An Introduction to General Relativity using Interactive Symbolic Computation

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December 4, 2019

Declaration

I, Jonathan Esteves , hereby certify that the work done in this dissertation is wholly my own original work except where due references have been made. It is being submitted for the degree of Masters of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before at any other institution.

Signed: Date: December 4, 2019

Abstract

Just over a century after Einstein's writing of 1914 – 1917, *The Berlin years,* General relativity (GR) is still the most robust theory of physics. This theory became the first to approximately predict the perihelion behaviour of Mercury[1], as well as offer mathematical justification for photon deflection. During the year 1916 of Einstein's Berlin years, he predicted the existence of Gravitational waves, which was only confirmed a century later by the LIGO experiment[2]. His theory was later used by a Russian and soviet physicist and mathematician Alexander Friedmann in 1922, to explain that a homogeneous and isotropic model of the universe is an expanding one. Friedmann derived the equations of physical cosmology governing this expansion phenomenon. These Friedmann equations formed the foundation of the FLRW model[3, 4, 5], the first empirically plausible cosmological model of our universe.

This interactive text is focussed on building a foundation of GR tooling, grounded in computation, intended to give students a well-rounded approach to GR as well as insight to build and/or solve problems all the way up to complex scenarios. This tooling foundation comprises of three key areas: tensor calculus, physics, and the computation and visualisation of results. As most of the computation in GR is often cumbersome to do by hand, the approach of this text places emphasis on the use of modern symbolic computation in Mathematica to provide a direct framework for exploring the introductory implications of GR through computation interaction. As Mathematica is praised for its design consistency, aesthetics and automation, it became the logical choice of software for this endeavour.

We begin by introducing curvilinear coordinates, tensor calculus, investigate the Eulerian and Lagrangian formalisms for the equations of fluid mechanics in tensor notation. Now that the reader has been exposed to familiar material in this notation, we define and discuss the geodesic equations, making a link to the Euler-Lagrange equations. The reader is then introduced to the principles of GR and supplied with a discussion of how Einstein unified inertia and gravity. This sets the foundation on which different representations of GR are introduced and discussed. A chapter dedicated to exact and analytical solutions is supplied, addressing the Minkowski, Schwarzschild and Kerr space-times as well as the FLRW universe model. The final chapter is dedicated to the introduction of perturbation theory in GR and its role in the detection of gravitational waves. Lastly, an introduction to the foundation of numerical relativity with an application to an FLRW universe is supplied. Throughout the text the reader is asked to do several written and Mathematica exercises, of which the solutions are supplied in the two appendices.

In no way is this work to be interpreted as novel, as it is an amalgamation of several texts, but rather as a modern approach to the teaching of GR with symbolic computation at the postgraduate level. It is aimed at retaining the historic development of GR as a key approach to unifying Inertia and Gravity; rather than perspectives aimed at simplifying Gravity to be merely considered as an artefact of geometric representation.

The project was conducted under the supervision of Dr. R.S. Herbst and Dr. T. Gebbie, and was inspired by the detailed handwritten notes of Professor David P. Mason on GR and tensors. These notes, that were written in the 1990's, have been widely used by lecturers and graduate students from the School over the decades. Prof. Mason aimed at presenting a unified theory for fluid mechanics and elasticity, developing the theory using curvilinear coordinates, making his notes an invaluable resource. There is considerable pedagogical value in returning to the roots of fluid modelling and its influence on the development of ideas that led to GR, particularly through hands-on computation and manipulation of solutions to the Einstein Field Equations using a symbolic computation framework, such as Mathematica.

Dedication

To my parents, Miguel and Yvonne Esteves, and brother, Dominique Esteves, as well as my late grandmother and great-uncle, Adelaide Correia and Louis Morte.

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First and foremost, I am incredibly grateful to Prof. D. Mason for all the effort he put into writing the notes that inspired this dissertation. Your thorough, carefully compiled notes are testament to the quality of work you produce and your wholehearted interest in the success of your students. Your legacy of passion and dedication is an inspiration.

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To my family, friends and colleagues, I appreciate all of your support, advice and sacrifices. This has been a long journey with many helping hands and listening ears, thank you. This would not have been possible without you.

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Chapter 1

Introduction

This chapter is focussed on the introduction of tensor algebra and tensor analysis. A brief overview and notation introduction is supplied before diving into curvilinear coordinates where concepts are extended in a natural way from flat space to curved ones. Concluding the chapter, differentiation on a curved manifold is included to illustrate the subtle yet important differences between treatment of differentiation in a flat space and that of curvilinear ones. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]

1.1 Brief overview and notation introduction

Just over a century after Einstein's writing of 1914 – 1917, *The Berlin years*, General relativity (GR) is still the most robust theory of physics. This theory became the first to approximately predict the perihelion behaviour of Mercury[1], as well as offer mathematical justification for photon deflection. During the year 1916 of Einstein's Berlin years, he predicted the existence of Gravitational waves, which was only confirmed a century later by the LIGO experiment[2]. His theory was later used by a Russian and Soviet physicist and mathematician, Alexander Friedmann in 1922, to explain that a homogeneous and isotropic model of the universe is an expanding one. Friedmann derived the equations of physical cosmology governing this expansion phenomenon. These Friedmann equations formed the foundation of the FLRW model[3, 4, 5], the first empirically plausible cosmological model of our universe.

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In no way is this work to be interpreted as novel, as it is an amalgamation of several texts, but rather as a modern approach to the teaching of GR with symbolic computation at the postgraduate level. It is aimed at retaining the historic development of GR as a key approach to unifying Inertia and Gravity; rather than perspectives aimed at simplifying Gravity to merely be considered as an artefact of geometric representation.

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Given the nature of this text, all references used will be included in the italics introduction section of their respective chapters. In the case of long chapter, references will be included in the italics introduction sections of each chapter section.

Symbolic computation

Symbolic computation, also known as computer algebra, has been around for about half a century. R. A. D'Inverno published a paper in 1974 aimed at introducing potential users of the existence and power of an algebraic computer system[6]. The paper covers a description of the ALAM system and later versions which were developed by a relativity research group in London. Illustration solutions included solutions to the following metrics: Dingle, Schwarzschild, Bondi, Sach, Harrison, Crane, Jackson and Levy-Cohen. It is noted that solutions found with the aid of symbolic computation were often restricted to cases where axisymmetry could be assumed.

Later in 2016, C. G*ü*zelg*ü*n wrote an advanced physics project report on computational packages concerning GR[7]. G*ü*zelg*ü*n wrote about the use of four software systems with GR packages: Maple and GRTensor II, Sage and SageManifolds, Maxima and ctensor, and Python and GraviPy. In 2018, M. A. H. MacCallum wrote a review article on computer algebra in gravity research[8]. This thorough review describes what computer algebra is and how it is useful in the study of general relativity. Software systems described and discussed include: Macsyma (and Maxima), Maple, Mathematica, Reduce, Axiom, FORM, MuPAD, Redberry, Sage, SymbolicC++, Xcas, FORMAC, muMATH and SMP.

All software systems offer computer algebra capabilities, but require different levels of understanding/training before they can be used. The benefits of all of the options include: accuracy, speed, reduction repetitive/cumbersome tasks and computation of solutions that would otherwise be infeasible if done by hand. The disadvantages of such software systems include hardware limitations and software cost implications.

Python is a very good option for this kind of symbolic computation, being open source software makes it particularly desirable, but it does require some computer science knowledge to get up and running with. This is true for all the other open source software options as well. Propriety software on the other hand, often comes with support and user friendly interfaces. The one highly desirable feature of Mathematica is how user friendly it is for someone from a non-computer science background. Since this dissertation is aimed at producing a postgraduate introduction course for students from an applied mathematics background, Mathematica is a particularly good fit. It is a propriety software and as such has cost implications. Maple is also a good choice of software for symbolic computation, but it is more expensive than Mathematica (based on 2019 academic license pricing) and is slightly more complex to use. With all of this in mind, choosing to use Mathematica for symbolic computation was a convenient choice.

It is important to note that symbolic and numerical computation are distinct. Symbolic computation is used to arrive at the equations that are then solved either analytically or numerically. There is no shortage of software options for numerical computation but we choose to make use of Mathematica as it also offers this capability. The main motivation for the use of Mathematica lies with how user friendly it is for both symbolic and numerical computation.

Notation

The labelling of mathematical objects - vectors and tensors in this context - plays a vital role in keeping track of what mathematical operations are being performed; physically, it also assists in the way we represent and consequently, understand such objects. Index notation and the Einstein summation convention have proven to be a powerful tools for the component-wise analysis of tensor calculus. Before we dive right into setting up coordinate systems and how to transform in those coordinate systems, let us define the full notation of this text.

An index may be found in either the lower or upper positions on the right-hand side of a tensor quantity representation, thus defining the tensor as covariant or contravariant respectively. A tensor quantity that represents a scalar or vector quantity - which will be formally defined in the following section - is defined with either no index or a single index in either position, however a vector quantity is differentiated from a scalar quantity by its bold typography. Tensors of orders higher than one are defined by using two or more indices all in the covariant, contravariant or a combination of both positions.

The indices you will come across in the text will either be of Latin or Greek origin. Latin letters are specifically for pure spacial coordinate systems and will take on values of 1,2 and 3, where the Greek letters are specifically for time-space coordinate systems and will take on the values of 0, 1, 2 and 3. I.e. for cartesian coordinates the vector x^i is defined by

$$x^i = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

where, for cartesian coordinates in a pseudo-Riemannian manifold, the vector x^ν is defined by

$$x^\nu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}.$$

One should note that early Latin alphabet characters are used to denote the first set of Riemannian coordinates, e.g. x^i, and the late Latin alphabet characters are used to denote a different Riemannian coordinate system, e.g. $\overline{x}^s$.

Now, let us consider the Einstein summation convention. This convention requires a summation over two repeated indices, where one is covariant and the other is contravariant, i.e. consider the vector quantity $\mathbf{A}$ of the cartesian basis form

$$\begin{aligned}\mathbf{A} &= A^i\mathbf{e}_i\\ &= A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3\\ &\text{or}\\ \mathbf{A} &= A_i\mathbf{e}^i\\ &= A_1\mathbf{e}^1 + A_2\mathbf{e}^2 + A_3\mathbf{e}^3.\end{aligned}$$

If you are yet to appreciate the compact nature this notation, as you continue to work through this text it will become apparent why it is a convenient notation of choice.

1.2 Curvilinear coordinates

An introduction to tensor analysis is supplied by considering curvilinear coordinates, where the reader is exposed to the previously mentioned notation from a hands-on perspective. This is achieved by considering the curvilinear coordinaes as a function of rectilinear ones.

This section demonstrates the importance of covariant and contravariant components of vectors and tensors. It concludes with vector products and the triple scalar product, a convenient point to mention the surface element and volume element respectively.

1.2.1 Introduction to curvilinear coordinates

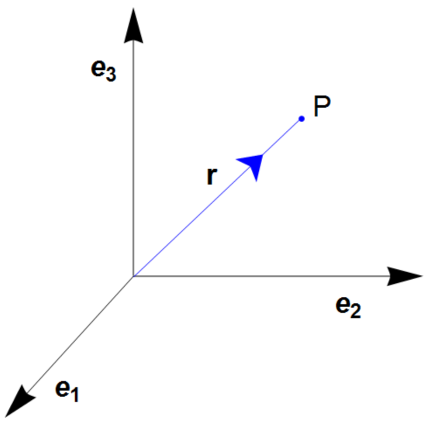


Figure 1.1: Vector r connecting the origin of a rectilinear coordinate system and a point at position P

Let (x^1, x^2, x^3) be the cartesian coordinates of a point P, as depicted above in [1] figure 1.1, referred to a right handed orthogonal cartesian set of axes.

$$\begin{aligned}\mathbf{r} &= x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3\\ &= x^n\mathbf{e}_n\end{aligned}$$

where

$$\mathbf{e}_n \cdot \mathbf{e}_m = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}.$$

[1]See Mathematica solution 1 for diagram creating instructions.

We introduce general coordinates θ^i by the transformation

$$\theta^i = \theta^i\left(x^1, x^2, x^3\right) \tag{1.1}$$

where θ^i are single-valued functions of the cartesian coordinates x^i which posses derivatives up to any required order. We assume that

$$det\left[\frac{\partial\theta^i}{\partial x^j}\right] = \left|\frac{\partial\theta^i}{\partial x^j}\right| \neq 0 \tag{1.2}$$

which implies that (1.1) is reversible and can be solved to give

$$x^i = x^i\left(\theta^1, \theta^2, \theta^3\right).$$

To each set of values of x^i there corresponds a set of values of θ^i and vice versa, this may not be unique: consider the spherical polar coordinates at the poles. Thus the variables θ^i determine points in 3d Euclideon space. We may represent the space by the variables θ^i instead of by the cartesian coordinates x^i.

Definition 1. Curvilinear coordinates

The variables $\left(\theta^1, \theta^2, \theta^3\right)$ are called curvilinear coordinates.

The reason for the name will become apparent soon. Consider now the geometrical significance of the coordinate θ^i,

$$\theta^i\left(x^1, x^2, x^3\right) = constant,$$

for $i = 1, 2, 3$ is the equation of a surface and as the value of the constant varies with each index i, a family of surfaces is obtained. I.e.

$$\begin{aligned}
\theta^1\left(x^1, x^2, x^3\right) &= constant,\\
\theta^2\left(x^1, x^2, x^3\right) &= constant,\\
\theta^3\left(x^1, x^2, x^3\right) &= constant.
\end{aligned}$$

The point of intersection of one member from each of these families determines a unique intersection point P of the space. Condition (1.2) expresses the fact that this point of intersection is unique.

Definition 2. Coordinate surfaces

The surface $\theta^i\left(x^1, x^2, x^3\right) =$ constant is called the θ^i-surface.

The intersections of these surfaces gives three curves through every point P, two of them lying on each coordinate surface.

Definition 3. Coordinate curves

The curves produced by the intersection of the three coordinate surfaces at a point P are called the coordinate curves.

Along the coordinate curve which is the intersection of the θ^2-surface and the θ^3-surface, θ^1 alone varies, the other coordinates being constant, and this coordinate curve is called the θ^1-curve. Similarly, the other coordinate curves are called the θ^2-curve and the θ^3-curve. The coordinate curves in this system of coordinates are therefore curved lines, justifying the name curvilinear coordinates.

Note: cartesian coordinates are a special case of curvilinear coordinates. Examples of curvilinear coordinates include: Sperical polar coordinates $\left(r, \theta, \phi\right)$ and Cylindrical polar coordinates (r, θ, z).

1.2.2 Base vectors

Let $\mathbf{r}$ be the position vector of a point P, as depicted in figure 1.1. Expressed in cartesian coordinates x^i,

$$\mathbf{r} = x^n \mathbf{e}_n.$$

Transform to curvilinear coordinates θ^i

$$x^n = x^n \left(\theta^1, \theta^2, \theta^3\right)$$

then the infinitesimal change in $\mathbf{r}$ is given by

$$\begin{aligned} d\mathbf{r} &= dx^n \mathbf{e}_n \\ &= \frac{\partial x^n}{\partial \theta^i} d\theta^i \mathbf{e}_n \\ &= \left(\frac{\partial x^n}{\partial \theta^i} \mathbf{e}_n\right) d\theta^i \\ &= \mathbf{g}_i d\theta^i \end{aligned}$$

where

$$\mathbf{g}_i = \left(\frac{\partial x^n}{\partial \theta^i} \mathbf{e}_n\right).$$

Consider now

$$\begin{aligned} \mathbf{g}_i \cdot \mathbf{g}_j &= \left(\frac{\partial x^n}{\partial \theta^i} \mathbf{e}_n\right) \cdot \left(\frac{\partial x^m}{\partial \theta^j} \mathbf{e}_m\right) \\ &= \frac{\partial x^n}{\partial \theta^i} \frac{\partial x^m}{\partial \theta^j} \mathbf{e}_n \cdot \mathbf{e}_m. \end{aligned}$$

However we have that $\mathbf{e}_n \cdot \mathbf{e}_m = 0$ if $n \neq m$ and $\mathbf{e}_n \cdot \mathbf{e}_m = 1$ if $n = m$, thus

$$\begin{aligned} \mathbf{g}_i \cdot \mathbf{g}_j &= \frac{\partial x^n}{\partial \theta^i} \frac{\partial x^n}{\partial \theta^j} \\ &= \frac{\partial x^1}{\partial \theta^i} \frac{\partial x^1}{\partial \theta^j} + \frac{\partial x^2}{\partial \theta^i} \frac{\partial x^2}{\partial \theta^j} + \frac{\partial x^3}{\partial \theta^i} \frac{\partial x^3}{\partial \theta^j} \end{aligned}$$

and so in general, the basis vectors are not orthonormal. I.e.

$$\mathbf{g}_i \cdot \mathbf{g}_j \neq \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}.$$

This translates to the understanding that $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$ are not orthogonal, contrary to the special case where rectilinear coordinates are considered. By definition the rectilinear / cartesian coordinates must have basis vectors that are orthogonal and unit.

As the basis vectors of the curvilinear coordinates are not orthogonal or unit, let us introduce a vector orthogonal to them. Consider

$$\mathbf{g}^i = \frac{\partial \theta^i}{\partial x^n} \mathbf{e}^n$$

and let us formally define these objects.[2]

[2]Calculate the spherical polar coordinate covariant and contravariant base vectors using equations (1.3) and (1.4) in Mathematica. See 2 for a possible Mathematica solution.

Definition 4. Covariant base vectors

The N base vectors $\mathbf{g}_i$, $i = 1, 2, \cdots, N$ are called the covariant base vectors

$$\mathbf{g}_i = \frac{\partial x^n}{\partial \theta^i} \mathbf{e}_n \tag{1.3}$$

Definition 5. Contravariant base vectors

The N base vectors $\mathbf{g}^i$, $i = 1, 2, \cdots, N$ are called the contravariant base vectors

$$\mathbf{g}^i = \frac{\partial \theta^i}{\partial x^n} \mathbf{e}^n \tag{1.4}$$

With the above definitions in mind, one should be comforted by the fact that $\mathbf{e}^n = \mathbf{e}_n$. This indicates the insignificance of index position when considering rectilinear / cartesian coordinates.

Now that the contravariant base vectors have been introduced, let us consider what is known about these vectors. These vectors are not necessarily unit, but are most definitely orthogonal to the covariant base vectors. So let us consider the scalar product of the curvilinear covariant base vectors and the curvilinear contravariant base vectors. One would expect this result to be related to the scalar product of the covariant and contravariant base vectors of rectilinear coordinates. This relation would be inferred by definitions 4 and 5. Before we investigate these properties, let us consider the scalar product of the covariant and contravariant base vectors of the rectilinear coordinate system.

$$\mathbf{e}_m \cdot \mathbf{e}^n = \mathbf{e}_m \cdot \mathbf{e}_n = \mathbf{e}^m \cdot \mathbf{e}_n = \mathbf{e}^m \cdot \mathbf{e}^n$$

since $\mathbf{e}^n = \mathbf{e}_n$, and since

$$\mathbf{e}_n \cdot \mathbf{e}_m = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m, \end{cases}$$

we may define the Kronecher delta, a symmetric second-order mixed tensor,

$$\delta_m{}^n = \mathbf{e}_m \cdot \mathbf{e}^n = \mathbf{e}^m \cdot \mathbf{e}_n = \delta^m{}_n.$$

Definition 6. Kronecker delta

The Kronecher delta, $\delta^i{}_k = \delta_i{}^k$ equal to 0 for $i \neq k$ and is equal to 1 for $i = k$.

Let us now consider the scalar product of the base vectors in curvilinear coordinates: [3]

$$\begin{aligned}
\mathbf{g}^i \cdot \mathbf{g}_j &= \left(\frac{\partial \theta^i}{\partial x^n}\mathbf{e}^n\right) \cdot \left(\frac{\partial x^m}{\partial \theta^j}\mathbf{e}_m\right) \\
&= \frac{\partial \theta^i}{\partial x^n}\frac{\partial x^m}{\partial \theta^j}\mathbf{e}^n \cdot \mathbf{e}_m = \frac{\partial \theta^i}{\partial x^n}\frac{\partial x^m}{\partial \theta^j}\delta^n_m \\
&= \frac{\partial \theta^i}{\partial x^n}\frac{\partial x^n}{\partial \theta^j} = \frac{\partial \theta^i}{\partial \theta^j} \\
&= \delta^i_j
\end{aligned}$$

Thus if $i \neq j$, $\mathbf{g}^i$ is orthogonal to $\mathbf{g}_j$. We note the following:

[3]Verify that $\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i{}^j = \delta^j{}_i = \mathbf{g}^j \cdot \mathbf{g}_i$ using spherical polar coordinates. See 3 for a possible Mathematica solution.

(i) $\mathbf{g}_i$ and $\mathbf{g}^i$ are not in general unit vectors.

(ii) $\mathbf{g}^1$ is orthogonal to the θ^1 - surface (plane constructed by $\mathbf{g}_2$ and $\mathbf{g}_3$)
$\mathbf{g}^2$ is orthogonal to the θ^2 - surface (plane constructed by $\mathbf{g}_3$ and $\mathbf{g}_1$)
$\mathbf{g}^3$ is orthogonal to the θ^3 - surface (plane constructed by $\mathbf{g}_1$ and $\mathbf{g}_2$)

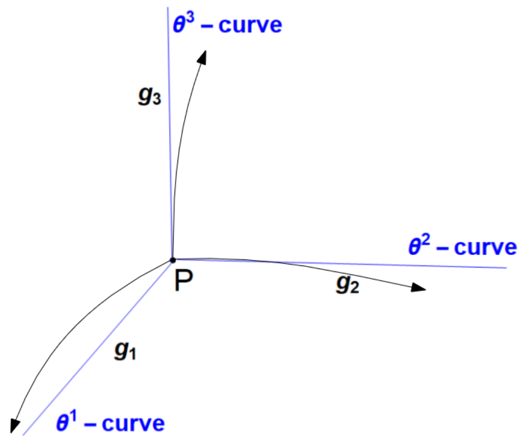


Figure 1.2: Point P at the origin of a curvilinear coordinate axes superimposed with the θ^i-curve axes

We have now derived definitions for curvilinear base vectors and covectors in terms of rectilinear ones. It may be shown that the definitions of rectilinear base vectors and covectors may be defined in terms of curvilinear ones:

$$\mathbf{e}_m = \frac{\partial \theta^i}{\partial x^m}\mathbf{g}_i \qquad \text{and} \qquad \mathbf{e}^m = \frac{\partial x^m}{\partial \theta^i}\mathbf{g}^i.$$

1.2.3 Vectors and tensors

We have established the representation of a point in space relative to frames of reference in their arbitrary coordinates, and now we will be considering representation of higher order tensors and how one may relate tensor quantities relative to different frames of reference in their arbitrary coordinates. Consider a vector **A** with components (A^1, A^2, A^3) with respect to a rectilinear basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$

$$\mathbf{A} = A^n \mathbf{e}_n.$$

This vector represented by the curvilinear basis vectors $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$:

$$\mathbf{A} = \overline{A}^i \mathbf{g}_i.$$

As both of these vector representations are equivalent, thus

$$\overline{A}^i = \frac{\partial \theta^i}{\partial x^n} A^n. \tag{1.5}$$

The above equation, (1.5), is the transformation law for components A^i between cartesian coordinates x^n and curvilinear coordinates θ^i. We want to go further and determine the transformation law of components A^i between two sets of curvilinear coordinates. Let $\overline{\theta}^i$ be another set of curvilinear coordinates, such that

$$\overline{\theta}^i = \overline{\theta}^i\left(\theta^1, \theta^2, \theta^3\right).$$

Building from the same argument, when finding the transformation law of the contravariant vector components between curvilinear and rectilinear coordinates, we define

$$\mathbf{A} = A^i \mathbf{g}_i = \overline{A}^k \overline{\mathbf{g}}_k. \tag{1.6}$$

Now, let us first determine the transformation for the basis covariant vectors in the new curvilinear coordinates, $\overline{\mathbf{g}}_k$, relative to the curvilinear ones $\mathbf{g}_i$. Recall the transformations for curvilinear coordinate covariant base vectors and rectilinear ones, and apply them to find the transformation between the new curvilinear coordinate base covectors in terms of the rectilinear ones.

$$\overline{\mathbf{g}}_i = \frac{\partial x^n}{\partial \overline{\theta}^i} \mathbf{e}_n \tag{1.7}$$

$$\mathbf{e}_n = \frac{\partial \overline{\theta}^k}{\partial x^n} \overline{\mathbf{g}}_k \tag{1.8}$$

Substitute equation (1.8) into the definition of the curvilinear base covectors, in terms of the rectilinear base covectors, to obtain:

$$\mathbf{g}_i = \frac{\partial \overline{\theta}^k}{\partial \theta^i} \overline{\mathbf{g}}_k. \tag{1.9}$$

Similarly, we can use the definition of the rectilinear base covectors, in terms of the first curvilinear base covectors, in equation (1.7) to obtain

$$\overline{\mathbf{g}}_i = \frac{\partial \theta^k}{\partial \overline{\theta}^i} \mathbf{g}_k. \tag{1.10}$$

Now, by making a substitution with equation (1.9) into equation (1.6) we obtain the definition of the contravariant components of a vector. By considering a similar argument, we may also obtain the definition of the covariant components of vector.

Definition 7. Contravariant components of a vector

If under the coordinate transformation $\overline{\theta}^i = \overline{\theta}^i \left(\theta^1, \theta^2, \cdots, \theta^N\right)$,

$$\overline{A}^i = \frac{\partial \overline{\theta}^i}{\partial \theta^k} A^k$$

then $A^i \left(\theta^1, \theta^2, \cdots, \theta^N\right)$ for $i = 1, 2, \cdots, N$, are the components of a contravariant vector (contravariant tensor of order one). They are referred to as the contravariant components of the vector $\mathbf{A}$.

Definition 8. Covariant components of a vector

If under the coordinate transformation $\overline{\theta}^i = \overline{\theta}^i \left(\theta^1, \theta^2, \cdots, \theta^N\right)$,

$$\overline{A}_i = \frac{\partial \theta^k}{\partial \overline{\theta}^i} A_k$$

then $A_i \left(\theta^1, \theta^2, \cdots, \theta^N\right)$ for $i = 1, 2, \cdots, N$, are the components of a covariant vector (covariant tensor of order one). They are referred to as the covariant components of the vector $\mathbf{A}$.

We note the following:

(i) $\mathbf{A} = A^i\,\mathbf{g}_i = \overline{A}^i\,\overline{\mathbf{g}}_i = A_i\,\mathbf{g}^i = \overline{A}_i\,\overline{\mathbf{g}}^i$, **A** is an invariant because it has the same value and form with respect to all variables.

(ii) The contravariant components A^i and the covariant A_i are two different representations of the same vector **A**. The word component is frequently ommitted and we refer to A^i and A_i as contravariant and covariant vectors respectively. They are not different vectors but two aspects of the single vector space.

(iii) When the vector **A** is represented by a directed line from point P at which the values of the vector are calculated, the contravariant components A^i are the components of **A** in the direction of the covariant base vectors $\mathbf{g}_i$, while the covariant components A^i are the components of **A** in the direction of the contravariant base vectors $\mathbf{g}^i$. This is illustrated in figure 1.3 for 2-dimensional vectors, where

$$\mathbf{A} = A_i\mathbf{g}^i = A^i\mathbf{g}_i = A_1\mathbf{g}^1 + A_2\mathbf{g}^2 = A^1\mathbf{g}_1 + A^2\mathbf{g}_2.$$

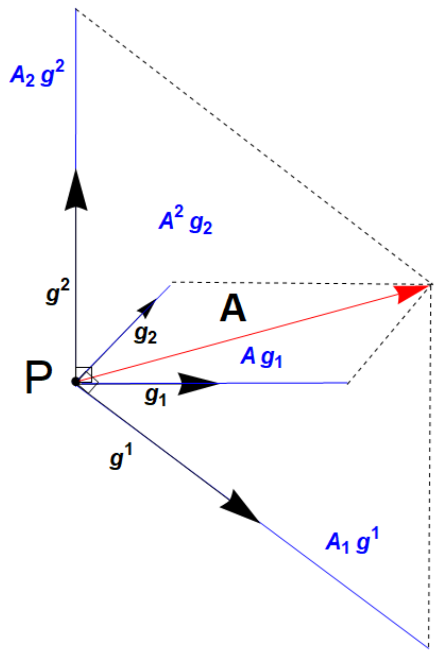


Figure 1.3: Vector **A** decomposed into covariant and contravariant component products of curvilinear base vectors and covectors

Definition 9. Scalar

If under the coordinate transformation $\overline{\theta}^i = \overline{\theta}^i\left(\theta^1, \theta^2, \cdots, \theta^N\right)$,

$$\overline{\phi}\left(\overline{\theta}^1, \overline{\theta}^2, \cdots, \overline{\theta}^N\right) = \phi\left(\theta^1, \theta^2, \cdots, \theta^N\right)$$

then ϕ is scalar (tensor of order zero), and ϕ is a scalar invariant or simply invariant.

Definition 10. Contravariant tensor of order two

If under the coordinate transformation $\overline{\theta}^i = \overline{\theta}^i\left(\theta^1, \theta^2, \cdots, \theta^N\right)$,

$$\overline{t}^{ij} = \frac{\partial\overline{\theta}^i}{\partial\theta^m}\frac{\partial\overline{\theta}^j}{\partial\theta^n}t^{mn}$$

then the N^2 functions $t^{ij}\left(\theta^1, \theta^2, \cdots, \theta^N\right)$ are the components of a contravariant tensor of order two.

Definition 11. Covariant tensor of order two

If under the coordinate transformation $\overline{\theta}^i = \overline{\theta}^i\left(\theta^1, \theta^2, \cdots, \theta^N\right)$,

$$\overline{t}_{ij} = \frac{\partial\theta^m}{\partial\overline{\theta}^i}\frac{\partial\theta^n}{\partial\overline{\theta}^j}t_{mn}$$

then the N^2 functions $t_{ij}\left(\theta^1, \theta^2, \cdots, \theta^N\right)$ are the components of a covariant tensor of order two.

Definition 12. Mixed tensor of order two

If under the coordinate transformation $\overline{\theta}^i = \overline{\theta}^i\left(\theta^1, \theta^2, \cdots, \theta^N\right)$,

$$\overline{t}^i_j = \frac{\partial\overline{\theta}^i}{\partial\theta^m}\frac{\partial\theta^n}{\partial\overline{\theta}^j}t^m_n$$

then the N^2 functions $t^i_j\left(\theta^1, \theta^2, \cdots, \theta^N\right)$ are the components of a mixed tensor of order two.

Definition 13. Mixed tensor of order three

If under the coordinate transformation $\overline{\theta}^i = \overline{\theta}^i\left(\theta^1, \theta^2, \cdots, \theta^N\right)$,

$$\overline{t}^i{}_{jk} = \frac{\partial\overline{\theta}^i}{\partial\theta^m}\frac{\partial\theta^n}{\partial\overline{\theta}^j}\frac{\partial\theta^p}{\partial\overline{\theta}^k}t^m{}_{np}$$

then the N^3 functions $t^i{}_{jk}\left(\theta^1, \theta^2, \cdots, \theta^N\right)$ are the components of a mixed tensor of order three, contravariant in i and covariant in j and k.

We note that if all the components of a tensor are zero in one curvilinear coordinate system then they are all zero in any curvilinear coordinate system. In particular if all cartesian components of a tensor are zero, then so are all its components with respect to any curvilinear basis.

Definition 14. Contraction

When an upper index is set equal to a lower index and summation is carried out on that index, by means of multiplication with the mixed metric tensor, the operation is called a contraction.

E.g. $A^{ij}{}_k$ has two possible contractions, $\delta^k{}_i A^{ij}{}_k = A^{kj}{}_k$ and $\delta^k{}_j A^{ij}{}_k = A^{ik}{}_k$.

A contraction may also be taken over two indices by making use of the metric and inverse metric tensors - these will be defined later in this section - in the following manner:

$$g_{ij}A^{ij} = A = g^{ij}A_{ij}.$$

It is very important to note that a contraction must be performed on one upper and one lower index unless specifically stated that cartesian coordinates are being used.

1.2.4 Tensor multiplication and the quotient theorem

There are two types of tensor multiplication that one may perform, namely the outer and inner products of two tensors.

Definition 15. Outer product

By particular example: If ${A^i}_j$ and ${B_m}^n$ are tensors then

$${C^i}_{jm}{}^n = {A^i}_j {B_m}^n$$

is a tensor of order four, contravariant in i and n covariant in j and m. It is called the outer product of ${A^i}_j$ and ${B_m}^n$. The outer product of two tensors is always a tensor.

Definition 16. Inner product

By particular example: If ${A^i}_j$ and ${B_m}^n$ are tensors and if ${A^i}_j\ {B_m}^n$ is contracted with respect to an upper index in one and a lower index in the other, the resulting tensor is called the inner product of A and B.

Definition 17. Tensor product

By particular example: If **U** and **V** are tensors, then the tensor product of **U** with **V** is defined by

$$\begin{aligned}\mathbf{U}\otimes\mathbf{V} &= (U\otimes V)^{ij}\\ &= U^i V^j.\end{aligned}$$

Theorem 1. ***Quotient theorem***

Any set of functions of the coordinates, the inner or outer product of which with an arbitrary tensor is always a tensor, are themselves the components of a tensor.

Proof. Suppose that A_{ij} is a system of n^2 functions of the coordinates $\theta^1, \theta^2, \cdots, \theta^n$ such that

$$A_{ij}B^i = C_j \tag{1.11}$$

for every covariant tensor B^i where C_j is known to be a covariant tensor. We prove that A_{ij} is a covariant tensor. Let A^*_{ij} be the quantities which replace A_{ij} under the coordinate transformation $\overline{\theta}^i = \overline{\theta}^i(\theta^1, \theta^2, \cdots, \theta^n)$. Then

$$A^*_{ij}\overline{B}^i = \overline{C}_j \tag{1.12}$$

where

$$\overline{B}^i = \frac{\partial\overline{\theta}^i}{\partial\theta^n} B^n$$

$$\overline{C}_j = \frac{\partial\theta^m}{\partial\overline{\theta}^j} C_m.$$

Introduce the n^2 quantities $\overline{A}_{ij}$ defined by

$$\overline{A}_{ij} = \frac{\partial\theta^r}{\partial\overline{\theta}^i}\frac{\partial\theta^s}{\partial\overline{\theta}^j} A_{rs}.$$

Then

$$\begin{aligned}
\overline{A}_{ij}\overline{B}^i &= \frac{\partial\theta^r}{\partial\overline{\theta}^i}\frac{\partial\theta^s}{\partial\overline{\theta}^j} A_{rs}\frac{\partial\overline{\theta}^i}{\partial\theta^n} B^n \\
&= \frac{\partial\theta^r}{\partial\theta^n}\frac{\partial\theta^s}{\partial\overline{\theta}^j} A_{rs} B^n \\
&= \delta^r_n \frac{\partial\theta^s}{\partial\overline{\theta}^j} A_{rs} B^n \\
&= \frac{\partial\theta^s}{\partial\overline{\theta}^j} A_{rs} B^r \\
&= \frac{\partial\theta^s}{\partial\overline{\theta}^j} C_s
\end{aligned}$$

by equation (1.11). Hence,

$$\overline{A}_{ij}\overline{B}^i = \overline{C}_j. \tag{1.13}$$

Subtract equation (1.13) from equation (1.12)

$$\left(A^*_{ij} - \overline{A}_{ij}\right)\overline{B}^i = 0. \tag{1.14}$$

But $\overline{B}^i$ is arbitrary since B^i is arbitrary. Choose in turn that $\overline{B}^i$ and B^i are both nonzero. Equation (1.14) then yields

$$A^*_{ij} - \overline{A}_{ij} = 0$$

$$A^*_{ij} = \overline{A}_{ij}.$$

Thus A_{ij} is a covariant tensor of order two. □

1.2.5 Symmetric and skew-symmetric tensors

The tensors A^{ik} and B_{ik} are symmetric if

$$A^{ik} = A^{ki} \qquad \text{and} \qquad B_{ik} = B_{ki}.$$

Definition 18. Symmetric

A tensor of any order is said to be symmetric in two contravariant indices or in two covariant indices if it is unaltered when these two indices are interchanged. The tensor is completely symmetric if the interchange of any contravariant indices or any covariant indices leaves it unaltered.

E.g. A tensor A_{ijk} of the third order which is completely symmetric satisfies the relations

$$A_{ijk} = A_{jki} = A_{kij} = A_{jik} = A_{kji} = A_{ikj}$$

Definition 19. Skew-symmetric

A tensor of any order is said to be skew-symmetric in two contravariant indices or in two covariant indices if the interchange of two indices alters only the sign of the components. The tensor is completely skew-symmetric if the interchange of any contravariant indices or any covariant indices changes only the sign of the components.

E.g. The tensors A^{ik} and B_{ik} are skew-symmetric if

$$A^{ik} = -A^{ki} \qquad \text{and} \qquad B_{ik} = -B_{ki}.$$

It follows directly that $A^{ii} = 0$, $B_{ii} = 0$, where i is not summed.

We note the following:

(i) If a tensor is symmetric or skew-symmetric with respect to two indices in one coordinate system then it is so in every coordinate system. For example, suppose that

$$A^{ik} = -A_{ki}.$$

We show that $\overline{A}^{ik} = -\overline{A}_{ki}$. For

$$\begin{aligned}
\overline{A}^{ik} &= \frac{\partial\overline{\theta}^i}{\partial\theta^r}\frac{\partial\overline{\theta}^k}{\partial\theta^s}A^{rs} \\
&= \frac{\partial\overline{\theta}^i}{\partial\theta^s}\frac{\partial\overline{\theta}^k}{\partial\theta^r}A^{sr} && \text{(renaming dummy indices, r→s, s→r)} \\
&= -\frac{\partial\overline{\theta}^i}{\partial\theta^s}\frac{\partial\overline{\theta}^k}{\partial\theta^r}A^{rs} && (A^{sr} = -A^{rs}) \\
&= -\frac{\partial\overline{\theta}^k}{\partial\theta^r}\frac{\partial\overline{\theta}^i}{\partial\theta^s}A^{rs} && \left(\text{order of } \frac{\partial\overline{\theta}^k}{\partial\theta^r} \text{ and } \frac{\partial\overline{\theta}^i}{\partial\theta^s} \text{ is not significant}\right)
\end{aligned}$$

therefore $\overline{A}^{ik} = -\overline{A}^{ki}$.

(ii) Every second order contravariant or covariant tensor may be expressed uniquely as the sum of a symmetric and skew-symmetric tensor. For,

$$\begin{aligned}
A_{ik} &= \frac{1}{2}(A_{ik} + A_{ki}) + \frac{1}{2}(A_{ik} - A_{ki}) \\
&= B_{ik} + C_{ik}
\end{aligned}$$

where

$$\begin{aligned}
B_{ik} &= \frac{1}{2}(A_{ik} + A_{ki}) && \text{(Symmetric)} \\
C_{ik} &= \frac{1}{2}(A_{ik} - A_{ki}) && \text{(Skew-symmetric).}
\end{aligned}$$

Suppose that the decomposition is not unique.

$$A_{ik} = D_{ik} - E_{ik} \qquad (D_{ik} = D_{ki},\ E_{ik} = -E_{ki}) \tag{1.15}$$

then

$$A_{ki} = D_{ik} - E_{ik} \tag{1.16}$$

Add equations (1.15) and (1.16)

$$\begin{aligned} D_{ik} &= \frac{1}{2}(A_{ik} + A_{ki}) \\ &= B_{ik} \end{aligned}$$

Subtract equation (1.16) from equation (1.15)

$$\begin{aligned} E_{ik} &= \frac{1}{2}(A_{ik} - A_{ki}) \\ &= C_{ik} \end{aligned}$$

Thus the decomposition is unique.

(iii) If $A^{ik} = A^{ki}$ and $B_{ik} = -B_{ki}$, then

$$A^{ik}B_{ik} = 0. \tag{1.17}$$

Verification of equation(1.17) is found in the following equation:

$$\begin{aligned} A^{ik}B_{ik} &= A^{ki}B_{ki} && \text{(renaming dummy indeces, } i \to k,\ k \to i\text{)} \\ &= A^{ik}(-B_{ik}) && (A^{ki} = A^{ik},\ B_{ki} = -B_{ik}) \\ &= -A^{ik}B_{ik} \\ \therefore\ \ 2A^{ik}B_{ik} &= 0. \end{aligned}$$

1.2.6 The metric and alternating tensors

In definition 6 we defined the second-order tensor, the Kronecker delta, contravariant in one index and covariant in another index, i.e.

$$\delta^i{}_j = g^i{}_j \tag{1.18}$$
$$\delta_i{}^j = g_i{}^j. \tag{1.19}$$

This result was obtained by taking the product of a contravariant base vector with a covariant base vector. Recall definition of the curvilinear base covectors in terms of rectilinear ones and consider the product of two covariant base vectors

$$\begin{aligned} \mathbf{g}_i \cdot \mathbf{g}_k &= \left(\frac{\partial x^n}{\partial \theta^i}\mathbf{e}_n\right) \cdot \left(\frac{\partial x^m}{\partial \theta^k}\mathbf{e}_m\right) \\ g_{ik} &= \frac{\partial x^n}{\partial \theta^i}\frac{\partial x^m}{\partial \theta^k}\mathbf{e}_n \cdot \mathbf{e}_m \end{aligned}$$

but since $\mathbf{e}_n$ are orthonormal, we may write[4]

$$g_{ik} = \frac{\partial x^n}{\partial \theta^i}\frac{\partial x^m}{\partial \theta^k}\delta_{nm}. \tag{1.20}$$

The inverse of this relation may be determined in a similar fashion by considering the definition of the rectilinear base covectors in terms of curvilinear ones,

$$\delta_{nm} = \frac{\partial \theta^i}{\partial x^n}\frac{\partial \theta^k}{\partial x^m}g_{ik}. \tag{1.21}$$

Similarly, by taking the product of two contravariant base vectors one may obtain

$$g^{ik} = \frac{\partial \theta^i}{\partial x^n}\frac{\partial \theta^k}{\partial x^m}\delta^{nm} \tag{1.22}$$

from the definition of curvilinear base vectors in terms of rectilinear ones, and

$$\delta^{nm} = \frac{\partial x^n}{\partial \theta^i}\frac{\partial x^m}{\partial \theta^k}g^{ik} \tag{1.23}$$

from the definition of rectilinear base vectors in terms of curvilinear ones.

Definition 20. Metric tensors

The tensors g_{ik}, g^{ik}, ${g^i}_k$ and ${g_k}^i$ are called the covariant, contravariant and mixed metric tensors, and their components are defined by equations (1.20), (1.22), (1.18) and (1.19).

We note that the metric tensors g_{ik} and g^{ik} are both symmetric tensors, i.e.

$$g_{ik} = g_{ki} \tag{1.24}$$
$$g^{ik} = g^{ki}. \tag{1.25}$$

This may be shown by considering equation (1.20)

$$\begin{aligned} g_{ik} - g_{ki} &= \frac{\partial x^n}{\partial \theta^i}\frac{\partial x^m}{\partial \theta^k}\delta_{nm} - \frac{\partial x^m}{\partial \theta^k}\frac{\partial x^n}{\partial \theta^i}\delta_{mn} \\ g_{ik} - g_{ki} &= \frac{\partial x^n}{\partial \theta^i}\frac{\partial x_n}{\partial \theta^k} - \frac{\partial x^n}{\partial \theta^i}\frac{\partial x_n}{\partial \theta^k} \\ g_{ik} &= g_{ki} \end{aligned}$$

and thus equation (1.24) is true. Similarly, one may consider equation (1.22) to verify equation (1.25). Let us now consider the third order tensors which are derived from the permutation tensor.

Definition 21. Permutation tensor

In cartesian coordinates x^i, the permutation tensor is defined by

$$\begin{aligned} e_{ijk} &= e^{ijk} \\ &= \begin{cases} 0 & \text{if any two indices } i, j, k \text{ are the same} \\ +1 & \text{if } i, j, k \text{ is an even permutation of } 1\ 2\ 3 \\ -1 & \text{if } i, j, k \text{ is an odd permutation of } 1\ 2\ 3 \end{cases} \end{aligned}$$

[4] Determine the metric and inverse metric tensors in spherical polar coordinates using equations (1.20) and (1.22), respectively, in Mathematica. See 4 for a possible Mathematica solution.

When speaking about i, j, k being an even or odd permutations of 123, reference is being made to the number of interchanges required to put the configuration in the form 123. An example of an even and odd permutation is the configuration 312 and 132 respectively. The configuration 312 would have to undergo two interchanges to attain the form 123, and the configuration 132 would have to undergo one interchange to attain the form 123.

Definition 22. Alternating tensor

The tensors ϵ_{ijk} and ϵ^{ijk} whose components are defined by

$$\varepsilon_{ijk} = \frac{\partial x^r}{\partial \theta^i}\frac{\partial x^s}{\partial \theta^j}\frac{\partial x^t}{\partial \theta^k} e_{rst}$$
$$\varepsilon^{ijk} = \frac{\partial \theta^i}{\partial x^r}\frac{\partial \theta^j}{\partial x^s}\frac{\partial \theta^k}{\partial x^t} e^{rst}$$

are called the covariant and contravariant alternating tensors.

The alternating tensor and determinants of matrices

Consider now the matrix representation of the second-order tensors A_{ik}, A^{ik}, $A^i{}_k$ and $A_i{}^k$. The first index, i, represents the row, and the second index, k, represents the column in the matrix representation of the aforementioned second-order tensors. We denote the determinant of a tensor A_{ij} as $\left|A_{ij}\right|$.

Exercise 1. *If we define $g = \left|g_{ij}\right|$ then, by making use of the above expressions for determinants, show that:*

i) $g = \left|\frac{\partial x^i}{\partial \theta^j}\right|^2$

ii) $\frac{1}{g} = \left|\frac{\partial \theta^i}{\partial x^j}\right|^2$

iii) $\varepsilon_{ijk} = \sqrt{g}\; e_{ijk}$

iv) $\varepsilon^{ijk} = \frac{1}{\sqrt{g}}\; e^{ijk}$

See solution 1.

1.2.7 The metric tensor and the scalar product

The line element ds

Let $\mathbf{r}$ be the position vector of a point P with cartesian coordinates x^i.

$$\mathbf{r} = x^n \mathbf{e}_n \qquad \Rightarrow \qquad d\mathbf{r} = dx^n \mathbf{e}_n$$

Let ds denote the magnitude of $d\mathbf{r}$, then we may refer to ds as the line element and define the squared line element as

$$ds^2 = g_{ij} d\theta^i d\theta^j.$$

Raising and lowering of indices

The inverse metric tensor and metric tensor may, respectively, be used to raise and lower indices in the following manner:

$$A^i = g^{ik} A_k$$
$$A_i = g_{ik} A^k$$

Scalar product

The scalar product and the magnitude of a vector both require the metric or inverse metric tensors and are formally defined as follows:

Definition 23. Scalar product

The scalar product of **A** with **B** is

$$\mathbf{A}\cdot\mathbf{B} = g_{ij}A^iB^j = g^{ij}A_iB_j = A_iB^i = A^iB_i$$

Definition 24. Magnitude of a vector

The magnitude of the vector **A** is

$$A = ||\mathbf{A}|| = \sqrt{\mathbf{A}\cdot\mathbf{A}} = \sqrt{g_{ij}A^iA^j} = \sqrt{g^{ij}A_iA_j} = \sqrt{A^jA_j}$$

1.2.8 Vector product and surface element

For rectangular cartesian base vectors

$$\mathbf{e}_r \times \mathbf{e}_s = e_{rst}\mathbf{e}^t \qquad \text{and} \qquad \mathbf{e}^r \times \mathbf{e}^s = e^{rst}\mathbf{e}_t.$$

So, if we define the vector **A** and **B** as

$$\mathbf{A} = A^m\mathbf{e}_m = A_m\mathbf{e}^m \qquad \text{and} \qquad \mathbf{B} = B^n\mathbf{e}_n = B_n\mathbf{e}^n,$$

then the vector product of the two vectors may be expressed as

$$\mathbf{A}\times\mathbf{B} = (A_m\mathbf{e}^m)\times(B_n\mathbf{e}^n) = A^mB^n\mathbf{e}_m\times\mathbf{e}_n = A^mB^ne_{mnp}\mathbf{e}^p.$$

Similarly

$$\mathbf{A}\times\mathbf{B} = A_mB_ne^{mnp}\mathbf{e}_p.$$

Now, consider the alternating tensor definition and the relations between cartesian and curvilinear base vectors.[5] It may be shown that the vector product of two curvilinear base covectors is defined by

$$\mathbf{g}_i\times\mathbf{g}_j = \varepsilon_{ijk}\mathbf{g}^k.$$

and similarly for the vector product of two curvilinear base vectors is defined by

$$\mathbf{g}^i\times\mathbf{g}^j = \varepsilon^{ijk}\mathbf{g}_k$$

Definition 25. Vector product

The vector product of vectors **A** and **B** is

$$\mathbf{A}\times\mathbf{B} = \begin{cases} \varepsilon_{ijk}A^iB^j\mathbf{g}^k = \sqrt{g}\,e_{ijk}A^iB^j\mathbf{e}^k \\ \varepsilon^{ijk}A_iB_j\mathbf{g}_k = \dfrac{1}{\sqrt{g}}e^{ijk}A_iB_j\mathbf{e}_k \end{cases}$$

[5] Define the Levi-Cevita symbol/alternating tensor in spherical polar coordinates using definitions 22 in Mathematica. See 5 for a possible Mathematica representation.

1.2.9 Scalar triple product and volume element

Definition 26. Scalar triple product

The scalar triple product, $[\mathbf{A},\mathbf{B},\mathbf{C}]$ of vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ is defined by

$$[\mathbf{A},\mathbf{B},\mathbf{C}] = \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$$

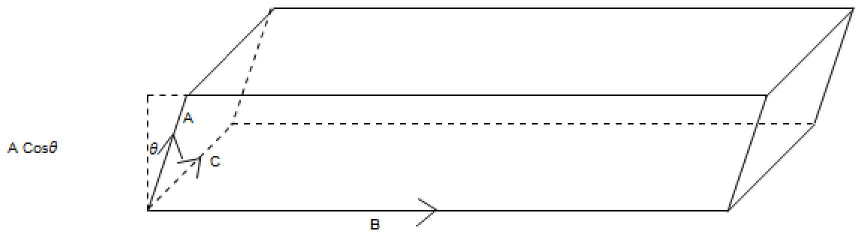


Figure 1.4: Parallelepiped diagram

$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ is the volume of the parallelepiped of which $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are adjacent sides. For

$$\begin{aligned}\text{Area of base} &= |\mathbf{B}\times\mathbf{C}|\\ \text{Vertical height} &= A\cos\theta\\ \text{Volume} &= \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})\end{aligned}$$

Consider the scalar triple product of vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ defined in curvilinear base covectors

$$\begin{aligned}[\mathbf{A},\mathbf{B},\mathbf{C}] &= \left[A^i\mathbf{g}_i, B^j\mathbf{g}_j, C^k\mathbf{g}_k\right]\\ &= A^iB^jC^k\,\mathbf{g}_i\cdot\left(\mathbf{g}_j\times\mathbf{g}_k\right)\\ &= A^iB^jC^k\,\mathbf{g}_i\cdot\left(\epsilon_{jkl}\,\mathbf{g}^l\right)\\ &= A^iB^jC^k\,\varepsilon_{jkl}\left(\mathbf{g}_i\cdot\mathbf{g}^l\right)\\ &= A^iB^jC^k\,\varepsilon_{jkl}\,\delta_i{}^l\\ &= \varepsilon_{ijk}\,A^iB^jC^k.\end{aligned}$$

In a similar fashion, it may be shown that the scalar triple product of vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ defined in curvilinear base vectors is

$$[\mathbf{A},\mathbf{B},\mathbf{C}] = \varepsilon^{ijk}\,A_iB_jC_k.$$

1.3 Differentiation on a curved manifold

The transformation laws of conserved tensorial quantities is now considered, where it is brought to the readers attention that differentiation on a curved manifold depends on the current position on the manifold. This section begins by addressing the differentiation of basis vectors, introducing the reader to Christoffel symbols/connection coefficients, $\Gamma^a{}_{bc}$, and then arriving at the definition of covariant differentiation. The Riemann-Christoffel curvature tensor, $R^a{}_{bcd}$, is introduced, a derivation of the derivative operator in curvilinear coordinates is supplied, and this section concludes with a look at the physical components of tensors and a derivation of the divergence theorem.

1.3.1 Differentiation of base vectors

Consider a tensor that is defined at every point of a space, then the tensor is said to form a tensor field. As we move to the tensor calculus realm, it is important that a notion convention is introduced. Let us define a comma as the partial derivative with respect to θ^i, i.e.

$$\phi_{,i} = \frac{\partial \phi}{\partial \theta^i}$$

and in the case where a second partial derivative has been taken, then we omit the second comma, i.e.

$$\frac{\partial^2 \phi}{\partial \theta^i \partial \theta^j} = \phi_{,i,j} \; = \; \phi_{,ij}$$

Now that we have described the notation to be following, let us consider the partial derivative of the covariant base vector $\mathbf{g}_i$

$$\begin{aligned}
\frac{\partial}{\partial \theta^j}\mathbf{g}_i &= \frac{\partial}{\partial \theta^j}\left(\frac{\partial x^n}{\partial \theta^i}\mathbf{e}_n\right) \\
\mathbf{g}_{i,j} &= \left(\frac{\partial}{\partial \theta^j}\frac{\partial x^n}{\partial \theta^i}\right)\mathbf{e}_n + \frac{\partial x^n}{\partial \theta^i}\left(\frac{\partial}{\partial \theta^j}\mathbf{e}_n\right) \\
&= \frac{\partial^2 x^n}{\partial \theta^i \partial \theta^j}\mathbf{e}_n + \frac{\partial x^n}{\partial \theta^i}\mathbf{e}_{n,j} \\
&= x^n{}_{,ij}\; \mathbf{e}_n + x^n{}_i\,(0) \\
&= x^n{}_{,ij}\; \left(\frac{\partial \theta^k}{\partial x^n}\; \mathbf{g}_k\right) \\
&= x^n{}_{,ij}\; \theta^k{}_{,n}\; \mathbf{g}_k
\end{aligned}$$

If we let $\quad \Gamma^k{}_{ij} = x^n{}_{,ij}\; \theta^k{}_{,n} \quad$ then we have

$$\mathbf{g}_{i,j} = \Gamma^k{}_{ij}\; \mathbf{g}_k \tag{1.26}$$

Alternatively, we may also define the derivative of the base covectors in terms of the contravariant base vectors

$$\begin{aligned}
\mathbf{g}_{i,j} &= \Gamma^k{}_{ij}\; g_{ks}\, \mathbf{g}^s \\
&= x^n{}_{,ij}\; \theta^k{}_{,n}\; g_{ks}\, \mathbf{g}^s \\
&= x^n{}_{,ij}\; \theta^k{}_{,n}\; x^n{}_{,k}\; x^m{}_{,s}\, \delta_{nm}\, \mathbf{g}^s \\
&= x^n{}_{,ij}\; x^m{}_{,s}\, \delta_{nm}\, \mathbf{g}^s \\
&= x^n{}_{,s}\; x_{n,ij}\; \mathbf{g}^s
\end{aligned}$$

Let $\quad \Gamma_{sij} = x^n{}_{,s}\; x_{n,ij} \quad$ and swap dummy indices s and k, then

$$\mathbf{g}_{i,j} = \Gamma_{kij}\; \mathbf{g}^k \tag{1.27}$$

and we may define

$$\Gamma_{kij} = g_{ks}\; \Gamma^s{}_{ij} \tag{1.28}$$

Now that we have established the derivative definitions of covariant base vectors, let us determine the derivative definition for contravariant base vectors.

$$\begin{aligned}
\mathbf{e}^n{}_{,j} &= x^n{}_{,jk}\; \mathbf{g}^k + x^n{}_{,k}\; \mathbf{g}^k{}_{,j} \\
\theta^i{}_{,n}\,(0) &= \theta^i{}_{,n}\left(x^n{}_{,jk}\; \mathbf{g}^k + x^n{}_{,k}\; \mathbf{g}^k{}_{,j}\right) \\
\theta^i{}_{,n}\; x^n{}_{,k}\; \mathbf{g}^k{}_{,j} &= -\theta^i{}_{,n} x^n{}_{,jk}\; \mathbf{g}^k \\
\delta^i_k\; \mathbf{g}^k{}_{,j} &= -\Gamma^i{}_{jk}\; \mathbf{g}^k \\
\left(\delta^i_k\; \mathbf{g}^k\right)_{,j} &= -\Gamma^i{}_{jk}\; \mathbf{g}^k
\end{aligned}$$

Thus

$$\mathbf{g}^i{}_{,j} = -\Gamma^i{}_{jk}\,\mathbf{g}^k$$

As done in the case of the base covectors derivative, we may look into determining the expression for the derivative of the base vectors in terms of the base covectors, however this would be done in vain. The symbols related by equation (1.28) will be formally defined soon, and in finding the definitions for $\mathbf{g}^i{}_j$ in terms of the base covectors arrives at an expression that doesn't include either Γ_{ijk} or $\Gamma^k{}_{ij}$.

Before we formally define these symbols let us consider the partial derivative of a second order tensor, or more specifically the partial derivative of the metric tensor. Consider

$$\begin{aligned}\frac{\partial}{\partial\theta^k}\left(g_{ij}\right) &= \frac{\partial}{\partial\theta^k}\left(\frac{\partial x^r}{\partial\theta^i}\frac{\partial x^s}{\partial\theta^j}\delta_{rs}\right)\\ g_{ij,k} &= x^n{}_{,j}\,x^m{}_{,ik}\,\delta_{mn} + x^m{}_{,i}\,x^n{}_{,jk}\,\delta_{mn} + x^m{}_{,i}\,x^n{}_{,j}\,\delta_{mn,k}\\ &= \Gamma_{jik} + \Gamma_{ijk} + 0\end{aligned}$$

therefore

$$g_{ij,k} = \Gamma_{jik} + \Gamma_{ijk}. \tag{1.29}$$

In equation (1.29) let $i \to j$, $j \to k$ and $k \to i$, then

$$g_{jk,i} = \Gamma_{kji} + \Gamma_{jki} \tag{1.30}$$

Also, for equation (1.29) let $i \to i$, $j \to k$ and $k \to j$, then

$$g_{ik,j} = \Gamma_{kij} + \Gamma_{ikj} \tag{1.31}$$

Add equations (1.29) and (1.31), and minus equation (1.30)

$$\begin{aligned}g_{ij,k} + g_{ik,j} - g_{jk,i} &= \left(\Gamma_{jik} + \Gamma_{ijk}\right) + \left(\Gamma_{kij} + \Gamma_{ikj}\right) - \left(\Gamma_{kji} + \Gamma_{jki}\right)\\ &= x^n{}_{,j}\,x^m{}_{,ik}\,\delta_{nm} + \Gamma_{ijk} + x^n{}_{,k}\,x^m{}_{,ij}\,\delta_{nm} + x^n{}_{,i}\,x^m{}_{,kj}\,\delta_{nm} - \Gamma_{kji} - \Gamma_{jki}\\ &= x^n{}_{,j}\,\frac{\partial^2 x^m}{\partial\theta^k\partial\theta^i}\,\delta_{nm} + \Gamma_{ijk} + x^n{}_{,k}\,\frac{\partial^2 x^m}{\partial\theta^j\partial\theta^i}\,\delta_{nm} + x^n{}_{,i}\,\frac{\partial^2 x^m}{\partial\theta^j\partial\theta^k}\,\delta_{nm} - \Gamma_{kji} - \Gamma_{jki}\\ &= x^n{}_{,j}\,\frac{\partial^2 x^m}{\partial\theta^i\partial\theta^k}\,\delta_{nm} + \Gamma_{ijk} + x^n{}_{,k}\,\frac{\partial^2 x^m}{\partial\theta^i\partial\theta^j}\,\delta_{nm} + x^n{}_{,i}\,\frac{\partial^2 x^m}{\partial\theta^k\partial\theta^j}\,\delta_{nm} - \Gamma_{kji} - \Gamma_{jki}\\ &= x^n{}_{,j}\,x^m{}_{,ki}\,\delta_{nm} + \Gamma_{ijk} + x^n{}_{,k}\,x^m{}_{,ji}\,\delta_{nm} + x^n{}_{,i}\,x^m{}_{,jk}\,\delta_{nm} - \Gamma_{kji} - \Gamma_{jki}\\ &= \Gamma_{jki} + \Gamma_{ijk} + \Gamma_{kji} + \Gamma_{ijk} - \Gamma_{kji} - \Gamma_{jki}\\ &= 2\,\Gamma_{ijk}\end{aligned}$$

to arrive at the definition of the Christoffel symbol of first kind.[6]

[6]Define the torsion-free connection coefficients/Christoffel symbols in spherical polar coordinates using Mathematica, by making use of definitions 27 and 28. Print all non-zero Christoffel symbols of second kind for the spherical polar coordinate system. See 6 for a possible Mathematica representation.

Definition 27. Christoffel symbol of first kind

Being expressed in terms of the metric tensor, the Christoffel symbol of first kind is given by

$$\Gamma_{ijk} = \frac{1}{2}\left(g_{ij,k} + g_{ik,j} - g_{jk,i}\right)$$

The Christoffel symbol of first kind is symmetric in the last two indices, i.e. in indices j and k.

Definition 28. Christoffel symbol of second kind

The more commonly used Christoffel symbol of second kind is defined by

$$\Gamma^{i}{}_{jk} = g^{is}\Gamma_{sjk}$$

The Christoffel symbol of second kind is symmetric in the covariant indices, i.e. in indices j and k.

Both the Christoffel symbols of first and second kind are not tensors, hence they are not referred to as such. It is also of importance to note that the definitions of the Christoffel symbols are in terms of the metric tensor. As the metric tensor prescribes a geometry, the space is assumed to be free of torsion. The tetrad formalism, described later in section 5.6, by definition allows for space with torsion.

Exercise 2. *Show that the Christoffel symbol of second kind, definition 28, is not a tensor. See solution 2.*

1.3.2 Covariant differentiation

We have established that the partial derivatives of base vectors do not necessarily vanish. Therefore, when taking the derivative of tensor quantities that comprise of covariant or contravariant vector and base vector or covector components, one should consider the partial derivative of the base vector or covector components as well. This is where we begin to see that a step taken in space is determined by the path direction and the curvature of that space. As before, consider first the partial derivative of a scalar

$$\phi_{,i} = \frac{\partial\phi}{\partial\theta^{i}}.$$

Now, consider how $\phi_{,i}$ transforms under the coordinate transformation $\theta^{i} \rightarrow \overline{\theta}^{i}$, i.e. $\overline{\phi}(\overline{\theta}^{i}) = \phi(\theta^{i})$

$$\begin{aligned} \frac{\partial\overline{\phi}}{\partial\overline{\theta}^{i}} &= \frac{\partial\phi}{\partial\theta^{j}}\frac{\partial\theta^{j}}{\partial\overline{\theta}^{i}} \\ \overline{\phi}_{,i} &= \phi_{,j}\,\theta^{j}{}_{,i}. \end{aligned}$$

One should notice that the partial derivative of a scalar transforms according to the covariant type of transformation and forms a covector. However, this is not true in the vector case. The partial derivative of the components of a vector do not transform like the components of a second order tensor, for

$$\overline{v}^{i}{}_{,j} = \left(\overline{\theta}^{i}{}_{,s}\, v^{s}\right)_{,r}\theta^{r}{}_{,j} \;=\; \left(\overline{\theta}^{i}{}_{,sr}\, v^{s} + \overline{\theta}^{i}{}_{,s}\, v^{s}{}_{,r}\right)\theta^{r}{}_{,j} \;=\; \overline{\theta}^{i}{}_{,sr}\,\theta^{r}{}_{,j}\, v^{s} + \overline{\theta}^{i}{}_{,s}\,\theta^{r}{}_{,j}\, v^{s}{}_{,r}.$$

Because of the first term ($\overline{\theta}^{i}{}_{,sr}\,\theta^{r}{}_{,j}\, v^{s}$), the partial derivatives of vectors do not transform like the components of a second order tensor. Consider the vector $\mathbf{v}$ in the curvilinear basis

$$\mathbf{v} = v^{r}\mathbf{g}_{r} \;=\; v_{r}\mathbf{g}^{r}$$

and take the partial derivative thereof. One should bare in mind the product rule of differentiation, as the base vector is not necessarily constant and may be a function of the curvilinear coordinate system.

$$\begin{aligned}
\mathbf{v}_{,i} &= \left(v^r \mathbf{g}_r\right)_{,i} \\
&= v^r{}_{,i}\, \mathbf{g}_r + v^r\, \mathbf{g}_{r,i} \\
&= v^r{}_{,i}\, \mathbf{g}_r + v^r\, \Gamma^t{}_{ri}\, \mathbf{g}_t \\
&= v^r{}_{,i}\, \mathbf{g}_r + \Gamma^r{}_{si}\, v^s\, \mathbf{g}_r \\
&= \left(v^r{}_{,i} + \Gamma^r{}_{si}\, v^s\right) \mathbf{g}_r
\end{aligned}$$

If we let

$$v^r{}_{;i} = v^r{}_{,i} + \Gamma^r{}_{ti}\, v^t \tag{1.32}$$

then $\mathbf{v}_{,i} = v^r{}_{;i}\, \mathbf{g}_r$. Similarly, we have

$$\begin{aligned}
\mathbf{v}_{,i} &= \left(v_r\, \mathbf{g}^r\right)_{,i} \\
&= v_{r,i}\, \mathbf{g}^r + v_r\, \left(-\Gamma^r{}_{it} \mathbf{g}^t\right) \\
&= v^r{}_{,i}\, \mathbf{g}^r - \Gamma^s{}_{ir}\, v_s\, \mathbf{g}^r \\
&= \left(v_{r,i} - \Gamma^s{}_{ir}\, v_s\right) \mathbf{g}^r
\end{aligned}$$

and if we let

$$v_{r\,;i} = v_{r,i} - \Gamma^s{}_{ir}\, v_s \tag{1.33}$$

then $\mathbf{v}_{,i} = v_{r\,;i}\, \mathbf{g}^r$.

Definition 29. Covariant derivatives

Equations (1.32) and (1.33) are called the covariant derivatives of the components v^r and v_r, respectively, of the vector $\mathbf{v}$. Note that covariant differentiation is denoted by a semi-colon.

Theorem 2. *The covariant derivatives $v^i{}_{;j}$ and $v_{i\,;j}$ of the components of a vector form a tensor of order two.*

Proof. We have

$$\begin{aligned}
\overline{\mathbf{v}} &= \overline{v}^r\, \overline{\mathbf{g}}_r \\
&= \frac{\partial \overline{\theta}^r}{\partial \theta^s}\, v^s\, \frac{\partial \theta^t}{\partial \overline{\theta}^r}\, \mathbf{g}_t \\
&= \delta^t_s\, v^s\, \mathbf{g}_t \\
&= v^s\, \mathbf{g}_s \\
&= \mathbf{v}
\end{aligned}$$

thus

$$\frac{\partial \overline{\mathbf{v}}}{\partial \overline{\theta}^i} = \frac{\partial \mathbf{v}}{\partial \overline{\theta}^i} = \frac{\partial \mathbf{v}}{\partial \theta^j} \frac{\partial \theta^j}{\partial \overline{\theta}^i} = \mathbf{v}_{,j}\, \theta^j{}_{,i}$$

Therefore, $\dfrac{\partial \mathbf{v}}{\partial \theta^i}$ transforms according to the covariant type of transformation. But we have that $\dfrac{\partial \mathbf{v}}{\partial \theta^i} = v^r{}_{;i}\, \mathbf{g}_r$ and $\dfrac{\partial \mathbf{v}}{\partial \theta^i} = v_{r\,;i}\, \mathbf{g}^r$. By the quotient theorem (theorem 1) , since $\mathbf{g}_r$ and $\mathbf{g}^r$ are arbitrary tensors and $\mathbf{v}_{,i}$ is always a vector, it follows that $v^r{}_{;i}$ and $v_{r\,;i}$ are the components of tensors.

□

We note the following:

(i) Covariant differentiation of a scalar is the same as partial differentiation: $\phi_{;i} = \phi_{,i}$

(ii) Covariant differentiation satisfies the product rule for differentiation. If T and S are arbitrary tensors then

$$(TS)_{;i} = T_{;i}\, S + T\, S_{;i} \tag{1.34}$$

Equation (1.34) is a tensor equation and thus is satisfied in all coordinate systems.

(iii) We can form covariant derivatives of a tensor of order two by considering the derivatives of scalars

$$\begin{aligned} T &= A^i{}_j B_i C^j \\ R &= A^{ij} B_i C_j \\ S &= A_{ij} B^i C^j \end{aligned}$$

where B^i and C^j are the components of arbitrary vectors.

As the covariant derivative of a scalar is equivalent to the partial derivative of that scalar and the product rule of differentiation is satisfied for covariant differentiation, let us consider the covariant derivative of the scalar quantity $T = A^i{}_j B_i C^j$.

$$\begin{aligned} T_{,r} &= T_{;r} \\ \left(A^i{}_j\, B_i\, C^j\right)_{,r} &= \left(A^i{}_j\, B_i\, C^j\right)_{;r} \\ A^i{}_{j,r}\, B_i\, C^j + A^i{}_j\, B_{i,r}\, C^j + A^i{}_j\, B_i\, C^j{}_{,r} &= A^i{}_{j;r}\, B_i\, C^j + A^i{}_j\, B_{i;r}\, C^j + A^i{}_j\, B_i\, C^j{}_{;r} \\ A^i{}_{j,r}\, B_i\, C^j + A^i{}_j\, B_{i,r}\, C^j + A^i{}_j\, B_i\, C^j{}_{,r} &= A^i{}_{j;r}\, B_i\, C^j + A^i{}_j\, \left(B_{i,r} - \Gamma^t{}_{ir} B_t\right) C^j + A^i{}_j\, B_i\, \left(C^j{}_{,r} + \Gamma^j{}_{rt} C^t\right) \\ A^i{}_{j,r}\, B_i\, C^j &= A^i{}_{j;r}\, B_i\, C^j - A^i{}_j\, \Gamma^t{}_{ir} B_t\, C^j + A^i{}_j\, B_i\, \Gamma^j{}_{rt} C^t \\ \left(A^i{}_{j,k} + \Gamma^i{}_{tk}\, A^t{}_j - \Gamma^t{}_{kj}\, A^i{}_t\right) B_i\, C^j &= \left(A^i{}_{j;r}\right) B_i\, C^j \end{aligned}$$

Since B_i and C^j are arbitrary, it follows that

$$A^i{}_{j;k} = A^i{}_{j,k} + \Gamma^i{}_{tk}\, A^t{}_j - \Gamma^t{}_{kj}\, A^i{}_t$$

By a similar argument, it can be shown that when considering the scalar quantity $R = A^{ij}\, B_i\, C_j$ the covariant derivative of the contravariant tensor A^{ij} is given by:

$$A^{ij}{}_{;k} = A^{ij}{}_{,k} + \Gamma^i{}_{tk}\, A^{tj} + \Gamma^j{}_{tk}\, A^{it}$$

and that when considering the scalar quantity $S = A_{ij}\, B^i\, C^j$ the covariant derivative of the covariant tensor A_{ij} is given by:

$$A_{ij;k} = A_{ij,k} - \Gamma^t{}_{ki}\, A_{tj} - \Gamma^t{}_{kj}\, A_{it}$$

The process of forming covariant derivatives is quite general and can be applied to obtain covariant derivatives of tensors of any type and order, for example

$$A^i{}_{jk;r} = A^i{}_{jk,r} + \Gamma^i{}_{tr}\, A^t{}_{jk} - \Gamma^t{}_{rj}\, A^i{}_{tk} - \Gamma^t{}_{rk}\, A^i{}_{jt}.$$

Theorem 3. ***Ricci's Lemma***

(i)

$$g_{i\,j\,;\,k} = 0$$

(ii)

$$g^{i}_{j\,;\,k} = 0$$

(iii)

$$g^{i\,j}{}_{;\,k} = 0$$

Proof. (i)

$$\begin{aligned} g_{i\,j\,;\,k} &= g_{i\,j\,,\,k} - \Gamma^{t}{}_{ki}\, g_{tj} - \Gamma^{t}{}_{kj}\, g_{it} \\ &= g_{ij,k} - g_{tj}\, g^{ts}\, \Gamma_{sik} - g_{it}\, g^{ts}\, \Gamma_{sjk} \\ &= g_{ij,k} - \delta^{s}_{j}\, \Gamma_{sik} - \delta^{s}_{i}\, \Gamma_{sjk} \\ &= g_{ij,k} - \left(\Gamma_{jik} + \Gamma_{ijk}\right) \\ &= g_{ij,k} - g_{ij,k} \\ &= 0 \end{aligned}$$

□

Exercise 3. *Show that part* (ii) *and* (iii) *of Ricci's Lemma, Theorem 3, are true. (See solution 3)*

We note the following:

(i) Since the covariant derivatives of tensors are tensors, the covariant derivative index can be raised or lowered by the rules for raising and lowering indices of tensors, e.g.

$$A_{ij}{}^{;r} = g^{rt} A_{ij;t}$$

The notation $A_{ij}{}^{;r}$ is not used often.

(ii) The covariant derivatives of the alternating tensor ϵ_{ijk} vanish for the same reason that the covariant derivatives of the metric tensor vanish. This may be verified by direct calculation. Thus, the metric tensor g_{ij} and the alternating tensor ϵ_{ijk} can be treated as constants for the purpose of covariant differentiation but not for the purposes of ordinary partial differentiation.

1.3.3 Riemann-Christoffel curvature tensor

Since the covariant derivatives of tensors are tensors we can find their covariant derivatives. The resulting tensors are called the second covariant derivatives of the original tensors and we can continue in this way to find covariant derivatives of any order.

Notation: When writing second covariant derivatives, we omit the second semi-colon. I.e.

$$A^{i}{}_{;\,j\,;\,k} = A^{i}{}_{;\,j\,k}$$

Theorem 4.

$$A_{i;jk} - A_{i;kj} = A_n R^n{}_{ijk} \tag{1.35}$$

where

$$R^n{}_{ijk} = \Gamma^n{}_{ik,j} - \Gamma^n{}_{ij,k} + \Gamma^t{}_{ik}\Gamma^n{}_{tj} - \Gamma^t{}_{ij}\Gamma^n{}_{tk} \tag{1.36}$$

Proof.

$$A_{i;j} = A_{i,j} - \Gamma^t{}_{ij}A_t$$

$$\begin{aligned} A_{i;jk} &= A_{i\,;\,j\,,\,k} - \Gamma^t{}_{ik}A_{t;j} - \Gamma^t{}_{jk}A_{i;t} \\ &= \left(A_{i,j} - \Gamma^t{}_{ij}A_t\right)_{,k} - \Gamma^t{}_{ik}\left(A_{t,j} - \Gamma^s{}_{tj}A_s\right) - \Gamma^t{}_{jk}\left(A_{i,t} - \Gamma^s{}_{it}A_s\right) \end{aligned}$$

$$A_{i;jk} = A_{i,jk} - \Gamma^t{}_{ij,k}A_t - \Gamma^t{}_{ij}A_{t,k} - \Gamma^t{}_{ik}A_{t,j} + \Gamma^t{}_{ik}\Gamma^s{}_{tj}A_s - \Gamma^t{}_{jk}A_{i,t} + \Gamma^t{}_{jk}\Gamma^s{}_{it}A_s \tag{1.37}$$

Replace j by k and k by j

$$A_{i;kj} = A_{i,kj} - \Gamma^t{}_{ik,j}A_t - \Gamma^t{}_{ik}A_{t,j} - \Gamma^t{}_{ij}A_{t,k} + \Gamma^t{}_{ij}\Gamma^s{}_{tk}A_s - \Gamma^t{}_{kj}A_{i,t} + \Gamma^t{}_{kj}\Gamma^s{}_{it}A_s \tag{1.38}$$

Subtract equation (1.38) from equation (1.37)

$$\begin{aligned} A_{i;kj} - A_{i;jk} = {} & A_{i,kj} - \Gamma^t{}_{ik,j}A_t - \Gamma^t{}_{ik}A_{t,j} - \Gamma^t{}_{ij}A_{t,k} + \Gamma^t{}_{ij}\Gamma^s{}_{tk}A_s - \Gamma^t{}_{kj}A_{i,t} + \Gamma^t{}_{kj}\Gamma^s{}_{it}A_s \\ & - \left[A_{i,jk} - \Gamma^t{}_{ij,k}A_t - \Gamma^t{}_{ij}A_{t,k} - \Gamma^t{}_{ik}A_{t,j} + \Gamma^t{}_{ik}\Gamma^s{}_{tj}A_s - \Gamma^t{}_{jk}A_{i,t} + \Gamma^t{}_{jk}\Gamma^s{}_{it}A_s\right] \\ = {} & A_n\left[\Gamma^n{}_{ik,j} - \Gamma^n{}_{ij,k} + \Gamma^t{}_{ik}\Gamma^n{}_{tj} - \Gamma^t{}_{ij}\Gamma^n{}_{tk}\right] \\ = {} & A_n R^n{}_{ijk} \end{aligned}$$

□

We note that since A_i is an arbitrary tensor and $A_{i;jk} - A_{i;kj}$ is always a tensor, it follows from the quotient theorem that $R^n{}_{ijk}$ is a tensor.[7]

Definition 30. Riemann-Christoffel curvature tensor

$R^n{}_{ijk}$ is called the Riemann-Christoffel curvature tensor, consisting only of components of the metric tensor and its derivatives up to second order.

Let us investigate the symmetry properties of the curvature tensor. We begin by lowering the first index,

$$\begin{aligned} R_{ijkl} &= g_{in}R^n{}_{jkl} \\ &= g_{in}\left(\Gamma^n{}_{jl,k} - \Gamma^n{}_{jk,l} + \Gamma^t{}_{jl}\Gamma^n{}_{tk} - \Gamma^t{}_{jk}\Gamma^n{}_{tl}\right) \end{aligned}$$

or

$$R_{ijkl} = \Gamma_{ijl,k} - \Gamma_{ijk,l} + \Gamma^t{}_{jl}\Gamma_{itk} - \Gamma^t{}_{jk}\Gamma_{itl} \tag{1.39}$$

Let $i \leftrightarrow j$ in equation (1.39)

$$\begin{aligned} R_{jikl} &= \Gamma_{jil,k} - \Gamma_{jik,l} + \Gamma^t{}_{il}\Gamma_{jtk} - \Gamma^t{}_{ik}\Gamma_{jtl} \\ &= \Gamma_{jil,k} - \Gamma^t{}_{ik}\Gamma_{jtl} - \left(\Gamma_{jik,l} - \Gamma^t{}_{il}\Gamma_{jtk}\right) \\ &= \left(\Gamma_{jil}\right)_{;\,k} - \left(\Gamma_{jik}\right)_{;\,l} \\ &= \left(-\Gamma_{ijl}\right)_{;\,k} - \left(-\Gamma_{ijk}\right)_{;\,l} \\ &= -\left[\Gamma_{ijl,k} - \Gamma^t{}_{jk}\Gamma_{itl} - \Gamma_{ijk,l} + \Gamma^t{}_{jl}\Gamma_{itk}\right] \end{aligned}$$

[7] Define the Riemann-Christoffel curvature tensor in Mathematica and determine all non-zero components for the spherical polar coordinate system. Make use of Theorem 4. See 7 for a possible Mathematica representation.

therefore

$$R_{jikl} = -R_{ijkl}$$

By a similar argument, one may show that

$$R_{ijlk} = -R_{ijkl}$$

and consequently

$$R_{jilk} = R_{ijkl}$$

Definition 31. Euclidean space

A Euclidean space is a space which admits a cartesian coordinate system.

We note the following:

(i) R_{ijkl} is skew-symmetric under interchange of i and j and under interchange of k and l. It is symmetric under interchange of the pair ij with the pair kl.

(ii) In three-dimensional space, R_{abcd} has only six independent components:

$$\begin{array}{ccc} R_{1212} & R_{1213} & R_{1223} \\ & R_{1313} & R_{1323} \\ & & R_{2323} \end{array}$$

(iii) Corresponding results for contravariant vectors are obtained by raising the index:

$$A_{i;jk} - A_{i;kj} = A_n R^n{}_{ijk}$$
$$A^i{}_{;jk} - A^i{}_{;kj} = A_n R^{ni}{}_{jk}$$

(iv) The second covariant derivative of a tensor of order two is given by

$$A_{ij;rs} - A_{ij;sr} = A_{nj} R^n{}_{irs} + A_{in} R^n{}_{jrs}$$
$$A^{ij}{}_{;rs} - A^{ij}{}_{;sr} = A_n{}^j R^{ni}{}_{rs} + A^i{}_n R^{nj}{}_{rs}$$

(v) In a cartesian coordinate system $\Gamma^i{}_{jk} = 0$ everywhere and therefore the Riemann-Christoffel curvature tensor is identically zero:

$$R^a{}_{bcd} = 0$$

But this is a tensor equation. Thus $R^a{}_{bcd} = 0$ in all curvilinear coordinate systems which can be established in Euclidean space.

(vi) When the space is Euclidean

$$A_{i;jk} = A_{i;kj}$$
$$A^i{}_{;jk} = A^i{}_{;kj}$$
$$A^{ij}{}_{;rs} = A^{ij}{}_{;sr}$$

Thus the order of covariant differentiation commutes.

(vii) The definition of the Riemann-Christoffel curvature tensor is not confined to three-dimensional spaces or to Euclidean spaces. For a n-dimensional non-Euclidean space, a cartesian coordinate system cannot be established throughout the space, i.e. $R^a{}_{bcd} \neq 0$ and covariant differentiation does not commute.

Theorem 5. *Bianchi identities*

$$R^a{}_{bcd;e} + R^a{}_{bde;c} + R^a{}_{bec;d} = 0 \tag{1.40}$$

Proof. Consider the second covariant derivative of a second-order tensor, C_{ab}, that is defined to be the outer product of two co-vectors ,$A_a B_b$, then

$$\begin{aligned} C_{ab;cd} &= (A_a B_b)_{;cd} = \left(A_{a;c} B_b + A_a B_{b;c}\right)_{;d} \\ &= A_{a;cd} B_b + A_{a;c} B_{b;d} + A_{a;d} B_{b;c} + A_a B_{b;cd}. \end{aligned} \tag{1.41}$$

Now, interchange indices c and d in equation (1.41) and subtract the result from equation (1.41).

$$\begin{aligned} C_{ab;cd} - C_{ab;dc} &= A_{a;cd} B_b + A_a B_{b;cd} - A_{a;dc} B_b - A_a B_{b;dc} \\ &= A_f R^f{}_{acd} B_b + A_a R^f{}_{bcd} B_f \\ &= C_{fb} R^f{}_{acd} + C_{af} R^f{}_{bcd} \end{aligned}$$

by equation (1.35). Now, take the second-order tensor, C_{ab}, to be the covariant derivative, $A_{a;b}$, so that

$$A_{a;bcd} - A_{a;bdc} = A_{f;b} R^f{}_{acd} + A_{a;f} R^f{}_{bcd}. \tag{1.42}$$

Take cyclic permutations of b, c and d in equation (1.42) and add all three equations to obtain

$$\begin{aligned} &A_{a;bcd} - A_{a;bdc} + A_{a;cdb} - A_{a;cbd} + A_{a;dbc} - A_{a;dcb} \\ &= A_{f;b} R^f{}_{acd} + A_{a;f} R^f{}_{bcd} + A_{f;c} R^f{}_{adb} + A_{a;f} R^f{}_{cdb} + A_{f;d} R^f{}_{abc} + A_{a;f} R^f{}_{dbc} \end{aligned}$$

$$\begin{aligned} &\left(A_{a;bc} - A_{a;cb}\right)_{;d} + \left(A_{a;cd} - A_{a;dc}\right)_{;b} + \left(A_{a;db} - A_{a;bd}\right)_{;c} \\ &= A_{f;b} R^f{}_{acd} + A_{f;c} R^f{}_{adb} + A_{f;d} R^f{}_{abc} + A_{a;f} \left(R^f{}_{dbc} + R^f{}_{bcd} + R^f{}_{cdb}\right). \end{aligned}$$

By equation (1.35) and proof of exercise 4 we have

$$\begin{aligned} \left(A_f R^f{}_{abc}\right)_{;d} + \left(A_f R^f{}_{acd}\right)_{;b} + \left(A_f R^f{}_{adb}\right)_{;c} &= A_{f;b} R^f{}_{acd} + A_{f;c} R^f{}_{adb} + A_{f;d} R^f{}_{abc} \\ A_f R^f{}_{abc;d} + A_f R^f{}_{acd;b} + A_f R^f{}_{adb;c} &= 0. \end{aligned}$$

The factor A_f occurs throughout the expression, thus, with a change in indices ($\{f, a, b, c, d\} \to \{a, b, c, d, e\}$), we have

$$R^a{}_{bcd;e} + R^a{}_{bde;c} + R^a{}_{bec;d} = 0.$$

□

Exercise 4. *Prove that*

$$R^a{}_{dbc} + R^a{}_{bcd} + R^a{}_{cdb} = 0$$

See solution 4.

Tensors related to the Riemann-Christoffel curvature tensor

Definition 32. Ricci tensor

The Ricci tensor, R_{bd}, is defined as

$$R_{bd} = \delta^c{}_a R^a{}_{bcd} \;=\; R^a{}_{bad}$$

We note that a contraction in indices other than the first and third, yields either zero or negative the Ricci tensor as a result of the symmetry and antisymmetry properties of the Riemann-Christoffel curvature tensor.[8]

Theorem 6. *R_{ab} is symmetric*

Proof.

$$R_{ab} = g^{ce} R_{eacb} \;=\; g^{ce} R_{cbea} \;=\; R_{ba}$$

□

Definition 33. Curvature scalar

The curvature scalar, R, is defined as

$$R = g^{ab} R_{ab} \;=\; \delta^a{}_b R_a{}^b \;=\; g_{ab} R^{ab}.$$

Definition 34. Einstein tensor

The Einstein tensor, G_{ab}, is defined as

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}.$$

Theorem 7.

$$G^{ab}{}_{;b} = 0$$

Proof. First we derive the expression for $R^{ab}{}_{;b}$ using the Bianchi identities. We have

$$\begin{aligned} g^{fe}\delta^c{}_a g^{bd}\left(R^a{}_{bcd;e} + R^a{}_{bde;c} + R^a{}_{bec;d}\right) &= 0 \\ g^{fe} R^{cd}{}_{cd;e} + R^{cdf}{}_{d;c} + R^{cdf}{}_{c;d} &= 0 \\ R^{fc}{}_{;c} &= g^{fe} R_{,e}. \end{aligned} \tag{1.43}$$

From the definition of the Einstein tensor, definition 34, we have

$$G^{ab}{}_{;b} = \left(R^{ab} - \frac{1}{2} R g^{ab}\right)_{;b} \;=\; R^{ab}{}_{;b} - \frac{1}{2} g^{ab} R_{,b}$$

as $g^{ab}{}_{;b} = 0$. Substitute equation (1.43) into this result

$$G^{ab}{}_{;b} = R^{ab}{}_{;b} - \frac{1}{2} g^{ab} R_{,b} \;=\; \frac{1}{2} g^{ab} R_{,b} - \frac{1}{2} g^{ab} R_{,b} \;=\; 0.$$

□

Definition 35. Weyl tensor

The Weyl tensor, C_{abcd}, in n dimensions is defined as

$$C_{abcd} = R_{abcd} + \frac{1}{n-2}\left(R_{ad} g_{bc} - R_{ac} g_{bd} + R_{bc} g_{ad} - R_{bd} g_{ac}\right) + \frac{1}{(n-1)(n-2)} R \left(g_{ac} g_{bd} - g_{ad} g_{bc}\right).$$

[8] Determine the components of the Ricci tensor and curvature scalar in spherical polar coordinates, using Mathematica. See 8 for a possible Mathematica representation.

1.3.4 The derivative operator in curvilinear coordinates

In cartesian coordinates the del operator is defined as

$$\nabla = \mathbf{e}^n \frac{\partial}{\partial x^n}.$$

Transform from cartesian coordinates x^i to curvilinear coordinates θ^i

$$\begin{aligned}
\nabla &= \mathbf{e}^n \frac{\partial}{\partial x^n} \\
&= \frac{\partial x^n}{\partial \theta^k}\mathbf{g}^k \frac{\partial \theta^i}{\partial x^n}\frac{\partial}{\partial \theta^i} \\
&= \frac{\partial x^n}{\partial \theta^k}\frac{\partial \theta^i}{\partial x^n}\mathbf{g}^k \frac{\partial}{\partial \theta^i} \\
&= \delta^i_k \mathbf{g}^k \frac{\partial}{\partial \theta^i} \\
&= \mathbf{g}^i \frac{\partial}{\partial \theta^i}
\end{aligned}$$

Definition 36. Del operator in curvilinear coordinates

The del operator ∇ in curvilinear coordinates is defined as

$$\nabla = \mathbf{g}^i \frac{\partial}{\partial \theta^i}.$$

We note the following:

(i) The partial derivative $\dfrac{\partial}{\partial \theta^i}$ operates on everything which follows the operator ∇.

(ii) $\dfrac{\partial}{\partial \theta^i}$ satisfies the transformation law for the components of a covariant vector between curvilinear coordinates θ^i and $\overline{\theta}^i$

$$\frac{\partial}{\partial \overline{\theta}^i} = \frac{\partial \theta^k}{\partial \overline{\theta}^i}\frac{\partial}{\partial \theta^k}.$$

Definition 37. Gradient operator

If $\phi = \phi\left(\theta^i\right)$ is a scalar field, then the gradient of ϕ is

$$\nabla \phi = \mathbf{g}^i \frac{\partial \phi}{\partial \theta^i}$$

The covariant components of the gradient of a scalar ϕ, $\frac{\partial \phi}{\partial \theta^i}$, satisfies the transformation law for the components of a covariant vector, for[9]

$$
\begin{aligned}
\frac{\partial \overline{\phi}}{\partial \overline{\theta}^s} &= \frac{\partial \phi}{\partial \overline{\theta}^s} \\
\overline{\phi}_{,s} &= \frac{\partial \phi}{\partial \theta^k} \frac{\partial \theta^k}{\partial \overline{\theta}^s} \\
\overline{\phi}_{,s} &= \phi_{,k}\, \theta^k{}_{,s}.
\end{aligned}
$$

Now, let us consider the divergence operator[10]

$$
\begin{aligned}
\nabla \cdot \mathbf{A} &= \mathbf{g}^i \cdot \left(A^j \mathbf{g}_j\right)_{,i} \\
&= \mathbf{g}^i \cdot \left(A^j{}_{,i} \mathbf{g}_j + A^j \mathbf{g}_{j,i}\right) \\
&= A^j{}_{,i}\, \mathbf{g}^i \cdot \mathbf{g}_j + A^j\, \mathbf{g}^i \cdot \mathbf{g}_{j,i} \\
&= A^j{}_{,i}\, \mathbf{g}^i \cdot \mathbf{g}_j + A^j\, \mathbf{g}^i \cdot \mathbf{g}^k \Gamma_{kji} \\
&= \delta^i_j\, A^j{}_{,i} + A^j\, g^{ik}\, \Gamma_{kji} \\
&= A^i{}_{,i} + \Gamma^i{}_{ji}\, A^j
\end{aligned}
$$

and the curl operator[11]

$$
\begin{aligned}
\nabla \times \mathbf{A} &= \mathbf{g}^i \times \left(A^j \mathbf{g}_j\right)_{,i} \\
&= \mathbf{g}^i \times \left(A^j{}_{,i}\, \mathbf{g}_j + A^j\, \mathbf{g}_{j,i}\right) \\
&= A^j{}_{,i}\, \mathbf{g}^i \times \mathbf{g}^k g_{kj} + \mathbf{g}^i \times \mathbf{g}^k A^j\, \Gamma_{kji} \\
&= \varepsilon^{ikj} \mathbf{g}_j \left(A^j{}_{,i}\, g_{kj} + A_k\, g^{jk}\, \Gamma_{kji}\right) \\
&= \varepsilon^{ikj} \mathbf{g}_j \left(A_{k,i} + A_k\, \Gamma^j{}_{ij}\right) \\
&= \varepsilon^{ikj}\, A_{k;i}\, \mathbf{g}_j
\end{aligned}
$$

Definition 38. Divergence operator

The divergence of a vector $\mathbf{A}$ is defined by

$$\nabla \cdot \mathbf{A} = A^i{}_{,i} + \Gamma^i{}_{ji}\, A^j$$

Definition 39. Curl operator

The curl of a vector $\mathbf{A}$ is defined by

$$\nabla \times \mathbf{A} = \varepsilon^{ijk} A_{j;i} \mathbf{g}_k$$

[9] Define the grad operator on an arbitrary vector function **f** in spherical polar coordinates using Mathematica. See 9 for a possible Mathematica representation.

[10] Define the divergence operator acting on an abritrary vector function **f** in spherical polar coordinates using Mathematica. See 10 for a possible Mathematica representation.

[11] Define the curl operator acting on an abritrary vector function **f** in spherical polar coordinates using Mathematica. See 11 for a possible Mathematica representation.

We go on further to define the Laplace operator, which is described by the divergence of the gradient

$$\begin{aligned}\nabla^2\phi &= (\nabla\cdot\nabla)\,\phi \\ &= \mathbf{g}^i\cdot\left(\mathbf{g}^j\phi_{,\,j}\right)_{,\,i} \\ &= \mathbf{g}^i\cdot\left(\mathbf{g}^j{}_{,\,i}\,\phi_{,\,j}+\mathbf{g}^j\,\phi_{,\,j\,i}\right) \\ &= \mathbf{g}^i\cdot\left(-\mathbf{g}^k\,\Gamma^j{}_{ik}\,\phi_{,\,j}+\mathbf{g}^j\,\phi_{,\,j\,i}\right) \\ &= g^{ij}\,\phi_{,\,j\,i}-g^{ik}\,\Gamma^j{}_{ki}\,\phi_{,\,j}.\end{aligned}$$

Replace dummy indices k with j and j with k in the second term of the above expression to obtain

$$\begin{aligned}\nabla^2\phi &= g^{ij}\,\phi_{,\,j\,i}-g^{ij}\,\Gamma^k{}_{ji}\,\phi_{,\,k} \\ &= g^{ij}\,\left(\phi_{,\,j\,i}-\Gamma^k{}_{ij}\,\phi_{,\,k}\right) \\ &= g^{ji}\,\phi_{;\,j\,i}\end{aligned}$$

Definition 40. Laplace operator

The Laplace operator is defined by

$$\begin{aligned}\nabla^2\phi &= (\nabla\cdot\nabla)\,\phi \\ &= g^{ji}\,\phi_{;\,j\,i}.\end{aligned}$$

1.3.5 Physical components of a tensor

The base covectors and vectors, $\mathbf{g}_i$ and $\mathbf{g}^i$ respectively, are not necessarily unit vectors. To obtain physical components of the tensor, components must be re-evaluated in terms of unit base vectors and covectors.

(i) The base covectors $\mathbf{g}_i$

$$\left|\mathbf{g}_i\right| = \sqrt{\mathbf{g}_i\cdot\mathbf{g}_i} = \sqrt{g_{ii}}$$

Define the unit base vector $\overline{\mathbf{g}}_i = \dfrac{\mathbf{g}_i}{\sqrt{g_{ii}}}$.

(i) The base vectors $\mathbf{g}^i$

$$\left|\mathbf{g}^i\right| = \sqrt{\mathbf{g}^i\cdot\mathbf{g}^i} = \sqrt{g^{ii}}$$

Define the unit base vector $\overline{\mathbf{g}}^i = \dfrac{\mathbf{g}^i}{\sqrt{g^{ii}}}$. Consider any vector $\mathbf{A}$

$$\mathbf{A} = A^i\mathbf{g}_i = A^i\sqrt{g_{ii}}\frac{\mathbf{g}_i}{\sqrt{g_{ii}}} = \overline{A}^i\overline{\mathbf{g}}_i$$

where $\overline{A}^i = \sqrt{g_{ii}}A^i$. Also

$$\mathbf{A} = A_i\mathbf{g}^i = A_i\sqrt{g^{ii}}\frac{\mathbf{g}^i}{\sqrt{g^{ii}}} = \overline{A}_i\overline{\mathbf{g}}^i$$

where $\overline{A}_i = \sqrt{g^{ii}}A_i$.

Definition 41. Physical components

The physical components $\overline{A}^i$ and $\overline{A}_i$ of the tensor components A^i and A_i respectively of the vector $\mathbf{A}$ are defined by

$$\overline{A}^i = \sqrt{g_{ii}} A^i \qquad \text{and} \qquad \overline{A}_i = \sqrt{g^{ii}} A_i$$

Theorem 8. *When the coordinate curves are orthogonal, we have*

$$\overline{\mathbf{g}}_i = \overline{\mathbf{g}}^i$$
$$\overline{A}_i = \overline{A}^i$$

Proof.

(i)

$$\begin{aligned}\overline{\mathbf{g}}_i &= \frac{\mathbf{g}_i}{\sqrt{g_{ii}}} \\ &= \frac{g_{ij}\ \mathbf{g}^j}{\sqrt{g_{ii}}} \\ &= \frac{g_{ii}\ \mathbf{g}^i}{\sqrt{g_{ii}}} \\ &= \sqrt{g_{ii}}\ \mathbf{g}^i \\ &= \frac{\mathbf{g}^i}{\sqrt{g^{ii}}} \\ &= \overline{\mathbf{g}}^i.\end{aligned}$$

(ii)

$$\begin{aligned}\overline{A}_i &= \sqrt{g_{ii}}\ A^i \\ &= \sqrt{g_{ii}}\ g^{ij}\ A_j \\ &= \frac{1}{\sqrt{g^{ii}}}\ g^{ii}\ A_i \\ &= \sqrt{g^{ii}}\ A_i \\ &= \overline{A}^i.\end{aligned}$$

□

Theorem 9. ***Divergence theorem***

Suppose **A** *is a vector defined throughout a volume* τ *which is bounded by a closed surface S and that* **n** *is a unit vector normal to the surface S at which it is defined. Then*

$$\int_\tau \nabla \cdot \mathbf{A} d\tau = \int_S \mathbf{A} \cdot \mathbf{n} dS.$$

To express the result in index notation let $\mathbf{A} = A^i \mathbf{g}_i$ *and* $\mathbf{n} = n_j \mathbf{g}^j$*, then*

$$\int_\tau A^i{}_{;i} d\tau = \int_S A^i n_i dS.$$

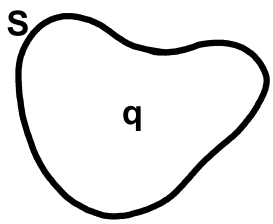
Chapter 2

Physical coordinate systems

From the ground up, the reader will be guided through both the Eulerian and Lagrangian formalism derivations. Setting the foundation of understanding the core difference between these two distinct formalisms. This chapter then finishes off with a derivation of the equations of fluid mechanics. [12, 22, 23, 24, 25, 26]

2.1 Eulerian formalism

The Eulerian formalism is built on the assumption of a global frame of reference, or in a more general sense it is built on the assumption of a preferred time frame of reference - to be discussed later in section 5.4. This formalism finds application in flow problems, seeking to converse some scalar quantity, q, enclosed by a surface, S.



The rate of change of this quantity, enclosed in the fixed volume, Ω, is equivalent to the flux of the fluid through the fixed boundary $\partial\Omega$,

$$\frac{d}{dt}\int_{\Omega} q\, dV = -\int_{\partial\Omega} q\mathbf{u}\cdot\mathbf{dS}. \tag{2.1}$$

Since Ω is a fixed volume, the total time derivative may be moved into the integral without having an effect on the equation. On the right-hand side of equation (2.1), the divergence theorem is applied so that both sides of the equation are volume integrals.

$$\int_{\Omega} \frac{\partial q}{\partial t} dV = \quad -\int_{\Omega} \nabla\cdot(q\mathbf{u})\, dV$$

Performing some algebra and integrating with respect to time, yields the strong integral form of the conservation law,

$$\int\int_{\Omega}\left[\frac{\partial q}{\partial t} + \nabla\cdot(q\mathbf{u})\right] dV\, dt = 0. \tag{2.2}$$

This equation, (2.2), always holds true, describing the conservation of some quantity, q, enclosed by some fixed volume, Ω, over some time domain. For some arbitrary time domain, equation (2.2) may be written in the weak integral form of the conservation law,

$$\int_{\Omega} \left[\frac{\partial q}{\partial t} + \nabla \cdot (q\mathbf{u}) \right] dV = 0. \tag{2.3}$$

Equation (2.3) only holds true for some quantity, q, being conserved in a fixed volume, Ω, for an unspecified time domain. When modelling an arbitrary volume over an arbitrary domain the conservation law is reduced to its differential form,

$$\frac{\partial q}{\partial t} + \nabla \cdot (q\mathbf{u}) = 0. \tag{2.4}$$

We note the following:

(i) The generalised quantity q has not been referred to as a particular mathematical object, i.e. it may be scalar, vector or tensor of higher order.

(ii) Equation (2.4) makes no reference to any particular spatial coordinate system.

(iii) The differential form of the conservation law does not admit discontinuous solutions, whereas the weak and strong integrals forms do. Therefore one's choice of form of the conservation law should always be considered with care.

2.2 Lagrangian formalism

The Lagrangian formalism is based on the assumption of a local frame of reference, i.e. the frame of reference follows the motion of some arbitrary parcel. From a more general perspective the Lagrangian formalism assumes a preferred space frame of reference, which is the key distinguishing property from the Eulerian formalism. The more general preferred space frame of reference will be discussed in detail later in section 5.3.

When no global frame of reference may be set up, or if one desires to model the path of motion of some parcel from its point of reference, then an application of the Lagrangian formalism is useful. It begins by describing some scalar quantity at some time in some point in space, $q = q(t,x,y,z)$ and then at some displaced time and position, $q = q(t+\delta t, x+\delta x, y+\delta y, z+\delta z)$. The rate of change of this quantity with respect to time is of interest, thus

$$\frac{Dq}{Dt} = \frac{q(t+\delta t, x+\delta x, y+\delta y, z+\delta z) - q(t,x,y,z)}{\delta t}. \tag{2.5}$$

Find the Taylor series expansion, to first order, of $q(t+\delta t, x+\delta x, y+\delta y, z+\delta z)$

$$\begin{aligned} q(t+\delta t, x+\delta x, y+\delta y, z+\delta z) = q(t,x,y,z) &+ q_t(t,x,y,z)\delta t + q_z(t,x,y,z)\delta z \\ &+ q_y(t,x,y,z)\delta y + q_x(t,x,y,z)\delta x. \end{aligned}$$

Substituting the above result into equation (2.5) yields

$$\frac{Dq}{Dt} = \frac{q_t\delta t + q_z\delta z + q_y\delta y + q_x\delta x}{\delta t}.$$

We manipulate the right-hand side of the above result

$$\frac{Dq}{Dt} = q_t + q_z\frac{\delta z}{\delta t} + q_y\frac{\delta y}{\delta t} + q_x\frac{\delta x}{\delta t},$$

and make a substitution for the derivative operator, ∇,

$$\frac{Dq}{Dt} = \frac{\partial q}{\partial t} + \mathbf{u} \cdot \left(\nabla q\right) \tag{2.6}$$

to obtain the generalised rate of change of the quantity, q, with respect to time. We note the following:

(i) The generalised quantity q has not been referred to as a particular mathematical object, i.e. it may be scalar, vector or tensor of higher order.

(ii) Equation (2.6) makes no reference to any particular spatial coordinate system.

2.3 Equations of fluid mechanics

In dynamic fluid model problems, the quantities one conserves include: mass density, momentum density and energy density. Let us begin with the mass density conservation of the system. Recall equation (2.4) with q representing the scalar mass density of the system

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u})$$

or

$$\frac{\partial \rho}{\partial t} + \left(\rho u^i\right)_{;i} = 0. \tag{2.7}$$

Equation (2.7), is known as the continuity equation of the system. Now, let us consider the conservation of the momentum density, $J^i = \rho u^i$. The quantity $\frac{D\mathbf{J}}{Dt}$ is the force density, so we recall Newton's second law of motion where one includes any relevant forces acting in the system. Thus using equation (2.4) and including possible forces within a fluid dynamic system would look something like

$$\frac{D\mathbf{J}}{Dt} = \mathbf{F}_{\text{Newton}} = \mathbf{F}_p + \mathbf{F}_v + \mathbf{F}_g + \mathbf{F}_B + \mathbf{F}_E + \cdots .$$

Let us consider the simplest case where the fluid if affected only by pressure and gravity,

$$\begin{aligned}
\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) &= -\nabla p - \rho \nabla \phi \\
\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) &= -\nabla \cdot \left(p \mathbf{g}\right) - \rho \nabla \phi \\
\frac{\partial \rho u^i}{\partial t} + \left(\rho u^i u^j\right)_{;j} &= -\left(p g^{ij}\right)_{;j} - \rho g^{ij} \phi_{,j} \\
\frac{\partial \rho u^i}{\partial t} + \left(\rho u^i u^j + p g^{ij}\right)_{;j} &= -\rho g^{ij} \phi_{,j}
\end{aligned}$$

thus

$$\frac{\partial \rho u^i}{\partial t} + T^{ij}{}_{;j} = -\rho g^{ij} \phi_{,j} \tag{2.8}$$

where the momentum-stress tensor, T^{ij}, is defined by

$$T^{ij} = \rho u^i u^j + p g^{ij}. \tag{2.9}$$

Now, let us consider the energy density conservation of the system. Similarly, from the laws of thermodynamics we arrive at

$$\rho E = \frac{1}{2} \rho\, g_{ij} u^i u^j + \rho \epsilon$$

where the internal energy is dependent on the internal pressure of the system. From Newton's second law of motion and the work-energy theorem, we have that the change in energy density is equivalent to negative the sum of the pressure gravitational contributions of the system:

$$\begin{aligned}
\frac{D\rho E}{Dt} &= -\nabla \cdot \left(p \mathbf{u}\right) - \rho \mathbf{u} \cdot \nabla \phi \\
\frac{\partial \rho E}{\partial t} + \nabla \cdot (\rho E \mathbf{u}) &= -\nabla \cdot \left(p \mathbf{u}\right) - \rho \mathbf{u} \cdot \nabla \phi \\
\frac{\partial \rho E}{\partial t} + \left(\rho E u^i\right)_{;i} &= -\left(p u^i\right)_{;i} - \rho u^i \phi_{,i}
\end{aligned}$$

Thus

$$\frac{\partial \rho E}{\partial t} + \left(\rho E u^i + p u^i\right)_{;i} = -\rho u^i \phi_{,i} \tag{2.10}$$

Equations (2.7), (2.8) and (2.10) are closed by the Poisson equation, forming the system of equations conserving mass density, momentum density and energy density of a fluid problem where viscosity, electric and magnetic affects are negligible.

Exercise 5. *Derive the equations of fluid dynamics on the surface of a doughnut. Make use of toroidal coordinates with the assumption that the distance from the center of the doughnut tube to center of doughnut is two times the radius if the doughnut tube. See solution 5.*

Chapter 3

Geodesics

Geodesics are extremized parametric curves, describing the shortest distance between two fixed points. In this chapter we derive the geodesic equation using three different approaches. [27, 28, 29, 30, 31, 32]

When optimizing the path between two points, the shortest distance between the two points is always a straight line.

Definition 42. Geodesic

A geodesic between two points P_1 and P_2 is that curve for which $\int_{P_1}^{P_2} ds$ is an extremum for variations which leave the end points P_1 and P_2 fixed.

Let us now use this definition to find the differential equation of a geodesic. Let

$$\begin{aligned} I &= \int_{P_1}^{P_2} ds \\ &= \int_{P_1}^{P_2} \frac{ds}{d\lambda} d\lambda \end{aligned}$$

for some parameter λ. From section 1.2.7 of the metric tensor and the scalar product, we know that we can describe the line element ds as $\sqrt{g_{\mu\nu} dx^\mu dx^\nu}$, and as such the rate of change of this line element with respect to some parameter λ as

$$\sqrt{\frac{ds^2}{d\lambda^2}} = \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}$$

arriving at

$$I = \int_{P_1}^{P_2} \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}\, d\lambda.$$

Let $g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = f(x^\kappa, \dot{x}^\kappa)$, then

$$I = \int_{P_1}^{P_2} \sqrt{f(x^\kappa, \dot{x}^\kappa)}\, d\lambda.$$

Now, for a geodesic we must have $\delta I = 0$. I.e. $\sqrt{f}$ satisfies Euler-Lagrange equation,

$$\frac{d}{d\lambda} \left(\frac{\frac{d}{d\dot{x}^k} f(x^k, \dot{x}^k)}{2\sqrt{f(x^k, \dot{x}^k)}} \right) - \frac{\frac{d}{dx^k} f(x^k, \dot{x}^k)}{2\sqrt{f(x^k, \dot{x}^k)}} = 0. \tag{3.1}$$

Exercise 6. *Show that equation (3.1) may be simplified to the geodesic equation*

$$\ddot{x}^\kappa + \Gamma^\kappa{}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0. \tag{3.2}$$

See solution 6.

The affine parameter

The differential equation of a geodesic, equation (3.2), with s as the variable parametrising the curve

$$\frac{d^2x^\kappa}{ds^2} + \Gamma^\kappa{}_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0$$

is not parameter-independent. This form is only valid if a particular class of parameter is used. Consider the parameter $p = p(s)$:

$$\begin{aligned}
\frac{d^2x^\kappa}{ds^2} + \Gamma^\kappa{}_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} &= 0 \\
\frac{d}{ds}\left(\frac{dx^\kappa}{dp}\frac{dp}{ds}\right) + \Gamma^\kappa{}_{\mu\nu}\frac{dx^\mu}{dp}\frac{dp}{ds}\frac{dx^\nu}{dp}\frac{dp}{ds} &= 0 \\
\frac{d^2x^\kappa}{dp^2}\frac{dp}{ds}\frac{dp}{ds} + \frac{dx^\kappa}{dp}\frac{d^2p}{ds^2} + \Gamma^\kappa{}_{\mu\nu}\frac{dx^\mu}{dp}\frac{dp}{ds}\frac{dx^\nu}{dp}\frac{dp}{ds} &= 0 \\
\frac{dx^\kappa}{dp}\frac{d^2p}{ds^2} + \left(\frac{d^2x^\kappa}{dp^2} + \Gamma^\kappa{}_{\mu\nu}\frac{dx^\mu}{dp}\frac{dx^\nu}{dp}\right)\left(\frac{dp}{ds}\right)^2 &= 0
\end{aligned}$$

The geodesic equations are left invariant only if

$$\frac{dx^\kappa}{dp}\frac{d^2p}{ds^2} = 0.$$

However the tangent vector $\frac{dx^\kappa}{dp} \neq 0$, so

$$\frac{d^2p}{ds^2} = 0$$

and $p(s) = As + B$, where A and B are arbitrary constants.

Definition 43. Affine parameter

An affine parameter satisfies the differential equation of a geodesic in conserved form, equation (3.2).

3.1 Parallel transport defined geodesics

Applying the definition of the covariant derivative of some vector quantity, A^μ, as a quantity under parallel transport in the direction of δx^κ with a first order Taylor series expansion to obtain the definition of the absolute derivative and ultimately the geodesic equation.

Recall equation (1.32), the definition of the covariant derivative of a vector quantity A^μ

$$A^\mu{}_{;\nu} = A^\mu{}_{,\nu} + \Gamma^\mu{}_{\kappa\nu}A^\kappa$$

and then in the direction δx^ν

$$A^\mu{}_{;\nu}\delta x^\nu = A^\mu{}_{,\nu}\delta x^\nu + \Gamma^\mu{}_{\kappa\nu}A^\kappa \delta x^\nu. \tag{3.3}$$

However the quantity $A^\mu{}_{,\nu}\delta x^\nu$ may be obtained from a Taylor series expansion of A^μ about the point x^ν. To second order

$$
\begin{aligned}
A^\mu(x^\nu + \delta x^\nu) &= A^\mu(x^\nu) + \delta x^\nu A^\mu{}_{,\nu} + O((\delta x^\nu)^2) \\
A^\mu{}_{,\nu}\delta x^\nu &= A^\mu(x^\nu + \delta x^\nu) - A^\mu(x^\nu) - O((\delta x^\nu)^2).
\end{aligned}
$$

Substituting into equation (3.3)

$$
\begin{aligned}
A^\mu{}_{;\nu}\delta x^\nu &= A^\mu(x^\nu + \delta x^\nu) - A^\mu(x^\nu) - O((\delta x^\nu)^2) + \Gamma^\mu{}_{\kappa\nu}A^\kappa \delta x^\nu \\
&= A^\mu(x^\nu + \delta x^\nu) - \left(A^\mu(x^\nu) - \Gamma^\mu{}_{\kappa\nu}A^\kappa \delta x^\nu\right).
\end{aligned}
$$

The expression $A^\mu(x^\nu) - \Gamma^\mu{}_{\kappa\nu}A^\kappa\delta x^\nu$ in a sense represents the value A^μ at $x^\nu + \delta x^\nu$. Subtracting this from $A^\mu(x^\nu + \delta x^\nu)$ then results in $A^\mu{}_{;\nu}\delta x^\nu$ vanishing.

Definition 44. Parallel transport definition

The vector A^μ is parallely transported in the direction δx^ν if and only if

$$
A^\mu(x^\nu + \delta x^\nu) = A^\mu(x^\nu) - \Gamma^\mu{}_{\kappa\nu}A^\kappa\delta x^\nu,
$$

alternatively, if and only if $A^\mu{}_{;\nu}\delta x^\nu = 0$.

Definition 45. Absolute derivative

Suppose the vector A^μ is defined everywhere along a curve C, $x^\mu = x^\mu(p)$. Then the absolute derivative of A^μ along C is defined as

$$
\frac{DA^\mu}{Dp} = A^\mu{}_{;\nu}\frac{dx^\nu}{dp}. \tag{3.4}
$$

We note the following:

(i) The left-hand side of equation (3.4) is a tensor as the right-hand side of equation (3.4) is a tensor.

(ii) A^μ is parallely transported along the curve C if at each point along C we have $A^\mu{}_{;\nu}\frac{dx^\nu}{dp} = 0$, and consequently $\frac{DA^\mu}{Dp} = 0$.

(iii) In general, any tensor that is parallely transported along some curve C must satisfy $\frac{D}{Dp}T^{\mu\kappa\cdots}{}_{\lambda\sigma\ldots} = 0 = T^{\mu\kappa\cdots}{}_{\lambda\sigma\cdots;\nu}\frac{dx^\nu}{dp}$.

(iv) Another property of a straight line is that a tangent vector along a straight line remains parallel to itself.

Definition 46. Parallel transport definition of a geodesic

A geodesic is that curve, the tangent vector, $\frac{dx^\mu}{dp}$, to which is parallely transported along the curve.

Theorem 10. ***Differential equation of a geodesic***

Making use of definition 46, the differential equation of a geodesic is of the form

$$
\frac{d^2x^\mu}{dp^2} + \Gamma^\mu{}_{\kappa\nu}\frac{dx^\kappa}{dp}\frac{dx^\nu}{dp} = 0.
$$

Proof. The tangent vector $\frac{dx^\mu}{dp}$ is parallely transported along a curve C if and only if

$$\begin{aligned} \frac{D}{Dp}\left(\frac{dx^\mu}{dp}\right) &= 0 \\ \left(\frac{dx^\mu}{dp}\right)_{;\nu}\frac{dx^\nu}{dp} &= 0 \\ \frac{d}{dx^\nu}\left(\frac{dx^\mu}{dp}\right)\frac{dx^\nu}{dp} + \Gamma^\mu{}_{\kappa\nu}\frac{dx^\kappa}{dp}\frac{dx^\nu}{dp} &= 0 \\ \frac{d^2x^\mu}{dp^2} + \Gamma^\mu{}_{\kappa\nu}\frac{dx^\kappa}{dp}\frac{dx^\nu}{dp} &= 0. \end{aligned}$$

□

3.2 Alternative approach to geodesics

Following Dirac's approach, a brief description of an alternative approach to geodesics is addressed in this section.

Let us define a vector A^μ at some point C. In rectilinear Euclidean space it is simpler to define a transport mechanism, such as parallel transport, than it is to in curved space. Making use of an approximation is thus pivotal when moving to curved space. One may approach this by considering the vector A^μ being displaced by some small amount from C to C', and assume an uncertainty of second order. The vector at C' is said to be parallel to the vector at C and of the same magnitude.

When using this approximate method of parallel transport it may be applied continuously, taking the vector A^μ from the point C to D. This would leave the displaced vector parallel to the vector at C with respect to the path taken to D. If a different path was to be taken, the displaced vector to D would be parallel to the vector at C with respect to the new path taken. Thus the absolute meaning of parallel transport does not exist.

To arrive at equations for the parallel displacement of a vector, one may immerse our four-dimensional physical space in a flat space of higher dimensions. Let the number of dimensions of this flat space be N and introduce rectilinear coordinates z^n to this N-dimensional space, where $n = 0, 1, 2, \cdots, N$. Then consider an invariant distance ds between two neighbouring points defined by

$$ds^2 = h_{mn}dz^n dz^m \tag{3.5}$$

where h_{nm} are constants and ds^2 is summed for $n, m = 0, 1, 2, \cdots, N$.

When the 4-dimensional vector has been immersed in the flat N-dimensional space, we allow the physical space of our four dimensions to form a "surface" in the higher dimensional realm. This means that a point in the surface x^μ maps a definite point f^n in the N-dimensional space, or that every coordinate f^n is a function of a point in the surface x^μ. There are $N f^n(x)$'s from which one may eliminate the four x's to arrive at the $N-4$ equations of the surface.

Let us consider two neighbouring points in the surface δx^μ apart. By taking the derivative of $f^n(x)$ with respect to the surface parameters x^μ

$$\frac{\partial f^n(x)}{\partial x^\mu} = f^n{}_{,\mu}$$

we are able to arrive at the difference in function values for the points in the surface,

$$\delta f^n = f^n{}_{,\mu}\delta x^\mu. \tag{3.6}$$

From equation (3.5), the squared distance between the function values, resulting from surface elements a distance δx^c apart, is

$$\delta s^2 = h_{nm}\delta f^n \delta f^m = h_{nm} f^n{}_{,\mu} f^m{}_{,\nu} \delta x^\mu \delta x^\nu.$$

As h_{nm} are constants we may write

$$\delta s^2 = f^n{}_{,\mu} f_{n,\nu} \delta x^\mu \delta x^\nu$$

or by the metric tensor form, $\delta s^2 = g_{\mu\nu}\delta x^\mu \delta x^\nu$. Thus the metric tensor of the N-dimensional space may be defined by

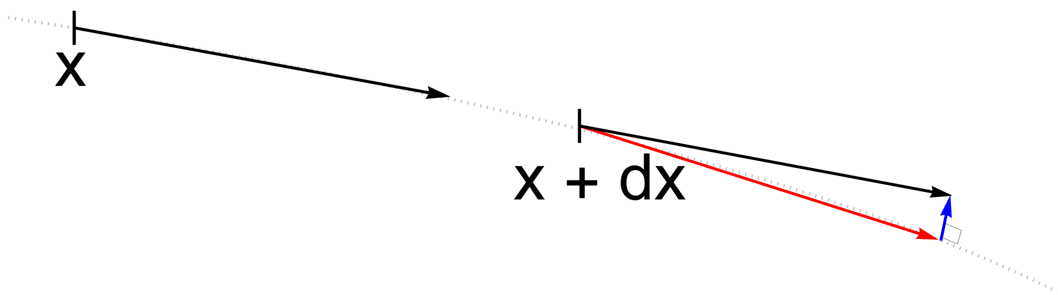
$$g_{\mu\nu} = f^n{}_{,\mu} f_{n,\nu} \qquad \text{or} \qquad g_{\mu\nu} = h_{nm} f^n{}_{,\mu} f^m{}_{,\nu}.$$

Now, let us take a contravariant vector in physical space, positioned at a point x, and denote it by A^μ. The components of this contravariant vector are similar to those of δf^n of equation (3.6), thus in N-dimensional space the contravariant vector A^μ provides a contravariant vector A^n

$$A^n = f^n{}_{,\mu} A^\mu. \tag{3.7}$$

Consider the situation where the vector A^n, which lies in the surface of the N-dimensional space, is shifted to neighbouring point $x + dx$ in the surface, such that the vector A^n remains parallel to itself.

Taking the curvature of the surface into account, the vector A^n will no longer lie in the surface at its displaced position $x + dx$. The vector displaced to the neighbouring position, $x + dx$, may therefore be described by the sum of the projected tangent vector in the surface and a vector normal to the surface.



Therefore,

$$A^n = A^n{}_{\text{normal}} + A^n{}_{\text{tangent}}. \tag{3.8}$$

If we define κ^μ as the components of $A^n{}_{\text{tangent}}$ with respect to the x coordinate system in the N-dimensional surface, then by equation (3.6) we have

$$A^n{}_{\text{tangent}} = \kappa^\mu f^n{}_{,\mu}(x + dx), \tag{3.9}$$

with coefficients $f^n{}_{,\mu}$ taken at the displaced position in N-dimensional surface.

The normal vector $A^n{}_{\text{normal}}$ is defined such that no matter the components $\kappa^m u$, this vector is orthogonal to the tangential vector. By multiplying equation (3.8) by $f_{n,\nu}$, at the displaced position $x + dx$,

$$A^n f_{n,\nu}(x + dx) = A^n{}_{\text{normal}} f_{n,\nu}(x + dx) + \kappa^\mu f^n{}_{,\mu}(x + dx) f_{n,\nu}(x + dx), \tag{3.10}$$

we are able to use the definition of the linear vector combination to show that $A^n{}_{\text{normal}} f_{n,\nu}(x + dx)$ will vanish, i.e.

$$A^n{}_{\text{normal}} f_{n,\nu}(x + dx) = 0.$$

Equation (3.10) therefore simplifies to

$$A^n f_{n,\nu}(x+dx) = \kappa^\mu g_{\mu\nu}(x+dx)$$

in metric tensor form. With some rearrangement and finding the Taylor series expansion of $f_{n,\nu}$ to first order

$$\begin{aligned}\kappa^\mu g_{\mu\nu}(x+dx) &= A^n f_{n,\nu}(x+dx),\\ \kappa_\nu|_{x+dx} &= A^n\left[f_{n,\nu}\big|_x + f_{n,\nu\sigma}\big|_x dx^\sigma\right].\end{aligned}$$

By equation (3.7),

$$\kappa_\nu|_{x+dx} \;=\; A^\mu f^n{}_{,\mu}\left[f_{n,\nu}+f_{n,\nu\sigma}dx^\sigma\right] \;=\; A^\mu g_{\mu\nu} + A^\mu f^n{}_{,\mu} f_{n,\nu\sigma}dx^\sigma \;=\; A_\nu + A^\mu f^n{}_{,\mu} f_{n,\nu\sigma}dx^\sigma,$$

we are able to define the vector κ_ν as the vector A_ν which has undergone parallel displacement from a point x to $x+dx$. Thus, the change of vector A_ν under parallel displacement is $dA_\nu \;=\; A_\nu - \kappa_\nu$, simplifying to

$$dA_\nu = A^\mu f^n{}_{,\mu} f_{n,\nu\sigma}dx^\sigma. \tag{3.11}$$

Exercise 7. *Show that*

$$\mathbf{g}_\mu \cdot \mathbf{g}_{\nu,\sigma} = \Gamma_{\mu\nu\sigma}$$

See solution at 7.

Thus equation (3.11) may be written as

$$dA_\nu = A^\mu \Gamma_{\mu\nu\sigma} dx^\sigma. \tag{3.12}$$

As the Christoffel symbol of first kind is completely described by the metric tensor of the physical space, the definition for the change in the vector A_ν under parallel displacement is described completely without any reference to N-dimensional space. Now, using equation (1.18) we have that

$$g^\alpha{}_{\nu,\sigma} \;=\; \delta^\alpha{}_{\nu,\sigma} \;=\; 0.$$

We may go further by making use of the rules developed in section 1.2.7 for raising and lowering indices,

$$\begin{aligned}0 &= g^\alpha{}_{\nu,\sigma}\\ 0 &= \left(g^\alpha{}_\nu\right)_{,\sigma}\\ 0 &= g^{\beta\nu}\left(g^{\alpha\mu}g_{\mu\nu}\right)_{,\sigma}\\ 0 &= g^{\beta\nu}\left(g^{\alpha\mu}{}_{,\sigma}g_{\mu\nu} + g^{\alpha\mu}g_{\mu\nu,\sigma}\right)\\ 0 &= g^{\beta\nu}g_{\mu\nu}g^{\alpha\mu}{}_{,\sigma} + g^{\alpha\mu}g^{\beta\nu}g_{\mu\nu,\sigma}\\ 0 &= g^\beta{}_\mu g^{\alpha\mu}{}_{,\sigma} + g^{\alpha\mu}g^{\beta\nu}g_{\mu\nu,\sigma}\end{aligned}$$

thus

$$g^{\alpha\beta}{}_{,\sigma} = -g^{\alpha\mu}g^{\beta\nu}g_{\mu\nu,\sigma}. \tag{3.13}$$

In order to prove that the length of a vector is unchanged under parallel displacement, we find the change in ds or equivalently ds^2.

$$\begin{aligned}d(ds^2) &= d(g^{\mu\nu}A_\mu A_\nu)\\ &= g^{\mu\nu}A_\mu dA_\nu + g^{\mu\nu}dA_\mu A_\nu + A_\mu A_\nu g^{\mu\nu}{}_{,\sigma}dx^\sigma\\ &= A^\nu dA_\nu + dA_\mu A^\mu + \left(g_\alpha{}^\mu g^\alpha{}_\mu\right)\left(g_\beta{}^\nu g^\beta{}_\nu\right)A_\mu A_\nu g^{\mu\nu}{}_{,\sigma}dx^\sigma\\ &= A^\nu dA_\nu + dA_\mu A^\mu + g_\alpha{}^\mu A_\mu g_\beta{}^\nu A_\nu g^\alpha{}_\mu g^\beta{}_\nu g^{\mu\nu}{}_{,\sigma}dx^\sigma\\ &= A^\nu dA_\nu + dA_\mu A^\mu + A_\alpha A_\beta g^{\alpha\beta}{}_{,\sigma}dx^\sigma\end{aligned}$$

Making use of equations (3.12) and (3.13)

$$\begin{aligned}
d(ds^2) &= A^\nu\left(A^\mu\Gamma_{\mu\nu\sigma}dx^\sigma\right)+\left(A^\nu\Gamma_{\nu\mu\sigma}dx^\sigma\right)A^\mu + A_\alpha A_\beta\left(-g^{\alpha\mu}g^{\beta\nu}g_{\mu\nu,\sigma}\right)dx^\sigma\\
&= A^\nu A^\mu\left(\Gamma_{\mu\nu\sigma}+\Gamma_{\nu\mu\sigma}\right)dx^\sigma - g^{\alpha\mu}A_\alpha g^{\beta\nu}A_\beta g_{\mu\nu,\sigma}dx^\sigma\\
&= A^\mu A^\nu\left(\Gamma_{\mu\nu\sigma}+\Gamma_{\nu\mu\sigma}\right)dx^\sigma - A^\mu A^\nu g_{\mu\nu,\sigma}dx^\sigma
\end{aligned}$$

and definition 27, the Christoffel symbols of first kind, we have

$$d(ds^2) = A^\mu A^\nu\left(g_{\mu\nu,\sigma}\right)dx^\sigma - A^\mu A^\nu g_{\mu\nu,\sigma}dx^\sigma. \tag{3.14}$$

We see that the right hand side of equation (3.14) vanishes. Thus, the length of vector, including the null vector case, is left unchanged under parallel displacement. This result is proven by making use of equation (3.12) which was determined through a geometrical argument. The argument was subject to an assumption of first order, i.e. the path of parallel displacement of a vector is an affine parameter.

We recall definition 28 of the Christoffel symbols of second kind, equation (3.12) and the definition 20 of the metric tensor,

$$dA_\nu = A^\lambda\Gamma_{\lambda\nu\sigma}dx^\sigma \ = \ A_\mu g^{\mu\lambda}\Gamma_{\lambda\nu\sigma}dx^\sigma$$

to arrive at the standard formula referring to covariant components,

$$dA_\nu = A_\mu\Gamma^\mu{}_{\nu\sigma}dx^\sigma. \tag{3.15}$$

Let us consider a second vector B^ν and consider the parallel displacement of its product with the covariant vector A_ν,

$$d(A_\nu B^\nu) = dA_\nu B^\nu + A_\nu dB^\nu.$$

As this vector product was transformed by parallel displacement we must have

$$\begin{aligned}
dA_\nu B^\nu + A_\nu dB^\nu &= 0\\
A_\nu dB^\nu &= -dA_\nu B^\nu\\
A_\nu dB^\nu &= -dA_\mu B^\mu \qquad \text{(renaming indices } \nu\to\mu\text{)}.
\end{aligned}$$

By equation (3.15) and for any A_ν,

$$A_\nu dB^\nu = -A_\nu\Gamma^\nu{}_{\mu\sigma}dx^\sigma B^\mu,$$

we obtain the standard formula for parallel displacement referring to contravariant components,

$$dB^\nu = -\Gamma^\nu{}_{\mu\sigma}B^\mu dx^\sigma. \tag{3.16}$$

Now, let us consider a point with coordinates z^μ and suppose it is a function of of some parameter p as it moves along a track. Therefore, the rate of change of the point in terms of the parameter p may be described by a vector u^μ, i.e.

$$\frac{dz^\mu}{dp} = u^\mu. \tag{3.17}$$

At each point along the track, one may define a vector u^μ. If we are to suppose that the vector is displaced in a parallel fashion from an initial point z^μ to $z^\mu + u^\mu dp$, where u^μ is the initial vector value, then we are able to determine the track as well as the parameter p. The track produced by this process is defined as a geodesic.

Recalling the standard formula for parallel displacement referring to contravariant components, equation (3.16), to find the rate of change of the vector u^μ with respect to the parameter p, we have

$$\frac{du^\mu}{dp} + \Gamma^\mu{}_{\nu\sigma} u^\mu \frac{dz^\sigma}{dp} = 0 \tag{3.18}$$

alternatively we have

$$\frac{d^2 z^\mu}{dp^2} + \Gamma^\mu{}_{\nu\sigma} \frac{dz^\mu}{dp} \frac{dz^\sigma}{dp} = 0 \tag{3.19}$$

by making use of equation (3.17).

Chapter 4

Inertia and Gravity

This chapter dives into the six principles of GR, with a brief description of each of them. Namely Mach's principle, the equivalence principle, the principle of covariance, the correspondence principle, the simplicity principle and the Cosmological principle.

These six principles form the foundation of the geodesic hypothesis, equipping the reader with the interpretation of GR as suggested in the notes of Einstein. Particular emphasis is placed on understanding that inertia and gravity are unified through GR, rather than gravity being reduced to a geometrical object interpretation. [33, 34, 35, 36, 37, 38, 39, 40, 41]

4.1 Principles of General Relativity

On 28 November 1919, Einstein wrote an elucidating note in the London *Times* on an outline of relativity theory. Placing emphasis on the distinction between "constructive" and "principle" theories, he explained the development of the theory.

In Einstein's discussion it was suggested that he preferred the constructive approach to theory development, as this gives one the opportunity to "build up a picture of the more complex phenomena out of materials of a relatively simple formal scheme from which they start out". He went on further to describe how a constructive theory is more clear, complete and adaptable, a consequence of its synthetic nature, whereas a principle theory is built from an analytical perspective, offering "logical perfection and security of the foundation".

This distinction was the cornerstone to his argument, the difference between true "understanding" of a group of natural phenomena in the construction of a theory, and the quest by analytic means to deduce the privileged conditions, separate events have to be subject to, in the construction of a theory.

Mach's principle

"Inertial forces are exerted by mass-energy sources, not by absolute space"

Indulging a deeper, more precise understanding of the nature of Newton's laws, one may appreciate the value Einstein saw in Ernst Mach's idea. Mach put forward the notion that one should refer to the inertia of a body being influenced by distant bodies, stars, rather than relative to absolute space. Einstein adopted this idea, naming it after Mach, developing it to become the first principle of GR.

This principle may be better understood by two core ideas:

- "Inertial forces have a dynamical rather a kinematical origin, and so must be derived from a field theory".
- "The whole of the inertial field must be due to sources, so that in solving the inertial field equations the boundary conditions must be chosen appropriately".

This essentially means that "all motion is relative", allowing for us to set up local patches with Lorentzian frames.

Equivalence principle

> "Physics for an observer at rest in a uniform gravitational field is identical to an observer undergoing uniform acceleration"

In the early 17^{th} century, Galileo experimentally showed that the velocities of mass-varied bodies rolling down an incline did not significantly vary from each other. A century later, Kepler illustrated knowledge of this principle by describing, in the scenario where orbital dynamics were not in play, how the earth and the moon would approach each other in distances proportional to each other's comparative mass'. He did this by assuming gravity and inertia to be equivalent. Later that century, Newton used these ideas and designed a pendulum experiment. He used bobs of different masses, with equal length rods, and measured the periods of these pendulums. He concluded that the effects of gravity are diminished with an increase in distance between objects. It was only in the early 20^{th} century, that Einstein proposed his version of the equivalence principle. He described the inability to differentiate, in a local frame, between the physics experienced due to a gravitational field and that of accelerated motion. This statement is mathematically embodied by the replacement of partial derivatives with covariant ones.

Covariance principle

> "All observers are equivalent"

This principle stresses the importance of formulating the physical laws of physical quantities that an observer in any frame of reference can unequivocally agree upon. This implies that the laws of nature must be expressed in a form which is independent of a particular coordinate system, making tensor notation particularly useful.

Correspondence principle

> "Relativistic theory of gravity must agree with experiment"

With the development of any physics theory, it is important to have experimental evidence of the phenomenon being described. In this instance, the theory expressed in GR should reduce to Newtonian theory under certain assumptions, i.e. in the case of weak gravitational fields and small velocities.

Simplicity principle

> "The equations of the laws of physics should be similar on any arbitrary manifold"

Complimentary to the covariance and correspondence principles, the mathematical representations of the laws of physics should, for the most part, look the same in curved space as in flat space. This principle is used to motivate the formation of the Einstein field equations and the general conservation equations.

Cosmological principle

> "Our solar system is not a privileged domain of the universe"

The cosmological principle, a more generalised version of the Copernican principle, defines our universe, when viewed on a sufficiently large scale, to be isotropic and homogeneous. The universe is defined to be finite in size but with no boundary.

4.2 Unifying Inertia and Gravity

When moving from a Riemannian manifold to the more general space-time pseudo-Riemannian manifold, the gravitational contribution in Newtonian problems surface as connection coefficients in the geodesic equations. As the Euler-Lagrange equations are equivalent to the geodesic equations, proven in chapter 3, a link is made between the Newtonian inertial contributions and Newtonian gravitational contributions in the equations of motion. The motivation of this link was debated over the better half of the 1920's and 1930's, arguing the geometrisation of gravity against the unification of gravity with inertia.

Around 1925, Einstein argued against Meyerson's stance of gravity geometrisation, saying that this view said nothing interesting about the physics. Einstein went further to say that it is a private matter should one choose to link the 'geometrical' intuitions with a theory (gravitation or electricity), this was a response in agreement with Hans Reichenbach letter to Einstein. Rechenbach and Einstein both agreed that the same way Weyl's and Eddington's theory unified electricity and magnetism, gravity and inertia were unified through GR. If one is to claim gravitation to be geometry, then equivalently electricity should be claimed to be geometry. One could describe this mathematically as instead of reducing gravitation to geometry,

$$\ddot{x}^{\mu} = -\Gamma^{\mu}{}_{\nu\lambda}\dot{x}^{\nu}\dot{x}^{\lambda}$$

there is a unification of inertia and gravity,

$$\ddot{x}^{\mu} + \Gamma^{\mu}{}_{\nu\lambda}\dot{x}^{\nu}\dot{x}^{\lambda} = 0.$$

This difference may seem subtle and somewhat superficial, but it has profound implications. It is in achieving this form of a unified field theory where one comes to the realisation that the observed phenomenon described as gravity in Newtonian theory, is simply a manifestation of space-time curvature. There is no logical reason to differentiate gravitation from inertia (or to differentiate inertia from gravitation), other than for the reconciliation of GR with prior theories.

Chapter 5

Representations of General Relativity

This chapter is aimed at introducing the different representations of GR consistent with the principles outlined in the previous chapter. It begins with an argument for the incorporation of frame of reference time as a coordinate. An introduction to the concept of geodesic deviation is included, showing the direct dependence of geodesic deviation is linearly dependent on the curvature of the space-time. The hydrodynamic equations are proposed based on the argument of local time in a frame of reference. This is then motivated in a section addressing the energy-momentum tensor and conservation equations. The Einstein field equations (EFEs) of gravitation are derived and are shown to reduce to classical equations of motion. A $1+3$ *threading of space-time and* $3+1$ *foliation of space-time are introduced as the general relativistic Lagrangian and Eulerian formalisms respectively. The EFEs are now derived from the Einstein-Hilbert Palantini formulation of the least action principle and lastly the tetrad formalism of GR is supplied. [24, 42, 43, 44, 45, 46, 47, 48]*

Let us investigate a derivation of a theory that satisfies the principles discussed in chapter 4. First consider the Covariance principle, where different observers have to unequivocally agree upon an observed measurable, and the means of observation available. One observational mean, the best known to date, is that of electromagnetic radiation detection.

One can only imagine how long humans have observed the optic radiation of celestial bodies. Many astronomers tried to predict the behaviour of these celestial bodies, forming beliefs about how these bodies influence our existence and how we can utilise their dynamics to better navigate our experience of existence. One important observation, in the context we prescribe to here, was that of Ole Rømer in 1676[42]. He was studying the periods of Jupiter's inner most moon, Io, when he noticed that they were different depending on how close by or far away Earth was from Jupiter. He described the suggestion of a finite speed of light, ultimately discrediting the idea of time universality, and became the first person to successfully calculate its speed. Two centuries later, in 1873[43], James Maxwell was able to determine the speed of electromagnetic wave propagation to be related to the inductance and capacitance of the vacuum, i.e

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

where μ_0 and ϵ_0 are the permeability and permittivity of free space, respectively. In 1887[44], Michelson and Morley designed an experiment to prove the existence of aether, believed to be the medium in which light propagated. They concluded that aether does not exist and that the speed of light is constant for all observers. This locally observed phenomenon suggests that any observer is able to measure the speed of light and unequivocally agree upon it. Let us investigate this further. Consider an observer, in an arbitrary Riemannian manifold, measuring the speed of light for some time interval δt for which a ray of light passes through some distance δx, the line element described in section 1.2.7, then

$$\begin{aligned} c^2\delta t^2 &= \delta x^2 \\ -c^2\delta t\delta t + \delta x\delta x &= 0. \end{aligned} \tag{5.1}$$

Equation (5.1) is a scalar invariant equation and it suggests that time could be included in the coordinate system. Thus, the metric would be

$$g_{\mu\nu} = \begin{pmatrix} -c^2 & 0 \\ 0 & 1 \end{pmatrix}$$

and we define the momentum of an observer in this pseudo-Riemannian manifold to be

$$p^\mu = mu^\mu \;=\; m\begin{pmatrix} c\dot{t} \\ \dot{x} \end{pmatrix} \;=\; m\dot{t}\begin{pmatrix} c \\ v \end{pmatrix}$$

where $v \;=\; \frac{dx}{d\tau}\frac{d\tau}{dt} \;=\; \frac{dx}{dt}$ and the parameter τ is the proper time. Define the squared momentum of an observer at rest, in a uniform gravitational field, to be

$$p^\mu p_\mu = -m^2c^2$$

and the squared momentum of an observer undergoing uniform acceleration, to be

$$\bar{p}^\mu \bar{p}_\mu = -m^2\dot{t}^2c^2 + m^2\dot{t}^2v^2.$$

By the equivalence principle, these two quantities must be identical

$$-m^2c^2 = -m^2\dot{t}^2c^2 + m^2\dot{t}^2v^2$$

therefore

$$\dot{t} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{and} \qquad p^\mu \;=\; \frac{m}{\sqrt{1-\frac{v^2}{c^2}}}\begin{pmatrix} c \\ v \end{pmatrix}$$

which suggests that the energy is defined by

$$p^0 \;=\; E = \frac{mc}{\sqrt{1-\frac{v^2}{c^2}}} \;=\; \frac{mc^2}{\sqrt{c^2-v^2}} \tag{5.2}$$

and the momentum is defined by

$$p^1 \;=\; p = \frac{mv^2}{\sqrt{1-\frac{v^2}{c^2}}}.$$

Let us take a Taylor series expansion of the energy equation, equation (5.2), with v as the expansion parameter, then

$$E = mc^2 + \frac{1}{2}mv^2\left(c^2-v^2\right)^{\frac{-3}{2}} + \cdots \tag{5.3}$$

we note that there is an mc^2 term in equation (5.3), that didn't come up in the classical energy definitions. This quantity is known as the rest mass energy. For $v << c$, in the case of Newtonian mechanics, the energy reduced to just the rest mass energy, a phenomenon observed in relativity. This means that the energy, E, described in the fluid equations of section 2.3 is the proper energy density, $\rho E \;=\; \rho_0\left(c^2+\epsilon\right)$.

We have now established that time is to be included in the coordinate system, and energy has been unified with momentum. This suggests that potentials, like gravitational potential, should be unified with some quantity in a similar way. Thus, let us investigate the relationship between the Euler-Lagrange equations and the geodesic equations by considering two physically identical problems, by the equivalence principle.

Consider a free particle moving in the equatorial plane of a spherical coordinate system, of a Riemannian manifold. Use the Euler-Lagrange equations for the derivation of the equations of motion. The Lagrangian of the free moving particle is therefore given by

$$L = \frac{1}{2} m g_{ij} \dot{x}^i \dot{x}^j = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2 \right) \tag{5.4}$$

the equations of motion are therefore[1]

$$\ddot{r}(t) = r(t) \dot{\phi}(t)^2 \tag{5.5}$$
$$\ddot{\phi}(t) = -\frac{2\dot{r}(t)\dot{\phi}(t)}{r(t)}. \tag{5.6}$$

From equations (5.5) and (5.6) we can read off the Christoffel components ${\Gamma^r}_{\phi\phi} = r(t)$ and ${\Gamma^\phi}_{r\phi} = {\Gamma^\phi}_{\phi r} = \frac{-1}{r(t)}$ respectively. Now let us repeat this for a particle in a central potential. The Lagrangian, in a Riemannian manifold, is given by

$$L = \frac{1}{2} m g_{ij} \dot{x}^i \dot{x}^j - V = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\phi}^2 \right) + \frac{GMm}{r} \tag{5.7}$$

and the equations of motion are therefore[2]

$$\ddot{r}(t) = r(t) \dot{\phi}(t)^2 - \frac{GM}{r(t)^2} \tag{5.8}$$
$$\ddot{\phi}(t) = -\frac{2\dot{r}(t)\dot{\phi}(t)}{r(t)}. \tag{5.9}$$

Again, we notice that from equations (5.8) and (5.9) we can read off the Christoffel components ${\Gamma^r}_{\phi\phi}$ and ${\Gamma^\phi}_{r\phi} = {\Gamma^\phi}_{\phi r}$ as in equations (5.5) and (5.6), however there is a term $-\frac{GM}{r(t)^2}$ representing the gravitational force that has not yet been associated with a Christoffel component. We note the absence of coordinate component derivative squared in the gravitational force term, which must be of the form of $\dot{t} = 1$ due to the assumption of absolute/universal time. We recall that the geodesic equations remain invariant under affine transformations. Let us, for investigative purposes, introduce a new time parameter, τ, so that $t(\tau) = \tau$. Then the gravitational force term is associated with a Christoffel symbol, namely ${\Gamma^r}_{tt}$. This association unifies inertia and gravity, by relating the potential forces to the Christoffel components of the geodesic equations. Gravity is simply a manifestation of space-time.

Geodesic deviation

Over longer periods of time, in the case where time is not included as a coordinate, the uniformity of acceleration and a gravitational field is no longer present. This means that the geodesics are bound to deviate from an expected path, in the classical sense. Let us consider the scenario where two point masses are inside of a windowless room that is approaching the surface of the Earth. From the perspective of an observer inside the room, the point masses approach each other - as depicted in figure 5.

It is important for us to further investigate this phenomenon, thus let us begin by defining a deviation vector, $\xi^a(t)$, as the difference between the expected geodesic position vector and the actual position vector:

$$\xi^a(t) = \tilde{x}^a(t) - x^a(t) \tag{5.10}$$

[1] Determine the equations of motion from the Euler-Lagrange equations in spherical coordinates using Mathematica. Using the Lagrangian equation (5.4), one must obatin equations (5.5) and (5.6). See 12 for a possible Mathematica representation.

[2] Determine the equations of motion from the Euler-Lagrange equations in spherical coordinates using Mathematica. Using the Lagrangian equation (5.7), one must obatin equations (5.8) and (5.9). See 13 for a possible Mathematica representation.

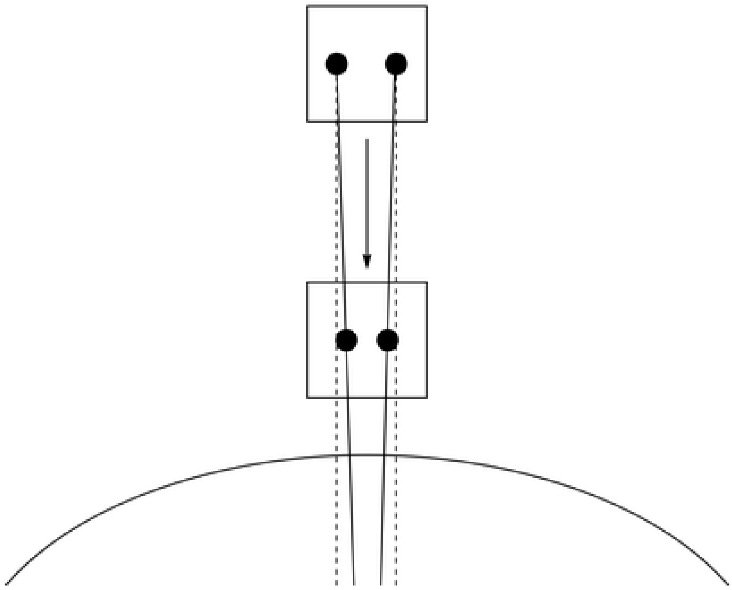


Figure 5.1: Two point masses inside a windowless room that is descending to the surface of the Earth

The geodesic equation, equation (3.2), must be satisfied for both the actual and expected position vectors, i.e.

$$\ddot{\tilde{x}}^a + \tilde{\Gamma}^a_{bc}\dot{\tilde{x}}^b\dot{\tilde{x}}^c = 0 \tag{5.11}$$

$$\ddot{x}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{x}^c = 0. \tag{5.12}$$

The deviation vector definition, equation (5.10), may be rearranged to obtain

$$\tilde{x}^a(t) = \xi^a(t) + x^a(t) \tag{5.13}$$

and the connection coefficient may be determined using a first-order Taylor series expansion of $\Gamma^a{}_{bc}$ with the deviation vector as the expansion parameter:

$$\tilde{\Gamma}^a_{bc} = \Gamma^a{}_{bc} + \Gamma^a{}_{bc,d}\xi^d. \tag{5.14}$$

Now, subtract equation (5.12) from equation (5.11), substitute both equations (5.13) and (5.14) into this result and neglect second-order terms of the expansion parameter. This yields

$$\ddot{\xi}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{\xi}^c + \Gamma^a{}_{bc,d}\dot{x}^b\dot{x}^c\xi^d + \Gamma^a{}_{bc}\dot{\xi}^b\dot{x}^c = 0 \tag{5.15}$$

where each dot above a symbol represents a derivative with respect to the line element s.

Exercise 8. *Show that*

$$\begin{aligned}\ddot{\xi}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{\xi}^c + \Gamma^a{}_{bc,d}\dot{x}^b\dot{x}^c\xi^d + \Gamma^a{}_{bc}\dot{\xi}^b\dot{x}^c = &\frac{d}{dt}\left(\dot{\xi}^a + \Gamma^a{}_{bc}\dot{x}^b\xi^c\right) - \Gamma^a{}_{bc,d}\xi^b\dot{x}^c\dot{x}^d \\ &- \Gamma^a{}_{bc}\xi^b\ddot{x}^c + \Gamma^a{}_{bc,d}\dot{x}^b\dot{x}^c\xi^d + \Gamma^a{}_{bc}\dot{\xi}^b\dot{x}^c\end{aligned}$$

See solution 8.

We make use of the proof in exercise 8 in equation (5.15), definition 45, the absolute derivative, and the geodesic equation, equation (3.2), to get rid of $\ddot{x}^c$. After some algebra, one will arrive at the equation of geodesic deviation

$$\frac{D^2\xi^a}{D\tau^2} + R^a{}_{bcd}\dot{x}^b\xi^c\dot{x}^d = 0.$$

In a flat space-time the Riemann curvature tensor vanishes and thus $\frac{D^2\xi^a}{D\tau^2} = 0$, which gives rise to deviations that are linear $\xi^a(\tau) = A^a\tau + B^a$. We note that geodesics are directly dependent on the curvature of the space-time.

Hydrodynamic equations

So far we have proposed the incorporation of time as a coordinate, thus found a connection between energy and momentum, and unified inertia and gravity. These all have an affect on the form of the fluid dynamic equations, equations (2.7), (2.8) and (2.10) which are closed by the Poisson equation. As such, let us combine the energy density and momentum density vector into one vector

$$J^{\mu} = \{\rho E, \rho u^1, \rho u^2, \rho u^3\}.$$

Since we have incorporated time as a component, combined the energy and momentum into one physical vector and unified inertia and gravity, then there must exist a more general stress-energy tensor, $T^{\mu\nu}$, such that all contributions to a dynamic system are incorporated in one general conservation law (similar to how the stress-momentum tensor was formed when deriving equation (2.8)), such that the five equations (one density, three momentum and one energy density) of fluid dynamics become hydrodynamic equations in relativity defined as

$$J^{\mu}{}_{;\mu} = 0 \tag{5.16}$$
$$T^{\mu\nu}{}_{;\nu} = 0. \tag{5.17}$$

From here, it is important to note that we will be choosing geometric units such that $c = 1$ and $G = 1$.

5.1 Energy momentum tensor

The reader is reminded of the discussion initiated in section 2.3 of a more general energy momentum tensor. This tensor is to be formally introduced for case of a perfect fluid, followed by the introduction of conservation laws for energy and momentum, and finishing off this section with the dust case investigation.

In relativity, matter and energy are equivalent. Our representation of matter must be covariant and also include all forms of energy, such as rest mass, kinetic energy, viscous and elastic stresses and pressure. Consider now a fluid.

Definition 47. Proper energy density

The proper energy density $\rho_0(1+\epsilon)$ of a fluid is its energy density as measured in the local inertial systems moving with fluid element at the instant of measurement. This scalar includes the rest mass density and internal energy density.

Definition 48. Pressure

The scalar pressure p of a fluid is its pressure as measured in the local inertial system moving with the fluid element at the instant of measurement.

Definition 49. Perfect fluid

A perfect fluid is one in which the only internal stress is an isotropic pressure. I.e. viscous stresses can be ignored. The universe, as well as the interior of stars, may be regarded as a perfect fluid.

We look for a tensor, called the energy-momentum tensor, which describes the energy (and matter) content of space-time but excludes any “gravitational energy" (because gravity is built into the structure of space-time). The divergence of this tensor must therefore give the rate of change in time and space of the energy distribution, i.e. give the GR analogue of the non-relativistic equations of conservation of energy and momentum. A second order tensor will achieve this as its divergence, $T^{ab}{}_{;b} = 0$, yields four equations. A convenient property of this second order tensor is its symmetry, a result of the vanishing divergence in either index. Thus, we are looking for a symmetric, second order tensor, with zero divergence which describes the energy and matter content of space-time, such that $T^{ab}{}_{;b} = 0$ reduces to the non-relativistic conservation equations in a local inertial system.

Theorem 11. *Energy-momentum tensor for a perfect fluid*

The energy-momentum tensor for a perfect fluid is

$$T^{\mu\nu} = \left(\rho_0(1+\epsilon)+p\right)u^\mu u^\nu + p g^{\mu\nu} \tag{5.18}$$

where $u^\mu = \frac{d\mathbf{x}^\mu}{d\tau}$ is the 4-velocity of the fluid element and we define the projection tensor, $h^{\mu\nu}$, to be

$$h^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu.$$

Proof. Consider special relativity and fluid at rest in an inertial system. Conservation of energy and momentum equations are

$$\frac{\partial}{\partial t}\left(\rho_0(1+\epsilon)\right) = 0 \tag{5.19}$$

$$-\frac{\partial p}{\partial x^i} = 0, \tag{5.20}$$

respectively, where $i = 1, 2, 3$. For this system, let us consider the energy-momentum tensor $\overline{T}^{\mu\nu}$ defined by

$$\overline{T}^{\mu\nu} = \begin{pmatrix} -\rho_0(1+\epsilon) & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}.$$

This symmetric, second-order tensor also satisfies the condition of vanishing divergence

$$\overline{T}^{0\nu}{}_{;\nu} = 0 \qquad \text{and} \qquad \overline{T}^{i\nu}{}_{;\nu} = 0$$

as required, yielding the conservation of energy and momentum equations, equations (5.19) and (5.20), respectively. Consider now GR and a local inertial system, S^0, moving with the fluid element. The energy-momentum tensor at the origin S^0 is defined by $\overline{T}^{\mu\nu}$, and let the coordinates at the origin be $\{\overline{x}^\mu\}$. Consider the transformation from these coordinates to an arbitrary coordinate system $\{x^\mu\}$ with energy-momentum tensor $T^{\mu\nu}$:

$$\begin{aligned} T^{\mu\nu} &= x^\mu{}_\kappa\, x^\nu{}_\lambda\, \overline{T}^{\kappa\lambda} \\ &= x^\mu{}_0\, x^\nu{}_0\, \overline{T}^{00} + x^\mu{}_1\, x^\nu{}_1 \overline{T}^{11} + x^\mu{}_2\, x^\nu{}_2 \overline{T}^{22} + x^\mu{}_3\, x^\nu{}_3 \overline{T}^{33} \end{aligned}$$

thus

$$T^{\mu\nu} = x^\mu{}_0\, x^\nu{}_0 \left(\rho_0(1+\epsilon)\right) + x^\mu{}_i\, x^\nu{}_i\, p. \tag{5.21}$$

Now consider the diagonal Lorentz/Minkowski metric tensor - to be discussed in section 6.1 - at the origin, which has the diagonal $\{-1, 1, 1, 1\}$, then

$$\begin{aligned} g^{\mu\nu} &= x^\mu{}_\kappa\, x^\nu{}_\lambda\, \overline{g}^{\kappa\lambda} \\ &= x^\mu{}_0\, x^\nu{}_0\, \overline{g}^{00} + x^\mu{}_1\, x^\nu{}_1 \overline{g}^{11} + x^\mu{}_2\, x^\nu{}_2 \overline{g}^{22} + x^\mu{}_3\, x^\nu{}_3 \overline{g}^{33} \\ &= -x^\mu{}_0\, x^\nu{}_0 + x^\mu{}_i\, x^\nu{}_i \end{aligned}$$

$$x^\mu{}_i\, x^\nu{}_i = g^{\mu\nu} + x^\mu{}_0\, x^\nu{}_0. \tag{5.22}$$

Equation (5.22) into equation (5.21)

$$T^{\mu\nu} = \rho_0(1+\epsilon)\, x^\mu{}_0\, x^\nu{}_0 + p\left(g^{\mu\nu} + x^\mu{}_0\, x^\nu{}_0\right) \tag{5.23}$$

But S^0 is the local inertial system moving instantaneously with the fluid element. Thus the time interval $d\overline{x}^0$ in S^0 is the proper time interval $d\tau$, i.e. $x^\mu{}_0 = \frac{\partial x^\mu}{\partial \tau} = u^\mu$. So equation (5.23) may be written as

$$\begin{aligned} T^{\mu\nu} &= (\rho_0(1+\epsilon)+p)u^\mu\, u^\nu + p g^{\mu\nu} \\ &= \rho_0(1+\epsilon)\, u^\mu\, u^\nu + p\left(g^{\mu\nu} + u^\mu\, u^\nu\right) \end{aligned} \tag{5.24}$$

or

$$T^{\mu\nu} = \rho_0(1+\epsilon)\, u^\mu\, u^\nu + p\, h^{\mu\nu} \tag{5.25}$$

for projection tensor $h^{\mu\nu} = g^{\mu\nu} + u^\mu\, u^\nu$. □

We note the that this tensor is symmetric and from equation (5.24) it can be seen that the coefficient of $u^a u^b$ is not merely $\rho_0(1+\epsilon)$, but $\rho_0(1+\epsilon)+p$. This is later shown to imply that p plays an inertial and dynamic role in the pressure gradient term.

Proposition 1. *Neglecting terms $O(v^2)$, $O(p\ v^i)$ in a local inertial system with respect to which fluid element has instantaneous velocity* **v**, *it can be shown that*

(i) $T^{0\nu}{}_{;\nu} = 0$ *reduces to* $\frac{\partial}{\partial t}\left(\rho_0(1+\epsilon)\right) + \left(\rho_0(1+\epsilon)\ v^i\right)_{;i} = 0$, *the conservation of mass and energy equations.*

(ii) $T^{i\nu}{}_{;\nu} = 0$ *reduces to* $\frac{\partial}{\partial t}\left(\rho_0(1+\epsilon)\ v^i\right) = -\frac{\partial p}{\partial x^i}$, *the conservation of momentum equation.*

Proof.

(i) On world line of fluid element in local inertial system,

$$\begin{aligned} ds^2 &= -dt^2 + dl^2 \\ d\tau^2 &= -dt^2(1-v^2) \\ \left(\frac{dt}{d\tau}\right)^2 &= 1 + O(v^2) \end{aligned}$$

where $ds^s = -d\tau^2$ and $v = \frac{dl}{dt}$. Now

$$\begin{aligned} T^{0\nu} &= \left(\rho_0(1+\epsilon)+p\right) u^0 u^\nu + p g^{0\nu} \\ &= \left(\rho_0(1+\epsilon)+p\right) \frac{dx^0}{dt}\frac{dx^\nu}{dt}\left(\frac{dt}{d\tau}\right)^2 + p g^{0\nu} \\ &= \left(\rho_0(1+\epsilon)+p\right) \frac{dx^\nu}{dt} + p g^{0\nu} + O(v^2). \end{aligned} \tag{5.26}$$

Since $g^{00} = -1$

$$T^{00} = \left(\rho_0(1+\epsilon)+p\right) + p(-1) \ = \ \rho_0(1+\epsilon)$$

thus

$$T^{00}{}_{,0} = \frac{\partial}{\partial t}\left(\rho_0(1+\epsilon)\right). \tag{5.27}$$

Also from equation (5.26)

$$T^{0i} = \left(\rho_0(1+\epsilon)+p\right)\frac{dx^i}{dt} + pg^{0i} \; = \; \rho_0(1+\epsilon)v^i$$

thus

$$T^{0i}{}_{,i} = \left(\rho_0(1+\epsilon)+v^i\right)_{,i}. \tag{5.28}$$

Neglect terms $g^{0i}=0$ and $p\frac{dx^i}{dt} = O\left(p\;v^i\right)$. Now combining equations (5.27) and (5.28), we obtain

$$T^{0\nu}{}_{;\nu} \; = \; 0 \Rightarrow \; \frac{\partial}{\partial t}\left(\rho_0(1+\epsilon)\right) \; + \; \left(\rho_0(1+\epsilon)\;v^i\right)_{;i} \; = \; 0.$$

(ii) From equation (5.28) we have that $T^{i0} \; = \; T^{0i} \; = \; \rho_0(1+\epsilon)v^i$, thus

$$T^{i0}{}_{,0} = \frac{\partial}{\partial t}\left(\rho_0(1+\epsilon)v^i\right). \tag{5.29}$$

Also

$$T^{ij} = \left(\rho_0(1+\epsilon)+p\right)\frac{dx^i}{dt}\frac{dx^j}{dt} + pg^{ij} \; = \; O\left(v^2\right) + p\delta_i{}^j$$

thus

$$T^{ij}{}_{,j} = \left(\delta_i{}^j p\right)_{,j} \; = \; 0 + \delta_i{}^j p_{,j} \; = \; \frac{\partial p}{\partial x^i} \tag{5.30}$$

since $\left(\frac{dt}{d\tau}\right)^2 = 1 + 0(v^2)$ and $g^{ij} = \delta^{ij} = \delta_i{}^j$. Combining equations (5.29) and (5.30), we obtain

$$T^{i\nu}{}_{;\nu} = 0 \Rightarrow \; \frac{\partial}{\partial t}\left(\rho_0(1+\epsilon)\;v^i\right) \; = \; -\frac{\partial p}{\partial x^i}.$$

□

We note the following:

(i) In GR, the following hypothesis is made - the total energy-momentum tensor, $T^{\mu\nu}$, of a system has zero divergence

$$T^{\mu\nu}{}_{;\nu} = 0 \tag{5.31}$$

and those 4 differential equations are the GR analogues of the non-relativistic equations of conservation of energy and momentum.

(ii) The conservation laws are given as a set of differential equations. Clearly defined expressions for the conserved quantities - energy and momentum - are not obtained. In order to obtain the GR analogue of energy and momentum, it is necessary to re-express equation (5.31) as an ordinary divergence.

5.1.1 Conservation equations

This subsection is dedicated to the conservation law of the total energy-momentum tensor, where covariant expressions for the equations of conservation of the energy and momentum are studied independently from the perspective of an observer along a world line.

We can split up the conservation equations, equation (5.31), using the 4-velocity vector $\mathbf{u}^\mu$ and the projection tensor $h_{\mu\nu}$ leads to covariant expressions for the equations of conservation of energy and momentum.

(i) Equations of conservation of energy along the world line of observer u^μ. Consider local co-moving inertial system S^0 in which

$$u^\mu = \frac{dx^\mu}{d\tau} = (1,0,0,0) \ = \ \delta^\mu{}_0$$

as $dx^i = 0$ and $dx^0 = d\tau$. Thus in S^0

$$u_\mu = g_{\mu\nu}u^\nu \ = \ g_{\mu 0} \ = \ -\delta^0{}_\mu$$

because $g_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. Hence in S^0

$$u_\mu T^{\mu\nu}{}_{;\nu} = -\delta^0{}_\mu T^{\mu\nu}{}_{;\nu} = -T^{0\nu}{}_{;\nu}.$$

But it was shown that $T^{0\nu}{}_{;\nu} = 0$ reduces to the energy conservation equation in S^0. Thus we interpret

$$u_\mu T^{\mu\nu}{}_{;\nu} = 0$$

as an equation of conservation of energy along the world line of observer moving with 4-velocity u^μ.

(ii) Equation of conservation of momentum along world line of observer u^μ. To write $T^{i\nu}{}_{;\nu} = 0$ in a covariant way, consider the mixed projection tensor

$$h^\mu{}_\nu = u^\mu u_\nu + g^\mu{}_\nu.$$

In S^0

$$h^\mu{}_\nu = u^\mu u_\nu + g^\mu{}_\nu \ = \ -\delta^\mu{}_0\delta^0{}_\nu + \delta^\mu{}_\nu$$

as $g^\mu{}_\nu = \delta^\mu{}_\nu$. Thus in S^0

$$h^0{}_\nu = -\delta^0{}_0\delta^0{}_\nu + \delta^0{}_\nu \ = \ 0$$
$$h^i{}_\nu = -\delta^i{}_0\delta^0{}_\nu + \delta^i{}_\nu \ = \ \delta^i{}_\nu$$

as $\delta^i{}_0 = 0$. Hence in S^0 we have

$$h^0{}_\nu T^{\nu\kappa}{}_{;\kappa} = 0 \qquad \text{and} \qquad h^i{}_\nu T^{\nu\kappa}{}_{;\kappa} \ = \ T^{i\kappa}{}_{;\kappa}.$$

However, it was shown that $T^{i\kappa}{}_{;\kappa} \ = \ 0$ reduces to the momentum conservation equation in S^0. Thus in GR we interpret

$$h^\mu{}_\nu T^{\nu\kappa}{}_{;\kappa} = 0$$

as an equation of conservation of momentum along the world line of observer u^μ.

Notation: The absolute derivative $\frac{D}{D\tau}$ of a tensor T along the world line $x^\mu = x^\mu(\tau)$ of observer moving with 4-velocity $u^\mu = \frac{dx^\mu}{d\tau}$ is usually denoted by a dot:

$$\frac{D}{D\tau}T^{\mu\nu} = T^{\mu\kappa}{}_{;\kappa}u^\nu = \dot{T}^{\mu\nu}.$$

The dot denotes the time rate of change of any quantity as measured by an observer moving with 4-velocity u^μ. It corresponds to the convective time derivative of non-relativistic theory:

$$\dot{f} = f_{,\mu}u^\mu = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{d\tau} = \frac{df}{d\tau} \qquad \text{and} \qquad \dot{u}^\mu \ = \ u^\mu{}_{;\nu}u^\nu.$$

5.1.2 Dust

In this section a brief description of non-interacting matter with zero pressure is supplied. Showing the reader that the world line of a free particle is geodesic, and consequently the geodesic hypothesis for free particles is a special case of a more general conservation law hypothesis.

Definition 50. Dust

Dust in GR is defined as a distribution of non-interacting matter with zero pressure.

The energy momentum tensor of dust is

$$T^{\mu\nu} = \rho_0 u^\mu u^\nu \tag{5.32}$$

where ρ_0 is its proper rest mass density - matter density as measured in local co-moving inertial system. To obtain equation (5.32) from (5.18), set $p = 0$ and $\epsilon = 0$. We set $\epsilon = 0$ as it is the internal energy of the fluid which is non-existent in the case of dust.

Proposition 2. *The world line of a particle of dust is a geodesic.*

Proof.

$$\begin{aligned} T^{\mu\nu}{}_{;\nu} &= 0 \\ \left(\rho_0 u^\mu u^\nu\right)_{;\nu} &= 0 \\ \rho_{0;\nu} u^\mu u^\nu + \rho_0 u^\mu{}_{;\nu} u^\nu + \rho_0 u^\mu u^\nu{}_{;\nu} &= 0 \end{aligned}$$

contracting with $h^\kappa{}_\mu$ gives

$$\begin{aligned} \rho_{0;\nu}\left(h^\kappa{}_\mu u^\mu\right) u^\nu + \rho_0\left(h^\kappa{}_\mu u^\mu{}_{;\nu} u^\nu\right) + \rho_0\left(h^\kappa{}_\mu u^\mu\right) u^\nu{}_{;\nu} &= 0 \\ \rho_{0;\nu}(0)\, u^\nu + \rho_0 \dot{u}^\kappa + \rho_0 (0)\, u^\nu{}_{;\nu} &= 0 \\ \dot{u}^\kappa &= 0 \end{aligned}$$

Thus world line is geodesic. □

This result is to be expected as fluid particles in dust do not interact - free particles. The geodesic hypothesis for free particles is thus a special case of the more general conservation law hypothesis.

5.2 Einstein's Field Equations of Gravitation

This section is focused on deriving, from a classical perspective, the EFEs by considering the requirements for the gravitational field equations. Central to this is understanding the choice of the Einstein tensor and the coupling of the energy-momentum tensor to the geometry as an approach to realising the equivalence principle using Riemannian geometry as its representation. In order to eliminate torsion, conformal flatness of the metric is argued with the correspondence principle.

Essential to the correct formulation of the EFEs are the constraints that encode conservation of energy-momentum, $T^{\mu\nu}{}_{;\nu} = 0$. This requirement has implications on the geometry via the Bianchi identities in order to derive the full eleven equations that have become known as the EFEs. [24, 45, 46, 49, 50, 51, 52, 53, 54, 55]

Before we jump into this section, let us define a few important terms:

Empty space	–	no matter or energy anywhere.
Free space	–	empty space between matter or energy distribution.
Distribution of energy	–	covers all forms of energy, including matter.

Now, let us consider the requirements of the gravitational field equations:

(i) The equations must be covariant.

(ii) Because matter and energy are equivalent, both must be included in a covariant way.

(iii) Solution of equations must give $g_{\mu\nu}$ corresponding to given distribution of matter and energy. At first sight, this requires 10 scalar equations - $g_{\mu\nu}$-symmetric. However, because we are free to set up a coordinate transformation, we have "4 degrees of freedom", so only 6 equations must be determined. The 10 scalar equations must therefore satisfy 4 identities such as $G^{\mu\nu}{}_{;\nu} \equiv 0$.

(iv) It is the variation of the gravitational acceleration which is significant rather than its actual intensity. It follows that it is $g_{\mu\nu,\kappa\lambda}$ which is significant. Thus the second derivatives of $g_{\mu\nu}$ should occur in the equations.

(v) Because empty space is flat, the field equations in free space must be some generalisation of the flat space condition

$$R^{\kappa}{}_{\lambda\mu\nu} = 0$$

which still admits the Lorentz metric as a particular solution. This is compared to Newtonian theory where the free space equations, $\nabla^2\phi = 0$, admit the empty space solution, $\phi = 0$, as a particular solution.

(vi) In the Newtonian limit, the field equations must reduce to Poisson's equation, $\nabla^2\phi = 4\pi k\rho$, where k is Newton's gravitational constant and ρ is the rest mass density. This shows

$$\sum_{i=1}^{3} \phi_{,ii} = 4\pi k\rho. \tag{5.33}$$

But in the Newtonian limit we have that $g_{00} = -(1+2\phi)$, thus equation (5.33) becomes

$$\sum_{i=1}^{3} g_{00,ii} = -8\pi k\rho. \tag{5.34}$$

(vii) Field equations should be quasi-linear - linear in highest derivatives. Although linear in highest derivative, we expect equations to be non-linear in derivatives of g_{ab}. For a gravitational field contains potential energy and so an effective mass which creates a further gravitational field. It serves as part of its own source, meaning equations which describe feedback must be non-linear.

Meeting the requirements

1. To satisfy (i) we look for a tensor equation, and (ii) is satisfied by a tensor equation of form

$$\begin{bmatrix} \text{tensor representing} \\ \text{geometry of space} - \text{time} \end{bmatrix} = \text{constant} \begin{bmatrix} \text{tensor representing matter and non}- \\ \text{gravitational energy constant of space} - \text{time} \end{bmatrix}.$$

   The gravitational energy is contained in non-linear terms in tensor on the left hand side.

2. On the right hand side we use energy-momentum tensor because:

- It describes energy content of space-time and is zero in empty space.
- In local rest frame the inertial system T_{00} for dust is $T_{00} = \rho_0$.

$T_{00} = \rho_0$ is the same as the right hand side of Poisson's equation which becomes

$$\sum_{i=1}^{3} g_{00,ii} = 8\pi k T_{00}.$$

3. Consider the left hand side and use requirement (v). Necessary and sufficient condition for space-time to be flat is

$$R_{abcd} = 0. \tag{5.35}$$

Free space field equations must be a generalisation of equation (5.35) which still admits Lorentz metric as a particular solution, because those will also be field equations in empty space - different boundary conditions. This means that equation (5.35) must be weakened to admit more general solutions.

4. Analogue of summation in $\sum_{i=1}^{3} g_{00,ii}$ is a contraction in a tensor equation. Weakening equation (5.35) can be done with two possible contractions

$$R^{\kappa}{}_{\mu\kappa\nu} = R_{\mu\nu} \qquad \text{and} \qquad R^{\mu\nu}{}_{\mu\nu} = R.$$

5. Because $T^{\mu\nu}{}_{;\nu} = 0$, must use tensor on left hand side with zero divergence. Choose

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \tag{5.36}$$

where $G^{\mu\nu}$ is symmetric, second order like $T^{\mu\nu}$ and constructed out of the contractions of $R^{\mu}{}_{\nu\kappa\lambda}$. It also depends on $g_{\mu\nu,\kappa\lambda}$ - requirement (iv) - is linear in $g_{\mu\nu,\kappa\lambda}$ - requirement (vii) - and non-linear in linear derivatives of $g_{\mu\nu}$. Because $G^{\mu\nu}{}_{;\nu} \equiv 0$, requirement (iii) is satisfied. Hence we arrive at Einstein's gravitational field equations

$$G^{\mu\nu} = K T^{\mu\nu}$$

where K is Einstein's gravitational constant and is determined later by requirement (vi).

We note the following:

- $T^{\mu\nu}$ plays a dual role in GR - inertial and gravitational - because it determines equations of motion through $T^{ab}{}_{;b} = 0$ and gravitational field by $G^{\mu\nu} = K T^{\mu\nu}$ and it explains equivalence of Newtonian concepts of inertial and gravitational mass.
- Because $T^{\mu\nu}$ and $G^{\mu\nu}$ are symmetric, we have 10 scalar equations for the 10 unknowns, $g_{\mu\nu}$. However, there are only 6 independent equations because of the 4 relations $(G^{\mu\nu} - K T^{\mu\nu})_{;\nu} = 0$.
- Tensor with zero divergence are usually associated with a conserved quantity. Such tensors are associated with persistent rather than transient phenomena. Hence gravitational field associated with $G^{\mu\nu}$ as $G^{\mu\nu}{}_{;\nu} \equiv 0$.

The cosmological constant

It can be shown that the most general 2[nd] order tensor with zero divergence and constructed entirely from $g_{\mu\nu}$, $g_{\mu\nu,\kappa}$, $g_{\mu\nu,\kappa\lambda}$ and linear in $g_{\mu\nu,\kappa\lambda}$ is

$$G^{\mu\nu} + \Lambda g^{\mu\nu}$$

where Λ is a constant. The field equations

$$G^{\mu\nu} + \Lambda g^{\mu\nu} = KT^{\mu\nu} \tag{5.37}$$

are commonly used, especially in cosmology. Considering the flat empty space gives

$$T^{\mu\nu} = 0, \quad R^{\mu}{}_{\nu\kappa\lambda} = 0, \quad R_{\mu\nu} = 0, \quad R = 0 \quad \text{and} \quad G^{\mu\nu} = 0.$$

If equation (5.37) is true then

$$\Lambda g^{\mu\nu} = 0.$$

Thus if $\Lambda \neq 0$, the empty space cannot be flat. In older cosmological theories, the condition that an empty universe be flat is sometimes dropped and equation (5.37) with $\Lambda \neq 0$ is used. Such a model universe has an intrinsic curvature not produced by matter distribution. Satisfactory modern theories - time dependent, not static like the older theories - with $\Lambda = 0$ have since been developed.

Definition 51. Cosmological constant

Λ is called the cosmological constant because Λ is a measure of the departure from flatness of an empty universe, it must be small, if not zero.

Cosmological theories with $\Lambda \neq 0$ give

$$\Lambda = \frac{1}{(\text{radius of universe})^2}.$$

Observations will eventually decide if $\Lambda = 0$ or $\Lambda \neq 0$, but it does seem unreasonable if $\Lambda \neq 0$ because it is a field which acts on everything but is not acted on by anything.

5.2.1 Equivalent forms of the field equations

Theorem 12.

$$G^{\mu\nu} = 0 \Longleftrightarrow R^{\mu\nu} = 0$$

Proof.

(i) Given $G^{\mu\nu} = 0$, then

$$\begin{aligned} R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R &= 0 \\ g_{\mu\nu}\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right) &= 0 \\ R - 2R &= 0 \\ R &= 0 \end{aligned} \tag{5.38}$$

as $g^{\nu}{}_{\nu} = \delta^{\nu}{}_{\nu} = 4$. Thus by equation (5.38) we have that

$$R^{\mu\nu} = 0.$$

(ii) Suppose that $R^{\mu\nu} = 0$, then $g_{\mu\nu}R^{\mu\nu} = R = 0$ and by equation (5.36) we have that

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 0 = G^{\mu\nu}.$$

□

Theorem 13.

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = KT^{\mu\nu} \Longleftrightarrow T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T = \frac{1}{K}R^{\mu\nu}$$

where $T = T^{\mu}{}_{\mu}$.

Proof.

(i) Suppose

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = KT^{\mu\nu} \tag{5.39}$$

$$g_{\mu\nu}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) = Kg_{\mu\nu}T^{\mu\nu}$$

$$R - 2R = KT$$

$$R = -KT.$$

Substituting this result into equation (5.39) yields

$$R^{\mu\nu} + \frac{1}{2}g^{\mu\nu}KT = KT^{\mu\nu}$$

$$T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T = \frac{1}{K}R^{\mu\nu}.$$

(ii) Conversely, suppose

$$T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T = \frac{1}{K}R^{\mu\nu} \tag{5.40}$$

$$g_{\mu\nu}\left(T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T\right) = \frac{1}{K}g_{\mu\nu}R^{\mu\nu}$$

$$T - 2T = \frac{1}{K}R$$

$$T = -\frac{1}{K}R.$$

Substituting this result into equation (5.40) yields

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = KT^{\mu\nu}.$$

□

We note that the field equations can be written as

$$T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T = \frac{1}{K}R^{\mu\nu}$$

which is very useful when $T^{\mu\nu}$ is simple, and this is indicative of a symmetry property in the field equations between $T^{\mu\nu}$ and $R^{\mu\nu}$.

5.2.2 Newtonian limit and evaluation of k

Theorem 14. *$G^{\mu\nu} = KT^{\mu\nu}$ is a generalisation of Poisson's equation $\nabla^2\phi = 4\pi kp$ provided*

$$K = 8\pi k.$$

If $c \neq 1$, $K = \frac{8\pi k}{c^2}$.

Proof. Suppose coordinate system exists with respect to which

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + \epsilon\gamma_{\mu\nu}dx^\mu dx^\nu$$

where $\gamma_{\mu\nu}$ is independent of x^0 and $g_{\mu\nu}$ is the Lorentz metric tensor. Consider dust with low density and small velocity such that ρ, ϵ and v are same order of smallness and terms $O(v^2)$ can be neglected. We use

$$R_{\mu\nu} = K\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right). \tag{5.41}$$

For dust we have

$$\begin{aligned} T^{\mu\nu} &= \rho\mathbf{u}^\mu\mathbf{u}^\nu \\ &= \rho\frac{dx^\mu}{dt}\frac{dx^\mu}{dt}\left(\frac{dt}{d\tau}\right)^2. \end{aligned}$$

But, for $dl^2 = \sum_{i=1}^3 dx^i dx^i$, $v = \frac{dl}{dt}$ and $ds^2 = -d\tau^2$, we have that

$$\begin{aligned} ds^2 &= -dt^2 + dl^2 + \epsilon\gamma_{ij}dx^i dx^j \\ -d\tau^2 &= -dt^2\left(1 - v^2 - \epsilon\gamma_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}\right) \\ d\tau^2 &= dt^2\left(1 - \epsilon\gamma_{00} + O(v^2)\right) \\ \left(\frac{dt}{d\tau}\right)^2 &= 1 + O(\epsilon) \end{aligned}$$

and with $g_{00} = -1 + O(\epsilon)$ we have that

$$\begin{aligned} T^{\mu\nu} &= \rho\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} + 0(\rho\epsilon) \\ T^{\mu\nu} &= \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + O(\rho\epsilon) \\ T^\mu{}_\nu &= \begin{bmatrix} -\rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + O(\rho\epsilon). \end{aligned}$$

Thus

$$T = \delta_\mu{}^\nu T^\mu{}_\nu = -\rho + O(\rho\epsilon).$$

Also

$$T_{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + O(\rho\epsilon).$$

Using equation (5.41) the gravitational field equations become

$$\begin{aligned}
R_{\mu\nu} &= K\left(\begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{2}\rho \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) + O(\epsilon\rho) \\
&= K\frac{\rho}{2}\delta_{\mu\nu}.
\end{aligned} \tag{5.42}$$

Now,

$$\begin{aligned}
R_{\mu\nu} &= \Gamma^{\lambda}{}_{\mu\nu,\lambda} - \Gamma^{\lambda}{}_{\lambda\mu,\nu} + \Gamma^{\lambda}{}_{\lambda\kappa}\Gamma^{\kappa}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\kappa}\Gamma^{\kappa}{}_{\mu\lambda} \\
&= \Gamma^{\lambda}{}_{\mu\nu,\lambda} - \Gamma^{\lambda}{}_{\lambda\mu,\nu} + O(\epsilon^2)
\end{aligned}$$

as $\Gamma^{\mu}{}_{\nu\kappa} = O(\epsilon)$. To obtain Poisson's equation consider $\mu = 0, \nu = 0$

$$\begin{aligned}
R_{00} &= \Gamma^{\lambda}{}_{00,\lambda} - \Gamma^{\lambda}{}_{\lambda 0,0} + O(\epsilon^2) \\
&= \Gamma^{i}{}_{00,i}
\end{aligned} \tag{5.43}$$

as the metric tensor is time independent. But, because $g_{\mu\nu,0} = 0$, we have

$$\begin{aligned}
\Gamma^{i}{}_{00} &= g^{i\kappa}\Gamma_{\kappa 00} \\
&= g^{i\kappa}\frac{1}{2}\left(g_{0\kappa,0} + g_{0\kappa,0} - g_{00,\kappa}\right) \\
&= -\frac{1}{2}g^{i\kappa}g_{00,\kappa} \\
&= -\frac{1}{2}g^{ij}g_{00,j} \\
&= -\frac{1}{2}\left(\delta^{ij} + O(\epsilon)\right)\left(-1 + \epsilon\gamma_{00}\right)_{,j} \\
&= -\frac{\epsilon}{2}\delta^{ij}\gamma_{00,j} + O(\epsilon^2) \\
\Gamma^{i}{}_{00} &= -\frac{\epsilon}{2}\delta^{ij}\gamma_{00,j} + O(\epsilon^2).
\end{aligned}$$

Thus from equation (5.43)

$$R_{00} = -\frac{\epsilon}{2}\sum_{i=1}^{3}\gamma_{00,ii} + O(\epsilon^2). \tag{5.44}$$

But by equation (5.42), with $\mu = 0, \nu = 0$

$$R_{00} = \frac{K\rho}{2} \tag{5.45}$$

Combining equations (5.44) and (5.45), we have

$$-\frac{\epsilon}{2}\sum_{i=1}^{3}\gamma_{00,ii} = \frac{K\rho}{2}. \tag{5.46}$$

But to some approximation we have that $\epsilon\gamma_{00} = -2\phi$, as $g_{00} = -1 + \epsilon\gamma_{00} = -(1 + 2\phi)$. Hence equation (5.46) becomes

$$\sum_{i=1}^{3}\phi_{,ii} = \frac{K\rho}{2}.$$

But Poisson's equation is

$$\sum_{i=1}^{3}\phi_{,ii} = 4\pi k\rho$$

where k is Newton's gravitational constant. Thus

$$K = 8\pi k.$$

□

One should note that sometimes units of mass are chosen such that $k = 1$, in which case $K = 8\pi$.

5.3 1 + 3 threading of space-time

This mapping allows one to consider all properties of motion experienced by each independent observer, as represented by the choice of co-moving four-velocity u^{μ}, with respect to each others observer frames. It may be viewed as the Lagrangian representation of GR and is analogous to the Lagrangian formalism of fluid mechanics. [56, 57]

We recall from section 2.2 that the Lagrangian formalism considers a change in position over time in a local frame of reference, so let us begin by defining a unique 4-velocity relative to the proper time of the fundamental worldlines.

Definition 52. Average 4-velocity

> The 4-velocity components are described by
>
> $$u^{\mu} = \frac{dx^{\mu}}{d\tau}$$
>
> where τ is the proper time measured along the fundamental world lines.

Now, consider this 4-velocity to have corresponding preferred world lines where $u^{\mu}u_{\mu} = -1$. Given this unique 4-velocity, there exist unique projection tensors

$$U^{\mu}{}_{\nu} = -u^{\mu}u_{\nu} \tag{5.47}$$

and

$$h_{\mu\nu} = g_{\mu\nu} + u_{\mu}u_{\nu} \tag{5.48}$$

where equation (5.47) is the projection tensor parallel to the 4-velocity vector and equation (5.48) is the projection tensor determining the orthogonal metric tensor properties of the observer moving with the 4-velocity vector, at some rest-space instant. From equations (5.47) and (5.48) we have

$$U^{\mu}{}_{\kappa}U^{\kappa}{}_{\nu} = U^{\mu}{}_{\nu}$$
$$U^{\mu}{}_{\mu} = 1$$
$$U_{\mu\nu}u^{\nu} = u_{\mu}$$

and

$$h^{\mu}{}_{\kappa}h^{\kappa}{}_{\nu} = h^{\mu}{}_{\nu}$$
$$h^{\mu}{}_{\mu} = 3$$
$$h_{\mu\nu}u^{\nu} = 0,$$

respectively. Let us also define a volume element for the rest-space of the 4-velocity vector, $\eta_{\mu\nu\kappa\lambda}$ such that

$$\eta_{\nu\kappa\lambda} = u^{\mu}\eta_{\mu\nu\kappa\lambda}$$

where $\eta_{\nu\kappa\lambda}$ is an antisymmetric tensor, i.e. $\eta_{\nu\kappa\lambda} = \eta_{[\nu\kappa\lambda]}$, and $\eta_{\nu\kappa\lambda}u^{\lambda} = 0$. The 4-dimensional volume element therefore is antisymmetric as well, i.e. $\eta_{\mu\nu\kappa\lambda} = \eta_{[\mu\nu\kappa\lambda]}$, and element $\eta_{0123} = \sqrt{g}$.

Definition 53. Covariant time derivative

> The covariant time derivative along the fundamental world lines for any tensor $T^{\mu\nu}{}_{\kappa\lambda}$ is denoted by a dot above the tensor and defined by
>
> $$\dot{T}^{\mu\nu}{}_{\kappa\lambda} = u^{\sigma}\nabla_{\sigma}T^{\mu\nu}{}_{\kappa\lambda}.$$

Definition 54. Fully orthogonal projected covariant derivative

The fully orthogonally projected covariant derivative of any tensor $T^{\mu\nu}{}_{\kappa\lambda}$ is defined by

$$\tilde{\nabla}_\sigma T^{\mu\nu}{}_{\kappa\lambda} = h^\mu{}_\alpha h^\nu{}_\beta h^\gamma{}_\kappa h^\zeta{}_\lambda h^\iota{}_\sigma \nabla_\iota T^{\alpha\beta}{}_{\gamma\zeta}.$$

We may expand the tensor quantity $T_{\mu\nu}$ into symmetric and antisymmetric parts by the transformation

$$T_{\mu\nu} = T_{(\mu\nu)} + T_{[\mu\nu]}.$$

Definition 55. Projected symmetric trace-free components

The projected symmetric trace-free (PSTF) components of a tensor is denoted by the use of angle brackets and defined by

$$v^{\langle\mu\rangle} = h^\mu{}_\nu v^\nu$$

for a first-order tensor and by

$$T^{\langle\mu\nu\rangle} = \left[h^{(\mu}{}_\kappa h^{\nu)}{}_\lambda - \frac{1}{3}h^{\mu\nu}h_{\kappa\lambda}\right] T^{\kappa\lambda} \tag{5.49}$$

for a second-order tensor.

Definition 56. Fermi derivative

The Fermi derivative is the covariant time derivative of the projected symmetric trace-free (PSTF) tensor, defined by

$$\dot{v}^{\langle\mu\rangle} = h^\mu{}_\nu \dot{v}^\nu$$

for a first-order tensor and by

$$\dot{T}^{\langle\mu\nu\rangle} = \left[h^{(\mu}{}_\kappa h^{\nu)}{}_\lambda - \frac{1}{3}h^{\mu\nu}h_{\kappa\lambda}\right] \dot{T}^{\kappa\lambda}$$

for a second-order tensor.

Consider now the covariant derivative of u_ν split into its irreducible parts, by definitions 53 and 54,

$$\nabla_\mu u_\nu = -u_\mu \dot{u}_\nu + h^\kappa{}_\mu h^\lambda{}_\nu \tilde{\nabla}_\kappa u_\lambda. \tag{5.50}$$

As we may expand a tensor quantity into symmetric and antisymmetric parts, we have that

$$h^\kappa{}_\mu h^\lambda{}_\nu \tilde{\nabla}_\kappa u_\lambda = h^{(\kappa}{}_\mu h^{\lambda)}{}_\nu \tilde{\nabla}_\kappa u_\lambda + h^{[\kappa}{}_\mu h^{\lambda]}{}_\nu \tilde{\nabla}_\kappa u_\lambda. \tag{5.51}$$

By definition 55, equation (5.49), we have

$$\tilde{\nabla}_{\langle\mu} u_{\nu\rangle} = \left[h^{(\kappa}{}_\mu h^{\lambda)}{}_\nu - \frac{1}{3}h^{\kappa\lambda}h_{\mu\nu}\right] \tilde{\nabla}_\mu u_\nu$$

therefore

$$h^{(\kappa}{}_\mu h^{\lambda)}{}_\nu \tilde{\nabla}_\mu u_\nu = \tilde{\nabla}_{\langle\mu} u_{\nu\rangle} + \frac{1}{3}h^{\kappa\lambda}h_{\mu\nu}\tilde{\nabla}_\kappa u_\lambda. \tag{5.52}$$

Substituting equations (5.52) into (5.51), and the result into equation (5.50)

$$\begin{aligned}
\nabla_\mu u_\nu &= -u_\mu \dot{u}_\nu + \tilde{\nabla}_{\langle\mu} u_{\nu\rangle} + \frac{1}{3} h^{\kappa\lambda} h_{\mu\nu} \tilde{\nabla}_\kappa u_\lambda + h^{[\kappa}{}_\mu h^{\lambda]}{}_\nu \tilde{\nabla}_\kappa u_\lambda \\
&= -u_\mu \dot{u}_\nu + \frac{1}{3} h^{\kappa\lambda} h_{\mu\nu} \tilde{\nabla}_\kappa u_\lambda + \tilde{\nabla}_{\langle\mu} u_{\nu\rangle} + \tilde{\nabla}_{[\mu} u_{\nu]} \\
&= -u_\mu \dot{u}_\nu + \frac{1}{3} \tilde{\nabla}_\kappa u^\kappa h_{\mu\nu} + \tilde{\nabla}_{\langle\mu} u_{\nu\rangle} + \tilde{\nabla}_{[\mu} u_{\nu]}.
\end{aligned}$$

Let us denote the trace $\tilde{\nabla}_\kappa u^\kappa$ by Θ, the trace-free symmetric tensor $\tilde{\nabla}_{\langle\mu} u_{\nu\rangle}$ by $\sigma_{\mu\nu}$, anti-symmetric tensor $\tilde{\nabla}_{[\mu} u_{\nu]}$ by $\omega_{\mu\nu}$. Then

$$\nabla_\mu u_\nu = -u_\mu \dot{u}_\nu + \frac{1}{3} \Theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}. \tag{5.53}$$

Definition 57. Hubble scalar

Hubble scalar which is defined by

$$H = \frac{1}{3}\Theta$$

where the rate of volume expansion scalar of a fluid is $\Theta = \tilde{\nabla}_\kappa u^\kappa$.

Definition 58. Rate of shear tensor

The rate of shear tensor, $\sigma_{\mu\nu}$, describes the rate of distortion of the matter flow, and is define by

$$\sigma_{\mu\nu} = \tilde{\nabla}_{\langle\mu} u_{\nu\rangle}$$

with the properties

$$\sigma_{\mu\nu} = \sigma_{(\mu\nu)}, \qquad \sigma_{\mu\nu} u^\nu = 0 \quad \text{and} \quad \sigma^\mu{}_\mu = 0.$$

Definition 59. Vorticity tensor

The vorticity tensor, $\omega_{\mu\nu}$, describes the rotation of matter relative to a non-rotating frame, and is defined by

$$\omega_{\mu\nu} = \tilde{\nabla}_{[\mu} u_{\nu]}$$

with the properties

$$\omega_{\mu\nu} = \omega_{[\mu\nu]} \quad \text{and} \quad \omega_{\mu\nu} u^\nu = 0.$$

The rate of shear and vorticity tensors follow from the evolution equation when considering the relative position vector.

Definition 60. Relative position vector

The relative position vector is defined by

$$\perp \eta^{\mu} = h^{\mu}{}_{\nu}\eta^{\nu}$$

where the position vector η^{ν} deviates from the family of fundamental world lines, i.e. $\eta^{\nu}\nabla_{\nu}u^{\mu} = u^{\nu}\nabla_{\nu}\eta^{\mu}$.

Definition 61. Generalised Hubble law

The generalised Hubble law is defined by the propagation equation

$$\frac{\dot{\delta l}}{\delta l} = \frac{1}{3}\Theta + \sigma_{\mu\nu}e^{\mu}e^{\nu}$$

where the relative distance δl is defined in the quantity $\perp \eta^{\mu} = \delta l e^{\mu}$ for $e_{\mu}e^{\mu} = 1$.

Definition 62. Evolution of relative vector equation

The evolution equation for the relative vector is defined by the propagation equation

$$\dot{e}^{\langle\mu\rangle} = \left(\sigma^{\mu}{}_{\nu} - \sigma_{\lambda\kappa}e^{\lambda}e^{\kappa}h^{\mu}{}_{\nu} - \omega^{\mu}{}_{\nu}\right)e^{\nu}$$

describing an observed rate of change of positions of distant galaxies.

Definition 63. Relativistic acceleration vector

The relativistic acceleration vector, $\dot{u}^{\mu}$, is the degree to which matter moves due to forces other than that of gravity and inertia, and is defined by

$$\dot{u}^{\mu} = u^{\nu}\nabla_{\nu}u^{\mu}$$

where $\nabla_{\nu}u^{\mu}$ is defined by equation (5.53).

Consider now the energy-momentum tensor, $T_{\mu\nu}$,

$$T_{\mu\nu} = \left(T_{\kappa\lambda}u^{\kappa}u^{\lambda}\right)u_{\mu}u_{\nu} - h_{\mu\sigma}T_{\kappa\lambda}u^{\kappa}h^{\lambda\sigma}u_{\nu} - u_{\mu}h_{\nu\sigma}T_{\kappa\lambda}u^{\kappa}h^{\lambda\sigma} + \frac{1}{3}\left(T_{\kappa\lambda}h^{\kappa\lambda}\right)h_{\mu\nu} + T_{\kappa\lambda}h^{\kappa}{}_{\langle\mu}h^{\lambda}{}_{\nu\rangle}.$$

Let us denote $T_{\kappa\lambda}u^{\kappa}u^{\lambda}$ by $\rho_0(1+\epsilon)$, $-h_{\mu\sigma}T_{\kappa\lambda}u^{\kappa}h^{\lambda\sigma}$ by q_{μ}, $-h_{\nu\sigma}T_{\kappa\lambda}u^{\kappa}h^{\lambda\sigma}$ by q_{ν}, $\frac{1}{3}\left(T_{\kappa\lambda}h^{\kappa\lambda}\right)$ by p, and $T_{\kappa\lambda}h^{\kappa}{}_{\langle\mu}h^{\lambda}{}_{\nu\rangle}$ by $\pi_{\mu\nu}$. Then

$$T_{\mu\nu} = \rho_0(1+\epsilon)u_{\mu}u_{\nu} + q_{\mu}u_{\nu} + u_{\mu}q_{\nu} + ph_{\mu\nu} + \pi_{\mu\nu}. \tag{5.54}$$

Definition 64. Relativistic energy density

The relativistic energy density, $\rho_0(1+\epsilon)$, relative to u^{a} is defined by

$$\rho_0(1+\epsilon) = T_{\mu\nu}u^{\mu}u^{\nu}.$$

Definition 65. Relativistic momentum density

The relativistic momentum density, q^μ, is also the energy flux relative to u^μ and is defined by

$$q^\mu = -T_{\nu\kappa} u^\nu h^{\kappa\mu}.$$

Definition 66. Isotropic pressure

The Isotropic pressure, p, is defined by

$$p = \frac{1}{3} T_{\mu\nu} h^{\mu\nu}.$$

Definition 67. Anisotropic pressure

The anisotropic pressure, $\pi_{\mu\nu}$, is defined by

$$\pi_{\mu\nu} = T_{\kappa\lambda} h^\kappa{}_{\langle\mu} h^\lambda{}_{\nu\rangle}.$$

Equation (5.54) is the general energy momentum tensor that may be reduced to that of the perfect fluid as described in Theorem 11. Let $q^\mu = \pi_{\mu\nu} = 0$, then

$$T_{\mu\nu} = \rho_0(1+\epsilon) u_\mu u_\nu + p h_{\mu\nu} \tag{5.55}$$

which is the equation of state for a perfect fluid.

Definition 68. Maxwell field strength tensor

The Maxwell field strength tensor is $F_{\mu\nu}$ of an electromagnetic field may be split into electric and magnetic field parts, relative to u^μ, by the relations

$$\begin{aligned} E_\mu &= F_{\mu\nu} u^\nu \Rightarrow & E_\mu u^\mu &= 0 \\ H_\mu &= \frac{1}{2} \eta_{\mu\nu\lambda} F^{\nu\lambda} \Rightarrow & H_\mu u^\mu &= 0 \end{aligned}$$

where $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ and A_μ is the 4-potential.

Definition 69. Weyl curvature tensor

The Weyl conformal tensor is $C_{\mu\nu\lambda\sigma}$, is analogous to $F_{\mu\nu}$ of the Maxwell field strength tensor, which may be split relative to u^μ into “electric" and “magnetic" Weyl curvature parts according to

$$\begin{aligned} E_{\mu\nu} &= C_{\mu\lambda\nu\sigma} u^\lambda u^\sigma \Rightarrow & E^\mu{}_\mu &= 0, E_{\mu\nu} = E_{(\mu\nu)}, E_{\mu\nu} u^\nu = 0 \\ H_{\mu\nu} &= \frac{1}{2} \eta_{\mu\sigma\kappa} C^{\sigma\kappa}{}_{\nu\lambda} u^\lambda \Rightarrow & H^\mu{}_\mu &= 0, H_{\mu\nu} = H_{(\mu\nu)}, H_{\mu\nu} u^\nu = 0. \end{aligned}$$

These represent the ‘free gravitational field’, enabling gravitational action at a distance (tidal forces, gravitational waves), and influence the motion of matter and radiation through the geodesic deviation equation for a timelike and null congruence.

Together with the Ricci Curvature tensor $R_{\mu\nu}$, these quantities completely represent the space-time Riemann curvature tensor $R_{\mu\nu\lambda\sigma}$, which in fully 1 + 3 - decomposition form becomes

$$
\begin{aligned}
R^{\kappa\lambda}{}_{\nu\iota} &= R_{P}^{\kappa\lambda}{}_{\nu\iota} + R_{I}^{\kappa\lambda}{}_{\nu\iota} + R_{E}^{\kappa\lambda}{}_{\nu\iota} + R_{H}^{\kappa\lambda}{}_{\nu\iota} \\
R_{P}^{\kappa\lambda}{}_{\nu\iota} &= \frac{2}{3}\left(\mu + 3p - 2\Lambda\right) u^{[\kappa} u_{[\nu} h^{\lambda]}{}_{\iota]} + \frac{2}{3}\left(\mu\Lambda\right) h^{\kappa}{}_{[\nu} h^{\lambda}{}_{\iota]} \\
R_{I}^{\kappa\lambda}{}_{\nu\iota} &= -2u^{[\kappa} h^{\lambda]}{}_{[\nu}\, q_{\iota]} - 2u_{[\nu} h^{[\kappa}{}_{\iota]}\, q^{\lambda]} - 2u^{[\kappa} u_{[\nu}\, \pi^{\lambda]}{}_{\iota]} + 2h^{[\kappa}{}_{[\nu}\, \pi^{\lambda]}{}_{\iota]} \\
R_{E}^{\kappa\lambda}{}_{\nu\iota} &= 4u^{[\kappa} u_{[\nu} E^{\lambda]}{}_{\iota]} + 4h^{[\kappa}{}_{[\nu} E^{\lambda]}{}_{\iota]} \\
R_{H}^{\kappa\lambda}{}_{\nu\iota} &= 2\eta^{\kappa\lambda\xi} u_{[\nu} H_{\iota]\xi} + 2\eta_{\nu\iota\xi} u^{[\kappa} H^{\lambda]\xi}.
\end{aligned}
$$

Definition 70. Vorticity vector

The vorticity vector is defined by

$$\omega^{\nu} = \frac{1}{2}\eta^{\nu\kappa\lambda}\omega_{\kappa\lambda}$$

such that $\omega_{\nu} u^{\nu} = 0$ and $\omega_{\nu\kappa}\omega^{\kappa} = 0$.

We also have that the magnitudes

$$\omega^2 = \frac{1}{2}\left(\omega_{\nu\kappa}\omega^{\nu\kappa}\right) \qquad \text{and} \qquad \sigma^2 = \frac{1}{2}\left(\sigma_{\nu\kappa}\sigma^{\nu\kappa}\right)$$

must not be negative and the average length scale, S, is determined by

$$\frac{\dot{S}}{S} = \frac{1}{3}\Theta$$

so the volume of a fluid element varies as S^3. Further it is helpful to define particular spatial gradients orthogonal to u^{ν}, characterising the inhomogeneity of space-time:

$$
\begin{aligned}
X_{\nu} &\equiv \tilde{\nabla}_{\nu}\rho_0(1+\epsilon) \\
Z_{\nu} &\equiv \tilde{\nabla}_{\nu}\Theta.
\end{aligned}
$$

The energy density $\rho_0(1+\epsilon)$ (and also Θ) satisfies the important commutation relation for the $\tilde{\nabla}$-derivative

$$\tilde{\nabla}_{[\kappa}\tilde{\nabla}_{\lambda]}\rho_0(1+\epsilon) = \eta_{\kappa\lambda\xi}\omega^{\xi} u^{\nu}\left(\rho_0(1+\epsilon)\right)_{,\nu}. \tag{5.56}$$

This shows that if $\omega^{\xi}\mathbf{u}^{\nu}\left(\rho_0(1+\epsilon)\right)_{,\nu} \neq 0$ in an open set, then $X_{\nu} \neq 0$ there, so non-zero vorticity implies anisotropic number counts in an expanding universe (this is because there are then no 3-surface orthogonal to the fluid flow).

Ricci identities and propagation equations

From the above definitions we arrive at the following equations, which are the 1 + 3 formalism equivalent of the EFEs. The first set arise from the Ricci identities for the vector field u^{μ}, i.e.

$$2\nabla_{[\kappa}\nabla_{\lambda]}u^{\nu} = R_{\kappa\lambda}{}^{\nu}{}_{\xi}u^{\xi}.$$

The Raychaudhuri equation,

$$\dot{\Theta} - \tilde{\nabla}_{\nu}\dot{u}^{\nu} = -\frac{1}{3}\Theta^2 + \dot{u}_{\nu}\dot{u}^{\nu} - 2\sigma^2 + 2\omega^2 - \frac{1}{2}(\rho_0(1+\epsilon) + 3p) + \Lambda,$$

is the basic equation of gravitational attraction, showing the repulsive nature of a positive cosmological constant, leading to the identification of $(\rho_0(1+\epsilon)+3p)$ as the active gravitational mass density, and underlying the basic singularity theorem.

The Vorticity propagation equation,

$$\dot{\omega}^{\langle\nu\rangle} - \frac{1}{2}\eta^{\nu\kappa\lambda} = -\frac{2}{3}\Theta\omega^{\nu} + \sigma^{\nu}{}_{\kappa}\omega^{\kappa},$$

together with constraint equation (5.57) below, shows how the conservation follows if there is a perfect fluid with acceleration potential Φ, since then, on using equation (5.56),

$$\eta^{\kappa\lambda\nu}\tilde{\nabla}_{\lambda}\dot{u}_{\nu} = \eta^{\kappa\lambda\nu}\tilde{\nabla}_{\lambda}\tilde{\nabla}_{\nu}\Phi = 2\omega^{\kappa}\dot{\Phi}.$$

The Shear propagation equation is

$$\dot{\sigma}^{\langle\kappa\lambda\rangle} - \tilde{\nabla}^{\langle\kappa}\dot{u}^{\lambda\rangle} = -\frac{2}{3}\Theta\sigma^{\kappa\lambda} + \dot{u}^{\langle\kappa}\dot{u}^{\lambda\rangle} - \sigma^{\langle\kappa}{}_{\nu}\sigma^{\lambda\rangle\nu} - \omega^{\langle\kappa}\omega^{\lambda\rangle} - E^{\kappa\lambda} + \frac{1}{2}\pi^{\kappa\lambda}$$

where the anisotrpoic pressure source term $\pi_{\kappa\lambda}$ vanishes for a perfect fluid; this shows how the tidal gravitational field, the electric Weyl curvature tensor $E_{\kappa\lambda}$, directly induces shear (which then feeds into the Raychaudhuri and vorticity propagation equations, thereby changing the nature of the fluid flow).

Three constraint equations surface, the 0α-equation, vorticity divergence identity and the $H_{\kappa\lambda}$-equation. The 0α-equation,

$$0 = (C_1)^{\kappa} = \tilde{\nabla}_{\lambda}\sigma^{\kappa\lambda} - \frac{2}{3}\tilde{\nabla}^{\kappa}\Theta + \eta^{\kappa\lambda\nu}\left[\tilde{\nabla}_{\lambda}\omega_{\nu} + 2\dot{u}_{\lambda}\omega_{\nu}\right] + q^{\kappa},$$

shows how the momentum flux (zero for a perfect fluid) relates to the spatial inhomogeneity of the expansion. The vorticity divergence identity is given by

$$0 = (C_2) = \tilde{\nabla}_{\kappa}\omega^{\kappa} - \dot{u}_{\kappa}\omega^{\kappa},$$

and the $H_{\kappa\lambda}$-equation,

$$0 = (C_3)^{\kappa\lambda} = H^{\kappa\lambda} + 2\dot{u}^{\langle\kappa}\omega^{\lambda\rangle} - \eta^{\nu\xi\langle\kappa}\tilde{\nabla}_{\nu}\sigma^{\lambda\rangle}{}_{\xi},$$

characterises the magnetic Weyl curvature as being constructed from the 'distortion' of the vorticity and the 'curl' of the shear, $\eta^{\nu\xi\langle\kappa}\tilde{\nabla}_{\nu}\sigma^{\lambda\rangle}{}_{\xi}$.

Twice-contracted Bianchi identities

The second set of equations arise from the twice-contracted Bianchi identities which, by EFEs, imply the conservation equations. Projecting parallel and orthogonal to u^{κ}, we obtain the propagation equations

$$\dot{u} + \tilde{\nabla}_{\kappa} q^{\kappa} = -\Theta\left(\rho_0(1+\epsilon)+p\right) - 2\left(\dot{u}_{\kappa}q^{\kappa}\right) - \sigma_{\kappa\lambda}\pi^{\kappa\lambda}$$

and

$$\dot{q}^{\langle\kappa\rangle} + \tilde{\nabla}^{\kappa}p + \tilde{\nabla}_{\lambda}\pi^{\kappa\lambda} = -\frac{4}{3}\Theta q^{\kappa} - \sigma^{\kappa}{}_{\lambda}q^{\lambda} - (\rho_0(1+\epsilon)+p)\dot{u}^{\kappa} - \dot{u}^{\kappa} - \dot{u}_{\lambda}\pi^{\kappa\lambda} - \eta^{\kappa\lambda\nu}\omega_{\lambda}q_{\nu}$$

which constitute the energy conservation equation and the momentum conservation equation, respectively. For perfect fluids, characterised by equation (5.55), these reduce to

$$u^{\nu}\left(\rho_0(1+\epsilon)\right)_{,\nu} = -\Theta\left(\rho_0(1+\epsilon)+p\right),$$

and the constraint equation

$$0 = \tilde{\nabla}_\kappa p + (\rho_0(1+\epsilon) + p)\,\dot{u}_\kappa. \tag{5.57}$$

This shows that $(\rho_0(1+\epsilon)+p)$ is the inertial mass density, and also governs the conservation energy. It is clear that if this quantity is zero (an effective cosmological constant) or negative, the behaviour of matter will be anomalous.

The third set of equations arise from the Bianchi identities

$$\nabla_{[\kappa} R_{\lambda\nu]\xi\iota} = 0.$$

On using the splitting of $R_{\kappa\lambda\nu\xi}$ into $R_{\kappa\lambda}$ and $C_{\kappa\lambda\nu\xi}$, the above 1+3 splitting of those quantities, and the EFEs, the once contracted Bianchi identities give two further propagation equations and two further constraint equations, which are similar in form to the Maxwell's field equations in an expanding universe. The propagation equations are the $\dot{E}$-equation,

$$\begin{aligned}\dot{E}^{\langle\kappa\lambda\rangle} + \frac{1}{2}\dot{\pi}^{\langle\kappa\lambda\rangle} - \eta^{\nu\xi\langle\kappa}\,\tilde{\nabla}_\nu H_\xi{}^{\lambda\rangle} + \frac{1}{2}\tilde{\nabla}^{\langle\kappa}\,q^{\lambda\rangle} = &-\frac{1}{2}(\mu+p)\sigma^{\kappa\lambda} - \Theta\left(E^{\kappa\lambda} + \frac{1}{6}\pi^{\kappa\lambda}\right)\\ &+ 3\sigma^{\langle\kappa}{}_\nu\left(E^{\lambda\rangle\nu} - \frac{1}{6}\pi^{\lambda\rangle\nu}\right) - \dot{u}^{\langle\kappa}\,q^{\lambda\rangle}\\ &+ \eta^{\nu\xi\langle\kappa}\left[2\dot{u}_\nu E_\xi{}^{\lambda\rangle} - \frac{1}{2}\sigma^{\lambda\rangle}{}_\nu q_\xi - \omega_\nu H_\xi{}^{\lambda\rangle}\right],\end{aligned}$$

and the $\dot{H}$-equation,

$$\begin{aligned}\dot{H}^{\langle\kappa\lambda\rangle} + \eta^{\nu\xi\langle\kappa}\,\tilde{\nabla}_\nu E_\xi{}^{\lambda\rangle} - \frac{1}{2}\eta^{\nu\xi\langle\kappa}\,\tilde{\nabla}_\nu \pi_\xi{}^{\lambda\rangle} = &-\Theta H^{\kappa\lambda} + 3\sigma^{\langle\kappa}{}_\nu H^{\lambda\rangle\nu} + \frac{3}{2}\omega^{\langle\kappa}\,q^{\lambda\rangle}\\ &-\eta^{\nu\xi\langle\kappa}\left[2\dot{u}_\nu E_\xi{}^{\lambda\rangle} - \frac{1}{2}\sigma^{\lambda\rangle}{}_\nu q_\xi - \omega_\nu H_\xi{}^{\lambda\rangle}\right],\end{aligned}$$

which are analogous to the Maxwell-Faraday equation and Ampere's circuit law (with Maxwell's addition), respectively. These equations show how gravitational radiation arises: taking the time derivative of the $\dot{E}$-equation gives a term of the form $(\eta^{\nu\xi\langle\kappa}\,\tilde{\nabla}_\nu H_\xi{}^{\lambda\rangle})$; commuting the derivative and substituting from the $\dot{H}$-equation eliminates H, on E; similarly the time derivative if the $\dot{H}$-equation gives a wave equation for H. There are two constraint equations, the (div$E$) and (div$H$) equations. The (div$E$)-equation is

$$\begin{aligned}0 = (C_4)^\kappa = &\tilde{\nabla}_\lambda\left(E^{\kappa\lambda} + \frac{1}{2}\pi^{\kappa\lambda}\right) - \frac{1}{3}\tilde{\nabla}^\kappa\left(\rho_0(1+\epsilon)\right) + \frac{1}{3}\Theta q^\kappa - \frac{1}{2}\sigma^\kappa{}_\lambda q^\lambda - 3\omega_\lambda H^{\kappa\lambda}\\ &-\eta^{\kappa\lambda\nu}\left[\sigma_{\lambda\xi}H_\nu{}^\xi - \frac{3}{2}\omega_\lambda q_\nu\right]\end{aligned}$$

with source the spatial gradient of the energy density, which can be regarded as a vector analogue of the Newtonian Poisson equation, enabling tidal action at a distance. The (div$H$)-equation is

$$\begin{aligned}0 = (C_5)^\kappa = &\tilde{\nabla}_\lambda H^{\kappa\lambda} + (\rho_0(1+\epsilon)+p)\omega^\kappa + 3\omega_\lambda\left(E^{\kappa\lambda} - \frac{1}{6}\pi^{\kappa\lambda}\right)\\ &+\eta^{\kappa\lambda\nu}\left[\frac{1}{2}\tilde{\nabla}_\lambda q_\nu + \sigma_{\lambda\xi}\left(E_\nu{}^\xi + \frac{1}{2}\pi_\nu{}^\xi\right)\right]\end{aligned}$$

with source the fluid vorticity. These equations show, respectively, that scalar modes will result in a non-zero divergence of $E_{\kappa\lambda}$ (and hence a non-zero E-field), and vector modes in a non-zero divergence of $H_{\kappa\lambda}$ (and hence a non-zero H-field).

Maxwell's field equations

Finally, we turn for completeness to the $1+3$ decomposition of Maxwell's field equations

$$\nabla_\lambda F^{\kappa\lambda} = j_e^\kappa$$
$$\nabla_{[\kappa} F_{\lambda\nu]} = 0$$

and the propagation equations can be written as

$$\dot{E}^{\langle\kappa\rangle} - \eta^{\kappa\lambda\nu}\tilde{\nabla}_\lambda H_\nu = -j_e^{\langle\kappa\rangle} - \frac{2}{3}\Theta E^\kappa + \sigma^\kappa{}_\lambda E^\lambda + \eta^{\kappa\lambda\nu}\left[\dot{u}_\lambda H_\nu + \omega_\lambda E_\nu\right]$$
$$\dot{H}^{\langle\kappa\rangle} + \eta^{\kappa\lambda\nu}\tilde{\nabla}_\lambda E_\nu = -\frac{2}{3}\Theta H^\kappa + \sigma^\kappa{}_\lambda H^\lambda - \eta^{\kappa\lambda\nu}\left[\dot{u}_\lambda E_\nu - \omega_\lambda H_\nu\right]$$

while the constraint equations assume the form

$$0 = C_E = \tilde{\nabla}_\kappa E^\kappa - 2\omega_\kappa H^\kappa - \rho_e$$
$$0 = C_H = \tilde{\nabla}_\kappa H^\kappa + 2\omega_\kappa E^\kappa$$

with $\rho_e = -j_{e\kappa} u^\kappa$.

5.4 3 + 1 foliation of space-time

This foliation of space-time allows for a global time approach to a problem when considering space-like hypersurfaces at each step in time. By considering a global time, the problem could be seen as analogous to the Eulerian formalism only now in the general covariant setting of space-time. This framework is introduced as the ADM formalism, a Hamiltonian formulation of GR. [58, 59, 60]

This formalism deals with projecting the four-dimensional space-time into a family of space-like three-surfaces with instantaneous universal time. Each of these non-intersecting three-surfaces are some scalar function in time, a foliation of the space-time. Let us begin by defining some vector field

$$\Omega_\mu = \nabla_\mu t$$

which is closed by

$$\nabla_{[\mu}\Omega_{\nu]} = 0 \;=\; \frac{1}{2}\left(\nabla_\mu\Omega_\nu - \nabla_\nu\Omega_\mu\right).$$

The norm of this vector field may be calculated using the metric, which is defined to be equivalent to $-\alpha^{-2}$:

$$||\Omega||^2 = g^{\mu\nu}\nabla_\mu t \nabla_\nu t \;\equiv\; -\frac{1}{\alpha^2}$$

where α represents the measure of proper time lapsing between neighbouring three-surfaces. This lapse function will be discussed in more detail later on, but it is important to notice that this lapse function is always positive so that Ω_μ is time-like and it normalises Ω_μ to form the one-form ω_μ.

Definition 71. Normalised one-form

The normalised one-form vector field is defined by

$$\omega_\mu = \alpha\Omega_\mu$$

which is rotation-free, i.e. $\omega_{[\mu}\nabla_\nu\omega_{\sigma]} = 0$.

Consider now the definition of a unit vector, n^μ, which by construction has to point in the positive t direction, be normalised and time-like.

Definition 72. Unit vector normal to hyper-slice

The unit vector normal to the three-surface is defined by

$$n^\mu = -g^{\mu\nu}\omega_\nu$$

which is points in the positive t direction, $n^\mu\omega_\mu = 1$, and is normalised and time-like, $n^\mu n_\mu = -1$. Therefore, it may be understood to be the four-velocity of a "normal" observer whose world line is always normal to the three-surface.

A spatial metric, $\gamma_{\mu\nu}$, may be constructed using the metric, $g_{\mu\nu}$, and the one-form normal vector, n_μ.

Definition 73. Four-tensor spatial metric

The four-tensor spatial metric is defined by

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$

defining a projection tensor, projecting out all geometric objects lying along n^μ into the three-surface.

This metric offers an appropriate means of calculating distances within the hyper-slice, as no geometric contributions are found in the normal direction ($n^\mu\gamma_{\mu\nu} = 0$). Now, the contra-variant metric must exist by virtue of the existence of $g^{\mu\nu}$.

Definition 74. Four-tensor spatial contra-variant metric

The four-tensor spatial contra-variant metric is defined by

$$\gamma^{\mu\nu} = g^{\mu\sigma}g^{\nu\kappa}\gamma_{\sigma\kappa} \;=\; g^{\mu\nu} + n^\mu n^\nu.$$

Also, the projection tensor may be defined, for the projection of other four-tensors into the three-surface hyper-slice.

Definition 75. Spatial projection tensor

The spatial projection tensor is defined by

$$\gamma^\mu{}_\nu = g^\mu{}_\nu + n^\mu n_\nu \;=\; \delta^\mu{}_\nu + n^\mu n_\nu$$

which may be used to project an arbitrary space-time vector, v^μ, into the three-surface hyper-slice such that the quantity $\perp v^\mu = \gamma^\mu{}_\nu v^\nu$ is purely spatial. It is important to note that this tensor has to be applied to each index of higher order tensors, i.e. $\perp T^\mu{}_\nu = \gamma^\mu{}_\sigma\gamma^\lambda{}_\nu T^\sigma{}_\lambda$.

Now that we have established a manner in which to project a four-tensor into the three-surface hyper-slice, let us consider the gradient of the unit normal vector projected into the hyper-slice

$$D_\mu n_\nu = \gamma^\sigma{}_\mu\gamma^\lambda{}_\nu\nabla_\sigma n_\lambda$$

and split it up into its symmetric and antisymmetric components

$$\gamma^\sigma{}_\mu\gamma^\lambda{}_\nu\nabla_\sigma n_\lambda = \gamma^\sigma{}_\mu\gamma^\lambda{}_\nu\nabla_{(\sigma}n_{\lambda)} + \gamma^\sigma{}_\mu\gamma^\lambda{}_\nu\nabla_{[\sigma}n_{\lambda]} \;=\; \theta_{\mu\nu} + \omega_{\mu\nu}.$$

Definition 76. Expansion tensor

The expansion tensor is defined as the symmetric part of the projection of the gradient of the normal vector

$$\theta_{\mu\nu} = \gamma_\mu{}^\sigma \gamma_\nu{}^\lambda \nabla_{(\sigma} n_{\lambda)}.$$

Definition 77. Twist tensor

The rotation two-form or twist tensor is defined to be the antisymmetric part of the projection of the gradient of the normal vector

$$\omega_{\mu\nu} = \gamma_\mu{}^\sigma \gamma_\nu{}^\lambda \nabla_{[\sigma} n_{\lambda]}$$

which must vanish as a consequence of n^μ being rotation-free.

So it is evident that when finding the projection of the gradient of the normal vector, n^μ, there are geometric contributions to curvature that surface. This curvature has to be accounted for and so we define the extrinsic curvature.

Definition 78. Extrinsic curvature

The extrinsic curvature is defined as the negative expansion

$$K_{\mu\nu} = -\gamma_\mu{}^\sigma \gamma_\nu{}^\lambda \nabla_{(\sigma} n_{\lambda)} \tag{5.58}$$

which is symmetric and purely spatial. It offers information about direction changes between common points across two consecutive spatial hyper-slices.

The extrinsic curvature is also an indication of the rate of deformation from one hyper-slice to another. It should be evident that the quantities $\gamma_{\mu\nu}$ and $K_{\mu\nu}$, a time rate of change in the metric $\gamma_{\mu\nu}$, are analogous to the a position and velocity, respectively, in a classical mechanics formulation. We may also define this curvature in terms of the acceleration of the unit normal vector.

Definition 79. Unit normal acceleration

The unit normal acceleration is defined by

$$a_\mu = n^\nu \nabla_\nu n_\mu$$

which is purely spatial, $n^\mu a_\mu = 0$, and may be related to the lapse function, α, by

$$a_\mu = D_\mu \ln(\alpha) = \gamma_\mu{}^\nu \nabla_\nu \ln(\alpha).$$

Recall the definition of the extrinsic curvature, equation (5.58), and substitute in the definition of the projection tensor and identity $n^\mu \nabla_\nu n_\mu = 0$:

$$K_{\mu\nu} = -\left(\delta_\mu{}^\sigma + n_\mu n^\sigma\right)\left(\delta_\nu{}^\lambda + n_\nu n^\lambda\right)\nabla_{(\sigma} n_{\lambda)} \; = \; -\left(\delta_\mu{}^\sigma + n_\mu n^\sigma\right)\delta_\nu{}^\lambda \nabla_{(\sigma} n_{\lambda)}$$

therefore the extrinsic curvature may be defined as

$$K_{\mu\nu} = -\nabla_{(\mu} n_{\nu)} - n_{(\mu} a_{\nu)}.$$

Another way of defining the extrinsic curvature is as negative half of the Lie derivative, $\mathscr{L}_n$, in the unit normal vector, n^μ, direction

$$K_{\mu\nu} = -\frac{1}{2}\mathscr{L}_n \gamma_{\mu\nu} \tag{5.59}$$

where the Lie derivative in the normal vector direction of an arbitrary second order covariant tensor, $A_{\mu\nu}$, is defined by $\mathscr{L}_n A_{\mu\nu} = n^\lambda A_{\mu\nu,\lambda} + A_{\lambda\nu} n^\lambda{}_{,\mu} + A_{\mu\lambda} n^\lambda{}_{,\nu}$.

Definition 80. Mean curvature

The mean curvature is defined by

$$K = g^{\mu\nu} K_{\mu\nu} = \gamma^{\mu\nu} K_{\mu\nu}$$

or using the Lie derivative definition as

$$K = -\mathscr{L}_n \ln(\sqrt{\gamma}).$$

Let us now consider the projection of the gradient of a spatial vector, V^μ into the hyper-slice

$$\begin{aligned}
D_\mu V^\nu &= \gamma_\mu{}^\sigma \gamma^\nu{}_\lambda \nabla_\sigma V^\lambda \\
&= \gamma_\mu{}^\sigma \left(g^\nu{}_\lambda + n^\nu n_\lambda\right) \nabla_\sigma V^\lambda \\
&= \gamma_\mu{}^\sigma \nabla_\sigma V^\nu - \gamma_\mu{}^\sigma n^\nu V^\lambda \nabla_\sigma n_\lambda \\
&= \gamma_\mu{}^\sigma \nabla_\sigma V^\nu - V^\kappa n^\nu \gamma_\mu{}^\sigma \gamma_\kappa{}^\lambda \nabla_\sigma n_\lambda \\
&= \gamma_\mu{}^\sigma \nabla_\sigma V^\nu + V^\kappa n^\nu K_{\mu\kappa},
\end{aligned}$$

where $n^\nu n_\lambda \nabla_\sigma V^\lambda = -\gamma_\mu{}^\sigma n^\nu V^\lambda \nabla_\sigma n_\lambda$, resulting from the identity $n_\mu V^\mu = 0$.

Definition 81. Projected covariant derivative

The projected covariant derivative of the spatial vector V^μ is defined by

$$D_\mu V^\nu = \gamma_\mu{}^\sigma \nabla_\sigma V^\nu + V^\kappa n^\nu K_{\mu\kappa}$$

where n^ν is the normal unit vector to the three-surface hyper-slice and $K_{\mu\kappa}$ is the extrinsic curvature.

From this definition, it may be shown for any spatial vector V^μ that

$$\nabla_\mu V^\mu = \frac{1}{\alpha} D_\mu \left(\alpha V^\mu\right)$$

and

$$D_\mu D_\nu V^\sigma = \gamma_\mu{}^\iota \gamma_\nu{}^\kappa \gamma^\sigma{}_\lambda \nabla_\iota \nabla_\kappa V^\lambda - K_{\mu\nu} \gamma^\sigma{}_\lambda n^\iota \nabla_\iota V^\lambda - K^\sigma{}_\mu K_{\nu\iota} V^\iota$$

In the pursuit of the projected EFEs, another tensor quantity that is necessary to derive is the projected Riemann-Christoffel curvature tensor. This tensor will be used to define the projected Ricci Scalar and projected curvature scalar. As done in chapter 1, recall that the Riemann-Christoffel curvature tensor is derived by considering

$$\nabla_\sigma \nabla_\nu A_\mu - \nabla_\nu \nabla_\sigma A_\mu = A_\lambda {}^{(4)}R^\lambda{}_{\mu\nu\sigma}$$

where the Riemann-Christoffel curvature tensor is defined by

$${}^{(4)}R_{\mu\nu\sigma\kappa} = g_{\mu\lambda}\left(\Gamma^\lambda{}_{\nu\kappa,\sigma} - \Gamma^\lambda{}_{\nu\sigma,\kappa} + \Gamma^\iota{}_{\nu\kappa}\Gamma^\lambda{}_{\iota\sigma} - \Gamma^\iota{}_{\nu\sigma}\Gamma^\lambda{}_{\iota\kappa}\right).$$

Therefore, the projected Riemann-Christoffel curvature tensor is determined by taking two times the anti-symmetric projected covariant derivative of an arbitrary spatial vector.

Definition 82. Projected Riemann-Christoffel curvature tensor

The projected Riemann-Christoffel curvature tensor, otherwise known as Gauss' equation, is defined as

$$R_{\kappa\lambda\mu\nu} = \gamma_\kappa{}^\beta \gamma_\lambda{}^\zeta \gamma_\mu{}^\xi \gamma_\nu{}^{\sigma\,(4)}R_{\beta\zeta\xi\sigma} - K_{\kappa\mu}K_{\lambda\nu} + K_{\kappa\nu}K_{\mu\lambda} \tag{5.60}$$

where $K_{\mu\nu}$ is the extrinsic curvature, $\gamma_\mu{}^\nu$ is the projection tensor and ${}^{(4)}R_{\kappa\lambda\mu\nu}$ is the four-Riemann-Christoffel curvature tensor, as derived in chapter 1.

Now we have all required to derive the EFEs in the $3+1$ formalism. However, what makes this formalism different is that instead of having the traditional constraint equations in our system, the constraint equations are manipulated to form dynamic constraint equations. For this purpose, the formalism makes use of two sets of dynamic equations. The first is known as the Codazzi equation which may be derived by considering the spatial derivative of the extrinsic curvature.

Definition 83. Codazzi equation

The Codazzi equation is defined by

$$D_\lambda K_{\kappa\mu} - D_\kappa K_{\lambda\mu} = \gamma_\kappa{}^\beta \gamma_\lambda{}^\zeta \gamma_\mu{}^\xi n^{\nu\,(4)}R_{\beta\zeta\xi\nu} \tag{5.61}$$

which describes projections of the four Riemann-Christoffel curvature tensor with the fourth index projected in the direction of the normal vector.

The second set of dynamic equations is known as the Ricci equations which may be derived by considering the Lie derivative of extrinsic curvature in the direction of the normal vector.

Definition 84. Ricci equation

The Ricci equation is defined by

$$\mathscr{L}_n K_{\nu\lambda} = n^\kappa n^\mu \gamma_\nu{}^\sigma \gamma_\lambda{}^{\zeta\,(4)}R_{\kappa\zeta\mu\sigma} - \frac{1}{\alpha} D_\nu D_\lambda \alpha - K^\mu{}_\nu K_{\lambda\mu} \tag{5.62}$$

which describes the derivative in the normal direction (or time direction) as a projection of the four Riemann-Christoffel curvature tensor with the first and third indices projected in the direction of the normal vector.

Now, using the Projected Riemann-Christoffel curvature tensor, equation (5.60), and Einstein's equation, the Hamiltonian constraint equation may be determined.

Definition 85. Hamiltonian constraint equation

The Hamiltonian constraint equation is defined by

$$R + K^2 - K_{\mu\nu}K^{\mu\nu} = \frac{16\pi G}{c^2} n_\mu n_\nu T^{\mu\nu}.$$

Second, the Codazzi equation, equation (5.61), may be used to determine the momentum constraint equation.

Definition 86. Momentum constraint equation

The Momentum constraint equation is defined by

$$D_\nu K^\nu{}_\mu - D_\mu K = -\frac{8\pi G}{c^2}\gamma^\nu{}_\mu n^\lambda T_{\nu\lambda}.$$

Lastly, the evolution equation for the extrinsic curvature, $K_{\mu\nu}$, and spatial metric, $\gamma_{\mu\nu}$, which may be derived using the Ricci equation, equation (5.62), and definition of extrinsic curvature as negative half of the Lie derivative in the normal direction of the spatial metric, equation (5.59). These evolution equations comprise of a Lie derivative along n^μ. This is not of a natural time derivative form because the normal vector is not dual to the surface one-form Ω_μ, i.e. $n^\mu\Omega_\mu \neq 1$. Thus, let us consider the vector

$$t^\mu = \alpha n^\mu + \beta^\mu$$

which is dual to the one-form Ω_μ for an arbitrary spatial shift vector β^μ, i.e. we have that $t^\mu\Omega_\mu = \alpha n^\mu\Omega_\mu + \beta^\mu\Omega_\mu = 1$. Before the definitions of the evolution equations are provided, an important definition of the Lie derivative of a second order covariant tensors, $A_{\mu\nu}$, along an arbitrary vector V^λ is given by

$$\mathscr{L}_V A_{\mu\nu} = V^\lambda A_{\mu\nu,\lambda} + A_{\lambda\nu}V^\lambda{}_{,\mu} + A_{\mu\lambda}V^\lambda{}_{,\nu}.$$

The above definition will be useful in the following evolution equation definitions.

Definition 87. Extrinsic curvature evolution equation

The Extrinsic curvature evolution equation is defined by

$$\begin{aligned}\mathscr{L}_t K_{\mu\nu} = &- D_\mu D_\nu\alpha + \alpha\left(R_{\mu\nu} - 2K_{\mu\lambda}K^\lambda{}_\nu\right)\\ &- \frac{8\pi G}{c^2}\alpha\left(S_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}\left(\gamma^{\xi\sigma}\gamma_\xi{}^\lambda\gamma_\sigma{}^\kappa T_{\lambda\kappa} - n^\kappa n^\lambda T_{\kappa\lambda}\right)\right) + \mathscr{L}_\beta K_{\mu\nu}.\end{aligned}$$

Definition 88. Spatial metric evolution equation

The Spatial metric evolution equation is defined by

$$\mathscr{L}_t\gamma_{\mu\nu} = -2\alpha K_{\mu\nu} + \mathscr{L}_\beta\gamma_{\mu\nu}.$$

5.5 Least action principle

We intend on deriving the EFEs in a vacuum by making use of a least action principle approach, namely the Einstein-Hilbert Palantini formulation of least action principle. [61]

As the Euler-Lagrange approach begins by considering the minimisation of a function joining two points, at which the bounds are stationary, we too begin by considering the minimization of a function in four-dimensional space-time, that is stationary at the closed bounds.

Consider the action I of some volume, V, of space-time described by the Ricci scalar

$$\begin{aligned}I &= \int_V R\,dV\\ &= \int_V \left(R\sqrt{-g}\right)d^4x\end{aligned}$$

for which $\delta I = 0$. Therefore

$$\begin{aligned}\delta I = 0 &= \int \delta\left(R\sqrt{-g}\right) d^4x \\ &= \int \delta\left(g^{\kappa\lambda} R_{\kappa\lambda}\sqrt{-g}\right) d^4x \\ &= \int \left(\delta\left(g^{\kappa\lambda}\right) R_{\kappa\lambda}\sqrt{-g} + g^{\kappa\lambda}\delta\left(R_{\kappa\lambda}\right)\sqrt{-g} + g^{\kappa\lambda}R_{\kappa\lambda}\delta\left(\sqrt{-g}\right)\right) d^4x\end{aligned}$$

but $\delta\left(\sqrt{-g}\right) = -\frac{1}{2}\sqrt{-g}\; g_{\kappa\lambda}\delta\left(g^{\kappa\lambda}\right)$, thus

$$0 = \int \left(\left(R_{\kappa\lambda} - \frac{1}{2}g_{\kappa\lambda}R\right)\sqrt{-g}\;\; \delta\left(g^{\kappa\lambda}\right)\right) d^4x + \int \left(g^{\kappa\lambda}\delta\left(R_{\kappa\lambda}\right)\sqrt{-g}\right) d^4x. \tag{5.63}$$

Let us concentrate on the term $\delta\left(R_{\kappa\lambda}\right)$ for a moment. Recall the definition of the Ricci tensor

$$R_{\kappa\lambda} = \Gamma^{\nu}{}_{\kappa\lambda,\nu} - \Gamma^{\nu}{}_{\nu\kappa,\lambda} + \Gamma^{\nu}{}_{\nu\xi}\Gamma^{\xi}{}_{\lambda\kappa} - \Gamma^{\nu}{}_{\lambda\xi}\Gamma^{\xi}{}_{\nu\kappa}$$

therefore we have that

$$\delta(R_{\kappa\lambda}) = \delta(\Gamma^{\nu}{}_{\kappa\lambda})_{,\nu} - \delta(\Gamma^{\nu}{}_{\nu\kappa})_{,\lambda} + \delta(\Gamma^{\nu}{}_{\nu\xi})\;\Gamma^{\xi}{}_{\lambda\kappa} + \Gamma^{\nu}{}_{\nu\xi}\;\delta(\Gamma^{\xi}{}_{\lambda\kappa}) - \delta(\Gamma^{\nu}{}_{\lambda\xi})\;\Gamma^{\xi}{}_{\nu\kappa} - \Gamma^{\nu}{}_{\lambda\xi}\;\delta(\Gamma^{\xi}{}_{\nu\kappa})$$

which may be compacted into

$$\delta(R_{\kappa\lambda}) = \delta\left(\Gamma^{\nu}{}_{\kappa\lambda}\right)_{;\,\nu} - \delta\left(\Gamma^{\nu}{}_{\kappa\nu}\right)_{;\,\kappa}.$$

Substituting this result into equation (5.63)

$$\begin{aligned}0 &= \int \left(\left(R_{\kappa\lambda} - \frac{1}{2}g_{\kappa\lambda}R\right)\sqrt{-g}\;\; \delta\left(g^{\kappa\lambda}\right)\right) d^4x + \int \left(g^{\kappa\lambda}\left(\delta\left(\Gamma^{\nu}{}_{\kappa\lambda}\right)_{;\,\nu} - \delta\left(\Gamma^{\nu}{}_{\kappa\nu}\right)_{;\,\kappa}\right)\sqrt{-g}\right) d^4x \\ &= \int_V \left(R_{\kappa\lambda} - \frac{1}{2}g_{\kappa\lambda}R\right)\;\delta\left(g^{\kappa\lambda}\right) dV + \int_V \left(g^{\kappa\lambda}\delta\left(\Gamma^{\nu}{}_{\kappa\lambda}\right)\right)_{;\,\nu} dV - \int_V \left(g^{\kappa\lambda}\delta\left(\Gamma^{\nu}{}_{\kappa\nu}\right)\right)_{;\,\kappa} dV\end{aligned}$$

which by the divergence theorem, for the bounded surface S, we have

$$0 = \int_V \left(R_{\kappa\lambda} - \frac{1}{2}g_{\kappa\lambda}R\right)\;\delta\left(g^{\kappa\lambda}\right) dV + \int_S \left(g^{\kappa\lambda}\delta\left(\Gamma^{\nu}{}_{\kappa\lambda}\right)\right) n_\nu\; dS - \int_S \left(g^{\kappa\lambda}\delta\left(\Gamma^{\nu}{}_{\kappa\nu}\right)\right) n_\kappa\; dS.$$

The surface integrals vanish by virtue of the least action principle formulation, a stationary solution is obtained at the surface, hence

$$0 = \int_V \left(R_{\kappa\lambda} - \frac{1}{2}g_{\kappa\lambda}R\right)\delta\left(g^{\kappa\lambda}\right)\; dV.$$

Since we are considering the vacuum case of this formulation, we have, for non-vanishing $\delta\left(g^{\kappa\lambda}\right)$, that

$$R_{\kappa\lambda} - \frac{1}{2}g_{\kappa\lambda}R = 0$$

which are the EFEs in a vacuum.

5.6 Tetrad formalism

A complete tetrad formulation of the EFE's is supplied by setting up orthonormal tetrads locally, which are understood to follow the worldline of the parcel being modeled. The choice of local orthonormal coordinates is in accordance with local observations and is central to the understanding that this representation of the principles of GR displays the truly relational nature of the theory, in terms of one local frame with respect to another. [57, 62, 63]

Let us consider a time space continuum where the basis vector fields are constructed from a set of mutually orthogonal unit basis vectors fields.

Definition 89. Tetrad

A tetrad is the set of mutually orthogonal unit basis vector fields in a 4-dimensional time-space continuum, $\{\mathbf{e}_a\}$, which forms part of a locally defined coordinate basis by means of tetrad components

$$\mathbf{e}_\mu = e_\mu{}^\kappa\left(x^\lambda\right)\frac{\partial}{\partial x^\kappa} \qquad \text{or} \qquad \mathbf{e}_\mu\left(f\right) = e_\mu{}^\kappa\left(x^\lambda\right)\frac{\partial f}{\partial x^\kappa}$$

and $e_\mu{}^\kappa = \mathbf{e}_\mu\left(x^\kappa\right)$.

By using the tetrad components to define the local coordinate basis, a method of changing tensor components of the standard tensorial form, i.e.

$$T^{\mu\nu}{}_{\kappa\lambda} = e^\mu{}_\alpha e^\nu{}_\beta e_\kappa{}^\gamma e_\lambda{}^\zeta T^{\alpha\beta}{}_{\gamma\zeta}.$$

This change of general basis vectors is one from an integrable basis to that of a non-integrable one, which means that non-tensorial relations are to be defined differently.

Definition 90. Change of tetrad basis

The change of tetrad basis is defined by

$$\mathbf{e}_\mu = \Lambda_\mu{}^{\mu'}\left(x^\nu\right)\mathbf{e}_{\mu'} \tag{5.64}$$

with inverse change of basis defined by

$$\mathbf{e}_{\mu'} = \Lambda_{\mu'}{}^{\mu}\left(x^\nu\right)\mathbf{e}_{\mu}$$

where all matrices represent a Lorentz transformation.

If equation (5.64) is the change of tetrad basis, then

$$T^{\mu\nu}{}_{\kappa\lambda} = \Lambda_{\mu'}{}^{\mu}\Lambda_{\nu'}{}^{\nu}\Lambda_\kappa{}^{\kappa'}\Lambda_\lambda{}^{\lambda'}T^{\mu'\nu'}{}_{\kappa'\lambda'}$$

thus the change of tetrad basis to another allows for the transformations of the standard tensor form for all tensor quantities.

Definition 91. Metric tensor components in tetrad form

The components of the metric tensor in tetrad form is defined by

$$g_{\mu\nu} = g_{\kappa\lambda}e_\mu{}^\kappa e_\nu{}^\lambda = \mathbf{e}_\mu\cdot\mathbf{e}_\nu = \eta_{\mu\nu}$$

where $\eta_{\mu\nu}$ is the Minkowski metric with positive trace, i.e. $\eta_{\mu\nu} = diag(-1,+1,+1+1)$.

The inverse transformation of definition 91 is given by

$$g_{\kappa\lambda}\left(x^\sigma\right) = \eta_{\mu\nu}e^\mu{}_\kappa\left(x^\sigma\right)e^\nu{}_\lambda\left(x^\sigma\right).$$

Definition 92. Commutation function

The commutation function is defined as

$$[\mathbf{e}_\mu, \mathbf{e}_\nu] = \Gamma^\kappa{}_{\mu\nu}(x^\lambda)\mathbf{e}_\kappa.$$

Definition 93. Ricci rotation coefficients

The Ricci rotation coefficients ('Christoffel symbols of second kind' for the tetrad formalism) are defined as

$$\Gamma^\mu{}_{\nu\kappa}(x^\lambda) = e^\mu{}_\lambda e_\kappa{}^\sigma \nabla_\sigma e_\mu e_\nu{}^\lambda$$

which are symmetric in the first two indices.

If we apply Ricci identities to the tetrad base, we obtain the following two definitions.

Definition 94. Riemann curvature tensor

The Riemann curvature tensor is defined as

$$R^\mu{}_{\nu\kappa\lambda} = \mathbf{e}_\kappa\left(\Gamma^\mu{}_{\nu\lambda}\right) - \mathbf{e}_\lambda\left(\Gamma^\mu{}_{\nu\kappa}\right) + \Gamma^\mu{}_{\sigma\kappa}\Gamma^\sigma{}_{\nu\lambda} - \Gamma^\mu{}_{\sigma\lambda}\Gamma^\sigma{}_{\nu\kappa} - \Gamma^\mu{}_{\nu\sigma}\gamma^\sigma{}_{\kappa\lambda}$$

where $\gamma^\mu{}_{\nu\kappa} = \Gamma^\mu{}_{\nu\kappa} - \Gamma^\mu{}_{\kappa\nu}$.

Definition 95. Ricci curvature tensor

The Ricci curvature tensor is defined as

$$\begin{aligned} R_{\nu\lambda} &= \mathbf{e}_\mu\left(\Gamma^\mu{}_{\nu\lambda}\right) - \mathbf{e}_\lambda\left(\Gamma^\mu{}_{\nu\mu}\right) + \Gamma^\mu{}_{\sigma\mu}\Gamma^\sigma{}_{\nu\lambda} - \Gamma^\mu{}_{\sigma\lambda}\Gamma^\sigma{}_{\nu\mu} - \Gamma^\mu{}_{\nu\sigma}\gamma^\sigma{}_{\mu\lambda} \\ &= T_{\nu\lambda} - \frac{1}{2}T g_{\nu\lambda} + \Lambda g_{\nu\lambda}. \end{aligned}$$

Tetrad formalism in cosmology

For cosmology models using the tetrad formalism, we choose the temporal base vector, $\mathbf{e}_0$, to be directed into the future, unit and tangent to the matter flow, u^α. This fixing reduces the number of parameter freedom of rotations of spatial frame $\mathbf{e}_\alpha$ from six to three. The space-time connection coefficients can be split into the following set of equations:

$$\Gamma_{\alpha 00} = \dot{u}_\alpha \tag{5.65}$$

$$\Gamma_{\alpha 0\beta} = \frac{1}{3}\Theta\delta_{\alpha\beta} + \sigma_{\alpha\beta} - \varepsilon_{\alpha\beta\kappa}\omega^\kappa \tag{5.66}$$

$$\Gamma_{\alpha\beta 0} = \varepsilon_{\alpha\beta\kappa}\Omega^\kappa \tag{5.67}$$

$$\Gamma_{\alpha\beta\kappa} = 2a_{[\alpha}\delta_{\beta]\kappa} + \varepsilon_{\kappa\lambda[\alpha}n_{\beta]}{}^\lambda + \frac{1}{2}\varepsilon_{\alpha\beta\lambda}n_\kappa{}^\lambda \tag{5.68}$$

Equations (5.65) and (5.66) contain the kinematic variables of the system. Equation (5.67) describes the rate of rotation, Ω^κ, of the spatial frame with respect to a fermi-propagated basis. In equation (5.68) there are quantities related to a^α and $n_{\alpha\beta} = n_{(\alpha\beta)}$ which determine the nine spatial rotation coefficients.

We define the commutator functions as

$$\left[\mathbf{e}_\mu, \mathbf{e}_\nu\right] = \gamma^\lambda{}_{\mu\nu}\left(x^\kappa\right)\mathbf{e}_\lambda$$

which implies that $\gamma^\lambda{}_{\mu\nu}\left(x^\kappa\right) = -\gamma^\lambda{}_{\nu\mu}\left(x^\kappa\right)$. This commutator function applied to an arbitrary space-time function f is therefore

$$\begin{aligned}\left[\mathbf{e}_0, \mathbf{e}_\alpha\right](f) &= \dot{u}_\alpha \mathbf{e}_0(f) - \left[\frac{1}{3}\Theta\delta_\alpha{}^\beta + \sigma_\alpha{}^\beta + \varepsilon_\alpha{}^\beta{}_\kappa\left(\omega^\kappa + \Omega^\kappa\right)\right]\mathbf{e}_\beta(f) \\ \left[\mathbf{e}_\alpha, \mathbf{e}_\beta\right](f) &= 2\epsilon_{\alpha\beta\kappa}\omega^\kappa \mathbf{e}_0(f) + \left[2a_{[\alpha}\delta_{\beta]}{}^\kappa + \varepsilon_{\alpha\beta\lambda}n^{\lambda\kappa}\right]\mathbf{e}_\kappa(f).\end{aligned}$$

We are now ready to determine the set of constraint equations. Consider the Ricci identity for u^α to obtain the equation

$$0 = (C_1)^\alpha = \left(\mathbf{e}_\beta - 3a_\beta\right)\left(\sigma^{\alpha\beta}\right) - \frac{2}{3}\delta^{\alpha\beta}\mathbf{e}_\beta(\Theta) - n^\alpha{}_\beta\omega^\beta + q^\alpha + \varepsilon^{\alpha\beta\kappa}\left[\left(\mathbf{e}_\beta + 2\dot{u}_\beta - a_\beta\right)(\omega_\kappa) - n_{\beta\lambda}\sigma_\kappa{}^\lambda\right]$$

the (0α)-equation$(C_1)^\alpha$, or momentum conservation equation in the Hamiltonian view of the EFEs. The Ricci scalar identities for u^α yield the vorticity divergence identity (C_2) defined by

$$0 = (C_2) = \left(\mathbf{e}_\alpha - \dot{u}_\alpha - 2a_\alpha\right)\left(\omega^\alpha\right)$$

and we use the Jacobi identities to determine

$$0 = (C_3)^{\alpha\beta} = H^{\alpha\beta} + \left(\delta^{\kappa\langle\alpha}\mathbf{e}_\kappa + 2\dot{u}^{\langle\alpha} + a^{\langle\alpha}\right)\left(\omega^{\beta\rangle}\right) - \frac{1}{2}n_\kappa{}^\kappa\sigma^{\alpha\beta} + 3n^{\langle\alpha}{}_\kappa\sigma^{\beta\rangle\kappa} - \varepsilon^{\kappa\lambda\langle\alpha}\left[\left(\mathbf{e}_\kappa - a_\kappa\right)\left(\sigma_\lambda{}^{\beta\rangle}\right) + n_\kappa{}^{\beta\rangle}\omega_\lambda\right]$$

the $H_\alpha\beta$-equation$(C_3)^{\alpha\beta}$. Consider now the once contracted Bianchi identities to obtain the (div E)-equation $(C_4)^\alpha$

$$\begin{aligned}0 = (C_4)^\alpha &= \left(\mathbf{e}_\beta - 3a_\beta\right)\left(E^{\alpha\beta} + \frac{1}{2}\pi^{\alpha\beta}\right) - \frac{1}{3}\delta^{\alpha\beta}\mathbf{e}_\beta\left(\rho_0(1+\varepsilon)\right) + \frac{1}{3}\Theta q^a - \frac{1}{2}\sigma^a{}_b q^b - 3\omega_b \\ &\quad - \varepsilon^{abc}\left[\sigma_{bd}H_c{}^d - \frac{3}{2}\omega_b q_c + n_{bd}\left(E_c{}^d + \frac{1}{2}\pi_c{}^d\right)\right]\end{aligned}$$

and the (div H)-equation $(C_5)^\alpha$

$$\begin{aligned}0 = (C_5)^\alpha &= \left(\mathbf{e}_\beta - 3a_\beta\right)\left(H^{\alpha\beta}\right) + \left(\rho_0(1+\epsilon) + p\right)\omega^\alpha + 3\omega_\beta\left(E^{\alpha\beta} - \frac{1}{6}\pi^{\alpha\beta}\right) - \frac{1}{2}n^\alpha{}_\beta q^\beta \\ &\quad - \varepsilon^{\alpha\beta\kappa}\left[\frac{1}{2}\left(\mathbf{e}_\beta - a_\beta\right)(q_\kappa) + \sigma_{\beta\lambda}\left(E_\kappa{}^\lambda + \frac{1}{2}\pi_\kappa{}^\lambda\right) - n_{\beta\lambda}H_\kappa{}^\lambda\right].\end{aligned}$$

From the Jacobi identities we obtain the equation $\left(C_J\right)^\alpha$

$$0 = \left(C_J\right)^\alpha = \left(\mathbf{e}_\beta - 2a_\beta\right)\left(n^{\alpha\beta}\right) + \frac{2}{3}\Theta\omega^\alpha + 2\sigma^\alpha{}_\beta\omega^\beta + \varepsilon^{\alpha\beta\kappa}\left[\mathbf{e}_\beta(a_\kappa) - 2\omega_\beta\Omega_\kappa\right]$$

and lastly from the EFEs we obtain the $(C_G)^{\alpha\beta}$ and (C_G) equations

$$\begin{aligned}0 &= (C_G)^{\alpha\beta} = S^{*\alpha\beta} + \frac{1}{3}\Theta\sigma^{\alpha\beta} - \sigma^{\langle\alpha}{}_\kappa\sigma^{\beta\rangle\kappa} - \omega^{\langle\alpha}\Omega^{\beta\rangle} - \left(E^{\alpha\beta} + \frac{1}{2}\pi^{\alpha\beta}\right) \\ 0 &= (C_G) = R^* + \frac{2}{3}\Theta^2 - \left(\sigma_{\alpha\beta}\sigma^{\alpha\beta}\right) + 2\left(\omega_\alpha\omega^\alpha\right) - 4\left(\omega_\alpha\Omega^\alpha\right) - 2\rho_0(1+\epsilon) - 2\Lambda\end{aligned}$$

where

$$
\begin{aligned}
S^{*}{}_{\alpha\beta} &= \mathbf{e}_{\langle\alpha}\left(a_{\beta\rangle}\right) + b_{\langle\alpha\beta\rangle} - \epsilon^{\kappa\lambda}{}_{\langle\alpha}\left(\mathbf{e}_{|\kappa|} - 2a_{|\kappa|}\right)\left(n_{\beta\rangle\lambda}\right)\\
R^{*} &= 2\left(2\mathbf{e}_{\alpha} - 3a_{\alpha}\right)\left(a^{\alpha}\right) - \frac{1}{2}b_{\alpha}{}^{\alpha}\\
b_{\alpha\beta} &= 2n_{\alpha\kappa}n_{\beta}{}^{\kappa} - n_{\kappa}{}^{\kappa}n_{\alpha\beta}.
\end{aligned}
$$

Making use of constraint equations $(C_1)^{\alpha}$ and $(C_3)^{\alpha\beta}$, we are able to eliminate $\mathbf{e}_{\alpha}$ frame derivatives of kinematic variables (Θ, $\sigma_{\alpha\beta}$ and ω^{α}), to obtain

$$
\begin{aligned}
\mathbf{e}_0\left(a^{\alpha}\right) = &-\frac{1}{3}\left(\Theta\delta^{\alpha}{}_{\beta} - \frac{3}{2}\sigma^{\alpha}{}_{\beta}\right)\left(\dot{u}^{\beta} + a^{\beta}\right) + \frac{1}{2}n^{\alpha}{}_{\beta}\omega^{\beta} - \frac{1}{2}q^{\alpha}\\
&-\frac{1}{2}\varepsilon^{\alpha\beta\kappa}\left[\left(\dot{u}_{\beta} + a_{\beta}\right)\omega_{\kappa} - n_{\beta\lambda}\sigma_{\kappa}{}^{\lambda} - \left(\mathbf{e}_{\beta} + \dot{u}_{\beta} - 2a_{\beta}\right)\left(\Omega_{\kappa}\right)\right] + \frac{1}{2}\left(C_1\right)^{\alpha}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{e}_0\left(n^{\alpha\beta}\right) = &-\frac{1}{3}\Theta n^{\alpha\beta} - \sigma^{\langle\alpha}{}_{\kappa}n^{\beta\rangle\kappa} + \frac{1}{2}\sigma^{\alpha\beta}n_{\kappa}{}^{\kappa} - \left(\dot{u}^{\langle\alpha} + a^{\langle\alpha}\right)\omega^{\beta\rangle} - H^{\alpha\beta} + \left(\delta^{\lambda\langle\alpha}\mathbf{e}_{\kappa} + \dot{u}^{\langle\alpha}\right)\left(\Omega^{\beta\rangle}\right)\\
&-\frac{2}{3}\delta^{\alpha\beta}\left[2\left(\dot{u}_{\kappa} + a_{\kappa}\right)\omega^{\kappa} - \left(\sigma_{\kappa\lambda}n^{\kappa\lambda}\right) + \left(\mathbf{e}_{\kappa} + \dot{u}_{\kappa}\right)\left(\Omega^{\kappa}\right)\right]\\
&-\varepsilon^{\kappa\lambda\langle\alpha}\left[\left(\dot{u}_{\kappa} + a_{\kappa}\right)\sigma_{\lambda}{}^{\beta\rangle} - \left(\omega_{\kappa} + 2\Omega_{\kappa}\right)n_{\lambda}{}^{\beta\rangle}\right] - \frac{2}{3}\delta^{\alpha\beta}\left(C_2\right) + \left(C_3\right)^{\alpha\beta}
\end{aligned}
$$

the equations for evolution of spatial connection components. From the Ricci identities for u^{α}, the evolution equations are obtained

$$
\begin{aligned}
\mathbf{e}_0\left(\Theta\right) &= -\frac{1}{3}\Theta^2 + \left(\dot{u}_{\alpha} - 2a_{\alpha}\right)\dot{u}^{\alpha} - \left(\sigma_{\alpha\beta}\sigma^{\alpha\beta}\right)2\left(\omega_{\alpha}\omega^{\alpha}\right) - \frac{1}{2}\left(\rho_0\left(1+\epsilon\right) + 3p\right) + \Lambda\\
\mathbf{e}_0\left(\omega^{\alpha}\right) - \frac{1}{2}\varepsilon^{\alpha\beta\kappa}\mathbf{e}_{\beta}\left(\dot{u}_{\kappa}\right) &= -\frac{2}{3}\Theta\omega^{\alpha} + \sigma^{\alpha}{}_{\beta}\omega^{\beta} - \frac{1}{2}n^{\alpha}{}_{\beta}\dot{u}^{\beta} - \frac{1}{2}\varepsilon^{\alpha\beta\kappa}\left[a_{\beta}\dot{u}_{\kappa} - 2\Omega_{\beta}\omega_{\kappa}\right]\\
\mathbf{e}_0\left(\sigma^{\alpha\beta}\right) - \delta^{\lambda\langle\alpha}\mathbf{e}_{\kappa}\left(\dot{u}^{\beta\rangle}\right) &= -\frac{2}{3}\Theta\sigma^{\alpha\beta} + \left(\dot{u}^{\langle\alpha} + a^{\langle\alpha}\right)\dot{u}^{\beta\rangle} - \sigma^{\langle\alpha}{}_{\kappa}\sigma^{\beta\rangle\kappa} - \omega^{\langle\alpha}\omega^{\beta\rangle}\\
&\quad-\left(E^{\alpha\beta} - \frac{1}{2}\pi^{\alpha\beta}\right) + \varepsilon^{\kappa\lambda\langle\alpha}\left[2\Omega_{\kappa}\sigma_{\lambda}{}^{\beta\rangle} - n_{\kappa}{}^{\beta\rangle}\dot{u}_{\lambda}\right].
\end{aligned}
$$

Lastly, we obtain the evolution of matter

$$
\mathbf{e}_0\left(\rho_0\left(1+\epsilon\right)\right) + \mathbf{e}_{\alpha}\left(q^{\alpha}\right) = -\Theta\left(\rho_0\left(1+\epsilon\right) + p\right) - 2\left(\dot{u}_{\alpha} - a_{\alpha}\right)q^{\alpha} - \left(\sigma_{\alpha\beta}\pi^{\alpha\beta}\right)
$$

$$
\begin{aligned}
\mathbf{e}_0\left(q^{\alpha}\right) + \delta^{\alpha\beta}\mathbf{e}_{\beta}\left(p\right) + \mathbf{e}_{\beta}\left(\pi^{\alpha\beta}\right) = &-\frac{4}{3}\Theta q^{\alpha} - \sigma^{\alpha}{}_{\beta}q^{\beta} - \left(\rho_0\left(1+\epsilon\right) + p\right)\dot{u}^{\alpha} - \left(\dot{u}_{\beta} - 3a_{\beta}\right)\pi^{\alpha\beta}\\
&-\varepsilon^{\alpha\beta\kappa}\left[\left(\omega_{\beta} - \Omega_{\beta}\right)q_{\kappa} - n_{\beta\lambda}\pi_{\kappa}{}^{\lambda}\right]
\end{aligned}
$$

from the once-contracted Bianchi identities and Weyl curvature variables

$$\begin{aligned}
\mathbf{e}_0\left(E^{\alpha\beta}+\frac{1}{2}\pi^{\alpha\beta}\right)-\varepsilon^{\kappa\lambda\langle\alpha}\,\mathbf{e}_\kappa\left(H_\lambda{}^{\beta\rangle}\right)+\frac{1}{2}\delta^{\kappa\langle\alpha}\,\mathbf{e}_\kappa\left(q^{\beta\rangle}\right)=&-\frac{1}{2}\left(\rho_0\left(1+\epsilon\right)+p\right)\sigma^{\alpha\beta}-\Theta\left(E^{\alpha\beta}+\frac{1}{6}\pi^{\alpha\beta}\right)\\
&+3\sigma^{\langle\alpha}{}_\kappa\left(E^{\beta\rangle\kappa}-\frac{1}{6}\pi^{\beta\rangle\kappa}\right)+\frac{1}{2}n_\kappa{}^\kappa H^{\alpha\beta}-3n^{\langle\alpha}{}_\kappa H^{\beta\rangle\kappa}\\
&-\frac{1}{2}\left(2\dot{u}^{\langle\alpha}+a^{\langle\alpha}\right)q^{\beta\rangle}+\varepsilon^{\kappa\lambda\langle\alpha}\Big[\left(2\dot{u}_\kappa-a_\kappa\right)H_\lambda{}^{\beta\rangle}\\
&+\left(\omega_\kappa+2\Omega_\kappa\right)\left(E_\lambda{}^{\beta\rangle}+\frac{1}{2}\pi_\lambda{}^{\beta\rangle}\right)+\frac{1}{2}n_\kappa{}^{\beta\rangle}q_\lambda\Big]
\end{aligned}$$

$$\begin{aligned}
\mathbf{e}_0\left(H^{\alpha\beta}\right)+\varepsilon^{\kappa\lambda\langle\alpha}\,\mathbf{e}_\kappa\left(E_\lambda{}^{\beta\rangle}-\frac{1}{2}\pi_\lambda{}^{\beta\rangle}\right)=&-\Theta H^{\alpha\beta}+3\sigma^{\langle\alpha}{}_\kappa H^{\beta\rangle\kappa}+\frac{3}{2}\omega^{\langle\alpha}q^{\beta\rangle}-\frac{1}{2}n_\kappa{}^\kappa\left(E^{\alpha\beta}-\frac{1}{2}\pi^{\alpha\beta}\right)\\
&+3n^{\langle\alpha}{}_\kappa\left(E^{\beta\rangle\kappa}-\frac{1}{2}\pi^{\beta\rangle\kappa}\right)+\varepsilon^{\kappa\lambda\langle\alpha}\Big[a_\kappa\left(E_\lambda{}^{\beta\rangle}-\frac{1}{2}\pi_\lambda{}^{\beta\rangle}\right)-2\dot{u}_\kappa E_\lambda{}^{\beta\rangle}\\
&+\frac{1}{2}\sigma_\kappa{}^{\beta\rangle}q_\lambda+\left(\omega_\kappa+2\Omega_\kappa\right)H_\lambda{}^{\beta\rangle}\Big]
\end{aligned}$$

from the twice-contracted Bicanhi identities.

Chapter 6

Exact and analytic solutions to the EFEs

This chapter finds its foundation in assuming torsion-free coordinate choices, in cases where the space-time can be mapped to Minkowski space-time.

6.1 Minkowski space-time

The first section of this chapter introduces the reader to the Minkowski metric with positive trace, formally known as space-like metric of special relativity. It is aimed at familiarising the reader with the nature of calculations once GR has been applied to a physical problem, as well as address any constraints of this metric choice. [64]

Let us consider the line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

describing a distance squared in an arbitrary coordinate system. In a Riemannian manifold, this distance is easily interpreted physically and always greater than or equal to zero. As we move to a pseudo-Riemannian manifold, the coordinate system incorporates a time that may or may not differ from the proper time of the system/object being modelled. The concept of a distance in a four-dimensional pseudo-Riemannin manifold is not as intuitively interpreted, but the definition of the line element in tensor form remains true in every coordinate system. In a pseudo-Riemannian mannifold there is no constraint on the distance squared to be greater than or equal to zero, therefore we have three cases:

(1) $ds^2 < 0$

(2) $ds^2 = 0$

(3) $ds^2 > 0$

Now, let us assume an empty, shear-free space-time with an arbitrary Riemannian manifold embedded in the space portion of the metric tensor. Then, the line element may be defined by

$$ds^2 = g_{00}dt^2 + g_{ij}dx^i dx^j \tag{6.1}$$

for $i, j = 1,2,3$ and all other other components are zero. It would be optimal if we had an equality equation to solve, so let us consider the line element, equation (6.1), and the second case for our pseudo-Riemannian manifold:

$$0 = g_{00}dt^2 + g_{ij}dx^i dx^j$$

for $dt \neq 0$

$$0 = g_{00} + g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}$$

where $g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}$ is the coordinate velocity squared. Let us now consider a velocity squared that all observers, objects and systems can agree on. A postulate of GR defines the speed of light as constant to all observers, making the speed of light squared a logical and convenient choice of the coordinate velocity squared in the above case. Thus

$$0 = g_{00} + c^2$$
$$g_{00} = -c^2$$

and we will define a generalised Minkowski metric tensor to be

$$\eta^*_{\mu\nu} = \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & g_{11} & g_{12} & g_{13} \\ 0 & g_{21} & g_{22} & g_{23} \\ 0 & g_{31} & g_{32} & g_{33} \end{pmatrix}. \tag{6.2}$$

The Line element definition, equation (6.1), thus becomes

$$ds^2 = -c^2dt^2 + g_{ij}dx^idx^j$$

which we may now use to consider the other two cases. When $ds^2 < 0$ we have

$$c^2dt^2 < g_{ij}dx^idx^j$$

illustrating that the space components of the line element are dominant. Similarly, when $ds^2 > 0$ we have

$$c^2dt^2 > g_{ij}dx^idx^j$$

illustrating dominance in the time component of the line element. Consider the future light cone illustrated in figure 6.1 depicting the three cases of allowed values for the line interval ds^2.

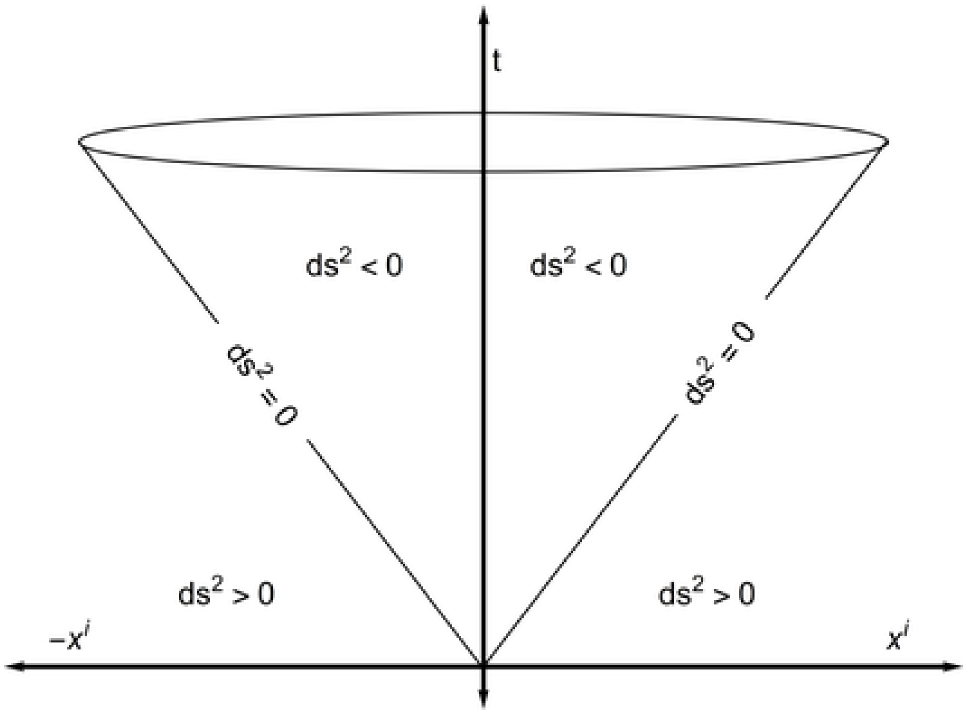


Figure 6.1: A future light cone with time-like, light-like and space-like regions illustrated

For simplicity, let us consider the line element to be unitary. Thus we define:

(1) $ds^2 = -1$ as a **space-like** geodesic line element

(2) $ds^2 = 0$ as a **null/light-like** geodesic line element

(3) $ds^2 = 1$ as a **time-like** geodesic line element

The Minkowski space-time is defined by our generalised equation (6.2) with cartesian spatial components and unitary constant c, i.e

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.3}$$

It is characterised by a constant, unit, diagonal matrix with both a positive trace and, in this case, a chosen characteristic determinant of 1.

6.1.1 Calculating $E = mc^2$

Consider the line element

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

describing the special relativistic, Minkowski, space-time with four-velocity vector

$$U^\alpha = \begin{pmatrix} Ut(\tau) \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The four-velocity is required to satisfy the condition $g_{\alpha\beta} U^\alpha U^\beta = -1$, producing two possible definitions for Ut, $Ut(t) = \pm\frac{1}{c}$. The positive option is chosen. Recall the Christoffel symbols of second kind and stress-energy tensor,

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa} \left(g_{\kappa\nu,\mu} + g_{\mu\kappa,\nu} - g_{\mu\nu,\kappa} \right) \qquad \text{and} \qquad T^{\mu\nu} = \left(\rho(t) + p(t)\right) U^\mu U^\nu + g^{\mu\nu} p(t).$$

Thus, the covariant stress-energy tensor is given by

$$T^{\mu\nu} = \begin{pmatrix} c^2\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

therefore the energy density, $\frac{E}{V}$, is T_{00} component

$$\frac{E}{V} = c^2\rho$$
$$\frac{E}{V} = c^2 \frac{m}{V}$$

which for $V \neq 0$, may be written as

$$E = mc^2.$$

There are pressure components present in the covariant stress-energy tensor,

$$p_x = p_y = p_z = p,$$

and the momentum density, momentum flux and shear stress components are all found to be zero.

6.2 Schwarzschild space-time

The Schwarzschild metric is determined by solving a system of differential equations when making an assumption on the function form of the metric in a vacuum. The Schwarzschild metric is determined and applied to motivate the perihelion motion of Mercury as well as photon path deflection. [1, 65, 66, 67, 68, 69, 70, 71, 72]

The Shwarzschild solution assumes static spherical symmetry with radial dependent components g_{00} and g_{11},

$$g_{\mu\nu} = \begin{bmatrix} -e^{2\lambda(r)} & 0 & 0 & 0 \\ 0 & e^{2\kappa(r)} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2(\theta) \end{bmatrix}. \tag{6.4}$$

Recall the Christoffel symbols of second kind,

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa}\left(g_{\kappa\nu,\mu} + g_{\mu\kappa,\nu} - g_{\mu\nu,\kappa}\right)$$

and calculate them. There are thirteen non-zero Christoffel symbols of second kind,[1]

$$\begin{aligned} \Gamma^{t}{}_{tr} &= \lambda'(r) \\ \Gamma^{r}{}_{tt} &= e^{-2\kappa(r)+2\lambda(r)}\lambda'(r) \\ \Gamma^{r}{}_{rr} &= \kappa'(r) \\ \Gamma^{r}{}_{\theta\theta} &= -e^{-2\kappa(r)} r \\ \Gamma^{r}{}_{\phi\phi} &= -e^{-2\kappa(r)} r\sin^2(\theta) \end{aligned}$$

$$\begin{aligned} \Gamma^{\theta}{}_{\theta r} &= \frac{1}{r} \\ \Gamma^{\theta}{}_{\phi\phi} &= -\cos(\theta)\sin(\theta) \\ \Gamma^{\phi}{}_{\phi r} &= \Gamma^{\theta}_{\theta r} \\ \Gamma^{\phi}{}_{\phi\theta} &= \cot(\theta). \end{aligned}$$

A condition of empty space is that the Einstein tensor vanishes, i.e. $G_{\mu\nu} = 0$, thus must have that the Ricci tensor and scalar vanish as well. Recall the definition of the Ricci tensor

$$R_{\mu\nu} = \Gamma^{\sigma}{}_{\nu\mu,\sigma} - \Gamma^{\sigma}{}_{\sigma\mu,\nu} + \Gamma^{\sigma}{}_{\sigma\gamma}\Gamma^{\gamma}{}_{\nu\mu} - \Gamma^{\sigma}{}_{\nu\gamma}\Gamma^{\gamma}{}_{\sigma\mu}$$

and calculate the components. Four non-zero Ricci tensor elements surfaced[2]:

$$R_{tt} = \frac{e^{2\lambda(r)-2\kappa(r)}\left(\lambda'(r)\left(-r\kappa'(r) + r\lambda'(r) + 2\right) + r\lambda''(r)\right)}{r} \tag{6.5}$$

$$R_{rr} = \frac{\kappa'(r)\left(r\lambda'(r)+2\right) - r\left(\lambda''(r) + \lambda'(r)^2\right)}{r} \tag{6.6}$$

$$R_{\theta\theta} = e^{-2\kappa(r)}\left(r\kappa'(r) - r\lambda'(r) - 1\right) + 1 \tag{6.7}$$

$$R_{\phi\phi} = R_{\theta\theta}\sin^2[\theta]. \tag{6.8}$$

As all four of these equations are required to vanish in empty space, for the non-trivial case with arbitrary r and θ, equations (6.5) and (6.6) lead us to $\lambda'[r] = -\kappa'[r]$. Using this result in equation (6.7),

$$\begin{aligned} 1 &= \left(1 - 2r\kappa'(r)\right)e^{-2\kappa(r)} \\ 1 &= \frac{d}{dr}\left(re^{-2\kappa(r)}\right) \\ r - C_1 &= e^{-2\kappa(r)} \\ e^{-2\kappa(r)} &= 1 - \frac{C_1}{r} \end{aligned}$$

[1] Determine the Christoffel symbols of second kind for the given ansatz metric in equation (6.4). See 14 for a possible Mathematica solution.

[2] Determine the Ricci tensor components for the given ansatz metric in equation (6.4). See 15 for a possible Mathematica solution.

yielding a solution that satisfies the remaining equation, equation (6.8). The constant C_1 is known as the Schwarzschild radius, r_s, and is defined as

$$r_s = \frac{2Gm}{c^2}.$$

Let us consider unitary constants G and c, then the Shwarzschild metric is given by

$$g_{\mu\nu} = \begin{bmatrix} -\left(1-\frac{2m}{r}\right) & 0 & 0 & 0 \\ 0 & \left(1-\frac{2m}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2(\theta) \end{bmatrix}.$$

Notice that the temporal component of the metric is undefined when $r = 0$ and the radial component is undefined when $r = r_s = 2m$. These instabilities are a consequence of the choice of coordinate system and could be interpreted because the center of the sphere experiences infinite time and at the Schwarzschild radius there is infinite space (the boundary of the black hole).

6.2.1 Geodesic equations in a Schwarzschild space-time

This subsection introduces the geodesic solution for a point mass, for modelling the perihelion behaviour of Mercury, and the null geodesic solution, for bending of light trajectory.

Recall the definition of the geodesic equations from Chapter 3, and the definition of the line element equation. Therefore, the line element equation is

$$ds^2 = -\left(1-\frac{2m}{r}\right)t'(\tau)^2 + \left(1-\frac{2m}{r}\right)^{-1} r'(\tau)^2 + r^2\theta'(\tau)^2 + r^2\sin^2(\theta)\phi'(\tau)^2 \tag{6.9}$$

and the geodesic equations are

$$t''(\tau) = \frac{2mr'(\tau)t'(\tau)}{r(\tau)(2m-r(\tau))} \tag{6.10}$$

$$r''(\tau) = \frac{2m-r(\tau)}{r(\tau)^3}\left(mt'(\tau)^2 - r(\tau)^3\left(\theta'(\tau)^2 + \sin^2(\theta(\tau))\phi'(\tau)^2\right)\right) - \frac{mr'(\tau)^2}{2mr(\tau)-r(\tau)^2} \tag{6.11}$$

$$\theta''(\tau) = \frac{r(\tau)\sin(\theta(\tau))\cos(\theta(\tau))\phi'(\tau)^2 - 2\theta'(\tau)r'(\tau)}{r(\tau)} \tag{6.12}$$

$$\phi''(\tau) = -\frac{2\phi'(\tau)\left(r'(\tau) + r(\tau)\theta'(\tau)\cot(\theta(\tau))\right)}{r(\tau)}. \tag{6.13}$$

Relaxing any variation on the θ direction, i.e. $\theta(\tau) = \frac{\pi}{2}$ yields the following simplified forms of equations (6.10) and (6.13) ,

$$\begin{aligned} t''(\tau) &= \frac{2mr'(\tau)t'(\tau)}{r(\tau)\,(2m-r(\tau))} \\ 0 &= t''(\tau)\left(1-\frac{2m}{r(\tau)}\right) + \frac{2mr'(\tau)}{r(\tau)^2}t'(\tau) \\ 0 &= \frac{d}{d\tau}\left(t'(\tau)\left(1-\frac{2m}{r(\tau)}\right)\right) \\ c &= t'(\tau)\left(1-\frac{2m}{r(\tau)}\right) \\ \therefore\quad t'(\tau) &= \frac{\epsilon}{1-\frac{2m}{r(\tau)}} \end{aligned}$$

and

$$\begin{aligned}
\phi''(\tau) &= \frac{-2r'(\tau)\phi'(\tau)}{r(\tau)} \\
r(\tau)^2\phi''(\tau) + 2r(\tau)r'(\tau)\phi'(\tau) &= 0 \\
\frac{d}{d\tau}\left(r(\tau)^2\phi'(\tau)\right) &= 0 \\
\therefore \quad \phi'(\tau) &= \frac{h}{r(\tau)^2}
\end{aligned}$$

where ϵ and h is a constants of integration. Equation (6.11) therefore becomes

$$r''(\tau) = -\frac{1}{2}\left(\frac{4h^2m}{r(\tau)^4} - \frac{2h^2}{r(\tau)^3} + \frac{\epsilon^2}{r(\tau)-2m} - \frac{\epsilon^2}{r(\tau)}\right) - \frac{mr'(\tau)^2}{r(\tau)(2m-r(\tau))}, \tag{6.14}$$

which will be used in the calculations that follow.

Particle Trajectories

This subsection introduces particle trajectories using the space-like geodesic assumption in a Schwarzschild space-time. An in-built numerical solver of Mathematica is used to solve the differential equation of this trajectory, then produce a polar plot of this solution with various mass and angular momentum values.

Consider the space-like geodesic case, i.e. equating equation (6.9) to $ds^2 = -1$, relaxing variations in the θ direction by setting $\theta = \frac{\pi}{2}$, using the results for $t'(\tau)$ and $\phi'(\tau)$, solving for ϵ^2:

$$\epsilon^2 = r'(\tau)^2 - \frac{\left(h^2 + r(\tau)^2\right)(2m - r(\tau))}{r(\tau)^3}$$

Using this result in equation (6.14) reveals

$$\frac{3h^2m}{r(\tau)} + mr(\tau) + r(\tau)^3 r''(\tau) = h^2. \tag{6.15}$$

One notes the similarities of equation (6.15) to that of classical Newtonian planetary motion, with a correction due to space-time curvature. A substitution is made in equation (6.15) for $r(\tau) = \dfrac{2m}{u(\phi(\tau))}$ and $r''(\tau) = -\dfrac{h^2u(\phi(\tau))^2u''(\phi(\tau))}{8m^3}$ yielding the equation

$$-2h^2u''(\phi) + 3h^2u(\phi)^2 - 2h^2u(\phi) + 4m^2 = 0. \tag{6.16}$$

Circular orbit trajectories would be expected when $u''(\phi) = 0$ and the incidental trajectory is tangential, $u'(0) = 0$. Consider $u''(\phi) = 0$, then from equation (6.16)

$$u(\phi) = \frac{1}{3} \pm \frac{1}{3}\sqrt{1 - \frac{12m^2}{h^2}}$$

which for unitary mass and $h = \sqrt{12}$ we have that $u(\phi) = \frac{1}{3}$. This may be used as the second initial condition, $u(0) = \frac{1}{3}$, and one could vary the values of h, to the left and right of $h = \sqrt{12}$, to produce particle trajectories that either fall into the black hole or get deflected by it. The result of this variation in h is expected as this parameter describes the energy of the particle. If a particle has enough energy then it will be able to escape the gravitational field of the black hole, otherwise the particle will fall into it.

Similarly, the value of h can be kept constant and by adjusting the value of m one could produce trajectory paths of particles that either orbit, get deflected by or fall into the black hole. Considering several different combinations of values for m, h and choice of initial conditions, one is able to produce a multitude of orbital trajectories. When considering the initial conditions $u(0) = 0.01$ and $u'(0) = 0$, unitary m and $h = 6$ the polar plot produced, in figure 6.2, illustrates a precession similar to that of the observed perihelion procession of Mercury. This model became a good approximation for Mercury's motion under the postulate that Mercury is much smaller than the Sun and thus, may be treated as a point particle.

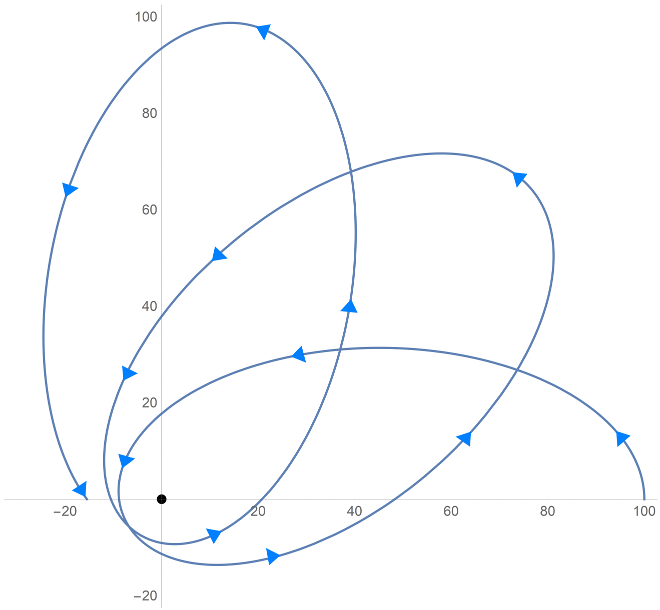


Figure 6.2: Particle trajectory with unitary mass, $h = 6$, $u[0] = 0.01$ and $u'[0] = 0$

Photon Trajectories

This subsection introduces photon trajectories using the null geodesic assumption in a Schwarzschild space-time. An in-built numerical solver of Mathematica is used to solve the differential equation of this trajectory, then produce a polar plot of this solution with various mass values.

Consider the null geodesic case, i.e. equating equation (6.9) to $ds^2 = 0$, relaxing variations in the θ direction by setting $\theta = \frac{\pi}{2}$, making a transformation of $\tau = \lambda$, using the results for $t'(\lambda)$ and $\phi'(\lambda)$, solving for h^2:

$$h^2 = \frac{r(\lambda)^3\left(r'(\lambda)^2 - \epsilon^2\right)}{2m - r(\lambda)}.$$

Now, using this result in equation (6.14) reveals

$$r''(\lambda) = \frac{(3m - r(\lambda))\left(\epsilon^2 - r'(\lambda)^2\right)}{r(\lambda)(2m - r(\lambda))}. \tag{6.17}$$

A substitution is made in equation (6.17) for

$$r(\lambda) = \frac{2m}{u(\phi(\lambda))}$$

$$r'(\lambda)^2 = \frac{\epsilon^2 u'(\phi(\lambda))^2}{u'(\phi(\lambda))^2 - u(\phi(\lambda))^3 + u(\phi(\lambda))^2}$$

and

$$r''(\lambda) = \frac{\epsilon^2 u(\phi(\lambda))^2 u''(\phi(\lambda))}{2m(u(\phi(\lambda)) - 1)u(\phi(\lambda))^2 - 2mu'(\phi(\lambda))^2}$$

yielding the equation

$$u''(\phi) = \frac{1}{2}u(\phi)(3u(\phi) - 2). \tag{6.18}$$

Note that this differential equation is independent of the constant ϵ and mass m. Equation (6.18) suggests that light paths are affected by curvature in a similar fashion to the trajectory paths of particles. This phenomena describes how the light path observed on earth from a distant star may have been affected by curvature of space-time, which is one of the discoveries that shot Einstein's theory into fame.

Now, let us consider the initial conditions to $u(0) = 0.1$ and $u'(0) = 0.2$ produces polar plots illustrating photon deflection by a black hole, as depicted in figure 6.3.

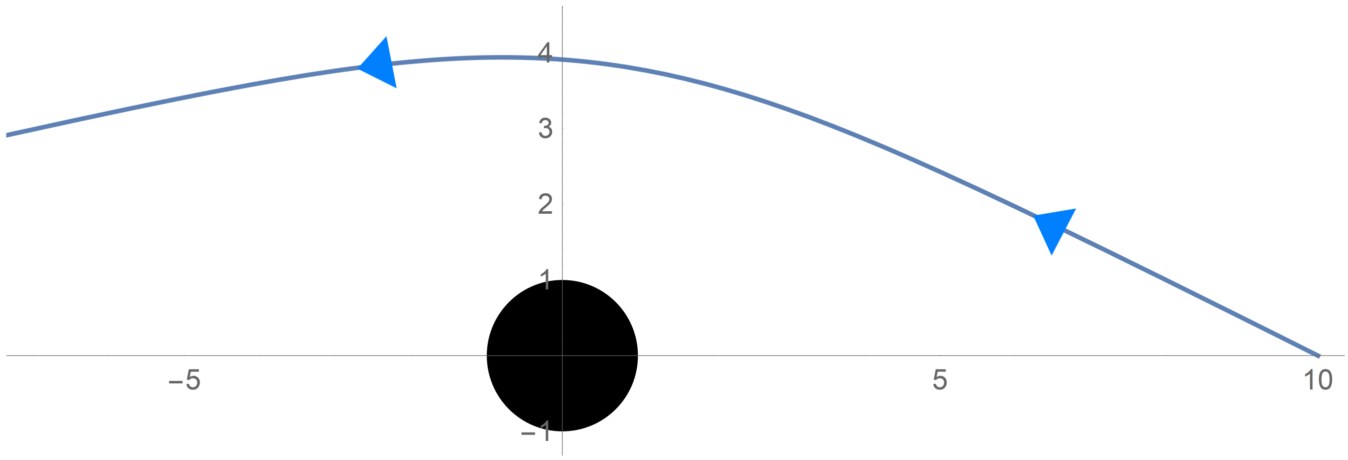


Figure 6.3: Photon trajectory deflected by black hole

6.2.2 Tolman Oppenheimer Volkoff (TOV) equation

The TOV solution is one of a spherically symmetric body of isotropic material in static gravitational equilibrium. Spherically symmetric metrics were first analysed by Richard C. Tolman in 1934 and 1939 and the TOV equation was derived in 1939 by J. Robert Oppenheimer and George Volkoff.

This model assumes a similar form to the Schwarzchild solution with the g_{tt} component being an arbitrary function in r and with mass being dependent on r as well. The form of this solution is assumed to be

$$g_{\mu\nu} = \begin{bmatrix} -a(r) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2m(r)}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2(\theta) \end{bmatrix},$$

with an assumed four-velocity of the form

$$U^\alpha = \begin{pmatrix} Ut(\tau) \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The four-velocity is required to satisfy the condition $g_{\alpha\beta}U^\alpha U^\beta = -1$, producing two possible definitions for Ut, $Ut(\tau) = \pm\frac{1}{\sqrt{a(r)}}$. The positive option was chosen. Recall and determine the Christoffel symbols of second kind and stress-energy tensor,

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2}g^{\lambda\kappa}\left(g_{\kappa\nu,\mu}+g_{\mu\kappa,\nu}-g_{\mu\nu,\kappa}\right) \qquad \text{and} \qquad T^{\mu\nu} = \left(\rho(r)+P(r)\right)U^{\mu}U^{\nu}+g^{\mu\nu}P(r).$$

Consider now the conservation law, $T^{\mu\nu}{}_{;\nu}=0$, which produced one non-zero equation of state:

$$a'(r)(P(r)+\rho(r))+2a(r)P'(r)=0 \tag{6.19}$$

when $r\neq 2m(r)$. Let us now consider the EFEs. Three non-zero, unique equations surfaced:

$$0=\frac{2m'(r)}{r^2a(r)}-\frac{8\pi\rho(r)}{a(r)} \tag{6.20}$$

$$0=\frac{2\left(m(r)+4\pi r^3P(r)\right)}{r}-\frac{(r-2m(r))a'(r)}{a(r)} \tag{6.21}$$

$$\begin{aligned}0={}&r^2(r-2m(r))a'(r)^2+4a(r)^2\left(rm'(r)-m(r)+8\pi r^3P(r)\right)\\&+2ra(r)\left(a'(r)\left(r\left(m'(r)-1\right)+m(r)\right)-r(r-2m(r))a''(r)\right).\end{aligned} \tag{6.22}$$

Now, solve for $P'(r)$, $m'(r)$ and $a'(r)$ in equations (6.19) to (6.21), respectively, and ensure that the solutions satisfy equation (6.22). Also, consider the mass-density function $\rho(r)$ to be constant, $\rho(r)=\rho_0$, and we have the following three first order coupled differential equations:

$$P'(r)=-\frac{\left(m(r)+4\pi r^3P(r)\right)(P(r)+\rho_0)}{r(r-2m(r))} \tag{6.23}$$

$$m'(r)=4\pi r^2\rho_0 \tag{6.24}$$

$$a'(r)=\frac{2a(r)\left(m(r)+4\pi r^3P(r)\right)}{r(r-2m(r))}. \tag{6.25}$$

At a radius of zero there is no mass, therefore we solve equation (6.24) with $m(0)=0$ to obtain the solution

$$m(r)=\frac{4}{3}\pi r^3\rho_0. \tag{6.26}$$

Make use of equation (6.26) in equation (6.23) and solve the differential equation to obtained

$$P(r)=-\frac{\rho_0\left(3e^{2c_1\rho_0}\sqrt{8\pi\rho_0r^2-3}-1\right)}{3\left(e^{2c_1\rho_0}\sqrt{8\pi\rho_0r^2-3}-1\right)}.$$

Now, let us assume that at $r=0$ we have that the pressure is P_0. Therefore, the above equation provides the constant of integration

$$c_1=\frac{\ln\left(-\frac{i(3P_0+\rho_0)}{3\sqrt{3}(P_0+\rho_0)}\right)}{2\rho_0}$$

and is now defined by

$$P(r)=-\frac{\rho_0\left(3P_0\left(\sqrt{9-24\pi\rho_0r^2}-1\right)+\rho_0\left(\sqrt{9-24\pi\rho_0r^2}-3\right)\right)}{3P_0\left(\sqrt{9-24\pi\rho_0r^2}-3\right)+\rho_0\left(\sqrt{9-24\pi\rho_0r^2}-9\right)}. \tag{6.27}$$

Now, we combine the solutions for $m(r)$ and $P(r)$, equations (6.26) and (6.27), into equation (6.25) and solve the differential equation for $a(r)$ to obtain the following solution

$$a(r)=c_1\left(3P_0\left(\sqrt{9-24\pi\rho_0r^2}-3\right)+\rho_0\left(\sqrt{9-24\pi\rho_0r^2}-9\right)\right)^2.$$

Let us now consider how to obtain the constant c_1. At the boundary of the star we must have that the pressure is zero. It is at this radius that the vacuum Schwarzchild solution will equal the TOV solution. Therefore, set the pressure equation (6.27) to zero and solve for the radius. Thus, the boundary of the star is found at a radius defined by

$$R = \sqrt{\frac{\frac{3}{2\pi}\left(2{P_0}^2 + P_0\rho_0\right)}{9{P_0}^2\rho_0 + 6P_0{\rho_0}^2 + {\rho_0}^3}}. \tag{6.28}$$

Now, the metric component g_{tt} of the vacuum Schwarzschild and TOV solutions must be equal. Therefore

$$-a(R) = -\left(1 - \frac{2M(R)}{R}\right)$$

and solving for the constant c_1 yields

$$c_1 = \frac{\left(P_0+\rho_0\right)^2}{9\left(3P_0+\rho_0\right)^2\left(\rho_0\left(\sqrt{\frac{(P_0+\rho_0)^2}{(3P_0+\rho_0)^2}}-3\right)+3P_0\left(\sqrt{\frac{(P_0+\rho_0)^2}{(3P_0+\rho_0)^2}}-1\right)\right)^2}.$$

Therefore, we have

$$a(r) = \frac{\left(P_0+\rho_0\right)^2\left(3P_0\left(\sqrt{9-24\pi\rho_0 r^2}-3\right)+\rho_0\left(\sqrt{9-24\pi\rho_0 r^2}-9\right)\right)^2}{9\left(3P_0+\rho_0\right)^2\left(\rho_0\left(\frac{P_0+\rho_0}{3P_0+\rho_0}-3\right)+3P_0\left(\frac{P_0+\rho_0}{3P_0+\rho_0}-1\right)\right)^2}. \tag{6.29}$$

Let us now consider plots of the functions $a(r)$, $m(r)$ and $P(r)$, equations (6.26), (6.27) and (6.29), respectively, for r $\in [0, R]$, where R is defined in equation (6.28). The following plot, figure 6.4, was generated with initial pressure and density of 0.4 and 0.25, respectively.[3]

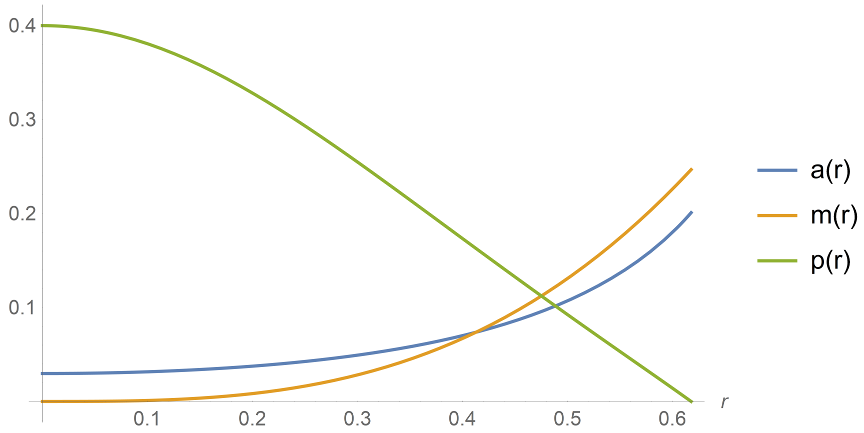


Figure 6.4: TOV solution of $a(r)$, $m(r)$ and $P(r)$ with $P_0 = 0.4$ and $\rho_0 = 0.25$

6.3 Kerr space-time

The Kerr metric is determined by solving a system of differential equations when making an assumption on the function form of the metric in a vacuum. This solution, from 1963, models a spherically axially-symmetric, rotating black hole in an empty geometry. [73, 74, 75, 76, 77, 78, 79]

[3] Create a Mathematica code that will plot functions $a(r)$, $m(r)$ and $P(r)$ on the same set of axes with $P_0 = 0.4$ and $\rho_0 = 0.25$. See 16 for a possible Mathematica representation.

The Kerr metric represents the gravitational field exterior to an axially symmetric rotating body. It was originally obtained by Kerr using the theory of algebraically special metrics, and so this metric will not be derived. It is

$$ds^2 = -\left(1-\frac{2mr}{\rho^2}\right)dt^2 + \frac{\rho^2}{\Delta}dr^2 + \frac{4mra}{\rho^2}\sin^2(\theta)dtd\phi + \rho^2 d\theta^2 + \left[\left(r^2+a^2\right)\rho m^2 + \frac{2mra^2\rho m^4\theta}{\rho^2}\right]d\phi^2 \quad (6.30)$$

where

$$\rho^2 \ = \ r^2 + a^2\cos^2(\theta), \qquad \Delta \ = \ r^2 - 2mr + a^2 \quad \text{and} \quad a \ = \ -\frac{KJ}{mc^2}$$

J reduces to the angular momentum of the central /body in the non relativistic limit. We note the following:

(i) If $a = 0$, then the Kerr metric reduces to the Schwarzschild metric

(ii) The Kerr metric has the general features expected of the metric of an axially symmetric rotating body:

  (a) For large r, it is approximately Schwarzschild.
  (b) It is axially symmetric, i.e. $g_{\alpha\beta}$ is independent of ϕ.
  (c) It is stationary, i.e. $g_{\alpha\beta}$ is independent of t.
  (d) It is invariant under the combined interchange $d\phi \to -d\phi$ and $dt \to -dt$.

Because the metric is non-diagonal, the location of the event horizon is more complicated than in the Schwarzschild case. We find that there are two event horizons:

1. The static limit horizon, inside which all objects are forced to rotate with the black hole.

2. The proper event horizon inside which r must decrease.

The static limit horizon

Consider particles at rest in a Kerr coordinate system. In its world line

$$ds^2 = -\left(1-\frac{2mr}{\rho^2}\right)dt^2$$

for a particle we have that $ds^2 < 0$. Thus the static limit is given by

$$1-\frac{2mr}{\rho^2} = 0$$

or

$$r^2 - 2mr + a^2\cos^2(\theta) = 0$$
$$r = m \pm \sqrt{m^2 - a^2\cos^2(\theta)}.$$

Let us consider only real solutions, such that $m > |a|$, then a particle can only be at rest in Kerr coordinate system outside S^+ or inside S^-.

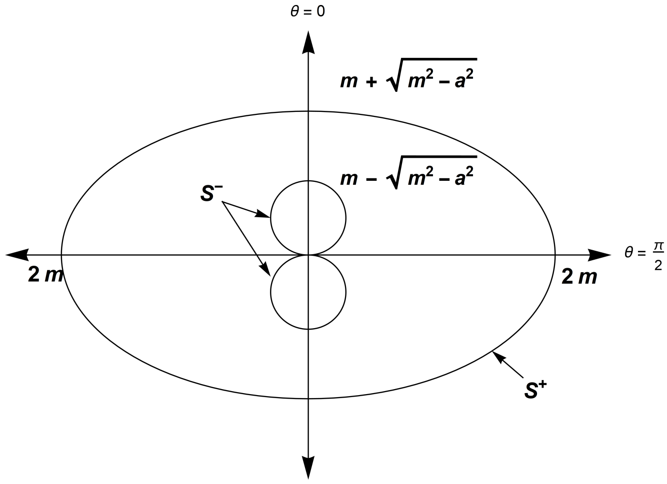


Figure 6.5: Static limit horizon of a Kerr black hole

The proper event horizon

Inside S^{+} a particle need not fall inwards but may be able to move circularly i.e. putting dr, $d\theta$ zero in the Kerr metric, it is possible to find $d\phi$ such that $ds^2 < 0$, i.e. such that

$$-\left(1-\frac{2mr}{\rho^2}\right)dt^2+\frac{4mra}{\rho^2}\sin^2(\theta)dtd\phi+\left[\left(r^2+a^2\right)\rho m^2+\frac{2mra^2\rho m^4\theta}{\rho^2}\right]d\phi^2<0. \tag{6.31}$$

The limiting case when the best $d\phi$ we can find will only give equality must give the condition for the event horizon. We will find that condition reduces to $\Delta = 0$, so that this horizon $g_{rr} < 0$ and $ds^2 < 0$ is satisfied by r necessarily decreasing.

The limiting case occurs when the left hand side of equation (6.31) is a perfect square, i.e. "$b^2 - 4ac = 0$".

$$\begin{aligned}
\frac{16m^2r^2a^2\sin^4(\theta)}{\rho^2}+4\left(1-\frac{2mr}{\rho^2}\right)\left[\left(r^2+a^2\right)\sin^2(\theta)+\frac{2mra^2\sin^4(\theta)}{\rho^2}\right]&=0\\
4\left(r^2+a^2\right)\sin^2(\theta)+\frac{8mra^2\sin^4(\theta)}{\rho^2}-\frac{8mr}{\rho^2}\left(r^2+a^2\right)\sin^2(\theta)&=0\\
\left(r^2+a^2\right)\rho^2+2mra^2\left(1-\cos^2(\theta)\right)-2mr\left(r^2+a^2\right)&=0\\
\left(r^2+a^2\right)\rho^2-2mr\left(r^2+a^2\cos^2(\theta)\right)&=0\\
\left(r^2+a^2-2mr\right)\rho^2&=0\\
\Delta\rho^2&=0
\end{aligned}$$

But $\rho^2 \neq 0$, thus $\Delta = 0$ and so $r = m \pm \sqrt{m^2 - a^2}$. We denote the event horizon by Σ_+ and Σ_-. The diagram illustrated in figure 6.6 is obtained. The region between Σ_+ and S_+, where motion is necessary but dr can be positive, is called the ergosphere. It can be shown that energy can be extracted from the ergosphere.

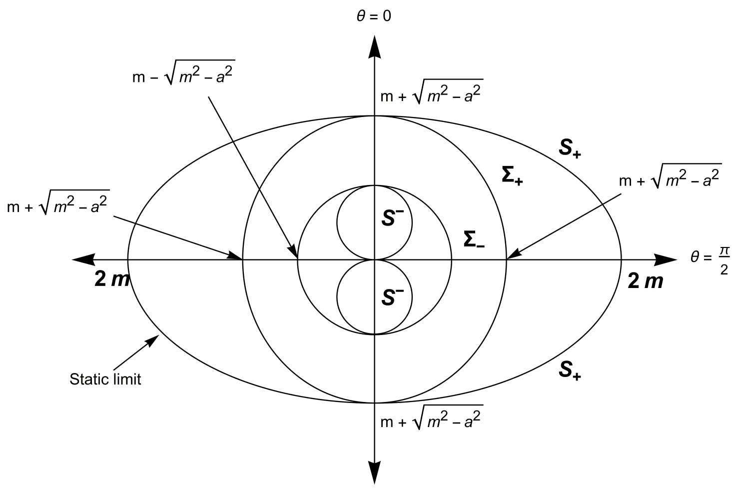


Figure 6.6: Proper event horizon of a Kerr black hole

6.3.1 Applications of Kerr solution

One of the bizarre features of a black hole is while it is static spacetime - i.e. the metric has no time dependence - it is not a stationary space-time - i.e. space-time is undergoing a continual process of distortion. In order to see this, one has to look at the $G^{t\phi}$ component. This component shows us how the space-time is being dragged around the black hole.

In order to create a visualisation of matter moving around a black hole, one would solve the hydrodynamic equations in a Kerr space-time or solve the geodesic equations for photon paths around a black hole. One such example can be found in the Interstellar film. More recently, the event horizon telescope collaboration used eight observatories across the globe to capture an image of the event horizon and a secretion disc around the M87 Kerr black hole.

6.4 FLRW model

The Friedmann-Lemaître-Robertson-Walker (FLRW) model, which is an exact solution to the EFE's, is considered. This solution is investigated as a viable model of the expanding universe. [57, 80, 81, 82, 83, 84]

Traditionally, the FLRW metric was obtained from the Weyl's postulate and later through geometric consideration only. Let us consider a complimentary derivation of this metric by solving the EFE's of a perfect fluid for gravitational collapse/expansion.

Let us consider the metric associated with a homogeneous, spherically symmetric, static fluid, described by co-moving coordinate t, with reduced circumference polar coordinates:

$$ds^2 = -dt^2 + a(t)^2\left(\frac{1}{1-kr^2}dr^2 + r^2 d\theta^2 + r^2\sin(\theta)^2 d\phi^2\right)$$

and four-velocity

$$U^\alpha = \begin{pmatrix} Ut(\tau) \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The four-velocity is required to satisfy the condition $g_{\alpha\beta}U^\alpha U^\beta = -1$, producing two possible definitions for Ut, $Ut(\tau) = \pm 1$. The positive option was chosen. Recall and calculate the Christoffel symbols of second kind and stress-energy tensor,

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2}g^{\lambda\kappa}\left(g_{\kappa\nu,\mu} + g_{\mu\kappa,\nu} - g_{\mu\nu,\kappa}\right) \qquad \text{and} \qquad T^{\mu\nu} = \left(\rho(t) + P(t)\right)U^\mu U^\nu + g^{\mu\nu}P(t).$$

The stress-energy tensor is

$$T_\mu{}^\nu = \begin{pmatrix} -\rho(t) & 0 & 0 & 0 \\ 0 & P(t) & 0 & 0 \\ 0 & 0 & P(t) & 0 \\ 0 & 0 & 0 & P(t) \end{pmatrix}.$$

The stress-energy tensor must obey the conservation law

$$T^{\mu\beta}{}_{;\beta} = T^{\mu\beta}{}_{,\beta} + \Gamma^\mu{}_{\gamma\beta}T^{\gamma\beta} + \Gamma^\gamma{}_{\beta\gamma}T^{\mu\beta} = 0.$$

The first conservation equation is found to be

$$\rho'(t) = -3\frac{a'(t)}{a(t)}(P(t) + \rho(t)). \tag{6.32}$$

Conservation equations in r,θ and ϕ where all found to be true. Recall now definitions from Chapter 5, let us define the Einstein tensor and the EFEs

$$R_{\mu\nu} = \Gamma^\kappa{}_{\nu\mu,\kappa} - \Gamma^\kappa{}_{\kappa\mu,\nu} + \Gamma^\kappa{}_{\kappa\gamma}\Gamma^\gamma{}_{\nu\mu} - \Gamma^\kappa{}_{\nu\gamma}\Gamma^\gamma{}_{\kappa\mu} \qquad R = R_{\beta\gamma}g^{\beta\gamma}$$

$$R^{\mu\nu} = g^{\mu\beta}g^{\nu\gamma}R_{\beta\gamma} \qquad G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R.$$

Solving the EFEs, $G^{\mu\nu} = 8\pi T^{\mu\nu}$, arrives at the following non-vanishing equations

$$\frac{3\left(a'(t)^2 + k\right)}{a(t)^2} = 8\pi\rho(t) \tag{6.33}$$

$$0 = 2a(t)\left(a''(t) + 4\pi a(t)P(t)\right) + a'(t)^2 + k. \tag{6.34}$$

Notice that there are three equations, and three unknowns. Traditionally, a substitution is made, describing the Hubble parameter, and a system of first order, coupled differential equations arise. Consider equations (6.33) and (6.34). They may be rearranged and combined to form equations:

$$\frac{a''(t)}{a(t)} = -\frac{4}{3}\pi(3P(t) + \rho(t)) \tag{6.35}$$

$$3\frac{a'(t)^2}{a(t)^2} = 8\pi\rho(t) - \frac{3k}{a(t)^2} \tag{6.36}$$

where $a'(t) \neq 0$ as equation (6.36) is a first integral of equation (6.35). Equations (6.35) and (6.36) are known as the Raychaudhuri and Friedmann equations respectively. Make the substitution

$$\Omega = \frac{8\pi\rho(t)}{3H(t)^2} = \frac{\rho(t)}{\rho_c} \tag{6.37}$$

where $H(t)$ and ρ_c are the Hubble parameter and critical density defined, respectively, by

$$H(t) = \frac{a'(t)}{a(t)} \tag{6.38}$$

and

$$\rho_c = \frac{3H(t)^2}{8\pi\rho(t)}.$$

We define the dimensionless quantity Ω as the density 'parameter' in cosmology, but mathematically this is a function. Equation (6.36) may therefore be written as

$$\Omega - 1 = \frac{k}{H(t)^2 a(t)^2} \tag{6.39}$$

where we define the following three cases:

(i) $\Omega > 1$ corresponding to closed three surfaces. Equivalently we choose $k = +1$ in this case.

(ii) $\Omega = 1$ corresponding to flat three surfaces. Equivalently we choose $k = 0$ in this case.

(iii) $\Omega < 1$ corresponding to open three surfaces. Equivalently we choose $k = -1$ in this case.

Let us now define a measure of the rate of expansion defined by

$$q(t) = -\frac{a''(t)a(t)}{(a'(t))^2} \tag{6.40}$$

which is a dimensionless quantity known as the deceleration parameter. Thus the rate of change in the Hubble parameter, equation (6.38), may be defined by

$$H'(t) = -\left(1 + q(t)\right) H^2(t). \tag{6.41}$$

6.4.1 Applications of FLRW solution

Let us define a dimensionless time variable τ from the length scale function $a(t)$ as

$$a\,(t(\tau)) = a_0 e^{\tau}. \tag{6.42}$$

Therefore

$$a'\,(t) = a'\,(\tau)\,\frac{d\tau}{dt}$$

and from equations (6.38) and (6.41) we have

$$\frac{dt}{d\tau} = \frac{1}{H} \tag{6.43}$$

and

$$\frac{dH}{d\tau} = -\left(1 + q\right) H, \tag{6.44}$$

respectively. Let us generally define two galactic epochs, as in [80], for matter-energy dominated ($\gamma = 1$) and radiation dominated ($\gamma = \frac{4}{3}$) universes by defining

$$p(t) = (\gamma - 1)\rho(t). \tag{6.45}$$

Substitute equations (6.37), (6.38), (6.39), (6.44) and (6.45) into equation (6.35) to obtain

$$q = \frac{1}{2}(3\gamma - 2)\Omega. \tag{6.46}$$

Now, substitute equations (6.37) and (6.38) into equation (6.36) and take the derivative with respect to time,

$$2ka'(t) + a(t)^3 H(t)\left(2(\Omega(t)-1)H'(t) + H(t)\Omega'(t)\right) = 0,$$

and then substitute equations (6.37), (6.39), (6.43), (6.44) and (6.46) into the result to obtain

$$\frac{d\Omega}{d\tau} = (3\gamma - 2)(\Omega - 1)\Omega. \tag{6.47}$$

Use equations (6.44), (6.46) and (6.47) to obtain

$$\frac{dH}{d\Omega} = \frac{H(\Omega+1)}{2\Omega(1-\Omega)} \tag{6.48}$$

which is to be integrated.

Radiation dominated universe

Consider the radiation dominated where $\gamma = \frac{4}{3}$ and integrate equation (6.48) to obtain

$$H = -\frac{A\sqrt{\Omega}}{\Omega - 1}$$

where A is a constant of integration and the solution is constrained by the condition

$$-2\pi < 2\arg(A) - 2\arg(1-\Omega) + \arg(\Omega) \leq 2\pi.$$

This solution is plotted parametrically, in figure 6.7, with several values for the constant A from integration.

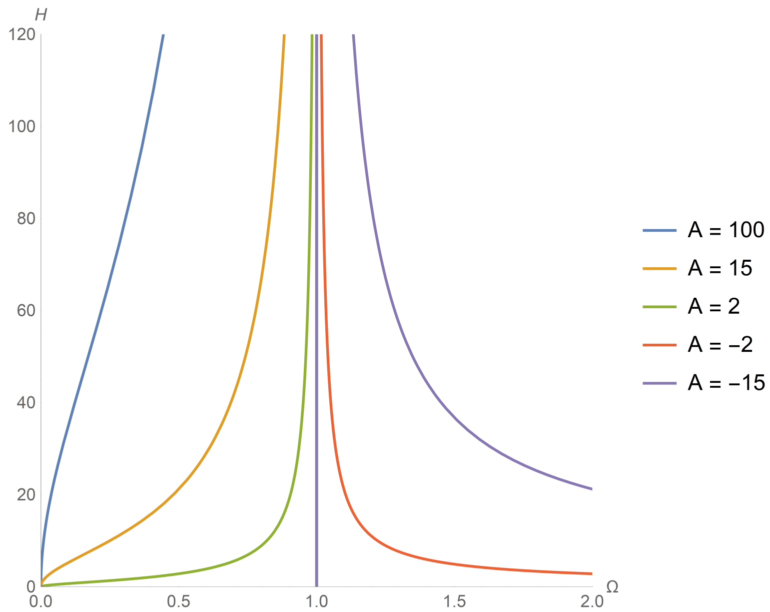


Figure 6.7: The ΩH-plane of FLRW radiation dominated universes

The phase-plane approach to the visualisation of solution to the Einstein Field Equations using a dynamical systems approach can be used to faithfuly represent the background dynamics of a broad variety of homogeneous cosmologies, this includes Λ-CDM concordance model that is based on the FLRW cosmology[85].

Secondly, we consider the expansion coefficient, $a(t)$, in FLRW open, flat and closed radiation dominated universes. Recall for a radiation dominated universe that the pressure may be expressed, as a function in density, by

$$P(t) = \frac{1}{3}\rho(t).$$

Solve for $a'(t)$ and $\rho'(t)$ in equations (6.33) and (6.32), respectively, differentiate the $a'(t)$ expression with respect to time and substitute all three of these equations into equation (6.34). The result is true, which shows that equation (6.34) is simply a second order representation of equations (6.32) and (6.33). Let us now solve the differential equations (6.32) and (6.33) to obtain

$$a(t) = \frac{\sqrt{-3kt^2 - 2\sqrt{3}t\sqrt{8\pi\rho_0 - 3k} + 3}}{\sqrt{3}} \tag{6.49}$$

$$\rho(t) = \frac{9\rho_0}{\left(3kt^2 + 2\sqrt{3}t\sqrt{8\pi\rho_0 - 3k} - 3\right)^2} \tag{6.50}$$

or

$$a(t) = \frac{\sqrt{-3kt^2 + 2\sqrt{3}t\sqrt{8\pi\rho_0 - 3k} + 3}}{\sqrt{3}} \tag{6.51}$$

$$\rho(t) = \frac{9\rho_0}{\left(-3kt^2 + 2\sqrt{3}t\sqrt{8\pi\rho_0 - 3k} + 3\right)^2}. \tag{6.52}$$

From all the solutions we see that in order to obtain real solutions, we must have that

$$\frac{3(1+k)k}{8\pi} \leq \rho_0 \leq \frac{3\left(k^2 + 2k + 1\right)}{32\pi}.$$

With this constraint, let us produce plots for the first set of equations with various values of ρ_0 and $k \in \{-1, 0, 1\}$, where $k = -1$, $k = 0$ and $k = 1$ represent open, flat and closed universes, respectively. This is depicted in figure 6.8.

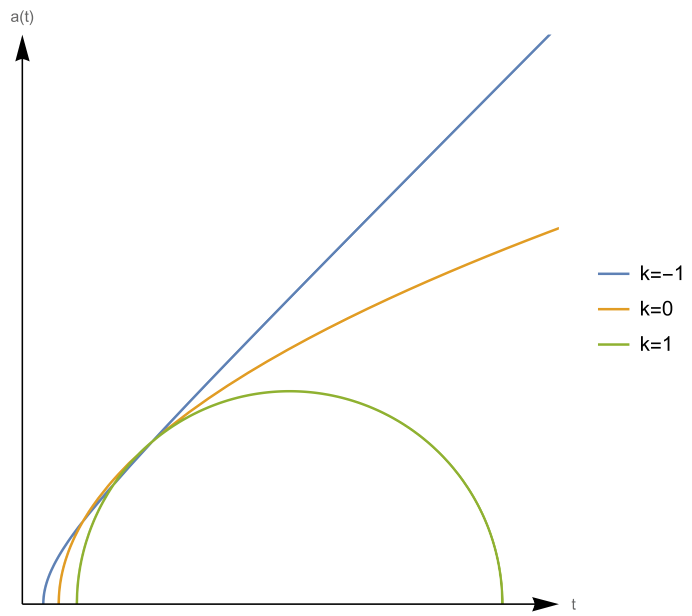


Figure 6.8: Expansion coefficient in FLRW open, flat and closed, radiation dominated universes

Chapter 7

Approximate solutions of the EFE

An analytical and then numerical approach to approximate solutions of of the EFEs is supplied in this chapter. The analytical approach encompasses a perturbation approach of the metric tensor to arrive at the theory of gravitational waves. The numerical approach investigates Runge-Kutte methods, Symplectic integrator method and the Hamiltonian, applications to hydrodynamics with given metric, numerical perturbation approach, the ADM formalism; and then covers some simple numerical relativity simulations.

7.1 Perturbation theory and gravitational waves

In the perturbation theory section, the reader will encounter the $1+3$ *formalism in the form of Newtonian Threading as well as the* $3+1$ *formalism as a Newtonian gauge theory. In perturbing both of these formalisms, this section will display how they both represent equivalent phenomenon. To conclude this section, a brief description of gravitational wave theory is to be addressed. [86, 87, 88, 89, 90, 91, 92]*

Consider a metric, $g_{\mu\nu}$, defined by a Minkowski space-time, $\eta_{\mu\nu}$, perturbed by a metric, $h_{\mu\nu}$, such that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{7.1}$$

Now, if we are to determine g

$$g = g^{\mu\nu} g_{\mu\nu} = \left(\eta^{\mu\nu} + h^{\mu\nu}\right)\left(\eta_{\mu\nu} + h_{\mu\nu}\right)$$

to first order

$$g^{\mu\nu} g_{\mu\nu} = \eta^{\mu\nu} h_{\mu\nu} + h^{\mu\nu}\eta_{\mu\nu}.$$

Thus, when considering raising and lowering indices, in first order perturbation, the Minkowski metric should be utilised. Now, recall the definition of the Christoffel symbols of second kind:

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa}\left(g_{\kappa\nu,\mu} + g_{\mu\kappa,\nu} - g_{\mu\nu,\kappa}\right)$$

which is now defined by

$$\Gamma^{\lambda}{}_{\mu\nu} \approx \frac{1}{2} \eta^{\lambda\kappa}\left(h_{\kappa\nu,\mu} + h_{\mu\kappa,\nu} - h_{\mu\nu,\kappa}\right)$$

as the Minkowski metric comprises of constant entries and is used to raise the indices.

Also recall the EFEs and stress-energy tensor,

$$G^{\mu\nu} = 8\pi T^{\mu\nu}$$

where

$$T^{\mu\nu} = \left(\rho(t) + p(t)\right) u^\mu u^\nu + g^{\mu\nu} p(t) \qquad \text{and} \qquad G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$$

therefore

$$\begin{aligned} R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R &= 8\pi T^{\mu\nu} \\ g_{\mu\nu}\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right) &= 8\pi g_{\mu\nu} T^{\mu\nu} \\ R - \frac{1}{2} 4R &= 8\pi T \\ R &= -8\pi T \end{aligned}$$

and so we may write

$$\begin{aligned} R^{\mu\nu} - \frac{1}{2} g^{\mu\nu}(-8\pi T) &= 8\pi T^{\mu\nu} \\ R^{\mu\nu} &= 8\pi\left(T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T\right) \end{aligned}$$

or equivalently

$$R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) \tag{7.2}$$

where the Ricci tensor is defined by

$$R_{\mu\nu} = \Gamma^\kappa{}_{\nu\mu,\kappa} - \Gamma^\kappa{}_{\mu\kappa,\nu} + \Gamma^\kappa{}_{\kappa\gamma}\Gamma^\gamma{}_{\nu\mu} - \Gamma^\kappa{}_{\nu\gamma}\Gamma^\gamma{}_{\kappa\mu}.$$

As we are only working to first order, the second order terms in the Ricci tensor are to neglected so that

$$\begin{aligned} R_{\mu\nu} &= \Gamma^\kappa{}_{\nu\mu,\kappa} - \Gamma^\kappa{}_{\mu\kappa,\nu} \\ &= \frac{1}{2}\eta^{\kappa\lambda}\left(h_{\nu\lambda,\mu\kappa} + h_{\lambda\mu,\nu\kappa} - h_{\nu\mu,\lambda\kappa}\right) - \frac{1}{2}\eta^{\kappa\lambda}\left(h_{\kappa\lambda,\mu\nu} + h_{\lambda\mu,\kappa\nu} - h_{\kappa\mu,\lambda\nu}\right) \end{aligned}$$

or

$$R_{\mu\nu} = \frac{1}{2}\eta^{\kappa\lambda}\left(h_{\nu\lambda,\mu\kappa} - h_{\mu\nu,\lambda\kappa} - h_{\lambda\kappa,\mu\nu} + h_{\mu\kappa,\nu\lambda}\right). \tag{7.3}$$

Gravitational waves

From the perturbed Ricci tensor, equation (7.3),

$$\begin{aligned} R_{\mu\nu} &= \frac{1}{2}\eta^{\kappa\lambda}\left(h_{\lambda\nu,\mu\kappa} - h_{\mu\nu,\lambda\kappa} - h_{\lambda\kappa,\mu\nu} + h_{\mu\kappa,\nu\lambda}\right) \\ &= -\frac{1}{2}\eta^{\kappa\lambda} h_{\mu\nu,\lambda\kappa} - \frac{1}{2}\left(h_{,\mu\nu} - h_\mu{}^\alpha{}_{,\nu\alpha} - h_\nu{}^\alpha{}_{,\mu\alpha}\right). \end{aligned}$$

The last term may be written as

$$h_{,\mu\nu} - h_\mu{}^\alpha{}_{,\nu\alpha} - h_\nu{}^\alpha{}_{,\mu\alpha} = \left(\frac{1}{2}\delta_\mu{}^\alpha h - h_\mu{}^\alpha\right)_{,\alpha\nu} + \left(\frac{1}{2}\delta_\nu{}^\alpha h - h_\nu{}^\alpha\right)_{,\alpha\mu} \tag{7.4}$$

where $h = h^\lambda{}_\lambda$. Equation (7.4) will vanish if we have

$$\left(\frac{1}{2}\delta_\mu{}^\alpha h - h_\mu{}^\alpha\right)_{,\alpha} = 0. \tag{7.5}$$

Thus, in choosing harmonic coordinates, the EFEs may now be defined by

$$-\frac{1}{2}\eta^{\kappa\lambda}h_{\mu\nu,\lambda\kappa}+\frac{1}{4}\eta^{\kappa\lambda}g_{\mu\nu}h_{,\lambda\kappa}=\frac{8\pi G}{c^4}T_{\mu\nu}. \tag{7.6}$$

Now, let us define

$$\phi_\mu{}^\nu=h_\mu{}^\nu-\frac{1}{2}\delta_\mu{}^\nu h \tag{7.7}$$

which must satisfy condition (7.5)

$$\phi_\mu{}^\nu{}_{,\nu}=0.$$

Raising the index in equation (7.6) and making a substitution of equation (7.7)

$$\Box\phi_\mu{}^\nu=-\frac{16\pi G}{c^4}T_\mu{}^\nu$$

where $\Box$ is D'Alembert operator.

Plane waves

Suppose gravitational waves are generated far away enough for the waves to be experienced as plane waves. The wave equation

$$\Box h_{\alpha\beta}=0 \tag{7.8}$$

is automatically satisfied by an function of the form

$$h_{\alpha\beta}=h_{\alpha\beta}(x^1-x^0)$$

which corresponds to a plane wave propagating in the positive x^1 direction with unit velocity and amplitude of which approached zero as $x^1\to\infty$ described by the metric

$$h_{\alpha\beta}=\begin{pmatrix} h_{00} & h_{01} & h_{02} & h_{03}\\ h_{10} & h_{11} & h_{12} & h_{13}\\ h_{20} & h_{21} & h_{22} & h_{23}\\ h_{30} & h_{31} & h_{32} & h_{33}\end{pmatrix}$$

satisfy the condition (7.5). Thus, when considering differentiation with respect to x^1-x^0, we have that

$$\begin{aligned} h_{\alpha\beta,0}&=-h'_{\alpha\beta}\\ h_{\alpha\beta,1}&=h'_{\alpha\beta}\\ h_{\alpha\beta,2}&=0\\ h_{\alpha\beta,3}&=0\end{aligned}$$

also, consider $h_\alpha{}^\beta$

$$\begin{aligned} h_\alpha{}^\beta&=\eta^{\beta\lambda}h_{\alpha\lambda}\\ &=\begin{pmatrix} -h_{00} & 0 & 0 & 0\\ 0 & h_{11} & 0 & 0\\ 0 & 0 & h_{22} & 0\\ 0 & 0 & 0 & h_{33}\end{pmatrix}\end{aligned}$$

then condition (7.5) for element $\mu=0$ becomes

$$\begin{aligned} h_\mu{}^\alpha{}_{,\alpha}-\frac{1}{2}\left(\delta_\mu{}^\alpha h\right)_{,\alpha}&=0\\ -h_{00,0}+h_{10,1}-\frac{1}{2}\left(h_0{}^0+h_1{}^1+h_2{}^2+h_3{}^3\right)_{,0}&=0\end{aligned}$$

therefore

$$h'_{00} + h'_{10} + \frac{1}{2}\left(-h'_{00} + h'_{11} + h'_{22} + h'_{33}\right) = 0 \tag{7.9}$$

and similarly for $\alpha = 1, 2, 3$ we have

$$h'_{01} + h'_{11} - \frac{1}{2}\left(-h'_{00} + h'_{11} + h'_{22} + h'_{33}\right) = 0 \tag{7.10}$$

$$h'_{02} + h'_{12} = 0 \tag{7.11}$$

$$h'_{03} + h'_{13} = 0. \tag{7.12}$$

Now, we are assuming that $h_{\alpha\beta} \to 0$ as $x^1 \to \infty$. Thus, equations (7.9) to (7.12) are simply integrated and the constants of integration vanish because of the boundary condition at ∞. Thus equations (7.9) to (7.12) become

$$h_{00} + h_{10} + \frac{1}{2}(-h_{00} + h_{11} + h_{22} + h_{33}) = 0$$

$$h_{01} + h_{11} - \frac{1}{2}(-h_{00} + h_{11} + h_{22} + h_{33}) = 0$$

$$h_{02} + h_{12} = 0$$

$$h_{03} + h_{13} = 0$$

and the following result is achieved by solving the system of equations above

$$h_{\alpha\beta} = \begin{pmatrix} h_{00} & -\frac{1}{2}(h_{00} + h_{11}) & -h_{12} & -h_{13} \\ -\frac{1}{2}(h_{00} + h_{11}) & h_{11} & h_{12} & h_{13} \\ -h_{12} & h_{12} & h_{22} & h_{23} \\ -h_{13} & h_{13} & h_{23} & -h_{22} \end{pmatrix}.$$

Note: It can be shown, correct to order ϵ, that the Riemann curvature tensor depends only on h_{22} and h_{23}. Hence, we suspect that h_{22} and h_{23} are the only elements with physical significance and that the other four elements h_{00}, h_{11}, h_{12} and h_{13} can be removed by a coordinate transformation.

A coordinate transformation involves 4 arbitrary functions. We try to choose those four functions so that the four quantities $\tilde{h}_{00}$, $\tilde{h}_{11}$, $\tilde{h}_{12}$ and $\tilde{h}_{13}$ are zero. Let us define $A_\alpha = h_\alpha{}^\mu{}_{,\mu} - \frac{1}{2}h_{,\mu}$. Other requirements of the coordinate transformation include:

- The coordinate transformation must leave the metric in the form $g_{\alpha\beta} = \eta_{\alpha\beta} + \epsilon h_{\alpha\beta}$, i.e. the coordinate transformation must be of the form
$$\tilde{x}^\alpha = x^\alpha + \epsilon f^\alpha$$
- The guage condition $A_\alpha = 0$, equation (7.5), must remain valid, i.e. we must have that $\tilde{A}_\alpha = 0$

Now, $\tilde{A}_\alpha = A_\alpha - \Box f_\alpha$, but $A_\alpha = 0$. Thus we are required to choose f_α such that $\Box f_\alpha = 0$ in order for $\tilde{A}_\alpha = 0$.

We now show that it is possible to find a coordinate transformation of form

$$\bar{x}^\mu = x^\mu + \epsilon f^\mu$$

$$\Box f_\mu = 0$$

such that

$$\bar{h}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h_{22} & h_{23} \\ 0 & 0 & h_{23} & -h_{22} \end{pmatrix}.$$

We begin by choosing $f_\alpha = f_\alpha\left(x^1 - x^0\right)$ which satisfies $\Box f_\alpha = 0$. Then

$$\bar{A}_\alpha = 0 \tag{7.13}$$

and we have

$$\bar{\Box}\bar{h}_{\alpha\beta} = 0. \tag{7.14}$$

As the above equations, equations (7.13) and (7.14), have exactly the same form as the unbarred system, we have

$$\bar{h}_{\alpha\beta} = \begin{pmatrix} \bar{h}_{00} & -\frac{1}{2}\left(\bar{h}_{00} + \bar{h}_{11}\right) & -\bar{h}_{12} & -\bar{h}_{13} \\ -\frac{1}{2}\left(\bar{h}_{00} + \bar{h}_{11}\right) & \bar{h}_{11} & \bar{h}_{12} & \bar{h}_{13} \\ -\bar{h}_{12} & \bar{h}_{12} & \bar{h}_{22} & \bar{h}_{23} \\ -\bar{h}_{13} & \bar{h}_{13} & \bar{h}_{23} & -\bar{h}_{22} \end{pmatrix}.$$

Now,

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - f_{\alpha,\beta} - f_{\beta,\alpha}$$

choose f_α so that $\bar{h}_{00} = \bar{h}_{11} = \bar{h}_{22} = \bar{h}_{33} = 0$. From $\bar{h}_{00} = 0$ we have

$$\begin{aligned} h_{00} - 2f_{0,0} &= 0 \\ f_{0,0} &= \frac{1}{2}h_{00} \end{aligned}$$

and similarly for $\bar{h}_{11} = 0$, $\bar{h}_{12} = 0$ and $\bar{h}_{13} = 0$ we have

$$\begin{aligned} f_{1,1} &= \frac{1}{2}h_{11} \\ h_{12} - f_{1,2} - f_{2,1} &= 0 \\ h_{13} - f_{1,3} - f_{3,1} &= 0, \end{aligned}$$

respectively. Now we have made the choice that $f_\alpha = f_\alpha\left(x^1 - x^0\right)$, and so f_α is a function of the same argument as $h_{\alpha\beta}$. Thus the above four conditions become

$$\begin{aligned} f_0' &= -\frac{1}{2}h_{00} & \qquad f_2' &= h_{12} \\ f_1' &= \frac{1}{2}h_{11} & \qquad f_3' &= h_{13} \end{aligned}$$

which for the chosen f_α becomes

$$\begin{aligned} f_0 &= -\frac{1}{2}\int^{x^1-x^0} h_{00}(u)du & \qquad f_2 &= \frac{1}{2}\int^{x^1-x^0} h_{12}(u)du \\ f_1 &= \frac{1}{2}\int^{x^1-x^0} h_{11}(u)du & \qquad f_3 &= \frac{1}{2}\int^{x^1-x^0} h_{13}(u)du. \end{aligned}$$

Also, as f_α is independent of x^2 and x^3 we have

$$\begin{aligned} \bar{h}_{22} &= h_{22} - 2f_{2,2} & \qquad \bar{h}_{23} &= h_{23} - f_{2,3} - f_{3,2} \\ &= h_{22} & \qquad &= h_{23} \end{aligned}$$

and hence we have that

$$\bar{h}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h_{22} & h_{23} \\ 0 & 0 & h_{23} & -h_{22} \end{pmatrix}.$$

Thus, if the space-time is flat except for a plane gravitational wave propagating in the x^1 direction, the metric of space-time is

$$ds^2 = -\left(dx^0\right)^2 + \left(dx^1\right)^2 + (1+\epsilon h_{22})\left(dx^2\right)^2 + (1-\epsilon h_{22})\left(dx^3\right)^2 + 2\epsilon h_{23} dx^2 dx^3.$$

We note that the above result is useful because of the following:

(i) It can be used to estimate the energy flux of a gravitational wave (very small).

(ii) It can be used to analyse the motion of a test particle in the wave. This shows that the wave causes a much larger transverse than longitudinal motion of the particle, which is consistent with Birkhoff's theorem (no monopole gravitational radiation).

7.2 Numerical relativity

Aimed at introducing the reader to the two numerical approaches of relativity. Brief descriptions of the general Runge-Kutta method, Symplectic method, hydrodynamic approach with a given metric, perturbation approach and the ADM formalism (R. Arnowitt, S. Deser and C.W. Misner) are given. [59, 60, 91, 93, 94, 95, 96, 97, 98]

Problem simulations in GR fall into two solution streams. The first comprises of 'easier' problems with known metrics. The numerical simulation difficulty is introduced when deciding on an appropriate numerical method. The second, more difficult, stream comprises of problems where the metric is to be determined. This stream requires more robust methods, with a high regard for computational efficiency and a generous availability of computational power.

7.2.1 Runge-Kutta methods

The Runge-Kutta (RK) methods were formulated around 1990, by two well-known German mathematicians Carl Runge and Wilhelm Kutta, that use Euler's method as a foundation for initial value problems. This method begins by looking to solve a system of first order differential equations described by

$$\dot{\mathbf{U}} = \mathbf{F}(\mathbf{U})$$

where $\mathbf{F}$ be a vector function. Let $\mathbf{U}_0$ be the initial value vector at an initial time t_0. Let n denote the step number in our scheme, $\mathbf{U}_n$ represent the solution vector at step n, δt be the step size in t and $\mathbf{k_1}$ to $\mathbf{k_4}$ be constant vectors calculated locally in each iteration. Then, the RK4 method is defined by

$$\begin{aligned}
\mathbf{k_1} &= \delta t \mathbf{F}(\mathbf{U}_n) \\
\mathbf{k_2} &= \delta t \mathbf{F}\left(\mathbf{U}_n + \frac{1}{2}\mathbf{k_1}\right) \\
\mathbf{k_3} &= \delta t \mathbf{F}\left(\mathbf{U}_n + \frac{1}{2}\mathbf{k_2}\right) \\
\mathbf{k_4} &= \delta t \mathbf{F}\left(\mathbf{U}_n + \frac{1}{2}\mathbf{k_3}\right) \\
\mathbf{U}_{n+1} &= \mathbf{U}_n + \frac{1}{6}(\mathbf{k_1} + 2\mathbf{k_2} + 2\mathbf{k_3} + \mathbf{k_4})
\end{aligned}$$

where $i = 0, 1, 2, \cdots, N$ and $N = t_{\text{final}} \div \delta t$. The RK methods generalise conveniently, even in higher orders, are particularly useful in GR when simulating solutions for the following two cases:

(i) Finding trajectories of photons and particles, by solving the geodesic equations, to investigate phenomena like curvature lensing and perihelion motion, respectively. This is accomplished by letting

$$v^{\lambda} = \dot{x}^{\lambda} \tag{7.15}$$

in the geodesic equations, described in equation (3.2), to obtain the following system of first order differential equations

$$\frac{d}{dt}\begin{pmatrix} x^{\lambda} \\ v^{\lambda} \end{pmatrix} = \begin{pmatrix} v^{\lambda} \\ -\Gamma^{\lambda}{}_{\mu\nu} v^{\mu} v^{\nu} \end{pmatrix}$$
$$\frac{d\mathbf{U}}{dt} = \mathbf{F}(\mathbf{U})$$

where $\lambda = 0, 1, 2, 3$, as such the vector $\mathbf{U}$ is an array of eight entries.

(ii) To solve the EFEs in one dimension, where all governing equations can be written systematically and solved accordingly, i.e. the system of differential equations must have the form

$$\frac{d\mathbf{U}}{dt} = \mathbf{F}(\mathbf{U}).$$

When higher order differential equations are encountered, one would make use of similar transformations to equation (7.15).

7.2.2 Symplectic integrator method and the Hamiltonian

The symplectic integrator method is one of the canonical integration techniques derived in 1983 by Roland Ruth. Unlike the RK methods, this scheme conserves the area of the phase space plot which ultimately bounds the error of numerical computation. This method is used to solve Hamiltonian systems, making it a suitable scheme for modelling GR in one dimension. We recall that the geodesic equations and the Euler-Lagrange equations are equivalent when defining the Lagrangian as

$$L = g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu}.$$

The Legendre transformation of the Lagrangian yields the Hamiltonian

$$H = p_{\mu} \dot{x}^{\mu} - L \tag{7.16}$$

where p_{μ} is the conjugate or canonical momentum in general, defined by

$$p_{\mu} = \frac{\partial L}{\partial \dot{x}^{\mu}}.$$

Alternatively, one may choose to define the Hamiltonian as

$$H = \frac{1}{2} g^{\mu\nu} p_{\mu} p_{\nu}. \tag{7.17}$$

Both equations (7.16) and (7.17) are valid forms of the Hamiltonian and are governed by the following evolution equations:

$$\dot{x}^{\mu} = \frac{\partial H}{\partial p_{\mu}} \tag{7.18}$$
$$\dot{p}_{\mu} = -\frac{\partial H}{\partial x^{\mu}}. \tag{7.19}$$

The symplectic integrator method uses equations (7.18) and (7.19), with four sub-iterations (in the fourth order case) for each time step iteration. Let both $x^{\mu}(n,m)$ and $p_{\mu}(n,m)$ be functions in m and n, where the value of $m = 1,2,3,4$ for each value of n. The value of n, where $n = 0,1,2,\cdots,N$, denotes the step number taken from the initial time t_0, in step sizes of δt, up until some final time t_f and $N = \frac{t_f - t_0}{\delta t}$. Then, the fourth-order symplectic integrator method is defined as

$$
\begin{aligned}
p_{\mu}(n,2) &= p_{\mu}(n,1) - c_1 \delta t \left.\frac{\partial H}{\partial x^{\mu}}\right|_{x^{\mu}=x^{\mu}(n,1)} \\
x^{\mu}(n,2) &= x^{\mu}(n,1) + d_1 \delta t \left.\frac{\partial H}{\partial p_{\mu}}\right|_{p_{\mu}=p_{\mu}(n,2)} \\
p_{\mu}(n,3) &= p_{\mu}(n,2) - c_2 \delta t \left.\frac{\partial H}{\partial x^{\mu}}\right|_{x^{\mu}=x^{\mu}(n,2)} \\
x^{\mu}(n,3) &= x^{\mu}(n,2) + d_2 \delta t \left.\frac{\partial H}{\partial p_{\mu}}\right|_{p_{\mu}=p_{\mu}(n,3)} \\
p_{\mu}(n,4) &= p_{\mu}(n,3) - c_3 \delta t \left.\frac{\partial H}{\partial x^{\mu}}\right|_{x^{\mu}=x^{\mu}(n,3)} \\
x^{\mu}(n,4) &= x^{\mu}(n,3) + d_3 \delta t \left.\frac{\partial H}{\partial p_{\mu}}\right|_{p_{\mu}=p_{\mu}(n,4)} \\
p_{\mu}(n+1,1) &= p_{\mu}(n,4) - c_4 \delta t \left.\frac{\partial H}{\partial x^{\mu}}\right|_{x^{\mu}=x^{\mu}(n,4)} \\
x^{\mu}(n+1,1) &= x^{\mu}(n,4) + d_4 \delta t \left.\frac{\partial H}{\partial p_{\mu}}\right|_{p_{\mu}=p_{\mu}(n+1,1)}
\end{aligned}
$$

where

$$
\begin{aligned}
c_1 = c_3 &= \frac{1}{2-2^{1/3}}, & c_2 &= \frac{-2^{1/3}}{2-2^{1/3}}, \quad c_4 = 0 \\
d_1 = d_4 &= \frac{1}{2\left(2-2^{1/3}\right)}, & d_2 = d_3 &= \frac{1-2^{1/3}}{2\left(2-2^{1/3}\right)}.
\end{aligned}
$$

The solution vectors are given by $p_{\mu}(n,1)$ and $x^{\mu}(n,1)$ for $n = 1,2,3,\cdots,N$, as values obtained when $m = 2,3,4$ are sub-iterations of each n iteration.

7.2.3 Applications to hydrodynamics with given metric

This approach is still considered to be one of the less complex numerical problems in GR as the metric is prescribed. It is different to the RK and symplectic methods as it involves a perturbation component, making it a bit more complex. The idea behind this approach, is to solve the hydrodynamic equations

$$
\begin{aligned}
\left(\rho u^{\mu}\right)_{;\mu} &= 0 \\
T^{\mu\nu}{}_{;\nu} &= 0
\end{aligned}
$$

with a given metric tensor. It requires knowledge of hyperbolic conservation laws coupled with appropriate numerical method, and form part of Riemann and approximate Riemann solvers. The benefit of this application is the abundance of numerical methods for solving differential equations of Hyperbolic conservation form and is particularly used to model plasmas and gasses around gravity sources, forming part of the broader field of General Relativistic Magneto Hydrodynamics. It is of great importance to develop appropriate numerical methods that are reliable in the extreme cases. A good example of this is the simulation of a Kerr black hole as depicted in Interstellar.

7.2.4 Perturbation approach

This approach is considered to be part of the complex numerical approaches in Numerical GR. It focusses on 'detecting' effects of other gravitational sources on the background metric, $H_{\mu\nu}(n)$, contributing to the metric of the gravitational body, $h_{\mu\nu}(n)$, making up the metric of the local system defined by

$$g_{\mu\nu}(n) = H_{\mu\nu}(n) + h_{\mu\nu}(n)$$

i.e. one works from some known manifold, $G^{\mu\nu}$, of the EFEs

$$G^{\mu\nu} = 8\pi T^{\mu\nu}.$$

These effects are incorporated in the governing equations and the following system of equations in n are utilised for each step until some relative error is satisfied.

$$\begin{aligned} \Box h^{\mu\nu}(n+1) &= KT^{\mu\nu}(n) \\ T^{\mu\nu}{}_{;\nu}(n+1) &= 0 \end{aligned}$$

This approach is computationally expensive and in each time step the number of sub-iterations will vary in general. For simple problems this is a very powerful approach, but in more complex problem solving there are other less computationally expensive approaches available.

7.2.5 ADM formalism

The $3+1$ formalism, section 5.4, is derived without specifying a particular choice of basis. The ADM formalism begins by making a choice of basis such that

$$\Omega_\mu e^\mu{}_i = -\frac{1}{\alpha} n_\mu e^\mu{}_i = 0$$

thus spatial components of the normal vector must vanish, i.e. $n_i = 0$. The fourth basis vector is determined by satisfying the duality condition, $t^\mu \Omega_\mu = 1$, and is defined to be

$$t^\mu = e^\mu{}_0 = (1,0,0,0).$$

Using Lie dragging along t^μ the spatial vectors are extended to neighbouring slices,

$$\mathscr{L}_t e^\mu{}_i = 0$$

and it may be shown that the spatial vector $e^\mu{}_i$ remains spatial, once Lie dragged along t^μ

$$\mathscr{L}_t\left(\Omega_\mu e^\mu{}_i\right) = 0.$$

It is expected that any spatial vector contracted with the normal vector will vanish, thus

$$n_\mu \beta^\mu = n_0 \beta^0 = 0$$

and we may define

$$\beta^\mu = \left(0, \beta^i\right).$$

We also have that

$$n^\mu = \left(\frac{1}{\alpha}, -\frac{\beta^i}{\alpha}\right)$$

and

$$n_\mu = (\alpha, 0, 0, 0)$$

from the normalisation condition, $n_\mu n^\mu = -1$. Therefore, we may define a metric in the $3+1$ form to be

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}\left(dx^i + \beta^j dt\right)\left(dx^j + \beta^i dt\right)$$

where $\alpha^2 dt^2$ is considered to be the proper time between neighbouring spatial hyper-slices and $\gamma_{ij}\left(dx^i + \beta^j dt\right)\left(dx^j + \beta^i dt\right)$ is considered to be the proper distance with a spatial hyper-slice. The EFEs are decomposed into the constraint and evolution equations, which are defined in the 3 + 1 formalism of chapter 5.4. First we define the Hamiltonian constraint equation

$$R + K^2 - K_{ij}K^{ij} = \frac{16\pi G}{c^2} n_\kappa n_\lambda T^{\kappa\lambda}$$

and momentum constraint equations

$$D_j\left(K^{ij} - \gamma^{ij}K\right) = -\frac{8\pi G}{c^2}\gamma^{ij} n^\nu T_{\nu j}.$$

Secondly, the extrinsic curvature evolution equation is defined by

$$\begin{aligned}\frac{\partial K_{ij}}{\partial t} =& -D_i D_j \alpha - \frac{8\pi G}{c^2}\alpha\left(S_{ij} - \frac{1}{2}\gamma_{ij}\left(\gamma^j{}_\kappa \gamma_{j\lambda} T^{\kappa\lambda} - n_\kappa n_\lambda T^{\kappa\lambda}\right)\right)\\ &+ \alpha\left(R_{ij} - 2K_{ik}K^k{}_j + KK_{ij}\right) + \beta^k \frac{\partial}{\partial x^k} K_{ij} + K_{ik}\frac{\partial}{\partial x^j}\beta^k + K_{kj}\frac{\partial}{\partial x^i}\beta^k\end{aligned}$$

and the spatial metric evolution equation is by

$$\frac{\partial \gamma_{ij}}{\partial t} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i.$$

Other useful contracted forms of the evolution equations are given by

$$\begin{aligned}\frac{\partial}{\partial t}\ln\left(\gamma^{\frac{1}{2}}\right) &= -\alpha K + D_i\beta^i\\ \frac{\partial K}{\partial t} &= -\gamma^{ij}D_iD_j\alpha + \alpha K_{ij}K^{ij} + \frac{4\pi G\alpha}{c^2}\left(n_\kappa n_\lambda T^{\kappa\lambda} + \gamma^j{}_\kappa\gamma_{j\lambda}T^{\kappa\lambda}\right) + \beta^i D_i K.\end{aligned}$$

The above system of equations is known as the 3 + 1 ADM formalsim. Many articles describing attempts to use this numerical scheme have described it to be unstable. In order to use this numerical scheme, one should investigate the use of the following:

(i) **Conformal mapping**

These transformations are local orientation and angle preserving, even in the case of curved manifolds. This is what makes it particularly useful when using numerical methods on curved space-time manifolds.

(ii) **Mesh refinement techniques**

As the name suggests, the mesh/step-size in a numerical simulation is refined to study a model's evolution more precisely. From experience with numerical methods, one would be aware that choice of step size is important when performing finite element analysis (FEA). If one is to choose either too small or too large a step-size, the results of the FEA are not reliable. Often the problem one encounters is related to needing to further refine the step-size/mesh, however this does come with an increase in computational cost. The finer your mesh/step-size, the more computational power and time is required to run the simulation. As such, in an attempt to optimise the computational and time cost of a simulation, mesh refinement methods have been developed. These methods aim to refine meshes where the refinement is required and leave the mesh less refined in areas where model behaviour is of less interest.

This does however, require in-depth knowledge of the physics of the system, for a strategic implementation of mesh refinement where necessary. One of the most popular mesh refinements is that of adaptive mesh refinement where the refinement of the mesh is implemented where it is needed in a very systematic (as such automated) fashion.

For a deeper context consider [98], where the binary black hole problem is addressed using the ADM formalism with conformal mapping and a mesh refinement method.

7.2.6 Simple numerical relativity simulation

We introduce the fourth-order RK method for application of FLRW universe solutions. Let us recall the FLRW model, section 6.4, and the system of coupled differential equations (6.32), (6.33) and (6.34) from the previous chapter. Now there are two unknown functions and three differential equations in our system of equations. Let us consider Euler's method to decrease the order of the differential equation (6.34) and consequently arrive at a system of first order, coupled differential equations and one constraint equation. Let

$$h(t) = a'(t) \tag{7.20}$$

then equations (6.32) and (6.34), respectively, become

$$\rho'(t) = -\big(P(t) + \rho(t)\big)\, h(t) a(t)^{-1} \tag{7.21}$$

$$h'(t) = -(2a(t))^{-1}\left(8\pi a(t)^2 P(t) + h(t)^2 + k\right) \tag{7.22}$$

subject to

$$h(t) = \pm\sqrt{\frac{8\pi}{3} a(t)^2 \rho(t) - k} \tag{7.23}$$

from equation (6.33). Another function of interest is the Hubble parameter, $H(t)$, defined by

$$H(t) = \frac{h(t)}{a(t)}$$

describing the rate of of expansion/collapse of the universe.

Closed radiation dominated universe

Assume these equations describe a universe that is radiation dominated, such that the pressure may be expressed as

$$P(t) = \frac{1}{3}\rho(t).$$

Substituting this result into equation (6.32) we have

$$\begin{aligned}
\frac{\rho'(t)}{\rho(t)} &= -4\frac{a'(t)}{a(t)} \\
\frac{d}{dt}\ln\rho(t) &= -\frac{d}{dt}4\ln a(t) \\
\frac{d}{dt}\left(\ln\rho(t) + \ln a(t)^4\right) &= 0 \\
\rho(t) a(t)^4 &= C_1
\end{aligned}$$

describing that there must be conservation of mass density in the universe. Therefore

$$\rho(t) = C_1 a(t)^{-4} \tag{7.24}$$

and now have the following system of equations to solve numerically using a Runge-Kutta fourth order method

$$\frac{d\mathbf{U}}{dt} = F(\mathbf{U})$$
$$\frac{d}{dt}\begin{pmatrix} a(t) \\ h(t) \end{pmatrix} = \begin{pmatrix} h(t) \\ -\frac{8\pi}{3}a(t)^5 C_1 \end{pmatrix}.$$

Two plots will be considered, $a(t)$ and $H(t)$, but initial conditions have to be decided on. From equation (7.23) at $t = 0$ we have

$$h_0 = \pm\sqrt{\frac{8\pi}{3}C_1 {a_0}^6 - 1}$$

which is only real when

$$C_1 \geq \frac{3}{8\pi {a_0}^6}$$

and the whole model is only valid for $a(t) \neq 0$ and so $a_0 > 0$. Using the RK4 method, going forward and backward in time, figure 7.1 depicts the expansion coefficient and rate of expansion coefficient functions from the beginning until the end of a closed universe.

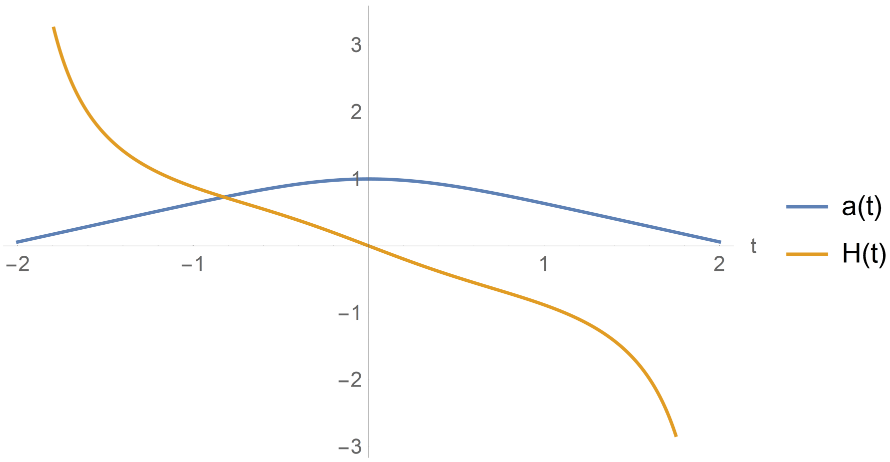


Figure 7.1: Expansion coefficient function and rate of expansion coefficient function over time of a closed universe

Exercise 9. *Using the RK4 method, analyse the model of an FLRW Dust filled/matter dominated universe, where the pressure may be expressed as*

$$P(t) = 0.$$

See solution 9.

Exercise 10. *Using the RK4 method, analyse the model of an FLRW Real gas universe, where the pressure may be expressed as*

$$P(t) = \rho(t)^\gamma.$$

See solution 10.

Chapter 8

Conclusion

In conclusion, this text ends with a chapter dedicated to listing conclusions of each chapter and assessing their significance.

Chapter 1

This chapter is concerned with setting a foundation of tensor algebra and calculus needed for the rest of the text. Curvilinear coordinates are used as a means of naturally extending flat Riemannian space to a curved pseudo-Riemannian manifold. The reader is introduced to the concept of covariant and contravariant tensors of various orders, including the metric and alternating tensors, as well as different tensor properties, like symmetry and antisymmetry of tensors. The first section concludes with the vector product and surface element, and the scalar triple product and volume element. Section two introduces differentiation on curved manifolds, the Riemann-Christoffel curvature tensor and the Weyl tensor before finishing up with the physical components of tensors.

Chapter 2

In Chapter two we address physical coordinate systems. It begins with the Eulerian formalism where a global frame of reference with a preferred time frame of reference is derived. The second section derives the Lagrangian formalism where local frame of reference is set up, following the motion of the material element, assuming a proffered space frame of reference. The chapter ends off with a derivation of the equations of fluid mechanics.

Chapter 3

Chapter three is focussed on the introduction of geodesics derived from three approaches: a variational principle, parallel transport along a curve, and Dirac's approach of immersing a four dimensional space-time surface in a higher dimensional flat space. The reader is introduced to geodesics as they are of particular importance in GR, describing the paths of photons and free particles in a space-time.

Chapter 4

This chapter is concerned with stating and briefly describing the six principles of GR: Mach's principle, equivalence principle, covariance principle, correspondence principle, simplicity principle and cosmological principle. Concluding the chapter, emphasis is placed on Einstein's true intention of unifying inertia and gravity. The need for this emphasis has stemmed from the naive regard of gravity as geometry over the past few decades.

Chapter 5

In Chapter five the representations of GR is covered, based on the principles covered in Chapter four. It begins with the introduction of the energy-momentum tensor and a derivation of the conservation laws for energy and momentum for a perfect fluid, based on the equations of fluid mechanics in Chapter two. A derivation of the EFEs is outlined, with equivalent forms presented. The 1 + 3 threading of space-time, the Lagrangian representation of GR, is supplied and analysed in detail. All properties of motion experienced by each independent observer, as represented by the choice of co-moving four-velocity, with respect to each other observer frames. It may be viewed as the Lagrangian representation of GR and is analogous to the Lagrangian formalism of fluid mechanics. Next, the 3 + 1 foliation of space-time is considered which allows a global time approach for space-like hypersurfaces at each time step. This formalism corresponds to the Eulerian formulation of the fluid mechanics. The framework is introduced as the ADM formalism, a Hamiltonian formulation of GR. The chapter is closed off with the Einstein-Hilbert Palantini formulation of the least action principle, and the tetrad formalism of cosmology. Orthonormal tetrads are set up in accordance with local observations, illustrating the truly relational nature of GR. In all representation, the propagation and constraint equations for the rate of expansion, rate of shear, vorticity and the electric and magnetic parts of the Weyl tensor are presented.

Chapter 6

Chapter six is focussed on exact and analytical solutions to the EFEs. It begins by introducing the Minkowski space-time of special relativity to familiarise the reader with the nature of calculations once GR has been applied to a physical problem. The Schwarzschild exterior solution and the Tolman, Oppenheimer, Volkoff interior solution for a static spherically symmetric body are considered next. The geodesic equations of the Schwarzschild space-time are solved to produce polar plots for the trajectory of a particle modelling the advance of perihelion of Mercury and photon modelling in the bending of light. A Kerr space-time is considered next, where a spherically axially-symmetric, rotating black hole solution is discussed. The static limit horizon and proper event horizon are discussed, and two applications of the Kerr solution are referenced. Closing off the chapter, the Friedmann-Lemaitre-Robertson-Walker cosmology solution is introduced and discussed. An application to a radiation dominated universe is supplied, illustrating the open, flat and closed universe cases of this model.

Chapter 7

Approximate solutions of the EFEs are considered in this chapter. First, an analytical solution of the perturbed Minkowski metric for the derivation of a plane gravitational wave solution. The second section is aimed at introducing the reader to the two numerical approaches of relativity. The first scheme is the solution for a known metric tensor, and the second, more difficult, scheme is to be applied in order to determine the metric tensor. This section covers Runge-Kutta methods, the symplectic integrator method and the Hamiltonian, Applications to hydrodynamics with given metric, numerical perturbation approach and the ADM formalism. The section concludes with a simple numerical relativity simulation of a closed, Friedmann-Lemaitre-Robertson-Walker, radiation dominated universe.

Appendices

Appendix A

Exercise solutions

Solution 1. *(for exercise 1)*
Recall equation (1.3) from definition 4, and multiply both sides by $\frac{\partial\theta^i}{\partial x^m}$ *to obtain*

$$\mathbf{e}_m = \frac{\partial\theta^i}{\partial x^m}\mathbf{g}_i \tag{A.1}$$

and in a similar fashion you may obtain

$$\mathbf{e}^m = \frac{\partial x^m}{\partial\theta^i}\mathbf{g}^i \tag{A.2}$$

which will both be useful in the following proofs.

Proof.

(i)

$$\begin{aligned} g &= \left|g_{ij}\right| \\ &= e^{ijk} g_{i1} g_{j2} g_{k3} \end{aligned}$$

but

$$g_{ij} = \frac{\partial x^r}{\partial\theta^i}\frac{\partial x^s}{\partial\theta^j}\delta_{rs}.$$

Thus

$$\begin{aligned} g &= e^{ijk}\frac{\partial x^a}{\partial\theta^i}\frac{\partial x^b}{\partial\theta^1}\delta_{ab}\frac{\partial x^c}{\partial\theta^j}\frac{\partial x^d}{\partial\theta^2}\delta_{cd}\frac{\partial x^r}{\partial\theta^k}\frac{\partial x^s}{\partial\theta^3}\delta_{rs} \\ &= \left(e^{ijk}\frac{\partial x^a}{\partial\theta^i}\frac{\partial x^c}{\partial\theta^j}\frac{\partial x^r}{\partial\theta^k}\right)\frac{\partial x^b}{\partial\theta^1}\frac{\partial x^d}{\partial\theta^2}\frac{\partial x^s}{\partial\theta^3}\delta_{ab}\delta_{cd}\delta_{rs} \\ &= \left|\frac{\partial x^n}{\partial\theta^m}\right| e^{acr}\frac{\partial x^b}{\partial\theta^1}\frac{\partial x^d}{\partial\theta^2}\frac{\partial x^s}{\partial\theta^3}\delta_{ab}\delta_{cd}\delta_{rs} \\ &= \left|\frac{\partial x^n}{\partial\theta^m}\right| e_{bds}\frac{\partial x^b}{\partial\theta^1}\frac{\partial x^d}{\partial\theta^2}\frac{\partial x^s}{\partial\theta^3} \\ &= \left|\frac{\partial x^n}{\partial\theta^m}\right|\left|\frac{\partial x^r}{\partial\theta^s}\right| \\ &= \left|\frac{\partial x^n}{\partial\theta^m}\right|^2. \end{aligned}$$

(ii)

$$
\begin{aligned}
\left|g^{nm}\right| &= e_{ijk} g^{i1} g^{j2} g^{k3} \\
&= e_{ijk} \frac{\partial \theta^i}{\partial x^a} \frac{\partial \theta^1}{\partial x^b} \delta^{ab} \frac{\partial \theta^j}{\partial x^c} \frac{\partial \theta^2}{\partial x^d} \delta^{cd} \frac{\partial \theta^k}{\partial x^r} \frac{\partial \theta^3}{\partial x^s} \delta^{rs}
\end{aligned}
$$

using a similar argument to part (i) of this proof, we obtain

$$
\begin{aligned}
\left|g^{nm}\right| &= \left|\frac{\partial \theta^p}{\partial x^q}\right| \left|\frac{\partial \theta^r}{\partial x^s}\right| \\
&= \left|\frac{\partial \theta^r}{\partial x^s}\right|^2
\end{aligned}
$$

Also

$$
\frac{\partial \theta^n}{\partial x^r} \frac{\partial x^r}{\partial \theta^m} = \delta^n{}_m
$$

Take the determinant of both sides of the above equation and use the determinant identity $|AB| = |A||B|$ to obtain

$$
\left|\frac{\partial \theta^n}{\partial x^r}\right| \left|\frac{\partial x^r}{\partial \theta^m}\right| = \left|\delta^n{}_m\right|
$$

but $|\delta^n{}_m| = 1$. Thus

$$
\left|\frac{\partial \theta^n}{\partial x^r}\right| = \frac{1}{\left|\frac{\partial x^r}{\partial \theta^m}\right|} = \frac{1}{\sqrt{g}}
$$

and hence

$$
\left|g^{nm}\right| = \frac{1}{g} = \left|\frac{\partial \theta^r}{\partial x^s}\right|^2 .
$$

(iii)

$$
\begin{aligned}
\varepsilon_{ijk} &= \frac{\partial x^r}{\partial \theta^i} \frac{\partial x^s}{\partial \theta^j} \frac{\partial x^t}{\partial \theta^k} e_{rst} \\
&= \left|\frac{\partial x^n}{\partial \theta^m}\right| e_{ijk} \\
&= \sqrt{g}\, e_{ijk}
\end{aligned}
$$

(iv)

$$
\begin{aligned}
\varepsilon^{ijk} &= \frac{\partial \theta^i}{\partial x^r} \frac{\partial \theta^j}{\partial x^s} \frac{\partial \theta^k}{\partial x^t} e^{rst} \\
&= \left|\frac{\partial \theta^n}{\partial x^m}\right| e^{ijk} \\
&= \frac{1}{\sqrt{g}} e^{ijk}
\end{aligned}
$$

□

Solution 2. *(for exercise 2)*

Proof. This proof is accomplished by determining the following equivalent form of the Christofel symbol of second kind, under the coordinate transformation $\theta^i \to \overline{\theta}^i$.

$$\overline{\Gamma}^i{}_{jk} = \overline{\theta}^i{}_{,m} \theta^n{}_{,j} \theta^p{}_{,k} \Gamma^m{}_{np} + \theta^m{}_{,jk} \overline{\theta}^i{}_{,m} \tag{A.3}$$

The above equation, equation (A.3), has the term $\theta^m{}_{,jk} \overline{\theta}^i{}_{,m}$ which would not be present if the Christoffel symbol of second kind were a tensor (a condition of the from the quotient theorem, theorem 1). We begin by recalling equation (1.26), thus

$$\begin{aligned}
\overline{\Gamma}^k{}_{ij}\, \overline{\mathbf{g}}_k &= \overline{\mathbf{g}}_{i,j} \\
&= \left(\theta^m{}_{,i}\, \mathbf{g}_m\right)_{,j} \\
&= \theta^m{}_{,ij}\, \mathbf{g}_m + \theta^m{}_{,i}\, \mathbf{g}_{m,n}\, \theta^n{}_{,j} \\
&= \theta^m{}_{,ij}\, \mathbf{g}_m + \theta^m{}_{,i}\, \theta^n{}_{,j}\, \Gamma^s{}_{mn}\, \mathbf{g}_{m,n} \\
&= \theta^m{}_{,ij}\, \overline{\theta}^k{}_{,m}\, \overline{\mathbf{g}}_k + \theta^m{}_{,i}\, \theta^n{}_{,j}\, \Gamma^s{}_{mn}\, \overline{\theta}^k{}_{,s}\, \overline{\mathbf{g}}_k
\end{aligned}$$

therefore

$$\left[\overline{\Gamma}^k{}_{ij} - \theta^m{}_{,ij}\, \overline{\theta}^k{}_{,m} - \theta^m{}_{,i}\, \theta^n{}_{,j}\, \overline{\theta}^k{}_{,s}\, \Gamma^s{}_{mn}\right] \overline{\mathbf{g}}_k = 0$$

yielding equation (A.3). □

Solution 3. *(for exercise 3)*

Proof. (ii)

$$\begin{aligned}
g^i_{j\,;\,k} &= \delta^i_{j\,;\,k} \\
&= \delta^i_{j\,,\,k} + \Gamma^i{}_{kt}\, \delta^t_j - \Gamma^t{}_{jk}\, \delta^i_t \\
&= 0 + \Gamma^i{}_{kj} - \Gamma^i{}_{jk} \\
&= 0
\end{aligned}$$

(iii)

$$\begin{aligned}
\delta^i_k &= g^{ij}\, g_{jk} \\
\delta^i_{k;r} &= \left(g^{ij}\, g_{jk}\right)_{;\,r} \\
0 &= g^{ij}{}_{;\,r}\, g_{jk} + g^{ij}\, g_{jk\,;\,r} \\
0 &= g_{jk}\, g^{ij}{}_{;\,r} \\
g^{ks}\,(0) &= g^{ks}\, g_{jk}\, g^{ij}{}_{;\,r} \\
0 &= \delta^s_j\, g^{ij}{}_{;\,r} \\
0 &= \left(\delta^s_j\, g^{ij}\right)_{;\,r} \\
0 &= g^{is}{}_{;\,r}
\end{aligned}$$

□

Solution 4. *(for exercise 4)*
Required to prove

$$R^a{}_{dbc} + R^a{}_{bcd} + R^a{}_{cdb} = 0. \tag{A.4}$$

Proof.
Recall equation (1.36) and substitute it into the right-hand side of equation (A.4)

$$\begin{aligned} R^a{}_{dbc} + R^a{}_{bcd} + R^a{}_{cdb} = &\left(\Gamma^a{}_{dc,b} - \Gamma^a{}_{db,c} + \Gamma^t{}_{dc}\Gamma^a{}_{tb} - \Gamma^t{}_{db}\Gamma^a{}_{tc}\right) \\ &+ \left(\Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^t{}_{bd}\Gamma^a{}_{tc} - \Gamma^t{}_{bc}\Gamma^a{}_{td}\right) \\ &+ \left(\Gamma^a{}_{cb,d} - \Gamma^a{}_{cd,b} + \Gamma^t{}_{cb}\Gamma^a{}_{td} - \Gamma^t{}_{cd}\Gamma^a{}_{tb}\right) \end{aligned}$$

which vanishes, as a result of the symmetry property in the lower indices of the Christoffel symbols of second kind.

□

Solution 5. *(for exercise 5)*
The surface of a doughnut may by described by a torus of the toroidal coordinate system. The transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (R + r\cos(\theta))\cos(\phi) \\ (R + r\cos(\theta))\sin(\phi) \\ r\sin(\theta) \end{pmatrix}$$

where R is the distance from the center of the torus tube to the center of the torus, and r is the radius of the torus tube. In order to obtain a doughnut shape we must have that $R > r$. Therefore, the following equations are to be determined using Mathematica

$$\begin{aligned} \frac{\partial\rho}{\partial t} + \rho_{,i}u^i + \rho u^i{}_{;i} &= 0 \\ \frac{\partial\rho}{\partial t}u^i + \rho\frac{\partial u^i}{\partial t} + T^{ij}{}_{;j} &= -\rho g^{ij}\Phi_{,j} \\ \frac{\partial\rho}{\partial t}E + \rho\frac{\partial E}{\partial t} + \rho_{,i}Eu^i + \rho E_{,i}u^i + \rho E u^i{}_{;i} + p_{,i}u^i + p u^i{}_{;i} &= -\rho u^i\Phi_{,i} \end{aligned}$$

where

$$\begin{aligned} E &= \frac{1}{2}g_{ij}u^i u^j + \epsilon \\ T^{ij} &= \rho u^i u^j + p g^{ij}. \end{aligned}$$

The following is a possible Mathematica code for determining the above equations in toroidal coordinates:

```
a = b = aa = bb = {r, \[Theta], \[Phi]};
c = d = {x, y, z};
R[r_] := 2 r;
e[x] = eu[x] = {1, 0, 0};
e[y] = eu[y] = {0, 1, 0};
e[z] = eu[z] = {0, 0, 1};
R = 2 r;
Xn[x] = (R + r Cos[\[Theta]]) Cos[\[Phi]];
Xn[y] = (R + r Cos[\[Theta]]) Sin[\[Phi]];
Xn[z] = r Sin[\[Theta]];
Do[g[i] = Sum[D[Xn[n], i] e[n], {n, c}], {i, a}];
gL[i_, j_] := g[i].g[j] // FullSimplify
GUT = Table[gL[i, j], {i, a}, {j, b}];
GLT = Inverse[GUT];
```

```
Do[gU[a[[i]], b[[j]]] = GLT[[i, j]], {i, 1, 3}, {j, 1, 3}];
\[CapitalGamma]1[i_, j_, k_] :=
 1/2 (D[gL[i, j], k] + D[gL[i, k], j] - D[gL[j, k], i]) // FullSimplify
\[CapitalGamma]2[i_, j_, k_] :=
 Sum[gU[i, l] \[CapitalGamma]1[l, j, k], {l, a}] // FullSimplify
Div0[v_, i_] := D[v, i]
Div1[U_, i_, j_] :=
 D[U[i], j] + Sum[\[CapitalGamma]2[i, k, j] U[k], {k, a}]
Div2[T_, i_, j_, k_] :=
 D[T[i, j], k] + Sum[\[CapitalGamma]2[i, l, k] T[l, j], {l, a}] +
  Sum[\[CapitalGamma]2[j, l, k] T[i, l], {l, a}]
Energy[t_, r_, \[Theta]_, \[Phi]_] :=
 Sum[1/2 gL[i, j] U[i] U[j] + \[Epsilon][t, r, \[Theta], \[Phi]], {i,
   a}, {j, b}]
T[i_, j_] := \[Rho][t] U[i] U[j] + p[t] gU[i , j]
EQN1 = D[\[Rho][t], t] +
    Sum[Div0[\[Rho][t], i] U[i], {i, a}] + \[Rho][t] Sum[
      Div1[U, i, i], {i, a}] == 0;
EQN2[i_] :=
 D[\[Rho][t] , t] U[i] + \[Rho][t]  D[U[i], t] +
   Sum[Div2[T, i, j, j] , {j, a}] == -\[Rho][t] Sum[
    gU[i, j] D[\[CapitalPhi][t, r, \[Theta], \[Phi]] , j], {j, a}]
EQN3 = D[\[Rho][t] , t] Energy[t, r, \[Theta], \[Phi]] + \[Rho][t]  D[
      Energy[t, r, \[Theta], \[Phi]] , t] +
    Sum[D[\[Rho][t] , i] Energy[t, r, \[Theta], \[Phi]]  U[i], {i,
      a}] + Sum[\[Rho][t] D[Energy[t, r, \[Theta], \[Phi]] , i] U[
       i], {i, a}] +
    Sum[\[Rho][t] Energy[t, r, \[Theta], \[Phi]] Div1[U, i, i], {i,
      a}] + Sum[D[p[t] , i] U[i], {i, a}] +
    Sum[p[t] Div1[U, i, i], {i, a}] == -\[Rho][t] Sum[
     U[i] D[\[CapitalPhi][t, r, \[Theta], \[Phi]] , i], {i, a}];
Print["The fluid equations on the surface of a doughnut are: "]
Print[EQN1] // FullSimplify
Do[Print[EQN2[i]], {i, a}] // FullSimplify
Print[EQN3] // FullSimplify
```

yielding the following system of equations:

$$\rho(t)\left(U'(r)+\frac{2U(r)}{r}+U'(\theta)+U'(\phi)-\frac{\sin(\theta)U(\theta)}{\cos(\theta)+2}-\frac{2\sin(\theta)U(\theta)}{2\cos(\theta)+1}\right)+\rho'(t)=0$$

$$
\begin{aligned}
&-\rho(t)\left(\frac{(4\cos(\theta)+5)\Phi^{(0,0,1,0)}(t,r,\theta,\phi)}{-4r^2\sin^2(\theta)+4r^2\cos(\theta)+5r^2}+\frac{2r\sin(\theta)\Phi^{(0,1,0,0)}(t,r,\theta,\phi)}{-4r^2\sin^2(\theta)+4r^2\cos(\theta)+5r^2}\right)\\
&= -\frac{2\sin(\theta)p(t)}{-4r^2\sin^2(\theta)+4r^2\cos(\theta)+5r^2}-\frac{2r\sin(\theta)p(t)\left(-8r\sin^2(\theta)+8r\cos(\theta)+10r\right)}{\left(-4r^2\sin^2(\theta)+4r^2\cos(\theta)+5r^2\right)^2}\\
&-\frac{(4\cos(\theta)+5)p(t)\left(-4r^2\sin(\theta)-8r^2\sin(\theta)\cos(\theta)\right)}{\left(-4r^2\sin^2(\theta)+4r^2\cos(\theta)+5r^2\right)^2}+\frac{\sin(\theta)(\cos(\theta)+2)\left(\frac{p(t)}{r^2(\cos(\theta)+2)^2}+\rho(t)U(\phi)^2\right)}{2\cos(\theta)+1}\\
&-\frac{\sin(\theta)\left(\frac{(4\cos(\theta)+5)p(t)}{-4r^2\sin^2(\theta)+4r^2\cos(\theta)+5r^2}+\rho(t)U(\theta)^2\right)}{\cos(\theta)+2}-\frac{4\sin(\theta)\left(\frac{(4\cos(\theta)+5)p(t)}{-4r^2\sin^2(\theta)+4r^2\cos(\theta)+5r^2}+\rho(t)U(\theta)^2\right)}{2\cos(\theta)+1}\\
&+\frac{4\left(\frac{2r\sin(\theta)p(t)}{-4r^2\sin^2(\theta)+4r^2\cos(\theta)+5r^2}+U(r)\rho(t)U(\theta)\right)}{r}\\
&+\rho(t)U(\theta)U'(r)+\rho(t)U(\theta)U'(\phi)+2\rho(t)U(\theta)U'(\theta)+U(\theta)\rho'(t)
\end{aligned}
$$

$$
\begin{aligned}
&-\rho(t)\left(\frac{r^2\Phi^{(0,1,0,0)}(t,r,\theta,\phi)}{-4r^2\sin^2(\theta)+4r^2\cos(\theta)+5r^2}+\frac{2r\sin(\theta)\Phi^{(0,0,1,0)}(t,r,\theta,\phi)}{-4r^2\sin^2(\theta)+4r^2\cos(\theta)+5r^2}\right)\\
&= -\frac{r^2p(t)\left(-8r\sin^2(\theta)+8r\cos(\theta)+10r\right)}{\left(-4r^2\sin^2(\theta)+4r^2\cos(\theta)+5r^2\right)^2}+\frac{2r\cos(\theta)p(t)}{-4r^2\sin^2(\theta)+4r^2\cos(\theta)+5r^2}+\frac{2rp(t)}{-4r^2\sin^2(\theta)+4r^2\cos(\theta)+5r^2}\\
&-\frac{2r\sin(\theta)p(t)\left(-4r^2\sin(\theta)-8r^2\sin(\theta)\cos(\theta)\right)}{\left(-4r^2\sin^2(\theta)+4r^2\cos(\theta)+5r^2\right)^2}-\frac{r(4\cos(\theta)+\cos(2\theta)+1)\left(\frac{p(t)}{r^2(\cos(\theta)+2)^2}+\rho(t)U(\phi)^2\right)}{4\cos(\theta)+2}\\
&-\frac{r\left(\frac{(4\cos(\theta)+5)p(t)}{-4r^2\sin^2(\theta)+4r^2\cos(\theta)+5r^2}+\rho(t)U(\theta)^2\right)}{2\cos(\theta)+1}-\frac{\sin(\theta)\left(\frac{2r\sin(\theta)p(t)}{-4r^2\sin^2(\theta)+4r^2\cos(\theta)+5r^2}+U(r)\rho(t)U(\theta)\right)}{\cos(\theta)+2}\\
&-\frac{2\sin(\theta)\left(\frac{2r\sin(\theta)p(t)}{-4r^2\sin^2(\theta)+4r^2\cos(\theta)+5r^2}+U(r)\rho(t)U(\theta)\right)}{2\cos(\theta)+1}+\frac{2\left(\frac{r^2p(t)}{-4r^2\sin^2(\theta)+4r^2\cos(\theta)+5r^2}+U(r)^2\rho(t)\right)}{r}\\
&+U(r)\rho(t)U'(\theta)+U(r)\rho(t)U'(\phi)+2U(r)\rho(t)U'(r)+U(r)\rho'(t)
\end{aligned}
$$

$$
\begin{aligned}
-\frac{\rho(t)\Phi^{(0,0,0,1)}(t,r,\theta,\phi)}{r^2(\cos(\theta)+2)^2} = {}&\rho(t)U(\phi)U'(r)+\frac{4U(r)\rho(t)U(\phi)}{r}\\
&+\rho(t)U(\phi)U'(\theta)+2\rho(t)U(\phi)U'(\phi)\\
&-\frac{3\sin(\theta)\rho(t)U(\theta)U(\phi)}{\cos(\theta)+2}-\frac{2\sin(\theta)\rho(t)U(\theta)U(\phi)}{2\cos(\theta)+1}+U(\phi)\rho'(t)
\end{aligned}
$$

$$
\begin{aligned}
&-\rho(t)\left(U(\phi)\Phi^{(0,0,0,1)}(t,r,\theta,\phi)\right.\\
&\left.+U(\theta)\Phi^{(0,0,1,0)}(t,r,\theta,\phi)+U(r)\Phi^{(0,1,0,0)}(t,r,\theta,\phi)\right)
\end{aligned}
=
\begin{aligned}
&\rho(t)U(\phi)\left(9\epsilon^{(0,0,0,1)}(t,r,\theta,\phi)+r^2(\cos(\theta)+2)^2U(\phi)U'(\phi)\right)\\
&+\rho(t)U(\theta)\left(9\epsilon^{(0,0,1,0)}(t,r,\theta,\phi)+r^2U(\theta)U'(\theta)\right.\\
&+r^2\sin(\theta)(-(\cos(\theta)+2))U(\phi)^2-2r\sin(\theta)U(r)U'(\theta)\\
&\left.-2\sin(\theta)U(r)^2-2r\cos(\theta)U(r)U(\theta)\right)\\
&+\rho(t)U'(r)\left(9\epsilon(t,r,\theta,\phi)+\frac{1}{2}r^2(\cos(\theta)+2)^2U(\phi)^2\right.\\
&\left.+\frac{1}{2}r^2U(\theta)^2-2r\sin(\theta)U(r)U(\theta)+\frac{1}{2}(4\cos(\theta)+5)U(r)^2\right)\\
&+\rho(t)\left(\frac{U(r)}{r}+U'(\theta)-\frac{2\sin(\theta)U(\theta)}{2\cos(\theta)+1}\right)\left(9\epsilon(t,r,\theta,\phi)\right.\\
&+\frac{1}{2}r^2(\cos(\theta)+2)^2U(\phi)^2+\frac{1}{2}r^2U(\theta)^2-2r\sin(\theta)U(r)U(\theta)\\
&\left.+\frac{1}{2}(4\cos(\theta)+5)U(r)^2\right)\\
&+\rho(t)\left(\frac{U(r)}{r}+U'(\phi)-\frac{\sin(\theta)U(\theta)}{\cos(\theta)+2}\right)\left(9\epsilon(t,r,\theta,\phi)\right.\\
&+\frac{1}{2}r^2(\cos(\theta)+2)^2U(\phi)^2+\frac{1}{2}r^2U(\theta)^2-2r\sin(\theta)U(r)U(\theta)\\
&\left.+\frac{1}{2}(4\cos(\theta)+5)U(r)^2\right)+\rho'(t)\left(9\epsilon(t,r,\theta,\phi)\right.\\
&+\frac{1}{2}r^2(\cos(\theta)+2)^2U(\phi)^2+\frac{1}{2}r^2U(\theta)^2-2r\sin(\theta)U(r)U(\theta)\\
&\left.+\frac{1}{2}(4\cos(\theta)+5)U(r)^2\right)+U(r)\rho(t)\left(9\epsilon^{(0,1,0,0)}(t,r,\theta,\phi)\right.\\
&-2r\sin(\theta)U(\theta)U'(r)+(4\cos(\theta)+5)U(r)U'(r)\\
&\left.+r(\cos(\theta)+2)^2U(\phi)^2+rU(\theta)^2-2\sin(\theta)U(r)U(\theta)\right)\\
&+9\rho(t)\epsilon^{(1,0,0,0)}(t,r,\theta,\phi)+p(t)\left(\frac{U(r)}{r}+U'(\phi)-\frac{\sin(\theta)U(\theta)}{\cos(\theta)+2}\right)\\
&+p(t)\left(\frac{U(r)}{r}+U'(\theta)-\frac{2\sin(\theta)U(\theta)}{2\cos(\theta)+1}\right)+p(t)U'(r)
\end{aligned}
$$

Solution 6. *(for exercise 6)*
Required to show that equation (3.1) may be written as

$$\ddot{x}^d+\Gamma^d{}_{ab}\dot{x}^a\dot{x}^b=0.$$

Proof. From equation (3.1), we have

$$\frac{d}{d\lambda}\left(\frac{\frac{d}{d\dot{x}^c}f(x^c,\dot{x}^c)}{2\sqrt{f(x^c,\dot{x}^c)}}\right)-\frac{\frac{d}{dx^c}f(x^c,\dot{x}^c)}{2\sqrt{f(x^c,\dot{x}^c)}}=0$$

$$\frac{d}{d\lambda}\left(\frac{\frac{d\left(g_{ab}\dot{x}^a\dot{x}^b\right)}{d\dot{x}^c}}{2\sqrt{f(x^c,\dot{x}^c)}}\right)-\frac{\frac{d\left(g_{ab}\dot{x}^a\dot{x}^b\right)}{dx^c}}{2\sqrt{f(x^c,\dot{x}^c)}}=0$$

but

$$
\begin{aligned}
\frac{d\left(g_{ab}\dot{x}^a\dot{x}^b\right)}{d\dot{x}^c}&=2g_{ac}\dot{x}^a,\\
\frac{d\left(g_{ab}\dot{x}^a\dot{x}^b\right)}{dx^c}&=g_{ab,c}\dot{x}^a\dot{x}^b
\end{aligned}
$$

and identifying λ with s we have

$$\sqrt{f(x^c, \dot{x}^c)} = \sqrt{\left(\frac{ds}{ds}\right)^2} \; = \; 1$$

and so

$$\frac{d}{ds}\left(g_{ac}\dot{x}^a\right) - \frac{1}{2}g_{ab,c}\dot{x}^a\dot{x}^b = 0.$$

Making use of the product rule for derivatives

$$\begin{aligned}
g_{ac}\ddot{x}^a + g_{ac,b}\dot{x}^b\dot{x}^a - \frac{1}{2}g_{ab,c}\dot{x}^a\dot{x}^b &= 0\\
g_{ac}\ddot{x}^a + \frac{1}{2}\left(2g_{ac,b}\right)\dot{x}^b\dot{x}^a - \frac{1}{2}g_{ab,c}\dot{x}^a\dot{x}^b &= 0\\
g_{ac}\ddot{x}^a + \frac{1}{2}\left(g_{ca,b} + g_{cb,a}\right)\dot{x}^b\dot{x}^a - \frac{1}{2}g_{ab,c}\dot{x}^a\dot{x}^b &= 0\\
g_{ac}\ddot{x}^a + \Gamma_{cab}\dot{x}^a\dot{x}^b &= 0.
\end{aligned}$$

Performing a contraction in the first term on indices a and c, and raising the index c in the second term

$$g^{cd}g_{ac}\ddot{x}^a + g^{cd}\Gamma_{cab}\dot{x}^a\dot{x}^b = 0$$

yields the differential equation for a geodesic

$$\ddot{x}^d + \Gamma^d{}_{ab}\dot{x}^a\dot{x}^b = 0.$$

□

Solution 7. *(for exercise 7)*
Required to prove

$$\mathbf{g}_\mu \cdot \mathbf{g}_{\nu,\sigma} = \Gamma_{\mu\nu\sigma} \tag{A.5}$$

Proof. Substitute equation (1.27) into the left-hand side of equation (A.5)

$$\mathbf{g}_\mu \cdot \mathbf{g}_{\nu,\sigma} = \mathbf{g}_\mu \cdot \left(\Gamma_{\lambda\nu\sigma}\,\mathbf{g}^\lambda\right) \; = \; \delta_\mu{}^\lambda\Gamma_{\lambda\nu\sigma} \; = \; \Gamma_{\mu\nu\sigma}.$$

□

Solution 8. *(for exercise 8)*
Show that

$$\begin{aligned}
\ddot{\xi}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{\xi}^c + \Gamma^a{}_{bc,d}\dot{x}^b\dot{x}^c\xi^d + \Gamma^a{}_{bc}\dot{\xi}^b\dot{x}^c = &\frac{d}{dt}\left(\dot{\xi}^a + \Gamma^a{}_{bc}\dot{x}^b\xi^c\right) - \Gamma^a{}_{bc,d}\xi^b\dot{x}^c\dot{x}^d\\
&- \Gamma^a{}_{bc}\xi^b\ddot{x}^c + \Gamma^a{}_{bc,d}\dot{x}^b\dot{x}^c\xi^d + \Gamma^a{}_{bc}\dot{\xi}^b\dot{x}^c.
\end{aligned} \tag{A.6}$$

Proof.

$$
\begin{aligned}
\text{RHS of equation (A.6)} = &\frac{d}{dt}\left(\dot{\xi}^a + \Gamma^a{}_{bc}\dot{x}^b\xi^c\right) - \Gamma^a{}_{bc,d}\xi^b\dot{x}^c\dot{x}^d \\
&- \Gamma^a{}_{bc}\xi^b\ddot{x}^c + \Gamma^a{}_{bc,d}\dot{x}^b\dot{x}^c\xi^d + \Gamma^a{}_{bc}\dot{\xi}^b\dot{x}^c \\
\\
=&\frac{d}{dt}\left(\dot{\xi}^a\right) + \Gamma^a{}_{bc}\dot{x}^b\frac{d}{dt}\left(\xi^c\right) \\
&\frac{d}{dt}\left(\Gamma^a{}_{bc}\right)\xi^c\dot{x}^b + \Gamma^a{}_{bc}\xi^c\frac{d}{dt}\left(\dot{x}^b\right) \\
&- \Gamma^a{}_{bc,d}\xi^b\dot{x}^c\dot{x}^d - \Gamma^a{}_{bc}\xi^b\ddot{x}^c \\
&+ \Gamma^a{}_{bc,d}\dot{x}^b\dot{x}^c\xi^d + \Gamma^a{}_{bc}\dot{\xi}^b\dot{x}^c \\
\\
=&\ddot{\xi}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{\xi}^c \\
&+ \Gamma^a{}_{bc,d}\xi^c\dot{x}^b\dot{x}^d + \Gamma^a{}_{bc}\xi^c\ddot{x}^b \\
&- \Gamma^a{}_{bc,d}\xi^b\dot{x}^c\dot{x}^d - \Gamma^a{}_{bc}\xi^b\ddot{x}^c \\
&+ \Gamma^a{}_{bc,d}\dot{x}^b\dot{x}^c\xi^d + \Gamma^a{}_{bc}\dot{\xi}^b\dot{x}^c \\
\\
=&\ddot{\xi}^a + \Gamma^a{}_{bc}\dot{x}^b\dot{\xi}^c + \Gamma^a{}_{bc,d}\dot{x}^b\dot{x}^c\xi^d + \Gamma^a{}_{bc}\dot{\xi}^b\dot{x}^c
\end{aligned}
$$

□

Solution 9. *(for exercise 9: **Dust filled/matter dominated universe**)*
Assume these equations describe a universe that is matter dominated, such that the pressure may be expressed as

$$P(t) = 0.$$

Substituting this result and equation (7.20) into equation (7.21) we have

$$
\begin{aligned}
\frac{\rho'(t)}{\rho(t)} &= -3\frac{a'(t)}{a(t)} \\
\frac{d}{dt}\ln\rho(t) &= -\frac{d}{dt}3\ln a(t) \\
\frac{d}{dt}\left(\ln\rho(t) + \ln a(t)^3\right) &= 0 \\
a(t)^3\rho(t) &= C_1
\end{aligned}
$$

describing that there must be conservation of mass density in the universe. Therefore

$$\rho(t) = \frac{C_1}{a(t)^3} \tag{A.7}$$

and now have the following system of equations to solve numerically using a Runge-Kutta fourth order method

$$
\begin{aligned}
\frac{d\mathbf{U}}{dt} &= F(\mathbf{U}) \\
\frac{d}{dt}\begin{pmatrix} a(t) \\ h(t) \end{pmatrix} &= \begin{pmatrix} h(t) \\ -\frac{4\pi C}{3a(t)^2} \end{pmatrix}
\end{aligned}
$$

Two plots will be considered, $a(t)$ and $H(t)$, but initial conditions have to be decided on. From equation (7.23) at $t = 0$ we have

$$h_0 = \pm\sqrt{\frac{8\pi}{3a_0}C_1 - k}$$

which is only real when

$$C_1 \geq \frac{3a_0 k}{8\pi}$$

and the whole model is only valid for $a(t) \neq 0$ and so $a_0 > 0$. The plot depicted in figure A.1 illustrates the Runge-Kutta 4 numerical simulation of the expansion coefficient and rate of expansion coefficient functions over time in a closed, dust filled universe. [1]

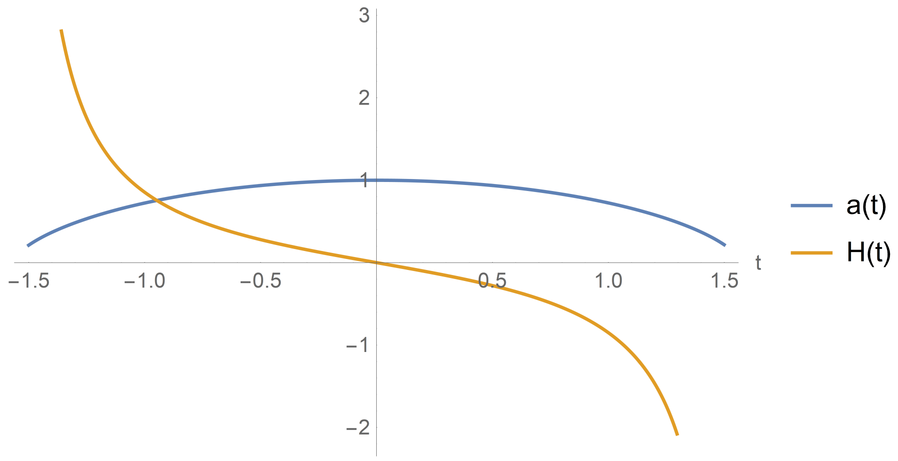


Figure A.1: Expansion coefficient function and rate of expansion coefficient function over time of a closed, dust filled universe

Solution 10. *(for exercise 10: **Real gas filled universe**)*
Assume these equations describe a universe that is radiation dominated, such that the pressure may be expressed as

$$P(t) = \rho(t)^{\gamma}.$$

This pressure equation is substituted into equations (7.20), (7.22) and (7.21) and then a Runge-Kutta fourth order method is considered to numerically integrate the resultant system of equations

$$\frac{d\mathbf{U}}{dt} = F(\mathbf{U})$$

$$\frac{d}{dt}\begin{pmatrix} a(t) \\ \rho(t) \\ h(t) \end{pmatrix} = \begin{pmatrix} h(t) \\ -a(t)^{-1}\left(3h(t)\left(\rho(t)^{\gamma} + \rho(t)\right)\right) \\ -\frac{4\pi}{3}a(t)\left(3\rho(t)^{\gamma} + \rho(t)\right) \end{pmatrix}$$

and produce plots of two functions of interest $a(t)$ and $H(t)$. Now, let us introduce some initial conditions. From equation (7.23) at $t = 0$ we have

$$h_0 = \pm\sqrt{\frac{8\pi}{3}a(t)^2\rho(t) - k}$$

which is only real when

$$\rho_0 \geq \frac{3(1+k)k}{16\pi a_0^2}$$

[1]Use Mathematica to generate a plot of the expansion coefficient function and rate of expansion coefficient function over time of a closed, dust filled universe. See 17 for a possible Mathematica representation.

and the whole model is only valid for $a(t) \neq 0$ and so $a_0 > 0$. The plot depicted in figure A.2 illustrates the Runge-Kutta 4 numerical simulation of the expansion coefficient and rate of expansion coefficient functions over time in a closed, real gas filled universe.[2]

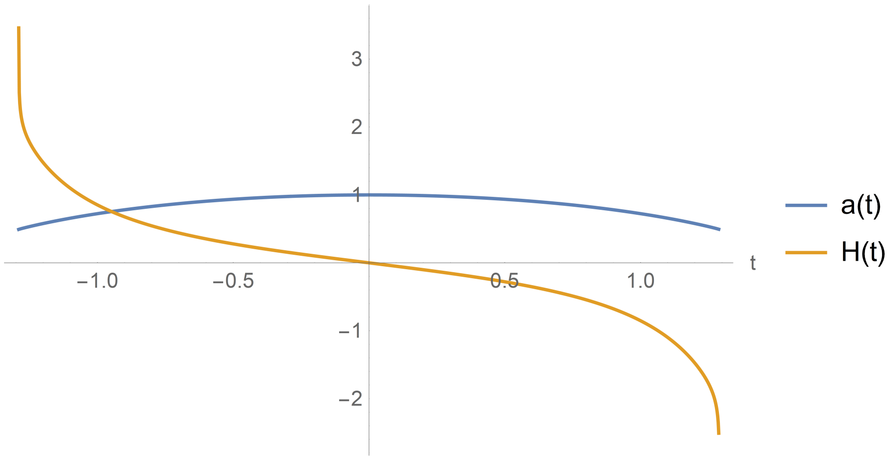


Figure A.2: Expansion coefficient function and rate of expansion coefficient function over time of a closed, real gas filled universe

[2] Use Mathematica to generate a plot of the expansion coefficient function and rate of expansion coefficient function over time of a closed, real gas filled universe. See 18 for a possible Mathematica representation.

Appendix B

Mathematica

Mathematica exercise solution 1. *For Mathematica exercise 1 of chapter 1.*
Steps for creating and exporting one's own diagrams in Mathematica:

1. *Open a notebook and add a new graphic by either pressing `Ctrl + 1` of choosing `Insert` → `Picture` → `New graphic` in the top menu. You will notice a blank box popping up in your notebook.*

2. *Open your drawing tools by pressing `Ctrl + d` or by selecting `Graphics` → `Drawing tools` in the top menu.*

3. *Use the drawing tools to add line, shapes, texts, formulae, etc. Note: you may adjust the size of the diagram by dragging the the sides of the image boundaries with your mouse.*

4. *To export your image, click to the left of the image and add the text `Export["ImageName.Extension",`. Now click to the right of the image and add the text `,formats]`. Then simply press `Shift + Enter` to export the image to your local documents folder. Some of the file extensions available for image export include 'jpg', 'png', 'pdf', 'gif' and many others that may be found by selecting `Help` → `Wolfram documentation` and then searching 'guide/ListingOfAllFormats'. Some of the typical formats for exporting include 'ImageSize', 'ImageResolution', 'ImageFormattingWidth', 'SampleRate', etc.*

Mathematica exercise solution 2. *For Mathematica exercise 2 of chapter 1.*
The following is an example code of how one can apply equations (1.3) and (1.4) to obtain the spherical polar coordinate basis vectors:

```
a = b = {r, \[Theta], \[Phi]};
c = {x, y, z};
e[x] = eu[x] = {1, 0, 0};
e[y] = eu[y] = {0, 1, 0};
e[z] = eu[z] = {0, 0, 1};
Xn[x] = r Sin[\[Theta]] Cos[\[Phi]];
Xn[y] = r Sin[\[Theta]] Sin[\[Phi]];
Xn[z] = r Cos[\[Theta]];
Do[g[i] = Sum[D[Xn[n], i] e[n], {n, c}], {i, a}];
Do[Print[g[i]], {i, a}]
\[Theta]i[r] = Sqrt[x^2 + y^2 + z^2];
\[Theta]i[\[Theta]] = ArcCos[z/r] /. r -> \[Theta]i[r];
\[Theta]i[\[Phi]] = ArcTan[y/x];
Do[gu[i] = Sum[D[\[Theta]i[i], n] eu[n], {n, c}], {i, a}];
Do[gu[i] =
   Evaluate[gu[i] /. x -> Xn[x] /. y -> Xn[y] /. z ->  Xn[z],
     r > 0 && Sin[\[Theta]] > 0] // FullSimplify, {i, a}];
Do[Print[gu[i]], {i, a}]
```

The basis vectors were determined to be

$$\mathbf{g}_r = \{\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)\}$$
$$\mathbf{g}_\theta = \{r\cos(\theta)\cos(\phi), r\cos(\theta)\sin(\phi), -r\sin(\theta)\}$$
$$\mathbf{g}_\phi = \{-r\sin(\theta)\sin(\phi), r\sin(\theta)\cos(\phi), 0\}$$

while the basis covectors were determined to be

$$\mathbf{g}^r = \{\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)\}$$
$$\mathbf{g}^\theta = \left\{\frac{\cos(\theta)\cos(\phi)}{r}, \frac{\cos(\theta)\sin(\phi)}{r}, -\frac{\sin(\theta)}{r}\right\}$$
$$\mathbf{g}^\phi = \left\{-\frac{\csc(\theta)\sin(\phi)}{r}, \frac{\csc(\theta)\cos(\phi)}{r}, 0\right\}.$$

Mathematica exercise solution 3. *For Mathematica exercise 3 of chapter 1.*
Using the results of Mathematica exercise 2, the following code may be run to show that $\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i{}^j = \delta^j{}_i = \mathbf{g}^j \cdot \mathbf{g}_i$*:*

```
Do[Print[g[i].gu[j] // FullSimplify], {i, a}, {j, b}]
Do[Print[g[i].gu[j] == gu[i].g[j] // FullSimplify], {i, a}, {j, b}]
```

Mathematica exercise solution 4. *For Mathematica exercise 4 of chapter 1.*
This following code is an example of how one might calculate the metric and inverse metric tensors in spherical polar coordinates:

```
a = b = {r, \[Theta], \[Phi]};
c = {x, y, z};
e[x] = eu[x] = {1, 0, 0};
e[y] = eu[y] = {0, 1, 0};
e[z] = eu[z] = {0, 0, 1};
Xn[x] = r Sin[\[Theta]] Cos[\[Phi]];
Xn[y] = r Sin[\[Theta]] Sin[\[Phi]];
Xn[z] = r Cos[\[Theta]];
Do[GL[i, j] = FullSimplify[Sum[D[Xn[n], i] e[n], {n, c}].
        Sum[D[Xn[n], j] e[n], {n, c}]], {i,a}, {j, b}];
Do[Print[GL[i, j]], {i, a}, {j, b}]
\[Theta]i[r] = Sqrt[x^2 + y^2 + z^2];
\[Theta]i[\[Theta]] = ArcCos[z/r] /. r -> \[Theta]i[r];
\[Theta]i[\[Phi]] = ArcTan[y/x];
Do[GU[i, j] = Sum[D[\[Theta]i[i], n] eu[n], {n, c}].Sum[
     D[\[Theta]i[j], n] eu[n], {n, c}], {i, a}, {j, b}];
Do[GU[i, j] = Evaluate[GU[i, j] /. x -> Xn[x] /. y -> Xn[y] /. z ->  Xn[z],
     r > 0 && Sin[\[Theta]] > 0] // FullSimplify, {i, a}, {j, b}];
Do[Print[GU[i, j]], {i, a}, {j, b}]
```

The above example code prints entries of the the metric and inverse metric. They are

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2(\theta) \end{pmatrix} \qquad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2(\theta)} \end{pmatrix}.$$

Mathematica exercise solution 5. *For Mathematica exercise 5 of chapter 1.*
The alternating tensor, ϵ_{ijk}*, and inverse alternating tensor,* ϵ^{ijk}*, in spherical coordinates are determined, as an example, in the following Mathematica code:*

```
a = b = {r, \[Theta], \[Phi]};
c = {x, y, z};
Xn[x] = r Sin[\[Theta]] Cos[\[Phi]];
Xn[y] = r Sin[\[Theta]] Sin[\[Phi]];
Xn[z] = r Cos[\[Theta]];
\[Epsilon][i_, j_, k_] := Sum[D[Xn[ii], i] D[Xn[jj], j] D[Xn[kk],k]
        LeviCivitaTensor[3][[Position[c, ii][[1, 1]], Position[c, jj][[1, 1]],
                Position[c, kk][[1, 1]]]], {ii, c}, {jj, c}, {kk, c}] // FullSimplify
\[Theta]i[r] = Sqrt[x^2 + y^2 + z^2];
\[Theta]i[\[Theta]] = ArcCos[z/r] /. r -> \[Theta]i[r];
\[Theta]i[\[Phi]] = ArcTan[y/x];
\[Epsilon]u[i_, j_, k_] := Sum[D[\[Theta]i[i], ii] D[\[Theta]i[j], jj] D[\[Theta]i[k],kk]
        LeviCivitaTensor[3][[Position[c, ii][[1, 1]], Position[c, jj][[1, 1]],
                Position[c, kk][[1, 1]]]], {ii, c}, {jj, c}, {kk, c}] // FullSimplify
```

Mathematica exercise solution 6. *For Mathematica exercise 6 of chapter 1.*
Under the assumption that the metric tensor, g_{ij}, and inverse metric tensor, g^{ij}, have been defined, as in 4, then the following code is an example of how to define the Christoffel symbols of first and second kind:

```
\[CapitalGamma]1[i_, j_, k_] :=
        1/2 (D[GL[i, j], k] + D[GL[i, k], j] - D[GL[j, k], i]) // FullSimplify
\[CapitalGamma]2[i_, j_, k_] :=
        Sum[GU[i, s] \[CapitalGamma]1[s, j, k], {s, a}] // FullSimplify
Do[Print[Subscript[(("\[CapitalGamma]")^i), i j] ==
        \[CapitalGamma]2[i, j, k]], {i, a}, {j, aa}, {k, b}]
```

The non-zero Christoffel symbols of second kind in the spherical polar coordinate system are:

$$\Gamma^{r}{}_{\theta\theta} = -r$$
$$\Gamma^{r}{}_{\phi\phi} = -r\sin^2(\theta)$$
$$\Gamma^{\theta}{}_{r\theta} = \frac{1}{r}$$
$$\Gamma^{\theta}{}_{\theta r} = \frac{1}{r}$$
$$\Gamma^{\theta}{}_{\phi\phi} = -\cos(\theta)\sin(\theta)$$
$$\Gamma^{\phi}{}_{r\phi} = \frac{1}{r}$$
$$\Gamma^{\phi}{}_{\phi r} = \frac{1}{r}$$
$$\Gamma^{\phi}{}_{\theta\phi} = \cot(\theta)$$
$$\Gamma^{\phi}{}_{\phi\theta} = \cot(\theta).$$

Mathematica exercise solution 7. *For Mathematica exercise 7 of chapter 1.*
The following Mathematica code is an example of how one may define and determine the Riemann curvature tensor components under the assumption that the Christoffel symbols of second kind have already been defined, Mathematica exercise solution 6.

```
R[n_, i_, j_, k_] :=
 D[\[CapitalGamma]2[n, i, k], j] - D[\[CapitalGamma]2[n, i, j], k] +
   Sum[\[CapitalGamma]2[ii, i, k] \[CapitalGamma]2[n, ii, j], {ii, a}]
        - Sum[\[CapitalGamma]2[ii, i, j] \[CapitalGamma]2[n, ii, k], {ii, a}]
                // FullSimplify
Do[Print[Subscript[(R^n), i j k] ==  R[n, i, j, k]], {n, a}, {i,
  b}, {j, aa}, {k, bb}]
```

All components of the Curvature tensor in spherical polar coordinates were determined to be zero.

Mathematica exercise solution 8. *For Mathematica exercise 8 of chapter 1.*
The following Mathematica code represents a possible way of defining and printing the components of the Ricci tenor in Spherical polar coordinates. This code assumes that the Riemann curvature tensor has been defined, from Mathematica exercise 7.

```
R[i_, j_] := Sum[R[ii, i, ii, j], {ii, a}]
Do[Print[Subscript[R, i j ] == R[i, j]], {i, a}, {j, b}]
R = Sum[GU[ii, jj] R[ii, jj], {ii, a}, {jj, b}]
```

All Ricci components as well as the curvature scalar in spherical polar coordinates were determined to be zero.

Mathematica exercise solution 9. *For Mathematica exercise 9 of chapter 1.*
Assuming you have defined the metric and inverse metric tensors (from exercise 5), and the Christoffel symbols of second kind (from exercise 6), then the grad operator acting on an arbitrary vector function **f***,*

$$\mathbf{f} = \begin{pmatrix} f(r,\theta,\phi) \\ g(r,\theta,\phi) \\ h(r,\theta,\phi) \end{pmatrix},$$

in spherical polar coordinates as a physical tensor (the dimensions of the vector function components agree), may be defined in Mathematica by the following code:

```
Fu[r] = f[r, \[Theta], \[Phi]];
Fu[\[Theta]] = g[r, \[Theta], \[Phi]];
Fu[\[Phi]] = h[r, \[Theta], \[Phi]];
Fl[i_] := Sum[GL[i, j] Fu[j], {j, aa}]
GradFu[i_, j_] :=
 D[Fu[i], j] + Sum[\[CapitalGamma]2[i, j, k] Fu[k], {k, aa}]
GradFl[i_, j_] :=
 D[Fl[i], j] - Sum[\[CapitalGamma]2[k, i, j] Fl[k], {k, aa}]
Table[GradFl[i, j], {i, aa}, {j, bb}] // MatrixForm
Table[GradFu[i, j], {i, aa}, {j, bb}] // MatrixForm
```

which yields the following tensor gradients when $\mathbf{f} = f_i$

$$
\begin{aligned}
f_{r;r} &= f^{(1,0,0)}(r,\theta,\phi) \\
f_{r;\theta} &= f^{(0,1,0)}(r,\theta,\phi) - r\,g(r,\theta,\phi) \\
f_{r;\phi} &= f^{(0,0,1)}(r,\theta,\phi) - r\,h(r,\theta,\phi)\sin^2(\theta) \\
f_{\theta;r} &= g^{(1,0,0)}(r,\theta,\phi)\,r^2 + g(r,\theta,\phi)\,r \\
f_{\theta;\theta} &= g^{(0,1,0)}(r,\theta,\phi)\,r^2 + f(r,\theta,\phi)\,r \\
f_{\theta;\phi} &= r^2 g^{(0,0,1)}(r,\theta,\phi) - r^2\cos(\theta)\,h(r,\theta,\phi)\sin(\theta) \\
f_{\phi;r} &= r\,h(r,\theta,\phi)\sin^2(\theta) + r^2 h^{(1,0,0)}(r,\theta,\phi)\sin^2(\theta) \\
f_{\phi;\theta} &= \cos(\theta)\,h(r,\theta,\phi)\sin(\theta)\,r^2 + \sin^2(\theta)\,h^{(0,1,0)}(r,\theta,\phi)\,r^2 \\
f_{\phi;\phi} &= \cos(\theta)\,g(r,\theta,\phi)\sin(\theta)\,r^2 + \sin^2(\theta)\,h^{(0,0,1)}(r,\theta,\phi)\,r^2 + f(r,\theta,\phi)\sin^2(\theta)\,r
\end{aligned}
$$

and the following when $\mathbf{f} = f^i$

$$f^r{}_{;r} = f^{(1,0,0)}(r,\theta,\phi)$$
$$f^r{}_{;\theta} = f^{(0,1,0)}(r,\theta,\phi) - r\,g(r,\theta,\phi)$$
$$f^r{}_{;\phi} = f^{(0,0,1)}(r,\theta,\phi) - r\,h(r,\theta,\phi)\sin^2(\theta)$$
$$f^\theta{}_{;r} = \frac{g(r,\theta,\phi)}{r} + g^{(1,0,0)}(r,\theta,\phi)$$
$$f^\theta{}_{;\theta} = \frac{f(r,\theta,\phi)}{r} + g^{(0,1,0)}(r,\theta,\phi)$$
$$f^\theta{}_{;\phi} = g^{(0,0,1)}(r,\theta,\phi) - \cos(\theta)\,h(r,\theta,\phi)\sin(\theta)$$
$$f^\phi{}_{;r} = \frac{h(r,\theta,\phi)}{r} + h^{(1,0,0)}(r,\theta,\phi)$$
$$f^\phi{}_{;\theta} = \cot(\theta)\,h(r,\theta,\phi) + h^{(0,1,0)}(r,\theta,\phi)$$
$$f^\phi{}_{;\phi} = \frac{f(r,\theta,\phi)}{r} + \cot(\theta)\,g(r,\theta,\phi) + h^{(0,0,1)}(r,\theta,\phi).$$

Mathematica exercise solution 10. *For Mathematica exercise 10 of chapter 1.*
Assuming you have defined the metric and inverse metric tensors (from exercise 5), and the Christoffel symbols of second kind (from exercise 6), then the divergence operator acting on an arbitrary vector function **f**,

$$\mathbf{f} = \begin{pmatrix} f(r,\theta,\phi) \\ g(r,\theta,\phi) \\ h(r,\theta,\phi) \end{pmatrix},$$

in spherical polar coordinates as a physical tensor (the dimensions of the vector function components agree), may be defined in Mathematica by the following code:

```
Fu[r] = f[r, \[Theta], \[Phi]];
Fu[\[Theta]] = g[r, \[Theta], \[Phi]];
Fu[\[Phi]] = h[r, \[Theta], \[Phi]];
Fl[i_] := Sum[GL[i, j] Fu[j], {j, aa}]
DivFu = Sum[D[Fu[i], i], {i, aa}] + Sum[\[CapitalGamma]2[i, i, k] Fu[k], {k, aa}, {i, bb}]
DivFl = Sum[GU[j, i] D[Fl[i], j], {i, aa}, {j, bb}] -
  Sum[GU[j, i] \[CapitalGamma]2[k, i, j] Fl[k], {k, aa}, {i, bb}, {j, a}]
```

which yields the to following when $\mathbf{f} = f_i$

$$g^{ji} f_{i;j} = f^{(1,0,0)}(r,\theta,\phi) + \frac{2f(r,\theta,\phi)}{r} + g^{(0,1,0)}(r,\theta,\phi) + \cot(\theta)\,g(r,\theta,\phi) + h^{(0,0,1)}(r,\theta,\phi)$$

and the following when $\mathbf{f} = f^i$

$$f^i{}_{;i} = f^{(1,0,0)}(r,\theta,\phi) + \frac{2f(r,\theta,\phi)}{r} + g^{(0,1,0)}(r,\theta,\phi) + \cot(\theta)\,g(r,\theta,\phi) + h^{(0,0,1)}(r,\theta,\phi).$$

Mathematica exercise solution 11. *For Mathematica exercise 11 of chapter 1.*
Assuming you have defined the base vectors and covectors (from exercise 1), the metric and inverse metric tensors (from exercise 5), and the Christoffel symbols of second kind (from exercise 6), then the curl operator acting on an arbitrary vector function **f**,

$$\mathbf{f} = \begin{pmatrix} f(r,\theta,\phi) \\ g(r,\theta,\phi) \\ h(r,\theta,\phi) \end{pmatrix},$$

in spherical polar coordinates as a physical tensor (the dimensions of the vector function components agree), may be defined in Mathematica by the following code:

```
Fu[r] = f[r, \[Theta], \[Phi]];
Fu[\[Theta]] = g[r, \[Theta], \[Phi]];
Fu[\[Phi]] = h[r, \[Theta], \[Phi]];
Fl[i_] := Sum[GL[i, j] Fu[j], {j, aa}]
CurlFu = Sum[
   LeviCivitaTensor[3][[Position[aa, i][[1, 1]],
      Position[aa, j][[1, 1]],
      Position[aa, k][[1, 1]]]] (Sum[GL[j, l] D[Fu[l], i], {l, aa}] +
      Sum[GL[j, l] \[CapitalGamma]2[l, i, m] Fu[m], {m, aa}, {l, bb}])
      g[k], {k, a}, {i, a}, {j, b}] // MatrixForm
CurlFl = Sum[
   LeviCivitaTensor[3][[Position[aa, i][[1, 1]],
      Position[aa, j][[1, 1]],
      Position[aa, k][[1, 1]]]] (D[Fl[j], i] -
      Sum[\[CapitalGamma]2[l, j, i] Fl[l], {l, aa}]) g[k], {k, bb},
      {i, a}, {j, b}] // MatrixForm
CurlFl - CurlFu // FullSimplify
```

which yields the to following when $\mathbf{f} = f_j$

$$\nabla \times f_j \;=\; \epsilon^{ijk} f_{j;i}\mathbf{g}_k = \begin{pmatrix} r\left(\sin(\theta)\left(\sin(\phi) f^{(0,1,0)}(r,\theta,\phi) - r\left(\cos(\phi)\left(g^{(0,0,1)}(r,\theta,\phi)\right.\right.\right.\right. \\ \left.\left. + \sin(\theta)\left(r\cos(\theta) h^{(1,0,0)}(r,\theta,\phi) - \sin(\theta) h^{(0,1,0)}(r,\theta,\phi)\right)\right) \right. \\ \left.\left. + r\sin(\phi) g^{(1,0,0)}(r,\theta,\phi)\right) - 2r\sin(\phi) g(r,\theta,\phi)\right) + \cos(\theta)\cos(\phi) f^{(0,0,1)}(r,\theta,\phi)\right) \\ \\ r\left(\sin(\theta)\left(-\cos(\phi) f^{(0,1,0)}(r,\theta,\phi) + r\left(r\cos(\phi) g^{(1,0,0)}(r,\theta,\phi)\right.\right.\right. \\ \left.\left. - \sin(\phi)\left(g^{(0,0,1)}(r,\theta,\phi) + \sin(\theta)\left(r\cos(\theta) h^{(1,0,0)}(r,\theta,\phi) - \sin(\theta) h^{(0,1,0)}(r,\theta,\phi)\right)\right)\right) \right. \\ \left. + 2r\cos(\phi) g(r,\theta,\phi)\right) + \cos(\theta)\sin(\phi) f^{(0,0,1)}(r,\theta,\phi)\right) \\ \\ r\left(\sin(\theta)\left(r\sin(\theta)\left(r\sin(\theta) h^{(1,0,0)}(r,\theta,\phi) + \cos(\theta) h^{(0,1,0)}(r,\theta,\phi)\right) - f^{(0,0,1)}(r,\theta,\phi)\right)\right. \\ \left. - r\cos(\theta) g^{(0,0,1)}(r,\theta,\phi) + 2r\sin(\theta) h(r,\theta,\phi)\right) \end{pmatrix}$$

and the following when $\mathbf{f} = f^j$

$$\nabla \times f^j \;=\; \epsilon^{ijk} g_{jl} f^l{}_{;i}\mathbf{g}_k = \epsilon^{ijk} f_{j;i}\mathbf{g}_k.$$

Mathematica exercise solution 12. *For Mathematica exercise 1 of chapter 5.*
The following Mathematica code is an example representation of how one could obtain the Euler-Lagrange equations/ equations of motion from a defined Lagrangian. This code assumes that the metric tensor has been defined, as in from Mathematica exercise 4.

```
\[Theta][t_] := \[Pi]/2;
dx[r] = r'[t];
dx[\[Theta]] = \[Theta]'[t];
dx[\[Phi]] = \[Phi]'[t];
L = Sum[1/2 m Evaluate[GL[i, j] /. r -> r[t]
               /. \[Theta] -> \[Pi]/2 /. \[Phi] -> \[Phi][t]] dx[i] dx[j],
                       {i, a}, {j, b}] // FullSimplify
Solve[D[D[L, r'[t]], t] - D[L, r[t]] == 0 , r''[t]]
Solve[D[D[L, \[Phi]'[t]], t] - D[L, \[Phi][t]] == 0 , \[Phi]''[t]]
```

Mathematica exercise solution 13. *For Mathematica exercise 2 of chapter 5.*
Lagrange equations/ equations of motion from a defined Lagrangian. This code assumes that the metric tensor has been defined, as in from Mathematica exercise 4.

```
\[Theta][t_] := \[Pi]/2;
dx[r] = r'[t];
dx[\[Theta]] = \[Theta]'[t];
dx[\[Phi]] = \[Phi]'[t];
L = Sum[1/2 m Evaluate[GL[i, j] /. r -> r[t]
                /. \[Theta] -> \[Pi]/2 /. \[Phi] -> \[Phi][t]]
                        dx[i] dx[j], {i, a}, {j, b}] + (G M m)/r[t] // FullSimplify
Solve[D[D[L, r'[t]], t] - D[L, r[t]] == 0 , r''[t]]
Solve[D[D[L, \[Phi]'[t]], t] - D[L, \[Phi][t]] == 0 , \[Phi]''[t]]
```

Mathematica exercise solution 14. *For Mathematica exercise 1 of chapter 6.*
For the given metric tensor in equation (6.4), the Christoffel symbols of second kind may be determined using the following example Mathematica code:

```
\[Alpha] = \[Beta] = \[Alpha]\[Alpha] = {t, r, \[Theta], \[Phi]};
Do[GL[\[Mu], \[Nu]] = 0, {\[Mu], \[Alpha]}, {\[Nu], \[Alpha]}];
GL[t, t] = -Exp[2 \[Lambda][r]];
GL[r, r] = Exp[2 \[Kappa][r]];
GL[\[Theta], \[Theta]] = r^2;
GL[\[Phi], \[Phi]] = r^2 (Sin^2)[\[Theta]];
Do[GU[\[Mu], \[Nu]] = 0, {\[Mu], \[Alpha]}, {\[Nu], \[Alpha]}];
GU[t, t] = GL[t, t]^-1;
GU[r, r] = GL[r, r]^-1;
GU[\[Theta], \[Theta]] = GL[\[Theta], \[Theta]]^-1;
GU[\[Phi], \[Phi]] = GL[\[Phi], \[Phi]]^-1;
\[CapitalGamma]1[\[Mu]_, \[Nu]_, \[Sigma]_] :=
 1/2 (D[GL[\[Mu], \[Nu]], \[Sigma]] + D[GL[\[Mu], \[Sigma]], \[Nu]] -
     D[GL[\[Nu], \[Sigma]], \[Mu]]) // FullSimplify
\[CapitalGamma]2[\[Mu]_, \[Nu]_, \[Sigma]_] :=
 Sum[GU[\[Mu], \[Iota]] \[CapitalGamma]1[\[Iota], \[Nu], \[Sigma]], {\
\[Iota], \[Alpha]}] // FullSimplify
Do[Print[Subscript[((
    "\[CapitalGamma]")^\[Mu]), \[Nu] \[Sigma]] == \[CapitalGamma]2[\
\[Mu], \[Nu], \[Sigma]]], {\[Mu], \[Alpha]}, {\[Nu], \[Beta]}, {\
\[Sigma], \[Alpha]\[Alpha]}]
```

Mathematica exercise solution 15. *For Mathematica exercise 2 of chapter 6.*
The following Mathematica code assumes that the code in exercise 14 has been run. It represents an example of how one can calculate the Ricci tensor components:

```
R[\[CurlyPhi]_, \[Mu]_, \[Nu]_, \[Sigma]_] :=
 D[\[CapitalGamma]2[\[CurlyPhi], \[Mu], \[Sigma]], \[Nu]] -
   D[\[CapitalGamma]2[\[CurlyPhi], \[Mu], \[Nu]], \[Sigma]] +
   Sum[\[CapitalGamma]2[\[Iota], \[Mu], \[Sigma]] \[CapitalGamma]2[\
\[CurlyPhi], \[Iota], \[Nu]], {\[Iota], \[Alpha]}] -
   Sum[\[CapitalGamma]2[\[Iota], \[Mu], \[Nu]] \[CapitalGamma]2[\
\[CurlyPhi], \[Iota], \[Sigma]], {\[Iota], \[Alpha]}] // FullSimplify
R[\[Mu]_, \[Nu]_] :=
 Sum[R[\[Sigma], \[Mu], \[Sigma], \[Nu]], {\[Sigma], \[Alpha]}]
Do[Print[Subscript[R, \[Mu] \[Nu]] ==
   R[\[Mu], \[Nu]]], {\[Mu], \[Alpha]}, {\[Nu], \[Beta]}]
```

Mathematica exercise solution 16. *For Mathematica exercise 3 of chapter 6.*

```
Sol1[r_, p0_, \[Rho]0_] := ((p0 + \[Rho]0)^2 (\[Rho]0 (-9 + Sqrt[9 - 24 \[Pi] r^2 \[Rho]0])
        + 3 p0 (-3 + Sqrt[9 - 24 \[Pi] r^2 \[Rho]0]))^2)/(9 (3 p0 + \[Rho]0)^2 (\[Rho]0 (-3
        + Sqrt[ 1 - (4 p0 (2 p0 + \[Rho]0))/(3 p0 + \[Rho]0)^2]) + 3 p0 (-1
        + Sqrt[ 1 - (4 p0 (2 p0 + \[Rho]0))/(3 p0 + \[Rho]0)^2]))^2)
Sol2[r_, p0_, \[Rho]0_] := 4/3 \[Pi] r^3 \[Rho]0
Sol3[r_, p0_, \[Rho]0_] := -((\[Rho]0 (3 p0 (-1 + Sqrt[9 - 24 \[Pi] r^2 \[Rho]0])
        + \[Rho]0 (-3 + Sqrt[ 9 - 24 \[Pi] r^2 \[Rho]0])))/( 3 p0 (-3
        + Sqrt[9 - 24 \[Pi] r^2 \[Rho]0]) + \[Rho]0 (-9 + Sqrt[ 9 - 24 \[Pi] r^2 \[Rho]0])))
Plot[{Sol1[r, 0.4, 0.25], Sol2[r, 0.4, 0.25], Sol3[r, 0.4, 0.25]}, {r,0,
        Evaluate[(Sqrt[3/(2 \[Pi])] Sqrt[2 Subscript[p, 0]^2
        + Subscript[p, 0] Subscript[\[Rho], 0]])/Sqrt[9 Subscript[p, 0]^2 Subscript[\[Rho], 0]
        + 6 Subscript[p, 0] Subscript[\[Rho], 0]^2 + Subscript[\[Rho],0]^3]
        /. Subscript[p, 0] -> 0.4 /. Subscript[\[Rho], 0] -> 0.25]},
        PlotLegends -> {"a(r)", "m(r)", "p(r)"}, AxesLabel -> Automatic,
        PlotRange -> All, ImageSize -> Medium]
```

Mathematica exercise solution 17. *For Mathematica exercise 1 of appendix A.*
The following is a Mathematica representation of Runge-Kutta 4 method applied to find a numerical solution of a dust filled, closed universe:

```
k = 1;
a0 = 1;
s = 1;
\[Delta]C = 0;
Time = 1.5;
\[Delta]t = 0.001;
C1 = (3 a0 k)/(8 \[Pi]) + \[Delta]C;
h0 = s Sqrt[(8 \[Pi])/(3 a0) C1 - k];
F[k_, U_] := {U[[2]], -((4 \[Pi] C1)/(3 U[[1]]^2)) }
tif = 0;
tff = Time;
tfb = -Time;
tib = tif;
U = {a0, h0};
Sola = {U[[1]]};
Solh = {U[[2]]};
SolHubble = {U[[2]]/U[[1]]};
While[ tib > tfb,
  {k1 = \[Delta]t F[k, U];
   k2 = \[Delta]t F[k, U - k1/2];
   k3 = \[Delta]t F[k, U - k2/2];
   k4 = \[Delta]t F[k, U - k3];
   U = U - 1/6 (k1 + 2 k2 + 2 k3 + k4);
   tib = tib - \[Delta]t;
   Sola = Append[Sola, U[[1]]];
   Solh = Append[Solh, U[[2]]];
   SolHubble = Append[SolHubble, U[[2]]/U[[1]]];}];
Sola = Reverse[Sola];
Solh = Reverse[Solh];
SolHubble = Reverse[SolHubble];
U = {a0, h0};
While[ tif < tff,
  {k1 = \[Delta]t F[k, U];
   k2 = \[Delta]t F[k, U + k1/2];
   k3 = \[Delta]t F[k, U + k2/2];
   k4 = \[Delta]t F[k, U + k3];
```

```
   U = U + 1/6 (k1 + 2 k2 + 2 k3 + k4);
   tif = tif + \[Delta]t;
   Sola = Append[Sola, U[[1]]];
   Solh = Append[Solh, U[[2]]];
   SolHubble = Append[SolHubble, U[[2]]/U[[1]]];}];
N1 = (tff − tfb)/\[Delta]t;
i = 1;
t = tfb;
SolaPlot = {{tfb, Sola[[1]]}};
SolhPlot = {{tfb, Solh[[1]]}};
SolHubblePlot = {{tfb, SolHubble[[1]]}};
While[i < N1, {
  t = t + \[Delta]t;
  i = i + 1;
  SolaPlot = Append[SolaPlot, {t, Sola[[i]]}];
  SolhPlot = Append[SolhPlot, {t, Solh[[i]]}];
  SolHubblePlot = Append[SolHubblePlot, {t, SolHubble[[i]]}];
  }]
PlotImage =
 ListLinePlot[{SolaPlot, SolHubblePlot}, AxesLabel −> {"t"},
  PlotLegends −> {"a(t)", "H(t)"}, ImageSize −> Medium]
Export["DustFilledRK4Plot.png", PlotImage, ImageResolution −> 1000]
```

Mathematica exercise solution 18. *For Mathematica exercise 2 of appendix A.*

```
k = 1;
a0 = 1;
s = +1;
\[Delta]\[Rho] = 0;
\[Gamma] = 10;
Time = 1.29;
\[Delta]t = 0.002;
\[Rho]0 = (3 k)/(8 a0^2 \[Pi]) ((1 + k)/2) + \[Delta]\[Rho];
h0 = s Sqrt[−k + 8/3 a0^2 \[Pi] \[Rho]0];
F[k_, U_, \[Gamma]_] := {U[[3]], −((
   3 U[[3]] (U[[2]] + U[[2]]^\[Gamma]))/
   U[[1]]), −(4/3) U[[1]] \[Pi] (U[[2]] + 3 U[[2]]^\[Gamma])}
tif = 0;
tff = Time;
tfb = −Time;
tib = tif;
U = {a0, \[Rho]0, h0};
Sola = {U[[1]]};
Sol\[Rho] = {U[[2]]};
Solh = {U[[3]]};
SolHubble = {U[[3]]/U[[1]]};
While[ tib > tfb,
  {k1 = \[Delta]t F[k, U, \[Gamma]];
   k2 = \[Delta]t F[k, U − k1/2, \[Gamma]];
   k3 = \[Delta]t F[k, U − k2/2, \[Gamma]];
   k4 = \[Delta]t F[k, U − k3, \[Gamma]];
   U = U − 1/6 (k1 + 2 k2 + 2 k3 + k4);
   tib = tib − \[Delta]t;
   Sola = Append[Sola, U[[1]]];
   Sol\[Rho] = Append[Sol\[Rho], U[[2]]];
   Solh = Append[Solh, U[[3]]];
   SolHubble = Append[SolHubble, U[[3]]/U[[1]]];}];
Sola = Reverse[Sola];
```

```
Sol\[Rho] = Reverse[Sol\[Rho]];
Solh = Reverse[Solh];
SolHubble = Reverse[SolHubble];
U = {a0, \[Rho]0, h0};
While[ tif < tff,
  {k1 = \[Delta]t F[k, U, \[Gamma]];
   k2 = \[Delta]t F[k, U + k1/2, \[Gamma]];
   k3 = \[Delta]t F[k, U + k2/2, \[Gamma]];
   k4 = \[Delta]t F[k, U + k3, \[Gamma]];
   U = U + 1/6 (k1 + 2 k2 + 2 k3 + k4);
   tif = tif + \[Delta]t;
   Sola = Append[Sola, U[[1]]];
   Sol\[Rho] = Append[Sol\[Rho], U[[2]]];
   Solh = Append[Solh, U[[3]]];
   SolHubble = Append[SolHubble, U[[3]]/U[[1]]];}];
N1 = (tff - tfb)/\[Delta]t;
i = 1;
t = tfb;
SolaPlot = {{tfb, Sola[[1]]}};
Sol\[Rho]Plot = {{tfb, Sol\[Rho][[1]]}};
SolhPlot = {{tfb, Solh[[1]]}};
SolHubblePlot = {{tfb, SolHubble[[1]]}};
While[i < N1, {
  t = t + \[Delta]t;
  i = i + 1;
  SolaPlot = Append[SolaPlot, {t, Sola[[i]]}];
  Sol\[Rho]Plot = Append[Sol\[Rho]Plot, {t, Sol\[Rho][[i]]}];
  SolhPlot = Append[SolhPlot, {t, Solh[[i]]}];
  SolHubblePlot = Append[SolHubblePlot, {t, SolHubble[[i]]}];
  }]
PlotImage =
 ListLinePlot[{SolaPlot,(*Sol\[Rho]Plot,*)SolHubblePlot}, AxesLabel -> {"t"},
  PlotLegends -> {"a(t)"(*,"\[Rho](t)"*), "H(t)"}, ImageSize -> Medium]
Export["RealGasFilledRK4Plot.png", PlotImage, ImageResolution -> 1000]
```

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