

CHAPTER 1: INTRODUCTION TO THE STUDY

1.1 INTRODUCTION

As a practising high school mathematics teacher, I have found time and again that many learners have limited cognitive resources with which to think about mathematics. Despite completing many daily exercises intended to develop their functioning in any mathematical domain, learners do not seem to develop real understanding of what they are doing or why. The rich, meaningful connections across different areas of mathematics that make mathematics so useful and interesting, seem to be lost on most students.

As a teacher, I would like to equip my learners with thinking skills that enable them to unpack problems, understand the problems and work through the problems efficiently. Simply teaching students the mathematics alone does not appear to do this. Learners seem to need additional instruction in *how* to think about problems; what strategies to use, and how, when and why to use those strategies. This ability to monitor and regulate one's own thinking is part of the concept "metacognition".

Metacognition is often referred to simplistically as "thinking about thinking," but is much more than that. Initially conceptualised by Flavell (1979; 1987) and Brown (1978; 1987), the idea of metacognition has been researched by many authors over the last 50 years (Schneider & Artelt, 2010). Although there are a number of modern interpretations, most researchers acknowledge either Flavell or Brown as their theoretical basis (Schneider & Artelt, 2010).

Flavell (1979) initially developed a model of monitoring cognitive processes that proceeds through the actions of, and interactions among, four classes of phenomena. He called these phenomena: 1) *metacognitive knowledge*, 2) *metacognitive experiences*, 3) *goals (or tasks)* and 4) *actions (or strategies)*. He defined *metacognitive knowledge* as a person's knowledge or beliefs about what factors interact to affect the course and outcome of cognitive endeavours (Flavell, 1979). *Metacognitive experiences* were defined as the conscious cognitive or affective experiences that accompany a mental enterprise; *goals* were the objectives of a cognitive endeavour and *strategies* were the cognitive behaviours used to achieve those goals (Flavell, 1979).

Brown (1987), on the other hand, described metacognition as comprised of two main components: *knowledge of cognition* and *regulation of cognition*. Over time, her conceptualisation has been expanded and explicated by a number of different authors. Knowledge of cognition is seen as divided into *declarative knowledge* (the knowledge about *what* one knows), *procedural knowledge* (the knowledge about *how* to use what one knows) and *conditional knowledge* (the knowledge about *when* and *why* to use what one knows) (Shraw & Dennison, 1994). Regulation of cognition, on the other hand, is concerned with predicting, planning, monitoring, reflecting and evaluating cognitions (Brown, 1987).

For learners to think effectively about mathematics, they may not only need metacognitive skills but also a connected understanding of mathematics. Mathematics may be conceived as being able to “do” mathematics (Schoenfeld, 1985), but as with metacognition, mathematics is much more than that. In their seminal study of American mathematics education, Kilpatrick, Swafford and Findell (2001) described the kinds of knowledge and skills required for competence in mathematics as *mathematical proficiency*. They conceptualised mathematical proficiency as comprised of five interrelated, interdependent strands: *conceptual understanding*, *procedural fluency*, *strategic competence*, *adaptive reasoning* and *productive disposition* (Kilpatrick, Swafford & Findell, 2001). The first four strands are concerned with cognitive abilities while the fifth strand concerns affect. The authors claim that for mathematics instruction to be effective, all five strands need to develop simultaneously over time and that the strands mutually support and strengthen one another (Kilpatrick et al., 2001).

For this dissertation, I sought to connect the two constructs of mathematical proficiency and metacognition. Using a small convenience sample of matriculants at a public high school, I researched if training in metacognitive knowledge and skills could possibly influence the learners’ mathematical proficiency.

1.2 CONTEXT OF THE STUDY

The study took place at a well-resourced public all-girls high school in Johannesburg, South Africa during the first and second terms of 2015. I was fortunate to have two small classes ($n = 15$ and $n = 13$) of matriculants, twelve of whom I had worked with in Grade 11 in 2014. The learners on the whole were of lower to moderate ability having achieved averages in their respective classes of 46,7% and 36,2% in their Grade 11 year; quite below the whole grade average of 51,5%. The learners represented approximately the racial demographics of the country (79,2% Black, 8,9% White, 8,9% Coloured, 2,5% Indian) (www.southafrica.info). However, as the school is a fee-paying school with fees over R30 000 per year, many of the learners may have been from a socioeconomic background higher than the national average.

The school has a 10-day cycle that includes eight 50-minute periods of mathematics plus a 50-minute test period. The content covered over the course of the two terms was Sequences and Series, Trigonometry and Euclidean Geometry. The curriculum was based upon the Curriculum and Assessment Policy Statement (CAPS) that came into effect for matriculants in 2014.

The learners were preparing for the national Department of Basic Education (DBE) school-leaving examinations that take place in November of every year. Not only do these examinations provide an exit from high school, but determine for which area of tertiary study the learners are eligible. Depending upon their results, learners may qualify for a higher certificate, studying toward a diploma at a college or studying toward a degree at a university. The final examinations of two 3-hour papers of 150 marks each form 75% of the final mark. The other 25%, termed School Based Assessment or SBA, is comprised of nationally specified tasks and tests that take place during school time. The national examinations are set externally

to the schools and externally moderated by a regulatory body. The SBA tasks are set within the school and moderated at district level.

Our mathematics department was fortunate to have one of the National Examiners as our Head of Department (HOD). Not only was our HOD very aware of the level of mathematics required by our learners, but she was particularly interested in developing all the strands of the learners' mathematical proficiency. We attended Professional Development meetings regularly to discuss ways to increase learners' conceptual understanding and strategic competence. Rather coincidentally, one of the reading materials she gave us was the chapter on mathematical proficiency from Kilpatrick et al. (2001). Thus, I was fortunate to be in a department that valued learner thinking.

1.3 RATIONALE

A number of factors motivated me to want to pursue the connection between metacognition and mathematical proficiency. Firstly, a great deal of literature exists indicating that metacognition is associated with effective learning. For example, in an extensive meta-review of the causes of school success, Wang, Haertel and Walberg (1990) identified metacognition as a significant predictor of learning in any subject. In addition, a number of studies indicate that metacognitive skills improve mathematics performance specifically (e.g. Aydin & Ubuz, 2010; Lucangeli & Cornoldi, 1997; Mevarech, 1999; Mevarech & Amrany, 2008; Mevarech & Fridkin, 2006; Mevarech & Kramarski, 1997; Mevarech & Kramarski, 2003). I wanted to see for myself in my particular context whether metacognition made a difference to mathematical ability.

Despite this large body of research, I found relatively few studies that have been done in South Africa or by South Africans (see Case & Gunstone, 2006; Case, Gunstone & Lewis, 2001; Maqsud, 1997; Davidowitz & Rollnick, 2003; Kung & Linder, 2007; Moseki & Schulze, n.d.). Of these, only one (Maqsud, 1997) worked with high school students on mathematics problems; the other researchers worked with university students in the sciences. Thus, there is very little research to date in South Africa on metacognition in general, and its relationship to mathematics specifically, which was my second reason for the pursuit of this study.

Thirdly, Veenman and Van Hout-Wolters (2006), in the inaugural issue of *Metacognition and Learning*, identified ten issues surrounding the study of metacognition itself. Among these included: the complex relation between metacognition and cognition, developmental processes in metacognition, assessment of metacognition and the conditions for the acquisition and instruction of metacognition (Veenman & Van Hout-Wolters, 2006). I hoped my study, which was concerned with training in metacognitive skills over approximately six months and the possible influence of those skills on mathematical cognitive processes, would shed some light on these issues concerning metacognition, specifically with regard to mathematical thinking.

Fourthly, I conjectured that the complexity of mathematical thinking may lend itself particularly to requiring metacognitive abilities. Later in this dissertation, I will describe the views of mathematical proficiency given by two different authors as an indication of the scope of the discussion surrounding mathematical proficiency. It is possible that metacognitive skills are the “missing link” between underdeveloped mathematical abilities and mathematical proficiency. While a number of studies have correlated metacognitive ability to mathematical performance (e.g. Lucangeli & Cornoldi, 1997; Mevarech, 1999; Mevarech & Amrany, 2008; Mevarech & Fridkin, 2006, etc.) few have used as complex an understanding of mathematical ability as that delineated by Kilpatrick et al. (2001).

Finally, the method of my study was unusual in metacognition research. Many studies have used large-scale quantitative studies to correlate mathematics success to metacognition (see: Aydin & Ubuz, 2010; Lucangeli & Cornoldi, 1997; Mevarech, 1999; Mevarech & Amrany, 2008; Mevarech & Fridkin, 2006; Mevarech & Kramarski, 1997; Mevarech & Kramarski, 2003). I used a case study methodology and worked intensively with two classes of matriculants in a public high school. My intention was to perform an in-depth, qualitative study of the ability of the learners to use metacognitive training in mathematical exercises.

To summarise, the reasons for undertaking this study were:

- A significant amount of research points to the efficacy of metacognition to improve learning in general and mathematics learning in particular. Thus, training in metacognition may be a considerable aid to learning and should be studied to improve the learning and teaching of mathematics in South Africa.
- Little research has been done to date on metacognition in South Africa, especially with regard to mathematics.
- There are areas of research in metacognition to which my study may contribute.
- Mathematical proficiency is a complex skill requiring more than just a knowledge of procedures; it may be that the meta-level thinking of metacognition could positively influence mathematical proficiency.
- Qualitatively-analysed, case-study designs are rare in metacognition research.

1.4 RESEARCH AIMS

There were four main aims for my study. Firstly, I wanted to investigate whether learners, who are provided with metacognitive training in a regular classroom setting, acquire and use the metacognitive skills at all.

The second aim of my study was to see if students, when given training in metacognitive skills, actually use them in written mathematical assessments. This aim is based upon the work of Mevarech, Kramarski, Fridkin and Amrany, who over the course of approximately 10 years used the IMPROVE system in Israel to train students in metacognitive self-questioning techniques (Mevarech, 1999; Mevarech & Amrany, 2008; Mevarech & Fridkin, 2006;

Mevarech & Kramarski, 1997; Mevarech & Kramarski, 2003). Their research indicates that students' mathematics performance improved as a result of the students' increase in metacognitive ability. Much of their research, though, centred on small-group cooperative learning.

In this study, written instruments were chosen to assess metacognitive ability and mathematical proficiency because of the heavy weighting of that type of assessment in South Africa. The South African National Senior Certificate: Curriculum Assessment and Policy Statement (2011) requires that written final examinations form 75% of a learner's final mark. For a South African matriculant, 75% of his/her final mark rests on an individual, silent, written, timed summative assessment. I wanted my instrument to have elements that mirror the type of assessment that is seen as most important in my country's educational system.

It needs to be noted that metacognition can be assessed in primarily two ways: *off-line* assessments and *on-line* assessments (Veenman, Bavelaar, De Wolf & Van Haaren, 2013). The terms "on-line" and "off-line" are not computer-related, but refer to whether the metacognition is happening *simultaneously* with another cognitive activity (*on-line* assessment) or before or after some other cognitive activity (*off-line* assessment) (Veenman et al., 2013). Off-line assessments primarily rely upon the respondents' answers to questions regarding metacognition and are usually in the form of a questionnaire with Likert-type scale answers (e.g. Shraw & Dennison Metacognitive Awareness Index, 1994). Thus, off-line assessments rely upon the respondents' self-report of metacognition and require a high level of honesty and self-awareness that may or may not be present in the respondent (Veenman et al., 2013). This is one area of concern regarding assessment of metacognition. Off-line assessments, though, have the advantage that they are relatively easy to administer and were used in the studies of Mevarech and Fridkin (2006) and Mevarech and Amrany (2008), although in the latter study interviews were also conducted.

On-line assessments, on the other hand, try to measure metacognitive activity while some other cognitive activity is occurring. On-line methods have included using think-aloud protocols (Desoete, 2008), observations (Schoenfeld, 1985) or eye movement registrations (Kinnunen & Vauras, 1995). Like the off-line methods, on-line methods have their advantages and disadvantages. For example, one disadvantage is the relative difficulty to administer and analyse on-line assessments (Artzt & Armour-Thomas, 1992). However, recent research recommends on-line methods as the more accurate measure of metacognitive activity (Veenman et al., 2013).

Thus, the third aim was to see if on-line assessment of metacognition is possible using an instrument that concurrently requires students to explicate their thinking while working through mathematics problems in a high school setting. There is a precedent for this approach in the research of Lucangeli and Cornoldi (1997) who studied the correlation between mathematical ability and the metacognitive processes of predicting, planning, monitoring and evaluating. Using an instrument that required 3rd and 4th graders to complete mathematical tasks

and answer questions about their metacognitive activities simultaneously, the authors discovered a strong correlation between metacognitive processes and mathematical ability.

Finally, my fourth aim was to see whether there was some connection between metacognition and mathematical proficiency. I hoped that the pre-test/intervention/post-test design would indicate some changes in the way the students approach problems.

1.4.1 Assessing Knowledge of Cognition vs. Regulation of Cognition

As mentioned earlier, metacognition can be divided into knowledge of cognition and regulation of cognition. In order to study metacognition, I had to choose between these two areas. My study focused on knowledge of cognition and more specifically, strategy use, for several reasons.

Firstly, assessing regulation of cognition is problematic. Regulation of cognition involves the activities of predicting, planning, monitoring, reflecting and evaluating cognitions (Brown, 1987). Regulation of cognition, an *interior, mental* process, needs to be inferred from *exterior, observable* behaviour. This creates two difficulties: a practical one and a theoretical one. On the practical side, studying the regulation of cognition would involve extensive video- and/or audio taping of classroom activities. As I was working as a teacher-researcher, this could have been impractical and distracting in an ordinary classroom setting.

Also, I specifically wanted to investigate the metacognition that is used in a written assessment as this type of assessment carries the greatest weight in my country's curriculum. To assess interior *processes* of predicting, planning, monitoring, reflecting and evaluating while students are also working through problems would mean an on-line assessment that requires students to simultaneously work through mathematics exercises, be aware of their cognitive regulating *and* write down their observations. It seemed a bit too much to ask of high school students.

On the theoretical side, operationalising externally-observed metacognitive behaviours such as regulation of cognition can also be problematic. Veenman and Van Hout-Wolters (2006) acknowledge that very little overt behaviour can be identified as metacognitive; metacognition usually needs to be inferred from certain cognitive activities. Some authors (e.g. Schoenfeld, 1985; Artzt & Armour-Thomas, 1992) have tried to operationalise observable behaviours. For example, Artzt and Armour-Thomas (1992), distinguish between cognitive and metacognitive behaviours in their Cognitive-Metacognitive Framework for Protocol Analysis of Problem Solving in Mathematics. However, both these researchers worked in teams which improved their inter-rater reliability. Working alone, I believed I may have had difficulty distinguishing cognitive activities from metacognitive ones. Therefore, for these reasons, I chose to investigate knowledge of cognition rather than regulation of cognition.

Because knowledge of cognition could be a very broad category, I focused specifically on developing strategies that are applicable across a range of content and problem-types. The

meaning of strategy that I am using is similar to the sense of the word “heuristic” used by Pólya in his book, *How to Solve It*; that is, heuristic is the “study of methods of solution” (Pólya, 1957). In his seminal book, Pólya (1957, p. 37) offers a “short dictionary of heuristic” in which he describes the many possible approaches with which a person can tackle a problem. These include: using an analogy; drawing a picture; working backwards; using a simpler example, and many others (Pólya, 1957). This collection of strategies becomes part of a person’s declarative metacognitive knowledge (*what* a person is aware of knowing). Shraw (2002) extends this idea to the other categories of metacognitive knowledge: procedural knowledge and conditional knowledge. An extensive knowledge of strategies is useless if one does not know *how* (procedural knowledge) to use them, or *when* and *why* to use them (conditional knowledge). Thus, in line with Shraw’s (2002) recommendation, part of the training process was to develop procedural and conditional knowledge of strategies.

1.5 RESEARCH QUESTIONS

From the foregoing considerations, my research questions were as follows:

1. Is there evidence that when given metacognitive skills training, students use those skills in an individual, written mathematical assessment? What is the nature of that evidence?
2. To what extent do learners utilize metacognitive knowledge, specifically strategies, if given metacognitive training in a regular, high school classroom?
3. If the metacognitive skills are used, do these skills affect the students’ mathematical proficiency in any way?

CHAPTER 2: LITERATURE REVIEW

2.1 INTRODUCTION

This literature review will begin with an exposition of the theoretical foundations of metacognition, mostly taken from Flavell (1979, 1987) and Brown (1978, 1987). Additional conceptualisations will also be presented briefly. As metacognition has been investigated for over forty years, a number of issues have been raised by various authors regarding its conceptualisation, its development and its assessment; these issues will be discussed next. A deeper look into how metacognition is assessed follows. Then, the relationship between metacognition and mathematics education is presented with a detailed look at the work of Alan Schoenfeld (1985, 1987a, 1987b, 1994); I will also consider the connection between cognitive science, mathematics and mathematics education which Schoenfeld used in his research. A review of some of the empirical literature discussing metacognition in mathematics education will be presented. A close look at the IMPROVE system designed by Mevarech et al. (1997, 1999, 2003, 2006, 2008), which has significantly informed this paper follows. Next, definitions of mathematical proficiency are explored as well as a description of Pólya's (1957) work. Finally, the conceptual framework for the data is presented.

2.2 THEORETICAL FOUNDATIONS OF METACOGNITION

Metacognition, which has been defined variously by a number of authors, has roots that can be traced as far back as 1910 to the writings of Dewey and Thorndike (Brown, 1987). However, many modern studies (e.g. Aydin & Ubuz, 2010; Veenman et al., 2013) seem to concentrate on the formulations of Flavell or Brown as the basis for their writings. As such, I will describe Flavell's and Brown's ideas as the theoretical foundations for metacognition before discussing additional interpretations.

Studies surrounding metacognitive-like behaviours began in the 1970s. Brown (1987) acknowledges Flavell as introducing the idea of "metamemory" and conducting modern studies of metamemorial processes in children. The results of studies comparing the monitoring abilities of older and younger children, particularly with regard to memory, made Flavell conclude that young children are "quite limited in their knowledge and cognition about cognitive phenomena . . . and do relatively little monitoring of their own . . . cognitive enterprises" (Flavell, 1979, p. 906). He used the term "metacognition" to describe this "knowledge and cognition about cognitive phenomena" (Flavell, 1979, p. 906). The term had appeared in the literature from around 1975 (Brown, 1987). Seeing the discrepancy between the metacognitive abilities of older and younger children prompted Flavell to ask what could children or adolescents learn that adults possibly know or toward what "developmental target" (Flavell, 1979, p. 906) could children progress?

To answer those questions, Flavell initially developed a model of monitoring cognitive processes that proceeds through the actions of, and interactions among, four classes of

phenomena (Flavell, 1979). He called these: 1) *metacognitive knowledge*, 2) *metacognitive experiences*, 3) *goals (or tasks)* and 4) *actions (or strategies)* (Flavell, 1979). He defined *metacognitive knowledge* as a person's "stored world knowledge that has to do with people as cognitive creatures" (Flavell, 1979, p. 906) and consists of the knowledge or beliefs about what factors interact to affect the course and outcome of cognitive endeavours. *Metacognitive experiences* were defined as any "conscious cognitive or affective experiences that accompany and pertain to any intellectual enterprise" (Flavell, 1979, p. 906). *Goals (or tasks)* were seen as the "objectives of a cognitive enterprise," while the *actions (or strategies)* were the "cognitions or other behaviours employed" to achieve those goals (Flavell, 1979, p. 907). Although goals and actions are listed as two specific classes of phenomena, Flavell tended to describe them as being subsumed within metacognitive knowledge and experience. He later only concentrated on the two constructs, metacognitive knowledge and metacognitive experiences, but expanded metacognitive knowledge to include psychological matters as well as purely cognitive ones (Flavell, 1987).

Another theorist who is often associated with the early formulation of metacognition is Ann Brown (Dinsmore, Alexander & Loughlin, 2008). Brown (1987, p. 65) states that metacognition "refers to understanding of knowledge, and understanding that can be reflected in either *effective use* or *overt description* [my emphasis] of the knowledge in question." Brown further separates metacognition into the "knowledge and control of one's own cognitive system" (Brown, 1987, p. 66) and acknowledged that research regarding metacognition branched into two distinct areas: knowledge of cognition and regulation of cognition. She sees knowledge of cognition as the "stable, statable, often fallible, and often late developing thinking" that people have about their own cognitions (Brown, 1987, p. 67). Regulation of cognition, on the other hand, concerns activities around planning, monitoring and evaluating cognitive processes and is unstable, not necessarily statable and age-independent (Brown, 1987). Brown acknowledged that because metacognition seems to have these two somewhat distinct, but overlapping constructs, a clear conceptualization of how metacognition is defined is often problematic (Brown, 1987).

2.3 ADDITIONAL CONCEPTUALIZATIONS OF METACOGNITION

Although there is still a great deal of debate about the definitions and components of metacognition (Veenman & Van Hout-Wolters, 2006), a number of researchers use Brown's framework of separating metacognition into knowledge of cognition and regulation of cognition (e.g. Shraw & Dennison, 1994; Aydin & Ubuz, 2010). Knowledge of cognition is further divided into declarative knowledge (the knowledge about *what* one knows), procedural knowledge (the knowledge about *how* to use what one knows) and conditional knowledge (the knowledge about *when* and *why* to use what one knows) (Shraw & Dennison, 1994). Regulation of cognition is conceived of slightly differently by different authors. Shraw and Dennison (1994) see regulation of cognition as comprised of the five component skills of planning, information management strategies, comprehension monitoring, debugging strategies and evaluation. Lucangeli and Cornoldi (1997) researched the regulatory skills of predicting,

planning, monitoring and evaluating. Some authors replace “evaluating” with “reflecting” (Mevarech & Kramarski, 1997). However, there is a common thread in many studies that metacognition is usefully divided into knowledge of cognition and regulation of cognition in order to aid in the investigation of metacognition. Aydin and Ubuz (2010) noted that many researchers have used Brown’s framework instead of Flavell’s because it is more applicable in academic settings.

2.4 ISSUES, PROBLEMATIC AREAS AND NEEDS FOR FURTHER RESEARCH IN METACOGNITION

From early in its conceptualization to the present day, the construct of metacognition has been surrounded by a number of problematic areas. In a recent summary, Veenman and Van Hout-Wolters (2006) identified ten. These include problems with regard to: a clear, operational definition of metacognition; the components of metacognition; the complex relation between cognition and metacognition; conscious vs. autonomic metacognitive skills; developmental processes; assessment and others (Veenman & Van Hout-Wolters, 2006).

In the 1980’s, Brown (1987, p.67) acknowledged that the construct of metacognition is complex, “fuzzy,” problematic and often poorly understood. She attributed the problems with this term to two sources: the difficulty to separate “meta” from cognitive activities, and the many historical roots from which this area of inquiry developed (Brown, 1987). With respect to the first problem, she explained that the same activity, for example looking for the main points in a chapter, can be seen as either a *cognitive* activity – to improve knowledge, or a *metacognitive* activity – to monitor knowledge (Brown, 1987). Thus, in this case, the same activity can be seen as the strategy itself (cognitive), its monitoring function (metacognitive) or a reflection of the knowledge that the strategy is appropriate (metacognitive) (Brown, 1987). Related to this is the fact that what are now called metacognitive skills, especially in reading, may have previously been seen as strategies (cognitive), such as modifying reading in relation to purpose or evaluating a text for clarity (Brown, 1987). She admitted that to separate the ‘meta’ from the ‘cognitive’ in these kinds of complex activities would be difficult.

Brown (1987) additionally indicated that the problematic nature of metacognition stems from four historically separate but interlinked problems within psychology that relate to metacognition. These include 1) using verbal reports of a person’s own cognitive processes as data, 2) explaining the executive mechanisms in an information processing model of human and machine intelligence, 3) ideas regarding self-regulation and concept reorganisation within developmental psychology, and 4) the idea of “other regulation” which is part of Vygotsky’s theory of development (Brown, 1987, p. 69). Because the sources for the ideas of metacognition were so varied, the construct has also developed variously and can be interpreted from different perspectives.

Veenman and Van Hout-Wolters (2006) concur that problems still surround the definition of metacognition. Acknowledging Flavell’s and Brown’s initial conceptualization of

metacognition as the “knowledge about and regulation of one’s cognitive activities in learning processes”, Veenman and Van Hout-Wolters (2006, p. 3) then list numerous other terms related to metacognition. These include, among others, metacognitive beliefs, metacognitive awareness, feeling of knowing, judgment of learning, theory of mind and metamemory (Veenman & Van Hout-Wolters, 2006). The proliferation of related concepts can result in a lack of coherence in the domain of metacognition (Veenman & Van Hout-Wolters, 2006).

Veenman and Van Hout-Wolters (2006) also acknowledge the difficulty of separating cognitive activities from metacognitive ones. For Veenman and Van Hout-Wolters (2006), part of this difficulty stems from the fact that metacognitive activities draw on cognition. For example, one has to have substantial domain-specific cognitive knowledge in order to have metacognitive knowledge about one’s competency in the domain (Veenman & Van Hout-Wolters, 2006). On the other hand, if metacognition is seen as self-instructions for regulating a task, those self-instructions are cognitive activities (Veenman & Van Hout-Wolters, 2006). Thus, a circular interplay exists between metacognition and cognition which make them very difficult to extricate from each other (Veenman & Van Hout-Wolters, 2006). Added to this is the fact that very little overt behaviour can be identified as metacognitive. Sometimes children verbalise self-instructions, but usually metacognition needs to be inferred from certain cognitive activities (Veenman & Van Hout-Wolters, 2006). Thus, researchers need to distinguish between explicitly observable metacognitive behaviour, metacognition inferred from cognition and cognitive activities (Veenman & Van Hout-Wolters, 2006).

Some researchers have tried to make clear distinctions between cognitive and metacognitive activities. For example, Shraw and Dennison (1994) had operational definitions for the three kinds of knowledge of cognition (declarative, procedural and conditional) and for five processes of regulation of cognition (planning, information management, monitoring, debugging and evaluation). Artzt and Armour-Thomas (1992) also identified operational definitions of cognitive behaviours (reading, exploring, implementing and verifying) and metacognitive behaviours (understanding the problem, analysing, exploring, planning, implementing and verifying). Notice that Artzt and Armour-Thomas (1992) also recognized that the same behaviours (exploring, implementing and verifying) could be indicative of either cognitive or metacognitive activity as Brown (1987) suggested.

Another aspect of metacognition that requires further study is the developmental trajectory of metacognition (Veenman & Van Hout-Wolters, 2006). According to previous research, the following picture has emerged. First, theory-of-mind develops somewhere between 3 and 5 years, followed by the development of metamemory and metacognitive knowledge which can continue throughout a person’s life (Veenman & Van Hout-Wolters, 2006). From 8 to 10 years, metacognitive skills (such as planning) emerge and expand thereafter, but certain skills such as monitoring and evaluation only mature later (Veenman & Van Hout-Wolters, 2006). However, there is also evidence that some metacognitive skills develop earlier in an elementary form (Veenman & Van Hout-Wolters, 2006). It is also possible that metacognitive knowledge and skills emerge earlier than recognized but become “more sophisticated and academically oriented” because of the requirements of formal education (Veenman & Van Hout-Wolters,

2006, p. 8). Thus, one of the needs in research of metacognition is to explore its development in a number of different areas. Although my research spanned a short, 6-month period, one of the aims of my research was to see if there were indications of development of metacognition over time in a formal schooling environment.

2.5 ASSESSING METACOGNITION

Assessing metacognition is another area requiring scrutiny. Metacognition has been assessed using either *on-line* or *off-line* methods (Veenman et al., 2013). As referred to earlier in this dissertation, on-line assessments measure metacognition *during* some task performance, while off-line assessments measure metacognition *before* or *after* task performance (Veenman & Van Hout-Wolters, 2006). On-line methods have included analysis of think-aloud protocols (e.g. Desoete, 2008), observations (e.g. Schoenfeld, 1985) or eye movement registrations (e.g. Kinnunen & Vauras, 1995). In a very recent article, Veenman et al. (2013) used an analysis of computer-log file reports made as students worked through computer-based inductive exercises.

Off-line methods, on the other hand, measure metacognition either before a task is performed (*prospective assessment*) or after the task is performed (*retrospective assessment*), and usually rely upon participant self-reports (Veenman et al., 2013). However, there are also questionnaires such as the Metacognitive Awareness Index (MAI) (Shraw & Dennison, 1994) which are off-line assessments of a person's general awareness of their metacognitive ability and which are not connected to any specific task performance. Interviews have also been used as off-line assessments of strategy use (e.g. Mevarech & Amrany, 2008).

However, the question remains as to which method, on-line or off-line, measures metacognition most accurately. Veenman et al. (2013) contend that on-line measures are more accurate because they assess actual behaviour according to some sort of externally-established criteria, whereas off-line methods rely only on the participants' self-reports. If the assessment is retrospective this means that the participant has to construct the metacognitive events from memory which can be subject to failure, distortion or reinterpretation (Veenman et al., 2013). In addition, there is evidence that students' off-line self-reports do not converge with their actual on-line metacognitive strategy use (Veenman et al., 2013). In other words, students do not actually perform as they predict they will perform and did not perform as indicated by their retrospective self-reports (Veenman et al., 2013). Thus, on-line assessment seems a more accurate measure of metacognitive activity.

While on-line methods are perhaps preferable to off-line methods, Veenman and Van Hout-Wolters (2006) also acknowledge the need for multi-method designs. For example, Desoete (2008) investigated the relationship between mathematical problem-solving and metacognitive skills. She wanted to know what kind of assessment was an appropriate and accurate measure for young children. She conducted a study using two standardized mathematics tests, two off-line assessments (one prospective and one retrospective), one off-line calibration rating, one

on-line assessment (a think-aloud protocol analysis), one combined (on-line and off-line) assessment and a teacher rating of children's metacognitive ability. Thus, metacognition, mathematical ability and their interrelationship were measured in a number of different ways. In my study, I chose an on-line assessment in addition to mathematical exercises.

2.6 METACOGNITION AND MATHEMATICS EDUCATION

2.6.1 Schoenfeld's mathematical problem solving and metacognition

One of the early explicators of metacognition within mathematics education was Alan Schoenfeld in the 1980s. Schoenfeld was investigating his version of metacognitive knowledge and skills during approximately the same time as Brown. While Brown was a developmental psychologist and conducted many studies related to memory (Brown, 1978), Schoenfeld was a mathematician and mathematics professor. As stated in his seminal book, *Mathematical Problem Solving*, the two main questions that preoccupied him at the time were: "What does it mean to 'think mathematically'?" and "How do we help students to do it?" (Schoenfeld, 1985, p. xi). These are questions I believe I also need to pose continually as a mathematics educator. Schoenfeld's ideas have strongly influenced this dissertation and will be discussed in some detail.

In *Mathematical Problem Solving*, Schoenfeld (1985) developed a framework for analysing complex problem solving and identified four aspects required for that kind of intellectual activity. He called his component parts: *cognitive resources*, *heuristics*, *control* and *belief systems* (Schoenfeld, 1985, p. xii). *Cognitive resources* consist of the content knowledge and procedures a student has at his disposal; *heuristics* are 'rules of thumb' for getting through difficulties; *control* has to do with the efficiency with which someone tackles a problem; and *belief systems* represent the student's understanding of the discipline in which he is working (Schoenfeld, 1985). After nearly a decade of research at university level, Schoenfeld concluded that instruction that only includes "mastery of facts and procedures" will not produce the higher-order thinking skills required for mathematics, but if instruction "focuses on those skills, students can learn them" (Schoenfeld, 1985, p. xiii).

Two of Schoenfeld's four components relate directly to metacognition: heuristics and control. "Heuristics" and "strategies" can be used somewhat synonymously and as strategy use is one of the focus points of this paper, I will go into Schoenfeld's explication of heuristics in some detail.

Schoenfeld was introduced to the idea of heuristics in 1974 through George Pólya's book, *How to Solve It*, when Schoenfeld was a practising mathematician himself. He read through it avidly and acknowledged that the strategies he used as a mathematician were the ones Pólya described (Schoenfeld, 1985). Pólya defines heuristics as the "mental operations typically useful for the solution of problem" [sic] (1957, p.2) and included in his book "a short dictionary of heuristic" (1957, pp.37 - 232). Schoenfeld defines heuristics as "strategies and techniques for making

progress on unfamiliar or nonstandard problems; rules of thumb for effective problem solving” (Schoenfeld, 1985, p.15). Examples of heuristics used by Schoenfeld (but sourced from Pólya) include:

- “Drawing figures; introducing suitable notation
- Exploiting related problems
- Reformulating problems; working backwards
- Testing and verification procedures” (Schoenfeld, 1985, p. 15).

Pólya devised a four-part plan to solve problems: 1) understanding the problem; 2) devising a plan; 3) carrying out the plan; and 4) looking back (Pólya, 1957, p. xvi). For Pólya (1957), heuristics are applied in all four parts of the problem-solving process. Pólya’s intention to “revive heuristic in a modest and modern form” (Pólya, 1945, p.113), eventually resulted in problem-solving being the focus of school mathematics in America in the 1980s and much subsequent research into heuristics (Schoenfeld, 1985).

Schoenfeld provides a rationale for studying heuristics and for using heuristics when teaching problem solving. He gives four main reasons. The first is that students of mathematics, and eventually practicing mathematicians, solve “thousands upon thousands of problems” (Schoenfeld, 1985, p. 70). Sometimes the person finds a particular method that succeeds more than once. That method now becomes a *strategy* to employ in the solution of future problems and the individual begins to accumulate a storehouse of these strategies (Schoenfeld, 1985). Secondly, although the collection of strategies may be somewhat idiosyncratic to the experts in mathematics, there is also likely to be considerable overlap in the ways various experts approach problems, thus it is likely that there are a number of strategies that would be relevant for all workers of mathematics (Schoenfeld, 1985). Third, by experts using introspection to record their strategies (as Pólya did) or by others observing experts, a sort of catalogue of strategies could be compiled which Pólya began with his short dictionary of heuristics and continued in his later works (Schoenfeld, 1985). The final step in the rationale is to provide instruction in these strategies so that individuals do not need to go through the trouble of discovering them for themselves (Schoenfeld, 1985).

However, Schoenfeld states that as sensible as this rationale appears, it does not work as planned. He cites a number of studies that demonstrate that heuristics are “far more complex and far less tractable than had been hoped or expected” (Schoenfeld, 1985, p. 72). In his further explication of heuristics, he states three reasons why this is so. Firstly, he claims that strategies such as those given by Pólya are actually “labels” for categories of “more precisely defined strategies” that need to be fully explicated before they can be useful (Schoenfeld, 1985, p.73). Secondly, training in the use of strategies is much more complex than it originally appears because using any particular strategy actually means implementing a number of sub-phases which in turn require their own training to the same degree that is needed for standard subject content (Schoenfeld, 1985). Lastly, the knowledge of heuristics cannot replace subject-matter knowledge and in fact, the implementation of the heuristics relies heavily on accurate domain-specific understanding (Schoenfeld, 1985).

Because heuristics are themselves so complex, Schoenfeld (1985) stresses that the control component of his framework is critical. He described *control* as the executive decisions that people make as they work through a mathematical problem (Schoenfeld, 1985). At nearly every step of the solution process, using either the general heuristics or the sub-strategies that Schoenfeld explicates, decisions are required by the problem-solver that may lead to success or failure (Schoenfeld, 1985). Included in executive decisions are activities such as making plans, selecting goals and sub goals, monitoring and assessing solutions as they evolve, and revising or abandoning plans (Schoenfeld, 1985). In a series of studies, he compared the problem solving abilities of novices and experts and found a notable difference in their respective executive control processes (Schoenfeld, 1987).

Schoenfeld (1985) described four different types of control decisions and how they would affect problem solving. In “Type A”, “bad decisions guarantee failure” with problem solvers going on “wild goose chases” that waste resources while ignoring “potentially helpful directions” (Schoenfeld, 1985, p.116). “Type B” decisions are somewhat neutral; while “wild goose chases are curtailed before they cause disasters,” the problem solvers do not exploit resources available to them (Schoenfeld, 1985, p.116). For “Type C” behaviour, “control decisions are a positive force in a solution; the problem-solver chooses resources carefully, continuing with those that are productive and abandoning others after careful consideration” (Schoenfeld, 1985, p.116). Finally, in “Type D” behaviour, control decisions are not needed at all because the appropriate information and procedures are readily accessed from long-term memory (Schoenfeld, 1985).

In a series of studies of both undergraduate students and fellow mathematics professors, Schoenfeld (1985) tested his hypotheses with regard to the relationship between problem solving via heuristics and control procedures. Schoenfeld (1985, p. 107) developed a “prescriptive control strategy” to guide college students in integration problems. He developed his control strategy from the uniformity he discovered watching good problem solvers select strategies. He found, firstly, that when students worked on college-level, algorithm-based exercises on integration using a prescriptive control strategy that the students made significant improvements in their performance (Schoenfeld, 1985, p.106). In addition, he found that students’ problem-solving performance also significantly improved when trained in the use of heuristic techniques, *if* that training was accompanied by a prescriptive control strategy (Schoenfeld, 1985, p.144). In other words, just as heuristic use is dependent upon the resources available to the problem-solver, it is also dependent on the control strategies that guide the resources and heuristics chosen.

2.6.2 Empirical research on metacognition and mathematics

Many studies have been performed that have investigated the relationship between metacognition and mathematics. Brief summaries of articles that have informed my thinking and are more relevant to my research are presented here.

Aydin and Ubuz (2010) investigated the effects of two metacognitive constructs, knowledge of cognition (KNOOFCOG) and regulation of cognition (REGOFCOG), on three knowledge constructs: declarative knowledge (DECKNOW), conditional knowledge (CONKNOW) and procedural knowledge (PROKNOW) in the area of geometry. The authors defined declarative knowledge as knowledge referring to factual information, procedural knowledge as knowledge that compiles declarative knowledge into algorithms and conditional knowledge as the knowledge that enables the accessing of certain facts or employing certain procedures (Aydin & Ubuz, 2010). While acknowledging that conceptual knowledge may be construed as one form of mathematical knowledge, these authors preferred to separate conceptual knowledge into declarative knowledge and conditional knowledge (Aydin & Ubuz, 2010).

Using structural equation modelling, the authors found the following results. Firstly, knowledge of cognition had a positive direct effect on procedural knowledge, indicating that students who displayed characteristics of knowledge of cognition such as judging how well something is understood, tended to perform more successfully on procedural knowledge questions (Aydin & Ubuz, 2010). On the other hand, knowledge of cognition had a direct negative effect on declarative knowledge, indicating that students who learn best by understanding a topic do not emphasise knowing mathematical facts when learning (Aydin & Ubuz, 2010). In contrast, regulation of cognition had a positive direct effect on declarative knowledge, indicating that students who pay attention to important information tend to recognize core knowledge and use that to make further generalizations (Aydin & Ubuz, 2010). At the same time, regulation of cognition had a direct negative effect on procedural knowledge, showing that students who displayed metacognitive behaviours such as thinking of different ways to solve a problem were less successful on procedural knowledge questions (Aydin & Ubuz, 2010). While some of the findings of these authors are unexpected, the study suggests that metacognitive behaviours have a significant effect on mathematical performance.

Artzt and Armour-Thomas (1992) developed a cognitive-metacognitive framework to distinguish between cognitive and metacognitive observable behaviours. The authors based their framework on Schoenfeld's (1985) protocol parsing and Garofalo and Lester's (1985) metacognitive aspects of performance of mathematical tasks. Artzt and Armour-Thomas (1992) identified eight problem-solving episodes: read, understand, analyse, explore, implement, verify, and watch and listen. Of these, they categorized reading as purely cognitive; understanding and analysing as purely metacognitive; and exploring, implementing and verifying as a mixture of cognitive and metacognitive (Artzt & Armour-Thomas, 1992). Watching and listening, although acknowledged as important in the problem solving process, could not be categorized as either cognitive or metacognitive as there is no verbalization observable to categorise (Artzt & Armour-Thomas, 1992). The framework developed by Artzt and Armour-Thomas (1992) is significant because it allows other researchers to distinguish between cognitive and metacognitive behaviours (one of the issues surrounding the study of metacognition); and it operationalises those behaviours. In addition, the research highlights the interplay between cognitive and metacognitive behaviours when solving mathematical problems and the important role metacognitive behaviours have in enhancing and propelling the problem-solving process (Artzt & Armour-Thomas, 1992).

Lucangeli and Cornoldi (1997) studied the correlation between mathematical ability and the metacognitive processes of predicting, planning, monitoring and evaluating. Using an instrument that required 3rd and 4th graders to complete mathematical tasks and answer questions about their metacognitive activities simultaneously, the authors discovered a strong correlation between metacognitive processes and mathematical ability (Lucangeli & Cornoldi, 1997). The children were tested on arithmetic, geometry and problem solving (Lucangeli & Cornoldi, 1997). The results showed a strong correlation between the metacognitive components and the geometry and problem solving tasks for both groups, while a correlation with arithmetic only existed for the 3rd graders (Lucangeli & Cornoldi, 1997). By the 4th grade, learners are much more familiar with arithmetic procedures and the authors concluded that the relationship between metacognition and mathematical performance is closer for tasks that are performed less automatically (Lucangeli & Cornoldi, 1997).

Desoete (2008) used multiple methods to investigate the relationship between mathematical problem solving and metacognitive skills in a group of third graders in Belgium. She distinguished between *on-line* and *off-line* assessments: on-line assessments measured metacognitive skills concurrently with mathematical problem solving, while off-line assessments measured metacognitive skills either before (prospective techniques) or after (retrospective techniques) the cognitive activity (Desoete, 2008). She investigated if prospective, retrospective, on-line and combined techniques or teacher ratings could be used to effectively assess metacognitive skills in relation to mathematical performance (Desoete, 2008). Desoete (2008) assessed four metacognitive skills: prediction, planning, monitoring and evaluating, and two aspects of mathematical performance, namely, mathematical reasoning and numerical facility.

Using a small sample ($N = 20$) of third graders in a large urban city in Flanders, Desoete (2008) administered two mathematics tests and three metacognitive tests: one prospective, two retrospective and one combined. She also conducted videotaped think-aloud protocols and surveyed the children's teacher. In addition, the participants' intelligence quotients (IQ) were established prior to the study (Desoete, 2008).

Desoete (2008) found that the teacher ratings on prediction skills correlated significantly with the combined metacognitive assessment prediction skills. The teacher's ratings on planning and monitoring skills also correlated significantly with the think-aloud protocol results measuring the same (Desoete, 2008). In addition, the teacher's ratings on evaluation skills correlated with the one retrospective test, the think-aloud protocols and the combined test (Desoete, 2008). Given that metacognitive skills testing is a time-consuming process, the study shows that an experienced, insightful teacher can be a significant resource for metacognitive assessment.

Interestingly, Desoete (2008) found that although there were some significant correlations for the metacognitive variables among the different metacognitive assessments, none of the metacognitive variables predicted mathematics performance once the IQ measures were covaried out (Desoete, 2008). These results are in alignment with a model of metacognition

conceived as being a manifestation of intelligence and not a separate construct on its own (Desoete, 2008). Lastly, Desoete also found that while significant correlations existed between the prospective and retrospective assessments, the metacognitive skills were not influenced by the actual performance of the children (Desoete, 2008). These findings confirmed the results of Desoete's previous studies that metacognitive skills need to be explicitly taught and do not develop freely just from exposure to mathematics (Desoete, 2008).

2.6.3 Metacognition training and the IMPROVE system

One series of studies investigating metacognition and its relation to mathematics performance that formed a significant springboard for my study involved the metacognitive skills training programme called IMPROVE. This programme was started as an innovative way to meet the challenges of diversity within non-tracked, heterogeneous classrooms (Mevarech & Kramarski, 1997). The programme comprised the elements indicated by the acronym: **I**ntroducing new concepts, **M**etacognitive questioning, **P**racticing, **R**eviewing and reducing difficulties, **O**btaining mastery, **V**erification, and **E**nrichment. The programme consisted of three interrelated components: metacognitive activities, peer interaction, and the provision of feedback, including correction and enrichment (Mevarech & Kramarski, 1997).

The authors investigated a number of aspects over the course of a decade, mostly using experimental designs. The initial study included the effect of IMPROVE as a whole on the information processing and mathematical reasoning of 7th graders in Israel using close to 300 participants (Mevarech & Kramarski, 1997). Later studies included investigating the effects of metacognitive training vs. strategy training on mathematical problem solving (Mevarech, 1999); the effects of metacognitive training vs. worked-out examples on students' mathematical reasoning (Mevarech & Kramarski, 2003); the effects of IMPROVE on mathematical knowledge, mathematical reasoning and metacognition (Mevarech & Fridkin, 2006); and the immediate and delayed effects of metacognitive instruction on the regulation of cognition and mathematics achievement (Mevarech & Amrany, 2008). In every study, their research showed that metacognition was linked to improved mathematical performance.

The original 1997 study investigated metacognitive questioning within small cooperative groups. As many aspects of this study are relevant to my own, I will go into some detail to describe it. According to Mevarech and Kramarski (1997), there is evidence to suggest that guiding students to formulate certain questions may have a number of beneficial effects. The questions may elicit elaborate explanations; they may encourage students to justify when, why and how to use specific strategies; and the questions may help students develop inferences about concepts and new perspectives on their existing knowledge. Drawing on the works of King, Schoenfeld, and Pólya, Mevarech and Kramarski (1997) initially developed three metacognitive questions that students would ask each other. *Comprehension questions* enable the learners to orient themselves to the main ideas of a problem, categorise the problem and elaborate new concepts; *connection questions* help learners make links to previous problems, identifying similar and different characteristics; *strategic questions* guide students to find

strategies appropriate to a problem (Mevarech & Kramarski, 1997). In a later study, *reflection questions* were added (Mevarech & Fridkin, 2006). These encouraged students to evaluate the solution process while working on the problem or after the problem was completed (Mevarech & Fridkin, 2006).

The 1997 paper addressed two studies. In the first, 99 Grade 7 learners were part of the treatment group in one district in Israel where the IMPROVE system had been in place for 2 – 3 years and 148 learners were members of the control group in another district that did not use IMPROVE (Mevarech & Kramarski, 1997). The treatment and control groups used the same textbook, covered the same content of Introduction to Algebra, were of similar mathematical abilities and were members of similar socioeconomic groups. The first study took place over one semester. The IMPROVE students were split into 4-member heterogeneous groups with one high, two middle and one low achieving members (Mevarech & Kramarski, 1997). The teacher would present new material by asking three metacognitive questions: “What’s in the problem? What are the differences/similarities between . . . and . . .? and What strategies/tactics/principles were appropriate for solving the problem at hand?” (Mevarech & Kramarski, 1997, p. 377). These represented the comprehension, connection and strategic questions respectively.

The students then worked in their groups on the prepared materials, asking and answering each other’s questions and the solutions were complete when all group members agreed (Mevarech & Kramarski, 1997). The students’ answers included not only the solution to the problem, but their mathematical explanations and some of their metacognitive responses as guided by their learning materials (Mevarech & Kramarski, 1997). Students could ask for teacher assistance if they came to an impasse. While the students worked, the teacher would join one team and work as a member, reading aloud the problem, using the metacognitive questions and explaining the steps to solve (Mevarech & Kramarski, 1997). After every unit, a formative assessment was given with a large percentage of the problems being at a high cognitive level according to Bloom’s taxonomy (Mevarech & Kramarski, 1997). Those that did not achieve mastery (80% or higher) were given corrective materials and allowed to rewrite a parallel version of the formative test. Those in the mastery category were given enrichment materials that focused on mathematical reasoning (Mevarech & Kramarski, 1997). In the feedback-corrective-enrichment phase, the students worked in homogeneous groups and the teacher assisted the different groups as necessary (Mevarech & Kramarski, 1997). The feedback-corrective-enrichment lesson was timetabled for one period every other week (Mevarech & Kramarski, 1997).

The first treatment lasted for one semester, after which a 36-item mathematics achievement test on Introduction to Algebra was administered to all participants. The test contained 25 multiple-choice items testing factual knowledge and computation problems; the remaining 11 items did not require computation but tested mathematical reasoning (Mevarech & Kramarski, 1997). Because the control group was in tracked classes, the analysis of the results compared the different ability groups within both the treatment and control groups. The results showed that the middle- to high achieving IMPROVE students outperformed their control counterparts

on the Introduction to Algebra test, but not the low achieving students. However, all three ability groups using IMPROVE outperformed their control counterparts for the mathematical reasoning section (Mevarech & Kramarski, 1997). The second study, which extended the research to more treatment participants ($N = 164$) and one full school year demonstrated similar results (Mevarech & Kramarski, 1997).

Veenman and Van Hout-Wolters (2006) claim that research shows that three principles need to be followed for effective instruction in metacognition: a) metacognition needs to be embedded in the content to ensure connectivity; b) learners need to be informed about the usefulness of metacognition to motivate them to make the effort to use it, and c) training needs to be prolonged to guarantee sustained application of the metacognitive activity. The self-questioning technique of the IMPROVE system followed those principles and I incorporated self-questioning as one of the strategies in my own intervention. However, as my assessments were individual, I used self-directed questions instead of peer-directed.

2.7 COGNITIVE SCIENCE, MATHEMATICS, AND MATHEMATICS EDUCATION

Cognitive science is a broad-ranging discipline that began in the 1960s and is concerned with many aspects ranging from representation of knowledge to learning and memory to problem solving to descriptions of computer programs that model human-like abilities (Schoenfeld, 1987b). According to Schoenfeld, a fundamental premise of cognitive science is that “mental structures and cognitive processes . . . are extremely rich and complex” but understandable; and understanding them will “yield significant insights into the ways that thinking and learning take place” (Schoenfeld, 1987b, p. 2). Cognitive science studies aim to discover the mental processes that lead to “productive thinking;” they tend to be very detailed and conducted on few subjects (Schoenfeld, 1987b, p. 2). These studies also focus on “process analyses,” that is, “on the means used to obtain a particular result,” and were in contrast to the “product analyses” of typical mathematical educational research of the 1960s and 1970s (Schoenfeld, 1987b, p. 8). Instead of merely looking at the result of problem-solving efforts, that is, whether the answer is right or wrong, cognitive scientists examine in minute detail what people do when they solve problems. This distinction between process vs. product was also a key element in how I analysed my data as well.

While there is no single methodology that uniquely defines cognitive science research, Schoenfeld (1987b) claims there are features common to many cognitive analyses. A cognitive science researcher is likely to videotape a few students who are asked to work aloud through a particular kind of problem; the videotape and the work produced is then “analysed in detail, with the intention of identifying consistent behaviour patterns” (Schoenfeld, 1987b, p. 9). From that analysis, the researcher will create a “process model” of the student’s understanding that specifies the particular knowledge of the student, the thinking strategies employed by the student and interactions between the two (Schoenfeld, 1987b, p. 9). From that process model, the cognitive scientist will design a computer program that models the subject’s methods and simulates his responses; the degree to which the computer program reproduces the subject’s

methods indicates the accuracy of the process model and the accuracy with which the researcher understands the subject's thinking (Schoenfeld, 1987b). In other words, if the computer program gets the same answer as the subject, the subject's thinking has been modelled correctly; if not, then the researcher knows he does not understand the subject's thinking processes (Schoenfeld, 1987b). The value in the method, from a research perspective, lies in the fact that empirical methods are created to verify the correct interpretation of internal, mental processes.

The worth of applying a cognitive science approach to mathematical problem solving is extensive. Firstly, cognitive science research demonstrates that a sufficiently detailed prescriptive method can result in accurately solved problems by developing machines that learned to play chess, solve cryptarithmic problems and prove theorems (Schoenfeld, 1987b). Schoenfeld used this information to develop what he called a "prescriptive control strategy" to teach college students how to use executive control strategies to solve integration problems (Schoenfeld, 1985). By *prescriptive*, Schoenfeld is not referring to a mechanical following of procedures, but providing descriptions that are in sufficient detail that they can be applied to a number of problem types. However, a key to the prescriptive procedures of the cognitive scientist lies in the very fine "grain size" of the analyses the cognitive scientist performs in order to produce accurate process models (Schoenfeld, 1987b). And it is this aspect of "grain size" that becomes very important when teaching students strategies.

The prominence of problem solving as a mathematical goal began in the 1980s but owed its roots to the publication of Pólya's seminal book, *How to Solve It* (Schoenfeld, 1987b). Pólya's work, which underlies much of the research in this paper, will be described in more detail below; however, it is now being considered with regard to Schoenfeld's ideas of mathematics education and cognitive science. Schoenfeld maintains that while Pólya's ideas were brilliant and his four-step method of problem solving became the foundation for problem solving in mathematics education, the use of Pólya's methods met with mixed, inconclusive results (Schoenfeld, 1987b). Schoenfeld attributes this to the distinction he makes between the *description* and *prescription* of strategies (Schoenfeld, 1987b). A description "merely characterises a procedure in sufficient detail for it to be recognised," while prescription "characterises a procedure in precise enough detail so that the characterisation serves as a guide for implementing the strategy" (Schoenfeld, 1987b, p. 18). Like with cognitive science process models, Schoenfeld found that Pólya's strategies needed to be broken down into the appropriate level of detail to be effective. If the "grain size" was too coarse, strategies were not implementable (Schoenfeld, 1987b). Herein lies one of the obstacles to effective strategy instruction.

2.8 MATHEMATICAL PROFICIENCY

2.8.1 The Strands of Mathematical Proficiency

Mathematical proficiency is not a simple construct and one must realize its complexity in order to fully appreciate what is required for successful mathematical thinking. Two views of mathematical thinking or proficiency will be outlined here, that by Kilpatrick, Swafford and Findell (2001) and Schoenfeld (1985). These two sets of authors have been foundational in the development of my research.

In order to understand the definition of mathematical proficiency advanced by Kilpatrick et al. (2001), one needs to know the context in which it arose. Mathematics instruction in the United States was fairly stable in the first half of the twentieth century and relied upon the pedagogical method of “drill and practice” as a result of Thorndike’s associationist learning theory (Schoenfeld, 1987b). However, in the succeeding years, mathematics instruction swung like a pendulum roughly every decade (Schoenfeld, 1987b). The “drill and practice” method of the 1950s was followed by the “new math” of the 1960s as an attempt to upgrade America’s mathematics instruction in response to a perceived threat of superior Soviet educational practices that had lead the Soviets to manned space flight before the US (Schoenfeld, 1987b). When the “new math” was perceived as a failure, a “back to basics” movement started in the 1970s (Schoenfeld, 1987b). As the Scholastic Aptitude Test scores declined steadily through the following decade, the next curricular swing was the “problem solving” and “mathematical power” reform movement in the 1980s and 1990s, supported fully by the National Council of Teachers of Mathematics (Schoenfeld, 1987b; Kilpatrick et al., 2001). Later, the pendulum swung yet again in reaction to the reform movement and the importance of memorizing mathematical facts, computing accurately and proving mathematical assertions were stressed (Kilpatrick et al., 2001).

In addition to the curricular swings, research in mathematics education started to change. Research through the 1950s to the late 1970s focused on large scale, statistical analyses which made the assumption that data provided by large numbers of participants were more accurate than data provided by single or small groups of participants (Schoenfeld, 1987b). Researchers also assumed, that like studies conducted in disciplines such as agriculture, variables could be rigorously controlled in educational research (Schoenfeld, 1987b). However, studies in the 1960s and 1970s could not show conclusively that given treatments had statistically significant effects; thus, describing how learning took place was more complicated than expected (Schoenfeld, 1987b). In addition, some research began to move from *product analyses* to *process analyses*; in other words, instead of looking at *what* a student may know by answering a multiple-choice test, research began looking at *how* the student knows (Schoenfeld, 1987b). At the same time, research in cognitive science began to grow and inform ideas about mathematical thinking (Schoenfeld, 1987b).

Taking the previous history of mathematics education into account and the new direction of research, the authors of the strands of mathematical proficiency drew their inspiration from many sources. After an extensive study of mathematics performance in the United States, Kilpatrick et al. (2001) developed an in-depth description of mathematical proficiency. Their “composite, comprehensive view of successful mathematics learning” was derived from cognitive psychology, the authors’ experiences as learners, teachers and educators of mathematics, and their analyses of the mathematics to be learned in school (Kilpatrick et al., 2001, p. 116). They chose the term *mathematical proficiency* to describe the interrelated competencies, knowledge and facilities required to learn mathematics successfully. They claim that mathematical proficiency is the result of five interdependent strands: *conceptual understanding*, *procedural fluency*, *strategic competence*, *adaptive reasoning* and *productive disposition* (Kilpatrick et al., 2001). These authors define their concepts in the following way:

- “conceptual understanding – comprehension of mathematical concepts, operations and relations
- procedural fluency – skill in carrying out procedures flexibly, accurately, efficiently and appropriately
- strategic competence – ability to formulate, represent and solve mathematical problems
- adaptive reasoning – capacity for logical thought, reflection, explanation and justification
- productive disposition – habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy” (Kilpatrick et al., 2001, p. 116).

The use of the metaphor “strand” is significant and emphasises the interrelatedness of the components: just as a rope is as strong as the strength and connectedness of its individual strands, so is mathematical proficiency.

Although Kilpatrick et al. (2001) describe five strands, I will concentrate on the three that are the focus of this paper. The first strand is *conceptual understanding*. Conceptual understanding is defined as the “integrated and functional grasp of mathematical ideas” (Kilpatrick et al., 2001, p. 118). Students with conceptual understanding have “organised their knowledge into a coherent whole” and they understand “why a mathematical idea is important and the kinds of contexts in which it is useful” (Kilpatrick et al., 2001, p. 118). Students with conceptual understanding retain what they learn more readily because their knowledge is connected and are less likely to remember incorrectly because they understand what they learn (Kilpatrick et al., 2001). Because their memory is based upon understanding, they can “monitor what they remember and try to figure out whether it makes sense” (Kilpatrick et al., 2001, p. 118). An important aspect of conceptual understanding is “being able to represent mathematical situations in different ways and knowing how different representations can be useful for different purposes” (Kilpatrick et al., 2001, p. 118). With conceptual understanding, students understand the connections between various representations and their similarities and differences; they can also use their understanding to generate new knowledge and solve unfamiliar problems (Kilpatrick et al., 2001). Students with conceptual understanding

encapsulate their knowledge into “compact clusters of interrelated facts and principles” which can be used efficiently in appropriate situations (Kilpatrick et al., 2001, p. 120). Developing conceptual understanding provides a foundation upon which the other strands can be built, and which in turn is improved through the strength of the other strands.

The second strand, *procedural fluency* is described as the “knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently” (Kilpatrick et al., 2001, p. 121). While at first glance this may appear as “drill and practice” in a new guise, the strand involves much more than that. Kilpatrick et al. (2001) maintain that students need to develop procedural fluency first with whole number calculations and be able to move flexibly between working mentally, with pen and paper, using tools such as calculators or computers or working with manipulative materials; they need to know which method to use and under what conditions each is effective. To promote procedural fluency over conceptual understanding or vice versa as often happened in the past was to create a “false dichotomy”: procedural fluency and conceptual understanding are mutually supportive (Kilpatrick et al., 2001). For example, working with algorithms benefits both procedural fluency and conceptual understanding because algorithms often provide efficient methods of solution and understanding the algorithm promotes understanding the mathematical structure from which the algorithm is derived (Kilpatrick et al., 2001). However, the authors warn that if procedures are introduced without promoting understanding, students may practice incorrect procedures or resist later instruction in penetrating the reasons underlying the procedure (Kilpatrick et al., 2001). They also claim that without sufficient procedural fluency, students cannot deepen their understanding of mathematical ideas or solve problems: so much attention is given to working out a problem that important relationships go unnoticed (Kilpatrick et al., 2001). In addition, learning skills without understanding has the consequence that information is learnt as unrelated bits of knowledge and provide no connected network upon which to build new knowledge (Kilpatrick et al., 2001). In short, conceptual understanding and procedural fluency are interdependent.

The third strand that is the focus of this paper is *strategic competence* which “refers to the ability to formulate mathematical problems, represent them, and solve them” (Kilpatrick et al., 2001, p. 124). This seemingly compact strand is similar to the heavily researched area of mathematical problem solving (Kilpatrick et al., 2001). Fundamental to the solving process is flexibility: students need to be able to have a repertoire of strategies and know which strategies are useful for solving specific problems. They may use “reasoning, guess-and-check, algebraic or other methods to suit the demands presented by the problem” (Kilpatrick et al., 2001, p. 127). Unpacking each of the sub-steps of strategic competence – formulating, representing and solving – reveals they rely heavily on both conceptual understanding and procedural fluency.

Although in school mathematics the problems are usually formulated for the students, in mathematics outside of school the first step in solving a problem would be to formulate it (Kilpatrick et al., 2001). Setting projects where students have an ill-defined problem would give them experience in formulating questions. The student’s conceptual understanding may

determine whether they are able to formulate a rich, interesting problem or a relatively superficial one.

Once the problem is formulated, the student then needs to represent it; this could be done numerically, verbally, graphically or symbolically and the student's conceptual understanding and procedural fluency come into play (Kilpatrick et al., 2001). In order to represent a problem, the student needs to create a mental image of the problem's essential parts and avoid "number grabbing" where students pick numbers out of a problem and perform arithmetic or algebraic operations on them (Kilpatrick et al., 2001). An indication of strategic competence is the ability to generate a problem model and representation that represents the essential features of the problem and ignores irrelevant elements (Kilpatrick et al., 2001). Students need to be able to distinguish what is known from what is required and recognise the similarity in mathematical structure that many problems share even if the surface features seem very different (Kilpatrick et al., 2001). All of this depends upon sound conceptual understanding.

Once the student begins to solve the problem, procedural fluency comes into play and a well-formulated, well-represented problem without the procedural skills to complete it leaves the student at a loss. On the other hand, strategic competence assists in developing procedural fluency (Kilpatrick et al., 2001). An example from Schoenfeld's research illustrates the intimate connection between the strands of strategic competence and procedural fluency. Schoenfeld (1987b) describes an experiment conducted by cognitive scientist Bundy in 1975. Bundy proposed that experienced mathematicians use high level strategies in the service of goals to solve complicated mathematical problems (Schoenfeld, 1987b). Two of the strategies they employ would be appropriate at high school level as well: 1) "collect like terms to simplify expressions," and 2) "try to replace 'nasty' terms with nicer ones" (Schoenfeld, 1987b). The mathematicians' strategic competence allowed them to fluently solve log, radical and trigonometric equations using similar fundamental strategies. And for the mathematicians, both their strategic competence and procedural fluency were supported by robust conceptual understanding.

2.8.2 Schoenfeld's conception of mathematical proficiency

Two questions that preoccupied Schoenfeld have also preoccupied me as an educator: "What does it mean to 'think mathematically'?" and "How can we help students to do it?" (Schoenfeld, 1985, p. xi). Those two questions have made me pursue the connection between mathematical proficiency and metacognition. Many students may agree with the perspective "almost everything students do in school is a problem" (Davidson & Sternberg, 1998, p. 47) and this is especially so for mathematics. If a problem can be defined as an "impasse en route to a particular goal state" (Declos & Harrington, 1991, p. 35), then students face that every day and need help to overcome those impasses. Even if exercises are distinguished from problems by the ease of access to a procedure that solves the problem (Schoenfeld, 1985), most learners are still faced with developing that "ease of access". Thus, as Schoenfeld (1985, p. 4) states,

problem solving processes are “absolutely central in *any* discussion of mathematical performance” [my emphasis].

As discussed above under metacognition, Schoenfeld (1985) developed his own framework for mathematical behaviour. Through his research he identified four aspects that affect mathematical problem solving: *resources*, *heuristics*, *control* and *belief systems* (Schoenfeld, 1985). *Resources* are seen as the inventory of mathematical knowledge the student brings to a problem and the ways in which he accesses that knowledge (Schoenfeld, 1985). *Heuristic strategies* are rules of thumb that lead to successful problem solving and help progress the solution process (Schoenfeld, 1985). Schoenfeld drew many of his ideas about heuristic strategies from Pólya (1957). *Control* refers to the critical executive decisions an individual makes during the problem solving process that can lead to success or failure; this is very similar to regulation of cognition in the metacognitive literature (Schoenfeld, 1985). Lastly, *belief systems* refer to the collection of beliefs about mathematics, held consciously or not, that provides the psychological context for the mathematical problem solving (Schoenfeld, 1985).

Schoenfeld’s views of what it means to *understand* mathematics evolved over time. Initially in his career he took the “ability to solve problems” as his operational definition of understanding mathematics; in other words, if the students could “do” mathematics, they understood mathematics (Schoenfeld, 1985, p. 12). He assumed that if students had the correct foundation of basic mathematical knowledge (resources), familiarity with a broad range of problem-solving techniques (heuristics) and were taught how to strategically use those resources and heuristics (control), then the students could solve many kinds of problems successfully – which they did – and his operational definition seemed true (Schoenfeld, 1985). However, once he began videotaping the students’ problem-solving activities, other issues came to light.

For example, the students’ resources were much weaker than their performance on standard tests would indicate. In the videos, the students demonstrated that they were not actually aware of, nor could use, mathematical heuristics. The students also showed little ability for executive control of their problem-solving processes. Finally, their belief systems, revealed in their problem-solving performance, showed serious misunderstandings about the nature of mathematics (Schoenfeld, 1985, p. 13). From this, Schoenfeld concluded that using the ability to solve problems as an operational definition of *understanding* is far too limited and in order to explain problem-solving or teach it, one must consider:

“(1) whatever mathematical information problem solvers understand or misunderstand, and might bring to bear on the problem; (2) techniques they have (or lack) for making progress when things look bleak; (3) the way they use, or fail to use, the information at their disposal; and (4) their mathematical world view, which determines the ways that the knowledge in the first three categories is used” (Schoenfeld, 1985, p. 14).

Schoenfeld, by focusing on understanding, rather than getting the right answer, emphasised a process-oriented approach rather than a product-oriented approach. His emphasis on process also guided the research in this paper.

Both sets of authors seem to have common ideas within their frameworks as well as distinct perspectives on mathematical thinking. For example, I would posit that Schoenfeld's idea of resources would include conceptual understanding and procedural fluency as well as adaptive reasoning. The strand of strategic competence could fall within resources and/or heuristics within Schoenfeld's framework. In addition, the strand of productive disposition could be a part of Schoenfeld's beliefs, but again there may be aspects that are distinct to both frameworks. However, both frameworks emphasise the importance that affect brings to any mathematical task. What the Kilpatrick et al. (2001) conception does lack, though, is the explicit mention of metacognition. Although, as will be discussed below, the very words that Kilpatrick et al. (2001) use to describe the strands indicate the presence of metacognition. Possible other ways to compare the two conceptions is to see Schoenfeld's framework as perhaps more descriptive of what *actually* occurs as students solve problems while the Kilpatrick et al. (2001) model is more descriptive of what *should* occur to promote success in mathematics. Or, lastly, the strands of mathematical proficiency may be more appropriate and reachable for school-level mathematics and Schoenfeld's four components apply to higher level mathematics. However, conceived, I believe both conceptualisations enrich the discussion surrounding mathematical proficiency.

2.9 PÓLYA'S PROBLEM SOLVING HEURISTICS

George Pólya's little book, *How to Solve It*, introduced in 1945 and issued as a second edition in 1957, was the introspective reflections of a prominent mathematician on the strategies he used to solve problems (Schoenfeld, 1987b). Pólya's four phase problem solving model – *understanding the problem, devising a plan, carrying out the plan, and looking back* – became the approach adopted by the mathematics education community in the 1980s (Schoenfeld, 1987b) and is still found in textbooks today. What receives less attention are Pólya's heuristics.

Pólya defines "heuristic" as the "study of methods of solution" (Pólya, 1957, p. vii), while Schoenfeld, using the plural "heuristics," sees them as "rules of thumb for making progress on difficult problems" (Schoenfeld, 1987b). On reading through Pólya's book, both definitions can be applied and it is the latter that this report uses.

In Pólya's "Short Dictionary of Heuristic" (1957, pp. 37 – 232), he introduces approximately 67 approaches or insights one could use to solve problems arranged in alphabetical order. Some, such as "Lemma" are specific to certain areas (in this case, proof), whereas others, such as "Draw a figure" can be applied across many problem types. In my research, I focused on eight that I felt would be generally applicable and helpful in a high school mathematics context. These were: *Introduce auxiliary elements, Introduce auxiliary problem, Draw a figure,*

Decompose and recombine, Generalization, Specialization, Work backwards and Variation of the problem. Each will be described briefly and how they can be applied at a high school level.

Pólya describes what he calls “*auxiliary elements*” as “an element we introduce in the hope that it will further the solution” (Pólya, 1957, p. 46). Typical examples from high school mathematics would include constructing a perpendicular to calculate the area of a triangle, or substituting a variable such as k to replace a complicated expression in order to work more fluently to solve an equation (often referred to as a “ k -substitution” in South African textbooks).

Similar to auxiliary element, an “*auxiliary problem*” is a “problem we consider, not for its own sake, but because we hope that its consideration may help us to solve another problem, our original problem” (Pólya, 1957, p. 50). At high school level, a typical example would be setting up an equation that solves for one variable that can be used to solve for another variable using simultaneous equations. Questions on both of my instruments required using an auxiliary problem strategy in order to solve.

“*Draw a figure*” is a strategy that I stress as a teacher again and again. Often relationships can be seen visually that numeric or symbolic expressions do not communicate as readily. Pólya (1957) stresses the importance of introducing suitable notation when drawing the figure to aid the solution process.

Pólya claims that “*decomposing and recombining*” are “important operations of the mind” that apply to scenarios as varied as renting a house, deciphering a cryptic telegram or solving a mathematical problem (Pólya, 1957, p. 75). A person begins by getting an idea of the object as a whole, then focuses his attention on one particular detail after another; when the person considers the object as a whole again, it is seen differently (Pólya, 1957, p. 76). As Pólya states, “you decompose the whole into its parts, and you recombine the parts into a more or less different whole” (Pólya, 1957, p. 76). This strategy is extremely useful for area, volume and surface area problems, trigonometry in two- or three-dimensions or analysing the elements in a geometry figure. From my experience as a teacher, I think it is a strategy that does not come naturally to most students.

Pólya defines “*generalization*” as “passing from the consideration of one object to the consideration of a set containing that object; or passing from the consideration of a restricted set to that of a more comprehensive set containing the restricted one” (Pólya, 1957, p. 108). Generalising is a fundamental activity of mathematics (Gravemeijer & Terwel, 2000), but at high school level, an example of content where the strategy of generalising can be used is patterns. Whether the pattern is visual, numeric or symbolic, being able to see the connection between all the elements is a result of generalising.

The definition of “*specialization*” is (as expected) the opposite of generalization: “passing from the consideration of a given set of objects to that of a smaller set, or of just one object, contained in the given set” (Pólya, 1957, p. 190). Specialization as a strategy includes looking at special cases of a certain set of objects; for example, these special cases may help produce a counter

example that refutes a statement or the special cases may shed light on the problem that results in generalisation and a solution (Pólya, 1957). While specialization may not be as universally applicable as some of the other strategies at high school level, it may provide a starting point to tackle a difficult problem.

Pólya (1957) gives a lengthy description of “*working backwards*” and attributes the exposition of the method to a Greek mathematician, Pappus, living in approximately 300 A.D. He admits there is a “certain psychological difficulty” in “not following the direct path to the desired end,” but states that “anybody can do it with a little common sense” (Pólya, 1957, p. 230). The strategy works by looking for successive antecedents of the desired end goal. Many problems in high school would benefit from a “working backwards” strategy: examples are proofs in Geometry (e.g. in order to prove proposition C, I need to demonstrate proposition B, in order to demonstrate B, I need to demonstrate A), trigonometric equations requiring identities, and volume and surface area problems to name a few. Often the “working backwards” strategy needs to be combined with “decomposing and recombining” or “auxiliary problem” strategies.

Pólya also describes the heuristic “*variation of the problem*” in some detail and states that a man, compared to an insect or a mouse, should be able to “explore the various possibilities with more understanding, to learn by his errors and shortcomings” (Pólya, 1957, p. 209). In a practical sense, I encouraged learners to vary the problems by introducing a similar, but much simpler problem. For example, students will often act intuitively on numeric situations, but have trouble with the same problem in variables. If they can understand the operations they applied mentally to numbers, they can often translate that into an equation with variables.

In addition to the “Short Dictionary of Heuristics,” Pólya also describes his steps for problem solving as a *process* and used verbs in the present continuous tense: *understanding the problem*; *devising a plan*; *carrying out the plan*; and *looking back* (Pólya, 1957, p. xvi). Pólya presented his steps as sets of questions to guide the solution process. As these corresponded to the self-questioning techniques of IMPROVE, I also used them in my classes for the first two problem-solving steps.

To begin to understand a given problem, Pólya (1957, p. xvi) asks three questions: “*What is the unknown? What are the data? What is the condition?*” These questions help to bring the particulars of the problem into focus. The *unknown* could be the quantity the problem calls for, or a quantity required in order to determine the end goal of the problem, or both (Pólya, 1957). The *data* are the givens in the question; they may be stated explicitly or inferred or both (Pólya, 1957). The *condition* can be the relationship between the data and unknown(s), a specific situation of the problem or something else unique to the problem (Pólya, 1957). By identifying the data, condition and unknown, the problem solver can begin to unpack the problem in order to solve. To devise a plan, Pólya (1957, p. xvi) uses questions such as, “*Do you know a related problem?*” “*Have you seen it before?*” These questions can aid the problem solver to access previously solved problems to help solve the current one.

2.10 CONCEPTUAL FRAMEWORK

The conceptual framework for this study draws from four main sources: Pólya's description of heuristics; three strands of mathematical proficiency from the model of Kilpatrick et al. (2001); Shraw and Dennison's division of metacognitive knowledge into three aspects; and my own experience as a learner and teacher of high school mathematics. These four components combined will form the lens through which to view my data.

In order to construct my conceptual framework, I began by adapting Shraw and Dennison's (1994) operational definitions of the component categories of their Metacognitive Awareness Index (MAI) to the domain of mathematics. As its name implies, Shraw and Dennison (1994) developed the Metacognitive Awareness Index (MAI) to assess adults' metacognitive awareness. It is a 52-item inventory developed around the two widely accepted (Brown, 1987) components of metacognition: knowledge of cognition and regulation of cognition. Statistical analyses conducted by Shraw and Dennison (1994) revealed that the 2-factor model (knowledge of cognition and regulation of cognition) was reliable and intercorrelated. Shraw and Dennison operationalised the three component categories of knowledge of cognition as follows:

1. Declarative knowledge: knowledge about one's skills, intellectual resources and abilities as a learner.
2. Procedural knowledge: knowledge about *how* to implement learning procedures (e.g. strategies).
3. Conditional knowledge: knowledge about *when* and *why* to use learning procedures. (Shraw & Dennison, 1994, p. 474)

In the MAI, Shraw and Dennison (1994) were specifically developing an instrument that could easily and accurately measure an individual's general metacognitive awareness. However, in another paper, Shraw (2002, p. 5) indicates that both knowledge of cognition and regulation of cognition appear to "span a wide variety of subject areas and domains –that is, they are domain general in nature". Thus, I believed I could adapt Shraw and Dennison's operational definitions of general knowledge of cognition to declarative, procedural and conditional knowledge relating to the *strands themselves* and *mathematical strategies*. Therefore, I will define metacognitive knowledge of strategy use for the purposes of this paper as follows:

1. Declarative knowledge: knowledge about *what* strategies are related to mathematical problems.
2. Procedural knowledge: knowledge about *how* to implement those strategies.
3. Conditional knowledge: knowledge about *when* and *why* to use the appropriate strategies.

In addition to assessing the learners' metacognitive knowledge, I will also be looking at their mathematical proficiency. For this purpose, I will concentrate on three of the five strands

discussed above: *conceptual understanding*, *procedural fluency* and *strategic competence*. The strand of adaptive reasoning and the affective strand of productive disposition will not be assessed as it is beyond the scope of this research.

To assess the learners' mathematical proficiency, I will use the summarised version of the strands as my operational definitions, namely: conceptual understanding as the "comprehension of mathematical concepts, operations and relations," procedural fluency as the "skill in carrying out procedures flexibly, accurately, efficiently and appropriately" and strategic competence as the "ability to formulate, represent and solve mathematical problems (Kilpatrick et al., 2001, p. 116).

Furthermore, the strategies that I introduced to the learners can be categorised as two broad types: the general strategies called "heuristics" by Pólya (1957) and what I called "content-specific" strategies. Pólya's heuristics included: *Did you use all the data?*, *Auxiliary elements*, (*Draw a figure*; *Auxiliary problems*), *Do you know a related problem?*, *Decomposing and recombining*, *Generalization*, *Specialization*, *Variation of the problem*, *Working backwards* and *Analogy*. Content-specific strategies were what Schoenfeld (1987b) referred to as "fine-grained" strategies relating specifically to certain kinds of problems.

In the table overleaf, I have sought to relate the strands of mathematical proficiency to the metacognitive knowledge required for effective use of the strands as well as strategies specific to the strands. It is interesting to note that the language used by Kilpatrick et al. (2001) to describe the strands, indicated here as direct quotations, reveals that all three strands require declarative, procedural and conditional metacognitive knowledge in order to develop into effective competencies. The metacognitive aspect of the strands is only briefly mentioned in the introductory comments to the strands in the work of Kilpatrick et al. (2001) and it is very possible that the role of metacognition has been underestimated.

As can be seen from the table overleaf, strategies are applicable for every strand; that is, there are strategies that support conceptual understanding, procedural fluency and strategic competence. At the same time, each strategy and each strand can have the three components of knowledge of cognition: declarative, procedural and conditional.

In my conceptual framework, I have included the knowledge of cognition that supports effective strand use and development, but my analysis of data did not include analysing the metacognitive aspects of the strands. I focused qualitatively on the level of conceptual understanding, procedural fluency and strategic competence evident in the learner responses to the questions in the two instruments. However, I did discuss the three aspects of knowledge of cognition with regard to strategy use when I could.

Strategy use*	Strand	Knowledge of cognition
Strategies used to <i>conceptually understand</i> a problem: (Pólya) • Did you use all the data? • Auxiliary elements (Draw a figure; Auxiliary problems) • Do you know a related problem? • Decomposing and recombining • Generalization • Specialization • Variation of the problem • Working backwards • Analogy	Conceptual understanding “integrated and functional grasp of mathematical ideas” p. 118	Declarative: “they have organised knowledge into a coherent whole” p.118 “able to represent mathematical situations in different ways” p. 119 [know] “how the various representations connect with each other, how they are similar, and how they are different” p.119 Procedural: [know] “how different representations can be useful for different purposes” p.119 Conditional: “understand why a mathematical idea is important and the kinds of contexts in which it is useful” p.118
Strategies used to <i>work through</i> a problem: • Content-specific, e.g. ➢ “Collect like terms to simplify problems” ➢ “replace nasty terms with nicer ones” ➢ See trig worksheet	Procedural Fluency the “knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately and efficiently” p. 121	Declarative: “knowledge of procedures” p. 121 Procedural: “skill in performing them flexibly, accurately and efficiently” p. 121 Conditional: “knowledge of when and how to use them appropriately” p. 121
Strategies for <i>problem formulation</i>: • Auxiliary elements (Draw a figure; Auxiliary problems) • Do you know a related problem? • Decomposing and recombining • Generalization • Specialization • Variation of the problem • Working backwards • Analogy Strategies for representation: • Verbal • Numerical • symbolic • Graphical Strategies for solving: • Content-specific	Strategic competence “the ability to formulate mathematical problems, represent them, and solve them” p.124	Declarative: “know a variety of solution strategies” p. 124 Procedural: know how to implement each strategy (connects to procedural fluency) Conditional: [know] “which strategies might be useful for solving a specific problem” p.124

Table 1.1 Conceptual Framework combining the strands of mathematical proficiency and strategies. Page numbers refer to Kilpatrick et al. (2001)

CHAPTER 3: RESEARCH DESIGN AND METHODOLOGY

3.1 PHILOSOPHICAL FOUNDATIONS OF RESEARCH

3.1.1 Introduction

Behind any research endeavour, whether stated explicitly or not, are philosophical foundations that inform and drive the research (Cohen et al., 2011). Cohen et al. (2011), borrowing from social science research, identify four sets of assumptions that underpin any research paradigm and position the researcher. From a researcher's perspective, assumptions about the nature of reality (ontology) give rise to assumptions about the nature of knowing (epistemology) which affect methodological choices which in turn affect instrumentation choices, data collection (Cohen et al., 2011) and analysis. From the perspective of a mathematics teacher, what mathematics *is* and what it means *to know* mathematics also affects what and how I choose to research as well as the mathematics presented in the class. Because I was working as both a researcher and teacher, the sets of assumptions informed both roles.

The first set of assumptions deals with the nature of reality which fall under the philosophical term, ontology. With regard to research, this concerns "the very nature or essence of the social phenomena being investigated" (Cohen et al., 2011). Two possible contrasting ontological approaches are termed *nominalist* and *realist* (Cohen et al., 2011). For the realist, there is an external world 'out there' independent of the individual's conception; for the nominalist, social reality is created by the individual's cognition and interpretation (Cohen et al., 2011). The second set of assumptions revolve around what is knowledge, how it is known, and how it is communicated; this falls under the philosophical term, epistemology (Cohen et al., 2011). This is a far-reaching assumption that directly affects how one goes about research (Cohen et al., 2011). For the *positivist*, knowledge is "hard, objective and tangible," the researcher takes on the role of observer and uses natural science methods; for the *post-positivist*, knowledge is "personal, subjective and unique" and the researcher is involved with his subjects and uses alternative methods to the natural scientist (Cohen et al., 2011, p. 6). The third set of assumptions involve views on human nature and specifically, the interactions of humans with their environment (Cohen et al., 2011). If humans are seen as responding mechanically like puppets to their environment, this is the *deterministic* view; if humans are seen as producers of their environment through their own initiative and creativity, this is termed *voluntarism* (Cohen et al., 2011). Finally, the three foregoing sets of assumptions have a direct impact on methodology; researchers with contrasting ontologies, epistemologies and views of human nature will choose drastically different research methods (Cohen et al., 2011). If the researcher chooses methods and procedures that seek to find general laws like a natural scientist, this is termed a *nomothetic* approach; on the other hand, if the researcher is more concerned with understanding and explaining the "unique and particular case rather than the general and the universal", this is an *idiographic* approach (Cohen et al., 2011, p. 6). These sets of assumptions divide research into fundamentally two contrasting paradigms: objectivist vs. subjectivist (Cohen et al., 2011).

These two paradigms also exist within mathematics and science education research and some authors claim that a shift is occurring from an objectivist view to a subjectivist view (e.g. Schoenfeld, 1987b). Just as a scientist develops tools to better understand a problem and better tools allow for better questions to be asked, so with research in mathematics and science education: the dialectic between the “evolution of tools and the refinement of problems” contributes to growth in the field (Kelly & Lesh, 2000, p. 35). Over time, field-based educational researchers have come to recognise the “complex and multifaceted world” presented by classrooms and workplaces for which traditional research techniques do not provide adequate descriptions (Kelly & Lesh, 2000, p. 35). This has resulted in a shift away from traditional experimental methods to alternative methods more conducive to studying the complexities of learners, learning, the classroom and the workplace (Kelly & Lesh, 2000). However, authors like Kelly and Lesh (2000) do not provide a “silver bullet methodology” but invite fellow researchers, including teachers, to “struggle with them” to discover methods that better capture the complexity of learning and teaching, while maintaining scientific rigour. At the same time, this shift in methods follows from underlying factors that have an ontological and epistemological character. A number of these factors resonate with the ontological and epistemological positioning of myself as researcher and the research in this dissertation.

3.1.2 Ontological and epistemological positioning as researcher

Firstly, from an ontological perspective, I place myself within this emerging paradigm for mathematics and science education where there is a moral commitment for research to improve teaching and learning (Confrey, 2000); in other words, the research is not seen as value-free. This type of mathematics research has “a desire to go beyond the identification and documentation of educational phenomena to their reform and improvement” (Kelly & Lesh, 2000, p. 41) and to create a “better connection between research and practice” (Confrey & Lachance, 2000). Also, within this paradigm, researchers position themselves as learners, and their “methods reflect their experiences and personalities” (Kelly & Lesh, 2000, p. 38). This is in contrast to the researcher as an external expert seen in other paradigms. Finally, researchers within this paradigm are usually within the teaching profession itself; either as teachers or mathematics education educators. Because of this, the researchers “take pride in their identities as teachers” and use their “direct experiences with the realities of school and workplaces” to guide their research (Kelly & Lesh, 2000, p. 38). Thus, as a researcher, I am an active participant in the research process and my subjective experiences and expertise are seen as assets.

From an epistemological perspective, this type of research acknowledges a “commitment to the position (expressed variably) that all learners (researchers included) are active constructors of knowledge” (Kelly & Lesh, 2000, p. 39); thus, the research assumes a constructivist perspective. The constructivist perspective further leads to “the legitimization of the study of cognition” and acknowledges the strength of cognitive science (Kelly & Lesh, 2000, p. 39); within this dissertation, various cognitive processes are the object of study and the analytical method presented by cognitive science is utilised. Finally, within mathematics and science

education research there is a “commitment to subject matter” (Kelly & Lesh, 2000, p. 40). The researcher’s knowledge of subject matter can affect the design and interpretation of assessment devices, the depth of mathematics taught, the questions that can be asked of the research and the development of theory regarding the research (Kelly & Lesh, 2000). Within this paper, my commitment to mathematics permeated the questions asked, the instruments developed, the analyses conducted on the data and the conclusions derived from the data.

3.1.3 Ontological positioning as mathematics teacher

As a learner and teacher of mathematics, I would have to agree with Milgram (2007) that I have difficulty delineating in exact terms what mathematics *is*; yet while I may not be able to define mathematics precisely, I do have underlying beliefs about mathematics that inform what mathematics I teach and how I teach mathematics. I see mathematics as a discipline that has developed over the course of at least 3000 years (Rudman, 2010) and is in the constant process of developing further. I believe that mathematics has developed differently in different social contexts and in response to, or driven by, different social needs (McLeish, 1991). I acknowledge that I am only able to appreciate a very small aspect of this discipline due to my own mathematical skills and greatly respect the rigour and scholarship that have enabled this discipline to develop. However, I can also accept that the paradigm of mathematics as an impartial, infallible, “rigidly hierarchical body of knowledge” is being challenged and replaced with the idea of mathematics as a social construction that needs to be examined in the light of power and social interests (Ernest, 1994, p. x).

At the same time, I would also agree with Milgram that we can get close to a definition of mathematics by describing some of its most important characteristics. One of these characteristics is *precision* which includes the “precise definition of all terms, operations and the properties of these operations” (Milgram, 2007, p. 33). Another of the most important characteristics of mathematics is its primary function of “stating well-posed problems and solving them” (Milgram, 2007, p. 33). Because I often view mathematics from the perspective of teaching it, I would also add that what mathematics *is* may not be separable from proficiency in it; and that proficiency is multi-faceted. I discussed earlier the strands of mathematical proficiency; Schoenfeld’s four components of resources, heuristics, control and beliefs; and Pólya’s process of problem solving and using heuristics. For the purposes of this paper, all of these would be part of my conception of the nature of mathematics and would inform many of the choices made in the research of my topic.

As a mathematics education researcher, I also have ontological beliefs regarding the nature of enquiry appropriate to the particular focus I have chosen. Because I am most interested in learner thinking, I strongly believe that a research approach needs to focus on *process* as opposed to *product* (Schoenfeld, 1987b); in order to understand learner cognition, one needs to investigate the thinking (i.e. process) used to obtain answers and not just the answers themselves (i.e. product). Statistical analyses of answers alone, while informative in other respects, may not necessarily measure the thinking behind the answers (Schoenfeld, 1987b).

3.1.4 Epistemological positioning as mathematics teacher

As stated before, ontological biases give rise to epistemological assumptions and orientations. As both a teacher of mathematics and as a researcher, I work within the constructivist paradigm. I resonate with von Glasersfeld's description of learning from a constructivist perspective and believe it is especially applicable to mathematics:

Learning is not a passive receiving of ready-made knowledge but a process of construction in which the students themselves have to be the primary actors. But ... this principle in no way precludes the notion that the students' individual constructing is constantly stimulated, constrained, and thus directed by interactions with each other and with a teacher. (von Glasersfeld, 1995, p. 1)

Constructivism as a learning theory had Jean Piaget and Lev Vygotsky as its early proponents; both were psychologists concerned with development and learning, but had differences in the way they saw the relationship between, and the role played by, the two processes (Gauvain & Cole, 1997). Of the two, Piaget's theories had more bearing upon this study.

Piaget's theory countered the behaviourist learning theory which dominated the first half of the 20th century. Piaget (1964) saw development as spontaneous, part of the process he called embryogenesis and through a series of four, specific, age-related stages. Learning, on the other hand, he saw as provoked by an external source, which could be a teacher, a psychological experimenter or an external situation and is *dependent on* development (Piaget, 1964). For Piaget, "to know an object is to act on it"; knowing is not a passive process but requires what he called an "operation," that is, an "interiorised action which modifies the object of knowledge" (Piaget, 1964, p. 20). Examples of operations are counting, ordering, classifying and measuring (Piaget, 1964). In contrast to the behaviourists, he maintained that for a stimulus to initiate a response, there had to be an *internal structure* which can integrate the stimulus and set off a response; the structure is the intermediary between the two (Piaget, 1964). He also claimed that the fundamental relation between a stimulus and response was not association, as proposed by the behaviourists, but assimilation. He defined assimilation as the "integration of any sort of reality into a structure" and asserted that assimilation was fundamental in learning, didactics and pedagogy (Piaget, 1964, p. 26). In assimilation, "external data are fit with currently known ways of understanding or *schema*" (Gauvain, 2001, p. 25). In accommodation, assimilation's twin process, "schemas are altered to fit with external data" (Gauvain, 2001, p. 25). For Piaget, the dialectical relationship between the two is always present when learning happens (Gauvain, 2001). From the point of view of mathematics, Piaget claimed that logical-mathematical structures do not develop through external reinforcement (as proposed by the behaviourists), but only through "internal equilibration" (his term for self-regulation) and only on the condition that the structure a person wants to teach is supported by "simpler, more elementary, logical-mathematical structures" (Piaget, 1964, p. 25).

In addition to explaining the process of learning, one of the important consequences of constructivism is that it facilitates interventions in the learning and teaching of mathematics (Ernest, 1994). Constructivism explains “individual idiosyncratic constructions of meaning, systematic errors, misconceptions and alternative conceptions in the learning of mathematics” (Ernest, 1994, p. 2). This means that as a teacher, I can diagnose and remediate errors in thinking (Ernest, 1994). It also means that as a teacher, I need to be most concerned with the mental operations of my students and focus on their understanding (von Glasersfeld, 1994).

Constructivism, as an epistemological perspective, also underpinned my decisions and understanding as a researcher. The anti-positivist, constructivist paradigm within research is concerned with the individual and strives to “get inside the person and to understand from within” (Cohen et al., 2011, p. 17). Not only did the learners construct their knowledge of mathematics, I also constructed my understanding of their responses to my instruments. I engaged in the hermeneutic task of “uncovering of meaning” (Brown, 1994) as I proceeded through my analysis.

3.2 RESEARCH METHODOLOGY

Given the foregoing ontological and epistemological underpinnings, my research approach was purely qualitative as I attempted to investigate learner thinking in detail. This is in line with the “growing legitimization of field-based and qualitative methods” represented by the shift in mathematics and science education research (Kelly & Lesh, 2000, p. 39). I use the word “qualitative” from both its ontological and methodological perspectives; a qualitative approach is appropriate within the constructivist paradigm within which I worked since it allows for multiple interpretations (McMillan & Schumacher, 2006). Also, the learner’s thinking as revealed in their comments and working-out of problems yielded rich data. Since there is an “immense loss of information that occurs when numbers are used to represent rich phenomena” (Kelly & Lesh, 2000, p.41), using qualitative methods to analyse the data also suited the research aims.

My research methodology borrowed from several sources including action research and teacher experiment studies, but was fundamentally a case study. This is in line with the shift in mathematics education research that the “legitimacy of educational research questions should not be predetermined by or constrained to the potentialities and capabilities of a particular research methodology” (Kelly & Lesh, 2000, p. 35). My study shared the moral commitment and practitioner-based elements of action research (Confrey, 2000; Feldman & Minstrell, 2000) and closely resembled this definition: action research is a “small-scale intervention in the functioning of the real world and the close examination of the effects of such an intervention” (Cohen et al., 2011, p. 345). However, this study lacked the “systematic cycles of inquiry” of action research where “new questions emerge, new actions are taken, and new data collected” (Doerr & Tinto, 2000, p. 408). As with teaching experiments, in my research there was the “conscious breaking down of the researcher-teacher divide” (Kelly & Lesh, 2000, p. 192) as I was a co-learner with my students as I adapted my activities in class to promote their strategy

use. My study also shared with teaching experiment research a “focus on development that occurs within conceptually rich environments explicitly designed to optimize the chances that relevant developments will occur in forms that can be observed” (Kelly & Lesh, 2000, p. 192). The classroom is a conceptually rich environment and the structure of my instruments allowed the development in strategy use to be made apparent.

However, as my intention in this study was to look in detail into the relationship between metacognitive strategy use and mathematical proficiency for a particular group of students in a particular context, a case study methodology seemed most appropriate. Case studies can be defined variously, and several are applicable to this research: a case study is a “single instance of a bounded system” or the “study of a case in a context” (Cohen et al., 2011, p. 289). The case in question was a group (two classes) of Grade 12 girls, the context was a well-resourced South African public all-girls high school, and the bounded system was the curricular demands placed upon Grade 12 students preparing for the National Senior Certificate. A case study is also a “specific instance that is frequently designed to illustrate a more general principle” (Cohen et al., 2011, p. 289). It is very possible that what my learners experienced with regard to implementing strategy use is not unique, but represents other instances; this is borne out by research by Schoenfeld (1987).

The context of a case is important and a case study provides “a unique example of real people in real situations” (Cohen et al., 2011, p. 289); a twelfth grade classroom at an all-girls public school in South Africa may be very different (or very similar) to its counterpart in another country. A case study of student learning can reveal the “messiness and complexity of mathematical learning as it occurs in classroom situations” (Cobb & Bauersfeld, 1995, p. ix). Also, case studies “recognize and accept that there are many variables operating in a single case” (Cohen et al., 2011, p. 289). Later on in this paper I will describe just some of variables encountered working in a regular classroom that may have affected the learners’ ability to incorporate strategy use.

Finally, case studies “can penetrate situations that are not always susceptible to numerical analysis” (Cohen et al., 2011, p. 289), and as mentioned earlier, much information can be lost if numerical methods are used for rich data (Kelly & Lesh, 2000). The case study, with its flexible approach, allows for working “from the inside,” examining teaching and learning with the ultimate goal of improving both (Ball, 2000, p. 366). Thus, it was an ideal match for the ontological and epistemological commitments underpinning the initial impetus of this research.

3.3 PARTICIPANTS AND CONTEXT

As stated in the introduction, the participants in my research were two classes of Grade 12 learners at a relatively well-resourced all-girls public high school in a large urban area of South Africa. The school was well known in the area for its high academic standard and received many awards and accolades for the performance of its learners. This school, however, is not typical of public schools in South Africa. In a recent presentation at UNISA, a researcher from

Department of Economics at the University of Stellenbosch found that the South African school system is actually two school systems: a functional system comprising twenty-five percent of schools in South Africa that perform at the same level as the bottom-end of developed countries, and a dysfunctional system comprising seventy-five percent of schools that perform at the bottom end of African countries (www.unisa.ac.za). This is despite a relatively high expenditure by the South African government on education (<http://www.unisa.ac.za>). The problems that render schools dysfunctional include incompetent school management, the lack of a culture of learning, discipline and order, weak teacher content knowledge, slow curriculum coverage and high teacher absenteeism (<http://www.unisa.ac.za>). My school, however, did not have these problems due to very strong and competent senior management, a dedicated staff, and policies and procedures that ensured learning and teaching took place consistently. The school may also have been helped by the relatively high school fees of approximately R31 500 per year compared to the national average estimated for high schools of R8 000 to R12 000 per year (<http://www.news24.com/SouthAfrica>).

To further understand the context of the situation, the matriculation system needs to be explained. Grade 12 learners in South Africa write their matriculation examinations for the National Senior Certificate in November of the school year (which starts in January). In Grade 10, all students in South African high schools have to choose the subjects they will write for their matriculation examinations. South African students write a minimum of seven subjects, four of which are compulsory: a home language, a first additional language, Mathematics or Mathematical Literacy, and Life Orientation, and three others that are chosen from several options. The results on the South African National Senior Certificate determine the eligibility of tertiary placement of the learners; according to their results, the learners can either go on to university, a vocational college or simply start to work. To receive their National Senior Certificate, students can fail one subject (except Life Orientation and Home Language); they need to attain at least 40% in the home language and 30% in all other subjects in order to pass. However, to be accepted at university, and particularly certain faculties, requires much higher results. Because of the consequences of the matriculation examinations, they can be seen as a high-stakes examination.

The students were grouped in two relatively small ($n = 15$ and $n = 13$) classes with learners in the average to weaker ability level. The promotion mark from Grade 11 for one class was 46,7% and 36,2% for the other class. Two learners were particularly at risk for failing in Grade 12; one learner attained a mark of 16% and the other, 25%, in Grade 11. In South Africa, all learners are required to take some type of mathematics every year in high school, but can choose between Mathematics and Mathematical Literacy. “Maths Lit” concentrates on more practical applications of mathematics in “real world” contexts and uses content up to approximately a grade 9 level; Mathematics is designed to prepare learners for university. Both of these learners were encouraged to move to Mathematical Literacy to prevent failing matric, but both felt they could bring their mark up to a passing level by the end of their Grade 12 year.

3.4 RESEARCH METHOD AND INSTRUMENTS

3.4.1 Research Method

As stated earlier, I was working as a researcher/practitioner in my own class. Although the design of this study was not experimental in the sense of random selection of participants and using a control group (Cohen et al., 2011), I followed a pre-test/intervention/post-test design in order to analyse any changes in metacognition and/or mathematical proficiency after a period of training in metacognitive strategy use. The sample was a convenience sample of two classes of Grade 12 learners. A pre-test was administered shortly after the school year began in January and was followed by training in metacognitive strategy instruction. The post-test was administered in June of that year. The period of the research from pre-test to post-test lasted approximately six months over the first two terms of the 2015 school year.

3.4.2 Instruments

As indicated previously, the crux of my research was to investigate whether and to what extent learners acquire metacognitive knowledge in the form of strategies and use that knowledge when working on mathematical problems. Another important aspect of my research was assessing learners in a manner similar to their heavily-weighted written assessments. Thus, my instrument was comprised of two sections: one section being mathematical problems that needed to be worked through and another section alongside which learners described their declarative, procedural and conditional knowledge of metacognitive strategies.

In line with recommendations of current research (Veenman et al., 2013), my instrument was thus an on-line assessment of metacognition that occurs simultaneously with mathematical problem solving. My instrument was very similar in concept to the one developed by Lucangeli and Cornoldi (1997); however, their instrument offered respondents multiple-choice prompts to assess the participants' regulation of cognition. This may have been suitable in their context as their participants were 3rd and 4th graders. For my purposes, though, an open questionnaire seemed more appropriate for high school learners.

Each instrument had four carefully chosen problems from the domains of data handling, measurement, sequences and applications of quadratic functions. I chose these questions because they were not purely procedural questions, but required various strategies and strands of proficiency. The problems mostly fell within the "problem-solving" cognitive level of the South African Curriculum and Assessment Policy (CAPS) (Department of Basic Education, n.d.). Grade 11 content was chosen because it was assumed to be fairly stable prior knowledge that would not be particularly affected by the content taught in class. The instruments had four columns: one for the mathematical working-out and three for the learners to explain what strategies they were using, how they used them and why they were using them. (See Appendix A – 1)

Both a pre-test and a post-test were constructed, complete with suggested memoranda, possible strategy use and strands of mathematical proficiency required for each question. The post-test also had four questions similar to the pre-test with regard to content, strategy use, strands of proficiency and cognitive levels (See Appendix A – 2, B – 1 and B – 2). I administered a different post-test to mitigate the possible testing effect of a pre-test (McMillan & Schumacher, 2006).

Within a week of the first term starting, I administered the first instrument to my two Grade 12 classes. Two learners were absent from the two classes and the total number of respondents was 26. A few learners happened to have notes from the previous year and were allowed to refer to them to answer questions. Except for the “open-book” aspect, the instrument was conducted under normal school examination conditions: individual, silent and under time constraints. The learners appeared willing to attempt to solve the problems, although some learners complained of forgetting topics, and one learner in particular felt very discouraged by the prospect of trying to solve “word problems.”

Instrument 2 was written late in the second term after approximately five months of strategy instruction. The instrument was written on the last day of regular classes, just before the June examinations started for the matriculants. There was one absentee in one class so the total number of respondents was 27. This instrument was not administered in my classroom due to scheduling conflicts. During the course of the two terms I had put up posters in my classroom to remind the learners of the different strategies available; these were not on display during the administration of the second instrument. A few learners commented that having them would have been helpful.

Instrument 2 covered the same content as Instrument 1 (data handling, measurement, sequences and applications of quadratic functions). When I administered Instrument 1, I assumed (very incorrectly) that the Grade 11 content would be accessible; I did not take into account the element of forgetting. For Instrument 2, I gave the learners some formulae as an *aide memoire* and allowed them to refer to their textbooks or notes; however, a specific study guide with detailed step-by-step instructions was not allowed. In all other respects, Instrument 2 was administered under the same conditions as Instrument 1: the learners were instructed to complete it individually, silently and within one class period.

The attitude of the learners toward completing Instrument 2 varied markedly between the two classes and between individuals. In one class, three learners in particular appeared discouraged and did not want to attempt the instrument; the timing of the instrument may have been at fault here. By this time of the year, the matriculants were generally discouraged after months of trying to improve in mathematics but not showing much success. They were moving into examinations that were part of the school based assessment marks that are factored into their final mark – so they were feeling pressurised. Also, one of these learners still felt daunted by any sort of “word problem” that faced her (the same learner as in Instrument 1). A couple of the other learners tried to talk with each other about the problems, even though they were not

supposed to. Several complained that they could not remember certain content. On the other hand, in that same class, several of the learners got out notes and tried to tackle the problems. In contrast, the learners in the other class, except one, set to work diligently and independently. One learner even proactively produced a “metacognitive self-questioning prompt sheet” that I had given them the previous year. However, one particular learner did not attempt the questions, claiming that she needs to study a section before she writes on it. She had shown disinterest in the strategy instruction throughout the process, and preferred to study her own way, working through a study guide with detailed step-by-step instructions for different kinds of problems. Both classes illustrate that although the strand of productive disposition is not within the scope of this paper, its importance cannot be ignored and may be behind the quality of some of the responses.

3.4.3 Intervention

The intervention in this study was to train learners to develop metacognitive knowledge specifically with regard to strategies. As mentioned previously, according to Shraw (2002) metacognitive knowledge has three components: declarative, procedural and conditional knowledge. The initial intent of the training was to try and demonstrate which strategy is appropriate for which situation and why. However, in practice, I often only managed to introduce the strategies and explain how they worked. The strategies came from a number of sources because I wanted the learners to find strategies that were meaningful for them.

The main strategies I modelled, demonstrated and reinforced were several heuristics from Pólya (1957), the self-questioning strategy of the IMPROVE system and strategies required by specific topics or content. I chose these because I believe learners do not usually possess a repertoire of strategies, and more importantly, I believe learners could benefit from them. I also needed to choose a metacognitive skill that could be expressed in writing because that was the form my instrument required.

3.4.4 Training procedure

As I consider self-interrogation an important strategy in itself, I introduced the learners to self-directed *comprehension*, *connection* and *strategy* questions over the course of the first term. I based the questions on the work of Pólya (1957) and Mevarech and Kramarski (1997).

The first type of questions IMPROVE students are trained to ask are *comprehension questions* which help students to articulate the main ideas in a problem (e.g. What is the question all about?) (Mevarech & Fridkin, 2006, p.

HOW TO SOLVE IT: A METACOGNITIVE APPROACH	
UNDERSTANDING THE PROBLEM: COMPREHENSION QUESTIONS	
FIRST. You have to understand the problem	• WHAT IS THE UNKNOWN? • WHAT ARE THE DATA? • WHAT IS THE CONDITION? ➤ DRAW A FIGURE ➤ INTRODUCE SUITABLE NOTATION

87). To help my students articulate their questions more precisely, I used Pólya's specific questions: *What are the data? What is the unknown? What is the condition?* (Pólya, 1957, p. xvi). I put posters, such as the one above, up at the front of the classroom, to help children remember to use the questions (see Appendix C – 1 to C – 3).

As discussed earlier in the literature review, to begin to understand a given problem, Pólya (1957, p. xvi) asks three questions: “*What is the unknown? What are the data? What is the condition?*” These questions help to bring the particulars of the problem into focus. The unknown could be the quantity the problem calls for, or a quantity required in order to determine the end goal of the problem, or both (Pólya, 1957). The data are the givens in the question; they may be stated explicitly or inferred or both (Pólya, 1957). The condition can be the relationship between the data and unknown(s), a specific situation of the problem or something else unique to the problem (Pólya, 1957). By identifying the data, condition and unknown, the problem solver can begin to unpack the problem in order to solve it.

At the time I introduced the questions, we were studying sequences and series and the questions were very useful for those problems. Identifying the condition was helpful in identifying if the sequence or series was arithmetic or geometric. However, sometimes in other kinds of problems, the condition was difficult to explain. I spent about 3 weeks on the comprehension questions. Although the teachers in Mevarech and Kramarski's (1997) study were able to present their new material *through* metacognitive questioning, I found I had to present the content and then use the metacognitive questioning when consolidating. For me, the students seemed to need the cognitive foundations *before* introducing metacognitive activities. As informal evidence that some students were using the strategy, I found a couple of students had written “DUC” (data, unknown, condition – our acronym developed in class) while they were working through their class test on sequences and had written down the data, the unknown and the condition.

The next question IMPROVE students are trained to ask are *connection questions* which lead learners to connect a current problem with one solved in the past (e.g. What are the similarities and/or differences between this question and one solved before and why?) (Mevarech & Fridkin, 2006). I tried to introduce the connection questions, again using suggestions from Pólya (1957). We were

busy at that point revising Grade 11 trigonometry and introducing Grade 12 trigonometry. I thought the connection questions would help with understanding. However, for most of the learners, their grasp of trigonometry foundations was so tenuous, that making deeper connections was almost impossible. If connections are ropes joining poles, the poles were far too flimsy to hold the ropes. As time is precious in the matric year, I did not concentrate on the connection questions, which unfortunately seemed too complicated to be helpful, and moved

CONNECTION QUESTIONS	
SECOND. Find the connection between the data & the unknown. You may need to make connections to other areas of mathematics or previously solved problems.	• DO I KNOW A RELATED PROBLEM? • DID I USE ALL THE DATA? • WHAT IS THIS PROBLEM RELATED TO?

on to the strategy questions. I was also motivated to move on because strategy was the area that I would be specifically assessing for this research.

The next questions IMPROVE students are trained in are *strategic questions* (e.g. What strategies are useful for solving the problem and why?) (Mevarech & Fridkin, 2006). However, from my observation of students, few have strategies at their disposal, so I introduced a number of the heuristics recommended by Pólya (1957). These included: *Introduce auxiliary elements*, *Introduce auxiliary problem*, *Draw a figure*, *Decompose and recombine*, *Generalization*, *Specialization*, *Work backwards* and *Variation of the problem*.

STRATEGY QUESTIONS	
THIRD. Think of appropriate strategies to tackle the problem. You may have to consider an auxiliary problem. Obtain a plan of the solution.	● WHAT STRATEGIES CAN YOU USE? ➤ TOPIC-SPECIFIC STRATEGIES ● HEURISTICS? ➤ INTRODUCE AUXILIARY ELEMENTS ➤ INTRODUCE AUXILIARY PROBLEM ➤ DRAW A FIGURE ➤ DECOMPOSE AND RECOMBINE ➤ GENERALISATION ➤ SPECIALISATION ➤ WORK BACKWARDS ➤ VARIATION OF THE PROBLEM

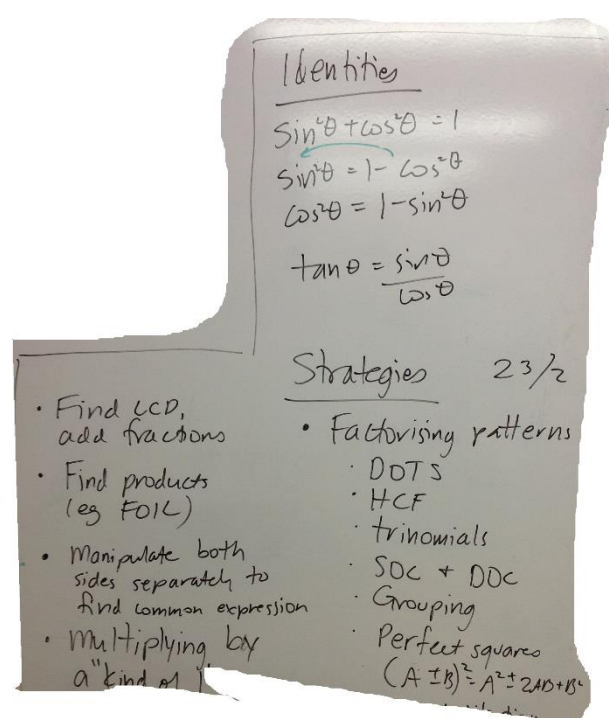
This was near the end of the first term and we were studying trigonometry in three dimensions. Quite near the end of the term I made posters to remind the students of the strategies that we were focusing on and referred to them fairly frequently during class. I felt the 3-dimensional trigonometry was particularly useful for illustrating the “draw a figure”, “work backwards” and “decompose and recombine” strategies. I also introduced my own category of strategies which I called *content-specific strategies*.

The April holiday gave me a chance to go back and review some literature and I focused on Schoenfeld’s works. He maintains that Pólya’s strategies are too broad to use; they need to be reduced to a “finer” grain of analysis to be useful (Schoenfeld, 1987b). While I can accept his propositions at his level of study (very competent college mathematics majors) I disagreed with him for high school students. I think they need simplicity and I continued to focus on the nine strategies mentioned earlier and did not break them down into further subroutines as suggested by Schoenfeld (1987b, 1985).

At the beginning of the second term, I decided that I would not be able to test strategy explication if I did not give the students a chance to practice it. I gave the students several take-home revision exercises that I used for informal formative assessment of their ability to use strategies.

I started with a textbook exercise at the end of the chapter on 3-dimensional trigonometry and gave a verbal instruction to just jot down what strategies they used, and how and why they used them (See Appendix D – 1 and D – 2). As this was a “non-compulsory” assignment and was not for marks, I could not force all students to hand them in, and 23 out of the 31 students complied. I was initially very happy with the engagement with the diagrams and began to feel that strategy use – at least with regard to diagrams – was beginning to be used. However, only a very few gave explicit indications that they used any strategies.

Once the second term began, and we moved onto trigonometric equations, the strategy use modelled by me focused on content- or topic-specific strategies. I explained the algebraic resources that the learners had at their disposal in order to solve various kinds of trigonometric equations. Strategies such as using a “k-substitution” for repeated terms in an equation, recognising the quadratic structure of certain trigonometric equations, using various factorising strategies (common factor, difference of squares, perfect squares, trinomials), using trigonometric identities, multiplying by “kinds of one” were all discussed and demonstrated. A section of the whiteboard summarising the strategies is reproduced alongside.



Over the rest of the second term, I tried to provide opportunities for the learners to practice incorporating strategies in their work and note their strategy use. However, from informal formative assessments, the learners did not appear to indicate that strategies had become part of their repertoire. I gave them two revision sheets and modelled strategy use in my memoranda. But the learners either did not do the revision sheets, or showed no indication of understanding how strategies can be applied when working through any kind of problems. The revision sheets and my memoranda are included as Appendices E and F. Note the strategy use on my memoranda.

A “snapshot” of one time period within the second term is illustrative of my difficulties. The whole Grade 12 had just written two tests in the course of one week, one on trigonometry and one on geometry and had done very poorly. The mathematics department as a whole decided to retest the grade on a combined trigonometry/geometry paper that would be included in the students’ school-based assessment; thus it was important. The learners had just under two weeks to prepare for the test, which included 2 weekends and approximately 8 class periods.

For my learners, I decided to initiate a “Paper 2 Challenge” that would last 12 days (see Appendix G – 1 and G – 2). I gave them a pack of material and assigned a section a day for revision. The pack covered trigonometric equations, trigonometry in two- and three-dimensions, Gr. 11 “Circle Geometry” and Grade 12 “Similarity Geometry.” I intended to complete all 12 pages of revision myself and join them on the challenge. I also intended to spend 15-20 minutes of each class period (we were on Calculus by that time) going over questions. Initially, almost everyone (including myself) kept up. We discussed the trigonometry equations quite thoroughly. However, as the days progressed, fewer students were able to keep to the revision timetable and I fell behind myself.

However, one significant result came to light from that revision process. The first revision section was trigonometric equations and when I worked through the beginning thirteen

questions, I wrote down 19 content-specific strategies that I employed at various times during those questions (see Appendix G – 3). I discussed them in class, and handed the sheet out with the strategies and asked the learners to write down more strategies as they solved the other equations. I had hoped to collect their lists as evidence of strategy use, but not one learner added to that list. It seemed at that point that the learners could not recognise which strategies to use in which problems nor could they articulate them. Shortly after that, we moved on to calculus and I administered the second instrument. However, as my analysis will show, the learners did learn something about strategy use.

3.5 DESCRIPTION OF DATA ANALYSIS

3.5.1 Introduction

My strategy for the data analysis was to proceed through it in layers, as it were, refining my understanding of what the data presented through each perusal. As I went through the data multiple times, the picture the data presented became clearer and clearer, much like bringing a very blurry picture into focus. As more information came to light on later questions, I would often need to go back and re-examine my previous interpretations.

I took each question from each instrument as a set of responses. I began by perusing the responses, looking for similarities between the responses which I could use to establish codes. Then, I went back again through the responses and coded them. Often in that process new trends would be discovered and I would have to go back and recode the question again. This process was followed for each question from two perspectives according to my conceptual framework: looking at the responses from the point of view of the mathematical proficiency present and then examining strategy use.

Sometimes coding the responses did not give what I thought was an accurate portrayal of the data. In those cases, I went back and “marked” the question as if it were for examination purposes. Other times, marking would not have shed light on the responses and I just used the coding.

While coding Instrument 2, Question 1, a further “layer” was noticed because in this question the strategy use described by some learners did not match the mathematical working presented. For example, one learner wrote, “- Determine whether the pattern is linear, geometric or quadratic – Then you create a simultaneous equation.” Her strategy use involved patterns but her working out involved the variance formula; thus, her strategy use did not match her working out. Also, if the strategy use matched the working out, the solution process may have been correct or not. This created a set of categories that I used to finally compare the results from the first instrument to the second: I looked at match/correct and match/incorrect from both instruments to see if any differences presented.

Research of this type, which is examining learner thinking, depends upon a qualitative analysis of the data; it is an emphasis on *process over product* (Schoenfeld, 1987b). Simplifying complicated thought processes to numerical tallies or percentages or even factor analyses (Schoenfeld, 1987b) has the possibility of losing important nuances evident only when the data is examined in its entirety and in fine detail. Although I employed some use of tallies and frequencies in my analysis, it was more to guide my understanding of common trends.

3.5.2 Data sets

My data sets consisted of responses to two instruments administered at the beginning of the school year and at the end of strategy instruction. I called them Instrument 1 and Instrument 2, respectively. Each instrument had one mathematical question each on four different aspects of mathematics: data handling, measurement, sequences and functions. The instruments were designed in a “landscape” format and had three columns with headings for learners to state explicitly their strategy use when completing the mathematical problems (see Appendices A-1 to B-2 for Instruments with memoranda).

I qualitatively analysed the instruments, examining the three strands of mathematical proficiency and strategy use present. I will describe my method of analysis and my coding decisions.

3.5.3 Method of analysis: Instruments 1 and 2

In order to more easily view the data, I scanned all the learners’ scripts and then “cut and pasted” each response from each learner to make composite documents of all the learners’ responses per question. I then used these documents for coding. I had columns for the learners’ mathematical working, their self-described strategies and blank columns for codes. These I referred to as “Learner responses with codes” (see Appendix H – 1 as an exemplar). These documents were created for every question on both instruments.

My first object of analysis was the learners’ mathematical proficiency; as mentioned previously, I focused on three of the five strands: conceptual understanding, procedural fluency and strategic competence. Some questions required all three strands and some required fewer.

For each question, my initial analysis involved a global perusal of the learner responses to each question. I listed the kinds of errors or correct responses made by the learners and took note of exceptional answers. Often, the same errors were repeated or errors could be grouped together into similar types of errors. These errors or error types were then assigned a number and recorded. I created two working documents for this process for each question. The first I called “Criteria for Coding Values per Question” (see Appendix I – 1 and I - 2). I also “cut and pasted” examples from learner responses to illustrate the codes; this was called “Learner examples to determine codes” (see Appendix J – 1 as an exemplar).

I then analysed each learner response in detail, noted the codes on the “Learner responses with codes” sheets (see Appendix K – 1 as an exemplar) and recorded the frequency of the codes for the group as a whole in a “Tally of Codes (Whole Group)” excel spreadsheet (see Appendix L – 1 and L – 2). I also tallied the number of ratings per learner in “Tally of Mathematical Proficiency and Strategy Use ratings (per learner)” (see Appendix M – 1 to N – 2). Finally, I wrote a qualitative explication of the learners’ responses using all the aforementioned sources. Those findings will be reported in Chapter 4. A diagram illustrating my data analysis process is below.

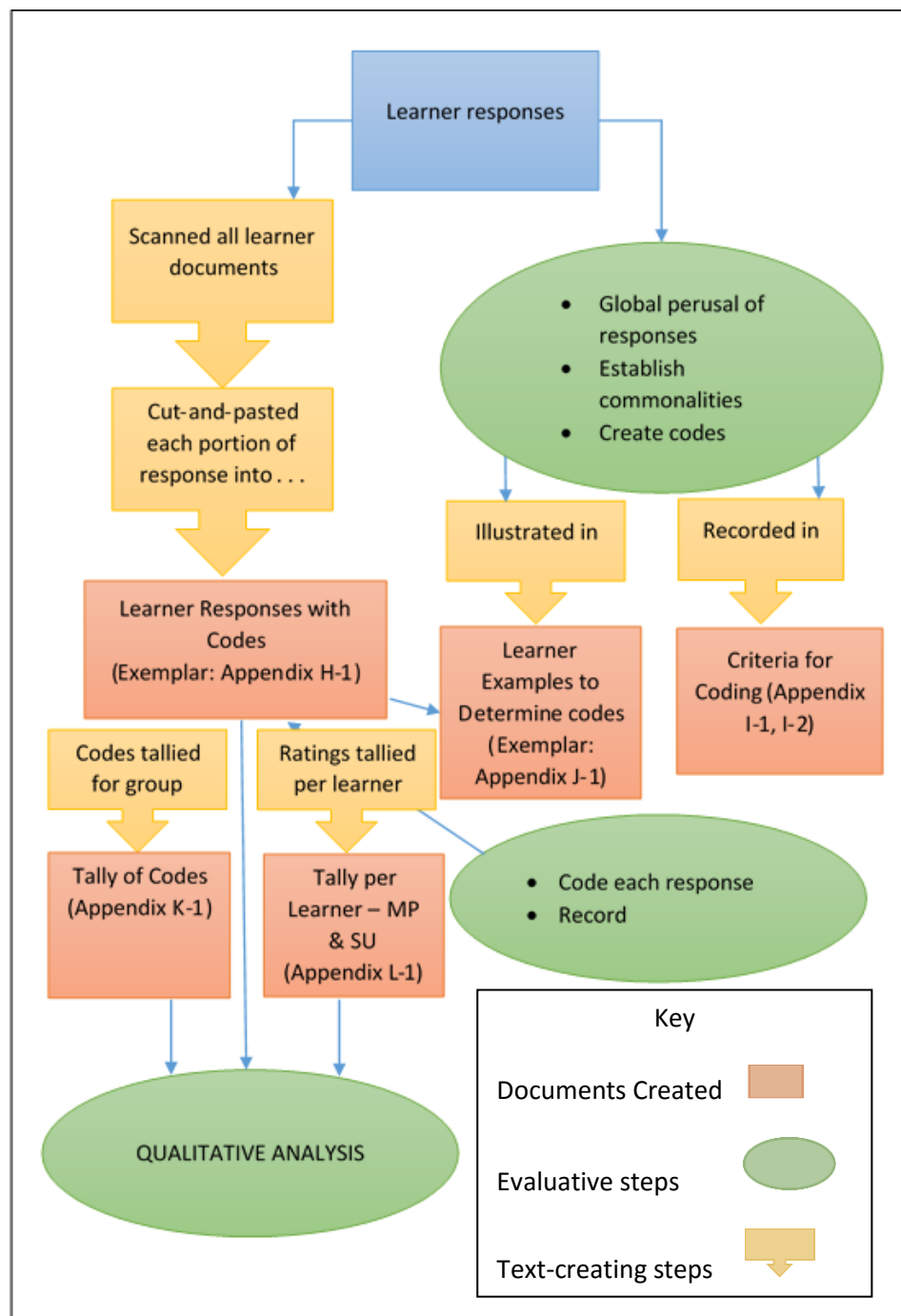


Figure 3.1 Diagram of Data Analysis Steps

3.5.4 Operationalising the three strands of mathematical proficiency

My qualitative analysis of the strands of proficiency for each learner response comprised three levels: each response was analysed within a *category*, was *rated* and each rating was given a *code* specific to the question and based upon the learner responses. Details are given below.

The *categories* assessed were taken from the strands of mathematical proficiency and included *conceptual understanding* (CU), *procedural fluency* (PF) and *strategic competence* (SC). As stated earlier in the paper, I used the summarised version of the strands as my operational definitions, namely: conceptual understanding as the “comprehension of mathematical concepts, operations and relations,” procedural fluency as the “skill in carrying out procedures flexibly, accurately, efficiently and appropriately” and strategic competence as the “ability to formulate, represent and solve mathematical problems (Kilpatrick et al., 2001, p. 116). Not every question required each strand. In the memoranda for each question, I described the strand(s) necessary to complete the question competently and possible strategy use that could be employed (see Appendices A – 2 and B – 2).

As I analysed the learners’ answers, I saw that the learner responses revealed more of a continuum of answers than just right/wrong responses, therefore, I chose to *rate* the learner responses within each category and established specific criteria for each rating, again based upon the learners’ answers. The ratings reflected the competency of the learner responses and responses showing *competence* were given a (+) rating, responses showing evidence of *partial competence* were given a (+/-) rating and responses showing *incompetence* were given a (-) rating. The foundation for these codes originated in research done for my BSc Hons degree and from the PhD thesis of my supervisor (Price, 2012; Van Jaarsveld, 2007).

In addition, even within the ratings of (+), (+/-) and (-) learners showed varieties of competence and errors. Some of the errors could be grouped together as a single type of error; for example, multiplying or raising to a power instead of using addition as required, would be coded as “wrong use of operation” and assigned a number. For each question, I established codes based upon the learner responses (see Appendix I – 1 and I - 2). A flow diagram of the method used to code the strands of proficiency is given on the next page (see Figure 3.2).

For example, in Question 1.1 of Instrument 1, learners were given the mean, median, standard deviation and range of a certain fictitious class’s test results. The question asked the learners to write down the new values if four marks were added to each student’s results. Because all students received the four marks, the range and standard deviation would not be affected but the mean and median would increase by four. Very little calculation was required for this question and it was based mostly upon conceptual understanding. However, a common learner response for this question was to *add 4 to each data value* (not just the median and the mean). A response such as this was categorised under the *category* CU, *rated* as (+/-) and *coded* 1 (2 answers correct) and 2 (added 4 to range and SD).

Steps to coding Mathematical Proficiency

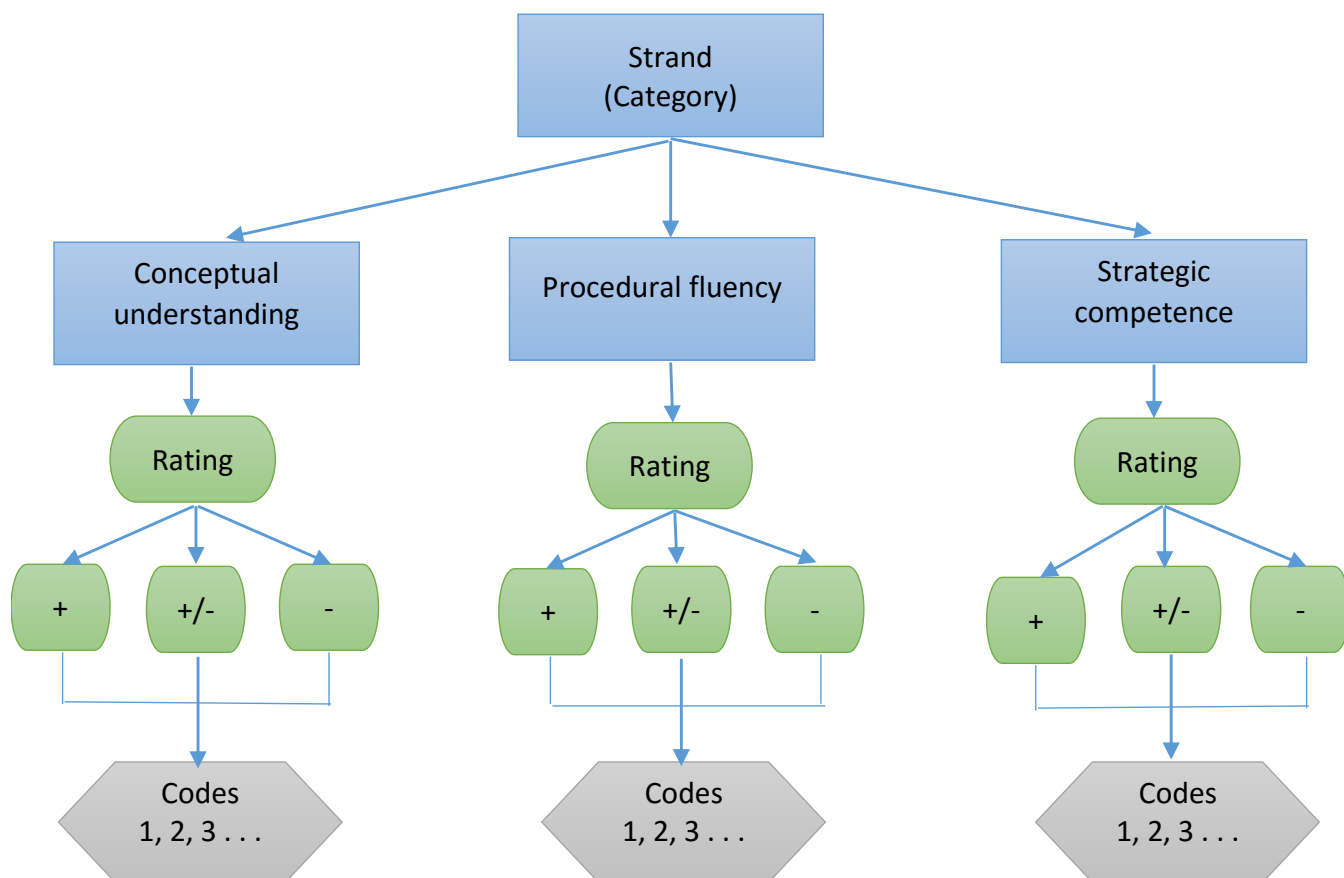


Figure 3.2 Steps to coding Mathematical Proficiency

3.5.5 Operationalising strategy use

During the period of strategy training in class, I concentrated on a number of the heuristics described by Pólya. On the board for several months I had posters which stated *draw a figure*, *work backwards*, etc. I also concentrated on content-specific strategies – particularly when working with trigonometry.

My first step in operationalising strategy use, was to find patterns in the learners’ responses. The first aspect I discovered was that strategy use could be implied or stated explicitly. For example, a learner may draw a figure to aid in the solution of a problem without explicitly stating that “draw a figure” was a strategy employed. Also, learners seemed to employ a strategy I came to call “explicate the problem” when they annotated the problem or their working – again without explicitly self-reporting that action as a strategy.

However, if we go back to Brown’s definition of metacognition, she states that metacognition “refers to understanding of knowledge, and understanding that can be reflected in either

effective use or *overt description* [my emphasis] of the knowledge in question” (Brown, 1987, p. 65).” These two descriptions - *effective use* or *overt description* – became the criteria for deciding whether metacognitive knowledge had been used by the learners. Thus, combining Pólya’s and Brown’s definition, I would define a strategy as metacognitive if it fulfilled any (or a combination of) the following characteristics:

- There was an *overt description* of knowledge, that is, a written statement indicating knowledge of what one is cognising. This could include a “method of solution” which may be general and applicable to many situations (like Pólya’s “dictionary of heuristics”) or content-specific, such as factorising a polynomial in order to solve. However, if the method of solution or procedure were overtly stated, I included that as a metacognitive strategy.
- There was an indication of an *additional cognitive tool* (Vygotsky, 1975) utilised to aid in solving the problem but was not overtly stated. These are strategies which I inferred from diagrams, annotations to problems, etc.
- The strategy involved *effective use of knowledge* – which would include being aware of a procedure and using it to effectively solve the problem (this relates to conceptual understanding, procedural fluency and strategic competence)

If the strategy use was *implied*, it needed to be effectively used to qualify as metacognitive. Alternatively, the strategy use may have been self-reported and thus qualify as an *overt description*. So the first level of classification was the *category* “implied” (I) or “explicit” (Ex). If the strategy was self-reported, then the three kinds of metacognitive knowledge as discussed by Shraw and Dennison (1994: 474) – declarative (D), procedural (P) and conditional (C) – needed to be *coded*. Once coded, the *strategy type* needed to be acknowledged: the strategy could have been *heuristic* (H), *content-specific* (CS) or *other* (O). As in the case with the mathematical proficiency, effectiveness for strategy use also exists on a continuum and the responses, whether implied or explicit, needed to be rated. The *rating* of effectiveness was “+” (effective), “+/-” (partial effectiveness) and “-” (ineffective). The effectiveness, however, was dependent on how well the student answered the question and therefore was intimately linked with the student’s mathematical proficiency. Thus, as with case of mathematical proficiency, the strategy use was *categorised*, *coded* and *rated* with the additional step of noting the *type* of strategy. A flow diagram of the process is given on the following page (see Figure 3.3).

For example, in Question 2 of Instrument 1, a number of learners drew pictures of rectangles to represent the problem. This would be *categorised* as an *implied* strategy if the learner did not specifically mention “I drew a picture.” This drawing would be *typed* as a *heuristic* and *rated* according to its effectiveness to correctly represent the problem and would receive a “+”, “+/-” or “-” rating accordingly. However, also in Question 2 of Instrument 1, a number of learners self-reported strategies, often writing an explanation in all three columns. These responses would be *categorised* as an *explicit* strategy. Next, the strategy would be coded for declarative, procedural, or conditional knowledge; the strategy typed and finally rated.

Steps to coding Strategy Use

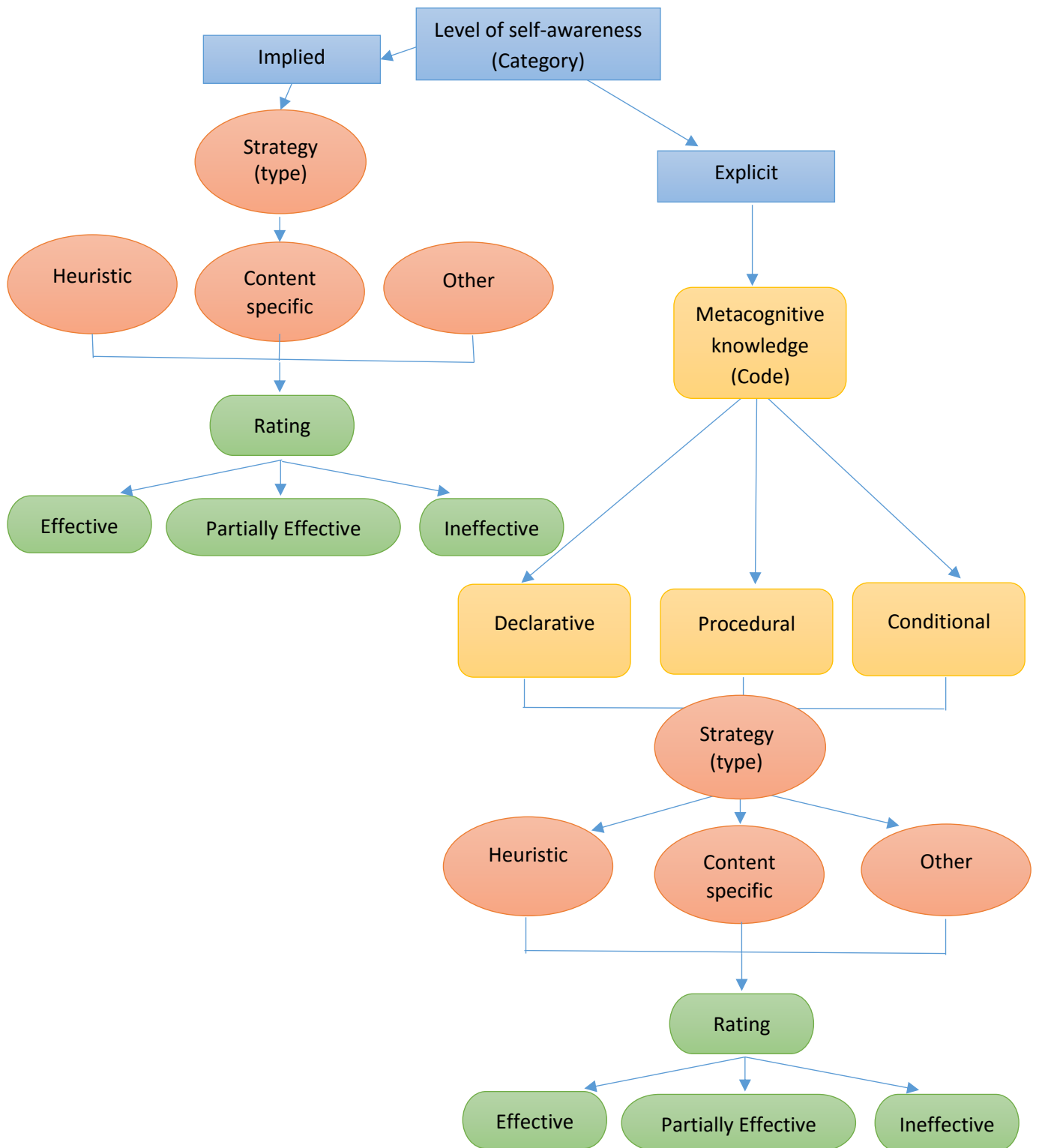


Figure 3.3 Steps to coding Strategy Use

3.6 VALIDITY, RELIABILITY AND ETHICS

3.6.1 Validity and Reliability

Because teaching situations are “inherently non-reproducible” and the distance between the “inquiring subject and object of study has been reduced to zero” (Feldman & Minstrell, 2000, p. 435), there can be a concern regarding the validity of the products of teacher research. This can be countered through several techniques borrowed from the social sciences. The techniques used in this paper were triangulation and the consideration of alternative perspectives (Feldman & Minstrell, 2000). Triangulation occurs when collected data represents several views of the same situation (Feldman & Minstrell, 2000). Although beyond the scope of this dissertation, I collected data informally through observation of the students and marking of their regular classroom assessments and revision assignments, looking for evidence of strategy use and their mathematical proficiency. I also kept a journal of my experiences. The students’ record of their own strategy use through the instruments, the classroom tests and revision exercises, and my journal entries were examples of the same situation recorded from different views and helped to ensure that my inferences of the situation were valid. Also, the data were reviewed multiple times and inferences were adjusted as more information came to light. Although the interpretation of the data was purely qualitative, I used simple quantitative methods to help track trends. Thus, in these ways I enhanced the validity of my study.

Because my instruments were informal and of my own devising, I was not be able to control for typical areas of reliability such as stability, equivalence, internal consistency or agreement (McMillan & Schumacher, 2006).

3.6.2 Ethical Considerations

The learners’ anonymity and confidentiality were assured by referring to each learner in the dissertation with a number, not a name, and the instruments are in my sole possession and will not be shared. Learners and their parents signed a consent form and the principal and district also gave permission for the study. Participation in the study was voluntary. If learners chose not to participate in completing the instruments, they would still have participated in the other class activities; however, all learners chose to complete both instruments. The two instruments were not for marks. Thus, all work for this research did not disadvantage the learners in any way.

CHAPTER 4: ANALYSIS AND RESULTS

4.1 INTRODUCTION

In this chapter, I analyse the two instruments from the perspectives of mathematical proficiency and strategy use as per my conceptual framework. This analysis is presented in three parts. The first part gives a qualitative description of the learners' responses so that the reader can feel immersed in the learners' thinking. The second part is a concise summary based upon the frequencies of the codes and summaries of trends. The third part looks at the results of these two analyses in the light of the original research questions.

In the first part, I will work question by question from Instrument 1, and then Instrument 2, comparing first the mathematical proficiency demonstrated by the learners on each instrument and then the strategy use. Instrument 1 acted as a pre-test, baseline assessment, while Instrument 2 was a post-test after the intervention period. In the Appendix section are tables of the frequencies of codes and ratings (see Appendix L – 1 to N – 2); certain portions are only included in this chapter if particularly illustrative of some results. For compactness, I sometimes refer to the instruments and questions in an abbreviated form such as I1.Q2 (i.e. Instrument 1, Question 2).

To recap the context of the two instruments, Instrument 1 was written during the second week of school in Term 1, 2015. The four questions in the instrument covered data handling, measurement, sequences and applications of quadratic functions. The content of the questions was based upon Grade 11 work that was assumed to be prior knowledge. A few learners happened to have notes from the previous year and referred to them to answer questions. Except for the “open-book” aspect, the instrument was conducted under normal school examination conditions: individual, silent and under time constraints. Instrument 2 was written late in the second term after approximately five months of strategy instruction. Instrument 2 covered the same content as Instrument 1. For Instrument 2, I gave the learners some formulae as an *aide memoire* and allowed them to refer to their textbooks or notes. In all other respects, Instrument 2 was administered under the same conditions as Instrument 1: the learners were instructed to complete it individually, silently and within one class period.

On the whole, while the content of the questions of Instrument 2 matched the content on Instrument 1, the questions themselves were more difficult. This was due to several factors. I expected the learners, who were now 6 months older and had been in school consistently since the first instrument, to be able to tackle problems of a more difficult nature. They had worked through sequences and series, trigonometry and geometry by the time the second instrument was written, which are complicated topics. Also, as the finals were looming, I wanted to expose the learners to “unseen”-type questions for practice.

4.2 QUESTION 1: DATA HANDLING

Question 1 (reproduced alongside) gave the mean, median, standard deviation and range for a fictitious class's test results. Question 1.1 asked learners to write down the mean, median, range and standard deviation if the teacher added 4 marks to each learner's paper in the question. Because every learner in the question received 4 marks, the mean and median would increase by 4, but the standard deviation and range would remain unchanged.

Question 1.2 in Instrument 1 was a more difficult question than Question 1.1 and added the facts that 50 students were in the group and the marks for the top 10% were reduced by 3 marks. The learners were asked to write down the mean, median and range for this new data set.

To calculate the mean, the learners would need to understand that $(60,3 \times 50)/50$ equalled the original mean but that $(10\%) \times (50) \times (3)$ would need to be subtracted from the $60,3 \times 50$ in order to calculate the new mean. They would also need to understand that as the alteration only occurred to the top 10% of the data set, the median would not be changed. Finally, since the top mark was now 3 marks less, the range would be reduced by 3.

As with Instrument 1, Question 1 in Instrument 2 focused on data handling; however, Instrument 2 introduced the idea of variance in addition to the mean. The question is reproduced alongside.

1. The table below contains the mean, median, standard deviation and range of the exam results of Paper 1 and Paper 2 in Mathematics for a group of students.

Exam	Mean	Med	SD	Range
P 1	46.1	46	15.2	91
P 2	60.3	60	11.6	70

1.1 The teacher added 4 marks to each of the students' marks for Paper 1. Write down the mean, the median, standard deviation and the range for Paper 1.

1.2 There were 50 students in the group and the marks of the top 10% of the students were reduced by 3 marks.

Write down the mean, the median and the range for Paper 2.

1. Consider the set of numbers:

3; x; y; 10; 12.

The mean of the set is 8 and the variance is 10.

Determine the numbers x and y.

4.2.1 Question 1: Mathematical Proficiency

To briefly restate my analysis method of mathematical proficiency, I first *categorised* responses as requiring *conceptual understanding* (CU), *procedural fluency* (PF) and/or *strategic competence* (SC). I then *rated* the strand use as *competent* (+), *partially competent* (+/-) or *incompetent* (-). Within the ratings, I developed *codes* that described responses that were similar. The codes were derived from the patterns I observed in the learners' data and were different for every question. The summaries of the frequencies of codes appears in Appendix L – 1 and L – 2, the summaries of ratings appears in Appendix M – 1 to N – 2, and an exemplar of code examples taken from learner responses appears in Appendix J – 1.

4.2.1.1 Question 1, Instrument 1: Mathematical Proficiency – Trends

Question 1 on Instrument 1 tested *conceptual understanding* of data handling. The question included determining the mean, median, range and standard deviation of a data set to which certain alterations were applied. The learners needed to understand how different data values are calculated in order to answer the questions correctly. The question required little *procedural fluency* as most calculations could be done mentally. However, a number of learners demonstrated a lack of procedural fluency through superfluous or inappropriate calculations.

Out of the two classes ($n = 27$), there were six correct responses for the mean, five for the median, one for the range and no correct responses for the standard deviation. Eleven respondents did not attempt the question at all. The errors that occurred with the greatest frequency were the wrong use of operation or simply not being able to proceed.

Some responses are particularly noteworthy. One learner wrote down the correct formula for the mean, but then *averaged* all the *given data quantities* (see illustration alongside). This same learner calculated the median by adding together the given mean, standard deviation and teacher increase, then divided the sum by 2. For the range, she subtracted the given standard deviation from the given range and added 4 to the result.

Mean
$$= \frac{\sum x_i}{n}$$
$$= \frac{46.1 + 46 + 15.2 + 9.1 + 4}{5}$$
$$= 40.46$$

Median
$$= 32.6$$

because $46 + 15.2 + 4$
$$= 65.2 \div 2$$
$$= 32.6$$

Range
$$= 9.1 - 15.2 + 4$$
$$= 79.8$$

In all three cases she indicated some understanding of the *procedure*, but does not understand the concept behind the procedure. She appears to know that the mean results from adding certain values and dividing by the number of values, but “grabbed” the only values at hand, i.e. the given mean, median, standard deviation and range. For the median, she seems to have some sense that the median is the middle of a data set, but finds the *midpoint of the sum* of the median, the standard deviation and the additional value of four. For the range, she seems to remember that subtraction is needed, but as she does not have the highest and lowest value of the data set, she uses the highest and lowest *numbers* given and then increases that difference by four.

Another learner (see illustration alongside) did the following operations for all values (mean, median, standard deviation and range – only the mean and median is illustrated): she changed the given data values and the additional value of 4 to percentages then decimals, added those values as decimals, changed those values back to percentages and then added the percentages. While she shows some understanding of

Mean
$$46.1\% = 0.461$$
$$4\% = 0.04$$
$$0.461 + 0.04 = 0.501 \times 100$$
$$= 50.1\%$$
$$\therefore 46.1\% + 50.1\%$$
$$= 96.2\%$$

Median
$$46\% = 0.46$$
$$4\% = 0.04$$
$$0.46 + 0.04 = 0.5 \times 100$$
$$= 50\%$$

converting between decimals and percentages, she shows no understanding of the applicability of this skill to this problem.

The aspect of percentages was echoed by other learners as well. One learner multiplied the mean, median and standard deviation by 0,4 (perhaps meaning 4%?), found that value and added it to the original data values (see figure alongside). Another learner added 4 to the given data values and changed the final answer into a percentage (see figure below). Lastly, another learner multiplied the given data values by 4 and concluded her calculations with that number divided by 100, but not simplified.

1.1 Mean	
46.1×0.4	
$= 18.44 + 46.1 = 64.54$	
Median	
46×0.4	
$= 18.4 + 46 = 64.4$	
SD	
15.2×0.4	
$6.08 + 15.2 = 21.28$	

When I coded the responses, I grouped these kinds of errors as “misused numbers given” or “wrong use of operation”; however, the codes do not give the gravity of the conceptual misunderstanding that was evident. This question revealed what

$46.1 + 4 = 50.1\%$ Mean
$46.1 + 4 = 50.1\%$ Mean
$46 + 4 = 50\%$ Median
$15.2 + 4 = 19.2$ Range Standard Deviation
$91 + 4 = 95\%$ Range

would be a common theme in both instruments namely, not understanding the *meaning of numbers*. In Question 1.1 of Instrument 1, four “numbers” were given that had specific meanings. They were the mean, median, standard deviation and range of an unknown data set. Each of these “numbers” has a prescribed way to be calculated. But the way the learners operated on them indicated they had little or no understanding of what those numbers *meant* or how they would have been calculated in the first place. Very often, what appears to be random operations with little intelligible connection to the problem were performed on the given numbers.

Question 1.2 was a more difficult question than Question 1.1 and added the facts that 50 students were in the group and the marks for the top 10% were reduced by 3 marks. The learners were asked to write down the mean, median and range for this new data set.

The codes are summarised in the table alongside. In this question, 11 learners did not attempt the question at all, and of the 16 who attempted it, 7 made a brief start but could not continue. Eight students correctly calculated 10% of 50 and three knew to multiply that 5 by 3, but no one was able to use that value to correctly find the mean.

Question 1.2		Frequencies		
Strand		Conceptual Understanding		
Rating		+	+/-	-
Codes	Description/Totals	0	12	27
1.	all 3 answers correct	0		
1.	2 answers correct		1	
2.	Correctly found 10% of 50		8	
3.	Multiplied 5 x 3 but used incorrectly		3	
1.	0 - 1 answer correct			8
2.	Misused numbers given			3
3.	Wrong use of operation			4
4.	Found 10% of another number			1
5.	Did not multiply 5 by 3			1
6.	subtracted 3 from mean and/or median			3
7.	little or no progress			7

Table 4.1 Frequency of CU Codes I1.Q1.2__

Only one learner knew that the median would not be changed and that same learner was the only one who correctly subtracted 3 from the range. Two learners (the same ones from Question 1.1) incorrectly applied decimal and percentage operations. Two learners subtracted 15 from all three values and three learners subtracted 3 from the mean and/or median. (Please note that in all tables, the totals may not match the sample size as many responses had several characteristics which were tallied separately.)

The last point highlights a recurrent solution method in this question which was *applying the same operation(s) to all data values* – a sort of one-size-fits-all approach. This happened in both Question 1.1 and 1.2. A total of seven learners in Question 1.1 and four learners in Question 1.2 applied the same operations to all four data values. Sometimes this approach gave them some correct answers, but it makes one wonder if those answers were not just luck. Using the same procedure indiscriminately also shows that very few learners understand that the mean, median, standard deviation and range are calculated by four different processes and represent four different summaries of data.

Although I did code for procedural fluency for Question 1.2, I could not find general patterns. Many learners could not get past finding 10% of 50 and then using that information effectively. However, some learners' responses are notable because their procedural fluency was so flawed.

The example alongside shows a number of procedural errors performed by a particular learner. Firstly, she started to find 10% of 70 (i.e. the *range*, not the number of learners) which she calculated by writing 10/100 *divided by* 70. On the line below, she implies the equivalency of this value (which is actually $\frac{1}{10} \div 70 = \frac{1}{700}$) with 70

$\div 10$. Notice that she has “cancelled” the zero of the 100 in the denominator with the zero of 70, which according to her mathematical statement, is also in the denominator. Her final answer is “7,” which is 10% of 70, but her calculations along the way do not reflect accurate reasoning. She then goes on to correctly find 10% of 50, but subtracts 3 from the 5 resulting in 2. She then *divides* both the mean and the median by 2. These operations make little sense in the context of the problem. Lastly, I cannot understand at all the learner's calculations for the range. The lines of text from this student emphasise the interconnectedness of conceptual understanding and procedural fluency. It may be the case that her lack of conceptual understanding affects her procedural fluency and her faulty procedural fluency affects her conceptual understanding.

Handwritten student work showing calculations for 10% of 70, 10% of 50, Mean, Median, and Range with various errors.

10% of 70
 $= \frac{10}{100} \div 70$
 $= 7$

10% of 50
 $= 5$
 $= 5 - 3$
 $= 2$

Mean
 $= 60.2 \div 2$
 $= 30.15$

Median
 $= 60 \div 2$
 $= 30$

Range
 $= 70 - 30 - 30.15$
 $= 9.85 \div 2$
 $= 4.925$

The first question on the first instrument revealed the importance of conceptual understanding to solving a problem. Almost all the learners did not have the correct understanding of mean, median, range and standard deviation to answer the question at all sensibly – even though the calculations could practically be done mentally. Many learners resorted to “number grabbing”

(selecting numbers and performing arithmetic operations on them) (Kilpatrick, et al, 2001, p. 124) in order to get an answer.

4.2.1.2 Question 1, Instrument 2: Mathematical Proficiency – Trends

In terms of the strands of proficiency, this question (see reproduction alongside) required *conceptual understanding* of how variance and mean is calculated. Learners would need to know that the sum of the given data values, divided by 5, resulted in 8; they would also need to know that the difference between each data value given and 8 would need to be squared, summed and that total divided by 5 to equal the variance of 10. As two data values were unknown, the learners would need *strategic competence* in order to set up the mean and variance equations that would be solved simultaneously. *Procedural fluency* would also be needed to work efficiently with the simultaneous equations. My memorandum answer is reproduced below to indicate the complexity of the question. I marked this question as well as coded it for the three strands.

1. Consider the set of numbers:

3; x; y; 10; 12.

The mean of the set is 8 and the variance is 10.

Determine the numbers x and y.

$\frac{3+x+y+12+10}{5} = 8$	✓ set up mean
$x+y=15$	✓ simplify
$\therefore y=15-x$A	✓ solve for one variable
$\frac{(-5)^2+(x-8)^2+(y-8)^2+2^2+4^2}{5} = 10$	✓ set up variance equation
$(x-8)^2+(y-8)^2=5$B	✓ simplify
Substituting A into B:	✓ substitution
$(x-8)^2+(15-x-8)^2=5$	✓ simplify
$x^2-16x+64+x^2-14x+49=5$	✓ standard form
$2x^2-30x+108=0$	✓ factorise
$x^2-15x+54=0$	✓ solve for one variable
$(x-6)(x-9)=0$	✓ solve for other variable
$x=6$ or $x=9$	
$y=9$ $y=6$	

Figure 4.1 Memorandum: Question 1, Instrument 2

A number of trends emerged from the coding. The first set of trends concerned how the learners conceptualised the problem. The learners' responses were divided into three categories:

learners who did not know how to proceed at all ($n = 9$), learners who saw the list of numbers as a *number pattern* ($n = 6$), and those who treated the list of numbers as a *data set* ($n = 10$). Within the latter two categories, trends emerged as well. Many of the subsequent competencies or errors were dependent on the initial conceptualisation of the problem. Only one learner out of the two classes was able to come close to the memorandum answer. The codes for conceptual understanding that emerged from the data are given in the table alongside.

Learners who saw the list of numbers as indicating a pattern approached the problem in different ways. For example, Learner 3.4 may have understood that two equations in two unknowns were needed and tried to set up simultaneous equations from the differences between terms. She made a number of conceptual errors that may have indicated the practice of applying procedures without understanding (Skemp, 1976). Her solution is reproduced below. She begins by seeing the list of five data elements separated by a semicolon as a sequence. She determines the first, second and third differences between the supposed terms, working in a procedurally accurate way but based upon the conceptual error that this list of numbers represents a cubic function.

In her resulting simplifications she ends with two expressions in two variables: $-3y + 3x + 7$ and $-22 + 3y - x$; these are expressions for the first two third differences. The first expression is an accurate simplification based upon her preceding calculations; the second expression contains an arithmetic error involving integers. She then sets the second expression equal to zero (for no conceptually accurate reason) and solves for x in terms of y . She then sets the other expression equal to zero, substitutes in the preceding expression for x and solves for y . She finally takes that value for y and substitutes it back into the second expression and solves for x .

Question 1 (I2)	Frequencies		
	CU		
	+	+ /-	-
Description/Totals	15	6	32
Saw numbers as data set	10		
used mean formula correctly	4		
used variance formula correctly	1		
recognised need for mean formula, but used incorrectly		1	
recognised need for variance formula, but used incorrectly		5	
Saw list of data values as pattern			6
tried to find differences between "terms" of "sequence"			5
little or no progress			9
used quadratic pattern $2a = \dots$ for "2nd difference"			2
Assumed <i>linear</i> pattern and found x and y in terms of 1st differences			1
ignored the "sum of" aspect of variance equation			4
equating unequal quantities			7

Table 4.2 Frequency of Conceptual Understanding Codes for I2.Q1

Handwritten mathematical work showing a learner's attempt to solve a problem. The work includes several equations and calculations, some of which are crossed out or corrected. The final result is $x = -\frac{27}{4}$.

The learner in this example is capable of quite complicated and relatively accurate calculations; in nearly a whole page of working-out, she only makes one small arithmetic mistake. However, she does not fully understand when to use the procedures or why they exist. She does not take context into account when she sees a list of numbers separated by semicolons; it is true that sequences are represented like this but a list of data values is not a sequence. She simplifies her differences until she has two expressions in two variables, but sets one expression equal to zero as if she were finding the roots of an equation, but equating to zero plays no part in this problem. Putting a polynomial equation in standard form works because equating to zero lets the one side be fully factorised into linear factors so that the linear factors can be equal to zero by the zero-product property. However, her expressions are linear to begin with, so the zero-product property does not apply. This response clearly indicates the interconnectedness of the strands but shows that serious conceptual errors lead to nonsensical, if accurately calculated, results.

Three learners tried to use the second differences to find the general term. One example is represented alongside. Learner 3.10 correctly finds the first and second differences but then tries to use the second differences to find the “a” coefficient of $T_n = an^2 + bn + c$ using the fact that $2a$ equals the second difference. However, she runs into difficulty because a quadratic sequence should have a *constant* second difference; in her formulation it does not. However, she works around this by using only her first second difference, setting it equal to $2a$ and solving for a in terms of x and y . She has now introduced a third unknown and stops because she cannot proceed further.

Handwritten student work for Learner 3.10:

3 ; x ; y ; 10 ; 12

Mean = 8

Variance = 10

3 ; x ; y ; 10 ; 12

x-3 ; y-x ; 10-y ; 2

2a = y-x - (x-3) ; 10-y - (y-x) ; 2 - (10-y)

2a = y-x - (x-3)

2a = y-x - x + 3

2a = y - 2x + 3

a = $\frac{y-2x+3}{2}$

Amongst the learners who attempted to solve this question by conceptualising it as a data handling question, most learners could not set up a correct variance equation. For example, Learner 3.6's solution is illustrated alongside. She starts her solution correctly by finding an expression for x and y that was derived from an equation utilising the definition of the mean. However, when she uses the variance equation, she only writes down $\frac{(x-8)^2}{5} = 10$ and

Handwritten student work for Learner 3.6:

Variance = $\frac{(x-8)^2}{5}$

$\frac{(x-8)^2}{5} = 10$

$(x-8)(x-8) = 50$

$x^2 - 16x + 64 = 50$

$x^2 - 16x + 64 - 50 = 0$

$x^2 - 16x + 14 = 0$

solves for x . She does not seem to realise that “ x ” in the variance formula represents *every* data value, not just one unknown. Also, she completely ignored the summation of the differences. It is possible that she confused the “ x ” in the variance formula as being the *same* “ x ” as in the question. From there she correctly simplified to a quadratic equation under the principle of continued accuracy. It is possible this learner, like Learner 3.4, demonstrated utilising procedures without understanding; in this case the lack of understanding springs from not knowing what the variables represent in a formula.

An important point to mention is that the learners were given a “notes” page with a brief summary of the content to accompany the instrument so that they did not need to access formulas from memory. The section in the notes on data had a description of the meaning of the variables in the formulas and an example of how to work with grouped data. Thus, even with the notes, Learner 3.6 did not have perhaps enough conceptual understanding of the variance formula to interpret the formula in light of the question asked in the instrument.

In addition, all three learners demonstrated the conceptual error of *equating unequal quantities* that was common to the classes regardless of whether they used the numbers as a data set or a number pattern. It seemed as if the learners knew an equation was needed somewhere to solve for the variables, but did not recognise which quantities were equal. Learner 3.6 set the third differences equal to zero; Learner 3.10 set the first second differences equal to $2a$ and Learner 3.4 ignored the summing of the differences and set one unknown data value minus the mean, squared and divided by 5 equal to the actual variance. In all three cases quantities that were unequal were equated.

In contrast, Learner 3.9 made full use of the notes and even modelled her answer on the tabulation method used in the example in the notes. Her solution is represented alongside. She uses the table to organise the given information into the correct categories of data (x) , $(x - \bar{x})$ and $(x - \bar{x})^2$, substituting the variables as data values and operating on them. She successfully finds an expression for y in terms of x using the formula for the mean. She also successfully sets up the simultaneous equation. Her only error is a small procedural one of factorising the quadratic incorrectly. Otherwise her solution is completely correct.

$3+x+y+10+12$			
$25+x+y$			
			$(x-8)(x-8)$
Data(x)	$x - \bar{x}$	$(x - \bar{x})^2$	$x^2 - 16x + 64$
3	-5	25	$(y-8)(y-8)$
x	$x-8$	$x^2 - 16x + 64$	$y^2 - 16y + 64$
y	$y-8$	$y^2 - 16y + 64$	
10	2	4	$25 + x^2 - 16x + 64 + y^2 - 16y + 64$
12	4	16	$+ 4 + 16$
$25+x+y$		$173+y^2+x^2-16y-16x=173$	
$\therefore 10 = \frac{173+y^2+x^2-16y-16x}{5}$			
$50 = 173 + x^2 - 16x + y^2 - 16y$			
$50 = 173 + (15-y)^2 - 16(15-y) + y^2 - 16y$			
$50 = 173 + 225 - 30y + y^2 - 240 + 16y + y^2 - 16y$			
$50 = 2y^2 - 30y + 108$			
$0 = 2y^2 - 30y + 108$			
$0 = y^2 - 15y + 54$			
$0 = (y-9)(y-4)$			
$\therefore y = 9 \text{ or } y = 4$			
for $y = 9$	for $y = 4$	9×4	
$15-y = x$	$15-y = x$		
$15-9 = 4$	$15-4 = 11$		
$\therefore x = 4$			
$\therefore x = 4 \text{ and } y = 9$			

It is significant to note that only one other learner in the two classes even attempted to use the table as presented in the notes and that learner could not proceed past putting down a blank table. What is surprising is that even when given a solution method, many learners could not see that the solution method applied to the particular problem on which they were working and use it.

Another trend that the coding revealed was that certain errors permeated all three strands of conceptual understanding, procedural fluency and strategic competence. For example, seeing the data set as a number pattern was an error in conceptual understanding, but resulted in an error in strategic competence by learners trying to establish equations related to patterns. Some

managed to perform their procedures accurately from that incorrect basis, but others did not. The conceptual error of misinterpreting the variance formula lead to a strategically flawed variance equation which resulted in a procedural dead-end. Thus, the interconnectedness of the strands was demonstrated again.

On the positive side, while many learners did demonstrate conceptual errors, for some, the complexity of their mathematical thinking and their procedural skills improved considerably over the first instrument. A few learners attempted to set up simultaneous equations and several learners demonstrated competence with quadratic equations. Five learners made zero or one procedural errors. On the whole, the solutions were more sophisticated than in Instrument 1 and showed greater procedural fluency, even if the conceptual foundations were faulty.

4.2.2 Question 1: Strategy Use

Although the first instrument preceded the strategy instruction, I looked for the same aspects of metacognitive knowledge as I hoped to find after the strategy instruction; that is, I looked for self-reports of *declarative knowledge*, *procedural knowledge* and *conditional knowledge*. As mentioned earlier in this paper, I have defined declarative knowledge as learners' knowledge of *what* strategies they used for mathematical problems; procedural knowledge is the learners' knowledge about *how* to implement those strategies and conditional knowledge is the knowledge about *when* and *why* to use the appropriate strategies. The learners were prompted to use these categories by the headings in the columns in the instrument. However, learners sometimes wrote notes, drew a picture or annotated a problem from which strategies could also be inferred, even without the learners explicitly stating their strategies. These were also considered in addition to the explicit self-reports and were categorised as implied strategy use.

To restate my analysis steps, I first determined the *category* of strategy used as either *implied* (I) or *explicit* (Ex). If the strategy was implied, it was defined by *type* of strategy: *heuristic* (H), *content-specific* (CS) or *other* (O); lastly, it was *rated* as *effective* (+), *partially effective* (+/-) or *ineffective*, depending upon whether the strategy lead to the correct calculations; however, effectiveness was intimately connected with mathematical proficiency. If the strategy was explicitly stated, it was *coded* as *declarative* (D), *procedural* (P), and/or *conditional* (C); defined by *type* of strategy as *heuristic* (H), *content-specific* (CS) or *other* (O) and *rated* using the same rating system as the implied strategies. For the most part, the type of strategy remained consistent across the code; for example the same content-specific strategy would be referred to declaratively, procedurally and conditionally, but occasionally the learners changed their mind, so the strategy was typed for every code.

4.2.2.1 Question 1, Instrument 1: Strategy Use – Trends

For Question 1.1, out of the 26 respondents for this instrument, only three learners attempted to report their strategies. Of those, three of the learners wrote down what strategy they used;

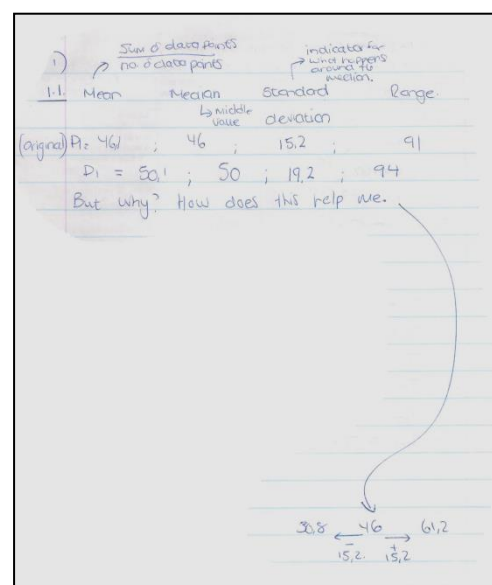
two wrote down how they used that strategy and two attempted conditional knowledge. Their strategies are reproduced on the next page. However, all three indicated metacognitive knowledge because of providing an overt description of knowledge (Brown, 1987). The second row of responses merits some discussion.

What strategy/ies did you use?	How did you use the strategy?	Why did you use the strategy or how do you know when to use the strategy?
I added 4 to each value.	By adding 4	
I just multiplied thought about it.	I multiplied everything by 4	I did not know what else to do.
Median, middle term, if 4 marks added to each kid - then $46 + 4 = 50$		General reasoning?

Table 4.3 Explicit Strategies: Question 1.1, Instrument 1

One learner wrote down responses in all columns, indicating possibly a sense of what constitutes declarative, procedural and conditional knowledge. In the “what” column she writes “I just ~~multiplied~~ thought about it” [sic]. The strike-through of “multiplied” may indicate that she was going to describe a procedure and then thought better of it and changed it to show that she reflected on the problem. This could be an indication of several processes: she may have realised that “multiplying” may not count as a strategy and that reflection did; or she may have realised that prior to multiplying she had to “think about” the problem. Either way metacognition is indicated because there is an overt description of her knowledge and she reveals that she is thinking about her thinking. Also, the statement “I just thought about it” could be considered a strategy because reflection, and not just jumping into a problem without thought, would be an additional cognitive tool brought to bear on a problem. However, the response in the second column, “I multiplied everything by 4” may indicate that her strategy was simply multiplying. This is confirmed by the third column “I did not know what else to do,” thus, indicating her strategy is not a heuristic but a procedure. From the point of view of effectiveness, neither the reflection nor the multiplying led to a computationally accurate result.

Learner 3.15's response, illustrated alongside, indicates metacognitive reflection on the problem as well, but was coded as implied strategy use. She writes down the self-directed question, "But why? How does this help me?" and draws an arrow to an interval of values between the given *median* and one standard deviation on either side. At the top of the problem, she also writes down the meanings of the terms mean, median and standard deviation. She is correct with her definitions of mean and median but defines standard deviation as "indicator for what happens around the median" which she then uses later on in her problem. Both the annotations and the self-directed question were categorised as "implied," typed as "other," and both were partially effective as two out of the four requested values were correct.

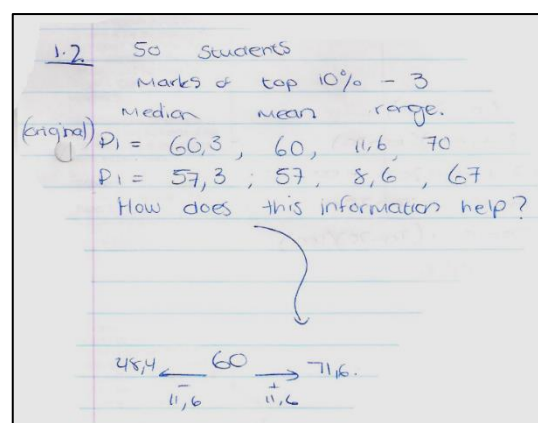


Evidence of strategy use was very minimal with Question 1.2. A table with learner responses in the strategy columns is reproduced below. Only two learners wrote something in the "what strategy" columns and one of those responses may have simply been the start of procedural steps. One learner did state "tried to find what 10% of 50 was, then multiplied that by 3" which was coded as a declarative, content-specific strategy. However, because she could not complete the problem, it was a partially effective strategy. She also wrote "used a calculator then" under the "how" column, which was coded the same.

What strategy/ies did you use?	How did you use the strategy?	Why did you use the strategy or how do you know when to use the strategy?
50 students $46 \cdot 4 + 4 \times 10\%$ $50 \times 10\% = 3$ mean: 60.3 median: 66 Range: 91	Given	Given.
First Tried to find what 10% of 50 was, then multiplied that by 3	Used a calculator then	

Table 4.4 Explicit Strategies: Question 1.2, Instrument 1

Learner 3.15 reproduced her thinking process while working, as she did in Question 1.1 (see illustration alongside). She wrote down the given original data values, then adds three to all values and writes down the self-directed question: “How does this information help?” Like before, she draws an arrow to an interval of values one standard deviation around what she has labelled as the mean, but what is actually the median in the question. Although this indicates the learner is reflecting on her solution process, she only wrote



down the correct answer for the range; this was coded as “other” and received a partially effective rating. Thus, a few learners in Instrument 1 show some idea of using strategies, but were rather telegraphic in their responses and their strategies did not impact significantly on their solution process.

4.2.2.2 Question 1, Instrument 2: Potential strategy use

As described earlier, conceptual understanding, strategic competence and procedural fluency were needed to conceptualise and complete this problem correctly. A number strategies could have been used by the learners to help them conceptualise the problem. Firstly, they could have used the self-directed question, “*Did I use all the data?*” and its further subdivisions of “*What is the data, what is the unknown and what is the condition?*” If they had done this, they could have seen that there were two unknowns, but also two givens. This could mean that a system of two equations in two unknowns could be set up and solved simultaneously. A number of learners demonstrated this understanding, even if they were working with patterns instead of data. Alternatively, they could have used the strategy, “*Do I know a related problem?*” Students have worked with the mean since at least Grade 8, and the idea of variance was introduced in Grade 11. They may have encountered similar problems before. Also, the strategy of “*working backwards*” may have prompted the learners to work backwards from the given mean and variance to the individual data values that gave rise to that mean and variance.

Strategies could also have been employed by the learners to support their strategic competence and to think of the type of equations needed to formulate the problem into a mathematically correct and workable model. Again, “*Did I use all the data?*” could come to the aid of the learners: they were given the value of the mean and the variance, and formulas exist for finding those quantities. If they had forgotten the formulas, they were on the notes page. “*Working backwards*” could also lead to formulating the equations.

In their next step, the learners needed to substitute into the formulas the specific information from this question. Several learners were able to do so for the mean, but the learners’ conceptual understanding of the variance formula and this particular question may not have been sound enough to realise that $\sum (x - \bar{x})^2$ represented that the mean is subtracted from *every data*

value, squared and then summed. They may not have realised that the “x” in the formula represents every data value and thought it was the same “x” as in the question. But again, use of “*Did I use all the data?*” could have helped the learners to reflect on the meaning of the variance formula.

Once the equations were established, *content-specific* strategies could have been employed to aid procedural fluency. For example, strategies such as *collecting like terms* could have helped the learners simplify complicated expressions – even if the learner had chosen to work with patterns.

4.2.2.3 Question 1, Instrument 2: Strategy Use - Trends from the coding

After tallying the frequency of responses by category, code, type and rating, the first most notable quantitative difference between strategy use in the first instrument and the second instrument was an *increase in the quantity* of attempts to articulate strategies: twelve learners using Instrument 2 compared to five learners in Instrument 1 wrote something in at least one of the three columns. The tables below indicate the frequency of the types and ratings of the strategies. In addition, there were a total of 28 separate responses in Instrument 2 compared to 7 in Instrument 1. Also notable was an increase in the number of heuristic strategies mentioned: 11 instances in the second instrument compared to zero in the first instrument. However, although the quantity of strategies increased, the effectiveness of the solution methods did not apparently increase as well; in fact the opposite seemed to occur: 57,1% of the articulated strategies did not lead to effective solutions.

Instrument 1 – Question 1.1 and 1.2 combined										
Frequency of Strategies by category, code, type and rating										
Category	Explicit						Implied			
Code	Declarative		Procedural		Conditional				Total	%
Rating/type	H	CS	H	CS	H	CS	H	CS		
E	0	1	0	0	0	0	0	0	1	14.3
PE	0	2	0	1	0	0	0	1	4	57.1
IE	0	0	0	1	0	0	0	1	2	28.6
Total	0	3	0	2	0	0	0	2	7	

Instrument 2 – Question 1										
Frequency of Strategies by category, code, type and rating										
Category	Explicit						Implied			
Code	Declarative		Procedural		Conditional				Total	%
Rating/type	H	CS	H	CS	H	CS	H	CS		
E	1	1	0	1	0	0	0	0	3	10,7
PE	2	2	2	2	1	0	0	0	9	32,1
IE	1	7	2	2	2	2	0	0	16	57,1
Total	4	10	4	5	3	2	0	0	28	

Table 4.5 Strategy Use Question 1: Instruments 1 and 2 Compared

However, while the frequencies give one description of the data, a qualitative look at the strategies reveals other trends. The self-reported strategies of the learners are discussed in more detail in Appendix O – 2. A representative sample of heuristic and content-specific strategies is given below and a summary of the highlights is presented here.

Type of Strategy	Rating	What strategy/ies did you use?	How did you use the strategy?	Why did you use the strategy or how do you know when to use the strategy?
Heuristic	Effective	None		
	Partially effective	I used trial and error.	I inserted numbers into my calculator in place of x and y until I got to one that equals a mean of 8.	I was given that the mean was 8 so I guessed until I got to an answer that worked.
	Ineffective	what type of problem is this	Used it to think about how to solve the Qs but didn't get to solve the Q.	when I don't know what to do, I think about the types of problems they are, and which is similar to mine
Content-Specific	Effective	Simultaneous equations. Tables	Solve for y , then solve for x by substituting	
	Partially effective	I attempted to find the common difference, but got stuck due to the missing numbers, tried to work it out backwards using the variance and mean.	used the mean formula and worked backwards.	i honestly did not know what else to do.
	Ineffective	I used the knowledge of number patterns	eliminate Simultaneous equations	when you have 2 unknown variables to solve for

Table 4.6 Representative Explicit Strategies: Question 1, Instrument 2

In addition to the increase in quantity of responses, the learners were more descriptive and less telegraphic in their responses in Instrument 2 than in Instrument 1, often writing their responses

in close to full sentences. This may show a greater awareness of their thinking processes. Another difference was that many learners seemed able to describe what strategies they used, how they used them and why they used them, indicating declarative, procedural and conditional knowledge, respectively. Out of the 13 learners who wrote responses, 6 attempted to write appropriate responses in all three columns, and two learners wrote in two columns. A final difference was the increase in the number of heuristic strategies mentioned compared with Instrument 1. Strategies such as “working backwards” and “trial and error” were mentioned by five learners.

Another unexpected trend displayed by the learners was that the strategies some learners described did not match their mathematical working-out. Six out of the 13 learners described strategies they did not employ, or they described strategies as if they employed them but did not in fact progress through the problem. This leaves me with the question that if the strategy described differs from the strategy actually used, is metacognitive knowledge truly present? I would have to answer that it is not in those cases.

On the other hand, comparing the mathematical working and the strategy use with those learners who achieved a correct or partially-correct rating revealed further trends. Three learners fell into this category: Learners 1.1, 1.10 and 3.9. This time, their strategy use is displayed alongside their mathematical working.

Learner 1.1's responses are reproduced alongside. Although her method may be considered less than rigorous, she is clear about what

she is doing, how she is doing it and why. And, incidentally, she discovered the correct answer. A more thorough trial-and-error approach would have been to double check if those numbers worked for the variance as well. She does demonstrate, though, that she understands how the mean is derived.

Handwritten mathematical working for finding the mean of x and y . At the top, x is written above 3 and y above 12, with a checkmark and the number 2 below them. Below this, $x = 6$ and $y = 9$ are written. Then, the numbers 3, 6, 9, 10, and 12 are added together, with a horizontal line underneath. The sum 34 is written below the line, and then 34 is divided by 5 to get 6.8. The final result is written as "mean = 6.8".

I used trial and error.

I inserted numbers into my calculator in place of x and y until I got to one that equals a mean of 8.

I was given that the mean was 8 so I guessed until I got to an answer that worked.

1. mean = $\frac{?}{5}$ $\neq 8$
 $\therefore 5 \times 8 = ?$
 $\therefore 5 \times 8 = 40$
 Sum of data/set = 40.

Data	$x - \bar{x}$	$(x - \bar{x})^2$
3		
x		
y		
10		
12		
	40	50

Variance = $\frac{?}{5} = 10$
 $\therefore 5 \times 10 = ?$
 $\therefore 5 \times 10 = 50$
 $\therefore$ Sum of $(x - \bar{x})^2 = 50$.

• I worked backwards to find the sum of the data altogether

• I said: mean = $\frac{?}{5} \neq 8$
 $\therefore 5 \times 8 = 40$
 • I said: deviance = $\frac{?}{5} \neq 10$
 $\therefore 5 \times 10 = 50$

Learner 1.10 (illustrated above) was one of only two learners who picked up that a solution method was actually provided by the notes and copied down the table. She realised that the last row of the columns represented sums and used the given information to “work backwards” to find the sum. Like Learner 1.1, she is very clear about what she is doing and how. Also, she may not have understood how variance is calculated (possibly indicated by the blank columns) but she did understand that it was a type of average, even if she did not grasp the details. She used what she *did* understand to further her progress in the problem.

Learner 3.9, although somewhat terse in her description of strategy use, again demonstrates that she knows what she is using and how: she employs simultaneous equations and tables successfully to solve the problem. In her case, she was also successful from a mathematical proficiency perspective as she was the closest to the memorandum answer. A few minor procedural errors kept her from a perfect score.

Simultaneous equations.
Tables

Solve for y , then solve for x by substituting

$3 + x + y + 10 + 12$
 $25 + x + y$

Data (x)	$x - \bar{x}$	$(x - \bar{x})^2$	$(x - \bar{x})(x - 8)$
3	-5	25	$x^2 - 16x + 64$
x	$x - 8$	$x^2 - 16x + 64$	$(y - 8)(y - 8)$
y	$y - 8$	$y^2 - 16y + 64$	$y^2 - 16y + 64$
10	2	4	$25 + x^2 - 16x + 64 + y^2 - 16y + 64$
12	4	16	$+ 4 + 16$
$25 + x + y$		$173 + y^2 + x^2 - 16x - 16y = 173$	

$10 = 173 + y^2 + x^2 - 16x - 16y$
 $5 = 25 + x + y$
 $50 = 173 + x^2 - 16x + y^2 - 16y$
 $50 = 173 + (15 - y)^2 - 16(15 - y) + y^2 - 16y$
 $50 = 173 + 225 - 30y + 158$
 $0 = 2y^2 - 30y + 108$
 $0 = y^2 - 15y + 54$
 $0 = (y - 9)(y - 6)$
 $\therefore y = 9 \text{ or } y = 6$

for $y = 9$ for $y = 6$ 9 & 6
 $15 - y = x$ $15 - y = x$
 $15 - 9 = 6$ $15 - 6 = 9$
 $\therefore x = 6$ $\therefore x = 9$ and $y = 6$

In all three cases illustrated, clear, self-aware thinking is evident in the articulation of their strategies and in the way they approach problems. In addition, a greater number of learners knew about appropriate strategies compared with Instrument 1.

4.3 QUESTION 2: MEASUREMENT

Question 2 of Instrument 1 (reproduced alongside) required learners to find the length and breadth of a patio if two separate transformations were applied to the dimensions resulting in two separate new areas. Neither of the

first two dimensions were given and that was part of the strategic competence required to solve the question: the learners had to realise that two equations were needed to solve for two unknowns.

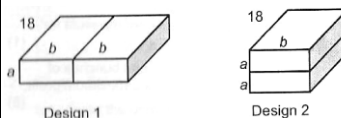
As in Instrument 1, Question 2 of Instrument 2 (reproduced alongside) dealt with the content area of measurement, but was concerned with the surface area of two rectangular prisms rather than the area of a flat surface. The question had structural similarities to the question in Instrument 1 in that the areas were affected by transformations and required simultaneous equations in order to solve. However, this question differed in that the surface areas were altered by a different stacking arrangement of the prisms.

2. Lesedi is keen to extend her rectangular patio. She has been told that if the length and breadth are each increased by 1 m, the area will be increased by 20 m^2 . If the length and breadth are each increased by 2 m, then the area will be doubled. Find the dimensions of the patio (correct to one d.p.).

2. The dimensions of each of two identical rectangular blocks are a , b , and 18 cm (for height, width and length) as shown below:

The two blocks are joined in two

different ways to form designs 1 and 2:



The sum of the height, width and length of Design 1 is 35 units, the sum of the dimensions of Design 2 is 31 units.

Calculate the surface area for each design.

4.3.1 Question 2: Mathematical Proficiency

Question 2 in Instrument 1 was a complicated question, requiring conceptual understanding, procedural fluency and especially strategic competence to set up and complete the question correctly. *Conceptual understanding* was needed to understand that the area of the patio was the product of the length multiplied by the width; however, the key information was equating the increase in length and width with the increase in area. *Strategic competence* was required to establish correct variables and set up correct equations. *Procedural fluency* was especially required for this question as it involved multiplying two binomials, solving using two simultaneous equations and solving the final problem with the quadratic formula.

Question 2 on Instrument 1 clearly illustrated the connection between the three strands of mathematical proficiency and resulted in one error appearing as a code for two or more strands (See Appendix L - 1). For example, Learner 3.1 correctly assigned two different variables to the original length and breadth, indicating the conceptual understanding that length and breadth are different for rectangles as compared to squares (see illustration alongside). She also sets up three (almost accurate) equations in three unknowns to use for solving simultaneously. However, she starts with one key error

that prevents her from making progress: she defines area as a^2 , possibly confusing area with the squared units used to measure it. This can be seen as a conceptual, procedural and strategic error. Conceptually, she is squaring a two-dimensional quantity which makes it now a 4-dimensional quantity. Strategically, she has introduced a third variable, a , which she could solve for because she has written three equations. However, she does not have sufficient procedural competence to follow her own reasoning and substitute in $l \times b$ for her a^2 term, which would give her two equations in two unknowns. This same error results in a procedural deadlock. She tries using the elimination method appropriate to solving systems of linear equations to solve two quadratics. While it is possible to solve non-linear systems of equations through elimination, the equations have to be set up correctly. In attempting this complicated method, she introduces a number of other errors: mixing up the two original equations; subtracting from inside brackets that are factors of products; and not distributing a negative. Thus, even with a hopeful beginning revealing strategic competence, her lack of procedural fluency and a critical error in conceptual understanding caused the student to not proceed past setting up equations.

4.3.1.1 Question 2, Instrument 1: Mathematical Proficiency – Trends

This was one of the best answered questions in terms of quantity – all but four learners attempted this question. However, no learner answered it correctly and only one learner's answer was near the memorandum answer.

My first attempt at analysis of this question was to look for patterns in the learners' responses and code them, just like I did in Question 1. Because conceptual understanding, procedural fluency and strategic competence were needed for this question, I coded for all three strands. The summary of

Instrument 1, Question 2				
Frequency of mathematical proficiency by category and rating				
Category	Conceptual Understanding	Procedural Fluency	Strategic Competence	Total
Rating				
+	13	15	24	52
+/-	18	10	27	55
-	17	10	15	42
Total	48	35	66	

Table 4.7 Frequency of Mathematical Proficiency Rating
Codes: I1. Q2 _

the frequencies of the codes appears in Appendix M - 1 and the totals of the ratings appears above. The coding did reveal some significant trends. For example, 14 out of the 26 respondents were able to assign separate variables for the two different dimensions, a “+” rating under the strategic competence strand and an important first step to solving the problem. However, the next step – setting up two correct equations – only occurred twice for both equations. Seven students were able to set up one side of the $(x + 1)(y + 1)$ equation, but not the other side. This was coded under strategic competence as a “+/-” rating. At the same time, eight students wrote their own variation of $(\text{original length} + 1)(\text{original breadth} + 1) = 20$: they used the 20 m^2 given in the question as the new area rather than the original area *increased by* 20 m^2 . This was coded as a conceptual error with a “+/-” rating. A similar number of students made the same mistake with the $(x + 2)(y + 2)$ equation. It was significant that nearly a third of the students could almost, but not quite set up the equations correctly.

However, in many ways the coding presented a “rosier” picture of the learner responses than what I experienced when coding them. For example, seven learners received a “+” rating in the procedural fluency category for the code “0 procedural errors (ca)”. I coded this way because in what the learners wrote, they made no procedural errors but that may have been because they did not proceed very far with the question. Because the codes were assigned to individual elements in each learner’s solution instead of viewing each answer as a whole, I felt that the codes did not accurately represent the learners’ responses to the question. I realised that if this were a school assessment and I were marking these, most learners would have received very few marks and many of these would have been from continued accuracy.

Before I begin a discussion of the learner solutions, I need to explain the method of marking I use, including the idea of “continued accuracy.” Continued accuracy is a principle that is recommended in curriculum documents of South Africa (DBE, 2014) to protect learners who make one error that affects the rest of the solution. It may be interpreted differently at different schools, but at my school our “rule of thumb” was to underline the first error, deduct that mark and then follow that error through the problem. If the learner completed the rest of the problem correctly using that faulty value, the learner only lost one mark. However, if another error occurred, the marking stopped. The principle of continued accuracy was a compromise between allowing for some human error (mostly computational or algebraic), an “anything goes” attitude and a rigid “it’s either 100% right or 100% wrong” perspective. I used this principle when marking the learners’ responses. My only deviation was to allow the setting-up of the two equations as one error. In other words, if the learner set up both equations incorrectly (two errors, therefore stop) I still continued marking the solution until the next error and then stopped.

While marking the question, I became aware of an aspect that the coding did not reveal; it was what I personally referred to as “points of derailment,” that is, critical points along the solution path where students could go wrong. I created a simplified flow-diagram (see Figure 4.2) that showed the relationship between 11 steps needed to complete this question correctly and where “derailments” could occur at the steps marked “NO”. Steps in the solution process are indicated by rectangles. The first steps (designated by “square” rectangles) are a combination of conceptual and strategic steps. The “rounded” rectangles indicate procedural steps. The ellipses represent a correct decision that leads to the next step.

I then went back and “marked” the question as if it were a school assessment, using the principle of continued accuracy, and I also noted at which step(s) the learner “derailed” and which steps they achieved. The marking scheme is reproduced below (see Figure 4.3). (Please note the line numbers in the marking scheme do not correspond to the steps shown in the flow chart above. In the succeeding discussion, “steps” refer to the steps in the flow chart.) It is important to note that the first two steps in the flow diagram, which are extremely important to the actual solution process, are not the first mark. Some learners derailed early in the solution process while other learners jumped several steps and solved certain aspects of the problem successfully if the principle of continued accuracy is taken into account. Several learner responses are discussed below to illustrate this idea of “derailment”.

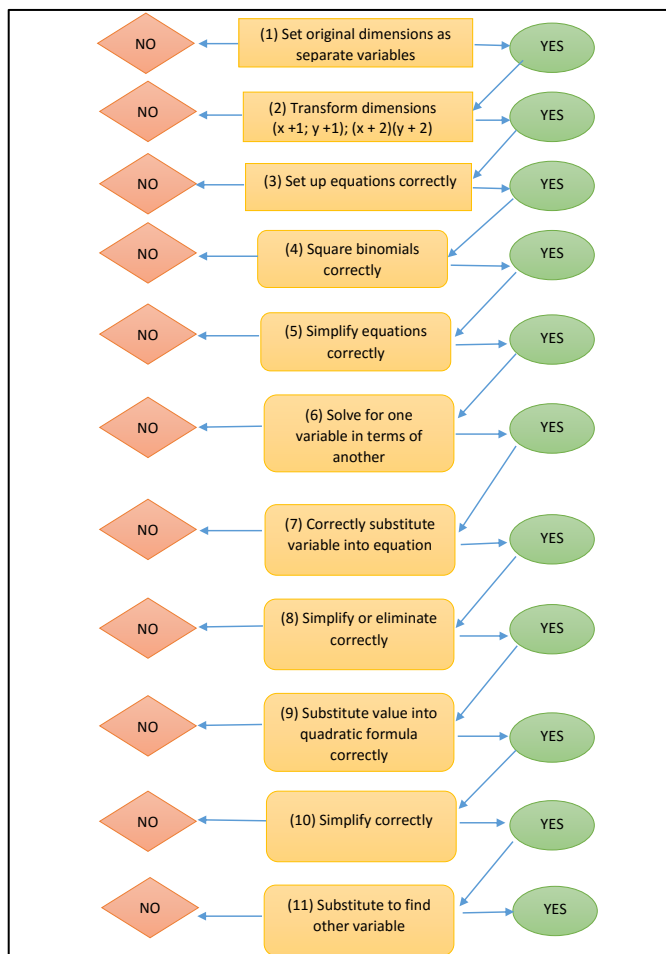


Figure 4.2 I1.Q2 Flow diagram of steps

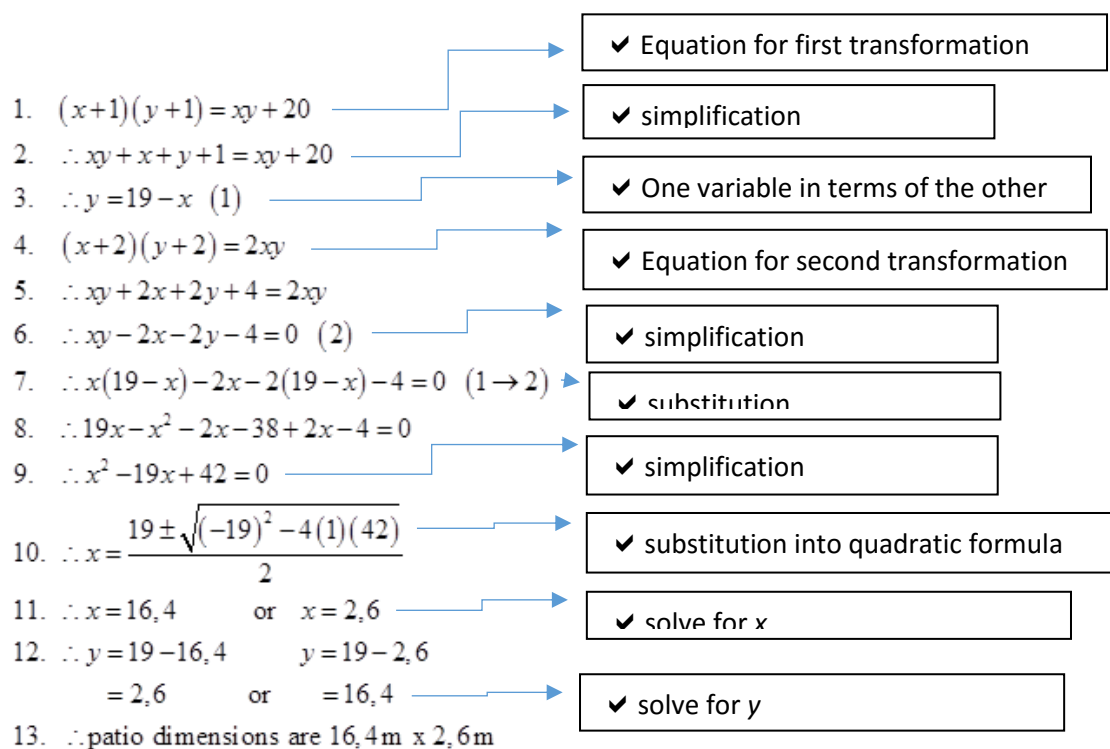


Figure 4.3 Memorandum for I1.Q2

As mentioned earlier, the learner responses clearly illustrated the co-dependence of the strands. For example, Learner 3.4's response is a good illustration of someone who exhibited procedural fluency, but made mistakes in her conceptual understanding and strategic competence (see solution alongside). She began by only illustrating one rectangle in which the sides have been increased by one. However, she makes the critical conceptual and strategic errors of assigning the same variable to the length and breadth. Then she equates that product of the two sides to " $x + 20$ ": she has made a further conceptual error by equating *area* on the left hand side (LHS) of the equation to a *single dimension* on the right hand side (RHS) – although she has increased that quantity by 20 as the problem requires. She then correctly solves the resulting quadratic equation using the quadratic formula and discards the negative quadratic answer. She has demonstrated procedural fluency in what she has attempted (although her response could be seen as somewhat inefficient) but has managed to avoid the more procedurally difficult elements of the problem by not introducing simultaneous equations. This is a learner who derailed at steps 1 – 3, but then partially achieved steps 4, 5, 9 and 10. In terms of my marking scheme, she achieved 3 out of 10 marks.

$(x+1) \times (x+1) = x+20$
 $x^2 + x + x + 1 = x+20$
 $x^2 + 2x + 1 = x+20$
 $x^2 + x + 1 = 20$
 $x^2 + x + 1 - 20 = 0$
 $x^2 + x - 19 = 0$
 $x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-19)}}{2}$
 $x = \frac{-1 \pm \sqrt{77}}{2}$
 $x = \frac{-1 + \sqrt{77}}{2}$

Learner 3.10 is another example of a learner who is procedurally fluent up to a point, but an early conceptual error followed by a later lack of procedural know-how prevents the completion of the solution process (see illustration alongside). This learner makes the strategically competent step of assigning two variables to the original two different dimensions, but then makes two conceptual

Diagram of a rectangle with dimensions $x+1$ and $y+1$. The perimeter is given as 40m.

$$A = 20 \text{ m}^2$$

$$A = l \times b$$

$$A = (x+1)(y+1)$$

$$20 = (x+1)(y+1)$$

$$20 = xy + x + y + 1$$

$$19 = (34-y)(y) + (34-y) + y$$

$$19 = 34y - y^2 + 34 - y + y$$

$$19 = 34y - y^2$$

$$-15 = y(34-y)$$

$$40 = (x+1) + 2 + (y+1) + 2$$

$$40 = x + 3 + y + 3$$

$$34 = x + y$$

$$34 - y = x$$

and strategic mistakes: she equates the increase in dimensions by one for each dimension to an overall area of 20, then adds 2 to the already increased dimensions and equates that sum to 40. She may be confusing area and perimeter with this step. However, she correctly, in terms of continued accuracy, solves for one variable in terms of another, and manages to simplify, but not solve, the quadratic equation. Compared to Learner 3.4, her strategic competence may be better because she may have realised that she had two unknowns, needed two equations to solve for two unknowns and needed to solve for one variable in terms of another. However, her procedural fluency was such that she did not rewrite the quadratic equation in standard form and solve. This learner derailed at step 3 and 9, but with continued accuracy completed steps 1 – 2 and 4 – 8 correctly. She received a score of 4 out of 10.

Learner 3.9's response was the closest to the memorandum answer in the whole group of students (see illustration alongside). However, using the continued accuracy principle, she only received 5/10 because she made two small procedural errors at step 8 and thus, according to my marking guidelines, the marking stopped. She showed full understanding, though, from the beginning of the problem to the end. This learner correctly set up the two equations in the two unknowns taking cognisance of the conditions of the increase in area and the related increase in dimensions. She then correctly solved both equations in terms of x . However, when she attempted to solve simultaneously, she copied her one equation incorrectly

Diagram of a rectangle with dimensions x and y . The perimeter is given as 40m.

$$(x+1)(y+1) = xy + 20$$

$$(x+2)(y+2) = 2xy$$

$$xy + x + y + 1 = xy + 20$$

$$x + y + 1 = 20$$

$$x + y = 19$$

$$x = 19 - y$$

$$x = 19 - (19 - \sqrt{197})$$

$$= 52.04 \text{ m}$$

$$x = 19 - y$$

$$x = xy - y - 2$$

$$0 = xy + 21$$

$$21 = xy$$

$$x = 41$$

$$y = 41$$

$$(x+1)(y+1) = 21 + 20$$

$$= 41$$

$$(x+2)(y+2) = 2(41)$$

$$= 82$$

$$y = -19 \pm \sqrt{197}$$

$$y = 52.04$$

(1st mistake), then eliminated her simultaneous equations incorrectly (2nd mistake). She made a further error when she moved the variable to the other side. However, from there her work is accurate. In this learner's case, her conceptual understanding, strategic competence and procedural fluency are strong. The only procedural errors she made were copying and assigning the incorrect sign, but if this were a school test, she would have only received half of the required marks. This is one case where the continued accuracy principle does not adequately reflect the learner's competence.

Learner 3.15, on the other hand, received more marks than Learner 3.9 even though her strategic competence was less than Learner 3.9's (see illustration alongside). This highlights the fact that memoranda are often faulty instruments and can overestimate or underestimate a learner's competence. In this case, the learner correctly sets up one equation but not the other. She also makes the strategic error of introducing a third variable into her equations. She then solves these equations correctly, and the two simultaneous equations, but cannot proceed because she does not have a third equation for her third unknown. She received 6/10 marks and derailed at step 3 but continued steps 4 – 8 correctly.

$(d+1m)(b+1m) = A+20m^2$
 $db+d+b+1 = A+20$
 $db+d+b = A+19$
 $(d+2)(b+2) = A+40$
 $db+2d+2b+4 = A+40$
 $db+2d+2b = A+36$
 $db+d+b = A+19$
 $-(db+2d+2b = A+36)$
 $(-d-b) = -17$
 $d+b = 17$

Table 4.8 below is a summary after marking the question, which shows a complementary picture to the coding. Firstly, if this were part of a classroom test marked with the principle of continued accuracy, 11 students would receive 0 marks (42% of the respondents). Only three students received 50% or more on this question (11.5% of respondents). While 11 learners could transform the original dimensions, 12 could not set up the equations correctly – step 3 had the highest frequency of being the first point of derailment. Only one student was able to correctly set up both equations without an error. Steps 1 – 3 were the steps requiring the most conceptual understanding and strategic competence. However, several students ($n = 8$) were able to use their procedural skills and continue with the problem as far as they could. This may also indicate a level of strategic competence because strategic competence includes solving problems, perhaps even if based upon faulty foundations.

Marks	Frequency	Steps achieved	Frequency	Point of 1 st derailment	Frequency
No attempt	4	0	6	1	8
0	11	1	9	2	1
1	1	2	11	3	12
2	2	3	1	4	1
3	3	4	8	5	0
4	1	5	7	6	0
5	1	6	4	7	1
6	2	7	3	8 - 11	0
7 - 10	0	8	3		
		9	3		
		10	3		
		11	2		

Table 4.8 Frequency of Steps Achieved and “Points of Derailment”

4.3.1.2 Question 2, Instrument 2: Mathematical Proficiency – Trends

All three strands of mathematical proficiency were required to solve this problem. Firstly, *conceptual understanding* was required for the understanding of surface area, but more specifically how the linear dimensions in the question were related to the total surface area. The key to this question was seeing the relationship between the given *sum* of the dimensions on each prism and using that information to find the two missing dimensions. *Strategic competence* was needed to set up meaningful simultaneous equations in order to find the dimensions required to calculate the surface area. *Procedural fluency* was also required for correctly and efficiently working through all the algebraic steps. A memorandum is reproduced below.

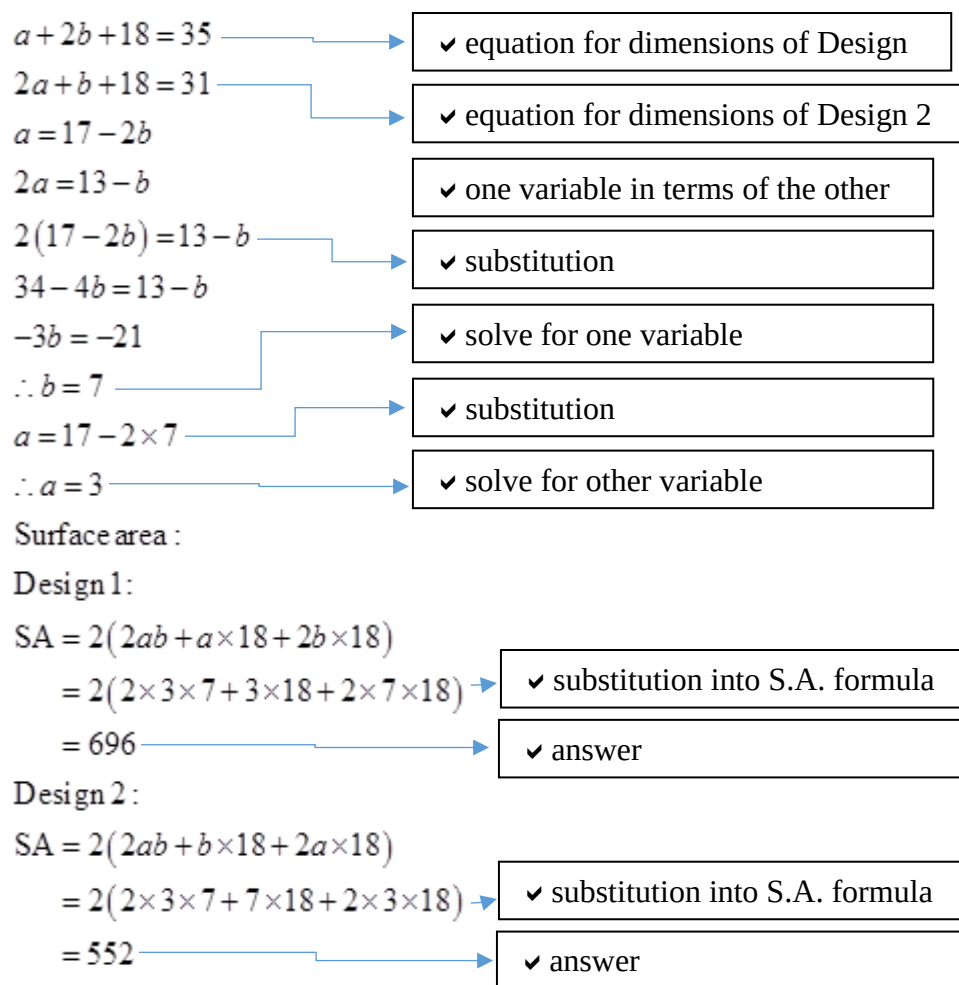


Figure 4.4 Memorandum for I2.Q2

This question made me reflect upon the strands of proficiency needed to utilise a formula effectively because so many learners were not able to do so. As I was coding, I defined writing down a formula for surface area as a procedural fluency step with a “+” rating. I did this because I see writing a formula as an important first *procedural* step in structuring one’s calculation. However, *choosing* a correct formula (from a list, for example) and *using* a formula correctly

I see as requiring *conceptual understanding*; the person has to understand what quantities are expressed in the formula and the operations performed on those quantities. However, if the quantities are not easily available in a given question, *strategic competence* is required to find a relationship amongst the given pieces of information that can be exploited resulting in a quantity or relationship that can then be utilised in the formula. This process guided my thinking while establishing codes.

Unlike I1.Q2, the learners did not have to remember the formula for surface area; it was given on the notes page and indicated as such. There were also two diagrams on the notes page showing a rectangular prism in 3-D and a 2-D net of the prism to visually show the relationship between the variables in the formula. Because that information was available so readily, I did not code writing down the formula as requiring conceptual understanding as I did in I1.Q2. However, learners needed to realise that for Designs 1 and 2, the heights and widths were different and had to be calculated before substituting into the surface area formula. Recognising the *need for the simultaneous equations* was coded as conceptual understanding; *setting up (correct) equations* was coded as strategic competence.

Many learners ($n = 16$) *started* with the formula for surface area from the notes without first trying to find the values of a and b . This may have been due to several reasons, and brings up the question of how much a formula sheet inadvertently “nudges” learners in a certain direction. One reason why the learners may have worked this way was because of the repetition of the letter “ b ” on both the instrument and the notes page (see Appendix P – 1); but the variable has a slightly different meaning on the two documents. In the question in the instrument, the “ b ” represents one dimension, which could be interpreted as breadth; however, as the other dimension is indicated by “ a ”, the “ b ” may be seen as simply a generic dimension. In the notes page, the “ b ” more clearly represents “breadth” because the other dimensions are given the variables “ l ” for length and “ h ” for height. If the students did not understand that subtle distinction, they may not have replaced the “ b ” in the formula with an expression for “ b ” from the instrument. Also, the formula given in the notes may have caused confusion because of the “2’s” in the formula and the fact that each of the stacked prisms contained one variable twice. Very few learners were able to double the correct quantity.

Learner 3.1’s response illustrates an incorrect understanding of the problem relative to the given formula (see solution alongside). She

$SA = 2(l \times b) + 2(b \times h) + 2(l \times h)$	$35 = 18 + a + b$
$= 2lb + 2bh + 2lh$	$17 = a + b$
$= 36(17-a) + 2(17-a)(7-b) + \frac{36}{2}(7-b)$	$17-b = a$
$= 612 - 36a + 2(289 - 17a - 17ab + 17ab) + 0(2-3b)$	$17-a = b$

starts by writing down the first two lines of the formula as given in the notes page. She also realises that she has been given information about the quantities a , b and 18 and correctly sets up an equation. However, she does not use the other information given to set up a second equation so creates a circular description of a and b : $17 - a = b$ and $17 - b = a$. She then substitutes this expression for b into the surface area formula, but appears to confuse the “ b ” of the formula (which stands for “breadth”) with the b in the question. Comparing line 3 of her

working, it is very difficult to determine which face she is describing and on which diagram. Ultimately, she ends with a complicated expression in a and b and cannot proceed further.

Learner 3.6 seems to have the opposite problem: she realises there is a relationship between the length, breadth and height of the prism that is related to their sum but a flawed approach in finding the relationship prevents her from attempting to use the surface area formula (see solution alongside). In this instance, the learner uses the information given about the sum of the dimensions and establishes algebraic relationships between the length, breadth and height of the prisms and she creates two equations in two variables, equating the height (h) in terms of the width (w). Conceptually, the learner was on the right track, but her strategic error has several components. Firstly, she introduced new variables into the problem: h , w and l instead of the a , b and 18 which represent the height, width and length respectively of the unstacked prisms. Possibly because of this, she does not see that in Design 1, “one height” + “two widths” + $18 = 35$, and in Design 2, “two heights” + “one width” + $18 = 31$. The learner simply calculates that height + width + length equals either 35 or 31, and ends with the illogical answer that $h = 17 - w$ and $h = 13 - w$. Because $h = h$, she equates both right-hand-sides of the initial equations. However, because in reality the heights are not the same, the simultaneous equation is unsolvable and she cannot proceed further. Introducing new variables might have confused the learner, as well as simply not seeing that dimensions were different in the two prisms.

Design 1 $h = \text{height}$ $w = \text{width}$

$$\begin{aligned} \text{height} + \text{width} + \text{length} &= 35 \\ \text{height} + \text{width} + 18 &= 35 \\ h + w &= 17 \\ \therefore h &= 17 - w \quad (1) \end{aligned}$$

Design 2 $h = \text{height}$ $w = \text{width}$

$$\begin{aligned} \text{height} + \text{width} + \text{length} &= 31 \\ h + w + 18 &= 31 \\ h + w &= 13 \\ \therefore h &= 13 - w \quad (2) \end{aligned}$$

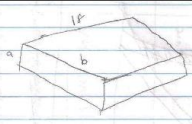
$$\begin{aligned} (17 - w) + w &= 13 & 13 - w &= 17 - w \\ 17 - w + w &= 13 & 13 - 17 &= -w + w \end{aligned}$$

Learner 3.9 correctly expressed the surface area for the two figures, but in terms of a and b and not using the numerical information given (see solution alongside). The verb “calculate” in the question should have given the clue that the solution was a numerical one. However, this is one out of 4 learners whose conceptual understanding of surface area was accurate. She is also one out of 11 who calculated the surface area (correctly or incorrectly) in terms of a and b without finding values or expressions for a and b .

$$\begin{aligned} SA_1 &= 2(l \times b) + 2(l \times a) + 2(a \times b) \\ &= 2(18 \times b) + 2(18 \times a) + 2(2b \times 18) \\ &= 2(2ab) + 2(18a) + 2(36b) \\ &= (4ab + 36a + 72b)u^2 \end{aligned}$$

$$\begin{aligned} SA_2 &= 2(2a \times b) + 2(2a \times 18) + 2(18 \times b) \\ &= 2(2ab) + 2(36a) + 2(18b) \\ &= (4ab + 72a + 36b)u^2 \end{aligned}$$

2)



$$\begin{aligned} 35 &= 2(18 \times a) + 2(18 \times b) + 2(a \times b) \\ 35 &= 36a + 36b + 2ab \\ 35a &= 36a + 36b + 2ab \\ a &= 1 + 36b + 2b \end{aligned}$$

$$\begin{aligned} 35 &= 2(18 \times (1 + 36b + 2b)) + 2(18 \times b) + 2((1 + 36b + 2b) \times b) \\ 35 &= 2(18 + 648b + 36b) + 36b + 2b + 72b^2 + 4b^2 + 2b \\ 35 &= 36 + 1296b + 36b + 36b + 2b + 72b^2 + 4b^2 \\ 35 &= 1 + 1334b + 76b^2 \\ -1 &= 1334b + 76b^2 \\ 0 &= 667b + 38 + \frac{1}{2} \end{aligned}$$

Learner 1.2 demonstrated a number of serious procedural errors after she began her solution with a conceptual error (see solution alongside). To

begin, she equated the surface area of the unstacked prism to the sum of the dimensions for Design 1. She simplified that correctly, then factorised a out of the bracket, even though a was not a common factor. She swaps a across the equal sign (but without changing its sign) and subtracts 35 from the RHS, resulting in an (incorrect) expression for a . Then, she substitutes this value of a back into her *original* equation from which she has just derived a . This she simplifies with errors and ends up with a quadratic equation that she cannot solve. Aside from the procedural errors, probably the most important conceptual mistake is the circular reasoning involved in determining one variable in terms of another and then plugging that value back into the same equation. Because of procedural errors along the way, the learner did not conclude something along the lines of $0 = 31$, which may have alerted her to a problem with the reasoning.

The critical steps in this problem were 1) setting up equations in a and b in order to find the areas of the individual surfaces; and 2) correctly identifying the separate areas of the stacked prisms in order to find the total surface area. In my first round of coding, I went through the learner responses to code what they did. However, I then went back to code again the codes concerning these critical steps. (Similar to having to “mark” I1.Q2 because the first round of coding did not accurately describe the learners’ responses.) Codes concerning these steps were conceptual understanding code CU 1: “understand the need to find values of a and b in order to calculate surface area,” code CU 2: “correctly expressed the separate areas of the faces of both figures,” and strategic competence code SC code 1: “created two equations to find the value of one variable in terms of another.” I restricted myself to responses where the learner attempted to find the surface area. If the learner received code CU 7: “very little progress,” I did not apply CU 1, CU 2 or SC 1. Between eight and ten learners either did not attempt an expression or equation in terms of a and b .

Only one learner in the two classes was able to complete the question correctly. Her solution is discussed in detail under Strategy Use because metacognition is very evident.

4.3.2 Question 2: Strategy Use

4.3.2.1 Question 2, Instrument 1: Strategy Use – Trends

For this question, the learners were much more prolific in their strategy use than in Question 1 on Instrument 1. Because the content of the question dealt with area, many learners ($n = 16$) employed the implied, heuristic strategy of drawing a figure. In addition, eight learners explicitly stated some combination of declarative, procedural and conditional knowledge. However, the effectiveness of the strategies varied as much as the mathematical proficiency. A summary table is provided below (see Table 4.10 on page 83). If the strategy statements are taken individually, 3 out of the 38 incidents of strategy use were effective in solving the problem (two of which were implied), while 15 were ineffective. In the middle of the continuum, 20 instances of strategy use were partially effective. However, the effectiveness of the strategy use was related to the mathematical working. A closer look at the specific incidents

of the strategy use reveals other information. The learners' explicit strategies are reproduced in Appendix O - 1, with a representative sample of effective, partially effective and ineffective implied strategies given below.

Type of Strategy	Rating	What strategy/ies did you use?	How did you use the strategy?	Why did you use the strategy or how do you know when to use the strategy?
Heuristic	Effective	None		
	Partially effective	I draw sketched a rectangle and tried to fill in the missing dimensions	I filled in what I knew on the tri rectangle.	This was the only strategy I could think of.
	Ineffective	I drew up the patio  = 20m ²	$20\text{m}^2 - 4\text{m} = 16\text{m}^2$ $l = 12\text{m}$ $b = 4\text{m}$	I used the information given and worked backwards
Content-Specific	Effective	Length x Breadth = Area	Substitute x as one formulae and y as the other formulae. Both x and y are the unknown.	Leant the method in grid 11
	Partially effective	Simultaneous	2 Simultaneous equations realised that might need 3	Given to expressions normally results in Simultaneous.
	Ineffective	I made the length and breadth equal to x.	I put the x values in an equation.	Because you are trying to solve for an unknown value

Table 4.9 Representative Explicit Strategies: Question 2, Instrument 1

Instrument 1: Question 2									
Frequency of Strategies by category, code, type and rating									
Category	Explicit						Implied		
Code	Declarative		Procedural		Conditional				Total
Rating/type	H	CS	H	CS	H	CS	H	CS	
E	0	1	0	0	0	0	2	0	3
PE	1	3	1	3	1	3	8	0	20
IE	1	2	0	3	1	2	6	0	15
Total	2	6	1	6	2	5	16	0	38

Table 4.10 Frequency of Strategy Use by Categories: I1.Q2

Several trends were noticeable about the strategy use for this question that require some detailed explanation. First, one common response was the recognition by the learners that simultaneous equations were required. This content-specific strategy is explicitly stated by four learners as declarative knowledge. (See Appendix O – 1 for all strategies related to the succeeding discussion.)

Learner 3.1 states “2 pieces of information was given [sic] and I decided to use simultaneous equations.” She did not go on to elaborate how to use the simultaneous equations, but the “why” is implied in her declarative statement: “2 pieces of information was given.” However, if you look at the question itself, much more than “2 pieces of information” were given: the patio is rectangular; if each dimension is increased by 1 unit the original area increases by 20 square units; and if each dimension is increased by 2 units, the original area is doubled. One wonders if the student meant two

Handwritten student work showing a diagram of a rectangle with dimensions l and b . Below the diagram, the student has written several equations:

$$a^2 = l \times b$$

$$a^2 + 20 = (l+1) \times (b+1) = a^2 + 20 = (l+1) \times (b+1)$$

$$2a^2 = (l+2) \times (b+2) = 2a^2 = (l+2) \times (b+2)$$

$$\therefore 2a^2 = (l+1) \times (b+1)$$

$$- (a^2 + 20 = (l+2) \times (b+2))$$

$$a^2 + 20 = (l-1) \times (b-1)$$

$$a^2 + 20 = (l+2) \times (b+2)$$

$$- 2a^2 = (l+1) \times (b+1)$$

$$-a^2 + 20 = 1 \times 1 \quad \therefore -a^2 + 20 = 1 \quad \therefore 20 - 1 = a^2 \quad 19 = a^2$$

unknowns. In actual fact there were three unknowns: each original dimension and the original area. A reproduction of her solution shows that she did in fact recognise the three unknowns, and her misrepresentation of area as a^2 was precisely what caused her to derail at step 3. (This learner’s solution is discussed in detail on p. 72.) Her strategy was rated as partially effective.

Learner 1.2 also stated “Simoultaneous [sic] equations” in answer to “what strategy/ies did you use, and was more accurate than Learner 3.1 in answering the question “why?”: she states “There are two unknowns.” However, her description of “how” the procedure was implemented, although fairly articulate, “used x and y and equaled [sic] them both +1 to 20,” shows her inability to implement the strategy effectively. This may have been due to a lack of conceptual understanding, strategic competence, procedural fluency or a combination of the three strands. In her solution alongside, she set up one equation in two unknowns, and transformed the dimensions correctly, but did not make that expression equal the area increased by 20. Then, she tried to use that equation to substitute back into itself, creating a sort of circular reasoning. She realised she had two unknowns and needed to use simultaneous equations but did not make the critical step of setting up two equations. Her strategy was rated as partially effective.

$20 = (x+1) \times (y+1)$
 $20 = xy + x + y + 1$ (2)
 $20 = xy + x + y + 1$ (1)
 $-xy - y - 1 - 20 = x$
 $-xy - y - 21 = x$
 $20 =$
 $20 = xy + x + y + 1$
 $20 = (-xy - y - 21) + y + 1$
 $= -xy$

Learner 3.14 simply wrote “Simultaneous” in the “what” column of strategy use and, in answer to “why” wrote: “given to [sic] expressions normally results in simultaneous.” Also, she stated in the “how” column “2 simultaneous equations realised might need 3.” Although this is not a clear statement of how to implement her strategy, it does show the learner’s insight that there are actually three unknowns. As her solution reveals (see figure alongside) she introduced a third variable, a , but was unable to solve her equations. Like the previous learners, her strategy was partially effective.

$(b+1)(c+1) = 20m^2$
 $bc + b + c + 1 = 20$
 $bc + b + c = 19$
 $(b+1) = 20/(c+1)$
 $b = 19 - c - bc$
 $b + c = a$
 $a - 2 - bc = 19 - c - (19 - c - 2 - bc)$
 $19 - c - bc = 19 - c - 2 - bc$
 $17 - bc = a = a - c - bc$
 $17 = a + bc = a + c + bc$
 $17 = 2bc + c$

Learner 3.8 reveals an ineffective strategy (illustrated alongside). She states in the “what” column, “– initially strategy was to work with the patio of 20 m² and then doubling to find the next patio’s dimensions – equating both patio’s dimensions.” Her first conceptual error was to take the original area as 20 m² and the second area to be 40 m² (i.e. 20 m² doubled). However, when she sets up the simultaneous equations, a critical further error occurs: She doubles her original equation and then equates the original equation to the doubled equation, in effect stating that 1 = 2. What follows as a solution is an exceptionally long algebraic attempt to solve unsolvable equations, complicated further by her addition of the m² units into the equation. The solution continues for a page and a half and ends as a cubic equation.

Handwritten mathematical work by Learner 3.8. The work shows a flawed strategy for solving a problem. The learner starts with two rectangles, one labeled 20m² and the other 40m², with dimensions a+1m and b+2m. She sets up equations: 20m² = (a+1m)(b+2m) and 40m² = 2(20m²) = 2(a+1m)(b+2m). She then equates these two equations, leading to a series of algebraic manipulations that result in a cubic equation: 20bm² - 2b³ - 2bm² + 20m² - 2b - 2m² = 2(20m² - 2b³ - 2bm² + 20m² - 2b - 2m²) + 20m² - 2b - 2m².

Although the learner can clearly state how she is implementing the strategy: “by equating the dimensions of both patios,” her explanation of “why” reveals her confusion. She writes, “– I was given values for two different patios – I assumed that the unknown would apply to one of the patios - ∴ I immediately equated the two,” and ends up equating the wrong quantities.

However, if one of the hallmarks of metacognitive knowledge is that it leads to effective use of knowledge, then the metacognitive nature of the learners’ knowledge needs to be examined. When these explicitly-stated strategies are compared with the learners’ solutions, then a discordancy is apparent: five out of the eight students listed above received a score of 0/10 and three received 3/10 – with the aid of continued accuracy. As described in the section on mathematical proficiency, only one learner was able to correctly set up the equations from the given information, and she is not included above as someone who described strategy use. According to their explicit strategies, a number of the learners know *what* content-specific strategy to use, but none actually know *how* – even if they think they do according to their explicit statements of procedural knowledge. However, two of the learners above (3.12 and 1.1) do know *why* they need simultaneous equations: they have two unknowns.

In addition to the content-specific strategies discussed above, two learners also mentioned the heuristic strategy of drawing a figure to aid in representing the problem. Figure-drawing was a common, implied heuristic for this problem that met with varying degrees of success. The continuum spanned from accurate figures that effectively represented the problem to some that represented the problem with a few errors to others that were not helpful at all. A sample of figures is presented below to illustrate common trends.

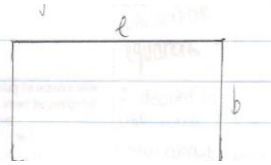
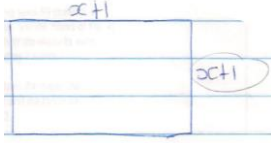
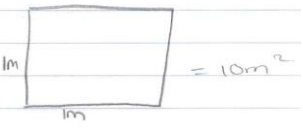
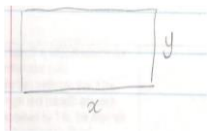
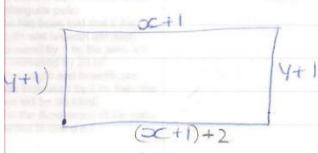
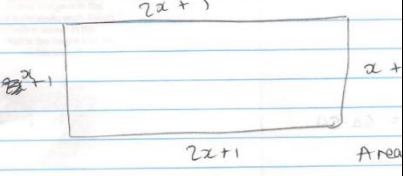
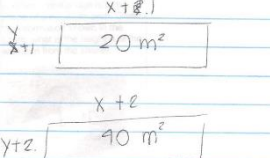
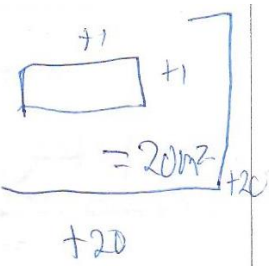
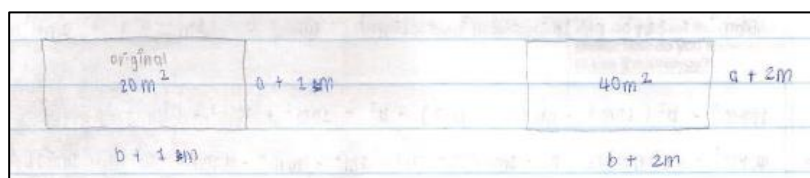
Effective	Partially effective	Ineffective
		
		
		

Table 4.11 Implied Strategy Use: I1.Q2

For a learner's drawing to be rated "effective," the learner needed to demonstrate through their drawing *and* their subsequent working through the problem that they understood three concepts: 1) that length and breadth needed separate variables, 2) that each dimension was increased by one metre for the first rectangle and 2 metres for the second rectangle, and 3) that the increase in dimension was related to an increase in area. As described in detail earlier in this paper, very few learners were able to make all three connections. The drawings also revealed misunderstandings or misconceptions. Only two learners' drawings were rated as effective (see above). This was because they assigned different variables to the length and the width, and in their solution process indicated an understanding of the increase in side length to the increase in area.

Eight of the learners received a "partially effective" rating. As can be seen from the examples above, the learners tended to make one of three mistakes: they assigned the same variable to both dimensions, they added on to the original dimension incorrectly, or they equated the new dimensions to the incorrect area. Six of the learners' drawings were rated ineffective because they gave little or no insight into the problem.

However, as stated above, metacognitive knowledge should indicate effective use of knowledge. Learner 3.8's drawings are reproduced below. On a first evaluation, her drawings should receive a partially effective rating because she has made one of the mistakes listed above: while she has identified two different variables for the dimensions and has added the correct amounts, she has equated these areas to 20 m^2 and 40 m^2 , respectively. However, her working, which was described in detail above, begins with a serious conceptual error that results in a two-page solution with a mark of 0/10. Although her drawing, taken on its own, was partially effective in its representation of the problem, it did not lead to an effective solution of the problem. Her solution was another example of how easily the gossamer threads of connection can be broken.



4.3.2.2 Question 2, Instrument 2: Strategy Use – Trends

From a strategy perspective, Question 2 in Instrument 2 could have utilised similar strategies as Question 1 to support the three strands. To conceptualise the problem, the learners could have used, “*Did I use all the data?*” and its further subdivisions of “*What is the data, what is the unknown and what is the condition?*” This question would have been helpful because in order to find the surface area of the stacked prisms, the learners needed to find the dimensions of the individual areas. The data given were sums of the dimensions of the stacked prisms, including a length of 18 cm that was constant for both figures. The learners needed to identify the unknowns as the dimensions and then the surface area. The condition was the difference in stacking the prisms.

Because simultaneous equations were needed, “*auxiliary elements – auxiliary problem*” could have been employed in conceptualising the problem. The learners needed to use the given information about the sums of the dimensions to solve for a and b simultaneously. Only then could the learners determine the total surface area. Another strategy they could have employed to help with understanding the problem is “*Do I know a related problem?*” Like the mean, students have worked with surface area since at least Grade 8. They may have encountered similar problems before. Also, the strategy of “*working backwards*” may have prompted the learners to work backwards from the given sum of dimensions to a relationship between the dimensions.

As with conceptualising the problem, strategies such as “*Did I use all the data?*”, “*working backwards*” and “*auxiliary elements – auxiliary problem*” could be used to support strategic competence. “*Did I use all the data?*” and “*auxiliary elements – auxiliary problem*” could help them look at the given information for the flat prisms, the stacked prisms and the sum of dimensions to conclude that two new equations could be developed incorporating two unknowns and that were different from the surface area formula. “*Working backwards*” might prompt them to think that the surface area formula requires values for the letters l , b and h and make them search for a way to find those values.

After conceptualising the problem, the learners could have started from two different points: find the total surface area in terms of multiples of 18, a and b and then find values for a and b , or vice versa. Once the initial steps were in place, the learners could have used *content-specific strategies* (distribution, combining like terms, etc.) to simplify their choice of equations.

A summary of the frequency and type of strategies used in Question 2, Instrument 1 and 2 is given below for comparison purposes. As with the strategies employed in Question 1, a detailed discussion is provided in Appendix O – 2; a summary of the qualitative trends is given here. As before, I focused on the heuristic and content-specific strategies.

Instrument 1: Question 2									
Frequency of Strategies by category, code, type and rating									
Category	Explicit						Implied		
Code	Declarative		Procedural		Conditional				Total
Rating/type	H	CS	H	CS	H	CS	H	CS	
E	0	1	0	0	0	0	2	0	3
PE	1	3	1	3	1	3	8	0	20
IE	1	2	0	3	1	2	6	0	15
Total	2	6	1	6	2	5	16	0	38

Instrument 2 – Question 2									
Frequency of Strategies by category, code, type and rating									
Category	Explicit						Implied		
Code	Declarative		Procedural		Conditional				Total
Rating/type	H	CS	H	CS	H	CS	H	CS	
E	0	1	0	1	0	0	0	0	2
PE	2	4	0	2	1	0	3	0	10
IE	3	2	0	3	0	2	3	0	15
Total	5	7	0	6	1	2	6	0	27

Table 4.12 Strategy Use Question 2: Instruments 1 and 2 Compared

For this question, 13 learners, compared to 8 in Instrument 1, self-reported some sort of strategy. Of those, three indicated the declarative, procedural and conditional aspects of their strategy and four learners described two of the three aspects. Unlike Question 1 of Instrument 2, only one learner's strategy did not match her working out, but this was because she did not proceed to answer the question. Many of the learners articulated their strategies clearly, even if in only a few words. A representative sample is given below.

Type of Strategy	Rating	What strategy/ies did you use?	How did you use the strategy?	Why did you use the strategy or how do you know when to use the strategy?
Heuristic	Effective	None		
	Partially effective	Best way to solve this problem? What do I know? Have I even done this before?		You know when to use the strategy when you are stuck. Ask yourself what is the best way you can do it or have you even done it before.
	Ineffective	- Redraw rectangular boxes. - colour code the rectangles. - Solve first design. - Solve second design.	Use the surface area formula.	It's better to see things in my hand-writing.
Content-Specific	Effective	→ simultaneous equations to solve for a and b → substitution methods → changing the subject of an equation → used the diagram as assistance		
	Partially effective	Area equation $A = l \times b$	Substitute values into area equation ↳ x 2 for each surface as there are 2 surfaces that are equal in area	
	Ineffective	The Formula for the surface area	I placed each number I saw into the formula.	When you hear surface area you know that you need to use the formula for the surface area.

Table 4.13 Representative Explicit Strategies: Question 2, Instrument 2

Three learner responses deserve a more detailed analysis and their strategy use needs to be examined in light of their mathematical working. I will start with two learners who achieved a partial solution and then the only learner who achieved a completely accurate solution.

A look at Learner 3.11's strategy use and working out (reproduced alongside) seems to reveal almost a dialogue between what she was asking herself and what she replied. She starts with asking three self-directed questions, "Best way to solve this problem?", "What do I know?" and "Have I ever done this before?" The latter two questions echo the "Did you use all the data?" and "Similar problem" heuristics, respectively. Her working seems to be a response to those

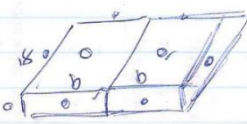
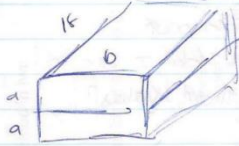
Best way to solve this problem?
What do I know?
Have I ever done this before?

questions. She starts by writing " $a + b + 18 = 35$ ", which should be the sum of the dimensions of Design 1. Unfortunately, she seems to have confused the dimensions of Design 1 with the flat prism. However, she correctly writes the sum of the dimensions for Design 2. On the right side of her working, she converts the height, breadth and length variables to those that match the variables in the question. She then tries to find an expression for the surface area in terms of multiples of 18, a and b . She does the same thing for Design 2. Unfortunately, because her initial dimensions

$a + b + 18 = 35$	$a = \text{height}$
$2a + b + 18 = 31$	$b = \text{width length}$
	$18 = \text{length}$
	Surface area =
$ \begin{aligned} T_1 &= 3 \\ T_2 &= \\ T_3 &= \\ T_4 &= \\ T_5 &= \\ T_6 &= \end{aligned} $	$ \begin{aligned} &2(b \times 18) + 2(18 \times b) + 2(a \times 18) \\ &= 2(b \times 18) + 2(18 \times b) + 2(a \times 18) \\ &= 2(18b) + 2(18b) + 2(18a) \\ &= 36b + 36b + 36a \end{aligned} $
Design 1	
$a = \text{height} = 18$	
$b = \text{breadth} = 2b$	
$18 = \text{length} = 18$	
	$ \begin{aligned} &2(b \times 18) + 2(18 \times b) + 2(a \times 18) \\ &= 2(18 \times 2b) + 2(2b \times 18) + 2(18 \times a) \\ &= 2(36b) + 2(36b) + 2(18a) \\ &= 72b + 72b + 36a = 35? \end{aligned} $

for Design 1 are incorrect, her surface area expression is also; however, she is on the right track with Design 2. She cannot continue, though, for two reasons: she does not have values for a and b , and she is setting the surface area equal to 35, which is the sum of the dimensions of Design 1. However, her response is very interesting because it shows this back-and-forth between the thinking about her strategies and her employment of them. Also, her statement of why to use the strategy is also interesting: "You know when to use the strategy when you are stuck. u Ask yourself what is the best way you can do it or have u even done it before." It may be she finds the strategy use helpful as a prompt to start a problem.

Learner 3.15 is another one who seems to be self-aware of her strategies and mental processes. She does not write her strategies in the columns but as a running commentary alongside her working (see illustration at right). This is an imitation of the way I modelled my thinking in class when I would explain the

2. Design 1  Sum = 35u Find surface area. ↳ Sum of areas of all the faces. $= 4(18 \times b) + 4(a \times b) + 2(18 \times a)$ $= 72b + 4ab + 36a$	Design 2.  • collect data and what is asked Sum = 31 units • equate the units.
---	--

strategies I employed as I worked through a problem (compare with an example of my strategy use in Appendix G – 2). First, she employs an implied heuristic strategy, “draw a figure.” Next to that she writes, “collect data and what is asked,” which could be seen as a variation on the “*did you use all the data – what is the unknown, what is the given, what is the condition*” heuristic. It may be noted that this particular learner is quite artistic and the activity of drawing a figure may have helped her note important information. She writes down the sums of the dimensions, the need to find the surface area and the definition of the surface area. Then she writes, “equations that link.” She knows she needs to find some way to connect the given information and the surface area. She creates a correct expression for the surface area of Design 1 in terms of a and b . For whatever reason, she does not proceed further. She does seem to show, though, that she is thinking about how to solve the problem and is trying to utilise strategies.

Learner 3.8 was the only learner to find the surface area of both figures almost correctly; however, two copying errors made her final answer incorrect. A look at her solution process (reproduced alongside) indicates that it was not a smooth process for her: two pages of working including several crossings-out. But the crossings-out indicate that she was thinking about her answers and double-checking as she went.

Learner 3.8 demonstrates a direct link between the strategies described and the strategies employed. Her first strategy she states as “simultaneous equations to solve for a and b .” She does this, but makes one small error (forgetting to copy down a “ b ”) when she is solving. She finds values for both variables, following her “substitution methods” and “changing the subject of an equation”

→ simultaneous equations to solve for a and b
 → substitution methods
 → changing the subject of an equation
 → used the diagram as assistance

surface area (design 1) = the sum of the areas of all faces
 $= 2(l \times b) + 2(b \times h) + 2(l \times h)$
 $= 2(18 \times 1,5) + 2(1,5 \times 1,5)$

design 1:
 length = 18 cm
 breadth = $2(7,75) = 15,5$ cm
 height = 1,5 cm

$\therefore SA = 2(18 \times 1,5) + 2(1,5 \times 1,5) + 2(18 \times 1,5)$
 $= 54 + 4,5 + 54$
 $\therefore SA = 112,5 \text{ cm}^2 \rightarrow \text{design 1}$

design 2:
 length = 18 cm
 breadth =

②
 $l = 18 \text{ cm}$
 $b = 2(5,25) = 10,5 \text{ cm}$
 $h = 6,5 \text{ cm}$
 $\therefore SA = 748,5 \text{ cm}^2 \rightarrow \text{design 1}$

③
 $l = 18 \text{ cm}$
 $b = 5,25 \text{ cm}$
 $h = 2(6,5) = 13 \text{ cm}$

design 1:
 $18 \text{ cm} + 0 + 2b = 35 \text{ cm}$

design 2:
 $18 \text{ cm} + 2a + b = 31 \text{ cm}$

① $\therefore a + 2b = 17 \text{ cm}$
 ② $\therefore 2a + b = 13 \text{ cm}$

③ $\therefore a = 17 - 2b$

Subst into ②:
 $2a + b = 13$
 $2(17 - 2b) + b = 13$
 $34 - 4b + b = 13$
 $-4b = -21$
 $\therefore b = 5,25 \text{ cm}$

$\therefore a = 17 - 2b$
 $= 17 - 2(5,25)$
 $\therefore a = 6,5 \text{ cm}$

check
 Design 1:
 $18 + a + 2b = 35$
 $18 + 1,5 + 2(7,75) = 35 \text{ cm}$

Design 2:
 $18 + 2(1,5) + b = 31 \text{ cm}$
 $\therefore 28,5 \text{ cm} \neq 31 \text{ cm}$

strategies. Although not self-reported, she checks her initial values to make sure they are correct. Unfortunately, another copying error prevents her seeing her mistake. However, the very fact that she checks her answer indicates a level of reflection on the problem. On her next page she crosses out her initial values for the dimensions of Designs 1 and 2; perhaps she is employing her “used the diagram as assistance” strategy to get the correct values (with continued accuracy) for length, breadth and height of the two designs. When she is sure they are correct, she uses the dimensions she has found and the surface area formula given (which has been written at the top of the page) and calculates the surface area. This was probably done on a calculator because there is no evidence of working out.

Like the learners mentioned above, her solution is interesting because it indicates she is thinking about how she is working and, according to her self-reported strategy use, she is consciously employing content-specific strategies.

4.4 QUESTION 3: SEQUENCES

Question 3 on both Instruments concerned the content area of sequence. Question 3 in Instrument 1 looked like a trigonometry problem but was actually about sequences and is reproduced alongside. Several patterns occurred simultaneously and were connected. On the left hand side of the equation, the “2’s” of the tangent function built in a pattern where each line had n number of 2’s forming a pattern of degrees. The tangent of those degrees is related to another pattern on the right hand side of the equation. The degrees on the right hand side form an arithmetic progression given by the general formula $T_n = 2 + (n-1)20$. However, because of the periodicity of the tangent function, the reduced angles form another pattern.

As in Instrument 1, Question 3 in Instrument 2 dealt with sequences. In this question, though, very little information is given and the students needed to generate a common difference from the common ratio of an embedded geometric sequence, which is also unknown.

3. Study the sequence below and answer the question that follows:

$$\tan 2^\circ = \tan 2^\circ$$

$$\tan 22^\circ = \tan 22^\circ$$

$$\tan 222^\circ = \tan 42^\circ$$

$$\tan 2222^\circ = \tan 62^\circ$$

$$\tan 22222^\circ = \tan 82^\circ$$

$$\tan 222222^\circ = \tan 102^\circ = -\tan 78^\circ$$

$$\tan 2222222^\circ = \tan 122^\circ = -\tan 58^\circ$$

Determine the 100th term of the sequence.

3. The first term of an arithmetic sequence is 3 while T_3 , T_6 and T_{10} form consecutive terms of a geometric sequence.

Find the value of the tenth term of the arithmetic sequence.

This question shared similarities with the previous question in that formulas exist that enable one to find specific terms of sequences. However, to use the formula, in this case $T_n = a + (n-1)d$, the common difference, d is missing. The first term, a , is given as 3 and, since the question is asking for the 10th term, $n = 10$. As in Question 2, the learners needed to establish a relationship between other given information in order to find a value for d . The

question states that the 3rd, 6th and 10th also form consecutive terms of a geometric sequence.

Because a geometric sequence is defined by its common ratio, $\frac{T_2}{T_1} = \frac{T_3}{T_2}$, the learners could have

exploited this fact to find a value for d . However, no values were given for the terms, so in order to set up the ratio, the learners needed to express the terms of the sequence in terms of 3 and multiples of d .

4.4.1 Question 3: Mathematical Proficiency

4.4.1.1 Question 3, Instrument 1: Mathematical Proficiency – Trends

My intention with this question was for the learners to connect the number of “2’s” in the tangent expression on the left hand side (LHS) with a tangent expression in terms of an acute angle on the right hand side (RHS). I was hoping that they might be a bit amazed that the tangent of (222222...100 2’s) could be the same as the tangent of a tiny acute angle. However, I did not ask for the answer in terms of an acute angle and no learner expressed their answer that way. Also, it was very difficult to tell if the learners connected the number of “2’s” in the LHS with the value of the tangent in the RHS; a number of learners noticed the linear pattern of the degrees in the RHS and used the general formula for an arithmetic sequence and found T_{100} .

A comparison of my memorandum answer with a learner’s answer may demonstrate what I see as a difference in conceptual understanding of the question (see Figure 4.5). Learner 3.1 took the first differences of the degrees on the RHS of the equations and found they formed an arithmetic sequence. She remembered the general term of an arithmetic sequence, substituted in the appropriate values and simplified the expression. The question did not ask for reduction formulae to be applied and she did not reduce 1982° to 2° . She definitely discovered the pattern on the RHS, but I’m not sure if she appreciated the periodicity of the problem. Also, she states that $T_{100} = 1982$, not $\tan 1982^\circ$. In other words, she found the linear pattern of the *numbers* of the degrees, but did not realise their connection to the tangent function. This turned out to be a question where learners could get the answer correct (almost) but with the perhaps mechanical application of a formula

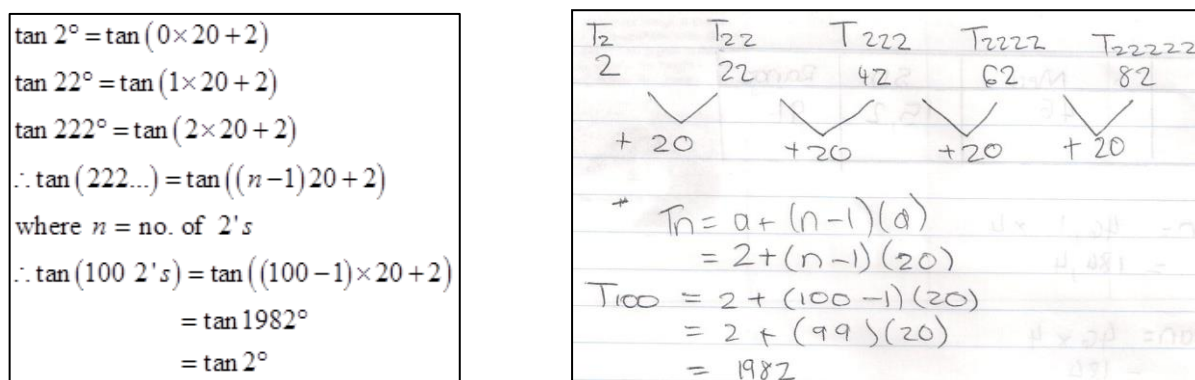


Figure 4.5 Memorandum for I1.Q3 Compared to Learner Response

Like Question 1, coding the question revealed more than marking the question. Several trends in the question emerged. Firstly, a number of students ($n = 11$) realised the degrees on the RHS formed a linear pattern and used the arithmetic progression formula $T_n = a + (n-1)d$, with $a = 2^\circ$ and $d = 20^\circ$. However, of those 11, only four simplified correctly to $\tan 1982^\circ$, another four simplified to just 1982° . Many students, though, left out the degree symbol in their solutions, so it is difficult to know if they really understood the nature of the question, or almost mechanically applied an arithmetic progression formula. A summary of the learners' responses by code and a tally of the frequency of the codes is found in Appendix L – 1 and M – 1, respectively.

Two learners made a critical procedural and conceptual error when substituting into the arithmetic progression formula; one learner's solution is presented alongside. The learner finds the first differences of the degrees as 20, but then substitutes $a = \tan 2$ and $d = \tan 20$. After substituting in $n = 100$, she then multiplies $\tan 20$ by 100 and gets $\tan 2000$ (this is inferred from her final answer). She then subtracts $\tan 18$ from $\tan 2000$ and gets the correct answer, $\tan 1982$, but through the wrong reasoning. She does not realise that $(\tan 20^\circ)(100) = 100 \tan 20^\circ$ not $\tan 2000^\circ$.

Also, $\tan 2000^\circ - \tan 18^\circ \neq \tan(2000^\circ - 18^\circ)$; $\tan(2000^\circ - 18^\circ)$ is a compound angle and has its own formula. The fact that the learner does not use any degree signs in the solution may also indicate she does not realise the difference between a degree of rotation, a real number and the trigonometric function of either.

Which (tan-something) is the Sequence?
How else I use the right one? Does tan have a function in the result?

$$\begin{aligned} T_1 &= \tan 2 \\ T_2 &= \tan 22 \\ T_3 &= \tan 42 \\ T_4 &= \tan 62 \\ T_5 &= \tan 82 \\ T_6 &= \tan 102 \\ T_7 &= \tan 122 \end{aligned}$$

20
20
20
20
20
20
20

$$\therefore T_n = a + (n-1)(d)$$

$$\therefore T_n = \tan 2 + (n-1)(\tan 20)$$

$$\therefore T_n = \tan 2 + \tan 20n - \tan 20$$

$$\therefore T_n = -\tan 18 + \tan 20n$$

$$\therefore T_{100} = -\tan 18 + (\tan 20)(100)$$

$$\therefore T_{100} = \tan 1982$$

Two learners also tried to apply a geometric progression formula to the problem; one learner's solution is presented alongside. This learner found the first differences of the degrees on the LHS of the equations and saw that they increase by 20, 200, 2000; that is, 2 times successive powers of 10. She tried to apply the geometric progression formula, $T_n = a.r^{n-1}$, to the differences and resulted in $T_{100} = 2 \times 10^{100}$.

This was another question in which the connection between the strands was strong. For example, I coded “using the arithmetic progression formula” as part of conceptual understanding, procedural fluency and strategic competence because that was a critical step for all three strands. Also, the errors regarding multiplying and subtracting the

$$\begin{aligned} T_1 &= \tan 2^\circ = \tan 2^\circ \\ T_2 &= \tan 22^\circ = \tan 22^\circ \\ T_3 &= \tan 222^\circ = \tan 42^\circ \\ T_4 &= \tan 2222^\circ = \tan 62^\circ \end{aligned}$$

$$T_n = a.r^{n-1}$$

① $\tan 2$ $\tan 22$ $\tan 222$ $\tan 2222$

20 200 2000

② $\tan 2$ $\tan 22$ $\tan 42$ $\tan 62$ $\tan 82$

20 20 20

③ $T_{100} = 2 \times 10^{100}$

tangent values mentioned above were also coded under conceptual understanding and procedural fluency, because they revealed a lack in both.

4.4.1.2 Question 3, Instrument 2: Mathematical Proficiency – Trends

With regard to the strands of mathematical proficiency required by this question, *conceptual understanding* was needed to understand how the terms of arithmetic and geometric sequences are connected by their respective differences and ratios. *Strategic competence* was needed to describe the terms of the sequence in terms of 3 and multiples of d , and then equate the ratios of the successive terms of the geometric sequence. *Procedural fluency* was required to successfully carry out the algebraic steps without error. My memorandum answer is provided alongside.

This was one question where very few students got very far, but one student did find the correct value for d . However, there was not the prevalence of misconception as in Question 1 nor the “grabbing” for numbers as in Question 2: many had a correct idea what to do, but could not see the next step. The summary tables for the conceptual understanding and strategic competence are presented in Appendix M – 2.

$$\begin{aligned}
 T_1 &= 3 \\
 \therefore \frac{T_6}{T_3} &= \frac{T_{10}}{T_6} \\
 \therefore \frac{3+5d}{3+2d} &= \frac{3+9d}{3+5d} \\
 (3+5d)^2 &= (3+9d)(3+2d) \\
 9+30d+25d^2 &= 9+33d+18d^2 \\
 7d^2-3d &= 0 \\
 d(7d-3) &= 0 \\
 d=0 \quad \text{or} \quad d &= \frac{3}{7} \\
 \text{N.V.} \\
 \therefore T_{10} &= 3+9 \times \frac{3}{7} \\
 &= \frac{48}{7}
 \end{aligned}$$

Learner 3.8 (whose solutions have been mentioned previously) produced a lengthy response that at one point began to take a promising direction, but due to a procedural error, could not be continued (see solution on p. 96). She starts with what seems to be almost a discussion with

herself about how an arithmetic progression works (see solution alongside). We had worked on arithmetic series several months earlier and it is possible she needed to remind herself about arithmetic progressions.

She begins with a circular argument defining the first term in terms of itself and correctly realises that T_1 , a and 3 all represent the same thing. By the end of the first page, she still has an error in reasoning because she is equating 3 to T_n for both the arithmetic and geometric formulas; in reality, 3 is the first term for both progressions.

On the next page, she makes a more hopeful start and sets up T_3 , T_6 and T_{10} both in terms of their differences and their ratios. She then sets up a simultaneous equation for T_{10} , equating the 10th term as an arithmetic progression to the 10th term as a geometric progression. Her equation is valid, but not solvable using the means we have studied in high school. She then makes a procedural error, but solves for r in terms of d as a surd. She next substitutes this value for r back into the expression for T_{10} as a geometric progression and tries to raise her surd to the 9th power. She incorrectly “distributes” the rational power to the two terms of the binomial and ends with an expression for T_{10} in terms of d raised to the 9/2. She then substitutes this back into the expression for T_{10} as an arithmetic progression and determines d by incorrect methods. She knows she is on the wrong track because she writes, “I’m not sure what this means.”

The procedural errors aside, her biggest mistake was a strategic one: she equated T_{10}

as an arithmetic progression term to T_{10} as a geometric progression term. This created a linear equation equal to an exponential equation, which cannot be solved in high school. She then tied herself up in a procedural knot because she did not have the means with which to solve the equation. She had sound conceptual understanding of the way arithmetic and geometric

$$\begin{array}{l}
 T_1 = 3 \quad T_n = a + (n-1)d \\
 T_1 = a \quad \therefore 3 = a + (1-1)d \\
 \quad \quad \quad 3 = a + 2d \\
 \quad \quad \quad 2d = 0 \\
 \quad \quad \quad \therefore d = 0 \\
 \text{but } d \text{ cannot equal } 0 \text{ as this is an} \\
 \text{arithmetic number pattern} \\
 \therefore \text{common difference is required} \\
 \\
 T_1 = a = 3 \quad \text{and} \quad T_1 = a + (1-1)d \\
 \quad \quad \quad 3 = a + (0)d \\
 \quad \quad \quad \therefore d = 0 \rightarrow \text{given!} \\
 \\
 \text{arithmetic: } T_n = a + (n-1)d \\
 \text{geometric: } T_n = a \cdot r^{n-1} \\
 \\
 \text{arithmetic: } T_1 = a + (1-1)d \\
 \quad \quad \quad \therefore T_1 = a \\
 \\
 \text{geometric: } T_1 = a \cdot r^{1-1} \\
 \quad \quad \quad \therefore T_1 = a \\
 \\
 T_n = a \cdot r^{n-1} = 3 \\
 T_n = a + (n-1)d = 3 \\
 \\
 T_{10} \text{ of arithmetic sequence} \rightarrow \text{FIND!}
 \end{array}$$

waffling
(sorry ma'am)

$$\begin{array}{l}
 T_{10} = 3 + (10-1)d \quad T_{10} = 3 \cdot r^{10-1} \\
 = 3 + 9d \quad = 3 \cdot r^9 \\
 \\
 T_3 = 3 + (3-1)d \quad T_3 = 3 \cdot r^{3-1} \\
 = 3 + 2d \quad = 3 \cdot r^2 \\
 \\
 T_6 = 3 + (6-1)d \quad T_6 = 3 \cdot r^{6-1} \\
 = 3 + 5d \quad = 3 \cdot r^5 \\
 \\
 \therefore T_6 = T_{10} \\
 3 + 9d = 3 \cdot r^9 \\
 1 + 3d = r^9 \\
 \text{Newton: } r = \sqrt[9]{1+3d} \\
 \\
 \therefore T_{10} = 3 \cdot r^9 \\
 = 3(\sqrt[9]{1+3d})^9 \\
 = 3[(1+3d)^{1/9}]^9 \\
 = 3(1+3d)^{9/2} \\
 = 3(1+140,89d^{9/2}) \\
 \therefore T_{10} = 3 + 420,89d^{9/2} \\
 \\
 3 + 9d = 3 + 420,89d^{9/2} \\
 9d = 420,89d^{9/2} \\
 \therefore d^{1/2} = 46,44 \\
 \therefore \text{I'm not sure what this means}
 \end{array}$$

Learner 3.9 also committed some serious procedural errors (see solution alongside). Her solution is novel because she set up the terms of the arithmetic sequence in terms of 3 and multiples of d , and then expressed the geometric terms in terms of the $3 + 2d$ (i.e. T_3) and r . Like Learner 3.8, she understands how to generate the terms of an arithmetic sequence and she almost understands the geometric terms. However, she makes the procedural and/or conceptual mistake of writing T_6 as $3 + 2dr^2$ instead of $(3 + 2d)r^2$.

Like Learner 3.8, she equates the T_{10} as an arithmetic progression term to T_6 . Because of the missing bracket, she solves for r . However, the quadratic she sets up is for an arithmetic progression in the question. However, her expression for T_{10} (as a geometric progression) is correct. Like Learner 3.8, she demonstrates a relatively sophisticated understanding of sequences, but her means of solving the problem and the procedures she employs.

On her first attempt, she writes down $T_2 - T_4$ in terms of 3 and multiples of d . (Her solution attempt is reproduced on the next page.) She also makes an equation equating the differences, but sees that it simplifies to an identity. She also sets up a ratio equation, but between the 10th, 9th and 7th terms. She solves until presumably she reaches what would simplify to $d = 0$, which is impossible in this case. On her second attempt, she systematically writes down $T_1 - T_6$ and T_{10} and expresses them in terms of 3 and multiples of d . She then sets up the ratio equation and solves for d with no procedural errors. Perhaps because of a lack of time or not knowing what to do next, she does not proceed any further, even though she has answered the hardest part of the problem.

$3 \cdot (31d) =$ $\rightarrow 3 + (9)d$ $\underline{3+8d}$	$T_1 = 3 \quad T_2 \quad T_3 \quad T_4 \quad T_7 \quad T_8 \quad T_9 \quad T_{10}$ $31d \quad 31+2d \quad 31+3d$ $T_3 = a + (n-1)d$ $= 3 + (n-1)d$ $=$ $T_n = 3(r)^{n-1}$	$31d - (3) = 31+2d - (3+d)$ $31d - 3 = 31+2d - 3 - d$
$T_3 = a + (n-1)d$ $= 3 + (n-1)d$ $=$ $T_n = 3(r)^{n-1}$	$a = 3$ $d = 2$ $\frac{3+9d}{3+8d} = \frac{3+8d}{3+7d}$ $(3+9d)(3+7d) = (3+8d)(3+8d)$ $9 + 21d + 27d + 63d^2 = 9 + \underbrace{24d + 24d}_{+64d}$ $9 + 48d + 63d^2 = 9 + 48d + 64d^2$	$31d - (3) = 31+2d - (3+d)$ $31d - 3 = 31+2d - 3 - d$

Learner 1.1 displayed another trend shared by several learners: she started off in the wrong direction, saw the error and started again more correctly (see solution alongside). However, like a number of learners, her lack of procedural skills did not allow her to finish.

In this case, the learner started off with the arithmetic series formula and finds the sum of three, six and ten terms. This she crosses out and starts again, this time with the 3rd, 6th, and 10th term in terms of 3 and multiples of d . She set up the ratio equation correctly, but made one strategic and possibly a conceptual mistake: she equated that ratio equation to r . Conceptually, the equation can be set up because both ratios on either side of the equal sign represent r and are equal by definition of a geometric sequence; the

ended with an expression for r . Like other learners discussed, her conceptual understanding of procedures is faulty.

Another trend shared by a number of learners is illustrated by the response from Learner 1.7 (see alongside). Firstly, like other learners mentioned previously, she started off in one direction, realised she made a mistake and started again. However, she begins again by trying to find a value for T_1 but simplifies her equation incorrectly and finds that $d = 1$, which she uses to find T_{10} . Five different learners used a variation of this method to use the first term to find the first term. A few realised that $T_1 = a$ and thus $d = 0$, but others found an answer for d through procedural errors and used that number to find T_{10} . This error had conceptual and procedural elements. Conceptually, the learners did not seem to understand that T_n represents the *general term*; that is, an expression for any term is derivable from the first term plus $(n - 1)$ differences. Several tried to “solve” for d by setting the first term equal to itself indicating a lack of conceptual understanding of how the general term works. Procedurally, if they did discover a value for d from the general formula, they had to make a mistake; thus indicating a lack of accuracy in their procedures.

The image shows handwritten mathematical work on lined paper. At the top, it says $T_1 = 3$. Below this, the general term formula $T_n = a + (n-1)d$ is written and then crossed out with a large diagonal line. To the right of the crossed-out formula, the equation $3 = 3 + (1-1)d$ is written, which is also crossed out. Below the crossed-out formula, the equation $T_n = 3 + (10-1)d$ is written. Further down, the equation $3 = 3 + (1-1)d$ is written. Below that, the equation $3 = 3 + 0 \cdot d$ is written. Then, the equation $\frac{3}{3} = \frac{3d}{3}$ is written, followed by $1 = d$. Below this, the general term formula $T_n = 3 + (n-1)(1)$ is written. Then, the equation $T_{10} = 3 + (10-1)(1)$ is written. Below that, the equation $T_{10} = 3 + 9$ is written. Finally, the equation $T_{10} = 12$ is written.

The responses to this question seemed to fall into three general categories. A few learners ($n = 4$) were able to exploit the relationship between the arithmetic progression and geometric progression to set up a ratio equation. However, of those, only one demonstrated the procedural fluency to complete the equation correctly. A few other learners ($n = 4$) simplified the problem considerably by finding a value for d incorrectly and then found a value for T_{10} . However, many students ($n = 16$) could not proceed very far at all.

On the other hand, another interesting trend was that several learners ($n = 5$) knew they started the question incorrectly, crossed out their work and started again. Although they may not have proceeded correctly or further on the second attempt, it does show reflection on their process.

Even though there was very little conceptual misunderstanding on this question compared to Question 1 and 2, the learners' lack of procedural fluency was another common trend. Even simple substitution into the arithmetic progression formula produced errors, as well as solving the more complicated ratio equation. As discussed, two learners, with reasonable conceptual

understanding of the problem, set up equations they could not solve with the methods available to them.

Like other questions, this one showed the strong links between conceptual understanding, procedural fluency and strategic competence. The same step, expressing the terms of the sequence in terms of a and multiples of d , was coded as a conceptual, procedural and strategic step. Likewise, the ratio relationship was connected to conceptual understanding, strategic competence and procedural fluency. A success or mistake in one strand usually resulted in a success or mistake in the others.

4.4.2 Question 3: Strategy Use

When trying to code these responses for Question 3 on Instrument 1, I needed to type them as heuristic or content-specific. One heuristic that could be applied to this problem is called “*generalisation*.” This heuristic is defined by Pólya as “passing from the consideration of one object to the consideration of a set containing that object; or passing from the consideration of a restricted set to that of a more comprehensive set containing the restricted one” (Polya, 1957, p. 108). In looking at this question from the heuristic perspective, I needed to define what I meant by “object” and “set.” If an object was each element on the RHS of the equation, $\tan 2^\circ$, $\tan 22^\circ$, etc. and the set contained all those objects, then I could say that the learners who employed number patterns used the generalisation heuristic, even if they did not state that heuristic explicitly. However, if I look at both the LHS and the RHS of each equation as an object, e.g. $\tan 2^\circ = \tan 2^\circ$, $\tan 22^\circ = \tan 22^\circ$, etc. and the list of equations as the set containing all those objects, I would say that employing arithmetic number patterns without recognising the relationship between the number of 2’s on the LHS to the n th term on the RHS, was not using the generalisation heuristic and was employing a content-specific strategy. I then used this distinction when coding. I will describe the coding process for most strategies represented below in conjunction with the learners’ solution steps because, as seen before, the strategy and the solution affected each other. Even though this question preceded the strategy instruction, the strategy use was quite prolific and will be discussed in some detail.

4.4.2.1 Question 3, Instrument 1: Strategy Use – Trends

In Question 3, 10 out of the 26 learners explicitly stated their strategy use in the strategy columns. Three learners demonstrated declarative, procedural and conditional knowledge, two of the

Instrument 1: Question 3									
Frequency of Strategies by category, code, type and rating									
Category	Explicit						Implied		
Code	Declarative		Procedural		Conditional				Total
Rating/type	H	CS	H	CS	H	CS	H	CS	
E	1	0	0	0	0	0	0	0	1
PE	1	4	1	1	1	2	0	0	10
IE	0	2	0	1	0	1	0	0	4
Total	2	6	1	2	1	3	0	0	15

Table 4.14 Frequency of Strategy Use by Categories: I1.Q3

ten learners demonstrated two of the three types of metacognitive knowledge and the rest indicated one form of metacognitive knowledge. A summary of the ratings and types is given in the Table 4.14 above. A representative sample of the learners' responses is reproduced in the table below and more details are in Appendix O - 1.

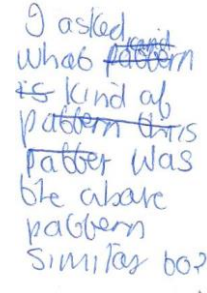
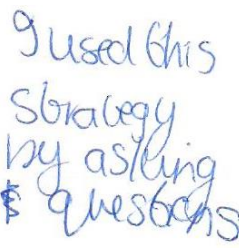
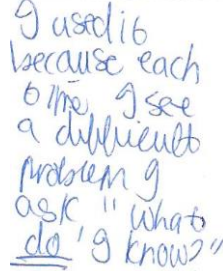
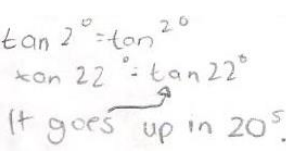
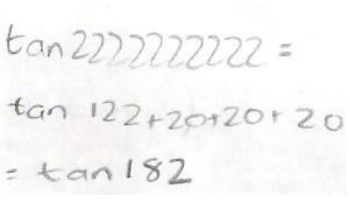
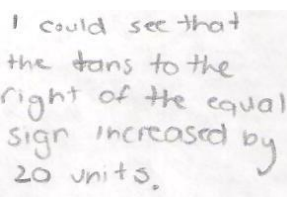
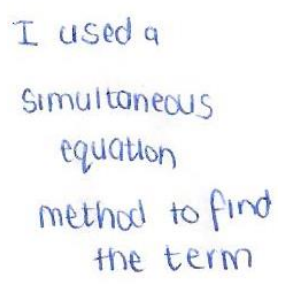
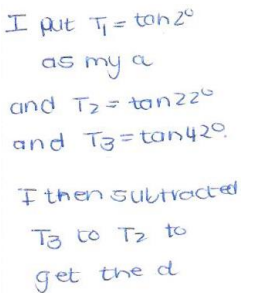
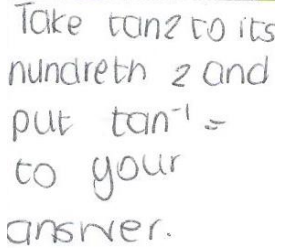
Type of Strategy	Rating	What strategy/ies did you use?	How did you use the strategy?	Why did you use the strategy or how do you know when to use the strategy?
Heuristic	Effective			
	Partially effective			
	Ineff.	None		
Content-Specific	Eff.	None		
	Partially effective			
	Ineffective			

Table 4.15 Representative Explicit Strategies: Question 3, Instrument 1

A few common trends were apparent. Most frequent was the association of the problem with arithmetic number patterns: in the “what strategy” column, five of the 10 learners specifically mentioned the word “pattern” or “progression,” and one learner mentioned “tried to find the ‘a’ or ‘d’ or ‘r’”. Two learners implied the recognition of patterns; one learner wrote “it goes up in 20’s” and another used the arithmetic progression formula in the strategy columns.

Three learners described their strategies in a similar manner and also used similar solution methods. All three mentioned that they used patterns to solve this problem and all three used the arithmetic progression formula, $T_n = a + (n-1)d$. The strategy and solution of Learner 3.1 is reproduced alongside as an example.

I decided to use what I know about number patterns and sequences in order to solve this question

Learner 3.1 seemed to be connecting the LHS of the equation to the RHS by writing T_2 , T_{22} , T_{222} , etc. above 2, 22, 42, etc. However, T_2 represents the first term with $n = \text{one } 2$, T_{22} represents the second term with $n = \text{two } 2$'s, etc. It is not evident from the learner's response that she understood this connection. What she seemed to see was the arithmetic progression of the *degrees* on the right hand side and used the

T_2	T_{22}	T_{222}	T_{2222}	T_{22222}
2	22	42	62	82
	+	+	+	+
	20	20	20	20
$T_n = a + (n-1)(d)$ $= 2 + (n-1)(20)$				
$T_{100} = 2 + (100-1)(20)$ $= 2 + (99)(20)$ $= 1982$				

formula to find the 100th term. Learner 3.1, though, makes the mistake that $T_{100} = 1982$ instead of $T_{100} = \tan 1982^\circ$. As there was no clear indication that the learner related the number of 2's in the LHS to the term number in the RHS, I coded this as a content-specific strategy, rather than a heuristic with a rating of partially effective.

Learner 3.2 stated as her strategy that she “tried to find the ‘a’ or ‘d’ or ‘r’”. Her solution is reproduced alongside. Unfortunately, she made the mistake of trying to find the common differences

Tried to find the 'a' or 'd' or 'r'

Because the difference is constant

of the *tangents* of the angle, instead of the differences of the angles themselves. Although this approach did not help with solving this question, it did indicate that the learner understood the difference between an angle and the tangent of that angle – a distinction that I am not sure if all students recognised. Because she focused on attempting to find an arithmetic progression on the RHS, I coded this as a content-specific, but ineffective strategy.

$a = \tan 2^\circ$						
$\tan 2^\circ$	$\tan 22^\circ$	$\tan 42^\circ$	$\tan 62^\circ$	$\tan 82^\circ$	$\tan 102^\circ$	$\tan 122^\circ$
0,37	0,50	0,99	5,23	-1,182	3,10	
	0,13					

Learner 3.5's declarative strategies were a little different from those of the previous learners discussed. She writes:

"I asked what ~~kind pattern~~ is kind of ~~pattern this~~ pattern was the above pattern similar to? [sic]" Quite a bit of metacognition may be happening in this response. If you ignore the

strikethroughs, the learner is possibly using two strategies: she is employing self-questioning and she is

using (if perhaps unwittingly) a strategy that Pólya calls "analogy." If you include the strikethroughs, she is also reflecting on what she is writing as she writes it. Her responses in the "how" and "why" columns seem to indicate self-questioning as the strategy she is employing. She writes in the "how" column, "I used this strategy by asking questions," and in the "why" column, "I used it because each time I see a difficult problem I ask 'what do I know.'" Taken together, her strategy seems to be self-questioning, which was coded as "other" and rated as effective given her solution steps reproduced above.

I asked
what ~~kind pattern~~
is kind of
~~pattern this~~
pattern was
the above
pattern
similar to?

$$\begin{aligned} T_n &= a + d(n-1) \\ &= 2 + 20(n-1) \\ &= 2 + 20n - 20 \\ &= -18 + 20n \\ T_{100} &= 20(100) - 18 \\ &= 1982 \\ \text{then } 1982 \end{aligned}$$

Two of the learners' strategies were difficult to code with my system. Learner 3.7 writes in the "what" column, " $\tan 2^\circ = \tan 2^\circ$, $\tan 22^\circ = \tan 22^\circ$ " and then draws an arrow with the caption, "it goes up in 20s." She seems to be on the right track with perceiving an arithmetic progression of the degrees, but then she writes in the "how" column, " $\tan 2222222222 = \tan 122 + 20 + 20 + 20 = \tan 182$ " which does not follow any given pattern, and she writes in the "why" column, "I could see that the tans to the right of the equal sign increased by 20 units." Like a number of learners, she confused tangent of an angle with the angle itself. She also did not do any further working. Her declarative, procedural and conditional strategies could only be typed as "other" and rated as ineffective.

Learner 3.12 did her working out in the "what" and "how" columns on the page as well as on lined paper. I do not know if she meant her procedural steps to be an implied strategy. Only her response "saw a common difference" in the "why" column could qualify as a content-specific, partially effective strategy taken in conjunction with her working which is reproduced alongside. Notice that this learner calculated the value of 1982, but also included another sequence with the first term as 78 and a common difference of -20 . The 100th term for this sequence was -1902 which she then combined for a final answer of $T_{100} = \tan 1982 - 1902$, demonstrating a poor grasp of the tangent function and order of operations when working with a trigonometric function. While half of her answer is on the right track, the other half and the final conclusion are not. Her strategy use was rated partially effective.

$$\begin{aligned} \textcircled{3} \quad T_n &= a + (n-1)d \\ T_n &= 2 + (n-1)20 \\ &= 2 + (n-1)20 \\ &= 2 + (100-1)20 \\ &= 1982 \\ T_n &= a + (n-1)d \\ T_n &= 78 + (n-1)(-20) \\ T_n &= -1902 \\ T_{100} &= \tan 1982 - 1902 \end{aligned}$$

Learner 1.4 wrote down a misapplied content-specific strategy. Her solution is reproduced alongside. Learner 1.4 states in the “what strategy” column that she “used a simultaneous method to find the term” and goes on to describe in the “how” column, “I put $T_1 = \tan 2^\circ$ as my a and $T_2 = 22^\circ$ and $T_3 = 42^\circ$. I then subtracted T_3 to T_2 to get the d .” In her working, she does set up simultaneous equations and with the incorrect order of operations, she finds $d = \tan 20^\circ$. The $\tan 2^\circ$ and $\tan 20^\circ$ are substituted into the arithmetic progression formula as a and d , respectively. She then conveniently “distributes” the $100 - 1$ to the $\tan 20^\circ$ to get $\tan 2^\circ + \tan 2000^\circ - \tan 20^\circ$, resulting in the “correct” answer of $\tan 1982^\circ$, but with completely the wrong understanding of simplifying trigonometric expressions. Her strategy was typed as content-specific and rated as partially effective.

$$\begin{aligned} T_1 &= \tan 2^\circ \\ T_2 &= a + d = \tan 22^\circ \quad (1) \\ T_3 &= a + 2d = \tan 42^\circ \quad (2) \\ d &= \tan 20^\circ \\ T_{100} &= a + (n-1)d \\ &= \tan 2^\circ + (100-1)\tan 20^\circ \\ &= \tan 2^\circ + \tan 2000^\circ - \tan 20^\circ \\ &= \tan 1982^\circ \end{aligned}$$

Learner 1.6 may have been the only learner who acknowledged that the LHS determined the number of two's, but

$$\begin{aligned} \tan 222222222222 &= 89,99^\circ \\ \tan 222222222222 \times 10 &= 90^\circ \end{aligned}$$

Unfortunately, her strategy, although correct, was impractical. She states in the “what strategy” column, “Take $\tan 2$ to its hundreth [sic] 2 and put $\tan^{-1}$ = to your answer.” Her method is correct in theory, but in practice, no handheld calculator could calculate that question. Hence the need to see a pattern and find the answer another way. Her strategy was typed as content-specific and rated as ineffective, particularly in light of her worked solution illustrated above.

4.4.2.2 Question 3, Instrument 2: Strategy Use – Trends

Question 3 in Instrument 1 again demonstrated the nearly universal applicability of the heuristic strategies to support conceptual understanding of a problem, the strategic competence to set up a mathematical model and the procedural fluency to solve a problem accurately and efficiently. In this particular question, “*Did I use all the data?*” and its

3. The first term of an arithmetic sequence is 3 while T_3 ; T_6 and T_{10} form consecutive terms of a geometric sequence.

Find the value of the tenth term of the arithmetic sequence.

further subdivisions of “*What is the data, what is the unknown and what is the condition?*” could have been very helpful questions to ask and several learners did seem to demonstrate something of a monologue with themselves [e.g. 3.3, 3.8, 3.10, 1.12]. Particularly because so little information appears to be given in the question, asking these strategy questions could have prompted looking for information that can be inferred from the question. It is very possible that those learners who were able to write out T_1 , T_2 , T_3 , etc. in terms of a and multiples of d were asking themselves a variant on these questions, even if they did not explicitly state them.

In conceptualising the problem, the “*Did I use all the data?*” and its subdivisions could have helped identify the data, condition and unknown. In this question, the data given were that an arithmetic sequence existed whose first term was 3 and three of its terms formed a geometric sequence as well. The condition was that the 3rd, 6th term and 10th term of the arithmetic sequence were the 1st, 2nd and 3rd terms, respectively, of the geometric sequence. The unknown was the 10th term which belonged to both sequences.

Drawing on their conceptual understanding of the problem, the learners could then use the “*Did I use all the data?*” strategy to also support their strategic competence to model the problem. The link between the two sequences was the ratio equation utilising the arithmetic sequence terms written in terms of a and multiples of d . This also required the “*auxiliary elements – auxiliary problem*” strategy because the ratio equation needed to be solved first for d (the common difference) before finding the value of the tenth term.

Procedural fluency could have been supported with concept-specific strategies such as multiplying through by the lowest common denominator in the ratio equation to end with a quadratic equation. This could be solved using factorising and the zero product property. Because two solutions would result from the quadratic, one solution that did not make sense from the context would need to be discarded.

A summary of the frequency and type of strategies used in Question 3, Instrument 1 and 2 is given alongside for comparison purposes. As can be seen in the table, the total number of strategies reported remained the same in Instrument 2 as in Instrument 1, with a slight increase in ineffective strategies and a slight decrease in partially effective strategies.

For this question, nine students self-reported strategy use; 3 learners

Instrument 1: Question 3									
Frequency of Strategies by category, code, type and rating									
Category	Explicit						Implied		
Code	Declarative		Procedural		Conditional				Total
Rating/type	H	CS	H	CS	H	CS	H	CS	
E	1	0	0	0	0	0	0	0	1
PE	1	4	1	1	1	2	0	0	10
IE	0	2	0	1	0	1	0	0	4
Total	2	6	1	2	1	3	0	0	15

Instrument 2 – Question 3									
Frequency of Strategies by category, code, type and rating									
Category	Explicit						Implied		
Code	Declarative		Procedural		Conditional				Total
Rating/type	H	CS	H	CS	H	CS	H	CS	
E	0	1	0	0	0	0	0	0	1
PE	2	2	0	4	0	0	0	0	8
IE	2	1	0	2	0	1	0	0	6
Total	4	4	0	6	0	1	0	0	15

Table 4.16 Strategy Use Question 3: I1 and I2 Compared

described declarative, procedural and conditional knowledge, two learners reported two of the three types of knowledge, and the rest reported one of the three. Many of the learners were quite articulate and detailed in their self-reported strategies. Seven out of the nine learners’ working matched their strategy use, with four of those being correct. The remaining two learners’ strategy use did not match their working, nor was it correct, but because they did not

progress with their working. A representative sample of effective, partially effective and ineffective strategy use is tabulated below. A table with additional analysis is provided in Appendix O - 2.

Type of Strategy	Rating	What strategy/ies did you use?	How did you use the strategy?	Why did you use the strategy or how do you know when to use the strategy?
Heuristic	Eff.	None		
	Partially effective			
	Ineffective			
Content-Specific	Effective			
	Partially effective			
	Ineffective			

Table 4.17 Representative Explicit Strategies: Question 3, Instrument 2

Learner 3.7's self-reported strategies and mathematical working are illustrated alongside. She employed a strategy that was shared by six other learners: she wrote down the list of terms as $T_1, T_2, T_3, \dots, T_{10}$. This may have been a way for the learners to investigate "what are the data," although this learner's self-reported strategy was coded as content-specific because she mentions using the formula for the arithmetic sequence. What is also important is that of the six learners who wrote down the list of terms, only two were then able to express all three important terms (T_3, T_6 and T_{10}) in terms of a and multiples of d .

Handwritten notes for Learner 3.7:

T_1 (circled with 3) T_2 T_3 T_4 T_5 T_6 T_7 T_8 T_9 T_{10}

$T_n = ar^{n-1}$

$T_n = a + (n-1)d$

$= 3 + (n-1)d$

Formulas for the arithmetic sequence.

Subbing in the given values
(although not much was given)

The question is clearly a patterns question and mentions the arithmetic sequence.

Learner 3.7 mentions in the "how" column that she substituted-in values; however, in her working, she has only substituted the 3 for a . However, this was coded as her strategy use matching her mathematical working because she did use the arithmetic formula. Lastly, her statement in the "why" column, "the question is clearly a patterns question and mentions the arithmetic sequence," while correct, might have nudged her into thinking the question was a simple patterns question. The fact that she could not continue, indicates that it was not.

Learner 3.8's mathematical working was considered in detail in the Mathematical Proficiency section; however, comparing her

→ Geometric and arithmetic sequences (number patterns)

→ lots of trial and error

→ write down everything I know

Handwritten mathematical working for Learner 3.8:

$T_1 = 3$ $T_n = a + (n-1)d$

$T_1 = a$ $\therefore 3 = a + (1-1)d$

$3 = 3 + 2d$

$2d = 0$

$d = 0$

but d cannot equal 0 as this is an arithmetic number pattern

$\therefore$ common difference is required

$T_1 = a = 3$ and $T_1 = a + (1-1)d$

$3 = a + (0)d$

$\therefore a = 3 \rightarrow$ given!

arithmetic: $T_n = a + (n-1)d$

geometric: $T_n = a \cdot r^{n-1}$

arithmetic: $T_1 = a + (1-1)d$

$\therefore T_1 = a$

geometric: $T_1 = a \cdot r^{1-1}$

$\therefore T_1 = a$

$T_n = a \cdot r^{n-1} = 3$

$T_n = a + (n-1)d = 3$

TIO of arithmetic sequence $\rightarrow$ FIND!

mathematical working with her self-reported strategies shows her engagement with her solution process. She mentions in the "what strategies" column that she used "geometric and arithmetic sequences (number patterns)" and "lots of trial and error". The latter is particularly evident on her first page where she uses circular reasoning and tries to discover a value for d and a . However, she tries again more successfully on the second page, doing exactly as she describes: "re-arranging/manipulating general formulae, through substitutions and changing the subject of the formula." She is not entirely successful because the valid expressions she finds do not make a workable equation. However, she does demonstrate an admirable self-awareness of her process. Even her response in the "why strategy" column, shows that she looks for key words

and key details to help her solve the problems. Her methods show reflection and evidence of declarative, procedural and conditional knowledge. She seems to be one learner who adopted the training in metacognitive strategies and used them to the best of her abilities.

→ Re-arranging / manipulating general formulae, through, substitutions and changing the subject of the formula

→ look for key words e.g. arithmetic, geometric sequence
→ mention of term number

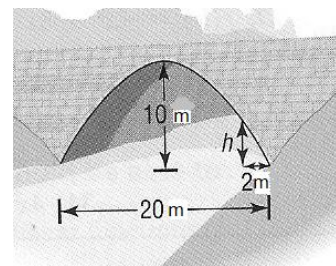
$$\begin{aligned}
 T_{10} &= 3 + (10-1)d & T_{10} &= 3 \cdot r^{10-1} \\
 &= 3 + 9d & &= 3 \cdot r^9 \\
 T_3 &= 3 + (3-1)d & T_3 &= 3 \cdot r^{3-1} \\
 &= 3 + 2d & &= 3 \cdot r^2 \\
 T_6 &= 3 + (6-1)d & T_6 &= 3 \cdot r^{6-1} \\
 &= 3 + 5d & &= 3 \cdot r^5 \\
 \therefore T_6 &= T_{10} \\
 3 + 9d &= 3 \cdot r^9 \\
 1 + 3d &= r^9 \\
 \text{Method 1: } r &= \sqrt[9]{1+3d} \\
 \therefore T_{10} &= 3 \cdot r^9 \\
 &= 3(\sqrt[9]{1+3d})^9 \\
 &= 3[(1+3d)^{1/9}]^9 \\
 &= 3(1+3d)^{9/9} \\
 &= 3(1+140 \cdot 3d^{1/2}) \\
 \therefore T_{10} &= 3 + 420 \cdot 89d^{1/2} \\
 3 + 9d &= 3 + 420 \cdot 89d^{1/2} \\
 9d &= 420 \cdot 89d^{1/2} \\
 \therefore d^{1/2} &= 46, \text{ etc.} \\
 \text{I'm not sure what this means}
 \end{aligned}$$

4.5 QUESTION 4: APPLICATIONS OF QUADRATIC FUNCTIONS

Question 4 from Instrument 1 is reproduced alongside. The question was an application of quadratic functions, and particularly how to set up an equation, given the x-intercepts and/or the turning point of a parabola. The question first of all requires learners to locate the parabola on the Cartesian plane in such a way that they could set up an equation. Once an equation was established, the learners could solve for h .

Question 4 in Instrument 2 was also an application of quadratic functions question; however, this time no picture was given to guide the solution process. The learners needed to translate the given information into an equation in order to solve for the height given a distance from the centre. Drawing a figure could have aided the solution process. There are a number of ways to orient the given information on the Cartesian plane; some orientations make working with the resulting equation easier.

4. A horizontal bridge is in the shape of a parabolic arch. Given the information shown in the figure, what is the height h of the arch 2 m from the shore?



4. A suspension bridge with weight uniformly distributed along its length has twin towers that extend 75 meters above the road surface and are 400 metres apart. The cables are parabolic in shape and are suspended from the tops of the towers. The cables touch the road surface at the centre of the bridge.

Find the height of the cables at a point 100 metres from the centre

4.5.1 Question 4: Mathematical Proficiency

4.5.1.1 Question 4, Instrument 1: Mathematical Proficiency – Trends

For Question 4 in Instrument 1, *conceptual understanding* was needed to translate the picture into a graph on the Cartesian plane using the knowledge of the properties of a parabola, *strategic competence* was needed to set up a meaningful equation, and *procedural fluency* was required for correctly and efficiently working through all the algebraic steps. As there were several ways to set up the original equation, a suggested memorandum is provided alongside.

Unfortunately, this question was the most poorly answered on the first instrument. This could have been due to a number of reasons: it was the last question and learners ran out of time and/or energy; they did not remember the properties or formulae for parabolas; or they could not interpret the question in a way that resulted in a meaningful answer. Only four learners wrote down any response at all and none were able to even begin an equation. So few attempts were made by the learners that I did not code this response, but have only given a verbal description.

One of several possible ways to solve depending on how the quadratic equation is set up:

$$y = a(x - 10)(x + 10)$$

$$\therefore 10 = a(0 - 10)(0 + 10)$$

$$\therefore 10 = a(-100)$$

$$\therefore a = -\frac{1}{10}$$

$$\therefore y = -\frac{1}{10}(x^2 - 100)$$

$$\therefore y = -\frac{1}{10}(8^2 - 100)$$

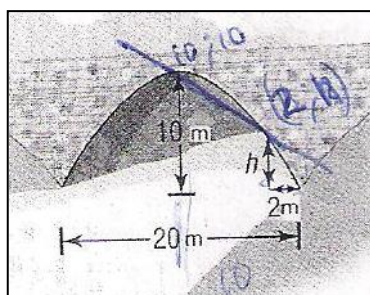
$$\therefore y = 3,6 \text{ m}$$

Figure 4.6 Memorandum for I1.Q4

Learner 3.12 tried to remember a formula for a parabola, but was unsuccessful. This was written in the “what strategy” column. Learner 3.14 started to find the average gradient between the turning point and the point (2, h). Learner 1.9 tried to work with the volume of a rectangular prism – possibly because of the variable h. And lastly, Learner 1.2 concluded that $h = 1 \text{ cm}$ because of ratios. Her solution will be discussed further in the strategy section. All four attempts are illustrated below.

Learner 3.12

Learner 1.2



Learner 3.14

Learner 1.9

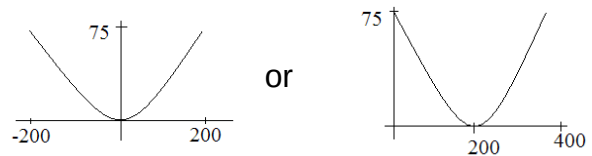
Figure 4.7 Learner Responses to I1.Q4

4.5.1.2 Question 4, Instrument 2: Mathematical Proficiency – Trends

Question 4 in Instrument 2 was also an application of parabolas question but this time no picture was given to guide the solution process. Only one learner attempted a quadratic function and one learner incorrectly used the slope of a line. However, many learners attempted to draw a figure for the scenario and also described their strategy use, so this question will be discussed in more detail in Strategy Use.

For this question, *conceptual understanding* was needed to translate the written information into a graph on the Cartesian plane using the knowledge of the properties of a parabola in order to create a function that models the situation. They would need to know the general shape of the parabola and make use of its symmetry. General knowledge should inform them that a suspension bridge is *u*-shaped and thus, the function would be a minimum-value parabola. This question can be modelled in many ways depending upon where the turning point is placed relative to the origin. The learners would need to connect the height of the cables to the horizontal distance. *Strategic competence* was needed to orient the parabola on the Cartesian plane so that the function was easy to work with. Strategic competence was also needed to create an equation for the function and an equation relating height to distance. *Procedural fluency* was required for correctly and efficiently working through all the algebraic steps. My memorandum answer is reproduced alongside.

One of several possible ways to solve depending on how the quadratic equation is set up:



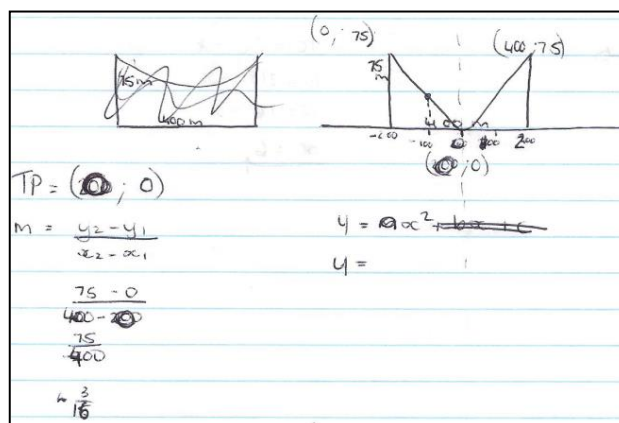
$$\begin{aligned}
 y &= ax^2 \\
 \therefore 75 &= a(200)^2 \\
 \therefore 72 &= 40000a \\
 \therefore a &= 0,00187 \\
 \therefore y &= 0,00187x^2 \\
 \therefore y &= 0,00187(100)^2 \\
 \therefore y &= 18,75m
 \end{aligned}$$

Figure 4.8 Memorandum for I2.Q4

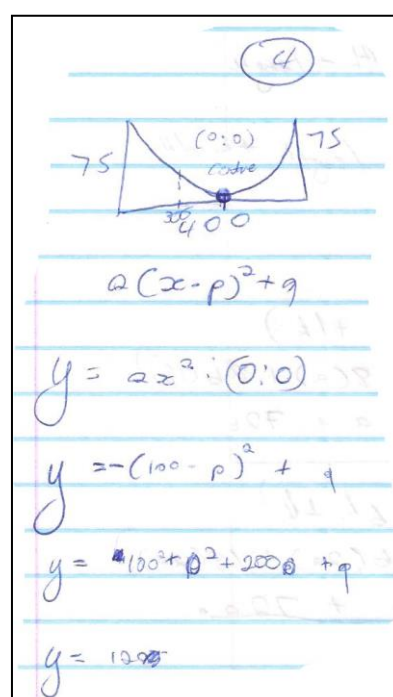
As mentioned above, only two learners attempted any numeric or algebraic response to this question and these will be discussed below. However, it is interesting to note *how* the learners drew the figure, if they were to attempt to mathematically model it.

For this question, 14 learners attempted to draw a figure. Of those 14, one interpreted the question as relating to 2d and 3d trigonometry and drew a triangle. Four learners drew shapes that were not parabolas: one had an upside-down cusp shape in the centre instead of a smooth curve, one was a sharp v-shape, one was rectangular and one had very vaguely curved lines between vertically straight lines. Of the nine parabola shapes, two learners placed them on the Cartesian plane; the others were accurate drawings of the situation described, but no intention to mathematically model the situation was evident. As “*draw a figure*” is considered a heuristic strategy, these will be discussed in the Strategy Use section.

Learner 3.14's solution is reproduced alongside. She is one of the learners to model the parabola on the Cartesian plane and her self-reported strategies will be discussed under the strategy section. She did not get too far with her mathematical modelling, but did know to make the turning point of the parabola at (0, 0) and also knew to use a quadratic function. However, she tried to use the slope formula to get some sort of ratio. This does not make sense for this question and she did not proceed further.



Learner 1.12 is the only other student to attempt to formulate an equation for the function and writes down both the general form and turning point forms for a quadratic function. However, while she seems to indicate that the turning point is at (0, 0), she also seems to employ a horizontal shift (of zero), but with a negative parabola and a vertical shift of q units. She shows no knowledge of the need to solve for a , the vertical dilation and just comes up with an answer that does not follow from her previous working. While she shows some understanding of the mathematics of the situation, her conceptual understanding and procedural fluency seem to be inadequate to continue correctly.



4.5.2 Question 4: Strategy Use

4.5.2.1 Question 4, Instrument 1: Strategy Use – Trends

For Question 4 in Instrument 1, three of the learners who attempted the question tried to use the strategy of drawing a figure and one learner tried to annotate the figure; their attempts are illustrated below. These were categorised as implied strategies.

Learner 3.1 was the most successful in terms of her drawing because she abstracted the parabolic shape to a Cartesian plane almost correctly: she introduced x - and y -axes with the parabola centred over the y -axis. She also indicated the 10 units to the right and left of the y -axis representing where the bridge reaches the water and which could be interpreted as the x -intercepts, although she does not indicate the negative sign of the one x -intercept. However,

she is not able to proceed further. Learner 3.3 redrew the information from the question, but did not introduce axes or additional details. Like Learner 3.1, Learner 3.12 drew the parabola shape on the Cartesian plane with the turning point on the y-axis. She also graphically indicated the line $y = 2$; however, she did not designate the x-intercepts and she was the learner above who tried to remember a formula for a parabola but was unsuccessful; despite her attempts, she was unable to proceed further. Learner 3.14 annotated the picture on the page and interpreted the drawing in terms of the gradient between the point $(2, h)$ and $(10, 10)$. If she mentally set the left-hand point where the bridge touches the water as $(0, 0)$, the x-value of 10 makes sense; however, there is no way to know if this was her intention or if she misinterpreted the value of 10. Interpreted correctly or not, her intention was to use this point to find the average gradient which was not applicable to this problem. In all cases, the drawings were categorised as implied strategies, typed as a heuristic and rated as ineffective.

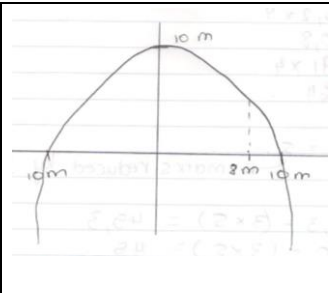
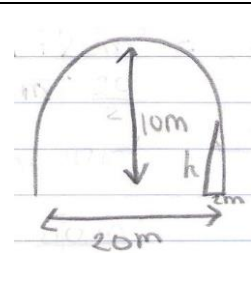
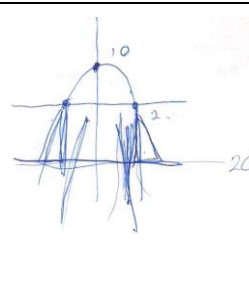
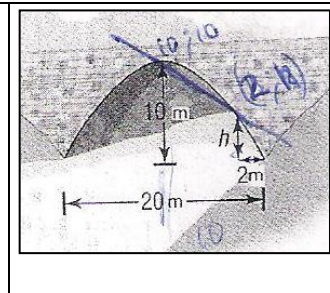
			
Learner 3.1	Learner 3.3	Learner 3.12	Learner 3.14

Table 4.18 Heuristic Strategy Use I1.Q4

Lastly, two learners explicitly described their strategies; their responses are illustrated alongside. Learner 3.7 wrote down, “I drew up a parabola and worked from there” in the “what strategy” column;

I drew up a parabola and worked from there

however, the learner did not turn in a drawing or any further working. She also wrote, “The question mentioned that the arch was in the shape of a parabola so it is bound to have the same characteristics as a parabola”; this was written in the “why” column. She demonstrated that she reflected on the problem, but there is no evidence that she could proceed further. This would fall into the further classification of not match/incorrect.

The question mentioned that the arch was in the shape of a parabola so it is bound to have the same characteristics as a parabola.

Learner 1.2 used an unusual line of reasoning for this question. In answer to “What strategy/ies did you use?” she wrote down, “Ratios.” She explains in the “why column,” “because the height is 10 cm to the breadth of 20 cm the ratio is 1:2 so if the breadth is 2 then the height is 1.” Aside from confusing the units of metres and centimetres, the learner demonstrates understanding of neither ratio nor parabolas. Ratios apply to quantities that are directly or indirectly proportional to each other, which

Because the height is 10 to the breadth of 20 cm the ratio is 1:2 so if the breadth is 2 then the height is 1

are modelled by linear and reciprocal functions, respectively. It is not correct to say $\frac{x}{y} = \frac{x^2}{y^2}$.

For example, $\frac{1}{4} \neq \frac{1}{16}$. However, the learner applies this strategy to the problem. It was categorised as explicit, typed as content-specific and rated as ineffective.

4.5.2.2 Question 4, Instrument 2: Strategy Use – Trends

As with Question 4 in Instrument 1, Question 4 in Instrument 2 was not answered to a great degree. However, a common response was to draw a figure to represent the information given in the problem. As “*auxiliary element – draw a figure*” was one of the heuristics I particularly emphasised in class, I would like to take some time to discuss the drawings.

As background information, it has been my experience as a teacher that learners are loathe to draw figures. Even in problems requiring visual information, such as analytical geometry where all that is given are coordinates and a question to accompany them, many learners will not draw the picture instinctively. When we were working as a class in the Euclidean geometry unit and the three-dimensional trigonometry unit, I stressed over and over the need to either draw a figure or engage with one if it is presented. I was surprised to learn that many learners have difficulty with the three-dimensional trigonometry precisely because they cannot visualise three dimensions or make sense of a diagram that represents three dimensions. However, I did find over time, as I corrected their tests, that there was evidence of more engagement with figures by the learners.

Coding the strategy use on this question was a little different from previous questions. Some of the learners stated “draw a figure” as a strategy and those were coded as declarative knowledge. However, most learners just drew a figure without stating it. These were coded as implied. Because only two learners attempted any algebraic solution to the question, I could not rate the strategy use as effective if the learners solved the problem correctly. This time, the strategy was rated effective if the learners placed a parabola on the Cartesian plane; partially effective if the parabola was the right shape and orientation and had the given information but was not on the Cartesian plane, and ineffective if the drawing did not represent a parabolic shape.

Fifteen learners drew a figure; five were rated ineffective, eight were rated partially effective and the remaining two were rated effective. The learner attempts are grouped below according to their rating and in order of increasing accuracy. Where possible, I have included three from each rating category, working from the weaker interpretations to the stronger ones. I have also provided a brief annotation with each drawing and included the examples of explicit strategy use if the learner provided them. I have included the large number of examples because the variation of interpretation of the question is notable. The remaining examples are included in Appendix O – 2 and the reader is encouraged to view them.

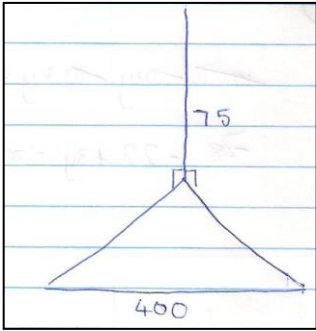
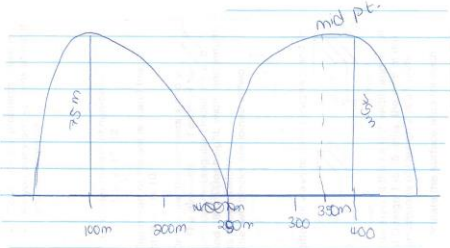
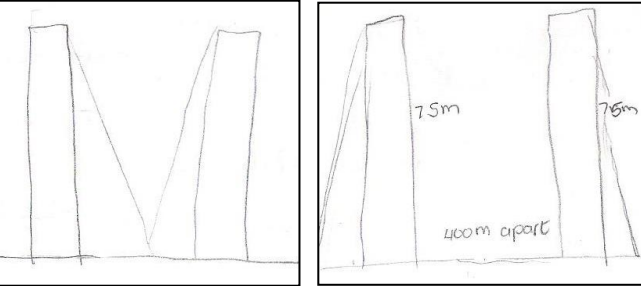
Ineffective Drawings		
LC	Learner drawing	Reported strategy use
3.4		the 3d and 2d equations trig. (in “what” column) I saw height and assumed that it was related to 3d and 2d trig. (in “why” column)
This learner demonstrates declarative and conditional knowledge that is conceptually unsound. She appears to be taking key words out of context. For some reason, she does not focus on the idea of a parabolic shape, but rather the word “height” catches her attention and she proceeds to draw a 3-d triangular space.		
3.1		None
This learner has created a cusp at the turning point instead of a smooth curve; or she has created two upside-down parabolas that meet. She has not interpreted the dimensions correctly.		
3.3		used a picture.
This learner has made the parabola with a sharp v-shape, but has interpreted the height of the towers and the distance between them correctly. She indicates declarative knowledge of her heuristic strategy.		

Table 4.19 Heuristic Strategy Use (Ineffective): I2.Q4

Partially Effective Drawings		
LC	Learner drawing	Reported strategy use
1.8		none
This learner, after two attempts at upside-down parabolas, settles on the correct orientation. She correctly interprets the height of the towers and the distance between them and seems to indicate by a dot that the turning point may be important. She crosses out “(200; y)” at the turning point, indicating she does know about the symmetry of the parabola, but has not interpreted the phrase from the question that the height at that point is zero.		
3.7		I drew out what the question was asking. I think that you had to apply the parabola theorem somewhere,
This learner has correctly interpreted the shape of the cables, the width between the towers and the height of the towers. She adds the additional information that the centre is 200 metres from the first tower. In her self-reported strategy, she states the strategy of drawing out “what the question was asking.” She also thinks that the “parabola theorem” needs to be applied “somewhere,” but does not seem to know how.		
3.10		none
In addition to interpreting the parabolic shape, tower height and distance between towers correctly, this is one of the few learners who correctly interpreted the height of the cables 100 m from the centre. She also made the turning point of her parabola sit at a height of zero metres.		

Table 4.20 Heuristic Strategy Use (Partially effective): I2.Q4

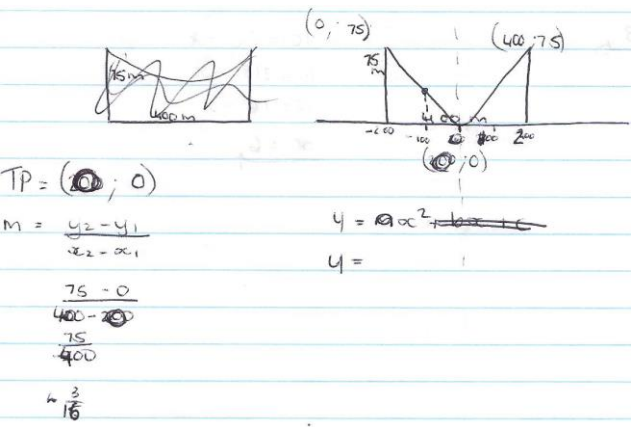
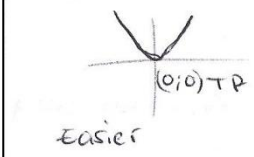
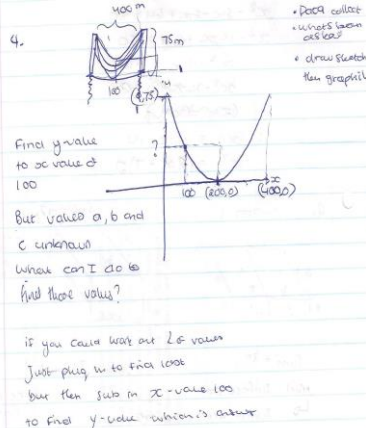
Effective Drawings		
LC	Learner drawing	Reported strategy use
3.14	 TP = (0, 0) $m = \frac{y_2 - y_1}{x_2 - x_1}$ $\frac{75 - 0}{400 - 0}$ $\frac{75}{400}$ $\approx \frac{3}{16}$ $y = ax^2 + bx + c$ $y =$	Tried to bend the equation to how I want it  Easier I set my equation based on the turning point being (0, 0) I thought it would work out nicely.
This learner showed declarative, procedural and conditional knowledge of the strategy she employed. She states that she “tried to bend the equation to how I want it” by basing the turning point at (0, 0) because she thought “it would work out nicely.” This was a well-applied strategy and would have been helpful if she had the conceptual understanding to apply a quadratic equation, which she did start with, but made the mistake of employing the linear slope formula. However, she shows an accurate understanding about the height of the towers and the distance between them. She also shows understanding of how the horizontal distance of 100 m from the centre relates to the height value asked for in the question. She also has a sense of directed numbers, which only one other learner demonstrated. She was one of two learners to place the parabola on the Cartesian plane and her strategy was rated as effective.		
3.15	 Final y-value to x-value at 100 But values a, b and c unknown What can I do to find those values? is you could work out 26 values Just plug in to find 100 but then sub in x-value 100 to find y-value which is answer	None
Learner 3.15, whose work has been discussed previously, has her “bullet point” list that seems to be a variant of “what are the data, what is the unknown, what is the condition?” substrategy. As well, she writes down her thinking process: “Find y-value to x-value at 100 But values a, b and c unknown What can I do to find those values?” By asking that question of herself, she indicates that she realises she is working with a parabola which in general form would be $ax^2 + bx + c$. She also orients her parabola on the Cartesian plane with the turning point marked at (200,0) and marks the y-value of $x = 100$ as “?”. She knows what she is looking for, but not how. However, she demonstrates understanding of the particulars of the question and her strategy was rated effective.		

Table 4.21 Heuristic Strategy Use (Effective): I2.Q4

In the learner responses to this question, there was a noticeable increase in both the quantity and quality of the “*draw a figure*” heuristic over the related question in the previous instrument (see Table 4.22 below). This could be due to the fact that no picture was provided in this question whereas in Question 4 of Instrument 1, a drawing of a parabola-shaped bridge was given. However, both questions required translating the information given either visually or verbally into a mathematical model. In Instrument 1, only four learners made an attempt to do this, with only moderate success with the drawings. In Instrument 2, fifteen learners made the attempt with ten of them demonstrating at least partial effectiveness. As mentioned before, I stressed in class the importance of diagrams; the responses to this question may be an indication that repeated exposure to a heuristic encourages learners to use them.

Instrument 1: Question 4									
Frequency of Strategies by category, code, type and rating									
Category	Explicit						Implied		
Code	Declarative		Procedural		Conditional				Total
Rating/type	H	CS	H	CS	H	CS	H	CS	
E	0	0	0	0	0	0	0	0	0
PE	0	0	0	0	0	0	0	0	0
IE	1	1	0	0	1	1	4	0	8
Total	1	1	0	0	1	1	4	0	8

Instrument 2 – Question 4									
Frequency of Strategies by category, code, type and rating									
Category	Explicit						Implied		
Code	Declarative		Procedural		Conditional				Total
Rating/type	H	CS	H	CS	H	CS	H	CS	
E	2	0	0	0	0	0	2	0	4
PE	2	0	0	0	0	0	8	0	10
IE	4	1	0	0	1	1	5	0	12
Total	8	1	0	0	1	1	15	0	26

Table 4.22 Strategy Use Question 4: I1 and I2 Compared

4.6 RESULTS

4.6.1 Concise summary of trends

The preceding discussion shows the level of detail available to qualitative analysis. I will now, in a concise format, recap the trends from all the questions. These are taken from the higher frequency codes and qualitative evidence in the questions. For brevity, Mathematical Proficiency is abbreviated (MP), Strategy Use is (SU) and the questions are referred to in abbreviated form: I1.Q1 is Instrument 1, Question 1. Instrument 2 headings are in bold to distinguish the instruments. Some trends are italicized because I believe they are important. Declarative, procedural and conditional knowledge is abbreviated D, P and C, respectively.

I1.Q1: (MP)

- Not understanding the *meaning of numbers*.
- Random operations with little intelligible connection to the problem were performed on the given numbers (i.e. *number grabbing*)
- *Applying the same operation(s) to all data values* – a sort of one-size-fits-all approach.

I2.Q1 (MP)

- *Not understanding what the variables represent* in a formula.
- Several examples of *equating unequal quantities* when setting up equations (particularly simultaneous equations)
- The *choice of the conceptual foundation* for answering a question drastically affected the question's outcome.
- Certain *errors permeated all three strands* of conceptual understanding, procedural fluency and strategic competence.
- *Greater sophistication* with solution methods (although often on faulty foundations)

I1.Q1: (SU)

- *A few ($n = 3$) self-report strategies* and demonstrate some declarative, procedural and conditional knowledge
- Self-reported strategies somewhat *simplistic* and telegraphic
- Most strategies content-specific, a few “other”, *no heuristics*

I2.Q1: (SU)

- *Larger number ($n = 13$) of self-reported strategies*, many indicating declarative, procedural and conditional knowledge or at least two of the three
- Self-reported strategies *more detailed and specific*
- Evidence of *heuristic strategy use as well as content-specific strategies*
- Phenomenon of *strategy use not matching working-out*
- Three learners had strategies that matched their working out *and* achieved a partially-effective or effective rating on their mathematical working

I1.Q2: (MP)

- Learners had points where they “*derailed*” off solution pathway
- The *first step* – setting up the initial equations correctly – was the *point of highest “derailment”*
- *Confusion between dimensions and area*
- Some learners *continued to solve problem*, albeit on faulty foundation
- Question illustrated *co-dependence of strands*

I2.Q2: (MP)

- Not finding values of dimensions (*a* and *b*) first *affected all three strands*.
- Many learners *started with surface area formula* (perhaps nudged by presence of formula sheet)
- *Difficulty identifying separate areas* for figures ($n = 6$)
- *Many procedural errors*
- “*Circular*” *substitution* (occurred in other problems as well)

I1.Q2: (SU)

- *Eight learners self-reported strategies*, 7 of whom wrote in all three columns indicating declarative, procedural and conditional knowledge
- *16 learners used implied heuristic* of drawing a figure; 2 stated drawing a figure explicitly
- *Drawings varied* from effective ($n = 2$) to partially effective ($n = 8$) to ineffective ($n = 6$)
- 4 learners indicated use of the *content-specific strategy* of simultaneous equations
- *Strategy use was mostly partially effective or ineffective* with regard to mathematical working; can indicate learners know *what* strategies to use, but not *how*.

I2.Q2: (SU)

- *Thirteen learners self-reported strategies*, 3 demonstrated D, P, and C knowledge. Four learners showed 2 of 3 kinds of knowledge.
- 3 learners particularly showed an *engagement between their strategy use and working out*
- 5 learners utilised heuristic strategy (*draw a figure*), 7 utilised content-specific strategies

I1.Q3: (MP)

- Learners *recognised linear pattern* but *not necessarily periodicity of the tangent* values.
- Many learners *used arithmetic progression correctly*
- *None correctly reduced* $\tan 1982^\circ$ to $\tan 2^\circ$

I1.Q3: (MP)

- *Little or no progress* ($n = 17$) highest frequency code
- *A few* ($n = 5$) *able to make correct expressions* for sequence terms before solving
- *Most* ($n = 13$) *not able to make correct expressions* for sequence terms before solving
- Several ($n = 8$) recognized need for arithmetic progression formula; *few* ($n = 4$) *substituted-in correctly*

I1.Q3: (SU)

- *10 learners explicitly stated their strategy use*
- *3 learners showed D, P and C knowledge*, 3 learners showed 2 of the three kinds of knowledge and 4 reported 1 of the three
- 2 learners mention *heuristic strategy*, 8 mention *content-specific*

I2.Q3: (SU)

- *9 students self-reported strategy use*; 3 learners described D, P and C knowledge, two learners reported 2 of the 3 types of knowledge, and 4 reported 1 of the three
- *7 out of the 9 learners' working matched their strategy use*, with 4 of those being correct.
- 2 learners' strategy use *did not match their working* because they made no progress
- 3 learners mention *heuristic strategies*, 4 mention *content-specific*, and 1 is a mixture of both

I1.Q4: (MP)

- 4 attempts at some response
- No attempts at equation

I2.Q4: (MP)

- 2 numeric or algebraic attempts

I1.Q4: (SU)

- 3 attempts at re-drawing figure: 2 on Cartesian plane (1 accurate, 1 not), 1 sketch
- 1 annotated figure (incorrectly)

I2.Q4: (SU)

- 15 learners used “draw a figure” as a heuristic; effective ($n = 2$), partially effective ($n = 8$), and ineffective ($n = 5$)
- 5 learners self-reported “draw a figure” heuristic; for 10 learners it was implied
- 1 (incorrect) content-specific strategy

4.6.2 Results

In my conceptual framework, I related strategy use to three strands of mathematical proficiency; I also illustrated how the description of each strand reveals it to be composed of declarative, procedural and conditional metacognitive knowledge. I gave examples of heuristics and strategies that could be used to support the three strands by helping to conceptualise the problem (conceptual understanding), set up a mathematical model of the problem (strategic competence) and work through the problem (procedural fluency). This conceptual framework was developed many months before I analysed the learners’ responses in Instrument 2.

The preceding section of this paper discussed at length the learners’ responses in terms of mathematical proficiency and strategy use. My initial analysis focused on the question-by-question responses and discussed each question as a single set of data for which I discovered trends. I had to go through that lengthy process in order to refine my focus on what was important. Also, the question-by-question analysis enabled me to see trends across questions as a whole in both mathematical proficiency and strategy use.

In order to discuss the results, I need to restate my original research questions. To distinguish the research questions from questions in the Instruments, I will call them Research Question 1, etc.

Research Question 1:

Is there evidence that when given metacognitive skills training, students use those skills in an individual, written mathematical assessment? What is the nature of that evidence?

Research Question 2:

To what extent do learners utilize metacognitive knowledge, specifically strategies, if given metacognitive training in a regular, high school classroom?

Research Question 3:

If the metacognitive skills are used, do these skills affect the students' mathematical proficiency in any way?

Research Questions 1 and 2 need to be answered together because the *evidence* (Research Question 1) was in the form of *strategy use* (Research Question 2). In the second instrument, there was evidence, in the form of explicit, self-reported strategies and strategies that could be implied from the learners' responses, that the metacognitive skill of strategy use was employed after a period of strategy instruction. However, the learners also self-reported strategies *before* the intervention. This could be for several reasons. It may be that if asked a question about strategy use without any training, a person will give a common-sense response. It is also possible that some of the learners, who I also taught in Grade 11 and gave a brief introduction to metacognitive self-questioning, utilised that information the following year. That being said, I still believe that the evidence shows there was both, in general, a greater number of responses and an increase in quality of the strategy self-reports in the second instrument.

The nature of that evidence was in the form of self-reports of strategy use as well as strategy use that could be inferred from the learners' working. With regard to the self-reported strategies, one aspect that generally increased after instruction was the ability of many learners to state quite articulately *what* strategy they used (declarative knowledge), *how* they used that strategy (procedural knowledge) and *why* they used that strategy (conditional knowledge). The interesting fact about this is that while I modelled my thinking process as I worked through problems, I did not explicitly teach the declarative, procedural and conditional aspects of the strategies – I quite frankly did not have time. I mostly explained what strategy I used, and I may have sometimes explained “I am using this because”, but had little time to go into excessive detail. In other words, the learners “picked up” the *what*, the *how* and the *why* quite naturally, and more importantly, were able to distinguish the difference between them in their self-reported strategies. This can be seen as evidence of metacognition.

Another difference between the first and second instruments which can be taken as evidence was, in general, the greater mention and use of heuristic strategies, particularly “*work backwards*” and “*draw a figure*.” In Instrument 2, there were definite signs of refinement of “drawing a figure.” A comparison between Question 4 on the two Instruments shows this: on Instrument 1, where the figure is given in context, only three learners attempted to redraw the figure in a “mathematised” way (Gravemeijer & Terwel, 2000), that is, on the Cartesian plane and making use of the properties of the parabola. On Instrument 2, where there was only a description of the situation, fifteen learners attempted to draw the figure. Of those, two managed to interpret the situation effectively and eight partially effectively. However, just the increase in the number of attempts shows more applications of the heuristic.

Thus, I do think the answer to Research Question 1 was that evidence did exist to show that learners did utilise metacognitive skills after a period of training. My only hesitation is the aspect of *individual, written mathematical assessment*. On the one hand, the assessments were individual, written and mathematical, but the intent behind that question was to see if students

utilised the strategies *naturally*. Several aspects come into play here which need to be discussed. I have mentioned earlier in my analysis this aspect of what I call “nudging,” that is, the inadvertent prompting of a direction. I have not come across literature on this, but that may be due to my lack of knowledge of the field. I mentioned the idea with regard to Question 2 on Instrument 2 where the learners were given a formula for surface area and many used that, almost mechanically (and without success). It seemed as if the very presence of the formula “nudged” them in a certain direction – and, anecdotally, I have definitely seen this before in my working with students. My concern is that the very presence of the strategy use columns on the instrument “nudged” the learners into both using and reporting strategies; if the columns were not there, the learners may not have thought about strategies at all. This is also borne out by the example, given in the Method chapter, where I asked learners to write down their strategies when completing trigonometric equations and none could. In that case, there were no columns, no nudging and no strategy self-reports.

That being said, two learners stand out who seemed to significantly internalise strategy use and metacognition: Learners 3.8 and 3.15. Learner 3.8’s solution to Question 2 on Instrument 2 was discussed in detail and showed clear evidence of reflection. Learner 3.15, on every solution in Instrument 2, used her strategy use “bullet points” to guide her solution process. In the end, this may be evidence of more *natural* strategy use.

The answer to Research Question 2, the *extent* to which learners utilise strategies, is less straightforward and relates to Research Question 3 because, if Pólya defines heuristics as the “mental operations typically useful for the solution of problem” [sic] (1957, p.2), strategy use should improve mathematical proficiency. Unfortunately, the evidence from this research is not conclusive on that point. I will discuss this aspect in light of my research and then discuss the possible reasons for this in the concluding chapter.

As I began to analyse Instrument 2 and particularly focused on the relationship between strategy use and mathematical working, I began to see further relationships in the data. Sometimes the data really surprised me. For example, I never expected the learners to misinterpret the data set in I2 Q1 as an arithmetic sequence; nor did I expect learners to write one strategy but perform completely different mathematical operations. From those results, I developed two-way contingency tables to describe the data. I described strategy use from what I called “matches working/doesn’t match working”, which was abbreviated to “match/not match”; and the mathematical proficiency as “correct/incorrect.” By “correct,” I included answers that were coded as “+/-“ as well as “+”, and “incorrect” included “-“ ratings.

When I tallied those responses, comparing the responses in Instrument 1 to those in Instrument 2, question by question, new patterns emerged and these are highlighted now. The tallies for each question are given below. The numbers are much smaller than given in the analysis section because they express each learner’s response as a whole: they are not divided up into declarative, procedural and conditional aspects. I consider the match/correct cells as the answer to Research Question 3. Question 4 was not included in the tables because so few learners attempted significant responses beyond drawing a figure.

Results Tables

Instrument: 1 Question: 1			Instrument: 1 Question: 2			Instrument: 1 Question: 3		
Strategy use			Strategy use			Strategy use		
Mathematical working	Match	Not match	Mathematical working	Match	Not match	Mathematical working	Match	Not match
Correct	2	0	Correct	6	0	Correct	7	0
Incorrect	2	0	Incorrect	2	0	Incorrect	3	0
Total	4	0	Total	8	0	Total	10	0

Instrument: 2 Question: 1			Instrument: 2 Question: 2			Instrument: 2 Question: 3		
Strategy use			Strategy use			Strategy use		
Mathematical working	Match	Not match	Mathematical working	Match	Not match	Mathematical working	Match	Not match
Correct	3	4	Correct	7	0	Correct	4	0
Incorrect	3	3	Incorrect	5	0	Incorrect	3	2
Total	6	7	Total	12	1	Total	7	2

Figure 4.9 Two-way contingency tables connecting mathematical working and strategy use

From the numbers in the tables as well as the qualitative analysis, I cannot state unequivocally that the strategy use affected the mathematical proficiency. As mentioned earlier, the learners generally demonstrated more complexity in their mathematical responses in Instrument 2 than Instrument 1, but that could have been due to a number of factors (e.g. maturity, six months of practice on school problems, extra tutoring, etc.). The fact that there was even one instance of strategy use not matching the working out indicates a disconnection between the strategies and the method of solution. Also, there were learners (e.g. Learner 3.9 in I2.Q1 and 3.12 in I2.Q3) who achieved very accurate solutions with little or no reported strategy use. Even a learner such as Learner 3.15 who seemed to internalise the strategy use process, did not perform that well mathematically. The possible reasons for this result are discussed in the next chapter.

CHAPTER 5: CONCLUSION

In the end, the data presented quite a puzzle and raised more questions than it answered. On the one hand, there seemed to be evidence of the increase in strategy use after instruction: both the quantity and quality generally increased. Also, a few students appeared to ‘take up’ the idea of metacognition and seemed either to reflect more actively on their working (Learner 3.8) or structure their responses with strategies in mind (Learner 3.15). But two big questions remained: 1) Did the strategy use improve the mathematical proficiency of the learners? and 2) How could students state one strategy but clearly not employ that strategy when working out their mathematical problems? These two ideas are related and will be discussed, amongst others, in this chapter.

An important trend displayed by the responses to several questions in the Instruments can be likened to learners being at a crossroads and choosing which path to follow. The learners tend to display one of three choices: they do not know which way to “go” and do nothing; they proceed down an incorrect “path,” or they proceed down the correct “path”. As learners stand at the crossroad of a problem, trying to figure out which way to “go,” they rely most heavily on their conceptual understanding of the structure of the problem and their strategic competence to turn the content of the problem into a mathematically workable model. If their conceptual understanding is sound and if their strategic competence is strong enough, they can start with an appropriate mathematical model and begin the steps down the correct path. How far they continue often depends upon their procedural fluency and their underlying conceptual understanding of the procedures. However, as was demonstrated by Question 2 in Instrument 1, opportunities for “derailment” can occur at almost every step.

While the foregoing may seem a somewhat fanciful interpretation of learner thinking, cognitive science research supports this idea of multiple decision pathways. Schoenfeld (1987b) reproduced a diagram (citing Brown and Burton, 1978) detailing the mental steps needed to solve a subtraction problem. The diagram is reproduced on the next page (see Figure 5.1). The complexity of the thinking process is made evident in the drawing. The computer program that models working through the subtraction algorithm is similarly complex (Schoenfeld, 1987b). Cognitive process analyses are even more detailed when they examine all the possible places “something can go wrong when a student tries to implement the procedures and trace the possible consequences of such mishaps” (Schoenfeld, 1987b). Even my simple diagram of “points of derailment” from Question 2 of Instrument 1 demonstrated this. From the complexity of the diagram illustrated below can be inferred the complexity of thinking behind even what is generally thought to be a straightforward algorithm. Thus, for complicated problems, such as those presented in both instruments, the number of decisions a learner needed to make was substantial. Recognising the complexity of thinking involved to correctly answer the questions in the instruments may help to explain why an improvement in mathematical proficiency was not generally observed. Exploring the idea of utilising metacognition and mathematical proficiency as a means of negotiating multiple decision pathways could be a source for future research.

If I reflect on my underlying intention for conducting this research, it was to help learners to make the *right* choices. In some respects, I think there was progress: learners did show evidence of some strategy use and in some respects their mathematical proficiency may have improved. However, the two may not be connected. More importantly, though, I do not know if introducing the strategies actually helped the learners to make the right choices and this could be due to a number of factors.

The first factor has to do with the metacognitive skill of *regulation of cognition* (Brown, 1978) or *control* (i.e. allocation of resources) as Schoenfeld (1985) calls it. Schoenfeld (1985) described four different types of control decisions and how they would affect problem solving. In “Type A”, “bad decisions guarantee failure” with problem solvers going on “wild goose chases” that waste resources while ignoring “potentially helpful directions” (Schoenfeld, 1985, p.116). This was very evident in I2.Q1 where the students saw the data set as a pattern and proceeded to find the general term. “Type B” decisions are somewhat neutral; while “wild goose chases are curtailed before they cause disasters,” the problem solvers do not exploit resources available to them (Schoenfeld, 1985, p.116). An example of this is learners who used the surface area formula in I2.Q2 but did not utilise the information about the dimensions. For “Type C” behaviour, “control decisions are a positive force in a solution; the problem-solver chooses resources carefully, continuing with those that are productive and abandoning others after careful consideration” (Schoenfeld, 1985, p.116). Learner 3.8 demonstrated this control functioning in her solution to I2.Q2: she reflected on her solution as she proceeded and made the necessary changes. This was in marked contrast to her solution in I1.Q2 where she tried to find the area of the patio and ended up with an unsolvable, excessively complicated cubic equation. Finally, in “Type D” behaviour, control decisions are not needed at all because the appropriate information and procedures are readily accessed from long-term memory (Schoenfeld, 1985). This may have been the situation with Learner 3.9 who worked accurately through I2.Q1 with little recourse to strategies.

It may be that strategy use instruction without concomitant control function instruction may render the strategy almost useless and may have indicated a limitation in the research process. Two examples from the literature support this. In the various studies of Mevarech, Kramarski, and Fridkin, (1997, 1999, 2003, 2006, 2008) the authors reported repeated success with training learners in metacognition and its resultant improvement in mathematical problem-solving skills. They did have a very comprehensive “package” in their IMPROVE system, any element of which might have improved their learners’ abilities. However, from the beginning (Mevarech & Kramarski, 1997) the authors emphasised the need for control and built it into their instruction 1) by using small, cooperative group interactions so that the students regulated each other, and 2) by including reflection questions as part of their self-questioning repertoire. (It may be noted that I had intended to include reflection questions, but did not have time during my research period). Schoenfeld also found that firstly, when students worked on college-level, algorithm-based exercises on integration using a “prescriptive control strategy” (Schoenfeld, 1985, p.116) that the students made significant improvements in their performance. In addition, he found that students’ problem-solving performance also significantly improved when trained

in the use of heuristic techniques, *if* that training was accompanied by a prescriptive control strategy.

The lack of training in control functioning may explain why in my research students demonstrated *knowledge* of cognition, and even declarative, procedural and conditional knowledge, but sometimes did not *implement* that knowledge in their working out or implement it effectively. Perhaps more importantly, this may mean that while metacognition as a *construct* can be divided into knowledge of cognition and regulation of cognition, in order to be *effective*, *instruction in* and *use of the two* have to occur together. The constructs may also be more tightly interwoven than descriptions of separation would indicate.

Another reason why the strategy instruction may not have been as effective as I wanted, may be due to the complexity of strategies themselves and the need for detailed instruction. Schoenfeld claimed that strategies such as those given by Pólya are actually “labels” for categories of “more precisely defined strategies” that need to be fully explicated before they can be useful (Schoenfeld, 1985, p.73). He also stated that training in the use of strategies is much more complex than it originally appears because using any particular strategy actually means implementing a number of sub-phases which in turn require their own training *to the same degree that is needed for standard subject content* (my emphasis) (Schoenfeld, 1985). In a regular classroom, where proceeding through standard subject content itself is paramount, particularly for matriculants, it is very possible that I did not go into sufficient depth with the strategy instruction.

A further reason why the strategy instruction may not have been as effective as possible lies with the learners’ level of content understanding. This has two aspects. Firstly, the results from Mevarech and Kramarski (1997) showed that the middle- to high-achieving IMPROVE students outperformed their control counterparts, but not the low achieving students. As mentioned earlier, many of these learners were not strong students in mathematics: coming out of Grade 11, one class had averaged 46,7% and the other class, 36,2%. The one class in particular is markedly below the Grade 11 average of 51,5%. It is very possible that introducing strategy use to low achieving students does not help them and may even make the subject content even more confusing. Two learner comments on the second instrument were notable. Learner 3.2 did not attempt the questions and wrote, “I need to study a section before I answer the questions. I really can’t remember what to do and how to do it.” While I assumed that the content (most of which was from Grade 11 and before) was “prior learning,” for this particular learner, it was not. Another learner, perhaps even more poignantly wrote,

“I am not a very good maths student and usually prefer to study a section
I am being tested on before I write an assessment/test. I found this task
difficult as I could not recall what steps to follow to get the answer. I usually
don’t follow the “strategies” table, as I find it complicates my thoughts [sic]
process”.

This learner, like Learner 3.2, needs advance preparation in order to achieve and also finds the strategy use burdensome.

In addition, sound content knowledge is a key element to strategy use in particular and metacognition in general. Veenman and Van Hout-Wolters (2006) claim that one has to have substantial domain-specific cognitive knowledge in order to have metacognitive knowledge about one's competency in the domain. This is echoed by Schoenfeld (1985) who states that knowledge of heuristics cannot replace subject-matter knowledge and in fact, the implementation of the heuristics relies heavily on accurate domain-specific understanding. In other words, just as heuristic use is dependent upon the control strategies that guide the resources, it is also dependent upon the resources available to the problem-solver.

Another factor in effective strategy use is time. I have touched on time in the classroom, but the problem is broader than that. Veenman and Van Hout-Wolters (2006) claim that research shows three principles that need to be followed for effective instruction in metacognition: a) metacognition needs to be embedded in the content to ensure connectivity; b) learners need to be informed about the usefulness of metacognition to motivate them to make the effort to use it, and c) training needs to be prolonged to guarantee sustained application of the metacognitive activity. One possible reason the original Mevarech and Kramarski (1997) study was effective was that the IMPROVE system had been in place in the treatment district for two to three years. This gave everyone, teachers and students, time to refine the techniques. Trying to fit effective strategy use into a few months was probably unrealistic; given another chance, I would extend the study over at least a year.

However, another possibility for the perceived lack of improvement in mathematical proficiency may lie in the description of the construct itself. When I was first introduced to the idea of the strands of mathematical proficiency as an honours student, I had one of those “aha” moments. This model seemed to explain all the questions I had about why students could not grasp mathematics easily or quickly: they needed to develop each of the strands individually and collectively and this would take time and practice. It also resonated with me because this model was the first I had come across that explicated the complexity of learning mathematics by dividing the competency into five distinguishable strands that could each be developed from novice to expert level.

I still believe in the strands as *one way* to conceptualise mathematical proficiency, particularly through high school. However, I now have some problems seeing the strands as a complete description of mathematical proficiency. Firstly, as indicated in my conceptual framework, appropriate use of the knowledge required by each strand requires metacognitive awareness and this is not explicitly stated in the description of the strands; the very language used to describe competency in the strands is the language used to describe metacognitive skills or knowledge. Secondly, as students get older and as their mathematical repertoire grows, I think they need further skills to define and tackle problems. It is almost as if the learners' “toolbox” has gotten so large and cluttered that they cannot find the right tool at the right time. Training in metacognitive strategies and executive control could help with this. However, this comes

with its own caveat: training in the use of heuristics is a study of its own and may be too complicated to include in a regular classroom. Schoenfeld (1985) was able to explore heuristics by a) using college students, b) using students that had previously demonstrated competence in mathematics, and c) by having a class that focused only on problem-solving; he was not trying to teach content, but trying to teach effective use of content already obtained. This was very different from my situation.

The language used to describe the strands implies metacognition; Schoenfeld's (1985) conceptualisation of mathematical proficiency makes the need for metacognition explicit. In his framework, *control* and *heuristics*, which can be interpreted as metacognitive skills and knowledge, respectively, form half of his framework. Schoenfeld's (1985) *beliefs* are analogous to the strand of productive disposition, and *resources* could be comprised of conceptual understanding, strategic competence, procedural fluency and adaptive reasoning, as well as a number of other capabilities or even experiences the learner could use to his advantage. A possible weakness in the Fitzpatrick et al. (2001) conceptualisation of mathematical proficiency may be the absence of the explicitly stated importance of metacognition. Thus, Schoenfeld's (1985) conceptualisation of mathematical proficiency may be the better of the two. In my situation, the lack of control and resources may explain some of the more ambiguous results in my research.

In light of a number of ideas mentioned above, the following questions could be sources for further research:

- Does strategy use improve for high school learners if they are also given training in control strategies?
- Does strategy use differ amongst learners with different ability levels?
- To what "grain size" do strategies need to be explained in order to be effective?
- How can students move from "Type A" strategy use to "Type D"?
- At what stages of a solution process do "points of derailment" occur? How can students learn to be aware of them?
- What resources provided to a student "nudge" them in a certain direction when solving problems? Can strategies be employed by students to "nudge" them in a productive direction?

In conclusion, from working with the learners' responses to unseen-type questions, I have begun to see the solution of mathematical problems as analogous to reaching a destination by negotiating complicated pathways. Every problem represents a crossroad of possible paths to follow or a set of choices to be made with consequences at every step. As in trailblazing, the first step down a pathway is probably the most important: a wrong choice and the person will not arrive at their destination, or will face more than necessary obstacles. However, every step along a mathematical pathway represents a choice. And that choice is dependent upon the conceptual understanding, procedural fluency and strategic competence the learner has for that particular question. However, to support, prompt, monitor and direct/guide those choices, the learner also needs metacognitive knowledge and skills at every step.

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