F. R. N. 10

Reprinted from the
Physical Society Bristol Conference Report, p. 1, 1948

DISLOCATION THEORY AND TRANSIENT CREEP

By N. F. Mott and F. R. N. Nabarro
Bristol

§1. Properties of Dislocations

The purpose of this paper is to put forward certain advances in the theory of dislocations, and in particular to discuss their application to the theory of transient creep, in the sense in which the term is used by Andrade (1911, 1914, 1932) and by Orowan (1947).

It is now usual to base any theory of the strength of solids on the assumption that a *perfect* crystal is very strong, and that it will not slip unless subjected to a shear strain of the order of 10°. A stress giving a shear of this magnitude is necessary in order to make one plane of atoms slide simultaneously over the one below it. This may be shown as follows. Consider two adjacent planes of atoms in the solid distant h apart. Then the force per unit area necessary to move one plane a distance x relative to the next is $\mu x/h$, if x is small. μ is here the shear modulus. But after displacing them by a certain distance a, a new position of equilibrium is reached. Thus for the force for displacements x that are not small we may write approximately

$$F=(\mu a/2\pi h)\sin(2\pi x/a), \qquad \ldots\ldots(1)$$

provided the crystal does not approach another equilibrium configuration during the deformation. This force has a maximum value $\mu a/2\pi h$, which is equal to the shear stress for simultaneous slip. For two close-packed planes of a f.c.c. or hexagonal close-packed structure sheared in the twinning direction, $h=\sqrt{2}a$. Thus the stress for simultaneous slip is $\mu/2\sqrt{2}\pi$ and the strain is $\tan^{-1}(1/4\sqrt{2})=10°$. This assumption of a sine curve in (1) probably gives slightly too large a value for the maximum stress. Moreover, as Zener (1947) has pointed out, the face-centred lattice passes through the body-centred configuration during this shear, and this may be a structure in stable mechanical equilibrium. This would reduce the theoretical yield strength by a factor of about two.

In the hardest technical alloys the ratio between Y (the yield stress) and μ, i.e. the shear strain at which slip begins, is of the order 0·02, as table 1 shows. In

Table 1

	Shear modulus μ (dynes/cm²)	Elastic limit Y (dynes/cm²)	μ/Y
Pure Al	$2{\cdot}5\times10^{11}$	$2{\cdot}6\times10^{8}$	900
Commercial drawn Al	*c.* $2{\cdot}5\times10^{11}$	$9{\cdot}9\times10^{8}$	250
Duralumin	*c.* $2{\cdot}5\times10^{11}$	$3{\cdot}6\times10^{9}$	70
Special Al with 10% alloys	*c.* $2{\cdot}5\times10^{11}$	$5{\cdot}4\times10^{9}$	45
Single crystal of Al	$2{\cdot}5\times10^{11}$	4×10^{6}	60,000
Single crystal of Sn	$1{\cdot}9\times10^{11}$	$1{\cdot}3\times10^{7}$	15,000
Single crystal of Ag	$2{\cdot}8\times10^{11}$	6×10^{6}	45,000
Soft iron	$7{\cdot}7\times10^{11}$	$1{\cdot}5\times10^{9}$	500
Heat-treated carbon steel	*c.* 8×10^{11}	$6{\cdot}5\times10^{9}$	120
Nickel chrome steel	*c.* 8×10^{11}	$1{\cdot}2\times10^{10}$	65

single crystals it is very much less. In most modern theories of slip, it is assumed that slip begins at one end of the crystal and travels across it, thus avoiding the concept of simultaneous slip. The discontinuity in a crystal when slip has travelled part of the way across it is called a "dislocation".

The concept of dislocations was introduced into physics by Prandtl (1928) and Dehlinger (1929) and applied first to explain the deformation of crystals by Taylor (1934), Orowan (1934) and Polanyi (1934). It has now become fairly familiar, and the mathematical theory has been developed in detail by Burgers (1939) and Koehler (1941). From the point of view of this paper a "dislocation of edge type" (Burgers) is the discontinuity which exists near A if that part of an initially perfect crystal which lies above TA slips one atomic distance along the glide plane TT′, while there is no slip over the remaining portion A′T′ of the glide plane. We shall recapitulate here some of the properties of dislocations of edge type.

(*a*) A dislocation is a *line* discontinuity, extending perpendicular to the paper from one boundary of the crystal to the other; it is in no way similar for instance to a vacant lattice point.

(*b*) Dislocations on one plane can be of two kinds, positive or negative (figures 1 (*a*) and (*b*)).

(*c*) The material just above a positive dislocation is in compression, that below in extension.

(*d*) As first pointed out by Taylor (1934), dislocations of unlike signs attract and those of like signs repel each other.

(*e*) The axis of the dislocation need not extend in an unbroken straight line perpendicular to the plane of the paper, but may be broken into segments, the ends of which are joined by "screw dislocations" also lying in the glide plane. Figure 2 shows the position of the dislocation of figure 1 in the glide plane through TT′ perpendicular to the plane of figure 1.

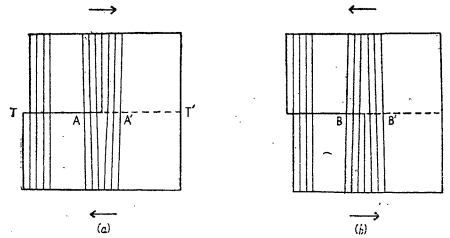

Figure 1. Positive and negative dislocations. The dotted lines are planes of atoms.

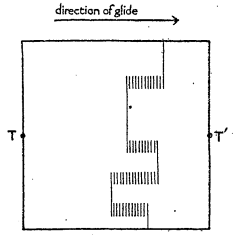

Figure 2. A single dislocation with some parts of edge and some of screw type. The slip plane is the plane of the paper.

The theory of cohesion in solids is sufficiently far advanced to enable one to estimate with some certainty how dislocations should behave. Results of calculations show :

(*a*) That a single dislocation in an otherwise stress-free solid will move under an applied stress very much less than the shear modulus. A theoretical estimate (Nabarro 1947) suggests that this critical stress will be about $\mu/2000$, which is much greater than the experimentally determined strength of some single

crystals. The theoretical estimate depends critically (through an exponential) on the details of the model used and on the calculated width of a dislocation (AA′ in figure 1) which is deduced from the model. The greater AA′ is, the smaller is the critical stress. It seems certain that the model used in the calculations (a simple cubic lattice) gives too small a value of AA′. The true value is probably two or three times that estimated, and the corresponding critical stresses are of the order of $10^{-7}\mu$ and $10^{-10}\mu$. The calculations of §3 suggest that, when this stress is very much less than μ, the line of a dislocation is not bound to those positions in the lattice where the energy of misfit is smallest. Shockley (1947) considers that the external stress required to move the dislocation is reduced by an averaging factor similar to that employed in §6 to calculate the yield strength of solid solutions, because not all parts of the dislocation have to be lifted simultaneously from potential troughs. If the calculated stress is not itself very small (as might occur in a crystal held by directed bonds), the line of the dislocation is bound to preferred positions, and the averaging process does not occur.

(*b*) The energy of a dislocation is of the order 1–5 ev. *per atomic plane*, and thus, for a crystal of width 0·1 mm., of the order 10^7 ev. It is thus out of the question that dislocations exist as a result of thermal equilibrium at any temperature at which the substance is solid.

(*c*) As already stated, dislocations are not necessarily straight; if the crystal in figure 1 is rotated about TT′ it might appear as in figure 2; but, since the energy increases with the length, the dislocation may be thought of as having a tension of the order μa^2 which tends to straighten it out, where a is the interatomic distance.

(*d*) An external shear stress σ tending to cause the crystal to slip along the glide plane is equivalent to a force σa in the direction of slip acting on unit length of the dislocation. This may be seen as follows. Suppose that the crystal of figure 1 is a cube of side L. Then if one dislocation moves across the crystal, the strain produced is a/L and the work done $\sigma a L^2$. The motion of a dislocation through a distance L results in a change of energy $\sigma a L^2$; hence the force on it is $\sigma a L$, or σa per unit length.

§2. CAUSE OF RESISTANCE TO FLOW

If, as seems highly probable, slip in crystalline solids takes place through the motion of dislocations, then resistance to slip must be due *either* to the difficulty in forming them *or* to the difficulty in moving them.

The problem of their formation is at present the furthest from solution in the whole theory. The extensions obtained in normal engineering tests are of the order of 50%, corresponding to the motion of one dislocation across each atomic plane. With a specimen of cross-section about 1 cm² this represents 10^8 dislocations per cm², and it would not be unreasonable to assume that such a concentration of dislocations was already present in the metal. Evidence from the extinction and breadth of x-ray lines, in conjunction with observations of surface markings after various types of treatment, suggests that an annealed single crystal is composed of blocks some 5000 A. on a side tilted with respect to one another by angles of the order of 10–15 minutes of arc. This tilt corresponds to the presence of dislocations in the boundaries of the blocks separated by about 300 atoms, giving a density of dislocations of about 10^{10} cm^{-2}. It is

doubtful if this density of dislocations could account for the very large shears observed in slip-bands in single crystals at high temperatures.

It seems highly probable, as will be seen below, that one dislocation, once it starts moving, generates others. Frank (1948) has recently suggested a possible mechanism for this process.

The experimental facts themselves indicate that the difficulty of forming dislocations is not usually the factor which determines strength. Deformed, cold-worked crystals are usually harder than annealed ones, and internal strains should help with the formation of dislocations but hinder their motion. Thus we may attempt to explain yield points and perhaps creep in terms of the stress required to set in motion a small number of dislocations which are already there.

Taylor first introduced the idea that the yield point of a crystal is determined by internal strains. Figure 3 (*a*) represents a cubic crystal containing random internal strains. All the lines are supposed to be drawn along, say, (100) planes, and the small parallelograms show the local distortions. Slip cannot take place along the line AB until an external strain is applied which is so great that all internal strains have the same sign (figure 3 (*b*)). From this concept it follows that if Y is the stress at which yield along AB occurs and σ_i the maximum internal shear stress at any point on AB, then $Y = \sigma_i$.

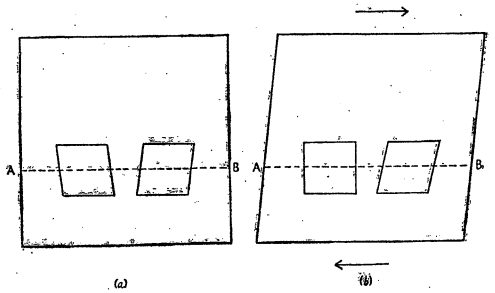

Figure 3. Internal strain in metals.

This model leads us to introduce the concept of the potential energy $V(x)$ of a dislocation, which can be plotted as a function of position x along a line such as AB in figure 3. It has already been shown that the presence of a stress σ exerts a force σaL on a dislocation. We may represent the internal stress σ_i by some oscillating function of x. The potential energy is given by

$$V(x) = a\int^{x} \sigma_i L\, dx.$$

An oscillating function of the type shown in figure 4 (*a*) is to be expected. In the presence of an external stress σ, the potential becomes

$$V(x) + a\sigma\int^{x} L\, dx, \qquad \ldots\ldots(2)$$

as shown in figure 4 (*b*). The dislocation will move right across the plane AB when the external stress is great enough to remove all minima.

In Taylor's theory of work hardening, he supposes that dislocations do not travel right across a slip plane, but get stuck in the crystal; their number thus increases with increasing strain. He then ascribes the field $V(x)$ in which any dislocation moves to the strains due to all the other dislocations. There is little doubt that something of the kind does actually happen during work hardening; also that the stuck dislocations are not rigidly stuck, but move a

certain distance themselves under the influence of an applied strain. All these effects may be seen in the "bubble model" developed by Bragg (1947).

Experiments, therefore, on creep or deformation of soft single crystals or pure annealed materials must be interpreted in terms of the motion of dislocations in a stress field of potential $V(x)$ which not only changes during work hardening but probably is changed by the applied stress. It is one of the aims of this paper to emphasize that an important first step for the study of slip and creep should be the investigation of materials where this is not so, and where the internal stresses giving rise to $V(x)$ change as little as possible during the test. One thinks at once of the strains round precipitates (which either have or have not separated from the matrix) responsible for age hardening. The ideal case would be one in which aging took place only at a temperature higher than the creep test, so that the centres of internal strain could be considered constant during the experiment. It thus appears to the authors that tests on age-hardened materials—among which many materials of technical importance are included—may be simpler to interpret and more likely to lead to advancement of the theory of slip than experiments with pure single crystals. In the remainder of this article the deformation of materials of this type, and of solid solutions, will be discussed.

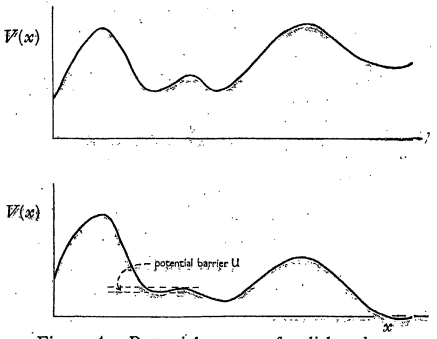

Figure 4. Potential energy of a dislocation, as a function of position.

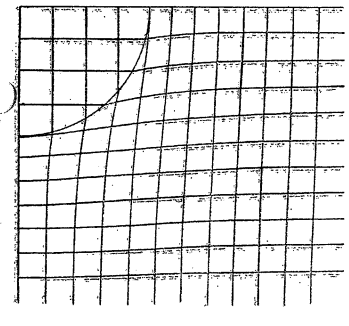
Figure 5. Strain round a precipitate.

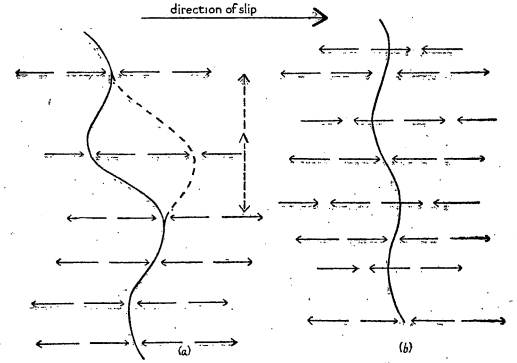

Figure 6. A curved dislocation. The plane of the paper is the slip plane.

The strains round precipitates have been discussed by various authors (Nabarro 1940 a, b, László 1943–5), and in figure 5 the type of strain envisaged is illustrated. In most alloys plate-like precipitates are formed; but the resulting

strains will be not dissimilar. What is immediately apparent, therefore, is that the theory illustrated in figure 3 must be made three-dimensional.

The reason for this is illustrated in figure 6 (*a*), which shows again the curved dislocation of figure 3; the arrows show the directions of the force which the internal stress exerts on the dislocation. Λ is of the order of the distance between precipitates.

A dislocation in equilibrium under these internal stresses will take up a curved or zig-zag form as shown.

§ 3. THE FLEXIBILITY OF DISLOCATIONS

We now have to answer the following questions:

(1) Given that the dislocation line is constrained to depart not more than one atomic distance from a given curve (of curvature everywhere much greater than $1/a$), is its equilibrium form a smooth curve or a zig-zag?

(2) By how much does the energy of the equilibrium form exceed that of a straight dislocation?

We consider only the case in which the tangent to the given curve is inclined at a small angle to the axis of a free straight dislocation. We use the same notation and formulae as before (Nabarro 1947, referred to as I), except that the interatomic spacing is represented by a instead of d as previously.

Figure 7 represents one complete tooth of a zig-zag dislocation which is

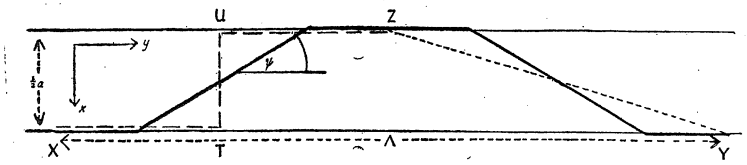

Figure 7. A single tooth of a zig-zag dislocation passing through the points X, Z and Y. The full line represents its equilibrium form; the broken lines XTUZ and ZY are possible extreme forms.

constrained to lie along one equilibrium line in the lattice at a series of points such as X and Y, separated by a distance Λ much greater than a, and to lie along the next equilibrium line at a distance $\frac{1}{2}d$ at a series of equidistant points such as Z. We allow it to take any form intermediate between that of the dotted line XTUZ and that of the dashed line ZY. We vary the angle ψ at which the dislocation line crosses from one equilibrium line to the other, and minimize the energy. In fact we carry out the calculation only for small values of ψ, and observe whether the energy increases or decreases as ψ increases from its least possible value, viz. a/Λ.

The energy of the zig-zag dislocation exceeds that of a straight dislocation lying along XY because the energy of misfit in the glide plane is increased, and because the elastic energy of the blocks above and below the glide plane is increased. As ψ increases, the region in which the misfit is unnecessarily bad becomes smaller, and the energy of misfit diminishes, but the same elastic displacements are concentrated into a smaller space, and the elastic energy increases.

We first estimate the change in the energy of misfit. We assume that an element of length dy at a distance x from the line XY has the same energy of misfit as an element of the same length in a dislocation lying parallel to the line

XY (the y-axis in figure 1 of I), and passing through the centre of the element. This energy exceeds that of an equal element lying in the line XY by an amount given by equation (34) of I, and approximately equal to

$$\frac{\mu a^2}{2\pi(1-\sigma_0)} \exp[-2\pi/(1-\sigma_0)] \cdot (1-\cos 4\pi x/a)\, dy, \qquad \ldots\ldots(3)$$

where σ_0 is Poisson's ratio for the crystal, assumed isotropic.

This expression vanishes over the parts of the zig-zag parallel to XY. The sloping parts of the zig-zag occupy a fraction $a/\Lambda\psi$ of its length, and the average misfit energy per unit length of the zig-zag dislocation exceeds that of a straight dislocation lying along XY by

$$W_m = \frac{\mu a^3}{2\pi\Lambda\psi(1-\sigma_0)} \exp[-2\pi/(1-\sigma_0)]. \qquad \ldots\ldots(4)$$

Next we estimate the elastic energy of the zig-zag dislocation. For a straight dislocation along the y-axis the displacements u and w are given by equations (4), (14) and (15) of I. There is no v component of displacement along the y-axis, and u and w are independent of y. We assume that the zig-zag dislocation also produces no v component of displacement, but, near the sloping parts of the zig-zag, u and w now depend on y. The shear strains between neighbouring planes perpendicular to Oy are given by

$$e_{xy} = \tfrac{1}{2}\psi\, \partial u/\partial x; \quad e_{zy} = \tfrac{1}{2}\psi\, \partial w/\partial x. \qquad \ldots\ldots(5)$$

The strain energy per unit length near the sloping parts exceeds that of a straight dislocation by

$$\tfrac{1}{2}\mu \int\int (e_{xy}^2 + e_{zy}^2)\, dx\, dz = \tfrac{1}{8}\mu\psi^2 \int\int \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] dx\, dz. \qquad \ldots\ldots(6)$$

If this integral were extended to infinite values of x and z it would diverge logarithmically. It is, however, clear that our approximation of considering that the displacement of atoms in any plane perpendicular to Oy depends only on the point at which the dislocation meets that plane, and not at all on the displacements of atoms in neighbouring planes, must break down in regions of the plane distant Λ or more from the dislocation. The variations from plane to plane become smaller at these distances, and the shear-strain energy (6) is reduced. An estimate of the rate at which the shear energy falls off with distance is given in the Appendix. The result is to multiply the integrand in (6) by a factor $\exp[-4\pi(x^2+z^2)^{\frac{1}{2}}/\Lambda]$. This new integral is taken over a fraction $a/\Lambda\psi$ of the total length of the dislocation, and the average elastic energy per unit length of the zig-zag dislocation exceeds that of the straight dislocation by

$$W_e = (\mu a\psi/8\Lambda)\, I(\Lambda/a),$$

where

$$I(\Lambda/a) = \int\int \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \exp\left[\frac{-4\pi(x^2+z^2)^{\frac{1}{2}}}{\Lambda}\right] dx\, dz. \qquad \ldots\ldots(7)$$

The values of $\partial u/\partial x$ and $\partial w/\partial x$ must be inserted from I. The approximate value of the integral $I(\Lambda/a)$ is derived in the Appendix, and we finally obtain

$$W_e \simeq \frac{1\cdot 3\, k\mu a^3\psi}{32\pi^2\Lambda} \ln\left(\frac{\Lambda}{4\pi\zeta}\right), \qquad \ldots\ldots(8)$$

where $(3-2\sigma_0)^2 > k > (1-2\sigma_0)^2$, and $\zeta = a/2(1-\sigma_0)$.

We shall take $k=1$ as an estimate of the true value.

The total energy is

$$W=W_m+W_e=\frac{\mu a^3}{2\pi\Lambda}\left[\frac{e^{-2\pi/(1-\sigma_0)}}{1-\sigma_0}\frac{1}{\psi}+\frac{1\cdot3\ln(\Lambda/4\pi\zeta)}{16\pi}\psi\right].$$

Varying ψ, we obtain a minimum of W when

$$\psi=\left[\frac{16\pi\exp[-2\pi/(1-\sigma_0)]}{1\cdot3(1-\sigma_0)\ln(\Lambda/4\pi\zeta)}\right]^{\frac{1}{2}}. \qquad \ldots\ldots(9)$$

To obtain a numerical value, we take $\sigma_0=\frac{1}{3}$, $\Lambda=100a$, giving

$$\psi=1\cdot06\exp[-\pi/(1-\sigma_0)]\simeq\exp[-\pi/(1-\sigma_0)]. \qquad \ldots\ldots(10)$$

This value of ψ is small, so that our original assumption is self-consistent. Moreover, it is greater than the least possible value a/Λ, provided Λ is of the order $100a$, as we have assumed. The energy is then

$$W=\frac{\mu a^3}{\pi\Lambda(1-\sigma_0)}\exp[-\pi/(1-\sigma_0)]. \qquad \ldots\ldots(11)$$

A straight dislocation of length Λ has become a zig-zag of length $\Lambda+a$ (cosec ψ $-\cot\psi)\simeq\Lambda+\frac{1}{2}a\psi$, and its energy has increased by ΛW. The tension T in it is, therefore, $2\Lambda W/a\psi$, which from (10) and (11) is

$$T=2\mu a^2/\pi(1-\sigma_0)\simeq\mu a^2. \qquad \ldots\ldots(12)$$

We conclude that a dislocation in a strained lattice will approximate to a smooth curve rather than to a square zig-zag, and that its effective tension is approximately μa^2. The shape of the dislocation in figure 6 will thus be given by

$$T/\rho=\sigma_i a, \qquad \ldots\ldots(13)$$

where ρ is the radius of curvature. This gives

$$\rho/a\simeq\mu/\sigma_i. \qquad \ldots\ldots(14)$$

A distinction must now be drawn between the case in which Λ, the distance between particles of precipitate, is greater than $\rho=\mu a/\sigma_i$, and the case in which Λ is less than this quantity. In the first case the internal stresses are great enough to force the dislocation into the regions of low stress between the particles of precipitate, as in figure 6 (*a*). It takes up a wavy shape in which both the wavelength and the amplitude are of order Λ. This is the case we shall consider first. In the other case the internal stresses are not large enough to force the dislocation into the regions of lowest stress, and the appearance of the dislocation will be as illustrated in figure 6 (*b*).

We believe that many commercial alloys correspond to the case in which Λ is greater than $\mu a/\sigma_i$.

Table 1 shows that for hard alloys the ratio μ/Y is of order 100, and on our assumption that $Y=\sigma_i$ this means $\mu/\sigma_i\simeq100$. The distance between precipitates is usually greater than 100 atomic diameters. On the other hand, Guinier has shown that the distance between nuclei in age-hardened alloys such as duralumin is only about $50a$, so that for these alloys neither the treatment of the following section, which assumes $\Lambda\gg\mu a/\sigma_i$, nor the treatment given for solid solutions, which assumes $\Lambda\ll\mu a/\sigma_i$, is a good approximation. That this

should happen for the strongest technical alloys can hardly be an accident: it seems that the equation $\Lambda=\mu a/\sigma_i$ defines the "critical dispersion" of the early theories of age-hardening. Too fine a dispersion allows the opposed internal stresses acting on different parts of a dislocation to compensate one another, as will be seen from the discussion of the hardness of solid solutions at the end of this paper. Too coarse a dispersion will leave large regions of low stress in which dislocations can move freely. Orowan (1948) suggests that in this case dislocations do not cross regions of high opposing stress, but form closed loops round them.

§ 4. MOTION OF DISLOCATIONS WHEN THE SCALE OF THE INTERNAL STRESSES IS LARGE

In the case where $\Lambda\gg\mu a/\sigma_i$, the equilibrium form of a dislocation (in the plane of slip) will be as shown in figure 6 (*a*); both the wavelength and the amplitude of the oscillation will be of order Λ. The elementary step in the motion forwards of a dislocation from one position of equilibrium to another will be its motion from a position such as that shown by the full line to that shown by the dotted line.

One can easily estimate the order of magnitude of the potential barrier to be surmounted by a loop of a dislocation in moving from one of these positions to another in the absence of an applied stress; it will be of order

$$\Lambda a\int\sigma_i(x)\,dx,$$

where $\sigma_i(x)$ is the mean stress along the loop when displaced a distance x from its position of equilibrium. Taking $\sigma_i(x)$ to be of the form $\sigma_i\sin(2\pi x/\Lambda)$, and integrating from a minimum to a maximum, we obtain

$$\sigma_i a\Lambda^2/\pi. \qquad \ldots\ldots(15)$$

The numerical magnitude of the quantity $\sigma_i a\Lambda^2/\pi$ will concern us in the remainder of this paper. We have seen that for the considerations of this section to be valid, $\Lambda\gg\mu a/\sigma_i$, so that (15) is great compared with $\mu a^2\Lambda/\pi$. Taking $\mu a^3\sim5$ ev. and $\Lambda\sim100a$, a value of 150 ev. seems to be a minimum. We shall base our arguments on the following values:

$$\Lambda=100a, \qquad \sigma_i=\mu/100, \qquad \mu a^3=10\text{ ev}.$$

Then

$$\sigma_i a\Lambda^2=1000\text{ ev}. \qquad \ldots\ldots(16)$$

We see, therefore, that heat motion will not give appreciable assistance to a dislocation in surmounting the barrier (15). At first sight this conclusion seems an obstacle to the understanding of creep. However, it should be pointed out:

(i) that values of σ_i extending over a wide range must exist for the various loops of the various dislocations present in a solid at any one time;

(ii) as emphasized by Orowan, transient creep only takes place for external stresses σ nearly great enough to cause slip at low temperatures, i.e. motion of dislocations without the help of heat motion. For these the potential barrier U is very much reduced, as shown in figure 4.

We shall therefore evaluate the height of the barrier in the case for which the external stress σ is nearly equal to the internal stress σ_i, so that the height

of the barrier is small. We shall make the calculation first for a straight dislocation of length L.

We denote the force on a dislocation due to the internal strains by

$$\sigma_i aL \cos(2\pi x/\Lambda).$$

The force in the presence of an applied stress σ is then

$$aL[\sigma_i \cos(2\pi x/\Lambda) - \sigma]. \qquad \ldots\ldots(17)$$

This will vanish when $\cos(2\pi x/\Lambda) = \sigma/\sigma_i$, or, if two roots of this equation are close together, $1 - \frac{1}{2}(2\pi x/\Lambda)^2 = \sigma/\sigma_i$. This gives for the two roots x_0

$$2\pi x_0/\Lambda = \pm[2(1-\sigma/\sigma_i)]^{\frac{1}{2}}. \qquad \ldots\ldots(18)$$

The required activation energy U is thus, by (17),

$$aL\int[\sigma_i \cos(2\pi x/\Lambda) - \sigma]\,dx,$$

where the integration is between the two roots given by (18). The integration gives

$$2aL\left[\sigma_i \frac{\Lambda}{2\pi}\sin\frac{2\pi x_0}{\Lambda} - \sigma x_0\right],$$

or, expanding in powers of x_0,

$$\frac{aL\Lambda}{\pi}\left[\frac{2\pi x_0}{\Lambda}(\sigma_i - \sigma) - \tfrac{1}{6}\sigma_i\left(\frac{2\pi x_0}{\Lambda}\right)^3 \ldots\right].$$

Substituting from (18), we obtain for the activation energy U shown in figure 4

$$U(\sigma) = (2\sqrt{2}/3\pi)\,\sigma_i aL\Lambda[1-\sigma/\sigma_i]^{\frac{3}{2}}. \qquad \ldots\ldots(19)$$

For the motion forward of a loop of the curly dislocation of figure 6 (a), a correct order of magnitude will be obtained if we substitute $\frac{1}{2}\Lambda$ for L in formula (19); thus

$$U(\sigma) \simeq 0{\cdot}15\sigma_i a\Lambda^2[1-\sigma/\sigma_i]^{\frac{3}{2}}. \qquad \ldots\ldots(20)$$

§ 5. THEORY OF CREEP

We now assume that the chance α per unit time that a loop of a dislocation jumps forward is given by

$$\alpha = \nu \exp[-U(\sigma)/kT], \qquad \ldots\ldots(21)$$

where ν is the frequency of vibration of a dislocation in its potential trough. This is to replace the corresponding formula in the Becker-Orowan theory, for which

$$U(\sigma) = V(\sigma_i - \sigma)^2/2\mu, \qquad \ldots\ldots(22)$$

V being the volume through which thermal fluctuations must raise the stress in order that the dislocation shall move forward. We may estimate ν as follows:

The force on a dislocation of length L may be taken to be

$$aL[\sigma_i \cos(2\pi x/\Lambda) - \sigma],$$

which vanishes for $x = x_0$, where $\cos(2\pi x_0/\Lambda) = \sigma/\sigma_i$. Putting $x = x_0 + \delta$, we find that the force for a displacement δ from a position of equilibrium is

$$(2\pi aL\sigma_i\delta/\Lambda)(1-\sigma/\sigma_i)^{\frac{1}{2}}.$$

Taking the effective mass to be Pa^2L, where P is the density of the material,

we see that the frequency ν with which a dislocation vibrates about its mean position is given by

$$\nu = \frac{1}{2\pi}\left(\frac{2\pi\sigma_i}{a\Lambda P}\right)^{\frac{1}{2}}\left(1-\frac{\sigma}{\sigma_i}\right)^{\frac{1}{2}}. \qquad \ldots\ldots(23)$$

Comparing this with the frequency ν_0 of atomic vibrations, which is of order $(\mu/a^2P)^{\frac{1}{2}}$, we see that if σ differs from σ_i by, say, 4%, ν is less by a factor 100–1000 than ν_0, and thus equal to 10^8–$10^9\,\mathrm{sec}^{-1}$.

We have assumed here that the statistical weight of the state in which the loop is passing over the potential barrier is the same as the weight of the state in which the loop lies in its potential trough. For we may describe the state of the loop, like that of a stretched string, in terms of its normal modes of vibration. Only the fundamental mode helps the loop to overcome the barrier, and the distribution of energy among the other modes is almost the same for the loop in the trough and for the loop at the top of the barrier.

In order to calculate creep rates we require to know the increase in strain which will result when each loop of a dislocation moves forward. This increase δs is given by $\delta s = a\Lambda^2/(\text{volume of crystal})$, unless each motion of a loop sets off an "avalanche" of other dislocations, forming a slip band. To take account of this possibility, we shall set $\delta s = a\Lambda^2 p/(\text{volume of crystal})$, where p is a numerical factor which may be great compared with unity.

We are now in a position to calculate creep rates with various models. Consider first the hypothetical case where every loop of every dislocation in the material has the same value of σ_i. Then if there are N dislocations per unit area (crossing a plane containing the direction of flow and perpendicular to the glide plane), there are N/Λ loops per unit volume, and one obtains for the creep rate

$$ds/dt = Na\Lambda p\nu \exp[-U(\sigma)/kT]. \qquad \ldots\ldots(24)$$

This is of course independent of the time but strongly dependent on stress.

Some values of the numerical constants must now be inserted: $\nu \sim 10^9\,\mathrm{sec}^{-1}$; $N \sim 10^{10\pm2}\,\mathrm{cm}^{-2}$; $\Lambda \sim 10^{-6}\,\mathrm{cm}.$; $a \sim 10^{-8}\,\mathrm{cm}.$ Thus $ds/dt \sim 10^{5\pm2}p\exp[-U(\sigma)/kT]$.

It is of interest to see how much σ must differ from σ_i (the yield stress at $T=0$) to obtain an observable creep rate (say $ds/dt = 10^{-6}\,\mathrm{sec}^{-1}$) at temperature T. We must then have

$$U(\sigma)/kT = \ln(10^{5\pm2}p/\dot{s}).$$

One cannot estimate the right-hand side accurately without knowing the (large) numerical constant p, but it may safely be assumed to be in the range 20–40. Taking 30 as a round value, and substituting for $U(\sigma)$, we see that

$$1-\sigma/\sigma_i = \left(\frac{200\,kT}{\sigma_i a\Lambda^2}\right)^{\frac{2}{3}}.$$

In (16) we have estimated $\sigma_i a\Lambda^2$ to be of the order 1000 ev., though the estimate may be out by a factor of 10. Accepting it, however, and putting $kT = 1/40$ ev., we see that

$$1-\sigma/\sigma_i = 0{\cdot}03.$$

Thus, according to this model, the yield strength (if defined as the stress at which a flow at the rate of $10^{-6}\,\mathrm{sec}^{-1}$ occurs) will not differ at room temperature by more than a few per cent from the value at the absolute zero. No reasonable choice of values for $\sigma_i a\Lambda^2$ seems to alter this conclusion.

The large observed dependence on temperature of the yield strength of many materials cannot therefore be explained on any hypothesis in which σ_i is independent of temperature. Cottrell (1948) has proposed a mechanism by which σ_i, the internal stress hindering the motion of a dislocation, may depend on T, but the decrease in σ_i with increasing T is likely to be appreciable only at high temperatures. Lawson (1947) has pointed out that in a metal hardened by cold-work or by precipitation it is the internal strains and not the internal stresses which remain constant if the temperature varies. Therefore σ_i and the yield strength decrease proportionately to the decrease in elastic modulus as the temperature increases. The increase in Young's modulus in going from room temperature to liquid air temperatures is about 5% for metals of high melting point, 10% for Mg, Al and Ag, and 30–50% for Na and Zn. The yield strength of annealed metals and single crystals increases by 40–100% over the same range. In iron, where a special mechanism operates, an increase in yield strength of 100–180% is accompanied by an increase in modulus of less than 5%.

The assumption of a single value of σ_i is of course not sufficient to explain transient creep. If a stress $\sigma = OB$ is applied to a material with a stress–strain curve as shown in figure 8, an instantaneous strain $s = OA$ results. This is followed by rapid creep which slows down roughly as $t^{\frac{1}{3}}$, as first shown by Andrade.

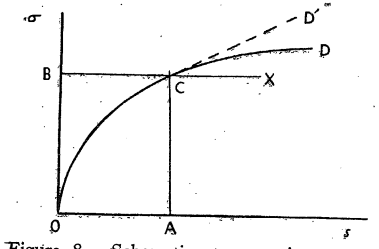

Figure 8. Schematic stress–strain curve of a material. CD′ is the tangent to the curve OCD at C.

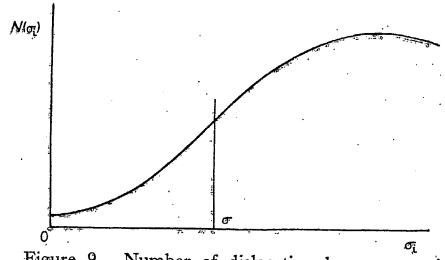

Figure 9. Number of dislocation loops per unit volume which at absolute zero temperature would start to move under stresses between σ and $\sigma_i + d\sigma_i$ is $N(\sigma_i)d\sigma_i$.

As Orowan (1947) in particular has pointed out, at C on the curve $\sigma = \sigma_i$, and so the activation energy is initially zero and rapidly increases.

One must now carefully distinguish between two assumptions. One is that, as we go up the stress–strain curve CD, the material hardens *physically* through the increase in the internal stresses. Making the further very drastic assumption that *at any point on the stress–strain curve* only a single value of σ_i is operative, i.e. σ_i is the same for all dislocations, the creep rate is still given by (24), but we may write $\sigma_i - \sigma = \beta s$, where s is the strain measured from A in figure 8 (the end of instantaneous extension) and β is the tangent of the angle D′CX. Thus

$$ds/dt = \text{const.} \exp\left[-\gamma s^{\frac{3}{2}}/kT\right],$$

with $\gamma = 0{\cdot}15\, a\Lambda^2\beta^{\frac{3}{2}}/\sigma_i^{\frac{1}{2}}$. The analogous formula with s^2 instead of $s^{\frac{3}{2}}$ was obtained by Orowan; it was found not to give the observed rapid initial increase of s with t. Orowan therefore assumed that, when $\sigma_i - \sigma$ was very small, any motion

of a dislocation would set into motion an additional number proportional to $1/(\sigma_i - \sigma)^2$. With this assumption he found

$$ds/dt = \{\text{const.}/s^2 \exp\left[-\gamma s^{\frac{3}{2}}/kT\right]\}, \qquad \ldots\ldots(25)$$

which gives good agreement with experiment.

It is, however, much more reasonable to assume that at any moment the creep is determined by a whole range of values of σ_i, and that the early parts of the hardening curve are determined at least partly by the *exhaustion* of dislocations for which $U(\sigma)$, the activation energy, is small. We shall then work out the dependence of creep rate on time on the following assumptions:

(i) At the moment when instantaneous slip under applied stress stops, a function $N(\sigma_i)$ can be defined so that $N(\sigma_i)d\sigma_i$ gives the number of dislocation loops per unit volume that would move under a stress between σ_i and $\sigma_i + d\sigma_i$. $N(\sigma_i)$ may be expected to appear as in figure 9. We are only interested in its value for $\sigma_i > \sigma$, loops for which $\sigma > \sigma_i$ will have moved during the instantaneous extension.

(ii) During creep $N(\sigma_i)$ does not change appreciably—i.e. hardening is due mainly to the exhaustion of dislocations.

(iii) A dislocation loop, once it has moved from a position of particularly low $(\sigma_i - \sigma)$, moves in general into a position where $(\sigma_i - \sigma)$ is much bigger, and does not again take part in the creep.

(iv) The chance αdt per time interval dt that a dislocation loop with a given value of σ_i moves forward is given by

$$\alpha = \nu \exp\left[-U(\sigma_i)/kT\right], \qquad \ldots\ldots(26)$$

and this results in an increase of strain equal to v/(volume of crystal). We shall not attempt to estimate v; it depends whether or not the motion of a single loop sets off an avalanche.

With these assumptions, the number of remaining dislocations in the range σ_i to $\sigma_i + d\sigma_i$ which have not moved after time t is $N(\sigma_i)d\sigma_i\, e^{-\alpha t}$, and their contribution to the creep rate is obtained by multiplying by αv. Thus the creep rate is given by

$$\frac{ds}{dt} = \alpha v \int_\sigma^\infty N(\sigma_i) e^{-\alpha t}\, d\sigma_i, \qquad \ldots\ldots(27)$$

and the total extension, measured from the time when instantaneous extension stops,

$$s = v\int_\sigma^\infty (1 - e^{-\alpha t}) N(\sigma_i) d\sigma_i. \qquad \ldots\ldots(28)$$

The integral (27) for the creep rate may be evaluated by making α the independent variable. We have from (19) and (26)

$$1 - \frac{\sigma}{\sigma_i} = \left[\frac{kT \ln(\nu/\alpha)}{0{\cdot}15\, a\Lambda^2\sigma_i}\right]^{\frac{2}{3}},$$

and hence, since $|\sigma - \sigma_i| \ll \sigma_i$,

$$\frac{d\sigma_i}{d\alpha} = \frac{2}{3}\sigma\left(\frac{kT}{0{\cdot}15\,\sigma_i a\lambda^2}\right)^{\frac{2}{3}}\left(\ln\frac{\nu}{\alpha}\right)^{-\frac{1}{3}}\frac{1}{\alpha}.$$

The simplest assumption is that $N(\sigma_i)$ is constant over the small range of σ_i which is of importance, but to be as general as possible we set $N(\sigma_i) = N_0(1 - \sigma/\sigma_i)^m$. Then

$$\frac{ds}{dt} = vN_0\int_0^\nu \alpha e^{-\alpha t}\left(1 - \frac{\sigma}{\sigma_i}\right)^m\left(-\frac{d\sigma_i}{d\alpha}\right)d\alpha.$$

A short reduction gives for this

$$\frac{ds}{dt} = A\int_0^1 e^{-\eta\tau}\left(\ln\frac{1}{\eta}\right)^{m-\frac{1}{3}} d\eta, \qquad \ldots\ldots(29)$$

where $\tau = \nu t$ and $A = \frac{2}{3}v\sigma N\nu(kT/0{\cdot}15\, a\Lambda^2\sigma_i)^{m+\frac{1}{3}}$.

Actually, since $\nu \sim 10^8\,\mathrm{sec}^{-1}$, $\tau = \nu t$ is large for all values of t of practical importance. We have thus to find an asymptotic form for the function

$$F(\tau) = \int_0^1 e^{-\eta\tau}\{\ln(1/\eta)\}^n d\eta. \qquad n = m - \tfrac{1}{3}.$$

Putting $\log(1/\eta) = z$, this gives

$$F(\tau) = \int_0^\infty \exp\{-\tau e^{-z} - z\} z^n\, dz.$$

If we set $z = \ln\tau + \zeta$, the integral becomes

$$\frac{1}{\tau}\int_{-\ln\tau}^\infty \exp(-e^{-\zeta} - \zeta)(\ln\tau + \zeta)^n d\zeta.$$

The term $-e^{-\zeta} - \zeta$ tends to $-\infty$ as ζ tends to $\pm\infty$, and has its maximum value when $\zeta = 0$. Thus if $\ln\tau$ is large, this may be written

$$\frac{(\ln\tau)^n}{\tau}\sum_0^\infty \frac{A_n}{(\ln\tau)^n}, \qquad \ldots\ldots(30)$$

where

$$A_n = C_n\int_{-\ln\tau}^\infty \exp(-e^{-\zeta} - \zeta)\left(\frac{\zeta}{\ln\tau}\right)^n d\zeta.$$

Here C_n is the nth binomial coefficient. Taking the first term only of the expansion, and replacing the lower limit in the integral by $-\infty$, we have $F(\tau) = \beta(\ln\tau)^n/\tau$, where β is a numerical factor of order unity. Thus on integration

$$s = \frac{A\beta}{n+1}(\ln\tau)^{n+1}. \qquad \ldots\ldots(31)$$

Thus, finally, with $n = -\frac{1}{3}$, we expect the extension to be given in terms of the time t by a formula of the type

$$s = \text{const.}\; T^{\frac{2}{3}}(\ln\nu t)^{\frac{2}{3}}, \qquad \ldots\ldots(32)$$

where $\nu \sim 10^8$.

It is characteristic of this formula, as of any derived from a pure "exhaustion" hypothesis, that the total extension before creep becomes observably slow must be larger than the total creep extension. Thus between 10 sec. and 10^6 sec. ($\sim$10 days) the factor $(\ln\nu t)^{\frac{2}{3}}$ will increase by about 30% only. Any extension *larger* than the initial instantaneous creep must, we believe, be associated with physical hardening, i.e. a change in $N(\sigma_i)$ during extension due to distortion of the lattice. It is hoped to treat this case in a later paper.

Pure "exhaustion" creep of the type described here will be observed primarily at low temperatures and particularly at low stresses, where the increase in the number of trapped dislocations, and particularly the change in $N(\sigma_i)$, will not be important.

Curves of very much the required type are observed at low temperatures for lead, cf. for example the curves obtained by Andrade (1914) for this material at $-180°$ and $-78°$ c.

It is possible to design distribution functions $N(\sigma_i)$ which give temperature dependences of the yield strength either more or less violent than that predicted by formulae (20) and (24). It seems unlikely that any reasonably smooth distribution function will give a temperature dependence as great as that observed experimentally.

§ 6. SLIP IN SOLID SOLUTIONS

We now turn to the case in which the internal stresses are not large enough, compared with tension of a dislocation, to force the dislocation into the region of low stress. The condition for this is $\Lambda\sigma_i \ll \mu a$. In this case lengths of the dislocation large in comparison with Λ run almost straight across the crystal, being bent by individual regions of stress into arcs with radii of curvature $\rho = \mu a/\sigma_i$. The amplitude of the wavy curve into which the dislocation is bent is of order $\Lambda^2/\rho = \Lambda^2\sigma_i/\mu a$.

If the dislocation showed no large-scale deviations from straightness it would move under the influence of practically any external stress, however small. For a dislocation of length L much greater than Λ is composed of L/Λ elements, on each of which the internal stress exerts a force of order of magnitude $\sigma_i a\Lambda$ and of random sign. The resultant force is of order $\sigma_i a\Lambda(L/\Lambda)^{\frac{1}{2}}$, while the force due to an external stress σ is σaL, which can always be made greater than $\sigma_i a\Lambda(L/\Lambda)^{\frac{1}{2}}$ by taking L large enough.

In fact we must not assume that the mean path of the dislocation will not depart from a straight line, however large L becomes. The force $\sigma_i a\Lambda(L/\Lambda)^{\frac{1}{2}}$ represents an average force $\sigma_i a(\Lambda/L)^{\frac{1}{2}}$ on unit length of the dislocation. This will bend it into a curve of radius

$$\rho = \frac{\mu a^2}{\sigma_i a(\Lambda/L)^{\frac{1}{2}}} = \frac{\mu}{\sigma_i}\left(\frac{L}{\Lambda}\right)^{\frac{1}{2}} a. \qquad \ldots\ldots(33)$$

If L is so large that this radius ρ is comparable with L, the dislocation is effectively flexible under the internal stresses. The condition for this is

$$L = (L/\Lambda)^{\frac{1}{2}} a\mu/\sigma_i \quad \text{or} \quad L = \mu^2 a^2/\sigma_i^2\Lambda. \qquad \ldots\ldots(34)$$

This is much greater than Λ.

We now assume that the length L of the dislocation moves under the influence of an applied stress σ. The force on it due to the internal stresses is $\sigma_i a(\Lambda L)^{\frac{1}{2}}$, and varies more or less periodically with wavelength Λ as the dislocation moves across the crystal. Its energy as a function of its displacement x is given by

$$W = \sigma_i a(\Lambda L)^{\frac{1}{2}}(\Lambda/2\pi)\sin(2\pi x/\Lambda) - \sigma aLx.$$

As before, the positions of maximum and minimum energy are given by

$$\sigma_i a(\Lambda L)^{\frac{1}{2}}\cos(2\pi x_0/\Lambda) = \sigma aL,$$

and if these are close to one another the roots are

$$\frac{2\pi x_0}{\Lambda} = \pm\left[2\left\{1 - \frac{\sigma}{\sigma_i}\left(\frac{L}{\Lambda}\right)^{\frac{1}{2}}\right\}\right]^{\frac{1}{2}}.$$

The activation energy is

$$\frac{2\sqrt{2}}{3\pi}\sigma_i aL^{\frac{1}{2}}\Lambda^{\frac{3}{2}}\left[1 - \frac{\sigma}{\sigma_i}\left(\frac{L}{\Lambda}\right)^{\frac{1}{2}}\right]^{\frac{3}{2}}. \qquad \ldots\ldots(35)$$

Since for the values of L and Λ concerned the factor outside the bracket is very much larger than kT, slip will only begin when

$$\sigma \simeq \sigma_i(\Lambda/L)^{\frac{1}{2}}.$$

Inserting the value of L, this gives

$$\sigma=\sigma_i^2\Lambda/\mu a. \qquad \ldots\ldots(36)$$

Before applying this formula, we must estimate the appropriate value of σ_i. In a solid solution of atomic concentration f the value of Λ, the mean distance between solute atoms, is

$$\Lambda=af^{-\frac{1}{3}}. \qquad \ldots\ldots(37)$$

In a precipitation-hardened alloy the dislocation is sufficiently flexible to avoid the regions of maximum stress, and the appropriate value of σ_i is that obtaining at a distance of order Λ from a particle of precipitate. In a solid solution the dislocation is not appreciably deviated by the local stresses, and the appropriate value of σ_i is a volume average of the stresses. If the misfit of the foreign atom in the lattice is ϵ, the tangential strain in the matrix at a distance r from a foreign atom is (Mott and Nabarro 1940) given by $\epsilon a^3/r^3$, and the shear stress by $\mu\epsilon a^3/r^3$. The mean value of this through the volume from $r=a$ to $r=\Lambda$ is

$$\sigma_i=\int_a^{af^{-\frac{1}{3}}}\frac{\mu\epsilon a^3}{r^3}4\pi r^2dr\Big/\int_a^{af^{-\frac{1}{3}}}4\pi r^2dr$$
$$=\frac{\mu\epsilon f}{1-f}\ln\frac{1}{f}\simeq-\mu\epsilon f\ln f. \qquad \ldots\ldots(38)$$

The yield strength is given by (36), (37) and (38) as

$$\sigma=\mu\epsilon^2f^{\frac{1}{3}}(\ln f)^2. \qquad \ldots\ldots(39)$$

We may simplify this by noticing that over the practical range of concentrations, from $f=0{\cdot}01$ to $f=0{\cdot}20$, the quantity $f^{\frac{1}{3}}(\ln f)^2$ varies only between the limits 0·89 and 1·22, reaching a maximum near $f=0{\cdot}07$. To a good approximation we may replace this factor in (39) by unity, obtaining

$$\sigma=\mu\epsilon^2f. \qquad \ldots\ldots(40)$$

As before (Nabarro 1946), we may estimate ϵ from the change in lattice parameter with concentration as

$$\epsilon=(1/a)(da/df). \qquad \ldots\ldots(41)$$

The dependence of the yield strength in (40) linearly on concentration and quadratically on the change of lattice parameter agrees rather satisfactorily with the experimental observations.

APPENDIX

The elastic energy of a bent dislocation

To estimate the rate at which the shear strain falls off with distance, we imagine the zig-zag disturbance replaced by a sinusoidal disturbance of the same wavelength. We shall find that the law of decay with distance is exponential: the exact nature of the disturbance can affect the decay law only by factors which are inverse powers of the distance, and we shall therefore consider a type of disturbance which is easily treated mathematically. The disturbance we consider is a body force in the x direction, localized along the line Oy, and alternating in sign according to the law

$$F_x=\frac{8\pi\mu(\lambda+2\mu)}{\lambda+\mu}F_0\sin\frac{2\pi y}{\Lambda}, \qquad \ldots\ldots(A1)$$

where λ and μ are Lamé's elastic constants for the solid. The displacements $(u_0v_0w_0)$ produced at any point (x, y, z) by this force distribution are given (Love 1927, §131) by

$$\left.\begin{aligned}u_0&=F_0\int_{-\infty}^{\infty}\left(\frac{x^2}{r^3}+\frac{\lambda+3\mu}{\lambda+\mu}\frac{1}{r}\right)\sin\frac{2\pi y'}{\Lambda}dy',\\ v_0&=F_0\int_{-\infty}^{\infty}\frac{x(y-y')}{r^3}\sin\frac{2\pi y'}{\Lambda}dy',\\ w_0&=F_0\int_{-\infty}^{\infty}\frac{xz}{r^3}\sin\frac{2\pi y'}{\Lambda}dy',\end{aligned}\right\} \qquad \ldots\ldots(A2)$$

where

$$r^2=x^2+(y-y')^2+z^2. \qquad \ldots\ldots(A3)$$

To evaluate the integrals, we write

$$\sin\frac{2\pi y'}{\Lambda}=\sin\frac{2\pi y}{\Lambda}\cos\frac{2\pi(y-y')}{\Lambda}-\cos\frac{2\pi y}{\Lambda}\sin\frac{2\pi(y-y')}{\Lambda}$$

and obtain

$$\left.\begin{aligned}u_0&=F_0\sin\frac{2\pi y}{\Lambda}\int_{-\infty}^{\infty}\left(\frac{x^2}{r^3}+\frac{\lambda+3\mu}{\lambda+\mu}\frac{1}{r}\right)\cos\frac{2\pi(y'-y)}{\Lambda}dy',\\ v_0&=-F_0\cos\frac{2\pi y}{\Lambda}\int_{-\infty}^{\infty}\frac{x(y'-y)}{r^3}\sin\frac{2\pi(y'-y)}{\Lambda}dy'\\ &=-F_0\cos\frac{2\pi y}{\Lambda}\int_{-\infty}^{\infty}\frac{2\pi}{\Lambda}\frac{x}{r}\cos\frac{2\pi(y'-y)}{\Lambda}dy',\\ w_0&=F_0\sin\frac{2\pi y}{\Lambda}\int_{-\infty}^{\infty}\frac{xz}{r^3}\cos\frac{2\pi(y'-y)}{\Lambda}dy'.\end{aligned}\right\} \qquad \ldots\ldots(A4)$$

Now (Watson 1922, §6.6)

$$\left.\begin{aligned}&\int_{-\infty}^{\infty}\frac{1}{r}\cos\frac{2\pi(y'-y)}{\Lambda}dy'=2K_0(2\pi\rho/\Lambda)\\ \text{and}\quad&\int_{-\infty}^{\infty}\frac{1}{r^3}\cos\frac{2\pi(y'-y)}{\Lambda}dy'=\frac{4\pi}{\rho\lambda}K_1(2\pi\rho/\Lambda),\end{aligned}\right\} \qquad \ldots\ldots(A5)$$

where

$$\rho^2=x^2+z^2, \qquad \ldots\ldots(A6)$$

and K_0 and K_1 are Bessel functions.

When $2\pi\rho\gg\Lambda$, K_0 and K_1 both have the asymptotic form (Watson §7.23)

$$K(2\pi\rho/\Lambda)\sim(\Lambda/4\rho)^{\frac{1}{2}}\exp(-2\pi\rho/\Lambda), \qquad \ldots\ldots(A7)$$

and the components of displacement all contain this factor $\exp(-2\pi\rho/\Lambda)$. The same factor remains when the displacements are differentiated to give the components of strain, and the elastic energy contains a factor $\exp(-4\pi\rho/\Lambda)$.

In the case of the zig-zag dislocation we therefore modify the integral (4) to the form

$$\tfrac{1}{8}\mu b^2\int\int\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2\right]\exp\left[\frac{-4\pi(x^2+z^2)^{\frac{1}{2}}}{\Lambda}\right]dx\,dz.$$

This integral is taken over a fraction $a/\Lambda\psi$ of the total length of the dislocation, and the average elastic energy per unit length of the zig-zag dislocation exceeds that of the straight dislocation by

$$W_e=(\mu a\psi/8\Lambda)I(\Lambda/a), \qquad \ldots\ldots(A8)$$

where

$$I(\Lambda/a)=\int\int\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2\right]\exp\left[\frac{-4\pi(x^2+z^2)^{\frac{1}{2}}}{\Lambda}\right]dx\,dz. \qquad \ldots\ldots(A9)$$

The values of $\partial u/\partial x$ and $\partial w/\partial x$ obtained from I are

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=-\frac{a}{4\pi(1-\sigma)}\int_0^\infty \exp\left\{\frac{-ma}{2(1-\sigma)}\right\}[mZ-2(1-\sigma)]e^{-mZ}\cos mx\,dm,\\ \frac{\partial w}{\partial x}&=\frac{a}{4\pi(1-\sigma)}\int_0^\infty \exp\left\{\frac{-ma}{2(1-\sigma)}\right\}[mZ-1+2(1-\sigma)]e^{-mZ}\cos mx\,dm,\end{aligned}\right\} \qquad (A10)$$

where

$$Z=z-\tfrac{1}{2}a. \qquad \ldots\ldots(A11)$$

These may be written

$$\left.\begin{aligned}\frac{4\pi(1-\sigma)}{a}\frac{\partial u}{\partial x}&=-\frac{Z}{R^2}+\frac{2(1-\sigma)(Z+\zeta)}{R^2},\\ \frac{4\pi(1-\sigma)}{a}\frac{\partial w}{\partial x}&=\frac{2Z(Z+\zeta)x}{R^4}+\frac{(1-2\sigma)x}{R^2},\end{aligned}\right\} \qquad \ldots\ldots(A12)$$

where

$$\zeta=a/2(1-\sigma) \qquad \ldots\ldots(A13)$$

and

$$R^2=(Z+\zeta)^2+x^2. \qquad \ldots\ldots(A14)$$

Remembering that Z is positive and $\frac{1}{2}>\sigma>-1$, we may write

$$\frac{2(1-\sigma)(Z+\zeta)}{R^2}>\frac{4\pi(1-\sigma)}{a}\left|\frac{\partial u}{\partial x}\right|>\frac{(1-2\sigma)(Z+\zeta)}{R^2},$$

giving *a fortiori*

$$\frac{(3-2\sigma)(Z+\zeta)}{R^2}>\frac{4\pi(1-\sigma)}{a}\left|\frac{\partial u}{\partial x}\right|>\frac{(1-2\sigma)(Z+\zeta)}{R^2}, \qquad \ldots\ldots(A15)$$

and

$$\frac{(3-2\sigma)(Z+\zeta)^2+(1-2\sigma)x^2}{R^4}|x|>\frac{4\pi(1-\sigma)}{a}\left|\frac{\partial w}{\partial x}\right|>\frac{1-2\sigma}{R^2}|x|,$$

giving *a fortiori*

$$\frac{(3-2\sigma)|x|}{R^2}>\frac{4\pi(1-\sigma)}{a}\left|\frac{\partial w}{\partial x}\right|>\frac{(1-2\sigma)|x|}{R^2}. \qquad \ldots\ldots(A16)$$

From (A15) and (A16),

$$\frac{(3-2\sigma)^2}{R^2}>\frac{16\pi^2(1-\sigma)^2}{a^2}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2\right]>\frac{(1-2\sigma)^2}{R^2}. \qquad \ldots\ldots(A17)$$

We may substitute these limits in (A9), where the integral is to be extended over all values of x and all positive values of Z, and then doubled to allow for the half of the crystal below the glide plane. In the exponential term we substitute Z for z. The result is

$$2(3-2\sigma)^2J(\Lambda/a)>(16\pi^2(1-\sigma)^2/a^2)I(\Lambda/a)>2(1-2\sigma)^2J(\Lambda/a), \qquad \ldots\ldots(A18)$$

where

$$J(\Lambda/a)=\int_0^\infty\int_{-\infty}^\infty \frac{\exp[4-\pi(x^2+Z^2)/\Lambda]}{x^2+(Z+\zeta)^2}\,dx\,dZ. \qquad \ldots\ldots(A19)$$

To evaluate $J(\Lambda/a)$, we write $x=r\cos\theta$; $Z=r\sin\theta$, and obtain

$$J(\Lambda/a)=\int_0^\infty\int_0^\pi \frac{e^{-4\pi r/\Lambda}}{r^2\cos^2\theta+(r\sin\theta+\zeta)^2}\,r\,d\theta\,dr$$

$$=2\int_0^\infty \frac{e^{-4\pi r/\Lambda}\tan^{-1}\left(\dfrac{r^2-\zeta^2}{2r\zeta}\right)}{r^2-\zeta^2}\,r\,dr.$$

Writing $r=\zeta\tan s$, this becomes

$$J(\Lambda/a)=2\int_0^{\frac{1}{2}\pi}\frac{2s-\frac{1}{2}\pi}{\sin(2s-\frac{1}{2}\pi)}e^{-(4\pi\zeta/\Lambda)\tan s}\tan s\,ds.$$

The factor $(2s-\frac{1}{2}\pi)/\sin(2s-\frac{1}{2}\pi)$ varies only between the values 1 and $\frac{1}{2}\pi$ in the range of integration, and we may replace it by a mean value 1·3. Since we are concerned only with very small values of ζ/Λ, the exponential factor is practically unity except near $s=\frac{1}{2}\pi$, and then falls rapidly to zero. We therefore take as an approximate value $J(\Lambda/a)=2\cdot6\int_0^{s_0}\tan s\,ds=-2\cdot6\ln\cos s_0$, where $(4\pi\zeta/\Lambda)\tan s_0=1$; $s_0\simeq\frac{1}{2}\pi-4\pi\zeta/\Lambda$. So

$$J(\Lambda/a)\simeq 2\cdot6\ln(\Lambda/4\pi\zeta). \qquad \ldots\ldots(A20)$$

Combining (A8), (A9), (A18) and (A20), we obtain equation (8) of the text.

REFERENCES

ANDRADE, E. N. DA C., 1911, *Proc. Roy. Soc.* A, **84,** 1 ; 1914, *Ibid.*, **90,** 329.
ANDRADE, E. N. DA C., and CHALMERS, B., 1932, *Proc. Roy. Soc.* A, **138,** 348.
BRAGG, W. L., and NYE, J. F., 1947, *Proc. Roy. Soc.* A, **190,** 474.
BURGERS, J. M., 1939, *Proc. Kon. Ned. Akad. Wet.*, **42,** 293, 378.
COTTRELL, A. H., 1946, this *Report*, p. 30.
DEHLINGER, U., 1929, *Ann. Phys., Lpz.*, **2,** 749.
FRANK, F. C., 1948, this *Report*, p. 46.
KOEHLER, J. S., 1941, *Phys. Rev.*, **60,** 397.
LÁSZLÓ, F., 1943 a, *J. Iron Steel Inst.*, **147,** 173 ; 1943 b, *Ibid.*, **148,** 137 ; 1944, *Ibid.*, **150,** 183 ; 1945, *Ibid.*, **152,** 207.
LAWSON, A. W., 1947, Private communication.
LOVE, A. E. H., 1927, *A Treatise on the Mathematical Theory of Elasticity* (Cambridge : University Press).
MOTT, N. F., and NABARRO, F. R. N., 1940, *Proc. Phys. Soc.*, **52,** 86.
NABARRO, F. R. N., 1940 a, *Proc. Phys. Soc.*, **52,** 90 ; 1940 b, *Proc. Roy. Soc.* A, **175,** 519 ; 1946, *Proc. Phys. Soc.*, **58,** 669 ; 1947, *Ibid.*, **59,** 256 .
OROWAN, E., 1934, *Z. Phys.*, **89,** 634 ; 1947, *J. West Scotland Iron Steel Inst.* ; 1948, *Symposium on Internal Stresses in Metals and Alloys, Institute of Metals* : Discussion.
POLANYI, M., 1934, *Z. Phys.*, **89,** 660.
PRANDTL, L., 1928, *Z. angew. Math. u. Mech.*, **8,** 85.
SHOCKLEY, W., 1947, Private communication.
TAYLOR, G. I., 1934, *Proc. Roy. Soc.* A, **145,** 362.
WATSON, G. N., 1922, *A Treatise on Bessel Functions* (Cambridge : University Press).
ZENER, C., 1947, Private communication.