



The c -numerical range of a quaternion skew-Hermitian matrix is convex

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Abstract

We show that the c -numerical range of a non-scalar skew-Hermitian quaternion matrix is convex. In fact, included in our result is that the c -numerical range of a skew-Hermitian matrix is a rotation invariant subset of the quaternions with zero real parts.

Keywords Quaternion c -numerical range · Convexity · Skew-Hermitian

Mathematics Subject Classification 15A60 · 15A63 · 15B33 · 47A12

1 Introduction

The quaternionic numerical range was introduced by Kippenhahn [10] and several basic properties were derived in [2, 3, 10, 14]. More recent papers include [1, 5, 12, 15]. The classical C -numerical range goes back to [7], which corresponds to the c -numerical range when the matrix C is Hermitian. A comprehensive account on C -numerical ranges can be found in the paper by Li [11]. In [16] it was shown that the c -numerical range of a complex matrix is always convex. It is easily seen that this does not extend to the case of quaternions in general (see, e.g., [8]). In this paper we will show that for skew-Hermitian quaternion matrices of size at least 2×2 the c -numerical

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range is always convex. This question of convexity was earlier pursued in the paper by Aghamollaei, et al. [1], where the k -numerical range (which is a special case of the c -numerical range) of a quaternion matrix was studied. Among the results in [1] it was shown that for an even size skew-Hermitian quaternion matrix its k -numerical range is convex. It was also conjectured there that the same is true for non-scalar odd size matrices. Our main result resolves this conjecture in the positive. We present the main result in the next section.

2 The main result

For notation regarding basic quaternion linear algebra, we refer to the monograph by Rodman [13]. The real skew-field of quaternions, denoted by $\mathbb{H}$, contains elements of the form:

$$x = x_0 + x_1i + x_2j + x_3k,$$

where $x_0, x_1, x_2, x_3 \in \mathbb{R}$ and the multiplication of quaternions follows the following rules:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

The real number x_0 is referred to as the *real part* of x , and we write $x_0 = \operatorname{Re} x$. Furthermore, $x_1i + x_2j + x_3k$ is called the *vector part* of x and we denote the set of quaternions that only have a vector part by $\mathbb{P} := \{x \in \mathbb{H} : \operatorname{Re} x = 0\}$. We denote the *conjugate quaternion* of x by $\bar{x} = x^* = x_0 - x_1i - x_2j - x_3k$ and the *norm* of $x \in \mathbb{H}$ by $\|x\| = \sqrt{x^*x} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2} \in \mathbb{R}$. For a quaternion matrix A , let $\bar{A}$ denote the matrix where each entry is the conjugate of the corresponding entry in A , and A^* the transpose of $\bar{A}$, as usual. Also, recall that an $n \times n$ matrix A is *unitary* if $AA^* = A^*A = I_n$. We use $\operatorname{Tr}(A)$ to denote the trace of the matrix A , i.e., the sum of the diagonal entries of A . Finally, it is useful to note that $j^*zj = \bar{z}$ for any complex number z .

A quaternion λ is called a *right eigenvalue* of A if $Ax = x\lambda$ for some nonzero $x \in \mathbb{H}^n$. For every right eigenvalue of A , say λ , the quaternion $a\lambda a^{-1}$, $0 \neq a \in \mathbb{H}$, is also a right eigenvalue of A . Moreover, the matrix A has exactly n right eigenvalues which are complex numbers with nonnegative imaginary parts, see for example [17, Theorem 5.4]. These complex eigenvalues are called the *standard right eigenvalues* of A .

We denote the *numerical range* of a quaternion matrix $A \in \mathbb{H}^{n \times n}$ by

$$W_{\mathbb{H}}(A) = \{x^*Ax : \|x\| = 1, x \in \mathbb{H}^n\} \subset \mathbb{H},$$

where $\|(x_j)_{j=1}^n\| = \left(\sum_{j=1}^n \|x_j\|^2\right)^{\frac{1}{2}}$. Note that if $q \in W_{\mathbb{H}}(A)$, and $u \in \mathbb{H}$ where $\|u\| = 1$, then $u^*qu \in W_{\mathbb{H}}(A)$. In general, $u^*qu \neq q$. For basic information on the quaternionic numerical range, see for instance [13, Section 3.3].

Let $c = (c_1, \dots, c_n) \in \mathbb{R}^n$. The c -numerical range of a quaternion matrix A is defined as

$$\begin{aligned} W_{c, \mathbb{H}}(A) &= \left\{ \sum_{i=1}^n c_i u_i^* A u_i : \{u_1, \dots, u_n\} \text{ is an orthonormal set in } \mathbb{H}^n \right\} \\ &= \left\{ \text{Tr}(\text{diag}(c_j) U^* A U) : U \in \mathbb{H}^{n \times n} \text{ unitary} \right\}. \end{aligned}$$

Note that if $V \in \mathbb{H}^{n \times n}$ is unitary, then $W_{c, \mathbb{H}}(A) = W_{c, \mathbb{H}}(V^* A V)$. Consequently, if A is normal, we may assume that A is diagonal; see [6]. Basic properties of the c -numerical range of quaternion matrices have been developed in [8].

When $c = (1, 0, \dots, 0)$, the c -numerical range reduces to the classical numerical range. When $c = (\frac{1}{k}, \dots, \frac{1}{k}, 0, \dots, 0)$, where $\frac{1}{k}$ appears k times with k a positive integer, it is the k -numerical range as defined in [1]. When A is Hermitian (i.e., $A = A^*$), then clearly $W_{c, \mathbb{H}}(A) \subseteq \mathbb{R}$ (as $\text{Tr} C U^* A U = \text{Tr} U^* A^* U C = \text{Tr} C U^* A U$ where in the last step we use that $C = \text{diag}(c_j)$ is real and $A = A^*$), while for a skew-Hermitian A (i.e., $A = -A^*$) we have $W_{c, \mathbb{H}}(A) \subseteq \mathbb{P}$; see Proposition 2.3 below.

For a skew-Hermitian quaternion matrix A of size $n \times n$ with $n \geq 2$, it follows from [2, Theorem 4] that its numerical range $W_{\mathbb{H}}(A)$ is convex. More recently, in [1] there are several results on the k -numerical range of skew-Hermitian matrices, including its convexity in the even size case and a conjecture regarding the odd size case. In our main result below we show that the c -numerical range of any skew-Hermitian quaternion matrix A of size $n \geq 2$ is convex, which also resolves the conjecture in the last section of [1].

Theorem 2.1 *Let $n \geq 2$ and $c = (c_i)_{i=1}^n \in \mathbb{R}^n$. If $A \in \mathbb{H}^{n \times n}$ is skew-Hermitian, then $W_{c, \mathbb{H}}(A)$ is convex. In fact, if the standard right eigenvalues of A are $ib_1, \dots, ib_n$, then we have*

$$W_{c, \mathbb{H}}(A) := \{x \in \mathbb{P} : \|x\| \leq \sum_{j=1}^n |c_j b_j|\},$$

where we order b_i, c_i so that $|c_1| \geq \dots \geq |c_n|$ and $|b_1| \geq \dots \geq |b_n|$.

Remark 2.2 Note that if $A = [a]$, $0 \neq a \in \mathbb{P}$, is a nonzero 1×1 skew-Hermitian matrix, then its numerical range consist of all $x \in \mathbb{P}$ with $\|x\| = \|a\|$, so then its numerical range is not convex.

We will first develop some auxiliary results before proving this main result.

Proposition 2.3 *Let $c \in \mathbb{R}^n$ and $A \in \mathbb{H}^{n \times n}$. If A is skew-Hermitian, then $W_{c, \mathbb{H}}(A)$ is a rotation invariant subset of $\mathbb{P}$ (i.e., if $x \in W_{c, \mathbb{H}}(A) \cap \mathbb{P}$ and $y \in \mathbb{P}$ satisfies $\|x\| = \|y\|$, then $y \in W_{c, \mathbb{H}}(A)$).*

If $c \notin \text{span}\{(1, \dots, 1)\}$ and $\sum_{j=1}^n c_j \neq 0$, then $W_{c, \mathbb{H}}(A) \subseteq \mathbb{P}$ implies that A is skew-Hermitian.

Proof For the first part, let A be skew-Hermitian. If U is unitary, we have that

$$\begin{aligned}\overline{\text{Tr}(\text{diag}(c_j) U^* A U)} &= \text{Tr}(\text{diag}(c_j) U^* A U)^* = \text{Tr}(U^* A^* U \text{diag}(c_j)) \\ &= \text{Tr}(\text{diag}(c_j) U^*(-A)U) = -\text{Tr}(\text{diag}(c_j) U^* A U),\end{aligned}$$

where we used that real numbers commute with quaternions. Thus the real part of $\text{Tr}(\text{diag}(c_j) U^* A U)$ is 0. This shows that $W_{c, \mathbb{H}}(A) \subset \mathbb{P}$. Recall that if $a, b \in \mathbb{P}$ and $\|a\| = \|b\|$, then there exists a $\gamma \in \mathbb{H}$ so that $a = \gamma b \gamma^{-1}$. By [8, Theorem 2.4(a)] we have that if $x \in W_{c, \mathbb{H}}(A)$, then $\gamma x \gamma^{-1} \in W_{c, \mathbb{H}}(A)$ for all $\gamma \in \mathbb{H}$. This proves the first part.

For the second part, suppose that $W_{c, \mathbb{H}}(A) \subseteq \mathbb{P}$. Let us write $A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*) =: H + S$. We need to show that $H = 0$, from which it follows that A is skew-Hermitian. Suppose to the contrary that $H \neq 0$, and let $\lambda_1 \geq \dots \geq \lambda_n$ be the eigenvalues of H . We claim that there exists a unitary U so that $\text{Re}(\text{Tr}(\text{diag}(c_j) U^* H U)) \in \mathbb{R} \setminus \{0\}$. Note that we may diagonalize H , and thus we may assume without loss of generality that $H = \text{diag}(\lambda_j)$. Let us also order c so that $c_1 \geq \dots \geq c_n$. We recall from [11, Result 5.1] that

$$\{\text{Tr}(\text{diag}(c_j) U^* \text{diag}(\lambda_j) U) : U \in \mathbb{C}^{n \times n} \text{ unitary}\} = [\alpha, \beta],$$

where

$$\alpha = c_1 \lambda_n + \dots + c_n \lambda_1, \quad \beta = c_1 \lambda_1 + \dots + c_n \lambda_n.$$

We now claim that $[\alpha, \beta]$ contains a non-zero number, which is equivalent to not both α and β being 0. If $\lambda_1 = \dots = \lambda_n$, we are done as we must have that $\lambda_1 \neq 0$ (since $H \neq 0$), and thus $\alpha = \lambda_1 \left(\sum_{j=1}^n c_j \right) \neq 0$. When λ contains two different values, we have that $\sum_{j=1}^n c_j \lambda_{\sigma(j)}$ has at least two different values when we let σ range through all permutations of $\{1, \dots, n\}$; here we used that $c \notin \text{span}\{(1, \dots, 1)\}$. But then we have that the interval $[\alpha, \beta]$ contains two different values (as $\sum_{j=1}^n c_j \lambda_{\sigma(j)} \in [\alpha, \beta]$), proving that $[\alpha, \beta]$ contains a nonzero real number y . Thus there exists a complex unitary $U \in \mathbb{C}^{n \times n}$ so that $\text{Tr}(\text{diag}(c_j) U^* \text{diag}(\lambda_j) U) = y$. Since $\text{Tr}(\text{diag}(c_j) U^* S U) =: z \in \mathbb{P}$, we get that $\text{Tr}(\text{diag}(c_j) U^* A U) = y + z \notin \mathbb{P}$. Thus $H \neq 0$ leads to a contradiction and therefore $H = 0$. $\square$

It follows from Proposition 2.3 that in order to prove convexity of $W_{c, \mathbb{H}}(A)$ for A an $n \times n$ skew-Hermitian matrix ($n \geq 2$), it suffices to show that $0 \in W_{c, \mathbb{H}}(A)$.

To illustrate the ideas in the upcoming proof of Theorem 2.1 we first provide an example. We will use the Schur–Horn theorem (see, e.g., [9, Chapter 4]), which states that the diagonal entries of a Hermitian matrix are majorized by its eigenvalues. Recall that $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ majorizes $d = (d_1, \dots, d_n) \in \mathbb{R}^n$ if and only if

$$d \in \text{conv}\{(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(n)}) : \sigma \in S_n\},$$

where $\text{conv}\{X\}$ is the convex hull of a set X and S_n is the symmetric group of permutations on $\{1, \dots, n\}$. Other characterizations of majorization may be found, for

instance, in [9, Chapter 4] and [4, Chapter II]. We let $\langle \cdot, \cdot \rangle$ denote the Euclidean inner product on $\mathbb{R}^n$.

Example 2.4 Let

$$A = \begin{bmatrix} i & 0 & 0 & 0 & 0 \\ 0 & 2i & 0 & 0 & 0 \\ 0 & 0 & 4i & 0 & 0 \\ 0 & 0 & 0 & 5i & 0 \\ 0 & 0 & 0 & 0 & 11i \end{bmatrix}$$

and $c = (1, 1, 1, 0, 0)$. We would like to find a 5×5 unitary matrix V so that

$$\text{Tr}(\text{diag}(c_j) V^* A V) = 0.$$

This means that the vector c is orthogonal to the vector of diagonal entries of $V^* A V$. We will also use that $j^*ij = -i$, which provides us with a way to change signs for both the c_i 's and the diagonal entries of A . To reach our objective, we will find a diagonal real unitary $\tilde{J}$ and a vector v majorized by $(1, 2, 4, 5, 11)$ so that $v \perp \tilde{J}c$. For instance, we take

$$\tilde{J} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad v = \begin{bmatrix} 4 \\ 6 \\ 10 \\ 1 \\ 2 \end{bmatrix},$$

giving that

$$\langle v, \tilde{J}c \rangle = 4 + 6 - 10 = 0.$$

We next find a real unitary U so that the diagonal of $U^* A U$ equals iv . The existence of such a U is ensured by the Schur-Horn theorem. Here we may take

$$U = \begin{bmatrix} I_3 & 0 & 0 \\ 0 & \sqrt{\frac{5}{6}} & \sqrt{\frac{1}{6}} \\ 0 & -\sqrt{\frac{1}{6}} & \sqrt{\frac{5}{6}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

which is unitary and the diagonal of $U^* A U$ equals iv . Next, we let

$$\tilde{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & j & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

which is obtained from $\tilde{J}$ by replacing -1 by j . Now $V = U\tilde{Q}$ has the property that $\text{Tr}(\text{diag}(c_j) V^* A V) = 0$. $\square$

Lemma 2.5 *Let $b, c \in \mathbb{R}^n$. Then there exist diagonal matrices $J, \tilde{J} \in \mathbb{R}^{n \times n}$ with ± 1 on the diagonal and permutations $\sigma_1, \sigma_2 \in S_n$ so that*

$$\langle P_{\sigma_1}(Jb), \tilde{J}c \rangle \geq 0 \quad \text{and} \quad \langle P_{\sigma_2}(Jb), \tilde{J}c \rangle \leq 0.$$

Here $P_{\sigma_i}(x)$ rearranges the coordinates of any tuple $x \in \mathbb{R}^n$ according to the permutation $\sigma_i \in S_n$.

Proof In the case where n is even, say $n = 2m$, choose J and $\tilde{J}$ such that $(\tilde{J}c)_i \geq 0$, $(Jb)_i \geq 0$, for $i = 1, \dots, m$, and $(\tilde{J}c)_i \leq 0$, $(Jb)_i \leq 0$, for $i = m+1, \dots, n$. Let σ_1 be the identity permutation, then

$$\langle P_{\sigma_1}(Jb), \tilde{J}c \rangle = \langle Jb, \tilde{J}c \rangle \geq 0.$$

Let $\sigma_2 = (1 \ m+1)(2 \ m+2) \cdots (m \ 2m)$, where we use cycle notation for permutations. Then

$$\langle P_{\sigma_2}(Jb), \tilde{J}c \rangle \leq 0.$$

For the odd case, let $n = 2m + 1$. Without loss of generality we may assume for the entries of c and b that

$$|c_1| \geq \cdots \geq |c_n| \quad \text{and} \quad |b_1| \geq \cdots \geq |b_n|.$$

Choose

$$J = \text{diag}(-1, I_m, -I_m) \text{diag}(\text{sign}(b_1), \dots, \text{sign}(b_n))$$

and

$$\tilde{J} = \text{diag}(I_m, -I_{m+1}) \text{diag}(\text{sign}(c_1), \dots, \text{sign}(c_n)).$$

Here we set $\text{sign}(0) = 1$. Then

$$\begin{aligned} & (\tilde{J}c)_1 (Jb)_2 + \cdots + (\tilde{J}c)_m (Jb)_{m+1} + (\tilde{J}c)_{m+1} (Jb)_{m+2} \\ & + \cdots + (\tilde{J}c)_{2m} (Jb)_{2m+1} + (\tilde{J}c)_{2m+1} (Jb)_1 \geq 0. \end{aligned}$$

For the other permutation, note that

$$\underbrace{(\tilde{J}c)_1(Jb)_1 + (\tilde{J}c)_{2m+1}(Jb)_{2m+1}}_{\leq 0} + \underbrace{(\tilde{J}c)_{m+1}(Jb)_2 + \cdots + (\tilde{J}c)_{2m}(Jb)_{m+1}}_{\leq 0} \\ + \underbrace{(\tilde{J}c)_2(Jb)_{m+2} + \cdots + (\tilde{J}c)_m(Jb)_{2m}}_{\leq 0} \leq 0.$$

Then we have opposite signs in the two dot products as required. $\square$

As usual, for $x \in \mathbb{R}^n$ we let

$$x^\perp = \{y \in \mathbb{R}^n : y \perp x\} = \{y \in \mathbb{R}^n : \langle x, y \rangle = 0\}.$$

Corollary 2.6 *Let $b, c \in \mathbb{R}^n$. Then there exist diagonal unitary matrices $J, \tilde{J} \in \mathbb{R}^{n \times n}$ such that*

$$\text{conv}\{P_\sigma(Jb) : \sigma \in S_n\} \cap (\tilde{J}c)^\perp \neq \emptyset. \quad (2.1)$$

Proof Using Lemma 2.5 we know we can find matrices $J, \tilde{J} \in \mathbb{R}^{n \times n}$ and permutations $\sigma_1, \sigma_2 \in S_n$ such that

$$\langle P_{\sigma_1}(Jb), \tilde{J}c \rangle \geq 0 \quad \text{and} \quad \langle P_{\sigma_2}(Jb), \tilde{J}c \rangle \leq 0.$$

Then there exists an $x \in \text{conv}\{P_{\sigma_i}(Jb) : i = 1, 2\} \subseteq \text{conv}\{P_\sigma(Jb) : \sigma \in S_n\}$ so that $\langle x, \tilde{J}c \rangle = 0$, yielding (2.1). $\square$

We will use the following notation in the proof. Let Δ be the map $\mathbb{H}^{n \times n} \mapsto \mathbb{H}^n$ that extracts the diagonal entries from a matrix; that is, $\Delta(M) = (M_{jj})_{j=1}^n$.

Proof of Theorem 2.1 Let A be skew-Hermitian with eigenvalues $b_1i, \dots, b_ni$, where $b_j \geq 0$ for all j . Without loss of generality, we may assume that b_i, c_i are ordered so that $|c_1| \geq \cdots \geq |c_n|$ and $|b_1| \geq \cdots \geq |b_n|$. Let $D = \text{diag}(b_j)_{j=1}^n$. Due to the invariance of the c -numerical range under unitary similarities, we may assume that $A = iD$. Choose $J, \tilde{J}$ such that (2.1) holds. Take a vector v so that

$$v \in \text{conv}\{P_{\sigma_i}(Jb) : \sigma_i \in S_n\} \cap (\tilde{J}c)^\perp,$$

the existence of which is guaranteed by Corollary 2.6. It follows from the Schur-Horn theorem that there exists a unitary matrix $U \in \mathbb{R}^{n \times n}$ such that

$$\Delta(U^*JDU) = v.$$

Let Q be a diagonal matrix where $Q_{rr} = 1$ if $J_{rr} = 1$ and $Q_{rr} = j$ if $J_{rr} = -1$. Similarly, let $\tilde{Q}$ be a diagonal matrix where $\tilde{Q}_{rr} = 1$ if $\tilde{J}_{rr} = 1$ and $\tilde{Q}_{rr} = j$ if $\tilde{J}_{rr} = -1$. Then $Q^*AQ = iJD$ and

$$\text{Tr}(\text{diag}(c_j) \tilde{Q}^*U^*Q^*AQU\tilde{Q}) = \text{Tr}(\text{diag}(c_j) \tilde{Q}^*i \text{diag}(v_j) \tilde{Q})$$

$$= i \sum_{r=1}^n (\tilde{J}c)_r v_r = i \langle v, \tilde{J}c \rangle = 0.$$

Thus $0 \in W_{c, \mathbb{H}}(A)$ and therefore by Proposition 2.3 the convexity of $W_{c, \mathbb{H}}(A)$ follows.

Finally, notice that one may choose $\tilde{Q}$, Q and U a permutation, so that

$$\text{Tr}(\text{diag}(c_j) \tilde{Q}^* U^* Q^* A Q U \tilde{Q}) = i \sum_{r=1}^n |c_r b_r|,$$

showing that $i \sum_{r=1}^n |c_r b_r| \in W_{c, \mathbb{H}}(A)$.

Finally, let $x \in W_{c, \mathbb{H}}(A)$. We need to show that $\|x\| \leq \sum_{r=1}^n |c_r b_r|$. For this, let $V = (v_{kl})_{k,l=1}^n \in \mathbb{H}^{n \times n}$ be unitary so that $x = \text{Tr}(\text{diag}(c_j) V^* A V)$ and let $(d_k)_{k=1}^n$ denote the diagonal entries of $V^* A V$. Then, for $k = 1, \dots, n$,

$$\|d_k\| = \left\| \sum_{l=1}^n v_{lk}^* i b_l v_{lk} \right\| \leq \sum_{l=1}^n \|v_{lk}^*\| \|b_l\| \|v_{lk}\| = \sum_{l=1}^n \|v_{lk}\|^2 \|b_l\| =: \beta_k.$$

Consequently,

$$(\beta_k)_{k=1}^n = (\|v_{lk}\|^2)_{k,l=1}^n (|b_l|)_{l=1}^n,$$

where the matrix $(\|v_{lk}\|^2)_{k,l=1}^n$ is doubly stochastic (since V is unitary). Thus the vector $(\beta_k)_{k=1}^n$ is majorized by the vector $(|b_l|)_{l=1}^n$. But then (e.g., by using [4, Section II.4]) it follows that

$$\|x\| = \left\| \sum_{k=1}^n c_k d_k \right\| \leq \sum_{k=1}^n |c_k| \|d_k\| \leq \sum_{k=1}^n |c_k| \beta_k \leq \sum_{k=1}^n |c_k| |b_k|.$$

□

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