^2}, \qquad \frac{1}{2}\frac{u'^2}{u\,(u-1)^2}.$$

Since there are two additional nonzero gauge symmetries, there exists two more nontrivial first integrals. The ode is linearizable.

XLIII. $u'' = \frac{3}{4}(\frac{1}{u} + \frac{1}{u-1})u'^2$, has Lagrangian $L = \frac{u'^2}{2(u(u-1))^{\frac{3}{2}}}$ which generate the zero gauge (variational) symmetries

$$\left[x(u-\tfrac{1}{2})hypergeom([\tfrac{1}{2},1],[\tfrac{5}{4}],u) + \tfrac{4xu(u-1)hypergeom([\frac{3}{2},2],[\frac{9}{4}],u)}{5}\right]\frac{\partial}{\partial x} + u(u-1)hypergeom([\tfrac{1}{2},1],[\tfrac{5}{4}],u)\frac{\partial}{\partial u}$$

which corresponds to a complicated first integral and the Noether symmetries

$$(u(u-1))^{\frac{3}{4}}\partial_u, \qquad \partial_x$$

have correspnding first integrals

$$\frac{-u'(u(u-1))^{\frac{1}{4}}}{u(u-1)}, \qquad \frac{u'^2}{2u(u-\sqrt{u(u-1)})}.$$

Here too, there are two additional nonzero gauge symmetries with nontrivial first integrals. The ode is linearizable.

XLIX. $u'' = \frac{1}{2}(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-\alpha})u'^2 + (u^2-u)(u-\alpha)(\beta + \frac{\gamma}{u^2} + \frac{\delta}{(u-1)^2} + \frac{\epsilon}{(u-\alpha)^2})$ has Lagrangian $L = \frac{1}{2}\frac{u'^2}{u(u-1)(u-\alpha)} + \beta\left(u + \frac{\alpha}{\alpha+1}\right) - \gamma\left(u^{-1} + (\alpha+1)^{-1}\right) - \frac{\delta\,(u+\alpha)}{(u-1)(\alpha+1)} + \frac{\epsilon\,(u+1)}{(-u+\alpha)(\alpha+1)}$

with a single variational symmetry ∂_x generating a first integral

$$T = -\frac{1}{2}\frac{u'^2}{u\,(u-1)\,(-u+\alpha)} - \frac{\beta\,(\alpha\,u+\alpha+u)}{\alpha+1} + \frac{\gamma\,(\alpha+1+u)}{u\,(\alpha+1)} + \frac{\delta\,(u+\alpha)}{(u-1)\,(\alpha+1)} - \frac{\epsilon\,(u+1)}{(-u+\alpha)\,(\alpha+1)}$$

5.3 Discussion and Conclusion

In this section, we constructed the first integrals of a large class of the well known second-order Painlevé equations. In some cases, variational and/or gauge symmetries have additional applications following a known Lagrangian in which case the first integral is obtained by Noether's theorem. In some convenient cases, we also adopted the 'multiplier' approach to find the first integrals.

Chapter 6

Conclusion

In this dissertation, our utilised a novel approach to dealing with the integrability of certain classes of reductions of some type equations.

In the second chapter, we dealt with the integrability of certain classes of reductions of some diffusion type equations, viz., the Fisher-Kolmogorov and Nagumo equations. The method utilises a knowledge of a Lagrangian of the reduced (ode) to obtain conditions under which gauge symmetries exists. There is an alternative way of obtaining values of parameters for which equations possess Painleve properties and, therefore, are integrable.

In chapters three and four, using Emden and Liénard equations as examples, we presented a general method for computing the first integrals of some well-known second-order ordinary differential equations. The separate chapters differentiated between linearizable and nonlinearizable cases. The 'multiplier approach' was used rather than a knowledge of lagrangian. Next, we showed how conservation laws and invariance properties can be used to reduce equations twice.

In our final chapter, we constructed first integrals of a class of well-known second-order Painlevé equations.

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