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Investigating mathematics teacher learning using a research-designed resource in a lessoning study

A thesis submitted to the Faculty of Humanities, University of the Witwatersrand, Johannesburg, in fulfillment of the requirements for the degree of Doctor of Philosophy

By

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March 2022

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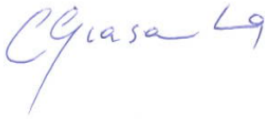
Declaration

I declare that this research report is my own work, except as suggested in the acknowledgements, the text and references. It is being submitted to the faculty in fulfilment of the requirements for the degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other institution.



4 March 2022

…….. ……

Signed Date

Lizeka Constance Gcasamba

Dedication

To my great-great-grandfather, Gcasamba

Nkabi ebethwa ingezwa. Wena malwa yedwa ongenamlamleli. The Nguni warrior eyafela empini. Ikroti elafa nomkhonto esifubeni. Mlomo ongatheth' imanga otheth' iqiniso. Ah-Malizo kaGcasamba!!!!!!!!!!!!

Acknowledgements

I would like to first thank my supervisor, Professor Jill Adler for placing her confidence in me to undertake this journey. I am grateful for the considerable role she played in guiding and advising me throughout my candidature; without her, this work would not have been possible. *Ndithi kuwe ungadinwa nangomso.*

The financial assistance of the National Institute for the Humanities and Social Sciences, in collaboration with the South African Deans' Association (SAHUDA) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at are those of the author and are not necessarily to be attributed to the NIHSS and SAHUDA. I am also thankful for the financial support that was provided by the Wits Maths Connect Secondary project at the initial stages of this research.

My profound gratitude also goes to the preservice teachers who participated in this study.

I am very glad to acknowledge my two dear friends, Dr Khohliso and Dr Moshugi for their tireless moral support . *Ngiyonikhohlwa ngifile.*

Ndibulela kumntu wonke ondinike inkxaso kolugqatso lunzima. Ndenze umbulelo omkhulu kubazali bam, uTony unyana kaMakuleni noNomaphethelo intombi kaDyobha; my children-uNomzingisi,Hlumela,Nosipho and Nontando; my dearest sister-Nonzuzo; umntuwam-Buyelekhaya; umthakathi-Mpendulo; nomakaStrike. *Cishe ngafa*, ngaphandle kwenu, I would not have made it.

In memory of my academic mentor (Dr Sikhumbuzo Manhood Yani) who believed in the power of education. Your words of wisdom definitely made a footprint in my mind : "Read, read and read....and do more research, research and research.....and do more in-depth analysis, and it is then that you'll be more empowered. I promise you ,you'll possess POWER!" SM Yani. I did not know that when you advised me to go for postgraduate studies, you were prophesying that one day I would inherit the calabash of knowledge. *Rhabula ungafinci*.

Ngibuye ndicamagushise kwizihlwele ezihlambulukileyo zakwaGcasamba nakwaMbucela. Camagu, booZizi, Cubungulashe nani Makwayi, booNgconde.Ndincome ubuhle bezibeleko zamakhehlekazi am : uMantambo, uBhelelazi, uNomayeza intombi kaMhluthwa, noDeliwe intombi kaMdukiswa. *Amaqobokazana angalala endleleni uboyazi kunyembelekile*.

Thixo kaGcasamba akunangqaleko, akunangqibeko, ukho eendaweni zonke, ukho ngamaxesha onke.

Abstract

The study reported in this thesis explored how mathematics preservice teachers used a research-designed resource named the Mathematics Teaching Framework (MTF) in the context of a particular lesson study called lessoning study (L/S), and the implications thereof for the nature of teacher learning. This study was located within the field of teacher collaborations that seek to understand the relationship between teacher-researcher collaborations and resources designed for collaborations in a collaborative environment and so to gain insights about the research–practice gap.

This study drew from ideas of Wenger's (1998) community of practice theory supplemented by both Sfard's (2008) communicational theory and Mercer's (1996) modes of talk to investigate what happens when mathematics preservice teachers collaborate with a resource for collaborations. What happens to this resource? How do collaborations offer opportunities to learn? What kinds of opportunities arise? What enables learning opportunities to arise?

This case study involved an L/S community of practice of four mathematics preservice teachers and the researcher. When a group of individuals collaborate in a community they partake in discursive activities of that community (Sfard, 2008). Building from what is known about discursive activity of mathematising (talk about mathematical objects), this study focused on a particular discursive activity of MTFying: a construct I developed and defined as talk about MTF objects.

This study investigated the relationship between teachers' MTFying discursive activity and the opportunities for teacher learning that arise through participation in L/S activities. The results showed that teachers who participated in this L/S study had locally defined practices guiding their work. These practices included lesson planning by identifying what was critical for learners to learn, teaching lessons in classrooms for addressing learners' needs, observing those lessons, and reflecting on those implemented lessons. As teachers engaged in this work, collaborative, MTFying, reflective and problematising practices were established. The results of this study further highlighted the important role played by the knowledgeable other (the researcher) for

these practices to emerge. In addition, this study discovered that the process of using MTF collaboratively enabled opportunities to learn how to capture learners' needs, content to be learnt, and teaching strategies related to critical aspects of object of learning, instructional coherence and mathematical teaching. At the same time, the results from the analysis also revealed that the knowledgeable other had opportunities to learn how to scaffold the MTFying discursive activities during the lessoning study sessions. These were categorised into high scaffolding, moderate scaffolding and low scaffolding.

This study has also revealed that a tool designed for collaboration may be transformed and become a tool from collaboration reflecting teachers' practice in new ways.

Keywords: Critical aspects of object of learning; MTF; discursive activities; collaboration; scaffolding; Lessoning study; opportunities for teacher learning

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Abbreviations and Acronyms

CA	Critical Aspect
CLS	Chinese LS
CoP	Communities of Practice
CoP-L/S	Communities of Practice-Lessoning Study
CPD	Continuing Professional Development
CPTD	Continuing Professional Development for Teachers
CT	Communicational Theory
CUMM	Cumulative
DISPU	Disputational
DoE	Department of Education
DPD	Delivery Pedagogical Discourse
EPD	Explorative Pedagogical Discourse
EXPLO	Exploratory
FA	Framework Analysis
IAF	Initial A Priori Framework
IF	Interaction Function
JE	Joint Enterprise
JLS	Japanese LS
KO	Knowledgeable other
LP	Lesson Planning
LS	Lesson Study
L/S	Lessoning Study
MD	Mathematics Discourse
MDI	Mathematical Discourse in Instruction
MD-MTF	Mathematical Discourse informed by Mathematics Teaching Framework
ME	Mutual engagement
MTD	Mathematical Teaching Discourse
MTF	Mathematics Teaching Framework
MTF-TD	Mathematics Teaching Framework Teaching Discourse
MTRD	Mathematics Teachers' Reflective Discourse
MU	Managing Understanding
OoL	Object of Learning
OTL	Opportunities-To-Learn
PD	Professional Development
PLC	Professional Learning Community
RD	Reflective Discussion
RFA	Reflection-for-action
RI	Routine Interaction
RIA	Reflection-in-action
ROA	Reflection-on-action

RTA	Realization Tree Assessment
SA	South Africa (n)
SC	Structuring Conversation
SCDA	Sociocultural Discourse Analysis
SLS	Swedish Learning Study
SR	Shared repertoire
T1	First Teaching
T2	Second Teaching
TD	Teaching Discourse
TE	Teaching experience
TM	Teaching Mathematics
UK	United Kingdom
USA	United States of America
WMCS	Wits Maths Connect Secondary
WSoE	Wits School of Education

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Chapter 1: Introduction

1.1 Introduction

The study reported in this thesis explored how preservice teachers used a research-designed resource named the Mathematics Teaching Framework (MTF) in the context of a particular lesson study (LS) called a lessoning study (L/S), and the implications thereof for the nature of teacher learning.

This study was located within the field of teacher collaborations that seek to understand the relationship between teacher-researcher collaborations and resources designed for collaborations in a collaborative environment and thus to gain insights about the research–practice gap. Within this field of research, Krainer and Spreitzer (2020) reported on current research on teacher collaborations in terms of collaborative environments and collaborative interactions of various participants in mathematics education such as teachers, teacher educators and researchers in these collaborative environments.

I further focused on the nature of teacher learning that is afforded by participation in teacher collaborative contexts while using the resource for collaborations. Putman and Borko (2000) called for the use of sociocultural and situated theoretical perspectives in teacher learning research. In their study they underlined a number of aspects to be considered such as: (1) situating teachers' learning experience; (2) the nature of the discourse within teacher teaching communities; and (3) the importance of teacher learning resources in collaborative environments. Following this call, Prediger (2020) reinforces it and the mathematics education research has seen an increase in studies related to teacher collaborations with resources (e.g., Adler & Alshwaikh, 2019; Brodie, 2020; Pepin et al., 2013; Weingarden & Heyd-Metzuyanim, 2020). A common theme addressed in these studies is related to the relationship between resources for collaborations and teacher learning or opportunities to learn. However, according to Robutti et al. (2016), little is known about what happens when teachers collaborate with resources for collaborations. What happens to these resources? How do collaborations offer opportunities to learn? What kinds of opportunities arise? What enables learning opportunities to arise? These are just some of the underexplored questions this study began to address.

In this study, I further extended this call by integrating Wenger's (1998) communities of practice theory, as well as Sfard's (2008) and Mercer's (1996) sociocultural frameworks, where I put under microscopic

lenses the use of the research-designed resource in a situated context of lessoning study and described the nature of emerging discourse and the nature of learning opportunities afforded by this environment. In this way, this study focused on theorising teacher learning with a research-designed resource, thus contributing to theoretical perspectives; this research built upon—the situated and sociocultural perspectives.

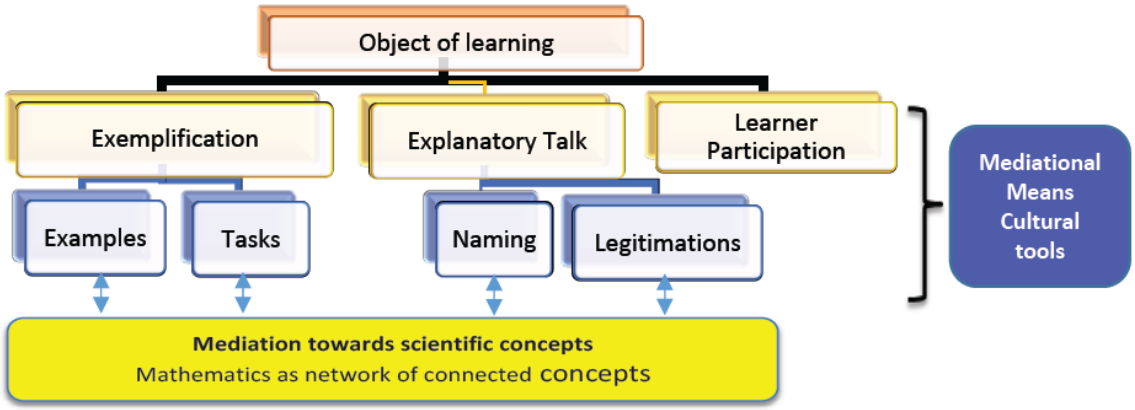
In this chapter I give a brief discussion of the development of the mathematics teaching framework (MTF) tool, followed by a discussion about the rationale for the study, significance of this study and the outline of the chapters.

1.2 The development of the MTF tool

MTF has its roots in what Venkat and Adler (2012) refer to as teachers' mathematical discourses in instruction (MDIs). According to them, MDIs are "aspects of what teachers say, do and write as they interact with learners in mathematics classrooms, [and] are a key feature of classroom practice" and include "a problem, a selected representation that is subsequently transformed, and explanations and justifications for the representations selected and transformations performed" (Venkat & Adler, 2012, p. 1). The aim of this early work was to understand how the MDIs, teacher explanations in particular, enabled connections between representations and transformation activity, and coherence with the stated problem. The notion of mathematical discourse in instruction (MDI) was later further developed in the Wits Maths Connect Secondary (WMCS) project into an analytic framework for describing and interpreting differences in key elements of mathematics teaching (Adler & Ronda, 2015), specifically: exemplification (including tasks and examples), explanatory talk, learner participation and the object of learning (see Figure 1):

Figure 1

The WMCS Mathematics Discourse in Instruction



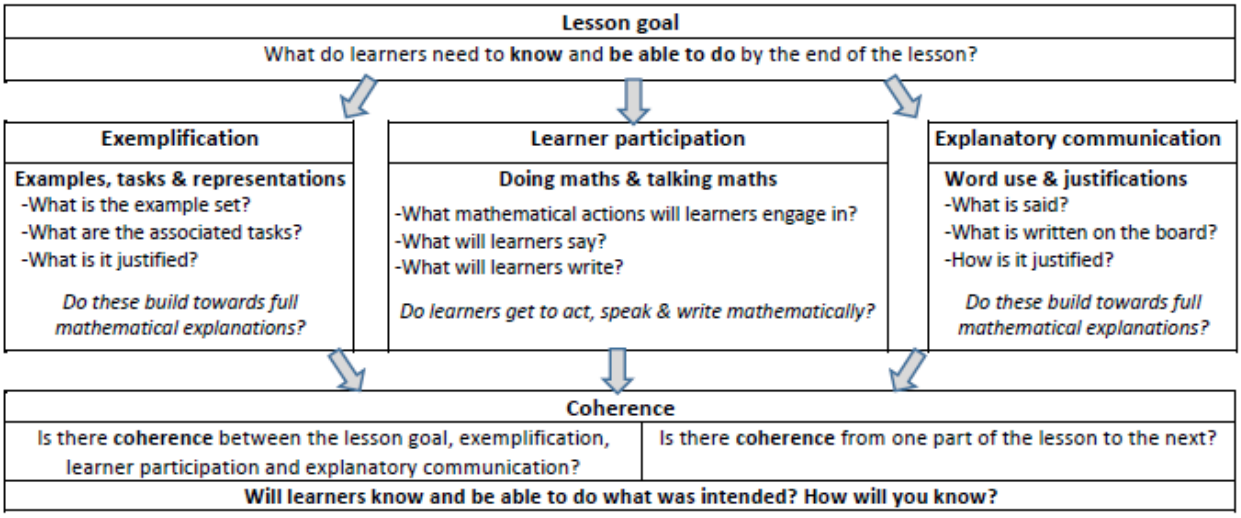
Note. Extracted from Adler & Ronda (2015)

As argued in Adler (2017), the MDI framework enables a description of mathematics made available to learn in a lesson through its highlighting of exemplification, explanatory talk and learner participation as instructional practices that are key in mathematical teaching. Adler (op cit) further described the MDI as a boundary object (Adler, 2017) as it could be, and had been, used in multiple contexts such as research, professional development and teaching communities. For instance, when used in research, it was elaborated into an analytical framework for analysing the quality of mathematics expanding to include a broad range of indicators for the quality of mathematics made available to learn in a lesson (Adler & Ronda, 2015). The framework has been used to investigate whether and how exemplification and explanatory talk enable or constrain learners' opportunities to learn (Adler & Ronda, 2017a), and adapted and to analyse mathematics made available in textbook "lessons" (Ronda & Adler, 2016).

When used in the professional development community, the MDI was elaborated as a teaching framework and used in conjunction with the designed course for the WMCS professional development programme called the Transition Mathematics (TM) course. In this professional development (PD) programme, the participating teachers were provided with opportunities to strengthen their mathematical knowledge for teaching. In order to achieve this goal, attention was focused on all elements of MDI. For instance, when teachers were working with the exemplification element, they were expected to consider the quality of selected examples and their sequencing and using principles of variation theory as described in the MDI framework (Adler & Pournara, 2020). This exemplifying process afforded teachers with opportunities to develop a knowledgeable use of examples that were varied in such a way that they foregrounded mathematical ideas that learners needed to learn. When the MDI was used in a teaching context, it was used as a teaching tool called Mathematics Teaching framework - MTF (see Figure 2).

Figure 2

The WMCS Mathematics Teaching Framework (MTF)



Note. Extracted from Adler (2021a)

The MTF tool was designed by WMCS researchers to foreground the valued mathematics and mathematics teaching practices in the project, and further to support teachers' work of teaching and learning about their mathematics teaching. For Adler (2021, p. 1210), "the knowledge and practices of the community, with all its history, are 'encoded' in the resource, creating a 'cultural organization of access'". These mathematical teaching practices include exemplification (examples and tasks), explanatory communication (word use and legitimations) and learner participation (what learners are invited to do). Selling et al. (2016) would call these "mathematical instructional objects" that teachers encounter and interact with while teaching mathematics. The MTF components can be viewed as instructional objects. In this study I referred to MTF components as MTF objects. In order to provide for teachers' use of the MTF in their classroom practice, and as a research and development project an additional context for the further development of the MTF, the WMCS also offered a form of LS for some of the teachers who had come on the TM course. This particular form of LS was driven by the same goals as other lesson studies and is discussed further below.

According to Adler (2021), the design of the MTF tool was inspired by their previous observations where they identified the lack of coherence in mathematics teaching. In Adler's (2021) words, "the mathematical object of learning was out of focus. Despite an announced topic, frequently there was no apparent mathematical story linking what learners were to know and be able to do" (Adler, 2021, p.

1211). For Adler (2021), the main goal of designing the MTF tool was to provide teachers with resources to develop learning opportunities for their learners. Adler (2021, p. 1215) defined opportunities to learn in terms of WMCS goals "for more coherent lessons, where example sets and accompanying explanatory talk cohered with the object of learning, and learners were invited to participate in doing and talking mathematically, of course in relation to the object of learning". The process of developing coherent mathematics teaching includes two steps. The first step relates to the articulation of what needs to be learned—the object of learning in a lesson. This step is then followed by the second step which concerns the connection of what is critical to learn through three teaching practices that are used as mediational means in the mathematical teaching context: exemplification, explanatory communication and learner participation.

Since its development, the MTF tool has been reported in a number of studies within the WMCS project (e.g., Adler & Alshwaikh, 2019; Leshota, 2020; Ntow & Adler, 2019; Ronda & Adler, 2019) with different foci. Of particular relevance for this current study, is the study by Adler and Alshwaikh (2019) who investigated in-service teachers learning to use the MTF through participation in LS. They adapted the Japanese LS where teachers were planning lessons, teaching the lessons and reflecting on lessons with the MTF as a guide through these processes. The LS was implemented in three different school clusters, with research focused in one of the clusters with teachers participating from three different schools. The goal of this localised LS was to provide a collaborative environment between teachers and the leaders of the WMCS (including the researchers) where both teams were exploring the role of the MTF tool in the work of teaching. The focus of the research reported in Adler and Alshwaikh (2019) was on teachers' take up of exemplification through LS cycle, and how it evolved through lesson planning, teaching and reflective discussion sessions. The findings showed evidence of change of example sets which were shaped by the principles of variation throughout the LS cycle. Such an outcome has implications for opportunities of teacher learning with the MTF tool—opportunities for learning about choosing and using examples.

Adler and Alshwaikh (2019) are in line with research on resource for collaborations (e.g., Kieran et al., 2012; Pepin et al., 2013). An important contribution to research is the focus on resource use for teacher learning where they illuminate the resources-teacher-researcher collaborations. From this, the current study extended Adler and Alshwaikh's (2019) study and lifted it to a different context of preservice teaching. I discuss in detail the MTF tool in Chapter 2, where I locate it in the field of tools for collaborations and discuss its features in detail, drawing from mathematics education research.

It is important to note that this current study did not use the elaborated MDI analytical framework for a number of reasons, and worked selectively with the MTF which evolved over time, and while this current study was in process. From an MDI perspective the object of study is mathematics and the relationship between MTF instructional practices. The MDI has a potential in describing and evaluating the mathematical content made available to learn, and therefore not directly relevant for this study. In the context of this study, the main objects of study are the MTF objects and humans rather than mathematical objects. My goal in this study was to examine teaching practices (processes) when teachers use the MTF tool and better understand how these practices take place in L/S setting and what learning opportunities were afforded through such participation.

1.3 The rationale and purpose of this study

1.3.1 Why did the study focus on a research-designed resource— MTF?

My journey into this research study started as a doctoral fellow in the Wits Maths Connect Secondary Project (WMCS). The WMCS Project is a ten-year research and development project, working in collaboration with the Gauteng Department of Education at both district and provincial level to support mathematics teachers. In phase 1 (2010–2014), WMCS worked with teachers from 11 secondary schools in one district in the greater Johannesburg area, and in phase 2 (2015-2019) this was extended to more teachers from a wider range of districts. A central goal of the project was to improve the teaching and learning of mathematics within each school through ongoing professional development (PD). The PD was structured into extended mathematics-for-teaching courses and classroom-based practice. An interest in teacher learning in classroom-based practice was the inspiration for this study. The WMCS classroom-based model used an adapted version of the well-known Japanese LS (Stigler & Hiebert, 1999) where teachers used the MTF tool (Adler & Alshwaikh, 2019). The MTF tool was developed within the project and has its basis in key needs and challenges in mathematics teaching in South Africa that were identified in the project.

One of the key problem areas for me was the problematic relationship between mathematics education research and practice. This notion has been widely reported in the literature and has even been coined the "research-practice gap" (e.g., Cheng et al., 2010; Goos, 2014; Kempe, 2019). In this study, I was working in this gap and attempted to understand the use of the research-designed tool (MTF) in a particular context and the nature of teacher learning that results from this use. This fits largely within the studies on research-designed resources (e.g., Kieran et al., 2012). According to Kieran et al. (2012),

while over the years mathematics education research has produced a number of resources aimed at healing the ills of practice, there is nevertheless a dearth in literature about ways in which teachers use such researcher-designed resources in practice. There have been calls in mathematics education research for the research that lends an ear to research-based resources and how they are used by teachers in practice (e.g., Adler & Ronda, 2015; Kazemi & Hubbard, 2008; Kieran et al., 2012; Pepin et al., 2013). For example, Pepin et al. (2013, p. 929) explicitly mentioned that future research needs to pay attention to the following: "how teachers may work in collectives and with resources, and in which ways 'productive' collectives may form and work together; … and how particular resources may support teachers in their work and learning". It was the aim of this study to contribute to this knowledge base, by investigating teachers' use of a research-designed tool in a collaborative practice and what implications can be drawn with regard to teacher learning.

The issue of the disconnection between research and practice has been frequently addressed in mathematics education research. Kempe (2019) describes it as the research-practice gap. Research is often seen by teachers as divorced from the practical realities of teaching practice (Carlgren, 2012). Silver and Lunsford (2017, as cited in Kempe, 2019, p. 250) stated,

> On the one hand, theorists/researchers claim that results from educational research (i.e., theoretical concepts, models or proposals) are relevant for teachers and can inform practice. Teachers, on the other hand, have stated that educational research was far from their everyday problems, too theoretical and not easy to access.

Studies of preservice teacher education have also revealed a disparity between the theory taught in teacher education programmes and the subsequent practice of these teachers in the classroom (e.g., Cheng et al., 2010). Cheng et al. (2010) defined this gap as the "inconsistencies between the selection of the best teaching strategies and the most commonly employed teaching strategies" (p. 94).

In attempts to break down the dichotomy between research and practice, researchers have offered different ways of integrating the two. For instance, some of the studies in the mathematics education research focused on the collaboration between teachers and researchers (Adler & Alshwaikh, 2019; Lewis & Perry, 2017; Pericleous, 2020). These studies examined the teacher-researcher collaborations where both participants have a common goal of working with and learning from the resources for teaching.

Lewis and Perry (2017, p. 263) argued that the gap between research-based knowledge and teacher knowledge is caused by the tension between five features of professional learning: “content focus, active learning, coherence, duration and collective participation”. For instance, they explained that when the research-based knowledge identifies the content that is not aligned with current teachers' knowledge in terms of curriculum standards and instructional strategies, tension prevails and the research-practice gap widens. In order to improve instruction, “professional learning must bridge the gap between teachers’ current knowledge and beliefs and research-based knowledge” (Lewis & Perry, 2017, p. 263). They further argued for an environment of collective participation by both researchers and teachers as co-learners in order to ease the tensions.

Grossman and McDonald (2008, p. 2) also noted that “the field of research on teaching still lacks powerful ways of parsing teaching that provide us with the analytic tools to describe, analyze, and improve teaching”. They argued for a framework that would help to identify strategies of teaching and ways to engage students and so improve teaching and that is able to inform both research and teaching. They suggested that these frameworks need to encompass common language between research and teaching, common examples of teaching as well as common contextualised demands of teaching. Nuthall (2004) once lamented the slow pace in the development of such frameworks in educational research. Up to this day, still not enough has been done in mathematics research to address this gap, as Adler (2021, p. 1208) noted: “Out of view and under-explored across this expansive body of work are ideational resources, particularly frameworks that foreground key elements of mathematics teaching”.

One contribution to such work that I have reviewed is the framework developed by Adler and Ronda (2015), called mathematics discourse in instruction (MDI). As already noted, the significance of this framework is its twofold character: its intentional use and movement between theoretical knowledge and practical skills, that is, between the best teaching practices as investigated by the research and the practical knowledge of teachers.

Over the past two decades of curriculum change, the South African education system has focused attention increasingly on the teaching of critical subjects such as science and mathematics to improve learner performance in these subjects (Jansen, 1998; Stols, 2013). However, not enough attention has been paid to the preparation of teachers to use the tools for teaching such subjects during teaching practice (Adler, 2000). In addition, the South African government has prioritised education in mathematics and science as part of the means to close the gap of scarce skills (National Skills Development Strategy III, DHET, 2013). Universities have since been identified as the higher education

institutions to address the development of work-ready graduates and training of teachers in the country. I, thus, present the study on the nature of the use of a research-designed tool (MTF) extended to a group of secondary mathematics preservice teachers in the context of a lessoning study. Building from what is known about teacher collaboration with resources for collaborations, this study focused on a particular act of collaboration, a construct I developed called MTF-use. The concept MTF-use can be understood in similar ways described by Remillard (2005) when referring to "curriculum-use", where she defined it as "how individual teachers interact with, draw on, refer to, and are influenced by material resources designed to guide instruction" (p. 212).

1.3.2 Why did the study focus on lessoning study?

This study addressed the opening in research, the under-examined issue of interrelatedness of exchange of knowledge between teachers and researchers (e.g., Goos, 2014). According to Pericleous (2020), there is a growing number of studies in mathematics education research focusing on the collaboration between teachers and researchers. Some of these studies examined the collaboration through a LS professional development programme (e.g. Adler & Alshwaikh, 2019; Coenders & Verhoef, 2018).

A response to the research-practice gap in preservice teacher education has been to incorporate LS into the preservice teacher education programmes (e.g., Carrier, 2011; Chassels & Melville, 2009; Lewis, 2019). The LS in preservice programmes is useful as it "provides opportunities for planning before lessons and analysis of artifacts coming out of lessons, but the actual interactive work with children is distant" (Lewis, 2019, p. 485). In this study I drew from ideas from Japanese LS (Stigler & Hiebert, 1999), Swedish Learning study (Lo & Marton, 2012), and Chinese LS (Huang & Bao, 2006) and developed a new adapted form of LS which I called the lessoning study. From Wenger's (1998) social learning perspective, I argue for a "lessoning study model" as an instance of a boundary encounter (Wenger, 1998) between researchers and teachers; and as a premier space for researchers and teachers, impacting both communities of practice.

Pepin et al. (2013) have argued for collaborative engagements of teachers with resources in different learning communities such as classrooms and professional development contexts. Hence the consideration of the three mentioned types of lesson studies as they all provide a space where different communities of practice come together (Wood, 2017). For example, the learning study pulls together learners, teachers and researchers (Pang & Marton, 2005). According to Pang and Marton (2005), a learning study is a learning investigation guided by the variation theory where it can be viewed in three

senses. First, the students in the study are expected to learn about the mathematics content. Second, the teachers are expected to learn about teaching mathematics content, and lastly, the researchers are expected to learn about how the variation theory works. According to Goos (2014), teachers and researchers are members of different but interrelated communities of practice. However, it is possible to bring these two communities of practice together through the establishment of a boundary encounter (Wenger, 1998). In this sense, learning study is a boundary encounter that bridges the boundaries between different but interrelated communities of practice (Goos, 2014). Kazemi and Hubbard (2008) have highlighted an important goal for future research where they questioned the role of the researcher in the classroom context. They pleaded for future research to capture the interactions between researchers and teachers in the classroom context. This study has heeded this call.

The collective work of teachers is emphasised in most studies investigating teachers' use of resources (e.g., Brodie, 2020; Gueudet & Trouche, 2012b; Pepin, 2012). According to Wood (2017), the Japanese LS, Swedish Learning Study and Chinese LS are increasingly respected as forms of classroom-based teachers' professional development. For example, the LS foregrounds the use of resources and has produced positive results in both teacher professional development and student learning in Japan where it originated (Yoshida, 2012) and in America where it has been adopted and adapted (Perry & Lewis, 2008). These forms of lesson studies highlight the importance of working collaboratively. They all share many common features, and are both concerned with teachers working together collaboratively to discuss lessons, which they previously jointly planned and then observed as they unfolded in actual classrooms. In all these studies, resources were perceived as beneficial for studying a particular topic. Some of the resources that are used in LS include lesson plans and curriculum resource documents (Lewis, 2019). These are all mathematics teaching resources, "which are developed and used by teachers (and pupils) in their interaction with mathematics in/for teaching and learning, inside and outside the classroom" (Pepin et al., 2013, p. 929). The MTF is a framework for working on teaching located in these types of resources (Adler, 2021).

In light of all these characteristics of various forms of LS, this current study has developed a hybrid of Japanese LS, Swedish learning and Chinese LS principles and adapted these to the South African (SA) context, as elaborated more fully in the literature review and methodology sections. I have referred to this new version as a "lessoning study (L/S)" model. It was set up and employed in this study with a focus on studying how teachers used a research-designed resource—the MTF tool—and what meanings they attached to the tool in terms of exemplification, explanatory talk and learner participation and how

these were manifested in their practice. Taking from Adler and Alshwaikh (2019), the MTF tool is a tool that guides teachers when planning, implementing and reflecting on their teaching during the LS. Hence, in this study I focused on how teachers made use of this tool during lesson planning, teaching, observing and reflecting in the lessoning study. This was another reason for choosing a lessoning study approach as my empirical field. The lessoning study provided me with a setting where the researcher and teachers work together collaboratively, share/negotiate knowledge and reflect on ways that a research-designed tool can be used in teaching practice. This study was specifically aimed at seeking to understand how teachers used a research-designed resource and adapted it to their own needs in lessoning study practice. This justifies why the lessoning study provided an ideal empirical field for me.

1.3.3 Why research teacher learning with preservice teachers?

At the start of this chapter I noted my initial engagement with the WMCS project. As the study progressed, I later joined the Wits School of Education (WSoE) as the teacher educator of the preservice mathematics teachers. Preservice teacher learning has long been the backbone of teacher education programmes. Participation in the teaching experience (TE) programme is integral to preservice teacher professional development. Teaching practice is an important component of the teacher education programme that provides opportunities for preservice teachers to learn in the context of a real classroom. Such a practice is a social practice (Wenger, 1998). According to Shulman (1986) and Darling-Hammond (2015), teaching experience/practice is a platform for preservice teachers to apply the theoretical knowledge learned from their coursework (methodology courses) in the teaching of specific subjects in schools.

Preservice teachers learn to teach learners during TE. In the WSoE preservice teacher education programmes, the conventional model of teaching practice involves sending trainee teachers to schools for a period of six weeks to practice what they have learnt during teaching methodology courses at the university, under the supervision of teacher educators and mentors—school teachers. During teaching practice sessions, preservice teachers are expected to plan their daily lessons individually and their teaching is observed occasionally by their teacher educators and mostly by their mentor teachers in schools, who will guide and evaluate their performance. My role in the TE programme was that of a supervisor—a knowledgeable other from the university. Before the student teachers go out for their teaching experience, a briefing meeting is convened between the supervising teacher and the students. The main aim of this meeting is to brief the student teachers on what is expected of them during the TE, and also advise them on teaching strategies to use. I used this opportunity to informally introduce them

to teaching strategies informed by MTF (i.e., identifying the object of learning, choosing a set of examples, constructing tasks, explanations and learner participation). During the visits I made to schools, I noticed that the student teachers were drawing ideas of MTF into their lesson planning, and also in their post-lesson reflections with me as the supervising teacher. A motivation to use preservice teachers was rooted from this background.

Informed by my exposure at the WMCS project, and their model of LS being informed by MTF (Adler & Alshwaikh, 2019), I was motivated to embark on an investigation of preservice teacher learning in the LS context. This decision was further cemented by my engagement with literature on preservice teacher learning in the context of LS (e.g., Carrier, 2011; Chassels & Melville, 2009; Lewis, 2019; Nakamura 2019). For example, Lewis (2019) has noted that there is a disconnect between preservice teacher methodology courses and the realities of the classroom. The LS model has been linked to help preservice teachers to develop teaching competencies (Chassels & Melville, 2009) and a model of learning to teach in a real context (Hiebert & Stigler, 2000). The lessoning study developed in this study was aimed at assisting preservice teachers to develop insights into teaching mathematics through the use of MTF.

According to Nakamura (2019), in order to understand how student teachers learn from LS, the research needed further considerations on (1) preservice teachers' observation of lessons, (2) preservice teachers' reflective discussion (3) the role of the knowledgeable other, and (4) teaching resources. In this study I paid attention to teaching resources for further consideration. The L/S model developed in this study is a mechanism through which preservice teachers in this study had the opportunity to connect with innovative specific components of mathematical teaching (work of teaching) such as identifying objects of learning (OoL)—anticipating students' responses, exemplifying, explaining and constructing tasks.

Reflection is a vital component of learning how to teach. LS has been reported as the model to help preservice teachers to develop reflective practice. However, research on the use of LS in initial teacher education is still in its infancy (Grosser-Clarkson, 2015), and there is no significant information about "how, and the extent to which the implementation of this practice facilitates reflection in its participants" (Myers, 2012, p. 1). This study contributes to this gap by showing the nature of teachers' reflective discourse process.

In the literature chapter, I describe in detail how the LS model works, and how it speaks about the collective participation and enquiry into teaching—and a development or change that follows. Since this

resonated strongly with the work of Wenger (1998), I drew on this work for the core theoretical framework of this study. This framework influenced the nature of the research and indeed the nature of preservice teachers' community of practice.

1.3.4 Why focus on Wenger's social learning theory?

Many studies referring to resources for collaborations have considered teachers' work within communities of practice (Brodie, 2020; Gueudet et al., 2013; Sabra & Trouche, 2011). The relationship between resources and communities of practice (CoP) can be linked to an important aspect characterising the CoPs—the shared repertoire of resources. For Wenger (1998), the shared repertoire of resources in a CoP is a product of the CoP's practice and at the same time it contributes to shaping this practice. Brodie (2020) has argued that since the resources in the CoP are socio-culturally situated, this implies that they will afford different learning situations. This idea is very important for this study, since this study has adapted a version of LS—the lessoning study which is characterised by major components of LS: lesson planning, teaching of a lesson and post-lesson reflections. According to Doig and Groves (2011), each of these offers different learning opportunities.

Secondly, Wenger's (1998) community of practice theory has been applied in most of the research on teacher collaborations (Robutti et al., 2016). For instance, it has been used to investigate teacher learning in professional development and/or professional learning communities that promote innovative learning (e.g., Chauraya, 2013; Graven, 2004; Pausigere, 2014). The current study extended this work by developing a concept of communities of practice in preservice teacher education using preservice teaching of secondary school mathematics. According to Goos and Bennison (2008), there are few studies that have applied social theory of learning to preservice teacher education. Although Wenger's (1998) theory has widely been used by researchers in education to understand the process of teacher learning, some caution needs to be observed when applying this perspective on learning in formal and informal education settings, including preservice teacher education (Graven & Lerman, 2003). I elaborate on these cautionary measures in Chapter 3 and discuss how I dealt with them in my study.

1.3.5 The critical questions of the study

This study was guided by one overarching question:

- How can mathematics teacher learning be understood through participation in a form of a L/S CoP—in relation to use of the research-designed tool—MTF?

As the research proceeded, new organising sub-questions emerged:

- How do teachers use the research-designed tool, MTF?
- What are the implications that can be drawn from the use of the MTF tool with regard to teacher learning?

There were three features specific to this research that distinguished it from other research relating to mathematics teacher education. First, it was about learning of preservice teachers in the context of LS, and not on in-service teachers which have in the past dominated teacher education literature. Secondly, it was a specific type of teacher learning programme that aimed to enhance participation in a community of practice. And thirdly, it was structured around a research-designed teaching resource.

1.3.6 The 'gap' in South African literature in which this study is located

Teachers' continuing professional development (CPD) is increasingly becoming a priority in most countries throughout the world. It is widely viewed as the most effective job-embedded approach to prepare in-service teachers adequately and improve their instructional practice (Sowder, 2007). In an attempt to address the need for effective CPD programmes, the Department of Education developed the National Policy Framework for Teacher Education and Development in South Africa. As a result of this policy (DoE, 2006), a continuing professional development for teachers (CPTD) was proposed. The Department of Education promotes setting up professional learning communities (PLCs) in schools, which are made up of clusters of schools where teachers from different schools work together in subject groupings: "PLCs are based on the vision that learning is more effective when it takes place within a community of professionals. Schools can provide the structure and culture to facilitate such collective learning" (DoE, 2015, p. 6).

There is increasing literature locally focusing on professional learning communities (e.g., Brodie, 2013; Chauraya, 2013) and communities of practice (e.g., Graven, 2004; Pausigere, 2014). Further review of how such communities work is part of this study; however, my particular focus was on what I call "lessoning" study CoP. Thus my focus in this study was on a hybrid version of lesson and learning study.

LS, as already noted, is a CPD model introduced in Japan, and adapted by various countries as a process through which teachers improve teaching and learning (Lewis, 2019). There are a number of studies in South Africa which have reported positive effects of LS to teacher learning (e.g., Chikamori et al., 2013; Ogegbo et al., 2019; Ono & Ferreira, 2010; Pillay & Adler, 2015; Posthuma, 2012). The University of Pretoria has reported on its website its collaboration with the Department of Science, Mathematics and Technology Education related to the adaptation of LS in their university programmes ranging from undergraduate studies, continuous teacher development to post-graduate studies[1]. These studies alluded to the challenges of implementing LS PD in South Africa. For instance, Ogegbo et al. (2019) have noted that challenges such as lack of time, lack of institutional support and insufficient instructional materials pose a threat to teachers' participation in LS.

According to Robutti et al. (2016), adaptation of lesson and learning studies are not mere issues of empirical interest, but issues that have significant bearing on the professional development of teachers and, indirectly, quality of mathematics learning. In light of this, research on the implementation of the LS, and the nature of teacher learning within the lessoning study would contribute to growing knowledge in this identified gap in the South African context.

1.4 Significance of the study

The main contribution of this study is the development of a framework for describing the process where researchers and teachers gather and share a resource to achieve goals of both communities. This contribution will be made by explicating on the following: 1) Theoretical perspectives on studying mathematics teacher collaboration; 2) collaborative contexts of mathematics teacher collaboration; 3) roles and interactions of various participants in mathematics teacher collaboration; and 4) the resources designed for teacher collaboration.

In this study I argue for the lessoning study as a platform to work within the research-practice gap. Hence the significance of my study should be understood in that context. I view this study as contributing to the current knowledge about research-designed tools in practice. It is assumed that in lessoning study, teachers learn to collaborate with each other and the researcher and use the knowledge embedded in the tool to learn teaching practices. Since this study is an extension of WMCS's

[1] To my surprise there were no research papers from the University of Pretoria LS (UP-LS) that discussed teachers' collaborative work during LS. Perhaps UP-LS researchers have other more immediate concerns and publishing research about collaboration may not have a high priority.

model into preservice teaching, the findings are likely to provide the leaders of WMCS with more insights into the efficacy of the project when implemented in different contexts other than in-service teachers' professional development programmes.

I also hope my study will contribute to existing literature on the adaptation of LS in the preservice teacher education context by considering the factors of the adapted LS version and explaining how they could have contributed to the nature of collaborative engagements and teacher learning. Theoretically, this study will contribute to the developing theoretical foundations by explaining the process of teacher collaborative interactions and teacher learning in collaborative contexts by building upon Wenger's (1998) CoP theory complemented by sociocultural constructs from Sfard (2008) and Mercer (1996).

Methodologically, as will be shown in Chapters 4 and 5, it has been necessary to draw from a wide range of research methods from several methodological traditions. Furthermore, it was necessary to focus on a range of issues concerning ethics, reliability, validity and dependability in relation to this study. This study seeks to offer insights into an approach to researching lessoning study in which the researcher is simultaneously a central part of that lessoning study.

1.5 Outline of Chapters

This thesis is made up of ten chapters starting with the introductory discussion in this chapter.

Chapter 2: Literature review

Here I review literature on teacher collaborations, collaborative environments and tools for collaborations—the research-designed resources. This is necessary because my research looked into the teachers' use of a research-designed tool in the context of a particular type of LS, and described the possible teacher learning.

Chapter 3: Theoretical framework

In this chapter I present Wenger's (1998) communities of practice theoretical framework which informed this study. This framework is further supplemented with Sfard's (2008) communicational theory together with Mercer's (1996) typologies of modes of talk. This chapter is central to my study because it provides a background on the origin of my analytical tools.

Chapter 4: Research design

In this chapter I describe the research design for the study which includes the research participants. It outlines the empirical field of study (lessoning study community), my role or positioning in the research, the data collection and analysis procedures; ethical considerations; and how rigour was attained in the study.

Chapter 5: Methodology

This chapter discusses the methodology underpinning the study, and also the analytic tools. I further show how a methodological contribution was made.

Chapter 6: Analysis of the first lesson planning session—LP1

This chapter presents an analysis of the teachers' reflective discourse about the first lesson planning session. The mathematical object in focus was effects of gradient and y-intercept in the linear function.

Chapter 7: Analysis of the first reflective discussion session—RD1

This chapter presents an analysis of the teachers' reflective discourse about the first teaching session. The mathematical object in focus was the effects of gradient and y-intercept in the linear function.

Chapter 8: Analysis of the second reflective discussion session—RD2

This chapter presents analysis of the teachers' reflective discourse about the second teaching session. The mathematical object in focus was the effects of gradient and y-intercept in the linear function.

Chapter 9: The nature of teacher learning

This chapter looks across the the three LS sessions—LP1, RD1 and RD2 and discusses the learning opportunities afforded to teachers when they used the MTF tool.

Chapter 10: Conclusion

This concluding chapter of the research contains contributions, strengths and weaknesses of the study, implications and ideas for possible new research.

Chapter 2: Literature Review

2.1 Introduction

In this chapter I review selected mathematics education research with a focus on what the research says about the research problem and what theoretical tools are used to talk about this problem. This study was informed by relevant literature relating to three main aspects: perspectives on teacher collaborations and how teacher collaborative environments and resources are implicated. In this chapter these main aspects are discussed as well as how they are linked to the theoretical framework of the study.

2.2 Teacher collaborations

Robutti et al. (2016, p. 652) defined collaborations as "co-working (working together) and can also imply co-learning (learning together). It involves teachers in joint activity, common purpose, critical dialogue and inquiry, and mutual support in addressing issues that challenge them professionally". This definition suggests that mutual engaging, and mutual relating to each other with shared purpose produce learning. This definition can further be related to Wenger's (1998) notion of community of practice (CoP). For instance, according to Wenger et al., (2002, p. 4) "CoP is a group of people who share a concern, a set of problems, or a passion about a topic, and who deepen their knowledge and expertise in this area by interacting regularly". Furthermore, according to Wenger (1998, p. 93), CoP can be conceived of as "shared learning stories".

The mathematics research field is abuzz with new ideas about teacher collaborations and learning. Terms like "resources for collaborations", "communities of practice", "professional learning communities" and "LS" abound when reading literature related to teacher collaborations. These new ideas about the nature of teacher collaborations and learning are interacting with the current reform movement in teacher education. For example, Krainer and Spreitzer (2020) reported on current research on teacher collaborations in terms of collaborative environments and collaborative interactions of various participants in mathematics education such as teachers, teacher educators and researchers in these collaborative environments.

What I have further learned is that most discussions of these new ideas and their implications for educational practice have been cast in terms of teacher collaborative environments and resources.

Furthermore, Robutti et al. (2016) reported that most of the research on teacher collaborations was informed by Wenger's (1998) community of practice theory. However, less attention has been paid to how teachers collaborate with resources. In this study I paid attention to this process. Thus, I discuss visited literature in terms of collaborative environments and collaborative resources. In Chapter 3 I engage with Wenger's (1998) theory of community of practice that I adapted for this current study.

2.3 Teacher collaborative environments

Amongst various forms of teacher collaborative programmes used across the globe is the LS (Chen & Zhang, 2019). This programme has been used as a form of professional development programme where teachers and researchers collaboratively engage in the LS activity (Isoda, 2020). However, there is little research that attempts to explain the processes on how the LS supports teacher learning (Ding & Jones, 2019; Huang & Shimizu, 2016; Wood, 2017). The most relevant studies that I reviewed were that of Dudley (2013) and Dudley et al. (2019), where both studies investigated the nature of teachers' collaborative discursive interactions in the context of the LS, and the type of learning that was revealed by these collaborative interactions. In these studies, they examined teachers' exchange of information through talk and how this process developed professional knowledge. In order to gain insights into the nature of teachers' collaborative interactions, both studies used Mercer's (1996) sociocultural discourse analysis and divided their analysis into three categories: "(1) discourse-related features detailed important elements of talk that contributed to a collaborative and productive learning environment; (2) content-related features pointed to teachers' mental processes in relation to specific content; and (3) learning points[2] " (Dudley et al., 2019, p. 290).

These two studies were important for this study since they are concerned with collaborative interactions in the context of the LS. As noted in the previous chapter, it was the aim of this study to examine how teachers interacted with each other and the researcher as they engaged in collaborative practice of the LS.

In the following section, I thus focus on the reported literature by focusing on the LS itself and its methodologies, and further discuss different adaptations of LS. I then conclude this section by offering a discussion on the adapted form of the LS used in this current study.

[2] Learning points refer to teachers' direct reference to teaching mathematics, understanding students' learning and learning about the subject of mathematics (Dudley, 2013).

2.3.1 Lesson Study

2.3.3.1 History of LS

The origins of LS can be traced to educational practice in the Meiji Era[3] of Japan. It is a scientific activity for teachers based on the methodology introduced in the 1880s (Isoda, 2010). The methodology was based on the observation of teaching methods (how to teach, conducting teaching and learning process) in whole classroom teaching. From the 1880s, LS was implemented with different study themes, as illustrated in the table below taken from Isoda (2010, p. 21).

Table 1

Topics of lesson study

Period	Lesson Study topic
1880s	Pestalozzi Method and Dialog Method (including argumentation/ discussion/dialogue between teacher and students)
1910s	Mathematics for Life (including problem posing)
1930s	Curriculum Integration in Mathematics (including Open-Ended Problems)
1950s	Core curriculum movement based on social studies
1960s	Mathematical Thinking (Japanese way of New Math)
1970s	Open-Ended Approach and Problem Solving Approach
1980s	Problem Solving

Note: Taken from Isoda (2010, p. 21)

During the 1980s the LS theme was based on problem solving. According to Isoda (2010), the problem solving approach is known as the process of posing a problem and independent solving. Its aim is to provide students with the ability to learn mathematics independently. The LS is well known globally as the problem-solving approach. It has been used as one of the local theories in mathematics education (Stigler & Hiebert, 1999). In this current study, the lessoning study theme was based on the use of a research-designed resource, the MTF-use.

[3] The Meiji era is an era of Japanese history which extended from October 23, 1868 to July 30, 1912 (Isoda, 2010).

In the late 1990s LS drew the attention of the United States of America (USA) mathematics education community (e.g., Lewis & Tsuchida, 1997; Stigler & Hiebert, 1999), and later spread globally. The LS has been adapted in different countries where it is used for educational purposes by teachers in the schools. As teaching is a cultural activity (Stigler & Hiebert, 1999), a number of different conceptualisations of LS and associated forms of activity have been developed in the United Kingdom (UK), Hong Kong, the USA, Japan and Singapore to improve teachers' teaching and the quality of student learning. According to Lewis et al. (2009), the LS is characterised by a situated approach and is a locally designed process; therefore, some variation in the LS is expected. In these different variations, LS groups use the features of the LS to develop localised LS versions. Lewis (2018) warned that LS should not be viewed as "recipe" (p. 2), rather it should be regarded as a flexible learning system that can be adapted to local settings while transforming these settings. Hence, I proposed a lessoning study for this study.

A number of variations of a LS have been reported, such as the Japanese LS (Chokshi & Fernandez, 2004), the Chinese LS (Huang & Bao, 2006), and the Swedish learning study (Lo & Marton, 2012), each reflecting different philosophies and implementation in different contexts. At the same time, according to Wood (2017), these three versions of LS are similar to each other as they all "focus teachers' attention on the difference in their students' experience of the object of learning, and both go to some lengths to uncover these differences and use them to design tasks for effective teaching" (p. 120). With this view in mind, they can be adapted in different contexts without any loss of the effects of the original LS.

In the following section I will be comparing and contrasting different variations of LS used in mathematics lesson studies. This will enable me to expound on the developed variation of the LS used in this current study—the lessoning study (L/S).

2.3.3.2 Japanese LS

The Japanese LS (JLS) is the systematic cyclic process of collaborative planning and reviewing of a research lesson by a community of teachers together with content experts known as knowledgeable others (Fernandez, 2002). The "knowledgeable other (KO)" brings theoretical and practical learning in an authentic way (Lewis & Tsuchida, 1997; Stigler & Hiebert, 1999). This suggests that a knowledgeable other plays a vital role in the LS. Such a claim has been supported in a number of reports on the LS (e.g., Chokshi & Fernandez, 2004; Yoshida, 1999).

According to Lewis et al. (2006), the JLS can be conceptualised as the process of teacher learning where teachers engage in learning about students' mathematical thinking by collecting evidence of student learning in order to improve it (Lewis et al., 2006). This learning opportunity is amplified by engaging in various cycles of collaborative activities such as lesson planning, teaching planned lessons along with the team observing classroom teaching, followed by reflective discussions about the previous teaching of the planned lesson (Stigler & Hiebert, 1999). From this review it seems the JLS puts at its heart the focus on student learning, collaborative practice, and the role of the knowledgeable other. These ideas will be raised when I discuss the design of the type of LS used in this study.

2.3.3.3 Chinese LS

The Chinese LS (CLS) is similar to the Japanese LS in terms of collaborative cycles of activities. The main aim of engaging in these activities is to produce exemplary lessons that reflect an effective teaching strategy of a specific content (Huang & Shimizu, 2016). In other words, it is both content focused and strategy oriented. The CLS is conceptualised as a form of deliberate practice (Huang et al., 2017). The process of a deliberate practice "emphasizes the necessity of repeated teaching and immediate input of knowledgeable others, and highlights the unique impact of the dynamic between enactments and reflection on teachers' learning" (Huang et al., 2019, p. 706). This practice highlights the role of the KO, who is involved throughout the collaborative activities of the CLS. According to Yang and Ricks (2013), the KO uses a protocol called the three-point framework to ascertain whether the lesson planning and reflective discussion activities are focused on the deliberate practice. The three-point framework includes the "key point: identifying what is to be learned; the difficult point: potential misconceptions; and the critical point: the means by which the learning of the key point is accomplished" (Yang & Ricks, 2013, p. 84).

2.3.3.4 Swedish Learning study

The Swedish learning study (SLS) model adopts the Japanese LS model (Stigler & Hiebert, 1999) that involves teachers and researchers working in collaborative cycles of planning a lesson, then teaching, observing, evaluating, and modifying the lesson. The main feature of the learning study is learning by the students, learning by the teachers and the researchers, where "students learn the object of learning better than they otherwise would have; teachers learn how to deal with the specific object of learning better and widen their teaching repertoire; and researchers learn whether and how the adopted theory works at the core" (Pang & Runesson, 2019, p. 163).

It is distinguished from the Chinese LS and Japanese LS in that it is underpinned by the theory of variation (Lo & Marton, 2012) which views teaching as a continuous process of changing learners' ways of seeing. Lo and Marton (2012) explained that the learning study model allows researchers and teachers to see how the objects of learning are dealt with in the classroom through lenses of variation theory. At the same time, it can be viewed as a pedagogical tool and a resource which provides teachers with shared lenses to analyse, observe and evaluate their teaching (Pang & Runesson, 2019).

The primary focus of a learning study is the object of learning and how that object of learning can be taught in the most effective way so as to increase learners' understanding (Pang & Runesson, 2019). Wood (2019, p. 600), stated that the object of learning "appears to be the elevation of students' mathematical thinking". According to Pang and Runesson (2019), the learning study is initiated by first establishing learners' previous learning and existing knowledge through the administration of pre-tests and post-tests or interviewing learners in order to identify the appropriate object of learning. The nature of the learners' learning knowledge and previous learning are then analysed in depth. Through this process, with the help of the knowledgeable other, teachers learn to identify the object of learning and the critical aspects related to the object of learning. They further learn to design lessons by incorporating variation theory to help learners discern the critical aspects of the object of learning. The lessons are then observed by the participating teachers, videotaped and reviewed to identify what it is possible for learners to learn. A post-test is administered soon after the lesson and the learners' results are analysed in terms of the dimensions of variation that were made possible for learners to experience in the lesson. This stage is followed by teacher reflections on the effectiveness of the learning study, what they have learnt and what their learners have learnt. The report is then prepared and later disseminated and shared with other teachers. The two main features of the learning study that informed this current study are the focus on the object of learning and the linkage of the theory and practice.

2.3.3.5 Lessoning study (L/S)[4]

The JLS, CLS and SLS are variants of LS. Comparing these different types of lesson studies, commonalities and differences can be found. The commonality among them is the focus on the lesson, and collaborative engagements make LS a strong candidate for being an ultimate teacher collaborative programme. The three types of LS, all emphasised the involvement of the knowledgeable other.

[4] The abbreviation L/S and lessoning study will be used interchangeably.

In terms of differences, all three variants embody different conceptualisations. For example, the JLS is characterised by collaborative practice, while the CLS features deliberate practice, with SLS characterised by the practice of identification of OoL. Another most important attribute of the SLS is its theoretical grounding for LS which helps teachers to pay close attention to necessary aspects of the object of learning in order to help the learners to learn (discern appropriate aspects of object of learning). On the other hand, the CLS is guided by the three-point framework.

The strategies used for focusing on students' learning differ. The JLS focuses on anticipating students' responses, collecting students' work during a research lesson (implementation), and discussing evidence of student learning during a reflective discussion session. On the other hand, the CLS uses a strategy of the "three-point framework" for planning and reflective discussions, where teachers analyse their students' work and identify the key points, difficult points and critical points. The SLS focuses on student learning through interviews, pre-tests and post-tests, and employs a variation theory for both lesson planning and reflective discussions.

What this analysis suggests is that the different variations of the LS may have different purposes and as such may have a different impact on the examination of teacher learning from different variations. However, combining the core attributes found in two varieties of LS—CLS and SLS—may be used to enrich and benefit teachers by providing them with the opportunity to engage with theoretical resources of their work (Pang & Marton, 2005).

The purpose of this analysis was twofold. First, I set out to advance the thesis that lessoning study should be considered as an ultimate teacher learning programme. Second, I argue that combining core attributes of CLS and SLS could provide teachers with a platform to gain access to different learning opportunities.

I chose to use six attributes for the design of a form of a LS used in this study—the lessoning study. The six attributes are compared in the table below with JLS, CLS and SLS.

Table 2

Comparison between Japanese LS, Chinese LS and Swedish LS

Attributes	Japanese LS	Chinese LS	Swedish LS	Lessoning study
1. Conceptualisation	Collaborative practice	Deliberate practice	Identification of OoL practice	Collaborative and Reflective practice
2. Student learning	Anticipating student response	Three-point Framework	Pre-test and post-test	Conjecturing Critical Aspect (CA) of OoL
3. Focus	Students' mathematical thinking	Content and teaching strategies	OoL	MTF practice
4. Lesson planning and reflective discussion informed by	Research problem and student thinking	Three-point Framework	Variation theory	MTF tool
5. Teaching	Repeated teaching	Repeated teaching	Repeated teaching	Repeated teaching
6. KO	Active participation throughout	Active participation throughout	Active participation throughout	Active participation throughout

The attributes I chose aligned mostly with both CLS and SLS. A detailed description of each attribute of the lessoning study can be found in Chapter 4.

2.3.3.6 LS as a form of CoP

Situated learning theory (Lave & Wenger, 1991) is the sound theoretical framework that supports the use of LS. This theory advances the premise that learning is situated in the specific activity and is embedded within a particular context and culture. Learning, according to Lave and Wenger (1991), is a social process in which individuals co-construct knowledge rather than transmit knowledge from one individual to the next. In the case of the LS, learning occurs as teachers interact with each other and collaborate on lessons for their actual classrooms. Situated learning is a theory of learning that emerges in a community of practice (Lave & Wenger, 1991). As teachers participate in the process of LS, they are collectively examining their practice; they are operating as a community of practice. Through these engagements, the members of the community develop a shared practice, a repertoire of shared experiences and understandings (Wenger et al., 2002).

During the LS, teachers are given an opportunity to form a community of practice around collaborative planning, teaching and joint post-reflective discussions. In these communities, teachers collectively

negotiate knowledge construction. Specifically, the intellectual process of the LS is the collaborative engagements (Chokshi & Fernandez, 2004).

Wenger (1998) suggested a number of conditions in order to develop and nurture communities of practice. These include the legitimatisation of participation and provision of support (Wenger, 1998). Legitimising participation entails giving members of the community time to participate in collegial activities and creating an environment that acknowledges the value of communities. Providing support comes in the form of resources such as meeting space and outside experts. The critical role of the knowledgeable other/expert has been echoed in the discussion of the LS above. Such a role resonates with that described in the communities of practice (Lave & Wenger, 1991; Wenger, 1998).

To sum up, the LS in general can be viewed as a form of CoP in terms of structural elements such as the domain, community and shared practices (Wenger et al., 2002). The domain of LS, that is, forms of knowledge, can be exemplified by analysing curriculum documents, teaching tasks, etc. The MTF can be categorised as the domain of lessoning study. In terms of community, the LS has a number of participants from teachers, and the KO. The common shared practices of LS CoP include lesson planning, teaching, and reflective discussions.

2.3.3.7 Learning in a LS

Learning in a LS in general

A number of studies, such as Lewis et al. (2006), have demonstrated the positive impact of LS in the professional learning communities. However, Dudley (2013) and Lewis et al. (2006) are amongst the scholars who have lamented the lack of theoretical research for providing explanations on why and how teachers learn in the context of LS and how the features of LS procedures and contexts support and contribute to teacher learning in LS. This call was also echoed by Robutti et al. (2016) where they noted that most of the studies they reviewed which were using LS for teacher collaborations had failed to relate the process of teacher learning to collaborations within a LS. This current study was an attempt to address such a call.

Some researchers believe that LS in the preservice settings may promote professional growth among preservice teachers due to their engagement in collaborative discussions about the teaching practice (Chassels & Melville, 2009). For example, Lewis (2019) reported three broad categories of preservice teacher learning that were apparent in the data from the LS model implemented in her study:

mathematical content, teaching strategies and development of the teacher identity. Also, it has been revealed that the LS methodology classes impacted positively on the delivery of lessons during the teaching experiment field work (Chassels & Melville, 2009; Lewis, 2019).

The major components of LS are lesson planning, teaching of a lesson, and post lesson reflective discussions offering different learning opportunities (Coenders & Verhoef, 2018; Lewis et al., 2006). For example, Coenders and Verhoef (2018) reported that during the lesson planning stage, teachers had an opportunity to learn how to select meaningful activities for their students. The post lesson reflective discussion stage exposed teachers to engaging with the analysis of the previous teaching session where they investigated the relationship between teacher instruction, student activities, and student learning.

With regard to the lesson planning session, according to Lewis et al. (2006), this session affords teachers with opportunities to collaboratively plan and produce a structured lesson plan. The lesson plan forms the backbone of a LS since it serves as: " i) a *teaching tool* because it provides the script for the activities of the lesson; ii) a *communication tool* since it conveys to others the thinking of the teachers who planned the lesson; and iii) an *observation tool* because it provides the guidelines for what to look for in the enacted lesson" (Pillay, 2013, p. 22) .

Prior research suggests that collaborative lesson planning characterised by structured protocol and talking provide teachers with a positive contribution to teacher learning. For example, Horn (2010) found that lesson planning processes guided by structured protocols contributed to teacher learning (opportunities to learn). Different structured protocols have been identified in the different variations of LS, as reported earlier: for example, the three-point framework (CLS) and the variation theory (SLS). The MTF in this current study can be viewed as a structured protocol guiding the planning. Whether it enhances opportunities of learning or not will be evidenced in the analysis chapters. Horn (2005) described how different types of talk enabled teachers to learn as they engaged in collaborative lesson planning.

Teaching practice involves the enactment of the collaboratively planned lesson and links teachers to real practices (Kazemi & Hubbard, 2008). The observation of enactment of lessons provides teachers access to real time teaching practices. From the CoP theory perspective, lesson enactments are noted as being effective for context-bound understanding and enable learning experientially (Lave & Wenger, 1991). In addition to teaching practices, teachers who are observers have an opportunity to "deprivatize" practice

(Levine & Marcus, 2010, pg. 396). For Levine and Marcus (2010), the practice of deprivatising refers to opportunities for teachers to observe each other's practice.

Fernandez and Chokshi (2002) further stated that the ultimate purpose of engaging in LS is to nurture the "deep and grounded reflection about the complex activities of teaching" (p. 10). With the knowledgeable others' input, the LS stimulates teacher reflective practice by affording opportunities for teachers to reflect on their teaching approaches through a cyclical process of collaborative lesson planning, lesson observation, reflection, and revision of research lessons (Fernandez, 2005; Lewis et al., 2009). Thus, this encourages teachers to make changes through reflections and enactment (Fernandez, 2005). Like with Adler and Alshwaikh's (2019) study, this highlights the importance of the knowledgeable other and teacher reflective practice where they call for research-driven reflective practice in the LS; in this case the research-based tool of MTF. The use of a guiding tool for reflective discussion in a LS context can be noted in the CLS, where a three-point framework is used as a protocol for conducting reflective discussions (Yang & Ricks, 2013).

Different theoretical frameworks have been explored to examine how interactions during reflective discussions have contributed to teacher learning (e.g., Dudley, 2013; Dudley et al., 2019; Runesson, 2013). For example, to examine the nature of interactions in the LS, Dudley (2013) built on Mercer's typology of talk and developed a theory of learning through talk. Through a discourse analysis of reflective discussions, Dudley (2013) showed how teachers used their teaching knowledge to develop their knowledge and promote student learning. Drawing on the theory of variation, Runesson (2013) demonstrated how teacher learning may be enhanced through participation in a process of identifying critical aspects (CA) of the object of learning in a context of learning study. The findings of her study highlighted that different ways of identifying CA in different sessions of a learning study could explain what teachers can learn in terms of what is critical for their students' learning. These various perspectives or frameworks used to examine LS reflect sociocultural perspectives which resonate with current trends in teacher learning (Jaworski & Huang, 2014).

In the following section, I describe the two important constructs that I identified as essential for this current study in relation to teacher learning in the context of LS. These are teacher awareness of the object of learning (SLS) and teacher reflections (JLS).

Teacher learning in terms of focus on OoL (Critical Aspects)

Gustavsson's work (as cited in Holmqvist, 2011) noted that teacher learning in a learning study context could be linked to the following:

a. The teachers directed their awareness first to different methods without connection to the lesson content.

b. The teachers directed their awareness first toward the content and how to handle the object of learning.

c. The teachers discussed how to handle the object of learning in terms of the variation theory

I now focus on the aspect of OoL as an important indicator of teacher learning. According to Runesson (2013), learning study can be described as a collaborative research approach with a major focus on exploring an object of learning. OoL is derived from variation theory, and from a variation theoretical perspective, the object of learning is the capability teachers want the learners to learn. An object of learning has two constituent parts: the direct and indirect objects of learning. The first part is defined in terms of content, that is, functions, geometric proofs, etc., and the latter refers to the specific capability students are expected to develop, for example, being able to calculate, being able to pronounce words, being able to prove theorems.

Objects of learning can be differentiated according to context. They may be "intended", "enacted" or "lived". The intended object of learning is the teachers' perspective on what is to be learnt, their thoughts and intentions with the learning situation. Thus, teachers' statements and actions concerning the object of learning establish the possibilities and limitations for learners to learn in a given situation (Holmqvist, 2011). Secondly, there is the enacted object of learning. This can be said to consist of processes of how the teacher structures the conditions of learning, and how the object of learning is shaped by the teacher and learners in cooperation during a teaching situation. Accordingly, the intended object of learning could change as the learners' participation in the classroom discussion might contribute dimensions not planned or offered by the teacher (Holmqvist, 2011). Lastly, the lived object of learning is what the learners actually learn. Learners do not always learn what is intended. According to Runesson (2013), the teacher may enact an "object of learning" in a lesson that does not express the "intended object of learning" before the lesson, and a student may encounter an "object of learning" as a lived experience that was not intended or enacted by the teacher. Since the focus of this study was not on learners but on teachers, the lived object of learning was understood as referring to the "existed"

object of learning, that is, the comparison between the intended and enacted objects of learning, to determine whether what was being taught matched what was intended to be taught.

According to Kullberg et al. (2017) the object of learning provides answers to the question "What is to be learned?" in three ways. It defines: (1) the content (direct OoL), (2) the lesson objective (indirect OoL), and (3) what needs to be learned (critical aspects). In terms of variation theory, learning implies seeing or experiencing critical features of the object of learning (Marton, 2015).

Critical aspect (CA) is a powerful tool for investigating teacher awareness of the OoL in the learning study (e.g., Kullberg et al., 2017; Pillay & Adler, 2015; Runesson, 2013). These authors seem to agree that the theoretical concepts such as the object of learning and its critical aspects can help teachers focus on the ability to be learned, and on what students need to learn in order to do so.

According to Runesson (2013), the implication for teaching is to make the critical aspects of the OoL visible and learnable. Furthermore, the learner should get the possibility to discern and simultaneously focus on the critical aspects. Marton and Tsui (2004) argued that these critical aspects can be found for instance during interviews with learners, analyses of lessons, together with analyses of what students have learned from the lessons. Lo (2012) urged that there are different methods for identifying the critical aspects of an object of learning and noted that these include:

> a literature review, sharing past experiences of teaching a particular topic with other teachers, interviewing students, administering a carefully designed diagnostic pre-test and post-test and analysing the results of students' responses, and observing lessons taught by another teacher and listening carefully to students during the lesson to find out how they see the object of learning. (Lo, 2012, pp. 71-72)

Runesson (2013) also suggested different ways of identifying critical aspects of the object of learning and categorised these into potential and real CA. Potential CA is the CA of the intended OoL, that is, what teachers believe to be critical aspects. These are identified through conjecturing, analysing learners' diagnostic tests, or even from teachers' previous knowledge about learner difficulties (Runesson, 2013). Real CA is the CA of enacted OoL, that is, what teachers discover as critical aspects after giving a lesson to their learners. Runesson (2013) believed this is confirmed as the real CA during the post-lesson reflective discussion.

The preservice teachers participating in this study had never come across the notion of critical aspects. Therefore, the one objective of L/S was to develop their capability to identify critical aspects of OoL.

Hence there was a need to conduct a workshop before the implementation of the lessoning study, as will be discussed in detail in chapter 4. The critical aspects (CA) have been categorised into different stages of the L/S study: CA—during lesson planning, CA—during teaching, CA—during the post-lesson reflective discussion session. This is in line with how Runesson (2013) has categorised the critical aspects.

OoL is one of the important components of MTF. It is used in a similar fashion as in the variation theory. I paid attention to teacher learning with respect to OoL from the learning study perspective because this idea runs parallel with the aims of the MTF. The aim of the MTF is to foreground what mathematics is made available to learn (OoL) through teaching practices such as exemplification and explanations (Adler & Ronda, 2015).

Teacher learning in terms of teacher reflection

The studies that investigate teacher learning in the LS contexts examine discourse interactions amongst the participants (e.g., Dudley, 2013; Vrikki et al., 2017). According to Vrikki et al. (2017, p. 211), "learning takes place as the result of interactions between individuals, or between individuals and cultural tools, with knowledge within groups often being co-constructed as a result of spoken interactions". Posthuma (2012) and Gutierez (2015) captured these verbal interactions through a concept of teacher reflections/reflective discussions. In these studies, through reflections, teachers have increased their self-knowledge and found new ways of teaching mathematics to learners. Olteanu (2016, p. 63) has noted important points about reflections in teacher learning:

> First, reflection is necessary for the teacher to recognise his or her own ideas on teaching concepts and to link them to the more scientific theories about teaching (Gallacher, 1997). Second, it allows the teacher to become more aware of his or her own teaching by making it explicit to colleagues. Third, through reflection processes, teachers have the opportunity to develop a way of teaching that is more focused on the content they teach and the learning of the students. Fourth, reflection needs to include a confrontation with scientific knowledge about teaching, learning and the content to be taught.

For Myers (2012, p. 1), the "ability to persistently and carefully consider what and how we are teaching, and to reflect on our actions as teachers to determine what works best for our students, is central to successful teaching". This definition of reflection is linked to actions. For example, according to Myers (2012), acts of reflection involve recalling, describing and explaining what happened, as well as thinking

about the consequences of what you do. Another example of the relationship between reflection and actions can be evidenced in Mathew et al.'s (2017) study, where they suggested that reflective practice is the ability to reflect on an action so as to engage in a process of continuous learning.

Schon (1983) introduced the reflection-action dyad by using the following constructs: "reflection-for-action", "reflection-in-action", and "reflection-on-action". The term "reflection-on-action" refers to looking back after a given event that took place: this entails thinking about the experience, and making suggestions of how things could have been done differently (Posthuma, 2012). "Reflection-for-action" refers to teachers investigating their plans and intentions. It involves thinking about the goals for student learning and development when planning a lesson. It also entails reflecting, ahead of the action, on the problems one might encounter and possible solutions for those problems as well as the misconceptions that might occur (Chikamori et al., 2013). "Reflection-in-action" refers to teacher's thinking about the teaching/learning process while directly engaged in teaching. It is the professional's ability to improvise on the spot, or to reflect immediately during action and then respond to uncertainties, conflict or surprises (Posthuma, 2012). The reflection-in-action is the ability for one to describe knowledge that is constructed or reconstructed during practice (Posthuma, 2012). Furthermore, this type of knowledge is interactive and situational and not confined by the rules and procedures. According to Posthuma (2012), reflection-in-action can be observed when a teacher reflects about changing his/her teaching approach during teaching after recognising that the approach is not working.

It is important to highlight that the notion of reflection is defined differently in the context of LS, where it is often described in terms of the time allocated to reflection. It is referred to as the "post-lesson discussion" *(*Yoshida & Jackson, 2011*)*, and the "debriefing session" (Hart & Carriere, 2011) with a certain procedure to be followed. Meyer and Wilkerson (2011) explained that the teacher who taught the lesson would first give feedback on his or her experience of that lesson. In the feedback the teacher would point out what went well, what needed improvement and the difficulties experienced. This would be followed by the report back by the group of observers who helped plan the lesson. The group would also point out what went well and what needed either improvement or changing completely. Finally, the group facilitator would provide feedback and would lead the discussion further. This view can be linked to Schon's (1983) aspect of reflection-on-practice described above.

Olteanu (2016), when discussing the notion of time allocated to reflection, introduced a process called systematic reflection, where she categorised time for reflection in terms of the reflection-action dyad:

reflection-for-action (RFA), reflection-in-action (RIA) and the reflection-on-action (ROA). Olteanu knitted the moments of reflection with the ideas of OoL. According to her, ROA is characterised by features in the enacted object of learning that are linked to the process of identifying the critical aspects related to the group of students taught, RIA is related to features of the intended object of learning that are linked to implementing the intended objects which results in identifying the potential critical aspects. RFA considers reflections on both enacted OoL and intended OoL. It also relates to reflecting on one's actions and thoughts and reflection on practice undertaken after the practice is completed—reframing (Posthuma, 2011). Cobb et al. (1997) located systematic reflections in the context of mathematics. They referred to reflective discourse as a type of discourse "in which mathematical activity is objectified and becomes an explicit topic of conversation" (p. 258). This discourse "is characterized by repeated shifts such that what the students and teacher do in action subsequently becomes an explicit object of discussion" (p. 258).

In the context of my study, the systematic reflections were discursive in nature, since they involved teachers communicating with each other in specific discursive activities. Thus, Olteanu's categorisation of time of reflection resonated with the different sessions of this current study. In the L/S study cycle of this study, reflection-for-action was a vital part of the lesson planning stage, where the teachers collectively worked together identifying the CA of OoL. The reflective discussion stage is the reflection-on-action phase where teachers looked back on the lesson that was taught by evaluating the teaching experience and collectively finding ways of improving or changing the lesson and the action required. I named this process "reframing" as used in a similar way by Posthuma (2011). In the L/S before the lesson was re-taught, reflection-on-action may have led to reflection-for-action, followed by reflection-in-action. The lessoning study structure will be discussed in detail in the research design chapter.

In conclusion, from the discussion above about the reflection-action dyad, one can link this pair to the notion of talk-discourse pair—a driving analytical construct in this thesis. Vrikki et al. (2017) related reflections to spoken interactions, and these can be linked to Mercer's (1996) conceptualisation of talk (modes of interaction). With regard to actions, the description offered by Myers (2012) and Cobb et al. (1997) with respect to the acts of reflections can be related to discursive acts. The discursive acts are components of discourse in terms of Sfard's (2008) theory. Furthermore, taking from Cobb et al's. (1997) ideas of reflective discourse discussed above, the L/S discourse was understood as the L/S reflective discourse which embodied a practice of reflective actions. From this, the lessoning study could be conceptualised as the reflective discourse where the participants engaged in a practice of

reflective actions during lesson planning, teaching and reflective discussions. At the core of lessoning study is a practice of reflective actions. This description runs parallel with key features of other variations of LS: the JLS (Collaborative practice), the CLS (Deliberate practice) and the SLS (Identification of OoL).

2.3.3.8 Teacher learning in this study

As already noted in Chapter 1, the view of teacher learning that was adapted for this study is from the situated learning perspective (Lave & Wenger, 1991). This perspective highlights three principles that characterise teacher learning in a community: (1) learning as participation—where participants work together interchanging ideas in order to solve the problem; (2) learning as social in nature—the group of participants create and negotiate meanings in a specific situation or context and learn from their interactions with group members and gain opportunities to learn to do something or to improve and do it better, and (3) learning as co-learning between novice and experts—the situated learning questions the ways the expert is positioned in the learning environment. From this view, it can be said that the learning from situated perspective is bounded by the opportunities of learning afforded by the context, community and participation.

As has already been noted in the previous chapter, Wenger's (1998) theory is insufficient in providing tools for analysing learning in the community. Thus, I drew from the sociocultural perspective in order to describe teacher learning in the L/S community where I adopted the ideas of "opportunities for learning" and "discursive learning".

Opportunities for learning can be referred to as "a conceptual lens for describing certain classroom environments but also as an important factor that predicts learning outcomes" (Heyd-Metzuyanim et al., 2016, p. 550). They further elaborated that opportunities for learning can be defined as the "circumstances that enable engagement with certain academic tasks" (Heyd-Metzuyanim et al., 2016, p. 550). For instance, in the case of a teacher learning programme, opportunities for learning can be regarded as the complete focused conversation in which there is a deep and extended engagement with the teachers' understandings of the topic under discussion such as mathematical content, pedagogical practice and resources for teaching. An example of an opportunity for learning was given by Brodie (2012) where teachers talked about the learners' thinking in error analysis. In those conversations the teachers tried to understand why particular learners made a particular pattern of errors on a mathematical test item. According to her, in those conversations the teachers had opportunities for

deepening their understanding of why learners make errors and viewed learner errors as reasoned and reasonable. In the context of this study, opportunities of learning can be understood as the construction of teaching strategies that foreground what learners need to know—those related to MTF.

Since this current study was focused on describing teacher learning from a LS community, from the reviewed literature in the previous sections it can be concluded that the LS context can provide various opportunities of learning such as teaching practices (e.g., mathematical teaching practices), participating in a LS practice (e.g., collaborative reflective practices and identifying what mathematics needs to be learnt—in the case of the learning study in particular) and learning from teaching resource (e.g., designing a structured lesson plan).

Another rationale for choosing a construct of opportunities for learning was that the unit of analysis for opportunities for learning is the group rather than the individual (Borko, 2004). The research that is focused on describing learning usually looks at the individual student's processes of learning (Heyd-Metzuyanim et al., 2016).

The notion of discursive learning will be dealt with extensively in chapter 3 (section 3.4.2.3) and will not be presented here. What needs to be highlighted however is that the focus on this construct was helpful in highlighting teaching or learning with special focus on a specific object of learning; this could be mathematical content or teaching resources (e.g., curriculum documents or MTF tool in the case of this study).

2.4 Tools for collaboration

The interest in this study was the MTF tool which is located in the vast resources for mathematical teaching. Pepin et al. (2013, p. 929) noted that "mathematics teaching resources are all the resources which are developed and used by teachers (and pupils) in their interaction with mathematics in/for teaching and learning, inside and outside the classroom". Brodie (2020, p. 37) located these tools in the teacher collaborative context and explained that "tools and resources are important for collaborative professional work and learning among teachers. They provide a means for the design of learning environments, for joint work in pursuit of learning goals, and can support links between teacher learning and teaching practice".

Brodie (2020) further provided a framework for thinking about resources, drawing on two intertwined constructs: "resources for" and "from collaboration". According to her, resources for collaboration are

those tools that are created or designed for support of joint work and learning. The MTF falls into this category. The resources from collaboration are those tools that emerge as a result of joint working. She argued that,

> Tools created for collaboration can become resources from collaboration, i.e. an outcome of collaboration, if they change their meaning and use through the collaboration, i.e. the tools support teachers to see elements of their practice in new ways, and teachers come to see the tools and resources as helping them to work differently (Brodie, 2020, p. 39).

This statement highlights a dialectical relationship between resources and teachers' practice. Most of the studies investigating the interaction between teachers and resources analysed a two-way process (e.g., Gueudet & Trouche, 2012a; Kieran et al., 2012; Remillard, 2009). According to Kieran et al. (2012), on one hand the designed resources (i.e., resources for collaboration) can reshape teaching practices. On the other hand, the consistent teachers' practices can lead teachers to shaping up a designed resource. According to Gueudet and Trouche (2012a), the collaborative interaction with resources for collaboration results in "shared repertoire of resources and shared associated knowledge (what teachers learn from conceiving, implementing, discussing resources)" (p. 309).

From a theoretical point of view, Wenger (1998) talked about tools (artifacts) in terms of reification as both the physical object and a process of attributing meaning over time. According to him, when a tool moves across boundaries from one community to another, as in the case of MTF from research community to teacher community (a research-designed resource), the process of reification could change within each community. The reification further suggests that a tool can take a life of its own beyond context of origin. For example, it will be shown in the analysis chapters that the MTF tool changed its form and developed new identity in the L/S community. For Wenger, a reification process provides a setting for negotiation of meaning. In Wenger's (1998) terms, the negotiation of meaning becomes organised around points of focus. For example, the MTF components OoL, tasks, examples and explanations could serve as points of focus around which negotiation of meaning becomes organised. These components become a focus for the negotiation of meaning, as teachers use a tool to perform an action. According to Rock (2005), these are stated goals around a reified object and teachers negotiate a response around them—an aspect of Wenger's (1998) joint enterprise.

Within this body of work is a common argument that when resources are created or designed for teacher collaborations, these resources may be transformed and become resources "from" collaborations, and opportunities for teacher learning may be witnessed. This implies that the resources

for teacher collaborations and teacher learning are interrelated, and thus must be studied at the same time. Drawing from this, this study explored teachers' use of the tool in order to explain the implied teacher learning and also describe how the tool was shaped.

I have identified two resources that can be categorised as resources for collaboration: Adler's (2017) MTF and Weingarden and Heyd-Metzuyanim's (2020) Realization Tree Assessment (RTA) Tool. As noted, the MTF tool represents the mathematical teaching practices such as exemplification, explanatory communication and learner participation (Adler, 2021). On the other hand, Weingarden and Heyd-Metzuyanim (2020) presented a resource for collaboration named the Realization Tree Assessment (RTA) tool which codified what they call representations of mathematics teaching. This tool is used as a resource to enable teachers to participate in explorative pedagogical discourse where they explicate the mathematical object together with its different realisations. Their aim was to examine the opportunities that this resource affords preservice teachers with developing awareness of mathematical teaching that promotes explorative participation. The results showed that the RTA tool afforded preservice teachers to identify and communicate the mathematical aspects that are visible in a lesson. Furthermore, it gave preservice teachers opportunities to be aware of different links between different realisations of mathematical objects. The authors called for more research on in-depth examination of preservice teachers' discourse and the learning process afforded by using a resource for collaboration tool. This current study heeded to this call.

Both the MTF and RTA agree that the research for collaboration visibilize the mathematical teaching practices and have the potential to afford teachers opportunities to learn the best mathematical teaching practices. However, although the two studies are grounded in the teacher collaboration context, there was no explanation of how the collaborations take place and no clear discussion about the analytical lenses used in order to analyse these collaborations. The current study attempted to explicate the rigourous methodological tools that are grounded in both situative and socio-cultural perspectives in order to describe and explain the nature of teacher collaborations with a specific tool.

2.4.1 The MTF

As noted in Chapter 1, the Mathematical Teachers' Framework (MTF) is a research-designed tool which was developed as the mediating tool to support teachers during instruction (see Figure 2 in Chapter 1). The MTF tool mirrors the components of the MDI framework, except that the object of learning in MDI

is renamed lesson goal which plays a similar role to the object of learning. In this study I refer to the lesson goal as the object of learning (OoL).

2.4.1.1 Object of learning (OoL)

The MTF has the object of learning as the focus. From the MTF perspective, teaching has a purpose (a goal), and learning is always about something. Adler and Ronda (2015) argued that learning is always the learning of something, and one cannot talk about learning by focusing only on what progressive teaching strategies are to be employed but rather must also pay attention to what is being learned, that is, the object of learning. The object of learning is described in a similar way as described in the variation theory (see section 2.3.3.7). Adler and Ronda (2015) developed the MTF to illustrate how a specific OoL can be used in bringing about student learning. The capability that teachers want learners to develop is brought to focus via three aspects: exemplification (examples and tasks); explanatory talk (explanations and talk) and learner participation (learner activity), as illustrated in Figure 1.

2.4.1.2 Examples

The discussion on the concept of "examples" is restricted to "instructional examples" (Zaslavsky, 2010) used in the mathematics teaching context. Examples are offered by a teacher within the context of learning a particular topic. From the MTF perspective, examples are seen as a key resource for teachers to introduce and develop concepts. Within this perspective, the examples, together with accompanying representations, and the tasks in which they are embedded are selected in a particular fashion so as to reflect visible critical aspects of the OoL in focus and thereby provide learners with opportunities to learn that specific mathematical object. Adler and Ronda (2015) recruited principles of "variation theory" (Marton & Pang, 2006) to illuminate how structured example sets may bring the object of learning into focus. Variation theory sees learning as discernment of key aspects of what is being learned, which can be described in terms of three patterns of variations in the examples, namely: contrast, generalisation and fusion.

Leinhardt (2001, p. 347) elaborated on the notion of OoL and argued that "for learning to occur, several examples are needed, not just one; the examples need to encapsulate a range of critical features and examples need to be unpacked, with the features that make them an example clearly identified". For example, in a teaching situation, a teacher may use three different examples to illustrate the critical

aspect of classifying a linear function (i) $5x = 6 + 3y$ (ii) $\frac{y}{2} = 3 - x$ and (iii) a graph representing a linear function.

The notion of critical aspects that I have used aligns with Leinhardt (2001)—and not with variation theory because the WMCS was developing the MTF and its use of variation theory while I was developing this study and my attention was drawn away from the specificities in the framework. Instead, I focused on mathematical routines together with the performer of those routines as means to identify examples and how they were varied as will be shown in the methodology chapter.

The examples are central to the planning of the mathematics lesson. As explained by Bills et al. (2006) the "use of examples is an integral part of the discipline of mathematics and just an aid for teaching and learning" (p. 126). Examples like these appear in classroom practice and play a role in developing conceptual understanding (Alcock & Inglis, 2008); hence, the value in studying their use in instruction.

The examples carry fundamental elements of explanations. Leinhardt (2001) documented that teachers built their explanations on examples and suggested that examples are used as a communication device during teachers' mathematical explanations and classroom discussions. This was expanded by Zazkis and Chernoff's (2008,p.196) discussion that "examples are an important component of expert knowledge and that examples are used to verify statements, to illustrate algorithms and procedures, and to provide specific cases that fit the requirements of the definition under discussion".

Zaslavsky (2010) also argued that examples carry a potential exploratory power. They carry important elements of explanations, as described by Leinhardt et al. (1990):

> Explanations consist of the orchestrations of demonstrations, analogical representations, and examples. [. . .]. A primary feature of explanations is the use of well-constructed examples, examples that make the point but limit the generalization, examples that are balanced by non- or counter-cases (p. 6).

It is important to note that MDI/MTF separates exemplification and explanation—as instructional practices— with the latter focused on talk/communication. Leinhardt (2001) had a different focus—she was concerned with instructional explanations and considered examples to be part of this. As noted above Leinhardt's research was to be able to elaborate mathematical instructional explanations (and these include examples). In MDI/MTF, concern is with the related but also distinct instructional

practices of exemplification and explanatory talk (that together need to illuminate and elaborate an object of learning).

2.4.1.3 Explanations

Explanations happen in various ways, relying upon the sort of inquiry that evokes them, that is, they can point to the premise of a wonder when responding to the inquiry "why", they can be portrayals of a structure when addressing the inquiry "how", or they can be meanings of importance of ideas or translations when responding to the inquiry "what" (Keil, 2006).

The explanations in a classroom context appear natural from a pedagogical approach. Leinhardt (2001) referred to this type of explanation as instructional explanations. Leinhardt et al. (1991, pg.89) defined the act of providing instructional explanations as "the activity in which teachers communicate subject-matter content to students". The activity includes not only literal explanations, but also "the systematic arrangement of experiences so that the student can construct a meaningful understanding of a concept or a procedure" (Leinhardt et al., 1991, p. 89). This argument links closely to the object of learning as clarified by Adler and Ronda (2015, p.241) that "analysing how objects focused on are named, and what is legitimated in an episode is key to being able to describe the mathematics made available to learn through explanatory talk". From the MTF perspective, explanatory communication refers to what teachers name and legitimate with the goal of foregrounding what is critical to learn—OoL. Adler and Ronda (2015) distinguished between non-mathematical and mathematical explanatory talk. The non-mathematical talk includes the use of informal words to refer to mathematical objects, while the mathematical talk refers to use of mathematical words to refer to mathematical objects. Adler and Ronda (2015) argued that explanatory communication in any lesson is the interplay between these two types of talk. In this view, both informal and mathematical talks are important considerations when investigating instructional explanations.

In the case of school mathematics, explanations may be "justifications for actions" (Leinhardt, 2001, p. 341). Actions on examples of mathematical objects are often accompanied by justifications, and in school mathematics, these will include mathematical words in use and different routines. Sfard (2008) refers to such justifications as substantiations. According to Leinhardt (2001), instructional explanations are "natural and frequent pedagogical actions that occur in response to implicit or explicit questions—whether posed by students or teachers" (p. 340). In the case of this study, explanations were viewed from different perspectives—the teachers and the learners.

2.4.1.4 Tasks

Tasks serve a communicative purpose between teacher and student by conveying what the teacher intends the learners to learn and the students' conception of that intent. Rezat and Strässer (2012) argued that learners' interaction with mathematics is mediated by artifacts, such as mathematical tasks. "Task" is what students are asked to do. The design and use of tasks for teaching purposes is at the core of mathematics education (Artigue & Perrin-Glorian, 1991). Tasks provide activities for learners that afford them opportunities to encounter mathematical concepts, ideas, and techniques, as well as develop mathematical thinking. Task selection, adaptation, design, sequencing, installation, observation, and evaluation are all part of teaching. This work of teaching is often done with the help of a textbook and/or other teaching resources created by outsiders.

Adler and Ronda (2015) argued that the given task needs to match the OoL, and elaborated that "while examples mediate mathematics learning by foregrounding what is the mathematical object to be learned, tasks are crafted to mediate the capability with respect to the object—what learners are expected to be able to do" (Ronda & Adler, 2014, p. 5). For example, the teacher may give the learners a classification task (critical aspect to be learnt) and the learners are required to classify the following function: $y = 2x + 1$. In a typical classroom setting, teachers typically assign exercises for class work, homework, projects, and investigations with the goal of putting a general idea to the test through practice.

2.4.1.5 Learner participation

This particular component of MTF looks at what opportunities are given to the learners in the class to participate in the lesson. During the presentation of the concept, questions to ask related to learner participation could be: "What mathematical actions will the learners engage in? What will they say? What will they write?"(Adler, 2021, p. 1212). The interactions viewed here are between the learners and the teacher as well as the learners interacting with the examples or tasks given. Teacher and learner interactions were not a focus of this present study. However, this will be discussed when analysing how teachers have negotiated their response as they reify the concept of learner participation.

In conclusion, in the foregoing discussion, I have highlighted three important points. First, how all the components of MTF are glued together by the notion of OoL. From the MTF perspective, the overarching goal of the framework is the emphasis on the coherence of the lesson and how all elements interact, how they link back to the OoL and open up opportunities to learn mathematics. Second, the

review has evidenced that the teachers use examples and tasks to develop explanations. Third, the explanations can be categorised from both the teachers' and the learners' perspectives. It will be the work of the analysis chapter to unpack how teachers in this study have used the MTF components.

2.5 Conclusion

A number of issues arose from reviewing the relevant literature for my study. These included the conceptual constructs and questions to consider and reflect on as the study unfolded. Concepts drawn included the L/S reflective discourse; reframing and identifying the CA of OoL actions; the dialectical relationship between resources and teaching practices; and the MTF tool. These concepts seemed to hold a promise for enabling the explanation of teacher collaborations in collaborative environments informed by a tool for collaboration. Having pulled the conceptual map from the literature review, one needs to gaze on this through the theoretical framework that informed this study. In the next chapter I turn to Wenger's (1998) CoP theory.

Chapter 3: Theoretical Framework

3.1 Introduction

Drawing from the previous chapter, the central issue in this thesis was teacher collaborations and reflections; how teachers collaborate with resources in the collaborative environment. In this chapter, I first draw from the Wenger's perspective to map out important features of collaboration in relation to my research question: How do preservice teachers use the research-designed tool—MTF, as they participate in the lessoning study community of practice? To do this, I use Wenger's (1998) concept of community of practice (CoP) as my starting point. This theoretical review discusses collaborations through the lens of Wenger's (1998) socially informed framework. More specifically, Wenger's (1998) community of practice theory is used to frame the study. While Wenger's (1998) theory provided useful and important handles for my study, it did not suffice with respect to fully describing how collaborations emerge. I thus recruited additional resources, particularly Sfard's (2008) communicational theory and Mercer's (1996) typologies of modes of talk.

3.2 The Situated Learning

According to Borko (2004), situative perspectives of teacher learning can provide a multi-directional research lens, thus affording an exploration of multiple units of analysis such as the individuals, the community context and the social interactions. Another view from situated learning theorist Lave (1996) is that when researchers study learning through a situated approach, the focus is on the social interactions together with physical contexts. Lave and Wenger (1991) viewed situated learning as a process of social engagement or participation in a community of practice.

With this view in mind, in this study I explored the use of the communities of practice (CoP) concept as developed by Wenger (1998) and Wenger et al. (2002) as a way to consider the process of social learning in teacher collaborative environments. This is in line with most of the research in mathematics education research focusing on teacher collaborations (e.g.,Pericleous, 2020; Robutti et al., 2016).

3.3 Communities of practice theory

Wenger et al (2011, p. 9) defined CoP as a "learning partnership among people who find it useful to learn from and with each other about a particular domain. They use each other's experience of practice

as a learning resource". Taking this definition as my starting point, below I explore the important concepts that underpin the principles of CoP. These are: domain; practice; community; negotiation of meaning; dimensions of CoP (mutual engagement, joint enterprise, and shared repertoire); and learning in a CoP. There are several other elements that underpin the theory of CoP, but these six inform the aspects of the discussion relevant for this thesis. The domain, practice and community are the primary structures of a CoP but the negotiation of meaning and the dimensions of CoP are central processes that cement the structure of the CoP.

3.3.1 Structure of the CoP: The domain, practice and community

Wenger et al. (2002) explained that all CoPs share a basic structure consisting of three fundamental elements: domain, community, and practice. These elements help to distinguish the exterior of a CoP and lay the basis for the lessoning study CoP model developed for this study.

Domain

The *domain* according to Wenger et al. (2002, p. 31) creates "common ground" that "brings people together and guides their learning". It constitutes "the area of knowledge that brings the community together, gives it its identity, and defines the key issues that members need to address" (Wenger, 2004, p. 1). Combined, these descriptions suggest that for a group of people to constitute a CoP, its members must come together around ideas or topics of interest (the domain) and interact with each other to learn together. In the same way, the preservice teachers and the researcher were members of lessoning study's domain. Thus, lessoning study's domain was the mathematics teaching framework (MTF). The element of domain relates to CoP identity and distinguishes it from a club of friends or a network of connections between people.

Community

For Wenger (2004, p. 1), the community constitutes "the group of people for whom the domain is relevant, the quality of the relationships among members, and the definition of the boundary between the inside and the outside". Snyder and Wenger (2010) pointed out that a key factor to a successful CoP is the leadership. This means the CoP theory recognises the role of the leader of the discourse. In this study, the community was made up of a group of preservice teachers and a researcher.

Practice

Wenger et al. (2002, p. 34) referred to practice as the "body of shared knowledge and resources that enables the community to proceed efficiently in dealing with its domain". In fact, according to Consalvo et al. (2015, p. 3), practice denotes "a way of acting in the world" and as "a field of endeavor and expertise". Thus, a community's practice explores the existing body of knowledge and the latest advances in the field which enable participants to learn the craft and become practitioners. Wenger et al (2002) mentioned a range of tools, resources, concepts, symbols and documents which members share as being part of practice. The resources attach meanings to practice which results from the "interplay of codification and interactions" (Wenger et al., 2002, p. 39), which Wenger (1998, p. 63) interpreted as the "duality of reification and participation". Relating all these aspects to the context of this study, one can say lessoning study practice is the MTF practice: exemplification, explanatory communication and learner participation (Adler, 2021).

In the context of my study I viewed lessoning study as a model through which teachers may negotiate their practice by participating in the activities of the lessoning study such as lesson planning, teaching, observing and reflecting. This conceptualisation of participating in lessoning study activities as a practice in Wenger's theory helped me explore teachers' MTF practice, that is, how they used examples/tasks; explanatory talk and learner participation when they planned, taught, observed and reflected on their teaching.

3.3.2 Negotiation of meaning

Wenger (1998) contended that an individual's engagement in a CoP always entails a process of negotiation of meaning which takes place in the convergence of two processes: participation and reification. Participation involves acting and interacting, and reification involves producing artifacts (such as tools, words, symbols, rules, documents, concepts and theories) which the negotiation of meaning is organised. Participation and reification are complementary processes in that each has the capacity to make up for the limitations of the other. I will return to this point later.

3.3.2.1 Participation

Participation in a social practice implies constant negotiation through an active process where participants shape each other's experience of meaning. It involves social relations amongst participants. For Wenger (1998, pp. 55), "participation refers to a process of taking part and also to the relations with

others … it suggests both action and connection". In this regard, participation in a CoP connects the participants with one another. Wenger (1998) discussed three important points relating to participation in CoPs:

(i) Participation which involves different kinds of relations among members of CoPs—harmony, conflict, intimate, political, cooperation and even competitiveness.

(ii) Participation in CoPs which shapes our experiences and also the experiences of those communities we belong to.

(iii) Meanings which are continuously negotiated as we participate in shared practices of CoPs.

In this study, participation as a constituent of meaning involved broader engagement in different parts of the lessoning study. First, the lesson-planning time was a time of intense engagement with teaching work and with one another. During the teaching session, teachers did not cease to be participants. Their participation took a different form, where one teacher taught and others were observers. During this session they still engaged with MTF through their observations since their teaching and observations were guided by the MTF. In this sense, participation went beyond collective direct engagement with each other. It placed the negotiation of meaning in the contexts of forms of engagement in different lessoning study situations (sessions). From this perspective, teachers' engagement in the lessoning study was the participation in different sessions of L/S even when it did not clearly involve collective/joint interactions with each other. Observing a teaching lesson may not seem like a social interaction, yet its meaning is fundamentally social. Taking notes by observers involved interaction with one's actions. This discussion highlighted the notions of collaborative interactions and different roles played by the participants at different stages of the lessoning study. Krainer and Spreitzer (2020) noted that in collaborative interactions the contributions of all participants are important as they all play a part in influencing the results of collaborative activities. In order to capture these interactions, I considered Mercer's (1996) analytical tools to examine how teachers' social relations related to MTF-use during their participation in lessoning study activities.

3.3.2.2 Reification

Wenger (1998) used the term reification "to refer to the process of giving form to our experience by producing objects that congeal this experience into 'thingness'. In so doing we create points of focus around which the negotiation of meaning becomes organized" (p. 58). It is the process by which communities of practice produce concrete representations of their practices, such as tools, rules, documents, concepts and theories. For Wenger (1998), reification can refer to both a process and its

resulting form (products). As a process, it gives form to experience by producing representational objects that can clarify and explain meanings within a community of practice (to its individual members and outsiders). These can become important points of focus around which negotiation of meaning is organised. A practitioner uses a tool to perform an action (e.g., in legal practice lawyers use the law to argue a point). This means the tool and the law in these examples become the point of focus through which negotiation of meaning is organised. In my study, the MTF's components (e.g., CA of OoL, examples, tasks, explanations) were the points of focus for negotiation of meaning.

According to Wenger (1998), reification is not only a process of objectification as it does not end in an object. It covers "a wide range of processes that include making, designing, representing, naming, encoding, and describing, as well as perceiving, interpreting, using, reusing, decoding, and recasting" (Wenger, 1998, p. 59). From these processes I have added a process of "routine-use", a very important concept used in this thesis to describe a process of applying discursive routines. This will be discussed under Sfard's (2008) communicational theory. According to Wenger (1998), reification is a process that can change our experience of the world by focusing our attention in a particular way and enabling new kinds of understanding. Wenger (1998, p. 61) explained that "the products of reification are … reflections of these practices, tokens of vast expanses of human meanings". Wenger (1998) gave the example of reflective narratives as products of reification. Mercer (1996) stressed that when describing and evaluating the interactions in any collaborative activity, the investigator needs to incorporate not only the way the speakers are interacting with each other, but also the process or actions which reflect the ground rules of a specific discourse they discuss. Thus, I focused on Sfard's (2008) communicational theory to describe how the process of routine-use enabled the construction of narratives. For instance, in the analysis chapter I am going to show how teachers integrated different routines to construct MTF narratives.

3.3.2.3 Participation and reification are dualities

According to Wenger (1998), participation and reification are seen as a duality; they do not oppose each other but are complementary. They both require and enable each other. Wenger (in Farnsworth et al., 2016, p. 9) advised that one has "to look for both processes whenever you try to understand a moment of meaning making". He further warned that "reification can capture more or less of the experience of participation" (p. 9) and "reification is a bit easier to investigate than participation because it is usually easier to collect visible evidence. But it is only half of the story" (p. 10).

With this view in mind, the reification of the MTF is just a form, representing a frozen aspect of teaching practice. Yet it is meaningless or empty without the participation of teachers. Conversely, the production of such reification is critical to the kind of negotiation that is necessary for teachers to develop learning opportunities for their learners (Adler, 2021) which is an aim of MTF practice. The reification and participation are complementary processes, in that each has a capability to repair the ambiguity of meaning the other engenders. So, for example, a particular strategy related to an example may be discussed or developed during lesson planning activities and summarised in a set of lesson plans—providing a codified reification of intended practice. However, the meaning and practical implications of the strategy only become clear as it is tried out during the teaching session and discussed later amongst teachers during the reflective discussion session. In this way, participation results in social learning that could not be produced solely by reification alone. At the same time, the reified products, such as the lesson plan, serve as a kind of memory of practice, cementing in place new learning. Such an analysis provides a way of describing the means by which practices develop within L/S.

In this study, participation and reification based on CoP perspectives were used to describe teachers' mutual engagement in practices of MTF. It is through their experiences of participating in practices such as identifying CA of OoL, exemplifying, choosing tasks and explaining, that teachers come to a collective understanding of what it is to use MTF in the context of L/S. This process is what is called joint enterprise. Through the interplay between the mutual engagements and joint enterprise, the CoPs' shared repertoire emerges. In the following section, I link the process of negotiation of meaning and its constructs to the dimensions of CoP.

3.3.3 Dimensions of CoP—Joint Enterprise, Mutual Engagement, and Shared Repertoire

Wenger (2010) emphasised that over time, through participation and reification, participants of a CoP develop and negotiate "a set of criteria and expectations by which they recognize membership" (p. 180). These criteria include (i) joint enterprise: a collective understanding of what the community is about, its purpose; (ii) mutual engagement: interacting and establishing norms, expectations, and relationships; and (iii) shared repertoire: shared resources, routines and tools for and from negotiation of meaning.

These dimensions of practice are important as they indicate how members of a CoP participate in shared practices, negotiate meanings and provide a context in which learning can occur (Wenger, 1998). It is important to note that a CoP is characterised by the mutuality of engagement or the relationship that binds its members together. It is through mutual engagement that members of a CoP are related to

one another. Members of a CoP might differ in ethnicity, gender or age, but they come together to participate in their community's shared practices. This signifies how diversity is acknowledged by CoP members. Goos and Bennison (2008) noted that productive relationships arise from diversity. However, it is important to understand that mutual engagement between members of a CoP is not always harmonious; similarities, differences and conflict occur between community members (Wenger, 1998). Nevertheless, mutual engagement is the basis for relationships necessary for the functioning of a CoP.

Wenger argued that mutual engagement relates to a process in which participants are engaged in actions whose meanings they negotiate with one another. According to Wenger (1998, p. 74), "[w]hatever it takes to make mutual engagement possible is an essential component of any practice" (1998, p. 74). This implies that anything that is enabling CoP engagement is welcomed; this may include discourse-specific talks that may help CoP members handle their work activity/tasks efficiently. In this study it was talk around the MTF discourse which included both mathematical and pedagogical discourses. For Wenger (1998), "engagement" is a very powerful process that involves relationships, interactions, sharing of practices, negotiation of meanings and learning among members of CoPs.

The participants of this study included a researcher and preservice teachers from different years of study. They were expected to work within a newly developed community of L/S, and make complementary contributions when using the MTF tool. Mutual engagement required sharing their understanding of MTF practice and the creation of relationships between the members that could work to the benefit of the community.

Wenger (1998) emphasised that interaction or engagement is the condition for the negotiation of meaning. Even though Wenger highlighted the important role the engagement plays in the negotiation of meaning, some scholars such as Creese (2005) and Tusting (2005) have pointed out that Wenger offers little insight into how meanings are negotiated and interpreted. Tusting (2005, pp. 39-40) elaborated as follows:

> Despite the centrality of negotiation of meaning to the communities of practice model, and the key role of language [discourse] within processes of participation and reification, Wenger does not draw out ideals about the relationship between language and meaning making more generally, beyond stating that meaning making cannot be reduced to language alone. However, while Wenger is careful to make clear that he is not just talking about language when talking about meaning, language is clearly central to much of the experience of negotiation of meaning

> we encounter in communities of practice … […] language has a privileged place in human communication. […]. Language is one of the principal means by which meaning is reified.

In order to address this shortcoming, in this study, the sociocultural tools provided by both Sfard (2008) and Mercer (1996) have been incorporated into Wenger's CoP theory. I will return to this point later in this chapter (section 3.4).

As members participate in the shared practices of their community, they become mutually engaged with one another. Participating in shared practices of a CoP also involves a series of negotiations among members. Joint enterprise is the response to such negotiations. As Murugaiah et al. (2016, p. 95) remarked, "it involves negotiation and accountability—a conscious concern of members about their engagement in CoPs. They negotiate ways of working towards communally agreed enterprise for the benefit of their CoP". A joint enterprise was described by Wenger (1998) as:

> The result of a collective process of negotiation … it is defined by the participants … it is their negotiated response to their situation…. it is not just a stated goal, but creates among participants' relations of mutual accountability that become an integral part of the practice. (pp. 77-78)

A joint enterprise is seen as a process in which members of a CoP share (a) a negotiated enterprise, (b) an indigenous enterprise and (c) a mutual accountability. These elements were explained by Wenger (1998) as follows:

a. A negotiated enterprise—in CoPs, members continuously negotiate their enterprise with each other which is their response to the situation in which they find themselves. Similar to mutual engagement, which does not necessarily mean harmony, the negotiation of an enterprise does not entail simple agreement among CoPs members.

b. An indigenous enterprise—even though a practice can be partly influenced either directly or indirectly by individual members or by other external entities and conditions. It has no absolute power to determine an enterprise of a CoP. The enterprise of the community is determined from within through the process of negotiation that is performed by members themselves.

c. A mutual accountability—the continuous process of negotiating an enterprise in a CoP creates the sense of mutuality among all its members. It is through their participation in the negotiation of a joint

enterprise that members of a CoP determine their course of action, deciding what is significant for them and what is not.

The mutual accountability is the important analytic component for the joint enterprise. This is accountability in the sense of being responsible to each another. The notion of accountability is linked to relations amongst the members. As Wenger puts it, negotiation of the joint enterprise "produces relations of accountability that are more than norms" (Rock, 2005, p. 85; also see Wenger, 1998). As Rock (2005, p. 85) argues, "these relations are manifested not [necessarily] as conformity, but as the ability to negotiate actions as accountable to an enterprise".

Research by Rock (2005) from sociolinguist perspective showed how negotiating joint enterprise and mutual management processes are played out through language. Rock's study examined the aspect of "the statement and explanation of the right to silence" for people under police arrest in the United Kingdom (UK). She drew on interviews with officers as they reflected on their practice. To unpack the notion of joint enterprise she used two constructs "stated, shared goal" and "mutual accountability". She further argued that these two are relevant to analyze the negotiated joint enterprise especially when combined with other aspects of Wenger's framework such as reification objects. The results of the analysis of the shared stated goal evidenced that officers negotiated response towards the reified object showed conflict between institutional goals (which are stated shared goals) with interactional goals. The officers did not simply adopt these stated shared goals instead they negotiated a response to them. The results with respect to mutual accountability were reported under two themes: prioritizing goals and problematizing practice. With respect to prioritizing goals, officers were seen to be prioritizing institutional goals over interactional or interpersonal ones. This result is in agreement with Wenger's assertion that, "relations of accountability include what matters and what does not . . . what to pay attention to and what to ignore . . . when actions and artefacts are good enough and when they need improvement or refinement" (Wenger 1998, p.81). Some officers evidenced the notion of mutual accountability by problematizing their practice and reviewed their practice and decided not to improve it in relation to a specific goal. Rock (2005) concluded that these different relations evidence officers' ability to negotiate actions as accountable to an enterprise. When providing the explanatory account in relation to the notion of joint enterprise, this study focused on the three constructs offered by Rock (2005): stated goals, problematizing and prioriting practices.

Wenger locates joint enterprise squarely with community members who define it in pursuing it. The community members described in this study include preservice teachers and the knowledgeable other.

The KO played an important role of bringing the research knowledge to the L/S CoP through the MTF tool. It is interested to understand to what extent her role promotes the shared goals of the MTF. Wenger (1998) also places the role of the knowledgeable other in the process of negotiated response, as someone who guides the execution of the MTF practice by highlighting "what to pay attention to and what to ignore" (Wenger 1998, p.81). Furthermore, as noted by Wenger, the role of the KO also involves "being responsible to others by not making life harder for others" (p. 81).

In the L/S context, this would require participants negotiating ways of working towards communally agreed enterprise, within the constraints of the L/S context. This does not mean all the participants must share the same view, but must negotiate their enterprise. Negotiating joint enterprise manifests relations of mutual accountability within the CoP. Working in a mutually accountable way would require a conscious concern about their engagement with MTF. There should be a sense of responsibility as individuals and as a community, with members working to the benefit of the CoP and with concern for themselves and other members. Mutual accountability might be reified by ground rules set by the KO at the start of the L/S.

It is through members' engagement with one another in the shared practices of their community, and their active participation in the continuous process of negotiation of a joint enterprise, that they produce a shared repertoire. In this regard, shared repertoire refers to the artefacts that are developed, produced and shared by members of a CoP. Wenger (1998) explained that "the repertoire of a community of practice includes routines, words, tools, ways of doing things, stories, gestures, symbols, genres, actions or concepts" (p. 83). These artefacts are produced as members actively participate in shared practices of CoP. In other words, it is thought to include aspects of participation and reification, as noted by Chauraya and Brodie (2018, p. 2) that "the shared repertoire consists of the meanings that are developed and ways of dealing with, or resolving, situations that are jointly negotiated and adopted as part of the community's practice". Actions and artefacts have histories of interpretation though it is suggested that they do not constrain meaning, but allow negotiation of new meaning and dynamic development through sustained engagement in the community. A shared repertoire indicates members' common views of the world, their expression of membership and their projection of identities (Wenger, 1998).

Wenger (1998, p. 137) posited that it is through joint enterprise, mutual engagement, and shared repertoire that a community establishes guidelines as to "what it is to be a competent participant, an outsider, or somewhere in between" and further adds that establishing such guidelines is crucial for

learning to take place in a CoP. In relation to my study, these dimensions of practice were adopted as the main framework to investigate teachers' use of MTF practice in the L/S. The three dimensions of practice: mutual engagement, a joint enterprise and a shared repertoire, as described earlier, provided a useful lens through which to explore how teachers become mutually engaged with each other through their shared practices of MTF in the L/S—the joint enterprise, how they negotiate these practices together, and how they produce a shared repertoire.

3.3.4 Learning in Communities of Practice

From CoP perspectives, knowledge is both individual and social. Instead of seeing learning as an end product, CoP views learning as a social process (Wenger, 1998). The core concept of learning in CoP "implies a mutual engagement in the search of a joint enterprise with a shared repertoire" (Gomez, 2007, p. 20).

In CoP, the dimensions of practice comprising mutual engagement, a joint enterprise and a shared repertoire, are essential for learning to occur. These three dimensions of learning are "interdependent and interlocked into a tight system" (Wenger, 1998, p. 96). According to Wenger (1998), "members interact, do things together, negotiate new meanings, and learn from each other" (p. 102). This implies that learning takes place as we participate in the shared practices of CoP. Learning occurs as we negotiate and create new meanings based on our participation and reification in CoP. Wenger (1998) maintained that, in order for learning in community of practice to occur, communities must be involved in the following three processes:

- *Evolving forms of mutual engagement:* discovering how to engage, what helps and what hinders; developing mutual relationships; defining identities, establishing who is who, who is good at what, who knows what, who is easy or hard to get along with.

- *Understanding and tuning the enterprise:* aligning the engagement with the enterprise, learning to become and hold each other accountable to the enterprise; struggling to define the enterprise and reconciling conflicting interpretations of what the enterprise is about.

- *Developing the repertoire, styles, and discourses:* renegotiating meaning; producing and adopting tools, artefacts, representations; recording and recalling events; inventing new terms and redefining or abandoning old ones; telling and retelling stories; creating and breaking routines. (p. 95).

Based on my literature review of the CoP perspective, I hypothesised that learning takes place as teachers participate in shared MTF practices with other members of their communities of practice. For example, learning may occur as teachers work together and negotiate with each other the ways to identify CA of OoL and connect them to examples and tasks. They may learn from one another the ways to effectively use critical aspects of OoL in order to gain access to learners' knowledge they need to accomplish their teaching tasks (e.g., of lesson planning). This would require them to participate actively in the shared practices of CoP and become mutually engaged with other members of their community, continuously negotiating their enterprises and learning. This kind of learning, according to Lave and Wenger (1991) and Wenger (1998), is social and is situated in our participation as members of CoP.

3.4 Complementing Wenger with socio-cultural approaches

Those within sociocultural research treat communication, thinking and learning as processes shaped by cultural factors, and view knowledge as shared and understandings being jointly constructed. As already noted earlier, this study engaged with the work of Mercer (1996) and Sfard (2008) in order to examine the linguistic aspects of the notion of participation and reification. Mercer's (1996) theoretical constructs helped to uncover the aspect of interacting. In addition, Sfard (2008) grounded the analysis in the mathematics context and elaborated on how communication plays out in understanding language in use through routine-use. Most recently, communicational theory has been raised to the field of mathematics teaching (e.g., Heyd-Metzuyanim et al., 2018; Tabach & Nachlieli, 2016). In this study I extended this focus on mathematics teaching and focused on a mathematical teaching tool—the MTF. The combination of sociocultural and situated perspectives is in line with calls that have been made by several researchers who are concerned with teacher learning in collaborative environments (e.g., Putnam & Borko, 2000; Robutti et al., 2016).

Littleton and Mercer (2013) argued that when members of the CoP are interacting with each other while engaging in the activities of the CoP, they engage in a process called interthinking. According to them, it is this interthinking which allows people to work on an idea together by pooling their thoughts, in negotiating shared understanding through talk. The main idea of interthinking is similar to Sfard's (2008) notion of thinking. However, Mercer's view of interthinking is concerned with the social, while Sfard is focused on the individual. For instance, Sfard (2008) defined thinking as "an individualized version of (interpersonal) communicating" (p. 81).

Interthinking process

Interthinking is the process to explain how spoken language can be used to think creatively and productively together. Mercer (2004, p. 137) view language as "a social mode of thinking—a tool for teaching-and-learning, constructing knowledge, creating joint understanding and tackling problems collaboratively". This implies that language is a cultural and psychological tool in making and negotiating meaning. This approach is concerned with the analysis of how spoken language is used as a tool for thinking collectively. It also focuses on how people pursue joint enterprise in the community.

Mercer (2001, p. 648) further elaborated that,

> With language, we are able not only to share or exchange information, but also to work together on it. We are not only able to influence the actions of other people, but also to alter their understandings. By talking with a teacher, you may learn; through hearing a good argument, you may change your mind. Language does not only enable us to interact, it enables us to "interthink".

Mercer's theoretical stance rests on three arguments. First, specific ways of talking permit certain social modes of thinking. Second, particular social modes of thinking are produced in particular kinds of collaborative relationships. Third, the collaborative relationships are shaped by participants' culturally-based definitions of the situation. These three arguments seemed to be linked to ways in which individuals interact with each other—which speaks to a concept of mutual engagement. Thus, Mercer's theoretical constructs of social interaction (meaning making) such as interthinking, with language being the principal means of establishing shared understanding was used to close the gap in Wenger's theory with respect to the process of negotiation of meaning.

Negotiating meaning as an interthinking process

Mercer (1996) developed an analytical framework that relates to two different levels when analysing and characterising the actual talk which takes place in any collaborative educational activity. The first level of the analysis, termed linguistic analysis, focuses on the character and purpose of participants' talk in group interaction. It seeks to answer the following questions:

> What kinds of 'speech acts' do the students perform? (Do they assert, challenge, explain, request . . .?) What kinds of exchanges take place? (That is, how do speakers build their conversations, how do they respond and react to each other's talk?) What topics are discussed? (Mercer, 1996, p. 369)

The second level, psychological analysis, characterises the talk as thought and action. It examines the ways in which participants approach and process learning tasks in their social interactions. It is aimed at highlighting participants' situated working strategies geared towards learning. It seeks to answer the following questions:

> What kinds of ground rules do the speakers seem to be following? How do the ways speakers interact, the topics they discuss and the issues they raise reflect their interests and concerns? (Mercer, 1996, p. 370)

This level of analysis seems to be attempting to unpack the rules of a specific discourse that seemed to be followed by the participants. One can relate this to Sfard's (2008, p. 91) concept of discourse: "Different types of communication, set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors, were called discourses".

Sfard's (2008) definition of discourse indicates how ways of communicating in a specific discourse are guided by special rules that distinguish that discourse from the rest. One can therefore conclude that Mercer's psychological analysis grounds the interaction in localised ways of doing things, for example, mathematical ways of talking. This level links interthinking to communicational actions (Sfard, 2008). This study paid attention to Sfard's (2008) ideas of routine-use as discussed at length in section 3.4.2.

In order to demonstrate the link between Mercer's (1996) linguistic and psychological analysis, the researchers interested in function of talk have adapted a methodology known as sociocultural discourse analysis (Mercer, 2004). The specifics of this method, and how it relates to this thesis, will be discussed in more detail in the methodology section. However, at this point it is important to establish the ethos behind its use in relation to the existing literature that relates typology of talk to language in use discourse.

Mercer's (1996) two dimensional analytical approach corresponds roughly to what Dudley (2013) calls (1) interaction function—types of talk and (2) knowledge types—type of new knowledge being constructed. In his analysis, Dudley was interested in the nature of interactions and how teachers developed and constructed new knowledge about how to improve learning of their learners in a particular classroom context while participating in LS. Through a discourse analysis of teachers' interactions, he considered types of talks and their impact upon the construction of knowledge during group interactions. Dudley claimed that this methodological approach enabled him to uncover patterns

of teacher learning emerging from complex teachers' discussions. His work makes an important contribution to the analysis of a group of teachers' discourse interactions in a specific context of LS.

I now refine Mercer's analytical ideas in order to gain deeper microanalyses to unearth different types of talks and their functions to learn and understand teachers' negotiation of meaning processes. From this I combine Mercer's (1996) interaction construct and Sfard's (2008) discourse construct to reflect a consolidated framework to analyse the process of negotiation of meaning:

Table 3

Adapted Mercer's analytical framework

	Aspects of analysis	
Level 1	Interaction	Typology of talk
Level 2	Discourse	Communicational theory

Note. Adapted from Mercer (1996)

The adapted Mercer's analytical framework reflects both participation and reification processes in its typology of talk and communicational approach dimensions respectively.

In the next two sections I give a detailed description of the two-dimensional analytical framework, namely typology of talk and communicational approach.

3.4.1 Typology of talk

According to socioculturalists, meanings are continually renegotiated through talk and interaction over variable periods of time. This approach is concerned with the analysis of how talk is used for the joint construction of knowledge and how speakers construct the contextual foundations of their talk (Mercer, 1996). A useful model in this sense is provided by a typology elaborated by Mercer (2004) that offers a useful frame of reference for making sense of the variety of talk in the context of the mathematics classroom. Three archetypical forms have been identified as cumulative, disputational, and exploratory.

In cumulative conversation, according to Mercer (1996), speakers build constructively, but uncritically, on what each other has stated. Repetition, confirmation, and elaboration are common characteristics. Disputational conversation occurs when there is dispute and individualized decision-making, and it is characterized by short exchanges of assertions and counter-assertions. Exploratory conversation refers to a situation in which speakers elaborate each other's reasoning in a collaborative manner, rather than

compete. It is distinguished by critical and productive dialogue. There are also other studies with similar findings, which affirmed that when the group of peers engaged in exploratory talk, they also learned more (e.g., Dudley, 2013; Patterson, 2009; Wegerif et al., 1999).

Dudley's (2013) work on discourse interactions in LS contexts as teachers plan and discuss research lessons made a significant contribution to my analysis and took it even a step further. In his work he proposed the coding scheme of analysing participant interactions in the discussions, which he named interaction function (IF). Dudley's coding framework reflected Mercer's (1996) categories of collaborative talk: cumulative, disputational, and exploratory. He then further extended these three categories to include two extra categories: structuring conversation and managing understanding.

According to Dudley (2013), structuring conversation is characterised by initiating or introducing a new idea, moving the conversation on, returning to an earlier subject. For example, managing the talk to make sure that goals are met and keeping the talk focused on the goal. McKay (2017) explained that the structured conversation is a communication tool that fosters understanding and provides direction. This communication is characterised by a targeted and purposeful undertaking (principles) for speaking and includes focused key issues and priorities for action, time limits and a description of how the conversation will be shared with others. McKay (2017) further added that structured conversations are also characterised by the inclusion of discussion prompts (e.g., reminders, warnings) with sharing expert knowledge and the authority to make decisions.

According to Dudley (2013), managing understanding is an active process of achieving understanding through eliciting, asking or answering factual information (making sure that everyone in the group understands), requesting clarity or greater detail, explaining to a group member some background, curricular, pedagogic or other content-related fact. In order to unpack the social modes of talk I adopted Dudley's (2013) analytical categorisation which included the following categories:

i. Exploratory talk
ii. Cumulative talk
iii. Disputational talk
iv. Structuring conversations talk, and
v. Managing understanding talk.

I adopted the coding from Dudley's work and cautiously applied it, helping to generate categories of the talk in use with respect to a notion of participation. How I used Dudley's work will be evidenced in both the methodology and the analysis chapters.

In the next section, I give details of Sfard's (2008) communicational theory and highlight the constructs which were useful for this current study in illuminating the notion of reification.

3.4.2 Communicational approach

Participating in collaborative activities is discursively constructed, that is, through communication. This means the discourse is a social action. One of the principles of Mercer's (1996) interthinking processes is to answer the question related to the focus of the discourse the speakers are talking about. In the case of this study, the MTF integrated both the mathematical and mathematical teaching discourses. I therefore needed a discursive theory that would foreground both the mathematics and mathematical teaching discourses and enable the discourse of MTF to be theorised. Sfard's (2008) communicational theory was promising.

Sfard (2008) argued that communication is at the heart of her framework. She developed a communicational approach based on the view that there is a strong relationship between communication and thinking. From this perspective, thinking is conceptualised as an activity of communication. Communication is understood by Sfard to be any form of interaction allowing individuals to coordinate their actions. According to Sfard (2008, p. 89), communicational actions are:

> *about* objects; that is, they may *lead to* an action on an object or to another communicative action about an object. In any case, *the object of a communicational act* is a thing to which the actor drives the re-actor's attention.

For example, a phrase "Ask them to identify this function" is an instance of communicative action in which a speaker expects the interlocutor to perform a discursive action of asking learners to identify (classify) a particular function. The object of exchange in this instance is that of a function.

In education studies, different types of communication that bring people together while at the same time excluding others are referred to as discourses. Sfard (2008, p. 93) referred to the term "discourse" as "different types of communication set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors". According to Sfard (2001, p. 28), the mediating tools are "the shapers of the content, that is, of the

object-level aspects of discourse". These communicational mediators assist the actor in performing the prompting action and the re-actor in being prompted. "Although any material object can be adapted to serve in this communicational role, communicational mediators are often objects produced specially for the sake of communication" (Sfard, 2008, p. 90). Symbols are one such example of specially created communicational mediators.

The rules of the discourse are classified in object-level rules, "narratives about regularities in the behavior of objects of the discourse" and metarules which "define patterns in the activity of the discursants trying to produce and substantiate object-level narratives" (Sfard, 2008, p. 201). In this study the distinguishing features of any discourse such as objects, communicational mediators and rules were applied in three different types of discourses: mathematical, MTF and pedagogical discourses. Since the communicational theory is grounded in mathematical discourse, the next section takes an in-depth look at a special kind of discourse—the mathematical discourse.

3.4.2.1 Mathematical discourse

As suggested earlier in this section, what distinguishes one discourse from another are their objects, mediators and rules. Mathematics is a discourse about mathematical objects—numbers, functions, and geometrical shapes. Mathematical objects are discursive constructs. For Sfard (2008) a mathematical object is a mathematical signifier together with its realisation tree. Sfard (2008) use a notion of realisation tree to refer to representations. In this study, I do not talk about realisation tree, and will not elaborate further on this construct. Instead, I talk about names of mathematical objects and their representations to refer to mathematical objects. I now extend the idea of talk about mathematical objects into the context of MTF objects where these will be described in terms of names of MTF components together with MTF routines.

Sfard (2008) has presented four features of any discourse, and mathematical discourse in particular: word use; visual mediators; routines and narratives. Following is a description of each of these features.

Word use

One defining feature of a discourse is its words (vocabulary). A discourse counts as mathematical if it features mathematical words. In mathematics, vocabulary refers to mathematical terminology. In contrast to colloquial discourse, words in mathematics are highly specialised. There are many words that learners meet in a mathematics classroom context that also appear in non-specialised colloquial

discourse (e.g., the word “function”), but these words take on different meanings when used in mathematical discourse. Sfard (2008) further referred to word use as discursive use, where “discursive use, in turn, means the totality of proper combinations in which the word may appear” (Sfard & Lavie, 2005, p. 247). These combinations include discursive routines, visual mediators and narratives.

Sfard (2008) categorised word use into a four stage model such as passive driven, routine driven, phrase driven and object driven. During the passive driven stage, an individual is first introduced to the word and cannot contribute to the conversation. For example, in the function discourse, learners name the graphs or equations based on their appearance. The process of naming the graph is an act of matching the graph with a given name. When a learner is asked for verification of why such a graph is called for example “parabola”, the course of action includes direct recognition that is self-evident. Some may use rote memorisation such as “I learnt it at school”.

According to Sfard (2008), routine-driven word use is the early stage of word use development where learners are action-oriented and their word use is driven by routine procedures. In the routine driven stage, an individual uses the word within particular discursive routines which they associate with tasks featuring the new word. For example, in a function discourse, word use is routine driven when naming of a graph involves not just matching a graph with a name, but referring to it with a common descriptive narrative according to some visual properties. When a learner is asked to give an explanation of why a given graph is called a “parabola”, the course of action includes direct recognition, scanning or interpretation of visual properties of that graph. A possible response would be “It looks like it has two x-intercepts” or using colloquial descriptions found in school mathematics discourse such as “It is a frowning/smiling face”.

During the phrase-driven stage, an individual uses the word more flexibly with a limited number of phrases (e.g., “a quadratic function is described by the algebraic expression where the highest exponent of x is 2”)

In the final object driven stage of word use, the word is used as if it has a life of its own. That is, as if they were occurring by themselves, without the participation of human beings. In her earlier work, Sfard (1991; 1992) elaborated on operational to structural modes of thinking about mathematical objects. Where the structural mode of thinking treats “mathematical notions as if they referred to object-like entities” (Sfard, 1992, p. 60), the operational mode of thinking (processual) “speaks about processes, algorithms and actions rather than about objects” (Sfard, 1991, p. 4). According to Sfard (2008), for any

mathematical object, such as function, these processes are often blended together to help the learner create the meaning and to implement the objectification process. For example, when the learner is asked why the graph is called a "parabola", the course of action is to check the defining conditions of the graph by interpreting the global features of the graph. A possible response is that "The graph has one turning point (local maxima/minima), axis of symmetry". The naming of the graph depends on its visual properties and common descriptive narrative accompanying the name of the graph (i.e., definition of the parabola).

Visual mediators

Visual mediators are "the providers of the images with which discussants identify the object of their talk and coordinate their communication" (Sfard, 2008, p. 147). In other words, visual mediators enable participants in a discourse to identify visually the objects of their discussion. This enhances mathematical communication (Sfard, 2008). Visual mediators are visible objects that may be created or operated upon as a part of the discursive process (Sfard, 2008). For example, they may be drawn, built from matchsticks or operated upon in the mind. The most common examples of visual mediators include algebraic symbols, tables, formulae, graphs, drawings, diagrams and numbers. The visual mediators also include concrete mediators found in the mathematical classroom such as use of gestures - hands, rulers, emoticons.

In summary, three types of visual mediators have been discussed. These visual mediators play a very important role in function discourse by highlighting the various facets of a function. According to Nachlieli and Tabach (2012), it is important that learners display mediational diversity in order to appreciate a concept of function, and this notion has a positive correlation with a process of objectification.

Narratives

Narratives are descriptions or accounts of objects. They are any written or spoken text that is used within the discourse and can be subject to endorsement, that is, narratives can be judged as true or false. Within the communicational framework, truth is packaged in endorsed narratives. In mathematics, the endorsed narratives are rules generally accepted by the mathematical community and narratives that become mathematical truths or facts. For example, axioms, definitions and theorems are all endorsed narratives. The statements, "The x-intercept is the point where the graph intersects the x-axis and where the value of y equals zero" and "The y-intercept is the point where the graph intersects the

y-axis and where the value of x equals zero" are endorsed narratives of an intercept, defining what an intercept is mathematically.

Drawing from Sfard's (2008) communicational theory, there are different types of substantiating narratives: derivation, construction, rule, visual, and recall (Ben-Yehuda et al., 2005). Visual refers to a substantiating narrative that has resulted from scanning the visual mediator with own eyes or mind's eye (imagination). Recall refers to narratives that are related to summoning previously endorsed narratives (e.g., recalling properties of the quadratic graph). Derivation refers to narratives that result from manipulating the symbolic equation through calculating. Rule refers to narratives that are related to use of the general definition. Construction relates to a performance of discursive procedures resulting in new narratives (e.g., constructing points resulting in a visual mediator-graph).

To sum up, from the discussion in this section, it should be noted that the four categories of mathematics discourse (word use, visual mediator, routines and narratives), while analytically distinct, are interrelated. For example, the identification of a quadratic graph (visual mediator) can be used by pronouncing the particular words associated with it (e.g., parabola), or one may interpret its visual properties and compare it with the established endorsed narratives of the quadratic function (e.g., "A u shaped curve with one turning point with axis of symmetry"). The interesting part of this study is the examination of each mathematical feature and their interrelatedness, and how this connectedness contributed to claims about explorative mathematical discourse-bondedness, a term used to describe a process where "the output of any given step in its procedure, if not yet the desired final product, feeds in (is used as an input) in latter steps" (Lavie et al., 2018, p. 76). It is also important to note that during the process of negotiation of meaning, the CoP need to participate in the mathematical discourse as this particular discourse is part of its joint enterprise—the MTF.

In the following section I delve into how routines are talked about in the context of mathematical discourse, and focus my discussion on the recent developments of this construct.

Routines

Routines are central to the discourse, since discourses are by their very definition repetitive and patterned. In fact, the "quest for discursive patterns is the gist of commognitive research" (Sfard, 2008, p. 200). These patterns are the result of processes governed by rules. Sfard distinguishes between object-level and meta-discursive rules of discourse. The former regard the properties of the objects of the discourse, while the latter govern the actions of the discursants. A routine, then, "is a set of meta-

rules describing a repetitive discursive action" (Sfard, 2008, p. 208). Specifically, mathematical regularities can be noticed whether one is watching out for the use of mathematical words and mediators, or following the process of creating and substantiating narratives. The common routines found in function discourse in particular include calculating a function in symbolic form, interpreting a graph or a table and constructing a graph. A growing number of studies within communicational theory have been adopting the notion of routines to describe the process of learning mathematics and learning in general. Thus, I will need to develop the construct "routine" further in the following section and refer to this process as "routine-use".

Routine-use

Lavie (2019, p. 140) defined routines as the "repetition-generated patterns of actions". She further described learning as a process of routinisation of our actions, "modeling our present actions on prior relevant experiences, gradually individualizing socially endorsed routine performances" (Lavie, 2019, p. 140).

I am going to discuss three categorisations of routines. First, I discuss the routines in terms of task-procedure pair. Second, I distinguish between two forms of situations where the routines can be found. Third, I make a distinction between ritual and explorative routines.

In their recent work, Lavie et al. (2018) redefined routines as a task-procedure pair, where the task relates to a set of conditions or situations under which a certain procedure is evoked by a person/task performer and indicates what needs to be achieved by executing the routine (e.g., one has to produce an endorsable narrative about the quotient of the divisor and the dividend). The procedure which is implemented by the task performer to a given task situation is a prescription for how the routine could be done (for example, an algorithm for long division). According to Lavie et al. (2018), the task-procedure pair is linked to a particular task situation[5] or condition and to a particular person. This way of looking at routine contextualises the routine. In this study I adapted the notion of the task-procedure pair into procedure-performer pair so as to be able to identify whether the performer is the teacher or learner. How I operationalised this construct will be explained in Chapter 4.

[5] "Actions are not performed in a vacuum; rather, they are performed as a response to some perceived need or request. The person's perception of the action-eliciting situation is what we refer to as task situation" (Lavie, 2019, p. 140).

According to Nachlieli and Tabach (2018), routines can be found in different situations or contexts: social and mathematical contexts. For example, in social contexts a routine can be described as a situation “when entering your home and placing the keys at a particular place”, and in mathematical contexts, it can be described as a situation “in a mathematics classroom, when a student refers to a certain problem as similar to one performed earlier, and adopts the same procedure to solve the problem” (Nachlieli & Katz, 2017, p. 2964). In the context of this study, I worked with the notion of forms of situations to study the types of routines emerging from different situations that emerged in the data such as mathematical context; MTF context; pedagogical context; lessoning study context; and social context.

Sfard and Lavie (2005) differentiated between two types of discursive routines: ritual routines, whose goals are socially oriented—acts of solidarity with co-performers, that is, to connect with (or please), others; and explorative routines[6], whose goal is to produce mathematical narratives about mathematical objects. Sfard (2008) defined ritual routines as routines "whose goal (closing condition) is alignment with others and social approval" (p. 301). In recent developments, rituals have been defined as the discursive performance which “is process oriented, that is, the learner is satisfied with having completed a certain set of actions (routines) that they have seen other, more skillful participants perform” (Heyd-Metzuyanim et al., 2018, p. 3). This means that the learners’ routine performance is confined to imitation of the more expert participants. The ritual performance is characterised by routines which demonstrate imitation and lack of independent decision making and are performed for the sake of pleasing others. From this it is noted that the process of using ritual routines highlights the central role of the knowledgeable others in the lessoning study. This idea is noted for future reference throughout this study.

Explorative routines are defined as routines "whose goal (closing condition) is production of an endorsed narrative" (Sfard, 2008, p. 298) and these goals “are posed by the actor and self-justified” (Heyd-Metzuyanim et al., 2018, p. 4). This means that the aim of performing explorative routines is to produce endorsed narratives about mathematical objects. The explorative routine performance is characterised by independent decision making where the learners’ performance is for the sake of solving a genuine problem posed to oneself by the actor.

[6] Within the communicational theory studies, the words explorative and exploratory are used interchangeably. In this study I opted to use the term ‘explorative’ as the word ‘exploratory’ conflicts with Mercer’s (1996) exploratory talk.

According to Sfard (2008), more often the rituals are performed with others or scaffolded by others. In other words, they are performed for the sake of others and according to these other performers' rules. In contrast, explorative routines may be performed individually with no need for scaffolding by others. In other words, they are performed for oneself and others. Sfard and Lavie (2005) and later Heyd-Metzuyanim (2013) differentiated ritual from explorative participation according to several attributes. Ritual routines often rely on syntactic rules (e.g., calculations) without reference to mathematical objects being manipulated. Explorative routines, on the other hand, are aimed at producing narratives about mathematical objects (talk about mathematical symbols in an objectified way). A learner who learns exploratively flexibly applies different routines. In contrast, someone who performs a ritual routine is concerned with rigid use of routines. Lavie et al. (2018) further claimed that explorative routine use differs from ritualistic routine performances by the level of flexibility, applicability, objectification of the discourse and substantiability. They described these attributes as follows:

Flexibility refers to the variation of routines when performing a specific task. For example, one can analyse the effects of the gradient by manipulating a linear equation through calculating procedure. One can also interpret the global features of the equation. On the other hand, a non-flexible routine performance refers to the rigid use of routines.

Applicability refers to an act of applying the same routine in a range of situations. For example, an action of applying a calculating routine in the contexts such as tasks, examples and explanations. The absence of the applicability rule indicates ritual performance (e.g., when the application of a particular procedure is limited to one task situation).

Performer's agentivity refers to a situation where the performer is able to make a decision without being helped. For instance, in the case of a classroom discourse, the presence of performer agentivity can be witnessed when teachers consider their learners as independent thinkers who can make their decisions about learning of mathematics or endorsing narratives. On the other hand, the absence of perfomer agentivity can be seen in situations whereby the teacher is the only authority of the discourse where she makes decisions about endorsing narratives. Such is indicative of ritual performance.

Substantiability refers to a process through which a performer becomes convinced that the narrative can be endorsed. It is a set of criteria used to assess own performance either through performing routines or authority (Lavie et al., 2018). For example, in the case of the classroom discourse, substantiability is an ability to ask or answer questions such as "How would you explain that?", "Why is

that?" In these instances, learners usually respond with a word "Because" (Ben-Yehuda et al., 2005). According to Ben-Yehuda et al. (2005), the routine procedure is also a prerequisite for substantiation. In other words, one can observe learners' substantiations through their discursive routines. Lavie (2019) argued that the learners' response to a substantiation request such as "Why is that?" could indicate both ritual, explorative performance. For example, when the learners' response imitates the one used by the teacher or used in the mathematical community, such as definitions and theorems that could be indicative of ritual performance. However, when the response shows performers' independent ability to substantiate the narratives that could be associated with explorative performance.

Objectification refers to a situation whereby the performer has a tendency to use nouns, that is, processes and actions being replaced by talks about mathematical objects (Sfard, 2008). According to Lavie et al. (2018), objectification is also characterised by a tendency to derive procedures from properties of the mathematical objects and word use is driven by definitions (mathematical facts, theorems etc.). The tendency of performers to rely on calculations is an indication of ritual performance. The ritual and explorative routines can be summarised according to goals, authority and mathematical objects, as illustrated in the table below:

Table 4

Categorisation of mathematical routines in terms of ritual-explorative

	Ritual	**Explorative**
Goals	Socially oriented	To produce mathematical narratives about mathematical objects
Authority	Learner seldom makes independent decisions (Non-performer agentivity)	Learner makes independent decisions (perfomer agentivity)
Mathematical objects	Non-flexibility	Flexibility
	Non-substantiability	Substantiability
	Non-applicability	Applicability
	Non-objectification	Objectification

Ritual and explorative routines play an important role in the development of mathematical discourse. Former studies suggest that ritual routines are inevitable and necessary during the initial stages participating in the new discourse (Nachlieli & Tabach, 2012; Sfard & Lavie, 2005). In the recent work on communicational theory, several researchers have used routines as the unit of analysis in order to unearth the process of learning of mathematics and learning in general (e.g., Heyd-Metzuyanim et al.,

2018; Lavie & Sfard, 2016; Viirman & Nardi, 2019). This study has also adapted the use of routines as a unit of analysis in order to examine the process of negotiation of meaning of MTF.

To sum up, within communicational theory the ritualistic engagement is to be expected from the students' initial participation in the discourse. According to Sfard (2017), teaching that is aiming to shift from ritualistic participation to explorative participation in the discourse can be supported by the teachers (or experts of the discourse). Sfard (2017) illustrated the two ways this support can take place as follows: "First they can model such discourse by demonstrating the type of explorative discourse they would like their students to develop. Second, they can explicitly encourage the desired kind discourse by appropriate pedagogical moves" (p. 44).

This idea was very useful for the current study, as it foregrounded the role of the expert of the discourse (the KO in this context). In the current study, the KO played the role of introducing the teachers to the MTF which integrated both the mathematics and mathematics teaching discourse. It is therefore important for the KO to deliberately negotiate actions as accountable to enterprise—that is, align teachers' engagement with the discourse of MTF.

3.4.2.2 Mathematical teaching routines

In this section, I discuss the notion of teaching routines and highlight how researchers have begun to talk about them from a communicational theory perspective. This discussion was useful for this study as it contributed to the talk about teaching routines which formed part of the pedagogical discourse which was a meta-discourse of the MTF—a joint enterprise.

Within the communicational framework, teaching is defined as "the communicational activity the motive of which is to bring the learners' discourse closer to a canonic discourse" (Tabach & Nachlieli, 2016, p. 299). According to them, this communicational activity includes a range of communicational actions performed by the teacher with aims of organising mathematical discourse. These can include explaining mathematical concepts and asking the learners questions. Some of the researchers, who have studied teaching practice from the communicational approach, define teaching as a routine activity (e.g., Heyd-Metzuyanim et al., 2016; Tabach & Nachlieli, 2016).

Mosvold (2016) attempted to define teaching through a notion of the work of teaching by proposing Sfard's communicational framework. In his work, he focused on teachers' discursive actions while engaging in what Ball et al. (2008) called tasks of teaching. He argued that, "when focusing on the

discursive aspects of the tasks of teaching mathematics, there is a potential to create a more manageable decomposition of the tasks of teaching mathematics into core acts of discourse" (Mosvold, 2016, p. 194). He described the discursive actions of teaching as actions that a teacher engages with, with the aim of helping learners learn mathematics content. In this study he translated Ball and colleagues' (2008) list of 16 tasks of teaching into discursive terms.

Mosvold further argued that the core tasks of teaching are discursive acts of teaching, for example, one of Ball et al's. (2008, p. 400) core tasks of teaching: "asking productive mathematical questions", is directly linked to an act of questioning. He further connected Adler and Ronda's (2015) MDI components: exemplification, explanations with discursive acts of teaching, and referred to them as acts of exemplifying and explaining. Mosvold's (2016) work was useful in distinguishing mathematical teaching routines from the other types of pedagogical routines that emerged during analysis as will be demonstrated in the methodology chapter.

A study by Viirman (2012) was also helpful in bringing awareness to different categories of routines that came into play during the teaching activity. For instance, this study reported on the examination of the teaching of functions in undergraduate courses in mathematics at three Swedish universities. Characteristic features of the teachers' discourses about functions were presented, focusing on routines—repetitive patterns characteristic of the discourse. The analysis of the discourse of mathematics teaching resulted in a categorisation of the mathematical routines and teaching routines: "didactical routines" (Viirman, 2012, p. 1*)*. The findings evidenced that the discursive practices contained two intertwined discourses: mathematical discourse and discourse of mathematics teaching. In this study I extended this focus on routines related to different situations such as MTF, contextual, and L/S process, with the aim of describing teachers' repetitive patterns as they interacted with the MTF tool in the context of L/S.

The role of the teacher

Heyd-Metzuyanim et al. (2016) defined a teacher or the instructor as a person who is a leading participant in the discourse, and the learner as the follower in the discourse. This definition addresses two ways of looking at the teacher/leader of the discourse, particularly in the case of this study where the preservice teachers assumed the role of learners while the KO played a leading role as the facilitator. From this understanding, this study added another element to the roles played by the participants which were different from a normal classroom context. In the L/S setting, the preservice teachers were

learning how to teach using a particular tool, and the KO played the role of facilitating this learning. From this I argue that the teaching role depends on the leading discourse, and I draw from Tabach and Nachlieli's (2016) definition of teaching as "the communicational activity which motive is to bring a change in the learners' discourse towards a leading discourse" (p. 1448). In the context of this study, L/S teaching can be understood to be the communicational activity whose goal is to usher teachers towards a leading discourse of MTF, where the KO plays the role of the leader of discourse (facilitator) and the preservice teachers play the role of learners.

Ritual-explorative teaching

It has already been noted that the communicational activity of teaching is captured through teaching routines. As seen with mathematical discourse, teaching routines can also be captured in ritual-explorative routines. Ritual and explorative teaching refer to teachers' actions that aim to provide students with ritual and explorative learning opportunities respectively. In other words, both ritual and explorative teaching actions are linked to ritual-explorative learning. Similar to mathematical learning, ritual-explorative teaching routines can also be categorised into goals, routines and authority. I now discuss studies that have used ritual-explorative teaching routines. In these studies, teaching activity is described along a "ritual-explorative dyad".

Heyd-Metzuyanim et al. (2018) reported on the learning of two middle school teachers participating in teaching practices of a professional development (PD) programme: The 5 Practices / Accountable Talk. The 5 Practices is defined as "a set of practices for orchestrating productive discussions around tasks", while the Accountable Talk is "a set of specific talk moves that hold students accountable to the learning community" (Heyd-Metzuyanim et al., 2018, pp. 276-277). This study illustrated the analysis of two lessons that the two teachers enacted. The results of the first lesson showed features of the ritual engagement in the PD practices where the two teachers imitated practices of the PD facilitators. On the other hand, the second lesson evidenced that the two teachers engaged in an explorative manner as they used the PD practices. Their engagement was characterised by choosing the tasks in accordance with the goals of the lesson, flexibly changing them and adapting them to the abilities and needs of their students.

This study was similar to the current study in many ways. First it examined the use of a teaching tool (The 5 Practices / Accountable Talk) by the teachers as reflected in their teaching discourse. In the current study the focus was on the use of the MTF teaching tool as teachers participated in a particular

type of teaching discourse, the L/S reflective discourse. Secondly, the main analytical tools of Heyd-Metzuyanim et al. (2018) were ritual-explorative teaching routines, similar to the context of my study.

The aim of Nachlieli and Tabach's (2018) study was to search for possible goals of ritual teaching. For this purpose, they operationalised the ritual-explorative dyad into two categories: ritual-enabling Opportunities-To-Learn (OTLs) and exploration-requiring Opportunities-To-Learn (OTLs). They define ritual-enabling OTLs as "teachers' actions that provide students with tasks that could be successfully performed by rigid application of a procedure that had been previously learned" (Nachlieli & Tabach, 2018, p. 5). At the same time, the term exploration-requiring OTLs is defined as "teachers' actions that provide students with tasks that could not be successfully solved by performing a ritual" (Nachlieli & Tabach, 2018, p. 5). In this study they looked at mathematics lessons from three different countries: Australia, the USA and Hong Kong. Their results reported that teachers' actions showed common threads across the three countries, where teachers' actions interweaved both ritual-enabling OTLs and exploration-requiring OTLs and noted that each of these types of OTLs were nested within each other. For example, they argued that the ritual-enabling OTLs served the purpose of facilitating possible exploration-requiring OTLs. Through their results, they claimed that both ritual-enabling OTLs and exploration-requiring OTLs supported the object-level and meta-level learning necessary for initiation into a new discourse. They hypothesised that a mixture of the two OTLs in teaching could be a catalyst for explorative mathematical discourse.

Heyd-Metzuyanim et al. (2015) delved into the domain of preservice mathematics teacher education, where they examined an instructor's underlying assumptions about what it means to do mathematics as reflected in the instructor's discourse. One of their aims was to incorporate analytical tools that enable the exploration of mathematical aspects together with social aspects of the discourse. Thus, they used ritual-explorative lenses by adding constructs such as mathematising and identifying. Mathematising is a talk about mathematical objects. The identifying talk is the communicational activity about the participants' identifying narrative. For example, "I am a mathematics teacher … You are an excellent student" (Heyd-Metzuyanim et al., 2015, p. 552). The findings of this study point to the instructor's goals which implied a tug of war between explorative mathematical goals. They clarified in these findings that at one level the goals are aligned with ritual talk, and simultaneously at other level with explorative talk. For instance, the instructor's talk provided opportunities for mathematising explorative mathematical discourse. However, the focus on identifying narratives limited opportunities to participate in the mathematical activity. The identifying narratives underline ritual participation. This study made an

important contribution to the current study, as it showed how the ritual-explorative constructs could be used in conjunction with social aspects of the discourse (e.g., the identifying activity).

The study reported by Viirman and Nardi (2017) also focused on the use of ritual-exploration pair where they examined the discourse of university biology students participating in mathematical modelling activities. The results showed that the students' participation was characterised by a mixture of both ritual and explorative engagements and this was due to the lecturer's scaffolding actions. Their discussion was twofold. First, they illustrated how a lecture's highly scaffolded teaching invited a more ritualised routine use. At the same time, they also demonstrated that when the lecturer's actions showed reduced scaffolding, the students' engagements showed signs of explorative routine use. From this, they argued that although the expert's scaffolding moves seem to invite ritualised routine use, they are necessary for the development of explorative participation.

Sfard (2017) studied the relationship between instruction and learning by looking at the ritual-explorative mathematising activity of both the teacher and the learners. She examined a case of South African classroom discourse. The results of her study showed that the learners participated in a ritualised mathematical discourse which was due to the teacher's ritualised teaching in terms of a number of attributes: (1) objects—his talk about mathematical objects lacked objectification; (2) roles—where he played the role of being the authority of the mathematical discourse where learners were just recipients; (3) goals—where the main goal of the mathematical activity was to create harmony amongst learners instead of encouraging construction of mathematical narratives and substantiating them; (4) sources of narratives—where he encouraged learners to imitate his mathematical narratives and to rely on memorisation of previously endorsed narratives. Sfard's (2017) categorisation of the ritual-explorative mathematising activity in terms of objects being talked about, goals of the mathematical activity, sources of narratives and roles of participants were useful for this current study as will be demonstrated in the analysis chapters.

The reviewed studies above highlight two important ideas which have a bearing on this current study. First, some of the studies have demonstrated how they operationalised the constructs of ritual-explorative. For example, Nachlieli and Tabach (2018) operationalised this pair into ritual-enabling OTLs and exploration-requiring OTLs, while Heyd-Metzuyanim et al. (2015) extended the two constructs by adding mathematising and identifying constructs. Secondly, most of these studies, except the one by Sfard (2017), highlighted the mixture of ritual-explorative participation differing in terms of goals, talk about mathematical objects, and roles of participants and sources of narratives. Lastly, what is further

highlighted in these studies is the role of ritual teaching, that of supporting the initiation to a new discourse and the development of explorative learning. However, Sfard's (2017) results show it is not necessarily the case, and that the ritual teaching can promote only ritual learning.

This study employed Sfard's concept of ritual-explorative routines to refine the theorising of teacher learning of the MTF tool (discursive practices). I did this by extending the concepts that have been used to describe mathematical / teaching discourse to the domain of learning MTF tool practices as done by Lavie and colleagues (2018). The analysis of this study integrated the indicators of ritual explorative routines from two perspectives: mathematics discourse and teaching discourse. The analysis further extended three categories for characterising ritual-explorations: goals, roles, and talk about mathematics objects into the MTF teaching discourse.

Drawing from Adler (2021) and Heyd-Metzuyanim et al. (2018), I wish to argue that explorative MTF discourse can be conceptulised as the integration of mathematical and pedagogical routines aimed at constructing narratives about MTF objects; coupled with the development of the connection between what is critical for learners to learn to the MTF objects—aspect of coherence.

Before concluding this section, I wish to highlight a very important construct that was useful in this study—the discursive activity. Within communicational theory, the process of learning involves participating in discursive activities, that is, the "patterned collective doings" (Sfard, 2006, p. 157). The main discursive activity found in mathematics discourse is called mathematising—talking about mathematical objects (Heyd-Metzuyanim, 2017). Hunter and Anthony (2014) argued that although mathematising activity is an essential component of mathematical learning, other social factors need to be considered. This view has been highlighted in the research fostering mathematical discourse in classrooms. For example, Lemke (2001) argued that sociocultural research needs to take into consideration both academic and social factors of the interactions in mathematical classrooms. Some of the researchers that have foregrounded both academic and social aspects of participating in the mathematical discourse include Hunter and Anthony (2014) and Wegerif et al. (1999); they have highlighted how this may advance mathematical learning. For example, Hunter and Anthony, informed by Sfard's (2008) communicational theory, illustrated ways in which a group of students use different types of talk based upon its focus. These included a focus on mathematical objects—mathematising—and talk about participants of the mathematical discourse—subjectifying. In this study they showed that the subjectifying talk was focused on mathematical actions of the participants. This means they used the construct of subjectifying in relation to the speaker's mathematising activities which were characterised

by features of mathematical discourse such as objects, routines and mediators. They coined this process as "on task subjectifying" and further concluded that this type of process supported mathematising actions of a group of students. A similar study was reported by Heyd-Metzuyanim (2017) where she extended Sfard's (2008) constructs of mathematical discourse such as words, routines, visual mediators and narratives by adding mathematising and subjectifying constructs in order to articulate how learners engage in the mathematical discourse.

Since the focus of this study was more on the social than the individual, ideas of mathematising and subjectifying discursive activities were taken into consideration when exploring the collective doings. I also extended the conceptualisation of the notion of discursive activity in the context of the MTF discourse, and defined a talk about MTF objects as MTFying discursive activity.

3.4.2.3 Learning from the communicational perspective-discursive learning

Under communicational theory, learning is considered as a change in one's discourse. In other words, the operational definition of learning under this framework implies looking for a change in at least one of the four discourse characteristics (i.e., words, routines, visual mediators and endorsed narratives). In a recent development on communicational theory, Lavie et al. (2018) defined mathematical learning as routinisation of people's action. This means that "to learn mathematics, one should individualize historically developed, well established routines" (Nachlieli & Tabach, 2018, p. 4). Such individualisation could include both the object-level and meta-level learning. The object-level learning "expresses itself in the expansion of the existing discourse, attained through extending a vocabulary, constructing new routines, and producing new endorsed narratives; this learning, therefore results in endogenous expansion of the discourse" (Sfard, 2008, p. 254). In addition, the meta-level learning

> is usually related to exogenous change in discourse. This change means that some familiar tasks, such as defining a word or identifying geometric figures, will now be done in a different, unfamiliar way and that certain familiar words will change their uses. (Ioannou, 2017, p. 2099)

This study adopted the notion of object-level and meta-level learning to describe the nature of the discursive learning. In the context of this study the object-level change refers to the process of extending objects and constructing routines related to the MTF tool. At the same time, the change in meta-rules is related to the process of integration of different routines such as mathematical and pedagogical routines during interaction with the MTF tool.

Conditions for learning

Sfard (2008) claimed that for learning to take place (change in meta-rules), the learner needs to be exposed to communicational conflicts[7]. According to Sfard (2007), the meta- level learning is evidenced by changes in meta-rules of a discourse through a process of resolving a communicational conflict. Communicational conflict is defined as "a situation in which communication is hindered by the fact that different discursants are acting according to different metarules" (Sfard, 2007, p. 574). In other words, this conflict prevails when participants of a discourse communicate according to diverging or conflicting discursive rules. For example, discourse about algebraic equations and trigonometric equations: $sinx = 2$, a learner may be tempted to apply the rule from algebra of making x a subject of a formula (e.g. $x = 2/sin$); this rule conflicts with those used in trigonometric discourse. According to Ioannou (2017), these conflicts are welcomed as they result in explorative participation as they are "a gate to the new discourse rather than a barrier to communication, both the newcomer and the old-timers must be genuinely committed to overcoming the hurdle" (Sfard, 2008, p. 282). This highlights a relationship and the interaction between learners of the discourse and the experts of the discourse. To gain insight into this process, Sfard (2008) proposed a notion of learner-teaching agreement. This process addressed three basic aspects of communicational process: the leading discourse, learners' and expert's respective roles and the nature of expected change.

Agreement on leading discourse refers to an agreement about a set of discursive routines (used for setting standards/goals) relevant to that specific discourse. In terms of this study, using Wenger's (1998) language, these are MTF stated goals. The notion of the leading discourse also highlights the leadership of the discourse that is a leading form of communication. For example, in a classroom context these could be textbooks, curriculum documents, and teachers' discourse. In case of this study, the MTF tool was a leading form of communication.

With regard to agreement on discursive roles, according to Sfard (2008), those who are new to the discourse need to be willing to play the role of learners, and the experts the role of teacher/facilitator. Such an agreement is a first step for establishment of harmonious relations critical for learning (e.g., mutual engagement). In case of this study, the KO played the role of being responsible for the change of preservice teachers' discourse (learners). However, Sfard (2008) stressed that being a leader of the discourse does not mean imposing ideas of the new discourse on the newcomers, and for the

[7] Communicational conflict and commognitive conflict are used interchangeably

newcomers to mindlessly imitate the new discourse. Rather, the presence of leadership needs to allow the newcomers to negotiate the new discourse and play the role of scaffolding newcomers to the rules of the new discourse. The notion of learning-teaching agreement can be linked to Wenger's (1998) aspect of joint enterprise which concerns the idea of working in mutually accountable ways. This implies participants' roles and engagements that directly affect their contribution to the enterprise (MTF).

Agreement on the course of the discourse change refers to a situation whereby a newcomer to the discourse makes the new discourse for themselves, that is, a type of discourse that one can use to solve their own problems. In the case of this study, when the teacher used the MTF tool outside of L/S for their own teaching practice, this could be an indication of discourse for oneself. It would have been possible to observe this scenario in a different teaching context other than of this study.

In conclusion, Sfard (2008) claimed that the commognitive conflict and learning-teaching agreement are the basic conditions for meta-level mathematics learning. First the communicational conflict highlights the discursants' actions according to diverging or conflicting discursive rules. Second the, learning-teaching agreement highlights the relationship and interaction between the participants of the discourse. The analysis of this study paid attention to two aspects of learning: communicational conflict and teaching agreement (leading discourse, and participants' roles) which have an impact on the nature of learning that emerges. These ideas are directly linked to Mercer's (1996) notion of interaction which is concerned about collaborative interactions between the participants of the collaborative activities and how their interactions may contribute to the process of negotiation of meaning.

The commognitive conflict has been discussed in the context of mathematical discourse which involves talk about mathematical objects. In this study I extended the notion of commognitive conflict to a talk about MTF objects where I showed how this conflict supported opportunities for learning the MTF discourse. It is important to note that recently the idea of commognitive conflict has been extended to a field of pedagogical discourse (Nachlieli & Heyd-Metzuyanim, 2021), where commognitive conflict is described as the misalignment between teachers' Delivery Pedagogical Discourse (DPD) and the Explorative Pedagogical Discourse (EPD). These two are related to teacher-centred and learner-centred instruction respectively. This study did not take this route; instead it focused on the commognitive conflict in relation to MTF objects.

3.4.2.4 What can Sfard's communicational approach help me to talk about?

The communicational framework enabled me to talk about the mathematical objects and actions on these objects (mathematical activity/process of mathematising). Furthermore, it also helped me to zoom out of talk about mathematical objects (Adler & Lerman, 2003) into talk which was not "tightly on mathematical activity" (Adler & Lerman, 2003, p. 445). In other words, a talk about wider educational practices which teachers in the L/S were exposed to, such as mathematical teaching, and MTF teaching practices.

In this study, the "construction of the theoretical" (Brown & Dowling, 1998, p. 11) involved re-contextualising theoretical constructs from Sfard's (2008) communicational theory in interaction with empirical data in order to explore discursive educational practices emerging in the context of L/S. This process, which I call the mathematical work of this study, has three parts: firstly, developing the notion of mathematical discourse—attending specifically talk about mathematical objects; secondly, developing the notion of mathematical teaching discourse—talk about teaching actions on mathematical objects; and thirdly, developing the notion of MTF teaching discourse—talk about specialised mathematical teaching practice. The mathematical work further focused on developing (i) meta-rules that defined routines, the procedure-performer pair, and (ii) extending/adapting categories of routines into ritual-explorative pair. Sfard's work has also helped me to extend the notion of discursive activity in the context of the MTF discourse, and talk about MTF objects as MTFying activity whose overall goal is to produce narratives about MTF objects.

In summing up, this study closely followed Mercer's dimension conceptual framework. The full categories of the analyses method are summarized in the table below which I call the Routine-Interaction (RI) model.

Table 5

Initial conceptual framework: Routine-Interaction (RI) model

Dimension	Analytical categorisation
Interaction Function	Exploratory talk
	Cumulative talk
	Disputational talk
	Structuring conversation talk
	Managing understanding talk
Routine use	Mathematical routines
	Teaching routines

The two aspects of the framework above, the interaction function and routine use were helpful when dealing with the analysis of both participation and reification constructs (Wenger, 1998) which are directly linked to the notion of mutual engagement (social relations and discursive actions).

The conceptual framework suggested in this chapter provided integrated lenses to address the following aspects/issues in the context of L/S:

- Ways of talking about mathematical objects;
- Ways of talking about mathematical teaching actions;
- Ways of talking about teaching actions (including MTF);
- Ways of interacting with each other.

It is going to be the work of methodology to demonstrate how the constructs defining these ways of talking were developed in interaction with empirical data using appropriate extracts from different episodes of the transcripts to develop the analytical framework. The analysis chapter will further present how/why these ways of acting are interrelated.

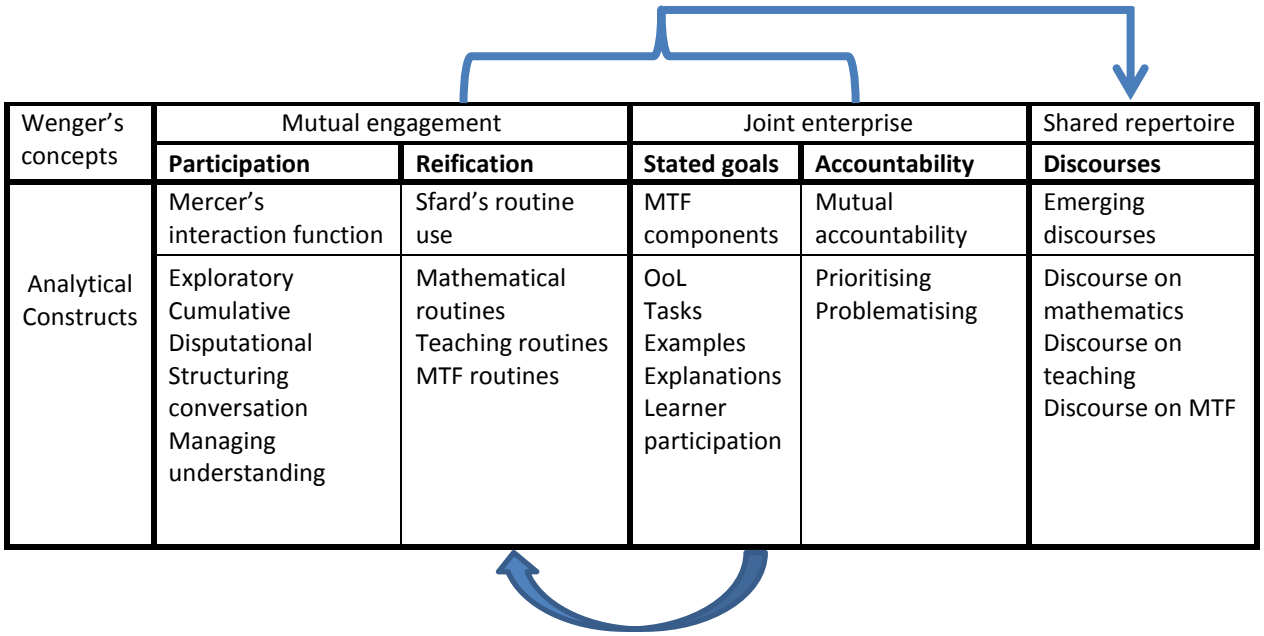
3.5 Summary of this chapter

3.5.1 Synopsis of the theoretical field for this study

Table 6 is a summary of the theoretical field for my study which includes the theories and the analytic resources from the literature.

Table 6

A summary of the theoretical field



Wenger's concepts	Mutual engagement		Joint enterprise		Shared repertoire
	Participation	**Reification**	**Stated goals**	**Accountability**	**Discourses**
Analytical Constructs	Mercer's interaction function	Sfard's routine use	MTF components	Mutual accountability	Emerging discourses
	Exploratory Cumulative Disputational Structuring conversation Managing understanding	Mathematical routines Teaching routines MTF routines	OoL Tasks Examples Explanations Learner participation	Prioritising Problematising	Discourse on mathematics Discourse on teaching Discourse on MTF

Overall, as already stated, my concern in this study was to explore teacher use of the MTF tool. Critical to this were the discursive actions at play. Therefore, I highlight the construct mutual engagement (participation and reification) as the heart of investigating the use of MTF. As shown in Table 6, an attempt towards developing a conceptual framework for my study included concepts from Sfard's (2008) theory of routine-use. These included mathematical and teaching routines (including MTF routines). The other resources for the analysis drawn from the literature that enabled an explanation of how teachers related to each other referred to Mercer's (1996) three typology of talk extended with Dudley's types of talk. These include exploratory; cumulative; disputational; structuring conversation; and managing understanding talk.

3.5.2 Revisiting my research questions

In Chapter 1, I indicated that my study was guided by one overarching question which read as follows: How can mathematics teacher learning be understood through participation in a form of a L/S CoP—in relation to use of the research-designed tool—MTF? This question highlights two important keywords that were used to guide the analysis: (1) Use of MTF and (2) Teacher learning. In this chapter, after an elaboration of the overall theoretical framework that framed this study, that is, use of Wenger's communities of practice theory complemented by Mercer's constructs of interaction and Sfard's

communicational approach—routine-use, I re-express below the guiding question in the form of analysis questions:

Table 7

The guiding questions in the form of analysis questions

Research Question	Key aspects		Analysis questions
HOW CAN MATHEMATICS TEACHER LEARNING BE UNDERSTOOD THROUGH PARTICIPATION IN A FORM OF A L/S CoP—IN RELATION TO USE OF THE RESEARCH DESIGNED TOOL—MTF?	**USE OF RESEARCH DESIGNED TOOL—MTF?**	**Mutual engagement Joint enterprise**	1. What is the focus of their reflections?
			2. Which objects of the MTF are in use? How are they used? Why are they used in this way? 2.1 Which routines are used in the process of construction of MTF narratives? 2.2 What is the relationship between the MTF objects and other routines? 2.3 How can teachers' process of constructing narratives related to these MTF objects be described in terms of ritual and/or explorative participation?
			3. Which social modes of talks are used in the L/S community? How are they used? Why are they used in this way? 3.1 What patterns of interactions did teachers engage with as they used the MTF tool? 3.2 How did these interactions enable and/or constrain teachers' construction of narratives as they used the MTF tool? In other words, how can teachers' construction of narratives in relation to the MTF be explained in terms of their accountability to each other (including the KO) and the joint enterprise?
		Shared repertoire	4. What are the specialised common discourses within the L/S CoP?
	TEACHER LEARNING		5. What implications can be drawn from the use of the MTF tool with regard to teacher learning?

These analysis questions will be further elaborated in Chapter 4 where I deal with the coding process transacting between my theoretical and empirical fields.

Chapter 4: Research Design

4.1 Introduction

This study investigated how teachers used a research-designed resource, MTF, in the context of a lessoning study and the nature of the learning opportunities created when teachers do so. The study was grounded in situated learning theory and drew on constructs from Wenger's (1998) Communities of Practice (CoP) perspective, as well as constructs within sociocultural learning, such as Sfard's (2008) communicational theory and Mercer's (1996) typologies of modes of talk.

The research questions, prior research literature, and conceptual framework all informed the study design and methodology described in this chapter. In this chapter, I first explain the rationale for designing a qualitative case study. I then describe the participants and the data collection methods used. The data analysis process is outlined in the following chapter (Chapter 5). Through this chapter, I elucidate the scientific process that led to the findings presented in this thesis.

4.2 Rationale for Qualitative Research

This study was a qualitative case study. Each aspect of this design was intentionally chosen to help answer my research questions and add to the body of research on the use of a research-designed tool in a specific teacher learning context. In this section, I explain these decisions.

I chose to conduct a qualitative study because I was interested in going "inside teacher communities" (Little, 2003, p.913) to understand the nature of the use of the MTF tool and the learning opportunities presented when teachers engage with it. In addition, a situated learning theory lens suggests the importance of understanding the dynamic interplay between individuals and their environments, thus requiring research methodologies that offer insights into this interplay. These points led me to a qualitative study, where "researchers study things in their natural settings, attempting to make sense of, or interpret, phenomena in terms of the meanings people bring to them" (Denzin & Lincoln, 2000, p. 3).

Since the main purpose of this study was to describe and explore qualitatively the process of teacher learning in the lessoning study, which is grounded by the use of a research-designed resource, MTF, a case study emphasising rich and detailed interpretation was thus used. According to McMillan (2004), a

case study is “an in-depth analysis of one or more events, settings, programs, social groups, communities, individuals, or other ‘bounded systems’” (p. 271).

The use of the MTF tool and the teachers who engage in L/S CoP can be thought of as a bounded system, or a specific case. Case studies allow a researcher to examine a phenomenon within a real-life context. As Yin (2009) explained, “A case study is an empirical inquiry that investigates a contemporary phenomenon in depth and within its real-life context, especially when the boundaries between phenomenon and context are not clearly evident” (p. 18). According to Yin (2009), there are two variations of case study designs, namely, single-case designs and multiple-case designs. In this current study I have chosen a single case study design as one instance of a lessoning study. This L/S focused on the effects of the y-intercept and gradient in the linear function with two cycles.

Epistemologically the qualitative approach emphasises that knowledge and understanding can be obtained through the experiences of the actors in their social context (Merriam 1998; Neuman, 2006). Many writers have traced the philosophical roots of qualitative research to phenomenology that provides evidence on a particular phenomenon based on the exploration of specific contexts and particular individuals (Merriam, 1998).

The choice of a qualitative research for this study coheres with various philosophical, epistemological and ontological assumptions inherent in this research, in L/S, and in the chosen theoretical framework. At the basis of my choice of qualitative methods were the assumptions that reality is constructed by individuals interacting in their social worlds and that meaning is embedded in individuals’ experiences and is mediated by the researcher’s perceptions (Merriam, 1998) and that knowledge is both personal and social (Lave & Wenger, 1991; Wenger, 1998).

4.3 The research contexts and the participants

This study was focused on mathematics preservice teachers participating in a lessoning study context. Due to the nature of the design of this study, the enactment of the lesson (teaching) needed to be carried out in the real classroom context. For this reason, I needed to select a school in order to conduct the teaching sessions. Furthermore, the formation of a lessoning study group lay at the core of the study and so too did teachers. Thus, in the discussion that follows I give a brief account of how the schools and the teachers were chosen.

4.3.1 Selection of school

To identify a school, two conditions needed to be satisfied. First, the school needed to be located in the Johannesburg area. Secondly, the school needed to be willing to welcome student teachers. With this in mind I opted to work with school J which is located in the Johannesburg area. For practical reasons, I chose school J because it was reasonably close to the university. This enabled regular visits to the school. In this sense the chosen school was a purposive target group for the lessoning study model (L/S). Furthermore, school J was a convenience sample, because I had access to the school since it was part of the participating schools within the Wits Maths Connect project. The "convenience" sampling type is also known as opportunity sampling which involves choosing a sample from those whom the researcher has an easy access to (Creswell, 2012). To ensure that my study did not cause any disruptions at the school, I decided that the L/S lessons be taught after school hours.

The choice of the grade was grade 10 and the content area of interest was the linear function topic. This decision was influenced by a number of reasons. First, the mathematical focus of the WMCS project was established as the topics of algebra and functions at grade 9/10. Secondly, the implementation of a L/S was done right at the beginning of the year (9 March-29 March 2017) so the learners would not have done anything on linear function. However, they would have had an introduction to linear function in grade 9, but as a straight line graph rather than as a function. This meant that choosing a grade 10 class was an important consideration in relation to the identification of CA of OoL (learners' previous knowledge in terms of difficulty and misconceptions with a linear function) as they would have gone through this topic in grade 9. Another reason for choosing functions was my own personal inclination towards the topic. In my previous Master's research, I conducted an investigation on grade 10 learners' function discourse where I noted an interest in extending the research to teachers for future studies. Thus, working with functions would provide me a space for furthering research on the function topic and also provide me with an opportunity to learn about teaching this topic at secondary level.

The grade 10 mathematics class in school J on average had a total of 50 learners. It was using traditional resources such as chalkboard, whiteboard and textbooks. There were no technological resources available. These attributes represent a typical classroom context found in South African schools.

4.3.2 Participants

This study had two types of participants: four preservice teachers and a researcher/KO. I therefore discuss each type of participant separately. It is important to note that school teachers did not participate in this study.

Preservice teachers

Marton and Pang (2006) provided a rough guideline when they described a learning study as a group of between two and six teachers working together. In the case of my study, the L/S group was made up of four mathematics preservice teachers[8] from the University of the Witwatersrand (Wits). These teachers were selected from a group of teachers who participated in the teaching practice programme (TE) during which I was their university mentor. During the TE programme, an initial meeting with the teachers whom I was mentoring was conducted. The aim of these meetings was to debrief the teachers about what was expected of them during the TE programme, discuss the ground rules of conducting the TE and also to suggest any teaching strategies that the teachers could use during their teaching practice. It was at this stage where I informally introduced the teachers to the MTF practice. As noted, the lessoning study was focused on preservice teachers specifically set up for this research and was not part of a TE programme.

I intentionally selected preservice teachers who used the ideas of MTF during their TE practice; these were observed during the sessions I conducted to evaluate their teaching in schools. Additionally, these teachers were willing to be part of this research study. I thus purposively sampled the four teachers who formed part of this study (McMillan & Schumacher, 2006). In my quest to have a relatively representative sample of the general population of secondary mathematics preservice teachers, I chose my sample to be representative of different years of study ranging from second year, third year and four years: one from fourth year, one from third year and two from second year. The rationale to select teachers from second through fourth year and not first year was that I needed teachers who had been exposed to teaching practice once or twice, and had been exposed to a real classroom context as practicing teachers.

The selected teachers were invited to participate in my research, after I explained that participation was voluntary and could be terminated at any time. I also explained the nature of my research and what

[8] Preservice teachers, student teachers and teachers will be used interchangeably

would be involved. For confidentiality purposes the teachers in the study were assigned pseudonyms: Teacher T, Teacher S, Teacher J, and Teacher M. The participants' attributes have been summarised in the table and below:

Table 8

The participants' attributes

Name	Academic Level	Majors
Teacher T	BEd 2nd year	Maths, Biology
Teacher S	BEd 2nd year	Maths, Physics
Teacher J	BEd 4th year	Maths, Physics
Teacher M	BEd 3rd year	Maths, Biology

It is important to note, none of the teachers were knowledgeable about the lessoning study nor were they familiar with the principles inherent in the MTF tool. Although the MTF principles were shared with the teachers during TE practice, they were not formally introduced to the detailed workings of the MTF. This meant they had to be initiated into the L/S process and the MTF. This was done in the workshop that was conducted at the start of the implementation of the lessoning study as will be discussed in section 4.4.2.

The researcher/ knowledgeable other (KO)

In this study, I took the stance of "participant observer" (Merriam, 1998) who was a complete participant with active membership role in the CoP (Spradley , 1980). As a participant observer the "researcher is fully immersed in the participant role but uses their position to conduct research" (Opie, 2004, p. 129). In undertaking research involving participant observation, the researcher needs to be concerned with balancing the dual roles of being both a CoP participant and a researcher (Spradley, 1980). This involves positioning oneself from both an insider and outsider perspective (Labaree, 2002). An insider position refers to participating in the activities of the community being researched. It involves interaction between the researcher and the practitioners who also engage in research into teaching (Jaworski, 2003). According to Jaworski (2003), outsider researchers are people who work as educators and conduct research with teachers. In this study I was an outsider in the sense that I was a researcher with an interest in exploring teacher learning in L/S underpinned by MTF. It was with this drive in mind that I set out to establish an L/S CoP and on this journey I positioned myself as the insider where I took on additional roles.

The first role was that of a *CoP coordinator*, where I formed an L/S community of practice with four preservice teachers. As a coordinator, I had several responsibilities such as documenting L/S administration, guiding L/S implementation, organising meeting space and times, preparing materials and activities for each L/S session. Within the L/S, I also assumed a second role of *knowledgeable other (KO),* with the function of providing the relevant support in terms of the mathematics where I played the role of a more experienced mathematics teacher. The role of KO also entailed giving support to the application of the theoretical constructs from the MTF that had a bearing on this study. My third role was that of a *facilitator.* Ronda (2013) used the term scaffolding to refer to the processes used to initiate teachers into the process of LS and facilitation provided to support teachers' own construction of knowledge for teaching. According to Ronda, this scaffolding is provided only when needed, and is gradually withdrawn as the teacher gains more skills and knowledge to become an independent learner. Graven (2002) also noted the initial importance of the expert leading the activities within the communities of practice. With this in mind, in this study, facilitation was provided during the workshop and throughout the implementation of the L/S study with slight withdrawal towards the end of the L/S, as will be shown in the analysis of the second reflective discussion in Chapter 8. The fourth role I played was that of a *discussion leader* where I was making suggestions for the focus of discussion, inviting participants' contributions, developing the ideas, ensuring the focus was sustained, concluding and summing up at the appropriate point.

Although I took on these insider roles, it is important to note that I was a relative newcomer to the key constructs that underpinned this study such as L/S, MTF and CoP. Here too, it was through my participation in the planning and implementation of the professional development work of the WMCS project in 2015 that I was exposed to, and gained experience with, some of the constructs associated with MTF and L/S. In addition, I am also not implying that I have extensive knowledge of teaching the function topic or that I possess expert facilitation skills. The additional experience that I brought to the L/S study was that I have been a secondary school mathematics tutor for 9 years and a mathematics educator at university level for six years. This experience included mathematics teaching experience mentor. Furthermore, I have had some previous research exposure in teaching of functions at a secondary level. Although I commenced this study with this background and took on the role of knowledgeable other, I was seen by the teachers participating in the study as the leading authority, possessing all the relevant skills and knowledge. With this, Saadi (2016) warns that such a perception may discourage productive interaction—exploratory talk, as the novice teachers do not consider themselves to be in a position to critically engage with the expert.

Having taken on this role (KO), I made it clear to the teachers that I was also a novice in the process of L/S and I was participating in the study to learn from the process and from them. As we went about implementing the lessoning study, it was a co-learning process (Jaworski, 2003). According to Jaworski (2003), co-learning is a process whereby researchers and teachers jointly engage in the action of conducting research into teaching, and hence develop their own teaching.

These different roles that I played implied that I became an integral part of my own data. On many occasions I found it difficult to separate my researcher's role and CoP participant. Baker (2006) experienced the same tension and advised that, "despite the level of involvement with the study group, the researcher must always remember her/his primary role as a researcher and remain detached enough to collect and analyse data relevant to the problem under investigation" (p. 172). Labaree (2002) described this as the dilemma of insider/outsider positions. He further warned that the insiderness contains hidden dilemmas relating to over-identifying with other participants and conflicting negotiations concerning involvement and detachment from the field.

Positioning myself as a researcher during L/S participation resulted in some tension. For instance, during the initial stages of data analysis I backgrounded my role as the KO, until it was queried by my supervisor. During the data collection process, there were some instances where I reminded myself of my role as a researcher, as noted in one of my journal notes: "I think I was too loud today, I need to let the teachers discuss amongst themselves". This was also evidenced during the lesson planning session LP1_Cycle 1 in lines 29-32 (shown below):

Episode	Speaker	What is said
EP1	M	29. Cause I was a bit concerned about the approach, if you say both of them are steep, the other one can be negative the other one positive.
	J	30. Yes
	T	31. You have not introduced the effects yet?
	J	32. This is part of introducing the effects, it says if the gradient is zero therefore it's that, if it's positive it's steep, but they don't know the directions. 33. I have introduced the positive sign and negative sign. But I have introduced the importance of gradient 'm'; it's for the steepness of the graph. Now, this is where I got confused.

In this extract teacher T (T) line 31, enquired about the effects of the gradient. Line 32, J reflected about the introduction of the lesson by interpreting the features of a gradient in the equation and compares it with the behaviour of the graph by using a mathematical justification. His word use was both informal and formal. J seemed to still associate the word "steepness" with positive gradient. This issue has been raised by M previously (line 29). J's explanation (line 32) as warned by M seemed to also have caused

confusion to J himself, where he admitted to experiencing some tension with the explanation he had chosen. During this interaction I wanted to come in and help J with his misconception; however, I decided not to comment, fearing interfering with data. Such a decision has implications on the nature of opportunities of learning provided to teachers and so could put teachers in a disadvantaged position.

Again, during the initial data analysis, I was annoyed by the heavy presence of my voice; it seemed I was leading the discussion and leading through the teachers. However, from a situated learning perspective which highlights an important role-leading discussant, I was happy to be an active participant. The problem came when I had to switch to a researcher's role.

According to Labaree (2002), the insider position is better characterised as a continual process of introspective enquiry which researchers can use to monitor their position, views and conclusions. He further advised that the researcher needs to be reflexive. Reflexivity is "the constant awareness, assessment, and reassessment by the researcher of the researcher's own contribution / influence / shaping of inter-subjective research and the consequent research findings" (Patnaik, 2013, p. 100). I have taken this advice, and will discuss my reflexivity in the conclusion chapter.

Having said this, my insider positioning was useful during the L/S activities. First, I took a stance of not holding back my participation because that could have intimidated the teachers; I wanted them to feel relaxed and not see me as their lecturer or evaluator. This perspective enabled the development of relationships in an L/S that promoted trust. Second, the role of the KO and/or facilitator that I played in the L/S was useful because it brought a research-designed resource—MTF. I was inserting research knowledge into the practice. This confirmed the field where this study is located, the research-practice gap. However, when I switched into my researcher's role, I started asking myself the following questions which I noted in my field journals:

> If the same person works very closely with the L/S group prior to the L/S cycles, and throughout the cycles, can I really bring a fresh perspective? Moreover, L/S as some kind of a CoP is supposed to be a practice where all participants truly collaborate as equals. What, then, should be an appropriate relationship between knowledgeable other and L/S group members? And what about compromising rigour of the data analysis?

As this study unfolded, during the data analysis these questions became more illuminated and will be reflected upon in the reflexivity section in the conclusion chapter.

The four teachers, together with the researcher, embarked on a journey where teachers were newcomers to the entire concept of L/S study and MTF, and the researcher possessed only some theoretical knowledge about L/S study and MTF, and so was a novice in this regard.

4.4 Procedure

4.4.1 Designing the Lessoning study model

The L/S CoP for this study had a domain, that is, a shared interest, something that the community (four preservice mathematics teachers and the KO/researcher) were keen to learn about the MTF-use in teaching and learning mathematics. As the L/S community, we were going to learn about our domain (MTF) by collectively engaging in activities that were defined by the lessoning study (lesson planning, teaching, reflective discussions). Furthermore, the lessoning study formed the empirical field of research for this study. For Dowling and Brown (2010, p. 147) the empirical field "may be glossed as the broad range of practices and experiences to which the research relates […] the empirical field also constitutes a community or communities". As the empirical field, the lessoning study provided access to data, and so set part of the context of the study.

Drawing from Japanese LS, Chinese LS principles, and Swedish LS principles in this study, I designed a lessoning study model where teachers collectively worked together with a tool through lessoning study cycles. The aim of L/S was not on student learning but on teacher learning where they learn how to use the MTF tool and negotiate its meaning, taking into consideration other factors influencing the use of MTF. However, student learning was considered when discussing issues around what learners need to learn—the object of learning. Secondly, the focus was on reflective discourse during the lesson planning and post-lesson discussions driven by the MTF tool.

4.4.2 The process of implementing the lessoning study

In this section I provide an overview of how I proceeded with the implementation of the lessoning study. This process was divided into two stages: the preparation stage and the actual implementation of the L/S.

Table 9 provides an overview of how the lessoning study model was designed. The table shows the sequencing of events building up to the implementation of the lessoning study cycle (preparation stage) and the sequencing of events that constitute the lessoning study cycle itself (implementation stage).

Table 9

An overview of the lessoning study model

Stage	DATE	L/S	ACTIVITY	VENUE	WHO	DATA COLLECTED	TIME
Preparation	20 Feb 17		Creation of WhatsApp group	WITS	KO/RESEARCHER	n/a	
	2 Mar 17		Workshop	WITS	RESEARCHER AND THE TEACHERS	n/a	
Implementation	9 Mar 17	CYCLE1	LP1	WITS	ALL	Video	104.16
	16 Mar 17		T1	SCHOOL J	TEACHER T	Video	44.47
	17 Mar 17		RD1/LP2	WITS	ALL	Video	56.34
	29 Mar 17	CYCLE2	T2	SCHOOL J	TEACHER S	Video	63.19
	29 Mar 17		RD2	WITS	ALL	Video	62.92

4.4.2.1 Preparation stage

This stage served as the first life cycle of a community (Wenger et al., 2002), where members of the community got to know each other. After teachers had agreed to participate in the study, a WhatsApp group was created for communication purposes. This was followed by the workshop. In this step, the four student teachers were invited to a workshop to study the lessoning study process, together with the MTF tool. The aim of conducting a workshop was to raise awareness of implementing the lessoning study guided by the use of the MTF. Importantly, this meeting was intended to create a conductive environment for the L/S group to enable a smooth take off. The workshop was conducted by myself where I played the role of a facilitator. As indicated earlier the content area of interest for this study was grade 9 linear functions which could be revised in grade 10 function, as in the case of this study. During the workshop, I discussed the function concept as described in the curriculum or textbook and further moved towards discussing a notion of function concept from a mathematics education research perspective (e.g., literature reviewed in function). I also used this opportunity to share with the teachers the learners' difficulties with functions as documented in the National Diagnostic Report of learners' performance: NDR (DoE, 2016), and also shared literature from the previous study I conducted on function concept (Gcasamba, 2014). The motivation for deciding to share this study was that the study was reflecting the South African learners' challenges/difficulties related to a function concept. Taking

Lo's (2012) advice, reviewing research literature about a specific concept is helpful when identifying critical features of the object of learning.

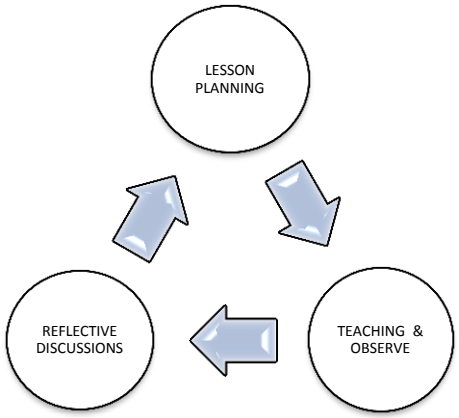
Details of this process (workshop) have been described in Appendix B. After the workshop, the L/S study was implemented by the members of the L/S community: Teacher T, S, J, M and the KO.

4.4.2.2 Implementation stage

During this stage a L/S was implemented with two cycles: cycle 1 and 2. Included in each cycle were the following L/S activities: lesson planning, teaching and post-lesson reflective discussions (see Figure 3). The First cycle included Lesson Planning session (LP1), teaching session (T1) and Reflective Discussion/lesson planning session (RD1/LP2). The first reflective discussion (RD1) included the discussions about the previous enacted lesson and discussions about editing a lesson plan for the next teaching session, that is lesson planning 2 (LP2). The second cycle included a teaching session (T2) and a reflective discussion (RD2). The object of learning in focus during the L/S study was on the effects of y-intercept and gradient of the linear function.

Figure 3

The lessoning study cycle



In the following section I will discuss in detail the events that took place in each L/S session to help the reader understand the structure of the L/S and how it unfolded.

First Lesson Planning (LP1)

At the beginning of this cycle, all L/S group members met for a lesson planning session. This meeting allowed us to select the topic and the object of learning (OoL) for this cycle. It was agreed that the first topic to attempt was on linear functions. Since this was our first cycle, I played a leading role in suggesting the first topic. This decision was influenced by my teaching and research experience about learners' difficulties with linear functions. After agreeing about the choice of topic (the linear functions) we then decided on an object of learning (What is it we wanted the learners to learn about the linear functions?).

At this stage all members shared their experiences of their understanding of the object of learning under investigation. We were guided by the following questions: (1) What difficulties do learners face in their attempt to understand their respective object of learning? (2) What are the critical aspects to consider when teaching the selected object of learning? (3) What means do teachers use to help learners resolve the difficulties they had encountered? To respond to these questions, it is a normal practice in the learning study to identify critical aspects by administering a pre-test which serves as a diagnostic "test" comprising several tasks teachers believe can reveal learners' difficulties and misconceptions. The aim of this test is to confirm and/or identify critical aspects to be taught in the lesson. Based on an analysis of the pre-test, the first lesson in the cycle is then jointly planned with a particular focus on how to teach the critical aspects. In this study, the pre-tests were not conducted because of the availability of learners to write the test and the fact that the head of the mathematics department at School J felt that the learners would be overloaded with writing extra tests. I was not troubled by not conducting a pre-test because I was aware of Lo's (2012) advice that there are different methods of identifying the critical feature of an object of learning which include:

> a literature review, sharing past experiences of teaching a particular topic with other teachers, interviewing students, administering a carefully designed diagnostic pre-test and post-test and analysing the results of students' responses, and observing lessons taught by another teacher and listening carefully to students during the lesson to find out how they see the object of learning. (Lo, 2012, pp. 71-72)

Following Lo's (2012) advice, in this study the critical aspects were identified by conducting the content analysis, sharing teachers' experiences and research analysis as follows:

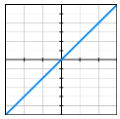
- Discussing relevant literature that indicates some common misconceptions or difficulties

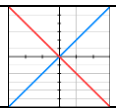
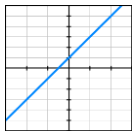
- Members sharing experiences in understanding the selected object of learning
- Members conjecturing critical aspects (learners' common errors or difficulties and misconceptions)
- Discussing content analysis of the chosen topic, for example, linear function

Following the process of identifying the critical aspects of OoL, the lesson plan was designed to show the introduction, implementation and conclusion. The lesson plan also reflected the ideas of the MTF such as OoL, set of examples, tasks, explanations and learner participation. The table below illustrates the designed lesson plan:

Table 10

The designed MTF lesson plan

<table>
<tr><td></td><td colspan="2">OoL: EFFECTS OF THE GRADIENT AND Y INTERCEPT IN THE LINEAR FUNCTION</td></tr>
<tr><td rowspan="3">Introduction</td><td>EXAMPLES AND TASKS</td><td>EXPLANATION AND LEARNER PARTICIPATION</td></tr>
<tr><td>Public announcement on the board, write $y = 2x + 1$</td><td>Ask learners to identify/classify the coefficient of x i.e. 2 and the constant i.e. 1</td></tr>
<tr><td>Write on the board the following examples/tasks:
• $y = \frac{x}{2} + 1$
• $\frac{y}{2} = 3 - x$
• $5x = 6 + 3y$
• $y = \frac{3}{x} + 2$</td><td>Ask learners to identify the gradient and the y-intercept</td></tr>
<tr><td></td><td>Generalise the standard form of the linear function, i.e. $y = mx + c$</td><td></td></tr>
<tr><td></td><td></td><td></td></tr>
<tr><td rowspan="3">Implementation</td><td><u>Write on the board:</u>
The lesson is about the effect of the sign of m and the value of c in $y = mx + c$</td><td></td></tr>
<tr><td><u>Examples:</u>
1. $y = 2x$
<table><tr><td>x</td><td>-3</td><td>-2</td><td>-1</td><td>0</td><td>1</td><td>2</td><td>3</td></tr><tr><td>y</td><td>-6</td><td>-4</td><td>-2</td><td>0</td><td>2</td><td>4</td><td>6</td></tr></table></td><td><u>Draw a table: [-3;3]</u>
Ask the learners to identify the y-intercept from the table
Ask the learners to analyse the behaviour of the range
<u>Draw a graph</u>
Ask the learners to identify the y-intercept from the graph
Ask the learners to analyse the slope of the graph (i.e. the behaviour and the direction)
<u>Do a comparison between</u>
the table and the graph, focusing on the gradient and the y-intercept</td></tr>
<tr><td>2. $y = -2x$
<table><tr><td>x</td><td>-3</td><td>-2</td><td>-1</td><td>0</td><td>1</td><td>2</td><td>3</td></tr><tr><td>y</td><td>6</td><td>4</td><td>2</td><td>0</td><td>-2</td><td>-4</td><td>-6</td></tr></table></td><td><u>Draw a table, domain [-3;3]</u>
Ask the learners to identify the y-intercept from the table
Ask the learners to analyse the behaviour of the range
<u>Draw a graph (on the same set of axes</u>)
Ask the learners to identify the y-intercept from the graph
Ask the learners to analyse the slope of the graph (i.e. the behaviour and the direction)</td></tr>
</table>

		**<u>Do a comparison between</u>** the table and the graph focusing on the gradient
	3. $y = 2x + 1$ x | -3 | -2 | -1 | 0 | 1 | 2 | 3 y | -5 | -3 | -1 | 1 | 3 | 5 | 7 	**<u>Draw a table-[-3;3]</u>** Ask the learners to identify the y-intercept from the table Ask the learners to analyse the behaviour of the range **<u>Draw a graph</u>** Ask the learners to identify the y-intercept from the graph Ask the learners to analyse the slope of the graph (i.e. the behaviour and the direction) **<u>Do a comparison between</u>** the table and the graph focusing on the y-intercept the table for $y = 2x + 1$ and $y = 2x$ the graph for $y = 2x + 1$ and $y = 2x$
Conclusion	Give learners the following task: Analyse the effects of gradient and the y-intercept in the following function: $y = -2x - 1$	Put learners in groups and discuss their answers in a group discussion

During the introduction, examples were chosen with the aim of triggering learners' previous knowledge about the object of learning. This was done to help to collect learners' mathematical discourse (words, word use, routines and endorsed narratives) and learners' difficulties, errors and misconceptions. The hope was that this knowledge would help the teacher during teaching and other members to collect data to be discussed during reflective discussions and re-planning of the next re-teaching session.

In cycle 1, the members dealt with linear function and object of learning: effects of m and c in $y = mx + c$. Four examples were designed to explore the learners' experiences with the OoL. The examples are reflected in the introduction row as shown in Table 10. These examples were intended to determine whether learners could identify the given linear function and further interpret its features such as gradient and y-intercept. What words were they using? How were they used (in everyday or mathematical language)? What were the definitions? Basically, what was their mathematical discourse?

Following the introduction was the implementation of the OoL (i.e., effects of m & c). During the implementation the teacher would announce the OoL, followed by the set of examples in different representations (e.g. symbolic, tabular, and graphical). Different mathematical routines related to linear functions were taken into account i.e. drawing, classifying and interpreting. It was agreed that tasks should be done through questioning throughout the lesson from introduction right through to the conclusion.

The final step, that is, the conclusion, was meant for classwork where learners would be given a task to work on as a group. As a way of concluding this session a teacher (teacher T) was selected to teach the designed lesson plan and other logistics discussed and the meeting was adjourned. How this lesson plan emerged through teacher collaborations will be discussed in Chapter 6. It is necessary to note that the analysis of this lesson planning session is rather lengthy, which is expected since the transcription was 104.16 minutes (Table 9).

First Teaching (T1)

This step was a very critical stage in an L/S cycle. During this step the selected teacher taught the lesson guided by the lesson plan that had been collectively planned by all members. The other members observed the lesson and noted their observations on the lesson observation schedules. These schedules were aimed at providing guidance on what to look for while observing. The guiding questions that were included in the observation schedule were:

- Did the teacher follow the lesson planned concerning the intended OoL?
- What difficulties were encountered?
- How would you amend the examples/tasks, explanations in relation to the OoL in order to promote learning in the next lesson?
- Any other opinion?

All participants' observational notes were used to facilitate discussions during post-lesson discussion sessions.

Description of the enactment of a first teaching lesson

At the end of the lesson planning meeting, T was chosen to be the person enacting the lesson, lesson 1. This lesson was scheduled to be taught at school J on Thursday, 16 March 2017 at 2 pm. The lesson was conducted to a grade 10 class by T, the rest of the team members were present during the lesson and played a role of being the observers. The duration of the lesson was 30 minutes. At the start of the lesson, T announced that the lesson was on linear function, and the aim of that lesson was to interpret the effects of the gradient and y-intercept. He wrote on the board his first example, $y = 2x + 1$. The summary of the lesson enactment by T (first teaching session) is provided in Appendix E (Table T1).

First reflective discussion and second re-planning (RD1/LP2)

The post-lesson discussion was designed to give teachers an opportunity to engage in systematic reflection, in order "to become aware of what worked and what did not in a teaching situation and to find strategies in order to attract the students' attention towards the object of learning" (Olteanu, 2016, p. 63). After the teaching session, all members came together to conduct a post-lesson discussion to reflect on how the teacher who conducted the lesson handled the OoL. Using our lesson observation notes and comments, we shared our views and suggestions on how to further improve the next re-teaching session. The original lesson plan was then modified to take into account the suggested changes. It needs to be highlighted that the lesson plan was similar to the first lesson plan in terms of structure (introduction, implementation, conclusions) and MTF components (examples, tasks, OoL, explanations, learner participation), instead the changes were discussed during the reflective discussions. These included discussions about focusing on word use, and teaching strategies that seek to promote learner engagement. At the end of this meeting, another teacher (teacher S) was selected for the re-teaching session.

Re-teaching and final post-lesson discussion

The re-teaching and final post-lesson discussion followed the same form as the first teaching and post-lesson discussion, but focused on the revised lesson plan. The end of the post-lesson discussion meeting marked the end of the L/S.

Description of the enactment of a second teaching lesson

At the end of the second lesson planning meeting, teacher S was chosen to be the person enacting the second lesson—lesson 2. This lesson was scheduled to be taught at school J on Wednesday, 29 March 2017 at 2 pm. The lesson was conducted by S with a different grade 10 class from the previous first teaching session; the rest of the team members were present during the lesson and played the role of observers. The duration of the lesson was 40 minutes. S announced that the lesson was on linear function, and the aim of that lesson was to interpret the effects of the gradient and y-intercept. He wrote on the board his first example, $y = mx + c$. The summary of the lesson enactment by S (second teaching session) is provided in Appendix E (Table T2).

4.4.2.3 Data selected for analysis

It should be noted that the data that was used for this study came from the lesson planning and post-reflective discussion sessions. The teaching session did not form part of the data analysis.

In Chapter 5, I provide a more detailed description of my data analysis process where I embark on a process of organising and interrogating data in such a way that "allow researchers to see patterns, identify themes, discover relationships, develop explanations, make interpretations, mount critiques, or generate theories '' (Hatch, 2002, p. 148).

4.4.3 Data collection methods

Qualitative research methods include interviews, observations, video recordings and field notes. According to Lave and Wenger (1991), the data collected informed by a theory of social practice should include what teachers say about practice and do within practice. Such data can be accessed through sources of data that describe meanings in an individual life, such as varieties of empirical materials, personal experiences, life stories, interviews, case studies, artifacts, cultural artifacts as well as observational texts (Denzin & Lincoln, 1994).

For me to be able to collect valuable data, which would enable the provision of answers to the research questions for my study as outlined in Section 1.3.5, the observation data collection approach was used. Observation is a data gathering method used to understand a specific phenomenon by watching people's accounts and actions in their natural settings (Creswell, 2012). The observational data was captured by videos and my personal field notes (journal). The field notes included a description of what I found to be interesting such as emerging hunches, questions that arose, and overall personal reactions to the meetings. These were not transcribed but were used for my own reflections about the overall lessoning study process.

Since lessoning study formed the heart of this study, the collection of the observation data was primarily focused on the use of a resource for collaborations (the MTF) when engaging in collaborative planning of lessons, the teaching of the collaboratively planned lessons and the collective post-lesson reflective discussions. During the lesson planning sessions, teachers produced the MTF lesson plan which was used as a structured protocol guiding the enactment of the lesson.

The focus of the data was on collective discussions done by the teachers during L/S activities. The purpose of the observations which were captured in the video recordings was to go inside the work of

L/S to collect data that would provide insights into the nature of the talk of teachers while engaging with MTF-use and the nature of opportunities for learning that were presented when teachers engaged in L/S.

As explained earlier, in this study, I took the stance of "observer as participant" (Merriam, 1998) in that I recorded videos and compiled notes of the observational accounts in my field work journal, yet I established myself as both a mathematics educator with experience of teaching secondary mathematics, teaching with MTF in order to support the learning environment and an active participant in the L/S activities, like lesson planning, observing a lesson and reflective discussions.

I acknowledge that it is the normal practice to collect data in the form of learners' performance in pre-test and post-test in a learning study methodology. However, these were not collected in this study due to administrative challenges at the schools. The pre-test is central to the identification of the critical features for the intended object of learning (Runesson, 2013). The omission of this created some limitations during lesson planning. How the lessoning study group dealt with this limitation will be illuminated during the data analysis of the first lesson planning session. The post-test is central to describing the lived object of learning and since it was not a focus of this study to investigate how learners experience OoL, such data was not of interest for the research.

Video recording of L/S activities

The video recording formed the main source of data collection, as indicated in the preceding section. All the L/S sessions were video recorded and transcribed. In total, there were five L/S activities. In each case, the video recording of the activities was done by making use of a digital video recorder that was mounted on a tripod. I made use of only one video recorder, which I operated. During classroom teaching, the video recorder was positioned at the back of the classroom so as not to obscure any learners' vision of the chalkboard or the teacher, as the teacher moved around the classroom during the lesson. The video recorder was focused on the teacher because the pedagogic practice in relation to the fundamental aspects of MTF was the object of study. During the time when the teacher got the learners to work on their own or in a group, the camera was focused on the teacher as he worked with the individual learner or group. It was not my goal to capture the interaction between teacher and/or other teacher-observers; the focus was on the teacher as he was conducting a lesson and when working with the learners. The video recordings were further collected during lesson planning and post-lesson discussion to enable me to revisit the participants' interactions.

I chose to video record the L/S activities because it provided me with a more permanent record of the activities as I could replay it as many times as was necessary and this allowed me to focus on events more closely. The video recording of the L/S activities also provided me with the ability to capture the verbal and non-verbal responses, thus capturing a more complete record of each activity than would have been possible if I had relied solely on using an observation schedule or a mere audio recording of the lesson.

A total of five videoed data was collected as depicted in Table 9. However, only three were transcribed (excluding teaching session data) as will be discussed in detail in the following chapter.

4.5 Considerations concerning rigour

Rigour in research deals with "presenting insights and conclusions that ring true to readers, educators and other researchers" (Merriam, 1998, p. 199). The phenomenon of rigour has often been explained as an element of trustworthiness (Lincoln & Guba, 1985; Merriam, 1998; Opie, 2004). In qualitative research, the researcher is the main instrument (Patton, 1990), hence the credibility of the research depends on the ability and efforts of the researcher (Golafshani, 2003). Rigour in my study was supported through a number of strategies that enhanced the validity and reliability of the findings. I have also included some discussions in section 4.5.2 on how rigour can be maximised from the L/S perspective.

4.5.1 Dependability

In quantitative research, reliability refers to the consistency of research processes in producing similar results on all occasions (Merriam, 1998). Applying such a conception to qualitative studies is problematic since qualitative research is not easily replicable. With reference to qualitative research, Lincoln and Guba (1985) used the term "dependability" instead of reliability. According to Merriam (1998), dependability refers to the extent to which research findings can be replicated (p. 205) with similar subjects in a similar context. It emphasises the importance of the researcher accounting for or describing the changing contexts and circumstances that are fundamental to the consistency of the research outcome.

To ensure information dependability I have provided readers with the opportunity to assess the validity of my interpretation of data, I have used direct quotes. Verbatim accounts or direct quotations enhance data corroboration and helps the study to provide a rich thick description of events and to illustrate

participant's experiences and meaning (McMillan & Schumacher, 2006) within L/S. This resonates with the qualitative educational interpretive methodology.

During the data collection process dependability was enhanced by the collection of similar data at different points during the study. Collecting data about the teachers' discussions with respect to MTF during the lesson planning session, teaching session and reflective discussion session enabled the identification of recurring and non-recurring practices and responses, which enhanced the reliability of claims about use of the MTF tool. The consistency and recurrence of responses minimised chances of oversimplification or undersimplification in reporting the data (Lewis et al., 2009).

Dependability in research findings is also enhanced through consistency and accuracy in data interpretation. Analysing the teachers' use of MTF involved coding data using the analytical tools mentioned earlier in section 4.1. I then developed a coding framework. In developing the framework, I held a number of meetings with my supervisor and this resulted in a number of changes to the coding framework. The process of establishing inter-rater reliability with fellow doctoral students involved two coding cycles. The first cycle involved discussion of the framework, coding some data, and then discussing the coding results and the framework again. In the second cycle, each coder coded two transcripts of teacher discussions. Where there were differences we looked again at the video, the transcript and the code identification rules until an agreement was reached. I will elaborate more on the codes and the coding principles in the next chapter.

Maxwell (1992) identified three types of validity in qualitative research: descriptive validity, interpretative validity and theoretical validity.

Descriptive validity is about the "factual accuracy" (Maxwell, 1992, p. 285) of the account, that is, the data must reflect what was said or done in the social event. According to Maxwell (1992), the reporting of the data must also reflect the same accuracy, which means that the transcription is an accurate account of what was said or the transcription of the videotapes portrays the unfolding of events in an accurate manner. In Section 4.4.3 it was noted that the video recordings in this study are a re-presentation of the actual social actions in which the teachers were reflecting on the use of the MTF tool. Thus in this study primary descriptive validity relates to the accuracy of the written transcripts (and where necessary the translations) as re-presentations of these video recordings. Consider for example [in line 91, episode 4, LP1: "Yes they do" (looking in CAPS doc)]. I have addressed this first form of *descriptive validity* in this study by successively revisiting the relationship between the written

transcripts and the video recordings; this took place both during the transcription itself and during the analysis when the video footage was revisited for clarity. Maxwell (1992) noted, however, that no descriptive account can include everything and that accuracy may be relative to the purposes of the study. So when re-presenting teacher T's utterance [in line 75, RD1: "Let's not get deep into it"]. I may have omitted a tone or gesture that was relevant to the study (that is a dismissive and defensive interaction related to disputational talk). I addressed this aspect of descriptive validity in this study by doing the initial transcription of two out of the five video recordings personally, in interaction with the initial development of appropriate analytic tools for the study. As I developed the tools, I was able to decide what detail was required in the transcripts and develop a consistent transcription notation: [Who? What is said? What is done?]. This attention to the relationship between the transcripts and the video recordings in the interests of descriptive validity goes some way to addressing reliability issues, that is, whether I applied my analytic tools consistently across transcription. Silverman (2010) has suggested that careful transcription of recordings in a theoretically informed way can contribute to such consistency.

According to Maxwell (1992, p. 290) *interpretive validity* captures "how well the researcher reports the participants' meaning of events, objects and/or behaviours". The key here is that the interpretations are not based on the researcher's perspective but that of the participants: "Interpretive validity is inherently a matter of inference from the words and actions of participants in the situations studied' (Maxwell, 1992, p. 49). Consider again teacher T's utterance [in line 75, RD1: "Let's not get deep into it"]. With the help of my journal notes, I was able to capture the gesture and tone of his voice which enabled me to interpret his utterance in relation to the social relations aspect.

Theoretical validity, sometimes referred to as construct validity, refers to the validity of the concepts as they apply to the phenomenon under study, and the validity of the relationships between these concepts (Maxwell, 1992). Hence, theoretical validity refers to the extent to which the operational definitions of variables reflect the actual theoretical meanings of the concepts. In Chapter 5 I have developed a methodological approach for understanding and interpreting my data, taking particular care to define the terms used. The framework used my theoretical underpinnings—Wenger's Community of Practice theory—as the core backbone to understand the L/S CoP of mathematics pre-service teachers, complemented with sociocultural framing from Sfard (2008) and Mercer's (1996). In doing this, I developed codes, defined them, and provided examples of the codes with regard to how they were used in my study.

4.6 Ethical Considerations

Ethical issues in research concern beliefs about what is wrong and what is right from a moral perspective when conducting research (McMillan & Schumacher, 2006). Research ethics therefore imply compliance with acceptable research norms, morals, standards and principles. I had to satisfy the university's Human Ethics Research Committee (Non-Medical) that I had done everything possible to ensure proper informed consent. The informed consent was sought and granted by the principals of the schools, the parents on behalf of the learners involved in the project, the preservice teachers and the Head of the School of Education. Herein the right to privacy of the learners was assured. Also, the participants were guaranteed protection from harm, and I took the necessary precautions to guard against treating participants as objects.

In my role as a researcher, I designed the intervention and then approached the principal, parents and learners in grades I was going to be working with, preservice teachers and the head of Wits School of Education for consent regarding the intervention identified in my study. To conform and comply with the university's research ethical codes, guidelines, protocols and practices, this research proposal was forwarded for approval to the Ethics and the Higher Degrees Committees. The application was approved (protocol number 2016ECE013D see Appendix A).

In line with the principle of "informed consent", I explained honestly and openly to all participants the nature, aims, purpose and educational benefits of my study and further elaborated to participants that participation in this research was voluntary and that the participants could withdraw from the study at any time. The preservice teacher participants and the learners (parents) signed the Participant Information and Informed Consent, the Interview Consent and the Recording Consent forms. I secured prior voluntary consent before video recording the participants.

The study was based on the principle of voluntary participation. This protects people from being coerced into participating in research. The preservice teachers were reassured that they could withdraw their permission at any time during this project without any penalty. They were further reassured that their marks would not depend on participating in the research. Furthermore, they were informed that they had the right to take any concerns about research participation to the Chair of Wits School of Education Ethics Committee.

In this study I also needed to comply with ethical issues of confidentiality, anonymity and privacy. To ensure confidentiality, names of the participants are not to be disclosed; the names of the schools, teachers, learners, principals (heads) remain anonymous and do not appear in the thesis. Instead, fictitious names were used throughout the study. The data gathered in this exercise was solely and strictly used for the purpose of this research project. During and after completion of the study I appropriately stored the raw research under lock and key. Such confidentiality initiatives and data storage measures were all in the interest of ensuring and protecting the privacy and anonymity of participants. McMillan and Schumacher (2006) and Neuman (2006) agree that guaranteeing privacy, anonymity and confidentiality means that access to participants' responses, behaviour and information is restricted to the researcher and kept secret from the public. I have made all the necessary efforts and commitment to ensure and uphold both the informants' privacy and research ethics principles during the fieldwork and in the compilation of my thesis.

4.7 Conclusion

In this chapter, I have provided a description of my empirical field, and given reasons for the choices I made in the population and sample of my study. A qualitative approach was adopted in order to address the guiding questions. This chapter also explored issues of ethics, validity and reliability. In the next chapter, I provide a description of the methodology used to analyse the data.

Chapter 5: Methods of Analysis and Presentation of Data

5.1 Introduction

In this chapter I explain the methods of analysis I adopted and their rationale based on existing research and the research questions on MTF-use that my analysis aimed to address, as outlined in Chapters 2 and 3. I therefore discuss the theoretical influences on my choice of methods and research questions in framing my analytic approach. In order to exemplify how the analytical framework was developed, I draw extracts from various episodes coming from the first cycle of the L/S—the lesson planning. Following this I introduce the methods I adopted, before offering more detail on the methods themselves, how I applied them to my data and how they were appropriate in my investigation of the research questions.

5.2 Background to my analysis methods

In approaching my data, I used sociocultural discourse analysis (SCDA) and framework analysis (FA) as complementary methods to address the use of talk and discursive actions to understand the use of the MTF tool over a series of L/S sessions. The SCDA is a methodological tool to examine the quality of discourse between participants working together in a joint activity (Mercer, 2004). It uses both qualitative and quantitative methods of data analysis to describe and assess the quality of social interaction, (e.g.) in a classroom or a lessoning study. Framework analysis complements the SCDA by offering a systematic quantitative method of reduction of large amounts of data. In this and the following two sections I offer the theoretical background of these methods, and how I applied them in my research.

5.3 The choice of sociocultural discourse analysis

My interest in this study was exploring the reificative and participative aspects of the L/S repertoire which are partly linguistic or entirely linguistic such as, "routines, words, tools, ways of doing things, stories, gestures, symbols, genres, actions, or concepts" (Wenger, 1998, p. 83). The elements of the L/S repertoire included teachers' talk and discursive actions. According to Tusting (2005), the participants of the community reify the elements of the repertoire through linguistic communication. The discourse analysis is therefore a methodology that allows one to analyse language, both spoken and written (Mercer, 2010).

As discussed, the sociocultural discourse analysis was chosen for this study. There are of course a number of methods to analyse talk, including conversational analysis, critical discourse analysis etc. But why, in particular, was SCDA necessary in this study?

According to Wu (2010, p. 130), SCDA,

> regards discourse as interactional activities and emphasizes the social function of language. This method not only analyses word and sentence expression form and meaning, but also analyzes all kinds of social cultural factors related to discourse.

The teachers' reflective talk and actions are discursive in nature and they provide clues about the "nature and function of language, thinking and social interaction" (Mercer, 2004, p. 138). Through the SCDA approach, I was able to examine the nature and function of the talk and take into consideration all interacting sociocultural factors influencing the process of negotiation of meaning of the MTF tool.

SCDA adopts a view of language as "a social mode of thinking—a tool for teaching-and-learning, constructing knowledge, creating joint understanding and tackling problems collaboratively" (Mercer, 2004, p. 137). This idea runs parallel with the aims of this study of analysing teachers' collaborative engagements. SCDA has frequently been used by researchers addressing the use of computers in education, whereby "both the computer-mediated communication system and the ideas from other participants that influence an individual's thinking can be regarded as mediating tools" (Mercer et al., 2004, p. 203). This is certainly relevant with regard to this study, where I focused on MTF—mediated communication instead of computer—mediated communication and consideration of participants' influences, particularly the KO. It can be concluded therefore that the SCDA "allows the information of each participant to be retained and for interactions between individual members to be noted as part of the recording of the group dynamic" (Ritchie & Lewis, 2003, p. 258).

Another important principle of SCDA is the dialectical relationship between discourse and context. According to Mercer (2004), the interaction is located within a particular social context. It is therefore critical to take into consideration the context in which the discourse is produced. The context could include social aspects of the participants, that is, social relations and social aspects of the practice, to name a few.

In this study, the SCDA was used to describe the notion of MTF-use in the actual context of L/S as it occurs spontaneously during interactions. This approach is based on the collection, description and analysis of data mainly collected from observations. The aim of this approach was to identify and

describe sequential patterns of the discourse produced by the participants during their interactions. From this perspective, the recorded data needed to be transcribed and analysed from the participants' perspective. I will engage with this issue in the coming sections.

The SCDA is a mixed method of both qualitative and quantitative analysis. The preceding discussion was related to qualitative analysis. Mercer (2010) outlined the use of concordance software such as Wordsmith Tools for the quantitative element within SCDA to identify themes or codes which occur particularly frequently and reveal which themes/codes tend to occur together. This view is useful in handling a large amount of data. However, Mercer warned about the risk of losing the actual data when using this method. To address this concern, I employed a framework analysis method which seemed to hold the promise of retaining and staying close to the original data.

5.4 The choice of framework analysis

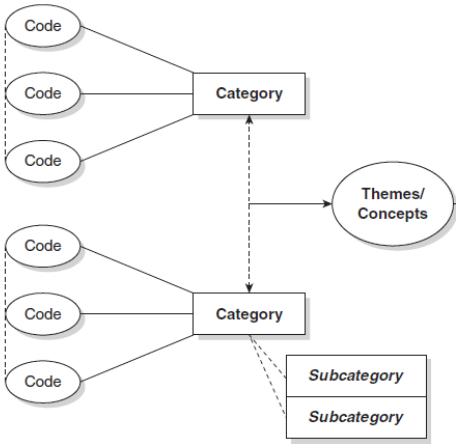
5.4.1 Overview

An adapted version of the framework method of analysis (FA) developed by Ritchie and Spencer (1994) was also used for the analysis of the L/S data. FA provides transparency, facilitating a clearly defined process that can be replicated. The method enabled me to systematically develop key themes which described and mapped the data. FA was selected because it facilitated a combined approach to deductive and inductive data analysis, allowing existing and specific issues to be explored, but also provided an opportunity for new experiences and interpretations to emerge (Gale et al., 2013). This was essential, given that the aim of this study was to describe and then explain the findings of the qualitative observations (L/S reflective actions). This method is also orientated towards the management of large data sets, and promotes "within-case and between-case analysis" (Srivastava & Thomson, 2009, p. 77). This meant that the participants' reflections on a particular L/S session could be compared within the same L/S.

A key part of FA is the development and refinement of the analytical framework used to code the data, which is discussed in the following sections. I tested the initial framework by applying it to the data. Following this activity and discussion about the framework, it was refined to form the final analytical framework to use in the subsequent steps of data analysis, which led to the development of key themes. Essentially, data analysis led me to move from codes to categories to themes as depicted in Figure 4.

Figure 4

Example of developing themes



Note. Extracted from Saldaña (2009)

The framework method involved undertaking six key stages of qualitative data analysis:

- Familiarisation
- Developing a working framework
- Development of codes and categories
- Applying the codes to the transcript
- Charting data into the framework matrix
- Interpreting the data

In the following sections I discuss each of these stages.

5.4.2 Familiarisation

Familiarisation was the first step of the analysis, and required me to immerse myself in the data that was gathered from the L/S (Richie & Spencer, 1994). Immersion and familiarisation began with the implementation of the L/S. Further familiarisation occurred from repeated listening to audio recordings of the observations in the process of transcription, and repeated reading of field notes.

5.4.2.1 Transcription level

In my research I used video recordings of L/S observations of the different sessions of reflective actions in the L/S. Hence my data analysis was focused on what was actually seen and heard from the participants. Maxwell (1992) located the main threat to valid description of what was seen and heard to

inaccuracy or incompleteness of the data. However, these, he said, are solved by audio or video recording the observations and interviews, and following this with verbatim transcriptions. According to Setati (2003), transcripts are a re-presentation of an interaction in a different form.

Once the video recordings had been made, the recordings were transcribed into written text re-presenting what the participants said and what actions ensued (e.g., looking at the textbook, writing on board, and gestures) in order to present a full and candid picture of what occurred. I described these actions, particularly the gestures, in brackets. The transcriptions of the video data were compared against the videos in order to increase reliability of auditory interpretations of data.

I initially transcribed the first two video data coming from the first lesson planning and reflective discussion sessions of the L/S. In interaction with my initial analysis of the transcripts I was able to determine how much detail was required and select a transcription notation that was appropriate for the study. This interactive process also allowed me to check the initial transcription a number of times.

Once I was satisfied that the transcripts were completed in enough detail and that the application of the analytic tools in the analysis of these transcripts was productive, I requested help from a professional transcriber to produce the first draft of the transcripts for the remaining video recordings. This transcriber was recommended to me by colleagues who were also analysing video recordings of mathematics classroom interaction. I then produced a second draft of each transcript based on these first drafts and my review of the recordings. At times during the analysis it was necessary to refer back to the video recordings, with the result that the transcripts were refined even further during the analysis process.

Sfard (2008) emphasised that it is very important that during the transcription, participants' utterances are reported as uttered by the participants. She described this as a principle of verbal fidelity, which minimises the loss of meaning. For example, there were a few times the teachers used their home language—isiZulu. As the transcriber was an English speaker, I reviewed the transcripts and was able to translate to English those aspects of discussion that were in teachers' home language since I am conversant in most of the teachers' home languages. Once the recordings were transcribed into written text re-presenting what the participants said, I chunked the transcripts into episodes of utterances.

5.4.2.2 Chunking transcripts into episodes of utterances

The utterances (which were chunked into episodes) formed a basic unit of analysis, because it is from the utterances that one can interpret the focus of the conversation. According to Sfard (2000, p. 302), "one of the basic properties of our utterances and thoughts is their aboutness, being about something".

The chunking of episodes was done by following Sfard's (2002) notion of discursive focus which enables the interpretation of the focus of the conversation with respect to both private and public aspects of the communication. Based on this assumption, Sfard (2002) distinguished three components of focus involved in grasping the object of attention discussed in the discourse: pronounced focus (what is explicitly named), attended focus (made visible by procedures and visual mediators), and intended focus (non-visible, private aspects of the conversation). These three foci allow for treatment of both private and public aspects of communication.

For the process of chunking, what I looked at was pronounced and attended focus (the public aspects of the communication). For Sfard (2002), the attended focus is the most important construct used to identify the focus of the conversation as it is linked to both the pronounced and intended foci. She gives two scenarios. First, in the mathematical talk the pronounced words often evoke the attending procedure. It is thus possible to decide on the focus of the conversation by linking the two. Secondly, the attended focus is linked to the intended focus. It is "the public exponent of inherently private intended focus and can also count as the operative version of the intended focus" (Sfard, 2002, p. 327). Based on this assumption, it then becomes the interpretive work to reconstruct the attended focus from the intended focus in order to decide what should count as a focus of a given utterance in the conversation.

In the discussion that follows I show how the three foci are interrelated and how both the pronounced and intended foci converged towards the attended focus to allow the identification of the discursive focus which resulted in dividing the transcript into episodes. Before beginning that discussion, I first exemplify by selecting an extract of utterances comprised of Episode 5 and the beginning of Episode 6 of the transcript, as shown in Table 11 below**.** This re-presents the interactions of the participants during the first lesson planning session of the lessoning study. The interaction is between three participants: KO (the knowledgeable other and the researcher), teacher J and teacher T (preservice teachers).

Table 11

Extract: Episode 5 and start of Episode 6

Episodes		Speaker	What is said
EP5	**B**	KO	104. So, in your introduction, we know they know **linear function**. Your introduction should speak to the **object of learning**. You *publicise* $y = mx + c$. 105. Ask them *to identify* this **function**, then quickly guide them towards **object of learning.** 106. *Classification*, when you write that equation and *ask them* what is this type of **function**, and then you are speaking to *classification.* 107. What are the problems? Some might not be able to *classify* **it.** 108. So already now when you are speaking to *classification*, we are now coming to **example space**. 109. We know that some of the problems have something to do with *calculations*, so it means some of the **examples** must focus on *calculation.* 110. So **our example space** is guided by errors, content analysis and also our previous knowledge. 111. So what we are saying is that we need **examples** where they are *calculating*, and we also want them to *calculate* maybe **gradient or intercept**, these could be **tasks.** 112. Other **examples** could be dealing with *translation,* i.e. moving from graph to table to *equation*. 113. In our planning we need **examples** for *classification.*
		J	114. Also include **counter examples** like $y = 3/x + 2$, and ask*: Is it* a **linear graph?** *Why are you saying that*?
		KO	115. We are now working with **learner engagement**. We also need to vary our **examples**.
		KO	116. The **examples**, they are based on *classification*. Do you understand? So every time you are coming with these kinds of **examples**, you've got **counterexamples**, **examples** in general form, 117. and then you say: now we have got the general formula in this form $y = mx + c$. By doing this, you are *confirming* that this is indeed a **linear function.**
		J	118. You can also use the graph. You can *draw* different *graphs* and *ask* them to *identify* them. 119. Because comparing to a standard form will not always work, like $y = 1$ does not look like $y = mx + c$. What interest me, we know that if $y = 1$ we have a line cutting $\boldsymbol{y}$**-axis** at one,it is a **linear function** but once we have $x = 1$,we have a line cutting at 1 on the $\boldsymbol{x}$**-axis**, that is not a **function**. (*demonstrating with hands the vertical line test-showing horizontal line when uttering 'y=1' and vertical line when uttering 'x =1'*
		T	120. So it can be part of your **examples**
EP6	**B**	KO	121. So the introduction is used to strengthen our **critical features**. 122. We need introduction, in your introduction you will announce on the board $y = 2x + 1$. *What is this?* You want to find out from them. 123. That means one of your **critical aspects** is *classification*.

Transcription key

S = teacher; M = teacher; T = teacher; J = teacher; KO = knowledgeable other
Italics is used for routines
Underlined evidences visual mediators
Bold is used for identification of the objects in focus
B marks the beginning of the episode
The gestures are marked by the use of brackets
I use the word lines to refer to utterances

During the transcription process each speech turn was given a line number (these refer to the utterance), for example, in the line numbered 114, J's comment represented one speech turn. This was followed by another speech turn, that is, KO's comment which was numbered as line 115. However, some of the lengthier turns that represented more than one speech function (e.g., a statement followed by an explanation) were also divided into lines numbers for example, KO's speech turns (104-113).

Following Sfard's (2002) description of discursive focus, as described above, in the case of this study the pronounced focus referred to the naming of mathematical and/or MTF objects and the attended focus involved different attending discursive routines. The emerging discursive routines were categorised into six different themes, namely:

- Mathematical routines (including visual mediators)
- MTF routines
- Pedagogical routines
- L/S pedagogical routines
- Lesson process routines
- Contextual routines

According to Sfard (2002), in mathematical talk the pronounced words often evoke the attending procedure. It is thus possible to decide on the focus of the conversation by linking the two. In this study, in most of the episodes, the focus of the conversation expressed itself in having a good sense of what mathematical and MTF objects were being named and what was being attended to in terms of routines used. For example, from lines 104-113, in the above extracts (Table 11), KO was talking about two categories of objects (**as bolded above**) such as the MTF objects and the mathematical object of a linear function and attended to different routines (*in italics above*) both mathematical routines (e.g., identifying, classifying, calculating and different representations such as graphs and equations) and pedagogical routines (e.g., identifying critical aspects of OoL). The intention of these utterances emphasised the development of the example space during the introduction stage of the lesson. While there were other MTF objects named, the discursive focus on examples was maintained from line 114 up to line 120 of the transcript. This constituted one episode. In line 121, the KO talked about the notion of critical aspects (**MTF object**) together with attended routines in relation to the pronounced MTF object. This and the discussion that followed represented a change in discursive focus (from examples to critical aspects) and hence a new episode.

Sfard (2002) contended that the interpretation of the discursive focus does not necessarily rely on the relationship between the two (pronounced and attended foci), it may rely on one. For example, in the absence of the pronounced words related to the objects of the conversation, it then becomes the researcher's interpretive efforts by looking for attending procedures to determine the focus of the conversation. In this instance, the researcher may analyse the speakers' direct statements and then consider the attending procedures if she finds them making sense in order to interpret the discursive focus. I illustrate this scenario in the following extract of utterances from Episode 2 in Table 12 below:

Table 12

Extract: Episode 2

Episodes	Speaker	What is said
EP2	KO	34. Ok now, at this point now I am going *to take a lead* on this planning session, so at least you understand how *we are playing the game*, otherwise I expect you next time that you are going to *take a lead*. 35. So, I think what is happening here, I did not know this thing; this is one of the things that is going to strengthen up *how the MTF can be used.* 36. The MTF does not allow any room *for introduction*. This is one of the things I am going to be analysing. 37. You come up with something that is so powerful that before we can even start *with implementation*, we need *to introduce*. 38. And when *we are introducing*, we are *reviewing from previous knowledge*, and I don't think that we need to emphasise so much *on introduction*. 39. It will depend *on the person* who is going to be conducting a lesson. We can throw in some ideas, because *an introduction* is more of a *personal* preference. 40. You *are trying to break an ice* with the learners, it's all about you. It's not something we can rehearse, it's more *interpersonal*. 41. We can just throw ideas. And from the critical aspects we are going to discuss can be part *of the introduction*. 42. How you *break ice with your learners* is more of an *individual preference.* 43. But this is important, when doing introduction, you want to get *previous knowledge*, you either do it using real life situation. 44. But in your mind you should know what are *critical aspects, what are the problems.* 45. So now the issue we need to talk about is *implementation, the actual MTF.* 46. We will see how we do the *conclusion*. So you have already taken us through how you are going to do *the introduction*. 47. I don't know if anyone wants to add some things that might be useful for him when he is doing *the introduction* because at the back of his mind he is thinking **of linear functions**. 48. And we are assuming that we are *introducing* **linear functions** for the first time. 49. We are guided by the CAPs documents. *What we are doing now is called curriculum analysis*

In this episode, the conversation was characterised by mainly attending procedures without any pronounced objects in focus. These attending procedures were related to roles, ground rules (e.g., lines 34-35), structuring (e.g., 36-37) and identifying CA (e.g., lines 38 and 44). From these attending procedures it became possible to decide on the focus of the conversation and to chunk this into an episode.

The intended focus was less tactile than the other two; it depended on the researcher's interpretive efforts. This process may involve matching of pronounced and attended focus as exemplified above.

I have already noted that in the absence of the pronounced words, the attended focus could be used as the basis of deciding the focus of the conversation. However, there was an inherent difficulty, particularly when the conversation displayed a cluster of experiences or statements evoked by either the pronounced or attended focus which sometimes made it vague to identify the focus of conversation. In such cases, it became the researcher's job to bring in other interpretative lenses. My interpretive work involved drawing in other analytical ideas such as: (1) different indicators for identifying the object of learning as suggested in the MTF literature (e.g., doing curriculum analysis, mathematical analysis and analysing learners' previous knowledge); (2) modes of talk in order to detect the KO's patterns of interaction which highlight her discursive actions; (3) and procedure-performer pair which was adapted from Lavie et al.'s (2018) task-procedure pair, and enabled me to identify the MTF components in discursive form by distinguishing between mathematical routines and the performer of those routines. The workings of all these interpretive lenses will be evidenced and discussed in detail in the coming sections.

What needs to be noted is that these interpretive lenses were used to help me reconstruct teachers' attended focus related to identifying CA, scaffolding and MTF routines. In this case, the choice of identifying the discursive focus was made under the guidance of the intended focus. For example, in line 107 in Table 11 ("What are the problems? Some might not be able to *classify* **it."),** none of the pieces of information translated automatically into clear-cut attended or intended focus, however it seems the intention was on identification of learners' difficulties with respect to linear function. It then became my interpretive process to reconstruct the attending procedure by consulting MTF literature's suggested ways of identifying the critical aspects of object of learning. This process enabled me to identify attending procedures related to identification of critical aspects (L/S pedagogical routines). From these attending procedures it became possible to decide on the focus of the conversation and to chunk into an episode.

What this discussion has highlighted was that the main construct used for chunking of data was the attended focus. I have shown with different scenarios how the other components of discursive focus (pronounced and intended) were integrated into the attended focus. The notion of discursive focus was further used in the analysis of the data, including coding, as will be demonstrated in section 5.4.4.

5.4.3 Developing a working analytical framework

The analytical framework described by Gale et al. (2013) is composed of categories and codes. A code is a descriptive or conceptual label that is assigned to excerpts of raw data in a process called coding. Codes can be grouped together to form categories that are clearly defined. In qualitative data analysis, categories can then be compared and consolidated into themes (Saldaña, 2009). The codes and categories that form the analytical framework can be developed either 1) through open coding of the data (inductively); 2) using existing theory, where the codes and categories of the analytical framework are predefined based on previous literature and/or existing theories (deductively), or 3) using a combination of both. I used a combination of both, as discussed below.

Before I proceeded with any data coding, I worked with a conceptual framework derived from the literature and theoretical framework chapters which I called the Routine-Interaction (RI). The RI was then used as the initial a priori framework. As mentioned in the literature chapters, the RI was identified to drive part of the analysis, particularly of the aspect of mutual engagement (ME). The RI constructs were therefore used to develop a priori categories of the analytical framework which were categorised into discursive routines foci: Maths, MTF, Pedagogical and Modes of talk: cumulative(CUMM), exploratory(EXPLO), disputational(DISPU), structuring conversation(SC) and managing understanding(MU). The initial a priori framework is illustrated in Table 13. A deductive approach was used to identify utterances in the data which were related to these main categories. This process resulted in an identification of a number of open codes aligned to these a priori categories. This means the categories of the initial a priori framework were a priori and the codes within them were open codes, grouped into these categories.

Table 13

Initial a priori framework

	Initial Categories of a priori Framework				
Discursive Routines	Maths		MTF		Pedagogical
Modes of talks	CUMM	EXPLO	DISPU	SC	MU

Once the initial framework was established, using an inductive approach since I wanted my framework to be open to issues arising from the data and issues which were participant orientated, I piloted this initial framework on one episode in order to refine my a priori categories using the data. Through this

analysis, new emerging categories such as *Lesson process* and *contextual* were included in the main category list of the discursive routines. The Maths category included two subcategories: routines and visual mediators. The MTF category was extended into two emerging subcategories: MTF naming and MTF routines. The pedagogical category was also extended by the inclusion of three emerging subcategories: general pedagogical, mathematical teaching and L/S pedagogical routines. The modes of talk category was expanded into five a priori categories derived from Dudley's (2013) coding framework, as described in the theoretical framework chapter.

I further tested the framework on different episodes of the transcript. I coded each episode individually, line by line, to compare how I had applied the categories. Where discrepancies between each coded episode occurred, this flagged up difficulty in applying my categories, which led to tweaking the framework. I refined the framework as I went along and continued to clarify how each framework category should be used. This helped me to develop a set of guidelines for using each of my framework categories. It was important during this stage to develop clarity about these categories because at subsequent stages they were used to code the remaining data set. I found that developing a framework required me to move slowly and tentatively, and that one should expect the framework to go through several iterations. The final framework used in my study of teachers' use of the MTF tool consisted of the following categories and subcategories in Table 14 below:

Table 14

Emerging analytical framework

Initial A priori Framework										
Discursive Routines	**Categories**	**Maths**		**MTF**		**Pedagogical**			**Lesson process**	**Contextual**
	Sub-category	**Maths Routines**	**Visual Mediators**	**MTF naming**	**MTF Routines**	**General Peda-gogical routines**	**Maths teaching routines**	**L/S peda-gogical practice routines**	**L/S process routines**	**Contextual factors routines**
Modes of talk	**Categories**	**Cumulative**	**Exploratory**		**Disputational**		**Structuring Conversation**		**Managing understanding**	

The constructs reflected in the emerging analytical framework were a manifestation of the SCDA perspective which allowed me to identify all sociocultural and contextual factors impacting the use of the MTF tool.

The main categories and subcategories of the emerging analytical framework were used alongside codes emerging from the data. As already noted, the discursive routine recorded codes were the results of

open coding, that is, they emerged from the data though, of course, not wholly independent from my understanding of mathematics education research. The codes recorded under the modes of talk category were a priori codes drawn from Dudley's (2013) coding framework. The emerging codes were grouped into relevant subcategories. For example, the utterances that were describing the exemplifying actions (e.g., line 112 in Table 11 above: "Other **examples** could be dealing with translation, i.e. moving from graph to table to equation".) were coded as translating examples and classified under the exemplifying subcategory.

The final analytical framework consisted of 67 codes, clustered into 14 subcategories and six main categories as illustrated in Table 15.

Table 15

Final analytical framework

Categories	**Subcategories**	**Codes**	
MATHS FOCUS	Routines	Interpreting	
		Constructing	
		Classifying	
		Translating	
		Calculating	
		Demonstrating	
		Defining	
	Visual Mediator	Verbal	
		Symbolic	
		Tabular	
		Graphical	
		Concrete	
MTF FOCUS	MTF routines	Exemplifying	5 codes
		Generating tasks	5 codes
		Teacher Explaining	6 codes
		Learner Explaining	6 codes
		CA of OoL	7 codes
	MTF Naming	Examples	
		OoL	
		Tasks	
		Explanations	
		Learner participation	
PEDAGOGICAL FOCUS	Mathematical teaching	Questioning	
		Generalising	

		Explaining
	Pedagogical Strategies	Structuring
		Revising
		Other
	L/S pedagogical practice	Reframing
		Scaffolding
		Identifying CA
LESSON PROCESS FOCUS	L/S process	Lesson planning
		Ground rules
		Roles
CONTEXTUAL FOCUS	Contextual factors	Learners
		Resources
		School teacher
		Time
MODES OF TALK	Cumulative talk	CUMM
	Disputational talk	DISPU
	Exploratory talk	EXPLO
	Structuring Conversation	SC
	Managing Understanding	MU

The first column and second column of the final analytical framework in Table 15 show the main categories which were adapted from the initial a priori framework (Table 13). The codes in the third column emerged from the data analysis, that is, inductively.

In the following section I give a brief description of the meaning and examples of what ideas or elements might be summarised under each code.

5.4.4 Development of codes and categories of the analytical framework

The goal of this study was to describe the nature of teachers' MTF-use and the nature of the opportunities for learning. The MTF-use in particular can be evidenced in teachers' narratives, which in turn are seen in teachers' discursive routines and patterns of interaction. I therefore needed to be able to describe the discursive routines and social modes of talk in a lessoning study, which is a complex social practice. As teachers plan and reflect on their lessons, they discuss mathematics, mathematics teaching, the MTF framework, the context of their work, and all of these occur within complex social relations and so a range of interaction modes. As is evident in Table 15 above, and then further in the examples given below, the result of this goal was an extensive set of analytic distinctions that in turn produced a very large data set comprising multiple episodes, each with numerous utterances, all of

which were coded and categorised. Following the analytic process described above, the large data set was reduced into frequency tables. These allowed for within and cross-case analysis (i.e., within each L/S session and across sessions), and so an exploration of how various components of the teachers' talk interacted.

In the following section I will first discuss principles guiding the coding of discursive routines. The coding process of modes of talks follows later in section 5.4.4.2. It should be noted that the episodes and utterances attracted numerous codes and some utterances multiple codes. I cannot show them all at the same time, I will be showing them separately and together when appropriate.

5.4.4.1 Discursive routines coding process

The analysis of the data was driven by the focus on discursive routines. In this study routines referred to repetitive patterns of actions (Lavie, 2019). Five sets of discursive routines have been categorised in this study: 1) those with a mathematical focus and these include routines found in the function discourse such as constructing, interpreting, calculating, translating; 2) those with an MTF focus, that is, routines related to examples, tasks etc.; 3) those with a pedagogical focus, that is, routines related to discursive acts of teaching such as questioning, strategising, and structuring a lesson; 4) those with a contextual focus, that is, with factors such as learner interaction, use of resources and 5) those focused on the lessoning study process, such as, ground rules, roles.

(1) Coding Mathematics focus

In each episode I noticed that repetitive actions related to mathematics were either explicit naming of mathematical objects or use of mathematical routines. The identification of mathematical naming was guided by pronounced focus since this action was publicised. The identification of mathematical routines was guided by the attended focus since these were visible in the data.

(i) Mathematical routines

Table 16 shows examples of utterances I coded as mathematical routines in all of the episodes. In the first column is the description of the code, the second column is the code, the third column provides the data source from where the example came from in the fourth column was taken. For example, in row 2 the source from which I took the example is KO/LP1/EP4/L74 where KO is a code for a speaker, LP1 refers to the first lesson planning session, EP refers to episode 4, and L74 refers to line 74. All the sources of emerging codes in this study are described in this manner.

The coding process of mathematical routines was guided by attended focus. In the following table, the attended procedures relate to both routines (*marked with purple italics*) and visual mediators (underlined). The objects in focus relate to either mathematical or MTF objects (**bolded**).

Table 16

Codes for Mathematical Routines

Description	Code	Source	Example
Routines	Interpreting Symbolic	KO/LP1/EP4/L74	74. Now you can come with your spider diagram. Or you can just ask them to *interpret* properties of $y = mx + c$
Visual mediators	Constructing Graphical	S/LP1/EP10/L165	165. I think draw *with them* the first three or the first two and maybe *ask them to draw the last one*…
	Classifying, Constructing Graphical	J/LP1/EP1/L118	118. You can also use the graph. You can *draw* different graphs and ask them to *identify* them.
	Translating Graphical, tabular	KO/LP1/EP7/L112	112. Other **examples** could be dealing with *translation*, i.e. moving from graph to table to equation.
	Calculating Symbolic	J/LP1/EP1/L10	10. the **output** which they can either write, choose any **inputs,** *multiply by 2* and it gives out the output.
	Demonstrating Concrete	J/LP1/EP1/L19	19. I hope they say this is horizontal and now on real life. On your horizontal line *you just walk, you don't feel any pain.*
	Defining Verbal	J/LP1/EP1/L5	5. Which is *defined as* a set of inputs and outputs that put the relationships and all that I would *define* **functions** as a whole.

In analysing mathematical discourse, I drew from Sfard's (2008) conceptions of mathematical discourse such as words/word use, routines, visual mediators and endorsed narratives, and focused particularly on one feature, word use. According to Sfard (2008), word use includes routines and visual mediators and these two can be used as a window to gain insights about the nature of mathematical discourse.

When coding mathematical utterances, I first looked for the mathematical routines (attending procedure) used when teachers were talking about mathematical objects. I then looked further in the same utterance for the implied visual mediator.

When a mathematical procedure involved the process of identifying a special kind of a function being in any representation (graphical, algebraic, and tabular) by analysing its properties, it was coded as

classifying[9] (see line 118). When a mathematical procedure involved a process of scanning through a specific representation (e.g., graph, symbolic equation) by analysing visual properties of that representation and also reading off points or values specifically in a tabular representation, it was coded as *interpreting* (see line 74). It is important to note that there is an overlap in the code identification rules for interpreting and classifying. This is because the process of classifying involves interpreting properties of the function/representation. The main difference between the two is that classifying ends with identification of a special kind of function or categorising that function into a particular family of functions. In contrast, the interpreting procedure does not go beyond analysing behaviour or pattern of a function.

When the procedure involved an action of drawing or producing a graph or a table, it was coded as *constructing* (see line 165). When the procedure involved an action of giving a definition (using verbal form), it was coded as *defining* (see line 5). A procedure was coded as *translating* when it involved a process of recognising one function in different representations. If it also involved moving from one representation to another, which may include a process of constructing or producing one representation of a function given another one, then it was coded as translating. It is important to note that the translating procedure may involve either interpreting or constructing procedures. However, what differentiates it from these two is moving between different representations (see line 112).

If the procedure involved manipulating and solving algebraic equations/using a formula or method, it was coded as *calculating* (see line 10). When a procedure involved an action of using routines from everyday discourse accompanied by concrete representations in order to speak about mathematical objects, it was coded as *demonstrating*. For example, in line 19 the "horizontal line" which is the concrete mediator is accompanied by an action of walking and is used to illustrate the behaviour of the graph (increasing/decreasing).

(ii) Mathematical objects

The mathematical objects were clearly visible and were identified by looking at pronounced names of different mathematical objects. The mathematical objects included linear functions, gradient and y-intercept. For example, lines 74 and 5 in Table 16, show mathematical objects related to linear functions.

[9] According to Leinhardt et al., (1990, p. 15), "by classification we refer to actions (a) deciding whether a particular relation is a function (the relation can be represented in any form, e.g., a graph, an algebraic rule, an arrow diagram); identifying a function among other relations; or (c) identifying a function among other functions".

(2) Coding MTF focus

In each episode I noticed that repetitive actions related to MTF were either the explicit naming of MTF objects or the use of mathematical routines related to MTF objects. The identification of MTF naming was guided by pronounced focus since this action was publicised. However, the identification of elements of MTF routines required additional attention since they were not visible in the talk. Therefore, analysis with respect to MTF routines was guided by the intended focus with the aim of reconstructing the attending procedure. This process focused on the mathematical routines in focus together with the performer of that routine. The coding process of mathematical routines was discussed in detail in the previous section and will not be repeated here. What this analysis will show is how the mathematical routines were related to the performer in order to produce the MTF routines.

(i) MTF routines

The coding process of MTF routines was guided by the intended focus with the aim of reconstructing the attended focus since these types of routines were not openly visible. I therefore needed to do more interpretive work by drawing other analytical lenses such as perfomer-procedure pair. In order to illustrate the process of coding the MTF routine, I use Table 17 below where I only highlight the mathematical routines (in italics) and visual mediators (underlined) and the performer of the mathematical procedure (highlighted in blue). Also bolded are objects related to either MTF or mathematics. In the table, I mark in red font the keywords that helped me identify the explaining codes. Table 17 shows examples of utterances I coded during MTF routines.

Table 17

Codes for MTF Routines

Description	Code	Source	Example
Exemplifying	Interpreting_T	J/LP1/EP1/L33	33. I have introduced the **positive sign** and **negative sign**. But I have introduced the importance of **gradient 'm'.**
	Constructing_T Constructing_L	S/LP1/EP7/L165	165. I think *draw* for them the first three or the first two and maybe ask them to *draw* the last one…
	Classifying_T	KO/LP1/EP5/L113	113. In our planning we need **examples** for *classification.*
	Translating_T	KO/LP1/EP5/L112	112. Other **examples** could be dealing with *translation*, i.e. moving from graph to table to equation.
	Calculating_T	KO/RD1/EP2/L31	31. I had to do some *calculations* on the side for them to see that $(x/2 \ is \ the \ same \ as \ 1/2\ x)$[10]
Generating Tasks	Interpreting_L	KO/LP1/EP3/L74	74. Now you can come with your spider diagram. Or you can just ask them to *interpret* **properties of** $\boldsymbol{y = mx + c}$.
	Constructing_L	J/LP1/EP1/L15	15. They *plot* the key points that they got from the spider or the

[10] The fractions in all equations in the coded transcripts that need underlining are not in equation editor format.

			table. Obviously, I have to be clever and choose the points that are going to give me **a linear function**.
	Classifying_L	J/LP1/EP5/L118	118. You can also use the graph. You can *draw* different graphs and ask them to identify them.
	Translating_L	J/LP1/EP1/L14	14. …from there they can transfer the table without telling them that this is the linear equation or what
	Calculating_L	J/LP1/EP1/L10	10. the output which they can either write, choose any inputs, *multiply by 2* and it gives out the output
Teacher-Explaining	Interpreting_TS	J/LP1/EP1/L20	20. I will probably say it's because the *slope is zero, its flat it is not steep*."
	Constructing_TS	S/LP1/EP7/L144	144. *Drawing* the graph could be another way of convincing them
	Classifying_TS	S/LP1/EP6/L138	138. So, you might find it necessary also, they might not even be convinced why you are saying $y = 3/x + 2$ is not a **linear function**
	Translating_TS	KO/LP1/EP9/L224-226	224. So now you are trying to *generalise* something here. And the most generic way would be from the graph. 225. Maybe here (*looking at the table*) we won't be able to see but can *interpret* from a different representation. 226. There is nothing we can see here (*looking at the equation*), they can only see that you have *multiplied by minus*.
	Calculating_TS	KO/LP1/EP8/L194	194. This is the **linear function**, ask why and *simplify the equations by making y a subject of the formula*
	Defining_TS	J/LP1/EP1/L5	5. Which is *defined* as a set of **inputs and outputs** that put the relationships and all that I would *define* **functions** as a whole.
Learner-Explaining	Interpreting_LS_u	J/LP1/EP1/L23	23. They will decide and then they might say *it's less than,* they might say *it's greater than*. They might probably say *it's greater than because* they are *thinking of going up*, they don't know anything about *directions of signs*
	Classifying_LS_u	KO/LP1/EP7/L162	162. Asking them to *draw* them, you want them to confirm if these are **linear functions**
	Translating_LS_u	KO/LP1/EP9/L227	227. Here (*looking at the table*), they can start to see what is it is the same here. Here they can see everything is the same
	Demonstrating_LS_u	J/LP1/EP1/L19	19. I hope they say this is horizontal and now on real life. On your horizontal line *you just walk, you don't feel any pain*
	Defining_LS_u	KO/LP1/EP3/L81	81. Listen to their words, what are they naming this thing, is it the **gradient**, is it the **slope,** is it the **y intercept**, are they saying **y intercept**, or are they saying **y intersect?**
CA of OoL[11]	CA-Interpreting_T	J/RD1/EP4/L64	64. When this **example** came, it became clear that there was a lot of confusion with the **word increasing**. There were like you are *increasing with one and the graph is shifting'*
	CA-Constructing_T CA-Interpreting_T	KO/RD1/EP9/L139	139. We must remember our **object of learning** is not about *plotting* the graph but *analysing the* ***effects of c****, the shift*.
	CA-Classifying_T	KO/LP1/EP5/L107	107. What are the problems? Some might not be able to *classify* it.
	CA-Translating_T	KO/LP1/EP9/L243-244	243. In the first two **example**s the **object of learning** is **the effects** of once you are happy that they have seen the **effect of m** you can *compare* it with the table. 244. And you can say now that you know the **effects of m** let us now go and have a look in your graph can you see where your graph is intersecting, what are the coordinates there, zero-zero.
	CA_Word use_T	KO/RD1/EP4/L65	65. The **word increasing** in the textbooks is associated with the **gradient.** What is happening with **the range** is associated with **the gradient**.

[11] The category of CA of OoL routines records the identified critical aspects of OoL. This category is linked to mathematical routines. At the same time the process of identifying these critical aspects is also strongly linked to the pedagogical routines linked to L/S. I will therefore discuss in detail this process when discussing the pedagogical focus codes.

	CA_Calculating_T	KO/LS_A/LP1/EP5/L109	109. We know that some of the problems have something to do with *calculations*, so it means some of the **examples** must focus on *calculation.*
	CA_Defining_T	KO/LS_A/LP1/EP3/L76	76. We are assuming they know this *definition*

I defined MTF routines as procedure-performer pair, where a procedure referred to a discursive action performed. For example, a procedure in the MTF context was related to a mathematical routine performed and included calculating, constructing procedure etc. The performer part was related to the person performing the procedure. For example, when a routine was done from the learners' or teachers' perspective, the performer was regarded as the learner or the teacher respectively as was evidenced by use of keywords "them" and "they" as highlighted in blue in Table 17 above.

The examples were used by the teacher as instructional resources to illustrate or demonstrate a technique about a particular mathematical concept. The examples were identified when a particular mathematical routine is attached to the teacher as the performer and with intentions to illustrate a specific aspect of a mathematical object. For example, in line 31 in Table 17 ("I had to do some *calculations* on the side for them to see that $x/2$ is the same as ½ x"), the KO, as the teacher reflected on what she had done (performer) in relation to the mathematical routine of calculating through symbolic representations. The intended focus in this utterance was about illustrating the effects of concept of gradient. This analysis therefore illustrated the reconstruction of attended focus which was an exemplifying routine. In all episodes, all utterances which were characterised by mathematical routines attached to a teacher with implied goals of providing illustrative mathematical procedure were coded as exemplifying routines (coded as Procedure_T, where T referred to the performer who was the teacher) as illustrated in lines 33, 165, 113, 112, 31 in Table 17 above.

On the other hand, the tasks related to what learners were asked to do. Therefore, the performers of the routines related to tasks were the learners. When the mathematical routines were attached to the learner as the performer, the intended focus was related to performing a mathematics activity related to a specific mathematical object. For example, in line 15 ("They *plot* the **key points** that they got from the spider or the table"), J reflected about what the learners (as highlighted by the keyword "they" referring to the learners) were expected to do in relation to the mathematical routine of constructing a visual mediator of the table. The intended focus in this utterance was on the generation of tasks about the mathematical object of linear function. This analysis therefore illustrated the reconstruction of attended focus which was generating tasks routine. In all episodes, all utterances which were characterised by mathematical routines attached to a learner with implied goals of providing what

learners needed to do, were coded as generating tasks routines (coded as Procedure_L, where L referred to the performer who was the learner) as illustrated in lines 74, 15, 118, 14, 10 in Table 17 above.

In the case of mathematics, explanations may be "justifications for actions" (Leinhardt, 2001, p. 341). They may be accompanied by various substantiating prompts such as "why", "because" (Ben-Yehuda et al., 2005). In the case of this analysis there were a number of keywords which enabled me to identify explanations such as "can see", "convince", "because" and "confirm". The explanations were done from both the teacher and the learner perspective. The mathematical utterances were coded as explanations if they were characterised by substantiating mathematical routines.

When the mathematical routines were attached to the teacher as the performer, then the intended focus was substantiating a specific mathematical object. For example, in line 144 ("*Drawing* the graph could be another way of convincing them"), S was reflecting about what teachers may do in relation to a mathematical routine or constructing a visual mediator of a graph. The intended focus in this utterance was on justifying actions about the mathematical object of a concept of gradient. This analysis therefore illustrated the reconstruction of attended focus which was a teacher-explaining routine. In all episodes all utterances which were characterised by substantiating mathematical routines attached to a teacher were coded as teacher-explaining routines (coded as Procedure_TS, where TS referred to the performer who was the teacher) as illustrated in lines 20, 144, 138, 224, 194, 5 in Table 17.

When the mathematical routines were attached to the learner as the performer, the intended focus was substantiating a specific mathematical object. For example, in line 162 ("Asking them to *draw* them, you want them to confirm if these are **linear functions"**), KO was reflecting about what learners were expected to do in relation to a mathematical routine or constructing a visual mediator. The intended focus in this utterance was on justifying actions (seen by the use of the keyword "confirm") about the mathematical object of a linear function. This analysis therefore illustrated the reconstruction of attended focus which was a learner explaining routine. In all episodes all utterances which were characterised by substantiating mathematical routines attached to a learner were coded as learner-explaining routines (coded as Procedure_LS_u, where LS_u being the performer who was the learner) as illustrated in lines 23, 162, 227, 19, 81 in Table 17.

The emerging Critical Aspects of OoL will be described in more detail when speaking to the process of Identifying CA. They were also coded following the same principle of procedure-performer and were

specifically coded as CA_procedure_T to distinguish them from other routine procedures. The performers for CA_Procedures were only teachers.

(ii) MTF naming

The analysis of MTF naming was guided by the pronounced focus since they were visible in the transcript. All utterances whereby teachers were calling out names of MTF objects were coded as Explanations, OoL, Tasks and Learner participation and are bolded, as shown in Table 18.

Table 18

Codes for MTF Naming

Description	Code	Source	Example
Calling out names of MTF	Explanations	KO/LP1/EP9/L210	210. Your **explanations** are going to be coming directly from your **examples.**
	Tasks	KO/LP1/EP5/L111	111. So what we are saying is that we need **examples** where they *calculate* these could be **tasks**.
	Examples		
	OoL	KO/LP1/EP11/L281	281. No, remember what your **object of learning** is. Our **object of learning** is **m and c.**
	Learner participation	KO/LP1/EP5/L115	115. We are now working with **learner participation**. We also need to vary our **examples**.

(3) Coding Pedagogical focus

The procedure-performer principle not only produced MTF routines, but also produced procedures that were pedagogical. This was the work of teaching at a pedagogical level. In these codes I have dropped the performer part since all pedagogical routines are from the teachers' perspective. Mosvold's (2016) work on tasks of teaching from a commognitive perspective was of particular relevance in this study. His study on discursive actions of the work of teaching resulted in a categorisation of mathematical teaching routines. In this study, I extended this focus on teaching routines related to strategies aiming to describe teachers' repeated actions thatwere linked to execution or organisation of a lesson during teaching, and teaching routines related to L/S context such as reframing and Identifying CA.

The coding process of pedagogical routines was guided by the intended focus with the aim of reconstructing the attending procedures since these types of routines were not openly visible. However, there were some instances where the procedures were publicly evident; in those cases, where I needed to do more interpretive work by drawing other analytical lenses I will provide explanation. The

pedagogic routines have been broken down into three parts: (i) discursive actions related to teaching mathematics; (ii) general pedagogical strategies; and (iii) L/S pedagogical practice.

(i) Teaching mathematics routines codes

The first part included discursive tasks for teaching mathematics as taken from Mosvold (2016), where these acts of teaching are actions that a teacher engages in with the aim of helping learners learn mathematics content. These include "presenting mathematical ideas, responding to students' 'why' questions, finding an example to make a specific mathematical point" (Ball et al., 2008, p. 400).

The coded tasks of mathematics teaching included discursive routines such as questioning, generalising and explaining (giving explanations). It must be noted that generalising were routines that were also present in the talk about mathematical objects. I defined questioning as a pedagogical practice whereby a teacher checked for understanding and included use of questioning prompts such as "why", "what", "how" with the aim of eliciting learners to engage with a particular mathematical activity. These questioning prompts are highlighted in yellow as shown in Table 19 below.

Table 19

Codes for mathematical teaching routines

Description	Code	Source	Example
Questioning	Questioning	J/LP1/EP5/L114 J/LP1/EP1/L20-21	114. Also include **counterexamples** like $y = 3/x + 2$, and ask: Is it a linear graph? Why are you saying that? 20. "What do you think if you had *to walk* on this line?" 21. "Do you think you would *feel the pain*?"
Generalising	Generalising	KO/LP1/EP9/L224-227	224. So now you are trying to generalise something here. And the most generic way would be from the graph. 225. Maybe here (*looking at table*) we won't be able to see but can *interpret* from a different representation. 226. There is nothing we can see here (*looking at the equation*), they can only see that you have *multiplied* by minus. 227. Here (*looking at the table*), they can start to see what is the same here. Here they can see everything is the same.
Explaining	Defining	J/LP1/EP1/L4	4. I thought the first thing I would do is *define* **functions**. Before I can talk about **the objective of the lesson**, I would *define* **functions**, this is the main topic, and not that it's an **objective of the lesson**

The questioning routines were characterised by a number of prompts:

(1) prompts learners to give an answer to a problem: "Is it a linear graph?"

(2) prompts learners to explain or justify their mathematical reasoning: "Why are you saying that?"

For example, in line 114 (Also include counterexamples: "Is it a linear graph?" "Why are you saying that?"), J was prompting the learners to interpret the linear graph and provide the answer. This was followed by questioning prompts ("Why are you saying that?") aimed at eliciting learners to justify their answer (mathematical reasoning).

(3) prompts learners to share their other ideas (including mathematical ideas), but not about explaining the mathematical process: ("What do you think?"). For example, in line 20-21, J used questioning prompts to elicit learners to share other ideas about their everyday experience of walking on the line, aiming to reinforce the behaviour of the gradient of the line.

I used the generalising activity to refer to teachers' actions to draw from different representations to formulate a general mathematical statement[12]. The act of generalising was closely related to a mathematical procedure of translating, as evidenced in lines 224-227, where the speaker (KO) switched between different representations (graphical, tabular, symbolic). The generalising procedure was also identified in instances where teachers were explicitly referring to the word "generalising" (line 224).

I used "explaining" as a pedagogical procedure when the teacher explicitly referred to an act of giving explanations. When this was non-explicit but the intention suggested that the teacher was engaging in an act of explaining, I used intended focus and brought in other lenses to determine the attended focus. Most of the time, these were coded as MTF routines of teacher explaining. For example, in line 225 in Table 19: ("Maybe here (*looking at the table*) we won't be able to see but can *interpret* from a different representation"), it seems the KO was intending to provide an explanation with respect to the notion of gradient interpreting its behaviour in different representations. The explanation act was signalled by the use of keywords such as "to see", but she was not explicit that she was doing explanation.

(ii) Pedagogical routines coding

The second part included discursive actions related to general pedagogical strategies. These were understood to be patterns of general teaching actions that serve to attain a specific goal of teaching. The emerging routines under this category included revising and structuring as illustrated in Table 20.

[12] For example, in a linear function $y = mx + c$ the teacher may use a different representation to formulate a mathematical statement about the gradient (m): the value of m affects the slope of the graph. As m increases, the gradient of the graph increases. If $m > 0$, then the graph increases from left to right (slopes upwards). If $m < 0$ then the graph increases from right to left (slopes downwards). (Siyavula, 2010, p.115)

Table 20

Codes for general teaching routines

Sub-category	Code	Source	Example
General Pedagogical	Structuring	KO/LP1/EP6/L122	122. We need introduction, in your introduction you will announce on the board $y = 2x + 1$. What is this? You want to find out from them.
		J/LP1/EP81/L3-4	3. I thought how I would go about teaching **linear functions.** Let's say we were not collective. Like I just had to go to class *and teach* **linear functions**. 4. I thought the first thing I would do *is define* **functions.** Before I can talk about the **objective of the lesson**, I would *define* **functions**, this is the main topic, and not that it's an **objective of the lesson.**
Revising	Revising	J/LP1/EP1/L9	9. I thought of *going back to grade 9 whereby* they know a spider diagram, as **inputs and outputs** and whatever is here.
Other	Other pedagogic strategies	M/LP1/EP1/L29	29. Because I was a bit concerned about the approach, if you say both of them are steep, the other one can be negative, the other one positive.
		KO/LP1/EP9/L246	246. They might say it's a y**-intercept**. If they are not saying it, you need to *scaffold them.*
		KO/LP1/EP12/L297	297. Ask them to draw it, and *ask them to discuss, maybe in groups*. This could be a way of *assessing.*

Structuring routines included explicit reference to keywords such as "introduction", "implementation" and "conclusion" of a lesson as highlighted in orange font in line 122. It also included comments about the sequence and the flow of the lesson. The sequencing of a lesson was concerned with how a teacher could execute the teaching of a specific mathematical concept. For example, in lines 3-4, J discussed how he would introduce an aspect of linear function. He reflected that he would first define a function before talking about the object of the learning. Revising routine comprised actions of reviewing pre-knowledge or knowledge from previous grade as shown in line 9.

When none of the teachers' routines resonated with any of the identified pedagogical strategies, I provided a category that I named "other pedagogic strategies". For example, in one utterance it was clear that the teacher's intention was related to a teaching strategy but that strategy could not fit into the identified strategies, for example, in line 29 ("Because I was a bit concerned about the approach"). Some of the strategies in this category "other" were those used sparingly such as "scaffolding learners" and "assessing" (shown in lines 246 and 297).

(iii) Pedagogical L/S routines

The third part included the pedagogical discursive actions from the L/S perspective which I called "reframing", "identifying CA" and "scaffolding". The act of reframing was associated with the practice of L/S, that is, reflection-on-action (retrospective reflection). The identification of CA was one of the L/S practices where teachers identified CA of the OoL by analysing curriculum, mathematical analysis and analysing research literature about functions (Lo, 2012). The scaffolding role of the KO was highlighted in both the lesson and learning study context.

Identifying CA

In the section above, where I was discussing the coding process of MTF routines, it was noted that since the critical aspects were not visibly identifiable from the utterances, I needed to recruit other lenses guided by the intended focus in order to reconstruct the attended focus related to MTF routines of OoL. The process of identifying CA facilitated the process of reconstructing the attended focus related to MTF routines—CA of OoL. As noted in Chapter 2, this process comprises a number of procedures such as paying attention to curriculum analysis, mathematical analysis and awareness of learners' current or previous knowledge with specific attention to learner difficulties and misconceptions. The process of identifying CA resulted in two categories of attended procedures: pedagogical routine-identifying CA, and MTF routine-CA of OoL. The MTF routine related to OoL was related to a mathematical routine accompanied by the performer, as seen with other MTF routines above. Such procedures were coded as CA of OoL_Procedure_Performer. This procedure resulted from the process of reconstruction of attended procedure illustrated in Table 21, below. In the table, I illustrate the process of identification of critical aspects of OoL. Highlighted in green are the keywords used to indicate the critical aspects.

Table 21

Codes for identifying CA and CA of OoL routines

Description	Code	Source	Example
Knowledge of learners	CA_interpreting Identifying CA	KO/RD1/EP6/L63 J/RD1/EP6/L64	63. You did, but it was vague, we need to be clear, there was a lot that could cause misconceptions. 64. When this **example** came, it became clear that there was a lot of confusion with the **word increasing**. There were like you are *increasing* with one and the graph is *shifting*. *The excerpt above is concerned with learners' knowledge through discussion about difficulties and misconceptions specific to the interpreting procedure (we can see that through the reference of graph shifting)*

	CA_calculating Identifying CA	KO/LP1/EP5/L109	109. We know that some of the problems have something to do with *calculations*, so it means some of the **examples** must focus on *calculation.*
	CA_classifying Identifying CA	KO/LP1/EP5/L107	107. What are the problems? Some might not be able to *classify* it.
Curriculum analysis	**CA_word use** Identifying CA	KO/RD1/EP6/L65	65. The **word increasing** in the textbooks is associated with **the gradient**. What is happening with **the range** is associated with **the gradient**.
Concept analysis	**CA_interpreting** Identifying CA	J/LP1/EP1/L26	26. Ja, I understand, I am approaching from grade 9 work, so I am planning to start with a **positive gradient** cause they don't know **it's positive**, it's increasing. If **it's negative** it's decreasing. The only thing I am talking about **it's flat** and **steep.** *In this extract, a reference is made to the curriculum (e.g., grade 9 work); at the same time the mathematical ideas about the concept of gradient are discussed. This excerpt shows that the identification of CA resulted from curriculum analysis and analysis of big ideas of functions that are related to a mathematical procedure-interpreting (it's positive, it's increasing).*

Knowledge of learners could be observed when the teacher was concerned about learners' comprehension of concepts or learners' knowledge and/or previous knowledge. This also included situations when the discussion was around learners' difficulties about a certain mathematical aspect. This can be observed when the discussion was about learners' confusion, explicit discussion about learner difficulties or problems, errors, and learners' misconceptions (lines 63-64).

Curriculum analysis could be observed when there was reference to curriculum documents; this could include CAPS documents, exam question papers and textbooks (lines 65 and 26).

Content/Mathematical analysis could be observed when attention was paid to the concept analysis, that is research done about that specific concept (e.g., errors, difficulties), concept definitions in terms of different representations, key mathematical ideas (big ideas) or basics associated with that concept and endorsed narratives by the mathematics community (line 26).

Reframing

Reframing was one of the L/S practices (reflection-on-action) which was about retrospective reflection on teaching (reported action). It involved reflections from both the individual teacher who was teaching and from a group of teachers who were observing the teaching lesson. Specifically, its focus was on the individual teacher who executed the lesson; thus, the teacher became an object of discussion. And such an act became more evident through the focus on referencing keywords "I" and "you". For example, in line 6 (Table 22), T reflected about his previous teaching session and reflected about the challenges he

faced related to contextual factors (chalkboard and the learners). In table 22 I illustrate the process of coding reframing routines. Highlighted in blue are the keywords used to indicate the individual teacher.

Table 22

Codes for reframing routines

Sub-category	Code	Source	Example
L/S pedagogical practice	Reframing	T/RD1/EP1/L6	6. Yesterday's lesson, for me it went well, the only challenge I faced there was the use of chalkboard and with the engagement of the learners. At the start they were very quiet and I got off my tune.
		T/RD1/EP2/L28	28. They were getting the idea. I was *questioning* repeatedly. I asked them to repeat with regards to what the standard formula.
		KO/RD1/EP8/L118	118. The strategy that T has used is the one of questioning and he noticed that the learners were not responding to his questions and he changed his questioning so to motivate them to participate.

The reframing procedure is evidenced in the other two utterances above. In these two utterances one can witness a use of keywords "I was" and "I asked them" (line 28) and "The strategy that T has used" (line 118), which provided clues of reflection of the individual teacher's reflection-in-action, that is, reflections about the decisions made during teaching.

Scaffolding

Scaffolding is used as a metaphor to describe how the KO facilitated learning to support teachers' participation in the L/S with MTF. Since this action was not easily identifiable from the utterances, I used Sfard (2002) construct of intended focus in order to reconstruct the attended procedure related to scaffolding actions. I was able to identify the presence of scaffolding routines by constructing a number of indicators with the help of different modes of talk such as managing understanding, exploratory, cumulative and structuring conversation, as illustrated in Table 23. A full description of how I coded these modes of talk in general appears in section 5.4.4.2.

Table 23

Modes of talk vs scaffolding routine

Modes of talk	Indicators related to scaffolding procedures
Managing understanding	**Scaffolding questions** (systematic) by probing for justification. Probing questions were aimed at getting teachers to think more deeply (probing for deeper thinking). **Help/assistance, guidance, support by providing expertise** (subject matter expert). Expertise is contained in an individual (KO) who knew something more about the subject or pedagogy of the content of a specific L/S team's work. **Giving clarity by explaining** some background, curricular, pedagogic or content-related fact

	(expertise knowledge)
Structuring conversation	**Prompts (warn, remind, reassure)** moving the conversation on **Introducing new idea** (facilitates and guides direction)
Cumulative	**Suggestions on what needs or should be done**
Exploratory	**Synthesising, summarising ideas (help reformulate interpretations)** **Proposing an idea** as a way of giving guidance on how things should be done

How these indicators were used will become evident in the analysis chapters when dealing with use of scaffolding routines.

(4) Coding L/S process focus

The L/S process was related to the comments about lesson planning, ground rules of participating in L/S including roles of each participant. Lesson planning was related to explicit talk about lesson plan and explicit talk about lesson planning process. I was able to identify these procedures by using attended focus. The lesson planning attending procedures were identifiable when the talk was about lesson planning and there was also a clear comparison between the execution and lesson planning (line 180 in Table 24).

Ground rules included explicit/implicit norms and expectations necessary in order to participate successfully in the L/S reflective discourse. These were related to comments about the rules of engagement in the L/S including the roles to be played by the participants. For example, these included specific rules that needed to be followed in terms of actions to be executed by specific individuals. For example, in lines 134 and 82, in table 24 the KO specifically commented about particular activities that needed to be done by the members of the L/S during the teaching session, such as collecting data, and observing learner participation. The roles included specific roles that were played by the individuals during the L/S. In other words, it was about who was doing what. For example, the KO became a facilitator; the teachers either played the role of the teacher during a teaching session or were observers (line 34).

Table 24

Codes for L/S process routines

Sub-category	**Code**		**Source**	**Example**
L/S process	Ground Rules	Rules	KO/LP1/EP6/L134 KO/LP1/EP3/L82	134. What are they bringing in? We are *collecting data* during this time. 82. We are going to be *observing* that. You as the teacher you might be overwhelmed, you may not even hear what they are talking about, we must *also look at their gestures*, those are

				colloquial visual mediators—they may be using their hands.
		Roles	KO/LP1/EP2/L34	34. Ok now, at this point now I am going to *take a lead* on this planning session; so at least you understand how we are *playing the game*, otherwise I expect you next time that *you are going to take a lead.*
	Lesson planning		KO/LP1/EP8/L180	180. Let us **proceed** with what *we originally planned* because we are already pre-empting on our introduction.

(5) Coding contextual focus

The contextual focus category included social routines that were related to factors such as time management, classroom management/arrangement, school teachers' participation, learner engagement, resources—chalkboard, computer. These routines were easily identifiable using attended focus. In Table 25 below I provide examples of different categories of codes related to social routines.

Table 25

Codes for contextual routines

Sub-category	Code	Source	Example
Contextual	Cont_L	KO/LP1/EP6/L133 KO/LP1/EP6/L135	133. Then we listen to what they are saying, their explanations, their words, the naming, how are they justifying their answers? 135. Now as a teacher, during introduction, the introduction can take longer. Things can change during teaching you may find out that our planning might change completely; it will depend on what learners' level is at.
	Cont_Reso	KO/LP1/EP7/L166-167	166. This is where technology is useful. 167. Maybe do PowerPoint
	Cont_Class	KO/RD1/EP8/L117	117**. Learner participation** is affected about how you interact with them, how they are seated in your classroom maybe in groups etc.
	Cont_Time	S/RD1/EP9/L143	143. If you come with enough **examples** you won't have enough time to give them to write.
	Cont_Teacher	KO/LP1/EP7/L150	150. And I am thinking coming to our context as preservice teachers, me as someone who is a preservice educator. I am not their teacher. I don't know what went on in that school the previous year.

- Time management (Cont_time): concerns about time to finish the lesson (line 143).
- Classroom management (Cont_Class): comments about arranging learners' working in groups (line 117).

- Learner engagement (Cont_L): comments about learner engagement, learner levels in terms of grade or knowledge as specified in the curriculum (lines 133 and 135).
- Resources (Cont_Reso): comments about resources used in the classroom such as textbooks or any artifacts, technology etc. (line 166).
- School teacher (Cont_T)-this concerns comments about the role/participation of the school teacher (see line 150).

5.4.4.2 Modes of talk coding process

The identification of modes of talk resonated with Dudley's (2013) broader categories of codes of what the interaction entails. These categories included cumulative (CUMM), disputational (DISPU), exploratory (EXPLO), structuring conversation (SC) and managing understanding (MU) types of talk.

The analysis of modes of talk followed Schütte et al.'s (2019) principle of turn by turn analysis. This focus was helpful in unpacking how the teachers interacted with each other in terms of talk. The social relation (notion of interactions) is a very private aspect not easily accessible straight from the utterances, and the interpretation can often be subjective. However, through turn by turn analysis informed by Dudley's mode of talk indicators, one can detect to some degree how teachers interact with each other.

In Table 26 below, I present a glossary of statements and their descriptions which enabled me to identify possible modes of talk from teachers' interactions when they engaged with each other during L/S activities. This rubric was used for analysing each utterance within an episode and the findings were then synthesised. Of importance was that more than one mode of talk could occur within the same utterance. Such occurrence was considered during coding. As the analysis chapters unfold in Chapters 6, 7 and 8 of this thesis I will demonstrate how I have dealt with this issue. Furthermore, I will also show how these codes helped in carrying out the data analysis.

Table 26

A glossary of statements and their descriptors for modes of talk

Sub-Category	Code	Description	Examples
Cumulative	**CUMM**	**Description**	
		Agreeing or accepting	A participant agrees or accepts something someone has said, e.g. "Yes", "Okay", "I know", "Mmm", "Ja"
		Rephrasing, echoing, playing back words	Repeats or rephrases another's comment without elaborating or qualifying, usually to mark agreement.

			KO in line 16, was paraphrasing J's comments about a strategy of revisiting previous knowledge. [KO/LP1/EP1/L16]
		Supporting/valuing another person	Usually used to positively underline another's suggestion, acknowledging by agreeing, recognising importance, "Even for me just to support M, I think there was a learning moment" [J/RD1/EP5/L79]
		Recounting or describing an event	Recounting, either events from a lesson or observation or recounting from other notes the guidance on L/S from the workshop notes or when someone describes the structure of his lesson. Used in scene setting. KO, in line 50 recounted a previous meeting where there was a discussion about the rules of engagement and details about the roles of the team members and her role as the knowledgeable other. [KO/LP1/EP3/L50]
		Emotional expression	When someone is expressing an emotional expression such as feelings: "Yesterday's lesson for me it went well, the only challenge I faced there was the use of chalkboard and with the participation of the learners. At the start they were very quiet and I got off my tune." [T/RD1/EP1/L6]
		Acknowledging	Recognising or showing awareness of something In line 33, J admitted to being confused where he said, "… Now this is where I got confused." [J/LP1/EP1/L33]
		Requirement/need	Refers to participants' concerns about what needs are required to complete a specific task. Keywords used include we need, we want, etc. (necessity) "So what we are saying is that we need examples where they calculate these could be tasks." [KO/LP1/EP5/L111]
Disputational	**DISPU**		
		Evaluative and defensive statements	Evaluative and defensive talk is evidenced in line 75 "Let's not get deep into it" where the general orientation of T's response is defensive about his teaching actions during the previous teaching session. T/L73. I don't know, cause M/L74. Cause interrogating them can work T/L75. Let's not get deep into it [T,M/RD1/EP5/L73-75]
		Disagreeing, with little or no justification	In the following excerpt, KO is in disagreement with the use of bible, instead she suggeted a way forward without justifying what is wrong with the use of bible. "I don't think we should say this is a bible; this is what we should do." [KO/RD1/EP6/L88]
Exploratory	**EXPLO**		
		Rehearsing teaching (scenario)	In lines 17-24 where J commented on his awareness of learners' knowledge, and gave different scenarios to that effect or different questioning (i.e., when learners know and when they do not know).

			[J/LP1/EP1/L17-24]
		Developing a point, extending an idea	Picking up on a point and adding further ideas or reasoning. In line 114, in addition to KO's suggestions, J suggested the inclusion of counterexamples, and that example space needs to be varied. [J/LP1/EP5/L114]
		Making an evidenced, insightful point/interpretation	In lines 255 where M argued against J's use of the word "shifting", and gave a mathematical account of the effects of y-intercept and that it is about intersecting the y-axis, not about the shifts. [M/LP1/EP10/L255]
		Challenging the validity of an idea or action	In lines 255-256 where M argued against J's use of the word "shifting", and gave a mathematical account. [M/LP1/EP16/L255-256]
		Qualifying somebody's comment to improve it or express slight doubt	When someone legitimises someone's comment and attempts to improve it. Or when someone seems not convinced (in doubt) with another person's ideas/suggestion and attempts to improve it. For example, in line 213, where M did not think (doubted) that a table would be sufficient and suggested construction of a graph: "Not really, but the drawing a graph will show because table and the equation is more or less the same" [M/LP1/EP9/L213]
		Justifying an idea (defending an idea)	In lines 9-11 where J justified his teaching approach, as informed by his awareness of learners' knowledge and curriculum. [J/LP1/EP1/L9-11]
		Suggesting an idea (proposing an idea)	In lines 68-78, KO suggested the task that could be given to the learners through questioning such as interpreting a linear equation. [KO/LP1/EP3/L68-78]
		Reflecting, remarking	When the discussion has provoked a thought or a memory from previous teaching. In lines 3-5, J remarked about how he was going to go about his teaching approach of linear function. [J/LP1/EP1/L3-5]
		Hypothesising or speculating with reason	In line 107, KO conjectured that classification is one of the critical aspects. [KO/LP1/EP5/L107]
		Summarising, synthesising, or concluding	Drawing together a series of points or suggestions and making a summary point. In lines 290-293, KO explained the issue of generalisation by sharing a brief summary of executing of the lesson—teaching strategy through a metaphor of storytelling. [KO/LP1/EP12/L290-293]
		Making a reasoned proposal (requesting others' opinions)	When someone makes a concrete suggestion and requests others' opinions. In lines 57-59, S seemed concerned about learners' responses and made a reasoned proposal (seeking reassurance) from other teachers about the routine to use for justification purposes, like calculating using the method.

			[S/LP1/EP4/L57-59]
		Highlight, emphasise, stress	When someone places emphasis or draws attention to a certain aspect and maintains their position by emphasising or highlighting a point giving reasons. In lines 40-42, KO highlighted the authority and freedom resting on the person who is teaching, on how to break the ice in his interaction with the learners. [KO/LP1/EP2/L40-42]
		Asserting idea	There is sense of insisting one's idea repeatedly, maintaining one's position. In lines 13-15, where J insisted on the strategy of revisiting previous knowledge by explaining how the example of "spiderweb" could be the best. [J/LP1/EP1/L13-15]
Structuring Conversation	**SC**		
		Initiating or introducing new idea	In line 36, where KO took a lead role of a facilitator where she was explaining features of MTF, its strengths and weaknesses and how MTF could be used in an L/S. [KO/LP1/EP2/L34-36]
		Moving conversation on	Usually to refocus the group or to speed things up. In line 104, where the KO moved on a discussion by scaffolding the use of MTF with respect to example space. [KO/LP1/EP5/L104]
		Returning to an earlier subject	Picking up an earlier thread which either needed finishing off or upon which a subsequent discussion may have shed new light or reflecting on a discussion from earlier episodes or discussions. In line 253, S made a remark about the word use of "shifting" as used by J in the earlier episodes. [S/LP1/EP10/L253]
		Inviting participation	Encouraging others to contribute. In line 53, KO invited the teachers to suggest more examples to be included in the implementation stage of the lesson plan to increase example space. [KO/LP1/EP3/L53]
		Warn, remind, reassure	Use of prompts as a way of keeping focus on the goal. KO (lines 177-184) was warning the team members about pre-empting the issues that could emerge during teaching like learner ability with regard to classification of linear functions. She also reminded the teachers about OoL in focus—gradient and y-intercept. She further reassured the teachers that any critical aspects that could emerge during the teaching session would be used as a resource for another planning session of the cycle. [KO/LP1/EP8/L177-184]
Managing Understanding	**MU**		
		Requesting clarity or greater detail	In line 61, where M enquired about inclusion of other learner difficulties/errors with linear functions.

			[M/LP1/EP4/L61]
		Explaining	This is used when a group member needs to explain some background, curricular, pedagogic or other content-related facts to another group member. In lines 35-49, where the KO explained features of MTF, its strengths and weaknesses and how MTF could be used in an L/S. [KO/LP1/EP2/L35-49]

Once I had developed a framework which had been piloted on LP1 episodes and felt confident in using it, I was then ready to move on to the next stage of framework analysis and began coding the data. This involved the indexing and charting stages outlined by Ritchie and Spencer (1994).

5.4.5 Applying the codes to the transcript

Microsoft Doctools was used to index the transcripts. This software was useful in extracting coded comments in word documents, thus organising and sorting qualitative data following the framework method. Computer software has previously been used to organise and sort qualitative data following the framework method (Ritchie & Spencer, 1994) and SCDA (Mercer, 2010). At this stage it was easier to store, sort, and organise the data in this way due to the volume of paper there would have been if doing the indexing by hand.

I worked through the transcript text, highlighting a chunk of the text and deciding which code and the relevant category (or categories) from the framework to assign the text to. The highlighted text was then assigned the code. After indexing the transcript, the Doctools software enabled me to extract all data coded to a category for each utterance, as shown in Table 27 below. This table shows a similar extract from episode 5 that was used to illustrate the chunking of data in section 5.4.2.2.

Table 27

Example of the indexed data

		Maths object	MTF Objects	Maths routines	MTF Routines	Pedagogical routines	Lesson process routines	Contextual routines	M of TALK
KO	104. So, in your introduction, we know they know linear function. Your introduction should speak to the object of learning. You publicize y=mx+c. 105. Ask them *to identify* this function, then quickly guide them towards object of learning. 106. *Classification*, when you write that equation and *ask them* what is this type of function, and then you are speaking to *classification.* 107. What are the problems? Some might not be able to *classify* it. 108. So already now when you are speaking to *classification*, we are now coming to example space. 109. We know that some of the problems have something to do with *calculations*, so it means some of the examples must focus on *calculation.* 110. So our example space is guided by errors, content analysis and also our previous knowledge. 111. So what we are saying is that we need examples where they are *calculating*, and we also want them to *calculate* maybe gradient or intercept, these could be tasks. 112. Other examples could be dealing with *translation,* i.e. moving from graph to table to *equation*. 113. In our planning we need examples for *classification*..	Linear function Gradient intercept	Examples Ool tasks	ROUTINE *Classifying* *Translating* *calculating* VM Graphical symbolic	Exemplifying classifying_T translating_T calculation_T Identifying OoL CA_calculating CA_classify Generating tasks Classifying_L classifying_L calculating_L	scaffolding Identifying CA Structuring	lesson planning	Cont_L	SC EXPLO CUMM
J	114. Also include counter examples: *Is it* a linear graph? *Why are you saying that*?	Linear function	Examples	ROUTINE Classifying VM Graphical	Generating tasks Classifying_L	Questioning			EXPLO
KO	115. We are now working with learner engagement. We also need to vary our examples.		examples			Scaffolding		Cont_L	CUMM
KO	116. The examples, they are based on *classification*. Do you understand? So every time you are coming with these kinds of examples, you've got counter examples, examples in general form, 117. and then you say: now we have got the general formula in this form y=mx+c. By doing this, you are confirming that this is indeed a linear function	Linear function	examples	ROUTINE Classifying VM Symbolic	Exemplifying Classifying_T Teacher explaining Classifying_TS	Scaffolding			EXPLO
J	118. You can also use the graph. You can *draw* different graphs and *ask* them to*identify* them. 119. Because comparing to a standard form will not always work, like y=1 does not look like y=mx+c.. What interest me, we know that if y=1 we have a line cutting y axis at one, it is a linear function but once we have x=1 , we have a line cutting at 1 on the x axis, that is not a function. (*demonstrating with hands the vertical line test-showing horizontal line when uttering 'y=1' and vertical line when uttering 'x =1'*	Linear function	Examples	ROUTINE *Classifying demonstrating* VM Graphical Concrete	Exemplifying constructing_T Teacher explaining classifying_TS demonstrating_TS				EXPLO
T	120. So it can be part of your examples	Linear function	Examples						CUMM

Table 27 illustrates the list of categories on the top row. Within each column there are emerging codes. The transcript text is in the first and second columns, showing the comments or statements of the respective speakers. What is highlighted in the text column is only the aspects that were visible (pronounced and attended focus). The intended focus is the private aspect of the communication and depends on the interpretive work and is not shown in the text. I worked through each episode of the entire transcript in this way. This process led to the generation of an Excel matrix table, shown in the following section.

5.4.6 Charting data into the framework matrix

Ritchie and Spencer (1994) discuss charting the data by lifting it from the original textual context (the transcript) and placing it into charts or matrices consisting of headings and sub-headings which are actually the names of the codes and categories that comprise the analytical framework.

When the indexed text within the transcripts together with the accompanying codes was extracted to DocTools (as illustrated in Table 27 above) it was possible to create a matrix within Excel using tables. The categories (including subcategories) and codes of the analytical framework were used to create the headings of the matrix for the data in a table and the data was lifted (copied) from each "code" in DocTools and pasted under the relevant heading in the Excel table (an example can be seen in Table 28 below). This framework matrix table enabled me to view frequency counts of the codes which supported the process of identification of patterns of discursive routines and modes of talk within each episode, that is, key characteristics of the data in terms of mathematics discourse, MTF discourse, pedagogical discourse and modes of interaction.

Table 28

Framework matrix

CATEGORIES	SUB-CATEGORIES		CODES	EP1	EP2	EP3	EP4	EP5	EP6	EP7	EP8	EP9	EP10	EP11	EP12	TOT	
MATHS	Routines		Interpreting	12		4	2			1	4	11	4	2		40	
			Constructing	1				1	1	6	2	5	3	2	1	22	
			Classifying				1	8	2	4	5					20	
			Translating	2				1			1	4	2	1	1	12	108
			Calculating	1		3		2			2			2		10	
			Demonstrating	1				1								2	
			Defining			2										2	
	Visual Mediator		Verbal	2		2										4	
			Symbolic	3		5	4	9	2	2	13	10		2		50	
			Tabular	5		1		1		3		9	3			22	126
			graphical	8		1	1	3	1	6	5	8	7	5	2	47	
			Concrete	2				1								3	
MTF	MTF NAMING	OoL	OoL	1		3		1		2	2	1		1		11	
		Tasks	Tasks					1					1		6	8	
		Examples	Examples					10	2	3		5	1			21	44
		Explanations	Explanations									3			1	4	
		Learner participation	Learner participation													0	
	MTF ROUTINES	Exemplifying	Interpreting_T	3							1	3				7	
			Constructing_T					1		2	1	5	3	2		14	
			Classifying_T					4		2						6	28
			Translating_T					1			1					2	
			Calculating_T					1								1	
		Generating tasks	Interpreting_L	6		1		1				2	1			11	
			Constructing_L	1					1	4	1				1	8	
			Classifying_L	6				3	1	1	3					14	38
			Translating_L	2							1					3	
			Calculating_L	1				1								2	
		Teacher explaining	Interpreting_TS	4				0			5	6	2	1	1	19	
			Constructing_TS							1		2				3	
			Classifying_TS					4	1		1					6	
			Translating_TS									5	1	1	1	8	44
			Calculating_TS			3					1			2		6	
			Defining_TS	2												2	

	Learner explaining	Interpreting_LS$_u$	3		1				1		1	1			7	14
		Constructing_LS$_u$													0	
		Classifying_LS$_u$					1		1						2	
		Translating_LS$_u$										1			1	
		Demostrating_LS$_u$	1												1	
		Defining_LS$_u$			2			1							3	
	Learner_CA															37
	Teacher_CA	CA-Interpreting_T	8							2		1			11	
		CA-Constructing_T													0	
		CA-Classifying_T			1	1	1	1	3	7					14	
		CA-Translating_T					1				1				2	
		CA_calculating_T					1			1					2	
		CA_Word use			1						1	3	1	1	7	
		CA_defining			1										1	
PEDAGOGICAL	General Pedagogical	Structuring	3	3	4		1	5	1	6	1	1		1	26	231
		Revising	4	1	1	2				1					9	
	Mathematical teaching	Questioning	6				3	1		3	2			2	17	
		Generalising								1	4	2	1	3	11	
		Explaining	1												1	
	L/S pedagogical practice	Identifying CA	6	2	8	4	3	1	3	8	2	3	1	1	42	
		Reframing	0												0	
		Scaffolding	0	12	12	1	11	8	17	16	22	6	6	8	119	
	Other	Other	4								1			1	6	
LESSON PROCESS	L/S process	Ground Rules		1	1			2							4	15
		Roles		3	1								1		5	
		Lesson planning	1				1		1	2			1		6	
CONTEXTUAL	Contextual factors	Cont_L	3	1	3	2	2	2	3	2				1	19	26
		Cont_Reso		3				1	1			1			6	
		Cont_Class													0	
		Cont_Time													0	
		Cont_school teacher							1						1	
MODES OF TALK	Cumulative	CUMM	3		4	7	3	3	5	4	4	3	3	2	41	
	Exploratory	EXPLO	9	1	4	3	7	2	8	9	9	7	4	3	66	
	Disputational	DISPU	0	0	0	0	0	0	0	0	0	0	0	0	0	
	Structuring Converstion	SC	1	1	2		1	1	1	6	2	1	2	1	19	
	Managing Understanding	MU	1	1	2	3	3	0	2	2	0	2	2	2	20	

The first two columns of the framework matrix reflect the components of the initial a priori framework (IAF) as discussed in detail in section 5.4.2.2. The table makes provision in columns 4-16 for displaying the frequency count of emerging codes from each episode (episodes 1-12). The last column shows the total codes under each sub-category.

The framework matrix shows patterns of data across all twelve episodes during LP1. These were related to pronounced focus—objects of attention such as mathematical and MTF objects. It further illustrates the attended focus—various types of routines teachers were performing such as mathematical, pedagogical, L/S process and contextual routines. The intended focus was not shown in the table since this is the private aspect of the communication. For example, EP1 shows that teachers were reflecting about various MTF objects related to the linear function. They also attended to different sets of routines categorised into mathematical, pedagogical and contextual routines. The EP1 column further shows that the teachers were interacting with each other mostly in exploratory talk.

In Appendix F, I attach the summary of what was in focus in each episode in terms of the discursive focus tripartite.

Once all data was charted in this way and the numerous sheets printed out and laid out, I looked across the codes and categories to begin forming descriptive themes. Both RD1 and RD2 followed the similar process. These are presented with discussions in Chapters 7 and 8 and also in Appendices D[13]. The way that the data was presented in the framework matrix (Table 28) also made it possible to look within each L/S session (e.g. LP1, RD1, RD2) at episodic level across L/S sessions in terms of the data that had been coded to each code of the framework.

The framework matrix was used as a tool in the management of complex data, identification of patterns of discursive routines and modes of talk and facilitating the interpretation of data. However, the thinking judging, deciding, and interpreting were still performed by me. As Tesch (1991, p. 77) explained, "The computer does not make conceptual decisions, such as which words or themes are important to focus on, or which analytical step to take next. These analytical tasks are still left entirely to the researcher". The aim of charting data in a framework matrix was to refine and develop the codes and categories into themes; key themes and sub-themes. Having the data organised in such a way facilitated the next stage of interpreting the data.

[13] The framework matrix for RD1 and RD2 are named as Table 28_A and Table 28_B respectively.

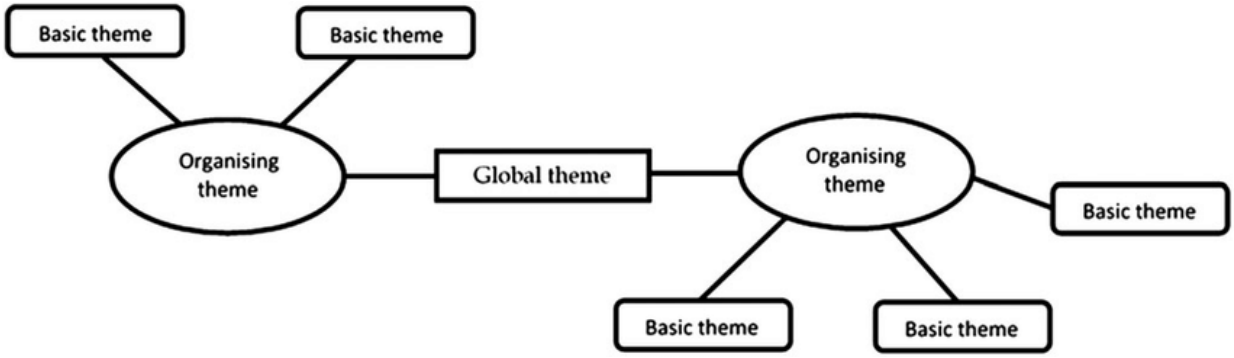
5.4.7 Interpreting data

Once all of the data was charted into the framework matrix, I now moved beyond data management towards understanding it. With the data organised in this way I started identifying initial and eventually key themes. The process of generation of themes requires a movement beyond a description of a number of categories. According to Saldana (2009, pg.13), a theme "is an outcome of coding, categorization, and analytic reflection, not something that is, in itself, coded". It is "a phrase or sentence describing more subtle and tacit processes" (Rossman & Rallis, 2003, cited in Saldaña, 2013, p. 13). In the case of this study a theme was understood to be the association between different categories and how they are in conversation with each other. Ritchie and Spencer (1994) described this process as pulling together key characteristics of the data, identifying associations within the data to map and interpret the data set as a whole, and developing explanations for these. Thus, the theme must do more than provide a category—it must provide a link between one or two categories. For example, in episode 5 (Table 27) the emerging theme: pedagogical routines, were used to support the process of generation of MTF objects and reflected two emerging routines: one related to the pedagogical category (questioning) and the other to the contextual factors category (learners). The analysis showed that there was a connection between the two which resulted in the generation of MTF objects—tasks, for example, in line 114 where J used a questioning routine prompting learners to give an answer to a question: "Also include counterexamples like $y = 3/x + 2$, and ask: Is it a linear graph? Why are you saying that?" This utterance was inviting the learners to classify the given function presented in symbolic form. In other words, learners were expected to engage in a task. The significance of the pedagogical routine is that it enables production of MTF narratives.

Following the analysis of associations between the emerging patterns in each category which resulted in the establishment of emerging themes, I borrowed from Attride-Stirling's (2001) thematic network ideas where I classified the emerging (basic) themes according to the underlying story they were telling and these became the organising themes (organises the emerging themes into clusters of similar issues). The organising themes were then reinterpreted in light of their emerging themes, and were brought together to illustrate a single conclusion or macro theme that became the global theme (Figure 5 below).

Figure 5

Structure of thematic network



Note. Adapted from Attride-Stirling (2001)

The next chapter covers the analysis of these findings, illustrating relevant findings for LP1 drawn from the teachers' discussions through thematic networks.

5.5 Conclusion

Having explained how the SCDA and FA were developed and used to analyse the data in this chapter, in the following four chapters I present detailed analysis that enabled the answering of the research questions. In Chapters 6, 7 and 8, I illuminate how teachers have used the MTF tool in the context of L/S. In Chapter 9, I discuss the nature of teacher learning in terms of the opportunities for learning that emerged through the use of the MTF tool in L/S.

Chapter 6: The Reflective Actions during Lesson Planning 1—LP1

6.1 Introduction

In this chapter I present the qualitative analysis of the reflective actions during the first lesson planning session following themes emerging from the quantitative analysis. Firstly, a brief summary of the key themes is given in terms of a thematic network. This is followed by a presentation and discussion of the organizing themes. The organizing themes (together with accompanying emerging themes) represent an exploration of teachers' mutual engagement and joint enterprise (negotiated responses-construction of narratives) of the MTF tool. Lastly the main discussion, I take the key findings of the organizing themes pulling them together into a global theme which represents the shared repertoire of the L/S CoP. I will follow the same structuring with RD1 and RD2 analysis. In the presentation of the results across this and Chapters 7 and 8, extracts from a single episode or different episodes of the transcripts (LP1, RD1 and RD2) have been used to support the findings reported, and to illuminate teachers' discursive talk/actions to illustrate a point as they used the MTF tool. Extracts from the three transcripts were selected with two main principles. Firstly, the extracts were selected for their richness in making the overall argument. Secondly, the discursive action in an extract had to be identifiable following the focal analysis triple: pronounced, attended and intended foci.

6.2 Summary of identified themes

As noted in the previous chapter, the emerging themes resulted from the associations between categories. As shown in Table 29 (second column), a total of 11 themes emerged. Thereafter, the data supporting the emerging themes were searched for similarities or shared ideas that could be used to interconnect the themes. Four similarity clusters were identified and used to define four higher level "organising" themes (Table 29, first column).

Table 29

Summary of emerging and organising themes for LP1

Organising the themes	Emerging themes
1. Use of MTF was linked with participation in mathematics discourse	▪ Teachers used mathematical and informal mathematical words ▪ Routine- and object-driven word use of mathematical objects ▪ Use of different routines/visual mediators
2. The use of MTF was linked to participation in pedagogical discourse	▪ Questioning routines used to generate tasks and explanations ▪ Identifying CA routines used to generate CA of OoL ▪ Scaffolding routines used by KO to support construction of MTF narratives ▪ Structuring routines enabled re-modification of MTF lesson plan tool
3. The use of MTF meant participating in the MTF discourse	▪ MTF Naming—routine- and object-driven ▪ MTF routines showed similar type of routines in each MTF object ▪ MTF routines linked to both mathematical and pedagogical routines
4. The modes of interaction supported the MTF-use	▪ Hybrid between exploratory talk and other talks enabled teachers' construction of narratives related to MTF-use

Finally, the supporting data were examined again in order to determine the global theme representing the key points, or main meanings of the data.

One global theme emerged, namely, that discursive activities performed when lesson planning about teaching of "effects of gradient and y-intercept" using the MTF tool actually belong to three distinct but intertwined discourses: a mathematical discourse, MTF discourse and a pedagogical discourse. In this chapter, this global theme will be explained with the help of summaries from different organising themes. These again emerged from the emerging themes that were directly derived from the coded text. The first three organising themes shown in Table 29 related to teachers' discourses as they used the MTF tool: mathematical, pedagogical and MTF discourses. These discourses were characterised by different discursive activities, namely, mathematising, teaching and MTFying respectively. The fourth organising theme related to how teachers interacted with each other when using the MTF tool.

In order to be able to study discursive activity, one needs to know more of what characterises different discourses in general and mathematical and MTF discourse in particular in the case of this study. Thus I will take into account four elements of analysing discursive activity offered by Sfard (2017): (1) the objects that are being talked about; (2) sources of narratives; (3) perceived goals of discursive activities; and (4) participants' (KO, teachers) roles in these activities. In addition to this, I will also analyse how teachers talk about pedagogical actions and explore how these actions are related to both mathematising and MTFying discursive activities.

LP1 discourse is ideally a combination of teachers' and KO's discourse. However, the data from the LP1 observation showed that the LP1 discourse mainly consisted of the KO's discourses. Nevertheless, extracts from all participants will be provided. As a way of concluding, I will comment on the roles

played by each. It is important note that as the KO I played several roles in this study: a full participant (teacher); an expert of the discourse and a researcher.

6.2.1 Organising theme 1: Use of MTF was linked with teachers' participation in mathematical discourse

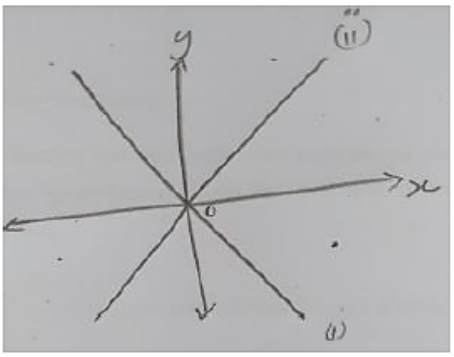
This organising theme pertains to the construction of mathematical narratives characterised by features of mathematical discourse, specifically words and word use. As discussed in Chapter 3, in order for the learners to participate in mathematical discourse they need to be exposed to the mathematical discourse of the mathematical community. It was therefore important to investigate the nature of the teachers' mathematical discourse. More specifically, one aim was to describe the discourse of mathematics as it appeared through the discursive activities of the teachers (including the KO). In the following three sections, I will describe and explore the emerging themes in support of this organising theme. I will then conclude this section by providing a summary of the main findings emerging from the three emerging themes in order to describe the nature of the mathematical discourse during LP1.

6.2.1.1 Emerging theme: Source of mathematical narratives included the mathematical and colloquial word use

This emerging theme focused on the nature of the mathematical objects teachers referred to, that is, what was pronounced. The analysis of the mathematical utterances showed that the teachers were using mostly mathematical words to refer to mathematical objects. However, there were some instances where teachers used a combination of informal mathematical and mathematical words to refer to mathematical objects. I summarise and exemplify with extracts from two different episodes below. In these extracts I highlight the routines (marked with *purple italics*), visual mediators (underlined) and the objects in focus (**bolded)**. I also highlight in blue the implied performers of the discursive activity, which will be discussed in section 6.2.1.4.

Extract 1

An example showing combination of mathematical and informal mathematical word use

Episode	Teacher	What is said	What is done
EP1	J	17. Yes, from that, y equals to two x ($y = 2x$) which is a **linear function**. I would be having a graph with a **y-intercept of zero** , but with a **gradient of 2**. 18. So now I can talk about the **slope**, meaning I am going to my **objective of learning** which are the **effects of m and c**. So now I can talk about the slope: "what do you notice about this line and this line?" (*pointing to line (i) and (ii) in the sketch shown on the right-hand-side*) 19. I hope they say this is horizontal. "And now on real life. On your horizontal line you just *walk*, you don't feel any pain, why is it that?" If they do Geography they will say. If they don't, (*using hands to demonstrate a horizontal line*) 20. I will probably say it's because the slope is zero, it's flat, it is not *steep* or what. "But now that, that's the case. What do you think if you had to *walk* on this line?" (*pointing to line (ii) in the picture on the right hand side*). 21. "Do you think you would feel the pain? Yes, why?" *It inclines*, *steepness* all those. That's how I'm gonna introduce my notion on **gradient**. That's the case. Because where our **gradient** is zero, why is it zero? Because it's horizontal, all that.	
EP5	KO	104. So, in your introduction, we know they know **linear function**. Your introduction should speak to the **object of learning**. You publicise y equals to mx plus c ($y = mx + c$) 105. Ask them to *identify* **this function**, then quickly guide them towards **object of learning** 106. *Classification*, when you write that equation and ask them "what is this type of ***function?***" And then you are speaking to *classification*.	

Mathematical utterances in EP1 (lines 17-21) showed a combination of informal mathematical and mathematical words and word use. For example, in J's mathematical utterance in line 17, the pronounced mathematical object was the "linear function" with its properties "gradient" and "y-intercept", and he used mathematical words to refer to the object and its properties and attended to the mathematical procedure of interpreting the behaviour of the symbolic representation ($y = 2x$) in terms of its gradient and y-intercept.

However, J then switched to informal mathematical words in line 18, and referred to the gradient as the "slope" of the line, and referred to the linear function as the "line". He then linked the word slope with a metaphor of pain when walking on a horizontal line, and then linked this experience with slope and used the following phrases "the slope is zero" and "it's flat, it is not steep". This suggested that J was thinking about a mathematical description of a gradient of a horizontal straight line (flat line) viewed from the

Cartesian plane which is equal to zero, as explicitly uttered by J in line 21 “where our gradient is zero, why is it zero? Because it’s horizontal, all that”. What is interesting is that the word “slope” was linked to an action found in informal talk: “walking on a horizontal line”. The imagined demonstration of walking on the line was used to substantiate the narrative about the gradient: “gradient is zero”, indicating a shift from informal to a mathematical way of substantiating the effects of a gradient.

In another example, where the word use was mathematical only was seen in lines 104-106, where the KO used the mathematical phrase “linear function” to refer to a mathematical object and linked the word to a visual mediator in a symbolic form which was further manipulated using a mathematical routine of classifying. The attending procedure of classifying was used with the intention of substantiating narratives about “linear function”.

6.2.1.2 Emerging theme: Source of mathematical narratives included action- and object-driven word use

This emerging theme further described the nature of the objects that the teachers referred to (pronounced objects).

In Extract 1, an immediate interpretation is that teachers’ mathematical utterances were action oriented. This means the observed teachers’ mathematical talk showed that they were applying mathematical procedures to visual mediators as sources of constructing and substantiating mathematical narratives about mathematical objects. This was reinforced by a strong connection between features of teachers’ mathematics discourse (i.e., words, routines, visual mediator and substantiated narratives). For example, those who were referring to mathematical objects in a mathematical way were observed linking the mathematical words (e.g. linear function) by applying a mathematical action (e.g., classifying) to representations (e.g., symbolic form). Those who used a combination of informal and mathematical ways were observed linking the mathematical words (e.g., gradient) to informal mathematical words (e.g., slope) and informal actions/routines (e.g., demonstrating) to imagined visual mediator (horizontal line). In both scenarios, the mathematical procedures applied to visual mediators were used to substantiate the narratives about “linear functions” and “gradient” respectively.

Looking across all the LP1 episodes, there were a number of instances where the mathematical objects such as “linear function”, and their properties “y-intercept”, were talked about as if they were occurring

by themselves without any human intervention. I exemplify with the following utterances from different episodes:

Extract 2

An example of object-driven word use of mathematical objects

Episode	Teacher	What is said
EP1	J	5. I thought the first thing I would do is *define* **functions**. Before I can talk about the **object of learning**, I would *define* **functions**, this is the main topic, and not that it's an **object of learning** 6. Which is *defined* as a set of inputs and outputs that put the relationships and all that I would *define* **functions** as a whole.
EP3	KO	67. Those are the two things that we should be thinking about. They have been introduced to **linear functions**, they know **it**, and they should know **it.** 68. They know **linear function** is y equals mx plus c ($y = mx + c$). 69. So, in your introduction you should write $y = mx + c$.
EP9	KO	247. Where your line is *cutting the y- axis*, you can see **its coordinates,** that is, your **y-intercept**.

In line 247, KO referred to a line (linear function) as if it was doing an act by itself, that is, it is "cutting the y axis".

There were other instances where teachers' mathematical narratives reflected object-driven word use. For example, J in lines 5–6 talked about the mathematical object "function" in a mathematical way only. In these utterances the naming of the object "function" was accompanied by the endorsing narratives usually found in school mathematical discourse, that is, signifiers of function (verbal form "set of inputs and outputs"). Similarly, in line 68, the naming of the mathematical object "linear function" was accompanied by its common signifier of a linear function (symbolic form:"$y = mx + c$"). These signifiers are commonly used in school mathematical discourse in order to construct and substantiate mathematical narratives related to both functions in general and linear functions in particular. The object-driven word use signals objectification—where talk of human actions (processes) is replaced with talk about mathematical objects (and mathematical signifiers). According to Sfard and Lavie (2005), objectification is a necessary process towards developing an explorative mathematical discourse.

6.2.1.3 Emerging theme: Sources of mathematical narratives included the use of various mathematical routines

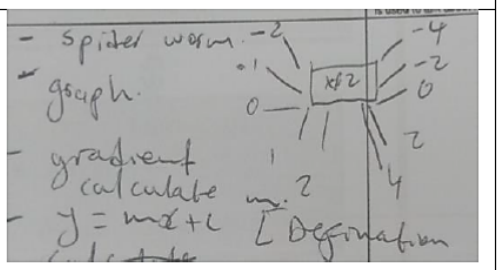
This emerging theme was concerned with the nature of the attending procedures with respect to mathematical objects. Considering mathematical routines (see Table 28), a classification of

mathematical routines is presented including interpreting, constructing, classifying, translating, calculating, demonstrating and defining. These routines are specifically connected to the topic of function.

The analysis showed that the teachers' mathematical narratives were characterised by use of various attending procedures in order to substantiate mathematical narratives with respect to mathematical objects. This happened when the teachers used more than one mathematical routine to substantiate one narrative. I illustrate with the following extract:

Extract 3

An example of use of various mathematical routines

Episode	Teacher	What is said	What is done
EP1	J	9. I thought of going back to grade 9 whereby they know a spider diagram, as inputs and outputs and whatever is here *(pointing to the picture on the right)* 10. it gives the relationship from the input and get the output which they can either write, *choose any inputs, multiply by 2 and it gives out the output* 11. So I can say, now that you have done this they can *either tabulate* whether it would be the x and the y; the x being the input and the y being the result they get from the spider web diagram. Maybe the resulting relationship can confirm the **linear function**.	

In lines 9-11, J's attending procedures offered two alternative mathematical procedures in order to substantiate a narrative about the linear function (in terms of an input/output relationship). First, he suggested that a learner may do calculations or they may construct the table in order for them to see the relationship between the x and y variables. This meant the attending procedures of both calculations and constructions were used in conjunction with the input-output relationship as sources to construct and substantiate mathematical narratives about the linear function. In Lavie et al.'s (2018) terms, the use of different routines signals the aspect of flexibility which is the characteristic of teachers' explorative mathematical discourse.

6.2.1.4 Summary

The mathematical discourse emerging during LP1 was a special kind of discourse with function discourse features: words (linear function, gradient, y-intercept, and slope); routines (classifying, interpreting)

visual mediators (graphs, equation, tables) and narratives being described by constructions, derivations and substantiating narratives. From the above it is further possible to describe the nature of participation in the LP1 mathematical discourse in terms of ritual-explorative principles, as illustrated in Chapter 3.

1. Nature of maths objects (in terms of words and word use)

Teachers have demonstrated combined use of informal mathematical and mathematical words to refer to mathematical objects. This combination was evidenced in the analysis in section 6.2.1.1 above, where I provided an example of teacher J's narratives in lines 17-21. What needs to be noted, as much as teacher J has used informal mathematical words, his word use was mathematical. In other words, he was referring to mathematical ideas but expressing them more informally. This indicates teachers' awareness of informal words that are present in a normal mathematical classroom discourse. It is interesting to note that teacher J's talk demonstrated efforts to shift his discourse (combination of informal and formal mathematics) towards a more literate mathematical discourse. Such a shift could be an indication of explorative mathematical discourse (Sfard, 2008). Teachers' awareness of both informal and formal mathematical word use has important implications for learning mathematical legitimising criteria for mathematical teaching (Adler & Ronda, 2015). Another important observation to be noted is that teachers' word use in relation to mathematical objects was both object- and routine-driven. In Sfard's (2008) terms, these two modes of communicating, when blended together, support the initiation of the objectification process—an element of explorative discourse.

2. Sources of mathematical narratives

The analysis in section 6.2.1.3 has indicated the use of various routines as sources of constructing and substantiating mathematical narratives. This finding suggests that various routines were used to strengthen the substantiating process. For Sfard (2008), the application of different routine procedures such as calculating, constructing, scanning etc., and also using different but equivalent representations (translating), in an effort to substantiate the same narrative, implies an aspect of flexibility, an indication of explorative participation.

The analysis has shown that the substantiating narratives resulted from a series of interconnections between words and discursive word use. Discursive word use includes routine use, visual mediator use and substantiated narratives. For example, when teachers were participating in the classification

mathematical activity of a linear function, they were observed uttering the mathematical words (e.g., "linear function") and constructing a linear graph in order to confirm whether the given function was a linear function or not. Sfard (2008) posited that a word and its discursive use should function in concert with each other. Lavie et al. (2018) referred to this process as "bondedness", a necessary characteristic of an explorative discourse, which implies that each move in the discursive activity gives rise to another move and towards a certain final outcome, which are substantiations in this study.

3. Goals of discursive activity

I have identified two types of discursive goals: object-level and meta-discursive goals. The former relates to the goals related to mathematical discursive activity; the latter relates to the goals that are developed as a result of participating with others in the mathematising activity. In other words, these are culturally bound goals.

*Meta-discursive goals***:** An important observation worth mentioning is that the goal of participating in mathematical discursive activities was to construct mathematical narratives for either teachers' or learners' use. This was heavily evidenced in all teachers' mathematising utterances that to the learners (see keywords "they") or the teachers, including the KO (see keywords "I", "you", "we") as highlighted in blue in the extracts 1-3. In these mathematical utterances, the mathematising talk was accompanied by a performer or intended performer of a specific mathematical activity. Take for example in line 247 (Extract 2), the KO associated a mathematical object with the teacher and further linked the discursive performance of seeing (interpreting in mathematical terms) to the same performer, that is the teacher: "Where your line is cutting the y-axis, you can see its coordinates that is your y-intercept". These utterances and many of the teachers' mathematical utterances show a strong relation between mathematical routines and the learners and teachers. Such a relationship is further explored in the upcoming section of the MTF discourse, where the analysis shows what this relationship means in terms of MTF-use.

*Object-level goals***:** Another mathematical activity goal during LP1 was to construct and substantiate mathematical narratives about linear function. For example, the analysis of the three emerging themes suggests that the mathematical routines were used as sources of constructing mathematical narratives and substantiating them, for instance in the pronounced narrative in line 106 (Extract 1): "*Classification*, when you write that equation and ask them *what is this type of function*, and then you are speaking to

classification". In this narrative, the mathematical routine to be applied was classifying routine. Classification is related to an action aimed at deciding whether a particular relation is a function (Leinhardt et al., 1990). An action of deciding is related to substantiating narrative. One can conclude that the mathematical activity in this mathematical narrative was about substantiating of mathematical narratives related to a linear function.

4. Roles

It is worth noting that in this mathematising discursive activity, it appears that both teachers and KO were seen as equal contributors to the mathematising talk, as evidenced in the heavy use of keywords related to collectiveness such as "we", especially from the KO's utterances. At the same time, the KO was seen to be positioning herself as the leading interlocutor. This is amplified by her authoritative voice which seemed to suggest that the KO was taking up a role of providing direction on how things need to be done, as shown in one of her utterances, for example, in line 69 (Extract 2): "So, in your introduction you should write $y = mx + c$". This finding helped me develop the argument later in this chapter about modes of talk and mutual accountability.

In summing up, from the discussion above it is possible to conclude that teachers' word use was characterised by properties of explorative mathematical discourse such as flexibility, objectification and substantiability. The L/S activity thus provided a context for such use. How this related to their teaching discourses will emerge later. As these were pre-service mathematics teachers, it was not surprising that their mathematical discourse of school-related mathematics was objectified, though there was evidence that this was not necessarily the case. For example, Sfard's (2017) analysis of a practicing secondary mathematics teacher in South Africa showed that the teacher's own discourse was not explorative. The goal of mathematical activity during the LP1 session was on construction of mathematical narratives about linear function and substantiating them, with respect to both teachers and learners as performers. It can then be argued that the teachers' mathematising talk suggests that they had the opportunity to participate in explorative discourse and so reinforce their own mathematics, and at the same time this reflects what and how they were able to talk mathematics in their teaching. It is useful to note that teachers' participation in mathematics discourse did not enable their learning but afforded opportunities that could promote learners' explorative mathematical learning.

6.2.2 Organising theme 2: The use of MTF meant participating in MTF discourse

While the L/S group was using MTF to plan a mathematics lesson, and so talking about mathematics and teaching while engaging with the tool, they were also learning to use the MTF. The teachers were newcomers to the discourse of the MTF, and so negotiating the meaning of its elements while planning the lesson. Hence, I focused on the MTF discourse itself—how the L/S group worked to make sense of the tool.

In this study, the main focus was on research into what constitutes the MTF discourse, and the relation between MTF discourses, mathematical discourses, and teaching discourses. The opportunities to participate in the MTF discourse were co-constituted between the KO and teachers. Teachers were learning how to use the MTF tool in the L/S teaching activities such as lesson planning, enactments of the lesson and reflective discussions. As such, the teachers were learners. On the other hand, the KO was playing a role of introducing teachers to the MTF tool and its use, where she was bringing knowledge of how the MTF tool may be used. As such, the KO was the facilitator, as well as the leading interlocutor with respect to the MTF.

As mentioned earlier, in this study the MTF was viewed as a discursive activity which I have called "MTFying", where MTF objects were defined as names (words) of MTF objects together with their discursive use, and using the MTF was viewed as engaging in the MTF discourse.

This organising theme explored how the MTF tool can be viewed as a unique discourse characterised by words and routines specific to the MTF discourse, as done with the mathematics discourse in the previous section. Thus, in this section, I will investigate the nature of MTF objects present in the participants' utterances. In addition to that, I also explored the sources of MTF narratives. The analysis will further describe the goals of participants' (i.e., KO and teachers) discursive activities, in other words, the utterances and actions implied with regard to goals of MTF activities and, finally, what the data suggests in terms of different roles played by the participants.

Looking at the MTF category (refer to row titled 'MTF' in Table 28[14]) one is able to see the MTF category showing the two subcategories developed during the analysis: MTF naming and MTF routines. The MTF

[14] Table 28 refers to the Framework matrix illustrated in Chapter 5 section 5.4.6. I will constantly be referring to this table throughout the analysis in this chapter.

naming category shows names of the components of the MTF tool. The MTF routines were characterised by various mathematical routines.

In the following section I will attend to these categories in turn, starting with MTF naming.

6.2.2.1 Emerging theme: MTF naming

Looking across the MTF row in the last second column of Table 28, it is evident that the names of MTF objects were used explicitly. These included the names of MTF components such as OoL, examples and tasks. Interestingly, there was no reference to "learner participation"; later I will reflect on how this element of MTF was treated during teacher engagements. Interestingly, the explicit naming of MTF objects was mostly from the KO's utterances, with few instances coming from the teachers. In those few noted instances, teachers explicitly referred to the MTF names only as a way of trying to get more clarification of what the KO had said, as shown in the following extract.

Extract 4

Example of teachers' explicit naming of MTF components

Episode	Speaker	What is said
EP12	KO	285. Now when you move to the more mathematical **explanation**, that's when you bring in the language of shift. 286. Now, as a way of concluding your lesson, you *generalise* that every time you have plus c it shifts your graph c units up.
	S	287. Are you going to generalise after you have given them **the task?**
	T	288. No, just here
	S	289. Mina I would choose *to generalise* after **the task**

This is not surprising, since it was the teachers' first experience with using the MTF tool. It was interesting to anticipate whether this would change in the next L/S sessions (RD1 and RD2) and whether the teachers would begin using the MTF names more after having been exposed to the MTF tool and its elements during LP1.

When one analyses extracts in instances where the MTF objects are named explicitly, it seems the explicit naming served three purposes: (1) to highlight the type of MTF objects being attended to; (2) to stress the strategy to be used to construct a specific narrative about the MTF object; and (3) to objectify

the MTF objects, which means the MTF objects are talked about as if they have a life of their own without any human action. The following extracts from different episodes demonstrate this.

Extract 5

Example of different purposes of MTF naming

Episode	Speaker	What is said
EP5	KO	109. We know that some of the problems have something to do *with calculations*, so it means some of the **examples** must focus on *calculation*. 110. So our **example space** is guided by errors, content analysis and also our previous knowledge. 111. So what we are saying is that we need **examples** where they are *calculating,* and we also want them *to calculate* maybe **gradient or intercept**, these could be **tasks**. 112. Other **examples** could be dealing with *translation*, i.e. *moving from graph to table to equation.* 113. In our planning we need **examples** for *classification*.
EP8	KO	172. If we are saying our **object of learning** is *classification*, then we need to prepare our graphs, 173. we *draw* them, then they will need to *compare* which equation belongs to which graph
EP9	KO	210. The **explanations** are going to be coming directly from the **examples**.
EP12	KO	293. Even with **tasks**, you don't need to go and *design them*. The way you *question* your learners during teaching serves as **tasks**.

The analysis showed firstly that MTF objects—examples, tasks, explanations and the OoL—were explicitly named, and that the talk related to each of these was linked to mathematical routines. For example, in lines 109, 111 and 112, the explicit naming of the MTF object-example was accompanied by the mathematical actions related to "calculations", "translations" and "classifications".

The explicit naming of examples was also used to bring to attention the pedagogical strategy to be used to construct the MTF narratives. For example, in line 293, the KO highlighted that the questioning routine was related to the MTF object of tasks. Another instance is shown in line 110 where the KO utterance suggested that the construction of examples was guided by the process of identifying CA as indicated in the previous chapter on methodology. These two utterances suggested that both the pedagogical routines were sources of construction of MTF narratives. This relationship will be dealt with in detail in the section on the analysis of pedagogical discourse.

The link between the explicit naming of MTF objects with specific discursive routines suggested routine-driven word use as seen with mathematical word use. In other words, the naming of MTF objects was accompanied by discursive actions from either mathematical or pedagogical discourses.

The analysis also exposed a number of instances where the MTF components were talked about as if they had a life of their own without any human action, as highlighted in line 210: "The **explanations** are going to be coming directly from the **examples**". This way of talking about explanations implied an object-driven word use of MTF objects as seen with mathematical objects. The combination of object- and routine-driven word use signalled a possible process of objectification, a property of explorative discourse, which in this instance was MTF.

As already noted, in most of the extracts the explicit naming of MTF components was from the KO utterances, which suggests the KO's intention to foreground the MTF tool and its objects. This resonates with Sfard's (2017) suggestion that the expert could support newcomers to the discourse (teacher in this case) by explicitly encouraging the desired discourse. I suggest that the explicit naming of MTF objects served to highlight the discursive focus and the structuring of the conversation to enable the teachers to use the MTF. This was done by focusing on MTF objects (examples, tasks, explanation, OoL), and using these in their teaching activities in the L/S context such as lesson planning, enactment of lesson (teaching) and post-lesson reflective discussions. In Wenger's (1998) terms, it can be argued that the process of MTF naming seem to have provided points of focus through which negotiation of meaning could be organised.

6.2.2.2 Emerging theme: MTF Routines

The third column of Table 28 in the MTF routine row shows that five types of routines were included: exemplifying, generating tasks, teacher explaining, learner explaining and CA of OoL. These routines are pedagogical in nature specifically connected to the components of MTF. These five types of routines played the same role in the description of the discourse of MTF as the mathematical routines did for mathematical discourse. Like them, they highlighted the function they were meant to serve within the discourse. They described the discursive acts of teaching (Mosvold, 2016). The description of the five MTF routines also contributed to a methodological development since it provided categorisation of MTF objects in communicational theory terms.

The in-depth analysis further evidenced that both teachers' and KO's utterances showed similar patterns with regard to MTF talk, as seen with mathematical talk in section 6.2.1.1 above. I will therefore provide evidence from all participants and highlight any deviations. The analysis with respect to discursive routines related to MTF objects revealed several findings. First, each narrative about each MTF

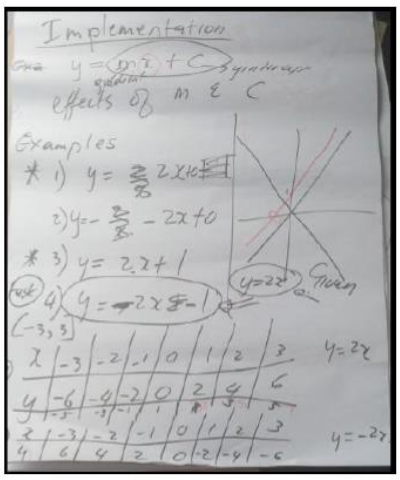
component was accompanied by mathematical routines. These mathematical routines were presented in various forms and were connected to each MTF object. Second, there was strong evidence of interconnectedness amongst MTF routines. Third, each MTF discursive routine was attached to a specific performer, either the teacher or the learner differing in their perceived goals. Lastly, each MTF narrative was linked to some pedagogical routines. In other words, when the participants talked about the objects of the MTF they did so with reference to mathematical routines together with the accompanying performer of those routines and/or teaching actions/pedagogical routines. I now discuss these different analyses for each MTF object. I also discuss what these different analyses imply in terms of the goals of teachers' discursive activities and sources of their MTF narratives.

(i) Examples

I use Extract 6 to discuss the nature of these MTF routines. In this extract, I also mark the objects in focus (**bolded**), the routine procedures (italics in *purple*), and mathematical representations (underlined). I also highlight the performer (or intended perfomer) of mathematical procedures in blue. All the extracts used to provide evidence in all MTF objects are highlighted in this manner.

Extract 6

An example of MTF routines related to examples

Episode	Speaker	What is said	What is done
EP9	KO	208. So let us start with our implementation. In our implementation we need **examples** **y equals two** **x** ($y = 2x$) I am thinking of kind of **explanations**	
	T	209. Let us change **the signs** *(writing on board, see picture on the right-hand side)*	
	KO	210. The **explanations** are going to be coming directly from the **examples.** 211. We can start off the **explanations** by saying, "Now look at the value of a **gradient is 2**", so we would like *to illustrate* with more **example**s, with more **explanations**. 212. For illustrations we can use tables. Will the **table** *show* any **effect on m and c?**	
	M	213. Not really, but the *drawing* a graph *will show* because **table and the equation** is more or less the same	

	KO	214. The table is going to help us to be able *to draw* a graph. It is part of our **explanation.** 215. In a table we will be able *to see* when we are changing **the sign of your gradient.** 216. It is not that we will be wasting time with our table; we can still use it to *draw* the graph. So let's *draw* it.	
	ALL	217. Yes	
	KO	218. An **interval of -3 to 3** 219. So we are going to start off by having a table here *(writing on the board)*. This is one of our **example** -3,-2,-1,0,1,2,3; now we are doing it together, -6,-4,-2,0,2,4,6 220. So we can do for the **second one**, then here we are going to have 6,4,2,0,-2,-4,-6 221. So what we did here was to *multiply* the **gradient** by minus 2. Can we see the difference between these two?	

An analysis of the extract above showed that the examples of linear functions were offered in different representations. For example, in the series of interactions from lines 208-221, the KO first highlighted an example of $y = 2x$ in a symbolic form. This was followed by T's proposal in line 209 where he suggested changing the sign of the coefficient of x which resulted in yet another example now of the linear function $y = -2x$ (as shown in the picture). Further representational forms of the examples were proposed. For example, in line 212, the KO suggested an example in the form of a tabular representation, followed by M in line 213 who further suggested an example in the form of a graphical representation. This whole set of representations implied a set of different examples of linear functions aimed at helping the teachers with illustrations intended at providing substantiations about the effect of the gradient in the linear function. The different representations further implied different mathematical routines. For example, the graphical and tabular representation implied either constructing or interpreting routine, and symbolic implied either calculating or interpreting routine, depending on the goal of the mathematical activity. Such an occurrence was evidenced in different utterances in the extract above. For example, in lines 218-221, the KO was demonstrating the process of construction of a tabular representation of the linear functions $y = 2x$ and $y = -2x$. As noted, the use of various mathematical routines signals an element of flexibility linked to explorative mathematical discourse. Hence, it is possible to conclude that MTF routines related to examples could be explorative in nature.

What needs to be noted is that the exemplifying talk across all participants' utterances evidenced that the examples were aimed for teachers' use. Each of these utterances highlighted the mathematical procedure together with the performer (the teacher, in this instance) for illustrative purposes. The

combination of these two elements signified an exemplifying routine. Furthermore, this type of interwoven talk about mathematical procedure and performer as the teacher for illustrative purposes could be important for the identification of exemplifying routines. This finding has important methodological implications for developing an alternative lens for analysing MTF objects in discursive form without relying only on explicit naming of MTF objects. In other words, it provides an operational definition of MTF objects as discursive objects.

This analysis further showed that the exemplifying routines were connected to other MTF routines such as teacher explaining routines. It seems the teachers used exemplifying routines as teachers' resources to illustrate a specific idea about mathematical objects, as evidenced in lines 210-212, where the KO discussed how a chosen example $y = 2x$ could be used in the form of a tabular representation for illustrative purposes to enhance explanatory talk about the effects of the gradient and the y-intercept. This was further well evidenced in line 214: "The table is going to help us to be able to draw a graph. It is part of our explanation". This finding agrees with Zaslavsky (2010) who claimed that examples carry a potential exploratory power, and that they carry important elements of explanations through illustrative representations and demonstrations of discursive activities, to name a few (Leinhardt et al., 1990). It is interesting to note that although the MTF components were separated analytically, they were intertwined in the discursive talk. The implication for this current study is that it highlights the blurred nature of boundaries between discursive acts of teaching related to MTF.

The analysis has also revealed that "examples" were associated with CA of OoL routines. For example, in line 109 (in Extract 5 above): "We know that some of the problems have something to do with calculations, so it means some of the examples must focus on calculation". In this utterance the KO remarked about the problems related to calculating that learner might experience. This suggested that the calculating routine was a critical aspect. Given the consideration of CA, KO further suggested a consideration of examples that were related to calculating. The established association between CA of OoL and examples runs parallel with Leinhardt's (2001) report which showed that examples encapsulate a range of critical features. She further noted that such an association is a positive contributing factor to learning. It is worth noting two things from this KO's utterance. First, what was identified as critical aspect was what the teacher assumes was problematic. This finding suggests that critical aspects are conjectured by the teachers. Secondly, after identifying the critical aspects, these were integrated into a

set of examples. This finding seems to provide evidence of the connection between CA of OoL and other MTF objects.

The analysis of exemplifying routines implies that the goal of participating in this discursive activity was to construct new narratives related to examples. This was well evidenced in the analysis which showed the interwoven relationship between mathematical routines and the performer (teacher) for purposes of creating illustrations which resulted in a set of exemplifying routines.

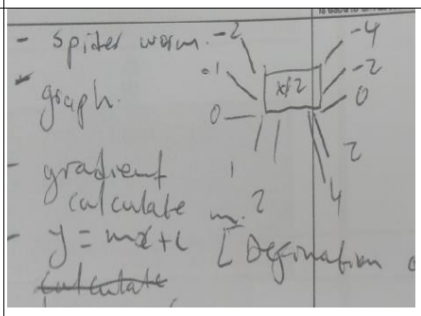
In terms of roles played by all participants, the analysis showed that both teachers and the KO were engaging in the discursive activity of exemplifying as a collective, as seen in the interaction illustrated in Extract 6. The repeated reference to keywords "we" and "our" confirmed this collective engagement. In this extract, teachers and the KO were seen to be contributing to the mathematizing actions aimed for illustrative purposes—discursive activity of contruction of MTF narratives related to examples. Nevertheless, the strong authoritative voice of the KO cannot be ignored. Looking closely at these utterances, particularly from the KO, although the KO seemed to be creating an environment of collective engagement through reference to keywords "we" and "our", her consistent reference to keywords such as "we would like", "we can start" suggested acts of instructing what needed to be done in terms of routines. This suggests that the KO was positioning herself as the bearer for what teachers needed to do or focus on. How the nature of interaction related to the nature of discursive activities will be discussed in detail in the upcoming section (6.3.4).

(ii) Tasks

With respect to generating task routines, the third column of Table 28 in the MTF routine row shows that this category also included various mathematical routines such as interpreting, constructing, classifying, translating and calculating routines. In order to provide evidence about the routines related to tasks, I am going to use extracts from different episodes.

Extract 7

An example of MTF routines related to tasks

Episode	Speaker	What is said	What is done
EP1	J	7. As I was saying, if I was about to teach **linear functions** alone, I would firstly *define* **functions** that what is a **function** as a whole. 8. From there, I am assuming they were not introduced to **linear functions** or whatever, but then, since I have *defined* this as input and output and all the relationship, 9. *I thought of going back to grade 9* whereby they know a spider diagram as inputs and outputs and whatever is here *[see picture on the right-hand side]* 10. it gives the relationship from the input and get the output which they can either write, *choose any inputs, multiply by 2 and it gives out the output* 11. So I can say, now that you have done this they can *either tabulate* whether it would be the x and the y. The x being the input and the y being the result they get from the spider web diagram. Maybe the resulting relationship can confirm the **linear function**.	

EP5	KO	104. So, in your introduction, we know they know **linear function**. Your introduction should speak to the object of learning. You publicise y equals mx plus c ($y = mx + c$). 105. Ask them *to identify* this **function**, then quickly guide them towards **object of learning** 106. *Classification*, when you write that equation and ask them "What is this type of function?", and then you are speaking to *classification.* 107. What are the problems? Some might not be able to *classify* it. 108. So already now when you are speaking to *classification*, we are now coming to **example** space. 109. We know that some of the problems have something to do *with calculations*, so it means some of the **examples** must focus on *calculation*. 110. So our **example** space is guided by errors, content analysis and also our previous knowledge. 111. So what we are saying is that we need **examples** where they are *calculating*, and we also want them to *calculate* maybe **gradient or intercept**, these could be **tasks**.	

An analysis of the data suggested that the process of generating tasks was aimed at foregrounding what learners were expected to do in relation to a specific mathematical object as evidenced in lines 9-11, where teacher J suggested that learners may either do calculations or they may construct the table in order for them to see the relationship between the x and y variables. These mathematical routines included calculations and constructions. It seems that the focus was on what mathematical actions the learners were expected to do. This could be seen in the blue highlighted keywords in the above extract (line 11): "So I can say, now that you have done this they can *either tabulate* whether it would be the x and the y".

Highlighted in each of these utterances was the association between mathematical routines together with the activities for learners. The combination of these two elements pointed to a generating task routine. As seen with the exemplifying routines, this relationship contributed to analytical ways of identifying the analysing MTF objects in discursive form.

The use of different mathematical routines (calculating and constructing) in order to substantiate a narrative about the linear function (in terms of input/output relationship) as evidenced in lines 9-11 indicated the presence of various mathematical routines as suggested in the beginning of this section. Since there was evidence of various mathematical routines, this implies that generating task routines could also be explorative.

The analysis of generating task routines has shown that these routines were linked to a pedagogical routine—questioning strategy—as evidenced in line 106 where the KO used a questioning strategy, prompting learners to give an answer to a question: "What is this type of function*?*" This prompt seems to be encouraging the learners to engage and apply a mathematical routine of "classifying" using a given representation. This analysis implies an association between a pedagogical strategy and mathematical routines related to learner tasks as reported in the following section, 6.2.3.1, where a full illustration will be given on how the questioning routines facilitated the process of construction of routines related to tasks.

The analysis also showed that the MTF routines related to tasks were related to other MTF objects, particularly the CA of OoL and learner participation. For example, the KO (line 109) remarked that some of the reported learner difficulties with linear functions are related to calculating routines. She then suggested (in line 111) that some of the tasks needed to highlight this critical aspect. As seen with the exemplifying routine, this finding seems to suggest that critical aspects are proposed by the teachers, and the identification of these prompted integration to other MTF components—tasks in this instance.

Furthermore, the analysis of the utterances in Extract 7 suggested that the routines related to tasks served a communicative purpose of conveying what learners are expected to do; that is, how they are expected to participate in the mathematical activity. This can be summarised as discursive activity related to the construction of narratives related to learner participation. This finding serves to reinforce the interconnection between discursive acts of teaching related to the MTF.

The analysis of the utterances showed that both teachers and the KO engaged in the discursive activity of generating tasks where each member was seen contributing to the mathematising actions in relation to what learners were expected to do. However, the KO's commanding voice could once again not be missed (in lines 105-106). As already indicated, how participants interacted with each other will be evidenced in the modes of talk section.

In summing up, the analysis of routines related to tasks suggests that the main discursive activity was to construct narratives related to tasks, including narratives about learner participation.

(iii) Explanations

The data revealed that the explanations were done from both teachers' and learners' perspectives, as evident in Table 28. Here too, the explaining routines recorded various mathematical routines.

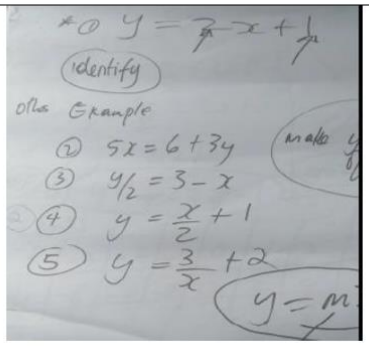
Learner explanations

The learners' explanations were not coming directly from the learners since this session was focused on discussions about the intended interactions for the upcoming teaching session. What is shown in the extracts is the learners' anticipated substantiations. In these extracts there was little indication of what should comprise these explanations—only that they should be given by the learners. In the reflective discussions in the next chapter teachers might talk about what learners actually said since they had an opportunity to engage with the learners in the real classroom discourse.

The utterances which were captured as learner explanations highlighted what learners were expected to do as illustrated in the following extract. Highlighted in red font are keywords related to justifications.

Extract 8:

An example of MTF routines related to learner explanations

Episode	Speaker	What is said	What is done
EP1	J	22. But then now "what do you think, is it less than zero or greater that zero?" 23. They will decide and then they might say, *"It's less than"*, they might say *"It's greater than"*. 24. They might probably say, "It's greater than" because they are thinking of going up, they don't know anything about directions of signs.	
EP5	J	114. Also include counterexamples like y equals three over x plus 2 ($y = 3/x + 2$), and ask: "Is it a linear graph ?" "Why are you saying that?"	
EP6	KO	133. We will be listening to their **explanations**. We are going to be listening to their words from previous learning. 134. What are they bringing in? We are collecting data during this time. So this is part of our introduction.	
EP7	T	159. Let's just focus on *classification*	
	KO	160. A table is not going to help us to *classify*. It's not a good **example.** 161. Or let's ask them to *draw* these **functions**, how they *draw* them. Maybe we can give them an interval 0-5. 162. Asking them to *draw* them, you want them to confirm if these *(looking at picture alongside)* are linear functions.	

Lines 162 and 23 show that the learners were expected to make their own decisions on how to substantiate mathematical narratives. The keywords used: "confirm" and "will decide" support this analysis. It is evident in line 133 that the teacher expected the explanations from learners' perspectives. Learners are seen as capable contributors to the discourse. The same pattern was observed across the episodes with all the utterances that were categorised under learner explanation.

Furthermore, Table 28 reflects that learner explanations carried different mathematical routines. For example, in line 162 the learners were expected to confirm whether the given functions (see the picture) were linear functions or not. They were expected to classify these functions. In order to successfully classify these, the learners would need to draw the graphs representing these functions.

This suggests that the drawing routine was used as a substantiating procedure. Another example of learner explaining can be seen in lines 23-24, where teacher J conjectured the kind of substantiating narrative that could come from the learners in response to: "But then now what do you think, is it less than zero or greater than zero?" These substantiating narratives were related to interpreting routines by learners interpreting the behaviour of the gradient evidenced by reference to mathematical phrases such as "It's less than", "It's greater than". These further pointed to flexible use of various routines signalling explorative discursive activity in relation to the learner explanations.

The analysis of learner explaining further showed that these routines were connected to pedagogical routines. For example, line 114 evidences that learner explanations were evoked through the use of questioning prompts such as "Why are you saying that?". Furthermore, this question suggests that teacher J was hoping that in response the learners might justify their mathematical actions for "Is it a linear graph?" Given this scenario, one may be tempted to conclude that the teachers' questioning routines for learner justification produced narratives related to learner explanations. The relation between questioning routines and learner explaining routines will be shown in section 6.2.3.1.

What emerged from this analysis is that the goal of discursive activity related to learner explaining routines was to invite learners to participate in the construction of mathematical narratives and then substantiate them. Such a finding links learner explaining routines to learner participation—MTF object. This further shows an interconnection between MTF components.

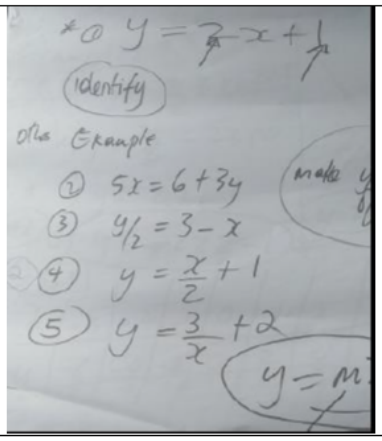
Here both teachers and KO are seen as equal contributors to the mathematising and MTFying talk, as evidenced in heavy use of keywords related to collectiveness such as "we", especially from the KO's utterances. Once again, the KO's utterances highlight her authoritative voice (see 133-134 and 160-162).

Teacher Explaining

With respect to teacher explaining routines, the fourth column of Table 28 in the MTF routine row showed that the teacher explaining category involved different mathematical routines: interpreting, constructing, classifying, translating, calculating and defining. Teacher explaining occurred as part of the process of communicating mathematical justifications to the learners. And this activity was characterised by substantiating routines. I use the following extract to illustrate this point.

Extract 9:

An example of MTF routines related to teacher explanations

Episode	Speaker	What is said	What is done
EP5	J	114. Also include **counterexamples** like y equals three over x plus 2 ($y = 3/x + 2$), and ask: "Is it a linear graph?" "Why are you saying that?"	
	KO	115. We are now working with learner engagement. We also need to vary our examples.	
		116. The examples, they are based on classification. Do you understand? So every time you are coming with these kinds of **examples**, you've got **counterexamples**, examples in general form, 117. and then you say: "Now we have got the general formula in this form y equals mx plus c ($y = mx + c$)". By doing this, you are confirming that this is not a linear function.	
	J	118. You can also use the graph. You can draw different graphs and ask them to identify them. 119. Because comparing to a standard form will not always work, like y equal one ($y = 1$) does not look like y equals mx plus c ($y = mx + c$). What interest me, we know that if $y = 1$ we have a line cutting y axis at one; it is a **linear function** but once we have x equals 1 ($x = 1$), we have a line cutting at 1 on the x axis; that is not a function.	(*demonstrating with hands the vertical line test-showing horizontal line when uttering "y=1" and vertical line when uttering "x =1"*

The interactions in lines 114-119 were centred on classification of a linear function represented in a graphical form. The discussion began in line 114 where J introduced an equation $y = 3/x + 2$ which he referred to as a special type of example called a counterexample, and invited the learners to participate in the mathematical activity by asking them to classify this equation in terms of the function family it belongs where he said: "Is it a linear graph?" This was followed by another questioning prompt: "Why are you saying that?" aimed at encouraging learners to justify their answer to the first question: "Is it a linear graph?" This was followed by KO, in line 115, extending on J's remarks by confirming that counterexamples were used to invite learner participation. She went further by linking the notion of counterexamples to substantiating activity in lines 116-117. The substantiating discursive activity was evidenced by the use of the keyword "confirming".

In line 118, J further extended on KO's substantiating narratives by highlighting gaps in KO's substantiating narrative related to matching to a standard form of a linear function, and suggested the use of a graph rather as a way of further demonstrating whether the equation ($y = 3/x + 2$) was a function or not. In his substantiating narrative he used two examples in symbolic form: $y = 1$ and $x = 1$. He was observed using his hands to demonstrate the behaviour of the function in order to justify

whether the two equations $y = 1$ and $x = 1$ could be classified as a function or not. This action is linked to the informal routine use of demonstrating using hands—a concrete mediator, but used in a mathematical way. Furthermore, this utterance used mathematical words such as y- and x-axis which implies interpretation of a graphical representation. It seems the use of these two mathematical routines: demonstrating and interpreting were intended for constructing mathematical substantiating narratives related to linear functions.

This analysis suggested two findings. First, the link between examples and the teachers' substantiating actions. Second, the analysis of this extract further showed that the teachers used three different mathematical routines such as classifying, demonstrating and interpreting in order to construct their substantiating narratives related to classification of the given equation ($y = 3/x + 2$). This shows various mathematical routines and implies explorative mathematical discourse: teacher explaining is also explorative in nature.

The analysis has further shown evidence of connection between teacher explaining routine and pedagogical routine. This is evident in KO's utterance (line 286) where she used the endorsed narrative commonly found in school mathematical textbooks[15] to describe a generalised way of taking about a notion of vertical shifts. This supports an earlier claim that the pedagogical routines support the construction of MTF narratives.

Extract 10

An example of connection between teacher explaining and pedagogical routines

Episode	Speaker	What is said
EP12	KO	285. Now when you move to the more mathematical **explanation,** that's when you bring in the language of shift. 286. Now, as a way of concluding your lesson, you *generalise* that every time *you have plus c it shifts your graph c units up*.

In terms of roles played by all participants, the analysis of the utterances has shown that both teachers and the KO were engaging in the discursive activity of teacher explaining where each member was seen contributing to the construction of mathematical narratives aimed at constructing mathematical substantiations. Still, the KO could be seen playing the role of structuring discursive activity as evidenced

[15] Given the graph of $f(x)$ the graph of $g(x) = f(x) + c$ will be the graph of $f(x)$ shifted up by c units if c is positive and or down by c units if c is negative (Siyavula, 2010).

in how she directs what teacher should do in lines 285-286. The section on modes of talk will discuss this in detail.

(iv) CA of OoL

Table 28 showed that the CA of OoL category carries various mathematical routines such as interpreting, classifying, translating, calculating, word use and defining. These mathematical routines were similar to those seen in other MTF routines. However, they differed according to the function they served within the discursive activity. For example, these mathematical routines represented what was conjectured by teachers as learner difficulties as discussed in detail in section 6.2.3.2 below where it will be revealed that the CA of OoL routines emerged through a process of identifying CA of OoL.

The analysis that was carried out with respect to the different MTF objects above in this section revealed the presence of some of these conjectured critical mathematical routines (e.g., in tasks and examples). This occurrence implies that each mathematical routine found in the CA of OoL was applicable across different contexts: from tasks, examples and explanations. This confirms Adler and Ronda's (2015) assertions that examples, tasks and explanations mediate mathematics learning by foregrounding what is the mathematical object to be learned. This finding contributes to the earlier claims, where it was concluded that the MTF objects (discursive actions) were interconnected. Furthermore, it prompts one to conclude that the aim of engaging in CA of OoL discursive activity was to consider other MTF components that reflect what was critical to learn. It is important to note that the emerging CA of OoL routines were from teachers' perspective—they were conjectured by teachers.

(v) Learner participation

As noted in the MTF naming section, the learner participation was not in focus. Sfard (2017) gave an account related to mathematical object where she argued that in the absence of mathematical object, it is unlikely that teachers would construct any new narratives about this. Given this assertion, I am inserting the MTF object in place of the mathematical object, and argue that according to Sfard (2017)," it is unlikely to construct MTF narratives in the absence of talk about MTF names". Surprisingly, the analysis has shown that the learner participation manifested itself in the learner explaining and generating task routines, as already discussed. This means that the learner participation is detached from the naming of it; however, it is brought to life through other discursive routines related to MTF.

This too points to the inter-relation of the elements of the MTF that are possible to distinguish analytically but in MTF discourse they interweave.

6.2.2.3 Summary

As a way of summarising the analysis on MTF discourse, I argue that MTF-use was the participation in the MTF discourse through an engagement in the discursive activity called MTFying. This particular discourse was characterised by names used to describe MTF objects such as examples, tasks, explanations and OoL. The repetitive patterns characteristic of this MTF discourse included exemplifying, generating tasks, learner explaining, teacher explaining and teacher CA of OoL. This particular discourse used similar visual mediators as those found in mathematical discourse such as graphs and equations. For example, the teachers described MTF objects through mathematical visual mediators. Take for instance the case of examples in line 114, EP5 by teacher J: "Also include counterexamples… Is it a linear graph? Why are you saying that?" In this utterance the graphical representation was used to refer to a special kind of example—the MTF object. Furthermore, this discourse was characterised by different narratives such as MTF narratives (the relation between MTF and MTF routines), mathematical narratives (relation between MTF and mathematical routines) and pedagogical narratives (relation between MTF and pedagogical routines).

I now take and extend the discussion and deal with the nature of the MTF discourse in terms of ritual-explorative participation, using similar structuring principles as done with the mathematics discourse.

1. In terms of nature of MTF objects (words and word use): The MTF objects that teachers were explicitly referring to included examples, tasks, explanations and OoL. There was no reference to learner participation. Although not explicitly named, it manifested itself in discursive narratives about tasks and learner explanations.

The analysis has shown how the teachers' talk about MTF objects was routine-driven. This was evidenced in the process of interplay between mathematical routines and the performers, where teachers were inclined to associate these mathematical routines to the performer of these routines. As shown, this procedure contributed to the construction of MTF narratives and the expansion of original MTF objects into one or two MTF routines: learner participation—tasks and learner explaining; explanations—teacher and learner explaining routines, OoL—teacher CA of OoL routines.

The analysis has also pointed to object-driven word use where MTF objects were talked about as if they have their own life. As seen with mathematical discourse, the mixture of routine and object-driven word use suggests possible explorative MTF discourse.

2. **Sources of MTF narratives**: The analysis has shown various sources of MTF narratives. First, the MTF objects were signified by different MTF routines such as exemplifying, generating tasks, teacher explaining, learner explaining and teacher CA of OoL. Each of these MTF routines was characterised by various similar mathematical routines suggests a possible explorative MTF discourse.

Secondly, the analysis has further revealed that pedagogical routines facilitated the construction of MTF narratives which resulted in the identification of a number of MTF routines. It was shown that each MTF routine was associated with different pedagogical routines: tasks-questioning routines; teacher explanation—questioning; learner explaining—questioning routines; teacher CA of OoL—identifying CA of OoL routines and learner participation—questioning routines.

What is interesting is that the CA of OoL object was linked to all the other components. In other words, all MTF components were carrying similar mathematical routines that were identified as critical to learn. Similar mathematical routines were applied to different MTF components (e.g., examples, tasks, teacher explanations). This process is linked to what Lavie et al. (2018) and Sfard (2008) termed a notion of applicability. This kind of applicability can be described in terms of MTF/MDI literature to refer to the notion of coherence—all three components bring to focus what is critical for learners to know as reflected in the MTF tool as reflected in the MTF tool (see figure 2) and discussed in Adler (2021). Here too, the MTF discourse could be explorative.

To sum up, the established connection between MTF components, with different routines such as mathematical and pedagogical routines suggest that one cannot talk about the objects of the MTF without reference to doing mathematics (mathematical actions/routines) and/or teaching actions/pedagogical routines. Such a connection signals explorative participation in relation to the MTF discourse. Heyd-Metzuyanim et al. (2018) explained this connection as finer attunement of components of a particular discourse with other discourses in a different context, which implies explorative participation. In other words, the MTF objects have been integrated with other elements of the discourses found in the L/S context. This finding suggests that the L/S has a bearing on the negotiation of meaning of the MTF tool. This will be discussed further in the teacher learning chapter.

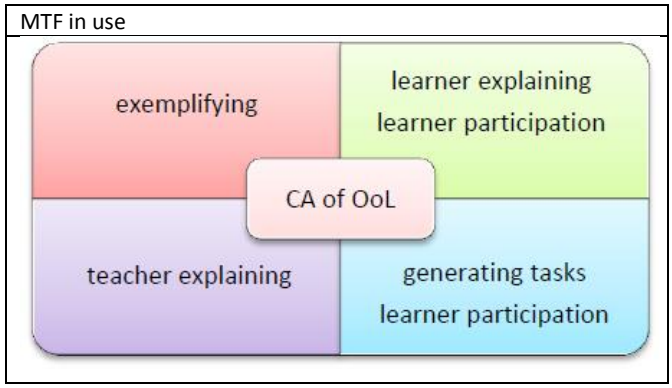
3. Goal of discursive activity: As seen with mathematical discourse, I have identified two types of discursive goals: object-level goals (MTFying) and meta-discursive goals (culturally bound).

*Meta-discursive goals***:** The analysis has evidenced that one of the goals of participating in MTFying discursive activities was to construct MTF narratives for either teachers' use or learners' use. This was heavily evidenced by the mathematical routine-performer pair which resulted in identification of examples and teacher explanations for the teachers' use, tasks and learner explanations for learners' use. Another meta-discursive goal was established in analysis of MTF naming where it was revealed that the KO named the MTF components explicitly in order to provide a point of focus with respect to different MTF objects—so to support the construction of MTF narratives.

Object-level goals: The analysis suggested that the object level goal was to construct MTF narratives by linking each MTF object with the mathematical object to be learned (critical aspects of object of learning) and expand the original MTF components into new MTF objects (examples, tasks, teacher explanations, learner explanations and learner participation). This finding has implications for the nature of the new emerging MTF tool which may help us understand the MTF in use—the discursive nature of the MTF tool. Furthermore, this finding will help me develop an argument about the reification process in Wenger's (1998) terms. It is important to note that the MTF discursive tool was not reified in the community, but it was produced in the community in terms of what the community focused on. However, the researcher reified it by producing a representation of it as illustrated in figure 6. I will pick up on this point in the teacher learning chapter. In the following figure I provide the transformation of the MTF tool into the MTF discursive tool (MTF-use).

Figure 6

Transformation of MTF tool to MTF discursive tool



4. Role: Since the MTFying activity built on the mathematising discursive activity, the MTFying talk echoed the same patterns of roles played by both the KO and the teachers. This was evident in the mathematising activity where they were seen as equal contributors to the MTFying talk, as evidenced in heavy use of keywords related to collectiveness such as "we", especially from the KO's utterances. However, the data in relation to the MTF narratives highlighted the leading role played by the KO in providing point of focus and how things should be done. I will elaborate on the significance of this finding in the upcoming modes of talk section.

To sum up, in this section I have provided a discussion of teachers' MTF discourse in terms of objects they were talking about, sources of their narrative, goals of their discursive activities and participants' roles. I have further presented how these elements suggest a possible explorative MTF discourse. However, in terms of the meta-discursive goal, particularly indicated by the leading role played by the KO, it implies ritualistic element of the MTF discourse. The KO's MTFying talk suggested that she was providing teachers with opportunities to participate in the new MTF discourse. It is also important to note that teachers' participation in MTF discourse has direct implications for teacher learning, as they are afforded opportunities to learn how to teach with the MTF tool. For example, the analysis has shown that the teachers were engaging in a discursive activity of identifying what learners needed to learn and considered a set of examples and tasks that reflected these critical aspects. This is a very important mathematics teaching practice (Adler, 2021). I will expand on this point in the teacher learning chapter.

6.2.3 Organising theme 3: Use of MTF was linked with participation in pedagogical discourse

As is visible in Table 28, pedagogical discourse was prevalent. This is to be expected as teachers were planning teaching. What is of significance and will be elaborated upon here is that while this was obvious, the discussions reveal the intertwinement between pedagogical routines, MTF and mathematical objects.

According to this organising theme, the analysis showed that teachers have talked about various pedagogical routines emerging from mathematical teaching discourse (characterised by questioning routines), L/S pedagogical discourse (characterised by scaffolding and identifying CA routines) and general teaching discourse (characterised by structuring routines). In the following sections, I discuss each of these emerging discourses in turn, beginning with mathematical teaching discourse.

6.2.3.1 Emerging theme: Mathematical teaching discourse

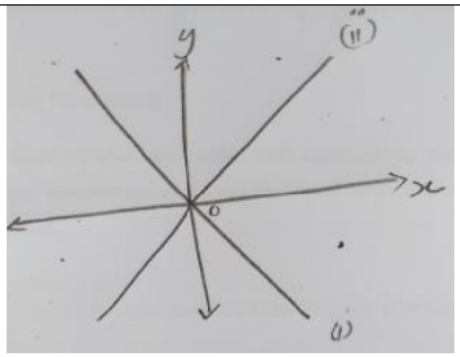
Questioning routines

With reference to Table 28, the questioning routine was the most prevalent mathematical teaching routine in the planning session of the L/S. The analysis of the narratives suggested that the questioning routines could be used for a purpose of constructing MTF narratives related to learner participation, tasks and learner explanations in relation to what learners were expected to do in terms of mathematical actions. This use of questioning routines complements those already highlighted in the literature (e.g., Ball et al., 2008).

Analysing these routines involved a consideration of questioning prompts. I identified a number of questioning prompts related to mathematical objects in the data and I use different extracts from different episodes as typical examples. In these extracts, MTF and mathematical objects are bolded; the routines are shown in purple italics, visual mediators are underlined; and the questioning prompts are highlighted in yellow.

Extract 11

An example of questioning routines

Episode	Teacher	What is said	What is done
FP1	I	17. Yes, from that, y equals two x ($y = 2x$) which is a **linear function**, I would be having a graph with a y**-intercept of zero**, but with **a gradient** of 2. 18. So now I can talk about the **slope**, meaning I am going to my **objective of learning** which are the **effects of** m **and** c. So now I can talk about the **slope**. That: "What do you notice about this line and this line?" *(pointing to line (i) and (ii) in the sketch shown on the right-hand side)* 19. I hope they say this is horizontal and now on real life. "On your horizontal line you *just walk*, you don't feel any pain, why is it that?" If they do Geography they will say. If they don't, (*using hands to demonstrate a horizontal line*) 20. I will probably say it's because the **slope** is zero, its *flat, it is not steep* or what. "But now that, that's the case what do you think if you had to walk on this line?" *(pointing to line (ii) in the picture on the right hand side).* 21. "Do you think you would feel the pain? Yes, why?" *It inclines, steepness* all those. That's how I'm gonna introduce my notion on **gradient**. That's the case. Because where our **gradient** is zero, why is it zero, because its horizontal, all that.	

EP12	KO	293. Even with tasks, you don't need to go and design them. The way you question your learners during teaching serves as tasks.	

The teachers' questioning prompts can be divided into two categories: (1) invitation for learner participation in mathematical actions, and (2) invitation for learner participation in presenting mathematical substantiating narratives. I provide evidence by drawing from teachers' imagined interactions between them and the learners in lines 18-21. For example, in line 18, J used questioning routine, prompting learners to give an answer to the question: "What do you notice about this line and this line?" (pointing to line (i) and (ii) in the sketch shown on the right-hand-side). This utterance was inviting the learners to interpret the behaviour of the slope of two different linear functions (lines) presented in graphical form. In line 19, J conjectured the learners' answers in the form of a real-life example using a metaphor of walking in a horizontal line. He then extended this narrative further by inviting learners to substantiate it by using a metaphor of pain: "you don't feel any pain, why is it that?" The expected response to this questioning was aimed at substantiating a narrative about the behaviour of a line in terms of a real-life example. According to J, learners' failure to respond would prompt him to further extend mathematical explanatory talk, in line 20, where he gave an account of this narrative in mathematical terms: "because the slope is zero". It seems the aim of this imagined interaction was to encourage the learners to participate in the mathematising activity by constructing mathematical narratives about mathematical objects and substantiating them. This implies an association between questioning routines and MTF narratives related to three MTF objects: learner participation, tasks and learner explanations.

6.2.3.2 Emerging theme: L/S pedagogical discourse

Scaffolding routines

In Table 28, the scaffolding routines were the most prevalent pedagogical routines. The scaffolding routines were related to KO's actions. In order to provide evidence, I will be focusing on extracts highlighting the KO as the speaker. The analysis showed that KO's scaffolding routines were used to convey mathematical teaching strategies as illustrated in the extract below. In this extract I have bolded both MTF and mathematical objects; however my focus here was the mathematical objects.

Extract 12

The relationship between scaffolding routines and mathematical teaching routines

Episode	Speaker	What is said
EP9	KO	214. The table is going to help you to be able to **draw.** It is part of your **explanation**. 215. In a table you will be able to see when you are changing **the sign of your gradient**. 216. It is not that you will be wasting time with your table; you can still use it to *draw* the graph. So, let's *draw* it.
EP12	KO	285. Now when you move to the more mathematical **explanation**, that's when you bring in the language of *shift.* 286. Now, you *generalise* that every time you have **plus c** *it shifts* your graph **c units** *up*.

Lines 285-286 showed that KO's talk was focused on mathematical teaching routine of explanation in relation to the y-intercept. This talk was characterisd by the general features of a discourse about the property y-intercept (e.g., "every time you have plus c it shifts your graph c units up"), as evidenced in line 286 where the KO explicitly commented about what teachers need to do in terms of a generalising routine. This pattern was also evidenced in lines 214-216, where she shared general properties related to a gradient concept when viewed from the tabular representation, and what this type of representation could mathematically enable in terms of different routines such as interpreting the behaviour of the gradient (e.g. line 215), and drawing a graph (e.g., line 214).

A second type of scaffolding routine focused on pedagogical routines found in the general teaching discourse—the structuring routine. I use the following extract to illustrate. In this extract, highlighted in orange are keywords indicating structuring routines, the other routines are shown in purple italics, MTF objects are bolded.

Extract 13:

The relationship between scaffolding routines and general teaching routines

Episode	Speaker	What is said
EP12	KO	290. I think you are telling a story. You have an introduction, then implementation, then climax. 291. When you reach climax you go down. 292. Tell your story and finish, have a conclusion. In your conclusion, you summarise, what is it you wanted to tell

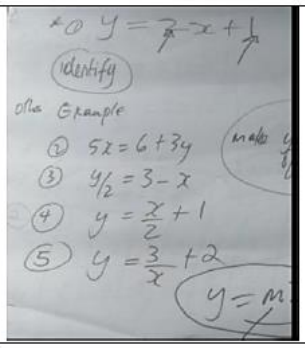
This extract from lines 290-292 showed how the KO summarised a general teaching procedure by using a metaphor of telling a story, demonstrating ways teachers may structure a lesson from introduction to

conclusion. This utterance was not related to construction of either the MTF or mathematical narratives, but rather to the general structure of the lesson. This analysis is significant as it contributed to the re-modification of the MTF lesson plan tool as will be shown in the section about structuring routines.

The analysis has also shown that KO's scaffolding actions supported the generation of MTF narratives. This is critical and appropriate as it is the new discourse.

Extract 14

The relationship between scaffolding routines and MTF narratives

Episode	Teacher	What is said	What is done
EP10	KO	260. So what you are looking at now, you will be focusing here $(y = 2x)$ and $(y = 2x + 1)$ that's our second part. And you go *and draw* the **two functions** ***(drawing the two functions)***	
EP11	KO	265. Now you can start by asking them to *interpret* the difference between **example** 1 and **example** 3 from the table *(pointing to the picture on the right-hand side)*	
EP12	KO	297. Ask them *to draw* it, and ask them to discuss, maybe in groups. This could be a way of assessing	

In all three utterances in Extract 14, the KO stated explicitly what teachers were supposed to do with respect to MTF components and even suggested what specific mathematical routines to invoke. For example, in lines 297 and 265 she stated that teachers needed to ask the learners to draw (a graph), and interpret a table. In these two utterances the MTF component in focus was tasks. Similarly, in line 260 she focused on two linear functions $y = 2x$ and $y = 2x + 1$ and suggested what actions the teachers needed to implement. The use of keywords "you go and draw" (line 260) could be linked to teachers' exemplifying actions. The analysis of these utterances further evidenced that the KO's talk about MTF

objects was linked to various mathematical routines, and attached to different performers such as teachers and learners. This relationship implies generation of new MTF narratives as seen in section 6.2.2.

In this section, I have examined how MTF narratives were constructed. The analysis points to the crucial role played by the KO in facilitating this process. The results of the analysis seem to suggest that the LP1 was characterised with KO's high scaffolding. For instance, through her narratives, the analysis showed how she (1) foregrounded points of focus (MTF names) around which the negotiation of meaning of MTF became organized,and (2) influenced the routine-use, not only in the obvious way of suggesting what routines to use, but also in the way to engage with these routines. Within this painted picture outlined above, it makes sense to interpret the KO's utterances as directives. Furthermore, her scaffolding actions placed her as the sole authority of the discourse, and reduced the need for teachers' reflections about what routines to use and why. From this, KO's scaffolding can be referred to as "prioritising" goals. This finding could be interpreted as an indication of what Sfard (2017) called the ritualised nature of the expert's discursive actions. Such a finding suggests that the KO's participation in MTF discourse had ritualistic elements due to KO's goals of introducing teachers to a new discourse. This is expected, since newcomers to the discourse are confined to the scaffolding actions of a more expert participant (Heyd-Metzuyanim et al., 2018). This finding has important implications for opportunities of learning for both the KO and the teachers. First, the KO had opportunities to learn how to scaffold the MTF discourse. Secondly, the teachers were afforded opportunies to learn how to participate in the MTFying activities, and also to participate in the explorative MTF discourse where they were guided by the KO on how to intergrate different routines such as questioning and various mathematical routines in order to construct MTF narratives. Furthermore, the analysis has shown how the KO's scaffolding actions facilitated the process of connecting the CA of OoL with all the other MTF objects.

Identifying CA routines

Looking across the second last column of the pedagogical focus row in Table 28, it was evident that teachers repeatedly used the identifying CA routines. The analysis has indicated that the process of identifying CA included two strategies. First, the teachers were observed looking at the curriculum documents in order to confirm what learners needed to learn at the current grade. For instance, the interactions illustrated in Extract 15 show both the KO and the teachers investigating whether the general formula of the linear function was introduced to the learners previously. Through the

interactions it appears that the symbolic form was what learners need to be introduced to, and therefore a critical aspect to be learnt at the current grade (grade 10).

Extract 15

An example of Identifying CA through curriculum analysis

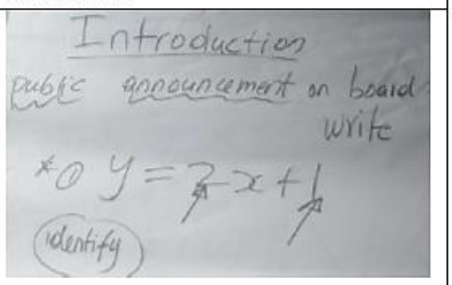
Episode	Speaker	What is said
EP4	J	85. Have they been introduced to this formula $y = mx + c$ before?
	T	86. From grade 9?
	J	87. Because I think they only start doing it from grade 10.
	KO	88. Is it?
	J	89. Yes
	M	90. The CAPS, let's look at CAPS
	KO	91. Yes they do *(looking in CAPS doc)*
	J	92. I was working on an assumption that I still need to introduce them to the formula $y = mx + c$.
	KO	93. Linear function was introduced to you from crèche through patterns
	T	94. That's why I went to the spider diagram because that is a linear function but they don't say $y = mx + c$ 95. Is it CAPS or NCS?
	KO	96. NCS specify clearly better than CAPS

Concerning the second strategy, the analysis suggested that teachers identified critical aspects of the OoL by conjecturing what they believed was critical to be learnt through their knowledge of learners' difficulties, common errors and even misconceptions about a particular mathematical object or process. The analysis further suggested that the process of identifying CA served two purposes: bringing into focus the learners' anticipatory thinking[16] and the construction of MTF narratives related to critical aspects of OoL. For example, the narratives which evidenced the identifying CA routines were characterised by the conjectured learners' difficulties in relation to mathematical routines (actions) which I referred to as critical aspects (CA). In the discussions that took place during LP1, one particular aspect related to CA arose: that a particular mathematical routine might be a problematic action that learners struggle with. For example, in the extract below the speculated problematic routines were related to classifying (line 107) and calculating (line 109):

Extract 16

An example of Identifying CA through learner knowledge

[16] "Anticipatory Thinking is ways of thinking about (strategies, approaches to) how learners may engage with content, processes, and concepts. It includes awareness of and responsiveness to student thinking" (Hauk et al., 2014, p. 26).

Episode	Teacher	What is said	What is done
EP5	KO	104. So, in your introduction, we know they know **linear function.** Your introduction should speak to the object of learning. You publicise y equal mx plus c ($y = mx + c$.) 105. Ask them *to identify* this **function**, then quickly guide them towards **object of learning** 106. *Classification*, when you write that equation and ask them what is this type **of function,** and then you are speaking to *classification.* 107. What are the problems? Some might not be able to *classify* it. 108. So already now when you are speaking to *classification*, we are now coming to **example** space. 109. We know that some of the problems have something to do *with calculations*, so it means some of the **examples** must focus on *calculation.* 110. So our **example** space is guided by errors, content analysis and also our previous knowledge. 111. So what we are saying is that we need **examples** where they are *calculating*, and we also want them to *calculate* maybe **gradient or intercept,** these could be **tasks.** 112. Other **examples** could be dealing with *translation*, i.e. moving from graph to table to equation. 113. In our planning we need **examples** for *classification.*	

The utterances (in lines 107 and 109) suggested that, according to the KO, in order for the learners to be able to construct narratives about linear function, they needed to manipulate a symbolic form of a mathematical object in question (linear function) through mathematical actions such as classifying and calculating. These routines were critical aspects for learning linear functions and the learners needed to discern these first. This finding corroborates Runesson's (2013) claims that what is critical for learners to learn about a specific mathematical object, needs to be derived from both the content and also from the learners, particularly the difficulties they are experiencing with that object. It therefore seems justifiable to conclude that the process of identifying CA is linked to mathematical actions which were identified as critical for the learning. This discursive activity could be classified as a special type of mathematical teaching routine found in the L/S context. In simple terms, the process of identifying CA facilitated the construction of MTF narratives related to critical aspects of OoL (MTF object-CA of OoL) which were related to different mathematical routines such a classifying and calculating. The analysis further highlighted that the identification of critical aspects led to consideration of MTF objects that reflected these critical aspects (line 109). This finding seems to suggest that the goal of engaging in the discursive activity of identifying CA was to construct the MTF objects.

It needs to be noted that in the KO's narratives in lines 107-109 she was mentioning the difficulties that the learners might have in relation to classifying and calculating, and did not elaborate about the nature of these. In most of the utterances from different episodes, the learners' difficulties were conjectured from both the teachers' and KO's experience and not from actual learner data; in other words, they could be classified as teachers' CA. What this analysis has highlighted is that there was a consideration of common difficulties that teachers were aware of (whether mathematical or not). The following session (RD1) will show whether the process of identifying CA is more elaborated after teachers had an opportunity of working with the learners in a real teaching context, in other words, they have the data of learners' difficulties.

In the painted picture thus far, it is possible to conclude that the teachers' and KO's pedagogical actions related to identification of CA enabled the construction of MTF narratives related to CA of OoL, and further supported the interconnection between the MTF objects.

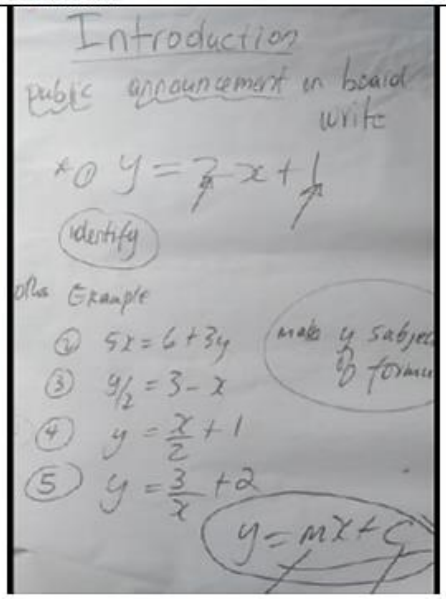
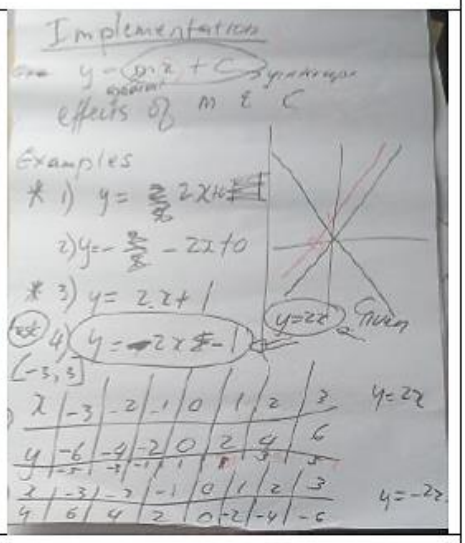
6.2.3.3 Emerging theme: General teaching discourse

Structuring routines

As shown in Table 28, the structuring routine also recorded high frequency. When analysing narratives related to structuring, I focused on keywords as highlighted in orange in the following extract that I use to provide evidence. As seen with the scaffolding and identifying CA routine, structuring routines are KO dominated, as shown from KO's utterances.

Extract 17

An example of structuring routines

Episode	Teacher	What is said	What is said
EP6	KO	121. So the introduction is used to strengthen our **critical features.** 122. We need introduction; in your introduction you will announce on the board y equals two x plus one $(y = 2x + 1)$. "What is this?" You want to find out from them. 123. That means one of your **critical** aspects is *classification*. 124. Then we listen to what they are saying, their **explanations**, their words, the naming, how are they justifying their answers? 125. So, can you see how MTF is coming to play? 126. So now, we need **examples**, following up on our introduction. Already, you are at your implementation stage, give **examples**, you are increasing your **example space.** 127. We need to include introduction in our lesson plan *(looking at original LP template).* 128. Our first **example** is y equals two x plus one $(y = 2x + 1)$ and other **examples** are 129. five x equals six plus three y $(\underline{5x = 6 + 3y})$ 130. y over two equals three minus x $(\underline{y/2 = 3 - x})$ 131 y equals x over two plus one $(\underline{y = x/2 + 1})$. This is linear but looks non-linear. 132. We can put another one, y equal three over x plus 2 $(\underline{y = 3/x + 2})$ and see how they *identify* it. 133. We will be listening to their **explanations**. We are going to be listening to their words from previous learning. 134. What are they bringing in? We are collecting data during this time. So, this is part of our introduction.	
EP9	KO	208. So let us start with our implementation. In our implementation we need **examples** $\underline{y = 2x}$. I am thinking of kind of **explanations**	
	T	209. Let us change the signs *(writing on board)* [get the picture]	
	KO	210. Your **explanations** are going *to be coming* directly from your **examples.** You can start off your **explanations** by saying, "Now look at the value of your **gradient** is 2", so you would like to illustrate with more **examples**, with more **explanations.** For illustrations you can use tables. Will the table show any **effect on *m* and *c*?**	
EP12	KO	286. Now, as a way of concluding your lesson, you *generalise* that every time you have **plus *c*** *it shifts* your graph ***c* units *up.***	

The patterns in structuring actions showed a common instructional function such as reflecting about activities during three different stages of the lesson: introduction, implementation and conclusion. The reflections in each stage were characterised by a talk about the aims of each stage and activities to be included in each stage. These emerged as: introduction (identifying CA), implementation (focused on explanatory talk) and conclusion (generalising routine).

The aim of conducting an introduction appeared to be on identification of critical aspects related to linear function as evidenced in line 121. The identified critical aspect was related to classifying routine (line 123). Interestingly, the introduction stage was accompanied by a number of examples aimed at foregrounding the critical aspect of classifications (see examples in lines 128-132, and shown in the picture on the right-hand side).

During the implementation stage it seems the focus was on constructing mathematical substantiating narratives through the use of MTF objects such as examples and teacher explanations. This was evidenced by exemplifying actions aimed at substantiating mathematical narratives (see section 6.2.2.2). According to the analysis in section 6.2.2.2, the aim of the implementation stage was to construct examples aimed at helping the teacher with illustrations intended to provide substantiations (explanations) about the effect of the gradient in the linear function (see interaction in lines 208-210, and the accompanying picture).

The conclusion stage appeared to be aimed at summarising the whole lesson by generalising the mathematical activity, as explicitly stated by KO in line 286 (Extract 17), where she commented about specific mathematics teaching routines such as generalising. This routine is often used to substantiate mathematical narratives by pulling all key ideas about mathematical objects in a generalised way.

From the discussions provided in the three stages, it seems that each stage foregrounded two discursive activities such as mathematising and MTFying, differing with functions of each stage, which were demonstrated through the use of different pedagogical routines. In the first instance, the analysis of the utterances in Extract 17 provided evidence of talk about mathematical objects throughout different stages of the structure of the lesson. Secondly, it seems that the structuring routines organised the MTF into stages, linking elements of the MTF into these stages. For example, the KO narratives suggested that the introduction linked with the OoL; implementation with examples and explanations; and conclusion with generality which was directly linked to explanations. Lastly, as already noted, the

function of each stage included use of different pedagogical routines (e.g., identifying CA—introduction; exemplifying and explaining—implementation and explaining—conclusion).

This identified dissection into different stages was not evident in the MTF tool as highlighted in lines 35-36 (Extract 18), where the KO reflected on the inadequacy of the MTF tool in terms of its failure in suggesting the structure of the lesson or lesson plan. She further suggested an insertion of three different stages of structuring a lesson: introduction, implementation and conclusion (see line 37). This became a feature of the way the MTF was scaffolded and then talked about in the L/S.

Extract 18

An example of KO reflecting about the structure of the MTF tool

Episode	Speaker	What is said	What is done
EP2	KO	35. So, I think what is happening here, I did not know this thing; this is one of the things that are going to strengthen up how the MTF can be used. 36. The MTF does not allow any room for introduction. This is one of the things I am going to be analysing. 37. You come up with something that is so powerful that before we can even start with implementation, we need to introduce, then finish off with conclusion.	*(Looking at the original lesson template see Appendix C)*

The use of the structuring routine showed some links to mathematising, MTFying actions and other pedagogical routines by highlighting how these could be arranged in different stages. However, the analysis suggested the most important purpose was facilitating restructuring of the original MTF lesson plan template (see Appendix C) to give it a discursive character, as such contributing to the development of a discursive lesson plan. This meant the modified tool gave clear mapping of actions to be executed at specific times during the teaching session. This finding is expected in the context of LS as confirmed by Lewis et al. (2006), who reported that participation in a form of LS, particularly during the lesson planning session, contributes to the refinement of a lesson plan.

6.2.3.4 Summary

After having analysed the pedagogical discourse, I argue that the pedagogical discourse emerging during the LP1 was a special form of discourse characterised by talk about both mathematical and MTF objects. Secondly, the routines characterising this type of discourse included mathematical teaching routines and MTF routines. Similar to the MTF discourse, the pedagogical discourse was characterised by three

different narratives: the mathematical, MTF and pedagogical. It seems the main discursive goal of this discourse was to consider several pedagogical actions in order to organise the MTF and mathematical discourses.

In the following discussion, I present the nature of pedagogical discourse in terms of ritual-explorative participation, using similar structuring principles as done with the mathematical and MTF discourses.

1. **What objects are they talking about?** The analysis has shown that the objects of the pedagogical discourse were the same as those found in both the mathematical and MTF discourses (i.e., linear function and explanations). This was not surprising since the pedagogical discourse in general was used by the teachers to perform several actions to organise or regulate teaching-learning activity. In this study there were two main discursive activities: MTFying (talking about MTF objects) and mathematising (talking about mathematical objects). However, differences between these discourses occurred at a level of routines—the sources of narratives.

2. **Sources of narratives**: The sources of pedagogical narratives included different pedagogical routines. Overall, four types of pedagogical routines were identified: questioning, scaffolding, identifying CA and structuring routines. These routines differed according to the functions they were meant to serve within the discourse. For example, regarding questioning routines, the results suggested that these routines were used by the teachers to invite learners to participate in the mathematising activity. On the other hand, the results with respect to scaffolding routines indicated that these were used by the KO to facilitate the use of mathematical teaching routines and other general teaching routines. In addition, it appears that the KO used these types of routines in order to facilitate the construction of MTF narratives. As for identifying CA routines, the analysis implied that these were used to open up a window to access learner mathematical thinking: the difficulties, errors or misconceptions the learners might have with regard to mathematics; that is, what was to be learned (CA of OoL). Furthermore, the analysis of identifying CA in particular suggested that this routine facilitated the interconnection between the MTF components through CA of OoL. The main argument with regard to these three pedagogical routines (questioning, scaffolding and identifying CA) is that the pedagogical routines supported the construction of MTF narratives.

Regarding the structuring routines, the findings showed that these were used to structure the mathematics lesson. These types of routines were interesting as they could help us to understand the

flow or structure to the lesson, with specific, practical suggestions for the things that needed to change in the enactment of the lesson.

The analysis of the pedagogical discourse in general, has shown the intertwinement between MTF, mathematical and pedagogical discourses. One cannot talk about pedagogical discourse in the planning session of the L/S without referring to MTF or mathematics. At the same time, the MTF discourse integrated both pedagogical and mathematical discourses. These interconnections will lead to a discussion about the nature of emerging discourse in section 6.2.5.

3. Goal of discursive activity: The analysis suggested that the pedagogical discourse was aimed at supporting both object-level and meta-discursive goals of participating in the MTF discourse.

Regarding the object-level goals, the analysis showed that the four identified pedagogical routines, with the exception of the structuring routine, seemed to be pointing to one common feature/purpose: supporting the expansion of MTF objects. For example, the questioning and identifying CA routines enabled the construction of MTF objects such as tasks, learner explanation and critical aspects of OoL respectively.

With respect to meta-discursive goals, it was evidenced how pedagogical routines, except for structuring routines, supported the construction of MTF narratives for both teachers' use (examples, teacher explanations) and learners' use (tasks, learner explanations)**.** Regarding the structuring routine, the analysis appeared to suggest that the main purpose of this type of routine was to support the modification of the MTF lesson plan template which was discursively mediated such that it took on its distinct meaning-reflecting discursive structure. Another meta-discursive goal was established in the analysis of KO's scaffolding routines which seemed to have facilitated the construction of MTF narratives by foregrounding the points of focus-MTF names and routine-use from varius discourses such mathematical and pedagogical discourses.

From the analysis of pedagogical discourse, it can be said that the teachers were exposed to explorative MTF discourse through KO's scaffolding routines in particular. However, the presence of KO's scaffolding actions with respect to keeping the MTF tool in focus implies her ritualistic goals in the MTFying discursive activity, where she was prioritising the MTF goals. The scaffolding routine further placed the KO as the authority of the MTF discourse, as someone initiating the teachers into the new discourse. According to James Gee (as cited in Sfard, 2008, p. 282), "one gains access to a discourse 'through

scaffolded and supported interaction with people who have already mastered the Discourse'". This finding has a bearing when dicussing meta-level learning in Chapter 9.

4. Role: As seen with both the mathematical and MTF discourses, the analysis of the participants' pedagogical narratives seemed to show a combination of KO's and teachers' narratives, which suggests that both KO and teachers were assumed to be equal contributors to the pedagogical talk, as evidenced in the heavy use of keywords related to collectiveness such as "we", especially from the KO's utterances. However, the data in relation to MTF narratives highlighted the leading role played by the KO in providing point of focus and how things should be done, particularly in the scaffolding routines. The implication of the KO's leading role in scaffolding routines has been covered in the previous section where I discussed the meta-discursive goals.

The main argument that could be advanced from the analysis of the pedagogical discourse was that in terms of goals of pedagogical talk, the analysis has shown that the goals were to support the construction of MTF narratives and expand the MTF objects. An implication of this was the possibility that pedagogical routines may open up opportunities to participate in an explorative MTF discourse. In addition, there was the complex role of the KO in this particular part of the L/S, indicating explorative participation by her most of the time with all things discussed, but explorative participation by the teachers with things other than elements of the MTF, where the discourse was there but dominated by the KO, thus obscuring what teachers might be able to do.

6.2.4 Organising theme 4: The modes of talk supported the MTF-use

The earlier sections explored the type of discourses that were integrated by teachers as they used and worked to make sense of the MTF tool as they planned a lesson. Beyond this, the data highlighted how teachers and the KO interacted with each other and this was associated with different roles played by these participants in the discursive activities. In the analysis the KO played a dominant role in the discourse. This finding had implications in how the analysis in this section was carried out—what was illuminated was the description of the KO's nature of talk in relation to the teachers.

As noted earlier in Chapter 5, the tools used in the analysis of teachers' talk were drawn from Dudley's (2013) analytical indicators for studying modes of talk, namely, cumulative; disputational; exploratory[17];

[17] This is a reminder, and particularly since exploratory talk in Mercer's terms is different from the use of exploratory discourse in Sfard's terms and how the word has been used to this point. Here exploratory talk is a

structuring conversation and mutual understanding. In Table 30 as a way of a reminder I provide the summary of indicators as discussed in Chapter 5:

Table 30

Categories of indicators explored in transcripts based on Dudley (2013, p. 111)

Cumulative	Disputational	Exploratory	Structuring Conversation	Mutual understanding
▪ Agreeing or accepting ▪ Rephrasing, echoing, playing back words. ▪ Supporting/valuing another person. ▪ Recounting or describing an event ▪ Emotional expression ▪ Acknowledging ▪ Requirement/need	▪ Evaluative and defensive statements ▪ Disagreeing with little or no justification.	▪ Rehearsing teaching (scenario). ▪ Developing a point, extending an idea. ▪ Making an evidenced, insightful point/interpretation. ▪ Challenging the validity of an idea or action. ▪ Qualifying somebody's comment to improve it or express slight doubt. ▪ Justifying an idea (defending an idea). ▪ Suggesting an idea (proposing is idea) ▪ Reflecting, remarking ▪ Hypothesizing or speculating with reason. ▪ Summarizing, synthesizing, or concluding. ▪ Making a reasoned proposal. (requesting others opinions) ▪ Highlight, emphasize, stress, ▪ Asserting idea	▪ Initiating or introducing new idea ▪ Moving conversation on. ▪ Returning to an earlier subject. ▪ Inviting participation ▪ Warn,remind, reassuring	▪ Requesting clarity or greater detail ▪ Explaining some background information

Note: My additional indicators are highlighted in blue font.

The purpose of this section was to identify the type of interaction in which teachers engaged with and to explore the relationship between those interactions and MTF-use. I specifically pursued the following analysis questions: (1) What patterns of interactions did teachers engage with as they used the MTF tool? (2) How did these interactions enable and/or constrain teachers' construction of narratives as they used the MTF tool?

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talk, "in which partners present ideas as clearly and as explicitly as necessary for them to become shared and jointly evaluated. Second, it is talk in which partners reason together – problems are jointly analyzed, possible explanations are compared, joint decisions are reached. From an observer's point of view, their reasoning is visible in the talk" (Mercer, 1996, p. 363). As noted in the theoretical review chapter, I used the term "explorative" to refer to Sfard's exploratory routines (see pg. 76).

As noted in Chapter 5, in order to evaluate the different kinds of talk that were present during different L/S sessions, I generated categories and coded each line in the transcripts where I identified all discursive routines and modes of talk as illustrated in Table 28. The in-depth analysis led me to notice that the modes of talk were related to various discursive activities. Figure 7 below uses a graphical representation to show the relationship between modes of talk and discursive activities.

Figure 7

The relationship between modes of talk and discursive activities

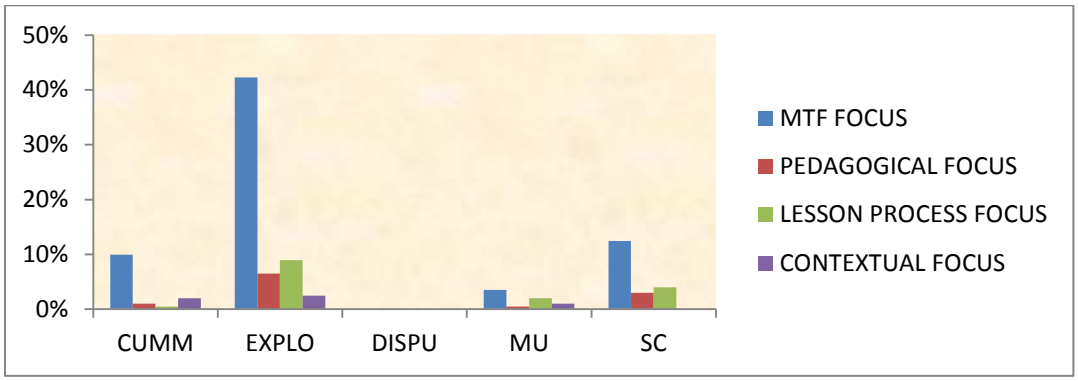


Figure 7 indicates that in the planning session, teachers mainly engaged in exploratory talk (EXPLO), followed by cumulative talk (CUMM), and that this type of talk was dedicated to MTF discursive activities, recorded as MTF focus[18]. These two types of interactional patterns usually appeared in utterances where all members tended to be more active participants in the talk. However, in these interactions, the KO was a leading person that dominated and continually focused on building positively, either critically or uncritically, on what the others have said.

There was also evidence of structuring conversation (SC) and managing understanding (MU) talks, but these were the least frequent in episodes. Similar to the exploratory and cumulative talks, these two types of talks were mainly focused on MTF discursive activity. More SC patterns were identified in utterances where less participation was recorded from the teachers, but more from the KO. On the other hand, MU was characterised by participation from both the KO and the teachers; however, in

[18] The Maths focus category is not reflected in the figure above because as per the analysis in the MTF discourse in the previous section, it was demonstrated that the use of mathematical routines implied MTF routines. In other words, the MTF routines integrated mathematical routines. For the analysis in this section I therefore record the MTF focus category only.

these interactions the KO took a central role in responding to other members' requests for clarification about a specific idea.

According to Figure 7, there was no disputational talk recorded. This is not surprising since the talk during LP1 in general was KO dominated with very few contributions from the teachers. This can be further justified by the fact that it was the teachers' first experience with the MTF tool, and they were not considered to be in a position to challenge the KO or to have any contradictory contributions; the KO could be seen as the source of knowledge. The KO largely appears to be giving directive orders to the group: "Ok now, at this point now I am going to take a lead on this planning session, so at least you understand how we are playing the game, otherwise I expect you next time that you are going to take a lead" (EP2, line 34). It was interesting to observe whether this would change in the next reflective discussion session.

As indicated in Chapter 5, the analysis of the interactions amongst teachers followed Schütte et al.'s (2019) principle of turn by turn analysis, where each utterance from one speaker is interpreted one after the other in the sequence of their occurrence in order to reconstruct the development of the interaction. Therefore, in order to provide evidence, I have used one lengthy extract in each section. In each extract I do not highlight the indicators related to modes of talk since these are interpretative and cannot be indicated directly in the utterances.

Some of the interactions on the MTF-use could not be divided into clear-cut types of talk provided by Dudley (2013) as illustrated in figure 7 above, since the exchanges consisted of different levels of talk. According to the results, four types of talk consisting of single level (exploratory), bi-level (exploratory-cumulative, exploratory-structuring conversation and exploratory-managing understanding) were identified. I have named these bi-level talks semi-exploratory talk, as they represent a development of the other talks towards the exploratory talk. What is also illuminated in this analysis is that social modes of talk were linked to talk about MTF objects (i.e., MTFying discursive activity). First, an example of each type is analysed, accompanied by the discussion with respect to the relation between each type of talk and the participants' MTFying talk.

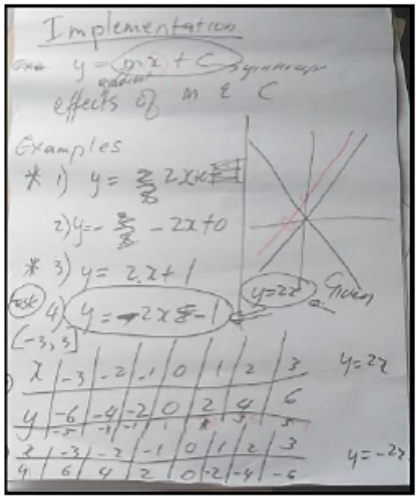
6.2.4.1 Emerging theme: Uni-level talk (Exploratory talk)

The interactions that featured exploratory talk showed explicit and more focus on discursive routines related to MTF. I use Extract 19 to show a sequence of interactions. This extract has characteristics of

exploratory talk in which both the KO and the teachers seem to be jointly constructing MTF narratives. What this sequence of interaction reveals is foregrounding of talk about MTF objects (MTFying) where the views of all participants are considered, the proposals are explicitly stated and evaluated, and finally the joint agreements are reached.

Extract 19

An example of MTF routines related to examples

Episode	Speaker	What is said	What is done
EP9	KO	208. So let us start with our implementation. In our implementation we need **examples** y equals two x ($y = 2x$) I am thinking of kind of **explanations**	
	T	209. Let us change **the signs** *(writing on board, see picture on the right-hand side)*	
	KO	210. The **explanations** are going to be coming directly from the **examples.** 211. We can start off the **explanations** by saying, "Now look at the value of a **gradient is 2,**" so we would like *to illustrate* with more **examples,** with more **explanations.** 212. For illustrations we can use tables. Will the **table** *show* any **effect on m and c?**	
	M	213. Not really, but the *drawing* a graph *will show* because **table and the equation** is more or less the same	
	KO	**214.** The table is going to help us to be able *to draw* a graph. It is part of our **explanation.** 215. In a table we will be able *to see* when we are changing **the sign of your gradient.** 216. It is not that we will be wasting time with our table; we can still use it to *draw* the graph. So let's *draw* it.	
	ALL	217. Yes	
	KO	218. An **interval of** -3 to 3 219. So we are going to start off by having a table here *(writing on the board)*. This is one of our **example** -3,-2,-1,0,1,2,3; now we are doing it together, -6,-4,-2,0,2,4,6 220. So we can do for the **second one,** then here we are going to have 6,4,2,0,-2,-4,-6 221. So what we did here was to *multiply* the **gradient** by minus 2. Can we see what is the difference between these two?	

In lines 208-213, the interaction showed that the teachers' talk was foregrounding the teachers' actions with respect to the MTF object of "explanations". At the beginning of the presented sequence, the KO

suggested the focus of discussion which was related to the type of examples and explanations to be included in the implementation stage. This is followed by the reaction from T (line 209), who suggested changing some of the examples by changing the sign of the coefficient of x in the equation. Following T's remark, KO (lines 210-212) elaborated on the process of developing explanations by focusing on the substantiating routine of interpreting the effects of the gradient in a tabular representation as a way of legitimising/endorsing the effects of the gradient. This was followed by a counterargument from M (line 213) who rather suggested a different procedure of constructing a graph. The KO, in turn, justified her idea (lines 214-216) by explaining the usefulness of a tabular representation in helping to construct the graph and also to interpret the effects of a gradient. Agreement was reached in line 217.

This sequence of interactions from lines 208-222 evidenced the joint engagement in discursive activity of constructing MTF narratives related to explanations. This implies presence of exploratory talk in Mercer's terms. At the same time, the analysis of the same extract in section 6.2.2.2 showed that the goal of this discursive activity was to construct MTF narratives related to explanations. It was concluded in that section that this discursive activity showed signs of explorative MTF discourse. Given this finding, from the analysis of the modes of interaction, the exploratory talk could be linked to explorative participation in MTFying discursive activity—interplay between interaction and MTFying. This finding seems to suggest that the exploratory talk could be a catalyst to a possible explorative participation in MTF discourses.

6.2.4.2 Emerging theme: Bi-level talks (Hybrid of exploratory talk and other talks)

The analysis of exploratory talk further revealed characteristics of more than one type of talk: cumulative-exploratory, structuring conversation-exploratory talk, managing understanding-exploratory.

(i) cumulative-exploratory

There were instances where teachers' interaction was characterised as both exploratory and cumulative talk. Further analysis of this type of talk suggested that the cumulative-exploratory interactions could be categorised into three purposes: (1) communicating what needed to be done; (2) communicating agreements; and (3) elaborating on construction of narratives mostly related to MTF.

The cumulative talk was mostly characterised by participants' concerns about what needed or was required to be done in order to complete specific tasks. These were particularly from the KO. In these

interactions I identified a number of keywords such as “we want”, “we need” (highlighted in grey in Extract 20 below) and these were related to specific MTF routines. The use of “we” by the KO seems to suggest that she was trying to develop a close rapport with the teachers in order to encourage them to take part in the conversations. Furthermore, these types of interactions seem to be developing towards exploratory talk where participants were elaborating on what needed to be done. For example, in lines 111-113, the KO stated explicitly what teachers, including herself (as a collective), needed to do with respect to examples (MTF objects) and further suggested what specific mathematical routines needed to accompany these examples (e.g., calculating, translating, classifying). This action resulted in the generation of different exemplifying routines as indicated in detail in section 6.2.2.2 above. Following KO’s utterance, J (line 114) was seen to be extending KO’s MTF narratives by suggesting an inclusion of particular type of example, counterexample, in the form $y = 3/x + 2$, and he further suggested questioning prompts to be used. According to the analysis in section 6.2.3, these questioning prompts facilitated the construction of MTF narratives related to tasks and learner explanations.

Extract 20

Example 1 of cumulative-exploratory talk

Episode	Teacher	What is said
EP5	KO	111. So what we are saying is that we need **examples** where they are *calculating*, and we also want them to *calculate* maybe **gradient or intercept**, these could be **tasks**. 112. Other **examples** could be dealing with *translation*, i.e. moving from graph to table to equation. 113. In our planning we need **examples** for *classification.*
	J	114. Also include **counterexamples** like $y = 3/x + 2$, and *ask*: “Is it a linear graph? Why are you saying that?”

This excerpt above further typified the cumulative talk which develops into exploratory talk where the KO explicitly proposed ideas related to MTF, highlighting the nature of MTF objects and giving guidance on how they could be used in terms of generating MTF narratives. For this, it could be said that cumulative-exploratory talk is linked to the process of generation of mathematical and MTF narratives. The analysis in section 6.2.2.2 has shown that this process implies explorative discourse (mathematical and MTF). It appears as if the cumulative-exploratory talk may be a catalyst to participation in explorative discourse.

Another example where there was evidence of exploratory-cumulative talk was in some interactions whereby teachers were collectively in agreement with exploratory opinions of others, the KO in this

particular instance. This was signalled by the use of keyword "yes", usually following a suggestion or an idea made by the KO:

Extract 21

Example 2 of cumulative-exploratory talk

Episode	Teacher	What is said
EP7	KO	160. A table is not going to help us to *classify*. It's not a good **example**. 161. Or let's ask them to *draw* these **functions**, how they *draw* them? Maybe we can give them an interval 0-5. 162. Asking them to *draw* them, you want them to confirm if these are **linear functions.**
	ALL	163. Yes

The first utterance in Extract 21 was made by the KO (line 160), proposing the type of tasks to be given to the learners. She elaborated further on the nature of these tasks by suggesting the accompanying mathematical routine to be used for constructing the graphs. KO's proposal was followed by agreement from all participants through the use of keyword "yes", suggesting that they were in agreement with KO's exploratory suggestions. What this implies is that the teachers were uncritically in agreement (cumulative) with the KO's construction of MTF narratives (exploratory). It can therefore be concluded that the cumulative-exploratory talk is associated with the process of construction of MTF narratives. As shown previously, this process (discursive activity) implies possible explorative MTF discourse. This finding further reinforces the claim that the cumulative-exploratory talk could be a catalyst to a possible explorative participation in MTF discourses.

(ii) Structuring conversation-exploratory talk

From the methodology chapter (Chapter 5), structuring conversation (SC) can be viewed as mutual understanding about the way in which exchanges should be structured. The analysis has revealed that the SC talk was strongly linked to KO's scaffolding routines. Furthermore, the analysis has revealed that this type of interaction was linked to the MTF.

Some of the interactions revealed a discussion directed by the KO, demonstrating her authority as a way of reminding the other participants about key features and priorities related to MTF. This was also done by inclusion of discussion prompts such as reminders and warnings not to divert from the object of learning, which was the key feature of the MTF. There was also evidence of exploratory talk in the same

interaction which was characterised by acts of rehearsing teaching and extending each other's ideas. The excerpt below exemplifies the exploratory-structuring understanding talk.

Extract 22

An example of structuring conversation-exploratory talk

Episode	Speaker	What is said
	KO	62. If you want to add it's okay. But remember, our **object of learning** is the **gradient and the y-intercept.** 63. We need to focus there, we must not divert. 64. However, you know what the problems are. One of the problems is that learners don't understand a **slope.** 65. You can see it in a **quadratic function** they will say *it's slender*, they use everyday language. 66. What helps is for us becoming aware of common problems that the learners are experiencing with the **gradient and y-intercept.** 67. Those are the two things that we should be thinking about. They have been introduced to **linear functions**, they know it, and they should know it. 68. They know **linear function** is $y = mx + c$. 69. So, in your introduction you should write $y = mx + c$. 70. How you introduce your lesson is important. What is it you announce publicly? You are aware that a **linear function** is in this form. 71. When you are doing introduction you don't want confuse your learners. What is it you want them to learn with **linear functions?** 72. You may say, "Here is the **linear function,** you know this, and you have been introduced to this before." 73. "This is what we are going to be doing." 74. Now you can come with your spider diagram. Or you can just ask them to *interpret* properties of y=mx+c. 75. Your introduction (J) is very generic; it is looking at the concept of **a function-input/output.** 76. We are assuming they know this definition. 77. So in your public announcement, you can say, "Today we are going to talk about **linear functions,**" let them tell you the properties of the **linear function.** 78. Direct them on what is your **object of learning.** Remember at the back of your mind, your focus is **object of learning.**

This conversation was about identification of critical aspects of OoL. The conversation started with KO (in lines 62-68) reminding the team members about the focus of the object of learning—effects of gradient and y-intercept—and warning the team members not to divert from this. She also shared research knowledge about the learner difficulty with regard to aspects of gradient and y-intercept. In lines 69-73, she further suggested the structure of the lesson introduction (this is a pedagogical action), where she emphasised what is announced during the introduction of a lesson, where the teacher brings learners' attention to linear functions. Afterwards, in lines 74-78, she discussed how a lesson could be executed in terms of examples and talked about the spider diagram example as suggested by J in the

first episode (returning to earlier subject). While discussing the structure she reminded the team members to always keep in mind what needed to be learnt.

These utterances indicated the presence of indicators related to both exploratory and structuring modes of talk, whereby the KO led the discussion by structuring the talk through the use of prompts such as "warning", "reminders". This talk was combined with the construction of MTF narratives particularly related to CA of OoL. This finding suggests that the structuring conversations talk has the possibility of increasing the incidence of exploratory talk by promoting explorative participation in the MTF discourse. The analysis of structuring conversation-exploratory talk seemed to further reinforce that semi-exploratory talk supports exploratory engagements, which could enable explorative participation in the MTF discourse.

(iii) Managing understanding-exploratory talk

Some of the interactions were characterised by requests for clarity or greater detail. In response to these requests, the interactions were followed by elaborations and explanations about some background either related to curricular, pedagogic or other object-related aspects such as mathematical and MTF objects. It should be noted that most of these elaborations were from the KO and the requests were from the teachers.

Extract 23

An example of managing understanding-exploratory talk

Episode	Speaker	What is said
EP10	S	253. If I may ask, I heard J commenting that using a word *shifting* of the graph can be confusing for the learners?
	KO	254. That is the language you are going to be using when you are talking about **the effects of c**.
	M	255. I want to agree with him. We have to *define* the term **intercept.** That is where *it cuts*. *It cuts* the y-axis, it's the y**-intercept**. We are not s*hifting* anything
	S	256. But what happens for it to *cut or shift?*
	KO	257. That will happen when you are *generalising*. When you are *generalising,* you are bringing in two things at the same time.
	S	258. Oh, you can say, that's where *it cuts* the y-axis, meaning it has been *shifted upwards*

The interactions reflected in the extract above were about MTF narratives related to the CA of OoL. For example, the discussion began with S requesting clarification about a comment previously made by J regarding confusion that could be caused through the use of the word "shifting". This narrative was

linked to the critical aspects of OoL—MTF object. This was followed by KO (line 254) explaining or giving some background about mathematical language related to the word "shifted". In line 255, M agreed with J's point about the use of the word "shifting", and further extended this idea by giving a mathematical account of the effects of the y-intercept and that it was about cutting the y axis, not about the shifts. In line 256, S seemed unhappy with M's argument and requested clarity. KO, in line 257, then intervened in this discussion and added to M's contribution that the word shifting could only be used when there was a comparison between two functions to demonstrate the effect of change in y-intercept as a way of generalising. S, in line 258, seemed to be in agreement with KO's explanation and expanded on it by adding another way of interpreting, talking about the word shift where he said "it cuts the y-axis". It is undeniable that the KO's strategy of giving clarity in terms of mathematical ways of talking were characterised by mathematical demonstrations about the nature of mathematical objects, that is, the exploratory ways of talking about mathematical objects. As seen with other types of talk, it can be concluded that managing understanding-exploratory talk possibly supported exploratory talk which was shown to be a possible catalyst to an explorative participation in the MTF discourse.

6.2.4.3 Summary

In summing up, the analysis showed that all three semi-exploratory talks were linked to the discursive activity of MTFying and served different purposes. First, the CUMM-EXPLO showed that this talk was characterised by solidarity and collaborative interaction. For example, the repeated use of keyword "Yes" implied solidarity amongst participants and the use of keywords "we need" implied collaborative engagements. Secondly, the SC-EXPLO highlighted the notion of scaffolding, which was used by the KO to facilitate the use of the MTF tool by demonstrating different MTF routines—implying prioritising interaction. Lastly, the MU-EXPLO showed that this talk was used for clarifying information, which was related to either MTF or mathematical objects—implying explicatory interaction.

Furthermore, it can be concluded that the analysis of semi-exploratory talk highlighted the role played by the KO in supporting the process of construction of narratives about the MTF tool. It can therefore be said that these types of talk were used by the KO in an attempt to promote exploratory talk. It can also be argued that these talks were necessary building blocks for exploratory engagements. This finding is in line with Arcidiacono and Gastaldi's (2011) study which confirmed that that the micro-sequences characterised by a combination of talks have the potential of developing into exploratory talk, where they argued that the presence of bi-level talks implies that each talk is a breeding ground/germinal stage

for exploratory talk. Furthermore, the analysis suggested that this exploratory talk could have supported the participation in explorative MTF discourse.

6.2.5 Global theme shared repertoire of the L/S CoP

As noted earlier, the global theme is the group of organising themes pulled together to represent the shared repertoire of the L/S CoP. The findings of the analysis of the organising themes in the previous sections led me to answer the research questions related to the shared repertoire dimension—questions related to the capability the L/S CoP has produced.

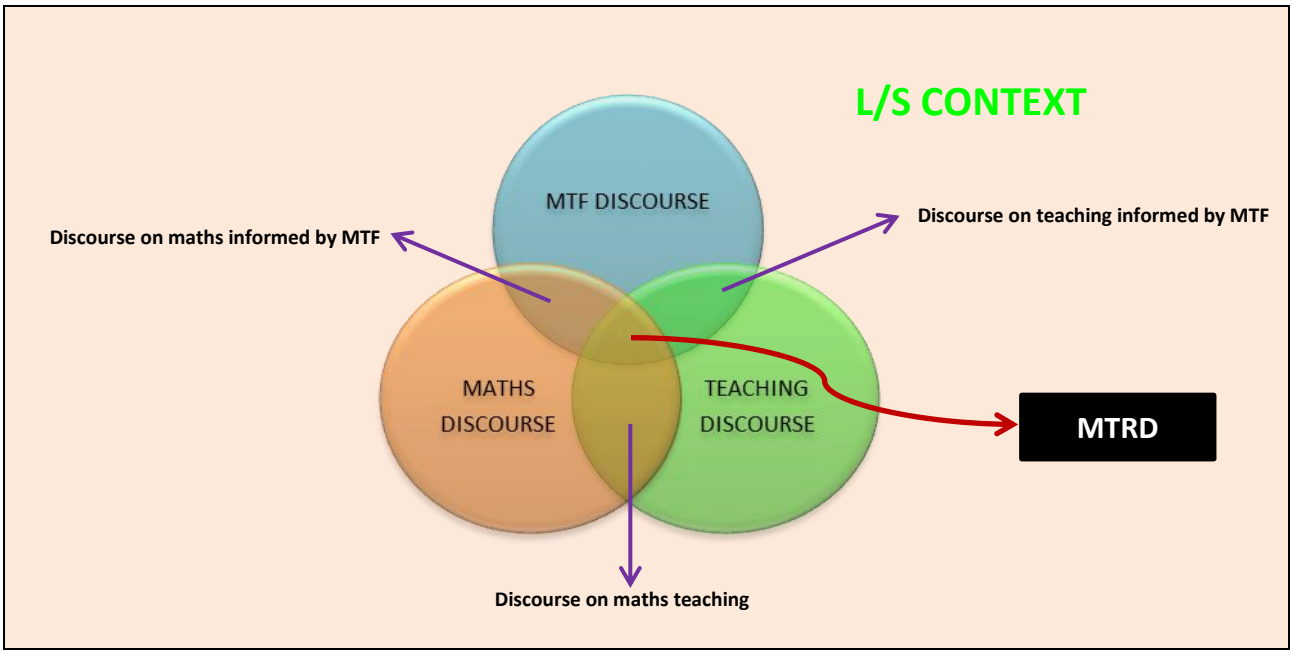
Wenger (1998) contended that when the community mutually engages in the pursuit of joint enterprise it creates shared history of shared repertoire which includes specialised discourses. This implies that both mutual engagement and the joint enterprise dimensions serve as a window through which I gained entry into the communities' shared repertoire. The shared repertoire of this L/S is reflected in the shared reflective discourse called mathematics teacher reflective discourse (MTRD). The MTRD is related to the ability to dynamically draw upon different routines for teaching mathematics such as mathematising, pedagogical and MTF routines. As discussed in Chapters 2 and 3, the routine-use is linked to the nature of the emerging discourse. It then prompts one to ask a question about the nature of these specialised discourses. In order to achieve this goal, I draw from the summaries of the organising themes in the previous sections. But first I will discuss the nature of the MTRD in general.

The main finding of this LP1 session was that the discursive activities performed when using the MTF tool actually belonged to three distinct but intertwined discourses: a *mathematical* discourse, *pedagogical* discourse and *MTF* discourse. Looking at the characteristics used within the communicational theory framework to distinguish different discourses, the mathematical and pedagogical discourses used the same words to name objects of the discourses, but differed to some extent regarding narratives and to a large extent regarding routines. However, the MTF discourse used unique words to refer to its objects (such as examples, tasks, OoL, explanations), with routines similar to the mathematical discourse, but differed slightly with regard to narratives.

I have identified six interrelated elements that are important in the emergence of the MTRD (see Figure 8).

Figure 8:

An illustration of MTRD



According to Figure 8, the MTRD is based on three primary forms of discourse. So, the first step should be to understand teachers' primary forms of discourse in the L/S context, which are: MTF discourse: **MTF**; teaching discourse: **TD**; mathematics discourse: **MD**. The nature of these three discourses has been dealt with individually in the previous sections.

Now that I have my primary focus of discourses, I now focus on where they interact; the aim is to understand their relationships. With reference to Figure 8, the interconnections are reflected in the following: Mathematics discourse informed by MTF (**MD-MTF**), Mathematical teaching discourse (**MTD)** and MTF teaching discourse **(MTF-TD).**

The centre of the diagram, otherwise known as MTRD represents a full understanding of how teachers talk about planning for teaching using the MTF tool. However, this is not the same as participating in each of the three primary discursive concepts individually. Instead, the point of MTRD is to understand how to use the MTF tool to teach mathematical concepts. The MTRD weaves all the primary discourses: MTF, teaching, and mathematical discourses. It is an emergent discourse that goes beyond all three primary components.

I now present the nature of the MTRD, referring to Figure 8. I will present the three intertwined discourses: MD-MTF, MTD and MTF-TD.

I specifically describe their nature in terms of four elements of analysing discursive activities: objects of the discourse; sources of narratives; goals of the discursive activities; and roles played by participants.

MTF-TD

The MTF teaching discourse organises the mathematical work of teaching with respect to mathematical instructional objects such as examples, tasks, explanations and OoL. The sources of narratives included MTF routines and pedagogical routines. The MTF routines included exemplifying, generating tasks, teacher explaining, learner explaining and identifying CA of OoL. The nature of these discursive routines was discussed extensively in section 6.2.2 and will not be repeated here. With regard to pedagogical routines, the analysis in the previous sections has shown that the pedagogical routines such as questioning facilitated the construction of MTF narratives and expansion of MTF objects. For example, the analysis has shown that the teachers' actions effected a modification of the MTF tool by dividing the explanation component into teacher explanations and learner explanations. Furthermore, the original MTF component of learner participation manifested itself in both learner explanations and tasks. This process of expansion was facilitated more specifically by questioning routines. In addition to questioning routines, other pedagogical routines that contributed to the construction of MTF narratives and expansion of MTF objects included identifying CA, where this particular routine produced the CA of OoL, and further facilitating the interconnection between MTF objects. At the same time, the analysis has evidenced how the KO's scaffolding routines supported the construction of MTF narratives. Another important pedagogical routine was the structuring routine where its use suggested a re-modification of the MTF tool by structuring it into different stages (such as introduction, implementation and conclusion) taking into consideration MTF narratives in each stage.

In terms of goals of participating in the discursive activities of MTF-TD, it can be concluded that the implied goals could be categorised into two levels: (1) the object level, where the analysis has highlighted that the pedagogical routines in use facilitated the expansion of MTF objects and their interconnection;(2) meta-level goals which were related to construction of MTF narratives and re-modification of the MTF tool.

Regarding the roles, the analysis has highlighted the dominant role of the KO through her scaffolding routines which facilitated the construction of MTF narratives.

The MTF discursive activity suggested a possible explorative discourse. However, the KO-dominated participation particularly through her scaffolding routines implied ritual goals. It can be concluded that the MTF-PD is characterised by a mixture of explorative and ritual elements.

MTD

The MTD shows the relationship between teachers' mathematical and pedagogical discourses in the context of L/S. The pedagogical discourse analysis from section 6.2.3 has shown that the objects of this discourse are similar to those found in the mathematical discourse. In other words, teachers organised their pedagogical talk about mathematical objects related to linear functions and its properties—the gradient and y −intercept. The most significant mathematical teaching routines were questioning routines. The analysis has further shown that the use of these routines facilitated the construction of mathematical narratives, which in turn facilitated the construction of MTF narratives related to MTF objects such as tasks, explanations and learner participation. Furthermore, it facilitated the expansion of MTF object— the explanation into learner explanations and teacher explanations. Another important mathematical teaching routine that was identified in the context of this study was identifying CA. Its aim was to identify a critical mathematical aspect that needs to be learned—CA of OoL, an MTF object.

Regarding the goals of the discursive activities within this type of discourse, the analysis has suggested that the object level goal was related to the expansion of MTF objects, and the meta-level goal was concerned with the construction of MTF narratives. In terms of the roles, the analysis has indicated that both teachers and the KO contributed equally to the discursive activities of this particular discourse.

MD-MTF

The objects in focus in this particular discourse were mathematical objects (linear function and its properties—gradient and y-intercept) and MTF objects (examples, tasks, OoL, explanations). The sources of narratives included both mathematical routines and MTF routines. With regard to mathematical routines, the analysis has shown that there was a consistent application of the same mathematical routine procedure across all MTF components, from tasks, examples, explanations and

OoL (critical aspects). This process resulted in an interconnection between MTF objects. Such a process is coined coherence (Adler, 2021).

The analysis exposed a relationship between the MTF and mathematical discourse and how this relationship was instrumental in producing MTF routines, where the MTF tool went through a transformation and inherited a discursive form. When the MTF interacted with mathematical routines together with the performer of those routines, new MTF narratives were constructed and these were referred to as MTF routines.

According to the analysis, the object level goal of participating in this discourse was to establish the interconnection between the MTF objects. On the other hand, the meta-level goal was to construct MTF narratives. In terms of the roles, both teachers and the KO contributed to the discursive activities, however the KO played a dominant role.

6.3 Summary of what opportunities to learn were made possible

By participating in the MTRD, teachers were afforded different opportunities of learning from three different discourses: mathematics, MTF, pedagogical and social relations. Furthermore, these opportunities of learning were supported by collaborative engagements with the KO. In Table 31, I summarise four themes that describe the possible opportunities for learning. I will discuss these themes in detail in the teacher learning chapter.

Table 31

Summary of what opportunities to learn were made possible

Type of discourse	**Opportunities to learn**
Mathematics discourse	Opportunities that could promote learners' explorative mathematical learning. Connection between word use and mathematical explanatory talk.
MTF discourse	Coherence between the CA of OoL and other MTF components.
Pedagogical discourse	Integrating other pedagogical strategies that are critical for the work of teaching.
Relations	Opportunities to learn from each other and from the KO

6.4 Conclusion

The MTRD reflected signs of explorative discourse in terms of sources of narratives, and goals of the discursive activity with the KO's ritualistic role. Furthermore, the analysis exposed that these discursive activities (construction of MTF narratives in particular) were supported by the KO's modes of talk which were characterised by semi-exploratory talk which highlighted collaborative, prioritising and explicatory interactional goals of participation in the LP1 reflective discourse. The analysis of LP1 points to the interplay between acts of mathematising, MTFying and social talk which have implications for the shaping of MTF-use, and teacher learning.

At the end of the LP1 session, a lesson plan was produced (see Table 10). As a way of concluding this session, teacher T was selected to teach the lesson in the first teaching session (T1). As noted in the methodology chapter, the teaching session itself was not a focus of this study, however the full description of this session was provided as it is the focused talk in the reflective discussion (RD1) (see section Appendix E ,Table T1). The aim of the teaching session was to enact the lesson plan that was produced during the first lesson planning session and to offer the teachers an opportunity to observe and then reflect on teaching the lesson. In the following chapter, I present the analysis of teachers' post-lesson reflective discussion (RD1).

Chapter 7: The Reflective Actions during Reflective Discussion 1— RD1

7.1 Introduction

The focus of this chapter is to describe and interpret MTF-use during the RD1 session. In RD1, teachers reflected on the first teaching session. As noted in the methodology chapter, the RD1 data also included LP2 data where discussions about the second lesson planning session (LP2) were recorded. The aim of this session was also to revise the lesson plan for the second teaching session (T2), taking into consideration any suggested changes to the lesson plan that was produced during the first lesson planning session (LP1).

The RD1 involved reflections from both the individual teacher who was teaching and from a group of teachers who were observing the teaching lesson. This process was guided by a specific rule as noted in the methodology chapter: the teacher who taught the lesson (teacher T) first gave feedback on his experience of the lesson. This was followed by the report back from the group members, including the KO. With reference to Table 28_A[19] (in Appendix D), a total of nine episodes were recorded. The reflections from teacher T were captured in episodes 1-3. The reflections from the others were captured in episodes 4-8. The talk about the second lesson planning session was captured in episode 9.

7.2 Summary of identified themes

The analysis of RD1 revealed similarities in the teachers' use of the MTF tool as seen with the LP1 analysis. Firstly, across all episodes, the use of MTF meant participating in a mathematical discourse which was characterised by object- and routine driven word use, a combination of mathematical and informal mathematical words, and use of various mathematical routines/mediators. Secondly, the use of MTF meant participating in a pedagogical discourse which was characterised by different pedagogical routines to support the process of construction of MTF narratives (including mathematical narratives). Thirdly, the use of the MTF meant participating in an MTF discourse which was characterised by explicit naming of MTF objects and MTF routines which showed coherence between MTF routines and pedagogical routines. Lastly, the modes of interaction supported the MTF-use.

[19]The process of the development of the framework matrix reflected in Table 28_A is similar to the one used in LP1 (Table 28)

Despite the articulated similarity in MTF-use, the pedagogical routines were characterised by reframing routines and disputational talk which were not present during LP1. Another difference between LP1 and RD1 was the integration of contextual factors related to learner engagement. All these differences highlight the different contexts under which the MTF was used.

In the table below I summarise the main findings that have emerged during RD1. As seen in the previous chapter, these findings are categorised according to the organising themes and associated emerging themes. The global theme reflected a similar finding as seen with LP1. Table 32 below shows that there were 12 emerging themes recorded alongside organising themes.

Table 32

Summary of the emerging and organising themes

Organising the themes	Emerging themes
1. Use of the MTF is linked with participation in mathematics discourse	▪ Teachers used mathematical and informal mathematical words ▪ Object/routine-driven word use of mathematical objects ▪ Use of different routines/visual mediators
5. The use of the MTF is linked to participation in pedagogical discourse	▪ Questioning routines used to generate learner participation and learner explanations ▪ Identifying CA routines used to generate CA of OoL ▪ Scaffolding used by the KO to support the construction of mathematical and MTF narratives ▪ Structuring used to highlight deviation from the planned lesson ▪ Reframing routine linked on-task subjectifying
6. The use of MTF meant participating in MTF discourse	▪ MTF Naming—object/routine driven ▪ MTF routines showed similar type of routines in each MTF object ▪ MTF routines linked to both mathematical and pedagogical routines
7. The modes of interaction supported the MTF-use	▪ Hybrid between exploratory talk and other talks enabled teachers' construction of MTF narratives

One global theme emerged, namely: MTF use is a type of discourse that interweaves several discourses—the mathematical, MTF and pedagogical.

7.2.1 Organising theme 1: Use of MTF was linked with teachers' participation in mathematical discourse

The analysis of the mathematical discourse followed a similar structure to LP1, and will be guided by the following principles: 1) the nature of mathematical objects present in teachers' utterances; 2) the sources of their mathematical narratives; 3) the goals of discursive activities; and (4) roles played by the participants (i.e., KO and teachers).

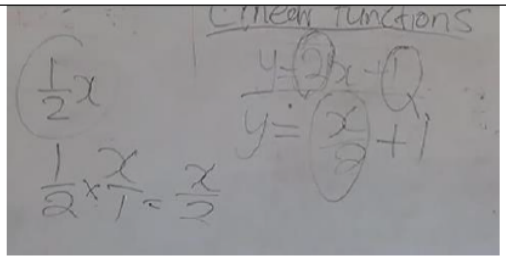
7.2.1.1 Emerging theme: Source of mathematical narratives included the mathematical and informal mathematical word use

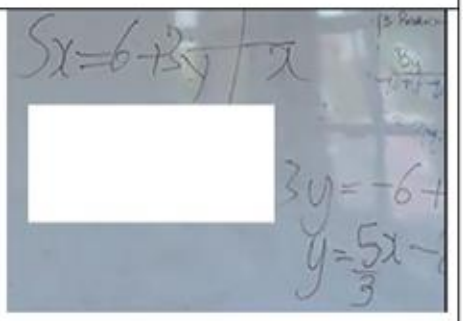
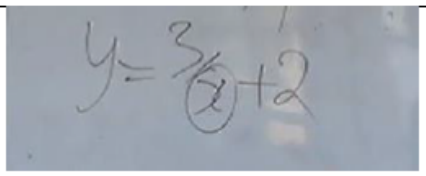
The analysis of the mathematical utterances showed that the teachers were using mostly mathematical words to refer to mathematical objects. However, there were some instances where there was the use of a mixture of informal mathematical and mathematical words, and such talk played an important role in construction of mathematical narratives, as shown below.

I summarise and exemplify with different extracts from different episodes in the table below. In these extracts I highlight the routines (marked with *purple italics*), visual mediators (underlined) and the objects in focus (**bolded)**. I also highlight in blue the implied performers of the discursive activity (I will only comment on this last point in section 7.2.1.4). Highlighted in green are keywords indicating learner difficulties.

Extract 24

An example showing combination of mathematical and informal mathematical word use

Episode	Speaker	What is said	What is done
EP2	T	30. Yes, remember there was that y equals x over 2 plus 1 ($y = x/2 + 1$) and I asked them to *identify* the **gradient** and they struggled with that and I tried to make an **example** just to see that if x over 2 ($x/2$) is half or not, they still struggled 31. and I had to do some *calculations* on the side for them to see that x over 2 ($x/2$) is the same as half x ($\frac{1}{2}x$)	

EP4	KO	49. One of my other comments is that we pay attention to words. There was a use of a word **slope**, we would like to use that everyday use of word and move it towards a mathematical use of a word and connect it to a **gradient**. 50. And with a word **fraction**, I would have loved to see how we can extend it to a more mathematical explanation when talking about a **gradient of a linear function** and discuss a notion of a **numerator** so that we can get a more generalised way of talking about it. 51. **A fraction** is a visual trigger to them and it's triggering a **hyperbola**.	
		52. But T, has emphasised the notion of a **coefficient.**Another thing I have noted is the use of words, such as "you want to put y on one side, you want to leave it on its own", "you are taking six to the other side'. (*see picture on the right-hand side*) 53. There is nothing wrong using this colloquial talk; but we need to guard against any future misconceptions, therefore we want to move towards a more mathematical talk.	
	T	54. I was trying to think about it, it was not coming, I could see I was saying same thing over and over. I think I said, "Make y **a subject of a formula**". Is that how you say it?	
	All	55. Yes	
	KO	56. Yes, but how do you do it. It's taking the additive inverse. The name of the process is making y **a subject of the formula**. The purpose of us doing this exercise is moving towards a more mathematical explanation.	
EP7	S	98. Another thing that I have noticed that could have led them to say 3, not all learners understand what we mean by **coefficient** because from where I was sitting there was a point where you were explaining to them that the **gradient** was the **coefficient of x** and only a few responded. But when you said, "Look at the number in front of x," they said okay.	

In line 30, the word "gradient" was matched with a process interpreting features of the symbolic representation $y = \frac{x}{2} + 1$. Similarly, in line 98, teacher S used the word "coefficient" to refer to the gradient of the symbolic equation $y = \frac{3}{x} + 2$ (see picture). The evoked routine was the interpreting

routine. The analysis of T's and S's mathematical narratives highlighted teachers' reliance on interpreting routine when attending to mathematical representations as recorded in Table 28_A.

With regard to the mixture of informal and mathematical talk, the KO in lines 49-51 used an informal mathematical word (slope) and a mathematical word (fraction) that were used in a mathematical classroom. She warned that these need to be moved into a more mathematical talk where a slope needed to be linked to the gradient, and careful attention had be paid to the word use of "fraction" when interpreting the equation in the form $y = \frac{x}{2} - 1$ as this could be confused with the hyperbola.

The KO went further in lines 52-53, highlighting the dangers of using colloquial talk when referring to the mathematical processes. For example, she referred to the use of phrases ("you want to put y on one side, you want to leave it on its own", "you are taking six to the other side"). These phrases could be interpreted as teachers' mathematical teaching actions. It is interesting that in all narratives showing the presence of informal mathematical word use, the implied action was related to teachers' mathematical teaching. Given this finding, it can be inferred that the informal mathematical word use was linked to pedagogical goals. In response to the KO's remarks (in lines 52-53), teacher T, in line 54, suggested a mathematical phrase to this process, where they talked about: "making y a subject of the formula". The results emerging from the analysis of the mixture of colloquial and mathematical talk suggests that the aim of this discursive activity was to highlight the possible learner misconceptions that the teachers' talk might create. In other words, one can say this discursive activity (mathematising) could be linked to the process of identifying critical aspects of the object of learning.

Another interesting interpretation that could be made with respect to the connection between the features of mathematical discourse was that both the mathematical and informal mathematical words were matched with mathematical routines and visual mediators. However, it appears as if the intended mathematical goal of the mathematising activity was not to substantiate mathematical narratives as seen during LP1 analysis. Instead it seems the goal was to identify the CA of mathematical objects under discussion—an important aspect of MTFying activity. Such a finding implies what I call interplay between mathematising and MTFying.

7.2.1.2 Emerging theme: Teachers' source of mathematical narratives: the use of various routines with respect to mathematical objects

The analysis of mathematical narratives has revealed that the source of narratives could be related to referencing of critical aspects of object of learning, talking about mathematical objects in objectified ways and an inclination to apply actions to mathematical objects, and use of various routines with respect to mathematical objects.

Looking closely at most of the teachers' mathematical narratives, it appears that the aim was to bring to light learners' difficulties related to interpretation of features of different representations of a linear function. For example, in lines 30 and 98 (Extract 24), there was a focus on learners' knowledge, as evidenced by use of phrases such as "they struggled" (line 30) and "not all learners understand" (line 98). This pattern was observed in most teachers' mathematical narratives. These two narratives suggest the activity of identifying learners' difficulties. This finding implies that the process of identifying critical aspects (CA) of the object of learning (OoL) was a source of construction of mathematical narratives. This presents a sharp contrast from the LP1 mathematising activity where the goal was to construct and then substantiate mathematical narratives.

Another interesting observation is that most of the teachers' mathematical talks were action oriented. This means the observed teachers' mathematical talk was focused on a performance of the procedure that constituted the discursive activity. For example in line 52 (Extract 24) the KO warned the teachers about using action-oriented phrases to refer to mathematical processes (e.g., "you want to put y on one side, you want to leave it on its own", "you are taking six to the other side"), and further suggested the inclusion of object-driven phrases to identify the mathematical processes such as "additive inverse" and "making y a subject of the formula" (line 56). The KO inserted formal mathematical naming and encouraged its use, which implies opportunities for teachers to learn mathematical explanatory talk.

Close scrutiny of the RD1 episodes resulted in instances which referred to mathematical objects using objectified talk, as shown in the following extract:

Extract 25

An example of object-driven word use of mathematical objects

Episode	Speaker	What is said
EP1	T	12. As I was doing the examples I felt like let me do just one example that has a **gradient** *being negative* and also include the constant. 13. From there I can show that from the prototype that the first thing is that the graph *is decreasing*. 14. Now the graph *is decreasing,* what is the **constant** *doing*? So they can see both actions at once. 15. Because the initial example, you just do one where we show the **gradient** and one where we show the **constant** 16. And the other one was supposed to be a class activity $y = 2x + 1$. It was never planned.

The mathematical objects talked about in this extract were properties of linear function (gradient and y-intercept). Also present in this extract were routines of interpretations ("gradient being negative", "graph is decreasing" and "what is the constant doing?"). These were talked about as if they were occurring by themselves with no human agency involved in the choice of the routine or performance.

The analysis also revealed that the source of mathematising activity included the use of various mathematical routines related to a function discourse such as interpreting, constructing and calculating routines. However, the interpreting routines were the most prominent routines used (see Table 28_A). This implies that teachers were inclined to use the global approach to functions—interpreting global features of representations. Take, for example, the following extract:

Extract 26

An example of use of various mathematical routines

Episode	Speaker	What is said
EP9	KO	132. I saw that he wanted it to *pass* on a certain **point** and we did not have that **point** in our table. So I am going to increase our **domain** so that it captures the **coordinates of the x intercept**. 133. We must remember that our **OoL** is about the **shifts**. So when we want to understand a **vertical shift,** *drawing* a graph from the equation will be an effective strategy. 134. Let's not use a table because we said it's confusing. In another words, you are showing them another strategy of *drawing* the graph by *calculating* the **y-intercept** using an **intercept method**, other than using a table.

In lines 132-134, KO's main focus was on the effects of the y-intercept (i.e., OoL is about the shifts). In her talk she integrated different mathematical routines from constructing the graph, interpreting the behaviour of the graph for the vertical shifts, and calculating the y-intercept. It appears as if these were used as resources for constructing and substantiating mathematical narratives related to the OoL of vertical shifts. Interestingly, in this utterance there was reference to learners' difficulty. For example, in line 134, KO commented about the limitation of one of the sources (i.e., construction of table) and what

difficulty it might create for the learners. This finding, as seen with the discussion above, seems to point to the goal of identifying critical aspects of the OoL (vertical shifts or effects of y-intercept).

The analysis of sources of mathematical narratives revealed that the goal of mathematising was to construct mathematical narratives. Hence, it could possibly be linked to the development of explorative discourse. Another interesting interpretation was that sources of constructing mathematical narratives were not only confined to mathematical objects as seen during LP1. Instead, there were considerations of users of these mathematical objects—the learners. This consideration was illuminated by referencing critical aspects of the object of learning. This aspect was strongly linked to a pedagogical routine of identifying CA. It can therefore be highlighted that participation in mathematical discourse with the MTF tool could offer opportunities to contextualise mathematical talk into the context of mathematical teaching with learners' thinking in mind.

7.2.1.3 Summary

The emerging mathematical discourse during RD1 reflected similar features as LP1's mathematical discourse. These features were those unique to the function discourse in terms of words, routines, visual mediators and narratives.

Similar principles were used during LP1 to describe the nature of participation in mathematical discourse during RD1.

1. Nature of maths objects (in terms of words and word use)

The analysis of word use brought three main findings. Firstly, it produced evidence showing that teachers' reflective talk had a combination of mathematical and informal mathematical words. Interestingly, the mathematical word use was aimed at describing mathematising actions. On the other hand, the informal mathematical word use described pedagogical actions. For example, the analysis has shown how the informal word use was used to develop arguments related to identification of CA (lines 49-51, Extract 24). Furthermore, the analysis has revealed that the word use was also used to link to mathematical explanatory talk (line 56, Extract 24). Secondly, the teachers' word use was linked to discursive routines such as mathematical routines and mathematical teaching routines. This was an indication of routine-driven word use. Thirdly, analysis has pointed to object-driven word use where mathematical objects were talked about as if they had their own life. In Sfard's (2008) terms, the

combination of both routine-driven and object-driven word use implies the initiation of the objectification process, an indication of explorative discourse.

The analysis has shown the connection between elements of word use, from words to attended procedures—mathematical routines and visual mediators. However, the intended focus was not only related to constructing mathematical narratives but also to highlight pedagogical narratives related to learner's mathematical difficulties—CA. This implies what I call interplay between mathematising and MTFying activities geared towards construction of MTF narratives. This finding has an important implication in describing the goals of participating in the mathematical discourse in the L/S context, which points to explorative discourse.

2. Sources of mathematical narratives

The analysis has indicated that the sources of mathematical narratives were from both the mathematical and mathematical teaching discourses. Looking at the lenses used within Sfard's (2008) communicational framework to distinguish discourses, these two discourses used mainly the same words and visual mediators but differed regarding routines. It is therefore expected that some features of these two could have served as sources for the construction of mathematical narratives.

From a mathematical discourse perspective, the results indicated the use of various mathematical routines. In terms of Sfard (2008), this can be referred to as "flexibility" in routine use—an element of explorative mathematical discourse.

3. Goals of discursive activity

The goals of mathematising discursive activity can be categorised into two: object and meta-level. The object level was evidenced by mathematising talk which was characterised by mathematical words, routines, and visual mediators from the function discourse.

Regarding the meta-level analysis, it seems that the goal of participating in the mathematising talk was to bring forward aspects of learner mathematical difficulties. For example, the analysis highlighted that the mathematical talk was frequently used in reference to learner difficulty or misconceptions in relation to mathematical object (y-intercept, gradient)—these are indicators of critical aspects. In turn, the learner difficulties were exemplified through learners' mathematical talk, which implies learner participation.

Another meta-discursive goal was related to the construction of mathematical narratives in relation to either teachers' use or learners' use. As seen with the analysis in LP1, this goal was evidenced by referencing keywords such as "they" (for the learners) and "we"/ "you" (for teachers). This finding is a pointer to a relationship between mathematising actions and MTFying actions.

4. Roles of participants

There were three distinct roles that group members undertook: the individual teacher reflections, the observers' reflections and the KO's expert knowledge reflections. The analysis of these roles indicated that these participants were talking about people's actions—subjectifying (Sfard, 2008). These were evidenced through the use of keywords "I", "my" from the individual teacher. On the other hand, the utterances from the group of observers were characterised by the use of a keyword "you", referring to the individual teacher. This type of talk seemed to be evaluative of the individual teacher's teaching. The subjectifying talk was expected since it was driven by the ground rules of participating in the RD context whereby participants were expected to reflect on the individual teacher's teaching. Interestingly, this subjectifying was referenced to the contents of reflections such as mathematising and pedagogical actions. In terms of Hunter and Anthony (2014), this subjectifying action can be referred to as "on-task" subjectifying. This means the talk was focused on the person's actions in relation to the objects of the discourse rather than on the person as such.

Similar to the analysis of LP1, the results showed signs of collaborative prompts which were evidenced by use of keywords such as "we", "us" by all of the participants, and not only from the KO as was seen in the LP1 analysis. The presence of these keywords seems to suggest the spirit of collectiveness.

In summing up, from the discussion above it can be concluded that teachers' word use was characterised by properties of explorative mathematical and pedagogical discourse. With respect to mathematical discourse, the analysis signalled properties of explorative mathematical discourse such as flexibility, objectification and with a goal of constructing mathematical routines. With regard to pedagogical discourse, the results have suggested that the teachers' talk about mathematical objects was combined with other routines found in the L/S context such as—identifying CA pedagogical routines. Such may be a signal of explorative mathematical teaching discourse (Heyd-Metzuyanim et al., 2018). This implies that while teachers reflected about mathematics in explorative ways, they also thought about their mathematical teaching, this is, a work of teaching.

In this section I have provided a discussion about teachers' mathematical discourse in terms of objects they were talking about, sources of their narrative, goals of their discursive activities, teachers' role and nature of the discourse in terms of ritual/explorative routines. This emerging mathematical discourse has implications for the nature of opportunities to learn that are afforded by participation in this type of discourse. This will be dealt with in section 7.3.

7.2.2 Organising theme 2: The use of the MTF meant participating in MTF discourse

Similar to LP1, this organising theme explored the nature of the MTF discourse in terms of the following: (1) words used to identify MTF objects; (2) sources of narratives; (3) goals of discursive activities; and (4) roles of the participants.

Looking at the MTF category (refer to row titled 'MTF' in Table 28_A in Appendix D), one is able to see the MTF category showed the two subcategories developed during the analysis: MTF naming and MTF routines. The MTF naming category shows names of the components of the MTF tool. The MTF routines were characterised by various mathematical routines. In the following section I will attend to these categories in turn, starting with MTF naming.

7.2.2.1 Emerging theme: MTF naming

Looking across the MTF row in the fourth column of Table 28_A, it is evident that all MTF objects were named explicitly. The examples were the most frequently named. The data of RD1 showed that there was reference to the name "learner participation". This is an interesting observation since this component was not explicitly referenced during LP1. In contrast to the analysis of LP1 data, the explicit naming of MTF objects was from all the participants, implying increased participation from the teachers. This observation could be due to a number of reasons. First, the teachers have had an opportunity to work with the MTF during the teaching session. Secondly, this could be due to the ground rules of the RD1, where the teachers were expected to reflect about MTF teaching.

An interesting feature of MTF naming is the relationship between MTF naming with: (1) other MTF objects; (2) pedagogical routines and (3) mathematical routines. The results showed that the teachers explicitly named one particular MTF object example as a focal point to talk about other MTF objects, as illustrated in the extract below:

Extract 27

Example of teachers' explicit naming of MTF components

Episode	Speaker	What is said
EP4	J	64. When this **example** came, it became clear that there was a lot of confusion with the word "**increasing**". There were like you *are increasing* with one and the graph is *shifting*. In that **example** where you had a graph of $y = 2x$, and you *drew* another for $y = 2x + 1$.

What was evidenced in line 64 was the relation between the examples and the aspects that learners seem to have a difficulty with. For example, teacher J referred to a word "confusion", which is the indicator of critical aspect of the object of learning. He further elaborated on the nature of this difficulty where he said: "There were like you are increasing with one and the graph is shifting". This discursive activity suggested that the explicit naming of the word example served to bring to focus the CA of OoL, which implies the relationship between the explicit naming of one MTF object to other MTF objects. It can therefore be assumed that the explicit naming of MTF objects served to facilitate the construction of MTF narratives related to CA of OoL which is related to what learners did or said—learner participation.

The second feature of MTF naming was the relationship between MTF naming and pedagogical routines, and this was present when the teachers referred to questioning strategies in relation to learner participation. As seen with LP1, the MTF naming was also characterised by pedagogical and mathematical routines. For example, the analysis of interactions between T and KO in Extract 28 below showed that the questioning routines were used as a strategy to encourage learner participation. Secondly, this talk highlighted what mathematical actions learners were invited to engage in. For example, in line 28, teacher T mentioned that he invited the learners to define the standard formula of a linear function and further interpret the gradient.

Extract 28

Example of different purposes of MTF naming—pedagogical and mathematical

Episode	Speaker	What is said
EP2	T	26. From **learners' participation,** at the beginning they were quiet; I felt that they did not know anything. I was bringing something new to them.
	KO	27. What made them to **engage or participate**?
	T	28. They were getting the idea. I was *questioning* repeatedly. I asked them to repeat with regards to what the standard formula says and what does the sign of the **gradient** means. 29. Every time I say, "If the gradient is negative, what does that mean?" and I kept on pushing. And they started understanding.

This analysis showed that the mathematical and pedagogical routines were used to foreground what learners get to do, and what strategies to use in order to encourage their participation. As seen in LP1, the tendency to link MTF names with either pedagogical or mathematical discursive routines suggests routine-driven word use. Furthermore, the interplay between the MTF names and discursive routines implies construction of MTF narratives—learner participation in this case.

To sum up, the analysis of MTF naming showed contributions from all participants. This contrasted with the findings from LP1 which indicated KO's dominance in MTF naming. This implies increased awareness of MTF components by the teachers. This means participation in the MTF naming activity has the potential to expose them to MTF components, which are important points of focus through which negotiation of meaning is organised.

7.2.2.2 Emerging theme: MTF Routines

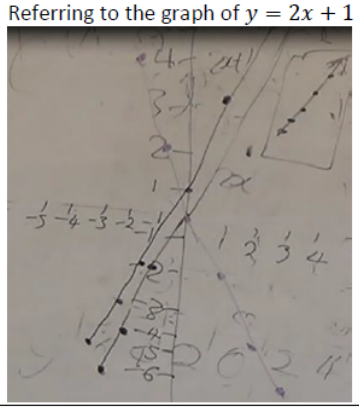
The fourth column of Table 28_A in the MTF routine row showed that the six types of routines that were included: exemplifying, generating tasks, teacher explaining, learner explaining, learner CA and teacher CA. Interestingly, the CA of OoL was divided into two subcategories: learner CA and teacher CA, which was not seen in LP1 where the CA was only from the teachers' perspective. This implies a different context where teachers are now focusing on learner participation and have been exposed to real classroom data which they collected during the teaching session. As with the LP1 analysis, the MTF routines played the same role of describing the discursive acts of teaching. In the following analysis, I describe the nature of each MTF routine in turn, following the same format used in the previous chapter.

Examples

The fourth column of Table 28_A in the MTF routine row shows that the exemplifying category involved different mathematical routine procedures from constructing, interpreting, translating and calculating. I exemplify using the following extract:

Extract 29

An example of MTF routines related to examples

Episode	Speaker	What is said	What is done
EP9	M	131. There was a question here in this **example** where one of the learners was confused where does the graph passes the x-axis cause it's not *showing*.	Referring to the graph of $y = 2x + 1$ 
	KO	132. I saw that, he wanted it to *pass* on a certain point and we did not have that point in our table. So I am going to increase our **domain** so that it captures **the coordinates of the x-intercept.** 133. We must remember that our **OoL** is about the shifts. So when we want to understand a **vertical shift,** *drawing* a graph from the equation will be an effective strategy. 134. Let's not use a table because we said it's confusing. Instead, let's show them a strategy of using **intercept method** to *draw* a graph.	
	All	135. For **x-intercept, $y = 0$ and y-intercept, $x = 0$**	
	KO	136. So you are running away from *drawing* a graph from the table and avoid changing our **domain** and include the **fractions like ½** and so on.	
	S	137. So when they are advanced they can flexibly use the table even if they have to work with **fractions or decimals**.	
	KO	138. This thing of **fractions** in a table will confuse them, rather introduce other method.	

The analysis of KO's narrative in lines 132-134 suggested that various mathematical representations and routines were used for teachers' use. The highlighted blue keywords ("we did not", "our domain") indicate special reference to teachers' use. The referenced mathematical representations included symbolic representation (equation), tabular representation and its features (points, domain, coordinates) and graphical representation and its features. The KO's talk was also characterised by different mathematical routines such as drawing, calculating, interpreting (e.g., "showing", see line 131) and translating routine (e.g., "drawing a graph from the equation" see line 133). This discursive activity suggested that teachers were afforded opportunities to illustrate mathematical ideas related to the effects of the y-intercept in flexible ways.

Furthermore, the analysis has shown a relationship between exemplifying routines and the CA of OoL. What is interesting, the CA of OoL was talked about from both the learner perspective (line 131) and the teacher perspective (line 138). For example, in line 131, teacher M commented about a particular example related to interpretation, and the problem that was evident from learners' talk during the teaching session. This was a real CA, and not conjectured from the teachers' perspective as shown in line 138, where the KO pointed out a difficulty that may be caused by using tabular representation for interpretation purposes. This utterance was what the teacher assumed might happen. This could suggest that learners' CAs were real critical aspects and teachers' CAs were conjectured CA. The

exemplifying discursive activity suggested that teachers were afforded opportunities to illustrate mathematical ideas related to the effects of the y-intercept in flexible ways and also to consider learners' thinking (CA of OoL) when talking about examples. This finding further pointed to the interplay between mathematising and MTFying.

Tasks

The discursive routines related to tasks were the least talked about, as shown in Table 28_A. The analysis revealed that the tasks were mostly in relation to the questioning routine as illustrated in the following extract:

Extract 30

An example of MTF naming related to tasks with pedagogical routine

Episode	Speaker	What is said
EP9	KO	141. Now about **the tasks**, as a teacher, *the* way you are *questioning* your learners is the task in its own right. I don't see that when you are teaching you need to wait for that moment you give them classwork. 142. You can give them **tasks** throughout teaching by *questioning* them and *interrogating* them further.

Extract 30 showed that the questioning routine was used by the KO as a means to bring to focus teacher pedagogical actions that were important to use to invite learner participation which was related to what learners were saying during the teaching session (learners' discursive responses during classroom interaction) and not necessarily what they were doing (classwork). During the RD1, the questioning routine was used not only to invite learner participation, but also as a means to construct discursive activities for the learners' tasks. This finding seems to suggest that the questioning routines during RD1 played a role of facilitating the construction of MTF narratives related to both tasks and learner participation as seen with LP1.

Explanations

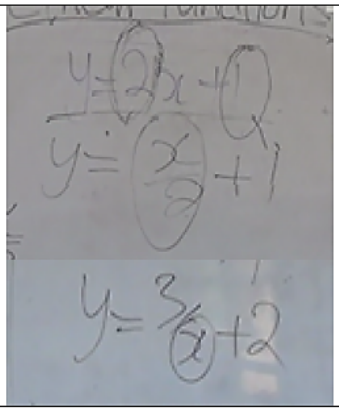
With respect to explanations, the data revealed that the explanations were from both teachers' and learners' perspectives, as evident in Table 28_A. As seen with previous routines, the explaining routines recorded various mathematical routines.

Learner explanations

The fourth column of Table 28_A in the MTF routine row shows that the learner explanations were characterised by interpreting routine only. I exemplify using the following extract:

Extract 31

An example of MTF routines related to learner explanations

Episode	Speaker	What is said	What is done
EP2	T	30. Yes, remember there was that $y = x/2 + 1$ and I asked them for the gradient and they struggled with that and I tried to make an example just *to see* that if $x/2$ is half or not, they still struggled 31. and I had to do some calculations on the side for them to see that $x/2$ is the same as $\frac{1}{2}x$	
	J	32. They said it's not the same	
	All	33. Ja	
	M	34. They said 2 is the **gradient**.	
	KO	35. So this is one part you can just write a **fraction** there (board). It's something we need to pay attention to, especially when dealing with **hyperbola** because you saw it in the end when I was *questioning* them, they came up with a **mathematical definition or property**, where they said when *you see* a **fraction, the x must be in the numerator**. That is more mathematical **explanation** (*referring to the equation* $y = \frac{x}{2} + 1$ *and* $y = \frac{3}{x} + 2$).	

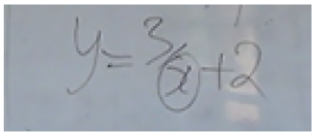
The learner explanations were talked about in relation to what was identified as learner difficulty. For example, teacher T in line 30 noted that the learners struggled with identifying the gradient in the equation $y = \frac{x}{2} + 1$, and in lines 32 and 34 they gave their responses which signalled misconceptions. What is interesting, the identification of learner difficulty motivated the teachers to suggest a questioning strategy to deal with this difficulty (line 35) and help learners to come up with a mathematical explanation. This finding suggests that the questioning strategy could be key in addressing the CA of OoL and also seemed to have enabled learners to participate in a more mathematical explanatory talk. The participation in learner explaining discursive activity appears to have the possibility of affording teachers with opportunities to identify the critical aspects from learners' substantiations and to further strategise pedagogical actions to deal with learner difficulties.

Teacher explanations

The fourth column of Table 28_A in the MTF routine row shows that this MTF routine was characterised by interpreting and calculating routines. However, the interpreting routines were more predominant than the calculating routines. The analysis of narratives containing teacher explaining routines suggested that these routines were aimed at presenting mathematical narratives. Still there were some examples of teacher explanations serving a pedagogical purpose. Consider the following extract:

Extract 32

An example of MTF routines related to teacher explanations

Episode	Speaker	What is said	What is done
EP7	S	98. Another thing that I have noticed that could have led them to say 3, not all learners understand what we mean by **coefficient** because from where I was sitting there was a point where you were explaining to them that the **gradient** was the **coefficient of x** and only a few responded but when you said, "Look at the number in front of x," they said okay (*referring to the equation* $y = \frac{3}{x} + 2$).	
		99. And the question becomes if you have a question like y equals 1 plus $2x$ ($y = 1 + 2x$) which one is next to x and y equals 1 plus $x2$ ($y = 1 + x2$) between these equations? 100. I think we should not lower the demand. I think you were trying to make it simpler for them but it becomes dangerous when you are using words like '**which one is closer to x' if you speak of *operating* signs, I think the one closer**, that the one that **is being multiplied by x is the coefficient** it becomes better. 101. So that's why here they saw an operating sign then they saw x, they saw **a gradient**.	

Starting with mathematical narratives, an analysis of S's utterances in lines 98-99 showed that he was communicating a mathematical idea related to a coefficient of x in relation to the gradient of a linear function. He did this by referring to three representations in a symbolic form: $y = \frac{3}{x} + 2$, $y = 1 + 2x$, $y = 1 + x2$ which were manipulated through the use of interpreting routines (reading the value next to x).

With regard to pedagogical narratives, the analysis of S's utterance in line 100 and KO's in line 52 (extract 24) evidenced reflections about teachers' word use which had elements of informal phrases such as ("which one is closer to x", and "you want to put y on one side, you want to leave it on its own",

"you are taking six to the other side"). These phrases were used for the purpose of naming the mathematical teaching procedures in order to explain the notion of the coefficient of x.

The most critical highlight of teacher explaining narratives in relation to word use is the link to learner misconception—a feature of critical aspect of OoL. For example in lines 100-101, teacher S elaborated on the use of an informal phrase such as –"which one is closer to x" and highlights misconceptions that this word use might cause. This is a pointer to critical aspects. What is interesting in S's utterances, the emerging critical aspects were conjectured by the teachers. In other words, they were assumed by teachers as problematic.

Participation in teaching explaining discursive activities appear to have afforded teachers with opportunities to construct and substantiate mathematical narratives which illuminate what is critical to learn by their learners (learner difficulties).

CA of OoL

The data revealed that the CA of OoL routines were categorised into two: learner CA and teacher CA as recorded in Table 28_A. In the analysis of the other MTF routines above, I have shown how the CAs of OoL were talked about in relation to other MTF routines, and the results revealed that the learner CAs emerged from teachers' reflective narratives about learners' mathematising talk. However, the teacher CAs emerged from teachers' reflective narratives about their mathematical teaching talk and conjecturing talk about what they believed were learner difficulties about a specific mathematical concept. The analysis has further indicated that after identifying these CAs, teachers further reflected on how to address these CAs in teaching such as inclusion of the questioning strategy and use of mathematical talk.

Interestingly, during LP1, the teachers conjectured that learners had difficulty specifically with interpreting and classifying a linear equation. Word use was also deemed as critical to be learnt. However, the analysis of RD1 data has shown that only two of these CAs were discussed and were categorised into two: learner CA (interpreting routines) and teacher CA (word use). It was not possible to draw any conclusions about the relationship between conjectured CAs from LP1 and reflected CAs from RD1 as done by others who studied the aspect of OoL in the learning study context (e.g., Runesson, 2013; Olteanu, 2016). These researchers were able to do this by conducting pre- and post-tests, and comparing the relationship between the intended CA (during lesson planning) and enacted CA (during

teaching). This was not the aim of this study. Instead, I considered what the identified CAs were during LP1, and reflected upon CAs during RD1, and how they were talked about during these two sessions.

In conclusion, it appears as if the L/S group reflected critically on an individual teacher's teaching considering the implications of his actions on learners' mathematising talk. This implied awareness of learners' needs and the importance of being sensitive to the learners' understanding of the OoL, and consequently adapting the teaching by paying attention to aspects of OoL that have been identified as critical. According to Adler and Ronda (2015), this awareness is the critical work of teaching. This finding seems to suggest that participating in a CA of OoL discursive activity could give teachers opportunities to focus on the object of learning—an important aspect of teacher learning (Adler, 2021).

Learner participation

Learner participation as an MTF routine manifested itself through tasks and learner explanations, as seen from analysis of LP1. In other words, I was able to identify routines related to these components through analysing the routines related to tasks and learner explanations. Furthermore, the analysis evidenced that learner participation was strongly linked to routines related to contextual factors connected to learners and questioning strategy.

Extract 33

An example of MTF routines related to learner participation

Episode	Speaker	What is said
EP2	T	26. From learners' participation, at the beginning they were quiet; I felt that they did not know anything. I was bringing something new to them.
	KO	27. What made them to engage or participate?
	T	28. They were getting the idea. I was questioning repeatedly. I asked them to repeat with regards to what the standard formula says and what does the sign of the **gradient** means. 29. Every time I say, "If the **gradient is negative** what does that mean?" and I kept on pushing. And they started understanding.

In line 26, teacher T reflected about learner participation in terms of their social interaction in the classroom context where he said: "they were quiet". T's awareness of this contextual factor motivated him to use a questioning routine in order to invite learners to engage in a mathematical activity (lines 28-29). This observation presented the interplay between teacher explaining and contextual factors. Teachers' focus on contextual factors is expected since the teachers had enough data of the classroom

discourse, so they were able to reflect about all possible contextual factors. The awareness of contextual factors motivated T to strategise ways to encourage participation. This finding seems to highlight the interplay between the context and the learning activities (Lave & Wenger, 1991).

7.2.2.3 Summary

From the analysis of the MTF discourse, this discourse can be understood as a form of discourse distinguishable by its word use, visual mediators, routines and narratives. The featured names included examples, tasks, OoL, explanations and learner participation. With regard to visual mediators, these were similar to those found in mathematical discourse—which were created and operated upon. For example, the analysis has shown that the examples were shown in different representations: symbolic, graphs. The narratives in this particular discourse included utterances which were characterised by mathematical, pedagogical and MTF naming routines. As seen with LP1, it seems the goal of the MTFying discursive activity was to construct MTF narratives.

Participation in the MTF discourse echoes similar characteristics as seen in LP1 in terms of the following:

(i) Nature of MTF objects (words and word use): The MTF objects that teachers were explicitly referring to included examples, tasks, explanations, OoL and learner participation.

(ii) Routine-use: The analysis has confirmed that the routine-use was both action- and object-driven.

(iii) The analysis has further revealed that the MTF narratives were produced from different sources such as MTF routines, mathematical routines, and pedagogical routines. These different sources had a similar function as seen during LP1. Regarding MTF routines, these included exemplifying, generating tasks, teacher explaining, learner explaining, learner CA, teacher CA and learner participation. What is interesting is that the CA of OoL was divided into two: learner CA and teacher CA, something that was not seen during LP1. Additionally, there was also talk about routines directly related to learner participation. This could be because during RD1 teachers were working with real learner data collected during the first teaching session, something that they did not have access to during the LP1, where they worked with conjectures of what they believed were learner difficulties, and what they imagined learner participation to be.

The MTF routines during RD1 had similar attributes as seen in LP1 where each MTF routine highlighted the critical aspects to be learnt. These were related to interpreting routines and word

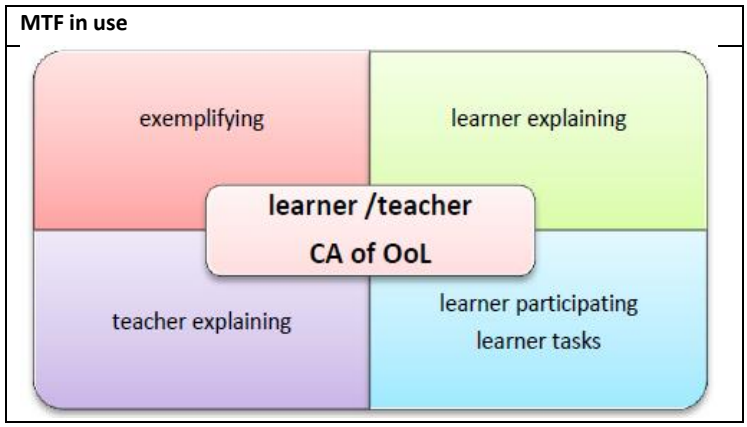
use. This finding seems to provide evidence of connection between CA of OoL and other MTF objects. According to Adler (2021), this process is linked to opportunities of teacher learning in relation to the MTF tool.

Concerning the pedagogical routines, the analysis of RD1 has confirmed the findings from LP1, which have evidenced that the pedagogical routines played a role in the construction of MTF narratives and facilitated the further expansion of the MTF tool. For example, questioning routines facilitated the construction of MTF narratives related to both learner participation and learner explaining. On the other hand, the analysis has also shown how the process of identifying the CA resulted in the construction of MTF narratives related to the CA of OoL—learner CA and teacher CA. According to the analysis of the MTF discourse, the process of identifying the CA further facilitated the intertwinement between various elements of the MTF. With regard to mathematical routines, as seen with LP1 analysis, the MTF routines during this session were also associated with various mathematical routines. As seen with LP1, this finding suggests possible explorative MTF discourse.

(iii) Concerning the goals of discursive activity, similar goals found during LP1 were echoed during this session. For example, the analysis has suggested that the meta-level goal was to construct MTF narratives. On the other hand, the object-level goal was linked to the expansion of MTF objects such as explanations and CA of OoL, where explanations were divided into teacher and learner explanations. At the same time, the CA of OoL was expanded into learner-teacher CA of OoL. Figure, 9, I provide the transformation of the MTF tool into the MTF discursive tool (MTF-use):

Figure 9

Transformation of MTF tool to MTF discursive tool



(iv) Regarding the roles, the results of participation in the MTF discourse have shown that all participants were contributors to the MTFying activity, with the KO still playing a leading role although slightly less compared to LP1. The participation of all participants was expected. This could be due to the design of the RD session, whereby one of the ground rules was to allow all participants to contribute to the reflective discursive activity. This finding seems to suggest that all participants were involved in the joint construction of MTF narratives.

In conclusion, the MTF discourse showed signs of explorative participation in terms of all defining features such as objects of discourse, sources of narratives, goals of discursive activity and roles of participants. More specifically, the MTF discourse intergrated other routines from both mathematical and pedagogical discourses. In addition to that, there seemed to be an established connection between CA of OoL and the other MTF objects—signalling an aspect of coherence.

7.2.3 Organising theme 3: Use of MTF was linked with participation in pedagogical discourse

The identified pedagogical routine during RD1 was similar to those identified in LP1 but extended to reframing routines.

7.2.3.1 Emerging theme: Mathematical teaching discourse

Questioning Routine

The data suggested that the main feature of the questioning routine was related to reflections about learner participation. The narratives evidenced explicit reference to the word "questioning" (or interrogate) which were linked to learner participation.

Extract 34

An example of questioning routines related to learner participation

Episode	Speaker	What is said
EP2	T	28. They were getting the idea. I was *questioning* repeatedly. I asked then to repeat with regards to what the standard formula says and what does the **sign** of the **gradient** mean. 29. Every time I say, "If the **gradient** is **negative,** what does that mean?" and *I kept on pushing*. And they started understanding.

Extract 34 showed that T reflected about the type of question he used in order to invite learner participation, such as: "what the standard formula" and "what does the sign of the gradient means". These questioning narratives seemed to serve as prompts for learners to engage in interpreting the gradient of the symbolic standard formula of the linear function. This suggests that the questioning routine was used to invite learners to participate in the mathematical activity—the discursive learner tasks. This finding further confirms that the questioning routines facilitated the construction of MTF narratives related to learner participation and tasks. Such occurrence can be linked to a possible explorative MTF discourse.

7.2.3.2 Emerging theme: L/S pedagogical discourse

Scaffolding routine

Similar to the LP1 discourse, scaffolding routines are an important part of teachers' reflective discursive repertoire as shown in Table 28_A. An interesting feature of these was the relationship between the KO's scaffolding routines with (1) mathematical teaching routines; (2) scaffolded questioning; and (3) ground rules. Starting with the scaffolding mathematical teaching activity, the results showed that the KO used this type of routine to convey mathematical teaching strategies, as illustrated in the extracts below:

Extract 35

An example of scaffolding routines—mathematical teaching

Episode	Speaker	What is said
EP9	KO	145. I think for me in the conclusion, it's not about the **task**. You can just use it for *generalisation*, and ask them, "What have we learnt today? We started off by learning that you have y equals mx plus c ($y = mx + c$). When m is greater than zero ($\boldsymbol{m > 0}$) your **gradient or slope** is *increasing*. And when m is less than zero ($\boldsymbol{m < 0}$) it means the **slope** is *decreasing*." 146. Then you also say, "In y equals mx plus c ($y = mx + c$), if your **c is greater than zero or positive,** it means you are going to have a **vertical shift upwards**. And when c is less than zero ($\boldsymbol{c < 0}$), your graph is going to **shift down**." And that is your conclusion. So there won't be any room or time to give them classwork. You can give them homework.

KO's utterances in lines 145-146 showed that her talk was focused on the mathematical teaching routine of generalisation in relation to the y-intercept and the gradient of the linear function. She commented about the generalised mathematical statement mostly found in the school mathematical textbooks about the behaviour of the y-intercept and the gradient when viewed from symbolic and graphical representations. The generalising routines were linked explanations. KO's comment suggested that the

use of scaffolding routines in combination with mathematical teaching routines facilitated the construction of MTF narratives.

The second function of KO scaffolding routines was scaffolded questioning, and this was present when the KO was probing the teachers (teacher T in particular) to reflect about their teaching experience:

Extract 36

An example of scaffolded questioning

Episode	Speaker	What is said
EP2	KO	23. In general did you enjoy the lesson?
	T	24. Yes, I did enjoy the lesson
	KO	25. What is the specific thing that you have learnt in that specific lesson?
	T	26. From learners' participation, at the beginning they were quiet; I felt that they did not know anything. I was bringing something new to them.
	KO	27. What made them to engage or participate?
	T	28. They were getting the idea. I was *questioning* repeatedly. I asked them to repeat with regards to what the standard formula says and what does the sign of the gradient means. 29. Every time I say, "If the gradient is negative, what does that mean?" and I kept on pushing. And they started understanding.

This extract contained evidence of the interaction between the KO and teacher T, who was conducting the lesson. This interaction was characterised by scaffolded questioning from the KO and teacher T's reactions to these questions. These reactions were related to learner participation and the questioning strategy that was used by the teacher in order to invite the learners to participate in the mathematical activity. Teacher T's discursive activity implies construction of MTF narratives linked to learner participation. It appears as if the KO scaffolded questioning served to encourage and invite increased teacher participation in the discursive activity of construction of MTF narratives; contrary to LP1, where the KO dominated, with no clear invitation for teacher participation. This could indicate that the KO, during this session, saw teachers as capable participants who could have a contribution to the discourse. Another function of KO's scaffolding routines was related to ground rules.

Extract 37

An example of scaffolding routines with ground rules

Episode	Speaker		What is said
EP1	KO	1.	What we are doing is, yesterday we were teaching our first lesson, now we are doing a reflective discussion, post-lesson discussion.
		2.	Those that observed the lesson will reflect on the lesson.
		3.	We will first allow the presenter to give us his feedback and comment on the lesson; after that we will all reflect.
		4.	After reflecting on what we saw on the lesson based on the MTF or what was planned, that is on the original lesson plan, were there any diversions and why and what are the things we need to be aware of for our next teaching session,
		5.	We are going to invite the presenter to give us feedback.

In this extract the KO commented about the general rules of engagement in the reflective discussions, including the roles to be played by the participants and the ground rules specific to the MTF. For example, in lines 2-3, the KO referred to the specific discursive activity to be done by the observers and the teacher who was conducting the lesson. The KO also encouraged the teachers to pursue the MTF tool in their reflective discussion (line 4). From the analysis of this line, it appears as if the KO's scaffolding routines served to communicate the ground rules related to L/S CoP and also to foreground the MTF tool as a guiding protocol in all L/S activities.

The analysis of KO scaffolding actions revealed that all participants were collaborating in the process of constructing MTF narratives with the KO playing a facilitative role of encouraging more teacher participation in the MTFying discursive activity, with a goal of helping them to extend their thinking with regards to this discursive activity, and to use explorative routines. It is also important to add that the KO's scaffolding was not focused on foregrounding the points of focus through MTF naming. Instead, this activity was a collaborative effort from all the participants, contrary to LP1 where the points of focus were provided by the KO only. Hence, I am referring to KO's scaffolding goal during RD1 as "facilitating" rather than "prioritising" as seen with LP1. This finding implies KO's moderate scaffolding compared to LP1 where she was seen to have been giving more directives on how to use different routines in order to construct MTF narratives. From this, it can be concluded that the KO's goal was also ritualistic. Interestingly, the analysis has revealed that this ritualistic goal was linked to explorative MTF discourse as seen with LP1. This finding, further suggests that the teachers were afforded opportunities to learn MTFying discursive activity from each other with the KO. Furthermore, the KO also had learning opportunites to scaffold explorative MTF discourse

Identifying CA

The identifying CA routines were amongst the most common pedagogical routines, as shown in Table 28_A. The data showed that this pedagogical routine was linked to learner difficulties with a specific mathematical procedure and teachers' word use that contributed to learner difficulties or misconceptions.

The following extract is an example of a narrative evidencing learner difficulties:

Extract 38

An example of Identifying OoL linked to learner difficulties

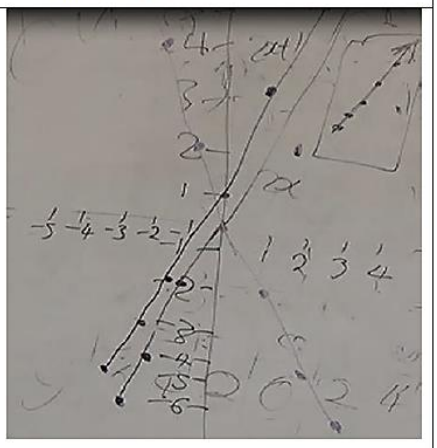
Episode	Speaker	What is done
EP2	T	30. Yes, remember there was that y equals x over 2 plus +1 ($y = x/2 + 1$) and I asked them *to identify* the **gradient** and they struggled with that and I tried to make an **example** just to see that if x over 2 ($x/2$) is half or not, they still struggled 31. and I had to do some *calculations* on the side for them to see that x over 2 ($x/2$) is the same as half x (½ x)

In the extract above, the identified learner difficulty was related to interpreting the gradient in the symbolic equation (line 30). Teacher T also elaborated on the action he took in order to address this difficulty where he gave an illustration through an example of $\frac{x}{2}$ and $½x$, and further discussed the mathematical routine he needed to use in order to address learner difficulty (line 31). Teacher T's reflections evidenced awareness of the learner difficulties during teaching, and also his response to address these identified difficulties.

A related type of routine aimed at illuminating teacher word use that could lead to learner misconceptions is exemplified in the following extract:

Extract 39

An example of Identifying OoL linked to teachers' word use

Episodes	Speaker	What is said	What is done
EP4	KO	61. There was also confusion on the word use of **increasing and shifting**.	
	T	62. I did not even use the word **shift.**	
	KO	63. You did, but it was vague. We need to be clear; there was a lot that could cause misconceptions.	
	J	64. When this **example** came, it became clear that there was a lot of confusion with the word **increasing**. There were like you are **increasing** with one and the graph is *shifting*. In that **example** where you had a graph of y equals $2x$ ($y = 2x$), and you *drew* another for y equals $2x$ plus 1 ($y = 2x + 1$) (*referring to the picture on the right hand side*). 65. You also used a **word increasing** when referring to the **gradient of the function** y equals $2x$ ($y = 2x$).	

In most of the utterances which showed presence of word use, there was a consistent pattern of reference to teachers' word use (e.g., increasing and shifting) and how the word use could possibly lead to misconceptions. For example, in the extract above, teacher J commented about teacher T's careless use of the word "increasing" when referring to the gradient and the increasing value of the y-intercept from zero ($y = 2x$) to 1 ($y = 2x + 1$) which could create misconceptions. An important point to make is that these kinds of misconceptions were due to teachers' mathematising talk, and were categorised as teacher CA.

To sum up, the analysis of the identifying CA routines suggested that this routine facilitated the construction of MTF narratives linked to the CA of OoL and learner participation. A very important result that emerged here was the categorisation of CA of OoL into learner CA and teacher CA.

This finding has important implications for the expansion of the MTF objects into both the learners' and teachers' use. Furthermore, this discursive activity has the potential to afford teachers opportunities to develop an awareness of the critical aspects of the object of learning and how to handle these critical aspects for the benefit of their learners.

It is also important to note that it seems the identification of the CA served different purposes in both LP1 and RD1. For example, during LP1, the analysis has shown that after identifying CA, the teachers went further to construct examples reflecting the identified CA. However, during RD1, after identifying the CA, the teachers engaged in talk about different teaching strategies to consider in addressing learner difficulties associated with the identified CA.

Reframing

The reframing routine was one of the most important pedagogical routines recorded in Table 28_A. As noted, reframing routines relate to actions whereby participants in the L/S reflect back on the previous teaching session. The data shows that the reframing routine could be categorised into individual teacher (T) who was teaching in the previous teaching session and the observers of the lesson, this included other teachers and the KO.

Individual teacher

T, who presented the previous lesson, reflected mainly about learner participation. His talk about learner participation was referenced to teaching strategies he used to invite or encourage learners to participate in the mathematical activity. These included questioning strategies and learner activities.

Extract 40

An example of reframing routines from individual teacher

Episode	Speaker	What is said
EP2	T	26. From **learners' participation**, at the beginning they were quiet; I felt that they did not know anything. I was bringing something new to them.
	KO	27. What made them **to engage or participate**?
	T	28. They were getting the idea. I was *questioning* repeatedly. I asked them to repeat with regards to what the **standard formula** says and what does the sign of the **gradient** means. 29. Every time I say, "If the **gradient is negative,** what does that mean?" and I kept on pushing. And they started understanding.
		30. Yes, remember there was that y equals x over 2 plus 1 ($y = x/2 + 1$) and I asked them *to identify* the **gradient** and they struggled with that and I tried to make an **example** just to see that if x over 2 ($x/2$) is half or not, they still struggled.

For example, in lines 28-29, T reflected about the questions he used which were directed as prompts to the learners to interpret the gradient of the standard formula of the linear function. Similarly, in line 30 he commented about the learner activity, asking them to identify the gradient in the linear function.

From the analysis of T's reframing narratives, one can possibly conclude that his narratives were focused on the construction of MTF narratives—specifically related to learner participation. Teacher T's reflective discourse was characterised by a talk about his mathematical actions, pedagogical actions and learners' mathematising actions. It can be argued that the RD1 gave teachers an opportunity to be

reflexive about important elements of their work of teaching. This finding has an important contribution in developing the argument related to modes of interaction as will be discussed in section 7.2.4.

Others, including the KO

The reflections from the group of observers were mainly focused on teacher T's teaching strategies, and these were characterised by exemplifying and questioning strategies. Starting with exemplifying strategies, consider, the KO's reflective narratives in the extract below, where she commented about the teaching decisions teacher T made during teaching which were different from what was collectively planned.

Extract 41

An example of reframing routines from observers' exemplifying

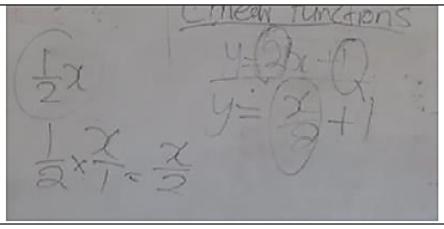
Episode	Speaker	What is said
EP4	KO	57. In our planning we agreed that during implementation we will start with the equation then move to the table and *interpret* the **behaviour of the range**. And move to *drawing* the graph. 58. I think our planning was limiting. What T did in his first **example**, he started with the equation and *drew* the table that is, an **example** of y equals $2x$ ($y = 2x$), then he had a table and compared the table. I think that worked best because you worked with y equals $2x$ ($y = 2x$) as if you want them to know how to *draw*, and the **OoL** was that you wanted them *to see* the **effect of the gradient**. 59. So if ever you decide to use one representation, you can stick to that representation and even if you did not go to the graph you would have known that you have implemented your original **OoL**. 60. So what he did, he used one representation that is the table to do the *comparison* and in the end used the graph to *generalise*. It worked for me because it was emphasising on the **OoL.**

Her reflections included teacher T's mathematical teaching actions which were characterised by making references to mathematical routines (interpreting, drawing, translating) and representations (tables, graphs, symbolic equations). In line 58, she emphasised that such a decision was due to teacher T's reactions to what he wanted his learners to know ("you want them to know how to draw, and the OoL was that you wanted them to see the effect of the gradient."). This could be interpreted as the relationship between mathematical teaching and learner participation, which implies that reframing routines contributed to the construction of narratives related to learner participation.

With regard to questioning routines, the analysis of data showed that the observers reflected on the questioning routines the individual teacher used during teaching. To illustrate this point, I use the following extract:

Extract 42

An example of reframing routines from observers' questioning

Episode	Speaker	What is said	What is done
EP5	M	70. For me the interaction with the learners was okay. Asking learners questions, testing their prior knowledge was good but there were insufficient **counterexamples.**	
	T	71. If there should be one, where would it be?	
	M	72. There was a point where they were *explaining* half (½ x) and x over 2 ($x/2$) and they gave their opinion which you did not *interrogate* further, "Why you say so?" That's when I focus on **counterexample.**	
	T	73. I don't know, cause	
	M	74. Cause *interrogating* them can work	
	T	75. Let's not get deep into it.	
	S	76. How long is the lesson?	
	All	77. 45 minutes	

Lines 70-77 show the interaction between the teachers who were observing the lesson (teachers M, S and T). This interaction highlighted that the observers were commenting about the limitations of teacher T's teaching in terms of learner participation and examples. For example, in line 72, teacher M commented about T's explanatory talk in relation to the interpretation of the coefficient of x in the example $\frac{x}{2}$. He further noted that teacher T had failed to engage the learners further by inviting them to participate in a mathematical activity. According to him, this could have been achieved by using a questioning prompt such as "Why you say so?" This analysis suggests the link between the questioning routines, learner critical aspects and learner participation.

What can be learned from the analysis of teacher T's and the observers' reframing routines is that their reflections about the individual teacher's actions were focused on mathematical and pedagogical actions. In terms of Hunter and Anthony (2014), this discursive activity can be explained as 'on-task' subjectifying, which has a positive contribution to teacher learning by affording teachers opportunities to problematise their practice and strengthen their reflective practice skills.

7.2.3.3 Emerging theme: General teaching discourse

Structuring routines

Structuring routines, these were not so frequent, with the most common one being referenced to factors influencing deviation from the planned lesson. These factors included contextual factors related to learners, as shown in the extract below:

Extract 43

An example of structuring routines

Episode	Speaker	What was done
EP8	KO	107. Now the practice is not the same as the research when you go and implement. We went with an introduction assuming that they know these things then planned to move to implementation; however, T spent most of the lesson on introduction. 108. One can sit on the introduction for the whole lesson, it depends on the context. I believe as a teacher you stayed on the introduction for the whole lesson because you noticed that there was so much that learners did not know.
	T	109. And there is nothing wrong with that?
	KO	110. Nothing wrong; you have the authority, why should you move forward when you notice that there is a problem with the learners' understanding? 111. This means you will be attending to contextual factors of your learners, they are not in the same level.

The KO attributed the teacher's deviation from the structured lesson plan in terms of prescribed stages (i.e., introduction, implementation, and conclusion) to teachers' decisions he needed to make during teaching. According to KO's reflections, teacher T stayed on the introduction for the rest of the lesson because he noticed that learners were struggling with a specific aspect of the mathematical concept under discussion. This finding was expected because during the teaching session the individual teacher was confronted with real learner challenges (critical aspects), and needed to respond to them; this could result in deviation from what was planned. According to Posthuma (2012), these deviations could also include the change of lesson content altogether.

7.2.3.4 Summary

The emerging pedagogical discourse during RD1 was characterised by talk about both the mathematical and MTF objects. The characterising routines of this discourse included teaching routines and MTF routines. The visual mediators were those from the mathematical discourse. Given that pedagogical discourse concerns talk about teaching of mathematical content, where mathematics is the subject of

mathematical discourse, this is hardly surprising. As for pedagogical narratives, they were mostly described as results of pedagogical routines such as questioning, identifying OoL, scaffolding and structuring routines.

The results of the RD1 pedagogical discourse showed similar results as those reported in LP1 to a large extent in terms of objects, sources of narratives and goals of the discursive activities, but differed regarding roles played by the participants and the functions of some of the routines. For example, the objects of this discourse were those from MTF and mathematical discourses. The sources of narratives were established through various teaching routines (e.g., questioning, scaffolding, identifying CA) and a pedagogical routine unique to L/S—the reframing routine. The reframing routine was not present during LP1. Interestingly, some of the pedagogical routines found during the RD1 analysis had a different function from the LP1 analysis. For example, the identifying CA illuminated the CA of OoL from teachers and the learners, but during LP1 the function of this pedagogical routine was on identification of the CA from the teachers' perspective only. The analysis of the reframing routine suggested that the function of this particular routine was to foreground teachers' talk about others while engaging in MTFying and pedagogical discursive activities. This process was referred to as "on task" subjectifying.

The new function established during the scaffolding routine analysis was that of scaffolded questioning. This particular routine illuminated the interaction between the KO and the teachers; something not found during LP1. This could be due to the KO's intention to encourage participation from the other members of the group, as specified by the ground rules of participating in the RD session. The findings of the analysis of KO scaffolding suggested her ritualistic goal to facilitate explorative MTF discourse. This finding implies moderate scaffolding to afford more teacher participation in the explorative MTF discourse. This finding also has a bearing when discussing the role of the KO in supporting teacher learning and also the situative nature of L/S teacher learning.

Regarding the roles played by the participants, analysis of the pedagogical discourse showed similar roles played during LP1, where all participants contributed to talk about pedagogical discursive activity. However, this talk was focused on talks about participants while engaging in the teaching activities related to MTF in particular, that is, on task subjectifying (Hunter & Anthony, 2014).

In conclusion, the findings suggest that the main function of participation in pedagogical discourse was to support the construction of MTF narratives—an important sign of explorative discourse. This was

evidenced in all features describing the discourse: objects, sources of narratives, goals of the discursive activities and the roles played by the participants.

7.2.4 Organising theme 4: The modes of talk supported the MTF-use

In this section, I characterise teachers' interaction with respect to the modes of talk used. Figure 10 summarises the results of teachers' talk in relation to the different categories in focus during RD1 session:

Figure 10

The relationship between modes of talk and discursive activities

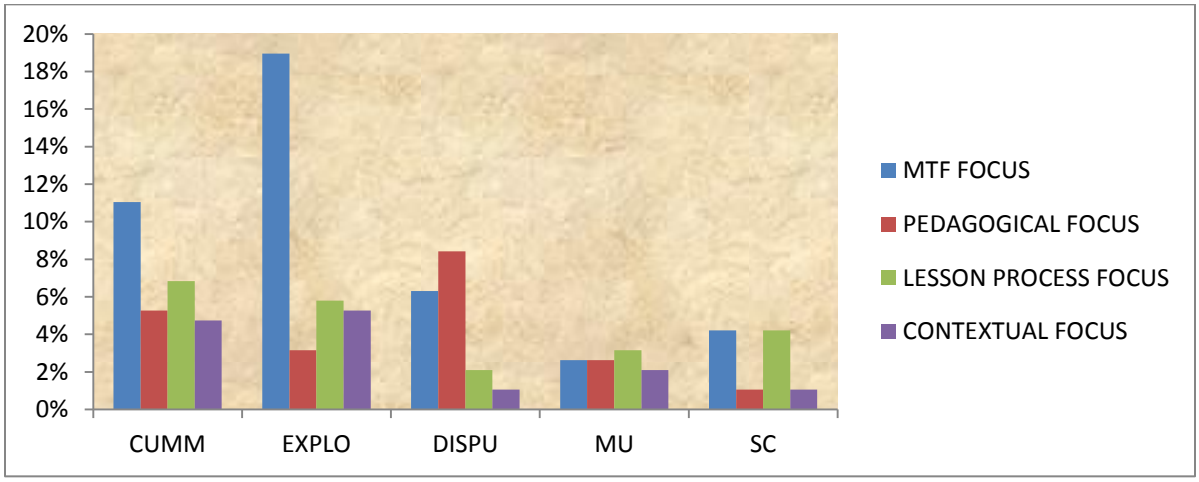


Figure 10 revealed that teachers' talk was dominated by the MTF focus, similar to LP1. The leading type of talk was exploratory, followed by cumulative talk. The RD1 revealed the presence of disputational talk, contrary to LP1. The MU and SC were also considered. As seen with the analysis of the previous session (LP1), the talk during RD1 can be divided into single level (disputational), and bi-level (structuring conversation-exploratory, cumulative-exploratory, disputational-exploratory and managing understanding- exploratory).

7.2.4.1 Emerging theme: Uni-level talk

Disputational talk

Figure 10 showed the pedagogical focus category had produced more disputational talk. An in-depth analysis of the pedagogical focus showed that the most prominent pedagogical focus subcategory was

reframing strategy. For example, analysis of reframing routines showed that the teachers' talk was more focused on the individual teacher's (i.e., teacher T) decisions that were made during a teaching session. The talk during RD1 was characterised by disputing—something that was not seen with LP1, where the teachers who were observers of a lesson reflected in a more evaluative manner, focusing on the 'teacher', that is, on the decisions the teacher made during teaching. Adding to what characterises disputational talk, the analysis showed evidence of interactions where the general orientation of T's response was defensive about his teaching actions during the previous teaching session. Moreover, the general relationship amongst the teachers appeared to be that of undermining and evaluative manner in a negative way. Extract 44 presents an illustrative example of disputational talk.

Extract 44

An example of disputational talk

Episode	Speaker	What is said
EP5	M	70. For me the interaction with the learners was okay. Asking learners questions, testing their prior knowledge was good, but there were insufficient counterexamples.
	T	71. If there should be one, where would it be?
	M	72. There was a point where they were explaining half x (½ x) and x over 2 ($x/2$) and they gave their opinion which you did not interrogate further, "Why you say so?" That's when I focus on counterexample
	T	73. I don't know, cause
	M	74. Cause interrogating them can work
	T	75. Let's not get deep into it.
	S	76. How long is the lesson?
	All	77. 45 minutes
	M	78. You were supposed to interrogate them; it's a confusing concept [visual trigger]. So if you interrogate them you would know how far they are, how much they know, why are you saying these are not the same, the other one is half x (½ x), the other is x over 2 ($x/2$)
	J	79. Even for me just to support J, I think there was a learning moment "is $x/2$ not the same as ½ x?" It seemed you looked for the correct answer. You went on your own on the board. I support him that after you have explained that x over 2 ($x/2$) is the same as half x (½ x) you could have asked that, "Why are they the same?"
	M	80. …is the same as 2 over x ($2/x$) 81. That's a counterexample, if I am not mistaken.

The extract above evidences the disputational interaction between teacher T and other teachers. This interaction showed the critiquing of teacher T's teaching strategy, highlighting the omission of questioning strategy, and followed by defensive talk. Such interaction was evidenced in line 72, where M was questioning T's teaching strategy of not questioning the learners. This was followed by T's response in line 75 where he seemed to be defensive: "Let's not get deep into it". Such an evaluative talk was evidenced further in lines 78 and 79. M and J, both disagreed with T's teaching strategy. Of

note, is that there was no attempt to develop a constructive argument by proposing how the questioning strategy could have supported teacher T's teaching in terms of construction of MTF narratives except for describing the omission and mentioning that this strategy was useful. This interaction seemed to suggest that in the disputational interaction there was a lack of evidence of joint construction of MTF narratives.

6.2.4.2 Emerging theme: Bi-level talks (Hybrid of exploratory talk and other talks)

Structuring Conversation-Exploratory talk

The in-depth analysis amongst different episodes revealed that the interaction indicated the KO's scaffolding actions and a sense of joint construction of MTF narratives.

Extract 45

An example of structuring conversation-exploratory talk

Episode	Speaker	What is said
EP4	KO	61. There was also confusion on the word use of "increasing" and "shifting".
	T	62. I did not even use the word "shift"
	KO	63. You did, but it was vague, we need to be clear, there was a lot that could cause misconceptions.
	J	64. When this example came, it became clear that there was a lot of confusion with the word "increasing". There were like you are increasing with one and the graph is shifting. In that example where you had a graph of $y = 2x$, and you drew another for $y = 2x + 1$. 65. You also used a word "increasing" when referring to the gradient of the function $y = 2x$.
	KO	66. You know what I don't know, but you can correct me. The word "increasing" in the textbooks is associated with the gradient (*showing an increasing slope with hands*). 67. When we want to talk about the shift let's not talk about what happens with the y values, the increasing, and only say, if we want to talk about the shift let's see what happens with the prototype, let's talk about what happens to y. When we compare the table, let's talk about the difference when we want to talk about the shift. 68. The difference between the y values is maybe 1 or minus 1. When there is a difference between y values of the prototype equation and the new equation then we can talk about the shift. 69. "Increasing", let's associate it with the gradient. When there is a difference between y values of two different functions then we talk about the shift. A difference of a certain constant.
	S	70. Can we maybe introduce a notion of vertical shift?

The extract above reflected the teachers' focus on the CA of the OoL—word use "shifting". This can be seen in line 61, where the KO focused the discussion about the problem with word use. This was followed by comments from KO and J (lines 63-65) expanding on the idea related to problems with word use. In lines 66-69, the KO started by requested others' opinions about the mathematical explanatory talk she was about to reflect on where she made a concrete suggestion on how to deal with the use of

word “shifting” in relation on the legitimate mathematical explanatory and requested others’ opinions. KO’s comment further evidences an atmosphere of collectiveness as referenced by keywords “we”. This extract appears to be characterised by exploratory interaction in which teachers engaged in a joint construction of MTF narratives related to the CA of OoL. All interlocutors gave their opinions which were taken for further consideration. This interaction further highlights the KO’s directive orientation, where she seemed to be scaffolding a mathematical talk related to the identified CA of OoL—word use “shifting”. This finding suggests that when the KO scaffolds participation in the mathematizing and MTFying activity, the exploratory talk could be realised and there is evidence of construction of MTF narratives—an indication of explorative participation. It is important to note that the KO’s structuring conversation was characterised by invitations for others’ opinions (in line 66) and not with warning or reminding prompts as seen with LP1,which implied KO’s scaffolding goals of providing points of focus in relation to MTF tool and its use. This has implication on the nature of her scaffolding goals.

Cumulative-exploratory talk

The data also revealed a type of interaction whereby the cumulative talk seemed to show a pattern of development towards exploratory talk; however this development was supported by other type of talk such as managing understanding, where one teacher would press for clarity.

Extract 46

An example of cumulative-exploratory talk

Episode	Speaker	What is said
EP2	T	24. Yes, I did enjoy the lesson.
	KO	25. What is the specific thing that you have learnt in that specific lesson?
	T	26. From learners’ participation, at the beginning they were quiet; I felt that they did not know anything. I was bringing something new to them.
	KO	27. What made them to engage or participate?
	T	28. They were getting the idea. I was questioning repeatedly. I asked them to repeat with regards to what the standard formula says and what does the sign of the gradient means. 29. Every time I say, “If the gradient is negative, what does that mean?” and I kept on pushing. And they started understanding
		30. Yes, remember there was that $y = x/2 + 1$ and I asked them to identify the gradient and they struggled with that and I tried to make an example just to see that if $x/2$ is half or not, they still struggled. 31. And I had to do some calculations on the side for them to see that $x/2$ is the same as ½ x

In lines 24 and 26, teacher T expressed his feelings about the lesson and through KO’s scaffolded questioning (lines 25 and 27) his talk appeared to have developed towards exploratory talk. This is

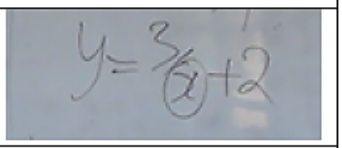
evidenced in lines 28-29 where he commented about a questioning strategy he used in order to invite learners to participate in the mathematical activity. In line 30, T mentioned the identified learner difficulty related to interpreting the gradient from the linear equation. He then concluded by commenting on the explanatory talk he used in order to address the learner difficulty. T's comments seem to provide evidence of engaging in the activity of constructing MTF narratives related to learner participation, the CA of OoL and teacher explanations. This finding highlights the KO's important scaffolding role. What started off as cumulative ended up being exploratory through the support of the KO's scaffolded questioning.

Disputational-Exploratory talk

The data showed evidence of interactions that were characterised by both exploratory and disputational talk. As seen in the previous analysis, the disputational talk seemed to have developed toward exploratory talk evidencing construction of MTF narratives. The following extract is an example of the apparent disputational talk that highlights elements of exploratory talk.

Extract 47

An example of disputational-exploratory talk

Episode	Speaker	What is said	What is done
EP7	J	93. In T's lesson there was no room for counterexamples during implementation but there was during introduction.	
	T	94. I have to work on that, if I see they are incorrect, I just correct them.	
	M	95. You need to interrogate them cause they had a reason to say 3 is the gradient	
	J	96. Cause of the position of 3, then go back to say, "Remember when talking about the coefficient".	
	M	97. If I am a teacher and I am observing that you are saying that the students are wrong, I would say, "No, no but sir you said the standard formula is $y = mx + c$".	
	S	98. Another thing that I have noticed that could have led them to say 3, not all learners understand what we mean by coefficient because from where I was sitting there was a point where you were explaining to them that the gradient was the coefficient of x and only a few responded. But when you said, "Look at the number in front of x", they said okay. 99. And the question becomes, "If you have a question like $y = 1 + 2x$, which one is next to x and $y = 1 + x2$ between these equations? "	
		100. I think we should not lower the demand. I think you were trying to make it simpler for them, but it becomes dangerous when you are using words like "which one is closer to x?" If you speak of operating signs, I think the one closer, that the one that is being multiplied by x is the coefficient, it becomes better. 101. So that's why here they saw an operating sign then they saw x, they saw a gradient.	

This conversation was about teachers' reflections on their observations of the learners' response when asked to identify the gradient in the equation $y = 3/x + 2$. The sequence was structured through alternation of evaluative interaction and counterarguments of the proposed claims. The first comment was made by teacher J, where he evaluated teacher T's lesson and its limitations. This was followed by T (line 94), defending the decision he had to make during teaching. In line 95, M further critiqued teacher T for not picking up on learner difficulty related to identifying the gradient. This was followed by teacher J (line 96) suggesting a mathematical explanation related to the interpretation of the properties of the equation and referring to the mathematical word: coefficient. M (line 97), further expanded on teacher J's remarks by offering substantiation where he summoned a general standard formula in order to talk about the gradient. Teacher S, in lines 98-101 intervened by further extending teacher M's assertion by giving a mathematical account with respect to the aspect of gradient. He used mathematical words such as "coefficient", and informal mathematical phrases such as "which one is next to x" as he interpreted

the properties of the linear equation. This interaction seems to provide evidence that teachers were involved in the process of identification of the CA which was related to interpreting the gradient. After commenting on the critical aspect, the interaction shows that teachers suggested mathematical explanations that the teacher may use to address the identified learner difficulty. This finding implies the construction of MTF narratives related to the CA of OoL and teacher explanations. This interaction seems to indicate that when the interaction is focused on constructing the MTF narratives, there is a possibility of developing exploratory talk. Furthermore, in this interaction there was an attempt to jointly engage in the construction of MTF narratives in order to address the limitations of teacher T's lesson, something that was not seen in interactions that were disputational only, as seen earlier on. This finding seems to suggest that teachers had an opportunity of learning from each other.

Managing Understanding-exploratory talk

The data also showed a conversation which consisted of two levels of talk, managing understanding (MU) and exploratory talk (EXPLO). The MU talk was evidenced in cases where one or two participants posed a query (requested clarity); in response the other participant contributed by giving an elaborative account on what was queried:

Extract 48

An example of managing understanding-exploratory talk

Episode	Speaker	What is said
EP1	KO	9. Why did you decide to make such a change?
	T	10. It just came through as I was teaching. I just found myself writing it down and started engaging learners with it.
	M	11. Was it because you saw something?
	T	12. As I was doing the examples I felt like let me do just one example that has a **gradient** being negative and also include the constant. 13. From there I can show that from the prototype that the first thing is that the graph is decreasing. 14. Now the graph is decreasing what is the constant doing? So they can see both actions at once. 15. Cause the initial example, you just do one where we show the gradient and one where we show the constant 16. And the other one was supposed to be a class activity $y = 2x + 1$. It was never planned.

In this extract, KO enquired about the change that was made by T concerning concerning editing examples from the lesson plan). In response, T gave a reason for the decision he made. It seemed teacher M was not satisfied with T's response, which prompted him to request more clarification (line 11). This request was then followed by teacher T's exploratory talk, where he gave a mathematical

account, explaining his choice of example. His mathematical talk was about the notion of gradient, where he pronounced mathematical words (such as “negative”, “constant”, “decreasing”) to interpret the behaviour of the graph. This interaction seems to provide evidence of the construction of MTF narratives related to examples and teacher explanations. Hence it is linked to explorative participation. This finding further confirms that exploratory talk could be linked to explorative participation.

7.2.4.3 Summary

The analysis of the interactions during RD1 suggests that the interactions were characterised by facilitating the goals of the MTF and its use, collaborative engagements and problematising teaching actions. With regard to disputational talk, this form of talk was linked to “reframing” which was found in RD1 and not in LP1; this could be due to the nature of the design of the L/S. The disputational talk resulted from the reflections from the teachers who were observers during the teaching session, and was based on their evaluation of the individual teacher’s decisions made during teaching which were different from what was agreed upon during the planning session (LP1). It has been shown that the disputational talk was linked to pedagogical narratives which featured the mathematical teaching actions. It can be argued that the teachers problematised the individual’s practice which resulted in a form of disputational kind of talk. However, through exploratory talk, teachers were able to jointly renegotiate or reconstruct their pedagogical narratives in relation to the MTF components, thus demonstrating collaborative engagement. As Tusting (2005, p. 85) explained, “defining a joint enterprise is a process, not a static agreement. It produces relations of accountability that are more than norms. These relations are manifested not as conformity, but as the ability to negotiate actions as accountable to an enterprise”.

Concerning the structuring conversation talk, the findings may help us to understand the nature of exploratory talk in relation to the concept of scaffolding and construction of narratives (MTF/mathematical/pedagogical). Through structuring conversation, the study has shown that the KO was positioned as the leader of the discourse, supporting teachers’ participation in the MTFying discursive activity, but at the same time inviting collaborative engagements from the teachers. KO’s interaction through the use of structuring-explorative talk can regarded as ritualistic-facilitative talk in the manner of encouraging and supporting teacher’s participation in the MTFying activity, with a goal of helping them to participate in the explorative MTF discourse.

It is important to note that, in both KO's MU and SC talks it seems that the KO's goal was that of inviting teacher participation (through scaffolded questioning—MU) and supporting the participation in MTFying activity (through SC) and not to provide points of focus in relation to MTF (prioritising the MTF goals) as seen with LP1.

To sum up, this study suggests a consideration of semi-exploratory talk in order to understand the important contribution of each talk in the development of exploratory talk and in joint construction of MTF narratives. The finding seems to suggest that semi-exploratory talk is linked to explorative MTF discourse. This study therefore argues for a hybrid form of talk between modes of talk as a powerful feature of L/S interactions.

7.2.5 Global theme shared repertoire of the L/S CoP

As seen with the LP1 results, the RD1 mathematics teachers' reflective discourse (MTRD) interwove several discourses: the mathematical, the pedagogical and the MTF (see figure 8). The interconnection between the various routines has been dealt with extensively in the concluding discussions of each emerging theme and will not be repeated here. Here, I summarise important points suggested by this interconnection.

The RD1 MTRD showed similar attributes to the LP1 MTRD in terms of the interconnection between mathematical, pedagogical and MTF sources of narratives and objects of the discourse. However, this discourse differed with that of LP1 regarding the goals of the discursive activities. For example, the analysis of the mathematising activity suggested that the goal of participating in the mathematical discourse seemed to point towards the construction of MTF narratives related to the CA of OoL rather than substantiating mathematical narratives, as seen with LP1. Furthermore, RD1's MTRD differed extensively from LP1's MTRD in terms of the roles played by the participants. The analysis suggested that the participants were engaging in subjectifying activity, where the focus was on talk about other teachers' MTFying actions. This finding points to a departure from the LP1 roles which suggested that the dominant role was that of KO's engagement in the scaffolding activity.

In this section I have discussed the nature of the emerging discourse, the MTRD, in terms of discursive objects, discursive routines, goals of discursive activity, and sources of narratives and roles of the participants.

7.3 Summary of what opportunities to learn were made possible

Participating in the MTRD meant that teachers were afforded different opportunities of learning from three different discourses: mathematics, MTF, pedagogical and social relations. Furthermore, these opportunities of learning were supported by collaborative engagements with the KO. In Table 33 below, I summarise four themes that describe the possible opportunities of learning. I will discuss these themes in detail in the teacher learning chapter.

Table 33:

Summary of what opportunities to learn were made possible

Type of discourse	**Opportunities to learn**
Mathematics discourse	Connection between the mathematics content and the mathematical capability related to the mathematics object that the learners need to learn. Connection between word use and mathematical explanatory talk
MTF discourse	Coherence between the OoL and other MTF components
Pedagogical discourse	Integrating other pedagogical strategies that are critical for the work of teaching
Relations	Opportunities to learn from each other and with the KO.

7.4 Conclusion

In the above analysis, two sources of narratives emerged from the data: the **reframing** that led to **"on-task" subjectifying talk**. An important finding worth mentioning that is unique to RD1 is that the goal of participating in the mathematical discourse in the L/S context that is informed by the use of a teaching tool does not necessarily point to that of substantiating mathematical narratives. Rather, the goal is to foreground the mathematics that needs to be learnt (CA of OoL). Such a process contextualises teacher participation in mathematical discourse in the teaching of mathematics—that is, instruction. At this point, one may draw in Adler and Ronda's (2015) assertions that the goal of participating in mathematical discourse in instruction is to bring to focus what needs to be learned by learners. Furthermore, the analysis confirmed a further expansion of MTF object: the **CA of OoL (for teachers and learners).** The analysis has also revealed that participation in the discursive activities of the RD1 reflective discourse was supported by the participants' **semi-exploratory talk** which highlighted the collaborative, facilitative and problematising interactional goals of participation in the RD1 reflective discourse.

At the end of the RD1 session, the suggestions to be considered in the next teaching session were noted in the lesson plan[20]. As a way of concluding this session, teacher S was selected to teach the lesson in the next teaching session (T2). In Chapter 8, I present the analysis of teachers' post-lesson reflective discussion (RD2).

[20] It needs to be noted that the original lesson plan was not edited. Instead, the considerations for the next teaching session were discussed and noted.

Chapter 8: The Reflective Actions during Reflective Discussions 2— RD2

8.1 Introduction

The focus of this chapter is to describe and interpret the MTF-use during RD2 session. RD2 provided teachers an opportunity for reflection on the second teaching session (T2). The RD2, which is the final post-lesson discussion, followed the same form as the first post-lesson discussion (RD1), now focusing on the teaching of the revised lesson plan. This session marked the end of the L/S.

As seen in the RD1 analysis, the reflections were from both the individual teacher (teacher S) who was teaching and from the group of teachers, including the KO, who were observers. Seven episodes were recorded. However, it was difficult to differentiate episodes from different groups (i.e., individual teachers and observers) as seen with RD1, as the reflections from these groups happened simultaneously. The RD2 was a shorter reflection; hence some of the extracts from episodes have been used more than once. Each section has it own extract, some copied from other sections but highlighting a different point. The quantitative analysis presented in Table 28_B (in Appendix D), shows that there was dominance of explicit naming of OoL and explanations, and the talk related to teacher explaining routines. There was also a dominance of talk about routines related to L/S pedagogical practices. The table further shows dominance about talks related to roles and contextual factors related to learners. The modes of interaction were dominated by cumulative talk with no recorded disputational talk. The table also shows that the mathematical talk was in focus with dominance of interpreting and graphical representations. As seen with the previous chapter, I first present the summary of the analysis in terms of the identified themes before I elaborate them.

8.2 Summary of identified themes

Looking at the attributes related to MTF-use that have been established in the previous chapter (of the RD1 session), the MTF-use in this chapter shows similar characteristics in terms of participation in MTF discourse, but differs regarding participation in the mathematical discourse and pedagogical discourse. The use of MTF meant participating in the MTF discourse which was characterised by explicit naming of MTF objects and MTF routines which showed the interplay between MTF routines and pedagogical routines. Participation in the mathematical discourse was characterised by object- and routine-driven word use; use of mathematical words and use of at least one mathematical routine, something which

was not seen in the past two L/S sessions. The pedagogical discourse was characterised by similar routines as those seen during RD1; however, these served a different function during RD2. The reflection during RD2 also focused on contextual factors related to learner participation, but this focus was also extended to a talk about the roles of participants, the KO in particular. Regarding the modes of talk, the analysis of RD2 also showed a combination of exploratory talk and other type of talks—semi-exploratory talk. All these attributes have implications for the nature of opportunities to learn that were afforded by participation in these discursive activities that are unique to the RD2 context.

In the table below I summarise the main findings that have emerged during RD2. According to Table 34, a total of 11 emerging themes have been recorded alongside the associated organising themes.

Table 34

Summary of emerging and organising themes

Organising the themes	Emerging themes
1. Use of MTF was linked with participation in mathematics discourse	▪ Teachers used mathematical words to refer to mathematical objects ▪ Object-driven word use of mathematical objects ▪ Use of only one mathematical routine
2. The use of MTF was linked to participation in pedagogical discourse	▪ Identifying CA routines used to generate CA of OoL ▪ Scaffolding used by KO to support off-task[21] actions ▪ Structuring linked to contextual factors ▪ Reframing routine linked on-task and off-task subjectifying
3. The use of MTF meant participating in the MTF discourse	▪ MTF Naming—object driven ▪ MTF routines showed similar type of routines in each MTF object ▪ MTF routines linked to both mathematical and pedagogical routines
4. The modes of interaction supported the MTF-use	▪ Hybrid between exploratory talk and other talks enabled teachers' construction of MTF narratives

8.2.1 Organising theme 1: Use of MTF was linked with teachers' participation in mathematical discourse

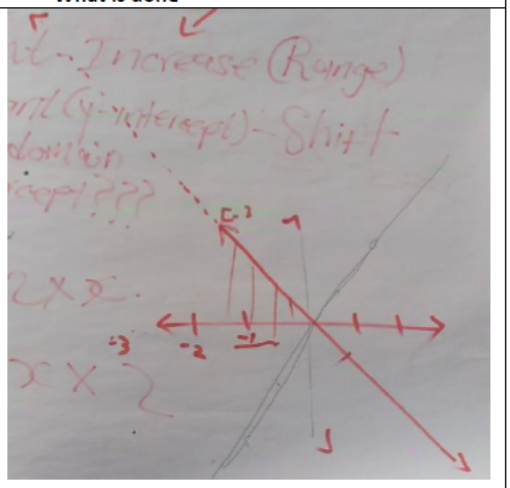
8.2.1.1 Emerging theme: Source of mathematical narratives included the mathematical word use

Teachers' reflections with respect to mathematical objects were characterised by mathematical words to refer to mathematical objects:

[21] I refer to all the actions which are not related to the construction of MTF and/or mathematical narratives

Extract 49

An example showing mathematical word use

Episode	Speaker	What is said	What is done
EP5	S	83. Ja, I noticed that the last part I think there was a problem there when the learner was asking as to "What do you mean that the graph *it's increasing* to somewhere using the arrows?"	
		84. I think I should have had a more mathematical explanation because honestly what I used I just used the concept of the **area under the graph**, which is something ... [intervened]	
	KO	85. I was, I lost that myself, I would not have been able to answer that question.	
	S	86. Ja, ja.	
	M	87. It's a very simple question. You see I once asked it when I was still a student. They are asking why does it have two end arrows?	
		88. The graph *is continuous*, that's what you tell them, "The graph *is continuous*, this **arrow** shows *it's continuous*, but now, we have to bear in mind that we have a restriction, which is **our interval**, **negative 3 to positive 3**, that's why now we cannot *draw* a huge graph."	
		89. That's why in our x **axis, and our** y **axis** there we put arrows to show them that, you know what, these things are going to **infinity**.	

The extract above shows that the teachers were reflecting about the linear function in a graphical form. They referenced mathematical words (such as "infinity", "x/y axis", "area under graph", "interval") and further used words such as "increasing", "continuous", and "arrows" to describe the behaviour of the graph. In other words, they were interpreting the global properties of the graph. The analysis of teachers' mathematical narratives foreground teachers' reliance on the interpreting routine when attending to mathematical representations, as highlighted in Table 28_B.

Interestingly, some of these mathematical words were not related to a function discourse at grade 10 level, instead these words were related to a calculus topic found in grade 12, for example, words like "infinity", "continuous" and "area under the graph". According to S in line 83, it seems the use of these words ("infinity", "area under the graph") somehow created confusion during teaching where learners questioned why S referred to the given figure (see figure in Extract 49) as the graph was increasing to somewhere as shown by the arrows going upwards, but at the same time he was saying the graph is a

decreasing function. One may conclude that the aim of the mathematical activity was to bring to focus learners' difficulties—an indicator of critical aspects. This finding implies that the goal of the mathematising activity was to identify critical aspects of the object of learning. This result confirms the findings related to the interplay between mathematising and the CA of OoL as established during the RD1 analysis. However, RD2 presented a different focus in relation to the use of mathematical words from mathematics and calculus discourses as opposed to the use of mathematical words from colloquial and mathematics discourses seen during RD1.

What is interesting is that S (line 84), extended his reflections by commenting about the type of mathematical teaching strategy—of providing a mathematical explanation that would be useful in order to resolve learner confusion. This was further extended by M in lines 87-89 who gave a detailed mathematical explanatory talk that a teacher may use to resolve any confusion with the learners. Through this talk, teachers had the potential to increase their collective understanding of what learners needed to learn and alter their plans accordingly. This finding further echoed those established during the RD1 analysis where it was revealed that the mathematical word use linked to the CA was also linked to pedagogical goals. Hence, this discursive activity could possibly afford teachers opportunities to learn mathematical explanatory practice.

8.2.1.2 Emerging theme: Source of mathematical narratives included object-driven word use of mathematical objects

Excerpt 49 further evidenced object-driven word use. For example, the words such as "infinity", "area under the graph" were used to describe the notion of arrows. Interestingly, these words were linked to the actions done by the graph itself, and did not refer to human action, for example, in line 83 ("graph it's increasing to somewhere using the arrows") and in line 88 ("the graph is continuous; this arrow shows it's continuous").

When analysing teachers' mathematising talk, it appeared as if it was more abstract and at a level of higher grades such as a calculus topic. This mathematising talk could explain why teachers commented mostly about learners' difficulties concerning the use of arrows in their explanatory talk. It seems the teacher did not reorganise his mathematical teaching activity to the level of learners according to the level of the curriculum. In other words, they used their own understanding to provide explanation, and the content was at grade 12 or university level which was related to a calculus topic. What needs to be

noted is in the analysis of all episodes of RD2 data, the KO did not intervene in order to structure the conversation, nor contribute to provide expert knowledge about relevant mathematical teaching at grade 10 level. Instead, her intervention only came later as a way of summarising, and commenting that the choice of word use about infinity/area under the graph was at a higher level, and she further suggested the relevant mathematical teaching with respect to the aspect of arrows, as will be shown in the pedagogical section that follows. There was also less focus on scaffolded questioning, which was seen in RD1 as a catalyst in promoting mathematical teaching strategies or a way of teaching a specific mathematics object. This suggests that the KO was withdrawing her participation, allowing teachers to reflect on their own. This finding suggests that opportunities to learn from the KO were limited during this session.

8.2.1.3 Emerging theme: Sources of mathematical narratives included the use of only one mathematical routine

The analysis showed that the source of the mathematising activity included at least one mathematical routine, the interpreting routine, which was recorded in Table 28_B. In addition, Table 28_B show that the most used representation was the graphical representation. This implies that teachers' mathematical talk was characterised by the inflexible use of routines. In other words, they were inclined to apply interpreting routines only. For example, in lines 83-84 (in Extract 49), teacher S's main focus was on the notion of arrows. In his talk, only one mathematical routine was in focus—the interpretation of the behaviour of the graph using the notion of arrows in order to substantiate the behaviour of the graph. These comments seem to provide evidence that teachers were more inclined to use the global approach to functions. Even (1998) warned of the dangers of using one approach as this constrains the flexibility of moving from one representation to another—a necessary aspect of developing and strengthening explorative participation in the functions discourse—thus, limiting learning opportunities for their learners.

8.2.1.4 Summary

The analysis of mathematical discourse during RD2 revealed that this particular discourse was characterised by features related to a function discourse in terms of words, routines, visual mediators and narratives. However, the words were from different levels of mathematical discourse: the grade 10

mathematics and grade 12 calculus topics. I utilise similar principles used during LP1 and RD1 to describe the nature of participation in mathematical discourse during RD2. These are the following:

1. In terms of nature of mathematical objects (words and word use)

The analysis of word use brought three main findings. Firstly, it produced evidence showing that teachers' reflective talk was characterised by mathematical words when talking about the mathematical object of gradient. However, this mathematical word use can be categorised into mathematical words describing function discourse at grade 10 level ("increasing", "decreasing") and mathematical words for describing calculus discourse (e.g., "are under the graph", "continuous", "infinity"). Importantly, functions and calculus mathematical word use was aimed at describing mathematising actions. On the other hand, the calculus mathematical word use served to describe pedagogical actions—identifying the CA of OoL (e.g., the teachers reflected about learner misconceptions related to the area under the graph and infinity). This finding suggests the interplay between mathematising and CA of OoL. Furthermore, it could mean that teachers were afforded opportunities of developing awareness of mathematics content in relation to what learners needed to learn.

Secondly, the findings showed that teachers' word use was linked to discursive routines such as interpreting routines. This was an indication of routine-driven word use. Thirdly, analysis has pointed to object-driven word use where mathematical objects were talked about as if they had their own life. The presence of both object- and routine-driven word use signalled elements of the objectification process which is linked to explorative mathematical discourse. It could also mean that teachers were afforded opportunities to reflect about the explorative discourse necessary for learners in order to participate in the mathematics discourse.

The analysis has further shown the connection between elements of word use, from words to attended procedures—mathematical routines and visual mediators. However, the intended focus was not related to substantiating mathematical narratives but to highlighting pedagogical narratives related to learners' difficulties in relation to specific mathematical ideas (e.g., arrows and gradient). Such implies the interplay between mathematising and CA of OoL. Furthermore, it could mean that teachers were afforded opportunities of developing awareness of mathematics content in relation to what learners needed to learn and modified their instruction based on what they believed would support learning for the learners.

2. Sources of mathematical narratives

The analysis indicated that the sources of mathematical narratives were from both the mathematical and L/S pedagogical discourses. The L/S pedagogical discourse was characterised by the process of identifying critical aspects of object of learning. According to Runesson (2013), the critical aspects have their roots in the mathematics content. It is therefore not surprising that this feature appears to be a source of mathematical narratives.

From the mathematical discourse perspective, the results indicated dependency on the use of only one mathematical routine—the interpreting routine. This limited the development of explorative discourse. In terms of Sfard (2008), this can be referred to as "rigid" routine use—an element of ritual mathematical discourse. This finding has implications for the KO, who was seen to have played an active role in facilitating the flexible use of routines during LP1.

3. Goals of discursive activity

As seen with the analysis of the two previous sessions, the goals of discursive activity can be categorised into two: object and meta-level. Regarding the object level, the analysis has suggested that the aim of engaging in mathematical activity was to construct mathematical narratives, as evidenced by mathematising talk which was characterised by mathematical words, routines, and visual mediators from function discourse.

With regard to the meta-level analysis, it seems that the goal of participating in the mathematising talk was to bring forward aspects of learner mathematical difficulties. For example, the analysis highlighted that the mathematical talk frequently referenced learner difficulty or misconceptions in relation to mathematical objects (y-intercept, gradient); these are indicators of critical aspects. This implies the interplay between mathematising and CA of OoL. Furthermore, it could mean that teachers were given opportunities to develop awareness of mathematics content in relation to what learners needed to learn.

Another meta-discursive goal was related to the construction of mathematical narratives in relation to either teachers' use or learners' use. As seen with the analysis in LP1, this goal was evidenced by referencing keywords such as "they" (for the learners) and "we"/" you" (for teachers). This finding points to a relationship between mathematising actions and MTFying actions. This interplay seemed to

have enabled opportunities of learning MTF components necessary for the negotiation of meaning such as examples for teachers and learner explanations.

4. Roles of participants

Analysis of the participation in the mathematical activities during RD2 did not show any distinct roles played by the participants as was seen with both the LP1 and RD1. Instead it showed that all participants played similar roles in contributing to the mathematising talk. Interestingly, the KO did not play any dominant role as seen in the two previous sessions. The passive role of the KO appears to have constrained learning opportunities for teachers through reflections about mathematical ideas at the relevant curriculum level and flexible use of routines necessary for the development of explorative discourse for their learners.

Furthermore, the analysis did not show any indications concerning talk about the other participants' mathematical activity—the on-task subjectifying as seen in RD1 which seemed to have enabled teachers to problematise their practice, thus learning from that process and from each other. Instead, the focus was on the mathematising activity itself. This discussion about the roles has an implication for the analysis related to the emerging talk in the upcoming section of modes of talk.

8.2.2 Organising theme 2: The use of MTF meant participating in MTF discourse

Looking at the MTF category (refer to row titled 'MTF' in Table 28_B, in Appendix D), one can see that both subcategories—MTF naming and MTF routines—developed during the analysis.

8.2.2.1 Emerging theme: MTF naming

The names of MTF objects were explicitly referenced when reflecting about the MTF tool as a resource:

Extract 50

Example of teachers' explicit naming of MTF components

Episode	Speaker	What is said
EP7	KO	154. I think, oh now you are taking up a general lesson. I want you to talk about MTF in terms of its use, like something like, maybe MTF is about knowing the **object of learning**, **gradient**, and **intercept**, **examples** ... [intervened]
	T	155. The participation?
	KO	156. Then ama explanations. And then the **participation**, what I am saying, do you think your lesson was reflecting that? Okay, I mean did you see yourself giving learners enough **examples**? 157. Giving them **explanations** which are making sense, which are connecting, to the **examples** together with the object of learning and the way **they participated**, do you think for your own, from where you are standing, can you say you have done justice to MTF?
	S	158. Ja no I think I have done justice to MTF, in a sense that I didn't not have, I didn't have to like to crack my head looking for **examples** from outside what I have here. It was just for me to, I don't know. Uhm ... [intervened]
	KO	159. You are still confused about it?
	S	160. Ja, eish.
	T	161. I think the question, is okay. Ja.
	M	162. But you know you need to understand what's here first, is what MTF is.
	S	163. No, I agree. That it has **learner participation**, enough **examples.**
	M	164. I agree the evidence of MTF. There was enough **examples** and then the **learner participation**, as the lesson was not teacher centred. It was also learner centred. 165. As some of the things he was using like **counterexamples,** like "Why do you say this is this?"

This extract shows that the names of MTF objects were referenced particularly when reflecting about purposes of MTF tool. What is interesting is the naming came later in the discussion; they were most evident in the last episode of the RD2. These were due to the KO's acts of scaffolded questioning aimed at eliciting teachers to reflect deeper about MTF names; otherwise this naming did not feature in teachers' reflections in the earlier discussions. This is surprising, as I was expecting to see teachers speaking about MTF names even more than they did with RD1 since they had received enough exposure with MTF-use in the previous L/S sessions. However, it was only after the KO's intervention that they started reflecting about these names; even so, these were not talked about in terms of discursive acts such as mathematical or pedagogical as seen with LP1 and RD1. This finding has an important implication in terms of the role of the KO. It seems her passiveness constrained learning opportunities for teachers related to naming of MTF objects which provide points of focus for the negotiation of meaning in relation to the MTF tool.

8.2.2.2 Emerging theme: MTF Routines

The fourth column of Table 28_B in the MTF routine row shows the MTF routines that were in focus were related to both learner and teacher explanations. The CA were also featured, as seen with RD1; these type of routines were divided into learner CA and teacher CA. Interestingly, neither tasks nor exemplifying routines were in focus. This could be due to the role of the KO who was seen in the previous two sessions to have been scaffolding construction of MTF narratives as a way of

foregrounding points of focus. This finding seems to suggest the withdrawal role of the KO during this session. The analysis has further revealed that the identified MTF routines reflected mostly the interpreting mathematical routine, that is, they were connected by this particular mathematical routine.

I now give evidence of the nature of these MTF routines, starting with those related to explanations.

(i) Explanations

With respect to explanations, the data revealed that the explanations were from both teachers' and learner's perspectives as evident in Table 28_B. In the following discussion, I will be commenting on both learner and teacher explanations as these were talked about simultaneously, that is, one evoked the other.

The learner explaining routines showed a pattern of inquiry posing prompts[22] from the learners addressed to the teachers. This means learners were requesting substantiations from the teachers. In answering these questions, the analysis showed that the teacher engaged in a process of constructing mathematical narratives and substantiating them. For example, lines 83 and 87 (Extract 49) showed the inquiring prompts from the learners: "What do you mean that the graph it's increasing to somewhere using the arrows?" and "why does it have two end arrows?"). It seems the learners were requesting teachers to substantiate their mathematical narratives about the "arrows". In response to these inquiring prompts, the analysis further showed teachers' reflections about their substantiations were mathematical, characterised by use of mathematical words, mathematical routines, and visual mediators (lines 88-90). This finding seems to suggest teachers' awareness of their learners' contributions to the mathematising activity and how these can be used as the resource for the development of their explanatory talk.

(ii) CA of OoL

The analysis showed that the CA of OoL was categorised into learner CA and teacher CA, as seen with RD1.

[22] The inquiry posing prompts could be seen similar to questioning prompts. However, these are used differently; the inquiring prompts are used by the learners with the aim of getting more clarity from the teachers, while the questioning prompts are used by the teachers deliberately in order to illicit learners to think more deeper about a particular mathematical idea.

Starting with learner CA, the analysis showed that teachers reflected about learners' knowledge with specific attention to learner difficulties about a particular mathematical idea. I exemplify using the following extract:

Extract 51

An example of the relationship between CA of OoL and learner difficulties

Episode	Speaker	What is said
EP4	S	71. number one because learners did not understand at first what is the **gradient** to begin with, what is a **coefficient,** what is a ***y*-intercept**, what is a **constant**.
		72. So I had to spend more time trying to engage them into understanding that

This extract shows that, S was reflecting about learners' difficulty with respect to interpreting the properties of gradient (coefficient) and y-intercept (constant). This suggests that the interpreting routine was a critical aspect which led the teacher to spend time explaining this aspect as explicitly remarked by S in line 72. This finding seems to confirm the connection between CA of OoL and teaching and further opens up learning opportunities for further development of explanatory talk.

With regard to teacher CA, the data revealed that teachers reflected about the key mathematical ideas of the OoL:

Extract 52

An example of relationship between CA of OoL and teacher mathematical ideas

Episode	Speaker	What is said
EP1	KO	9. There was so much more talk about **basics**. But we know that our **object of learning** was to do about the **shifts**.
		10. Maybe we can decide in the next new cycle, to say okay the first teaching lesson, is to say let's push the first part something like that, what is the **gradient**, and what is the **intercep**t because you want to check if ever they understand those two properties, 11. the second teaching lesson, then somebody can extend it, and then take it further to ama **shift,** I don't know.

This extract shows that the KO was discussing what the teachers viewed as critical, such as basic mathematical ideas about the gradient and intercept, in other words, the interpretation of these properties of the linear function. In addition, the other critical aspect was the notion of shifts (line 11). Further analysis of KO's reflections revealed that, in response to the identified critical aspects, the teachers needed to consider restructuring the lesson by dividing the OoL into two sessions with the first teaching session focusing on the basic aspects of the linear function and the second teaching session

focusing on the aspect of shifts. In other words, the identifying CA talk about CA led to a choice of pedagogical action related to future structuring of the lesson. This finding also seems to confirm the connection between the CA of OoL and teaching. Through talking about what teachers deemed to be critical for learning, teachers have opportunities to consider their future execution of the lessons.

To sum up, the analysis of the process of identifying CA with respect to CA confirmed similar findings as seen in RD1, which suggested that the aim of participating in this discursive activity was to identify CA of OoL and further reorganise the teaching strategies.

8.2.2.3 Summary

The emerging MTF discourse during this session reflected similar features of the discourse as seen with the other two sessions in terms of words, routines, visual mediators, narratives and goals of the discursive activities. However, these were used in a different way and so furthered opportunities of learning and constraints.

Looking at the characteristics used in the previous chapter to describe the nature of participation in the MTF discourse, the RD2 MTF discourse used similar words to name the MTF objects such as example, tasks, explanations, learner participation and OoL. The sources of narratives were similar to those found in both LP1 and RD1 such as MTF routines, mathematical routines, and pedagogical routines. However, in terms of MTF routines, the exemplifying and generating task routines were not in focus. As already noted, this could be due to the fact that this was the end of the cycle, and there was no intention of editing the lesson plan and developing it further, where these routines seemed to play a role of teachers' resources for teaching (mediating tools for teaching) the next lesson. Furthermore, there was no focus on the mathematical teaching routine such as the questioning routine which was shown in the previous two chapters to have been instrumental in facilitating the construction of tasks.

This finding has implications on constraints to teacher learning in terms of developing a connection between all MTF components—an important feature of learning with respect to the MTF tool and also explorative MTF-use. However, the explanation and CA OoL were in focus during this session, and these were related to how they were used during the teaching session, but the discussion did not focus on how these routines could be used for future teaching. This confirms that there was no intention of editing a lesson plan any further. The use of explanation and CA of OoL MTF routines echoed the similar function seen in RD1 analysis where the explanation category was divided into both teacher and learner

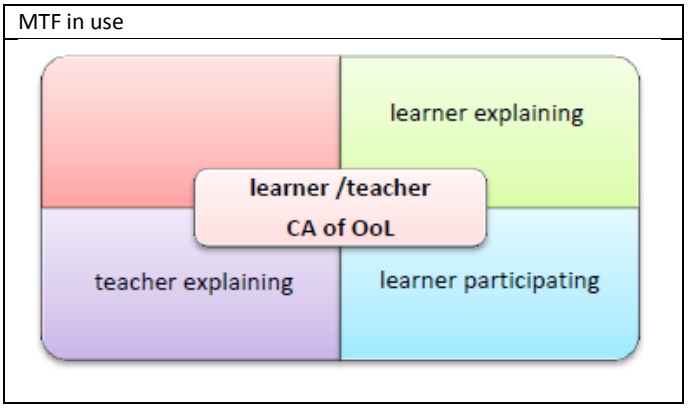
explanations, and the CA of OoL was divided into learner CA and teacher CA. The MTF routines used during this session were also characterised with mathematical routines as seen with the other two sessions. However, the mathematical routines were characterised mostly by interpreting routines.The emerging MTF routines have implications for the expansion of the MTF tool—an important aspect of discursive learning that will be discussed in the following chapter.

Regarding the pedagogical routines, the analysis of the RD1 confirmed that these types of routines played a role in facilitating the construction of MTF narratives and further expansion of MTF objects such as CA of OoL into learner CA and teacher CA. The results from the identifying CA category confirmed the intertwinement between MTF objects (e.g., teacher explanations-learner explanations; learner participation-learner explanation). This finding provides evidence of how pedagogical routines enabled opportunities for learning in relation to the construction of MTF narratives and the expansion of the MTF tool.

In terms of goals of participation in the MTF discourse, the results indicated that the object level goals were similar to those found in the RD1 session, that is, the expansion of MTF objects such as explanations (into teacher and learner explanations) and CA of OoL (into learner and teacher CA). In terms of meta-level goals, the implied goals were related to the construction of MTF narratives for either the learner or teacher use. However, the MTF narratives related to both exemplifying and generating tasks were not in focus. This implies limited participation in MTF explorative discourse. Figure 11 illustrates the transformation of the MTF tool into the MTF discursive tool (MTF-use):

Figure 11

Transformation of MTF tool to MTF discursive tool



The roles differed to a great extent from the LP1 and RD1. During this session all participants were contributing to the construction of MTF narratives with the KO in a less dominant role, with interesting consequences to the constraints with respect to foregrounding points of focus for negotiation of meaning.

In summing up, the foregoing discussion highlighted that teachers were involved in the construction of MTF narratives but with limitations where there was no focus on exemplifying and task routines. As much as participation in the MTF discourse had elements of explorative discursive activities, there were limitations in terms of developing coherence between MTF objects—an important aspect for teacher learning. This further signals a limited explorative MTFying process.

8.2.3 Organising theme 3: Use of MTF was linked with participation in pedagogical discourse

Table 28_B reflects similar types of pedagogical routines as seen with RD1. However, questioning routines did not feature during the RD2 session. This could explain why the tasks were not in focus since these were seen to be linked to questioning routines during LP1 and RD2 sessions.

8.2.3.1 Emerging theme: L/S pedagogical discourse

Scaffolding routine

The scaffolding routines did not feature as much in the teachers' reflective discourse as seen in the other two sessions and as further recorded in Table 28_B. Of particular interest was the relationship between KO's scaffolding routines with (1) structuring routines; (2) discussion about the roles. There was also one instance where teachers' talk was characterised by KO's scaffolded questioning and mathematical teaching strategy.

Structuring

Scaffolding was evident in utterances by the KO where she was giving guidance on how to restructure future implementation of a lesson with respect to the OoL:

Extract 53

An example of scaffolding routines in relation to structuring routines

Episode	Speaker	What is said
EP1	KO	14. Can we teach everything the first session 15. and then the second session repeat it as it is or we can break it down and say, 16. during the first teaching lesson we go through the **basics**, not to worry about ama shift or whatever, just focus on **the basics**, then the second teaching lesson the person is going to deal with **the shifts**?
	S	17. Ja that, it would, it would work.
	T	18. Do focus on **the basics** for the first lesson.
	KO	19. Of the same **object of learning**. 20. And then in the second teaching lesson, then you look at the **effects** 21. In the first teaching lesson, you are focusing on the word and the word use, and, the endorsed narratives, 22. what are **the definitions**. Then when we go to the second teaching session, we have armed them with these **basics**, so on the second teaching lesson the person we would want them to focus more on **the shifts.**

This extract shows that the KO was giving guidance about restructuring the future implementation of the lesson by suggesting the breaking down of the main OoL into sub-OoL. For example, in line 16, she suggested that the first teaching session of the cycle should deal with the identified critical aspects such as basics, followed by the second teaching session where the focus should be on the main OoL, that is, the shifts. She further elaborated on this in lines 21-22. KO's talk about what teachers may do in the future, with some mention of MTF strategies they may use when dealing with OoL, can possibly be explained from Wenger's CoP perspective. Having worked together in this L/S, CoP has many ways of talking and acting to "become a part of [the CoP's] practice" (Wenger, 1998, p. 83).

Roles of participants

The KO was observed discussing roles of the participants where she gave the background of a community of practice using a of a Stokvel[23] metaphor (extract 54) .This finding also points to ways of acting to become part of the L/S practice.

Extract 54

An example of scaffolding in relation to the roles

[23] Stokvels are a form of an informal communal financial insurance. There are different types of stokvels: investment/savings, grocery and burial societies.

Episode	Speaker	What is said
EP2	KO	38. So as a group, because of my role, I am a participant observer whereby, when you are a participant observer in your research, you say you are going to collect data, in classroom you also interviewing here and there but your job is to just to be an observer. 39. But now for a community of practice if I want to, you must always refer to a Stokvel example at eKasi. I am going to be a chairperson, and that person is an organiser and then there are members. 40. So at the same time each person as much as a chairperson takes a lead, he is also a member, you understand, all the benefits that are received by every member they do get. So, it is also important because now what you are trying to do here is to develop our own community of practice and the role of the expert. 41. I organise everything, I know so much, you understand, it is very important that at the end to say okay this community of practice, a role of an expert should be this one, because it makes, it changes the way things are.

Mathematical teaching strategy

Regarding the mathematical teaching activity, the results showed that the KO used this type of routine to convey mathematical teaching strategies, as illustrated in the extract below:

Extract 55

An example of scaffolding in relation to the mathematical teaching

Episode	Speaker	What is done
EP6	S	133. It's a *decreasing* **function** due to **negative gradient.**
	KO	134. You can't when you are answering this question and what does your d**omain** do? You can't say what is your ... [intervened]
	S	135. Ja, it's going to have a problem.
	KO	136. You can't say, "What does your **range** do?" you have to say ... [intervened]
	S	137. Talk about that
	KO	138. About the **change in y over**, you have to say, you are talking about the formula, for me really it makes sense that you say *its decreasing* but **the arrow** is *going up*
	S	139. Ja so **the gradient**, and it's the same throughout.
	KO	140. So now, it means now for their level, I mean this question would have been answered very quickly if it was grade 11 or 12. 141. Because they understand as **infinity**, as much as the curriculum says they should be understanding them, we know they don't.
	S	142. Ja.
	KO	143. So you can't be using idea curriculum and this thing of **infinity** and **closed brackets**, that is for a high level for them so you can just quickly say, "Remember there is infinity." So, for me, ja, I locked, this person is asking a very relevant question, how can you say, "This thing *is decreasing*?" So that's where you said okay fine, remember what is it we are talking about, we are looking at the **gradient.**
		144. Forget about the **arrows** *going up and down*, the only way for you to understand this thing is going down or up, look at **the change in y** and **the change in x**, that is the formula that you already introduced.

What needs to be noted, the KO only came in later with this facilitation of mathematical teaching relevant for word use from mathematical and calculus discourse. The analysis has already shown that

there was a lot of discussion about the issues related to word use about infinity in relation to gradient (Extract 49). However, the KO did not intervene during these discussions, which resulted in limitation to opportunities for teacher learning with respect to word use.

Scaffolded questioning

Another function of the KO scaffolding routines was scaffolded questioning, and this was present when the KO was probing the teachers (particularly the individual teacher who conducted the previous lesson) to reflect about their teaching experience.

Extract 56

An example of scaffolding in relation to scaffolded questioning

Episode	Speaker	What is said
EP7	KO	150. Okay, and how have you used the ideas of the MTF during your teaching session?
		151. According to you, do you say this, this teaching, your teaching, is reflecting the ideas of the MTF? If it does, to what extent would you say this is reflecting MTF, because you understand what is MTF?
	S	152. Okay, I don't know if I get your question, but what I would say, I would say it gives the whole, what can I say, it gives the whole overview of how the lesson should be carried out because it clarifies the introduction, the implementation and the conclusion. So it takes into account all elements of a lesson. You can't have a lesson without introduction, without implementation, without conclusion, 153. so at the end it can only also evaluate the lesson within the teaching because when you implement, now you are testing what you introduce, I mean when you conclude, looking at what you introduced and what you implement, implemented, so I think it does.
	KO	154. I think, oh now you are taking up a general lesson. I want you to talk about MTF, in terms of its use, like something like, maybe MTF is about knowing the **object of learning**, **gradient,** and **intercept**, **examples** ... [intervened]
	T	155. The participation?
	KO	156. Then ama **explanations**. And then the **participation,** what I am saying, do you think your lesson was reflecting that? Okay, I mean did you see yourself giving learners enough **examples?** 157. Giving them **explanations** which are making sense, which are connecting to the **examples** together with the **object of learning** and the way they participated, do you think for your own, from where you are standing, can you say you have done justice to MTF?
	S	158. Ja, no, I think I have done justice to MTF, in a sense that I didn't not have, I didn't have to like to crack my head looking for **examples** from outside what I have here. It was just for me to, I don't know. Uhm ... [intervened]

What the analysis of this discursive activity revealed is that, for the first time during the L/S cycle, the KO's scaffolding actions were not focused on supporting the construction of either the mathematical or MTF narratives. This could be explained by the KO's intention to slowly withdraw her scaffolding role with respect to MTFying activities and to allow teachers to take a dominant role, as earlier indicated in the research design chapter in terms of the design of the L/S roles of the participants.

The analysis of the KO's scaffolding actions indicated that the KO's goal during this session was to only provide ground rules related to L/S CoP in terms of structuring of the future lesson and the roles of the participants, and not of supporting construction of MTF narratives. It seems the KO played a role of that of a discussion leader. This can be regarded as KO's low scaffolding in relation to MTFying activities. Instead the analysis seemed to suggest that teachers were given opportunity to decide on what should be done and which routines to evoke without being helped by the KO's scaffolding actions. This implies that the teachers were viewed as independent participants in the MTFying discursive activities. This process is related to the notion of perfomer agentivity (Lavie et al., 2018)—an indication of explorative routine-use. From this, it can be argued that KO's low scaffolding was due to her goals of encouraging teachers to become agentive participants in the MTF discourse. This implies KO's explorative goals. It is interesting, that this goal implied KO reduced scaffolding, which constrained teacher engagement in explorative MTF discourse. Their participation in the MTF discourse was characterised with signs of ritualistic routine-use. This action presented a limitation for the development of a coherent practice, where teachers did not focus on all of the MTF components for the establishment of coherence instruction—an important aspect of teacher learning from the MTF perspective.

Reframing

The reframing routines were divided into three categories: (1) reframing about the KO's role; (2) reframing by the individual teacher; and (3) reframing by other teacher observers.

Reframing about the KO's role

During this RD2 session, the reframing routines evidenced reflections about the KO's role. For example, in the following extract the teachers were reflecting on the disadvantages of having the KO as the teacher executing the lesson citing that this could lead to misinformation about the nature of classroom discourse since the KO was viewed as the expert and had experience on how to handle classroom discourse and teaching. This idea was further elaborated by S in lines 56-58, where he used a doctor-patient metaphor, where a doctor as a healer could not see effects of the medicine unless they administered it to the patient and not themselves.

Extract 57

An example of reframing in relation to the KO's role

Episode	Speaker	What is said
EP3	M	54. Was it last week you were teaching ne *(referring to T).* And afterwards Ma'am came with the conclusion. Did you see the power in that conclusion? 55. The learners they were just standing, they were going to give us the wrong impression on our observation, the wrong impressions. ... [intervened]
	S	56. Ja it's like, it's like, it's like you see if you are creating, how can I put this, if you are a doctor or something, you make, you come up with this pill, this pill that can cure certain diseases, 57. I don't think it is worth it for you to drink it yourself, so you give other people and you observe how they react to it and then you take down, which is, the pill is this tool that we are having. 58. You understand, so it should be used by us. Then she will see what will, what implication, ja, it has.
	M	59. As I have said, I am not undermining any of the student teachers who taught there, if, if you are in a taxi, uhh and then your driver it's an old man who has been driving for seventy years and also in a taxi with someone who has been driving for like, just got his license, 60. you won't be comfortable in there. But for someone who has been driving, you are going to be comfortable, because you know there, and then those students the learners there, they can see that you have been doing this thing for quite a long time and then you see so ... [intervened]
	T	61. So basically we are saying you have an influence on how learners will learn.
	M	62. Ja. Ja it is going to give a wrong impression as he said. Because you remember the conclusion you are just were participating as if everything is normal but then things are not normal?

What this analysis implied was that the KO was not a suitable person to implement a lesson since she was viewed as someone who was very skilled at teaching and teachers would be left with nothing to critically reflect on since the KO's teaching could be seen as perfect. This finding seems to suggest that teachers viewed the KO's participation in execution of the lesson as a limiting factor for their learning as her participation did not represent a real classroom discourse since she was viewed as an expert. This finding can be explained in terms of subjectifying talk related to the roles and not related to construction of either mathematical or MTF narratives as seen with RD1. In terms of Hunter and Anthony (2014), such a talk is categorised as off-task subjectifying.

Individual teacher

As seen with the previous session, S, who was the teacher who presented the lesson, reflected mainly about the critical aspects from learner perspectives. For example, in line 71, he reflected about learners' knowledge with specific attention to learner difficulties about properties of gradient (coefficient) and y-intercept (constant). His reflections further showed evidence that he needed to change his teaching strategies in order to accommodate learners' current knowledge or lack of understanding of mathematical basics of these properties (lines 72-73).

Extract 58

An example of reframing in relation to individual teacher

Episode	Speaker	What is said
EP4	S	70. I executed the lesson with ease. However, there is something that couldn't, like I was hindered from preceeding into the **object of learning** which were the **effects of m and c** on the graph. Why?
		71. Number one, because learners did not understand at first what is **the gradient** to begin with, what is **a coefficient**, what is **a y-intercept** what is a **constant.**
		72. So, I had to spend more time trying to engage them into understanding that, which limited me uhh in instance where I had to implement the **object of learning** which is the **shift** and the change of **the gradient**.
		73. So, I can say over all the lesson was not the **object of learning** but **basics** of **gradient**, I don't know if I am making sense.

This finding highlights teacher awareness to what learners need to know, which led him to strategise on a teaching method to address this need. Through talking about what learners need to know (CA of OoL), S had opportunities to change the planned instruction and so the enacted OoL.

Other teacher observers

The reframing from other teacher observers was mostly focused on talk about mathematical objects (mathematising) that was observed during the previous teaching session and less about an individual teacher's mathematising actions (subjectifying) as seen during RD1 analysis:

Extract 59

An example of reframing in relation to other teacher observers

Episode	Speaker	What is done
EP5	S	90. Oh, I think what the learner's question was, we are seeing the graph is *decreasing,* yet this side is this *going up*, and this side it *is going down*, that was the learner's question.
	M	91. No, they said the **arrows**.
	S	92. Eh, those **arrows** why is it ja, and yet we are saying it's a *decreasing* **function**?
	M	93. But that one is a simple because you go then back to the table from the **negative 6 to 6**.
	J	94. So that means you will explain it using **the range**.
	M	95. The **range**.
	S	96. But they wanted ... [intervened]
	T	97. They are looking at the table, they are seeing the a**rrow** it is like ... [intervened]
	S	98. They want to see *decreasing*, so according to the learner, they want to see something.
	M	99. No, I understand.

This extract shows that the teachers were reflecting about learners' mathematical talk in relation to the aspect of arrows and its relation to the behaviour of the graph. This talk was characterised by mathematical words (decreasing, range,) linked with interpreting routines and different representations such as graphs and tables. Furthermore, these reflections evidenced teachers' explanatory talks in

relation to learners' mathematical talk (see line 90). It appears that the teachers' talk was focused on constructing mathematical narratives about arrows and substantiating them. It further illuminates teachers' awareness of learners' talk within the mathematical classroom discourse and problematising this talk. This finding seems to suggest that teachers' reflections were not on the talk about the person but on mathematising talk; contrary to RD1 where the reflections on the talk was based on the person in relation to his mathematical teaching actions. It could explain why there were no disputations during this session. These comments seem to provide evidence that teachers had opportunities to learn from each other and to learn about their learners' mathematical talk through reframing each other's practice.

Identifying CA

The analysis showed that the teachers' reflections with respect to identifying CA routines could be divided into two categories: learner CA and teacher CA. Concerning the learner critical aspects, the teachers' reflection showed referencing to what they had identified as learner difficulties with respect to a specific mathematical aspect during the lesson. For example, in lines 70-71, S commented about learners' poor background on notion of gradient and y-intercept which hindered him from executing the planned object of learning. The analysis has further shown that the identification of learner CA led the teacher to reorganise his teaching around this identified CA (line 72). This finding highlights the interplay between CA of OoL and general pedagogical strategies.

Extract 60

An example of relationship between Identifying CA with teaching strategies

Episode	Teacher	What is said
EP 4	S	70. I executed the lesson with ease. However, there is something that couldn't, like I was hindered from proceeding into the **object of learning** which were the **effects of** $\boldsymbol{m}$ **and** $\boldsymbol{c}$ on the graph. Why?
		71. Number one because learners did not understand at first what is the **gradient** to begin with, what is a **coefficient,** what is a $\boldsymbol{y}$**-intercept,** what is a **constant.**
		72. So, I had to spend more time trying to engage them into understanding that, which limited me uhh in instance where I had to implement the **object of learning** which is the **shift** and the change of, of the **gradient.**
		73. So, I can say over all the lesson was not about the planned **object of learning** but **basics** of **gradient**, I don't know if I am making sense.

With regard to teacher CA, the analysis further showed that the CAs were identified by teachers' reflections about teachers' choice of word use in relation to what was specified in the curriculum document.

Extract 61

Example of Identifying CA linked to curriculum

Episode	Speaker	What is done
EP6	KO	140. So now, it means now for their level, I mean this question would have been answered very quickly if it was grade 11 or 12. 141. Because they understand the concept of infinity, in topic of calculus
	S	142. Ja.
	KO	143. So you can't be using infinity and closed brackets ,that is a high level for them. 144. For me, ja I locked, this person is asking a very relevant question, "How can you say this thing is decreasing?", 145. You can just say, "Remember what is it we are talking about, we are looking at the gradient."
		146. Forget about the arrows going up and down, with words like infinity, and say, "The only way for you to understand this thing is going down or up, look at the change in y and the change in x." That is the formula that you already introduced.

According to the teacher reflections in the extract above, the chosen word use of "infinity" could have caused difficulties for the learners as this concept was outside of learners' curriculum level and was related to calculus. Interestingly, the identified critical aspect of word use of "infinity" was further linked to the relevant mathematical ways of talking about the aspect of gradient at the level of learners as elaborated by the KO in line 146. There she gave an account of the use of the gradient formula in order to explain the notion of decreasing function. This finding suggests that the use of identifying CA routines facilitated the construction of MTF narratives linked to CA of OoL, and teacher explanations.

An important result emerging from this section was that the process of identification of CA of OoL included word use from two different curriculum levels and learners' mathematical talk. By taking into consideration both what the curriculum wants taught and what learners are mathematising, the teachers engaged in conversations about what was critical to learn, what they were expected to teach, how to teach that content, and how learners were going to learn that content. This finding has important consequences for opportunities for teacher learning. The L/S group not only had opportunities to learn how to identify CA, but also had opportunities to learn how to strategise ways to deal with these CAs through their explanatory talk. This finding has important implications for developing the argument about the expansion of MTF objects into both the learners' and teacher use.

8.2.3.2 Emerging theme: General teaching discourse

Structuring routines

The structuring routines were seen when the teachers' talk referenced factors influencing deviation from the planned lesson. As seen with the RD1 analysis, these factors included learners' lack of knowledge, as shown in the extract below:

Extract 62

An example of structuring routines

Episode	Speaker	What is said
EP4	S	70. I executed the lesson with ease. However, there is something that couldn't, like I was hindered from proceeding into the object of learning which were the effects of m and c on the graph. Why?
		71. Number one, because learners did not understand at first what is the gradient to begin with, what is a coefficient, what is a y-intercept, what is a constant.
		72. So, I had to spend more time trying to engage them into understanding that, which limited me uhh in instance where I had to implement the object of learning, which is the shift and the change of, of the gradient.
		73. So, I can say over all the lesson was not about the planned object of learning but basics of gradient, I don't know if I am making sense.
	T	74. You do.
	M	75. Ja, I understand.
	T	76. And you stayed on **the introduction**
	S	77. Ja, ja I had to spend more time on **the introduction**.
	L	78. For the first twenty minutes?
	S	79. Yes.
	M	80. I know what you are saying.
	S	81. Ja, I had to spend more time there. So, 80% of the lesson lied there and only 20% was on the object of learning.
	KO	82. So that is why you stayed there, it's because you noticed something?

The interaction above revealed that the deviation from the structured lesson plan was due to identified learner critical aspects with basic knowledge about the gradient concept. Take, for example, teacher S's comments in line 71: "learners did not understand at first what the gradient is to begin with, what is a coefficient, what is a y-intercept, what is a constant". According to the analysis, the discovery of the learner critical aspects triggered the teacher to reorganise his teaching to address this gap in learners' knowledge which led to diversion from the planned lesson, in terms of activities to be executed at different stages of the lesson (i.e. introduction, implementation). These findings reveal the possibility that the lesson that was collaboratively planned can be edited during instruction. It further illuminates teacher's awareness of his learners' need. This finding is not surprising, as it corroborates Posthuma's (2012) prior research on reflective discussions who reported that when the individual teacher is faced

with a real problem during teaching they may change the planned teaching approach in order to resolve that problem. According to her, this action implies learning opportunities for the development of reflective practice.

8.2.3.3 Summary

The RD2 pedagogical discourse revealed similar features of the discourse as the RD1 in terms of the words used to name objects of the discourse, routines, visual mediators and narratives. Regarding the defining features of the participation in the pedagogical discourse, the similar attributes to those found in RD1 are noted, that is, the objects that are being named, sources of narratives, goals of discursive activities. However, the RD2 pedagogical discourse differed from the RD1 discourse regarding the roles played by the participants. Although all participants contributed to the pedagogical discursive talk, the general talk was not focused on the role of subjectifying the teacher who was conducting the previous lesson as seen with the RD1 and prescribed in the ground rules for participation in the reflective discussion. However, the subjectifying talk was focused on the KO's roles, and not specifically on her MTFying or mathematising actions. According to Hunter and Anthony (2014), this action can be categorised as off-task subjectifying. The analysis further revealed that the pedagogical discursive actions supported constructions of MTF narratives related to CA of OoL in particular, and new critical aspects, which were related to word use from mathematical and calculus discourses. It further supported the connection between CA of OoL to other MTF routines (e.g., explanations). For example, the analysis has shown how teachers based their pedagogical decisions on what they believed would best meet the needs of their learners. Some of these pedagogical decisions included mathematical explanations and word use relevant to learners' curriculum level. These findings seem to suggest that the pedagogical routines, the identifying CA in particular, were enablers for learning opportunities, and will be dealt with in detail in Chapter 9.

8.2.4 Organising theme 4: The modes of talk supported the MTF-use

As seen with RD1, the reflections were from the individual teacher who was conducting the previous lesson, the observers and the KO. What is interesting in the RD2 discussion, all participants took part in the co-construction of this reflective discursive activity, with no one playing a dominant role. This contrasts with the RD1 session where the three groups of participants (i.e., the individual teacher, the observers, the KO) played distinctive roles. For example, the KO played a directive role, the observers

played an evaluative role and the individual teacher was seen in a defensive position. However, the KO's talk did not show strong evidence of giving directives or facilitating the discussions through scaffolding, as seen during RD1 and LP1. Furthermore, during RD2 there was no strong evidence of subjectifying talk towards the individual teacher who was teaching the previous lesson as happened during RD1 where this talk was associated with evaluative-defensive interaction; instead the subjectifying talk was focused on the KO's role. This finding has implications for the emerging RD2 social interaction and collaborative talk in particular.

The following figure summarises the results of teachers' talk in relation to the different categories in focus during the RD2 session:

Figure 12

The relationship between modes of talk and discursive activities

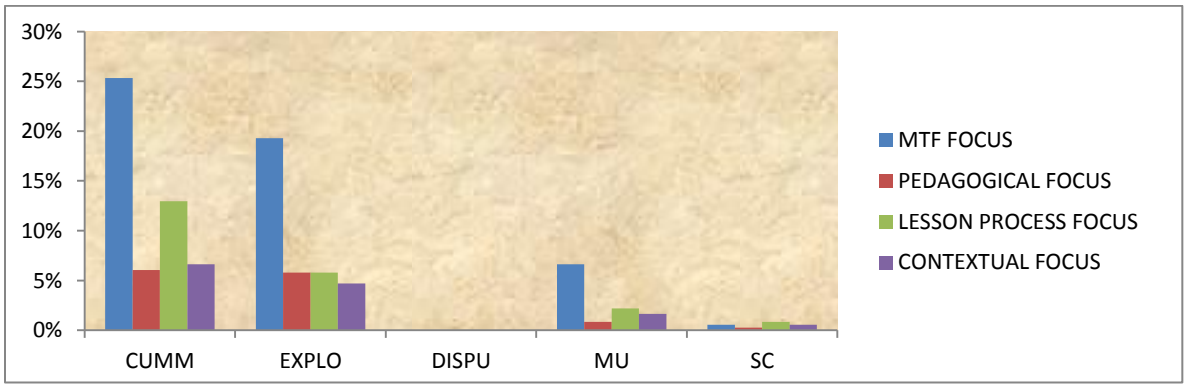


Figure 12 indicates that teachers mainly engaged in cumulative talk, followed by explorative talk, and that these talks were associated with MTFying activity-MTF focus. The RD2 presented a different scenario where cumulative talk was dominant over exploratory talk. Possible explanations for this trend could be attributed to the KO's role. For example, during RD2 the analysis has revealed that the KO slightly withdrew her scaffolding actions which were linked to exploratory talk. The dominant exploratory talk during LP1 and RD1 could be due to the presence of semi-exploratory by the KO, who used this combination through her scaffolding actions in order to develop the exploratory talk so as to create an environment of collaborative interactions. For instance, she was using both the cumulative and exploratory talk in order to encourage the teachers to explicitly publicise their opinions, and critically engage with each other's ideas.

There was also evidence of managing understanding talks, but these were the least frequent in episodes. The data produced less talk related to structuring conversation. This finding is not surprising because the interactions during RD2 were less driven by KO's scaffolding actions which were directly linked to structuring conversation talk. Similar to exploratory and cumulative talks, these two types of talks were focused on talking about people's actions in relation to MTFying discursive activity. What is interesting, according to Figure 12, there was no disputational talk recorded. This comes as a surprise, as I was expecting to see the presence of disputational talk since the RD2 was guided by the ground rules which relate to reflections of the individual teaching as seen in RD1. One reason for the absence of disputational talk could be that the teachers focused less on individual teacher's teaching activities but more on the mathematical activity.

As seen with the analysis of the previous two sessions, the talk during RD2 can be divided into single level (exploratory, cumulative), and bi-level (exploratory-cumulative, exploratory-managing understanding). My data did not offer me enough insight into the analysis of SC, since this type of talk was less privileged during this RD2 session.

8.2.4.1 Emerging theme: Uni-level talk

Exploratory talk

The comparative analysis amongst modes of talk categories reveals that when teachers engaged in exploratory talk they showed explicit and more focus on discursive routines related to MTF. I use Extract 63 below to exemplify.

Extract 63

An example of exploratory talk

Episode	Speaker	What is said
EP5	M	87. It's a very simple question. You see I once asked it when I was still a student. They are asking why does it have two **end arrows**.
		88. The graph is *continuous,* that's what you tell them. "The graph is *continuous,* this **arrow** shows it's *continuous*, but now, we have to bear in mind that we have a **restriction**, which is our **interval, negative 3 to positive 3**, that's why now we cannot draw a huge graph."
		89. That's why in our ***x*-axis,** and our ***y*-axis** there we put **arrows** to show them that, "You know what, these things are going to **infinity."**
	S	90. Oh, I think what the learner's question was, "We are seeing the graph is *decreasing,* yet this side is this *going up*, and this side it is *going down*," that was the learner's question.
	M	91. No, they said the **arrows.**
	S	92. Eh, those **arrows** why is it ja, and yet we are saying it's a *decreasing* **function**?
	M	93. But that one is a simple because you go then back to the table from the **negative 6 to 6.**
	J	94. So that means you will explain it using the **range.**
	M	95. The **range**.
	S	96. But they wanted ... [intervened]
	T	97. They are looking at the table they are seeing **the arrow,** it is like ... [intervened]
	S	98. They want to see d*ecreasing*, so according to the learner, they want to see something.
	M	99. No, I understand.

The extract above showed that the exploratory interaction was focused on construction of MTF narratives with respect to both teacher and learner explanations. For example, the conversation in Extract 63, started with M in lines 87-89 giving the teachers a mathematical account about the behaviour of the graph with respect to the notion of arrows where he used mathematical words (continuous, interval, x/y axis, infinity). These words were used to interpret the behaviour of the graphical representation. It seems M's intention was to substantiate the notion of arrows. In addition, the interactions further evidenced reflections about learner explanations. This can be seen in utterances by the teachers in lines 90-99 where S in lines 90 and 92 reflected on learners' concern about the behaviour of the graph, and how they were justifying their thinking. He provided evidence of how learners were focusing on interpreting the visual properties of the graph focusing specifically on mathematical words like "decreasing" and "increasing". This was followed by interactions in lines 93-97, where teachers M, J and T picked up on S's point (from lines 90 and 92). They extended the idea and focused rather on teacher explanation in response to learners' concerns or justifications where they commented about interpreting visual properties of a visual mediator—the table this time around. The use of mathematical words such as "range" was evident in their talk.

The interactions in this extract were mainly characterised by a typical form of exploratory talk where teachers jointly participated in constructing mathematical and MTF narratives related to learner and teacher explanations. The teachers not only presented their mathematical ideas but also elaborated on them. This finding further confirms that exploratory talk could be linked to explorative participation.

Cumulative

A fundamental characteristic of cumulative talk is the possibility to specify some contents through confirmations, rounds, and repetitions. The excerpt below concerns a typical sequence of cumulative talk:

Extract 64

An example of cumulative talk

Episode	Speaker	What is said
EP6	S	100. No, I can just use this part here. According to the learner they wanted that graph to start here and to show that it is *decreasing.*
	KO	101. It's *decreasing,* yes.
	S	102. Ja, for which this graph in real life it's from somewhere, even if the **restriction** here, but it's from *somewhere*, so that's why I opted to use **the area under the graph** for which in physics it would, it would work but for maths it becomes, ... [intervened]
	M	103. that's where you introduced the, the infinity, for this graph, you see you said it when it comes *from somewhere*, but you see ... [intervened]
	S	104. Ja, you see it comes from *somewhere*.
	M	105. And it goes there.
	S	106. Understand.
	M	107. But at this point we are **restricted** there
	S	108. Ja, so I think the only mathematical **explanation** here could be the **gradient.**

The discussion started with S giving an account of the learner interaction during the teaching session. This was followed by a number of interactions from other participants who were building positively on what S has said—supporting, in a way agreeing with teacher S's comments. For example, line 101 showed that KO was in agreement with S and she commented positively and uncritically. The interactions that followed in lines 102-108 were characterised by agreements.

A very important finding worth mentioning is that the teacher observers' reflections about S's teaching was marked by cumulative talk where the talk was characterised by agreement. Consider, for example, line 102 where teacher S was recounting decisions he made during teaching about the use of the real life example used in physics—area under the graph. This action was related to the reframing routine. Teacher S's reflections were followed by positive confirmations (lines 104-108). One may interpret this interaction as harmonious. Interetingly, the individual teacher's teaching reframing routine was linked to cumulative talk compared to RD1 where it was associated with disputational talk. Perhaps this could be due to the passive role played by the KO, who did not focus the group on any gaps in S's teaching or

reflections, thus opening up a platform for teacher evaluative talk. This kind of cumulative talk with respect to the reframing routine was witnessed throughout this stage of the L/S cycle. The relevance of this finding is the important role of the KO which was absent in order to move the talk towards more exploratory engagement (i.e. critically engaging with each other's ideas) as seen with LP1 and RD1. This finding has important implications in the nature of the emerging discursive learning, which will be discussed in the following chapter

8.2.4.2 Emerging theme: Bi-level talks (Hybrid of exploratory talk and other talks)

Managing understanding-exploratory talk

Another example which shows a form of mutual understanding talk developing towards exploratory talk can be illustrated in the following extract:

Extract 65

Example of managing understanding-exploratory talk

Episode	Speaker	What is said
EP3	KO	44. So now if I taught, what do you think was going to happen?
	M	45. Less misconceptions and in our observations and comments ... [intervened]
	KO	46. Wait, wait let me pick up that, and then the misconceptions?
	M	47. Also they are going to be less. It's like, it's not like I am undermining.
		48. Or everyone, but we are students, right, and we have been doing this thing for quite a long time. So you are highly experienced. 49. So sometimes that might, might have a limit for learners you see to, you know if something is just, you are going to spoon-feed them, you know, you are just going to get that pure knowledge.

In this extract, KO began by enquiring about her role as a teacher. In response, M gave a short response in line 45. In line 46, it seems KO was further pressing for an elaborated response which was followed by a form of exploratory talk from M in lines 47-49 where he gave evidence about KO's teaching strategy. It appeared to be pointing to a teacher-centred approach where he said "you are going to spoon-feed them". This utterance can be linked to the discursive activity of construction of pedagogical narrative related to general teaching. This finding could mean that teachers were afforded learning opportunities to talk about general teaching and not only mathematical teaching.

8.2.4.3 Summary

The analysis confirmed that some types of talk are associated with construction of narratives such as mathematical, pedagogical and MTF. For example, the analysis seemed to provide evidence that the exploratory talk supported the spirit of comradeship and joint construction of MTF narratives. It also confirmed the presence of semi-exploratory talk. The reframing routine was linked to cumulative talk, which created a harmonious environment different from RD1 where the climate was more disputational. The findings further confirmed the extent to which different categories of talk were useful and pertinent to support teachers as they jointly constructed narratives present in the RD2 discourse. Through cumulative and exploratory talks, data revealed a pervasive sense of mutual accountability among teachers with regard to operating in a spirit of comradeship and collegial relationships. When the talk was either cumulative or exploratory, the speakers collectively built positively on what others had said either uncritically or critically. Furthermore, the word "we" was used, which is the characteristic of collectiveness. However, the social interaction was more of a collaborative engagement, with a clear focus on construction of MTF narratives. This could be because the experience in a practical L/S environment for the fifth time influenced teachers positively in terms of trusting and being in solidarity with each other. This finding has an important implication for how collaborative interactions could support teachers' opportunities to participate in the MTF discourse and so opportunities or limitations to learn this new discourse.

Another important finding to be noted is that the social interaction during this session had no clear instances of problematising interactions as seen in RD1. Instead, the focus was on problematising the KO's role, and less of the individual teacher as seen with RD1. This was confirmed by the presence of "off-task subjectifying talk". Furthermore, it seems the teachers had developed enough confidence to be critical of the KO's role but not her MTFying or mathematising talk. The analysis has further confirmed that teachers viewed the KO as the expert whose withdrawal could constrain their learning opportunities.

Furthermore, the prioritising interactions were also not explicit. As this talk was due to the KO's scaffolding talk, the analysis suggested that the KO did not show rigorous scaffolding as seen during RD1 and LP1. For example, during RD1 there was heavy use of scaffolded questioning, and these were not present during RD2, except in the last episode, which showed that this routine was only used as a way of summarising or concluding the session, with no strong indication of foregrounding the MTF-use in terms

of MTFying discursive actions. The withdrawal of the KO from discursive activities was expected as per the ground rules of participating in the L/S where it was specified that for teachers' reflective discussions the KO would slowly withdraw her expert's input as suggested by Ronda (2013). This finding seems to suggest that the KO's substantial withdrawal during this session had some limitations to opportunities of learning. These emerging modes of interaction created the context in which discursive activities took place during the RD2 session. The analysis seems to provide evidence that these modes of interaction could have both supported and constrained the emerging learning opportunities.

8.2.5 Global theme shared repertoire of the L/S CoP

In this section, as in the two previous chapters, I present the characteristics of the RD2 emerging MTRD. The features of the RD2 MTRD are similar to those that emerged during the RD1 analysis in terms of the following: (1) interconnection between different discourses such as mathematical pedagogical and MTF; and (2) sources of narratives and objects of the discourse. The results of RD2 MTRD resonated with those of RD1 in terms of the goals of participating in the discursive activities: to bring to focus the CA of OoL. However, the RD2 MTRD differed extensively from RD1 in terms of the roles played by the participants, where the KO was playing a less dominant role. The defining feature of the roles in the RD2 discourse was both collaborative and off-task subjectifying.

8.3 Summary of what opportunities to learn were made possible

By participating in the MTRD, teachers were afforded different opportunities of learning from three different discourses: mathematics, MTF, pedagogical and social relations. These opportunities of learning were supported by collaborative engagements amongst each other. However, KO's passive role seemed to have constrained some of the learning opportunities. For example, the MTF routines were only focused on two components as point of focus: the CA of OoL and learner/teacher explanations, which implies ritualistic element of participation in MTF discourse.Thus, teachers had limited learning opportunities to develop the coherence between CA of OoL and all the other MTF objects. In Table 35, I summarise four themes that describe the possible opportunities of learning. I will discuss these themes in detail in the teacher learning chapter.

Table 35:

Summary of what opportunities to learn were made possible

Type of discourse	Opportunities to learn
Mathematics discourse	Connection between the mathematics content and the mathematical capability related to the mathematics object that the learners need to learn. Connection between word use and mathematical explanatory talk
MTF discourse	Coherence between the CA of OoL with learner/teacher explanations
Pedagogical discourse	Integrating other pedagogical strategies that are critical for the work of teaching—identifying CA in particular
Relations	Opportunities to learn from each other and constraints due to the KO.

8.4 Conclusion

The analysis of the RD2 discourse revealed similarities with RD1 in terms of MTF-use. For example, the analysis evidenced the expansion of the CA of OoL and the presence of subjectifying talk. The RD2 discursive talk was further supported by the semi-exploratory talk. However, this type of talk implied a different interaction function from the previous two sessions, that is, collaborative talk and no strong evidence of prioritizing, facilitating or problematising interactions. The specifics of how teachers participate in L/S sessions are part of what enables learning opportunities to arise, as will be discussed in Chapter 9. The analysis of RD2 marked the end of the L/S, and no lesson plan was produced.

Chapter 9: Nature of Teacher Learning

9.1 Introduction

The aim of this study was to understand how teachers used the MTF tool in the L/S reflective discourse and describe the nature of teacher learning that resulted when teachers used this tool. In order to attain this goal, I embarked on socio-cultural discourse analysis using different analytical lenses, where my first assignment was to understand the nature of the emerging discourse when using the MTF tool. The analysis in Chapters 6-8 presented my investigation of discursive activities during three L/S sessions—the LP1, RD1 and RD2. I also presented a visual summary of the overall reflective discourse emerging from these three sessions (see Appendix G). Table A, presented in this Appendix shows that the emerging MTRD in each session was characterised by similar attributes in terms of objects teachers were talking about, sources of their narratives and goals of their discursive activities. However, the analysis of the three sessions further revealed that these attributes served different purposes. For instance, although the object level goal was to expand the MTF tool in all three sessions, the expansion of the MTF components varied. For example, during LP1 the MTF tool was expanded to teacher and learner explanations. In RD1 and RD2 it was further expanded to reflect teacher and learner CA. At the same time, in terms of meta-level goals, the MTF tool was transformed into a discursive form, but was reified into different forms across all three sessions (see Figure A in Appendix G). What differentiated each reified form of MTF tool were the MTF routines that were foregrounded in each session. For instance, during LP1 all MTF routines were in focus. However, learner CA of OoL was not in focus, as seen during RD1 and RD2. At the same time, during RD1 and RD2, routines related to tasks were not present, and the exemplifying routines did not feature during RD2. Furthermore, the results of the analysis have revealed that the KO's semi-exploratory talk supported teachers' exploratory engagements which were linked to opportunities to learn the MTF practice.

There were also differences in terms of roles and ritual-explorative participation. Regarding the roles, the analysis has shown that all participants were participating as a collective, with some special roles being identified in each session. For instance, the KO played a leading role during LP1 through her scaffolding actions. However, during the two reflective discussion sessions there was a shift of roles where the teachers took the role of subjectifying. First they were talking about the individual teacher's

teaching using the MTF tool during RD1 (i.e., on-task subjectifying). Secondly, during RD2 they were talking about the KO's role (i.e., off-task subjectifying). Considering the ritual-explorative participation, in all sessions the participation was characterised by a mixture of ritual and explorative engagements. However, they differed in terms of the scaffolding goals of the KO. These differences seem to suggest that the MTF-use was situational. In addition, the analysis has revealed that in each session the MTF tool was the main leading discourse in focus. One important commonality between the sessions was that teachers' reflections focused on the learners as the heart of their reflections. Finally, the interactions across sessions were characterised by collaborative engagements. All these findings have implications for the nature of learning opportunities afforded in each session.

In this chapter, I extend my findings from previous analysis chapters and describe the nature of teacher learning that emerges from the MTF-use. To help analyse the nature of learning within the L/S CoP, I used CoP theory which views learning as a social practice. CoP theory provides three learning indicators—mutual engagement (ME), joint enterprise (JE) and shared repertoire (SR). Wenger (1998) noted that for learning in practice to occur, communities must be involved in the following three processes: "evolving forms of mutual engagement", "understanding and tuning their enterprise" and "developing their repertoire, styles, and discourses" (p. 95). These CoP theoretical constructs were supplemented by a number of theoretical constructs to help to explore the nature of preservice teacher learning. Firstly, I drew from Sfard's (2008) communicational theory which views learning as participating in a discourse. From this perspective, discursive actions are used to theorise learning, where learning is described through a notion of discursive learning—object and meta-level learning. Secondly, I used Mercer's (1996) modes of talk to explore how the talk (social interaction) supported learning.

In this chapter, I unpack the nature of L/S teacher learning in relation to the above mentioned theoretical aspects. To give the discussion in this chapter a structure and workable frame, I deal with each aspect separately. It should be noted, however, that the constructs related to the nature of teacher learning are closely intertwined. The discussion related to the nature of teacher learning is guided by the following themes:

What is the nature of teacher learning?

- Learning in relation to mutual engagement

- Learning in relation to the joint enterprise
- Learning in relation to the shared repertoire

However, first, I wish to discuss the nature of teacher learning from the situated learning perspective which grounded this study. One of the principles that underpin teacher learning from the situated perspective is learning as participation. I thus focus on learning as participating in L/S and highlight the notion of situated learning.

In this study, teacher learning was described through the notion of opportunities of learning. In Chapter 3 I explained the rationale for and meaning of the term "opportunities for learning" rather than "teacher learning". Opportunities for learning are understood to be certain situations or circumstances that enable certain discursive engagements (or practices).

9.2 Learning as participating in L/S situated learning

Chapters 6, 7 and 8 (three analysis chapters) illustrated how the MTF was used differently in different sessions in terms of shared ways of doing things (discursive actions) and mutual relations (modes of talking); negotiated responses; and shared discourses (the MTRD) as well as discursive artifacts (MTF discursive tool). Drawing from the Wenger's perspective, it can be argued that the L/S afforded teachers different learning opportunities from different L/S sessions. In support of this statement, Lenski and Caskey (2009) argued that from the LS perspective, learning is situated in specific activity and embedded within a particular situation. In this study, the learning was situated in different situations such as planning, teaching and reflective discussion. In terms of my conceptualisation of L/S learning, the process of learning can be seen as the negotiation and renegotiation of the MTF tool through the construction of MTF narratives by the teachers through their interaction and participation within the L/S setting. In associating learning with participation in a community of practice, Lave and Wenger (1991) asserted that "participation is always based on situated negotiation and renegotiation of meaning in the world" (p. 51).

Four important aspects had a bearing on situated learning which included ground rules, routine use and social relations. For instance, each L/S session was guided by specific ground rules. For example, in the lesson planning session the emphasis was on collaborative lesson planning informed by deliberate use of MTF tool components. However, in RD1 and RD2, the discussions were based on the individual teacher's reflections of the previous enacted lesson, followed by the reflections from the observers. These ground

rules were situational factors that had a bearing on the nature of interactions that took place in the L/S context. Furthermore, participation in the discursive activities through routine-use and social modes of interactions afforded teachers opportunities to learn different discursive practices in different L/S sessions.

This relationship between engagements in the activities of the community, the context of the community and social interactions were central to the study of situated learning. The researchers who have studied classroom interactions and situated learning, usually emphasise one or the other (Green & Dixon, 2007). This study explored the relationship between contextual interactions and situated learning simultaneously. It further illuminated how specific teaching practices (MTFying practices) together with actors of collaborations and L/S setting are useful and necessary for explaining the nature of teacher learning.

I now discuss the nature of learning that emerged in each L/S situations/sessions.

9.2.1 Lesson planning session—LP1

During LP1, teachers were collaboratively engaging in negotiating the meaning of lessons to be taught. They were participating in the process by which they came to understand and make sense of the lessons they anticipated implementing in the classroom context. They were also involved in collaborative planning, something they were new to as novice teachers. Thus, participation in lesson planning afforded teachers opportunities to learn to plan collaboratively. In normal teaching practice, teachers usually plan in isolation (Posthuma, 2012). Thus, collaboration itself is both a method and a learning opportunity for the teachers.

The analysis has shown that during LP1, the MTF structured the lesson planning process; teachers kept the MTF in focus when planning the lesson. Their intended focus was clearly articulated throughout the lesson planning process. For example, during L/S the object of learning was the effects of y-intercept and gradient in the linear function. This was kept in focus throughout, and the analysis has shown a tight connection between the OoL and other MTF components. This was depicted through the aspect of coherence (Adler, 2021). It was through this process that teachers were afforded opportunities to design mathematically coherent lesson plan by connecting what learners needed to learn through teaching practices of exemplifying, task generation, explanatory talks and learner participation.

The analysis has further shown how different kinds of talk supported the lesson planning process. These included semi-exploratory talks used by the KO to enable participation in the new MTF discourse, by foregrounding the points of focus and how the MTF tool may be used.

These findings corroborate prior research that suggests that collaborative lesson planning characterised by structured protocol guiding planning, talking, and intended focus provides teachers with a positive contribution to teacher learning. Horn (2010) found that lesson planning processes guided by structured protocols contributed to the opportunities to learn. Levine and Marcus (2010) also found that the intended focus of lesson planning influenced opportunities to learn. Horn (2005) described how different types of talk enabled teachers to learn as they engaged in collaborative lesson planning.

Teachers' participation was depicted through routine-use processes and their different ways of talk—relations with each other. Through this participation, teachers created a lesson plan—a reification object that was culturally mediated to an extent that it took on its distinct meaning. A routine-use process related to lesson plan was noted in the scenario where teachers diverted from the original MTF lesson plan template (Appendix C). In this regard, the lesson planning can be viewed as the negotiation of meaning of lessons to be taught—an aspect of learning (Wenger, 1998).

Findings from this study suggest that teachers who engage in collaborative lesson planning have locally defined practices that guide their work. By conceptualising a group of teachers engaged in collaborative lesson planning as a community that negotiates the meaning of upcoming lessons as they plan together, this study revealed how using a research-designed resource affords opportunities for teacher learning. According to Perry et al. (2003, p. 2), the collaboration pushes "teachers to come to joint understandings through sharing experiences". This relates to the common situative perspective of learning as "participation" (Sfard, 2008; Wenger, 1998).

9.2.2 Reflective discussion session: RD1-RD2

The nature of the reflective discussion session from the LS perspective is to open up opportunities for teachers to critically reflect on their teaching practice previously enacted lesson.

In this L/S, the RD sessions were characterised by features of critical reflection where teachers were involved in problematising practice. Problematising practice is related to critical enquiring on practice (Genor, 2005). It involved actions of looking back after a lesson: this entails thinking about the

experience, and making suggestions of how things could have been done differently and that could lead to refinement of the lesson plan. The problematising practice opened up opportunities for teachers to learn from each other, and they also had opportunities to learn by articulating their ideas, justifying their views and making valid arguments through semi-exploratory talk (Mercer, 1996).

When teachers were involved in post-lesson reflective discussions, they were afforded opportunities to identify problematic mathematical issues learners had with the OoL and then expanded on various ways of dealing with the intended focus of OoL. They were involved in negotiating and renegotiating the meaning of MTF, taking into consideration other pedagogical and contextual factors, thus providing an opportunity to make changes in their instruction and instructional materials—a refined MTF lesson plan. This process opened up opportunities for learning mathematical explanatory talk that incorporated learner thinking. Teachers' participation was guided by the MTF which kept their goals of teaching focused and monitored what was possible for them to learn. According to Genor (2005), when this process is guided by the explicit structured protocol, learning opportunities are optimised. These findings corroborate prior research that suggests that the act of problematising of practice offers teachers opportunities to learn through talk (Genor, 2005; Hickson 2011). In addition, the findings of this study offer credence to Rock's (2005) findings that problematising actions relates to the aspect of mutual accountability—the ability to negotiate actions as accountable to enterprise—that it aligns the engagement with the enterprise, an important aspect related to learning in the CoP (Wenger, 1998).

The L/S model has the potential to facilitate critical reflective practice, which promotes teacher learning. Research has reported that most preservice teachers normally reflect only on technical issues (Genor, 2005). In this study, however, the L/S focused on ways in which teachers problematised their actions, and then devised ways of modifying their instruction by connecting the mathematical content to the immediate learners' needs and what they believed their learners needed to know rather than focusing on the content only as done by others such as Chikamori et al. (2013) and Posthuma (2012).

9.3 Learning in relation to mutual engagement

With respect to mutual engagement, learning implies: "how to engage, what helps and what hinders; establishing who is who, who is good at what, who knows what, who is easy or hard to get along with" (Wenger, 1998, p. 95). From the analysis chapters it was established that the ME of the L/S was characterised by participation in discursive activities through modes of talk. Thus at the heart of ME are

the discursive activities and modes of talk. I now discuss the nature of learning that maybe realised by participating in these discursive activities and specific ways of talking.

9.3.1 Learning as participating in a discourse

When a group of individuals work together in a community, they engage in the "patterned collective doings" (Sfard, 2006, p. 157); in other words, they partake in discursive activities of that community. As noted, the MTF discourse emerged as a leading discourse throughout the three L/S sessions. It is thus the aim of this section to describe the type of learning that has been presented in discursive terms.

From communicational theory (CT) perspective, learning is a change in discourse moving towards specialised mathematical discourse. Recently, discursive learning has been defined as a process of routinisation of people's action at object-level and meta-level (Lavie et al., 2018). In this current study, the routinisation process was understood as the process of routine-use. The CT was used as a micro (local) theory within an L/S context to help identify changes in discourse, and this change defined learning on both the object level and meta-level. The object-level referred to MTF objects and the meta-level involved changes in the meta-level rules of the discourse. The object-level change "expresses itself in the expansion of the existing discourse attained through extending a vocabulary, constructing new routines, and producing new endorsed narratives; this learning, therefore, results in endogenous expansion of the discourse" (Sfard, 2008, p. 254). In addition, the change in meta-level rules "is usually related to exogenous change in discourse" (Ioannou, 2017, p. 2099). This change means that some different meta-rules related to teaching (e.g., pedagogical routines) were integrated during interactions with the MTF. It is important to note that from discursive learning (object-level and meta-level lenses) I was able to describe the learning as the change because these lenses enabled me to identify the changes at an object-level (MTF) and the meta-level routines that entered the discourse. However, it was through engaging in routine-use processes that opportunities for discursive learning were afforded.

9.3.1.1 Object level and meta-level discursive learning

In the L/S study, the learning situation seemed to promote both object-level and meta-level learning. The object-level learning was evidenced by the expansion of MTF objects through construction of new discursive MTF components. For example, the explanation and OoL components were divided into explanation for learners and teachers and CA for learners and teachers respectively. Secondly, the original component of learner participation was integrated into tasks, giving the MTF tool a new form,

reflecting: CA for learners, CA for teachers, examples; tasks; explanations for learners; explanations for teachers and learner participation. Each of these new MTF components gave teachers opportunities to learn different ways of dealing with MTF from different sessions. For instance, during LP1, the identified critical aspects were related to word use from colloquial and mathematical discourses (slope/gradient), while RD1 were from grade 10 mathematical and calculus discourses (increasing function/infinity).

The meta-level learning was established by integrating the MTF tool with different discourses such as mathematical, MTF and pedagogical discourses. For example, the analysis chapters have shown that the teachers constructed MTF narratives related to tasks by integrating questioning routines and mathematical routines. This process resulted in a new discourse—MTRD. According to Sfard (2015), an ability to develop new narratives implies discursive learning.

9.3.1.2 Basic conditions for discursive learning

From CT, the basic conditions for discursive learning include the communicational conflict and learning-teaching agreement. I wish to add another condition of discursive learning—the ritual/explorative participation. The analysis has evidenced that discursive learning was promoted by two basic conditions of meta-level learning: communicational conflict and learning-teaching agreement. The communicational conflict was present when teachers summoned rules from other discourses—mathematical rules, pedagogical rules and contextual rules. In fact, the analysis highlighted that some conflicts were particularly rich in opportunities for learning. For instance, when teachers summoned pedagogical routines related to identifying CA, the analysis has shown how this process afforded teachers with learning opportunities to analyse learners' previous knowledge in terms of identifying their misconceptions/difficulties. This further helped the teachers to identify what needed to be learnt (CA OoL). Furthermore, the CAs of OoL were connected to various MTF routines—allowing teachers to develop instructional coherence, where the set of examples, tasks, explanations were glued together by common mathematical routines which were identified as critical.

Regarding the learning-teaching agreement, the analysis highlighted the different roles played by the participants such as KO's scaffolding and teachers' on-task subjectifying talks. For example, the analysis has evidenced that KO's high scaffolding during LP1 gave teachers opportunities to participate in a new discourse of the MTF by constructing MTF narratives. Furthermore, her scaffolding role kept the MTF in focus, thus making MTF a leading discourse—a basic condition of learning-teaching agreement.

Regarding the subjectifying talk (on-task subjectifying) that was present during RD1, the analysis revealed that such a talk provided teachers with opportunities to construct subjectifying narratives that were linked to their practice (MTF and teaching); that is, they were problematising their practice. The problematising practice is an important instructional practice that is related to reflective practice. Heyd-Metzuyanim (2017) reported similar results but in the context of learners' mathematical learning, where she concluded that the interplay between mathematising and subjectifying talk supports explorative participation—an indication of learning. The notion of learning-teaching agreement can be linked to Wenger's (1998) aspect of joint enterprise which concerns the idea of working in mutually accountable ways which implies participants' roles and engagements that directly affect their contribution to the enterprise (MTF).

Considering ritual-explorative participation, the analysis has revealed that the mixture of ritual-explorative participation is necessary for object-level and meta-level discursive learning. The explorative participation was due to mathematical and pedagogical routines which entered the discourse of MTF across all L/S sessions and the established coherence between the CA of OoL and other MTF objects. The ritual engagement was due to the KO's heavy use of scaffolding routines during the LP1 session in particular. During this session, the KO's goal was to help teachers to gain access to explorative participation in the MTF discourse where she used different kinds of interactions such as MU and SC, which implied MTF discourse ritualistic nature. The KO needed to scaffold in order to maintain the ground rules of the CoP and also to help teachers in their initial steps of entering an MTF discourse. The analysis further showed that although the KO's heavy scaffolding was linked to ritual participation, the overall MTF discourse implied explorative participation. According to Nachlieli and Tabach (2018), ritual participation is essential at both object level and meta-level as it serves as a basis for exploration and helping teachers in their initial steps of entering a new discourse. The results of RD2 analysis revealed that the KO seemed to have significantly reduced her scaffolding roles, and this implied limited explorative participation in MTF discourse, where it was shown that the RD2 MTF discourse had ritualistic elements due to limited use of mathematical routines, and some of the MTF objects were not visible in teachers' talk, which implied absence of coherence between the CA of OoL and other MTF objects. This finding seems to suggest that teachers during RD2 had limited opportunities to learn MTF practice. At the same time, the results also revealed the integration of other discourses such as mathematical and pedagogical-an aspect of explorative participation in MTF discourse.

Interestingly, all L/S revealed a mixture of ritual-explorative participation with respect to MTF discourse but differed in terms of the KO's goals. This implies that routine-use was not uniform across all three sessions. I will elaborate on this in section 9.4.3.5. This finding corroborates earlier findings showing that the shift from ritual-explorative participation is not uniform (Heyd-Metzuyanim, 2017) and both pairs are necessary for the development of object- and meta-level learning (Nachlieli & Tabach, 2018). It further confirms the situative nature of L/S sessions; that is, routine-use process in each session may not reflect a shift from one session to another, each session is unique and distinct through its ground rules, modes of engagement, and social relations.

This finding aligns with Heyd-Metzuyanim et al. (2016), who reported a mixture of ritual-explorative from different levels and described this process as the tug of war between the goals of the instructor to offer students opportunities to learn mathematics and the goals of explorative participation in mathematics. This study has shown that discursive teacher learning in the context of L/S is not a linear view.

9.3.2 Learning as interacting in semi-exploratory talk

According to Wenger (1998), relationships are a critical aspect of engagement within a CoP. I now discuss the relations within L/S CoP that supported opportunities of learning.

The analysis of this study has shown and described the different types of talk that supported different learning opportunities for teachers—cumulative (CUMM), exploratory (EXPLO), disputational (DISPU), structuring conversation (SC) and mutual understanding (MU). Furthermore, it has evidenced how these different modes of talk seemed to have developed towards exploratory talk. The resulting talk was named semi-exploratory talk. I therefore argue that semi-exploratory talk amongst teachers and the KO also supported the process of construction of MTF narratives in particular. Specifically, learning is evidenced through the presence of exploratory talk (Mercer, 1996). Through exploratory talk, the analysis has shown how teachers jointly engaged in discursive activities of the L/S, thus constructing MTF narratives—an indicator of discursive learning.

The analysis has also revealed that teachers' interactions changed from LP1 to RD1 and to RD2. First, the interactions were mostly exploratory during LP1. This could be due to KO's goals to offer teachers opportunities to learn the MTFying practice in order to participate in the MTF discourse exploratively.

Second, some of the interactions during RD1 started off being disputational but later moved towards exploratory interactions. The disputational interaction was expected during this stage, as has been reported in Chikamori et al. (2013), Wits Connect project meetings, and in the workshop I attended on LS at the University of Pretoria in 2019 (as noted in my journal notes). These reports agree that teachers were critiquing each other during this stage. Another justification of disputational interaction can be explained through Wenger's (1998) ideas of collective engagement. Although the lesson was planned in collaboration with all members of the L/S group, according to the ground rules of teaching session, it was essential for one teacher to enact the lesson while others observed the lesson. During RD sessions, teachers engaged in the collective reflection about teaching that was done by the individual teacher. During the teaching session, it could seem like the teaching was not collective, resulting in the breakdown of collaborative interactions and collective engagement. This could explain why there were elements of disputational talk during RD1 session. Given these findings, Ramaligela and Essien (2013) suggested a notion of co-teaching, where a model is designed so that teachers share the teaching collectively, thus minimising disputational interactions. A noteworthy finding emerging from this study is that the disputational talk can be transformed to exploratory talk made possible by the presence of the KO, resulting in semi-exploratory talk. I therefore argue that the disputational talk was necessary during RD1 as it seemed to have offered teachers learning opportunities to reflect on their practice by problematising their practice (MTFying and teaching practices during RD1) through subjectifying talk. This finding resonates with Hunter and Anthony (2014), who investigated a group of students participating in a mathematical discourse, where they showed that an interplay between mathematising and subjectifying talks supported construction of mathematical narratives—an indicator of discursive learning.

Thirdly, the interactions during RD2 were characterised by cumulative and exploratory talks. The cumulative talk indicated the group's spirit of comradeship. This spirit that was fostered within L/S enabled participants to develop trust and mutual respect. Mercer (1996) also confirmed that cumulative talk can offer participants of the discourse an environment where they feel at ease to share ideas and develop understanding whilst working alongside their peers. He further added that, cumulative talk is necessary for the germination of exploratory talk, which promotes opportunities for learning. Finally, all L/S sessions reported elements of MU and SC. The analysis of all three sessions has shown how these developed towards exploratory talk (semi-exploratory). The semi-exploratory talk that was characterised by MU and SC talks were used by the KO during the LP1 session to initiate teachers into the new MTF

discourse through her scaffolding actions, thus affording the teachers learning opportunities to participate in MTFying practice.

9.4 Learning in relation to the joint enterprise

With respect to joint enterprise, learning implies: "aligning their engagement with it, and learning to become and hold each other accountable to it; struggling to define the enterprise and reconciling conflicting interpretations of what the enterprise is about" (Wenger, 1998 p. 95). In the theoretical review chapter, I noted that the process of joint enterprise entails negotiated response in relation to the enterprise involving relations of mutual accountability and taking into consideration external factors that supported the process of negotiating the enterprise. Thus, in order to present a discussion of how the process in which the L/S community "understanding and tuning their enterprise" (Wenger, 1998, pg. 95) implied learning opportunities, I will present my arguments guided by the following: (1) negotiated response in relation to the MTF tool; (2) negotiated accountability in relation to the MTF tool; and (3) factors that enabled the process of negotiated response.

9.4.1 Negotiated response in relation to the MTF tool

Joint enterprise revealed itself in negotiated responses around MTF components. For example, teachers were provided with MTF components such as OoL, examples, tasks, explanations and learner participation; they not only adopted them, but negotiated a response around them. The process of organising a negotiated response included the acts of construction of MTF narratives which were linked to various MTF routines. For instance, the explanations were presented as teacher and learner explaining routines; tasks were presented as generating tasks routines; the OoL was presented as teacher and learner CA routines; the learner participation was presented as routines of learner explaining and learner tasks. The process of construction of the MTF narratives was supported by integrating various routines from the mathematical discourse, mathematical teaching discourse and other aspects related to general teaching, lesson process and contextual routines.

This process of renegotiation was also supported by ways in which teachers related to each other. I discuss this aspect in the following section of mutual accountability.

9.4.2 Negotiated accountability in relation to MTF tool

For Wenger, "relations of accountability include what matters and what does not, . . . what to pay attention to and what to ignore" (Wenger, 1998, p. 81). Teachers' negotiated accountability to the enterprise manifested itself in three different ways.

First, the enterprise was renegotiated by the ground rules set by the KO through semi-exploratory talk which was characterised by SC and MU talks. These modes of talk were used by the KO during LP1 in order to scaffold the MTFying discursive activity, where she was prioritising MTF components (objects) over the other objects; consequently, offering learning opportunities of MTFying practice.

Secondly, the negotiation process was supported by participants' engagements in semi-exploratory talk characterised by cumulative modes of talk. The analysis revealed that these modes of talk were used mostly by the KO in order to create an environment of solidarity, and thus minimise a conflictual environment. This was mostly seen when there was presence of disputational talks during RD1 where the KO intervened in cumulative and exploratory ways in order to support teachers' collaborative engagement; thus, offering learning opportunities for collaborative practice.

Thirdly, the negotiation process was supported by participants' engagements in semi-exploratory talk characterised by disputational modes of talk during RD1. These modes of talk were used by the teachers as they were subjectifying the individual teacher's teaching practice, where they were deliberately questioning and critiquing the teacher's MTFying actions and teaching. In other words, they were problematising their practice. The analysis showed that this act of problematising led to a change where participants reviewed their practice and decided to change or not change their practice or actions. As earlier noted, the subjectifying talk afforded teachers learning opportunities for problematising practice which is linked to reflective practice.

In summing up, the aspect of negotiated accountability can be linked to the notion of learner-teaching agreement—a basic condition for discursive learning in section 9.3.1.2.

9.4.3 Factors that enabled the process of negotiated response

In this section, I discuss ways in which L/S supported the joint enterprise and so the opportunities for learning. These will be explained from the following perspectives from teacher education literature: (1) Forms of practice in the Learning Study (identifying CA of the OoL); (2) Forms of practices in LS

(collaborative, reflective, teaching practices); (3) Forms of relations with respect to the KO; and (4) L/S teaching resources—lesson plan.

9.4.3.1 Identifying CA of OoL

The identification of OoL is a practice related to learning study (Lo, 2012). The process of identification of CA of OoL provided teachers with learning opportunities for developing awareness of types of learning that learners need to learn. This study has shown that the CAs were foregrounded by connecting the identified CA with all MTF objects; illuminating the notion of coherence. The analysis has further revealed that the identification of CA of OoL was also situational. For example, the most prevalent CA of OoL identified during LP1 included classifying and interpreting routines. These were initially conjectured by the teachers. During the RD1 and RD2, new CA of OoL such as word use was identified. These were from both learners' and teachers' perspectives. This finding confirms Runesson's (2013) findings which categorised CA into different stages of the learning study: CA of intended OoL (during lesson planning), CA of enacted OoL (during teaching), and CA of lived OoL (post lesson discussion). According to Runesson (2013), the process of identifying critical aspects of OoL from different situations is an important indicator of teacher learning in the Learning Study context. This finding contributes to the literature about student thinking. The observations of all L/S sessions had multiple instances of talk about learners. As described in the analysis chapters, learners were at the heart of their collaborative work. This was reflected mostly in the process of identifying critical aspects. The talk about learner critical aspects implies learner thinking. The research has highlighted that an understanding of learners' mathematical thinking can be a catalyst for helping teachers to access opportunities of learning in teacher learning programmes (Brodie & Shalem, 2011). It is logical to assume that developing an understanding of learner's mathematical thinking is beneficial for teachers. The teachers in this study frequently discussed learners' thinking with respect to the OoL. This provided opportunities to learn about learner thinking which can influence decisions teachers make when working with learners.

The limitation of this study is on exploring whether the focus on a particular CA had an impact on student learning—this has implications for further research. Those who studied a notion of student thinking in the past, carried out post-tests to understand the impact of student learning resulting from strict attention to student thinking (Ronda, 2013; Runesson, 2013). This study, as mentioned earlier, did not conduct post-tests.

9.4.3.2 Collaborative practice

Collaborative practice is linked to the LS practice. Teachers were collaboratively involved in lesson planning. It has been shown in previous chapters how this collaboration has provided teachers with opportunities to learn. Through these collaborations teachers renegotiated the MTF components. Their mutual engagements helped this study demonstrate how the collaboration takes place. Research has reported that teachers who engage in collaborative lesson planning have locally defined practices guiding their work (Horn, 2010; Levine & Marcus, 2010). These practices included identification of CA, setting up examples, tasks and explanations. These were combined by the integration of mathematical and pedagogical routines. Through collaborative engagements teachers were afforded opportunities to collectively reflect on their engagements and also to learn from each other and the KO best ways of implementing the MTF tool in the L/S context, thus learning MTFying and problematising practice. By sharing what was done in one classroom, teachers had an opportunity to hear and learn beyond their own experience. In other words, practice was deprivatised.

This finding shows how collaborative L/S practice can serve as a form of preservice teacher learning programme, informing both the researchers and educators as they seek to support teacher learning through the act of collaboration in an L/S context. This research can help explain why and how teacher collaboration affords opportunities for teacher learning through different talks, discursive actions and protocol guiding the collaboration-MTF tool.

9.4.3.3 Affordances in relation to classroom teaching practice

The teaching practice involved the enactment of the collaboratively planned lesson. How classroom teaching practice contributed to participation and opportunities for learning was noted during two teaching sessions. For instance, in relation to the individual teacher who was enacting a lesson, the kind of learning that was afforded by this practice was evidenced during reflective discussions where teachers had an opportunity to problematise their practice which formed the basis for the learning and improvement of the lesson plans for future teaching practice. Teachers problematised their practice in terms of CA of OoL, strategies, and contextual factors (i.e., learners, physical context).

The enactment of a lesson links teachers to real practices and the observation of enactment of lessons provides teachers access to real-time teaching practices (Kazemi & Hubbard, 2008). From a CoP theory perspective, lesson enactments are noted as being effective for context-bound understanding and

enable learning experientially (Lave & Wenger, 1991). In addition to teaching practices, teachers who were observers had an opportunity to "see" the practice of the other teacher.

9.4.3.4 Reflective practice

The reflective discussions that followed the teaching and observation of a lesson became an intellectual activity for the teaching where teachers reflected through collaborative discussions by problematising their practice and exploring ways of improving the lesson plan, with particular reference to MTF practice (e.g. OoL, explaining, examples, and tasks); pedagogical strategies and other contextual factors. When teachers reflected on teaching sessions, they considered problems in their teaching and reflected on how those problems were related to learners' understanding of mathematical objects. Such a move afforded teachers learning opportunities to develop awareness of their teaching and how their own assumptions (e.g. conjecturing CA of OoL during LP) could influence their teaching.

9.4.3.5 Affordances in relation to KO

The social learning theory relates "experts" or the "KO" to the element of community (L/S) as CoPs consist of masters who are critical for ensuring learning and negotiation of meaning (Wenger, 1998). During L/S, teachers had an opportunity to engage with the KO. As has been shown in the analysis chapters, through structuring conversation, the KO pushed teachers to engage in a form of exploratory talk, which forms the basis for learning. Within participationist theory, experts are central in providing access to a community of practice (Lave & Wenger, 1991) and also in modelling the mathematical expertise and providing a leading discourse (Sfard, 2008). These contributions became the impetus for what opportunities for learning were present within the L/S CoP. The role of the KO has been discussed in detail in the previous chapters (through MU and SC modes of talk; and scaffolding actions).

In chapters 6-8, I analysed KO's scaffolding actions. The analysis of the three L/S sessions revealed three levels of KO scaffolding where LP1, RD1 and RD2 were characterised with high, moderate and low scaffolding respectively. These were explained in terms of KO's ritual-explorative goals where high scaffolding was due to KO's ritualistic-prioritising goals where she aimed to usher the teachers to a new explorative MTF discourse by prioritizing MTF goals—highlighting points of focus and to afford learning opportunities for teachers to participate in the explorative MTF discourse through MTFying discursive activities. The consequence was the use of explorative routines both by her and the teachers. This finding implies that the KO had learning opportunities to scaffold explorative routine-use, and

simultaneously the teachers were afforded opportunities to learn MTF objects and MTFying processes through explorative routine-use. The moderate scaffolding was due to KO's explorative facilitative goals where she was inviting teachers' participation in the MTFying discursive activities, and not that of bringing points of focus, and the consequence of that, the teachers' engagement showed signs of explorative routine-use. This finding seems to suggest that the KO had learning opportunities to facilitate explorative routine-use of MTF, and simultaneously the teachers were afforded opportunities to learn to participate in the MTFying activities through explorative routine-use. The low scaffolding was due to KO's explorative-facilitative goals, where the KO's goal was to withdraw her scaffolding to allow teachers to participate without guidance from her (teacher agency—explorative goal), and the consequence was teachers' limited explorative participation in the MTF discourse with limited routine-use which implied ritualistic MTF discourse. This finding seems to suggest that the KO's less scaffolding resulted in teachers' ritualistic routine-use with respect to MTF discourse, thus limiting their learning opportunities.

The results from KO's scaffolding show a mixture of ritual-explorative goals which have a bearing on the nature of teachers' participation in the MTF discourse. This finding resonates with other studies within CT research (e.g., Viirman & Nardi, 2017; Heyd-Metzuyanim et al., 2015) where these authors highlighted the role of the KO's scaffolding actions and its impact in participation of learners in the mathematising activity of the new mathematical discourse. They all demonstrated their findings through as a mixture of ritual-explorative pair in relation to KO scaffolding goals. For example, Heyd-Metzuyanim et al. (2015), reported that the instructor's ritualistic goals were in a tug of war with explorative mathematical goals, however, this tension resulted in learners' explorative participation in the new mathematical discourse. At the same time, Viirman & Nardi, 2017, reported that the KO's reduced scaffolding resulted in students' engagement in explorative routine-use.

In summary, it can be said that the KO's scaffolding goal was situational since it played a different function in each L/S session. Furthermore, through this discursive activity, teachers were afforded opportunities to learn how to participate in the MTF discourse (MTFying practice), and the KO also had learning opportunities to scaffold MTFying practice throughout the L/S. As such, KO's scaffolding was central to how the L/S made it possible for teachers to learn (opportunities to learn).

9.4.3.6 Teaching resources—lesson plan

Within the participationists' perspective, teaching artifacts are part of the community's shared resources. These are produced or adopted in the course of existence and become part of its practice (Wenger, 1998). In this study, the teachers foregrounded the use of L/S physical resources in their practice—MTF lesson plan. The lesson plan played a critical role in supporting participation and opportunities of learning. Firstly, during LP1, it served as a reified tool for the lesson. Secondly, during T1/T2 it served the role of protocol for enactment of a lesson. Thirdly, during RD1/RD2 it served as a resource for reflective discussions.

Wenger et al. (2002) also explained that communal resources such as "concrete objects" improve understanding of the community's knowledge, its practices and meanings (p. 38). Adler (2000) also argued that teaching resources or tools give teachers access to mathematics teaching practices. For Borko (2004), such tools are powerful in supporting/enabling teacher learning. Related to Wenger et al. (2002), these research findings indicate that the L/S physical resources gave the teachers opportunities to learn the MTF practices. When they were planning, they were involved in an inter-thinking process about teaching with MTF rather than writing down the traditional lesson plan. They were thinking about lessons, conjecturing learners' misconceptions and envisioning how they could implement those plans. During this collaborative planning they did not only think about content, they also considered other factors that are critical to produce a lesson plan in order to foreground what was made possible to learn by their learners and the processes to follow in order to achieve that goal. The revised lesson plan (from RD1) revealed and promoted student thinking through learner CA, an aspect which was absent in the first lesson (from LP1) since it carried only conjectured teacher CAs. The CA reflected in the revised plan reflected the confirmed CA, since they were tested during teaching. According to Lave and Wenger (1991), tools provide access to understanding and opportunities for participation in a practice of the community.

9.5 Learning in relation to the shared repertoire

Shared repertoire learning implies, "*Developing the repertoire, styles, and discourses:* renegotiating the meaning of various elements; producing or adopting tools, artifacts, representations; recording and recalling events; inventing new terms and redefining or abandoning old ones; telling and retelling stories; creating and breaking routines" (Wenger, 1998, p. 95). The shared repertoire of the L/S

community included both the reificative and participative aspects. The reificative aspect related to the creation of the MTF discursive tool. At the same time, the participative aspect related to the emerging discourse—the MTRD.

In this regard, the opportunities for learning were presented in terms of teachers' MTFying ways of MTF during different reflective actions of L/S. The changing ways of talking provided evidence of teacher learning in relation to their ability to experience the MTF tool as meaningful—a creation of the MTF discursive tool. The opportunities for learning were further achieved through participation in L/S reflective actions about MTF. This participation (that involved ways of talking) interacted with the process of routine-use.

9.5.1 The reificative MTF discursive tool

According to Binder (1996, p. 101) the reification process refers to processes by which the "community turns common understandings into meaningful artifacts". In this study, I argue that the MTF tool was reified through a process and objectification. For example, it was through a process of routine-use that the community transformed the MTF tool into meaningful artifact—the MTF discursive tool, in terms of what the community focused on. The routine-use process made it possible to manipulate the MTF components and combine them together. The L/S community through the process of negotiating their enterprise, the MTF tool became the discursive tool (Wenger, 1998). At the same time, it was the researcher who objectified the tool by illustrating the common understanding of the MTF discursive tool in a representational form (see Figure A in Appendix G). This finding is in line with Wenger's (1998) ideas, who suggested that the reification of a tool can either be a process or its resulting form (products). This finding highlights the different roles played by the participants of the CoP in the reification of the tool.

Furthermore, the KO's scaffolding actions facilitated the reification of the MTF tool, and helped the teachers to become active participants in the MTFying practice. The analysis further revealed that both the KO and teachers were involved in a co-learning environment where they were collaboratively engaging in the MTFying practice. Another important conclusion that can be taken from this study is that learning opportunities emerge out of collective and sharing, implying social participation. This places MTF in sharp contrast with those teaching resources known as concrete and being independent of human interaction (discourse) such as textbooks, and curriculum documents (Gueudet et al., 2013).

Furthermore, this finding is a contribution to the discursive lenses of analysing the work of teaching through its focus on the aspect of routine-use. Ball (2017) and Mosvold (2016) had flagged that scholars who study classrooms and analysing discourse, tasks, and interactions have failed to provide rigorous discursive lenses to unpack the work of teaching.

The routine-use process gave rise to an experience of meaning which is linked to Wenger's reification process: "the process of giving form to our experience by producing objects that congeal this experience into 'thingness'..." (Wenger,2008, p.58). This study has shown that the reification process from LP1 through RD2 was not similar or uniform (i.e., there was no clear form of change in pattern) from one situation to another. This confirms the principle of situative learning in which this study was grounded. According to Wenger, the reification process is understood as being situated within a particular context, shaped by the ground rules of that context.

9.5.2 The participative MTRD

The MTRD is a specialised L/S discourse that emerged as the result of processes of mutual engagement and joint enterprise. It presents shared ways of doing things (Wenger, 1998) that are unique to L/S CoP such as the following:

- Discursive activities that are used in the process of negotiation of meaning
- Forms of social relations
- Shared ways of communicating about the MTF tool
- Common discursive repertoire used within the L/S CoP.

There were three discursive activities that were used in the process of negotiation of meaning: MTFying, mathematising and teaching discursive activities. These were established in the process of meaning making—the routine-use process. There was interplay between the discursive activities and social modes of talking. As meaning making is also reshaping oneself, the analysis evidenced that teachers negotiated in active joint reflections with each other, the way they made sense not only of what was represented by the MTF tool but also how they experienced MTF through the discursive mechanisms (routine-use and talk) that supported that moment of social practice. In this process, teachers created their own routines (*"socially negotiated discursive routines"*) and their shared understanding deeply situated in the L/S context. The socially negotiated discursive routines reflected shared ways of communicating about the MTF tool. These routines included various MTF routines such as exemplifying,

generating tasks, teacher and learner explaining routines, teacher and learner CA routines. The MTF component of learner participation was commonly understood to be learner tasks and explaining routines.

The common shared discursive repertoire used in the L/S community included three intertwined discourses: mathematical, MTF and pedagogical discourses. These were characterised by a pool of routines that were used to shape a joint understanding of the L/S community's shared enterprise—the MTF discourse. Consequently, the interaction of these three discourses resulted in a shared discourse named MTRD.

To sum up, the findings in this section highlight a dialectical relationship between resources and teachers' practice. As Brodie (2020, p. 39) stated,

> Tools created for collaboration can become resources from collaboration, i.e. an outcome of collaboration, if they change their meaning and use through the collaboration, i.e. the tools support teachers to see elements of their practice in new ways, and teachers come to see the tools and resources as helping them to work differently.

It seems reasonable to conclude that participation in the L/S discourse with the MTF tool afforded teachers opportunities to reshape the MTF tool (into MTF discursive tool) and through this process they were afforded opportunities to modify their instruction, and so their learning.

9.6 Conclusion

The findings from this study offer credence to Wenger's perspective that a CoP engages in the negotiation of meaning through a process of participation and reification. In this chapter I have shown how learning from dimensions of CoP is intertwined. One cannot talk about learning from ME without linking it to the JE. Similarly, the analysis has also revealed that the SR is also interlinked to both the ME and JE. Furthermore, the discussions from this chapter have revealed that teacher learning can be described from two perspectives: the MTF (through discursive learning) and the teachers (the opportunities for teacher learning). This study not only supports Wenger's perspective, but has implications for future research on teacher collaboration, collaborative contexts and collaborative resources. In the following chapter, I discuss what these findings mean to the broader mathematics education field.

Chapter 10: Conclusion

10.1 Introduction

This study took into account all interacting factors present in the collaborative teaching environment: mathematics, mathematics teaching, general teaching practices, and contextual factors in order to provide a comprehensive answer to this research question:

- How can mathematics teacher learning be understood through participation in a form of a L/S CoP—in relation to use of the research-designed tool—MTF?

In so doing, it addressed the research-practice gap. It further highlighted the role of the actors of the collaboration as well as how they may be or become key resources for collaborative learning.

I used ten analysis questions based on the three dimensions of CoP by Wenger (1998) to unpack how teachers used the research-designed resource, MTF in lessoning study. The empirical findings around the use of MTF and the nature of teacher learning have been discussed in detail in the previous chapters. What can be highlighted is that the empirical findings are consistent with those of Kieran et al. (2012), who viewed teacher learning from research-designed resources from the idea of a dialectical relationship "involving both the teacher's shaping of the resource and her teaching practice being shaped by it" (p. 190). This current study has shown that when teachers participate in an L/S collaborative-reflective discourse using a research-designed tool—the MTF tool—the tool gets transformed and using this tool enables learning opportunities to arise.

In this chapter, after a brief summary of the study, I present a discussion on a number of issues: methodological and theoretical contributions; reflective discussion about the strengths and weaknesses in relation to methodological tools, insider/outsider roles, the MTFying discourse and the L/S CoP. The synthesis of the research findings related to the main research question of this study has been discussed in the previous chapter and therefore will not be repeated in this chapter.

10.2 Summary of the study

This study examined the teacher use of MTF through discursive lenses. The setting for the study was a L/S CoP comprised of four pre-service teachers and the researcher. A lessoning study was conducted

focusing on the OoL related to the effects of the gradient and the y-intercept of the linear function. The L/S comprised two cycles: LP1, T1, RD1-LP2 and T2, RD2. During the first week, pre-L/S, a workshop was conducted to introduce teachers to the L/S process and MTF tool. Data analysis included a transcription of three sessions of the L/S —the LP1, RD1 and RD2. After a close reading of the entire data set, the L/S teachers' reflective discourses (including LP1, RD1 and RD2) were chosen to undergo a high resolution analysis using socio-cultural discourse analysis (SCDA) (Mercer, 1996). The SCDA is concerned with two aspects: modes of talk evident in teachers' talk; kinds of processes of routines being discussed (e.g., discursive goals—routines, MTF) and the nature of the routine-use process. Taken together, the two aspects offered me insights into micro-level interactions, thus, offering microscopic lenses to view collaborative interactions with a resource designed for collaborations in a collaborative environment. Overall, the findings were presented qualitatively using thick descriptions.

10.3 Theoretical contribution

This study has made two main theoretical contributions and these are largely located in the analytical tools used in this study. These contributions are: (1) the extension of the notion of MDI in the preservice L/S CoP; and (2) extension of three dimensions of CoP to include the discursive language.

In their research, Adler and Ronda (2015) explored the MDI of teachers who participated in a professional development project by providing a description of mathematics made available to learn in a lesson. Building on this research, this study has shown that the MTF tool was negotiated and renegotiated to produce a new shared resource called the MTF discursive tool. In addition, this current study has extended the MDI in the context of preservice teachers' L/S CoP, and produced a specialised discourse called the MTRD.

The MTRD characterises teacher learning with the MTF tool as intertwined discourses combining MTF, mathematics and teaching discourses. Together these explain discursive learning emerging from the use of a tool together with particular patterns of collaborative interactions between teachers and the KO in the context of L/S. Through participation in the discursive activities of the MTRD, teachers negotiated the meanings of MTF objects as they collaboratively planned and reflected on their lessons. They negotiated the meaning of the MTF by designing mathematically coherent lessons by connecting what learners needed to learn through teaching practices of exemplifying, task generation, explanatory talks and learner participation. It is through this process that opportunities to learn emerged. The MTRD has

similarities with Adler and Ronda's (2015) MDI which characterises the teaching of a mathematics lesson as a sequence of examples together with tasks and accompanying explanations towards a particular goal—OoL—occurring within interactions in the classroom. The MTRD extends the idea within MDI by considering some of the interacting factors found in the classroom. Furthermore, this study contributes to MDI research by integrating an analytical tool that explains the nature of these interactions in terms of the interplay between mathematising and MTFying discursive activities in relation to the MTF. MDI has been critiqued for paying little attention to discursive acts of teaching (Mosvold, 2016). MTRD has further contributed to MDI research by providing sharp (microscopic) lenses on dealing with the important aspect of MDI—the CA. It explained the discursive processes that illuminate the notion of connection and coherence between different MTF components.

In this research, I also worked with Wenger's (1998) dimensions of CoP namely: mutual engagement; joint enterprise and shared repertoire. Wenger (1998) contended that these three dimensions of CoP are interrelated and interlocked into a tight system and are also mutually defining. This interrelationship between the dimensions of CoP became evident in the course of the analysis. This study has confirmed that mutual engagement and joint enterprise fed into each other and shaped the shared repertoire dimension of CoP in my study.

During the analysis it became evident that I could not analyse the mutual engagement and joint enterprise constructs separately. For deeper analysis, these two constructs were analysed simultaneously. Such a decision enabled me to provide a richer description of the emerging shared repertoire. Together these three dimensions enabled me to explain the opportunities for teacher learning. Of particular significance is that within the analytical language, a methodological contribution of this study is on highlighting that the negotiation of the joint enterprise (i.e., MTF) relied on discursive processes that were privileged in the community. In this CoP, teachers engaged in the L/S discursive activities which when viewed together formed a continuous process that supported their collective engagement throughout the study. From this perspective, Wenger's CoP theory presented some limitations. To deal with this challenge I needed to draw socio-cultural lenses into Wenger's theory to develop a methodological approach that captured the three dimensions of CoP in the mathematics preservice CoP which was investigated.

The theoretical contribution the study has made lies in the extension of Wenger's CoP theory to include discursive processes. According to Sfard (2008), when novices are faced with a new discourse they

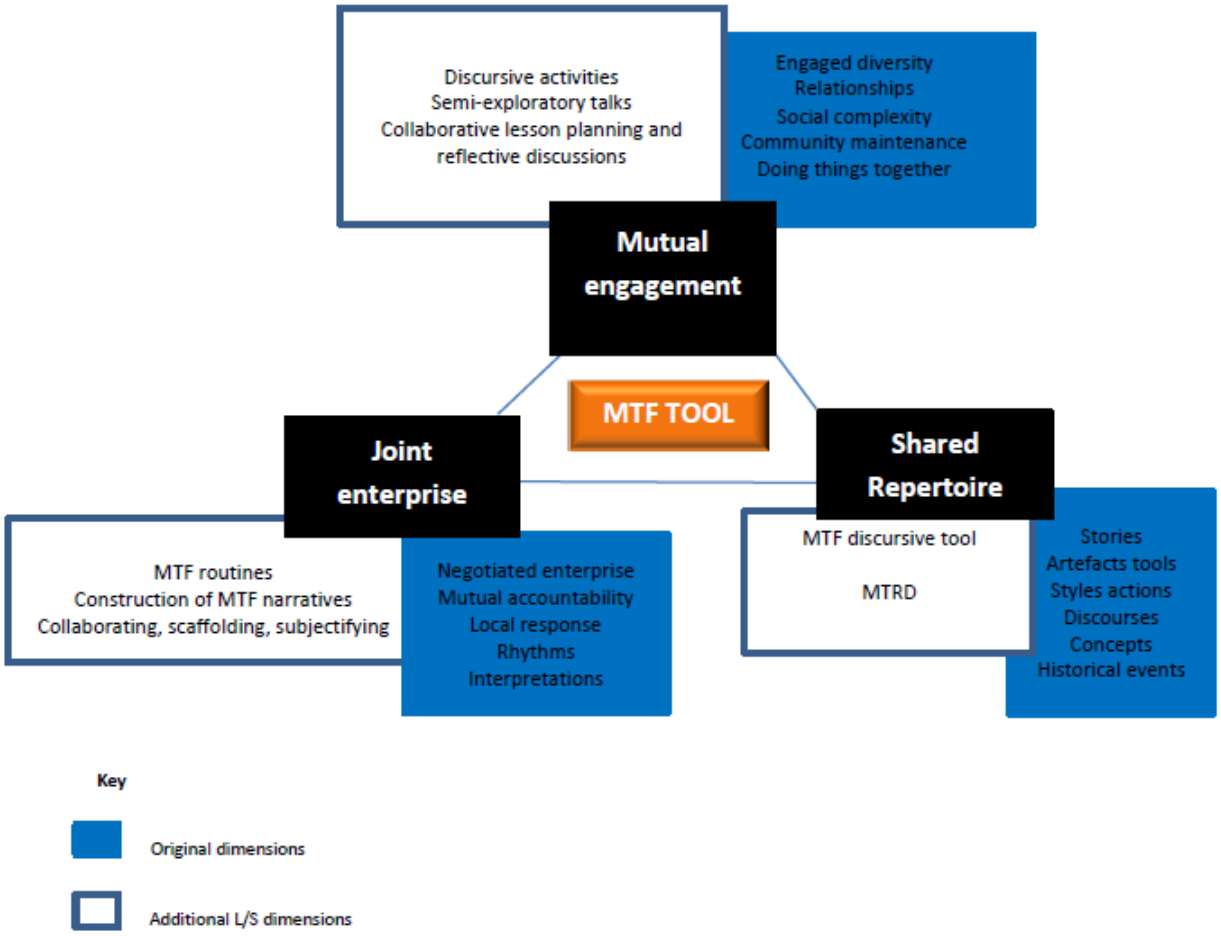
negotiate ways to manage the new discourse, to move from peripheral positions into active participants of that discourse. How do teachers organise their reflective discourse so that they can become more active participants in the MTF discourse—a new discourse? To answer this question, I analysed the teachers' reflective discourse that developed, by combining two analytical lenses: Mercer's (1996) mode of talk and Sfard's (2008) routine-use constructs.

The combination of Sfard (2008), Mercer (1996) and Wenger (1998) has enabled me to develop a new perspective with which to investigate L/S learning. Such a combination is a practical and coherent way of analysing the L/S reflective discourse to study learning processes, specifically of a collaborative tool. Thus, it provides a conceptual framework by which various aspects of L/S reflective discourse could be observed, identified and considered. As can be seen, it allowed one to focus on the nature of the discourses within the L/S CoP by illuminating what happens when teachers collaborate with the MTF tool, what happens to the MTF tool, how the collaborating actors interact with each other in relation to the MTF tool, and what kinds of learning opportunities arise. Its importance lies in the underlying assumptions that in order to improve our understanding of mathematics teacher learning, one should focus on processes that take place in a collaborative context (Prediger, 2020). Such a methodological approach also contributes to the enduring research-practice gap.

In conclusion, in developing Wenger's (1998) dimensions of CoP for an L/S collaborative environment, the above facets need consideration. The findings from this study offer the potential to develop Wenger's (1998) framework, expanding it for use in an L/S environment. Aspects of the social learning model and three essential dimensions of CoP include those factors requiring consideration when applying the framework in L/S contexts. The development of augmentation of the framework provided by Wenger (1998, p. 78) is shown in the figure below:

Figure 13

Augmented theoretical framework



Note. Adapted from Wenger, 1998, p. 78

The augmented framework presents the additional facets required of the CoP framework when applied to the L/S environment. The model includes the three main components arranged in a structure that adapts Wenger's (1998) original presentation with mutual engagement, joint enterprise and shared repertoire being positioned with key facets related to each other outlined in the squares. The model has been augmented to include the additional facets required within the L/S CoP. Mutual engagement includes shared ways of doing things: discursive activities, semi-exploratory talk, collaborative lesson planning and reflective discussions. Through this engagement, teachers negotiated the meaning of the MTF tool by constructing MTF narratives related to various MTF routines that emerged. Their negotiated response to the enterprise was supported by negotiating ways which involved acts of collaborating with each other, scaffolding by the KO and subjectifying by teachers who were observing the teaching/lesson.

The shared repertoire presents the MTF discursive tool and emergence of shared discourse—the MTRD. It is through members' engagement with one another in the shared discursive activities of the L/S CoP and their active participation in the continuous process of negotiation of the MTF tool that teachers produced the MTF discursive tool. In addition, the shared discourse, MTRD, emerged.

10.4 What does the research contribute to literature?

Literature on research in mathematics teacher education suggests a slow pace of research related to the relationship between resources for collaborations and teacher learning in collaborative contexts (e.g., Prediger,2020). These in turn inform research on the ongoing research-practice gap. Most of the research done thus far in the field of mathematics education research related to lesson and learning studies are studies in which the researchers study lesson/learning cycles that were implemented in schools by the in-service teachers. The lessoning study in this thesis was conducted in the context of preservice teacher education. This study contributes to the field by showing that the lessoning study may be used as the future teaching experiment programme for preservice teachers. The study demonstrates, and so confirms, that participating in collaborative engagements in the L/S context (in the form of a LS model guided by a teaching tool which was research-designed as a resource for collaboration) afforded various teachers learning opportunities. Furthermore, this study makes a contribution to the adaptation of LS in preservice teacher education.

The contributions of this study address the following: (1) What happens during the L/S interaction? (ii) How can researchers strengthen their understanding about learning within inherently social settings (e.g., L/S)? The L/S CoP model of social learning focuses specifically on the interaction between research designed resource, MTF, teaching practice and social interactions. Learning from this CoP is viewed primarily as a process of social interaction and negotiation of meaning. It is based on participation, and built from shared practice rather than being transferred in a linear fashion from experts to student teachers. The L/S model used in this study is a way of exploring the social learning process amongst preservice teachers, and is also a teaching practice tool that may interest teacher educators concerned with improving teaching practice programmes.

Previous research on teacher learning in the LS context has focused on what teachers learn from these engagements (e.g., Dudley, 2013; Lewis et al., 2006). However, there seems to be a lack of theoretical/analytical frameworks on how teachers learn in the LS context (Ding & Jones, 2019). This

study contributes to this strand of research by using Wenger's (1998) social learning theory as a theoretical framework complemented by other analytical tools for investigating how teachers learn in L/S and in the preservice context.

Another important contribution pertaining to teacher collaborations is a response to literature (e.g., Krainer and Spreitzer, 2020). These authors made a call for future research to elaborate more on collaborative interactions of various participants in mathematics education collaborative environments. This study is a contribution to research on teacher collaborations by illuminating the roles of the participants in collaborations in a number of ways. First, the analysis has shown the collaborative interactions between the teachers and the researcher who were engaging as co-learners and how the researcher who was also the KO supported the learning opportunities for the teachers through her scaffolding actions. Secondly, it highlighted the roles of these two in the reification of the resource for collaboration, where it noted that the MTF tool was reified through a process of routine-use by the community (teachers and the researcher); at the same time, it was objectified by the researcher who provided the representational form of the tool from collaborations.

Another theme that emerged was that teachers shaped the MTF by integrating pedagogical and contextual factors. Such factors are contributing to a research on understanding complexities of teacher learning as noted by Borko (2004), that research in teacher education needs to pay attention to developing lenses that draw out effects of social setting/factors to teacher learning.

This study has further made a contribution to a growing body of research on using communicational theory in the context of preservice teaching. Furthermore, this study contributes to the calls from research to pay attention to the interplay between academic discourse (e.g., mathematical, mathematical teaching) and other social factors (e.g., Hunter and Anthony, 2014), where it has shown interplay between discursive activities and social modes of talk.

As noted earlier, the purpose of the current study was to explore how teachers used a research-designed tool and what type of learning can be realised from such a use. The L/S CoP was a model where the MTF tool was in focus for teachers to integrate in their participation. Consequently, the thesis of my research is stated as follows:

When a new tool originates from a research context which codifies a type of teaching practice, the following can be realised: the tool gets reshaped; teacher learning can be described through various

types of learning opportunities. Such a dialectical relationship is organised in the MTRD. The MTRD is a type of discourse governed by certain goals—object level and meta-level goals. From this perspective, learning MTF requires taking part in the MTF discourse which is characterised by socially negotiated discursive routines. The learning from this perspective is described from different viewpoints: discursive learning and learning as participating in the L/S CoP. It is conceptualised in terms of: (i) a coherent connection between what learners need to learn—CA of OoL, exemplification, explanatory talk and learner participation and (ii) integration of different routines (mathematical and pedagogical) to construct MTF narratives.

10.5 Reflecting on methodological tools

As already noted, this study had recruited theoretical resources from Sfard (2008), Mercer (1996) and Wenger (1998). I have already discussed in detail the strengths and weaknesses of Wenger's (1998) theoretical framework and how I have dealt with these issues in Chapter 3.

Mercer's (1996) typologies of talk were productive by providing this study with social interactional analytical tools. The nature of the talk that emerged within the L/S community had provided clues as to why the MTFying activity had explorative elements—it was in the way the participants had collaboratively interacted with each other through talk (semi-exploratory talk), which seemed to have supported learning opportunities. The downside of these tools is the lack of methodological microscopic lenses for zooming in on an interaction characterised by multiple modes of talk. Further research is needed to strengthen these methodological tools.

The strength of Sfard's (2008) framework helped to unpack the "routine-use" by breaking it into procedures and the performer of those procedures. The focus on the procedure-performer aspect was helpful as it provided me with a clear description of MTF objects from two perspectives, that of learners and that of teachers, something missing in Adler and Ronda's (2015) MDI work. This focus on routine-use enabled the identification of a number of MTF routines which contributed to the expansion of MTF objects: examples, tasks, teacher-explanations, learner-explanations, teacher-CA of OoL, learner-CA of OoL and learner-participation.

What was gained from using this framework is the notion of discursive learning at a routine-use level, which enabled me to talk about the discursive MTF tool and how it was transformed from the codified

MTF tool. This transformation was highlighted in terms of the expansion of MTF routines and the integration of other routines such as pedagogical and mathematical routines.

The focus on ritual/explorative ideas helped me to discern the KO's discursive goals and so her conflicting roles that were experienced during the L/S, that of a teacher and expert, and how these interacted with the MTF objects in the discourse. This allowed me to illuminate the KO's role in supporting the teachers as they engaged in the collaborative-reflective discourse and so their learning opportunities.

While using Sfard's theory, I needed to deal with Sfard's notion of the ritual/explorative discourse. Originally this notion was discussed in the context of mathematical discourse. In recent years, there has been a growing number of studies in the context of pedagogical discourse drawing from Sfard. These studies have been attempting to provide various conceptualisations and/or operationalisations of what it means to participate in the ritual/explorative pedagogical discourse. However, it does not seem that the communicational theory researchers have come to a common understanding of this aspect. This current study has experienced a tension between the notion of ritual/explorative discourse with respect to the content (or objects of the discourse) and the ritual/explorative discourse with respect to pedagogical discourse. For example, the analysis of LP1 discourse revealed that the MTF discourse was explorative; however, it had ritualistic elements due to the KO's scaffolding actions. In this instance the discourse at an object level of MTF had features of explorative discourse; however, from a pedagogical perspective, the KO's scaffolding actions pulled it towards ritualistic participation. A question to ask is when does one categorise a pedagogical discourse as either explorative or ritualistic? This begs for more work to be done in conceptualisation of the notion of ritual/explorative pedagogical discourse.

10.6 Reflecting on insider/outsider roles

The challenge of this study from the data collection stage right through the analysis stage has been the multiple roles I had played: firstly, that of knowledgeable other within the lessoning study CoP; secondly, that of a researcher; thirdly, that of facilitating lessoning study processes; and fourthly, that of a co-learner whereby I also assumed a similar role to that of teachers. This created insider-outsider tension (Labaree, 2002). It became more acute during the analysis stage to separate the researcher-self and the participant-self to become an outsider in order to give interpretations. I practiced the principle of reflexivity to help me overcome these tensions. I further noted, in Chapter 4, that my researcher's

and CoP participant's (KO and teacher) roles created tension during the data collection stage and how I dealt with it. This tension was more exacerbated throughout the analysis as sometimes it became difficult to distinguish my role from that of a teacher and the KO. However, with my choice of discursive analytical tools such Sfard (2008) and Mercer (1996), I was able to distinguish these two roles.

Playing the role of a co-learner (i.e. teacher), was productive for me as it gave me an opportunity to learn how to use the MTF tool and participate in the L/S, as previously noted, that I was also a novice to this practice. It further allowed me to view teaching from a teacher's perspective. On the other hand, the KO's role was also productive in the L/S context. The analysis of the three L/S sessions has also illuminated the important role played by the KO. The first two sessions (LP1 and RD1) revealed that the KO's scaffolding (ritualistic goal) had a positive contribution in initiating teachers to participate in the explorative MTF discourse, where they integrated other routines such as pedagogical and mathematical to construct MTF narratives and further establish the connection between the MTF objects. However, the RD2 analysis evidenced less scaffolding from the KO, where I saw both absence of other MTF objects and the connection between MTF objects in teachers' discourse. This finding was linked to limited explorative MTF discourse. Hence, these results suggest that it is necessary to have the KO's scaffolding role throughout the first stages of the L/S, and then perhaps for it to be withdrawn as the teachers gain more skills and knowledge to become independent learners. This further suggests that teachers need to spend more time in L/S and the premature withdrawal of the KO role is discouraged.

Although to the best of my ability I have attempted to ease the tension that was caused by the KO's conflicting roles, for future research I would suggest that the schoolteacher take up the role of the KO and the facilitator in the L/S context instead of the researcher in order to minimise the insider-outsider tension. Since this study has highlighted the roles of the KO and the facilitator as essential components for the implementation of lessoning study when using a research-designed tool, it is critical that these individuals have some background of mathematics research knowledge and lesson/learning study facilitation knowledge in order to support the functioning of the L/S CoP and teacher learning.

10.7 Reflecting on the MTFying discourse

Returning to the three analysis chapters of this thesis and the discussion of the MTF discourse, this study has revealed that the teachers indeed MTFyed—constructed MTF narratives related to MTF objects.

Furthermore, the teachers' discourse showed a strong connection between MTF objects with CA of OoL—an important aspect of MTF learning.

I am aware that the analysis did not get any deeper in giving account of the nature of MTF narratives in terms of: (1) content and discursive change, and (2) quality— whether or not they were productive. The MTF analysis was at an abstract level—general. This was the result of the methodology. I used three theoretical resources together because I wanted to get the interactions of what is the negotiated enterprise and what is the shared repertoire. I wanted to do this discursively, but I also wanted to look at the social interaction and the consequence of that through something like a lesson study, which is a complex phenomenon which made the results move towards processes of the community rather than the products. In this research I was interested in how teachers were using the MTF tool in an L/S context. Hence I analysed the discursive practices—routine-use. Due to the focus and scope of my research, I did not conduct a detailed analysis of the MTF objects and highlight areas of problems and possibilities for improvement (good/bad) to investigate whether and how the chosen MTF objects enable or constrain students' opportunities to learn.

The analysis showed that the MTF tool was transformed by the L/S community into a discursive tool. However, it was my interpretation of what reified tool was, and what can be inferred as teacher learning in terms of learning opportunities that were afforded. Perhaps future research could look at how teachers talk about the codified MTF tool at the beginning of the study, and how they talk about the new transformed discursive tool—reified tool—at the end. In other words, how do they deliberately talk about the reified tool? In retrospect, this study could also examine teacher learning through the MDI framework and explore teachers' discourse in the L/S setting when using the MTF tool. One could ask the following analytical questions:

- What is the nature of MTF narratives?
    - Are tasks aligned with principles of cognitively demanding instruction as prescribed in the MDI analytical framework?
    - Are examples aligned with principles of variation theory as prescribed in the MDI analytical framework?
    - Is the explanatory talk in terms of MDI?

- Are CA of OoL examined enough to establish how they are connected to other MTF objects?

- What are the changes in teachers' MTF discourse through L/S sessions?
- And, what are the changes in teachers' capability of implementing the MTF tool through L/S sessions in foregrounding what learners need to learn? In other words, how do they reify the MTF tool in terms of MDI framework?

The scope of the study turned out to be much larger than had initially been anticipated. When I consider all MTF components: OoL, examples, tasks, explanations and learner participation, I regard CA of OoL as all important, integrating all the other components. In this study, CA of OoL provided significant information about all the other MTF components in terms of what learners were expected to learn and what mathematical instructional objects could be designed for these, that is, examples, tasks and explanations. Moreover, a careful analysis of how teachers identify and use CA of OoL throughout the L/S cycle could have shed light on how the CA of OoL has evolved from lesson planning, teaching and reflective discussion stages. This could have allowed one to examine the change in discourse and talk about what teachers have learnt instead of opportunities to learn as done by Adler and Alshwaikh (2019) where they focused on how examples have evolved throughout the LS sessions. The study could show the relationship between CA of OoL and other MTF objects and human participants, and further open up the opportunity to examine the nature of the mathematics that was provided by the teachers for their learners—by conducting pre-tests and post-tests. Therefore, if any other MTF object is considered when investigating teachers' MTF discourse, I recommend "CA of OoL" to be the main object of investigation. Thus, focusing on one MTF object could enable one to track down a change in teachers' discourse from lesson planning, teaching and reflective discussion stages.

10.8 Reflecting on the L/S CoP

I only did two cycles of L/S. For a community to learn this new practice of doing L/S and using a resource is not a spontaneous practice, and for teachers to grow and learn, it is a long-term goal. This practice (LS) in China and Japan has been in existence for over a century, it is a cultural activity. It is a complicated process masquerading as an easy-to-describe sequence of lesson planning, conducting a lesson and reviewing lessons. The way I structured this study with all the choice of theoretical tools, led the results that gave me a bigger picture of how complex and complicated this practice is. Learning this

practice is not trivial, not for the teachers, and not for the KO, particularly if the KO has conflicting roles and this study focused on the first steps in this practice. This study contributes to the field by showing that the L/S can be used to illuminate the interaction between teachers, the researcher and the collaborative resource. It also reveals that there are opportunities for teachers and researchers to learn together in the L/S CoP and how their social interaction may support their participation in the CoP.

The limitation of this study pertains to the absence of the expert teacher from the school. As seen through this study, the lessoning study model needed a school context in order to enact the lessons. This has implications for the role of the teacher from the school. The role of the expert teacher is very critical in preservice TE programmes as they act as the mentors who understand the context, including learners, better than the outsiders (i.e., preservice teachers and the researcher). These expert teachers hold valuable information, specifically of the learners in terms of their knowledge and previous knowledge, which is helpful in the process of identification of CA of OoL—an important aspect of the MTF discourse. Hence, this study recommends collaboration with expert schoolteachers for future L/S programmes.

10.9 Final remarks

Finally, it is evident that teacher collaborations are dynamic and complex and involve many other interacting factors that are present in the teaching and learning settings. From a research perspective, this affirms high demands for the use of unconventional and multi-directional research lenses in order to elaborately capture the complexity that lies at the heart of this relationship between the social interactions and practices. As such, it affords an exploration of multiple units of analysis such as the individuals, the community context, and the social interactions. This study, to its best of ability, attempted to use multiple frameworks. The question that has continued to permeate is, however, "when does one draw a line on which aspects of the units of analysis to focus on or ignore without doing injustice to the complex nature of teacher collaborations and teacher learning?" Hence, the focus here is in my view only a partial contribution in this regard.

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Appendices

Appendix A: Ethics Protocol number—2016ECE013D

Wits School of Education



27 St Andrews Road, Parktown, Johannesburg, 2193 Private Bag 3, Wits 2050, South Africa. Tel: +27 11 717-3064 Fax: +27 11 717-3100 E-mail: enquiries@educ.wits.ac.za Website: www.wits.ac.za

26 July 2016

Student Number: 9300068M

Protocol Number: 2016ECE013D

Dear Lizeka Gcasamba

Application for Ethics Clearance: Doctor of Philosophy

Thank you very much for your ethics application. The Ethics Committee in Education of the Faculty of Humanities, acting on behalf of the Senate, has considered your application for ethics clearance for your proposal entitled:

Investigating mathematics teacher learning using a research-designed resource in a lessoning study

The committee recently met and I am pleased to inform you that **clearance was granted.**

Please use the above protocol number in all correspondence to the relevant research parties (schools, parents, learners etc.) and include it in your research report or project on the title page.

The Protocol Number above should be submitted to the Graduate Studies in Education Committee upon submission of your final research report.

All the best with your research project.

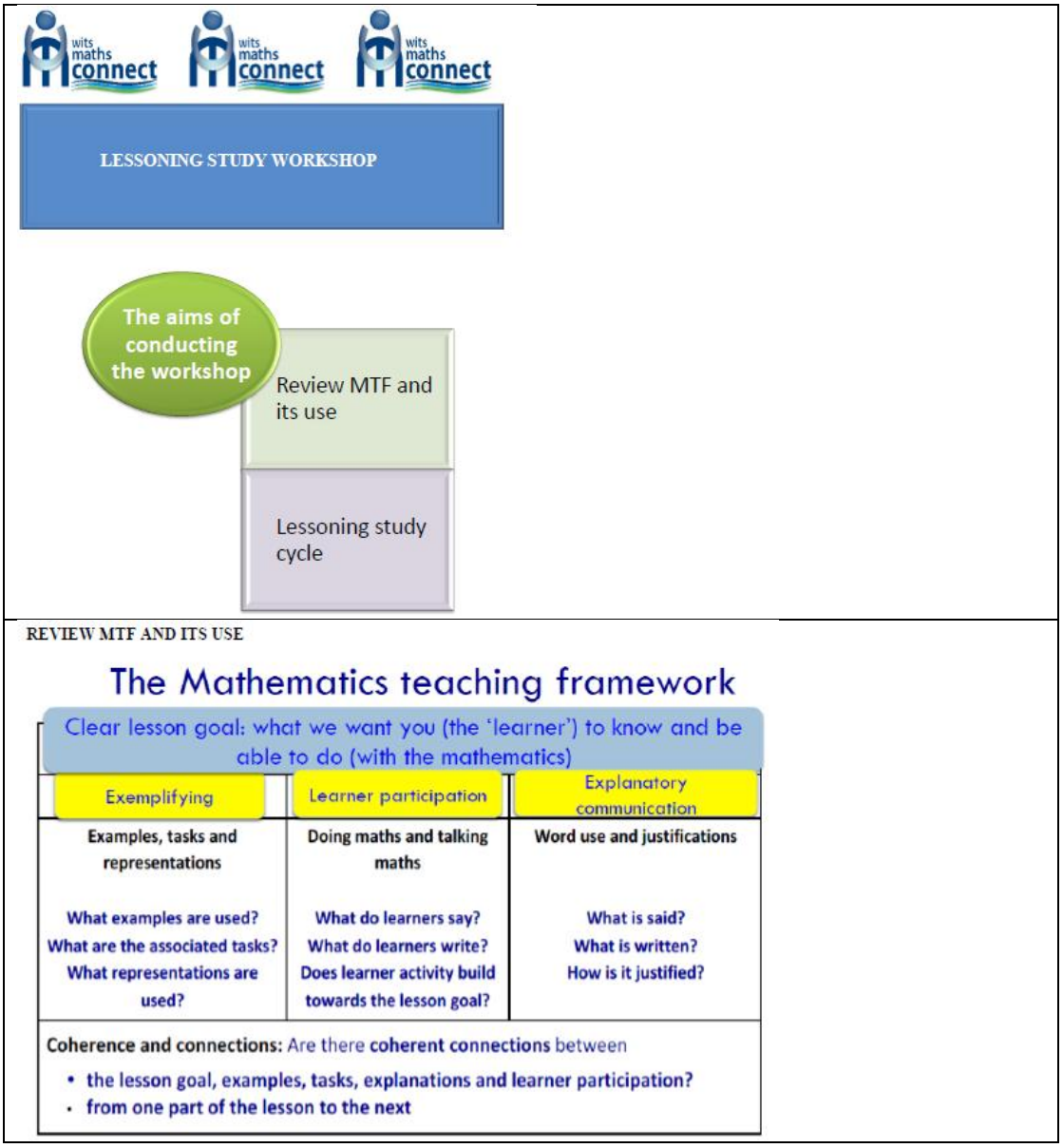
Yours sincerely,

Wits School of Education

011 717-3416

cc Supervisor - Prof. Jill Adler

Appendix B: L/S workshop

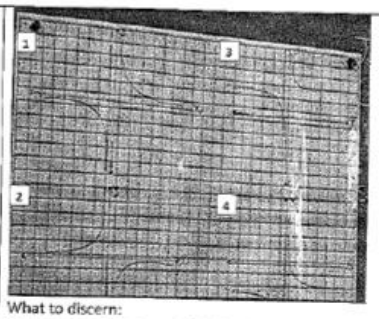


Mediation of the Object of Learning – Teacher's MDI

Analytic Framework

Examples

Teachers use examples as 'representatives' of the math objects and procedures that are made available to learn during a lesson. Taking from the point of view of variation theory that sees learning as discernment of the different features or aspects of what is being learned, the opportunities for learning offered by examples can be described in terms of the patterns of variation in the examples.

Levels	Criteria/Indicators	Examples
1	Examples provide opportunities for learners to experience *separation or contrast* within the lesson. *A set of examples provides opportunity to experience* ***separation*** *when the feature that needs to be discerned is invariant and the other aspects varies.* (L1-S) *A set of examples provides opportunity to experience* ***contrast*** *when it shows example and non-example of the object of learning for learners to compare.* (L1-C)	What to discern: The effect of the sign of a and the value of q in y=a/x+q Examples: 1. $y=\frac{2}{x}$ 2. $y=\frac{-2}{x}$ 3. $y=\frac{2}{x}+3$ 4. $y=\frac{2}{x}-3$ Separation: Invariant: All are equations of hyperbola Variation in sign of a = 2, -2, and sign of q = 3, -3 OR Structure of hyperbola function #3 was provided to contrast hyperbola and linear function 1. $y=\frac{4}{x}+5$ 2. $y=2-\frac{4}{x}$ 3. $y=\frac{x}{2}+3$ **Code: L1-S**
2	Examples provide opportunities for learners to experience at least any two of separation, fusion or contrast within the lesson *A set of examples provide opportunity to experience* ***fusion*** *when simultaneous discernment of critical aspects of the math object is possible.* (L3-F)	 What to discern: Properties and features of the function that varies and those that remains invariant
3	Examples provide opportunities for learners to experience generalization and fusion within the lesson. *A set of examples provide opportunity to experience* ***generalization*** *when there are other examples in different contexts so that learners can discern the invariants. This is an expanded form of separation.* (L3-G)	What to discern: Properties and feature of a hyperbola Variety of examples for hyperbola function in equation and graphical form presented in the lesson 1. $y=\frac{2}{x}$ 2. $y=\frac{-2}{x}$ 3. $y=\frac{2}{x}+3$ 4. $y=\frac{2}{x}-3$ 5. $y=\frac{-2}{x}+3$ 6. $y=3-\frac{2}{x}$ 7. $y=\frac{4}{x}+5$ 8. $y=2-\frac{4}{x}$

Tasks – (action on the examples)

Tasks describe how the learners are expected to engage with the examples. It can be categorized in terms of the opportunities for learners to use mathematics and make connections among mathematical objects and procedures.

Levels	Criteria/Indicators	Examples
1	Focus on operation and procedures: solve equation, simplify, graph by plotting points, translate, factor, find the derivative, etc.	Plot the graph of the following: *y=2/x; y=-2/x; y=2/x+3; y=2/x-3*
2	• Tasks where learners will have to make decision on what mathematical operation or procedure they should use. • Involves application of level 1 skills for e.g., arrange, classify, solve (one-step problem), compare/match representations; set-up equation; explain solution (procedure)	**T :** The pack that I gave you... you match the graphs, there are six equations, inside that pack Ls : Oh... T :You get the six functions, and there are also six graphs Ls :Ok... T: Make sure that you have six functions one up to six... Ls: inaudible... T :Ok now I want you to match the function together, the function goes together with a graph... inaudible ...
3	• Tasks where learners will have to make decision on what mathematical operation or procedure they should use. • Involves multiple concepts and more connections for e.g., solve problems in different ways; express in multiple representations; model; solve multi-step problems; pose/construct problems; generate example from given condition; compare/contrast 3 or more features/properties; prove/disprove; explain reasoning	T: *"Ok, first equation, number one ... number one the graph is y equals two over x and I want us to compare this graph with number three ...and how are we going to compare them? I want you to tell me* ***what are the things that are changing and what are the things that stay the*** *same between those two graphs."*

Talk/Naming

Talk refers to how learners and teachers refer to the math object, operations, procedures, etc.

Levels	Examples
Level 1 *Colloquial language* including ambiguous referents such as this, that, thing ... to refer to the objects	
Level 2 *Some mathematics language* to name individual objects, components or simply read string of symbols when explaining	T: *So that means that we have an extra* ***graph that does not have a function...*** *Ls: Yes...* *T: So can you* ***write the function for that*** *graph?* T: *"****Ok, first equation, number one ... number one the graph is y equals two over x and I want us to compare this graph with number three*** *...and how are we going to compare them? I want you to tell me what are the things that are changing and what are* ***the things*** *that stay the same between those two graphs."* (incomplete naming)
Level 3 *Uses appropriate names* of mathematics objects and procedures	

Substantiations/Legitimation – describes the type of substantiation or legitimation the teacher or the learner use in their explanation.

Levels	Examples
Level 1 NM: Visual Cues; use mnemonics; metaphor (relates to features of real objects	
Level 1M: Assertions/statements typically from teacher	
Level 2M: Use of specific/single case/local (real-life or purely mathematical context) OR Use of established shortcuts; procedural rules or conventions	
Level 3M (General but Partial) T/L attempts to justify using equivalent representations, definitions, procedures, previously established generalization but explanation is not clear or incomplete	
Level 4M: General, complete Clear and logical explanation using definitions, equivalent representations, procedures, theorems, axioms, properties	T: ...inaudible... another change is the symmetry. **The one symmetry is y is equals to x, the one with the zero asymptote. The symmetry will be y is equals to x...inaudible...** T: mm...mmmh Ls: And then for the second one it will be **y is equals to x plus three because the asymptote is three**

Interactional Pattern – describes the opportunities for learners to participate in the learning activity.

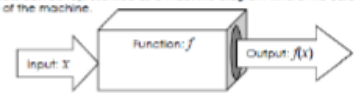
Teacher-directed: Teacher directs the thinking and analysis of the task

Types	Criteria/indicators	Examples
Level 1 Learners answer *yes/no questions or offer single words* to teachers unfinished sentence	A. Teacher-directed: lectures, answers own question, learners listen B. Teacher-directed: Learners answer yes/no questions only or supplied words to teachers unfinished sentence	
Level 2 Learners answer (what/ how) questions in phrases/sentences	C. Teacher directed: learners answer questions in words and sentences; learners are not given opportunity to explain	T: a over x plus q. So the general form is y equals to a over x plus q. So in all these equations what has been the value of, ... equation one two, one three and four, the value of a was? Ls: Two T: And it is a positive L: Yes... T: Now can you see that, can anyone tell us what happens if the value of a is positive Ls: ..inaudible... it changes the quadrant where it appears...inaudible... ya T: It changes the quadrant or the position where it appears like we discussed here like if it is a negative then it will appear where you have the x and the y that is neg... that is positive and negative... L: Negative... T: But if it is a positive therefore it will appear where we will have the x and the y that are positive and negative. ... Ok so the a determines the position where the graph will... Ls: ..inaudible... intersect T: It determines the position of the graph ... L: Appears... where it will appear
Level 3	D. Teacher use learners ideas in the discussion; teacher	27:12 [First part and last part qualify for Code D (level 3)]
Learners answer why questions; present ideas in/for discussion; teacher revoices/ confirms/ asks questions	revoices/ confirms/ asks questions based on learners answers;	L: Ma'am ... ma'am ... ma'am in the first one the horizontal asymptote is x equals to zero, no the vertical asymptote is x equals to zero Ls: Yes... T: Ok, now she says there is a vertical asymptote and it is x equals to... Ls: Zero... T: Zero... do you agree? Ls: Yes... T: So... ... if our x equals to zero we are talking about our y axis so we are saying that the y axis is also an asymptote, vertical asymptote Ls: Asymptote ... T: What about this one? Does it have a vertical asymptote Ls: Yes ...Inaudible... it's equal to zero as well T: So that means that the vertical asymptote is ...staying .. Ls: ...inaudible... the same T: So vertical asymptote is the same ...inaudible... T: ...inaudible... another change is the symmetry. The one symmetry is y is equals to x, the one with the zero asymptote. The symmetry will be y is equals to x...inaudible... T: mm...mmmha Ls: And then for the second one it will be y is equals to x plus three because the asymptote is three T: Good... that is very good... that is very good, can you clap hands?

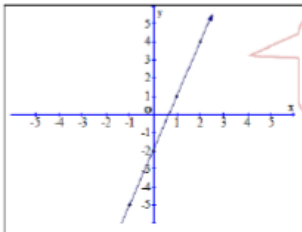
Grade 10 Mathematics Lesson Plans (*adapted from Gauteng Department of Education*)

GRADE	10	SUBJECT	MATHEMATICS	WEEK	1	TOPIC	Function –TIME: 60 MIN	Lesson	1

LESSON SUMMARY FOR:	DATE STARTED:		DATE COMPLETED:	
LESSON OBJECTIVES	The concept of a function, where a certain quantity (output value) uniquely depends on another quantity (input value). Work with relationships between variables using tables, graphs, words and formulae. Convert flexibly between these representations, with focus on the following concepts: • The concepts of functional relationships • Use of table to obtain a set of ordered pairs. • Draw a straight line graph with the use of a table.			

TEACHER ACTIVITIES	LEARNER ACTIVITIES	TIMING	RESOURCES NEEDED
1. Teaching methods: Explanation, Demonstration and Questions. 2. Lesson Development 2.1 Pre-knowledge required for this lesson. Explain to the learners the concepts of functional relationships • A function is a rule that takes an input (x value) and returns an output f(x). A function assigns exactly one output to each input. Functions can be given names like f or g etc. For example $f(x)=2x$ or $g(x)=5x$ .When a name is not needed the form $y=2x$ or $y=5x$ is used. • Mathematically, a function is a set of ordered pairs (x; y) in which no two ordered pairs has the same x-coordinate. • The process can be represented as a machine diagram where the calculator can serve the purpose of the machine.  Copy the baseline assessment on the board. Let learners verbally respond then show how you obtain the answer. Encourage learners to use brackets. 1.1 $f(2)=3(2)=6$ 1.2 $f(-3)=3(-3)=-9$ Explain the meaning of $f(2)$ and $f(-3)$ as follows: What is the value of y when x = 2 and -3	Learners must copy the notes in their workbooks. 1. Baseline assessment: If $f(x)=3x$ Evaluate the following: 1.1 $f(2)$ 1.2 $f(-3)$	Pre-knowledge and baseline: 15 min	• Text book, study guide and question papers from which selected exercises were taken. • Scientific calculator • If OHP is available, prepare graph (axis) before the time.

2.2 Main Body (Lesson presentation) The table method. Show the learners how to produce a table of values. Let $y=2x+1$ x: -2, -1, 0, 1, 2 y: -3, -1, 1, 3, 5 $2(-2)+1=-3$; $2(-1)+1=-1$; $2(0)+1=1$; $2(1)+1=3$; $2(2)+1=5$ In each case we obtained a set of ordered pairs. The ordered pairs are: $(-2;-3)(-1;-1)(0;1)(1;3)(2;5)$ Learners may use calculators to verify the above calculations calculation. Copy the Learner Activity 2 Question 2.1 on the board. Explain that $x\in\{-2;-1;0;1;2\}$ means the domain and represents the x values. Allow learners 5 minutes to solve the questions, and then show the solution on the board. 2.1 x: -2, -1, 0, 1, 2 y: -8, -5, -2, 1, 4 The ordered pairs are: $(-2;-8)(-1;-5)(0;-2)(1;1)(2;4)$ Emphasize that in each case the output value depend on the input value, and that no ordered pair has the same coordinates. Show learners how to plot the ordered pairs on the Cartesian plane and connect the points to obtain a straight line.	2. Learner activity. Simplify: 2.1 Draw up a table and determine the set of ordered pairs defined by the function: $y=3x-2;$ $x\in\{-2;-1;0;1;2\}$ 2.2 If $y=-2x+1$, determine the ordered pairs if $\{x:-3\le x\le 2; x\in R\}$ and sketch the graph.	Main Lesson presentation and learner activity 2 30 min	



Highlight the following:
All the points must be on the straight line.
Need only two points to draw the straight line.

Learners must solve the questions in their workbooks. Let them copy the solution from the board and make correction and notes in their workbooks.

Copy the Learner Activity 2 Question 2.2 on the board. Explain that $\{x : -3 \leq x \leq 2; x \in R\}$ means the domain (input values): all values where x is equal and bigger than -3 but smaller and equal to 2. Allow learners 5 minutes to solve the questions, and then show the solution on the board.

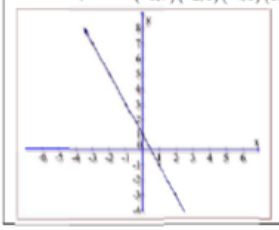
3. Class/Homework
On one system of axes, with the aid of a table, draw the graphs of:
1.3 $f(x) = 2x + 3$
1.4 $g(x) = 3x$
1.5 $h(x) = -2x - 2$

Class/ homework: 10 min.

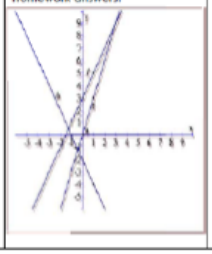
2.2

x	-3	-2	-1	0	1	2
y	7	5	3	1	-1	-3

The ordered pairs are: $(-3;7)\,(-2;5)\,(-1;3)\,(0;1)\,(1;-1)\,(2;-3)$



Homework answers:



Let the learners discuss the relationship between the function and the graph.
They must note that:
The straight line is a picture that represents the function.
The line shows what happens to the value of y if you apply the value of x to the function (rule).

Select similar questions from textbook.
Answers are in back of text book.

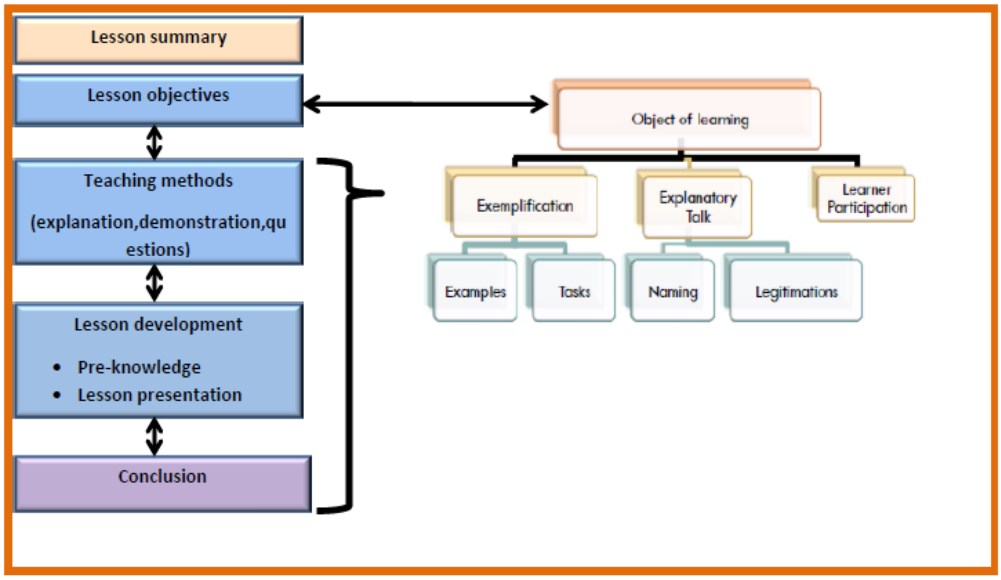
Conclusion 5 min.

If time allow copy the homework questions on the board one at a time. Give time limit and show learners how to solve the questions. Learners must correct their own work, ask questions and indicate problem areas. Continue until the end of the period, give remaining questions for homework. Select additional questions from exercise in text book and past question papers.

2.3 Conclusion

- Refer to chalkboard summary, transparency etc. and summarise the lesson. It may be necessary to restate the important concepts.
- Give learners informal assessment as homework (scaffolding towards formal assessment, where applicable.) to answer based on the presented lesson. See suggested homework in learner activity column.
- Learners must do homework in their workbooks for future reference.
- Learners must evaluate their own homework; the answers are at the back of the text book. They must redo the question if they do not get the right answer. Learners must show both their first and second attempts when they approach the teacher for help.

Lesson Plan and MTF



Week 2 : Lesson 1

Date: Topic:	School: Teacher:	Grade: No in class:
Object of learning		

Examples Identify all examples in the lesson. How are examples sequenced? Note what the learners have to do with the examples, i.e. what is the task for learners to do?	Explanations Do explanations focus on how and/or what? To attention given to why in explanations? What representations are used? Teacher and learner talk – note what maths language and what everyday language is used to talk about maths ideas	Learner activity and comment What are learners doing? Note particularly what learners have difficulty with and how this is noticed

LESSONING STUDY

Participants

- A group of at least four teachers, and myself included, will join to form into a lessoning study group.
- Three lessoning study cycles will be conducted
- Figure below illustrates the three cycles. The three teachers are labeled A, B, C. In all three cycles each teacher will be given an opportunity to teach and re-teach.

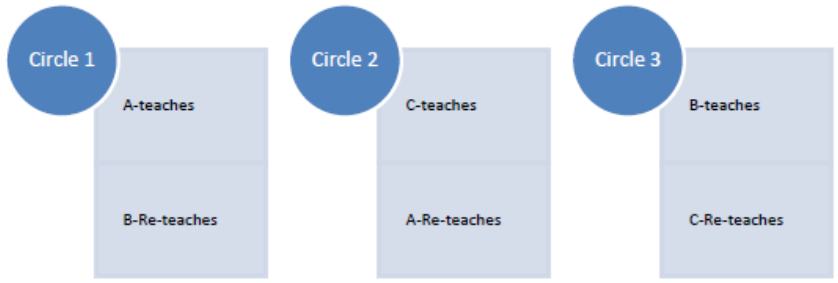
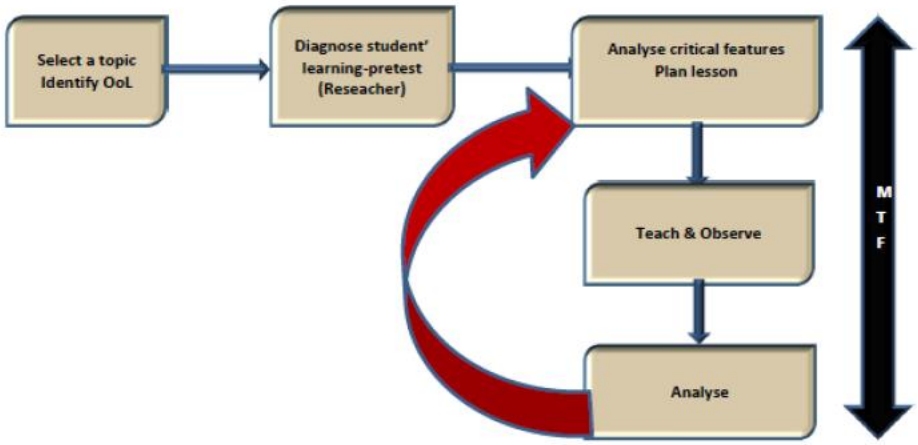


Figure: Example of three cycles

- The three teachers will use mathematics as the subject context. The teachers will choose the most suitable topic within grade 9-11 function topic.

Lesson study cycle



i. Plan lesson

In the planning session we will be guided by a Mathematics Teaching Framework (MTF). The MTF lesson plan tool will be used as guide for choosing examples, preparing explanations, and learner participation.

- Discuss learner difficulties from previous experience and from the pre-task
- Decide on OoL
- Identify critical features
- Plan lesson together using MTF tool
- Choose our first presenter
- Rationale for choice of examples/tasks

In selecting the examples we need to look for examples that would address learner misconceptions and learner difficulties (critical features) with respect to the OoL. Also, learners pre-knowledge about the OoL will be put into consideration in the process of selecting and choosing examples

ii. Teach and observe

- One teacher teaches
- Observers use observation schedule

iii. Analyze

1. Post-lesson discussion

   Immediately after the lesson, all those who observed the lesson spend some time in a short debrief of the lesson. The debriefing will start by allowing first the teacher(s) who taught the lessons to comment on their reactions to what happened, followed by the team who designed the lesson.

   - Debriefing-the presenter

   This part describes what you learned from implementing your lesson—i.e. what worked, what did not work, issues it raised for you about your goals (lesson goals, content goals, and lesson study goals), etc.

   - Group reflections
     - Guided by lesson plan(MTF)-are there any diversions

2. Revising

   - Any ideas to include in the lesson plan
   - Edit lesson plan
   - Choose the 2nd presenter, go through teach/observe and analyze sessions of the cycle.

Appendix C: MTF original lesson plan template

MTF lesson plan from WMCS project (2015)

Date: **Topic:**	**School:** **Teacher:**	**Grade:** **No in class:**
Object of learning		

Examples Identify all examples in the lesson. How are examples sequenced? Note what the learners have to do with the examples, i.e. what is the task for learners to do?	**Explanations** Do explanations focus on *how* and/or *what*? Is attention given to *why* in explanations? What representations are used? Teacher and learner talk – note what maths language and what everyday language is used to talk about maths ideas	**Learner activity and comment** What are learners doing? Note particularly what learners have difficulty with and how this is noticed.

Appendix D: Framework matrix

Framework matrix : RD1 Table 28_A

CATEGORIES	SUB-CATEGORIES		CODES	EP1	EP2	EP3	EP4	EP5	EP6	EP7	EP8	EP9	TOT	
MATHS	Routines		Interpreting	1	6	2	8	4	2	6		3	32	
			Constructing				2					7	9	
			Classifying										0	
			Translating				1						1	45
			Calculating		1		2						3	
			Demonstrating										0	
			Defining										0	
	Visual Mediator		Verbal										0	
			Symbolic	1	5	1	4	4	2	7		2	26	
			Tabular				4					8	12	49
			graphical	1		1	4					5	11	
			Concrete										0	
MTF	MTF NAMING	OoL	OoL			1	1		1			3	6	
		Tasks	Tasks									2	2	
		Examples	Examples	7	1	2	2	4	3	1		3	23	49
		Explanations	Explanations		3		2		1		1		7	
		Learner participation	Learner participation	2	1	1					7		11	
	MTF ROUTINES	Exemplifying	Interpreting_T			1	1						2	
			Constructing_T				2					4	6	
			Classifying_T										0	11
			Translating_T				1						1	
			Calculating_T									2	2	
		Generating tasks	Interpreting_L		1					0			1	
			Constructing_L									0	0	
			Classifying_L										0	1
			Translating_L										0	
			Calculating_L										0	
		Teacher explaining	Interpreting_TS				3	1		3			7	
			Constructing_TS										0	
			Classifying_TS										0	
			Translating_TS										0	10
			Calculating_TS		1		2						3	
			Defining_TS										0	

		Learner explaining	Interpreting_LS		5			1		2		1	9	
			Constructing_LS										0	
			Classifying_LS$_u$										0	9
			Translating_LS$_u$										0	
			Demonstrating_LS$_u$										0	
			Defining_LS$_u$										0	
		Learner_CA	CA-Interpreting_T		2		2	1	2	1		4	12	
			CA-Constructing_T				1					2	3	
			CA-Classifying_T										0	
			CA-Translating_T										0	25
			CA_calculating_T										0	
			CA_defining										0	
		Teacher_CA	CA_Word use				10						10	
PEDAGOGICAL		General Pedagogical	Structuring	1	3		2		2	1	1	1	11	
			Revising					1					1	
		Mathematical teaching	Questioning		2	1		5		1	2	1	12	
			Generalising		1		1					1	3	
			Explaining		1					1			2	123
		L/S pedagogical practice	Identifying CA		2		10	1	2	1	1	6	23	
			Reframing	5	6	6	4	5	1	4	3		34	
			Scaffolding	1	4	4	7				6	6	28	
		Other	Other						3	2	3	1	9	
LESSON PROCESS		L/S process	Ground Rules	1									1	
			Roles	1					8		1	1	11	32
			Lesson planning	3	4	3			3		3	4	20	
CONTEXTUAL		Contextual factors	Cont_L	5	1	2		1	1	3	11	4	28	
			Cont_Reso	1					2		1		4	
			Cont_Class	1							4		5	39
			Cont_Time					1				1	2	
			Cont_school teacher										0	
MODES OF TALK	Cumulative		CUMM	7	6	2	3	2	1	3		4	28	
	Exploratory		EXPLO	2	4	1	7		6	3	10	11	44	
	Disputational		DISPU	2			3	4		3			12	
	Structuring Conversation		SC	2	1	1	1		1		2	5	13	
	Managing Understanding		MU	2	3	3	2	2			2	1	15	

Framework matrix: RD2 Table 28_B

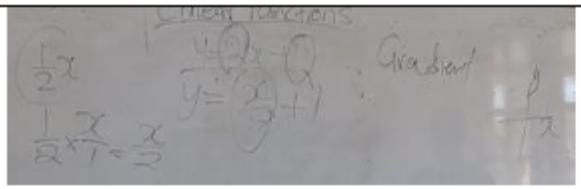
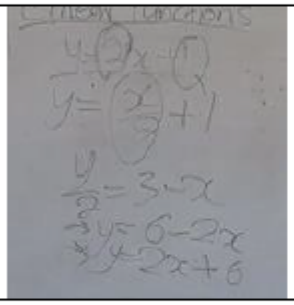
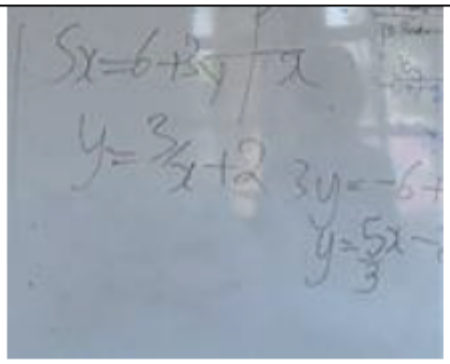
CATEGORIES	SUB-CATEGORIES		CODES	EP1	EP2	EP3	EP4	EP5	EP6	EP7	TOT	
MATHS	Routines		Interpreting	4			1	13	26		44	47
			Constructing								0	
			Classifying	1							1	
			Translating								0	
			Calculating					0	2		2	
			Demonstrating								0	
			Defining								0	
	Visual Mediator		Verbal								0	45
			Symbolic	1			1	0	2		4	
			Tabular				0	4	1		5	
			graphical				0	10	26		36	
			Concrete								0	
MTF	MTF NAMING	OoL	OoL	3			2			2	7	17
		Taska	Tasks									
		Examples	Examples							4	4	
		Explanations	Explanations				1	2	2		5	
		Learner participation	Learner participation							1	1	
	MTF ROUTINES	Exemplifying	Interpreting_T					2	1		3	4
			Constructing_T									
			Classifying_T									
			Translating_T									
			Calculating_T					1			1	
		Generating tasks	Interpreting_L								0	0
			Constructing_L								0	
			Classifying_L								0	
			Translating_L								0	
			Calculating_L								0	
		Teacher explaining	Interpreting_TS				4	5	30		39	43
			Constructing_TS									
			Classifying_TS									
			Translating_TS									

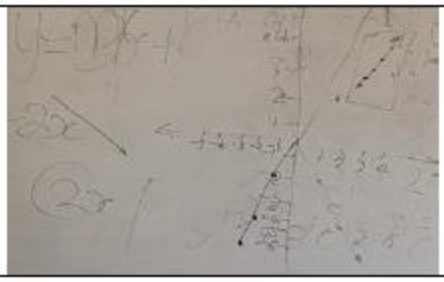
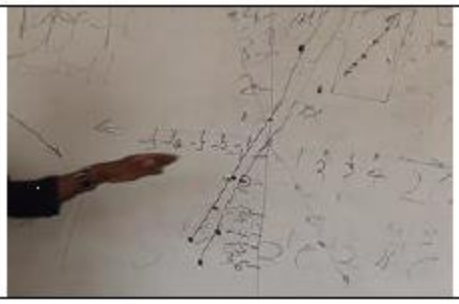
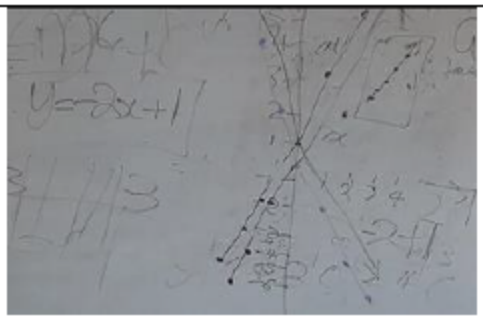
			Calculating_TS					2	2		4	
			Defining_TS									
		Learner explaining	Interpreting_LS				5	6	1		12	12
			Constructing_LS									
			Classifying_LS$_u$									
			Translating_LS$_u$									
			Demostrating_LS$_u$									
			Defining_LS$_u$									
		Learner_CA	CA-Interpreting_T	6		0	3				9	32
			CA-Constructing_T									
			CA-Classifying_T	1							1	
			CA-Translating_T									
			CA_calculating_T									
			CA_Word use					6	5		11	
			CA_defining									
		Teacher CA	CA_Word use					6	5		11	
PEDAGOGICAL	General Pedagogical		Structuring	4		1	6			1	12	45
			Revising	1							1	
	Mathematical teaching		Questioning									
			Generalising									
			Explaining			0		1			1	
	L/S pedagogical practice		Identifying CA	2		0	2	0	2		6	
			Reframing		1	1	7	1	3	2	15	
			Scaffolding	4	2	0	0	0	1	0	7	
	Other		Other	1						2	3	
LESSON PROCESS	L/S process		Ground Rules		1	2					3	22
			Roles		4	7	1				12	
			Lesson planning	5	1	1		0			7	
CONTEXTUAL	Contextual factors		Cont_L	1		7	2	0	1	3	14	28
			Cont_Reso			1	1	0		7	9	
			Cont_Class		1						1	
			Cont_Time	1		0	3				4	
			Cont_school teacher									

MODES OF TALK	Cumulative	CUMM	7	3	9	13	9	19	4	64
	Exploratory	EXPLO	3	4	5	2	5	17	3	39
	Disputational	DISPU	0	0	0	0	0	0	0	0
	Structuring Converstion	SC		1	0	1	0	0		2
	Managing Understanding	MU		1	2	5	1	6	5	20

Appendix E: Lesson enactment (description of of first and second teaching lessons)

1) Description of the enactment of a first teaching lesson: Table T1

	Written on the board	What was said
1.		Teacher T asked learners to identify the values of the gradient and y intercept in first equation $y = 2x + 1$. Learners gave a correct answer He then further elaborate on the what the y intercept was representing by constructing the Cartesian plane(as shown in the figure) Teacher T further asked them to identify the value of gradient in the equation $y = x/2 + 1$. Where learners gave an incorrect answer of 1. He then demonstrated on the side by breaking down the expression of ½ x into ½ multiplied by $x/1$ as shown in the figure. This lead learners to be able to identify the coefficient of x ,and gave the correct value of the gradient
2.		The following example was $y/2 = 3 - x$. Teacher T asked the learners to identify the value of the gradient and the y intercept. The learners failed to give the correct answer. This was followed by action where Teacher T guided learners through the simplifying procedure by making y the subject of the formula, a process which resulted in the equation $y = 2x + 6$. This simplified form made it possible for the learners to identify the gradient and the y intercept.
3.		The next example was $5x = 6 + 3y$. Teacher T followed the same procedure as the previous example, where he simplified the equation together with the learners. A process that resulted in a simplified form of $y = \frac{5}{3}x - 2$. From this equation teacher T requested learners to identify the values of the gradient and the y intercept. The next example was $y = 3/x + 2$. The teacher asked the learners to identify the value of the gradient, where learners said it was 3. Teacher T, further asked the learners to compare this particular equation with the previous examples, the learners said this equation was not the same as

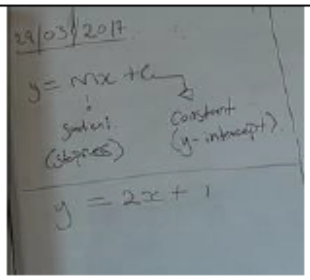
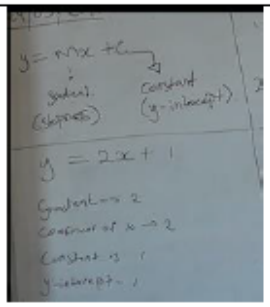
7.		The next examples were $y = 2x$ and $y = -2x$, where teacher T constructed the graphs of the two functions as shown in the picture. This process was done in order to illustrate the effects of the gradient graphically.
8.		Teacher T, on the same set of axes, he constructed the graph of $y = 2x + 1$ and compared it with that one of $y = 2x$ in order to illustrate the effects of the y intercept.
9.		The last example was that of the equation $y = 2x - 1$ which was used as learners' task, where he requested them to work in groups and compare the given function with $y = -2x$ and asked them to interpret the effects of the y intercept.. He moved around together with the other teachers who were observers. When the learners were finished with the group work, teacher T went back to the board and worked with the learners to construct the graph of $y = 2x - 1$, and illustrated the effects of the y intercept graphically. This marked the end of the lesson.

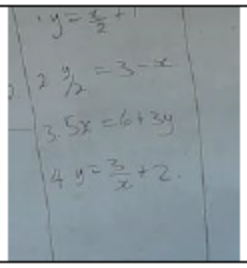
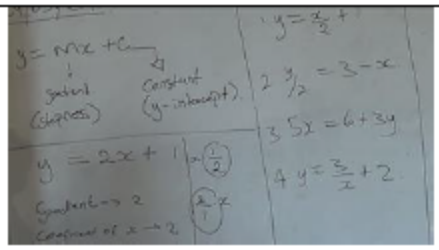
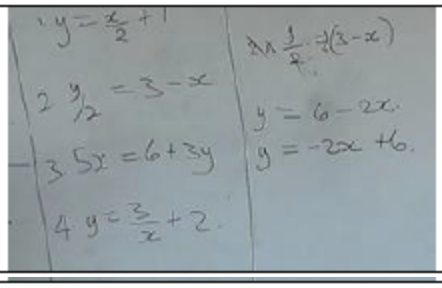
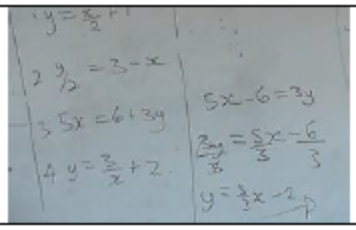
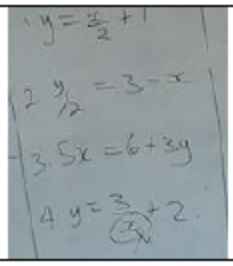
2) Description of the enactment of a second teaching lesson: Table T2

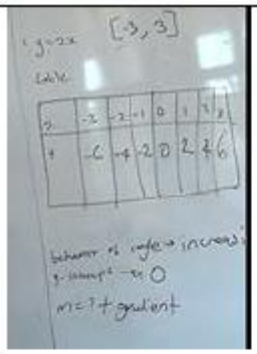
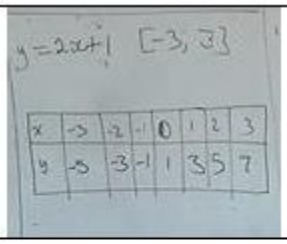
2)Description of the enactment of a second teaching lesson:

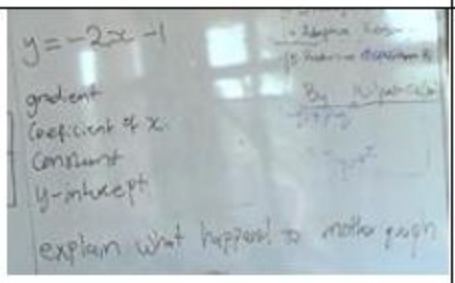
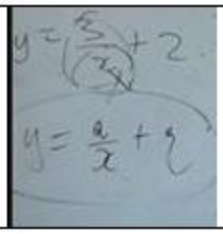
At the end of the second lesson planning meeting, teacher S was chosen to be the person enacting the second lesson-lesson 2. This lesson was scheduled to be taught at school J on a Wednesday, 29 March 2017 at 2pm. The lesson was conducted to a different grade 10 class from the previous first teaching session by teacher S, the rest of the team members were present during the lesson and played a role of being the observers. The duration of the lesson was forty (40) minutes. Teacher S announced that the lesson was on linear function, and the aim of that lesson was to interpret the effects of the gradient and y intercept. He wrote on the the board his first example, $y = mx + c$. Below, I give the summary of the lesson enactment by teacher S:

Summary of the enactment of the second teaching session:

	Written on the board	**What was said**
1.		Teacher S, asked learners to first classify the function $y = mx + c$. He then asked them to identify the values of the gradient and y intercept in terms of the x coefficient and constant respectively
2.		Teacher S asked learners to classify the function $y = 2x + 1$. He then asked them to identify the values of the gradient and $y\ intercept$ in terms of the x coefficient and constant respectively

3.		Teacher S, focused the learners on a set of examples (as shown in the picture), and asked them to classify the given functions. And further asked them to identify the values of the gradient and $y\ intercept$ in terms of the x coefficient and constant respectively in each of these functions. He asked the learners to work in group, while he moved around the class with the observers doing the same.
4.		After learners were done with their group work, teacher S, then asked them to classify the function $y = x/2 + 1$. The learners classified the equation as the non- linear function. Teacher S then wrote on the side by breaking down the expression $x/2$ into x multiplied by $1/2$ and compared it to the linear function $y = 2x + 1$ and took the expression $2x$ and broke it down into $2/1$ multiplied by x and used the notion of the coefficient of x to do the comparison in order to explain the concept of gradient. Then the learners were able to correct the answer they have given earlier and classified the equation $y = x/2 + 1$ as a linear function. He then further asked them to identify the values of the gradient and $y\ intercept$ in terms of the x coefficient and constant respectively.
5.		Teacher S, then moved to the following equation $y/2 = 3 - x$ and asked them to classify the equation .The learners classified the equation as the non- linear function. Teacher S then wrote on the side by simplifying the equation by making y a subject of the formula which resulted in a new equation $y = -2x + 6$. At this point, the learners were able to correct the answer they have given earlier and classified the equation $y/2 = 3 - x$ as a linear function. He then further asked them to identify the values of the gradient and $y\ intercept$ in terms of the x coefficient and constant respectively.
6.		Teacher S, then moved to the following equation $5x = 6 + 3y$ and asked them to classify the equation .The learners classified the equation as the linear function. He confirmed learners' response by simplifying the equation by making y a subject of the formula which resulted in a new equation $y = \frac{5}{3}x - 2$. He then further asked them to identify the values of the gradient and $y\ intercept$ in terms of the x coefficient and constant respectively.
7.		Teacher S, then moved to the following equation $y = 3/x + 2$ and asked them to classify the equation .The learners classified the equation as the non- linear function, because x was in a denominator. Teacher S, did not elaborate on this any further, instead he moved to the next example.

8.		The next example was $y = 2x$ .Teacher T went further to construct a table with an interval $[-3; 3]$ as shown in the picture. From the constructed table he then further asked the learners to interpret the behavior of the range, y intercept and the gradient.
9.		The next example was $y = -2x$ .Teacher T went further to construct a table with an interval $[-3; 3]$ as shown in the picture. From the constructed table he interpreted the behavior of the range, y intercept and the gradient. When explaining the notion of a gradient, he compared the gradients between $y = 2x$ and $y = -2x$.
10.		Following the previous examples, he went further and constructed the graphs of the two functions $y = 2x$ and $y = -2x$ on the same set of axes and demonstrated the effect of gradient graphically.
11.		Teacher T went further to construct the function of $y = 2x + 1$ in a tabular form with an interval $[-3; 3]$ as shown in the picture. He interpreted the behavior of the range, y intercept and the gradient
12.		Following the previous example, he went to the Cartesian plane and constructed graph of $y = 2x + 1$ alongside that of $y = 2x$, in order to interpret the effects of the y intercept.

13.		The last example was that of the equation $y = -2x - 1$ which was used as learners' task, where he requested them to work in groups and compare the given function with $y = -2x$ and asked them to identify the values of the gradient and y intercept in terms of the x coefficient and constant respectively. He further asked the learners to interpret the effects of the gradient and the y intercept by comparing the function $y = -2x - 1$ with the original function $y = -2x$. . He moved around together with the other teachers who were observers. After the learners were done with this group work, he invited one of the learners to come to the board and present their answers to the given task.
14.		Before sending off the learners, the KO took over the conclusion by doing a follow up on the function $y = 3/x + 2$, where she drawn in the standard formula of the hyperbola $y = a/x + q$ in order to enable the classification of the function $y = 3/x + 2$. This marked the end of the lesson

Appendix F: Discursive tripartite focus

	Pronounced		Attended						Intended
	Maths objects	MTF objects	Maths Routines/VM		MTF Routines	Pedagogical Routines	L/S Process Routine	Contextual Routines	
EP1	Linear function Gradient y intercept	OoL	Calculating Constructing Demonstrating Interpreting Translating	Verbal Symbolic Tabular Graphical Concrete	Exemplifying Explaining- Learner Explaining- Teacher Identifying OoL Generating Tasks	Explaining Identifying CA Questioning Revising Structuring		Learners	Identification of critical aspects
EP2						Identifying CA Revising Scaffolding Structuring	Ground rules Roles	Learners Resources	Ground rules and roles
EP3	Linear function Gradient y intercept	OoL	Calculating Defining Interpreting	Verbal Symbolic Tabular Graphical	Explaining- Learner Explaining- Teacher Identifying OoL Generating Tasks	Identifying CA Scaffolding	Ground rules Roles	Learners	Identification of critical aspects
EP4	Linear function Y intercept		Classifying Interpreting	Symbolic Graphical	Identifying OoL	Identifying CA Revising Scaffolding		Learners	Identification of critical aspects
EP5	Linear function Gradient y intercept	Examples OoL Tasks	Calculating Classifying Constructing Demonstrating Translating	Symbolic Tabular Graphical Concrete	Exemplifying Explaining- Learner Explaining- Teacher Identifying OoL Generating Tasks	Identifying CA Questioning Scaffolding Structuring	Lesson planning	Learners	Generating example space
EP6	Linear function Gradient y intercept	Examples	Classifying Constructing	Symbolic Graphical	Explaining- Learner Explaining- Teacher Identifying OoL Generating Tasks	Identifying CA Questioning Scaffolding Structuring	Ground rules	Learners Resources	Identification of critical aspects
EP7	Linear function Gradient y intercept	Examples OoL	Classifying Constructing Interpreting	Symbolic Tabular Graphical	Explaining- Learner Identifying OoL Generating Tasks	Identifying CA Scaffolding Structuring	Lesson planning	Learners	Contextual factors related to the learners
EP8	Linear function Gradient y intercept	OoL	Calculating Classifying Constructing Interpreting Translating		Exemplifying Explaining- Teacher Identifying OoL Generating Tasks	Generalizing Identifying CA Questioning Revising Scaffolding Structuring	Lesson planning	Learners	Structuring of the lesson
EP9	Linear function Gradient y intercept	Examples Explanations OoL	Calculating Constructing Interpreting	Symbolic Graphical	Exemplifying Explaining- Learner Identifying OoL Generating Tasks	Generalizing Identifying CA Scaffolding Structuring			Mathematical teaching
EP10	Linear function y intercept	Examples Task	Calculating Constructing Interpreting	Tabular Graphical	Exemplifying Explaining- Teacher Identifying OoL	Generalizing Identifying CA Scaffolding Structuring		Resources	Identification of critical aspects
EP11	Linear function y intercept	OoL	Calculating Constructing Interpreting Translating	Symbolic Graphical	Exemplifying Explaining- Teacher Identifying OoL	Generalizing Identifying CA Scaffolding	Lesson Planning Role		Identification of critical aspects

EP12	Linear function Gradient y intercept	Explanations Tasks	Constructing Translating	Graphical	Explaining- Teacher Identifying OoL Generating Tasks	Generalizing Identifying CA Questioning Scaffolding Structuring		Resources	Structuring of the lesson

The summary of what was in focus in each episode

From table above, one can begin to see that the identification of critical aspects was the predominant intended focus, which was evidenced in six out of twelve episodes. This finding confirms the hypothesis that the aim of this session (lesson planning) is about identifying critical aspects of OoL (Runesson, 2013). Furthermore, the objects of attention included both mathematical and MTF objects. Where, the mathematics objects in focus are both linear function and its properties-the gradient and y intercept. The most named MTF components included OoL and examples, and no learner participation. These two categories of objects were made visible by different categories of routines from mathematics, MTF and pedagogical routines. In addition to that, table 5.20 shows that the teachers' discussions during the LP1 session illuminated the talk related to aspects describing the environment - the L/S context. These included talks about with the L/S lesson process and general contextual factors. Such points to situative perspective in which this study is located. It will be the work of the analysis chapters to give the expanded descriptive and explanatory account of this finding.

Appendix G: L/S MTRD and MTF discursive tool

The L/S MTRD (Table A): the nature of the emerging reflective discourse-the MTRD across all three sessions:

MTRD		LP1	RD1	RD2
Objects		Linear function and its properties; Examples, tasks. explanations, OoL	Linear function and its properties; Examples, tasks. explanations, OoL, learner participation	Linear function and its properties; Examples, tasks. explanations, OoL, learner participation
Sources of narratives		Mathematical, MTF, Pedagogical	Mathematical, MTF, Pedagogical	Mathematical, MTF, Pedagogical
Goals of discursive activity	Object-level	expansion of MTF objects *teacher and learner explanations*	expansion of MTF objects *teacher and learner CA*	expansion of MTF objects *teacher and learner CA*
	Meta-level	Constructing mathematical, MTF, and pedagogical narratives, modification of MTF tool	Constructing mathematical, MTF, and pedagogical narratives, modification of MTF tool	Constructing mathematical, MTF, and pedagogical narratives, modification of MTF tool
Roles		Collective, with KO high scaffolding	Collective, with moderate KO scaffolding and with "on task" subjectifying	Collective, with low KO scaffolding and with "off task" subjectifying
ritual/ explorative participation		Explorative participation in MTF discourse with ritualistic KO goal	Explorative participation in MTF discourse with ritualistic KO goal	Ritualistic participation in MTF discourse with explorative KO goal

The MTF discursive tool (Figure A): the transformation of MTF tool into MTF discursive tool (MTF-use) across three L/S sessions:

