Spectral theory of differential operators on graphs [1][2]

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Abstract

The focus of this thesis is the spectral structure of second order self-adjoint differential operators on graphs.

Various function spaces on graphs are defined and we define, in terms of both differential systems and the afore noted function spaces, boundary value problems on graphs. A boundary value problem on a graph is shown to be spectrally equivalent to a system with separated boundary conditions. An example is provided to illustrate the fact that, for Sturm-Liouville operators on graphs, self-adjointness does not necessarily imply regularity. We also show that since the differential operators considered are self-adjoint the algebraic and geometric eigenvalue multiplicities are equal. Asymptotic bounds for the eigenvalues are found using matrix Prüfer angle methods.

Techniques common in the area of elliptic partial differential equations are used to give a variational formulation for boundary value problems on graphs. This enables us to formulate an analogue of Dirichlet-Neumann bracketing for boundary value problems on graphs as well as to establish a min-max principle. This eigenvalue bracketing gives rise to eigenvalue asymptotics and consequently eigenfunction asymptotics.

Asymptotic approximations to the Green's functions of Sturm-Liouville boundary value problems on graphs are obtained. These approximations are used to study the regularized trace of the differential operators associated with these boundary value problems. Inverse spectral problems for Sturm-Liouville boundary value problems on graphs resembling those considered in Halberg and Kramer, *A generalization of the trace concept*, Duke Math. J. **27** (1960), 607-617, for Sturm-Liouville problems, and Pielichowski, *An inverse spectral problem for linear elliptic differential operators*, Universitatis Iagellonicae Acta Mathematica **XXVII** (1988), 239-246, for elliptic boundary value problems, are solved.

Boundary estimates for solutions of non-homogeneous boundary value problems on graphs are given. In particular, bounds for the norms of the boundary values of solutions to the non-homogeneous boundary value problem in terms of the norm of the non-homogeneity are obtained and the eigenparameter dependence of these bounds is studied.

Inverse nodal problems on graphs are then considered. Eigenfunction and eigenvalue asymptotic approximations are used to provide an asymptotic expression for the spacing of nodal points on each edge of the graph from which the uniqueness of the potential, for given nodal data, is deduced. An explicit formula for the potential in terms of the nodal points and eigenvalues is given.

Declaration

I declare that this thesis is my own, unaided work. It is being submitted for the Degree of Doctor of Philosophy in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

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Sonja Currie

This __________ day of ____________________ , at Johannesburg, South Africa.

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Contents

Chapter 1

Introduction

Let G be a graph with finitely many edges, say K, each of finite length. Denote the edges by e_i, $i = 1, \ldots, K$, and the corresponding lengths of the edges by l_i, $i = 1, \ldots, K$. We consider the formal second order differential operator

$$ly := -\frac{d^2y}{dx^2} + q(x)y = \lambda y, \tag{1.1}$$

on G where, throughout this thesis, q is a real valued function on G. Constraints are placed on q as additional structure is needed: in Chapter 2, Chapter 3 (excluding Sections 3.5 to 3.7), Chapter 4 and Chapter 6, it is assumed that q is essentially bounded; in Sections 3.5 to 3.7 and in Chapter 7 we assume that $q \in \mathcal{C}^1(G)$; and, in Chapter 5 that $q \in \mathcal{C}^2(\bar{G})$. At the vertices or nodes of G we impose boundary conditions with respect to which l is formally self-adjoint (see [53] for the definition in the case of systems and Section 2.2 and [12] for graphs). Such boundary conditions will be called formally self-adjoint boundary conditions.

In particular by equation (1.1) we mean the system of equations

$$-\frac{d^2y_i}{dx^2} + q_i(x)y_i = \lambda y_i, \quad x \in (0, l_i),\ i = 1, ..., K,$$

where q_i and y_i denote q and y restricted to e_i, and e_i is identified with $(0, l_i)$. We consider boundary conditions of the form

$$\sum_{j=1}^{K} \left[\alpha_{ij}y_j + \beta_{ij}y'_j\right](0) + \sum_{j=1}^{K} \left[\gamma_{ij}y_j + \delta_{ij}y'_j\right](l_j) = 0, \quad i = 1, ..., 2K,$$

where the number of linearly independent boundary conditions is $2K$. This number of linearly independent boundary conditions is necessary (but not sufficient) for the self-adjointness of the boundary value problem on G. Self-adjoint boundary conditions for the Sturm-Liouville operator on a graph have been characterized by Harmer, Kostrykin and Schrader, and Kuchment in [34, 38, 45]. Carlson, in [12], gives a description of adjoints and domains of essential self-adjointness for a class of differential operators on a weighted graph. See Section 2.2 for more details on self-adjointness of boundary value problems on graphs.

Although our researches show that historically the first graph model was used in chemistry, see [29, 30, 55, 62], the development of the theory of differential operators on graphs is recent with most of the research in this area having been conducted in the last couple of decades. It should however be noted that both multipoint boundary value problems (less general than boundary value problems on graphs) and differential systems (more general than boundary value problems on graphs) were studied far earlier than this. Differential operators on graphs arise naturally in chemistry, physics and engineering (nanotechnology), and are mathematically interesting. Amongst these applications of differential operators on graphs are the free-electron theory of conjugated molecules in chemistry, see [29, 30, 62], quantum wires and quantum chaos, see [38, 39, 40, 41], and scattering theory and photonic crystals, see [25, 42, 46].

Other physical settings where differential operators on graphs are of interest are concerned with heat flows in a mesh, mechanical vibrations of networks of elastic strings and propagation of radiation in networks of optical fibres. We refer the reader to [43, 44] and the bibliographies thereof for an extensive survey of the physical systems giving rise to boundary value problems on graphs.

In order to proceed with a rigorous development various function spaces over graphs are needed. These are defined in Chapter 2, where certain of their useful properties are ascertained. In addition we show the operators associated with these boundary value problems to be lower-semi-bounded. A boundary value problem on the graph G can also be reformulated as a differential system. Neither self-adjointness nor formal self-adjointness guarantee regularity in the sense of Naimark, [53].

In [13], Carlson develops Floquet theory and its applications to spectral theory for periodic Schrödinger operators on graphs. Amongst other results he obtains eigenvalue asymptotics on graphs where all edges are of equal length. Spectral asymptotics for boundary value problems on graphs have also been obtained in [68]. In Chapter 3 we obtain rough spectral asymptotics with very mild conditions on the interaction at the nodes (more general than those in [13] and [68]). While in Chapter 4 we use a variational formulation which requires strong assumptions on the nodal boundary conditions but yields superior asymptotic accuracy.

The spectral structure of differential operators on general compact graphs is considered in Chapter 3. In order to find asymptotic bounds for the eigenvalues of the boundary value problem on G we make use of abstract Prüfer angle techniques similar to those of Atkinson, [4].

In [4], Atkinson provides asymptotic bounds for the eigenvalues of a differential system with very restrictive boundary conditions, namely

$$y(0) = 0 \quad \text{and} \quad y(1)\cos\frac{1}{2}\alpha = y'(1)\sin\frac{1}{2}\alpha.$$

These boundary conditions were also considered by Volkmer in [70], where by determining the asymptotic behavior of determinants of unitary solutions of matrix Riccatti differential equations containing a large parameter he proves results on the existence and asymptotic distribution of eigenvalues of indefinite matrix Sturm-Liouville problems.

Since we wish to consider more general boundary conditions than Atkinson and Volkmer, it is first necessary to find a matrix Prüfer angle formulation corresponding to our general boundary conditions. For this we consider the construction used by Etgen, [22] and Barrett, [6], where a matrix polar coordinate transformation is used to transform a second order differential system with separated boundary conditions to one of the same type but with more tractable boundary conditions. Etgen, in [22], shows the existence of eigenvalues but does not give eigenvalue asymptotics.

In order to make use of the matrix Prüfer angle of Etgen, the system boundary value problem given in Chapter 2 is shown to be equivalent to a formally self-adjoint system, of twice the dimension but with separated boundary conditions. Consequently all eigenvalues are semi-simple, asymptotic solutions for the system are found and an explicit form for the Green's function obtained. In addition an interlacing result for the eigenvalues is proved. Most of the results from Chapter 2 and Chapter 3 have be published by Currie and Watson, [20].

The approach used by Courant and Hilbert, [19], to obtain eigenvalue asymptotics for elliptic boundary value problems is adapted to the graph setting in Chapter 4. A variational formulation for boundary value problems on graphs is given. This requires a restriction of the class of admissable boundary conditions to what we have called co-normal boundary conditions, as they correspond to the similarly named boundary conditions for elliptic boundary value problems. A max-min property for Sturm-Liouville boundary value problems on directed graphs is then proved and, as a consequence, a type of Dirichlet-Neumann bracketing for the eigenvalues of the boundary value problem obtained. This in turn gives rise to eigenvalue and eigenfunction asymptotic approximations. The content of Chapter 4 appears in Currie and Watson, [21].

In parallel to the variational aspects of boundary value problems on graphs studied here and on trees in [68], the work of Pokornyi and Pryadiev, and Pokornyi, Pryadiev and Al-Obeid, in [59] and [60], should be noted for their extension of Sturmian oscillation theory to second order operators on graphs. The idea of approximating the behaviour of eigenfunctions and eigenvalues for a boundary value problem on a graph by the behaviour of associated problems on the individual edges, used here, has appeared previously in [71].

In Chapter 5 we obtain asymptotic approximations for the iterated Green's function of $l-\lambda$ on G where we have imposed Dirichlet boundary conditions at each node. Using the approach given by Gårding in [24] for elliptic partial differential equations and the above noted asymptotic approximations we obtain asymptotics for the iterates of the general Green's function. These are then used in order to study the regularized trace. We also refer the reader to [57] where analogous results for iterated Green's functions of elliptic differential operators can be found.

In [32], Halberg and Kramer solved a class of inverse spectral problems for Sturm-Liouville equations by considering the regularized traces of the operators. Analogous results for partial differential equations can be found in [56], where Pielichowski considers the inverse spectral problem, which originated in [3], for a Sturm-Liouville problem. This problem was generalized, by Bochenek in [9, 10], to boundary value problems for self-adjoint elliptic operators with constant principal part on bounded regions in $\mathbb{R}^n$ with $n \geq 2$. In [56] the generalization is taken one step further and the inverse spectral problem is analyzed for self-adjoint elliptic operators, of any order, with variable coefficients.

A variety of inverse problems have been posed and solved for boundary value problems on graphs. Gutkin and Smilansky show in [31] that the geometry of a non-commensurate (edges do not have equal lengths) simple graph is uniquely dependent on the spectrum of the Laplacian on the graph. Carlson, in [14], uses spectral data to reconstruct the geometry of the graph, while Pivovarchik, in [58], solves the uniqueness aspect of the inverse spectral problem for a star shaped graph with four nodes. In particular he shows that four spectra (corresponding to different boundary conditions) uniquely determine the potential on the graph. Yurko, in [77], investigates two inverse spectral problems on trees. He recovers the operator from the Weyl functions as well as from a system of spectra. In this context [17] should be noted for its work on the closely related problem of Borg-type theorems for matrix valued Schrödinger operators.

In Chapter 6 we consider solutions of non-homogeneous boundary value problems on graphs. Particular attention is paid to the relationship between the boundary norms of solutions to the non-homogeneous boundary value problem and the norm of the non-homogeneous term on the graph, see Theorem 6.2.

In addition the eigenparameter dependence of this relationship is explored. To complete the chapter an example is provided in Section 6.2, illustrating Theorem 6.2.

In [63, 64], Schauder considers interior estimates and estimates near the boundary for solutions of second order elliptic boundary value problems. His estimates near the boundary are for solutions of the Dirichlet problem. Estimates near the boundary for other than Dirichlet boundary conditions have been obtained by Miranda, [51], for second order elliptic boundary value problems and by Agmon, Douglis, Nirenberg and Browder, [2, 11], for arbitrary order elliptic operators. In the above references it should be noted that the estimates are given in a region near the boundary whereas our results provide estimates on the boundary.

In Chapter 7 we solve both the uniqueness and reconstruction aspects of the inverse nodal problem on graphs, see Theorem 7.2 for the explicit statement. This is achieved by considering eigenfunction and eigenvalue asymptotics so as to obtain an asymptotic estimate for the spacing of nodal points of an eigenfunction.

Inverse nodal problems for the scalar Sturm-Liouville equation have been studied by, amongst others, [33], [65] and [75]. In particular, in [50], McLaughlin proves the following uniqueness result:
Let $q_1, q_2 \in \mathcal{L}^2(0,1)$ and consider the eigenvalue problem

$$y'' + (\lambda - q_i)y = 0,$$

$$y(0) = y(1) = 0.$$

Suppose that the positions of the nodal points satisfy

$$x_n^{j(n)}(q_1) = x_n^{j(n)}(q_2)$$

where $j(n)$ is specifically chosen so that $\{x_n^{j(n)}(q_i)\}_{n=2}^{\infty}$, $i = 1, 2$, is dense in $(0, 1)$. Also suppose that

$$\int_0^1 q_1\,dx = \int_0^1 q_2\,dx$$

then $q_1 = q_2$ almost everywhere.

In other words if all eigenvalues are known and if the position of a particular node for each eigenfunction is known as well as the average of q on the interval, then there is at most one $q \in \mathcal{L}^2(0, 1)$ which can yield that set of nodes and spectrum.

A vectorial inverse nodal problem was considered by Shen and Shieh in [66]. They showed that if the potential, $P(x)$, is a 2×2 symmetric matrix-valued function on $[0, 1]$ and there exists an infinite sequence $\{y(x, \lambda_{n_j})\}_{j=1}^{\infty}$ of Dirichlet eigenfunctions of the operator $-\frac{d^2}{dx^2} + P(x)$ whose components all have zeros in common, then $P(x)$ is diagonalizable on $[0, 1]$. This result was extended by Cheng, Shieh and Law in [15] to more general boundary conditions of the form

$$\begin{aligned} Ay(0) + I_d y'(0) &= 0 && (1.2) \\ By(1) + I_d y'(1) &= 0 && (1.3) \end{aligned}$$

where they show the simultaneous diagonalizability of $(P(x), A, B)$. Our interest in the vectorial nodal problem stems from Chapter 3 where we showed that a boundary value problem on a graph can be reformulated as a system boundary value problem on $[0, 1]$ with separated boundary conditions. However when transforming the boundary value problem on a graph to a system boundary value problem we have immediately, from the graph structure, that the potential matrix is diagonal. Also, when transforming the general boundary conditions on the graph to boundary conditions for the system we find that the coefficient matrices of $y(0)$ and $y(1)$ are not symmetric, and those of $y'(0)$ and $y'(1)$ are not necessarily the identity. Therefore the inverse nodal problem which we consider is strictly more general then that solved in [15] and [66].

Chapter 2

Preliminaries

In this chapter the boundary value problem which forms the topic of this thesis is stated and the properties there of, required for our later development, given.

In particular, in Section 2.1, various function spaces are defined and certain properties are given.

An operator formulation of the boundary value problem is given in Section 2.2. Following this we show that the operator associated with the boundary value problem is lower-semi-bounded.

In Section 2.3, it is shown that a differential operator on a weighted directed graph can be considered as an ordinary differential system

$$\tilde{L}\tilde{Y} := -W\tilde{Y}'' + Q\tilde{Y} = \lambda\tilde{Y}$$

on $(0, 1)$ where the weight matrix W is a constant positive diagonal matrix and Q is a diagonal matrix with real valued entries from $\mathcal{L}^\infty((0, 1))$.

In the last section, Section 2.4, we prove that for differential operators on graphs, self-adjointness does not imply regularity. This is of importance since some of the results which we will prove in this thesis are well known for regular problems but not for irregular problems.

2.1 Function Spaces on Graphs

Let G denote a directed graph with a finite number of edges, say K, each of finite length and having the path-length metric. Each edge, e_i, of length say l_i can thus be considered as the interval $(0, l_i)$ where 0 is identified with the initial point of e_i and l_i with the terminal point.

The following classes of function spaces will be used in this thesis, the first three of which are Hilbert spaces when given Sobolev norms:

$$\begin{aligned}
\mathcal{L}^2(G) &:= \bigoplus_{i=1}^{K} \mathcal{L}^2(0, l_i), \\
\mathcal{H}^m(G) &:= \bigoplus_{i=1}^{K} \mathcal{H}^m(0, l_i), \quad m = 0, 1, 2, ..., \\
\mathcal{H}^m_o(G) &:= \bigoplus_{i=1}^{K} \mathcal{H}^m_o(0, l_i), \quad m = 0, 1, 2, ..., \\
\mathcal{C}^\omega(G) &:= \bigoplus_{i=1}^{K} \mathcal{C}^\omega[0, l_i], \quad \omega = \infty, 0, 1, 2, ..., \\
\mathcal{C}^\omega_o(G) &:= \bigoplus_{i=1}^{K} \mathcal{C}^\omega_o(0, l_i), \quad \omega = \infty, 0, 1, 2, ...
\end{aligned}$$

where by $f \in \mathcal{C}^\omega[0, l_i]$ we mean a function $f = \overline{f}|_{(0,l_i)}$ where $\overline{f} \in \mathcal{C}^\omega(\mathbb{R})$.

The inner product on $\mathcal{H}^m(G)$, denoted $(\cdot, \cdot)_m$, is defined by

$$(f, g)_m := \sum_{i=1}^{K} \sum_{j=0}^{m} \int_0^{l_i} f|_{e_i}^{(j)} \ \bar{g}|_{e_i}^{(j)} \ dt =: \sum_{j=0}^{m} \int_G f^{(j)} \ \bar{g}^{(j)} \ dt. \tag{2.1}$$

The inner products on $\mathcal{L}^2(G)$ and $\mathcal{H}_o^m(G)$ follow from noting that $\mathcal{L}^2(G) = \mathcal{H}^0(G)$ and $\mathcal{H}_o^m(G) \subset \mathcal{H}^m(G)$. For brevity we will write $(\cdot,\cdot) = (\cdot,\cdot)_0$, $\|f\|_m^2 = (f,f)_m$ and $\|f\| = \|f\|_0$.

Theorem 2.1 (Rellich's Theorem) *Let G denote a directed graph with a finite number of edges. Then the embedding of $\mathcal{H}^m(G)$ in $\mathcal{H}^n(G)$ for $n < m$ is compact.*

Proof: Let K denote the number of edges of G. Identify the edge e_i of the graph G with the interval $(0, l_i)$, which is obviously bounded. We may thus apply Theorem 7.2 of [74, p. 114] to the interval $(0, l_i)$ and thereby obtain the compact embedding of $\mathcal{H}^m(0, l_i)$ into $\mathcal{H}^n(0, l_i)$ for all $n < m$ and $i = 1, \ldots, K$. Thus $\bigoplus_{i=1}^{K} \mathcal{H}^m(0, l_i)$ is compactly embedded into $\bigoplus_{i=1}^{K} \mathcal{H}^n(0, l_i)$ for $n < m$. Hence the embedding of $\mathcal{H}^m(G)$ into $\mathcal{H}^n(G)$ is compact for $n < m$. ■

Theorem 2.2 (Sobolev's Lemma) *Let G be a directed graph with a finite number of edges, say K. For each $f \in \mathcal{H}^m(G)$ there exists a unique $g \in \mathcal{C}^{m-1}(G)$ such that $f^{(k)} = g^{(k)}$ a.e on G for all $k = 0, \ldots, m-1$. With this identification*

$$\mathcal{H}^m(G) \subset \mathcal{C}^n(G), \quad \textit{for} \quad n < m,$$

and there exists a constant $C(G, m) > 0$ such that

$$\sup_G |f^{(k)}| \leq C(G, m)\|f\|_m.$$

Proof: If we consider the single edge e_i, we may apply Sobolev's Lemma, [74, p. 107], to e_i, giving

$$\mathcal{H}^m(0, l_i) \subset \mathcal{C}^n(0, l_i), \quad \text{for} \quad n < m.$$

This holds for all $i = 1, \ldots, K$. Thus

$$\mathcal{H}^m(G) = \bigoplus_{i=1}^{K} \mathcal{H}^m(0, l_i) \subset \bigoplus_{i=1}^{K} \mathcal{C}^n(0, l_i) = \mathcal{C}^n(G)$$

for $n < m$.

We also have from [74, p. 107] that

$$\sup_{e_i} \left| f|_{e_i}^{(k)} \right| \leq C_i(e_i, m) \left\| f|_{e_i} \right\|_m .$$

Now

$$\begin{aligned} \sup_G |f^{(k)}| &= \max_i \left\{ \sup_{e_i} \left| f|_{e_i}^{(k)} \right| \right\} \\ &\leq \max_i \{ C_i(e_i, m) \left\| f|_{e_i} \right\|_m \} \\ &\leq C(G, m) \max_i \left\| f|_{e_i} \right\|_m \end{aligned}$$

where $C(G, m) = \max_i C_i(e_i, m)$ and

$$\begin{aligned} \left\| f|_{e_i} \right\|_m^2 &= \sum_{j=0}^{m} \int_0^{l_i} \left| f|_{e_i}^{(j)} \right|^2 dt \\ &\leq \sum_{i=1}^{K} \sum_{j=0}^{m} \int_0^{l_i} \left| f|_{e_i}^{(j)} \right|^2 dt \qquad (2.2) \\ &= \|f\|_m^2. \end{aligned}$$

Thus

$$\sup_G \left| f^{(k)} \right| \leq C(G, m) \|f\|_m$$

for all $f \in \mathcal{H}^m(G)$ and $k < m$. ■

Theorem 2.3 (Ehrling's Lemma) *Let G be a directed graph with a finite number of edges, say K. For each $\epsilon > 0$ there exists a constant $C(G, m, \epsilon)$ such that*

$$\|f\|_{m-1} \leq \epsilon \|f\|_m + C(G, m, \epsilon) \|f\|_0 \quad \textit{for all} \quad f \in \mathcal{H}^m(G),$$

Proof: We reason componentwise. Consider the edge e_i identified with the interval $(0, l_i)$. From [74, p. 114] we have that for each $\frac{\epsilon}{K} > 0$ there exists a constant $C_i\left(e_i, m, \frac{\epsilon}{K}\right)$ with

$$\|f|_{e_i}\|_{m-1} \leq \frac{\epsilon}{K}\|f|_{e_i}\|_m + C_i\left(e_i, m, \frac{\epsilon}{K}\right)\|f|_{e_i}\|_0 \quad \text{for all} \quad f \in \mathcal{H}^m(e_i).$$

Now

$$\begin{aligned}
\|f\|_{m-1} &= \sqrt{\sum_{i=1}^{K}\sum_{j=1}^{m-1}\int_0^{l_i}\left|f|_{e_i}^{(j)}\right|^2 dt} \\
&= \sqrt{\sum_{i=1}^{K}\|f|_{e_i}\|_{m-1}^2} \\
&\leq \sum_{i=1}^{K}\|f|_{e_i}\|_{m-1} \\
&\leq \sum_{i=1}^{K}\left(\frac{\epsilon}{K}\|f|_{e_i}\|_m + C_i\left(e_i, m, \frac{\epsilon}{K}\right)\|f|_{e_i}\|_0\right),
\end{aligned}$$

and using equation (2.2), we obtain that

$$\begin{aligned}
\|f\|_{m-1} &= \frac{\epsilon}{K}\sum_{i=1}^{K}\|f\|_m + \sum_{i=1}^{K}C_i\left(e_i, m, \frac{\epsilon}{K}\right)\|f\|_0 \\
&= \epsilon\|f\|_m + C(G, m, \epsilon)\|f\|_0
\end{aligned}$$

where $C(G, m, \epsilon) = K\max_i C_i\left(e_i, m, \frac{\epsilon}{K}\right)$. ■

Theorem 2.4 (Sobolev's Inequality) *Let G be a directed graph with a finite number of edges, say K. If $m \geq 1$, then $\mathcal{H}^m(G) \subset \mathcal{L}^\infty(G)$ and there exists a constant $K(G, m)$ such that for every $\epsilon \geq 1$ and each function $f \in \mathcal{H}^m(G)$ the inequality*

$$||f||_{\mathcal{L}^\infty(G)} \leq K(G, m)\epsilon^{\frac{1}{2m}-1}(||f||_m + \epsilon||f||_0)$$

holds.

Proof: From [57] we have that if $m \geq 1$ then $\mathcal{H}^m(0,l_i) \subset \mathcal{L}^\infty(0,l_i)$ and there exists a constant $K_i(e_i,m) > 0$ such that for every $\epsilon \geq 1$ and each function $f|_{e_i} \in \mathcal{H}^m(0,l_i)$ the inequality

$$||f|_{e_i}||_{\mathcal{L}^\infty(o,l_i)} \leq K_i(e_i,m)\epsilon^{\frac{1}{2m}-1}(||f|_{e_i}||_m + \epsilon||f|_{e_i}||_0)$$

holds.

Now

$$\begin{aligned} ||f||_{\mathcal{L}^\infty(G)} &= \max_i ||f|_{e_i}||_{\mathcal{L}^\infty(0,l_i)} \\ &\leq \epsilon^{\frac{1}{2m}-1} \max_i K_i(e_i,m)(||f|_{e_i}||_m + \epsilon||f|_{e_i}||_0) \\ &\leq K(G,m)\epsilon^{\frac{1}{2m}-1} \max_i(||f|_{e_i}||_m + \epsilon||f|_{e_i}||_0). \end{aligned}$$

where $K(G,m) = \max_i K_i(e_i,m)$.

Thus by equation (2.2)

$$||f||_{\mathcal{L}^\infty(G)} \leq K(G,m)\epsilon^{\frac{1}{2m}-1}(||f||_m + \epsilon||f||_0). \quad \blacksquare$$

2.2 Boundary Value Problems on Graphs

The differential equation (1.1) on the graph G can now be considered as the system of equations

$$-\frac{d^2y_i}{dx^2} + q_i(x)y_i = \lambda y_i, \quad x \in (0,l_i),\ i = 1,...,K, \tag{2.3}$$

where q_i and y_i denote $q|_{e_i}$ and $y|_{e_i}$.

The boundary conditions at the node ν are specified in terms of the values of y and y' at ν on each of the incident edges. In particular if the edges which start at ν are $e_i, i \in \Lambda_s(\nu)$ and the edges which end at ν are $e_i, i \in \Lambda_e(\nu)$ then the boundary conditions at ν can be expressed as

$$\sum_{j\in\Lambda_s(\nu)} \left[\alpha_{ij}y_j + \beta_{ij}{y'}_j\right](0) + \sum_{j\in\Lambda_e(\nu)} \left[\gamma_{ij}y_j + \delta_{ij}{y'}_j\right](l_j) = 0, \tag{2.4}$$

for $i = 1, ..., N(\nu)$, where $N(\nu)$ is the number of linearly independent boundary conditions at node ν.

Let $\alpha_{ij} = 0 = \beta_{ij}$ for $i = 1, ..., N(\nu)$ and $j \notin \Lambda_s(\nu)$ and similarly let $\gamma_{ij} = 0 = \delta_{ij}$ for $i = 1, ..., N(\nu)$ and $j \notin \Lambda_e(\nu)$. The boundary conditions (2.4) considered over all nodes ν, after possible relabelling, may thus be written as

$$\sum_{j=1}^{K} \left[\alpha_{ij}y_j + \beta_{ij}{y'}_j\right](0) + \sum_{j=1}^{K} \left[\gamma_{ij}y_j + \delta_{ij}{y'}_j\right](l_j) = 0, \quad i = 1, ..., 2K, \tag{2.5}$$

where $2K$ is the total number of linearly independent boundary conditions. It should be noted that the complete geometry of the graph G (other than the number of and length of the edges) is encapsulated in the boundary conditions.

To ensure formal self-adjointness we require the Lagrange form, $(lf, g)-(f, lg)$, to vanish for all $f, g \in \mathcal{C}^2(G)$ obeying (2.4). For formally self-adjoint boundary conditions $N(\nu) = \sharp(\Lambda_s(\nu)) + \sharp(\Lambda_e(\nu))$ and $\sum_\nu N(\nu) = 2K$. The formulation of self-adjoint boundary value problems on graphs was studied in detail in [12], and the class of self-adjoint boundary conditions was characterized in [34] and [38].

The boundary value problem (2.3)-(2.4) on G can be formulated as an operator eigenvalue problem in $\mathcal{L}^2(G)$, [1, 12, 67], for the closed densely defined operator

$$Lf := -f'' + qf \tag{2.6}$$

with domain

$$\mathcal{D}(L) = \{f \mid f, f' \in AC, Lf \in \mathcal{L}^2(G),\ f \text{ obeying (2.4) }\}, \tag{2.7}$$

or equivalently

$$\mathcal{D}(L) = \{f \mid f \in \mathcal{H}^2(G),\ f \text{ obeying (2.4) }\},$$

since $\mathcal{H}^m$ spaces can be defined in terms of absolutely continuous functions, see [52].

The formal self-adjointness of (2.3)-(2.4) ensures that L is a closed densely defined self-adjoint operator in $\mathcal{L}^2(G)$, see [76, p. 77-78].

Corollary 2.5 *[72, page 247, Corollary 2] If L_0 is a closed symmetric operator, with finite defect indices, bounded from below on a complex Hilbert space and L is a self-adjoint extension of L_0 then L is lower semibounded.*

Theorem 2.6 *The operator L is lower semibounded.*

Proof: From the above corollary as L is self adjoint, we need only show that L is lower semibounded on $\mathcal{C}_o^\infty(G)$. If $f \in \mathcal{C}_o^\infty(G)$ then

$$(Lf, f) = \int_G (-f''\bar{f} + q|f|^2)\ dx = \int_G (|f'|^2 + q|f|^2)\ dx \geq -\|f\|^2 \text{ess sup}\,|q|. \quad \blacksquare$$

2.3 System Formulation

We now show that the boundary value problem on a graph can be reformulated as a boundary value problem for a system on the interval $(0, 1)$.

Consider the edge e_i of length l_i, we then have

$$-y_i''(x) + q_i(x)y_i(x) = \lambda y_i(x) \qquad \text{on} \quad (0, l_i).$$

Let $t = \frac{x}{l_i}$ and $\tilde{y}_i(t) = y_i(l_i t)$. Then

$$-\frac{d^2}{dt^2}[\tilde{y}_i(t)] = -l_i^2 y_i''(l_i t) = l_i^2(\lambda y_i(l_i t) - q_i(l_i t) y_i(l_i t)) = l_i^2(\lambda - Q_i(t))\tilde{y}_i(t),$$

where $Q_i(t) = q_i(l_i t)$.

Thus for each $i = 1, ..., K$ our transformed equation is

$$-\tilde{y}_i'' + l_i^2(Q_i - \lambda)\tilde{y}_i = 0 \quad \text{on} \quad (0,1)$$

giving the system

$$\tilde{L}\tilde{Y} := -W\tilde{Y}'' + Q\tilde{Y} = \lambda\tilde{Y} \tag{2.8}$$

where $W = \operatorname{diag}\left[\frac{1}{l_1^2}, \ldots, \frac{1}{l_K^2}\right]$, $\tilde{Y} = \begin{bmatrix} \tilde{y}_1 \\ \vdots \\ \tilde{y}_K \end{bmatrix}$ and $Q = \operatorname{diag}[Q_1, \ldots, Q_K]$.

We now consider the boundary conditions. After performing the above transformation on each edge we have that all our edges are now of length 1 and thus we only have the endpoints at 0 and 1. Hence the boundary conditions may be written in matrix form as

$$\tilde{A}\tilde{Y}(0) + \tilde{B}\tilde{Y}'(0) + \tilde{C}\tilde{Y}(1) + \tilde{D}\tilde{Y}'(1) = 0 \tag{2.9}$$

where $\tilde{A} = [\alpha_{ij}]$, $\tilde{B} = \left[\frac{\beta_{ij}}{l_j}\right]$, $\tilde{C} = [\gamma_{ij}]$ and $\tilde{D} = \left[\frac{\delta_{ij}}{l_j}\right]$.

Thus our original boundary value problem on the graph G is equivalent to the system boundary value problem with differential equation (2.8) and boundary conditions (2.9).

Let $\mathcal{L}_K^2$ denote the weighted vector $\mathcal{L}^2$ space

$$\mathcal{L}_K^2 = \{F : (0,1) \to \mathbb{C}^K \mid F_i \in \mathcal{L}^2(0,1), i = 1, ..., K\}$$

with inner product

$$< F, G >_W = \sum_{i=1}^{K} l_i \int_0^1 F_i \bar{G}_i \, dt = \int_0^1 F^T W^{-\frac{1}{2}} \overline{G} dt. \qquad (2.10)$$

It should be noted that $\mathcal{L}_K^2$ is isometrically isomorphic to $\mathcal{L}^2(G)$ under the identification $\mathcal{L}^2(G) \to \mathcal{L}_K^2$ defined by

$$f(x) \mapsto \begin{bmatrix} f|_{e_1}(l_1 t) \\ \vdots \\ f|_{e_K}(l_K t) \end{bmatrix}$$

where $x \in G$ and $t \in (0, 1)$.

The boundary value problem (2.8) and (2.9) can be reformulated as an operator eigenvalue problem, [72], by setting

$$\tilde{L}F = -WF'' + QF$$

with domain

$$\mathcal{D}(\tilde{L}) = \{F| \ F, F' \in AC, \tilde{L}F \in \mathcal{L}^2(G), \ F \text{ obeying } (2.9) \ \}.$$

Theorem 2.7 *The system (2.8) and (2.9) is formally self-adjoint in $\mathcal{L}_K^2$ if and only if the boundary value problem (2.3) and (2.4) in $\mathcal{L}^2(G)$ is formally self-adjoint.*

Proof: Let $F, G : (0, 1) \to \mathbb{C}^K$ be C^2 and denote by f and g the functions on G defined by $f|_{e_i}(l_i t) = F_i(t)$ and $g|_{e_i}(l_i t) = G_i(t)$ for $i = 1, ..., K$ and $t \in (0, 1)$, then under this identification

$$< \tilde{L}F, G >_W - < F, \tilde{L}G >_W \ = \ -\sum_{i=1}^{K} l_i^{-1} \int_0^1 [F_i'' \bar{G}_i - F_i \bar{G}_i''] \, dt$$

$$\begin{aligned} &= -\sum_{i=1}^{K} l_i^{-1}[F_i'\bar{G}_i - F_i\bar{G}_i']_0^1 \\ &= \sum_{i=1}^{K}[(f'\bar{g} - f\bar{g}')|_{e_i}]_0^{l_i} \\ &= (Lf, g) - (f, Lg) \end{aligned}$$

and (2.4) holds if and only if (2.9). ■

In this setting the formal self-adjointness of (2.8) and (2.9) ensures the that the operator $\tilde{L}$ on $\mathcal{L}_K^2$ is a closed densely defined self-adjoint operator and thus the formal self-adjointness of (2.3) and (2.4) ensures that $\tilde{L}$ is a closed densely defined self-adjoint operator in $\mathcal{L}_K^2$, see [76, p. 77-78].

2.4 Irregularity

In this section we show that self-adjointness does not necessarily imply regularity, in fact in most cases it does not.

Without loss of generality we may assume that our boundary conditions are normalised, i.e. of the form

$$U_1(Y) = U_{10}(Y) - U_{11}(Y) = 0$$

$$U_2(Y) = U_{20}(Y) - U_{21}(Y) = 0$$

where

$$U_{10}(Y) = A_1Y'(0) + A_{10}Y(0)$$

$$U_{20}(Y) = A_2Y'(0) + A_{20}Y(0)$$

$$U_{11}(Y) = B_1Y'(1) + B_{10}Y(1)$$

$$U_{21}(Y) = B_2Y'(1) + B_{20}Y(1)$$

where for each $i = 1, 2$, at least one of the matrices A_i, B_i, is different from zero. If $A_i = 0$ then by the normalisation process given in [53, p. 120] we will obtain that A_{i0} will then become A_i and similarly for $B_i = 0$.

Following [53, p. 121], we define regularity of boundary conditions as follows.

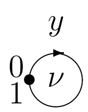
Definition 2.8 *The normalised boundary conditions, above, are said to be regular if both the numbers χ_- and χ_+ defined by*

$$\chi_- = i^{2n} \det \begin{bmatrix} W^{-\frac{1}{2}}B_1 & -A_1 \\ W^{-\frac{1}{2}}B_2 & -A_2 \end{bmatrix}, \quad \chi_+ = i^{2n} \det \begin{bmatrix} A_1 & -W^{\frac{1}{2}}B_1 \\ A_2 & -W^{\frac{1}{2}}B_2 \end{bmatrix}$$

do not vanish. Where W is the constant, positive, diagonal weight matrix of (2.8).

We make use of a counter example to show that even a simple self-adjoint boundary value problem on a graph need not be regular.

Consider the graph



with one node, ν, and the second order operator

$$\frac{-d^2y}{dx^2} + qy = \lambda y,$$

with boundary conditions of the form

$$y(0) = y(1),$$

$$y'(0) = y'(1),$$

at ν.

We then have that

$$\chi_+ = -\det \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} = 0$$

i.e. we don't have regularity.

Most self-adjoint problems on graphs are not regular, as is evident from the above example.

Chapter 3

Eigenvalue Asymptotics

We now consider the spectral structure of the boundary value problem (2.3), (2.4) on the graph G.

In Section 3.1, we show that the system (2.8) with boundary conditions (2.9) is equivalent to the formally self-adjoint system (3.1) on $(0, 1)$ with separated boundary conditions (3.2), (3.3).

Eigenvalue multiplicities are then considered in Section 3.2. Here we show that for self-adjoint differential operators on graphs the algebraic and geometric multiplicities are equal, i.e. the differential operator L defined in (2.6), (2.7) is semi-simple.

In Section 3.3, we develop asymptotic solutions for the system (3.1). These are then used in Section 3.4 where a general form for the Green's function of the boundary value problem (3.1)-(3.3) is given and thus implicitly for the boundary value problem on the graph. As a consequence, we obtain that the resolvent of our differential operator is compact.

In Section 3.5 we summarize, from Etgen, [22], the background material required in the final section, Section 3.6. Here, using abstract Prüfer methods, see [4] and [22], we find explicit asymptotic bounds for the eigenvalues.

3.1 Separated Boundary Conditions

In Section 2.3, we showed that the formally self-adjoint boundary value problem (2.3) and (2.4) could be reformulated as the formally self-adjoint boundary value problem for the system (2.8) with boundary conditions (2.9). In general the boundary conditions (2.9) are not separated, i.e. (2.9) cannot be equivalently written as

$$\begin{aligned} P\tilde{Y}(0) + Q\tilde{Y}'(0) &= 0, \\ R\tilde{Y}(1) + S\tilde{Y}'(1) &= 0 \end{aligned}$$

for suitable matrices P, Q, R and S.

In this section, we show that the system (2.8) with boundary conditions (2.9) can be replaced by a formally self-adjoint system of dimension $2K$, where K is as given in equation (2.3), on $(0, 1)$ with separated boundary conditions. This new system is equivalent to the system (2.8) with boundary conditions (2.9) generated by introducing a vertex, m_i, at the mid-point of each edge, e_i, and imposing the boundary conditions

$$\begin{aligned} y(m_i^-) &= y(m_i^+), \\ y'(m_i^-) &= y'(m_i^+) \end{aligned}$$

for $i = 1, ..., K$. It should be noted that these represent formally self-adjoint boundary conditions at the vertex m_i, and as the boundary conditions at each vertex ν of our original graph G are formally self-adjoint, the resulting boundary value problem is formally self-adjoint.

Definition 3.1 *The geometric multiplicity of an eigenvalue λ_0 of (3.1)-(3.3) is defined to be the number of linearly independent solutions of the boundary value problem for $\lambda = \lambda_0$.*

Let

$$< F, G >_M = \sum_{i=1}^{2K} l_i \int_0^1 F_i \bar{G}_i \, dt = \int_0^1 F^T M^{-\frac{1}{2}} \overline{G} dt,$$

where $l_i = l_{K+i}$ for $i = 1, \ldots, K$.

The following theorem provides a rigorous formulation of the above discussion in terms of the system (2.8) and its boundary conditions (2.9).

Theorem 3.2 *The system (2.8) with boundary conditions (2.9) formally self-adjoint (with respect to the inner product given in (2.10)), is equivalent to the formally self-adjoint (with respect to the inner product $< F, G >_M$) system*

$$\tau_P Y := -MY'' + PY = \lambda Y \tag{3.1}$$

with boundary conditions

$$A^* Y(0) - B^* Y'(0) = 0, \tag{3.2}$$

$$\Gamma^* Y(1) - \Delta^* Y'(1) = 0 \tag{3.3}$$

where $M = 4 \begin{bmatrix} W & 0 \\ 0 & W \end{bmatrix}$, $P = \begin{bmatrix} Q\left(\frac{t+1}{2}\right) & 0 \\ 0 & Q\left(\frac{1-t}{2}\right) \end{bmatrix}$, $A^* = \begin{bmatrix} I & -I \\ 0 & 0 \end{bmatrix}$, $-B^* = \begin{bmatrix} 0 & 0 \\ I & I \end{bmatrix}$, $\Gamma^* = [\tilde{C} \ \tilde{A}]$ *and* $-\Delta^* = 2[\tilde{D} \ -\tilde{B}]$. *By equivalent we mean* *that eigenvalues and their geometric multiplicites are preserved and eigenfunctions are mapped by*

$$\tilde{Y}(t) \mapsto Y(t) := \begin{bmatrix} \tilde{Y}\left(\frac{t+1}{2}\right) \\ \tilde{Y}\left(\frac{1-t}{2}\right) \end{bmatrix}.$$

Proof: Assume λ is an eigenvalue of (3.1)-(3.3) with multiplicity j. Then the solution Y of (3.1)-(3.3) can be written as a matrix made up of j columns where the columns are j linearly independent eigenvectors corresponding to the eigenvalue λ.

Let Y_1 and Y_2 be $K \times j$ matrices such that

$$Y := \begin{bmatrix} Y_1 \\ --- \\ Y_2 \end{bmatrix}$$

and define

$$\tilde{Y}(s) = \begin{cases} Y_1(2s-1), & \text{for all } \ \frac{1}{2} \le s \le 1 \\ Y_2(1-2s), & \text{for all } \ 0 \le s \le \frac{1}{2} \end{cases} \tag{3.4}$$

By equation (3.2) we have that $\tilde{Y}$ and $\tilde{Y}'$ are continuous at $s = \frac{1}{2}$ and therefore on the entire interval $(0, 1)$.

Now from equation (3.3) we obtain that

$$[\tilde{C}\ \tilde{A}] \begin{bmatrix} Y_1(1) \\ Y_2(1) \end{bmatrix} + 2[\tilde{D}\ \ -\tilde{B}] \begin{bmatrix} Y_1'(1) \\ Y_2'(1) \end{bmatrix} = 0,$$

giving

$$\tilde{C}Y_1(1) + \tilde{A}Y_2(1) + 2\tilde{D}Y_1'(1) - 2\tilde{B}Y_2'(1) = 0$$

which by (3.4) gives (2.9). Also

$$-M \begin{bmatrix} Y_1'' \\ Y_2'' \end{bmatrix} + P \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \lambda \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

so

$$-4WY_1'' + Q\left(\frac{t+1}{2}\right)Y_1 = \lambda Y_1$$

and

$$-4WY_2'' + Q\left(\frac{1-t}{2}\right)Y_2 = \lambda Y_2$$

which in terms of $\tilde{Y}$ gives equation (2.8).

From the above reasoning we can also conclude that the dimension, n, of the eigenspace of (2.8), (2.9) is at least equal to j, i.e. $n \geq j$, as the columns of $\tilde{Y}$ are linearly independent.

Conversely if λ is an eigenvalue of (2.8), (2.9) with multiplicity n, then the solution $\tilde{Y}$ of (2.8), (2.9) can be written as a matrix made up of n linearly independent columns each of which is an eigenvector of (2.8), (2.9) corresponding to the eigenvalue λ.

Using the mapping given in the statement of the theorem we have that

$$Y(0) = \begin{bmatrix} \tilde{Y}\left(\frac{1}{2}^{+}\right) \\ \tilde{Y}\left(\frac{1}{2}^{-}\right) \end{bmatrix}$$

and

$$Y(1) = \begin{bmatrix} \tilde{Y}(1) \\ \tilde{Y}(0) \end{bmatrix}.$$

Also

$$Y'(t) = \begin{bmatrix} \frac{1}{2}\tilde{Y}'\left(\frac{t+1}{2}\right) \\ \frac{-1}{2}\tilde{Y}'\left(\frac{1-t}{2}\right) \end{bmatrix}$$

giving that

$$Y'(0) = \frac{1}{2}\begin{bmatrix} \tilde{Y}'\left(\frac{1}{2}^{+}\right) \\ -\tilde{Y}'\left(\frac{1}{2}^{-}\right) \end{bmatrix}.$$

Therefore $\begin{bmatrix} I & -I \\ 0 & 0 \end{bmatrix} Y(0) = 0$ and $\begin{bmatrix} 0 & 0 \\ -I & -I \end{bmatrix} Y'(0) = 0.$

Hence $A^*Y(0) - B^*Y'(0) = 0$, i.e. (3.2), holds. Also

$$Y'(1) = \frac{1}{2}\begin{bmatrix} \tilde{Y}'(1) \\ -\tilde{Y}'(0) \end{bmatrix}$$

and from equation (2.9)

$$[\tilde{C} \quad \tilde{A}]Y(1) + 2[\tilde{D} \quad -\tilde{B}]Y'(1) = 0$$

giving that $\Gamma^*Y(1) - \Delta^*Y'(1) = 0$, i.e. equation (3.3) holds.

Lastly since

$$Y''(t) = \frac{1}{4}\begin{bmatrix} \tilde{Y}''\left(\frac{t+1}{2}\right) \\ \tilde{Y}''\left(\frac{1-t}{2}\right) \end{bmatrix}$$

we obtain that

$$\begin{aligned} -4\begin{bmatrix} W & 0 \\ 0 & W \end{bmatrix} Y''(t) &= \begin{bmatrix} -W\tilde{Y}''\left(\frac{t+1}{2}\right) \\ -W\tilde{Y}''\left(\frac{1-t}{2}\right) \end{bmatrix} \\ &= \begin{bmatrix} (\lambda - Q(\frac{t+1}{2}))\tilde{Y}\left(\frac{t+1}{2}\right) \\ (\lambda - Q(\frac{1-t}{2}))\tilde{Y}\left(\frac{1-t}{2}\right) \end{bmatrix} \\ &= \lambda Y(t) - \begin{bmatrix} Q\left(\frac{t+1}{2}\right) & 0 \\ 0 & Q\left(\frac{1-t}{2}\right) \end{bmatrix} Y(t). \end{aligned}$$

Thus equation (3.1) holds and the dimension, j, of the eigenspace of (3.1)-(3.3) is at least equal to n, i.e. $j \geq n$.

In other words if λ is an eigenvalue of (3.1)-(3.3) then it is an eigenvalue of (2.8), (2.9) of the same multiplicity and vice versa. ■

It should be noted that M is a diagonal matrix with constant positive entries and P is a diagonal matrix with real essentially bounded entries on the diagonal.

3.2 Eigenvalue Multiplicities

We begin by defining the algebraic and geometric mutiplicites of an eigenvalue, following which we show that for self-adjoint boundary value problems on graphs the algebraic and geometric multiplicites are the same.

In [22] it is shown that the eigenvalues of (3.1)-(3.3) are given by the zeros of

$$\Lambda(1, \lambda) := \det[\Gamma^* Y(1, \lambda) - \Delta^* Y'(1, \lambda)].$$

Definition 3.3 *If λ_0 is a zero of $\Lambda(1, \lambda_0)$ of order ν with respect to λ then the algebraic multiplicity of λ_0 is ν.*

Note that in [53], Naimark shows that ν (the order of this zero of $\Lambda(1, \lambda)$) coincides with the algebraic multiplicity. The usual definition of the algebraic multiplicity of λ_0 is the maximal dimension of $N(L - \lambda_0)^k$, $k \in \mathbb{N}$.

Remark The eigenvalues of (2.8), (2.9) are given by the zeros of

$$\det[\tilde{A}\tilde{Y}(0) + \tilde{B}\tilde{Y}'(0) + \tilde{C}\tilde{Y}(1) + \tilde{D}\tilde{Y}'(1)] = \Lambda(1, \lambda), \tag{3.5}$$

see [53].

Thus if λ is an eigenvalue of (3.1)-(3.3) with algebraic multiplicity m then by Theorem 3.2 λ is an eigenvalue of (2.8), (2.9) and by (3.5) its algebraic multiplicity is m.

Let

$$T_P Y = -MY'' + PY \tag{3.6}$$

with domain

$$\mathcal{D}(T_P) = \{Y \mid Y, Y' \in AC, T_P(Y) \in \mathcal{L}^2(G),\ U_\gamma Y = 0, \gamma = 1, 2\}, \tag{3.7}$$

where U_γ, $\gamma = 1$, 2 correspond to the boundary conditions (3.2), (3.3). Then $T_p = T_p^*$, see Lemma 3.12.

Definition 3.4 *Let λ_0 be an eigenvalue of T_P with eigenfunction $\Phi(x)$. The functions $\Phi_1(x)$, $\Phi_2(x)$ $\ldots$ $\Phi_m(x)$ are said to be associated with the eigenfunction $\Phi(x) = \Phi_0(x)$ if*

$$U_\gamma(\Phi_\mu) = 0, \quad \mu = 1 \ldots m;\ \gamma = 1,\ 2,$$

(we have this simple expression since none of our boundary conditions depend on λ),

and for $\lambda = \lambda_0$ the following relations hold:

$$\begin{aligned} T_P(\Phi_0) &= \lambda\Phi_0 \\ (T_P - \lambda_0)\Phi_1 &= \Phi_0 \\ &\cdots \\ (T_P - \lambda_0)\Phi_m &= \Phi_{m-1} \end{aligned}$$

See [53, p.16].

Theorem 3.5 *The algebraic and geometric multiplicities of an eigenvalue λ_0 of T_P are equal.*

Proof: From [53, p. 20] we have that at an eigenvalue λ_0 of (3.1)-(3.3) the system of eigenfunctions and their associated functions generate a subspace, the dimension of which is equal to the multiplicity of the zero of $\Lambda(\lambda)$ at $\lambda = \lambda_0$. I.e. the associated functions complete the set of eigenfunctions to form a space of the necessary dimension.

Hence, all we need to show is that (3.1)-(3.3) has no associated functions.

Assume Y_0 is an eigenfunction with eigenvalue λ, then $T_P Y_0 = \lambda Y_0$, $Y_0 \in \mathcal{D}(T_P)$ with an associated function Y_1. From the definition of associated functions we have that

$$(T_P - \lambda)Y_1 = Y_0.$$

Now Y_1 can not be identically zero (as then $Y_0 = 0$, which is an eigenfunction) and we have that $Y_0 \in R(T_P - \lambda)$. But

$$R(T_P - \lambda) = (N(T_P - \lambda))^{\perp}, \quad \text{for } \lambda \in \mathbb{R},$$

giving $Y_0 \in (N(T_P - \lambda))^{\perp}$ i.e. $Y_0 \in N(T_P - \lambda) \cap (N(T_P - \lambda))^{\perp} = \{0\}$, contradiction.

Hence all Jordan chains are of length 1. ■

From the above theorem we can now say that the eigenvalues are semi-simple.

We also note that our system (3.1) is in $\mathbb{C}^{4K}$ and we have $2K$ boundary condition constraints, (3.2), at $x = 0$, thus the maximum multiplicity of an eigenvalue is $2K$.

3.3 Asymptotic Solutions

Theorem 3.6 *Let $\rho^2 = \lambda$. The solution matrix Y of (3.1) obeying the initial condition* $\begin{bmatrix} Y(0) \\ Y'(0) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ *is entire in ρ and can be represented as $Y = [\mathbb{U} \;\; \mathbb{V}](\rho, t) = [C(\rho, t) \;\; S(\rho, t)] + [O(\rho, 1) \;\; O(\rho, 2)]$ with derivative (with respect to t), $Y' = [\mathbb{U}' \;\; \mathbb{V}'](\rho, t) = [-\rho^2 M^{-1} S(\rho, t) \;\; C(\rho, t)] + [O(\rho, 0) \;\; O(\rho, 1)]$ asymptotically for $|\rho| \to \infty$. Here C and S are the diagonal matrices*

$$\begin{aligned} C(\rho, t) &= \operatorname{diag}\left(\cos_1(\rho, t), \ldots, \cos_{2K}(\rho, t)\right), \\ S(\rho, t) &= \operatorname{diag}\left(\sin_1(\rho, t), \ldots, \sin_{2K}(\rho, t)\right), \end{aligned}$$

where $\cos_i(\rho, t) := \cos\left(\frac{l_i \rho t}{2}\right)$ and $\sin_i(\rho, t) := \frac{2}{l_i \rho} \sin\left(\frac{l_i \rho t}{2}\right)$ and the error order is $O(\rho, k) = diag\left(O\left(\frac{e^{|\Im(tl_1\rho)|/2}}{\rho^k}\right), \ldots, O\left(\frac{e^{|\Im(tl_{2K}\rho)|/2}}{\rho^k}\right)\right)$.

Proof: Let $Y(t)$ be the solution of (3.1) with initial conditions as stated above. If we denote $u(t) = Y_{ij}(t)$, where $j \notin \{i, i+2K\}$, then u is the solution of a second order linear differential equation with initial conditions $u(0) = 0 = u'(0)$ and is thus zero on $(0, 1)$. Hence all entries of Y other than Y_{ii} and $Y_{i\ i+2K}$, $i = 1, \ldots, 2K$, are identically zero.

Now consider $u(t) = Y_{ii}(t)$. Here u is the solution of

$$u'' + \frac{l_i^2}{4}(\rho^2 - P_{ii})u = 0 \tag{3.8}$$

obeying the initial conditions $u(0) = 1$ and $u'(0) = 0$. Thus from [35, Appendix],

$$\begin{aligned} u(t) &= \cos_i(\rho, t) + O\left(\frac{e^{|\Im(tl_i\rho|/2}}{\rho}\right) \\ u'(t) &= -\frac{l_i^2 \rho^2}{4}\left(\sin_i(\rho, t) + O\left(\frac{e^{|\Im(tl_i\rho|/2}}{\rho^2}\right)\right). \end{aligned}$$

Finally for $u = Y_{i,i+2K}$, u is the solution of (3.8) with P_{ii} replaced by $P_{i,i+2K}$ obeying the initial conditions $u(0) = 0$ and $u'(0) = 1$. Thus from [35, Appendix],

$$\begin{aligned} u(t) &= \sin_i(\rho, t) + O\left(\frac{e^{|\Im(tl_i\rho|/2}}{\rho^2}\right) \\ u'(t) &= \cos_i(\rho, t) + O\left(\frac{e^{|\Im(tl_i\rho|/2}}{\rho}\right). \quad \blacksquare \end{aligned}$$

Remark In the case of $P \equiv 0$, the $O(\cdot)$ terms in the above theorem are identically zero.

3.4 Resolvent Operators

In this section we give a general form for the Green's function of a formally self-adjoint differential operator on a finite graph. As consequences we obtain that the spectrum of the operator is countably infinite and consists purely of point spectrum. In order to achieve this end, we need only prove that our differential operator has compact resolvent, see [69, p. 343].

Theorem 3.7 *Let T_P be as defined in equation (3.6) with domain given by (3.7). By Theorem 2.6 T_P is lower semibounded so we may assume, without loss of generality, that 0 is not an eigenvalue of T_P. Then T_P^{-1} exists and is a compact operator on $\mathcal{L}^2_{2K}$. The spectrum of T_P consists only of point spectrum, is real, countably infinite and has $+\infty$ as its only accumulation point. If (λ_n) denote the eigenvalues of T_P in increasing order, repeated according to geometric multiplicity, then the corresponding sequence (F_n) of eigenfunctions can be chosen so as to be a complete orthonormal family in $\mathcal{L}^2_{2K}$. If λ is not* *an eigenvalue of* T_P *and* $F \in \mathcal{L}^2_{2K}$ *then*

$$\begin{aligned} T^{-1}_{P-\lambda}F(x) &= \sum_{k=1}^{\infty} \frac{< F, F_k >_M}{\lambda_k - \lambda} F_k(x) && (3.9) \\ &= \int_0^1 \mathbb{G}(x,t)F(t)\ dt && (3.10) \end{aligned}$$

where

$$\mathbb{G}(x,t) = [\mathbb{U}(x)\ \mathbb{V}(x)](J(\mathcal{N}J)^{-1}\mathcal{N} - I_{[0,x]}(t)) \begin{bmatrix} -\mathbb{V}(t) \\ \mathbb{U}(t) \end{bmatrix} \tag{3.11}$$

with $\mathbb{U}$ *and* $\mathbb{V}$ *as in Theorem 3.6,* $\mathcal{N} = [\Gamma^*\mathbb{U} - \Delta^*\mathbb{U}' \quad \Gamma^*\mathbb{V} - \Delta^*\mathbb{V}'](1)$ *and*
$J = \begin{bmatrix} I & 0 \\ I & 0 \\ 0 & I \\ 0 & -I \end{bmatrix}.$

Proof: The matrices M and P in Theorem 3.2 are diagonal, hence each component function z_i of Z on $(0,1)$, where $(T_P - \lambda)Z = F$, obeys a differential equation of the form

$$-M_{ii}z_i'' + (P_{ii} - \lambda)z_i = f_i \tag{3.12}$$

where f_i is the ith component of F. Thus, for $\lambda = \rho^2$, $-MZ'' + (P - \rho^2)Z = F$ has as a general solution,

$$\begin{aligned} Z(x) &= \mathbb{U}(x)C_0 + \mathbb{V}(x)C_1 + \mathbb{U}(x)\int_0^x \mathbb{V}(t)F(t)\ dt \\ &\quad + \mathbb{V}(x)\int_x^0 \mathbb{U}(t)F(t)\ dt, && (3.13) \end{aligned}$$

where C_0 and C_1 are $\mathbb{C}^{2K}$ constant vectors and $\mathbb{U}$ and $\mathbb{V}$ are diagonal matrix functions, as defined in Theorem 3.6 where their asymptotic forms are also given. This requires that $\mathbb{U}'\mathbb{V} - \mathbb{V}'\mathbb{U} = I$ which we may assume since $\mathbb{U}$ and $\mathbb{V}$ are both diagonal matrices and hence so are their derivatives, thus $\mathbb{U}'\mathbb{V} - \mathbb{V}'\mathbb{U}$

is a diagonal matrix with Wronskians on the diagonal. In addition we note that $Z(0) = C_0$ and $Z'(0) = C_1$. Imposing the boundary condition (3.2) gives that, if Z is to obey this boundary condition at $x = 0$ then C_0 and C_1 are of the form

$$C_i = \begin{bmatrix} I \\ (-1)^i I \end{bmatrix} H_i, \quad i = 0, 1,$$

where H_0 and H_1 can be any $\mathbb{C}^K$ vectors while the boundary condition at $x = 1$ imposes the condition that

$$\mathcal{N} \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} = \mathcal{N} \begin{bmatrix} -\int_0^1 \mathbb{V} F(t)\ dt \\ \int_0^1 \mathbb{U} F(t)\ dt \end{bmatrix}.$$

Imposing the boundary conditions at 0 and 1 is thus equivalent to requiring that H_0 and H_1 satisfy the equation

$$\mathcal{N} J \begin{bmatrix} H_0 \\ H_1 \end{bmatrix} = \mathcal{N} \int_0^1 \begin{bmatrix} -\mathbb{V}(t) \\ \mathbb{U}(t) \end{bmatrix} F(t)\ dt$$

which has a unique solution for non-eigenvalues. Routine computation now gives (3.10) and thus (3.11).

From (3.10) and (3.11) it follows that for λ not an eigenvalue of T_P the operator $(T_P - \lambda)^{-1}$ is a compact operator on $\mathcal{L}^2_{2K}$, it is in fact a Hilbert-Schmidt operator. Thus T_P is an operator with compact resolvent, from which the remaining claims of the theorem follow, see [69, p. 344]. ■

3.5 Matrix Prüfer Angles and Etgen's Approach

For the remainder of this chapter q is assumed to be real and continuous.

In this section we give the background material needed from [22] in order to obtain the eigenvalue asymptotics given later in the chapter.

In [22] the second order matrix system

$$Y' = K(x,\lambda)Z, \quad Z' = -G(x,\lambda)Y \tag{3.14}$$

where $x \in [a,b]$, $K(x,\lambda)$ and $G(x,\lambda)$ are $n \times n$ symmetric matrices of continuous real valued functions and $K(x,\lambda)$ is positive definite, is considered.

Separated boundary conditions of the following form are then imposed

$$A^*(\lambda)Y(a,\lambda) - B^*(\lambda)Z(a,\lambda) = 0, \tag{3.15}$$

$$\det[\Gamma^*(\lambda)Y(b,\lambda) - \Delta^*(\lambda)Z(b,\lambda)] = 0 \tag{3.16}$$

where $*$ denotes the transpose and $A(\lambda)$, $B(\lambda)$, $\Gamma(\lambda)$, $\Delta(\lambda)$ are $n \times n$ matrices of continuous, real valued functions for real λ, (otherwise * denotes the conjugate transpose).

The reason for considering the boundary conditions in this form is that the two point boundary problem (3.14), (3.15), (3.16) has a non-trivial solution if and only if the vector-matrix two point boundary problem

$$y' = K(x,\lambda)z, \quad z' = -G(x,\lambda)y, \tag{3.17}$$

$$A^*(\lambda)y(a,\lambda) - B^*(\lambda)z(a,\lambda) = 0, \tag{3.18}$$

$$\Gamma^*(\lambda)y(b,\lambda) - \Delta^*(\lambda)z(b,\lambda) = 0 \tag{3.19}$$

has a non-trivial solution.

In addition in [22] it is assumed that the coefficients in the system obey the following conditions:

(1) $A^*(\lambda)B(\lambda) = B^*(\lambda)A(\lambda)$
(2) $\Gamma^*(\lambda)\Delta(\lambda) = \Delta^*(\lambda)\Gamma(\lambda)$

(3) $A^*(\lambda)A(\lambda) + B^*(\lambda)B(\lambda) = I$

(4) $\Gamma^*(\lambda)\Gamma(\lambda) + \Delta^*(\lambda)\Delta(\lambda) = I$

where I is the identity matrix.

Definition 3.8 *For each λ, a solution pair $\{Y(x,\lambda), Z(x,\lambda)\}$ of (3.14) is conjoined provided*

$$Y^*(x,\lambda)Z(x,\lambda) \equiv Z^*(x,\lambda)Y(x,\lambda), \quad \forall\ x \in (a,b).$$

Definition 3.9 *A solution pair $\{Y(x,\lambda), Z(x,\lambda)\}$ of (3.14) is nontrivial provided* $\det Y(x,\lambda)$ *has at most a finite number of zeros for each fixed λ.*

The following results are then obtained in [22]:

A solution pair $\{Y(x,\lambda), Z(x,\lambda)\}$ of (3.14) satisfying

$$Y(a,\lambda) \equiv B(\lambda), \quad Z(a,\lambda) \equiv A(\lambda) \tag{3.20}$$

is nontrivial and conjoined. Clearly this pair satisfies the boundary condition at a given in (3.15) and hence this pair is the only solution which needs to be considered.

Theorem 3.10 *Let $\{Y(x,\lambda), Z(x,\lambda)\}$ be the solution pair of (3.14), (3.20), (3.16). The matrix $\theta(Y,Z)$ defined by*

$$\theta(Y,Z) = (Z + iY)(Z - iY)^{-1}$$

exists for all $x \in (a,b)$ and for each λ has the following properties on (a,b):

(i) θ is a unitary matrix;

(ii) θ satisfies the differential equation

$$\theta' = 2i\theta\Omega(x, \lambda)$$

where

$$\Omega(x, \lambda) = (Z^* + iY^*)^{-1}[Z^*KZ + Y^*GY](Z - iY)^{-1};$$

(iii)If $\phi_j(x, \lambda)$, $j = 1, \ldots, n$ are the characteristic roots of θ then $|\phi_j(x, \lambda)| = 1$, $j = 1, \ldots, n$, and for any fixed x, $\phi_j(x, \lambda) = +1$ for at least one j if and only if $\det Y(x, \lambda) = 0$;

(iv) The functions $\phi_j(x, \lambda)$ move monotonically and positively on the unit circle when they are at $+1$, as x increases;

(v) For each fixed x, the multiplicity of a zero of $\det Y(x, \lambda)$, i.e. the dimension of the null space of $Y(x, \lambda)$, is equal to the number of characteristic roots $\phi_j(x, \lambda)$ of θ having the value $+1$;

(vi) Let $\omega_j(x, \lambda) = \arg \phi_j(x, \lambda)$, $j = 1, \ldots, n$, where it is assumed that the functions $\omega_j(x, \lambda)$ are continued as continuous functions with respect to x. Then

$$\det \theta = \exp\{i \sum_{j=1}^{n} \omega_j(x, \lambda)\}$$

and

$$2\int_0^x tr\Omega(t, \lambda)dt = \sum_{j=1}^{n}[\omega_j(x, \lambda) - \omega_j(a, \lambda)].$$

The solution pair $\{Y(x, \lambda), Z(x, \lambda)\}$ of (3.14), (3.20) clearly satisfies the boundary condition at $x = a$ given in (3.15). Next a solution pair which satisfies the boundary condition at $x = b$ as given by (3.16) is established. To accomplish this a polar coordinate transformation is used. This is an extention of the work done by Barrett in [6].

Solution pairs $\{S(x), C(x)\}$ of the matrix differential system

$$Y'(x) = H(x)Z(x), \quad Z'(x) = -H(x)Y(x) \tag{3.21}$$

$$Y(0) = \Sigma, \quad Z(0) = \Xi, \tag{3.22}$$

where $H(x)$ is an $n \times n$ continuous symmetric matrix and Σ and Ξ are $n \times n$ constant matrices satisfying $\Sigma^*\Xi = \Xi^*\Sigma$, $\Sigma^*\Sigma + \Xi^*\Xi = I$ are considered. The solution pairs of systems of the form (3.21), (3.22) behave in a manner quite similar to trigonometric functions.

Theorem 3.11 *[22, Thm C]*
Let $\{Y(x,\lambda), Z(x,\lambda)\}$ be the solution pair of (3.14),(3.20). There exists a continuous, symmetric matrix $H(x,\lambda)$ and a nonsingular, continuously differentiable (in x) matrix $T(x,\lambda)$ such that

$$Y(x,\lambda) = S^*(x,\lambda)T(x,\lambda), \quad Z(x,\lambda) = C^*(x,\lambda)T(x,\lambda)$$

for each λ, where $\{S(x,\lambda), C(x,\lambda)\}$ is the solution of

$$S' = H(x,\lambda)C, \quad C' = -H(x,\lambda)S, \tag{3.23}$$

$$S(a,\lambda) = B^*(\lambda), \quad C(a,\lambda) = A^*(\lambda). \tag{3.24}$$

Moreover , $T(x,\lambda)$ is the solution of

$$T' = [SKC^* - CGS^*]T, \quad T(a,\lambda) = I$$

and

$$H(x,\lambda) = CKC^* + SGS^*. \tag{3.25}$$

Note Using the above theorem the boundary form (3.16) may be written in terms of the matrices S and C giving

$$\Gamma^*Y - \Delta^*Z = \Gamma^*S^*T - \Delta^*C^*T = [\Gamma^*S^* - \Delta^*C^*]T.$$

Also since T is nonsingular, the zeros of $\det[\Gamma^* Y - \Delta^* Z]$ coincide with those of $\det[\Gamma^* S^* - \Delta^* C^*]$.

The pair of matrices $\{U, V\}$ are defined by the equations

$$U(x) = S(x)\Gamma - C(x)\Delta, \quad V(x) = C(x)\Gamma + S(x)\Delta. \tag{3.26}$$

The tuple, $\{U, V\}$ is a trigonometric pair in the sense that it is the solution of (3.21)-(3.22), with $H(x, \lambda)$ as given in (3.25) and

$$\Sigma(\lambda) = B^*(\lambda)\Gamma(\lambda) - A^*(\lambda)\Delta(\lambda), \quad \Xi(\lambda) = A^*(\lambda)\Gamma(\lambda) + B^*(\lambda)\Delta(\lambda)$$

where $\Sigma^*\Xi = \Xi^*\Sigma$ and $\Sigma^*\Sigma + \Xi^*\Xi = I$.

Each initial value problem defining a trigonometric pair is, essentially, an initial value problem of the form (3.14), (3.20). Hence the matrices

$$E(x, \lambda) = (C - iS)^{-1}(C + iS),$$

$$F(x, \lambda) = (V - iU)^{-1}(V + iU), \tag{3.27}$$

with $\{S, C\}$ given by (3.23)-(3.24) and $\{U, V\}$ given by (3.26), exist and obey properties (i)-(iii), (v) and (vi) of Theorem 3.10. From Theorem 3.11, it can be seen that $E(x, \lambda) \equiv \theta(x, \lambda)$ and hence E also possesses property (iv) of Theorem 3.10. Let $f_j(x, \lambda)$, $j = 1, \ldots, n$, denote the characteristic roots of $F(x, \lambda)$ and let $\beta_j(x, \lambda) = \arg f_j(x, \lambda)$ for each j, with the assumption that $\beta_j(x, \lambda)$ is a continuous function and $\beta_j(a, \lambda) \in [0, 2\pi)$. Using property (vi) of Theorem 3.10 we have

$$\det E(x, \lambda) = \exp\{i \sum_{j=1}^{n} \omega_j(x, \lambda)\},$$

$$\det F(x, \lambda) = \exp\{i \sum_{j=1}^{n} \beta_j(x, \lambda)\} \tag{3.28}$$

and

$$\begin{aligned} 2\int_a^x \mathrm{tr}H(t,\lambda)dt &= \sum_{j=1}^{n}[\omega_j(x,\lambda)-\omega_j(a,\lambda)] = \sum_{j=1}^{n}[\beta_j(x,\lambda)-\beta_j(a,\lambda)] \\ &= 2\int_a^x \mathrm{tr}\Omega(t,\lambda)dt. \end{aligned} \tag{3.29}$$

The matrix $F(x,\lambda)$ is the matrix Prüfer angle associated with the boundary value problem (3.14), (3.15), (3.16).

The eigenvalues of (3.14)-(3.16) are the values of λ for which $\beta_j(b,\lambda) = 0(\mathrm{mod} 2\pi)$.

3.6 Eigenvalue Asymptotics

We find asymptotic bounds for the eigenvalues of the differential operator (3.1), which are in turn the eigenvalues of the differential operator on the graph G. This second order operator, (3.1), can be rewritten as a first order system as follows

$$Y' = Z \quad \text{and} \quad Z' = -G(x,\lambda)Y \tag{3.30}$$

where $G(x,\lambda) = M^{-1}(\lambda I - P)$. We consider general, separated, self-adjoint boundary conditions of the form (3.2), (3.3) where A, B, Γ, Δ are constant matrices with $[\Gamma^*, -\Delta^*]$ and $[A^*, -B^*]$ having maximal rank (i.e. $2K$).

In order to apply the background material given in Section 3.5 we first note that in our case, in equation (3.14), $K(x,\lambda) = I$ and is thus obviously positive definite. We still need to check that the following properties hold:

(1) $G(x,\lambda)$ is continuous and symmetric.

(2) $A^*B = B^*A$ and $\Gamma^*\Delta = \Delta^*\Gamma$.

(3) $A^*A + B^*B = I$ and $\Gamma^*\Gamma + \Delta^*\Delta = I$.

Since $Q(x)$ is continuous it is obvious that $G(x, \lambda)$ is continuous and it is easy to show that $G(x, \lambda)^T = G(x, \lambda)$, i.e. it is symmetric.

Condition (2) is not necessarily true for our original problem but there is an equivalent boundary value problem for which (2) is true as we are only interested in the null spaces of $[A^*, -B^*]$ and $[\Gamma^*, -\Delta^*]$ respectively.

Lemma 3.12 *The system (3.1)-(3.3) is equivalent to a system (3.1) but with the coefficient matrices in (3.2) and (3.3) obeying condition (2).*

Proof: Consider the inner product setting $\mathcal{L}^2_M$ with

$$< u, v >_M = \int_0^1 u^T M^{-1} \overline{v} dt.$$

Then

$$\begin{aligned} < T_P u, v > - < u, T_P v > &= \int_0^1 [(-u''^T M M^{-1} \overline{v} + u^T P M^{-1} \overline{v}) \\ &\quad -(-u^T M^{-1} M \overline{v}'' + u^T M^{-1} P \overline{v})] dt \\ &= \int_0^1 (u^T \overline{v}'' - u''^T \overline{v}) dt \\ &= [u^T \overline{v}' - u'^T \overline{v}]_0^1 \\ &= [u^T, u'^T] \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} \overline{v} \\ \overline{v}' \end{bmatrix}_0^1 . \end{aligned}$$

So T_P is formally self-adjoint on $\mathcal{D}(T_P)$ and the Lagrange form of T_P is

$$\mathcal{L}[u, v] = [u^T, u'^T] \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} \overline{v} \\ \overline{v}' \end{bmatrix}.$$

Now $[\Gamma^*, -\Delta^*]$ is of rank $2K$ and

$$\Gamma^* x - \Delta^* y = 0 = \Gamma^* z - \Delta^* w \quad \Rightarrow \quad [x^T, y^T] \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} \overline{z} \\ \overline{w} \end{bmatrix} = 0$$

$$\Leftrightarrow \quad [x^T, y^T] \begin{bmatrix} \overline{w} \\ -\overline{z} \end{bmatrix} = 0$$

$$\Leftrightarrow \quad x^T \overline{w} - y^T \overline{z} = 0$$

We may also write the null space of $[\Gamma^*, -\Delta^*]$ as $[D_1^T, D_2^T]^T p$, $p \in \mathbb{C}^{2K}$, for suitable $2K \times 2K$ matrices D_1 and D_2. In particular we require $\Gamma^* D_1 p - \Delta^* D_2 p = 0$ and $(D_1 q)^T (\overline{D_2 p}) - (D_2 q)^T (\overline{D_1 p}) = 0$.

I.e. for all p, $q \in \mathbb{C}^{2K}$

$$\begin{aligned} q^T D_1^T \overline{D}_2 \overline{p} - q^T D_2^T \overline{D}_1 \overline{p} = 0 \quad &\Rightarrow \quad D_1^T \overline{D}_2 = D_2^T \overline{D}_1 \\ &\Rightarrow \quad D_1^* D_2 = D_2^* D_1. \end{aligned}$$

So $D_2^* D_1 = D_1^* D_2$, i.e. $D_2^* D_1 p - D_1^* D_2 p = 0$ for all $p \in \mathbb{C}^{2K}$. Hence $[D_2^*, -D_1^*]$ has the same null space as $[\Gamma^*, -\Delta^*]$ and the same rank. As we are only interested in the null space of $[\Gamma^*, -\Delta^*]$ we can without loss of generality let $\Gamma^* = D_2^*$ and $\Delta^* = D_1^*$.

Since $A^* = \begin{bmatrix} I & -I \\ 0 & 0 \end{bmatrix}$, $-B^* = \begin{bmatrix} 0 & 0 \\ I & I \end{bmatrix}$, see Theorem 3.2, it it trivial to show that $A^* B = B^* A$. ■

Lemma 3.13 *The system (3.1)-(3.3) is equivalent to a system (3.1) but with the coefficient matrices in (3.2) and (3.3) obeying condition (3).*

Proof: First we consider the case of Γ and Δ as A and B are explicitly known.

Obviously $\Gamma^*\Gamma \geq 0$ and $\Delta^*\Delta \geq 0$ giving that $\Gamma^*\Gamma+\Delta^*\Delta \geq 0$. Since $\Gamma^*\Gamma+\Delta^*\Delta$ has rank 2K

$$\Gamma^*\Gamma + \Delta^*\Delta > 0$$

and is Hermitian symmetric. Therefore there exists Υ and $\Phi > 0$ such that $\Upsilon^* = \Upsilon^{-1}$ and

$$\Gamma^*\Gamma + \Delta^*\Delta = \Upsilon^*\Phi\Upsilon$$

giving

$$\Upsilon\Gamma^*\Gamma\Upsilon^* + \Upsilon\Delta^*\Delta\Upsilon^* = \Phi.$$

Thus

$$\Phi^{-\frac{1}{2}}\Upsilon\Gamma^*\Gamma\Upsilon^*\Phi^{-\frac{1}{2}} + \Phi^{-\frac{1}{2}}\Upsilon\Delta^*\Delta\Upsilon^*\Phi^{-\frac{1}{2}} = I.$$

Let $\Psi = \Phi^{-\frac{1}{2}}\Upsilon\Gamma^*$ and $\Omega = \Phi^{-\frac{1}{2}}\Upsilon\Delta^*$. Then

$$\begin{aligned} [\Psi\,,\,-\Omega]\begin{bmatrix} Y(1) \\ Y'(1) \end{bmatrix} &= [\Phi^{-\frac{1}{2}}\Upsilon\Gamma^*\,,\,-\Phi^{-\frac{1}{2}}\Upsilon\Delta^*]\begin{bmatrix} Y(1) \\ Y'(1) \end{bmatrix} \\ &= \Phi^{-\frac{1}{2}}\Upsilon[\Gamma^*Y(1) - \Delta^*Y'(1)] \\ &= 0 \end{aligned}$$

if and only if $\Gamma^*Y(1) - \Delta^*Y'(1) = 0$. Therefore $[\Psi\,,\,-\Omega]$ has the same null space as $[\Gamma^*, -\Delta^*]$ and the same rank. As we are only interested in the null space of $[\Gamma^*, -\Delta^*]$ we can without loss of generality let $\Gamma^* = \Psi$ and $\Delta^* = \Omega$ and then we have $\Gamma^*\Gamma + \Delta^*\Delta = I$. Also $(\Phi^{-\frac{1}{2}}\Upsilon\Gamma^*)(\Delta\Upsilon^*\Phi^{-\frac{1}{2}}) = \Phi^{-\frac{1}{2}}\Upsilon\Delta^*\Gamma\Upsilon^*\Phi^{-\frac{1}{2}}$, so (2) is preserved.

Now for the case of A and B we have that $\frac{1}{\sqrt{2}}A^*Y - \frac{1}{\sqrt{2}}B^*Y' = 0$. Let $\mathbb{A} = \frac{1}{\sqrt{2}}A$ and $\mathbb{B} = \frac{1}{\sqrt{2}}B$. Then $\mathbb{A}^*\mathbb{A} + \mathbb{B}^*\mathbb{B} = I$. Also $[\mathbb{A}^*, -\mathbb{B}^*]$ has the same null space and rank as $[A^*, -B^*]$ so we can without loss of generality let $A = \mathbb{A}$ and $B = \mathbb{B}$. ■

Thus we can, without loss of generality, assume that properties (1), (2) and

(3) hold for our boundary value problem and all the material given in Section 3.5 is applicable with $(a, b) = (0, 1)$ and $n = 2K$.

With $\{U, V\}$ as defined in equation (3.26) we have that the eigenvalues of our problem are precisely the values for which $\det U(1, \lambda) = 0$. I.e. a terminal Dirichlet problem in U.

From Section 3.5 the eigenvalues of (3.1)-(3.3) are the values of λ for which $\beta_j(1, \lambda) = 0(\text{mod}2\pi)$. Hence we are concerned with the behaviour of the functions $\beta_j(x, \lambda)$, $j = 1, \ldots, 2K$.

Theorem 3.14 *For fixed $x \in (0, 1)$ the functions $\beta_j(x, \lambda)$ are monotone increasing in $\lambda \in \mathbb{R}$.*

Proof: By the definition of F in equation (3.27) and Theorem 3.10, F is unitary, therefore by the note following Theorem 3.11

$$\begin{aligned} F &= (V^* + iU^*)(V^* - iU^*)^{-1} \\ &= (\psi + i\phi)(\psi - i\phi)^{-1} \end{aligned}$$

where $\phi = \Gamma^* Y - \Delta^* Z$ and $\psi = \Gamma^* Z + \Delta^* Y$ and where $\{Y, Z\}$ is solution pair of (3.30) satisfying $Y(0) = B$ and $Z(0) = A$.

Differentiating F with respect to λ, we obtain

$$\frac{\partial F}{\partial \lambda} = 2iFJ(x, \lambda)$$

where $J(x, \lambda) = (\psi^* + i\phi^*)^{-1}[\psi^* \phi_\lambda - \phi^* \psi_\lambda](\psi - i\phi)^{-1}$.

Since the arguments of the eigenvalues of F are of the form $2 \int \mu_j(\lambda)$ where μ_j are the eigenvalues of J we need only show that J is a positive definite matrix.

From the definitions of ϕ and ψ and the properties of Γ and Δ we have

$$\begin{aligned}\psi^*\phi_\lambda - \phi^*\psi_\lambda &= (Z^*\Gamma + Y^*\Delta)(\Gamma^*Y_\lambda - \Delta^*Z_\lambda) - (Y^*\Gamma - Z^*\Delta)(\Gamma^*Z_\lambda + \Delta^*Y_\lambda) \\ &= Z^*(\Gamma\Gamma^* + \Delta\Delta^*)Y_\lambda - Y^*(\Gamma\Gamma^* + \Delta\Delta^*)Z_\lambda \\ &= Z^*Y_\lambda - Y^*Z_\lambda.\end{aligned}$$

Now

$$[Z^*Y_\lambda - Y^*Z_\lambda]' = Y^*G_\lambda Y,$$

thus we have, on integrating from 0 to x, that

$$\int_0^x [Y^*G_\lambda Y] = [Z^*Y_\lambda - Y^*Z_\lambda](x).$$

Since $G_\lambda = M^{-1} > 0$ we have that $\psi^*\phi_\lambda - \phi^*\psi_\lambda > 0$, thus J is of the form $\mathcal{A}^*\mathcal{B}\mathcal{A}$ where $\mathcal{B}$ is positive and $\mathcal{A}$ is invertible, hence J is positive definite. Therefore the functions $\beta_j(x, \lambda)$ are increasing in λ. ■

The matrix F thus satisfies the boundary conditions and has eigenvalues with arguments monotonically increasing in λ.

Lemma 3.15 *For large λ, ($\lambda >$ Trace P), the arguments of the characteristic roots of $F(x, \lambda)$ are increasing in x for each fixed λ.*

Proof: Using the same reasoning as in Theorem 3.14 we have that

$$F = (\psi + i\phi)(\psi - i\phi)^{-1}$$

where $\phi = \Gamma^*Y - \Delta^*Z$ and $\psi = \Gamma^*Z + \Delta^*Y$.

Differentiating F with respect to x, we obtain

$$\frac{\partial F}{\partial x} = 2iFJ(x, \lambda)$$

where $J(x, \lambda) = (\psi^* + i\phi^*)^{-1}[\psi^*\phi_x - \phi^*\psi_x](\psi - i\phi)^{-1}$.

From the definitions of ϕ and ψ and the properties of Γ and Δ we have

$$\begin{aligned}\psi^*\phi_x - \phi^*\psi_x &= (Z^*\Gamma + Y^*\Delta)(\Gamma^*Y_x - \Delta^*Z_x) - (Y^*\Gamma - Z^*\Delta)(\Gamma^*Z_x + \Delta^*Y_x) \\ &= Z^*(\Gamma\Gamma^* + \Delta\Delta^*)Z - Y^*(\Gamma\Gamma^* + \Delta\Delta^*)(-GY) \\ &= Z^*Z + Y^*GY > 0.\end{aligned}$$

Thus by the same reasoning as in Theorem 3.14 we have the required result.

■

Lemma 3.16 *Let* $\lambda_0 \leq \lambda_1 \leq \lambda_2 \ldots$ *denote the eigenvalues of* (3.1)-(3.3) *repeated according to multiplicity, then*

$$2\pi(n+1) \leq \arg\det F(1,\lambda_n) < 2(n+1)\pi + 4K\pi, \tag{3.31}$$

for all $n = 0, 1, 2, \ldots$.

Proof: Denote by n_j the number of solutions of the congruence relation

$$\beta_j(1,\lambda) \equiv 0(\mathrm{mod} 2\pi), \quad \lambda \leq \lambda_n.$$

Then $\sum_1^{2K} n_j = n+1$, is the number of eigenvalues (with multiplicity) not exceeding λ_n, [4, p. 310].

Since $\beta_j(0,\lambda) \geq 0$ and the $\beta_j(x,\lambda)$ are increasing functions of x we have that for any fixed λ, $\beta_j(x,\lambda) > 0$ for $0 < x < 1$.

Now since the $\beta_j(x,\lambda)$ are positive and montone increasing in λ, we have

$$\beta_j(1,\lambda_n) \geq 2(n_j + 0)\pi$$

so summing over j,

$$\arg\det F(1,\lambda_n) = \sum_1^{2K} \beta_j(1,\lambda_n) \geq 2(n+1)\pi. \tag{3.32}$$

Similarly

$$\beta_j(1,\lambda_n) < 2(n_j+1)\pi$$

and so

$$\arg\det F(1,\lambda_n) < 2(n+1)\pi + 4K\pi. \quad \blacksquare \tag{3.33}$$

Theorem 3.17 *Let G be a graph with finitely many edges and let $\lambda_0 \leq \lambda_1 \leq \lambda_2 \ldots$ denote the eigenvalues of* (2.3), (2.4) *repeated according to multiplicity. Then there is a constant n_0 such that for $n > n_0$, the eigenvalues are given asymptotically by*

$$-\pi + \frac{\pi(n+1) - 4K\pi}{K_2} \leq \sqrt{\lambda_n} \leq \frac{\pi(n+1) - 4K\pi}{K_1} + \pi$$

where K_1 and K_2 are given by

$$K_1 = \sum_1^K \left\{ \left(1 + \frac{l_i^2}{4}\right) - \frac{1}{2}\left|1 - \frac{l_i^2}{4}\right| \right\}$$

and

$$K_2 = \sum_1^K \left\{ \left(1 + \frac{l_i^2}{4}\right) + \frac{1}{2}\left|1 - \frac{l_i^2}{4}\right| \right\}$$

Proof: From equation (3.31) we have that

$$0 \leq \sum_{j=1}^{2K} \beta_j(1,\lambda_n) - 2\pi(n+1) < 4\pi K.$$

Also $0 \leq \beta_j(0,\lambda_n) < 2\pi$ giving that

$$0 \leq \sum_{j=1}^{2K} \beta_j(0,\lambda_n) < 4\pi K.$$

Consequently

$$-4K\pi < \sum_{j=1}^{2K} [\beta_j(1,\lambda_n) - \beta_j(0,\lambda_n)] - 2\pi(n+1) < 4\pi K.$$

I.e.

$$\left|\sum_{j=1}^{2K}[\beta_j(1,\lambda_n)-\beta_j(0,\lambda_n)]-2\pi(n+1)\right|<4\pi K.$$

Now from equation (3.29)

$$2\int_0^1 \mathrm{tr}H=\sum_{j=1}^{2K}[\beta_j(1,\lambda_n)-\beta_j(0,\lambda_n)]$$

and we get that

$$\left|\int_0^1 \mathrm{tr}H-\pi(n+1)\right|<2\pi K.$$

Also by equation (3.29), $\int_0^1 \mathrm{tr}H=\int_0^1 \mathrm{tr}\Omega$, so

$$\left|\int_0^1 \mathrm{tr}\Omega-\pi(n+1)\right|<2\pi K. \tag{3.34}$$

It now remains to get an improved estimate on $\int_0^1 \mathrm{tr}\Omega(t,\lambda_n)\,dt$.

Let

$$\theta^+=(Z+i\sqrt{\lambda}Y)(Z-i\sqrt{\lambda}Y)^{-1}$$

then ϕ_j^+ and ω_j^+ are defined analogously to ϕ_j and ω_j for θ, see Theorem 3.10. I.e. $\phi_j^+(x,\lambda)$ are the characteristic roots of θ^+ and $\omega_j^+(x,\lambda)=\arg\phi_j^+(x,\lambda)$ for $j=1,\ldots,2K$.

If $\omega(x,\lambda)$ is an eigenvalue of $\theta(x,\lambda)$, defined in Theorem 3.10, with eigenvector $f(x,\lambda)\neq 0$ then $\theta f=\omega f$, i.e.

$$(Z+iY)(Z-iY)^{-1}f=\omega f.$$

Setting $g=(Z-iY)^{-1}f$ we get

$$(Z+iY)g=\omega(Z-iY)g$$

so

$$(1-\omega)Zg=-i(1+\omega)Yg.$$

Thus

$$(\omega-1)(Z+i\sqrt{\lambda}Y)g = i((1+\omega)+\sqrt{\lambda}(\omega-1))Yg$$

and

$$(\omega-1)(Z-i\sqrt{\lambda}Y)g = i((1+\omega)-\sqrt{\lambda}(\omega-1))Yg.$$

Hence

$$\frac{(Z-i\sqrt{\lambda}Y)g}{(1+\omega)-\sqrt{\lambda}(\omega-1)} = \frac{(Z+i\sqrt{\lambda}Y)g}{(1+\omega)+\sqrt{\lambda}(\omega-1)}$$

giving

$$(Z+i\sqrt{\lambda})g = \frac{(1+\omega)+\sqrt{\lambda}(\omega-1)}{(1+\omega)-\sqrt{\lambda}(\omega-1)}(Z-i\sqrt{\lambda}Y)g.$$

Therefore

$$\theta^+ f = \frac{(1+\omega)+\sqrt{\lambda}(\omega-1)}{(1+\omega)-\sqrt{\lambda}(\omega-1)}f.$$

I.e. $f^+ = f$ is an eigenvector of θ^+ with eigenvalue $\omega^+ = \frac{(1+\omega)+\sqrt{\lambda}(\omega-1)}{(1+\omega)-\sqrt{\lambda}(\omega-1)}$ which is a Möebius transformation preserving the points ± 1 and maps the upper half plane to the upper half plane and the lower half plane to the lower half plane.

Thus

$$\left|\omega_j(x,\lambda_n) - \omega_j^+(x,\lambda_n)\right| < \pi,$$

giving

$$\left|\sum_{j=1}^{2K}\omega_j(x,\lambda_n) - \sum_{j=1}^{2K}\omega_j^+(x,\lambda_n)\right| < 2K\pi$$

as we wanted.

Hence

$$\left|2\int_0^1 \mathrm{tr}\Omega - \sum_{j=1}^{2K}[\omega_j^+(1,\lambda_n) - \omega_j^+(0,\lambda_n)]\right| < 4K\pi. \tag{3.35}$$

Thus it only remains to show that

$$2\sqrt{\lambda}\int_0^1 \mathrm{tr}\Omega^+ = \sum_{j=1}^{2K}[\omega_j^+(1,\lambda_n) - \omega_j^+(0,\lambda_n)] \tag{3.36}$$

and to obtain an asymptotic formula for $\mathrm{tr}\Omega^+$.

Now by a straightforward calculation we obtain that

$$\theta^{+\prime} = 2i\sqrt{\lambda}\theta^+\Omega^+$$

where $\Omega^+ = (Z^* + i\sqrt{\lambda}Y^*)^{-1}(Z^*Y' - Y^*Z')(Z - i\sqrt{\lambda}Y)^{-1}$.

Therefore we get

$$\det\theta^+(1) = \exp\{2i\sqrt{\lambda}\int_0^1 \mathrm{tr}\Omega^+\, dt\}\det\theta^+(0),$$

i.e.

$$2\sqrt{\lambda}\int_0^1 \mathrm{tr}\Omega^+ = \sum_{j=1}^{2K}[\omega_j^+(1,\lambda_n) - \omega_j^+(0,\lambda_n)].$$

Now since

$$\Omega^+ = (Z^* + i\sqrt{\lambda}Y^*)^{-1}(Z^*Y' - Y^*Z')(Z - i\sqrt{\lambda}Y)^{-1}$$

it is possible to show by direct computation that

$$\begin{aligned}
&Z^*Z + Y^*GY \\
&= \frac{1}{4}[\{(Z^* + i\sqrt{\lambda}Y^*) + (Z^* - i\sqrt{\lambda}Y^*)\}\{(Z + i\sqrt{\lambda}Y) + (Z - i\sqrt{\lambda}Y)\} \\
&-\{(Z^* + i\sqrt{\lambda}Y^*) - (Z^* - i\sqrt{\lambda}Y^*)\}\frac{G}{\lambda}\{(Z + i\sqrt{\lambda}Y) - (Z - i\sqrt{\lambda}Y)\}].
\end{aligned}$$

Thus since θ^+ is unitary

$$\begin{aligned}
4\Omega^+ &= \{I + \theta^{+*}\}\{\theta^+ + I\} - \{I - \theta^{+*}\}\tfrac{G}{\lambda}\{\theta^+ - I\} \\
&= \{\theta^+ + I + I + \theta^{+*}\} - \{\tfrac{G}{\lambda}\theta^+ - \tfrac{G}{\lambda} - \theta^{+*}\tfrac{G}{\lambda}\theta^+ + \theta^{+*}\tfrac{G}{\lambda}\}.
\end{aligned}$$

Also $\mathrm{tr}\left(\theta^{+*}\frac{G}{\lambda}\theta^+\right) = \mathrm{tr}\left(\frac{G}{\lambda}\right)$, so it can easily be shown that

$$4\mathrm{tr}\Omega^+ = 2\left[\mathrm{tr}\left(I + \frac{G}{\lambda}\right) + \mathrm{Re}\ \mathrm{tr}\left(\left(I - \frac{G}{\lambda}\right)\theta^+\right)\right].$$

Now

$$-\mathrm{tr}\left|I - \frac{G}{\lambda}\right| \le \mathrm{Re}\ \mathrm{tr}\left(I - \frac{G}{\lambda}\right)\theta^+ \le \mathrm{tr}\left|I - \frac{G}{\lambda}\right|$$

thus

$$\left|4\int_0^1 \mathrm{tr}\Omega^+ - 2\int_0^1 \mathrm{tr}\left(I+\frac{G}{\lambda}\right)\right| \le \int_0^1 \mathrm{tr}\left|I-\frac{G}{\lambda}\right|.$$

Combining these results gives

$$\left|2\sqrt{\lambda}\int_0^1 \mathrm{tr}\Omega^+ - \sqrt{\lambda}\int_0^1 \mathrm{tr}(I+\frac{G}{\lambda})\right| \le \frac{\sqrt{\lambda}}{2}\int_0^1 \mathrm{tr}\left|I-\frac{G}{\lambda}\right|.$$

Thus from equations (3.35), (3.36) and equation (3.34) respectively we obtain

$$\left|\sqrt{\lambda}\int_0^1 \mathrm{tr}(I+\frac{G}{\lambda}) - 2\int_0^1 \mathrm{tr}\Omega\right| < 4K\pi + \frac{\sqrt{\lambda}}{2}\int_0^1 \mathrm{tr}\left|I-\frac{G}{\lambda}\right|$$

and

$$\left|2\int_0^1 \mathrm{tr}\Omega - 2\pi(n+1)\right| < 4K\pi$$

giving that

$$\left|\sqrt{\lambda}\int_0^1 \mathrm{tr}(I+\frac{G}{\lambda}) - 2\pi(n+1)\right| < 8K\pi + \frac{\sqrt{\lambda}}{2}\int_0^1 \mathrm{tr}\left|I-\frac{G}{\lambda}\right|.$$

Therefore

$$\left|\sqrt{\lambda_n}\left(K+\sum_1^K \frac{l_i^2}{4}\right) - \pi(n+1)\right| < 4K\pi + \frac{\sqrt{\lambda_n}}{2}\sum_1^K\left|1-\frac{l_i^2}{4}\right| + O\left(\frac{1}{\sqrt{\lambda_n}}\right).$$

Letting

$$K_1 = \sum_1^K\left\{\left(1+\frac{l_i^2}{4}\right) - \frac{1}{2}\left|1-\frac{l_i^2}{4}\right|\right\}$$

and

$$K_2 = \sum_1^K\left\{\left(1+\frac{l_i^2}{4}\right) + \frac{1}{2}\left|1-\frac{l_i^2}{4}\right|\right\}$$

we obtain that

$$O\left(\frac{1}{\sqrt{\lambda_n}}\right) + \sqrt{\lambda_n}K_1 - 4K\pi \le \pi(n+1) \le O\left(\frac{1}{\sqrt{\lambda_n}}\right) + \sqrt{\lambda_n}K_2 + 4K\pi.$$

Hence

$$\frac{\pi(n+1)-4K\pi}{K_2} + O\left(\frac{1}{\sqrt{\lambda_n}}\right) \le \sqrt{\lambda_n} \le \frac{\pi(n+1)-4K\pi}{K_1} + O\left(\frac{1}{\sqrt{\lambda_n}}\right),$$

and for large n these imply

$$-\pi + \frac{\pi(n+1)-4K\pi}{K_2} \le \sqrt{\lambda_n} \le \frac{\pi(n+1)-4K\pi}{K_1} + \pi$$

since the terms $O\left(\frac{1}{\sqrt{\lambda_n}}\right)$ will be less than π in absolute value if n is large enough. ■

Corollary 3.18 *Let G be a graph with finitely many edges. If each edge has a rational length (or each edge has length a rational multiple of $k > 0$), then the eigenvalues of (2.3), (2.4) are given asymptotically by*

$$\sqrt{\lambda_n} = \frac{\pi n}{\textit{total length}} + O(1).$$

Proof: Assume we have a graph with K edges that have rational lengths. It is then possible to find a least number l_s such that all the edges can be divided exactly into an integer number of smaller edges each of length l_s where at each new node η_i say, we have introduced boundary conditions of the form $y(\eta_i^-) = y(\eta_i^+)$ and $y'(\eta_i^-) = y'(\eta_i^+)$ which do not alter our problem. The total number of edges will then be given by

$$\sum_{i=1}^{K} \frac{l_i}{l_s} = L$$

say, where l_i for $i = 1, \ldots, K$ are the respective original lengths of the edges. We can then carry out the separation of the boundary conditions exactly as before by introducing a node in the middle of each edge and this gives us $2L$ edges of length $\frac{l_s}{2}$ each. So we have that

$$M = 4 \begin{bmatrix} W & 0 \\ 0 & W \end{bmatrix} = \frac{4}{l_s^2} I$$

and we can then divide through getting that

$$-IT'' + \frac{l_s^2}{4} PT = \frac{l_s^2}{4} \lambda T.$$

Hence from the theorem above we have that

$$\sqrt{\frac{l_s^2}{4}\lambda_n} = \frac{\pi n}{2L} + O(1)$$

giving us

$$\sqrt{\lambda_n} = \frac{2}{l_s}\left(\frac{\pi n}{2L} + O(1)\right)$$

and from the definition of L we have the required result. ∎

The case of finding better eigenvalue asymptotics for a graph with irrational edge length ratio is the motivation for Chapter 4.

3.7 Interlacing

In this section we describe the interlacing of the eigenvalues of (3.1)-(3.3) with those of (3.1), (3.2) but with (3.3) replaced by Dirichlet conditions and by Neumann conditions. This is a consequence of the matrix Prüfer angle considerations above.

We begin by considering the case of $\Delta = 0$ and $\Gamma = I$ in (3.3) i.e. Dirichlet boundary conditions at 1. The matrix Prüfer angle corresponding to this boundary value problem is given by

$$F^D(x, \lambda) = (C - iS)^{-1}(C + iS) = E(x, \lambda), \tag{3.37}$$

see equation (3.27) and Theorem 3.11, (since C and S do not depend on the boundary condition at 1).

If $\beta_j(x, \lambda)$ and $\omega_j(x, \lambda)$ are as previously defined, the arguments of the characteristic roots of F and F^D respectively, then, from equation (3.29), we have

$$\sum_{j=1}^{2K}[\omega_j(1, \lambda) - \omega_j(0, \lambda)] = \sum_{j=1}^{2K}[\beta_j(1, \lambda) - \beta_j(0, \lambda)]. \tag{3.38}$$

Since the boundary conditions are independent of λ we have that $\sum_{j=1}^{2K} \omega_j(0,\lambda)$ and $\sum_{j=1}^{2K} \beta_j(0,\lambda)$ are constants independent of λ.

Theorem 3.19 *Let λ_i denote the eigenvalues of the system (3.1)-(3.3) and let λ_i^D denote the eigenvalues for the boundary value problem*

$$-MY'' + PY = \lambda Y \tag{3.39}$$

with boundary conditions

$$\begin{aligned} A^*Y(0) - B^*Y'(0) &= 0, && (3.40)\\ Y(1) &= 0 && (3.41) \end{aligned}$$

where M, P, A^* and B^* are as defined in (3.1)-(3.3). Then there is at least one λ_i in the interval $[\lambda_n^D, \lambda_{n+4K-1}^D]$ and at least two λ_i's the interval $[\lambda_n^D, \lambda_{n+4K}^D]$.

Proof: For the the boundary value problem (3.39)-(3.41), the eigen-condition is

$$\omega_j(1,\lambda) = 0(mod 2\pi), \quad \text{for some } j \in \{1,\ldots,2K\} \tag{3.42}$$

while for the boundary value problem (3.1)-(3.3), the eigenvalues are given by

$$\beta_j(1,\lambda) = 0(mod 2\pi), \quad \text{for some } j \in \{1,\ldots,2K\}. \tag{3.43}$$

To ensure that at least one $\beta_j(1,\lambda)$, $j \in \{1,\ldots,2K\}$, has increased by 2π, by equation (3.38) and note thereafter, it is sufficient for $\sum_{j=1}^{2K} \omega_j(1,\lambda)$ to have increased by at least $4K\pi$.

From equation (3.42), we have that as λ increases from λ_n^D to λ_{n+2K}^D we are guaranteed that $\sum_{j=1}^{2K} \omega_j(1,\lambda)$ increases by at least 2π, since (3.42) has been solved $2K+1$ times. However if λ increases from λ_n^D to λ_{n+2K+1}^D then $\sum_{j=1}^{2K}\omega_j(1,\lambda)$ increases by at least 4π. Proceeding in this manner, as λ increases from λ_n^D to λ_{n+N}^D, $\sum_{j=1}^{2K}\omega_j(1,\lambda)$ increases by at least $2\pi(N-(2K-1))$. So, to ensure that $\sum_{j=1}^{2K}\omega_j(1,\lambda)$ increases by at least $4K\pi$ we need $2\pi(N-2K+1)\geq 4K\pi$, giving that $N \geq 4K-1$. Hence in the interval $[\lambda_n^D, \lambda_{n+4K-1}^D]$ we have at least one eigenvalue of (3.1)-(3.3).

Now consider the interval $[\lambda_n^D, \lambda_{n+4K}^D]$. Then by the above calculations $\sum_{j=1}^{2K}\omega_j(1,\lambda)$ increases by at least $2\pi(2K+1)$ as λ increases from λ_n^D to λ_{n+4K}^D. Hence $\sum_{j=1}^{2K}\beta_j(1,\lambda)$ increases by at least $2\pi(2K+1)$ as λ increases from λ_n^D to λ_{n+4K}^D. Then at least one β_j has increased by at least 2π, say β_1. If more then one β_j has increased by 2π, the result is proved. So assume that only β_1 has increased by at least 2π. Then β_j, $j=2,\ldots,2K$, each increase by at most $2\pi-\epsilon$ for some $0<\epsilon<2\pi$. Thus the increase in $\sum_{j=1}^{2K}\beta_j(1,\lambda)$ for λ increasing from λ_n^D to λ_{n+4K}^D is at most $(2K-1)(2\pi-\epsilon)$, making the increase in β_1 at least $2\pi(2K+1)-(2K-1)(2\pi-\epsilon)=4\pi+(2K-1)\epsilon>4\pi$, giving at least two eigenvalues of (3.1)-(3.3) in $[\lambda_n^D, \lambda_{n+4K}^D]$. $\blacksquare$

Next we consider the case where $\Delta = I$ and $\Gamma = 0$ in (3.3) i.e. Neumann boundary conditions at 1. The matrix Prüfer angle is then given by

$$F^N(x,\lambda) = (S-iC)^{-1}(S+iC) \tag{3.44}$$

where C and S are as previously defined in Theorem 3.11.

Let $f_j^N(x,\lambda)$, $j=1\ldots 2K$, denote the characteristic roots of $F^N(x,\lambda)$ and let $\beta_j^N(x,\lambda) = \arg f_j^N(x,\lambda)$ for each j, with the assumption that $\beta_j^N(x,\lambda)$ is a continuous function.

Theorem 3.20 *Let λ_i denote the eigenvalues of the system (3.1)-(3.3) and let λ_i^N denote the eigenvalues for the boundary value problem*

$$-MY'' + PY = \lambda Y \tag{3.45}$$

with boundary conditions

$$\begin{aligned} A^*Y(0) - B^*Y'(0) &= 0, & (3.46)\\ Y'(1) &= 0 & (3.47) \end{aligned}$$

where M, P, A^* and B^* are as defined in (3.1)-(3.3). Then there is at least one λ_i in the interval $[\lambda_n^N, \lambda_{n+4K-1}^N]$ and at least two λ_i's the interval $[\lambda_n^N, \lambda_{n+4K}^N]$.

Proof: First we relate $F^N(x, \lambda)$ and $F^D(x, \lambda)$:

$$\begin{aligned} F^N(x, \lambda) &= (S - iC)^{-1}(S + iC) \\ &= (-C - iS)^{-1}(C - iS) \\ &= -(C + iS)^{-1}(C - iS) \\ &= -(F^D(x, \lambda))^{-1}, \end{aligned}$$

and thus

$$e^{i\beta_j^N} = -e^{-i\omega_j} = e^{i(\pi - \omega_j)},$$

which implies that

$$\beta_j^N = (2\pi k_j - \pi) - \omega_j \tag{3.48}$$

where k_j is a constant.

The interlacing result now follows directly from the analysis used in Theorem 3.19 since adding an additional constant term into equation (3.38) does not change any of the reasoning.

Hence we obtain that in the interval $[\lambda_n^N, \lambda_{n+4K-1}^N]$ we have at least one eigenvalue of (3.1)-(3.3) and in the interval $[\lambda_n^N, \lambda_{n+4K}^N]$ we have at least two eigenvalues of (3.1)-(3.3). $\blacksquare$

Chapter 4

Dirichlet-Neumann Bracketing

In this chapter, q, given in equation (1.1), is assumed to be real valued and essentially bounded on G.

We give a variational formulation for a class of self-adjoint boundary value problems on graphs. This in turn, enables us to develop an analogue of Dirchlet-Neumann bracketing for the eigenvalues of the boundary value problem. This forms a theoretical structure from which spectral asymptotics can be found.

A variational reformulation of the boundary value problem is given in Section 4.1 along with the definition of co-normal boundary conditions. Two examples are given illustrating co-normal and non-co-normal boundary conditions. The variational formulation leads to a max-min characterization of the eigenvalues of the boundary value problem, Section 4.2, and hence to a type of Dirichlet-Neumann bracketing of the eigenvalues, see Section 4.3. For the analogue in the case of partial differential equations we refer the reader to [19].

In the final section, Section 4.4, spectral asymptotics are considered.

4.1 Variational Formulation

In this section we give an $\mathcal{H}^1(G)$, variational formulation for the boundary value problem (2.3)-(2.4) or equivalently for the eigenvalue problem associated with the operator L defined in (2.6)-(2.7). For details in the setting of partial differential equations we refer the reader to [19]. The variational formulation gives rise to a max-min characterization of the eigenvalues and eigenfunctions of the boundary value problem, developed in the next section. We conclude the section by proving that the $\mathcal{H}^1(G)$ eigenfunctions are in fact regular, i.e. are in $\mathcal{H}^2(G)$.

Without loss of generality, we assume the boundary conditions (2.4), or equivalently (2.5), to be in the form

$$\sum_{j=1}^{K}[\alpha_{ij}x_j(0)+\gamma_{ij}x_j(l_j)]=0 \tag{4.1}$$

for $i=1,\ldots,J$,

$$\sum_{j=1}^{K}[\alpha_{ij}x_j(0)+\beta_{ij}x_j'(0)+\gamma_{ij}x_j(l_j)+\delta_{ij}x_j'(l_j)]=0 \tag{4.2}$$

for $i=J+1,\ldots,2K$, where by $x_i=x|_{e_i}$. Here all possible Dirichlet-like terms are in (4.1), i.e. if (4.2) is written in matrix form then Gauss-Jordan reduction will not allow any pure Dirichlet conditions linearly independent of (4.1) to be extracted.

Let $F(x,y)$ to be the sesquilinear form given by

$$F(x,y):=\int_{\partial G} f x\overline{y}\,d\sigma+\int_G (x'\overline{y}'+xq\overline{y})\,dt, \tag{4.3}$$

with domain

$$\mathcal{D}(F) = \{y \in \mathcal{H}^1(G) \,|\, y \text{ obeying } (4.1)\},$$

where ∂G denotes the boundary of the graph G and

$$\int_{\partial G} y \, d\sigma := \sum_{i=1}^{K} [y_i(l_i) - y_i(0)] = \int_G y' \, dt.$$

Definition 4.1 *We say that the boundary conditions on a graph are co-normal with respect to l if there exists f defined on ∂G, such that $x \in \mathcal{D}(F) \cap \mathcal{C}^1(G)$ has*

$$\int_{\partial G} f x \overline{y} \, d\sigma = \int_{\partial G} x' \overline{y} \, d\sigma, \quad \textit{for all} \quad y \in \mathcal{D}(F)$$

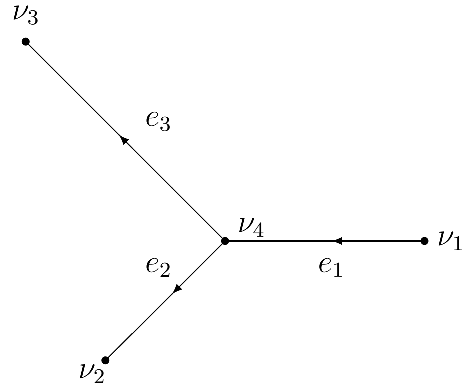
if and only if x obeys (4.2).

Note that every self-adjoint operator on a Hilbert space, which is bounded from below, has a form domain (denoted by $\mathcal{D}(F)$ in this thesis). The assumption that the boundary conditions are co-normal means that the form has a particular form.

Remark Co-normal boundary conditions on a graph correspond in nature to co-normal (non-oblique) boundary conditions for elliptic partial differential operators.

We will now give an example containing the three types of boundary conditions of most physical interest, i.e. Neumann, Kirchhoff and Dirichlet boundary conditions, at the nodes of the graph and show that this falls into the co-normal category. This class does not include all self-adjoint boundary value problems on graphs as the second example will illustrate.

Example 1 Consider the graph



with the following boundary conditions:

$$\begin{aligned} y_1'(0) &= 0, \\ y_3'(l_3) &= 0, \\ y_2(l_2) &= 0, \\ y_1'(l_1) - y_2'(0) - y_3'(0) &= 0, \\ y_1(l_1) = y_2(0) &= y_3(0), \end{aligned} \tag{4.4}$$

where $y_i = y|_{e_i}$.

The domain $\mathcal{D}(F)$ is then given by

$$\mathcal{D}(F) = \{y \in \mathcal{H}^1(G) \,|\, y_2(l_2) = 0,\, y_1(l_1) = y_2(0) = y_3(0)\}.$$

Let $f \equiv 0$. We now show that for $x \in \mathcal{D}(F) \cap \mathcal{C}^1(G)$, the condition

$$0 = \int_{\partial G} x'\overline{y}\, d\sigma, \quad \text{for all} \quad y \in \mathcal{D}(F),$$

is equivalent to x obeying the boundary conditions, (4.4).

Let $x \in \mathcal{D}(F) \cap \mathcal{C}^1(G)$. Suppose $0 = \displaystyle\int_{\partial G} x'\overline{y}\, d\sigma$ for all $y \in \mathcal{D}(F)$ then

$$\begin{aligned} 0 &= \int_{\partial G} x'\overline{y}\, d\sigma \\ &= \sum_{i=1}^{3} [x_i'\overline{y}_i(l_i) - x_i'\overline{y}_i(0)] \\ &= \overline{y}_1(l_1)[x_1'(l_1) - x_2'(0) - x_3'(0)] + \overline{y}_3(l_3)x_3'(l_3) - \overline{y}_1(0)x_1'(0). \end{aligned}$$

Since $\overline{y}_1(l_1)$, $\overline{y}_3(l_3)$ and $\overline{y}_1(0)$ are unrelated we obtain that $x_1'(l_1) - x_2'(0) - x_3'(0) = 0$, $x_3'(l_3) = 0$ and $x_1'(0) = 0$. As $x \in \mathcal{D}(F) \cap \mathcal{C}^1(G)$, $x_2(l_2) = 0$ and $x_1(l_1) = x_2(0) = x_3(0)$, i.e. the boundary conditions (4.4) are obeyed.

Conversely if $x \in \mathcal{D}(F) \cap \mathcal{C}^1(G)$ and obeys the boundary conditions (4.4), then from (4.4) we obtain that for $x \in \mathcal{D}(F) \cap \mathcal{C}^1(G)$, $x_1'(0) = 0$ and $x_3'(l_3) = 0$ and hence

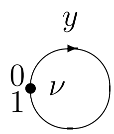
$$\begin{aligned} \int_{\partial G} x'\overline{y}\, d\sigma &= \sum_{i=1}^{3} [x_i'\overline{y}_i(l_i) - x_i'\overline{y}_i(0)] \\ &= x_1'(l_1)\overline{y}_1(l_1) + x_2'(l_2)\overline{y}_2(l_2) - x_2'(0)\overline{y}_2(0) - x_3'(0)\overline{y}_3(0). \end{aligned}$$

Since $y \in \mathcal{D}(F)$, $y_2(l_2) = 0$, $y_1(l_1) = y_2(0) = y_3(0)$ and thus by (4.4)

$$\begin{aligned} \int_{\partial G} x'\overline{y}\, d\sigma &= \overline{y}_1(l_1)[x_1'(l_1) - x_2'(0) - x_3'(0)] \\ &= 0. \end{aligned}$$

Therefore the boundary conditions are co-normal. ■

Example 2 Consider the single loop,



where we have the following boundary conditions

$$y(0) = y'(1), \quad y(1) = -y'(0). \tag{4.5}$$

In order for these boundary conditions to be co-normal we require that there exists f defined on ∂G, such that $x \in \mathcal{D}(F) \cap \mathcal{C}^1(G)$ has

$$\int_{\partial G} f x\overline{y}\, d\sigma = \int_{\partial G} x'\overline{y}\, d\sigma, \quad \text{for all} \quad y \in \mathcal{D}(F) \tag{4.6}$$

if and only if x obeys (4.5).

In matrix form the boundary conditions (4.5) are given by

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} y(0) \\ y(1) \\ y'(0) \\ y'(1) \end{bmatrix} = 0$$

and $\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$ is not equivalent to a matrix of the form $\begin{bmatrix} a & b & 0 & 0 \\ 0 & p & q & r \end{bmatrix}$.

Thus $\mathcal{D}(F) = \mathcal{H}^1(G)$, i.e. we have no domain conditions.

Assuming that there exists f such that (4.6) holds, then as x obeys the boundary conditions (4.5) we have

$$\int_{\partial G} f x \overline{y}\, d\sigma = x'(1)\overline{y}(1) - x'(0)\overline{y}(0) = x(0)\overline{y}(1) + x(1)\overline{y}(0).$$

Expanding the left hand side and rearranging terms we obtain

$$x(1)[f(1)\overline{y}(1) - \overline{y}(0)] - x(0)[f(0)\overline{y}(0) + \overline{y}(1)] = 0,$$

which must hold for all $x(0)$ and $x(1)$, giving

$$f(1)\overline{y}(1) = \overline{y}(0) \quad \text{and} \quad f(0)\overline{y}(0) = -\overline{y}(1). \tag{4.7}$$

Since $y(0)$ and $y(1)$ are unrelated, (as $\mathcal{D}(F) = \mathcal{H}^1(G)$), it is not possible to find an f for which (4.7) holds for all $y \in \mathcal{D}(F)$. Hence the boundary conditions (4.5) are not co-normal. ■

Theorem 4.2 *If T is a distribution on G with $\partial T \in \mathcal{L}^1_{loc}(G)$ then $T = f$ for some absolutely continuous f and $\partial T = Df$.*

Proof: Since $\partial T \in \mathcal{L}^1_{loc}(G)$ it follows that

$$\partial T(\varphi) = \int_G \varphi h \, dt, \quad \text{for all} \quad \varphi \in \mathcal{C}_0^\infty(G),$$

for some $h \in \mathcal{L}^1_{loc}(G)$.

Let φ be such that $\varphi|_{e_j} \equiv 0$ for all $j \neq i$ and $\varphi|_{e_i} \in \mathcal{C}_0^\infty((0,l_i))$, then $\varphi \in \mathcal{C}_0^\infty(G)$ and

$$\int_G \varphi h \, dt = \int_{e_i} \varphi h \, dt.$$

I.e.

$$\int_{e_i} \varphi h|_{e_i} \, dt = \partial T(\varphi).$$

Let J_i be the distribution

$$J_i \psi = \int_{e_i} \psi h|_{e_i} \, dt = \int_0^{l_i} \psi h|_{e_i} \, dt$$

then J_i is a distribution on $e_i = (0, l_i)$. So by [67, Theorem 1.5(a), p. 43] there exists a distribution T_i such that

$$\partial T_i = J_i.$$

Now $J_i \in \mathcal{L}^1_{loc}(e_i)$. Hence by [67, Theorem 1.6, p. 44], $T_i \in AC$, i.e.

$$T_i \psi = \int_{e_i} \psi H_i \, dt \quad \text{for all} \quad \psi \in \mathcal{C}_0^\infty(e_i)$$

for some $H_i \in AC$ and $\partial T_i = D H_i$.

Let $H \in AC$ be defined by $H|_{e_i} = H_i \in AC$ and let

$$S\varphi = \int_G \varphi H \, dt.$$

Then

$$\partial S(\varphi) \quad = \quad -S(D\varphi)$$

$$
\begin{aligned}
&= -\sum_{i=1}^{K} T_i(D\varphi|_{e_i}) \\
&= \sum_{i=1}^{K} \partial T_i(\varphi|_{e_i}) \\
&= \sum_{i=1}^{K} J_i(\varphi|_{e_i}) \\
&= \partial T(\varphi)
\end{aligned}
$$

for all $\varphi \in \mathcal{C}_0^\infty(G)$, giving $\partial S = \partial T$.

By definition of S, $S = H \in AC$ and

$$\partial S = \sum_{i=1}^{K} \partial T_i = \sum_{i=1}^{K} DH_i = D\left(\sum_{i=1}^{K} H_i\right) = DH.$$

Hence $\partial T = DH$.

So by [67, Theorem 1.5(b), p. 43] $T = S + k$, for some $k \in \mathbb{R}$. Therefore $T \in AC$ and $T = H + k$. Lastly $\partial T = DH = D(H + k)$ and setting $f := H + k$ gives the result. ■

The following lemma shows that a function is a variational eigenfunction if and only if it is a classical eigenfunction.

Lemma 4.3 *Suppose that (4.1)-(4.2) are co-normal boundary conditions with respect to l of (1.1), then $u \in \mathcal{D}(F)$ satisfies $F(u, v) = \lambda(u, v)$ for all $v \in \mathcal{D}(F)$ if and only if $u \in \mathcal{H}^2(G)$ and u obeys (1.1), (4.1)-(4.2).*

Proof: Assume that $u \in \mathcal{H}^2(G)$ and u obeys (1.1), (4.1)-(4.2). Then for each $v \in \mathcal{D}(F)$

$$F(u, v) \;=\; \int_{\partial G} f u\overline{v}\, d\sigma + \int_G (u'\overline{v}' + qu\overline{v})\, dt$$

$$
\begin{aligned}
&= \int_{\partial G} fu\overline{v}\, d\sigma + \int_G ((u'\overline{v})' - u''\overline{v} + qu\overline{v})\, dt \\
&= \int_{\partial G} fu\overline{v}\, d\sigma + \int_G (u'\overline{v})'\, dt + \lambda(u, v) \\
&= \int_{\partial G} (fu + u')\overline{v}\, d\sigma + \lambda(u, v).
\end{aligned}
$$

The assumption that (4.1)-(4.2) are co-normal boundary conditions with respect to l gives that $u \in \mathcal{D}(F)$ and

$$\int_{\partial G} (fu + u')\overline{v}\, d\sigma = 0, \quad \text{for all} \quad v \in \mathcal{D}(F),$$

completing the proof in this case.

Now assume $u \in \mathcal{D}(F)$ satisfies $F(u, v) = \lambda(u, v)$ for all $v \in \mathcal{D}(F)$. As $\mathcal{C}_o^\infty(G) \subset \mathcal{D}(F)$, it follows that

$$F(u, v) = \lambda(u, v), \quad \text{for all} \quad v \in \mathcal{C}_0^\infty(G).$$

Hence $F(u, \cdot)$ can be extended to a continuous linear functional on $\mathcal{L}^2(G)$. In particular this gives that

$$\partial u' \in \mathcal{L}^2(G) \subset \mathcal{L}^1_{loc}(G)$$

where ∂ denotes the distributional derivative. Then by Theorem 4.2, $u' \in AC$ and $u'' \in \mathcal{L}^1_{loc}(G)$ allowing integration by parts. Thus $lu = -u'' + qu \in \mathcal{L}^1_{loc}(G)$ and consequently $lu = \lambda u \in \mathcal{L}^2(G)$. Now $q \in \mathcal{L}^\infty(G)$ and $\mathcal{D}(F) \subset \mathcal{L}^2(G)$, giving $u, u'' \in \mathcal{L}^2(G)$ and hence $u \in \mathcal{H}^2(G)$.

The definition of $\mathcal{D}(F)$ ensures that (4.1) holds. Integration by parts gives

$$\int_{\partial G} (fu + u')\bar{y}\; d\sigma = 0, \quad \text{for all} \quad y \in \mathcal{D}(F),$$

which from the definition of f, (Definition 4.1), and the constraints on the class of boundary conditions allowed, is equivalent to u obeying (4.2). ■

4.2 Max-Min Property

In this section we give a maximum-minimum characterization for the eigenvalues of boundary value problems on graphs. We refer the reader to [19, p. 406] and [73] where boundary value problems for partial differential operators are considered, and analogous results for such eigenvalues developed.

Theorem 4.4 (F. Riesz' Representation Theorem) *[76, p. 90] Let H be a Hilbert space and ν a continuous linear functional on H. Then there exists a unique $f \in H$ such that $\nu(g) = (g, f)$ for all $g \in H$.*

In the following theorem $\{v_0, \ldots, v_{n-1}\}^\perp$ will denote the orthogonal complement in $\mathcal{L}^2(G)$ of $\{v_0, \ldots, v_{n-1}\}$. In addition, as is customary, it will be assumed that the eigenvalues, λ_n, are listed in increasing order and repeated according to multiplicity, and the eigenfunctions, y_n, chosen so as to form a complete orthonormal family in $\mathcal{L}^2(G)$.

Since we are dealing with co-normal boundary conditions we have that

$$\begin{aligned} F(y_i, y_j) &= \int_{\partial G} f y_i \overline{y}_j \, d\sigma + \int_G (y_i' \overline{y}_j' + q y_i \overline{y}_j) \, dt \\ &= \int_{\partial G} f y_i \overline{y}_j \, d\sigma + \int_G (y_i' \overline{y}_j)' \, dt + \lambda_i (y_i, y_j) \\ &= \lambda_i (y_i, y_j) \\ &= \lambda_i \delta_{ij}. \end{aligned}$$

Theorem 4.5 *For $v_j \in \mathcal{L}^2(G), j = 0, 1, \ldots,$ let*

$$d_n(v_0, \ldots, v_{n-1}) = \inf \left\{ \frac{F(\varphi, \varphi)}{||\varphi||^2} \;\middle|\; \varphi \in \{v_0, \ldots, v_{n-1}\}^\perp \cap D(F) \setminus \{0\} \right\}. \quad (4.8)$$

Then

$$\lambda_n = \sup\{d_n(v_0, ..., v_{n-1}) \,|\, v_0, ..., v_{n-1} \in \mathcal{L}^2(G)\}, \quad \textit{for} \quad n = 0, 1, \ldots,$$

and this maximum-minimum is attained for $\varphi = y_n$ *and* $v_i = y_i$, $i = 0, \ldots, n-1$.

Proof: Let $v_0, \ldots, v_{n-1} \in \mathcal{L}^2(G)$. As $\operatorname{span}\{y_0, \ldots, y_n\}$ is $n+1$ dimensional and $\operatorname{span}\{v_0, \ldots, v_{n-1}\}$ is at most n dimensional there exists

$$\varphi \in \operatorname{span}\{y_0, \ldots, y_n\} \setminus \{0\}$$

having

$$(\varphi, v_i) = 0, \quad \text{for all} \quad i = 0, \ldots, n-1.$$

In particular, this ensures that $\varphi \in \mathcal{D}(F)$ as each y_i is in $\mathcal{D}(F)$.

Denote $\varphi = \sum_{k=0}^{n} c_k y_k$, then

$$F(\varphi, \varphi) = \sum_{i,k=0}^{n} c_i \bar{c}_k F(y_i, y_k) = \sum_{i,k=0}^{n} c_i \bar{c}_k \lambda_i \delta_{i,j} = \sum_{i=0}^{n} |c_i|^2 \lambda_i \le \lambda_n \sum_{i=0}^{n} |c_i|^2 = \lambda_n \|\varphi\|^2,$$

thus showing that

$$d_n(v_0, \ldots, v_{n-1}) \le \lambda_n \quad \text{for all} \quad v_0, \ldots, v_{n-1} \in \mathcal{L}^2(G).$$

For brevity denote

$$m := \sup\{d_n(v_0, ..., v_{n-1}) \,|\, v_0, ..., v_{n-1} \in \mathcal{L}^2(G)\}.$$

The above reasoning has shown that $m \le \lambda_n$.

In order to complete the proof we require that there exists $\varphi \in \mathcal{D}(F)$ with $\|\varphi\| = 1$ and $(\varphi, v_i) = 0$ for all $i = 0, \ldots, n-1$ such that $F(\varphi, \varphi) =$

$d_n(v_0, \ldots, v_{n-1})$. From the definition of $d_n(v_0, \ldots, v_{n-1})$, there exists a sequence $(u_k) \subset \mathcal{D}(F)$ with $\|u_k\| = 1$ and $(u_k, v_i) = 0$ for all $i = 0, \ldots, n-1$ and $k \in \mathbb{N}$ such that

$$\lim_{k\to\infty} F(u_k, u_k) = d_n(v_0, \ldots, v_{n-1}).$$

As $\mathcal{H}^1(G)$ is compactly embedded in $\mathcal{L}^2(G)$, see [67, page 64], there exists a subsequence of (u_k), which we again denote by (u_k) which converges in $\mathcal{L}^2(G)$ to say u with $\|u\| = 1$.

To show that $u \in \mathcal{H}^1(G)$ we need only show that the distribution ∂u is in $\mathcal{L}^2(G)$. For each $\psi \in \mathcal{C}_o^\infty(G) \subset \mathcal{D}(F)$,

$$\partial u(\psi) = -\int_G u\psi'\,dt = -\lim_{k\to\infty}\int_G u_k\psi'\,dt = \lim_{k\to\infty}\int_G u_k'\psi\,dt.$$

Thus

$$|\partial u(\psi)| \le \limsup \|u_k'\|\|\psi\| \le [d_n(v_0, \ldots, v_{n-1}) + \text{ess sup}|q|]^{1/2}\|\psi\|$$

and ∂u can be extended to a continuous linear functional on $\mathcal{L}^2(G)$. By Theorem 4.4 this gives $\partial u \in \mathcal{L}^2(G)$ and then by Theorem 4.2, $u \in AC$ with

$$\|u'\|^2 \le d_n(v_0, \ldots, v_{n-1}) + \text{ess sup}|q|.$$

Thus $u \in \mathcal{H}^1(G)$ and as

$$(u_k', \psi) = -(u_k, \psi') \to -(u, \psi') = (u', \psi), \quad \text{for all} \quad \psi \in \mathcal{C}_o^\infty(G),$$

it follows, from [1] applied componentwise, that $u_k \to u$ in $\mathcal{H}^1(G)$.

Let $u \in \mathcal{D}(F) \cap \{v_0, \ldots, v_{n-1}\}^\perp \backslash \{0\}$ have $F(u, u) = d_n(v_0, \ldots, v_{n-1})$ and $\|u\| = 1$. We show that u is an eigenfunction of (1.1), (4.1)-(4.2) with eigenvalue $\lambda = d_n(v_0, \ldots, v_{n-1})$.

Let

$$J(\varphi,\epsilon)=\frac{F(u+\epsilon\varphi)}{\|u+\epsilon\varphi\|^2}\quad\text{for all}\quad \varphi\in\mathcal{C}_0^\infty(G),\epsilon\in\mathbb{R}.$$

Differentiation with respect to ϵ of $J(\varphi,\epsilon)$ gives

$$0=\frac{\partial}{\partial\epsilon}J(\varphi,\epsilon)|_{\epsilon=0}=2\Re[F(\varphi,u)-d_n(v_0,\ldots,v_{n-1})(\varphi,u)],$$

for all $\varphi\in\mathcal{C}_0^\infty(G)$. Thus u is a variational eigenfunction with eigenvalue $\lambda=d_n(v_0,\ldots,v_{n-1})$. Lemma 4.3 now gives that u is in $\mathcal{H}^2(G)$, obeys boundary condition (4.1)-(4.2) and the equation (1.1) with $\lambda=d_n(v_0,\ldots,v_{n-1})$.

In the case of $n=0$, d_0 does not depend on any v_i and d_0 is an eigenvalue having

$$m=d_0\le\lambda_0.$$

Thus, in this case, $m=d_0=\lambda_0$.

In general we have shown $d_n(v_0,\ldots,v_{n-1})$ to be an eigenvalue less than or equal to λ_n and $m\le\lambda_n$. But if $v_i=y_i, i=0,\ldots,n-1$, then for u to be orthogonal to $v_0,\ldots,v_{n-1}$ and an eigenfunction to an eigenvalue, μ, less than or equal to λ_n forces $\mu=\lambda_n$ and u to be in the eigenspace of λ_n and orthogonal to $y_0,\ldots,y_{n-1}$. ■

4.3 Eigenvalue Bracketing

If the boundary conditions (2.5) are replaced by the Dirichlet condition $y=0$ at each node of G, i.e.

$$y_i(l_i)=0\quad\text{and}\quad y_i(0)=0,\quad i=1,\ldots,K,\tag{4.9}$$

then the graph G becomes disconnected with each edge e_i becoming a component sub-graph, G_i, with Dirichlet boundary conditions at its two nodes (ends). The boundary value problem on each sub-graph G_i is equivalent to a Sturm-Liouville boundary value problem on a compact interval with Dirichlet boundary conditions.

Denote by $A(\lambda)$ the number of eigenvalues less than λ, counted according to multiplicity, of (1.1), (4.1)-(4.2). Let $A^D(\lambda)$ be the number of eigenvalues less than λ of (1.1) but with (4.1)-(4.2) replaced by Dirichlet boundary conditions as discussed above, and let $A_j^D(\lambda)$ be the number of eigenvalues less than λ of (1.1) on G_j with Dirichlet boundary conditions. Then

$$\sum_{j=1}^{K} A_j^D(\lambda) = A^D(\lambda), \quad \lambda \in \mathbb{R}.$$

Denote by λ_n^D the eigenvalues of (1.1) with Dirichlet boundary conditions, as discussed above.

Consider the boundary value problem (1.1), (4.1)-(4.2) with the boundary conditions (4.1)-(4.2) replaced by the non-Dirichlet conditions

$$y_i'(l_i) = f(l_i)y_i(l_i) \quad \text{and} \quad y_i'(0) = f(0)y_i(0), \quad i = 1, \ldots, K \tag{4.10}$$

where f is given in (4.3), then as in the Dirichlet case, above, G decomposes into a union of disconnected graphs $G_1, \ldots, G_K$. Let λ_n^N denote the eigenvalues of (1.1), (4.10) and $A^N(\lambda)$ the number of eigenvalues less than λ counted according to multiplicity.

Let $A_i^N(\lambda)$ denote the number of eigenvalues less than λ of (1.1) on G_i with boundary conditions

$$y_i'(l_i) = f(l_i)y_i(l_i) \quad \text{and} \quad y_i'(0) = f(0)y_i(0).$$

Then

$$\sum_{i=1}^{K} A_i^N(\lambda) = A^N(\lambda).$$

In the case of co-normal boundary conditions, Theorem 4.5 has as a consequence that the spectral counting functions defined above are related by

$$\sum_{i=1}^{K} A_i^D(\lambda) = A^D(\lambda) \leq A(\lambda) \leq A^N(\lambda) = \sum_{i=1}^{K} A_i^N(\lambda), \quad \lambda \in \mathbb{R}, \tag{4.11}$$

and hence the eigenvalues are ordered by

$$\lambda_n^N \leq \lambda_n \leq \lambda_n^D, \quad n = 0, 1, \ldots. \tag{4.12}$$

These results are the content of the following corollary to Theorem 4.5 which give an analogue of [19, p. 407-410] for graphs.

Corollary 4.6 *If the boundary conditions (4.1)-(4.2) are co-normal with respect to l, then the the spectral counting functions for (1.1), (4.1)-(4.2) and the related boundary value problems with the Dirichlet and non-Dirichlet boundary conditions given in (4.9) and (4.10) are related by (4.11) and their spectra are related by (4.12).*

Proof: Denote by F_D the restriction of F to $\mathcal{H}_o^1(G)$ and by F_N the continuous extension (with respect to the $\mathcal{H}^1(G)$ norm) of F to $\mathcal{H}^1(G)$. As $\mathcal{H}_0^1(G) \subset \mathcal{D}(F) \subset \mathcal{H}^1(G)$ it follows that

$$\begin{aligned}
&\left\{\frac{F_D(\varphi,\varphi)}{\|\varphi\|^2} \,|\, \varphi \in \{v_0, \ldots, v_{n-1}\}^\perp \cap \mathcal{H}_o^1(G)\backslash\{0\}\right\} \\
&\subset \left\{\frac{F(\varphi,\varphi)}{\|\varphi\|^2} \,|\, \varphi \in \{v_0, \ldots, v_{n-1}\}^\perp \cap \mathcal{D}(F)\backslash\{0\}\right\} \\
&\subset \left\{\frac{F_N(\varphi,\varphi)}{\|\varphi\|^2} \,|\, \varphi \in \{v_0, \ldots, v_{n-1}\}^\perp \cap \mathcal{H}^1(G)\backslash\{0\}\right\}.
\end{aligned}$$

Taking infima gives

$$\begin{aligned}
d_n^D(v_0, \ldots, v_{n-1}) &:= \inf\left\{\frac{F_D(\varphi,\varphi)}{\|\varphi\|^2} \,|\, \varphi \in \{v_0, \ldots, v_{n-1}\}^\perp \cap \mathcal{H}_o^1(G)\backslash\{0\}\right\} \\
&\geq d_n(v_0, \ldots, v_{n-1}) \\
&\geq \inf\left\{\frac{F_N(\varphi,\varphi)}{\|\varphi\|^2} \,|\, \varphi \in \{v_0, \ldots, v_{n-1}\}^\perp \cap \mathcal{H}^1(G)\backslash\{0\}\right\} =: d_n^N(v_0, \ldots, v_{n-1}).
\end{aligned}$$

Theorem 4.5 now gives

$$\begin{aligned}
\lambda_n^D &= \sup\{d_n^D(v_0, \ldots, v_{n-1}) \,|\, v_0, \ldots, v_{n-1} \in \mathcal{L}^2(G)\} \\
&\geq \quad \lambda_n = \sup\{d_n(v_0, \ldots, v_{n-1}) \,|\, v_0, \ldots, v_{n-1} \in \mathcal{L}^2(G)\} \\
&\geq \quad \sup\{d_n^N(v_0, \ldots, v_{n-1}) \,|\, v_0, \ldots, v_{n-1} \in \mathcal{L}^2(G)\} = \lambda_n^N
\end{aligned}$$

from which the claims of the theorem follow directly. ■

4.4 Spectral Asymptotics

The results of the previous section provide a means by which to approximate the spectrum of a boundary value problem on a graph with co-normal type boundary conditions by considering the spectrum of a finite family of Sturm-Liouville problems on a compact interval having separated boundary conditions. Sturm-Liouville problems on a compact interval with separated boundary conditions have been extensively studied, and consequently eigenvalue approximations for such problems are well known, see [35]. These eigenvalue approximations in turn provide information about the spectral counting function for each Sturm-Liouville problem. Corollary 4.6 can now be applied, giving bounds on the spectral counting function for the original boundary value problem on the graph, from which eigenvalue asymptotics can be deduced.

Theorem 4.7 *[35, Theorem A4] Consider the boundary value problem*

$$-u'' + qu = \lambda u,$$

$$u(0)\cos\alpha + u'(0)\sin\alpha = 0,$$

$$u(l_i)\cos\beta + u'(l_i)\sin\beta = 0.$$

The asymptotic form of the eigenvalues of the above problem are as follows.

If $\sin\alpha = \sin\beta = 0$, *i.e. Dirichlet boundary conditions, then*

$$\lambda_n = \frac{(n+1)^2\pi^2}{l_i^2} + O(1), \quad n = 0, 1, \ldots.$$

If $\sin\alpha \sin\beta \neq 0$ *then*

$$\lambda_n = \frac{n^2\pi^2}{l_i^2} + O(1), \quad n = 0, 1, \ldots.$$

Note that in [35, Theorem A4] equation (A20) should read as follows:

$$\sqrt{\lambda_n} = \frac{(n+1)\pi}{\xi(1)} + O\left(\frac{1}{n}\right).$$

Theorem 4.8 *Let G be a compact graph with finitely many nodes. If the boundary value problem (2.3), (2.5) has co-normal boundary conditions, then its eigenvalues obey the asymptotic development*

$$\lambda_n = \frac{n^2\pi^2}{L^2} + O(n), \quad \text{as } n \to \infty,$$

and its spectral counting function has asymptotic approximation

$$A(\lambda) = \frac{L\sqrt{\lambda}}{\pi} + O(1), \quad \text{as } \lambda \to \infty,$$

where $L = \sum_{i=1}^{K} l_i$ *is the total length of the graph.*

Proof: In this proof we use the notation of Section 4.3. If we denote by $\lambda_n^{D,i}, n = 0, 1, \ldots$ the eigenvalues of l operating on the graph G_i with Dirichlet conditions at both ends, then Theorem 4.7 gives that

$$\lambda_n^{D,i} = \frac{(n+1)^2\pi^2}{l_i^2} + O(1), \quad n = 0, 1, \ldots,$$

and consequently as $\lambda \to \infty$ we obtain

$$A_i^D(\lambda) \geq \frac{l_i\sqrt{\lambda - c_i^D}}{\pi} - 1, \tag{4.13}$$

for some constant $c_i^D > 0$.

Similarly if we denote by $\lambda_n^{N,i}, n = 0, 1, \ldots$ the eigenvalues of l operating on the graph G_i with separated boundary conditions given in (4.10), then Theorem 4.7 gives that

$$\lambda_n^{N,i} = \frac{n^2\pi^2}{l_i^2} + O(1), \quad n = 0, 1, \ldots,$$

and consequently for large λ

$$A_i^N(\lambda) \leq \frac{l_i\sqrt{\lambda + c_i^N}}{\pi} + 1, \tag{4.14}$$

for some constant $c_i^N > 0$.

Taking $c = \max_{i=1,\ldots,K}\{c_i^D, c_i^N\}$, equations (4.13) and (4.14) remain valid with c_i^D and c_i^N replaced by c. Thus (4.13) and (4.14) yield

$$\frac{l_i\sqrt{\lambda - c}}{\pi} - 1 \leq A_i^D(\lambda) \leq A_i^N(\lambda) \leq \frac{l_i\sqrt{\lambda + c}}{\pi} + 1, \quad \text{as} \quad \lambda \to \infty. \tag{4.15}$$

Corollary 4.6, equation (4.11), can now be combined with (4.15) to give

$$\frac{L\sqrt{\lambda - c}}{\pi} \leq \sum_{i=1}^{K} A_i^D(\lambda) - K \leq A(\lambda) \leq \sum_{i=1}^{K} A_i^N(\lambda) \leq \frac{L\sqrt{\lambda + c}}{\pi} + K.$$

This can be rewritten as

$$A(\lambda) = \frac{L\sqrt{\lambda}}{\pi} + O(1), \quad \text{as } \lambda \to \infty.$$

Solving the asymptotic equation $A(\lambda) = n$ as both λ and n tend to infinity gives

$$\sqrt{\lambda_n} = \frac{n\pi}{L} + \delta_n$$

where $\delta_n = O(1)$, from which the stated eigenvalue asymptotic approximation follows directly. ■

Chapter 5

Regularized Traces

The "generalized" trace of an operator was introduced by Halberg and Kramer, [32], to study trace type characteristic of non-trace class operator. More precisely they study bounded perturbations of self-adjoint operators whose resolvent operators were of trace class. Their main two theorems, [32, Theorems 1 and 2], form two of the foundations of which our work builds. They apply their techniques to study the Sturm-Liouville equation

$$ly = \lambda y, \tag{5.1}$$

where

$$ly = -\frac{d^2y}{dx^2} + q(x)y, \tag{5.2}$$

on a compact interval with general Lagrange self-adjoint boundary conditions. Javjan, [37], extends this approach to singular Sturm-Liouville equations, while Gilbert and Kramer treat various higher order problems in [26, 27].

In this chapter we study the regularized trace of the differential operator, L given in (2.6), (2.7), associated with the formal operator l of (1.1) subject to the formally self-adjoint boundary conditions of Kirchhoff, Dirichlet and Neumann types, see [13, p. S30] and [16, 23] for detailed studies of Kirchhoff and δ type boundary conditions. Here by Kirchhoff boundary conditions at a given node, we mean continuity at the node and that the derivatives at the node sum to zero. This enables us to solve some classes of spectral inverse problems. Along the way asymptotic approximations for the iterated Green's function are found.

This chapter is structured as follows. In Section 5.1 we consider the iterated Green's function of $(l-\lambda)$ on G where we have imposed Dirichlet boundary conditions at each node. An asymptotic approximation for this Green's function is then obtained and used in Section 5.2 to find asymptotics for the iterated Green's function under general boundary conditions, see [24] and [57] for analogues in partial differential equations. The regularized trace is studied in Section 5.3 and two classes of inverse spectral problem are solved in the last section.

Throughout this chapter we will assume that q, given in equation (1.1), is a real valued function from $\mathcal{C}^2(\bar{G})$, unless otherwise stated.

We now set L_0 to be the principal part of L and let V_p, for each real valued $p \in \mathcal{C}^2(\bar{G})$, denote the multiplier operator

$$V_p f = p \cdot f, \quad \text{for all} \quad f \in \mathcal{L}^2(G). \tag{5.3}$$

Observe that $L = L_0 + V_q$ and V_q is a bounded self-adjoint operator in $\mathcal{L}^2(G)$.

5.1 Green's Function - Dirichlet

As previously discussed in Section 4.3, if we impose Dirichlet boundary conditions at each node on the graph G then, from a boundary value problem perspective, the graph can be considered as a disconnected graph composed of the disjoint union of the edges e_i with Dirichlet boundary conditions at both ends. Equation (2.3) with Dirichlet boundary conditions at each node, has a particularly simple Green's function. This Green's function and its iterates form the subject of this section. Denote the iterates of this Green's function by $\Gamma^k(x, y, \lambda), x, y \in G, k \in \mathbb{N}$, i.e. the kernel of the operator $\Gamma^k_\lambda := (l - \lambda I)^{-k}$ where here l is restricted to the domain $\mathcal{H}^2(G) \cap \mathcal{H}^1_o(G)$.

For $\rho > 0$, denote $\sqrt{\lambda} = i\rho$.

Lemma 5.1 *The iterated Green's function* $\Gamma^k(x, y, \lambda)$ *of*

$$(l - \lambda)g = -\frac{d^2g}{dx^2} + qg - \lambda g,$$

with Dirichlet boundary conditions at each node is given by

$$\Gamma^k(x, y, \lambda) = \begin{cases} 0, & x \in e_i, y \in e_j \quad \textit{where} \quad i \neq j \\ \Gamma^k_i(x, y, \lambda), & x, y \in e_i, \end{cases}$$

where $\Gamma^k_i(x, y, \lambda)$ *is the iterated Green's function of* $(l - \lambda)$ *on the edge* e_i *with Dirichlet boundary conditions at both ends.*

Proof: Let $f \in \mathcal{L}^2(G)$ and $x \in e_i$, then

$$(l - \lambda) \int_G \Gamma(x, y, \lambda) f(y)\, dy = (l - \lambda) \int_0^{l_i} \Gamma_i(x, y, \lambda) f_i(y)\, dy = f(x).$$

Also for $x \in e_i$ and $f \in \mathcal{D}(l - \lambda)$ we have

$$\int_G \Gamma(x, y, \lambda)(l - \lambda) f(y)\, dy = \int_0^{l_i} \Gamma_i(x, y, \lambda)(l - \lambda) f_i(y)\, dy = f(x),$$

thus proving the claim for $k = 1$.

Assuming the result for k and letting $x \in e_i$ we obtain from the case of $k = 1$ that

$$\Gamma^{k+1}(x, y, \lambda) = \int_G \Gamma(x, z, \lambda)\Gamma^k(z, y, \lambda)\, dz = \int_0^{l_i} \Gamma_i(x, z, \lambda)\Gamma^k(z, y, \lambda)\, dz.$$

But the hypothesis that the result holds for k gives for $z \in e_i$ that

$$\Gamma^k(z, y, \lambda) = \begin{cases} 0, & y \in e_j \quad \text{where} \quad i \neq j \\ \Gamma_i^k(z, y, \lambda), & y \in e_i, \end{cases}$$

and hence

$$\Gamma^{k+1}(x, y, \lambda) = \begin{cases} 0, & y \in e_j \quad \text{where} \quad i \neq j \\ \int_0^{l_i} \Gamma_i(x, z, \lambda)\Gamma_i^k(z, y, \lambda)\, dz, & y \in e_i, \end{cases}$$

from which the result follows for $k + 1$ upon noting that

$$\Gamma_i^{k+1}(x, y, \lambda) = \int_0^{l_i} \Gamma_i(x, z, \lambda)\Gamma_i^k(z, y, \lambda)\, dz.$$

The theorem now follows by induction. ■

Lemma 5.2 *Let $\Gamma_i^k(x, y, -\rho^2)$, be as defined in Lemma 5.1. Then Γ_i^k has the following asymptotic approximations*

$$\Gamma_i^k(x, y, -\rho^2) = O\left(\frac{e^{-\rho|x-y|}}{\rho^k}\right), \tag{5.4}$$

$$\frac{\partial \Gamma_i^k(x, y, -\rho^2)}{\partial y} = O\left(\frac{e^{-\rho|x-y|}}{\rho^{k-1}}\right), \tag{5.5}$$

$$\Gamma_i(x, y, -\rho^2) = \frac{e^{-\rho|x-y|}}{2\rho}\left(1 + O\left(\frac{1}{\rho}\right)\right), \tag{5.6}$$

$$\Gamma_i^2(x, y, -\rho^2) = \frac{e^{-\rho|x-y|}}{4\rho^2}\left[\left(|x-y| + \frac{1}{\rho}\right)\left(1 + O\left(\frac{1}{\rho}\right)\right)\right], \tag{5.7}$$

where (5.4) and (5.5) hold uniformly in x and y as $\rho \to +\infty$ while (5.6) and (5.7) hold uniformly for (x, y) on compact subsets of $e_i \times e_i = (0, l_i) \times (0, l_i)$ as $\rho \to +\infty$.

Proof: By Lemma 5.1 we need only consider the case of $x, y \in e_i$. We proceed by induction on k.

k=1:

Let $S(x,\rho)$ be the solution $(l+\rho^2)S = 0$ on $e_i = (0, l_i)$ having $S(0,\rho) = 0$ and $S'(0,\rho) = 1$, then from [35, Appendix],

$$S(x,\rho) = \frac{\sinh \rho x}{\rho} + O\left(\frac{e^{\rho x}}{\rho^2}\right), \tag{5.8}$$

$$S'(x,\rho) = \cosh \rho x + O\left(\frac{e^{\rho x}}{\rho}\right). \tag{5.9}$$

Let $\sigma(x,\rho)$ be the solution of $(l+\rho^2)\sigma = 0$ on e_i with $\sigma(l_i,\rho) = 0$ and $\sigma'(l_i,\rho) = 1$, then from [35, Appendix]

$$\sigma(x,\rho) = \frac{\sinh \rho(x-l_i)}{\rho} + O\left(\frac{e^{l_i - x}}{\rho^2}\right), \tag{5.10}$$

$$\sigma'(x,\rho) = \cosh \rho(x-l_i) + O\left(\frac{e^{l_i - x}}{\rho}\right). \tag{5.11}$$

In (5.8)-(5.11) the approximations are uniform in x as $\rho \to +\infty$.

The Green's function Γ_i can be explicitly expressed in terms of the solutions $S(x,\rho)$ and $\sigma(x,\rho)$ by

$$\Gamma_i(x,y,-\rho^2) = \begin{cases} \frac{S(x,\rho)\sigma(y,\rho)}{W[\sigma,S]}, & x \le y \\ \frac{\sigma(x,\rho)S(y,\rho)}{W[\sigma,S]}, & x \ge y \end{cases}, \tag{5.12}$$

see [18] or [53, page 35-37]. Here $W[\sigma,S](\rho)$ denotes the Wronskian of $\sigma(x,\rho)$ and $S(x,\rho)$, which has the argument x omitted as it is independent of x, see [18, page 82]. It should be noted that $\Gamma_i(x,y,-\rho^2)$ is a continuous function of x and y, see [53, page 29].

Combining (5.8)-(5.11), direct computation gives

$$W[\sigma,S](0,\rho) = -\frac{e^{\rho l_i}}{2\rho}\left(1 + O\left(\frac{1}{\rho}\right)\right), \tag{5.13}$$

as $\rho \to +\infty$.

Equations (5.8), (5.10) and (5.13) substituted into (5.12) give

$$\Gamma_i(x,y,-\rho^2) = O\left(\frac{e^{-\rho|y-x|}}{\rho}\right),$$

uniformly in x and y for $\rho \to +\infty$, and uniformly for (x,y) on compact subsets of $e_i \times e_i = (0,l_i)\times(0,l_i)$ as $\rho \to +\infty$ we have the more precise estimate

$$\Gamma_i(x,y,-\rho^2) = \frac{e^{-\rho|x-y|}}{2\rho}\left(1 + O\left(\frac{1}{\rho}\right)\right).$$

We have thus established (5.4) for $k=1$ and (5.6).

Differentiating (5.12) with respect to y yields

$$\frac{\partial \Gamma_i(x,y,-\rho^2)}{\partial y} = \begin{cases} \frac{S(x,\rho)}{W[\sigma,S](\rho)}\,\frac{d\sigma(y,\rho)}{dy}, & x<y \\ \frac{\sigma(x,\rho)}{W[\sigma,S](\rho)}\,\frac{dS(y,\rho)}{dy}, & x>y \end{cases}, \tag{5.14}$$

for all $x \neq y$ and $\rho > 0$. Substituting the estimates (5.8)-(5.11) in (5.14) we obtain

$$\frac{\partial \Gamma_i(x,y,-\rho^2)}{\partial y} = O\left(e^{-\rho|x-y|}\right), \tag{5.15}$$

uniformly in x and y for $\rho \to +\infty$, thus proving (5.5) for $k=1$.

Induction step:

For the remainder of the proof we assume the lemma true for k. We begin by considering (5.4) for $k+1$. From the definition of Γ_i^{k+1} it follows that

$$\Gamma_i^{k+1}(x,y,-\rho^2) = \int_0^{l_i} \Gamma_i^k(x,z,-\rho^2)\Gamma_i(z,y,-\rho^2)\,dz. \tag{5.16}$$

The induction hypothesis and (5.4) for the case of $k=1$ applied to (5.16), where the uniformity of the approximations is noted, gives

$$\Gamma_i^{k+1}(x,y,-\rho^2) = O\left(\int_0^{l_i} \frac{e^{-\rho|x-z|}}{\rho^k}\,\frac{e^{-\rho|y-z|}}{\rho}\,dz\right).$$

Since $|x-z|+|z-y| \geq |y-x|$ the above equation yields

$$\Gamma_i^{k+1}(x,y,-\rho^2) = O\left(\frac{e^{-\rho|x-y|}}{\rho^{k+1}}\right),$$

uniformly in x and y as $\rho \to +\infty$, there by proving (5.4).

The proof of (5.5) follows from (5.4) and the case of (5.5) for $k=1$ since

$$\begin{aligned}\frac{\partial\Gamma_i^{k+1}(x,y,-\rho^2)}{\partial y} &= \int_0^{l_i} \Gamma_i^k(x,z,-\rho^2)\frac{\partial\Gamma_i(z,y,-\rho^2)}{\partial y}\,dz \\ &= O\left(\int_0^{l_i} \frac{e^{-\rho|x-z|}}{\rho^k}\,e^{-\rho|y-z|}\,dz\right),\end{aligned}$$

from which it follows, as in the case of the iterates of Γ_i, that

$$\frac{\partial\Gamma_i^{k+1}(x,y,-\rho^2)}{\partial y} = O\left(\frac{e^{-\rho|x-y|}}{\rho^k}\right),$$

uniformly in x and y as $\rho \to +\infty$, there by proving (5.5).

We now progress to the proof of (5.7). From (5.8), (5.10) and (5.12), observe that for $x \leq y$

$$\begin{aligned}&\frac{\rho^2}{4}e^{2\rho l_i}\Gamma_i^2(x,y,-\rho^2)\\ &= \frac{\rho^2}{4}e^{2\rho l_i}\int_0^{l_i}\Gamma_i(x,z,-\rho^2)\Gamma_i(z,y,-\rho^2)\,dz\\ &= \int_0^x\left(\sinh\rho(x-l_i)\sinh^2\rho z\sinh\rho(y-l_i)+O\left(\frac{e^{\rho(2z+2l_i-x-y)}}{\rho}\right)\right)dz\\ &\quad+\int_x^y\left(\sinh\rho x\sinh\rho(z-l_i)\sinh\rho z\sinh\rho(y-l_i)+O\left(\frac{e^{\rho(2l_i+x-y)}}{\rho}\right)\right)dz\\ &\quad+\int_y^{l_i}\left(\sinh\rho x\sinh^2\rho(z-l_i)\sinh\rho y+O\left(\frac{e^{\rho(2l_i-2z+x+y)}}{\rho}\right)\right)dz\\ &= \sinh\rho(x-l_i)\sinh\rho(y-l_i)\int_0^x\sinh^2\rho z\,dz\\ &\quad+\sinh\rho x\sinh\rho(y-l_i)\int_x^y\sinh\rho(z-l_i)\sinh\rho z\,dz\\ &\quad+\sinh\rho x\sinh\rho y\int_y^{l_i}\sinh^2\rho(z-l_i)\,dz\\ &\quad+O\left(\frac{e^{\rho(2l_i+x-y)}}{\rho^2}+|x-y|\frac{e^{\rho(2l_i+x-y)}}{\rho}\right).\end{aligned}$$

Straightforward computation gives

$$\int_0^x \sinh^2 \rho z\, dz = -\frac{x}{2} + \frac{\sinh 2\rho x}{4\rho}, \tag{5.17}$$

$$\int_y^{l_i} \sinh^2 \rho(z-l_i)\, dz = -\frac{l_i - y}{2} + \frac{\sinh 2\rho(l_i - y)}{4\rho}, \tag{5.18}$$

$$\begin{aligned}\int_x^y \sinh \rho(z-l_i) \sinh \rho z\, dz &= -\frac{y-x}{2}\cosh \rho l_i + \frac{\sinh \rho(2y-l_i)}{4\rho} \\ &\quad -\frac{\sinh \rho(2x-l_i)}{4\rho}.\end{aligned} \tag{5.19}$$

Combining (5.17)-(5.19) with the expression for $\Gamma_i^2(x,y,-\rho^2)$ gives

$$\begin{aligned}
&\frac{\rho^2}{4} e^{2\rho l_i}\Gamma_i^2(x,y,-\rho^2) \\
&= \sinh \rho(x-l_i)\sinh\rho(y-l_i)\left(-\frac{x}{2} + \frac{\sinh 2\rho x}{4\rho}\right) \\
&+ \sinh \rho x \sinh \rho(y-l_i)\left(-\frac{y-x}{2}\cosh \rho l_i + \frac{\sinh \rho(2y-l_i)}{4\rho} - \frac{\sinh \rho(2x-l_i)}{4\rho}\right) \\
&+ \sinh \rho x \sinh \rho y \left(-\frac{l_i-y}{2} + \frac{\sinh 2\rho(l_i-y)}{4\rho}\right) \\
&+ O\left(\frac{e^{\rho(2l_i+x-y)}}{\rho^2} + |x-y|\frac{e^{\rho(2l_i+x-y)}}{\rho}\right),
\end{aligned}$$

uniformly in $x \le y$ as $\rho \to +\infty$. If we relax the uniformity of the above estimates to uniformly in (x,y) on compact subsets of $e_i \times e_i = (0,l_i)\times(0,l_i)$ and use the symmetry of Γ_i, the above expression can be simplified to

$$\Gamma_i^2(x,y,-\rho^2) = \frac{e^{-\rho|x-y|}}{4\rho^2}\left[|x-y|\left(1+O\left(\frac{1}{\rho}\right)\right) + \frac{1}{\rho}\left(1+O\left(\frac{1}{\rho}\right)\right)\right],$$

thereby proving (5.7). ■

The following Corollaries follows from Lemma 5.2.

Corollary 5.3 *Let $q \in C^{2k-1}(\bar{G})$ and $\Gamma_i^k(x,y,-\rho^2)$, be as defined in Lemma 5.1, then*

$$\left|\frac{\partial^j}{\partial x^j}\Gamma_i^k(x,y,-\rho^2)\right| \le \frac{K}{\rho^{k-j}}e^{-\rho|x-y|} \tag{5.20}$$

uniformly in x and y as $\rho \to \infty$ for $0 \le j < k$, $k \in \mathbb{N}$, and K a constant.

Proof: By Lemma 5.1 we need only consider $x, y \in e_i$.

For $f \in \mathcal{D}((l-\lambda)^j)$ with Dirichlet boundary conditions we have

$$\Gamma_i^k b^j f = \Gamma_i^{k-j} f.$$

Thus

$$\int_0^{l_i} \Gamma_i^k(x, y, -\rho^2) b_y^j \varphi(y)\, dy = \int_0^{l_i} \Gamma_i^{k-j}(x, y, -\rho^2)\varphi(y)\, dy$$

for $\varphi \in \mathcal{C}_0^\infty(e_i)$.

Since the boundary value problem is formally self-adjoint with $q \in C^{2k-1}(\bar{G})$ and since $\varphi \in \mathcal{C}_0^\infty(e_i)$ we obtain from the above equation

$$\int_0^{l_i} b_y^j \Gamma_i^k(x, y, -\rho^2)\varphi(y)\, dy = \int_0^{l_i} \Gamma_i^{k-j}(x, y, -\rho^2)\varphi(y)\, dy.$$

Therefore

$$b_x^j \Gamma_i^k(x, y, -\rho^2) = \Gamma_i^{k-j}(x, y)$$

for $0 \le j < k$ and $b_x^k \Gamma_i^k(x, y, -\rho^2) = 0$ for $x \ne y$.

We now show the following inequality

$$\left| \frac{\partial^j}{\partial x^j} \Gamma_i^k(x, y, -\rho^2) \right| \le \frac{K_{k,j}}{\rho^{k-j}} e^{-\rho|x-y|} \tag{5.21}$$

for all $0 \le j < 2k-1$, where $K_{k,j}$ is a constant.

Lemma 5.2 gives immediately that (5.21) is true for $k = 1, 2$.

Suppose that (5.21) is true for all $1, 2, \ldots, k$, then by Lemma 5.2 we have that for k replaced by $k+1$ and $j = 0, 1$, (5.21) holds, i.e.

$$|\Gamma_i^{k+1}(x, y, -\rho^2)| \le \frac{K_{k+1,0}}{\rho^{k+1}} e^{-\rho|x-y|}$$

and

$$\left| \frac{\partial}{\partial x} \Gamma_i^2(x, y, -\rho^2) \right| \le \frac{K_{k+1,1}}{\rho^{k-1}} e^{-\rho|x-y|}$$

for $K_{k+1,0}, K_{k+1,1}$ constants.

Now assume that

$$\left|\frac{\partial^j}{\partial x^j}\Gamma_i^{k+1}(x,y,-\rho^2)\right| \leq \frac{K_{k+1,j}}{\rho^{k+1-j}}e^{-\rho|x-y|}$$

is true for all $j = 0, 1, \ldots J$, where $J \leq 2k-2$.

Then using Leibnitz rule, see [1, p. 9]

$$\begin{aligned}
\left|\frac{\partial^{J+1}}{\partial x^{J+1}}\Gamma_i^{k+1}(x,y,-\rho^2)\right| &= \left|\frac{\partial^{J-1}}{\partial x^{J-1}}(-b_x + q_i(x) + \rho^2)\Gamma_i^{k+1}(x,y,-\rho 2)\right| \\
&= \left|\frac{\partial^{J-1}}{\partial x^{J-1}}[-\Gamma_i^k(x,y,-\rho^2) + (q_i(x)+\rho^2)\Gamma_i^{k+1}(x,y,-\rho^2)]\right| \\
&= \left|-\frac{\partial^{J-1}}{\partial x^{J-1}}\Gamma_i^k(x,y,-\rho^2) + \rho^2\frac{\partial^{J-1}}{\partial x^{J-1}}\Gamma_i^{k+1}(x,y,-\rho^2)\right. \\
&+ \left.\sum_{m\leq J-1}\binom{m}{J-1}\left(\frac{\partial^m q_i(x)}{\partial x^m}\right)\frac{\partial^{J-1-m}}{\partial x^{J-1-m}}\Gamma_i^{k+1}(x,y,-\rho^2)\right| \\
&\leq e^{-\rho|x-y|}\left(\frac{K_{k,J-1}}{\rho^{k-J+1}} + \frac{K_{k+1,J-1}}{\rho^{k-J}} + O\left(\frac{1}{\rho^{k-J+2}}\right)\right) \\
&= e^{-\rho|x-y|}O\left(\frac{1}{\rho^{k-J}}\right) \\
&\leq K_{k+1,J+1}\frac{e^{-\rho|x-y|}}{\rho^{k-J}}
\end{aligned}$$

for $K_{k,J-1}$, $K_{k+1,J-1}$ and $K_{k+1,J+1}$ constants. Hence the Lemma now follows.

■

Corollary 5.4 *Let $q \in C^{2k-1}(\bar{G})$ and $\Gamma_i^k(x,y,-\rho^2)$, be as defined in Lemma 5.1, then*

$$\Gamma_i^2(x,x,-\rho^2) = \frac{1}{4\rho^3}\left[1 + O\left(\frac{1}{\rho}\right)\right],$$

uniformly for x on compact subsets of $e_i = (0, l_i)$ as $\rho \to +\infty$.

5.2 Green's Function - General

Formulating (2.3), (2.4) as a self-adjoint boundary value problem for a second order system with separated boundary conditions, see Chapter 3 for details, gives directly that the boundary value problem has a Green's function and that the Green's operator is a compact operator. Denote the iterated Green's function by $g^k(x, y, \lambda), k \in \mathbb{N}$, i.e. the kernel of the operator $G_\lambda^k := (L-\lambda I)^{-k}$. We give an analogue of the result [24, Section 3] for iterated Green's functions on a graphs.

Denote

$$\Gamma_\rho f(y) \quad = \quad \int_G \Gamma(y, x, -\rho^2) f(x)\, dx, \quad \text{for all} \quad f \in \mathcal{L}^2(G), \tag{5.22}$$

then by definition of Γ, given in the previous section,

$$(l+\rho^2)\Gamma_\rho f \quad = \quad f, \quad \text{for all} \quad f \in \mathcal{L}^2(G), \tag{5.23}$$

$$\Gamma_\rho(l+\rho^2) f \quad = \quad f, \quad \text{for all} \quad f \in \mathcal{H}^2(G) \cap \mathcal{H}_o^1(G), \tag{5.24}$$

and for $x \neq y$

$$(l_x + \rho^2)\Gamma(x, y, -\rho^2) = 0, \tag{5.25}$$

where l_x denotes l operating with respect to the variable x, with y held constant. As the boundary value problems considered are self-adjoint and as q and the coefficients in the boundary conditions are real

$$\Gamma(y, x, -\rho^2) = \Gamma(x, y, -\rho^2). \tag{5.26}$$

Note that (5.25) and (5.26) also hold for $\Gamma(x, y, -\rho^2)$ replaced by the Green's function $g(x, y, -\rho^2)$. Now set

$$G_\rho f(y) \quad = \quad \int_G g(x, y, -\rho^2) f(x)\, dx, \quad \text{for all} \quad f \in \mathcal{L}^2(G), \tag{5.27}$$

then (5.23) holds for Γ_ρ replaced by G_ρ, while (5.24) is replaced by

$$G_\rho(l+\rho^2)f \quad = \quad f, \quad \text{for all} \quad f \in \mathcal{H}^2(G) \quad \text{obeying (2.4)}. \qquad (5.28)$$

Lemma 5.5 *Let $k \in \mathbb{N}, q \in \mathcal{C}^{2(k-1)}(\bar{G})$ be real valued, and $V \subset\subset G$. Let r be a third of the distance from the boundary of the graph G to the closure of V, i.e. $r = \frac{1}{3}\mathrm{dist}(\partial G, \bar{V})$, then for $y, z \in V$*

$$|g^k(y,z,-\rho^2) - \Gamma^k(y,z,-\rho^2)| \leq \frac{C(V)}{\rho^2 e^{2r\rho}}.$$

where $C(V) > 0$ is independent of ρ and y, z.

Proof: Let $b_\rho := l + \rho^2$, then the inverse of b_ρ with boundary conditions (2.4) is generated by the Green's function $g(x,y,-\rho^2)$, of (2.3)-(2.4) with $\lambda = -\rho^2$.

The sesquilinear forms generated by $g^k(x,y,-\rho^2)$ and $\Gamma^k(x,y,-\rho^2)$ will respectively be denoted by

$$\begin{aligned} G_\rho^k(f,h) &= \int_G \int_G f(x) g^k(x,y,-\rho^2)\bar{h}(y)\, dx\, dy, \\ \Gamma_\rho^k(f,g) &= \int_G \int_G f(x)\Gamma^k(x,y,-\rho^2) g(y)\, dx\, dy, \end{aligned}$$

for all $f, h \in \mathcal{L}^2(G)$. Now $\lambda_0(g,g) \leq (Lg,g)$, for all $g \in \mathcal{D}(L)$, and consequently $(\rho^2+\lambda_0)(g,g) \leq ((L+\rho^2)g,g)$, from which it follows that for all $\rho^2 > \lambda_0$,

$$0 \leq (\lambda_0+\rho^2)||g||^2 \leq ||(L+\rho^2)g||\,||g||.$$

Thus

$$0 \leq (\lambda_0+\rho^2)||g|| \leq ||(L+\rho^2)g||, \quad \text{for all} \quad g \in \mathcal{D}(L).$$

Let $h \in \mathcal{L}^2(G)$ and $g := G_\rho h$, then the above display gives

$$||G_\rho h|| \leq \frac{||h||}{\lambda_0+\rho^2}$$

from which it follows immediately that

$$||G_\rho|| \leq \frac{1}{\lambda_0 + \rho^2}, \quad \text{for all } \rho^2 > \lambda_0.$$

Hence there exists $\kappa > 0$ such that, for $\rho > 0$ sufficiently large,

$$||G_\rho^k|| \leq \frac{\kappa}{\rho^{2k}},$$

and thus for $k \in \mathbb{N}$ and $\rho > 0$ large,

$$|G_\rho^k(f,h)| \leq \frac{\kappa}{\rho^{2k}}\|f\|\|h\|, \quad \text{for all} \quad f,h \in \mathcal{L}^2(G), \tag{5.29}$$

with a similar bound holding for Γ_ρ^k. Now let

$$C_\rho^k(f,g) = G_\rho^k(f,g) - \Gamma_\rho^k(f,g).$$

From (5.29) and its analogue for Γ_ρ^k, there exists a constant $\gamma > 0$ such that for large $\rho > 0$,

$$\begin{aligned} \|C_\rho^k\| &= \sup_{f,g\in\mathcal{L}^2(G)\setminus\{0\}} \frac{|C_\rho^k(f,g)|}{\|f\|\ \|g\|} = \sup_{f,g\in\mathcal{L}^2(G)\setminus\{0\}} \frac{|G_\rho^k(f,g) - \Gamma_\rho^k(f,g)|}{\|f\|\ \|g\|} \\ &\leq \frac{\gamma}{\rho^{2k}}. \end{aligned} \tag{5.30}$$

Thus $\|C_\rho^k\| = O(\rho^{-2k})$.

From the definitions of $g^k(x,y,-\rho^2)$ and $\Gamma^k(x,y,-\rho^2)$

$$G_\rho^k(f,b_\rho^k h) = (f, G_\rho^k b_\rho^k h) = (f,h) = (f,\Gamma_\rho^k b_\rho^k h) = \Gamma_\rho^k(f,b_\rho^k h)$$

for all $f \in \mathcal{L}^2(G)$ and $h \in \mathcal{H}_o^{2k}(G)$ and thus

$$C_\rho^k(f,b_\rho^k h) = G_\rho^k(f,b_\rho^k h) - \Gamma_\rho^k(f,b_\rho^k h) = 0, \tag{5.31}$$

for all $f \in \mathcal{L}^2(G)$ and $h \in \mathcal{H}_o^{2k}(G)$.

Let V be a non-empty open subset of G with compact closure in G and choose V_1, V_2 such that

$$\overline{V} \subset V_1 \subset \overline{V}_1 \subset V_2 \subset \overline{V}_2 \subset G.$$

Let $\varphi \in \mathcal{C}_o^\infty(G)$ with $\varphi|_{V_1} \equiv 1$ and $\varphi|_{G\setminus V_2} \equiv 0$. Now let $y, y^* \in V_1$ and

$$p_\rho^k(y,x) := b_{\rho,x}^k[\Gamma^k(y,x,-\rho^2)(1-\varphi(x))], \tag{5.32}$$

then $p_\rho^k(y,x)$ and $p_\rho^k(y^*,x)$ vanish everywhere except possibly for $x \in V_2\setminus V_1$.

For each $y, y^* \in V_1$ let

$$\begin{aligned} c^k(y,y^*,-\rho^2) &:= C_\rho^k(p_\rho^k(y,\cdot), p_\rho^k(y^*,\cdot)) \\ &= \int_G\int_G p_\rho^k(y,x)[g^k-\Gamma^k](x,w,-\rho^2)p_\rho^k(y^*,w)\,dx\,dw, \end{aligned}$$

and for each $f, h \in \mathcal{L}^2(G)$ with support in V_1,

$$c_\rho^k(f,h) = \int_G\int_G f(y)c^k(y,y^*,-\rho^2)h(y^*)\,dy\,dy^*.$$

From the continuity of p_ρ^k and c_ρ^k

$$\begin{aligned} &c_\rho^k(f,h) \\ &= \int_G\int_G f(y)c^k(y,y^*,-\rho^2)h(y^*)\,dy\,dy^* \\ &= \int_G\int_G \left[\int_G f(y)p_\rho^k(y,z)\,dy\right][g^k-\Gamma^k](x,w,-\rho^2)\left[\int_G p_\rho^k(y^*,w)h(y^*)\,dy^*\right]dw\,dx \\ &= C_\rho^k\left(\left[\int_G f(y)p_\rho^k(y,\cdot)\,dy\right], \left[\int_G p_\rho^k(y^*,\cdot)h(y^*)\,dy^*\right]\right) \\ &= C_\rho^k\left(b_\rho^k\left[(1-\varphi)\Gamma_\rho^k f\right], b_\rho^k\left[(1-\varphi)\Gamma_\rho^k h\right]\right) \\ &= C_\rho^k(f - b_\rho^k\varphi\Gamma_\rho^k f, h - b_\rho^k\varphi\Gamma_\rho^k h), \end{aligned}$$

for $f, h \in \mathcal{H}^{2k}(G)$ with $\mathrm{supp}(f)$, $\mathrm{supp}(h) \subset V_1$.

Since $\varphi\Gamma_\rho^k f, \varphi\Gamma_\rho^k h \in \mathcal{H}_o^{2k}(G)$, by (5.31), $C_\rho^k(f, b_\rho^k\varphi\Gamma_\rho^k h)$, $C_\rho^k(b_\rho^k\varphi\Gamma_\rho^k f, h)$ and $C_\rho^k(b_\rho^k\varphi\Gamma_\rho^k f, b_\rho^k\varphi\Gamma_\rho^k h)$ are zero and $C_\rho^k(f,h) = c_\rho^k(f,h)$. Thus

$$G_\rho^k(f,h) = \Gamma_\rho^k(f,h) + c_\rho^k(f,h), \tag{5.33}$$

for $f, h \in \mathcal{H}^{2k}(G)$ with $\mathrm{supp}(f)$, $\mathrm{supp}(h) \subset V_1$. By continuity of the forms, (5.33) holds for all $f, h \in \mathcal{L}^2(G)$ with supports contained in V_1. Consequently

$$g^k(z,w,-\rho^2) = \Gamma^k(z,w,-\rho^2) + c^k(z,w,-\rho^2), \tag{5.34}$$

a.e. for $z, w \in V_1$, and, since $g^k(z,w,-\rho^2), \Gamma^k(z,w,-\rho^2)$ and $c^k(z,w,-\rho^2)$ are continuous with respect to $z, w \in G$, (5.34) holds for all $z, w \in V_1$.

From (5.30) it follows that for $\rho^2 > |\lambda_0|$ for $y, y^* \in V$

$$|c^k(y,y^*,-\rho^2)| \leq \frac{\gamma}{\rho^{2k}} \, \|p^k_\rho(y,\cdot)\| \, \|p^k_\rho(y^*,\cdot)\|$$

Let $y \in V$, then

$$\|p^k_\rho(y,\cdot)\| \leq K(\varphi) \sum_{i=0}^{2k-1} \sup_{x \in V_2 \backslash V_1} \left| \frac{\partial^i \Gamma^k(y,x,-\rho^2)}{\partial x^i} \right|.$$

and by Corollary 5.3,

$$\|p^k_\rho(y,\cdot)\| \leq C(\varphi) \sup_{x \in V_2 \backslash V_1} e^{-\rho|x-y|} \rho^{k-1},$$

where $K(\varphi)$ and $C(\varphi)$ depend on φ and its derivatives.

Let $r = \operatorname{dist}(\bar{V}_2 \backslash V_1, \bar{V})$, then from the above bound and Lemma 5.2 there is a constant $C(\varphi) > 0$ such that for $y \in V$

$$\|p_\rho(y,\cdot)\| \leq C(\varphi) e^{-r\rho} \rho^{k-1}.$$

Hence for all $y, y^* \in V$

$$|c^k(y,y^*,-\rho^2)| \leq \gamma C^2(\varphi) \frac{e^{-2r\rho}}{\rho^2},$$

from which the lemma follows directly. ■

Combining Lemma 5.2 and Lemma 5.5 yields immediately the following corollary used later in the chapter.

Corollary 5.6 *For $\lambda < -|\lambda_0|$, where λ_0 is the least eigenvalue of (2.3)-(2.4), and $q \in C^2(\bar{G})$, the iterated Green's function $g^2(x,y,\lambda)$, of $(l-\lambda)$ with (2.4) has*

$$\lim_{\rho \to \infty} \rho^3 g^2(x,x,-\rho^2) = \frac{1}{4}, \quad \text{for each} \quad x \in G. \tag{5.35}$$

This limit holds uniformly on compact subsets of G.

5.3 Regularized Traces

A self-adjoint operator on a Hilbert space is said to be of trace class if it is compact and the sum of its eigenvalues (with repetition according to multiplicity) is absolutely convergent.

If L and $\hat{L}$ are lower semi-bounded self-adjoint differential operators with eigenvalues $\lambda_0 \leq \lambda_1 \leq \ldots$ and $\hat{\lambda}_0 \leq \hat{\lambda}_1 \leq \ldots$ listed in increasing order and repeated according to multiplicity, then the regularized trace of L with respect to $\hat{L}$ is $\sum(\lambda_j - \hat{\lambda}_j)$, if this summation converges . This summation is termed the regularized trace of L with respect to $\hat{L}$, see [32], since neither L nor $\hat{L}$ has finite trace as $\sum \lambda_j$ and $\sum \hat{\lambda}_j$ are both divergent to $+\infty$.

Let T be a self-adjoint operator on Hilbert space H with domain $\mathcal{D}(T)$, T be semi-bounded from below and $(T - \mu I)^{-1}$ be of trace class for $\mu \leq M$, for some $M < 0$. Denote the eigenvalues of T by $\mu_0 \leq \mu_1 \leq \ldots$, where eigenvalues are repeated according to multiplicity. Associate with this sequence of eigenvalues a corresponding complete orthonormal sequence of eigenfunctions $\varphi_0, \varphi_1, \ldots$.

In this context, Halberg and Kramer, [32, Theorems 1 and 2], prove the following theorem, on which this chapter relies.

Theorem 5.7 *[Halberg and Kramer]*
Let V be a bounded operator defined on $\mathcal{D}(T)$ such that the operator $T+V$ has a denumerable sequence of real eigenvalues $\lambda_0 \leq \lambda_1 \leq \ldots$ having the property that $\sum_{n=0}^{\infty}(\lambda_n - \mu_n)$ is convergent. Then $(T-\mu I)^{-1}V(T-\mu I)^{-1}$ is of trace class for $\mu \leq M$, and

$$\sum_{n=0}^{\infty}(\lambda_n - \mu_n) = \lim_{\mu \to -\infty} \mu^2 S[(T-\mu I)^{-1}V(T-\mu I)^{-1}], \tag{5.36}$$

where $S[(T-\mu I)^{-1}V(T-\mu I)^{-1}]$ *denotes the trace of the operator* $(T-\mu I)^{-1}V(T-\mu I)^{-1}$.

If, in addition, $\sum_{n=0}^{\infty}(V\varphi_n, \varphi_n)$ *is convergent, then*

$$\sum_{n=0}^{\infty}(\lambda_n - \mu_n) = \sum_{n=0}^{\infty}(V\varphi_n, \varphi_n). \tag{5.37}$$

It should be noted that in order to obtain (5.36) from the above theorem, we need to verify the following conditions for the self-adjoint operator T in H and the bounded operator V on H:

(a) T is semi-bounded from below;

(b) there exists $M < 0$ such that $(T - \mu I)^{-1}$ is of trace class for $\mu \leq M$;

(c) $T + V$ has a denumerable sequence of (real) eigenvalues;

(d) $\sum_{n=0}^{\infty}(\lambda_n - \mu_n)$ is convergent.

Lemma 5.8 *Let* L *and* V_p *be as defined in (2.6)-(2.7), (5.3) and* $\mu_0, \mu_1, ...$ *and* $\lambda_0, \lambda_1, ...$ *be the eigenvalues of* $L + V_p$ *and* L*, respectively, listed in increasing order and repeated according to multiplicity. If* $\sum_{n=0}^{\infty}(\mu_n - \lambda_n)$ *is convergent then*

$$\lim_{\lambda \to -\infty} \lambda^2 tr(V_p L_\lambda^{-2}) = \sum_{n=0}^{\infty}(\mu_n - \lambda_n). \tag{5.38}$$

Proof: Throughout this proof we assume $\sum_{n=0}^{\infty}(\mu_n - \lambda_n)$ to be convergent. We note that in our situation conditions (a) to (d) given above hold since (a) is proved in Theorem 2.6, (b) follows from eigenvalue asymptotics previously obtained, (c) follows from $L+V_p$ being self-adjoint and having compact resolvent and (d) holds by assumption. Thus Theorem 5.7 is applicable. Hence

$$\lim_{\lambda\to-\infty} \lambda^2 tr(L_\lambda^{-1}V_pL_\lambda^{-1}) = \sum_{n=0}^{\infty}(\mu_n-\lambda_n).$$

The lemma now follows upon noting that for $\lambda<\lambda_0$, the self-adjointness of L gives

$$\begin{aligned} tr(L_\lambda^{-1}V_pL_\lambda^{-1}) &= \sum_{n=0}^{\infty}(L_\lambda^{-1}V_pL_\lambda^{-1}\varphi_n,\varphi_n) \\ &= \sum_{n=0}^{\infty}(V_pL_\lambda^{-1}\varphi_n,L_\lambda^{-1}\varphi_n) \\ &= \sum_{n=0}^{\infty}\frac{(V_pL_\lambda^{-1}\varphi_n,\varphi_n)}{\lambda_n-\lambda} \\ &= \sum_{n=0}^{\infty}(V_pL_\lambda^{-2}\varphi_n,\varphi_n), \end{aligned}$$

where $\{\varphi_n\}$ is an orthonormal family of eigenfunctions of L corresponding to the eigenvalue sequence $\{\lambda_n\}$. ■

As in [56], the Mercer expansion, together with Corollary 5.6 and Lemma 5.8, shows that the convergence of the regularized trace of $L+V_p$ with respect to L implies that the mean value of p is zero, more precisely we obtain the following theorem.

Theorem 5.9 *Let L, V_p be as defined in (2.6)-(2.7), (5.3) and $\mu_0,\mu_1,...$ and $\lambda_0,\lambda_1,...$ be the eigenvalues of $L+V_p$ and L, respectively, listed in increasing order and repeated according to multiplicity. If $\sum_{n=0}^{\infty}(\mu_n-\lambda_n)$ is convergent, then $\int_G p(x)dx=0$.*

Proof: The Mercer expansion gives

$$g^2(x,y,\lambda)=\sum_{n=0}^{\infty}\frac{\varphi_n(x)\varphi_n(y)}{(\lambda_n-\lambda)^2}$$

where $\{\varphi_n\}$ is an orthonormal sequence of eigenfunctions of L corresponding to the eigenvalue sequence $\{\lambda_n\}$. In particular

$$g^2(x,x,\lambda) = \sum_{n=0}^{\infty} \frac{\varphi_n^2(x)}{(\lambda_n - \lambda)^2}$$

where the summation

$$b(x) := \sum_{n=0}^{\infty} \frac{\varphi_n^2(x)}{|\lambda_n - \lambda|^2}$$

converges both pointwise and in $\mathcal{L}^1(G)$ as there exists constants $0 < K_1 < K_2$ such that, for large n, $K_1 n^2 \le \lambda_n \le K_2 n^2$, see Section 3.6. Thus $b(x) \max|p(x)|$ is an $\mathcal{L}^1(G)$-bound for the pointwise convergent sequence of partial sums

$$\left\{ \sum_{n=0}^{N} \frac{\varphi_n^2(x)p(x)}{(\lambda_n - \lambda)^2} \right\}.$$

Hence Lebesgue's Dominated Convergence Theorem can be applied to give

$$\int_G g^2(x,x,\lambda)p(x)\ dx = \sum_{n=0}^{\infty} \int_G \frac{\varphi_n^2(x)p(x)}{(\lambda_n - \lambda)^2}\ dx = tr(V_p L_\lambda^{-2}).$$

Now as $\displaystyle\sum_{n=0}^{\infty}(\mu_n - \lambda_n)$ converges, from Lemma 5.8 we obtain

$$\lim_{\lambda\to-\infty} \lambda^{3/2} tr(V_p L_\lambda^{-2}) = \lim_{\lambda\to-\infty} \lambda^2 tr(V_p L_\lambda^{-2}) \lim_{\lambda\to-\infty} \lambda^{-1/2} = 0.$$

Hence

$$0 = \lim_{\lambda\to-\infty} \lambda^{3/2} tr(V_p L_\lambda^{-2}) = \lim_{\lambda\to-\infty} \int_G \lambda^{3/2} g^2(x,x,\lambda)p(x)\ dx.$$

The uniformity of the limit in Lemma 5.6 allows us to interchange the limit and summation, above, to give

$$0 = \int_G \lim_{\lambda\to-\infty} (-\lambda)^{3/2} g^2(x,x,\lambda)p(x)\ dx = \frac{1}{4}\int_G p(x)\ dx. \quad \blacksquare$$

5.4 Inverse Spectral Problems

In this section we apply Theorem 5.9 to inverse spectral problems for second order operators on graphs. The first theorem gives a simple consequence of

Theorem 5.9 while the second result utilizes the variational reformulation of (2.3), (2.4) to give a somewhat deeper result.

Theorem 5.10 *If L, V_p, $\{\lambda_j\}$ and $\{\mu_j\}$ are as defined in Theorem 5.9, if*

$$\sum_{j=0}^{\infty}(\mu_j - \lambda_j) \tag{5.39}$$

is convergent and if p is of constant sign on G, then $p = 0$ on G.

Proof: From Theorem 5.9, $\int_G p(x)\, dx = 0$ making $p = 0$ a.e. and thus identically zero, as $p \in \mathcal{C}^2(\bar{G})$. ■

The eigenvalue problem (2.3)-(2.4) or equivalently for L, has a variational or weak $\mathcal{H}^1(G)$ formulation which was studied in detail in Section 4.1.

Without loss of generality, we assume the boundary conditions (2.4) to be in the form (4.1), (4.2).

Let $F(x, y)$ to be the sesquilinear form given by (4.3), with domain

$$\mathcal{D}(F) = \{y \in \mathcal{H}^1(G)\,|\, y \quad \text{obeying} \quad (4.1)\},$$

where as before

$$\int_{\partial G} y\, d\sigma := \sum_{i=1}^{K}[y_i(l_i) - y_i(0)] = \int_G y'\, dt.$$

Most physically interesting boundary conditions on graphs fall into the co-normal category, where co-normal boundary conditions are defined in Definition 4.1. In particular, 'Kirchhoff' and Neumann boundary conditions are co-normal. Using the same method as in Example 1 in Chapter 4 it is easy to observe that if node ν has Kirchhoff boundary conditions then $f(x) = 0$ for all $x \in \nu$ and this node contributes the domain conditions $y(x) = y(z)$ for all $x, z \in \nu$, while if the node ν has a Neumann boundary condition then $f(x) = 0$ for all $x \in \nu$ and this node does not contribute any domain conditions.

If each node of G has boundary conditions either of Kirchhoff type or of Neumann type, let Λ_K denote the collection of all nodes with Kirchhoff boundary conditions and Λ_N denote the collection of all nodes with Neumann boundary conditions. Then

$$\mathcal{D}(F) = \{y \in \mathcal{H}^1(G) \,|\, y(x) = y(z) \text{ for all } x, z \in \nu, \text{ for each } \nu \in \Lambda_K\}, \tag{5.40}$$

and f is the constant 0 function on ∂G.

We recall that in Lemma 4.3 it was shown that if (4.1)-(4.2) are co-normal boundary conditions with respect to l of (1.1), then $u \in \mathcal{D}(F)$ satisfies $F(u, v) = \lambda(u, v)$ for all $v \in \mathcal{D}(F)$ if and only if $u \in H^2(G)$ and u obeys (1.1), (4.1)-(4.2).

Theorem 5.11 *Consider the boundary value problem on the graph G consisting of (2.3) and boundary conditions at each node which are of either Neumann of Kirchhoff type. Let $\tilde{L}$ be the operator generated from this boundary value problem and L be operator generated from this problem but with $q = 0$. If $\lambda_0 = \mu_0$ and $\sum(\lambda_n - \mu_n)$ converges, where $\mu_0, \mu_1, \ldots$ and $\lambda_0, \lambda_1, \ldots$ are the eigenvalues of $\tilde{L}$ and L respectively, then $q = 0$.*

Proof: From Theorem 5.9 with $p = -q$, we obtain that $\int_G q(x)\, dx = 0$.

Let $\tilde{F}$ and F denote the sesquilinear forms corresponding to the eigenvalue problems for $\tilde{L}$ and L respectively, and let $\mathcal{D}(\tilde{F}) = \mathcal{D} = \mathcal{D}(F)$ denote their domain as given in (5.40). Observe that

$$\tilde{F}(x, y) = \int_G (x'\overline{y}' + xq\overline{y})\, dt,$$

and

$$F(x,y) \quad = \quad \int_G x'\overline{y}'\,dt.$$

Hence F is positive definite on $\mathcal{D}$ making $\lambda_0 \geq 0$. In addition, from the definition of $\mathcal{D}$ it is apparent that the constant 1 function $\mathbf{1}$ is in $\mathcal{D}$. Also $F(\mathbf{1},\mathbf{1}) = 0$ and thus from the variational formulation of the boundary value problem in Chapter 4, zero is the least eigenvalue of L and has eigenfunction $\mathbf{1}$. The hypotheses of the theorem now enable us to conclude that zero is also the least eigenvalue of $\tilde{L}$, making $\tilde{F}$ positive definite on $\mathcal{D}$. But the definition of $\tilde{F}$ along with the mean value of q being zero, gives

$$\tilde{F}(\mathbf{1},\mathbf{1}) = \int_G q\ dt = 0.$$

Hence $\mathbf{1}$ is an eigenfunction of $\tilde{L}$ with eigenvalue zero, from Chapter 4.

In Chapter 2 it was noted that the eigenvalue problem for the operator L and the boundary value problem (2.3), (2.4) are equivalent. Consequently $\mathbf{1}$ is an eigenfunction of (2.3), (2.4) for the eigenvalue zero and so

$$q = -(\mathbf{1})'' + q \cdot \mathbf{1} = 0 \cdot \mathbf{1} = 0. \quad \blacksquare$$

Chapter 6

Boundary Estimates

In this chapter we consider solutions of non-homogeneous boundary value problems on graphs. Estimates to the norms of solutions to non-homogeneous boundary value problems on the boundary are obtained. These estimates are given in terms of the norm of the non-homegeneity and their eigenparameter dependence is studied. An example is then provided verifying our results. It should be noted that in this chapter the perturbation, q, is taken to be essentially bounded.

6.1 Boundary Estimates

Theorem 6.1 *Let $\lambda = -k^2$, $k > 0$, then for y a solution of the boundary value problem* (2.3), (2.4),

$$||y||_{\mathcal{L}^2(G)} = \frac{1}{\sqrt{2k}}||y|_{\partial G}||_{\mathcal{L}^2(\partial G)}\left(1 + O\left(\frac{1}{k}\right)\right) \tag{6.1}$$

as $k \to \infty$, where ∂G denotes the boundary of G.

Proof: Consider the second order Sturm-Liouville problem on the interval $(0, l_i)$ given by

$$-y_i'' + qy_i = \lambda y_i \tag{6.2}$$

with boundary conditions

$$y_i(0) = \alpha_i \quad \text{and} \quad y_i(l_i) = \beta_i. \tag{6.3}$$

Let λ_0^i denote the least eigenvalue of (6.2) on $(0, l_i)$ with Dirichlet boundary conditions, $y_i(0) = 0 = y_i(l_i)$. Taking $\lambda < \Lambda := \min_{i=1,\ldots,K} \lambda_0^i$ we have that (6.2), (6.3) has a unique solution for each α_i, β_i.

From [35, Appendix A1] we have that the fundamental solutions of (6.2) obeying the boundary conditions

$$\begin{aligned} u_1(0) = &\; 1 \;= u_2'(0), \\ u_1'(0) = &\; 0 \;= u_2(0) \end{aligned}$$

are given asymptotically for large $k > 0$, by

$$u_1(t) = \cosh kt + O\left(\frac{e^{kt}}{k}\right), \tag{6.4}$$

$$u_2(t) = \frac{1}{k}\sinh kt + O\left(\frac{e^{kt}}{k^2}\right) \tag{6.5}$$

with corresponding derivatives

$$u_1'(t) = k\sinh kt + O(e^{kt}), \tag{6.6}$$

$$u_2'(t) = \cosh kt + O\left(\frac{e^{kt}}{k}\right), \tag{6.7}$$

uniformly with respect to t.

It should be noted that the Wronskian of $u_1(t)$ and $u_2(t)$ is equal to 1 for all t, i.e.

$$u_1(t)u_2'(t) - u_2(t)u_1'(t) = 1, \quad \text{for all } t.$$

From equation (6.3) we have that

$$y_i(t) = \alpha_i u_1(t) + \gamma_i u_2(t)$$

where γ_i is determined by

$$\beta_i = \alpha_i u_1(l_i) + \gamma_i u_2(l_i). \tag{6.8}$$

Solving for γ_i in (6.8) gives

$$\gamma_i = -\frac{u_1(l_i)\alpha_i - \beta_i}{u_2(l_i)}$$

and substituting γ_i into the equation for $y_i(t)$ above we get

$$\begin{aligned} y_i(t) &= \alpha_i u_1(t) - \frac{u_1(l_i)\alpha_i - \beta_i}{u_2(l_i)} u_2(t) \\ &= \frac{1}{u_2(l_i)} \left[-\alpha_i(-u_1(t)u_2(l_i) + u_1(l_i)u_2(t)) + \beta_i u_2(t) \right]. \end{aligned} \tag{6.9}$$

Let

$$w(t) := -u_1(t)u_2(l_i) + u_1(l_i)u_2(t),$$

then w is the solution of (6.2) with

$$\begin{aligned} w(l_i) &= 0 \\ w'(l_i) &= -u_1'(l_i)u_2(l_i) + u_1(l_i)u_2'(l_i) = 1. \end{aligned}$$

Thus from [35, Appendix A1], for large $k > 0$,

$$w(t) = \frac{1}{k}\sinh k(t - l_i) + O\left(\frac{e^{k(l_i - t)}}{k^2}\right), \tag{6.10}$$

uniformly in t.

Upon substituting in for $u_2(t)$ and $w(t)$, equation (6.9) now becomes

$$\begin{aligned} y_i(t) &= \frac{1}{u_2(l_i)}[\beta_i u_2(t) - \alpha_i w(t)] \\ &= \frac{1}{ku_2(l_i)} \left[\beta_i \left(\sinh kt + O\left(\frac{e^{kt}}{k}\right) \right) + \alpha_i \left(\sinh k(l_i - t) + O\left(\frac{e^{k(l_i - t)}}{k}\right) \right) \right]. \end{aligned}$$

Squaring this, we obtain

$$\begin{aligned} y_i^2(t) &= \frac{1}{k^2 u_2^2(l_i)} \left[\beta_i^2 \left(\sinh kt + O\left(\frac{e^{kt}}{k}\right)\right)^2 \right. \\ &+ \alpha_i \beta_i \left(\sinh kt + O\left(\frac{e^{kt}}{k}\right)\right)\left(\sinh k(l_i - t) + O\left(\frac{e^{k(l_i-t)}}{k}\right)\right) \\ &+ \left. \alpha_i^2 \left(\sinh k(l_i - t) + O\left(\frac{e^{k(l_i-t)}}{k}\right)\right)^2 \right]. \end{aligned}$$

Now

$$u_2^2(l_i) = \frac{e^{2kl_i}}{4k^2}\left(1 + O\left(\frac{1}{k}\right)\right). \tag{6.11}$$

Hence for large $k > 0$, $y_i^2(t)$ is bounded on $(0, l_i)$ so Lebesgue's dominated convergence theorem can be used, and only the pointwise limit of $y_i^2(t)$ needs to be considered for $t \in (0, l_i)$. For $t \in (0, l_i)$ and $k \to \infty$,

$$\begin{aligned} y_i^2(t) &= \frac{1}{k^2 u_2^2(l_i)} \left[\frac{\beta_i^2 e^{2kt}}{4}\left(1 + O\left(\frac{1}{k}\right)\right) + \frac{\alpha_i\beta_i e^{kl_i}}{4}\left(1 + O\left(\frac{1}{k}\right)\right) \right. \\ &+ \left. \frac{\alpha_i^2 e^{2k(l_i-t)}}{4}\left(1 + O\left(\frac{1}{k}\right)\right)\right]. \end{aligned}$$

Integrating from 0 to l_i gives

$$\begin{aligned} \int_0^{l_i} y_i^2(t)\,dt &= \frac{1}{k^2 u_2^2(l_i)} \left[\frac{\beta_i^2}{4}\left(1 + O\left(\frac{1}{k}\right)\right)\frac{e^{2kl_i}-1}{2k} + \frac{l_i\alpha_i\beta_i e^{kl_i}}{4}\left(1 + O\left(\frac{1}{k}\right)\right) \right. \\ &+ \left. \frac{\alpha_i^2}{4}\left(1 + O\left(\frac{1}{k}\right)\right)\frac{e^{2kl_i}-1}{2k}\right] \\ &= \frac{e^{2kl_i}}{8k^3 u_2^2(l_i)}\left[\beta_i^2\left(1 + O\left(\frac{1}{k}\right)\right) + \alpha_i^2\left(1 + O\left(\frac{1}{k}\right)\right)\right. \\ &+ \left. 2\alpha_i\beta_i k l_i e^{-kl_i}\left(1 + O\left(\frac{1}{k}\right)\right)\right] \\ &= \frac{e^{2kl_i}}{8k^3 u_2^2(l_i)}\left[\beta_i^2\left(1 + O\left(\frac{1}{k}\right)\right) + \alpha_i^2\left(1 + O\left(\frac{1}{k}\right)\right)\right] \\ &= \frac{e^{2kl_i}}{8k^3 u_2^2(l_i)}(\alpha_i^2 + \beta_i^2)\left(1 + O\left(\frac{1}{k}\right)\right), \end{aligned}$$

and by substituting in (6.11) we get

$$\int_0^{l_i} y_i^2(t)\,dt = ||y_i||^2_{\mathcal{L}^2(0,l_i)} = \frac{\alpha_i^2 + \beta_i^2}{2k}\left(1 + O\left(\frac{1}{k}\right)\right).$$

Therefore

$$||y_i||^2_{\mathcal{L}^2(0,l_i)} = \frac{1}{2k}||y_i|_{\partial(0,l_i)}||^2_{\mathcal{L}^2(\partial(0,l_i))}\left(1+O\left(\frac{1}{k}\right)\right).$$

Summing over $i = 1, \ldots, K$ proves the theorem. ■

The following theorem gives bounds for the boundary norm of solutions to the non-homogeneous boundary value problem in terms of the non-homogeneous term.

Theorem 6.2 *There exists a constant $C > 0$ such that*

$$\frac{2C}{k^{\frac{3}{2}}}||f||_{\mathcal{L}^2(G)} \geq ||y|_{\partial G}||_{\mathcal{L}^2(\partial G)} \tag{6.12}$$

for all $f \in \mathcal{L}^2(G)$ and y the solution of

$$-y'' + qy = \lambda y + f \tag{6.13}$$

obeying the boundary conditions (2.4).

Proof: Let G_λ denote the Green's operator of the boundary value problem (2.3), (2.4) and let G_λ^D denote the Green's operator of the boundary value problem (2.3) but with Dirichlet boundary conditions at every node (i.e. y at all nodes is zero).

We note that

$$(l-\lambda)(G_\lambda - G_\lambda^D)f = f - f = 0$$

for $f \in \mathcal{L}^2(G)$ and where l is as given in (1.1). Then $(G_\lambda - G_\lambda^D)f$ obeys (6.1) and we obtain that, since $(1 + O(\frac{1}{k})) \geq \frac{1}{2}$ for large k,

$$||(G_\lambda - G_\lambda^D)f||_{\mathcal{L}^2(G)} \geq \frac{1}{2\sqrt{k}}||(G_\lambda - G_\lambda^D)f|_{\partial G}||_{\mathcal{L}^2(G)}, \tag{6.14}$$

for all $f \in \mathcal{L}^2(G)$. Now as G_λ and G_λ^D are both Green's operators we have

$$||(G_\lambda - G_\lambda^D)f||_{\mathcal{L}^2(G)} \leq C\frac{||f||_{\mathcal{L}^2(G)}}{|\lambda|} = C\frac{||f||_{\mathcal{L}^2(G)}}{k^2} \tag{6.15}$$

where $C > 0$ is a constant.

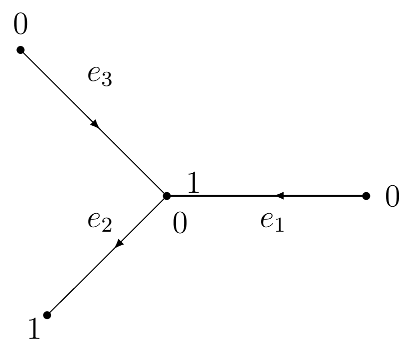
Hence combining (6.14) and (6.15) we obtain that

$$2C\frac{||f||_{\mathcal{L}^2(G)}}{k^2} \geq \frac{1}{\sqrt{k}}||(G_\lambda - G_\lambda^D)f|_{\partial G}||_{\mathcal{L}^2(\partial G)} = \frac{1}{\sqrt{k}}||G_\lambda f|_{\partial G}||_{\mathcal{L}^2(\partial G)}$$

since G_λ^D vanishes on the boundary of G. Taking $y = G_\lambda f$ gives (6.12). ■

6.2 Example

Assume that we have the following graph



where we have taken all edges to be length 1.

Now consider the second order differential equation

$$-y_i'' = \lambda y_i + 1 \tag{6.16}$$

for $i = 1, 2, 3$ on the above graph. In other words we have set $q_i = 0$ and $f_i = 1$ for $i = 1, 2, 3$. Here y_i, q_i and f_i denote y, q and f, of equation (6.13), restricted to the edge e_i.

Suppose that at the nodes we impose the following boundary conditions

$$\begin{aligned} y_1(0) &= 0, \\ y_2'(1) &= 0, \\ y_3(0) &= 0, \\ y_1'(1) - y_2'(0) + y_3'(1) &= 0, \\ y_1(1) = y_2(0) &= y_3(1). \end{aligned} \tag{6.17}$$

Using the method of variation of constants we obtain that the solutions of the boundary value problem (6.16), (6.17) on the above graph are of the form

$$y_i(t) = b_i \sinh kt - \frac{a_i}{k} \cosh kt + \frac{1}{k^2} \tag{6.18}$$

where a_i and b_i, $i = 1, 2, 3$, are constants.

From (6.18) and the boundary conditions (6.17), the constants a_i and b_i, $i = 1, 2, 3$, are given as follows

$$\begin{aligned} a_1 &= a_3 = \frac{1}{k}, \\ a_2 &= \frac{2\cosh k}{k(2\cosh^2 k + \sinh^2 k)}, \\ b_1 &= b_3 = \frac{3\cosh k \sinh k}{k^2(2\cosh^2 k + \sinh^2 k)}, \\ b_2 &= \frac{2\sinh k}{k^2(2\cosh^2 k + \sinh^2 k)}. \end{aligned}$$

Substituting the contants back into (6.18) we get that

$$\begin{aligned} y_1(t) &= y_3(t) = \frac{3\cosh k \sinh k}{k^2(2\cosh^2 k + \sinh^2 k)} \sinh kt - \frac{\cosh kt}{k^2} + \frac{1}{k^2} \\ y_2(t) &= \frac{2\sinh k}{k^2(2\cosh^2 k + \sinh^2 k)} \sinh kt - \frac{2\cosh k}{k^2(2\cosh^2 k + \sinh^2 k)} \cosh kt + \frac{1}{k^2}. \end{aligned}$$

Now evaluating y_i, $i = 1, 2, 3$, at 0 and 1 gives

$$y_1(0) = y_3(0) = 0$$

as expected from the boundary conditions and

$$y_1(1) = y_3(1) = y_2(0) = \frac{1}{k^2}\left(1 - \frac{2}{2\cosh k + \tanh k \sinh k}\right),$$

$$y_2(1) = \frac{1}{k^2}\left(1 - \frac{2}{3\cosh^2 k - 1}\right).$$

Thus

$$||y|_{\partial G}||^2_{\mathcal{L}^2(\partial G)} = \sum_{i=1}^{3}[y_i^2(0) + y_i^2(1)] = \frac{4}{k^4}\left(1 + O\left(\frac{1}{e^k}\right)\right).$$

Also

$$||f||^2_{\mathcal{L}^2(G)} = \sum_{i=1}^{3}\int_0^1 f_i^2 = 3$$

and

$$\frac{3}{k^3} \geq \frac{4}{k^4}\left(1 + O\left(\frac{1}{e^k}\right)\right).$$

Thus verifying Theroem 6.2. ■

Chapter 7

Inverse Nodal Problems

In this chapter we use eigenfunction and eigenvalue asymptotics in order to obtain an asymptotic approximation for the nodal points. These asymptotics then enable us to find a formula for the potential $q \in \mathcal{C}^1(G)$ and to show the unique dependence of the potential on the nodal points and eigenvalues, see Theorem 7.2.

7.1 Inverse Nodal Problems on Graphs

Let (λ_n) denote the eigenvalues of (2.3), (2.4) listed in increasing order and repeated according to multiplicity. Let (y_n) be an orthonormal sequence, in $\mathcal{L}^2(G)$, of eigenfunctions corresponding to the eigenvalue sequence (λ_n). For each edge e_i let $\sigma_i(n)$ be a one-to-one increasing map from $\mathbb{N}$ to $\mathbb{N}$ with $y_{\sigma_i(n)}|_{e_i} \not\equiv 0$ for each $n \in \mathbb{N}$ and $i = 1, \ldots, K$. In fact $\sigma_i(n)$ can be chosen such that $y_j|_{e_i} \not\equiv 0$ if and only if there exists $n \in \mathbb{N}$ such that $j = \sigma_i(n)$ and in this case $(y_{\sigma_i(n)}|_{e_i})_n$ is a basis for $\mathcal{L}^2(e_i)$.

Let the nodal points of $y_{\sigma_i(n)}$ on the edge $e_i = (0, l_i)$ be given in increasing order by

$$x_{i,n}^j, \quad j = 1, 2, \ldots, J_{i,n}.$$

For brevity, we denote $y_{\sigma_i(n)}|_{e_i}$ by $y_{i,n}$ and $\lambda_{\sigma_i(n)}$ by $\lambda_{i,n}$.

Theorem 7.1 *The set of nodal points* $\{x_{i,n}^j : j = 1, 2, \ldots, J_{i,n},\, n \in \mathbb{N}\}$ *is a dense subset of* e_i *and the distance between adjacent nodes satisfies the following asymptotic approximation for* n *large,*

$$x_{i,n}^{j+1} - x_{i,n}^j = \frac{\pi}{\sqrt{\lambda_{i,n}}} + O\left(\frac{1}{\lambda_{i,n}}\right). \tag{7.1}$$

Proof: As $x_{i,n}^j$ is a nodal point of $y_{i,n}$,

$$y_{i,n}(x_{i,n}^j) = 0.$$

Since $y_{i,n} \not\equiv 0$ we may, for convenience of notation assume

$$y_{i,n}'(x_{i,n}^j) = 1.$$

From [35], $y_{i,n}$ is given asymptotically, for large n, by

$$\begin{aligned} y_{i,n}(x) &= \int_{x_{i,n}^j}^x \frac{q_i(t)\sin\sqrt{\lambda_{i,n}}(x-t)\sin\sqrt{\lambda_{i,n}}(t-x_{i,n}^j)}{\lambda_{i,n}}\,dt \\ &\quad + \frac{\sin\sqrt{\lambda_{i,n}}(x-x_{i,n}^j)}{\sqrt{\lambda_{i,n}}} + O\left(\frac{1}{\lambda_{i,n}^{3/2}}\right), \qquad x \in (0, l_i), \end{aligned} \tag{7.2}$$

which can be rewritten as

$$y_{i,n}(x) = \frac{\sin\sqrt{\lambda_{i,n}}(x-x_{i,n}^j)}{\sqrt{\lambda_{i,n}}} + g_{i,n}(x), \tag{7.3}$$

where $g_{i,n}(x) = O\left(\frac{1}{\lambda_{i,n}}\right)$ and from Chapter 3 (for the special case of commensurate graphs see [13, 28]),

$$\sqrt{\lambda_{i,n}} = \sqrt{\lambda_{\sigma_i(n)}} = \frac{\sigma_i(n)\pi}{L} + o(\sigma_i(n)), \tag{7.4}$$

where L is the total length of the graph G.

Hence, for sufficiently large n, we may write

$$x_{i,n}^{j+1} = \frac{\pi}{\sqrt{\lambda_{i,n}}} + x_{i,n}^j + \delta_{i,n}^j,$$

where

$$|\delta_{i,n}^j| \leq \frac{\pi}{2\sqrt{\lambda_{i,n}}},$$

since, if

$$x_n^j := \frac{\pi}{2\sqrt{\lambda_{i,n}}} + x_{i,n}^j \quad \text{and} \quad \tilde{x}_n^j := \frac{3\pi}{2\sqrt{\lambda_{i,n}}} + x_{i,n}^j$$

then

$$y_{i,n}(x_n^j) = \frac{1}{\sqrt{\lambda_{i,n}}} + O\left(\frac{1}{\lambda_{i,n}}\right) > 0 > \frac{-1}{\sqrt{\lambda_{i,n}}} + O\left(\frac{1}{\lambda_{i,n}}\right) = y_{i,n}(\tilde{x}_n^j).$$

Thus $y_{i,n}$ has a nodal point in $(\tilde{x}_{i,n}^j, x_{i,n}^j)$ and by Prüfer considerations this is the subsequent nodal point to $x_{i,n}^j$ on e_i.

But

$$y_{i,n}(x_{i,n}^{j+1}) = 0 \quad \text{and} \quad g_{i,n}(x) = O\left(\frac{1}{\lambda_{i,n}}\right),$$

giving

$$\sin\sqrt{\lambda_{i,n}}(x_{i,n}^{j+1} - x_{i,n}^j) = -\sqrt{\lambda_{i,n}}\; g_{i,n}(x_{i,n}^{j+1})$$

and consequently

$$\sin(\pi + \delta_{i,n}^j\sqrt{\lambda_{i,n}}) = O\left(\frac{1}{\sqrt{\lambda_{i,n}}}\right).$$

Hence

$$-\sin(\sqrt{\lambda_{i,n}}\delta_{i,n}^j) = O\left(\frac{1}{\sqrt{\lambda_{i,n}}}\right). \tag{7.5}$$

As
$$-\frac{\pi}{2} \leq \delta_{i,n}^{j} \sqrt{\lambda_{i,n}} \leq \frac{\pi}{2},$$
equation (7.5) yields
$$\delta_{i,n}^{j} \sqrt{\lambda_{i,n}} = O\left(\frac{1}{\sqrt{\lambda_{i,n}}}\right).$$
Therefore
$$\delta_{i,n}^{j} = O\left(\frac{1}{\lambda_{i,n}}\right)$$
and consequently
$$x_{i,n}^{j+1} = \frac{\pi}{\sqrt{\lambda_{i,n}}} + x_{i,n}^{j} + O\left(\frac{1}{\lambda_{i,n}}\right),$$
proving (7.1). The fact that $\{x_{i,n}^{j} : j = 1, 2, \ldots, J_{i,n}, n \in \mathbb{N}\}$ is a dense subset of e_i comes directly from (7.1) and $\{\sigma_i(n), n \in \mathbb{N}\}$ being an infinite sequence tending to infinity. ■

We now prove the main result of this chapter.

Theorem 7.2 *The potential function, $q \in C^1(G) := \bigoplus_{i=1}^{K} C^1([0, l_i])$, is uniquely determined by the nodal data (i.e. spectrum and nodal points) and on e_i is given by*

$$q_i(x) := q|_{e_i}(x) = \lim_{n\to\infty} 2\lambda_{i,n}\left(\frac{\sqrt{\lambda_{i,n}}}{\pi}(x_{i,n}^{j_n+1} - x_{i,n}^{j_n}) - 1\right), \tag{7.6}$$

where σ_i is as defined at the beginning of this section and j_n is such that $x \in [x_{i,n}^{j}, x_{i,n}^{j+1})$, for each $n \in \mathbb{N}$.

Proof: Since $x_{i,n}^{j+1}$ is a nodal point of $y_{i,n}(x)$, we have that $y_{i,n}(x_{i,n}^{j+1}) = 0$. Therefore, from (7.2),

$$0 = \sqrt{\lambda_{i,n}} \sin \sqrt{\lambda_{i,n}}(x_{i,n}^{j+1} - x_{i,n}^{j}) + I_{i,n}^{j} + O\left(\frac{1}{\lambda_{i,n}^{1/2}}\right), \tag{7.7}$$

in which

$$I_{i,n}^{j}=\int_{x_{i,n}^{j}}^{x_{i,n}^{j+1}} q_i(t)\sin\sqrt{\lambda_{i,n}}(x_{i,n}^{j+1}-t)\sin\sqrt{\lambda_{i,n}}(t-x_{i,n}^{j})\,dt.$$

Asymptotically, for large $n\in\mathbb{N}$, from Theorem 7.1,

$$\begin{aligned}
\sin\sqrt{\lambda_{i,n}}(x_{i,n}^{j+1}-t) &= \sin\sqrt{\lambda_{i,n}}((x_{i,n}^{j+1}-x_{i,n}^{j})-(t-x_{i,n}^{j}))\\
&= \sin\sqrt{\lambda_{i,n}}(x_{i,n}^{j+1}-x_{i,n}^{j})\cos\sqrt{\lambda_{i,n}}(t-x_{i,n}^{j})\\
&- \cos\sqrt{\lambda_{i,n}}(x_{i,n}^{j+1}-x_{i,n}^{j})\sin\sqrt{\lambda_{i,n}}(t-x_{i,n}^{j})\\
&= O\left(\frac{1}{\sqrt{\lambda_{i,n}}}\right)+\left(1+O\left(\frac{1}{\lambda_{i,n}}\right)\right)\sin\sqrt{\lambda_{i,n}}(t-x_{i,n}^{j}).
\end{aligned}$$

Using the above estimate we have

$$I_{i,n}^{j}=\int_{x_{i,n}^{j}}^{x_{i,n}^{j+1}}\left[O\left(\frac{1}{\sqrt{\lambda_{i,n}}}\right)+\sin^2\sqrt{\lambda_{i,n}}(t-x_{i,n}^{j})\right]q_i(t)\;dt$$

Since

$$q_i(t)=q_i(x)+O(|x_{i,n}^{j+1}-x_{i,n}^{j}|),\quad \text{for}\quad t,x\in[x_{i,n}^{j},x_{i,n}^{j+1}),$$

it follows from Theorem 7.1 that

$$\begin{aligned}
I_{i,n}^{j} &= [q_i(x)+O(|x_{i,n}^{j+1}-x_{i,n}^{j}|)]\int_{x_{i,n}^{j}}^{x_{i,n}^{j+1}}\sin^2\sqrt{\lambda_{i,n}}(t-x_{i,n}^{j})\,dt\\
&+ O\left(\frac{x_{i,n}^{j+1}-x_{i,n}^{j}}{\sqrt{\lambda_{i,n}}}\right)\\
&= \left(q_i(x)+O\left(\frac{1}{\sqrt{\lambda_{i,n}}}\right)\right)\int_{x_{i,n}^{j}}^{x_{i,n}^{j+1}}\left(\frac{1}{2}-\frac{\cos 2(\sqrt{\lambda_{i,n}}(t-x_{i,n}^{j}))}{2}\right)dt\\
&+ O\left(\frac{1}{\lambda_{i,n}}\right)\\
&= \left(q_i(x)+O\left(\frac{1}{\sqrt{\lambda_{i,n}}}\right)\right)\left[\frac{1}{2}(x_{i,n}^{j+1}-x_{i,n}^{j})-\frac{\sin 2\sqrt{\lambda_{i,n}}(x_{i,n}^{j+1}-x_{i,n}^{j})}{4\sqrt{\lambda_{i,n}}}\right]\\
&+ O\left(\frac{1}{\lambda_{i,n}}\right).
\end{aligned}$$

Now applying Theorem 7.1 again, we obtain

$$I_{i,n}^{j}=O\left(\frac{1}{\lambda_{i,n}}\right)+q_i(x)\frac{x_{i,n}^{j+1}-x_{i,n}^{j}}{2}. \tag{7.8}$$

Hence, for each $x \in [x_{i,n}^j, x_{i,n}^{j+1})$, combining (7.7) and (7.8) gives

$$\sqrt{\lambda_{i,n}} \sin \sqrt{\lambda_{i,n}}(x_{i,n}^{j+1} - x_{i,n}^j) + q_i(x)\frac{x_{i,n}^{j+1} - x_{i,n}^j}{2} = O\left(\frac{1}{\lambda_{i,n}}\right). \tag{7.9}$$

Solving for $q_i(x)$ in (7.9) we obtain,

$$\begin{aligned} q_i(x) &= \frac{-2\lambda_{i,n} \sin \sqrt{\lambda_{i,n}}(x_{i,n}^{j+1} - x_{i,n}^j)}{\sqrt{\lambda_{i,n}}(x_{i,n}^{j+1} - x_{i,n}^j)} + O\left(\frac{1}{\lambda_{i,n}}\right) \\ &= \frac{2\lambda_{i,n} \sin[\sqrt{\lambda_{i,n}}(x_{i,n}^{j+1} - x_{i,n}^j) - \pi]}{\sqrt{\lambda_{i,n}}(x_{i,n}^{j+1} - x_{i,n}^j)} + O\left(\frac{1}{\lambda_{i,n}}\right), \end{aligned}$$

for $x \in [x_{i,n}^j, x_{i,n}^{j+1})$. By Theorem 7.1,

$$\sqrt{\lambda_{i,n}}(x_{i,n}^{j+1} - x_{i,n}^j) - \pi = O\left(\frac{1}{\sqrt{\lambda_{i,n}}}\right),$$

hence

$$\sin[\sqrt{\lambda_{i,n}}(x_{i,n}^{j+1} - x_{i,n}^j) - \pi]$$

can be approximated to order $O\left(\frac{1}{\sqrt{\lambda_{i,n}}}\right)$ by its argument, giving

$$\begin{aligned} q_i(x) &= \frac{2\lambda_{i,n}[\sqrt{\lambda_{i,n}}(x_{i,n}^{j+1} - x_{i,n}^j) - \pi]}{\sqrt{\lambda_{i,n}}(x_{i,n}^{j+1} - x_{i,n}^j)} + O\left(\frac{1}{\lambda_{i,n}}\right) \\ &= \frac{2\lambda_{i,n}[\sqrt{\lambda_{i,n}}(x_{i,n}^{j+1} - x_{i,n}^j) - \pi]}{\pi + O\left(\frac{1}{\sqrt{\lambda_{i,n}}}\right)} + O\left(\frac{1}{\lambda_{i,n}}\right) \end{aligned} \tag{7.10}$$

for $x \in [x_{i,n}^j, x_{i,n}^{j+1})$, from which (7.6) follows directly.

Solving for q_i on each edge e_i for $i = 1, \ldots, K$ uniquely gives the potential on the whole graph. ■

Chapter 8

Conclusions

In this thesis we have proved that a self-adjoint boundary value problem on a graph can be considered as a self-adjoint system. This system was then shown to be equivalent to a system of twice the dimension but with separated boundary conditions. Abstract Prüfer angle methods were then used to find eigenvalue asymptotics.

We then returned to the original graph structure, where, using techniques from partial differential equations, we set up a variational formulation for boundary value problems on graphs. From this we were able to give a type of Dirichlet-Neumann bracketing for boundary value problems on graphs. Consequently eigenvalue and eigenfunction asymptotic approximations were obtained. We note that the variational formulation used by [45] is distinct from ours. In particular, because of the definition of the Sobolev space $\mathcal{H}^1$ over a graph used in [45], the formulation in [45] does not yield Dirichlet-Neumann bracketing or eigenvalue asymptotics.

Next an asymptotic approximation to the Green's function was found in order to study the regularized trace. Two inverse spectral problems for boundary value problems on graphs were then solved, using the regularized trace.

Solutions of non-homogeneous boundary value problems on graphs were then considered. We found a relationship between the norms of solutions to the non-homogeneous boundary value problem restricted to the boundary of the graph and the norm of the non-homogeneous term on the graph.

Finally, the nodal structure of eigenfunctions to the boundary value problem was considered, and the inverse nodal problem solved (both uniqueness and construction).

Although the literature on boundary value problems on graphs is rapidly increasing there are still many undeveloped areas, for example eigenvalue ratios for differential operators on graphs and Harnack's inequality on graphs.

A problem which we have considered (not included here) is the M-matrix inverse problem. So far we have managed to give a well-defined M-matrix, relate the matrix Prüfer angle to the M-matrix using a technique similar to that in [8], and obtain asymptotics for the M-matrix as λ tends to negative infinity. Unfortunately even recovering the boundary conditions, which is the next step in solving the M-matrix inverse problem, is very difficult. This, along with the above mentioned undeveloped areas, will be considered in future research.

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