Closed Left Ideal Decompositions of G^* [1]

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Abstract

Let G be a countably infinite discrete group and let βG be the Stone-Čech compactification of G. The group operation of G extends to βG making it a compact right shift continuous semigroup. The semigroup βG has import applications to Ramsey theory and to topological dynamics. It has long been known that remainder $G^* = \beta G \setminus G$ can be decomposed into $2^{\mathfrak{c}}$ left ideals of βG (E. van Dowen 1980-s, D. Davenport and N. Hindman 1991). In 2005 I. Protasov strengthened this theorem by proving that G^* can be decomposed into $2^{\mathfrak{c}}$ closed left ideals of βG such that the corresponding quotient space is Hausdorff. Let $\mathcal{I}$ denote the finest decomposition of G^* into closed left ideals of βG with the property that the corresponding quotient space of G^* is Hausdorff and let $\mathcal{I}'$ denote the finest decomposition of G^* into closed left ideals of βG. If $p \in G^*$ is a P-point then $(\beta G)p \in \mathcal{I}$. We show that it is consistent with ZFC, the system of usual axioms of set theory, that if G can be algebraically embedded into a compact group, then every $I \in \mathcal{I}$ contains $2^{\mathfrak{c}}$ maximal principal left ideals of βG, in particular, neither member of $\mathcal{I}$ is a principal left ideal of βG. We also show that there is a dense subset of points $p \in G^*$ such that $(\beta G)\, p \in \mathcal{I}' \setminus \mathcal{I}$, in particular, $\mathcal{I}'$ is finer than $\mathcal{I}$.

Declaration

I declare that this thesis is my own work which was compiled while registered as a part-time student at the University of the Witwatersrand, Johannesburg and has is being submitted for the Degree of Doctor of Philosophy to the same university. The main results of this thesis were presented at two annual congresses of the South African Mathematical Society and have been published as two separate papers in the journal Topology and its Applications. It has not been submitted before for any degree or examination to any other University.

(Signature)

(Date)

Acknowledgement

I would like to dedicate this dissertation to my family, to my loving wife Shel and her endless supply of coffee and caring when the end seemed very far away and to Professor Yuliya and Yevhen Zelenyuk whose patience for this part-time student cannot be overstated.

"Life is like riding a bicycle. To keep your balance you must keep moving."

- Albert Einstein (1879-1955)

Contents

Chapter 1

Introduction

Let G be a countable infinite discrete group and let βG be the Stone-Čech compactification of G. The operation of G extends to the Stone-Čech compactification βG of G in such a way that for each $a \in G$, the left translation

$$\lambda_a : \beta G \ni x \to a \cdot x \in \beta G$$

is continuous, and for each $q \in \beta G$, the right translation

$$\rho_q : \beta G \ni x \to x \cdot q \in \beta G$$

is continuous. We take the points of βG to be the ultrafilters on G and identify the principal ultrafilters with the points of G.

For every $A \subset G$, let $\overline{A} = \{p \in \beta G : A \in p\}$ and $A^* = \overline{A} \setminus A$. The subsets $\overline{A}$ where $A \subset G$ form a base for the zero dimensional topology on βG. For $p, q \in \beta G$ the ultrafilter $p \cdot q$ has a base consisting of subsets of the form $\bigcup_{x \in A} x \cdot B_x$ where $A \in p$ and $B_x \in q$.

Theorem 1.0.1. *The compact space βG with the operation $\cdot : \beta G \times \beta G \to \beta G$ is a compact right shift continuous semigroup.*

The semigroup βG has important applications to Ramsey theory, topological dynamics and topological group theory. The first example of applications was provided by Hindman's theorem. It says that whenever the set $\mathbb{N}$ of natural numbers is finitely coloured there is an infinite $A \subset \mathbb{N}$ such that the set of all finite sums of distinct elements of A, denoted by $FS(A)$, is monochrome. Such A is derived from a certain ultrafilter on $\mathbb{N}$, an idempotent of the semigroup $\mathbb{N}^* \subset \mathbb{Z}^*$. See [HS12, Zel11] for more information about the semigroup βG and its applications.

Definition 1.0.2. *If $p \in G^*$ then $(\beta G)p$ is the principal left ideal of βG generated by p.*

A principal left ideal of βG is maximal if it is not properly contained in any other principal left ideal of βG. It is a remarkable fact that any two principal left ideals of βG are either disjoint or one of them is contained in the other ([HS12], Corollary 6.20), and consequently any two distinct maximal principal left ideals of βG are disjoint. An obvious example of a maximal principal left ideal of βG is that generated by a prime element, that is a point $p \in (G^*)^2 \setminus G^*$. The next argument shows that βG contains many such prime elements.

Definition 1.0.3. *A subset $A \subset G$ is called thin if for any distinct $x, y \in G$, $(xA) \cap (yA)$ is finite.*

Lemma 1.0.4. *[Cho69]Every infinite subset of G contains an infinite thin subset.*

Lemma 1.0.5. *If A is a thin subset of G then every $p \in A^*$ is prime.*

Corollary 1.0.6. *The set of prime elements of βG contains a dense open subset of G^*.*

The next question is one of the main open problems about βG (see [HS12] p.135)

Question 1.0.7. *Does every point of G^* lie in a maximal principal left ideal of βG? Of special interest is the case $G = \mathbb{Z}$.*

Equivalently, do the maximal principal left ideals of βG form a decomposition or partition of G^*? It has been known for a long time that G^* can be decomposed into $2^{\mathfrak{c}}$ left ideals of βG (see [DH91] and [HS12], Theorem 6.53). In [Pro05] this was strengthened by showing that G^* can be decomposed into $2^{\mathfrak{c}}$ closed left ideals of βG such that the corresponding quotient space of $G^*$is Hausdorff. The proof was complicated and based on balleans and slowly oscillating functions. It was simplified in [FS07] and [TZ12] and another direct proof was supplied in [Zel13]. Considering the diagonal of the quotient mappings justifies the following definition

Definition 1.0.8. *Let $\mathcal{I}$ denote the finest decomposition of G^* into closed left ideals of βG with the property that the corresponding quotient space of G^* is Hausdorff and let $\mathcal{I}'$ denote the finest decomposition of G^* into closed left ideals of βG.*

The decomposition $\mathcal{I}$ can be intrinsically characterized as follows (see [Zel13]):

Theorem 1.0.9. *For every $p \in G^*$ let $\mathcal{F}_p$ denote the filter on G with the base consisting of subsets of the form $\bigcup_{x \in G} x \cdot (A \setminus F_x)$ where F_x is a finite subset of G for each $x \in G$ and $A \in p$. If $I_p = \bigcap_{B \in \mathcal{F}_p} \overline{B}$ then $\mathcal{I} = \{I_p : p \in G^*\}$.*

It is clear that every maximal principal left ideal of βG is contained in a member of $\mathcal{I}$. The next example shows that assuming additional set-theoretic assumptions, some members of $\mathcal{I}$ may be maximal principal left ideals of βG.

Definition 1.0.10. *A non principal ultrafilter $p \in \omega$ is a P-point if the intersection of countably many neighbourhoods of $p \in \omega^*$ is again a neighbourhood of p.*

Equivalently p is a P-point if whenever $\{A_n : n < \omega\}$ is a partition of ω and $A_n \notin p$ for all $n < \omega$ there exists $A \in p$ such that $A \cap A_n$ is finite for all $n < \omega$. Martin's Axiom (denoted by MA) implies the existence of P-points. It is however consistent with ZFC, the standard axiomatic foundations of set theory, that there are no P-points (see [She98])

Lemma 1.0.11. *If $p \in G^*$ is a P-point then $I_p = (\beta G)p$.*

After these results the following two questions naturally arise:

Question 1.0.12. *Can it be shown in ZFC that there is $I \in \mathcal{I}$ which is a maximal principal left ideal of βG.*

Question 1.0.13. *[Zel13] Is $\mathcal{I} = \mathcal{I}'$?*

The aim of this dissertation is to answer Question 2 and Question 3. We show that it is consistent with ZFC that if G can be algebraically embedded into a compact group then every $I \in \mathcal{I}$ contains $2^{\mathfrak{c}}$ maximal principal left ideals of βG [BZ13]. We also show that there is a dense subset of points $p \in G^*$ such that $(\beta G)p \in \mathcal{I}' \setminus \mathcal{I}$, in particular $\mathcal{I}'$ is finer than $\mathcal{I}$ [BZZ14].

We proceed as follows: In Chapter 2 we will rapidly cover the Set theoretic and Topological results needed to underpin our further investigation of βG. In Chapter 3 a review of the literature and the general construction of the Stone-Čech compactification of a topological semigroup will be given and we will briefly investigate why the ultrafilter construction on a discrete semigroup is the natural framework for investigating semigroup compactifications. Topological and Algebraic properties required for the main results will also be given here. Finally in Chapter 4 the results in [BZ13] and [BZZ14] will be given.

Chapter 2

Preliminaries and Ground Cover

2.1 Set Theory and Topology

In this section we will cover the set theoretic topology and pure set theory required for the following chapters. We start by building the framework to describe Martin's Axiom.

Definition 2.1.1. *Continuum Hypothesis: There is no set whose cardinality is strictly between* $\mathbb{N}$ *and* $\mathbb{R}$*.*

Definition 2.1.2. *A filter* F *on a partially ordered set* X *is a subset of* X *such that*

1. *For all* $a, b \in F$ *there must exist* $c \in F$ *such that* $c \leq a$ *and* $c \leq b$*.*
2. *For all* $a \in F$ *and* $b \in X$*,* $a \leq b \implies b \in F$*.*
3. $F \neq \emptyset$*.*

Definition 2.1.3. *A filter* F *on a partially ordered set* X *is proper if* $F \neq X$*.*

Definition 2.1.4. *Two elements* $a, b \in X$ *are incompatible if there is no element* $c \in X$ *such that* $c \leq a$ *and* $c \leq b$*.*

Definition 2.1.5. *An anti chain on* X *is a subset of pairwise incompatible elements.*

Definition 2.1.6. *A partially ordered set has the countable chain condition if every anti chain on* X *is countable.*

Definition 2.1.7. *A subset* $Y \subset X$ *is dense in the partial order sense in* X *if for all* $x \in X$ *there exist some element* $y \in Y$ *with* $y \leq x$*.*

Definition 2.1.8. *Martin's Axiom: If X is a partially ordered set which satisfies the countable chain condition and $\mathcal{D}$ is a family of dense subsets of X with $|\mathcal{D}| < 2^\omega$ then there must exist a filter F on X with $F \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.*

Martin's axiom is not a natural set theoretic axiom and its origins arise from proofs of theorems in ZFC+CH where the continuum hypothesis was used via a combinatorial argument on the cardinality of some set. We can consider the family of Martin's axioms as follows

Definition 2.1.9. *MA(k): If X is a partially ordered set which satisfies the countable chain condition and $\mathcal{D}$ is a family of dense subsets of X with $|\mathcal{D}| < k$ then there must exist a filter F on X with $F \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.*

Definition 2.1.10. *A filter F is called D-generic if there exists a family $\mathcal{D} \subset 2^X$ consisting of dense subsets of the partially ordered set X such that $F \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.*

Theorem 2.1.11 (Rasiowa-Sikorski Lemma)**.** *If X is a partially ordered set with $x \in X$ and $\mathcal{D}$ is a countable family of dense subsets of X then there exists a D-generic filter F on X such that $x \in F$.*

Proof. Enumerate the set $\mathcal{D} = \{D_i\}_{i \in \mathbb{N}}$. Define a sequence on X as follows:

1. Choose $p_1 \in D_1$ such that $p_1 \leq x$, which exists as D_1 is dense in X
2. Assume we chosen $\{p_i\}_{i \leq n}$ such that $x \geq p_1 \geq p_2 \geq \ldots \geq p_n$ with $p_i \in D_i$ then since D_{n+1} is dense in X we can choose $p_{n+1} \in D_{n+1}$ such that $p_n \geq p_{n+1}$

Define a filter F on X where $F = \bigcup_{i \in \mathbb{N}} \{y \in X : q \geq p_i\}$. It is easy to see that F is a filter and by construction $p_i \in F \cap D_i$ for all $i \in \mathbb{N}$ □

This shows that MA(ω) is true in ZFC. The set $[0, 1] \subset \mathbb{R}$ with the normal topology is a counterexample in ZFC to MA(k) where $k \geq 2^\omega$.

Theorem 2.1.12. *The Continuum Hypothesis implies Martins Axiom.*

Proof. See [Kun14] □

Martins axiom's full strength only comes into play in models of ZFC where we have denied the Continuum Hypothesis as it can be shown that ZFC+¬CH+MA is still a consistent set theory. As a combinatorial set theoretic principal, Martin's axiom is used when we require that all cardinals less than 2^ω must behave as if they are almost countable. The pseudo-intersection number is a classical example of this behaviour.

Definition 2.1.13. *A pseudo-intersection of a family of infinite sets is an infinite set S such that each element of the family contains all but a finite number of elements S.*

Definition 2.1.14. *A family of subsets has the strong finite intersection property if the intersection of any finite subfamily is infinite.*

Definition 2.1.15. *The pseudo-intersection number $\mathfrak{p}$ is the minimum cardinality of a family of sets in 2^{2^ω} that has the strong finite intersection property but does not have a pseudo-intersection.*

Theorem 2.1.16. *ZFC implies that $\omega < \mathfrak{p} \leq c$.*

Proof. See [Hal11] □

So in ZFC we know that countable families of subsets of $\mathbb{N}$ cannot have pseudo-intersections while a continuum number of families definitely do have a pseudo-intersections.

Theorem 2.1.17. *Martins Axiom implies that $\mathfrak{p} = c$.*

Proof. See [Fre88] □

This demonstrates the previously mentioned principal, that Martin's axiom forces countable like behaviour on all cardinals between ω and $\mathfrak{c}$. Our set theoretic adventure will revolve around assumptions or denials of Martin's axiom and the Continuum hypothesis and the implications there of regarding the existence of certain types of filters on a set. We now proceed to the basic topological results required in this thesis.

Definition 2.1.18. *A topological space consists of a set X and a family of subsets $\mathcal{T} \subset 2^X$ such that*

- $\emptyset, X \in \mathcal{T}$.
- *if* $O_1, O_2 \in \mathcal{T}$ *then* $O_1 \cap O_2 \in \mathcal{T}$.
- $\{O_\alpha\}_{\alpha \in I} \subset \mathcal{T}$ *implies that* $\bigcup_{\alpha \in I} O_\alpha \in \mathcal{T}$.

We call the elements $O \in \mathcal{T}$ open sets and any subset $C = X \setminus A$ where $O \in \mathcal{T}$ a closed subset of X.

Definition 2.1.19. *A* base *B for a topological space is a family of open subsets $\mathcal{B} \subset 2^X$ such that $\forall U \in \mathcal{T}$ we can find a collection $\{B_i\}_{i \in I} \subset \mathcal{B}$ such that $U = \bigcup_{i \in I} B_i$.*

Definition 2.1.20. *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $f : X \to Y$ be a function*

1. *The function $f : X \to Y$ is* continuous *if $\forall O_Y \in \mathcal{T}_Y$, $f^{-1}(O_Y) \in \mathcal{T}_X$.*
2. *The function $f : X \to Y$ is an* open (closed) map *if for all $U \subset X$ with U open (closed) we have that $f(U)$ is open (closed) .*
3. *The function $f : X \to Y$ is a homeomorphism if it is one-to-one, onto, continuous and open .*

We will make frequent use of the closure operator in the results that follow, some basic results and definitions required are supplied below. From now on, if we say that X is some topological space, the topology is implied but $\mathcal{T}$ will be supressed.

Definition 2.1.21. *Let X be a topological space and let $U \subset X$. The closure of U, denoted by $\overline{U}$, is the smallest closed set containing U.*

Definition 2.1.22. *Let X be a topological space and let $U \subset X$. The subset U is called dense in X if $\overline{U} = X$.*

Definition 2.1.23. *Let X be a topological space and let $x \in X$ be some point in X. A subset $U \subset X$ is a* neighbourhood of the point x *if there exists an open subset V of X such that $x \in V \subset U \subset X$.*

Definition 2.1.24. *Let X be a topological space and let $A \subset X$ be some subset of X. A subset $U \subset X$ is a* neighbourhood of A *if there exists an open subset V of X such that $A \subset V \subset U \subset X$.*

Definition 2.1.25. *Let X be a topological space then X has the separation property:*

1. *T_0 if $\forall x, y \in X$ with $x \neq y$ there exists $U \in \mathcal{T}$ such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.*
2. *T_1 if $\forall x, y \in X$ with $x \neq y$ there exists $U \in \mathcal{T}$ such that $x \in U$ and $y \notin U$.*
3. *T_2 or Hausdorff if $\forall x, y \in X$ with $x \neq y$ we can find disjoint open subsets U and V such that $x \in U$ and $y \in V$.*

4. T_3 *or* regular *if it is* T_1 *and* $\forall x \in X$ *and closed subsets* $Y \subset X$ *with* $x \notin Y$ *there exists open disjoint subsets* U *and* V *such that* $x \in U$ *and* $Y \subset V$. *Equivalently, a topological space is called* T_3 *or* regular *if for all* $x \in X$ *and open neighbourhoods* $U \subset X$ *there exists an open set* $V \subset X$ *such that* $x \in V \subset \overline{V} \subset U$.

5. $T_{3\frac{1}{2}}$ *or* completely regular *if it is* T_1 *and* $\forall x \in X$ *and closed subsets* $Y \subset X$ *with* $x \notin Y$ *there exists a continuous function* $f : X \to \mathbb{R}$ *with* $f(x) = 0$ *and* $f(Y) = 1$.

6. T_4 *or* normal *if for all closed disjoint subsets*A *and* B *of* X *there exists open disjoint subsets* U *and* V *such that* $A \subset U$ *and* $B \subset V$.

Theorem 2.1.26. *If* C *is a closed subset of a regular space* X *and if for all neighbourhoods* U *of* C *we have* $\overline{A} \subset U$ *then* $\overline{A} \subset C$.

Proof. Since X is regular and C is closed we know that $C = \bigcap_{U \supset C} U$. Thus if for all U we have that $\overline{A} \subset U$ then $\overline{A} \subset C$ □

Theorem 2.1.27. *A compact,* $T2$ *space is in fact* $T4$*, and is thus* $T3$.

Proof. See [GL13] □

Definition 2.1.28. *Two subsets* A *and* B *of a topological space* X *are* completely separated *if we can find a continuous function* $f : X \to \mathbb{R}$ *with* $f(A) = 0$ *and* $f(B) = 1$.

Theorem 2.1.29 (Urysohn's Lemma)**.** *Suppose that* X *is a normal space and let* U *and* V *be disjoint closed subsets then* U *and* V *are completely separated.*

Proof. See [CN74] □

Definition 2.1.30.

- *A subset* A *of a topological space* X *is a* zero set *if there exists a continuous function* $f : X \to (-1, 1)$ *with* $A = f^{-1}(0)$.
- *The collection of all zero sets of a topological space* X *is denoted by* $\mathcal{Z}(X)$.
- *If* $A \subset X$ *such that* $X \setminus A \in \mathcal{Z}(X)$ *then* A *is a* cozero set.

Theorem 2.1.31. *Two subsets* A *and* B *of a completely regular space* X *are completely separated in* X *if and only if* A *and* B *are contained in disjoint zero sets.*

Proof. See [Wal74] □

Definition 2.1.32. *A topological space X is* compact *if every open cover contains some finite subcover.*

Definition 2.1.33. *A topological space X is* locally compact *if at every point $x \in X$ we can find an open neighbourhood $U \ni x$ such that the closure of U is compact.*

Definition 2.1.34. *A topological space X is pseudocompact if every every continuous function $f : X \to \mathbb{R}$ is bounded.*

Definition 2.1.35. *A* compactification of a topological space X *is a pair* $\left(\left(\hat{X}, \hat{\mathcal{T}}\right), h\right)$ *where* $\left(\hat{X}, \hat{\mathcal{T}}\right)$ *is a compact topological space and $h : X \to \hat{X}$ is a homeomorphism onto a dense subset of $\hat{X}$.*

Definition 2.1.36.

1. *A topological space X is* connected *if $X \neq U \cup V$ where both U and V are nonempty open subsets with $U \cap V = \emptyset$.*
2. *A* connected component *of a topological space is an equivalence class under the equivalence relation $x \sim y$ if and only if there exists a connected open subset U such that $\{x, y\} \subset U$.*
3. *A topological space is* totally disconnected *if all its components are singletons .*

Definition 2.1.37. *A topological space X is* zero dimensional *if there exists a basis for that topology consisting of clopen (i.e. both closed and open) subsets of X.*

Theorem 2.1.38. *A countable T_3 space is zero dimensional.*

Proof. See [Zel11] □

Theorem 2.1.39. *A totally disconnected, Hausdorff, locally compact space is zero dimensional.*

Proof. See [HR79] □

Theorem 2.1.40. *If X is a discrete topological space, then X is T_4, locally compact, totally disconnected and zero dimensional.*

Proof. Since every subset of a discrete space is both open and closed it is easy to see that it is locally compact, normal and thus Hausdorff and totally disconnected and thus zero-dimensional. □

2.2 Filters, Ultrafilters and P-points

Filters on a set and their corresponding ultrafilters form the basis for the majority of this line of study. For a more in-depth look into the theory of filters and ultrafilters please see [CN74]

Definition 2.2.1. *A* filter $\mathcal{F}$ *in a set* X *is a family of non-empty subsets of* X *such that*

1. *if* A *and* $B \in \mathcal{F}$ *then* $A \cap B \in \mathcal{F}$.
2. *if* $A \in \mathcal{F}$ *and* $A \subset B \subset X$ *then* $B \in \mathcal{F}$.

Definition 2.2.2. *A family of subsets* $\mathcal{C}$ *of a set* X *has the* finite intersection property *if for any finite collection of sets* $\{A_i\}_{i=1\ldots n}$ *we have that* $\cap_{i=1\ldots n} A_i \neq \emptyset$.

Theorem 2.2.3. *Every family of subsets* $\mathcal{C}$ *of a set* X *that has the finite intersection property is contained in some filter on* X.

Proof. We build the required filter in two steps. Firstly we close the family $\mathcal{C}$ under all finite intersections and then close the resulting family under supersets. It is easy to see that the generated family of sets is in fact a filter $\square$

Definition 2.2.4. *Let* $A \subset X$ *and denote by* $\langle A \rangle$ *the smallest filter on* X *such that* $A \subset \langle A \rangle$.

Definition 2.2.5. *An* ultrafilter $\mathcal{F}$ *in a set* X *is a filter on* X *which is maximal with respect to filter containment.*

The following theorem is a standard application of Zorn's lemma.

Theorem 2.2.6. *If* $\mathcal{C} \in 2^{2^X}$ *is a family of subsets with the finite intersection property then there exists an ultrafilter* $\mathcal{F}$ *such that* $\mathcal{C} \subset \mathcal{F}$.

Proof. Since $\mathcal{C}$ is contained in some filter we know that $F = \{\mathcal{F} : \mathcal{C} \subset \mathcal{F}\}$ is non-empty and it is easy to see that the union of any increasing chain of filters ordered by set inclusion is also a filter. By applying Zorn's lemma we immediately arrive at a maximal element in F which is the required ultrafilter $\square$

An ultrafilter is a filter that cannot be properly contained in any other filter on X. Ultrafilters can be characterised by the simple criteria that if $A \subset X$ then either $A \in \mathcal{F}$ or $X \backslash A \in \mathcal{F}$. This criteria suggests that an ultrafilter supplies a measure of largeness to a family of subsets of X. For example either a set is large or its complement is large, the union of two large sets is large etc. In fact an ultrafilter can be seen as a $\{0,1\}$ valued measure on 2^X, by the characterisation, so that if a property holds on the members of an ultrafilter then it holds *almost everywhere*.

Theorem 2.2.7. *If $\mathcal{F}$ is a filter on a set X and Φ is the set of all ultrafilters on X which contain $\mathcal{F}$ then $\mathcal{F} = \bigcap_{\mathcal{P}\in\Phi} \mathcal{P}$.*

Proof. We can easily see that $\mathcal{F} \subset \bigcap_{\mathcal{P}\in\Phi} \mathcal{P}$ since $\mathcal{F} \subset \mathcal{P}$ for all $\mathcal{P} \in \Phi$. So let $A \in \bigcap_{\mathcal{P}\in\Phi} \mathcal{P}$ but assume $A \notin \mathcal{F}$. The family of subsets of X, $\mathcal{F} \cup \{X \setminus A\}$ has the finite intersection property and is thus contained in some ultrafilter $\mathcal{Q}$ which also contains $\mathcal{F}$ by construction. However, then $A \in \mathcal{Q}$ and $X \setminus A \in \mathcal{Q}$ which is a contradiction. Thus $\mathcal{F} = \bigcap_{\mathcal{P}\in\Phi} \mathcal{P}$ □

Definition 2.2.8. *An ultrafilter $\mathcal{F}$ is called* principal or fixed *if we can find some $x \in X$ such that $\mathcal{F} = \{A \subset X : x \in A\}$. An ultrafilter is free if it is not fixed.*

As a consequence of Theorem 2.2.6 we have the following result:

Theorem 2.2.9. *If X is an infinite set then there exists a free ultrafilter on X.*

Proof. Consider the Fréchet filter F, consisting of all subsets of X with finite complements. It is easy to see that this is indeed a filter and thus by Theorem 2.2.6 we know there must exist some ultrafilter $\mathcal{F}$ containing the Fréchet filter. Now if $\mathcal{F}$ is principal then $\{x\} \in \mathcal{F}$ and $X \setminus \{x\} \in F \subset \mathcal{F}$ which is a contradiction. □

The following theorem characterises the subsets of a set X which are elements of an ultrafilter p. This theorem usually serves as the only leverage available when working with free ultrafilters. This is due to the fact that in general free ultrafilters are very hard to describe explicitly. For example any free ultrafilter on $\mathbb{N}$ would give an explicit construction of a non-measurable subset of $[0,1]$.

Theorem 2.2.10. *Let X be an infinite set and $U \subset X$. Let p be an ultrafilter on X. Then $U \in p$ if and only if $U \cap V \neq \emptyset$ for all $V \in p$.*

Proof. The necessity is easy, since filters are closed under intersection and the empty set is never an element of a filter. For the sufficiency let $U \subset X$ such that $X \setminus U \in p$ but $U \cap V \neq \emptyset$ for all $V \in p$. But then $U \cap (X \setminus U) \neq \emptyset$ which is a contradiction. □

The number of ultrafilters on a set X is trivially bounded above by $2^{2^{|X|}}$ since every ultrafilter is a collection of subsets of the set X. We can show that this bound is actually tight in the case of infinite sets X.

Theorem 2.2.11. *If X is a set then the number of ultrafilters on X is $|X|$ if X is finite, and $2^{2^{|X|}}$ if X is infinite.*

Proof. If X is finite then each ultrafilter on X is principal, and so there are exactly $|X|$ ultrafilters. In the rest of the proof we will assume that X is infinite.
Let F be the set of all finite subsets of X, and let Φ be the set of all finite subsets of F.
For each $A \subseteq X$ define

$$B_A = \{(f, \phi) \in F \times \Phi \mid A \cap f \in \phi\}$$

and $B_A^C = (F \times \Phi) \setminus B_A$. For each $\mathcal{S} \subseteq 2^X$ define

$$\mathcal{B}_\mathcal{S} = \{B_A \mid A \in \mathcal{S}\} \cup \{B_A^C \mid A \notin \mathcal{S}\}$$

Let $\mathcal{S} \subseteq 2^X$, and suppose $A_1, \ldots, A_m \in \mathcal{S}$ and $D_1, \ldots, D_n \in 2^X \setminus \mathcal{S}$, so that we have $B_{A_1}, \ldots, B_{A_m}, B_{D_1}^C, \ldots, B_{D_n}^C \in \mathcal{B}_\mathcal{S}$. For $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ let $a_{i,j}$ be such that either $a_{i,j} \in A_i \setminus D_j$ or $a_{i,j} \in D_j \setminus A_i$. This is always possible, since $A_i \neq D_j$. Let

$$f = \{a_{i,j} \mid 1 \leq i \leq m,\, 1 \leq j \leq n\}$$

and put

$$\phi = \{A_i \cap f \mid 1 \leq i \leq m\}$$

Then $(f, \phi) \in B_{A_i}$, for $i = 1, \ldots, m$. If for some $j \in \{1, \ldots, n\}$ we have $D_j \cap f \in \phi$ then $D_j \cap f = A_i \cap f$ for some $i \in \{1, \ldots, m\}$. But thus is impossible, as $a_{i,j}$ is in one of these sets but not the other. So $D_j \cap f \notin \phi$, and therefore $(f, \phi) \in B_{D_j}^C$. So

$$(f, \phi) \in B_{A_1} \cap \cdots \cap B_{A_m} \cap B_{D_1}^C \cap \cdots \cap B_{D_n}^C$$

This shows that any finite subset of $\mathcal{B}_\mathcal{S}$ has non-empty intersection, and therefore $\mathcal{B}_\mathcal{S}$ can be extended to an ultrafilter $\mathcal{U}_\mathcal{S}$.
Suppose $\mathcal{R}, \mathcal{S} \subseteq 2^X$ are distinct. Then, relabelling if necessary, $\mathcal{R} \setminus \mathcal{S}$ is nonempty. Let $A \in \mathcal{R} \setminus \mathcal{S}$. Then $B_A \in \mathcal{U}_\mathcal{R}$, but $B_A \notin \mathcal{U}_\mathcal{S}$ since $B_A^C \in \mathcal{U}_\mathcal{S}$. So $\mathcal{U}_\mathcal{R}$ and $\mathcal{U}_\mathcal{S}$ are distinct for distinct $\mathcal{R}$ and $\mathcal{S}$. Therefore $\{\mathcal{U}_\mathcal{S} \mid \mathcal{S} \subseteq 2^X\}$ is a set of $2^{2^{|X|}}$ ultrafilters on $F \times \Phi$. But $|F \times \Phi| = |X|$, so $F \times \Phi$ has the same number of ultrafilters as X. So there are at least $2^{2^{|X|}}$ ultrafilters on X, and there cannot be more than $2^{2^{|X|}}$ as each ultrafilter is an element of 2^{2^X}. □

Definition 2.2.12. *The* dispersion character *of a filter $\mathcal{F}$ is $\Delta(\mathcal{F}) = \min\{|A| :$

$A \in \mathcal{F}\}$.

The following theorem is also an easy consequence of Theorem 2.2.6

Theorem 2.2.13. *Let X be set with $|X| = \kappa$ and let $\omega \leq \beta \leq \kappa$. Then there exists an ultrafilter on X such that $\Delta(\mathcal{F}) = \beta$.*

Proof. Let $\mathcal{C} = \{U \subset X : |X \setminus U| < \beta\}$ which obviously has the finite intersection property so by Theorem 2.2.6 there exists an ultrafilter $\mathcal{F}$ such that $\mathcal{C} \subset \mathcal{F}$. Suppose $\Delta(\mathcal{F}) < \beta$ then there exists a $U \in \mathcal{F}$ such that $|U| < \beta$. However then $X \setminus U \in \mathcal{C}$ as $|X \setminus (X \setminus U)| = |U| < \beta$ and thus $U \cap X \setminus U = \emptyset \in \mathcal{F}$ which is a contradiction. □

Definition 2.2.14. *Let κ be a cardinal. An ultrafilter $\mathcal{F}$ is called* κ-uniform *if $\Delta(\mathcal{F}) \geq \kappa$.*

We are now ready to define the concept of a Ramsey Ultrafilter and a P-point. P-points were first investigated when questions about the homogeneity of the Stone-Čech compactification of the naturals were under scrutiny.

Definition 2.2.15. *An ultrafilter p is Ramsey if for all $k, r \in \mathbb{N}$ and $[\omega]^k = \{Y \subset \omega : |Y| = k\}$ r-coloured there exists $A \in p$ such that $[A]^k$ is monochrome.*

Ramsey ultrafilters are related to Infinite Ramsey theory which is a family of combinatorial facts about random colourings of infinite sets.

Theorem 2.2.16. *Martins Axiom implies the existence of Ramsey ultrafilters.*

Proof. See [Jec13] □

Definition 2.2.17. *A point $x \in X$ is a P-point if the intersection of countably many neighbourhoods of p is still a neighbourhood of p.*

Theorem 2.2.18. *Every non-principal Ramsey ultrafilter on ω is a P-point.*

Proof. See [Zel11] □

Theorem 2.2.19. *It is consistent with ZFC that there are no P-points in ω^*.*

Proof. See [She98] □

Theorem 2.2.20. *It is consistent with ZFC that there are not every P-point in ω^* is a Ramsey ultrafilter.*

Proof. See [Hal11] □

2.3 Topological Groups and Semigroups

Definition 2.3.1. *A* topological group *is a pair $(G, \mathcal{T})$ for which*

1. *$(G, \mathcal{T})$ is a topological space .*
2. *G is a group .*
3. *the function $g : G \times G \to G$ defined by $g(x, y) = x \cdot y^{-1}$ is continuous.*

For example, any group combined with the discrete topology on that group is a topological group. Less trivially, the additive group of rationals $\mathbb{Q}$ with the subset topology from $\mathbb{R}$ is a topological group. Not every group admits a non-discrete group topology though.

Theorem 2.3.2. *Assume CH. There exists a group G with $|G| = \omega^+$ such that the only group topology on G is discrete. Furthermore, any finite subset of G is contained in a topologizable subgroup of G.*

Proof. See [She80] □

Topological groups tend to be well behaved when it comes to their topological properties due to the restrictions placed on their topologies by the continuity of the group operations.

Theorem 2.3.3. *Any T_0 topological group is in fact a completely regular topological group.*

Proof. See [HR79] Theorem 8.2 and Theorem 8.4 □

A topological group is a homogeneous space and thus the family of open neighbourhoods of the identity element characterize the topology on the group G

Definition 2.3.4. *A base for a neighbourhood system is called* countably decreasing *if it is of the form $\{V_i : i \in \mathbb{N}\}$ such that $V_1 \supseteq V_2 \supseteq \ldots$*

Definition 2.3.5. *A base for a neighbourhood system is called* symmetric *if for any element U of that base it is true that $U = U^{-1}$.*

Theorem 2.3.6. *If G is a group with identity $e \in G$ then $\phi \subset 2^G \setminus \{\emptyset\}$ is the basis of the neighbourhood filter at the identity of a T_2 group topology $\mathcal{T}$ if and only if*

1. *$\cap\phi = \{e\}$.*

2. $\forall U, V \in \phi$ *there exists* $W \in \phi$ *such that* $W \subset U \cap V$.

3. $\forall U \in \phi$ *there exists* $V \in \phi$ *such that* $V^2 \subset U$.

4. $\forall U \in \phi$ *there exists* $V \in \phi$ *such that* $V^{-1} \subset U$.

5. $\forall U \in \phi$ *and* $g \in G$ *there exists* $V \in \phi$ *such that* $g^{-1}Vg \subset U$.

Furthermore, we may always assume that the base generated by ϕ *is symmetric and that if this topology is first countable then we can choose this base so that it is countably decreasing.*

Proof. Suppose $\phi \subset 2^G$ satisfies the conditions of the theorem, then by condition (2) ϕ is a basis for a filter on G (as for all $U, V \in \phi$ there exists a $W \in \phi$ such that $W \subset U \cap V \neq \emptyset$). Consider the family of subsets $\mathcal{N}_x = \{U \subset G : xV \subset U$, for some $V \in \phi\}$ where $x \in G$. Let $U \subset N_x$. There exists a $V \in \phi$ such that $xV \subset U$. Let $W \in \phi$ such that $W^2 \subset V$ and consider $xW \in \mathcal{N}_x$. Let $y \in xW$ i.e. $y = xw$ for some $w \in W$ then $yW = xwW \subset xW^2 \subset xV$ and thus $xV \in \mathcal{N}_y$ for all $y \in xV$ and thus $\mathcal{N}_x$ is the neighbourhood filter at $x \in G$ for some topology on G.

Now let $x, y \in G$ and $U \in \phi$ so that $xyU \in \mathcal{N}_{xy}$. Now there exist $V, W \in \phi$ such that $V^2 \subset U$ and $y^{-1}Wy \subset V$ but then $xW \in \mathcal{N}_x$ and $yV \in \mathcal{N}_y$ and thus $xW \cdot yV = xy\left(y^{-1}Wy\right)V \subset xyV^2 \subset xyU$. Consequently, the function $\cdot : G \times G \to G$ is continuous. It should be noted that for all $g \in G$ the function $\lambda_g : G \to G$ given by $\lambda_g(x) = gx$ is a homeomorphism. Now $x^{-1}U \in \mathcal{N}_{x^{-1}}$ and there exists $V, W \in \phi$ such that $xWx^{-1} \subset U$ and $V^{-1} \subset W$ but then $x\,(xV)^{-1} = xV^{-1}x^{-1} \subset xWx^{-1} \subset U$ and thus $(xV)^{-1} \subset x^{-1}U$. This implies that $x \to x^{-1}$ is continuous as well and thus ϕ is a base of the neighbourhood filter at the identity of a group topology on G.

Now this topology is T_1 as if $x \neq y \in G$ then there exists a $U \in \phi$ such that $e \in U$ but $x^{-1}y \notin U$ (as if this was not true then $\cap\phi \neq \{e\}$ but then $x \in xU \in \mathcal{N}_x$ and $y \notin xU$. However, if $U \in \phi \subset \mathcal{N}_e$ then there exists $V, W \in \phi$ such that $W^2 \subset U$, $V^{-1} \subset W$. But if $x \in \overline{W}$ then $xV \cap W \neq \emptyset$ which implies that $x = w_1 v_1^{-1} \subset WV^{-1} \subset W^2 \subset U$ i.e. $\overline{W} \subset U$ but this implies that the topology is in fact T_3 and thus Hausdorff.

Now if ϕ is a base of the neighbourhood filter at the identity of a Hausdorff group topology on G then both $\cdot : G \times G \to G$ and $x \to x^{-1}$ are continuous at the identity and thus $e^2 = e$ implies (3), $e^{-1} = e$ implies (4) and $geg^{-1} = e$ implies (5). (1) is implied by the fact that the topology is Hausdorff and (2) is implied by the fact that ϕ is a base for a filter.

Now by considering the subsets $\tilde{U} = U \cap U^{-1}$ if necessary, we can assume that this neighbourhood basis is in fact symmetric. Now suppose the resulting topology generated by ϕ is in fact first countable. We can then find a countable base $\tilde{\phi} = \{U_i : i \in \mathbb{N}\}$ of the neighbourhood system of identity which satisfies conditions (1) to (5). Define V_i as follows : $V_1 = U_1$ and assuming V_i is defined for $i \leq k$, define V_{k+1} to be the first U_i such that $U_i^3 \subset V_k \cap U_1 \cap ... \cap U_k$ which is well-defined since $e^3 = e$. This implies that $V_{k+1} = e^2 V_{k+1} \subset V_{k+1}^3 \subset V_k$ and furthermore that the V_i's form a base for the filter $\tilde{\phi}$. $\square$

The general theory of topological groups is vast and deep and covers multiple areas of amthematics. For a more general introduction I suggest [Pon66] or [HR79].

Definition 2.3.7. *A* topological semigroup *is a pair* $(S, \mathcal{T})$ *for which*

1. $(S, \mathcal{T})$ *is a topological space.*

2. S *is a semigroup.*

3. *the function* $m : S \times S \to S$ *defined by* $m(a, b) \to a \cdot b$ *is continuous.*

We can weaken the definitions further by not demanding joint continuity in the variables of the semigroup operator.

Definition 2.3.8. *Let* S *be a* semigroup *endowed with a topology* $\mathcal{T}$*.*

1. S *is a left topological semigroup or left shift continuous semigroup if the left translations* $\lambda_\alpha : S \to S$ *defined by* $\lambda_\alpha(x) = \alpha \cdot x$ *are continuous for all* $\alpha \in S$*.*

2. S *is a right topological semigroup or right shift continuous semigroup if the right translations* $\rho_\alpha : S \to S$ *defined by* $\rho_\alpha(x) = x \cdot a$ *are continuous for all* $\alpha \in S$*.*

3. S *is a semitopological semigroup if it is both a left topological semigroup and right topological semigroup.*

There appears to be some confusion in the literature about what left and right topological semigroup implies, with some papers referring to a left topological group as what another group would have defined as a left topological group. We prefer to be explicit in our choice and will generally refer to groups as left shift or right shift continuous as is appropriate.

2.4 T-Filters on arbitrary groups

Theorem 2.4.1. *Let X be a set and $a \in X$. Let (x_n) be a non-trivial sequence such that $x_n \neq a$ for all $n \in \mathbb{N}$. Then there exists a Hausdorff topology on X such that $x_n \to a$.*

Proof. Define $A_m = \{x_n : n > m\}$ Define a topology on X as follows: For all $x \in X \setminus \{a\}$ define $\mathcal{N}_x = \{U \subset X : x \in U\}$ and define $\mathcal{N}_a = \{U \subset X : A_m \subset U$, for some $m \in \mathbb{N}\}$. Now let $U \in \mathcal{N}_x$ for some $x \in X \setminus \{a\}$ then $U \cap (X \setminus \{a\}) \in \mathcal{N}_x$ and $U \cap (X \setminus \{a\}) \subset U$ and for all $y \in U \cap (X \setminus \{a\})$ we have that $U \cap (X \setminus \{a\}) \in \mathcal{N}_y$. Now if $U \in \mathcal{N}_a$ then $A_m \subset U$ for some $m \in \mathbb{N}$ but then $A_m \in \mathcal{N}_y$ for all $y \in A_m$ and thus we have defined a topology on X such that $x_n \to a$. Obviously, this topology is Hausdorff. $\square$

The situation is not as trivial for the case when X is in fact a group and we wish for the resulting topology to be a group topology. In fact there exists a group G and a non-trivial sequence (x_n) which does not converge to the identity for any non-discrete Hausdorff topology. Consider the p-quasicyclic group

$$\mathbb{Z}(p^\infty) = \{e^{\frac{2ni\pi}{p^m}} : n \in Z, m \in \mathbb{Z}^+\}$$

which is a subgroup of $\mathbb{T}$ and consider the sequence $x_n = e^{2\pi i\left(\frac{1}{p^n} - \frac{1}{p}\right)}$. Suppose that $x_n \to 1$ i.e. $e^{\frac{2\pi i}{p^n}} \to e^{2\pi i}$. Then $x_{n+1}^p = e^{2\pi i\left(\frac{1}{p^n} - 1\right)} = e^{-2\pi i} e^{\frac{2\pi i}{p^n}} \to 1$ and thus $\frac{x_{n+1}^p}{x_n} \to 1$. But $\frac{x_{n+1}^p}{x_n} = e^{2\pi i\left(\frac{1}{p} - 1\right)} = x_1$ and so x_1 is in every neighbourhood of the identity which contradicts the fact that the topology is Hausdorff. The discrete topology is always available but it is only useful or interesting in certain cases. We need to start from a more general position in an attempt to answer this question. That starting point is that topological groups are homogeneous spaces and this suggests another method of attack. Instead of trying to define an entire topology in one go we could instead focus on one point and build something there, allowing the homogeneity of the space to fill in the gaps. For starters that something has to be more general then a sequence.

Definition 2.4.2. *A filter $\phi \subset 2^G$ is said to converge to a point $x \in G$ i.e. $\phi \to x$ if and only if the neighbourhood filter at that point $\mathcal{N}_x \subset \phi$.*

The notion of a filter converging to a point is an extension of the standard notion of a sequence converging to a point but is better behaved topologically. Many results for convergent sequences extend to convergent filters, for example filter limits in a Hausdorff space are unique and a space is compact if and only if every ultrafilter converges on that space. Now it is very easy to define a Hausdorff topology on that set such that a given sequence converges to a given point in that topology.

Definition 2.4.3. *A filter $\phi \subset 2^G$ is called a* T-filter *if there exists a Hausdorff group topology on G for which $\phi \to e$. The strongest group topology for which $\phi \to e$ is denoted by $\mathcal{T}(G, \phi)$.*

Let $\mathcal{N}_e$ denote the neighbourhood filter of identity of a group topology on a topological group G then it is the case that $\mathcal{T}(G, \mathcal{N}_e)$ is the original topology on G.

Definition 2.4.4. *Let $\mathcal{F}$ be a filter on an arbitrary group G. $\mathcal{F}$ is called* multiplicative *if and only if for all $U \in \mathcal{F}$ there exists a $V \in \mathcal{F}$ such that $V \subset U$ and $V^2 \subset U$.*

For example, let $A_i = \{n^2 : n \geq i\}$ and $\mathcal{F} = \langle \{A_i : i \in \mathbb{N}\} \rangle$ be the filter generated by the sequence of square natural numbers. Now if $U \in \mathcal{F}$ then $V = \{k^2, (k+1)^2, ...\} \subset U$ for some $k \in \mathbb{N}$. However the set $V^2 = \{(mn)^2 : m, n \geq k\} \subset U$ and thus $\mathcal{F}$ is a trivial example of a multiplicative filter.

Definition 2.4.5. *Let G be a group and let $(F_n)_{n \in \mathbb{N}}$ be a sequence of non-empty subsets of G. Let S_n denote the n-element permutation group. Define*

$$SP_{i \leq n} F_i = \bigcup_{\sigma \in S_{n+1}} F_{\sigma(0)} F_{\sigma(1)} ... F_{\sigma(n)}$$

and

$$SP_{i \in \mathbb{N}} F_i = \bigcup_{n \in \mathbb{N}} (SP_{i \leq n} F_i)$$

The introduction of the permutation group is necessary as we are not guaranteed that our group is Abelian. We wish to construct the maximal multiplicative filter contained in a given filter $\mathcal{F} \subset 2^G$.

Lemma 2.4.6. *If $\mathcal{F}$ is a filter on a group G, then there exists a maximal multiplicative filter ϕ such that $\phi \subset \mathcal{F}$. The family of subsets*

$$SP(\mathcal{F}) = \{SP_{n \in \mathbb{N}} F_n : F_n \in \mathcal{F}, n \in \mathbb{N}\}$$

forms a basis for this multiplicative filter.

Proof. We need to show that $SP(\mathcal{F})$ generates a multiplicative filter contained in $\mathcal{F}$. So let F_n and G_n be two sequences of elements from $\mathcal{F}$. Then as $\mathcal{F}$ is a filter we know that $F_n \cap G_n \in \mathcal{F}$ for all $n \in \mathbb{N}$. Also it is easy to see that

$$SP_{n\in\mathbb{N}}(F_n \cap G_n) \subset SP_{n\in\mathbb{N}}F_n \cap SP_{n\in\mathbb{N}}G_n$$

and thus $SP(\mathcal{F})$ is a basis for some filter on G, say ϕ. So let $U \in \phi$ and let $(F_n) \subset \mathcal{F}$ be a sequence of subsets such that $SP_{n\in\mathbb{N}}F_n \subset U$ and $F_{n+1} \subset F_n$ for all $n \in \mathbb{N}$. We can construct the required sequence through induction. Let G_n be a sequence in $\mathcal{F}$ such that $SP_{n\in\mathbb{N}}G_n \subset U$. Define $F_0 = G_0$ and $F_{n+1} = G_n \cap F_n$ then F_n is a sequence with the required properties. Now define $V_0 = SP_{n\in\mathbb{N}}F_{2n}$ and $V_1 = SP_{n\in\mathbb{N}}F_{2n+1}$. Now $V_i \in \phi$ and since $SP_{n\in\mathbb{N}}F_n$ is defined using all the permutations of the permutation group S_{n+1} we have that $V_0V_1 \subset U$. Let $V = V_0 \cap V_1$ then

$$V^2 \subset V_0V_1 \subset U$$

i.e. ϕ is a multiplicative filter. Finally, since

$$\mathcal{F} \ni F_0 \in SP_{i\leq 0}F_i \subset SP_{n\in\mathbb{N}}F_i \subset U$$

we have that $U \in \mathcal{F}$ for all $U \in \phi$. □

We can now prove the existence of a maximal multiplicative filter contained in any filter $\mathcal{F} \subset 2^G$

Lemma 2.4.7. *Let $\mathcal{F}$ be a filter on a group G, then there exists a maximal multiplicative filter ϕ such that $\phi \subset \mathcal{F}$.*

Proof. Let $\mathcal{Z} = \{\mathcal{G} \subset \mathcal{F} : \mathcal{G} \text{ is a multiplicative filter }\}$. By Lemma 2.4.6, $\mathcal{Z} \neq \emptyset$. Let $(\mathcal{G}_\alpha)_{\alpha\in\Lambda}$ be a chain in $\mathcal{Z}$. We claim that $\cup\mathcal{G}_\alpha$ is a multiplicative filter. Now it is easy to see that $\cup\mathcal{G}_\alpha$ must be a filter on G. For suppose $U, V \in \cup\mathcal{G}_\alpha$ then there exists $\alpha_1, \alpha_2 \in \Lambda$ such that $U \in \mathcal{G}_{\alpha_1}$ and $V \in \mathcal{G}_{\alpha_2}$. Let $\alpha_3 = \max(\alpha_1, \alpha_2)$ then $U, V \in \mathcal{G}_{\alpha_3}$ and so

$$U \cap V \in \mathcal{G}_{\alpha_3} \subset \cup\mathcal{G}_\alpha$$

Also if $W \subset G$ and $U \subset W$ with $U \in \cup\mathcal{G}_\alpha$ then there must exist an $\alpha_1 \in \Lambda$ such that $U \in \mathcal{G}_{\alpha_1}$ which implies that

$$W \in \mathcal{G}_{\alpha_1} \subset \cup\mathcal{G}_\alpha$$

Thus we have shown that $\cup\mathcal{G}_\alpha$ is a filter. Also if $U \in \cup\mathcal{G}_\alpha$ then $U \in \mathcal{G}_{\alpha 1}$ for some $\alpha_1 \in \Lambda$. But as $\mathcal{G}_{\alpha_1}$ is a multiplicative filter, there exists a $V \in \mathcal{G}_{\alpha_1}$ such that

$V^2 \subset U$. But

$$V \in \mathcal{G}_{\alpha_1} \subset \cup \mathcal{G}_\alpha$$

which implies that $\cup \mathcal{G}_\alpha$ is a multiplicative filter on G which is trivially contained in $\mathcal{F}$. Thus the chain $(\mathcal{G}_\alpha)_{\alpha\in\Lambda}$ has an upper bound and thus by Zorn's lemma, $\mathcal{Z}$ contains a maximal element. Thus there exists a maximal multiplicative filter $\phi \subset \mathcal{F}$ □

We actually have constructed something stronger here, the filter constructed in Lemma 2.4.6 is in fact the maximal multiplicative filter contained in a given filter $\mathcal{F}$. This is the final step of our construction.

Theorem 2.4.8. *If $\mathcal{F}$ is a filter on a group G, then the family of subsets*

$$SP(\mathcal{F}) = \{SP_{n\in\mathbb{N}}F_n : F_n \in \mathcal{F}, n \in \mathbb{N}\}$$

forms a basis for the maximal multiplicative filter contained in $\mathcal{F}$.

Proof. Let $\mathcal{G}$ denote the filter generated by $SP(\mathcal{F})$. By Lemma 2.4.6 we know that this filter is in fact a multiplicative filter contained in $\mathcal{F}$. Also, by Lemma 2.4.7, the maximal multiplicative filter ϕ contained in $\mathcal{F}$ exists. Let $U \in \phi$. We now construct a sequence of elements F_n of ϕ such that $SP_{n\in\mathbb{N}}F_n \subset U$. This is sufficient to prove the theorem as then $U \in \mathcal{G}$ for all $U \in \phi$ and thus $\mathcal{G} = \phi$ since ϕ is the maximal multiplicative ultrafilter contained in $\mathcal{F}$. We proceed by induction. Let $U_{-1} = U = F_{-1}$ and let $U_0 \in \phi$ such that $U_0^2 \subset U_{-1}$. Let $V_0 \in \phi$ such that $V_0^2 \subset U_0$. Define $F_0 = V_0 \cap U_0$ then

$$F_0^2 \subset U_0^2 \subset U_{-1} = F_{-1}$$

and

$$F_0^3 \subset V_0^2 \cdot U_0 \subset U_0^2 \subset U_{-1} = F_{-1}$$

Proceed by induction to define a sequence F_n such that $F_n^2 \subset F_{n-1}$ and $F_n^3 \subset F_{n-1}$. Now let $(i_m)_{m=0...n}$ be a sequence in $\mathbb{N}$ and let $i_j = \min(i_0, i_1, ..., i_n)$. We now show that

$$F_{i_0} \cdot F_{i_1} \cdot ... \cdot F_{i_n} \subset F_{i^*-1}$$

We proceed by induction. If $n = 0$ then by the construction of F_{i_0} we have that $F_{i_0} \subset F_{i_0-1}$. So suppose that for all $k = 0...n-1$ we have that

$$F_{i_0} \cdot F_{i_1} \cdot ... \cdot F_{i_k} \subset F_{i^*-1}$$

where $i_j = \min(i_0, i_1, ..., i_k)$. Now consider $i_k = \min(i_0, i_1, ..., i_n)$. If $k \neq 0$ and $k \neq n$ then we can compute $i_p = \min(i_0, i_1, ..., i_{k-1})$ and $i_q = \min(i_{k+1}, i_{k+2}, ..., i_n)$ and we know that $i_k < \min(i_p, i_q)$. So we have that

$$\left(F_{i_0} \cdot F_{i_1} \cdot ... \cdot F_{i_{k-1}}\right) \cdot F_{i_k} \cdot \left(F_{i_{k+1}} \cdot F_{i_{k+2}} \cdot ... \cdot F_{i_n}\right) \subset F_{i_p} \cdot F_{i_k} \cdot F_{i_q} \subset F_{i_k}^3 \subset F_{i_k - 1}$$

If $k = 0$ then we have that $i_0 = i_k < i_p = \min(i_1, i_2, ..., i_n)$ and thus

$$F_{i_0} \cdot (F_{i_1} \cdot ... \cdot F_{i_n}) \subset F_{i_k} \cdot F_{i_p} \subset F_{i_k}^2 \subset F_{i_k - 1}$$

similarly for $k = n$ and $i_n = i_k < i_p = \min(i_0, i_1, ..., i_{n-1})$ we have that

$$\left(F_{i_0} \cdot F_{i_1} \cdot ...F_{i_{n-1}}\right) \cdot F_{i_n} \subset F_{i_p} \cdot F_{i_k} \subset F_{i_k}^2 \subset F_{i_k - 1}$$

We have shown that for any $\sigma \in S_{n+1}$ we have that

$$F_{\sigma(0)} \cdot F_{\sigma(1)} \cdot ... \cdot F_{\sigma(n)} \subset F_{-1}$$

and thus that $SP_{i \leq n} F_i \subset F_{-1}$ and finally that $SP_{n \in \mathbb{N}} F_n \subset F_{-1} = U$. □

Thus we have shown that every filter on a group admits a maximal multiplicative filter contained in that filter. This maximal multiplicative filters are important for the characterization of T-filters on countable groups and thus have a special name.

Definition 2.4.9. *Let $\mathcal{F}$ be a filter on a group G, we call the maximal multiplicative filter contained in $\mathcal{F}$ the multiplicative hull of $\mathcal{F}$ and denote it by ψ.*

Once again we wish to classify the T-filters on a given countable group. The lack of commutability in a general countable group complicates the proceedings considerably. The classification of a general T-filter requires three separate parts.

Definition 2.4.10. *Let ϕ be a filter on a group G. Define $\hat{\phi} = \{F^* | F \in \phi\}$ where $F^* = F \cup F^{-1} \cup \{e\}$.*

This first part of the construction is to extend the original filter to include the inverse information of the group in question.

Lemma 2.4.11. *If ϕ is a filter on a group G then $\hat{\phi}$ is a basis for a filter on G.*

Proof. Let $F, G \in \phi$ so that $F^*, G^* \in \hat{\phi}$. Also, since ϕ is a filter there exists a $H \in \phi$ such that $H \subset F \cap G$. We claim that $H^* \subset F^* \cap G^*$. But this is easy to see as

$$F^* \cap G^* = (F \cap G) \cup (F \cap G^{-1}) \cup (F^{-1} \cap G) \cup (F^{-1} \cap G^{-1}) \cup \{e\}$$

and $H \subset F \cap G$ implies that $H^{-1} \in F^{-1} \cap G^{-1}$ $\square$

To simplify notation, from now on ϕ^* will denote the filter generated by $\hat{\phi}$. We now introduce a trick (and the second part of the definition) which will be useful for another section of the discussion.

Definition 2.4.12. *Let G be a group and let ϕ be a filter on this group. Define*

$$\tilde{\phi} = \{\bigcup_{x \in G} x^{-1} \cdot f(x) \cdot x | f : G \to \phi\}$$

This second part encodes the multiplicative information of the original group as it pertains to the original filter, in particular it encodes the non commutative nature of the group multiplication.

Lemma 2.4.13. *If ϕ is a filter on a group G then $\tilde{\phi}$ is a basis for a filter on G.*

Proof. Let $f : G \to \phi$ and $g : G \to \phi$ so that

$$\bigcup_{x \in G} x^{-1} \cdot f(x) \cdot x \in \tilde{\phi}$$

and

$$\bigcup_{x \in G} x^{-1} \cdot g(x) \cdot x \in \tilde{\phi}$$

Define $h : G \to \phi$ by $h(x) = f(x) \cap g(x) \in \phi$ since ϕ is a filter. Then

$$\bigcup_{x \in G} x^{-1} \cdot h(x) \cdot x \subset (\bigcup_{x \in G} x^{-1} \cdot f(x) \cdot x) \cap (\bigcup_{x \in G} x^{-1} \cdot g(x) \cdot x)$$

$\square$

Once again, to simplify notation we will denote the filter generated by the basis $\tilde{\phi}$ as ϕ^G. We now have enough basic definitions to state the final part of the construction for the classification theorem.

Lemma 2.4.14. *Let ϕ be a filter on a group G. Denote by ψ the filter generated by the filter basis $SP\left((\phi^*)^G\right)$. Then the following statements are true*

1. *ψ is a multiplicative filter.*

2. *if $U \in \psi$ then $U^{-1} \in \psi$.*

3. *if $U \in \psi$ and $y \in G$ then $y^{-1} \cdot U \cdot y \in \psi$.*

Proof. 1. By Lemma 2.4.11 and Lemma 2.4.13 we know that $(\phi^*)^G$ is a filter on G and thus, by Lemma 2.4.8 we know that $SP\left((\phi^*)^G\right)$ generates a multiplicative filter

2. Let $U \in SP\left((\phi^*)^G\right)$, so there exists a sequence of elements of $(\phi^*)^G$, say $(F_i) \subset (\phi^*)^G$ such that

$$SP_{i\in\mathbb{N}}F_i = \bigcup_{n\in\mathbb{N}} (SP_{i\le n}F_i) \subset U$$

recalling that

$$SP_{i\le n}F_i = \bigcup_{\sigma\in S_{n+1}} F_{\sigma(0)}F_{\sigma(1)}...F_{\sigma(n)}$$

However, for all F_i there exists a sequence of functions $g_i : G \to \phi^*$ such that

$$G_i = \bigcup_{x\in G} x^{-1} \cdot g_i(x) \cdot x \subset F_i$$

However, for all $V \in \phi^*$ we know that $V^{-1} = V$ and thus $G_i \subset F_i^{-1}$ where $G_i \in (\phi^*)^G$. This implies that $SP_{i\le n}G_i \subset SP_{i\le n}F_i^{-1}$ and finally that $SP_{i\in\mathbb{N}}G_i \subset SP_{i\in\mathbb{N}}F_i^{-1} \subset U$ which was the claim we needed to show.

3. Similarly as before we can find a sequence G_i of elements of $(\phi^*)^G$ such that $SP_{i\in\mathbb{N}}G_i \subset U$. Define a sequence of functions $h_i : G \to \phi^*$ so that $h_i(x) = g_i(x \cdot y)$. Then

$$y^{-1}H_iy = \bigcup_{x\in G} y^{-1} \cdot x^{-1} \cdot h_i(x) \cdot x \cdot y = \bigcup_{x\in G} (xy)^{-1} \cdot g_i(xy) \cdot (xy) = G_i$$

and thus

$$y^{-1} \cdot (SP_{i\in\mathbb{N}}H_i) \cdot y \subset y^{-1} \cdot U \cdot y$$

□

We now have sufficiently many pieces to state and prove the promised classification theorem. Recall that a filter ϕ is generated by a set A if and only if $\cap\phi = A$

Theorem 2.4.15. *If ϕ is a filter on a group G, then ϕ is a T-filter if and only if the multiplicative hull of ϕ is generated by $\{e\}$. Also, if ϕ is a T-filter from some*

Hausdorff group topology $\mathcal{T}$, then the multiplicative hull of ϕ is a neighbourhood filter of the identity in that topology.

Proof. Firstly, let ϕ be a T-filter on a group G with Hausdorff topology $\mathcal{T}$ and let τ_e denote a neighbourhood filter of the identity in topology $\mathcal{T}$. Now by Theorem 2.3.6 we know that τ_e is a multiplicative filter and since $\phi \to e$ we know that $\tau_e \subset \phi$. But the topology $\mathcal{T}$ is Hausdorff and so $\cap\tau_e = \{e\}$. But since τ_e is multiplicative it must be contained in the multiplicative hull ψ of ϕ and thus $\cap\psi \subset \cap\tau_e = \{e\}$. Furthermore, $e \in U$ for all $U \in \psi$ and thus ψ is generated by the identity $e \in G$. Also, by Lemma 2.4.14 and Theorem 2.3.6 we know that ψ must also be a neighbourhood filter of the identity in the topology $\mathcal{T}$. Now let ϕ be a filter on a group G such that its multiplicative hull is generated by $\{e\}$. But once again by Lemma 2.4.14 and Theorem 2.3.6 we know that ψ must also be a neighbourhood filter of the identity for some Hausdorff topology $\mathcal{T}$. But $\psi \subset \phi$ and so $\phi \to e$ in that topology and thus ϕ is a T-filter $\square$

2.5 T-filters and T-sequences on Abelian Groups

Definition 2.5.1. *Let $\phi \subset 2^G$ and let (F_n) be a sequence in ϕ. Then $\sum_{n\in\mathbb{N}} F_n = \bigcup_{n\in\mathbb{N}} \sum_{i=1}^{n} F_i$. Denote the set of all such combinations by $\sum \phi$ i.e. $\sum \phi = \{\sum_{n\in\mathbb{N}} F_n : (F_n)$ is a sequence in $\phi\}$.*

The sum sets are useful for simplifying the previous framework we developed for T-filters on a general group. We start with a basic observation.

Lemma 2.5.2. *Let ϕ be a filter on an Abelian group G. Then $\sum \phi^* \subset SP\left((\phi^*)^G\right)$.*

Proof. Let $U \in \sum \phi^*$ so there exists a sequence $(F_n) \in \phi*$ such that $U = \sum_{n\in\mathbb{N}} F_n = \bigcup_{n\in\mathbb{N}} \sum_{i=1}^{n} F_i$. Define a sequence of functions $f_i : G \to \phi^*$ by $f_i(x) = F_i$ then $SP_{i\in\mathbb{N}}(\cup_{x\in G}(-x + f_i(x) + x)) \subset U$ since the group is Abelian and thus the introduction of the permutations does not change the final product. This implies that $\sum \phi^* \subset SP\left((\phi^*)^G\right)$ □

So we could apply all the theorems of the previous section and obtain the obvious characterization of a T-filter of an Abelian group. In particular, we know that if $\cap \sum \phi^* = \{e\}$ then ϕ is a T-filter by Theorem 2.4.15. However, simpler proofs can be given for this case and the converse can also be proved. Some of the techniques will be useful later so we reprove that which is useful in this regard.

Lemma 2.5.3. *Let ϕ be a filter on a group G.*

1. *$e \in A$ for all $A \in \sum \phi^*$.*
2. *if $A \in \sum \phi^*$ then $-A \in \sum \phi^*$.*
3. *if $A, B \in \sum \phi^*$ then there exists a $C \in \sum \phi^*$ such that $C \subset A \cap B$.*
4. *if $C \in \sum \phi^*$ then there exists $A, B \in \sum \phi^*$ such that $A + B \subset C$.*

Proof.

1. Since $e \in U \cup -U \cup \{e\}$ for all $U \in \phi$ implies that $e \in \sum_{i=1}^{n} F_i$ for any sequence $(F_i) \subset \phi^*$ and thus $e \in \sum_{i\in\mathbb{N}} F_i$.
2. Since $U \cup -U \cup \{e\} = -(U \cup -U \cup \{e\})$ for all $U \in \phi$ we know that $-\sum_{i=1}^{n} F_i = \sum_{i=1}^{n} F_i$ for any sequence $(F_i) \subset \phi$ and thus $-\sum_{i\in\mathbb{N}} F_i \in \sum \phi^*$.
3. Since $A = \sum A_i$ and $B = \sum B_i$ and there exists $C_i \subset A_i \cap B_i$ then $\sum \phi^* \ni C = \sum C_i \subset A \cap B$.

4. Let $C \in \sum \phi^*$ so $C = \sum C_i$. Define $A = \sum C_{2i} \in \sum \phi^*$ and $B - \sum C_{2i+1} \in \sum \phi^*$ then obviously $A + B \subset C$.

□

Lemma 2.5.4. *Let G be a topological group and ϕ a filter on G such that $\phi \to e$. Then for all $U \in \mathcal{N}_e$ there exists $V \in \sum \phi^*$ such that $V \subset U$.*

Proof. Now by Theorem 2.3.6 if $U \in \mathcal{N}_e$ then there exists $W_1 \in \mathcal{N}_e$ such that $W_1^* = W_1 \subset U$ and $W_2 \in \mathcal{N}_e$ such that $W_2 + W_2 \subset U$ and thus $W = W_1 \cap W_2 \in \mathcal{N}_e$ satisfies $W + W \subset U$ and $W^* = W$. Define $V_1 = W$. Now suppose that V_n has been defined for $n \leq k$ then there exists a $V_{k+1} \in \mathcal{N}_e$ such that $V_{k+1} + V_{k+1} \subset V_k$ and $V_{k+1}^* = V_{k+1}$. So by induction the sequence (V_n) is well defined. Now since $\phi \to e$ we have that $(V_n) \subset \mathcal{N}_e \subset \phi$ and so $\sum V_n^* = \sum V_n \in \sum \phi^*$ and $V = \sum V_n \subset U$ □

We come to the characterisation of a T-filter for an arbitrary Abelian group

Theorem 2.5.5. *A filter ϕ on a group G is a T-filter if and only if $\bigcap \sum \phi^* = \{e\}$. If ϕ is a T-filter then $\sum \phi^*$ forms a basis for the neighbourhood filter at the identity for the topology $\mathcal{T}(G, \phi)$.*

Proof. If ϕ is a T-filter then $\bigcap \sum \phi^* = \{e\}$ since the topology $\mathcal{T}(G, \phi)$ is T_1 and every neighbourhood of the identity is in $\sum \phi^*$ by Lemma 2.5.4. Now suppose ϕ is a filter on G such that $\bigcap \sum \phi^* = \{e\}$ then by Lemma 2.5.3 and Theorem 2.3.6 the family of subsets $\sum \phi^*$ forms a basis for the neighbourhood filter at the identity which generates the largest Hausdorff topology on G for which $\phi \to e$ □

We return now to the question of sequences generating Hausdorff group topologies on a given group G. Let $(x_n) \subset G$ be a sequence and define $A_m = \{x_n : n > m\}$ and $A_m^* = \{x_n : n > m\} \cup \{-x_n : n > m\} \cup \{e\}$. Any sequence generates a filter $\phi = \{U \subset G : A_n \subset U \text{ for some } n \in \mathbb{N}\}$ and we use this fact to define T-sequences on a group.

Definition 2.5.6. *Let $(x_n) \subset G$ be a sequence in a group G. Then $(x_n) \subset G$ is a T-sequence if the corresponding filter $\phi = \{U \subset G : A_n \subset U$ is a T-filter on G. We denote the strongest group topology on G for which the filter $\phi \to e$ by $\mathcal{T}(G, (x_n))$.*

Now Theorem 2.5.5 fully characterises T-sequences by there corresponding filters on G. However, there exists a more natural characterising for the T-sequence case that utilises the special structure of the filter ϕ.

Definition 2.5.7. *Let $(x_n) \subset G$ be a sequence on a group G. Define*

$$A(k,m) = \{\sum_{i=1}^{k} g_i : g_i \in A_m^*\}$$

Theorem 2.5.8. *A sequence $(x_n) \subset G$ is a T-sequence on G if and only if for every $g \in G \setminus \{e\}$ and $k \in \mathbb{N}$ there exists a $m \in \mathbb{N}$ such that $g \notin A^*(k,m)$.*

Proof. First suppose that $(x_n) \subset G$ is a T-sequence, then by Theorem 2.5.5 we know that $\bigcap \sum \phi^* = \{e\}$. Now if $g \in G$ and $g \neq e$ then there exists a sequence $\left(A_{n_i}^*\right) \in \phi^*$ such that $g \notin \sum A_{n_i}^*$. Let $m = \max\{n_1, n_2, ..., n_k\}$ then $g \notin A(k,m) \subset \sum A_n^*$. Now suppose that $(x_n) \subset G$ is a sequence on G such that for all $g \in G \setminus \{e\}$ and $k \in \mathbb{N}$ there exists a $m \in \mathbb{N}$ such that $g \notin A^*(k,m)$. Now there exists $n_1 \in \mathbb{N}$ such that $g \notin A_{n_1}^*$. We proceed by induction, suppose we have chosen $n_1, n_2, ..., n_k$ such that $g \notin \sum_{j=1}^{k} A_{n_j}^*$ but for any $n_{k+1} \in \mathbb{N} \setminus \{n_1, n_2, ..., n_k\}$ we have that $g \in \sum_{j=1}^{k+1} A_{n_j}^*$. Now by passing to sub-sequences if necessary we can assume that there exists sequences $(g_{i,j}) \in A_{n_i}^*$, $i = 1...k+1$ and $j \in \mathbb{N}$ such that $g = \sum_{i=1}^{k+1} g_{i,j}$ and one of the two following cases hold

Case 1 If $(g_{i,j}) \subset (x_n)$ or $(g_{i,j}) \subset (-x_n)$ for all $i \leq k$ then there exists a $m \in \mathbb{N}$ such that $g \notin A(k+1,m)$. However we have that $g = \sum_{i=1}^{k+1} g_{i,j} \in A(k+1,m)$ for sufficiently large $j \in \mathbb{N}$ which is a contradiction.

Case 2 Otherwise there exists $i \leq k$ for which $g_{i,j} = b_i \neq \pm a_n$ for all $n \in \mathbb{N}$. Let $J = \{j_1, ..., j_t\}$ be the set of all indexes with this property. But then $e \neq g - \sum_{l=1}^{t} b_{j_l} = \sum_{s \notin J} g_{i,j}$ furthermore there exists $m \in \mathbb{N}$ such that $g - \sum_{l=1}^{t} b_{j_l} \notin A^*(k+1-t,m)$ which contradicts the definition of the b_j's.

□

Definition 2.5.9. *A sequence (x_n) is called* trivial *if there exists a $m \in \mathbb{N}$ and $b \in G$ such that $x_n = e$ for all $n \geq m$.*

Now as noted for the p-quasicyclic group, not every sequence on a group is a T-sequence. Any trivial sequence is a T-sequence but the resulting topology is the discrete topology which is not very interesting in this regard. In fact it is easy to see that the discrete topology on a group G is generated by the trivial T-sequence $x_n = e$.

Lemma 2.5.10. *Let A be an infinite subset of a countable Abelian group G such that for every element $g \in G \setminus \{e\}$ the set $\{x \in A : kx = g\}$ is finite for every integer $k \in \mathbb{Z}$. Then there exists a T-sequence $(a_n) \subset A$.*

Proof. Let $G \setminus \{e\} = \{g_n : n \in \mathbb{N}\}$. We wish to define a T-sequence on G through the enumeration of $G \setminus \{e\}$ and induction. Define

$$X(a_0, a_1, ..., a_n) = \{x_0 a_0 + x_1 a_1 + ... + x_n a_n : x_i \in \{0, \pm 1, ..., \pm(i+1)\}, i \leq n\}$$

where $a_i \in G$. Note that

$$\sum_{n \in \mathbb{N}} A_n^* = \bigcup_{n \in \mathbb{N}} X(a_0, a_1, ..., a_n)$$

for any sequence $(a_i)_{i \in \mathbb{N}} \subset G$. Now since the set $\{x \in A : \pm x = g_0\}$ is finite there exists an a_0 such that $g_0 \notin X(a_0)$. Now suppose the sequence $a_0, a_1, ..., a_n$ has been defined so that $g_m \notin X(a_m, a_{m+1}, ..., a_n)$ for any $m \leq n$. Now the set

$$X(a_m, a_{m+1}, ..., a_n, x) = X(a_m, a_{m+1}, ..., a_n) + \{kx : k \in \{0, \pm 1, ..., \pm(n - m + 1)\}$$

and is thus finite by the definition of $X(a_m, a_{m+1}, ..., a_n)$ and the finiteness of the sets $\{x \in A : kx = g_m\}$ and thus there exists an a_{n+1} such that $g_m \notin X(a_m, a_{m+1}, ..., a_{n+1})$ for $m \leq n+1$. This is sufficient to show that $g_m \notin \sum_{n \geq m} A_n^* = \bigcup_{n \in \mathbb{N}} X(a_m, a_{m+1}, ..., a_n$. But by Theorem 2.5.5 this implies that $(a_n)_{n \in \mathbb{N}}$ is a T-sequence in G □

The next technical lemma allows us to extend the previous lemma to general Abelian groups. Note this result requires some flavour of the axiom of choice and thus we prefer to be explicit in its use as without the axiom the quotient space generated by using some countable infinite subset as a basis for a set of group words need not be countable.

Lemma 2.5.11. *Let G be an arbitrary Abelian group and $A \subset G$ an infinite set. Then their exists a countable subgroup H of G such that $|H \cap A| = \aleph_0$.*

Proof. If G contains an element $x \in G$ such that $x^n \neq e$ for all $n \in N$ (i.e. of infinite order) then we are done with $H = \{x^n : n \in \mathbb{Z}\}$. So suppose that for all $x \in G$ there exists an $n \in N$ such that $x_n = e$. Let $(a_n)_{n \in \mathbb{N}}$ be any non-repeating sequence of elements of $G \cap A$. Define $H = \{a_0^{p_0} a_1^{p_2} ... a_n^{p_n} : n \in N, p_i \in N\}$ then it is easy to see that H is a subgroup of G and that H, being the union of countably many finite sets, is countable by the axiom of choice. Also since $(a_n)_{n \in \mathbb{N}} \subset G \cap A$ we have that $|H \cap A| = \aleph_0$. □

We can now prove the result for general Abelian groups.

Theorem 2.5.12. *Let A be an infinite subset of an Abelian group G such that for every element $g \notin G \setminus \{e\}$ the set $\{x \in A : kx = g\}$ is finite for every integer $k \in \mathbb{Z}$. Then there exists a T-sequence $(x_n) \subset A$.*

Proof. If G is countable then we are can use Lemma 2.5.10. So suppose G is uncountable, then by Lemma 2.5.11 there exists a countable subgroup $H \subset G$ such that $|A \cap H| = \aleph_0$. But then by Lemma 2.5.10 there exists a T-sequence $(a_n)_{n\in\mathbb{N}} \subset |A \cap H| \subset A$ for the group H. Now define a topology $\mathcal{T}$ with the base $\big\{A \subset G : A \cap H \in \mathcal{T}(H, (a_n)_{n\in\mathbb{N}})\big\}$ then $(a_n)_{n\in\mathbb{N}}$ is a T-sequence for the group G with topology $\mathcal{T}$. □

For example the set $A = (\mathbb{R}\backslash\mathbb{Q}) \subset \mathbb{R}$ obviously contains at least one T-sequence. In fact, A contains an uncountable number of $T-sequences$ since we can continue to remove countable collections of elements from A and still have an uncountable set. The previous theorem admits a converse in the form that if A is an infinite subset of G and there exists a T-sequence taking values in A then the $\{x \in A : kx = g\}$ must be finite for and $k \in \mathbb{N}$ and $g \in G \setminus \{e\}$.

Theorem 2.5.13. *For every infinite group G the following statements are equivalent*

1. *For every infinite subset $A \subset G$ there exists a T-sequence $(x_n) \subset A$.*
2. *For every $g \in G \setminus \{e\}$ and $k \in Z$ the set $\{x \in A : kx = g\}$ is finite.*

Proof. (1) $\Rightarrow$ (2) Follows from Lemma 2.5.12

(2) $\Rightarrow$ (1) Suppose $\{x \in A : kx = g\}$ is infinite for some $k \in \mathbb{Z}$ and $g \in G \setminus \{e\}$ but A contains a T-sequence (x_n) then $g \in A(k, m)$ for every $m \in \mathbb{N}$ which implies $e \neq g \in \cap \sum \phi^*$ which contradicts the characterisation of T-sequences.

□

The previous theorem actually shows that there are $2^{\aleph_0}$ T-sequences on an infinite group and thus the example on $\mathbb{R}$ mentioned previously is not a special case and is in fact indicative of the general state of affairs, as the following corollary shows

Corollary 2.5.14. *Every infinite group admits at least one non-trivial T-sequence and thus admits at least $2^{\aleph_0}$ non-trivial T-sequences.*

Proof. The second result follows from the fact that any subsequence of a T-sequence is obviously also a T-sequence. Now if $\{x \in A : kx = g\}$ is finite then $A \subset G$ contains a T-sequence. Let S_p denote all elements of G with order p for some prime $p \in \mathbb{N}$. Now if $|S_p| < \infty$ for all $p \in \mathbb{N}$ then if $g \in G \setminus \{e\}$ the set $\{x \in A : kx = g\}$ is finite for $A = G$ and thus G contains a T-sequence. Now if $|S_p| = \infty$ for some prime $p \in \mathbb{N}$ then $\{x \in A : kx = g\}$ is finite for $A = S_p$ and thus G contains a T-sequence. $\square$

The previous corollary also shows that every Abelian group is topologizable, as every Abelian group admits at least one non-trivial T-sequence (x_n) and the group topology $\mathcal{T}(G, (x_n))$ is Hausdorff by the definition of a T-sequence and is non-discrete as (x_n) is not a trivial sequence. We have $2^{\aleph_0}$ non-trivial T-sequences to utilize though, but we are not guaranteed that each one generates a unique group topology. The next result, which we merely sketch, states that this is indeed the case.

Definition 2.5.15. *Two T-sequences (x_n) and (y_n) are called* strongly incomparable *if and only if there exist $U \in \mathcal{T}(G, (x_n))$ and $V \in \mathcal{T}(G, (y_n))$ such that $U \cap V = \{e\}$.*

Two strongly incomparable T-sequences cannot generate the same topology, for suppose that they did generate the same topology then U and V would be two open subsets of that topology and thus $\{e\} = U \cap V$ would be an open subset of that topology and thus the topology is discrete. This contradicts the claim that these two sequences are in fact T-sequences.

Theorem 2.5.16. *For every infinite group there exists $2^{\aleph_0}$ strongly incomparable non-trivial T-sequences.*

Proof. We call a non-trivial T-sequence $(a_n)_{n\in\mathbb{N}}$ alternative if for every partition of $\mathbb{N}$ into two infinite subsets N_1 and N_2 we have that $(a_n)_{n\in N_1}$ and $(a_n)_{n\in N_2}$ are strongly incomparable. We first prove that every T-sequence contains an alternative T-sequence and thus by corollary 2.5.14 that there exists an alternative T-sequence $(b_n)_{n\in\mathbb{N}}$ on any infinite group G. Recall that any family of subsets $\{W_\alpha \subset \mathbb{N} : \alpha < 2^{\aleph_0}\}$ is called almost disjoint if $|W_\alpha \cap W_\beta| < \infty$ for all $\alpha \neq \beta$. It can be shown that this non-disjoint family exists and thus the collection of sequences $\{(b_n)_{n\in W_\alpha} : \alpha < 2^{\aleph_0}\}$ is the required family of strongly incomparable T-sequences. $\square$

Thus every Abelian group admits $2^{\aleph_0}$ strongly incomparable, non-discrete group topologies. This result can be considerably strengthened, for not only does an Abelian group admit a continuum of strongly incomparable T-sequences but these resulting group topologies can be linearly ordered. Once again we merely sketch the proof.

Theorem 2.5.17. *Let G be an infinite Abelian topological group, then there exists a family of strongly incomparable T-sequences with cardinality that of the continuum such that the resulting group topologies can be linearly ordered.*

Proof. Once again, we construct an alternative T-sequence $(b_n)_{b\in\mathbb{N}}$ in G. Since $\mathbb{Q}$ is countable, there exists a bijection $f:\mathbb{N}\to\mathbb{Q}$. Denote by

$$W_r = \{n\in\mathbb{N} : f(n)\geq r\}$$

Where $r\in\mathbb{R}$. These subsets of $\mathbb{N}$ are obviously non-empty. Now if $r,t\in\mathbb{R}$ and $r<t$ then the group topology generated by $(b_n)_{n\in W_t}$ is stronger then the one generated by $(b_n)_{n\in W_r}$ since $W_t\subset W_r$. □

It should now be evident that Abelian groups are strongly topologizable. This is due to the nice nature of Abelian groups, in general the class of Abelian groups is free of many of the pathologies a general class of groups could have which limit the ability to generate group topologies for the members of that class. This can be seen in how a whole level of complexity is lifted when trying to characterise a T-filter on an Abelian group as compared to a general group. The whole framework developed in this section can be focused onto one specific Abelian group as well, $\mathbb{Z}$ being an obvious first port of call. There are many interesting results in this field, mainly concerning what types of sequences in $\mathbb{Z}$ are in fact T-sequences. For example

Theorem 2.5.18. *Let $(a_n)\subset\mathbb{Z}$ be a sequence such that $\frac{a_{n+1}}{a_n}\to r$ where r is a transcendental number. Then (a_n) is a T-sequence.*

Proof. See [PZ99] theorem 2.2.3 □

2.6 T-filters and T-sequences on Countable Groups

Definition 2.6.1. *Let G be a countable group with some enumeration $G = \{g_n | n \in \mathbb{N}\}$. Let ϕ be an arbitrary filter on G and and let $(F_i) \subset \phi^*$ be an arbitrary sequence. Define a sequence by*

$$U_k = (g_1^{-1} \cdot F_k \cdot g_1) \cup (g_2^{-1} \cdot F_{k+1} \cdot g_2) \cup ...$$

Then the subset $SP_{k\in\mathbb{N}}(U_k)$ is called a standard element of the family $SP((\phi^*)^G)$.

We can use these standard elements to give a different version of the T-filter classification theorem which is better suited to the class of countable groups.

Definition 2.6.2. *Let G be a countable group and let ϕ be a filter on that group. Denote by $\prod_{i\in\mathbb{N}} \phi^*$ the set of all standard elements of $SP((\phi^*)^G)$.*

Define a function $f_i : G \to \phi^*$ where $G = \{g_j\}$ is a countable group by $f_i(g_j) = F_{i+j-1}$. Then $\cup_{j\in\mathbb{N}} g_j^{-1} \cdot f_i(g_j) \cdot g_j \subset U_i$ and thus $SP_{k\in\mathbb{N}}(U_k) \in SP((\phi^*)^G)$. So once again a sufficient condition that ϕ is a T-filter is that $\cap \prod_{i\in\mathbb{N}} \phi^* = \{e\}$. We can show that this condition is necessary as well.

Theorem 2.6.3. *Let ϕ be a filter on a countable group G. Then ϕ is a T-filter if and only if $\cap \prod_{i\in\mathbb{N}} \phi^* = \{e\}$.*

Proof. We have already shown the sufficiency of the claim, so all that is left is to show that the condition is necessary for ϕ to be a T-filter. So let $V \in SP((\phi^*)^G)$, so there exists a sequence $V_k \in (\phi^*)^G$ such that $SP_{k\in\mathbb{N}}(V_k) \subset V$. But there exists a function $f_k : G \to \phi^*$ such that $\cup_{i\in\mathbb{N}} g_i^{-1} \cdot f_k(g_i) \cdot g_i \subset V_k$. Define a sequence of subsets $U_k \in \phi^*$ by

$$U_k = \bigcap_{m+n=k+1} f_m(g_n)$$

which is well-defined as ϕ^* is a filter. Now obviously, $SP_{k\in\mathbb{N}}(U_k) \in \prod_{i\in\mathbb{N}} \phi^*$ so all we need to show is that $U_k \subset V_k$ as then $SP_{k\in\mathbb{N}}(U_k) \subset V$. Now

$$U_{k+i-1} = \bigcap_{m+n=k+i} f_m(g_n) \subset f_m(g_n)$$

For all $m, n \in \mathbb{N}$ such that $m + n = k + i$ but then $U_{k+i-1} \subset f_k(g_i)$ which is the start of the union cascade we needed to prove. Thus $SP((\phi^*)^G) \subset \prod_{i\in\mathbb{N}} \phi^*$ and so if $\cap \prod_{i\in\mathbb{N}} \phi^* = \{e\}$ then $\cap SP((\phi^*)^G) = \{e\}$ and so by Theorem 2.4.15 we know that ϕ is a T-filter. $\square$

There is a more algebraic version of the previous theorem, which will be important for the next section.

Definition 2.6.4. *Let X be a set and Y another set. Denote by $X[Y]$ the group of words generated by $A \cup B$.*

So for example if $X = \{a, b, c\}$ and $Y = \{t\}$ then $a^2 \cdot t \cdot b^{-1} \in X[Y]$. The set Y is called the set of variables while X is called the set of constants. We use these group words to give an algebraic characterisation of T-filters.

Definition 2.6.5. *Let $X = \{x_{ij} | i, j \in \mathbb{N}\}$ be a set of indeterminants and let G be a countable group with some enumeration $G = \{g_i : i \in \mathbb{N}\}$. Define $f : X^n \to G[X]$ by*

$$f(x_{i_1,j_1}, x_{i_2,j_2}, ..., x_{i_n,j_n}) = g_{j_1}^{-1} \cdot x_{i_1,j_1} \cdot g_{j_1} \cdot g_{j_2}^{-1} \cdot x_{i_2,j_2} \cdot g_{j_2} \cdot ...$$

and $\mathcal{F}_n = \{f(x_{i_1,j_1}, x_{i_2,j_2}, ..., x_{i_n,j_n}) | i_k + j_k \leq n$ and $i_j = i_k \Leftrightarrow j = k\}$.

Lemma 2.6.6. *Let $G = \{g_n | n \in \mathbb{N}\}$ be a countable group and let ϕ be a filter on G. Let $(F_k)_{k\in\mathbb{N}}$ be a sequence in ϕ^* and let $SP_{k\in\mathbb{N}} U_k$ be the standard element of $SP((\phi^*)^G)$ generated by $(F_k)_{k\in\mathbb{N}}$. Then $g \in SP_{k\in\mathbb{N}} U_k$ if and only if there exists a $n \in \mathbb{N}$ and an element $f(x_{i_1,j_1}, x_{i_2,j_2}, ..., x_{i_n,j_n}) \in \mathcal{F}_n$ such that*

$$g \in f(F_{i_1+j_1}, F_{i_2+j_2}, ..., F_{i_n+j_n})$$

Proof. If $g \in f(F_{i_1+j_1}, F_{i_2+j_2}, ..., F_{i_n+j_n})$ then $g \in U_{\sigma(1)} \cdot U_{\sigma(2)} \cdot ... \cdot U_{\sigma(n)}$ for some $\sigma \in S_n$ and thus $g \in SP_{k\in\mathbb{N}} U_k$. So suppose that $g \in SP_{k\in\mathbb{N}} U_k$ and recall that $U_k = (g_1^{-1} \cdot F_k \cdot g_1) \cup (g_2^{-1} \cdot F_{k+1} \cdot g_2) \cup ....$ Now since $g \in SP_{k\in\mathbb{N}} U_k$ there exists a $m \in \mathbb{N}$ and a permutation $\sigma \in S_m$ such that $g \in U_{\sigma(1)} \cdot U_{\sigma(2)} \cdot ... \cdot U_{\sigma(m)}$ but then there exists a sequence $(j_l)_{l=1...m} \subset \mathbb{N}$ such that

$$g \in g_1^{-1} F_{\sigma(1)+j_1} g_1 \cdot g_2^{-1} F_{\sigma(2)+j_2} g_2] \cdot ... \cdot g_m^{-1} F_{\sigma(m)+j_m} g_m$$

Define $i_l = \sigma(l)$ for $l = 1...m$ and $i_l = l$ for $l = m+1... \max(i_1 + j_1, i_2 + j_2, ..., i_m + j_m)$. Define $j_l = 0$ for $l = m + 1... \max(i_1 + j_1, i_2 + j_2, ..., i_m + j_m)$ and let $f(x_{i_1 j_1}, x_{i_2 j_2}, ..., x_{i_n j_n}) \in \mathcal{F}$. Then

$$g \in f(F_{i_1+j_1}, F_{i_2+j_2}, ..., F_{i_n+j_n})$$

□

We use this lemma to give the obvious algebraic version of Theorem 2.6.3. Recall that for any family of subsets $\{W_\alpha\}$ of a set X such that $z \in W_\alpha$ for all α. We know that $\cap W_\alpha = \{z\}$ if and only if for all $x \in X \setminus \{z\}$ there exists a W_α such that $x \notin W_\alpha$. This trivial observation forms the basis of the next theorem.

Theorem 2.6.7. *Let $G = \{g_n | n \in \mathbb{N}\}$ be a countable group and let ϕ be a filter on G. ϕ is a T-filter on G if and only if for all $x \in G \setminus \{e\}$ there exists a sequence of elements of ϕ^* such that*

$$g \notin f(F_{i_1+j_1}, F_{i_2+j_2}, ..., F_{i_n+j_n})$$

for any $n \in \mathbb{N}$ and $f(x_{i_1 j_1}, x_{i_2 j_2}, ..., x_{i_n j_n}) \in \mathcal{F}_n$.

Proof. By Theorem 2.6.3 we know that ϕ is an ultrafilter if and only if $\cap \prod_{i\in\mathbb{N}} \phi^* = \{e\}$ but this is true if and only if for all $x \in G \setminus \{e\}$ there exists a $U \in \prod_{i\in\mathbb{N}} \phi^*$ such that $x \notin U$ but then by Lemma 2.6.6 this is true if and only if for all $f(x_{i_1,j_1}, x_{i_2,j_2}, ..., x_{i_n,j_n}) \in \mathcal{F}_n$ and $n \in \mathbb{N}$,

$$g \notin f(F_{i_1+j_1}, F_{i_2+j_2}, ..., F_{i_n+j_n})$$

□

We now return to the question of T-sequences on an arbitrary countable group. Recall that a sequence is a T-sequence if the filter corresponding to that sequence is a T-filter.

Definition 2.6.8. *Let (x_n) be a sequence in a group G. Define $A^*_m = \{e, x_m^{\pm 1}, x_{m+1}^{\pm 1}, ...\}$.*

These are the same star sets we encountered when we were studying Abelian groups, just written in multiplicative notation.

Theorem 2.6.9. *Let (x_n) be a sequence in $G = \{g_n | n \in \mathbb{N}\}$, a countable group. (x_n) is a T-sequence if and only if for any group word $f(x_1, x_2, ..., x_n)$ in $G[x_1, x_2, ..., x_n]$ such that $f(e, e, ..., e) = e$ and any $g \in G$ there exists a $m \in \mathbb{N}$ such that $g \notin f(A^*_m, A^*_m, ..., A^*_m)$*

Proof. The necessity of the statement follows directly from the fact that any topological group is a Hausdorff space. For let $f : G^n \to G$ be a group word in $G[x_1, x_2, ..., x_n]$ then f must be a continuous function. Let $e \neq g \in G$ then there exists an open neighbourhood of the identity V such that $g \notin V$ and $g^{-1} \notin V$. However, since $f(e, e, ..., e) = e$ we know that there must exist a symmetric open neighbourhood of the identity $U \subset G$ such that $f(U, U, ..., U) \subset V$ i.e. $g \notin f(U, U, ..., U)$ and $g^{-1} \notin f(U, U, ..., U)$. But $x_n \to e$ and thus there exists an integer $m \in \mathbb{Z}$ such that $x_n \in U$ for all $n \geq m$. But this implies that $g \notin A_m^* \subset U$ which gives the required result. The sufficiency follows from Theorem 2.6.7. $\square$

For example, consider the group G generated by the two element set $\{a, b\}$ with relation $a \cdot b \cdot a = b$. Consider the sequence, $x_n = a^n$ for $n = 1, 2, ...$ We can show that this sequence is a T-sequence and thus this group is topologizable. To see this consider any group word f on $G[x_1, x_2, ..., x_n]$ such that $f(e, e, ..., e) = e$ then by iterating the group relation we can show that $f(x_1, x_2, ..., x_n) = \prod x_i^{\alpha_i}$ for some sequence of $\alpha_i \in \{-1, +1\}$. But then $f(A_m^*, A_m^*, ..., A_m^*) = (A_m^*)^n$ and thus if $x \in f(A_m^*, A_m^*, ..., A_m^*)$ then $x = a^k$ for some $k \in \mathbb{N}$. This is sufficient to show that Theorem 2.6.9 is applicable and thus the given sequence is in fact a T-sequence.

2.7 Markov's criteria

Definition 2.7.1. *A group G is called* topologizable *if there exists a topology $\mathcal{T}$ such that*

1. *$(G, \mathcal{T})$ is a topological group.*

2. *$\mathcal{T}$ is a Hausdorff topology on G.*

3. *$\mathcal{T}$ is non-discrete.*

Markov [Mar50] posed the following question in this regard : Is every infinite group topologizable? It is a trivial fact of set theoretic topology that any group endowed with the discrete or trivial topology is a topological group and the Hausdorff and non-discrete conditions are meant to exclude these two possibilities. A topological space being Hausdorff is also a natural requirement as all but the weakest separation axioms result in a topological group being Hausdorff anyway. Furthermore, topological group theory has its genesis in analysis where Hausdorff separation conditions allow for the uniqueness of limits etc. We start off by linking the notion of a T-sequence with that of the topologiziablilty of a countable group.

Theorem 2.7.2. *A countable topological group $G = \{g_n\}$ is topologizable if and only if there exists a non-trivial T-sequence on G.*

Proof. If there exists a non-trivial T-sequence on G then trivially G must be topologizable. So suppose that G admits a non-discrete Hausdorff group topology and define the following sequence of open subsets. Enumerate G such that $g_0 = e$. Let $U_0 = G$ and choose V_1 such that $V_1 = V_1^{-1}$, $V_1^2 \subset U_0$ and $g_1 \notin V_1$. Now since the function $x \to g_0^{-1} \cdot x \cdot g_0$ is continuous we can find an open neighbourhood $U_1 \subset V_1$ such that $g_0^{-1} \cdot U_1 \cdot g_0 \subset U_0$. We can iterate this construction and thus obtain a sequence U_n of open subsets of G such that $U_n = U_n^{-1}$, $U_{n+1}^2 \subset U_n$, $g_n \notin U_n$ and $g_i^{-1} \cdot U_{n+1} \cdot g_i \subset U_n$ for $i \leq n$. By Theorem 2.3.6, the subsets V_n generate the neighbourhood filter of the identity for some group topology on G. Since this topology is first countable and G is a topological group it is metrizable, denote this metric by μ. Define a sequence a_n by choosing $a_n \in \{g \in G | \mu(g, e) < \frac{1}{n}\}$. Then a_n converges to the identity in this topology and thus is a T-sequence. $\square$

The next step is to link the existence of T-sequences on a countable group to some algebraic property of that group. Recall that any function $f : G \to G$ that is continuous must inverse map the identity to a closed subset of G, provided that the topology on G is T_1. This observation serves as the spring board into the classification. Before we begin, some clarification is required. There is confusion in the literature around the concept of an algebraic set. Markov termed what we will define as an algebraic set an additive algebraic set (see [Mar50]). This is contradicted in the modern literature, with Zelenyuk et al using the next definition (see [PZ99]) but Sipacheva using the original Markov terminology (see [Sip08]). We choose to use the Zelenyuk terminology and will refer to Markov's algebraic sets as weakly algebraic.

Definition 2.7.3. *A subset U of a group G is called* algebraic *if there exists a finite collection of group words $f_i \in G[x]$ such that*

$$U = \bigcup_{i \leq n} \{g \in G | f_i(g) = e\}$$

A subset U of a group G is called weakly algebraic *if it is the intersection of some family of algebraic subsets of G.*

It is easy to see that an algebraic set and weakly algebraic set must be closed in any Hausdorff group topology on G. For any group word on $G[x]$ defines a continuous function via the evaluation map on G, and thus the pre-image of $\{e\}$, a closed subset of any Hausdorff group topology, must be closed. Since finite unions and arbitrary intersections of closed sets are closed, we have the result. Secondly, both the empty set and the entire group are trivially algebraic sets[1].

Definition 2.7.4. *A point $g \in G$ is called* algebraically isolated *if the set $G \setminus \{g\}$ is algebraic.*

Lemma 2.7.5. *A point $g \in G$ is algebraically isolated if and only if $G \setminus \{g\}$ is weakly algebraic.*

Proof. The necessity of the claim is trivial, since any algebraic set must be weakly algebraic. So suppose that $G \setminus \{g\}$ is weakly algebraic, so there exist a family of algebraic subsets of G, say U_α such that

$$G \setminus \{g\} = \bigcap U_\alpha$$

Consider the subfamily of the U_α's which do not contain the point $g \in G$. Then it is easy to see that every element of this subfamily must be the same set as $G \setminus \{g\}$.

[1] By considering the empty collection of group words we can see that the entire group is algebraic

This implies the existence of a finite set of group words $f_i \in G[x]$ such that

$$G \setminus \{g\} = \bigcup_{i \leq n} \{y \in G | f_i(y) = e\}$$

i.e. $G \setminus \{g\}$ is algebraic. $\square$

The concept of algebraic isolation is what is required to classify those countable groups which admit group topologies.

Lemma 2.7.6. *Suppose $e \in G$ is algebraically isolated, then any point of G is algebraically isolated. Furthermore, any group topology on G must be discrete.*

Proof. Firstly, if $e \in G$ is algebraically isolated then every point of G is algebraically isolated. To see this consider any point $x \in G$, now since e is algebraically isolated we know that there exist group words f_i such that

$$G \setminus \{e\} = \bigcup_{i \leq n} \{g \in G | f_i(g) = e\}$$

Define $\hat{f}_i(g) = f_i(x^{-1} \cdot g)$ then if $g \neq x$ then $x^{-1} \cdot g \neq e$. But this implies the existence of a $i \leq n$ such that $\hat{f}_i(g) = f_i(x^{-1} \cdot g) = e$ which implies that $G \setminus \{x\}$ is algebraically isolated. But since any additively algebraic set is closed in a group topology this implies that $G \setminus \{x\}$ is closed for all $x \in G$ i.e. $\{x\}$ is an open subset of G for all $x \in G$. This implies that the topology is discrete. $\square$

The theme of the next lemma is more important then the lemma itself, but it is interesting in its own right.

Lemma 2.7.7. *Let G be a group with a non-discrete T_1 group topology and let $\{f_i : G \to G\}$ be a finite collection of continuous functions of this group such that $f_i(e) \neq e$ for all $i \leq n$. Then there exists a $g \in G \setminus \{e\}$ such that $f_i(g) \neq e$ and $f_i(g_{-1}) \neq e$.*

Proof. Firstly, extend the finite collection of functions $\{f_i : G \to G\}$ by including all functions of the form $\hat{f}_i(x) = f_i(x^{-1})$. We then have that $\hat{f}_i(e) = f_i(e^{-1}) = f_i(e) \neq e$. Also, since G is a topological group we know that the function $x \to x^{-1}$ is continuous and thus each $\hat{f}_i$ is continuous, being the composition of two continuous functions. So without loss of generality we may assume that the collection $\{f_i : G \to G\}$ contains all the required hat functions. Now consider the collection of subsets $A_i = f_i^{-1}(G \setminus \{e\})$. Since G is a T_1 space and each f_i is continuous we know that A_i is an open neighbourhood of the identity. Thus $\cap A_i$ is an open neighbourhood of the identity and since the topology is not discrete there must exist a $g \in \cap A_i$ such that $g \neq e$. $\square$

Lemma 2.7.8. *Let G be a group. If the identity is not algebraically isolated then for any finite collection of group works $f_i(x)$ with $f_i(e) \neq e$ there exists a $g \in G \setminus \{e\}$ with $f_i(g) \neq e$ and $f_i(g^{-1}) \neq e$.*

Proof. Once again extend the collection of group words by defining $\hat{f}_i(x) = f_i(x^{-1})$. Without loss of generality we may assume that the collection $\{f_i : G \to G\}$ contains all the required hat functions. Now suppose that for all $g \in G \setminus \{e\}$ there exists a i such that $f_i(g) = e$. This would imply that

$$G \setminus \{e\} = \bigcup_{i \leq n} \{g \in G | f_i(g) = e\}$$

Which contradicts the fact that $\{e\}$ is not algebraically isolated. $\square$

Theorem 2.7.9 (Markov Criteria)**.** *Let G be a countable group. G is topologizable if and only if the identity element $e \in G$ is not algebraically isolated.*

Proof. The necessity of the claim has been shown in Lemma 2.7.6 so we only require the sufficiency of the claim. So let G be a countable group such that its identity is not algebraically isolated. Enumerate G as $\{g_n\}$ with $g_0 = e$. Since G is countable, and the countable union of a countable family of sets is also countable, we can enumerate the following subset of the set of all words on G

$$\{f_k\} = \bigcup_{i \in \mathbb{N}} \{f \in G[x_1, x_2, ..., x_i] | f(e, e, ..., e) = e\}$$

Define a sequence in G as follows. Since $g_0 \in G$ satisfies $f_0(g_0, g_0, ..., g_0) = g_0 \neq g_1$ we know by Lemma 2.7.8 that there exists a $x_1 \in G \setminus \{g_0\}$ such that

$$f_0(\{e, x_1^{\pm 1}\}, \{e, x_1^{\pm 1}\}, ..., \{e, x_1^{\pm 1}\}) \neq g_1$$

We can continue iterating Lemma 2.7.8 to form a sequence x_i such that $x_{i+1} \in (G \setminus \{e\}) \setminus \{x_n\}_{n \leq i}$ and

$$f_m(\{e, x_1^{\pm 1} ..., x_{m+1}^{\pm 1}\}, \{e, x_1^{\pm 1} ..., x_{m+1}^{\pm 1}\}, ..., \{e, x_1^{\pm 1} ..., x_{m+1}^{\pm 1}\}) \neq g_{m+1}$$

But this implies that the sequence $\{x_m\}$ is a non-trivial T-sequence, by Theorem 2.6.9 and thus the group G admits a non-discrete Hausdorff topology. $\square$

We can use this theorem to show once again that every countable Abelian group is topologizable. For suppose that G is a countable Abelian group such that the identity e of G is algebraically isolated. Then there must exist a finite family of group words f_i such that

$$G \setminus \{e\} = \bigcup_{i \leq n} \{g \in G | f_i(g) = e\}$$

However, any group word on an Abelian group is of the form $f_i(x) = a_i \cdot x^{k_i}$ where $a_i \in G \setminus \{e\}$ and $k_i \in \mathbb{Z} \setminus \{0\}$. Without loss of generality we may assume that k_i is positive for if $k_i < 0$ we can consider the new group word $a_i^{-1} \cdot x^{-k_i} = e$ where $-k_i > 0$. So $y \in G$ is a solution to this equation if and only if y^{-1} is a solution to the original equation. So choose your favourite $f_i(x) = a_i \cdot x^{n_i}$ and consider two cases. If G has an element of order n_i for some $i \leq n$, say y then $f_i(y) \neq e$ for $f_i(y) = a_i \cdot y^{n_i} = a_i \neq e$. If for all $y \in G \setminus \{e\}$ and $i \leq n$, $y^{n_i} \neq e$ we have to argue slightly more carefully. So suppose that

$$G \setminus \{e\} = \bigcup_{i \leq n} \{g \in G | f_i(g) = e\}$$

It is obvious that $G \setminus \{e\}$ is a countable set, thus one of the $\{g \in G | f_i(g) = e\}$ must be countable. Choose n_i distinct elements of the countable set $\{g \in G | f_i(g) = e\}$, label them $\{y_k\}$ for $k = 1...n_i$. Then $a_i \cdot y_k^{n_i} = e$ for $k = 1...n_i$, but this implies that

$$a_i^{n_i} \cdot \left(\prod_{j=1}^{n_i} y_j \right)^{n_i} = \left(a_i \cdot \prod_{j=1}^{n_i} y_j \right)^{n_i} = e$$

But then $a_i \cdot \prod_{j=1}^{n_i} y_j$ is an element of order n_i which is a contradiction. So the identity e of the countable Abelian group G cannot be algebraically isolated, which implies that G is topologizable.

Definition 2.7.10. *A subset A of a group G is called* unconditionally closed *if it is closed in any Hausdorff group topology on G.*

The family of all unconditionally closed subsets of a group, as well as the set of all weakly algebraic sets of a group define two topologies on that group. Since the set of all weakly algebraic sets on a group G is closed under finite unions and arbitrary intersections, its forms a base for a topology on G. We call this topology the Zariski topology. Similarly, since the family of all unconditionally closed subsets is also closed under arbitrary intersections and finite unions, it to generates a topology on G. This topology is called the Markov topology.

Definition 2.7.11. 1. *The* Markov topology $\mathcal{T}_M$ *on a group* G *is the topology generated by all unconditionally closed subsets of* G.

2. *The* Zariski topology $\mathcal{T}_Z$ *on a group* G *is the topology generated by all weakly algebraic subsets of* G.

The Markov theorem can be rephrased as asking when these two topologies coincide. Markov showed that this was true for an arbitrary countable group.

Theorem 2.7.12 (General Markov's Theorem)**.** *Let* G *be a countable group. A set* A *of* G *is weakly algebraic if and only if it is unconditionally closed.*

Proof. We sketch a proof (for the full proof see [Mar50]). Since the arbitrary intersection of weakly algebraic sets is once again weakly algebraic, and any subset of a group G is contained in at least one weakly algebraic subset of G (namely G itself) we can define a weak algebraic closure operator on the group. We define

$$\mathrm{cl}_{\mathcal{T}_Z}(A) = \bigcap \{U \subset G : A \subset U \text{ and } U \text{ is weakly algebraic}\}$$

Now since the weak algebraic closure of a subset of G is a closed subset of G in any group topology $\mathcal{T}$ on G, the weak algebraic closure must contain the topological closure of that subset. Now using the theory of multi-norms on a group, as explored in [Mar50], it is possible to define a group topology on G on which the weak algebraic closure of a subset equals the topological closure of that same subset. This step depends strongly on the countability of the group in question. Now suppose the subset in question is in fact unconditionally closed, then it closed in the group topology where the topological closure and weak algebraic closure coincide, which implies that the subset must be weakly algebraic. □

Taking $A = G \setminus \{e\}$ shows that Theorem 2.7.9 is merely a special case of this theorem. A reasonable question to ask is if this theorem extends to general infinite groups, as this would serve as a characterization of topologizable groups. Sadly, this is not the case. It was shown in [Sip06] that there exists a group, assuming CH, which contains an unconditionally closed subset which is not algebraic.

Theorem 2.7.13. *Let* G *be a non-topologizable group such that any finite subset of* G *is contained in a topologizable subgroup of* G. *Then* $G \setminus \{e\}$ *is an unconditionally closed subset of* G *which is not weakly closed.*

Proof. Since G is non-topologizable, $\{e\}$ is open in every Hausdorff group topology on G and thus $G \setminus \{e\}$ is unconditionally closed. Suppose that $G \setminus \{e\}$ is also algebraic (and thus by Lemma 2.7.5 also weakly algebraic). This implies the existence of a finite collection of group words $\{f_i\}$ on G such that

$$G \setminus \{e\} = \bigcup_{i \leq n} \{g \in G | f_i(g) = e\}$$

Consider the word $f_k(x) = \prod_{j=1}^{n} \alpha_j x^{p_j}$ where $\alpha_j \in G$ and $p_j \in \mathbb{Z}$. Since the set $\{\alpha_j\}$ is finite for any word f_k, the collection of all the α_j's over all the words f_i is contained in some topologizable subgroup H of G. But then the restriction of the words to this subgroup satisfies

$$H \setminus \{e\} = \bigcup_{i \leq n} \{g \in H | f_i(g) = e\}$$

which contradicts the fact that H is topologizable. □

The existence of such a group will be expanded upon in the next section. Seeing that the general characterization is beyond our grasp, we can ask for which class of groups we can extend the Markov-Zariski equivalence and thus have a limited characterization of topologizable groups. Markov mentioned in [Mar50] the fact that his theorem can be extended to arbitrary infinite Abelian groups, this fact was credited to Perel'man but no proof was provided. Recently, Dikranjan and Shakhmatov ([DS08]) have given a complete characterization of the groups for which the Markov-Zariski equivalence does hold. Let $G^{<\omega}$ denote the family of all finite subsets of a group G and $G^{\leq\omega}$ denote the family of all countable subsets of a group G.

Definition 2.7.14. *Let $\mathcal{C}$ be a family of elements of $G^{\leq\omega}$ for some group G.*

1. *$\mathcal{C}$ is* closed *if for any increasing sequence of elements $\{C_i\}$ of $\mathcal{C}$, $\cup C_i \in \mathcal{C}$.*
2. *$\mathcal{C}$ is* unbounded *if for any element $U \in G^{\leq\omega}$ there exists a $V \in \mathcal{C}$ such that $U \subset V$.*
3. *$\mathcal{C}$ is a* club *if it is both closed and unbounded.*

Definition 2.7.15. *Let G be a group. Define $\mathcal{P}(G) = \{H \in G^{\leq\omega} : H$ is a subgroup of $G\}$.*

Since the union of an increasing family of countable subsets of a group G is still a countable subgroup of G and any countable subset of G is contained in the countable subgroup it generates, we have shown the following lemma.

Lemma 2.7.16. *$\mathcal{P}(G)$ is a club, for any group G.*

Recall that $\mathrm{cl}_{\mathcal{T}_Z}(A)$ denotes the weak algebraic closure of a subset A of G. This is the same as the topological closure operator of the Zariski topology (hence the notation). Let H be a subgroup of G, then $\mathrm{cl}_{\mathcal{T}_Z}\{H\}(A)$ denotes the topological closure operator of the subspace topology inherited from the Zariski topology on H. So $\mathrm{cl}_{\mathcal{T}_Z}\{G\}(A) = \mathrm{cl}_{\mathcal{T}_Z}(A)$ Similarly, let $\mathrm{cl}_{\mathcal{T}_M}\{H\}(A)$ denote the topological closure operator of the subspace topology inherited from the Markov topology on H.

Theorem 2.7.17. *Let G be an arbitrary group. Then $\mathcal{T}_M = \mathcal{T}_Z$ if and only if for all $A \subset G$, the family of subgroups $\{H \in \mathcal{P}(G) : \mathrm{cl}_{\mathcal{T}_M}\{H\}(A \cap H) = \mathrm{cl}_{\mathcal{T}_M}\{G\}(A) \cap H\}$ contains a club of elements from $G^{\leq\omega}$.*

Proof. See [DS08] $\square$

This theorem is sufficient to generalize the Markov criteria to arbitrary Abelian groups and the direct product of a family of countable groups. Interested readers are referred to [DS08] for the more set theoretic exposition of these ideas. Furthermore, with this theorem now in our arsenal we can extend the previous Markov-type proof of the existence of group topologies on countable Abelian groups to general Abelian groups. This is because that proof only required the countability of the Abelian group in question to apply Markov's criteria, and thus an extended criteria is immediately beneficial. We thus have this general Markov criteria

Theorem 2.7.18. *Let G be a group which is either the direct product of some family of countable groups or an Abelian group. There exists a non-discrete Hausdorff group topology on G if and only if the identity of that group is not algebraically isolated.*

Proof. If the subset $G \setminus \{e\}$ is unconditionally closed, then it is in $\mathcal{T}_M$. But by Theorem 2.7.17 we know that $\mathcal{T}_M = \mathcal{T}_Z$ and thus $G \setminus \{e\}$ is algebraic i.e. the identity of G is algebraically isolated. Similarly if the identity of G is algebraically isolated, then $G \setminus \{e\} \in \mathcal{T}_Z = \mathcal{T}_M$ and thus $G \setminus \{e\}$ is unconditionally closed. $\square$

2.8 Examples of non-topologizable groups

Given the strict conditions a natural topology places on the algebraic structure of a group it is surprising that many common classes of groups are topologizable. In fact every infinite Abelian group is extremely topologizable, as was previously mentioned. The general question of Markov regarding the existence of group topologies of infinite groups was answered in the negative by Shelah [She80] when he constructed an infinite group (assuming the continuum hypothesis) which only admits a discrete group topology using the techniques of Small Cancellation theory. The fact that all linear groups are topologizable vastly complicates the situation here as one must introduce amalgams of groups and other more exotic construction methods to find a counter example. His proof does not fall into the ZFC framework as well since the Continuum Hypothesis must be assumed.

Theorem 2.8.1. *Assume CH. There exists a group G with $|G| = \omega^+$ such that the only group topology on G is discrete. Furthermore, any finite subset of G is contained in a topologizable subgroup of G.*

Proof. See [She80] $\square$

This construction and Theorem 2.7.13 show the existence of groups whose topological nature cannot be determined solely on the finite algebraic structure of its subsets. It is possible to extend the notion of an algebraic set so that we have a partial strengthening of the equivalence between unconditionally closed and algebraic sets. The idea is to extend the number of words allowed to define the algebraic set from a finite limit to some cardinal number. The problem is that this generalisation breaks the correspondence between normal topological closure of a set and the algebraic nature of that set. Interested readers are directed to [Sip08].

Shelah also gave a sufficient condition for an uncountable group G to be non-topologizable. Since these conditions are much more tractable then the actual construction he presented in connection with Kurosh groups and since this conditions are easy to state it is worthwhile going through the short proof. The group constructed in [She80] essentially satisfies these two criteria, and thus cannot be topologizable.

Theorem 2.8.2. *An uncountable group G is non-topologizable if the following two conditions hold*

1. *There exists $m \in \mathbb{N}$ such that $A^m = G$ for every $A \subset G$ such that $|A| = |G|$.*

2. *If H is a subgroup of G such that $|H| < |G|$ then there exists $n \in \mathbb{N}$ and elements $x_1, x_2, ..., x_n \in G$ such that $\cap_{i=1..n} x_i^{-1} H x_i$ is a finite set.*

Proof. Let $\mathcal{T}$ be a group topology on G and let $V \subset G$ be an open neighbourhood of $e \in G$ such that $V \neq G$. Now since there exists an $m \in \mathbb{N}$ such that $A^m = G$ for every subset $A \subset G$ such that $|A| = |G|$ and $\mathcal{T}$ is a group topology there exists an open neighbourhood W of e such that $W^m \subset V$ and thus $|W| < |G|$. Let H be the subgroup generated by W. Since H consists of all finite products of elements of W we have that $|H| < |G|$. But then there exists an $n \in \mathbb{N}$ such that $\cap_{i=1..n} x_i^{-1} H x_i$ is a finite set, for some $x_1, x_2, ..., x_n \in G$. But $e \in x_i^{-1} H x_i$ for all $i = 1...n$ and thus

$$e \in \cap_{i=1..n} x_i^{-1} H x_i = \{e, y_1, y_2, ..., y_k\}$$

Where the y_i are distinct and k is some natural number. But $\mathcal{T}$ is a group topology and is thus Hausdorff so is T_1 and thus there exist open neighbourhoods Y_i of e such that $y_i \notin T_i$. But then

$$\{e\} = \bigcap_{i=1..k} Y_i \cap \bigcap_{i=1..n} x_i^{-1} H x_i$$

and thus $\mathcal{T}$ is discrete. □

The second condition is not required if we wish to extend this result to countable groups. This new sufficient condition should be compared with the characterisation theorem previously stated.

Theorem 2.8.3. *Let G be a countable group. If there exists $m \in \mathbb{N}$ such that $A^m = G$ for every countable $A \subset G$ then G is non-topologizable.*

Proof. Let $\mathcal{T}$ be a group topology on G and let $V \subset G$ be an open neighbourhood of $e \in G$ such that $V \neq G$. Now since there exists an $m \in \mathbb{N}$ such that $A^m = G$ for every subset $A \subset G$ such that $|A| = |G|$ and $\mathcal{T}$ is a group topology there exists an open neighbourhood W of e such that $W^m \subset V$ and thus $|W| < |G|$. But this implies that W is a finite open neighbourhood of e and thus by the proof of Theorem 2.8.2 we have that G is non-topologizable. □

The final demonstration of a countable non-topologizable group in *ZFC* was obtained by Hesse in [Hes79] and Ol'shanskii in [Ols80]. Hesse essentially showed that the continuum hypothesis was not a necessary requirement for the Shelah construction, at least in a special case sufficient to show the theorem. The Ol'shanskii construction is something completely different. Once again this constructed group is of rather exotic structure, as was to be expected. We give a version of his construction below, however we first require a "technical" lemma.

Lemma 2.8.4. *Let $m, n \in \mathbb{N}$ such that $n \geq 2$ and $m \geq 665$, with m an odd number. There exists a group $A(n,m)$ generated by n elements with the following three properties*

1. *$A(n,m)$ is a torsion free group.*
2. *The centre C of $A(n,m)$ is an infinite cyclic group.*
3. *$A(n,m)/C$ is an infinite group of period m.*

Proof. See Adian and Novikov [NA68a], [NA68b] and [NA68c] □

This lemma is only technical in the sense that its construction does not illuminate the question of topologizing arbitrary countable groups. The groups constructed in the above lemma are called Adian groups. These groups are in fact the counterexamples to the famous Bounded Burnside problem (Is every finitely generated groups of finite exponent finite?) and takes up over 330 journal pages to construct. We can now prove the existence of a countable non-topologizable group in ZFC.

Theorem 2.8.5. *Let $A(n,m)$ denote the Adian group and let $C_m = \{x^m : x \in C\}$. The infinite group $A(n,m)/C_m$ is not topologizable.*

Proof. Firstly, C_m is a subgroup of $A(n,m)$ for suppose $x, y \in C$ then $y^{-1} \in C$ and $xy^{-1} \in C$. Thus $x^m \cdot (y^{-1})^m = (x \cdot y^{-1})^m \in C_m$ since C is the centre of $A(n,m)$. Also since C is the centre of $A(n,m)$ we have that $x^{-1}C_m x \subset C_m$ for all $x \in G$ and thus we have that C_m is a normal subgroup of $A(n,m)$ and so $G = A(n,m)/C_m$ is well-defined. Since the centre C of $A(n,m)$ is an infinite cyclic group, we know that $C = \{c^n : n \in \mathbb{Z}\}$ for some $c \in C$. Now if $g \in A(n,m) \setminus C$ then $g \neq c^n$ for all $n \in \mathbb{Z}$. However, since $A(n,m)/C$ is an infinite group of period m we know that $g^m \in C$. But if $g^m \in C_m$ then $g^m = x^m$ for some $x \in C$. This implies that $(gx^{-1})^m = e$ with $gx^{-1} \neq e$ which contradicts the fact that $A(n,m)$ is a torsion free group. Thus if $g \in A(n,m) \setminus C$ then $g^m \in C \setminus C_m$. Now let $f : A(n,m) \to G$ be the natural homeomorphism. Let $a_1 = f(c)$, $a_2 = f(c^2), ..., a_{m-1} = f(c^{m-1})$ then for every $a \in G \setminus \{e\}$ we have that either $a \in \{a_1, a_2, ..., a_{m-1}\}$ or $a^m \in \{a_1, a_2, ..., a_{m-1}\}$. For suppose that $a \notin \{a_1, a_2, ..., a_{m-1}\}$ then for all $g \in f^{-1}(a)$ we have that $g \notin A(n,m) \setminus C$ which implies that $g^m \in C \setminus C_m$ which implies that $a^m = f(g)^m \in \{a_1, a_2, ..., a_{m-1}\}$. Now let $\mathcal{T}$ be any Hausdorff topology on G such that the mapping $h : G \to G$ where $h(x) = x^m$ is continuous. But then

$h^{-1}(\{a_1, a_2, ..., a_{m-1}\}) = G \setminus \{e, a_1, a_2, ..., a_m\}$. This implies that $\{e, a_1, a_2, ..., a_m\}$ is an open subset and thus that $\{e\}$ is an open subset i.e. $\mathcal{T}$ is discrete and thus G is a non-topologizable group. $\square$

So not every group admits a non-discrete Hausdorff topology, even if we limit ourselves to ZFC

Chapter 3

The Stone-Čech compactification of a Semigroup

3.1 The Stone-Čech compactification

In this section we will give the general construction of the Stone-Čech compactification of a completely regular space using z-ultrafilters as the basis of that construction. There are multiple ways of constructing the Stone-Čech compactification, see [GJ13]. We will then investigate the relationship between the topological properties of the compactification (namely certain separation properties) and the original topology on the initial space. We close with some set-theoretic topological properties that are important for later sections of this thesis. Recall the definition of a zero set. It is easy to see that any zero set must be closed as it is the inverse image of a closed set under a continuous function. However, not every closed set of an arbitrary completely regular space need be a zero set. Topologies that satisfy this criteria are called perfectly normal spaces.

Definition 3.1.1. *A family of subsets $\mathcal{F} \subset 2^X$ of a topological space X is a* z-filter *if*

1. $\emptyset \notin \mathcal{F}$.
2. $\forall A \in \mathcal{F}$, $A \in \mathcal{Z}(X)$.
3. *if* $A \in \mathcal{F}$ *and* $B \in \mathcal{F}$ *then* $A \cap B \in \mathcal{F}$.
4. *if* $A \in \mathcal{F}$ *and* $A \subset B$ *with* $B \in \mathcal{Z}(X)$ *then* $B \in \mathcal{F}$.
5. $\mathcal{F} \neq \mathcal{Z}(X)$.

Definition 3.1.2. *A maximal z-filter is a* z-ultrafilter*.*

Theorem 3.1.3. *Let X be a topological space and let $\mathcal{F} \subset \mathcal{Z}(X)$ have the finite intersection property. There exists a z-ultrafilter p on X such that $\mathcal{F} \subset p$.*

Proof. Let $\mathcal{C} = \{\mathcal{G} \in 2^{2^X} : \mathcal{G}$ be a z-filter and $\mathcal{F} \subset \mathcal{G}\}$. We order this family by set inclusion. Since finite intersections of open subsets are open we have that $\mathcal{F} \in \mathcal{C}$ and so $\mathcal{C} \neq \emptyset$. Let $\{\mathcal{G}_\alpha\}_{\alpha \in I} \subset \mathcal{C}$ be a chain, then $\mathcal{G}_\beta \subset \cup_{\alpha \in I}\mathcal{G}_\alpha \forall \beta \in I$ and it is easy to see that $\cup_{\alpha \in I}\mathcal{G}_\alpha \in \mathcal{C}$. This implies that the chain $\{\mathcal{G}_\alpha\}_{\alpha \in I}$ has an upper bound and thus by Zorn's Lemma $\mathcal{C}$ has a maximal element. Let p be such a maximal element of $\mathcal{C}$. Now if p is not a z-ultrafilter then there exists a filter $\mathcal{H}$ such that $p \subset \mathcal{H}$. But then $\mathcal{H} \in \mathcal{C}$ and so $p = \mathcal{H}$ i.e. p is an z-ultrafilter. □

Lemma 3.1.4. *Let p and q be z-ultrafilters on a topological space X.*

1. *if $B \in \mathcal{Z}(X)$ and $A \cap B \neq \emptyset$ for $A \in p$ then $B \in p$.*

2. *if $A, B \in \mathcal{Z}(X)$ and $A \cup B \in p$ then $A \in p$ or $B \in p$.*

3. *if $p \neq q$ then there are $A \in p$ and $B \in q$ such that $A \cap B = \emptyset$.*

Proof.

1. The family $\mathcal{F} = \{A \cap B : A \in p\} \subset \mathcal{Z}(X)$ has the finite intersection property and thus by Theorem 3.1.3 there must exist a z-ultrafilter r on X such that $\mathcal{F} \subset r$. Since $A \cap B \subset B$ and $A \cap B \in \mathcal{Z}(X)$ we have that $B \in r$. Similarly we have that $A \in r$, for arbitrary $A \in p$. But this would imply that $p \subset r$ and since p is a z-ultrafilter we must have that $B \in r = p$.

2. Let $A, B \in \mathcal{Z}(X)$ and assume that $A \notin p$ and $B \notin p$ then there exists $C, D \in p$ such that $A \cap C = \emptyset = B \cap D$. So $(A \cup B) \cap (C \cap D) = \emptyset$ and thus $A \cup B \notin p$.

3. Now there exists $B \in q$ such that $B \notin p$ so we can find some $A \in p$ such that $A \cap B = \emptyset$.

□

The previous results show that ultrafilters and z-ultrafilters are very close behaviourally. For example z-ultrafilters are also z-prime filters (property 2) the same way that normal ultrafilters on a set are prime filters on that set. However since in general $B \in Z(X)$ does not imply that $X \setminus B \in Z(X)$ we do not have that $B \notin p \implies X \setminus B \in p$ and thus z-ultrafilters do not necessarily satisfy the ultrafilter property on $Z(X)$.

Theorem 3.1.5. *If $x \in X$ then $\{A \in \mathcal{Z}(X) : x \in A\}$ is a z-ultrafilter.*

Proof. Let $A, B \in \{A \in \mathcal{Z}(X) : x \in A\}$. So there exists continuous functions $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ such that $A = f^{-1}(0)$ and $B = g^{-1}(0)$. Define $h : X \to \mathbb{R}$ by

$$h(t) = f^2(t) + g^2(t)$$

then $h^{-1}(0) = A \cap B$. Also $\emptyset \neq A \cap B \in \mathcal{Z}(X)$. Now let $A \in \{A \in \mathcal{Z}(X) : x \in A\}$ and let $A \subset B \in \mathcal{Z}(X)$. Since $x \in A \subset B$ and $B \in \{A \in \mathcal{Z}(X) : x \in A\}$ we have that $\{A \in \mathcal{Z}(X) : x \in A\}$ is a z-filter. Now by Theorem 3.1.3 there exists a z-ultrafilter p with $\{A \in \mathcal{Z}(X) : x \in A\} \subset p$. Now let $C \in p \backslash \{A \in \mathcal{Z}(X) : x \in A\}$ and thus $x \notin C \in \mathcal{Z}(X)$. However, X is $T_{3\frac{1}{3}}$ and so there exists a $D \in \mathcal{Z}(X)$ such that $x \in D$ and $C \cap D = \emptyset$. □

Definition 3.1.6. *If Y is a compact, Hausdorff topological space with a dense subspace X such that any continuous function $f : X \to Z$, where Z is a compact Hausdorff space, uniquely extends to a continuous function $\overline{f} : Y \to Z$ then Y is the Stone-Čech compactification of X.*

Lemma 3.1.7. *Let X be a topological space and denote by βX the set of all z-ultrafilters on X. Let $\mathcal{B} = \{\{p \in \beta X : A \notin p\} : A \in \mathcal{Z}(X)\}$. Define $e : X \to \beta X$ by $e(x) = \{A \in \mathcal{Z}(X) : x \in A\}$ then*

1. *$\mathcal{B}$ is a base for a compact, Hausdorff topology on βX.*
2. *$e[X \setminus A] = e[X] \cap \{p \in \beta X : A \notin p\}$ for $A \in \mathcal{Z}(X)$.*
3. *$e : X \to \beta X$ is a topological embedding of X into βX.*
4. *$\overline{e[A]} = \{p \in \beta X : A \in p\}$ for $A \in \mathcal{Z}(X)$.*
5. *$\overline{e[X]} = \beta X$.*

Proof.
1. Let $A_i \in \mathcal{Z}(X)$ and let $B_i = \{p \in \beta X : A_i \notin p\}$ for $i \in \{1, 2\}$. Now $A_1 \cup A_2 \notin p \Leftrightarrow A_1 \notin p$ and $A_2 \notin p$ and thus $B_1 \cap B_2 = \{p \in \beta X : A_1 \cup A_2 \notin p\} \in \mathcal{B}$ and thus $\mathcal{B}$ is a topological base on X. Now let $p, q \in \beta X$ such that $p \neq q$ so there exists $A, B \in \mathcal{Z}(X)$ such that $A \in p$, $B \in q$ and $A \cap B \neq \emptyset$. Now since disjoint zero sets are actually completely separated there are co-zero sets $C, D \subset X$ and zero sets $E, F \subset X$ such that $A \subset C \subset E$ and $B \subset D \subset E$ with $E \cap F = \emptyset$. But then $p \in \{t \in \beta X : X \setminus C \notin t\}$ and $q \in \{t \in \beta X : X \setminus D \notin t\}$ and so$\{t \in \beta X : X \setminus C \notin t\} \cap \{t \in \beta X : X \setminus D \notin t\} \subset \{t \in \beta X : E \in t\} \cap \{t \in \beta X : F \in t\} = \emptyset$ and thus $\mathcal{B}$ generates a

Hausdorff topological space. Now let $\{A_i : i \in I\} \subset \mathcal{Z}(X)$ and suppose the family $\{\{p \in \beta X : A_i \in p\} : i \in I\}$ has the finite intersection property then there exists $q \in \beta X$ such that $\{A_i : i \in I\} \subset q$ and thus

$$q \in \bigcap_{A_i \in q} \{p \in \beta X : A_i \in p\} \subset \bigcap_{i \in I} \{p \in \beta X : A_i \in p\} \neq \emptyset$$

and thus $\mathcal{B}$ generates a compact, Hausdorff topology

2. $p \in [X \setminus A] \Leftrightarrow \exists t \in X \setminus A \subset X$ such that $p = \{B \in \mathcal{B}(Z) : t \in B\} \Leftrightarrow p \in e[X]$ and $A \notin p$ as $A \in \mathcal{Z}(X) \Leftrightarrow p \in e[X] \cap \{q \in \beta X : A \notin q\}$

3. $e : X \to \beta X$ is obviously well defined. Now if $x \neq y$ then as $\{x\}$ and $\{y\}$ are closed subsets of X there exist subsets $A, B \in \mathcal{Z}(X)$ such that $x \in A$, $y \in B$ and $A \cap B = \emptyset$ but then $\{C \in \mathcal{Z}(X) : x \in C\} \neq \{C \in \mathcal{Z}(X) : x \in C\}$ and thus $e : X \to \beta X$ is an injection. Also as $e[X \setminus A] = e[X] \cap \{p \in \beta X : A \notin p\}$ and base for X is carried onto a base for the subspace topology of $e[X] \subset \beta X$

4. $\{p \in \beta X : A \in p\}$ is closed in βX and obviously $e[A] \subset \{p \in \beta X : A \in p\}$ and thus $\overline{e[A]} \subset \{p \in \beta X : A \in p\}$. Now suppose $p \in \{p \in \beta X : A \in p\} \setminus \overline{e[A]}$ then there exists $B \in \mathcal{Z}(X)$ such that $B \notin p$ and $e[A] \cap \{t \in \beta X : B \notin t\} = \emptyset$. However $e[A] \subset e[X] \cap \{t \in \beta X : B \notin t\} = e[B]$ and thus $B \in p$ which is a contradiction.

5. $\overline{e[X]} = \{p \in \beta X : X \in p\} = \beta X$

$\square$

Lemma 3.1.8. *Let X be a dense subset of a topological space Y and let $f : X \to K$ be a continuous function where K is a compact space.*

1. *If for $p \in Y$ there is a continuous function $f_p : X \cup \{p\} \to K$ such that $f \subset f_p$ then there exists a continuous function $g : Y \to K$ such that $f \subset f$.*

2. *If $p \in Y$ and there is no continuous function $g : X \cup \{x\} \to K$ such that $f \subset g$ then there exists $A, B \in \mathcal{Z}(K)$ such that $A \cap B = \emptyset$ and $p \in \overline{f^{-1}(A)} \cap \overline{f^{-1}(B)}$.*

Proof. 1. Define $g = \bigcup \{f_p : p \in Y\}$. Now let $p \in Y$ and let V and W be open neighbourhoods of $g(p)$ such that $\overline{W} \subset V$ (which is possible as X is a compact, completely regular space and is thus T_4) and let U be an open neighbourhood of $p \in Y$ such that $f(q) \in W$ for $q \in U \cap X$. If $q \in U$ then $q \in \overline{U \cap X}$ and thus $g(q) = f_q(q) \in \overline{f_q[U \cap X]} = \overline{f[U \cap X]} \subset \overline{W} \subset V$

2. The family $\mathcal{F} = \{\overline{f[U \cap X]} : U$ is a neighbourhood of $p \in Y\}$ is a family of closed sets with the finite intersection property and thus $\bigcap \mathcal{F} \neq \emptyset$. Now $|\bigcap \mathcal{F}| \geq 2$ as else there exists a continuous extension of f. So there exists $s, t \in \bigcap \mathcal{F}$ with $s \neq t$ and thus there exists $A, B \in \mathcal{Z}(Y)$ with $s \in A$, $t \in B$ and $A \cap B = \emptyset$ and so $p \in \overline{f^{-1}(A)} \cap \overline{f^{-1}(B)}$

$\square$

Definition 3.1.9. *Let X be a topological space.*

1. $C(X) = \{f : X \to \mathbb{R} : f$ *is continuous*$\}$.
2. $C^*(X) = \{f : X \to \mathbb{R} : f(X) \subset (-a, a)$ *for some* $a \in \mathbb{R}\}$.
3. *Let $A \subset X$ then A is C^*-embedded in X if for $f \in C^*(A)$ there exists a $g \in C^*(X)$ such that $f \subset g$.*

Lemma 3.1.10. *Let X be a dense subset of a compact space Y. The following statements are equivalent*

1. *For $f : X \to K$ with f continuous and K compact there is a continuous function $g : Y \to K$ such that $f \subset g$.*
2. *X is C^*-embedded in Y.*
3. *If $A, B \in \mathcal{Z}(X)$ and $A \cap B = \emptyset$ then $\overline{A} \cap \overline{B} = \emptyset$.*
4. *If $A, B \in \mathcal{Z}(X)$ then $\overline{A} \cap \overline{B} = \overline{A \cap B}$.*
5. *The family $\{A \in \mathcal{Z}(X) : p \in \overline{A}\}$ is a z-filter for $p \in Y$.*

Proof. **(1) $\Rightarrow$ (2)** This is essentially the definition of X being C^*-embedded in Y, as on a compact space $C(Y) \subset C^*(Y)$

(2) $\Rightarrow$ (3) Let $A, B \in \mathcal{Z}(X)$ with $A \cap B = \emptyset$. Now we can find a continuous function $f : X \to \mathbb{R}$ such that $A \subset f^{-1}(0)$ and $B \subset f^{-1}(1)$. However as X is C^*-embedded in Y there exists a continuous $g : Y \to \mathbb{R}$ with $f \subset g$. However then $g[\overline{A}] = \{0\}$ and $g[\overline{B}] = \{1\}$ and thus $\overline{A} \cap \overline{B} = \emptyset$

(3) $\Rightarrow$ (4) Now $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$ and suppose $p \notin \overline{A \cap B}$ then there exists a $f \in C(Y)$ such that $f(p) = 0$ and $f(\overline{A \cap B}) = \{1\}$ and thus $f(A \cap B) = \{1\}$. Now let $C = \{q \in Y : f(q) \leq \frac{1}{2}\}$ which is an open subset of Y as $f \in C(Y)$ and also $A \cap B \cap C = \emptyset$. Now $A \cap C \in \mathcal{Z}(X)$ as $A \in \mathcal{Z}(X)$ and A is a dense subset of Y. Similarly, $B \cap C \in \mathcal{Z}(X)$ and thus $\overline{A \cap C} \cap \overline{B \cap C} = \emptyset$ and thus $p \notin \overline{A} \cap \overline{B}$ and thus $\overline{A} \cap \overline{B} \subset \overline{A \cap B}$ so $\overline{A} \cap \overline{B} = \overline{A \cap B}$

(4) ⇒ (5) Let $A, B \in \mathcal{Z}(X)$ such that $p \in \overline{A}$ and $A \subset B$ then $p \in \overline{A} \subset \overline{B}$ and thus $B \in \{A \in \mathcal{Z}(X) : p \in \overline{A}\}$. Now let $C \in \mathcal{Z}(X)$ with $p \in \overline{A}$ then $p \in \overline{A} \cap \overline{C} = \overline{A \cap C}$ and $A \cap C \in \mathcal{Z}(X)$ and thus $A \cap C \in \{A \in \mathcal{Z}(X) : p \in \overline{A}\}$ and so $\{A \in \mathcal{Z}(X) : p \in \overline{A}\}$ is a z-filter on X.

(5) ⇒ (1) Now if we can find a continuous function $f : X \to K$ such that for any $g : Y \to K$, $f \not\subset g$ then there must exist a $p \in Y$ and $A, B \in \mathcal{Z}(K)$ such that $A \cap B = \emptyset$ and $p \in \overline{f^{-1}(A)} \cap \overline{f^{-1}(B)}$. But since $f^{-1}(A), f^{-1}(B) \in \mathcal{Z}(X)$ (since if $g \in C(K)$ such that $g(A) = \{0\}$ then $g \circ f \in C(X)$ and $g \circ f(f^{-1}(A)) \subset g(A) = \{0\}$) and $f^{-1}(A) \cap f^{-1}(B) = \emptyset$, the family $\{A \in \mathcal{Z}(X) : p \in \overline{A}\}$ cannot be a z-filter on X.

□

Theorem 3.1.11. *βX is the Stone-Čech compactification of X.*

Proof. By Lemma 3.1.7 the space βX is compact and $e[X]$ is a dense subset of βX. Let K be a compact, Hausdorff space and let $f : e[X] \to K$. Now let $p \in \beta X$ i.e. p is a z-ultrafilter on X and consider the family of subsets

$$\{A \in \mathcal{Z}(X) : p \in \overline{e[A]}\} = \{A \in \mathcal{Z}(X) : p \in \{q \in \beta X : A \in q\}\} = p$$

This shows that $\{A \in \mathcal{Z}(X) : p \in \overline{e[A]}\}$ is a z-filter on $e[X]$ and thus by Lemma 3.1.10 we can find $g \in C(\beta X)$ such that $f \subset g$. This implies that βX is the Stone-Čech compactification of $e[X]$ which is homeomorphic to X and thus βX is the Stone-Čech compactification of X. □

Lemma 3.1.12. *If A is a locally compact, dense subset of a Hausdorff space X then A is an open subset of X.*

Proof. For all $x \in A$ we can choose an open U_x such that $\overline{U_x} \cap A$ is compact. This implies that the set $A \cap U_x$ must be closed in the subspace topology of U_x which in turn means that $U_x \setminus (A \cap U_x)$ is open in the same subspace topology. This implies that

$$A = \bigcup_{x \in A} U_x \setminus \bigcup_{x \in A} U_x \setminus (A \cap U_x)$$

must be a closed subset relative to the subspace topology on $\bigcup_{x \in A} U_x$. But then $A = \overline{A} \cap \bigcup_{x \in A} U_x = X \cap \bigcup_{x \in A} U_x = \bigcup_{x \in A} U_x$ is open. □

Theorem 3.1.13. *If X is a locally compact, completely regular space then $\beta X \setminus e(X)$ is a compact subset of the Stone-Čech compactification of X .*

Proof. Since X is locally compact, we know that e(X) must also be locally compact. However, e(X) is also a dense subset of the Hausdorff space βX and thus $e(X)$ is open. This implies that $\beta X \setminus e(X)$ is closed and thus compact. $\square$

Definition 3.1.14. *Let $f : X \to K$ be a continuous function and K a compact Hausdorff space then the unique continuous function $\overline{f} : \beta X \to K$ such that $f \subset \overline{f}$ is the* Stone extension *of f.*

Corollary 3.1.15. *If X is a discrete topological space then $\beta X = \{p \subset 2^X : p$ is an ultrafilter on $X\}$ furthermore βX is a zero dimensional topological space.*

Proof. As X is a discrete topological space it is locally compact and completely regular and thus the Stone-Čech compactification exists. Let $A \subset X$ and define $f : X \to \mathbb{R}$ by $f(x) = 0$ if $x \in A$ and $f(x) = 1$ otherwise. Now as X is discrete this function is continuous and $f^{-1}(A) = \{0\}$ and thus A is a zero set. So any z-ultrafilter is in fact a standard ultrafilter on X and so $\beta X = \{p \subset 2^X : p$ is an ultrafilter on $X\}$. Now the family of sets

$$\mathcal{B} = \{\{p \in \beta X : A \notin p\} : A \subset X\}$$

forms a base for the compact Hausdorff topology on βX. Let $A \subset X$ and

$$U = \{p \in \beta X : A \notin p\}$$

then

$$\beta X \setminus U = \{p \in \beta X : A \in p\} = \{p \in \beta X : X \setminus A \notin p\} \in \mathcal{B}$$

as p is an ultrafilter on X. This implies that $\mathcal{B}$ is a clopen base for the compact topology on βX and thus is zero dimensional. $\square$

Definition 3.1.16. *A topological space X is strongly zero dimensional if βX is zero dimensional.*

Lemma 3.1.17. *Let X be a set and let $f : X \to X$ be a function such that $f(x) \neq x$ for all $x \in X$. There exist subsets A_0, A_1 and A_2 of X such that*

1. *$X = A_0 \cup A_1 \cup A_2$.*
2. *if $i \neq j$ then $A_i \cap A_j = \emptyset$.*
3. *$A_i \cap f[A_i] = \emptyset$.*

Proof. See [Kat67] $\square$

If X is a locally compact, Hausdorff topological space and $f : X \to X$ is a continuous function then we can extend the codomain of f to a compact, Hausdorff space by using the embedding $e : X \to \beta X$. The function $e \circ h : X \to \beta X$ is then a continuous function from X to a compact, Hausdorff space and thus has a Stone extension. We write $\overline{f}$ for the Stone extension of $e \circ h$.

Theorem 3.1.18. *Let X be a discrete topological space, $f : X \to X$ a function and let $p \in \beta X$. $\overline{f}(p) = p$ if and only if $\{x \in X : f(x) = x\} \in p$.*

Proof. Let $A = \{x \in X : f(x) = x\}$ and consider the set $e[A] \subset \beta X$. Now if $x \in A$ then $e(x) = \{C \subset X : x \in C\}$ and $\overline{f}(e(x)) = e(f(x)) = e(x)$. Now let $p \in \beta X$ and suppose $A \in p$ then $p \in \overline{e[A]}$ and since $\overline{f}$ is continuous we have that $\overline{f}(p) = p$.

Now if $A \notin p$ then $X \setminus A \in p$ as p is an ultrafilter on X and so $X \setminus A \neq \emptyset$ and if $x \in X \setminus A$ then $f(x) \neq x$. Now by Lemma 3.1.17 there exist subsets A_0, A_1 and A_2 of $X \setminus A$ such that

1. $X \setminus A = A_0 \cup A_1 \cup A_2$
2. if $i \neq j$ then $A_i \cap A_j = \emptyset$
3. $A_i \cap f[A_i] = \emptyset$

Now if $A_i \notin p$ for $i = 0, 1, 2$ then $X \setminus A_i \in p$ but then

$$\bigcap X \setminus A_i = X \setminus \bigcup A_i = X \ \setminus (X \setminus A) = A \in p$$

which is a contradiction and so with out loss of generality we may assume that $A_1 \in p$. If $A_2 \in p$ then $\emptyset = A_1 \cap A_2 \in p$ which is a contradiction, similarly $A_3 \notin p$. However, $p \in \overline{e[A_1]}$ and since $\overline{f}$ is continuous

$$\overline{f}(p) \in \overline{e[f[A_1]]} \subset \overline{e[A_2 \cup A_3]}$$

as $f[A_1] \cap A_1 = \emptyset$ and thus $\overline{f}(p) \neq p$ $\square$

We have used the more subtle concept of a z-ultrafilter to build the Stone-Čech compactification of an arbitrary completely regular topological space. We have also noted that z-ultrafilters are awfully close in behaviour to the traditional ultrafilter on a set. This suggests the question, when does the Stone-Čech compactification of an arbitrary completely regular topological space consist of the ultrafilters on that space.

Definition 3.1.19. *A point $x \in X$ is called isolated if $\{x\}$ is open in the topology of X.*

Definition 3.1.20. *A topological space X is called extremally disconnected if the closure of an open set is still open.*

These two topological properties are sufficient to characterize the types of topologies where the Stone-Čech compactification consists of ultrafilters on X.

Definition 3.1.21. *A compact Hausdorff spaceX is called projective if for all compact Hausdorff spaces Y and Z and every continuous onto function $f : Y \to Z$ and continuous function $g : X \to Z$ there exists a continuous function $h : X \to Y$ such that $g = f \circ h$.*

Theorem 3.1.22. *Any projective space X is extremally disconnected.*

Proof. Consider the product space $X \times \{0, 1\}$ where $\{0, 1\}$ is given the discrete topology. Being the finite product of compact spaces it is also compact, and thus the set

$$Y = (X \setminus G) \times \{0\} \cup \overline{G} \times \{1\}$$

is also compact, where $G \subset X$ is an arbitrary open subset. Also by construction the subset $\overline{G} \times \{1\}$ is open in the subset topology on Y. Define a continuous onto function $f : Y \to X$ by using the first projection $f(x, i) = x$ and another function $g : X \to X$ by $g(x) = x$. The projective property implies the existence of a function $h : X \to Y$ such that $g = f \circ h$. Now for $x \in G$ we have that $h(x) = (y,\text{i})$ for some $y \in (X \setminus G) \cup \overline{G}$ and $i \in \{0, 1\}$ but $f \circ h(x) = x$ and so $f((y, i)) = x \in G$ and thus $h(x) = (x, 1)$ for all $x \in G$. However since $\overline{G}$ is closed and h is continuous we must have that $h(x) = (x, 1)$ for all $x \in \overline{G}$ and so $\overline{G} = h^{-1}(\overline{G} \times \{1\})$ and so $\overline{G}$ is open □

Theorem 3.1.23. *Let X be a completely regular topological space and let βX be the Stone-Čech compactification of X. βX is extremally disconnected and contains a dense subset of isolated points if and only if the topology on X is discrete.*

Proof. If the topology on X is discrete then by 3.1.7 we know that $e(X)$ is a dense subset of points in βX. We also have that for all $x \in X$,

$$B_x = \{p \in \beta X : X \setminus \{x\} \notin p\} = \{p \in \beta X : \{x\} \in p\} = e(x)$$

is a basic open set and thus $e(X)$ is both dense and consists of isolated points. Now let Y and Z be compact Hausdorff spaces and let $f : Y \to Z$ be a continuous onto function and $g : \beta X \to Z$ be a arbitrary continuous function. Define a function $h : X \to Y$ by choosing

$$h(x) \in f^{-1}(g(e(x)))$$

which is trivially continuous as X is discrete. By the Stone-extension property 3.1.10 we can extend h uniquely to a continuous function $h : \beta X \to Y$ such that $g = f \circ h$ and so βX is projective and thus extremally disconnected. To prove the opposite implication we proceed as follows. Let A be the dense set of isolated points in βX and let $\hat{X}$ denote the set of all clopen subsets of βX. Define a map $f : 2^A \to \hat{X}$ by $f(B) = \overline{B}$ for $B \subset A$ and a map $g : \hat{X} \to 2^A$ by $g(Y) = Y \cap A$. f is well defined by the extremally disconnected property of the space βX and since A consists of isolated points we have that $f \circ g(Y) = Y$ and $g \circ f(B) = B$. Finally both maps are onto since A is dense in βX. This implies the existence of a bijection h from the set of all subsets of A and the set of all clopen subsets of βX. We can use this bijection to create a bijection from the set of all points of βX and the set of ultrafilters on the subset $A \subset \beta X$ as follows, let $x \in \beta X$ and define a filter base $\mathcal{F}_x$ on A by including all sets of the form $h(N)$ where N is a clopen neighbourhood of $x \in X$ and extend that filter base to an ultrafilter on A. Similarly every ultrafilter on A corresponds to a point of βX since the intersection of the closure of all subsets of the ultrafilter over a compact space must be a singleton. By the universal property of the Stone-Čech compactification we have shown that every z-ultrafilter on the base space X is in fact an ultrafilter on the homeomorphic subset A which is only possible if the original space X was in fact discrete. □

Definition 3.1.24. *A subset A of a topological space X is called σ-compact if it is a countable union of compact subsets of X.*

Being extremally disconnected is a useful property for the Stone-Čech compactification to have, since it allows us to control the growth of the closure of certain subsets.

Theorem 3.1.25. *Let A and B be σ-compact subsets of an extremally disconnected, compact Hausdorff space X. If $\overline{A} \cap B = A \cap \overline{B} = \emptyset$ then $\overline{A} \cap \overline{B} = \emptyset$.*

Proof. Let $A = \bigcup A_n$ and $B = \bigcup B_n$ where A_n and B_n are compact subsets of X. Since any compact Hausdorff space is normal we know that for all A_n and B_n

there must exist open subsets T_n,U_n, $V_n$and W_n such that

$$T_n \cap U_n = V_n \cap W_n = \emptyset$$

and $A_n \subset T_n$, $\overline{B} \subset U_n$, $\overline{A} \subset V_n$ and $B_n \subset W_n$. Define

$$C_n = \bigcap_{i=1}^{n} V_k \cap T_n$$

and

$$D_n = \bigcap_{i=1}^{n} U_k \cap W_n$$

and let $C = \bigcup C_n$ and $D = \bigcup D_n$. We can easily see that C and D are both open subsets of X and $C \cap D = \emptyset$ with $A \subset C$ and $B \subset D$. But since X is extremally disconnected we know that $\overline{C} \cap \overline{D} = \emptyset$ and thus $\overline{A} \cap \overline{B} = \emptyset$ □

Definition 3.1.26. *A topological space X is called totally separated if for all $x, y \in X$ with $x \neq y$ there exist open subsets U and V such that*

1. $x \in U$ *and* $y \in V$.
2. $X = U \cup V$.
3. $U \cap V = \emptyset$.

Theorem 3.1.27. *If X is an extremally disconnected Hausdorff space then X is totally separated.*

Proof. Since X is Hausdorff there exist disjoint open neighbourhoods U and V of x and y respectively. But since X is extremally disconnected we know that $\overline{U}$ and $X \setminus \overline{U}$ are also open disjoint neighbourhoods of x and y and thus X is totally separated □

We finally investigate the relationship between the closure of a z-filter on a topological space and the closure operator on βG if the completely regular topology satisfies some stronger properties.

Definition 3.1.28. *A topological space X satisfies Property C if every zero set of X is also a cozero set of X.*

Trivially every discrete space satisfies property C. Similarly every perfectly normal T_0 space which satisfies property C is discrete.

Definition 3.1.29. *Let X be a topological space:*

1. *A subset A is a G_δ set if there exists a countable family of open sets $\{O_i\}_{i\in\mathbb{N}}$ in X such that $A = \bigcap_{i\in\mathbb{N}} O_i$.*

2. *A subset A is a F_σ set if there exists a countable family of closed sets $\{C_i\}_{i\in\mathbb{N}}$ in X such that $A = \bigcup_{i\in\mathbb{N}} C_i$.*

Theorem 3.1.30. *Every zero set of a topological space X is a closed G_δ set. If X is normal then the converse is true.*

Proof. Let $A \subset X$ be a zero set then there exists a continuous function $f : X \to \mathbb{R}$ such that $f^{-1}(0) = A$. Since f is continuous we have that A is closed. Define

$$O_n = f^{-1}((\frac{-1}{n}, \frac{1}{n})) \subset X$$

and is open since $(\frac{-1}{n}, \frac{1}{n})$ is open in $\mathbb{R}$ we have that

$$A = \bigcap_{i\in\mathbb{N}} f^{-1}([0, \frac{1}{n})) = \bigcap_{i\in\mathbb{N}} O_n$$

since if $x \in \bigcap_{i\in\mathbb{N}} f^{-1}((\frac{-1}{n}, \frac{1}{n}))$ then $\frac{-1}{n} < f(x) < \frac{1}{n}$ for all $n \in \mathbb{N}$ and thus A is a G_δ set. Now suppose X is a normal space and that $A = \bigcap_{i\in\mathbb{N}} O_n$ is a closed G_δ space. Let $A_n = X \setminus \bigcap_{i\leq n} O_i$ which is obviously also a closed subset of X and $A \cap A_n = \emptyset$. Since X is normal we can find a family of continuous functions $f_n : X \to [0, 1]$ such that $f_n(A) = 0$ and $f_n(X \setminus A_n) = 1$. Define $f : X \to \mathbb{R}$ by

$$f(x) = \sum_{i\in\mathbb{N}} \frac{f_n(x)}{2^n}$$

then $x \in f^{-1}(0)$ implies that $f_n(x) = 0$ for all $n \in \mathbb{N}$ which implies that $x \in A$ and thus $f^{-1}(0) = A$ $\square$

Corollary 3.1.31. *If X is a normal space then a subset is cozero if and only if it is open F_σ.*

Proof. Since the cozero sets are complements of the zero sets and the complement of an open set is a closed one we have this result $\square$

Let X be a completely regular space that satisfies property C such that every singleton $\{x\}$ is a zero set then $X \setminus \{x\}$ is also a zero set and so $\{x\}$ is open which implies that the original topology had to be discrete.

Definition 3.1.32. *The cantor cube $C(\lambda) = \{0,1\}^\lambda$ for all ordinals λ is a topological space where $\{0,1\}$ is a discrete space and $C(\lambda)$ is endowed with the product topology.*

Since $\{0,1\}$ is compact and by Tychonoff's theorem the product of compact spaces is compact we have that $C(\lambda)$ is compact Hausdorff and thus normal. However, for $\lambda \geq \omega_2$ we have that the singletons of $C(\lambda)$ cannot be G_δ sets. Furthermore, since $\{0,1\}$ is zero-dimensional we have that $C(\lambda)$ is zero-dimensional. In fact every compact, zero-dimensional group is homeomorphic to some Cantor Cube (See [Shc81]). This suggests the existence of spaces that satisfy property C but are not discrete.

Theorem 3.1.33. *Let X be a topological space which satisfies property C and let p be a z-ultrafilter on X. If $B \notin p$ then $X \setminus B \in p$.*

Proof. We know that since $B \in \mathcal{Z}(X)$ that $X \setminus B \in \mathcal{Z}(X)$. Now suppose that $B \notin p$ then there exists $A \in p$ such that $A \cap B = \emptyset$ which implies that $A \subset X \setminus B$ and thus $X \setminus B \in p$ $\square$

Corollary 3.1.34. *If X is a topological space which satisfies property C then X is strongly zero dimensional.*

Proof. We know that if $U \in \mathcal{Z}(X)$ then $X \setminus U \in \mathcal{Z}(X)$ and then $\overline{U} = \{p \in \beta X : U \notin p\}$ is a basic open set. But then $\beta X \setminus \overline{U} = \{p \in \beta X : X \setminus U \notin p\}$ is also a basic open set and thus βX is zero-dimensional $\square$

We now assume that X has been given the discrete topology.

Definition 3.1.35. *Let $\mathcal{A} \subset 2^X$ be a filter on X and define $\overline{\mathcal{A}} = \bigcap_{A \in \mathcal{A}} \overline{e[A]}$.*

Theorem 3.1.36. *For every filter $\mathcal{A}$ of X the set $\overline{\mathcal{A}}$ is a non-empty closed subset of βX and $\overline{\mathcal{A}} = \{p \in \beta X : \mathcal{A} \subset p\}$ further more every closed non-empty subset of βX is of this form.*

Proof. Since βX is compact and any filter has the finite intersection property we know that $\overline{\mathcal{A}}$ must be closed and non-empty. Now

$$\overline{e[A]} = \{p \in \beta X : A \in p\}$$

for $A \in \mathcal{Z}(X)$ and so

$$\overline{\mathcal{A}} = \bigcap_{A \in \mathcal{A}} \overline{e[A]} = \bigcap_{A \in \mathcal{A}} \{p \in \beta X : A \in p\} = \{p \in \beta X : \mathcal{A} \subset p\}$$

Let Z be a closed subset of βX and consider the filter

$$\mathcal{F} = \{U \in \mathcal{Z}(X) : Z \subset \overline{U}\}$$

then $\overline{\mathcal{F}} = \{p \in \beta X : \mathcal{F} \subset p\}$ but then

$$p \in Z \subset \overline{U} = \{p \in \beta X : U \in p\}$$

for all $U \in \mathcal{F}$ and thus $p \in \overline{\mathcal{F}}$. So assume there exists $p \in \overline{\mathcal{F}}$ such that $p \notin Z$ then there $A \in p$ such that $\overline{A} \cap Z = \emptyset$ but then for all $q \in Z$ we have that $X \setminus A \in q$ which implies $X \setminus A \in \mathcal{F} \subset p$ which is a contradiction $\square$

There is a relationship between the filter closure operator $\overline{\mathcal{A}} = \bigcap_{A \in \mathcal{A}} \overline{e[A]}$ and the closure operator on βX.

Corollary 3.1.37. *For every subset A of βX we have that $\overline{\mathcal{F}} = \overline{A}$ where*

$$\mathcal{F} = \bigcap \{p \in \beta G : p \in A\}$$

Proof. Since $\{X\} \in p$ for all ultrafilters on X we know that $\mathcal{F} \neq \emptyset$. It is easy to see then that $\mathcal{F}$ must be a filter on the space X. By the previous result we also know that $\overline{\mathcal{F}}$ is a closed subset of βX. Let $p \in \overline{\mathcal{F}}$ then $\mathcal{F} \subset p$ and let $B \subset X$ with $B \in p$. We know that $p \in \overline{B}$ and that $\overline{B}$ is a basic open set of βX. Now suppose that for all $r \in A$ we have that $B \notin r$. Then we must have that $X \setminus B \in r$ for all $r \in A$ and thus $X \setminus B \in \mathcal{F}$. But then $B \in p$ and $X \setminus B \in p$ which is a contradiction. Thus there exists $r \in A$ with $B \in r$ and so $r \in \overline{B}$ and thus $\overline{B} \cap A \neq \emptyset$ for arbitrary basic open neighbourhoods of p i.e. $\overline{F} \subset \overline{A}$. Now let $p \in A$ then we know by construction that $\mathcal{F} \subset p$ and so $p \in \overline{\mathcal{F}}$. This implies that $A \subset \overline{\mathcal{F}}$ and thus $\overline{A} \subset \overline{\mathcal{F}}$ $\square$

We can give explicit constructions of the generating filters for the filter closure operator.

Theorem 3.1.38. *Let $\{\mathcal{F}_\alpha\}_{\alpha \in \mathcal{A}}$ be a family of filters on X such that $\bigcap \overline{\mathcal{F}_\alpha} \neq \emptyset$ and let $\mathcal{F}$ be the filter generated by $\bigcup \mathcal{F}_\alpha$. Then*

$$\bigcap_{\alpha \in \mathcal{A}} \overline{\mathcal{F}_\alpha} = \overline{\mathcal{F}}$$

Proof. In general, the arbitrary union of a family of filters does not have the finite intersection property (just consider the two filters generated by A and $X \setminus A$) so we first need to prove that $\mathcal{F}$ is well defined. Let $p \in \bigcap \overline{\mathcal{F}_\alpha} \neq \emptyset$ then $p \in \overline{\mathcal{F}_\alpha}$ for all $\alpha \in \mathcal{A}$ i.e. $\mathcal{F}_\alpha \subset p$ for all $\alpha \in \mathcal{A}$. Let

$$\{U_i\}_{i \leq n} \in \bigcup \mathcal{F}_\alpha$$

then $U_i \in \mathcal{F}_{\alpha_i} \subset p$ for all $i \leq n$ and so

$$\bigcap_{i \leq n} U_i \neq \emptyset$$

Thus $\bigcup \mathcal{F}_\alpha$ has the finite intersection property and $\mathcal{F}$ is well defined. This also shows that $p \in \overline{\mathcal{F}}$ since $\mathcal{F}$ is the smallest filter containing the family $\mathcal{F}_\alpha$ and $\mathcal{F}_\alpha \subset p$ for all $\alpha \in \mathcal{A}$. So let $p \in \overline{\mathcal{F}}$ then $\mathcal{F} \subset p$ but for all $\alpha \in \mathcal{A}$ and $A \in \mathcal{F}_\alpha$ we have that

$$A \in \mathcal{F}_\alpha \subset \mathcal{F} \subset p$$

which implies that $\mathcal{F}_\alpha \subset p$ for all $\alpha \in \mathcal{A}$ and thus $p \in \bigcap \overline{\mathcal{F}_\alpha}$ □

Lemma 3.1.39. *Let $\{\mathcal{F}_\alpha\}_{i \leq n}$ be a finite family of filters on X and let p be an ultrafilter on such $\bigcap_{i=1}^n \mathcal{F}_i \subset p$. There exists $i \leq n$ such that $\mathcal{F}_i \subset p$.*

Proof. Suppose that for all $i \leq n$ there exists $B_i \in \mathcal{F}_i$ such that $B_i \notin p$ then since p is an ultrafilter we know that $X \setminus B_i \in p$ and thus

$$B = \bigcap_{i \leq n} (X \setminus B_i) = X \setminus \bigcup_{i \leq n} B_i \in p$$

But $\hat{B} = \bigcup_{j \leq n} B_j \in \mathcal{F}_i$ for all $i \leq n$ and so $\hat{B} \in \bigcap_{i=1}^n \mathcal{F}_i \subset p$. So $\emptyset = B \cap \hat{B} \in p$ which is a contradiction □

Theorem 3.1.40. *If $\{\mathcal{F}_i\}_{i \leq n}$ is a finite family of filters on X then*

$$\bigcup_{i \leq n} \overline{\mathcal{F}_i} = \overline{\bigcap_{i \leq n} \mathcal{F}_i}$$

Proof. Let $p \in \bigcup_{i \leq n} \overline{\mathcal{F}_i}$ then $\mathcal{F}_j \subset p$ for some $j \leq n$. But

$$\bigcap_{i \leq n} \mathcal{F}_i \subset \mathcal{F}_j \subset p$$

and so $p \in \overline{\bigcap_{i \leq n} \mathcal{F}_i}$. Now suppose $p \in \overline{\bigcap_{i \leq n} \mathcal{F}_i}$ then $\bigcap_{i \leq n} \mathcal{F}_i \subset p$ but we know that there must exist some $j \leq n$ such that $\mathcal{F}_j \subset p$ since p is an ultrafilter thus

$$p \in \overline{\mathcal{F}_j} \subset \bigcup_{i \leq n} \overline{\mathcal{F}_i}$$

$\square$

Corollary 3.1.41. *Let $\{\mathcal{F}_\alpha\}$ be a family of filters on X. Then if for all ultrafilters p on X with $\bigcap \mathcal{F}_\alpha \subset p$ we can find an index α such that $\mathcal{F}_\alpha \subset p$ then $\bigcup \overline{\mathcal{F}_\alpha}$ is closed.*

Proof. If we follow the previous proof we see that this property is required to prove that $\overline{\bigcap \mathcal{F}_\alpha} \subset \bigcup \overline{\mathcal{F}_\alpha}$ which shows that $\bigcup \overline{\mathcal{F}_\alpha}$ is closed. $\square$

3.2 The semigroup βS

P-limits are the third way to study the idea of convergence in a general topological space, the first two being that of filters and that of nets. Recall that an ultrafilter encodes in some sense a choice of which subsets of a topological space X are large. The p-limit of a sequence converges to a point in the base space X if the subset of the index space where the sequence is close to the limit point is large. We can formalize this idea as follows.

Definition 3.2.1. *Let D be a set and p an ultrafilter on D. Let $\langle x_s \rangle_{s\in D}$ be an indexed family in some topological space X and let $y \in X$. We say that $p-lim_{s\in D} x_s := p-lim \langle x_s \rangle_{s\in D} := y$ if and only if for every neighbourhood U of y we have that $\{s \in D | x_s \in U\} \in p$.*

Theorem 3.2.2. *Let D be a set and let p be an ultrafilter on D. Let $\langle x_s \rangle_{s\in D}$ be an indexed family in a topological space X.*

1. *If X is Hausdorff and $p-lim \langle x_s \rangle_{s\in D}$ exists then it is unique.*
2. *If X is compact then $p-lim \langle x_s \rangle_{s\in D}$ exists.*
3. *Let X and Y be Hausdorff topological spaces and let $f : X \to Y$ be a continuous function. If $p-lim \langle x_s \rangle_{s\in D}$ exists then $f(p-lim \langle x_s \rangle_{s\in D})) = p-lim \langle f(x_s) \rangle_{s\in D}$.*

Proof. 1. Suppose $p-lim \langle x_s \rangle_{s\in D} = y_1$ and $p-lim \langle x_s \rangle_{s\in D} = y_2$ with $y_1 \neq y_2$. Now as X is Hausdorff, there exist open neighbourhoods of y_1 and y_2, say V and W respectively, such that $V \cap W = \emptyset$. But then $\{s \in D | x_s \in V\} \in p$ and $\{s \in D | x_s \in W\} \in p$ and so $\{s \in D | x_s \in V\} \cap \{s \in D | x_s \in W\} \neq \emptyset$, since p is a filter on D. This shows that $x_s \in V \cap W$ for some $s \in D$, which is a contradiction.

2. Now suppose that $p-lim \langle x_s \rangle_{s\in D}$ does not exist. For each $y \in X$ choose an open neighbourhood U_y of y such that $\{s \in D | x_s \in U_y\} \notin p$. Since the family $\{U_y\}_{y\in X}$ forms an open cover of the compact space X, we can choose a finite subcover $\{U_{y_n}\}_{n=1...m}$. However $D = \bigcup_{n=1...m} \{s \in D | x_s \in U_{y_n}\}$ and so we can find a $n \in \{1...m\}$ such that $\{s \in D | x_s \in U_{y_n}\} \in p$ as p is an ultrafilter. This contradicts the choice of U_{y_n}.

3. Let V be a neighbourhood of $f(p-lim \langle x_s \rangle_{s\in D})$ then there exists a neighbourhood $W \ni p-lim \langle x_s \rangle_{s\in D}$ such that $f(W) \subset V$ as f is continuous.

Now $\{s \in D | x_s \in W\} \in p$ and $\{s \in D | x_s \in W\} \subset \{s \in D | f(x_s) \in V\}$ so $\{s \in D | f(x_s) \in V\} \in p$

□

P-limits are useful to investigate other properties of a topological space as well.

Definition 3.2.3. *A topological space X is countably compact if every countable open cover of X has a finite sub cover.*

Not every countably compact space is a compact space, with the order topology on the first uncountable ordinal c being the standard textbook example of a countably compact space which is not compact. This is not a compact space as the open cover $\{(0, \alpha) : \alpha < c\}$ admits no finite sub cover (as the finite union of countable sets is countable). Since countably compact spaces are a generalization of compact spaces, the question of a Tychonoff type product theorem is an interesting one. The following theorem, stated without proof gives a characterisation of the product of countably compact spaces.

Theorem 3.2.4. *Let $\{X_\alpha : \alpha \in I\}$ be a family of spaces. Then the product space $X = \prod\{X_\alpha : \alpha \in I\}$ is countably compact if and only if for every sequence $f : \mathbb{N} \to X$, p–$\lim_{n\in\mathbb{N}} \pi_\alpha \circ f \in X_\alpha$ for all $\alpha \in I$.*

Proof. See [Ber70] □

We can use the Stone extension theorem to show that the binary operation of a group extends continuously to the entire compactification of that group. However, proving that the resulting operation is associative is rather difficult. Instead of journeying down that route we consider the family of p-limits on the space.

Definition 3.2.5. *Let S be a semigroup with binary operation $\cdot : S \times S \to S$. Define a binary operation $\cdot : \beta S \times \beta S \to \beta S$ as*

$$p \cdot q = p\text{–}lim_{s\in S} \left(q\text{–}lim_{t\in S} e\left(s \cdot t\right)\right)$$

where $p \in \beta S$ and $q \in \beta S$.

Lemma 3.2.6. *Let D be a set, X a compact Hausdorff space and $(x_s)_{s\in D}$ be a D indexed subset of X. Let p be an ultrafilter on D and let $U \subset X$ be a subset of X such that $\{s \in D : x_s \in U\} \in p$. Then $p\text{–}lim_{s\in D} x_s \in cl(U)$.*

Proof. Suppose that $p\text{–}lim_{s\in D} x_s \notin U$ then for all neighbourhoods $V \subset X$ of $p\text{–}lim_{s\in D} x_s$ we have that $\{s \in D : x_s \in V\} \in p$. But $\{s \in D : x_s \in U\} \in p$ and so $\{s \in D : x_s \in U \cap V\} \in p$ which implies that $U \cap V \neq \emptyset$ for all neighbourhoods $V \ni p\text{–}lim_{s\in D} x_s$ i.e. $p\text{–}lim_{s\in D} x_s \in cl(U)$ □

Theorem 3.2.7. *Let D be a discrete space, X a compact Hausdorff space and $(x_s)_{s\in D}$ be a D indexed subset of X. Then the function $f : \beta D \to X$ defined by $f(p) = p\text{–}lim_{s\in D} x_s$ is continuous.*

Proof. Since the space X is compact and Hausdorff, we have that f is well defined by Theorem 3.2.2 and that X is in fact T_3. All that is required is to show that $f : \beta D \to X$ is in fact continuous. So let $V \subset X$ be an open neighbourhood of $f(p)$ for some fixed $p \in \beta D$. Since X is T_3 there exists an open $U \subset X$ such that $f(p) \in U \subset \overline{U} \subset V$. Now the set $A = \{s \in D : x_s \in U\} \in p$ by the definition of a p-limit and thus $p \in \overline{A}$, which is an open subset of βD. But then $f\left(\overline{A}\right) \subset V$ as if $q \in \overline{A}$ then $A \in q$ i.e. $\{s \in D : x_s \in U\} \in q$ which by Lemma 3.2.6 implies that $f(q) \in cl(U) \subset V$. □

Corollary 3.2.8. *Let S be a semigroup endowed with the discrete topology. The extension of the operator to βS is right continuous on βS.*

Proof. Take X to be βS and let $q \in \beta S$. Define

$$\rho_q(p) = f(p) = p \cdot q = p\text{–}lim_{s\in S} x_s$$

where $x_s = q\text{–}lim_{t\in S} s \cdot t$. Then by Theorem 3.2.7, $f : \beta S \to \beta S$ is continuous. □

Lemma 3.2.9. *Let S be a semigroup and let X be a Hausdorff space with $< x_s >_{s\in S}$ an indexed family of elements of X. Let $p, q \in \beta S$. Then if all the limits involved exist then $(p \cdot q)\text{–}\lim_{v\in S} x_v = p\text{–}lim_{s\in S}(q\text{–}lim_{t\in S} x_{s\cdot t})$.*

Proof. Let $z = (p \cdot q)\text{–}\lim_{v\in S} x_v$ and for all $s \in S$ let $y_s = (p - lim x_{s\cdot t})$. Now if $q\text{–}\lim_{s\in S} y_s \neq z$ we can find two disjoint open neighbourhoods of z and $q\text{–}\lim_{s\in S} y_s$, say U and V respectively. Let

$$A = \{v \in S : x_v \in V\}$$

and

$$B = \{s \in S : y_s \in U\}$$

Then it can be easily shown that $A \in p \cdot q$ and $B \in p$. Let $C = \{s \in S : s \cdot p \in \overline{A}\}$ but $\overline{A}$ is a neighbourhood of $q\text{–}\lim_{v\in S} s \cdot p$ and so $C \in p$. But then there exists a

$s \in B \cap C$. Since $s \in B$ we know that $p - lim x_{s \cdot t} \in U$ so let

$$D = \{t \in S : x_{s \cdot t} \in U\} \in q$$

Also since $s \in C$ we know that $s \cdot p \in \overline{A}$ and so $\overline{A}$ is a neighbourhood of $p - lim s \cdot t$. Define

$$E = \{t \in S : s \cdot t \in \overline{A}\}$$

We know that $\emptyset \neq D \cap E \in p$ so choose $t \in D \cap E$. Since $t \in D$ we know that $x_{s \cdot t} \in U$. But then $s \cdot t \in A$ since $t \in E$ and so $x_{s \cdot t} \in V$ which implies that $U \cap V \neq \emptyset$. □

We can now prove the final theorem, that if S is a discrete topological semigroup then it is possible to extend the operation on S to the Stone-Čech compactification of S, βS in such a way that βS is a right shift continuous, zero dimensional, totally and extremally separated semigroup where the isolated points are exactly the original semigroup S.

Theorem 3.2.10. *Let S be a discrete topological semigroup. Then it is possible to uniquely extend the operation on S in such a way that βS is a right shift continuous, compact, Hausdorff, zero dimensional, totally and extremally separated topological semigroup such that the isolated points of βS are S.*

Proof. We have already shown that βS is a right shift continuous, compact and Hausdorff, zero dimensional, totally and extremally separated space if the starting topology on S is discrete. All that is required is to show that the extended shift operation is in fact associative. For this we use Lemma 3.2.9. For starters, since βS is compact we know that all $p - lim'$s exist by Theorem 3.2.2. Now let p, q and $r \in \beta S$ then

$$\begin{aligned} p \cdot (q \cdot r) &= p - \lim_{s \in S}((q \cdot r) - lim_{w \in S} s \cdot w) \\ &= p - \lim_{s \in S}(q - \lim_{t \in S}(r - lim_{v \in S} s \cdot (t \cdot v))) \\ &= p - \lim_{s \in S}(q - \lim_{t \in S}(r - lim_{v \in S}(s \cdot t) \cdot v)) \\ &= (p \cdot q) \cdot r \end{aligned}$$

The uniqueness of the extension comes from 3.1.23 and 3.1.10 and the universal property of the Stone-Čech compactification □

It is possible to give a more concrete description of the construction of the semigroup operator.

Definition 3.2.11. *For all $A \subset S$ and $q \in \beta S$, let $\Omega_q(A) = \{x \in S : x^{-1}A \in q\}$ where $x^{-1}A = \{y \in S : x \cdot y \in A\}$.*

Theorem 3.2.12. *If S is a discrete semigroup and $p, q \in \beta S$ then*

$$p\text{--}\lim_{s\in S}(q\text{--}\lim_{t\in S}(s \cdot t)) = p \cdot q = \{A \subset S : \Omega_q(A) \in p\}$$

Proof. Let $U \subset S$ such that $\{A \subset S : \Omega_q(A) \in p\} \subset \overline{U}$. This implies that:

$$\begin{aligned} & \Omega_q(U) \in p \\ \Leftrightarrow \quad & \{s : s^{-1}U \in q\} \in p \\ \Leftrightarrow \quad & \{s : \{t : s \cdot t \in U\} \in q\} \in p \end{aligned}$$

Now let $\alpha(s)$ denote the ultrafilter generated by the set $\{s \cdot B : B \in p\}$. We claim that $p\text{--}\lim_{t\in S} s \cdot t = \alpha(s)$. For suppose $\alpha(s) \in \overline{V}$ i.e. $V \in \alpha(s)$ then $V \cap (s \cdot B) \neq \emptyset$ for all $B \in p$ which implies that $\{t \in S : s \cdot t \in V\} \cap B \neq \emptyset$ for all $B \in p$ and thus $\{t \in S : s \cdot t \in V\} \in p$. This is sufficient to prove the first claim as well as implying that

$$\{s : p\text{--}\lim_{t\in S} s \cdot t \in \overline{V}\} = \{s : V \in \alpha(s)\}$$

Now let $s \in \{s : \{t : s \cdot t \in V\} \in q\}$ i.e. $\{t : s \cdot t \in V\} \in q$ and suppose that $V \notin \alpha(s)$ then as $\alpha(s)$ is an ultrafilter we have that $X \setminus V \in \alpha(s)$. This implies that $(X \setminus V) \cap s \cdot B \neq \emptyset$ for all $B \in q$. However, $\{t : s \cdot t \in V\} \in q$ and so $(X \setminus V) \cap s \cdot \{t : s \cdot t \in V\} \subset (X \setminus V) \cap V = \emptyset$ which is a contradiction. So

$$\{s : \{t : s \cdot t \in U\} \in q\} \subset \{s : U \in \alpha(s)\}$$

which implies that

$$\{s : p\text{--}\lim_{t\in S} s \cdot t \in \overline{U}\} \in p$$

$\square$

Theorem 3.2.13. *If S is a discrete semigroup with $A \subset S$ and $p, q \in \beta S$ then $A \in p \cdot q$ if and only if there exists a $U \in p$ and a family of sets $V_x \in q$ such that $\bigcup_{x\in U} x \cdot V_x \subset A$.*

Proof. We know that

$$p \cdot q = \{A \subset S : \Omega_q(A) \in p\}$$

so if $A \in p \cdot q$ then $\Omega_q(A) \in p$ so there exists a $U \in p$ such that

$$U \subset \Omega_q(A) = \{x : x^{-1}A \in q\}$$

but then for each $x \in U$ there exists a $V_x \in q$ such that

$$V_x \subset x^{-1}A \implies x \cdot V_x \subset A \implies \bigcup_{x \in U} x \cdot V_x \subset A$$

So suppose that we can find

$$\bigcup_{x \in U} x \cdot V_x \subset A \implies x \cdot V_x \subset A$$

for all $x \in U$ and so

$$V_x \subset x^{-1}A \implies x^{-1}A \in q \implies U \subset \Omega_q(A) \implies \Omega_q(A) \in p \implies A \in p \cdot q$$

$\square$

The Stone-Čech compactification of a completely regular topological semigroup allows us to extend the semigroup operations on the space continuously to the entire compactification. The space βS is quite rich both algebraically and topologically if S has the discrete topology. For example even though $\mathbb{N}^*$ contains many topological copies of $\beta\mathbb{N}$ it does not contain any algebraic copies ([Str92]). We now look into the question of which other group or semigroup topologies, other than the discrete topology, on the space S would preserve the algebraic structure when extended to the compact space βS. Recalling the left and right translations on the semigroup λ_α and ρ_α, it is easy to see that they satisfy the semigroup identity $\lambda_\alpha(\beta) = \rho_\beta(\alpha)$ for all $\alpha, \beta \in S$. This trivial identity is important as we would require it to hold for the Stone extensions $\overline{\lambda_\alpha} : \beta S \to \beta S$ and $\overline{\rho_\beta} : \beta S \to \beta S$ if we wish to preserve the semigroup operator on the space βS. It is possible to give a partial answer to this question.

Definition 3.2.14. *A semigroup S is said to satisfy the β-extension property if the resulting Stone extensions $\overline{\lambda_\alpha} : \beta S \to \beta S$ and $\overline{\rho_\beta} : \beta S \to \beta S$ satisfy $\overline{\lambda_\alpha}(\beta) = \overline{\rho_\beta}(\alpha)$ for all $\alpha \in S$ and $\beta \in \beta S$.*

Quite surprisingly, one of the simpler groups available to study is not β-extendable.

Theorem 3.2.15. *The additive group $\mathbb{R}$ with the normal topology on $\mathbb{R}$ does not satisfy the β-extension property.*

Proof. See [But75] $\square$

There exists a partial characterization of the topologies that are β-extendable for topological groups G.

Theorem 3.2.16. *A locally compact T_0 topological group G satisfies the β-extension property if and only if it is either compact or discrete.*

Proof. We have shown previously that any discrete topological semigroup satisfies the β-extension theorem and trivially since $\beta G = G$ for any compact group we trivially have that a compact group satisfies the β-extension property. So let us suppose that G is not discrete nor is compact and let U be a compact, symmetric neighbourhood of the identity $e \in G$. Define a sequence of points $\{x_n\}_{n\in\mathbb{N}}$ as follows:

1. Let $x_1 = e$

2. Suppose that $\{x_i\}_{i<n}$ have been chosen such that the family $\{U \cdot x_i\}_{i<n}$ is pairwise disjoint. Now since the continuous image of a compact space is compact and the finite union of compact sets is compact and$G$ is not a compact space there must exit $x_n \notin \bigcup_{i<n} U^{2\cdot} x_i$. We claim that $\{U \cdot x_i\}_{i\leq n}$ is still a pairwise disjoint family. For if $p \in U \cdot x_n \cap U \cdot x_i$ for some $i < n$ then $p = u \cdot x_n = v \cdot x_i$ for some $u, v \in U$ but then $x_n = u^{-1} \cdot v \cdot x_i \in U^2 \cdot x_i$ since U is a symmetric neighbourhood.

Now there must exist a $x \in \beta G$ such that the sequence $\{x_n\}_{n\in\mathbb{N}}$ has x as a cluster point since βG is a compact space. Suppose $x \in G$ then there exists $\mathcal{A} \subset \mathbb{N}$ such that $\{x_\alpha\}_{\alpha\in\mathcal{A}} \subset U \cdot x$. But then $x_\alpha = u \cdot x$ and $x_\beta = v \cdot x$ for some $\alpha, \beta \in \mathcal{A}$ which implies that $u^{-1} \cdot x_\alpha = v^{-1} \cdot x_\beta$ i.e.

$$U \cdot x_\alpha \cap U \cdot x_\beta \neq \emptyset$$

and so $x \in \beta G \setminus G$. Let $\{U_n\}_{n\in\mathbb{N}}$ be a family of open neighbourhoods of the identity e and $\{s_n\}_{n\in\mathbb{N}}$ be a sequence of points constructed as follows:

1. Let U_1 be an open neighbourhood of e such that $U_1^2 \subset U$ and let $s_1 \in U_1 \setminus \{e\}$. Now we can find U_2 an open neighbourhood of e such that $s_1 \notin U_2$ but $\overline{U_2} \subset U_1$

2. Suppose that the sequence $\{s_i\}_{i<n}$ and $\{U_i\}_{i\leq n}$ has been chosen such that $s_i \in U_i \setminus \{e\}$ and $s_i \notin U_{i+1}$ and $\overline{U_{i+1}} \subset U_i$ then we can a $s_n \in U_n \setminus \{e\}$ and U_{n+1} an open neighbourhood of e such that $s_n \notin U_{n+1}$ but $\overline{U_{n+1}} \subset U_n$

The sequence $\{s_n\}_{n\in\mathbb{N}}$ is contained in the compact set U by construction and thus has a cluster point $s \in \bigcap_{n\in\mathbb{N}} U_n \neq \emptyset$. Now since any T_0 group is in fact completely regular we can find a family of continuous functions

$$\{\phi_n : G \to [0,1]\}_{n\in\mathbb{N}}$$

such that $\phi_n(s) = 1$ but $\phi_n(G \setminus U_n) = 0$ for all $n \in \mathbb{N}$. Define a function $f : G \to [0,1]$ as

$$f(y) = \sum_{n=1}^{\infty} \phi_n(y \cdot x_n^{-1})$$

- $f : G \to [0,1]$ is well defined since if $\phi_n(y \cdot x_n^{-1}) \neq 0$ implies that $y \cdot x_n^{-1} \in U_n$ and thus $y \in U_n \cdot x_n \subset U \cdot x_n$ which is a pairwise disjoint family of subsets of G.

- $f : G \to [0,1]$ is continuous since if $y \in U \cdot x_n$ for some $n \in \mathbb{N}$ then $f(y) = \phi_n(u)$ where $y = u \cdot x_n$. Let $\epsilon > 0$ and choose a neighbourhood $V \ni u$ such that $V \subset U$ and $|\phi_n(u) - \phi_n(v)| < \epsilon$. Then $V \cdot x_n$ is a neighbourhood of y and $|f(y) - f(t)| < \epsilon$ for all $t \in V \cdot x_n$. If $y \notin \bigcup_{n\in\mathbb{N}} U \cdot x_n$ then

$$U_1 \cdot y \cap U_n \cdot x_n = \emptyset$$

  for all $n \in \mathbb{N}$ and so $|f(y) - f(t)| = 0 < \epsilon$ for all $t \in U_1 y$

Now let $\{x_{n_\alpha}\}_{\alpha\in\mathbb{N}}$ be a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ such that $\{x_{n_\alpha}\}_{\alpha\in\mathbb{N}}$ converges to $x \in \beta G$. Now

$$f(\overline{\lambda_{s_n}}(x)) = \lim_{\alpha\to\infty} f(\overline{\lambda_{s_n}}(x_{n_\alpha})) = \lim_{\alpha\to\infty} f(s_n x_{n_\alpha}) = \lim_{\alpha\to\infty} \phi_{n_\alpha}(s_n x_{n_\alpha} x_{n_\alpha}^{-1}) = \lim_{\alpha\to\infty} \phi_{n_\alpha}(s_n) = 0$$

but we also have that

$$f(\overline{\rho_{x_{n_\alpha}}}(s)) = f(s x_{n_a}) = \phi_n(s) = 1$$

Let $\{s_{n_\alpha}\}_{\alpha\in\mathbb{N}}$ be a subsequence of $\{s_n\}_{n\in\mathbb{N}}$ such that $\{s_{n_\alpha}\}_{\alpha\in\mathbb{N}}$ converges to $s \in G$.

Now since f is a continuous function if we have that $\overline{\lambda_\alpha}(\beta) = \overline{\rho_\beta}(\alpha)$ then

$$\begin{aligned}
1 &= \lim_{\alpha\to\infty} f(\overline{\rho_{x_{n_\alpha}}}(s)) = \lim_{\alpha\to\infty} f(\overline{\lambda_s}(x_{n_\alpha})) \\
&= f(\overline{\lambda_s}(x)) = f(\overline{\rho_x}(s)) \\
&= \lim_{\alpha\to\infty} f(\overline{\rho_x}(s_{n_\alpha}) = \lim_{\alpha\to\infty} f(\overline{\lambda_{s_{n_\alpha}}}(x)) \\
&= 0
\end{aligned}$$

Thus G does not satisfy the β-extension property. □

Definition 3.2.17. *A topological semigroup G is totally bounded if for every open neighbourhood U of the identity e there exists a finite set $F \subset G$ such that $G = F \cdot U$.*

Theorem 3.2.18. *A totally bounded topological group G has the β-extension property if and only if it is pseudocompact.*

Proof. See [FV+10] □

3.3 Ideals and Identities of βS

Definition 3.3.1. *Let I be a subset of a semigroup S.*

1. *I is a left ideal of S if $S \cdot I \subset I$.*

2. *I is a right ideal of S if $I \cdot S \subset I$.*

3. *I is an ideal of S if it is both a left ideal and a right ideal.*

Ideals form one of the basic structures when investigating the structure of an arbitrary semigroup and this is especially true for βS. We can characterize all closed left ideals of on βG for example. Recall that every closed subset of βG is of the form $\overline{\mathcal{F}}$ for some filter $\mathcal{F}$ on G.

Theorem 3.3.2. *Let $\mathcal{F}$ be a filter on a discrete group G and let $J = \overline{\mathcal{F}}$. Then the following statements are equivalent:*

1. *J is a left ideal of βG.*

2. *For every $x \in G$, $x\mathcal{F} = \mathcal{F}$.*

3. *For every $x \in G$, $x\mathcal{F} \subseteq \mathcal{F}$.*

Proof.

- (1) $\implies$ (3) Let $U \in \mathcal{F}$. For every $q \in J$, pick $B_q \in q$ such that $x^{-1}B_q \subseteq U$. The subsets $\overline{B_q}$, where $q \in J$, form an open cover of J, so there is a finite subcover $\overline{B_{q_i}} : i < n$. Let $V = \bigcup_{i<n} B_{q_i}$. Then $V \in \mathcal{F}$ and $x^{-1}V \subseteq U$, so $V \subseteq xU$, and consequently, $xU \in F$.

- (3) $\implies$ (2) From $x^{-1}\mathcal{F} \subseteq \mathcal{F}$ it follows that $x\mathcal{F} \supseteq \mathcal{F}$, and since also $x\mathcal{F} \subseteq \mathcal{F}$, one has $x\mathcal{F} = \mathcal{F}$.

- (2) $\implies$ (1) Let $q \in J$. Then for every $x \in G$, $x \cdot q \in J$, and so for every $p \in \beta G$, $p \cdot q \in J$.

$\square$

So every left shift invariant filter on G generates a closed left ideal in βG. We are interested in determining when $\beta S \setminus S$ is a closed left ideal of βS and if it is possible for us to generate new identities in $\beta S \setminus S$ which were not originally in S. For example, it is easy to see that for any infinite left-zero semigroup S with left zero $z \in S$ and the discrete topology that $\beta S \setminus S$ cannot be a left ideal of βS as $z \cdot p = z$ for all $p \in \beta S \setminus S$

Definition 3.3.3. *An element $e \in S$ is an identity of S if for all $s \in S$ we have that $s \cdot e = s$ and $e \cdot s = s$. The element is a right (left) identity if only the first (second) equation holds.*

Definition 3.3.4. *An semigroup S is left cancellative if $a \cdot x = a \cdot y$ implies that $x = y$.*

Definition 3.3.5. *A semigroup S is weakly left cancellative if for all $u, v \in S$ the set $A_{u,v} = \{x \in S : u \cdot x = v\}$ is finite.*

Obviously every left cancellative semigroup is also a weakly left cancellative semigroup. We wish to know when the remainder of the Stone-Čech compactification of a semigroup is itself a left ideal of βS.

Theorem 3.3.6. *If S is an infinite semigroup, then $S^* = \beta S \setminus S$ is a left ideal of βS if and only if S is weakly left cancellative.*

Proof. Suppose there exist $u, v \in S$ such that $A_{u,v}$ was infinite but that S^* was a left ideal of βS. That implies that any free ultrafilter p generated by $A_{u,v}$ is an element of S^* and so $e(u) \cdot p \in S^*$ but by the definition of $A_{u,v}$ we know that $e(u) \cdot p \in S$ which is a contradiction. So instead suppose that $S^*$was not a left ideal of βS but S was weakly cancellative. This would imply the existence of points $u, v \in S$ such that $e(u) \cdot p = e(v)$ for some $p \in S^*$ which by the properties of p-limits implies that $A_{u,v}$ must be infinite, which is a contradiction. □

Theorem 3.3.7. *Let S be a left cancellative discrete semigroup. If there exists a right identity $p \in \beta S$ then p is an identity for S and thus for βS.*

Proof. Since a left cancellative group is trivially weakly left cancellative we have that S^* must be an ideal of βS. But since p is a right identity we have that for $s \in S$ $s \cdot e = s \notin S^*$ and thus $e \notin S^*$ i.e. $e \in S$. But since S is left cancellative any right identity is also a left identity (since $s \cdot t = s \cdot e \cdot t$ and thus $e \cdot t = t$) and since S is a discrete semigroup all left and right shifts are trivially continuous and thus by Theorem 3.1.18 extend to the entire space βS □

The residual space $\beta S \setminus S$ is a closed left ideal of βS when S is weakly left cancellative, which is the case for any group G. Trivially, any non-finite left zero semigroup is not weakly left cancellative and thus there do exist semigroups where $\beta S \setminus S$ is not a left ideal of βS.

3.4 Compact Right Shift Continuous Semigroups

Definition 3.4.1. *An element $p \in S$ of a semigroup S is an idempotent if $p \cdot p = p$.*

One of the more powerful results in the study of compact right topological groups is that they contain idempotents.

Theorem 3.4.2 (Ellis)**.** *If S is a compact, right shift continuous, Hausdorff semigroup then S contains an idempotent.*

Proof. Let

$$\mathcal{A} = \{A \subset S : A^2 \subset A, A \neq \emptyset, \text{ and } A \text{ is compact}\}$$

with set inclusion as a partial ordering on $\mathcal{A}$. Now as S is compact, $\mathcal{A} \neq \emptyset$. Let $\mathcal{C} = \{A_\alpha\}$ be a chain in $\mathcal{A}$. Now $\cap\mathcal{C} \in \mathcal{A}$ and so $\mathcal{C}$ has a lower bound and thus by Zorn's Lemma, $\mathcal{A}$ has a minimal element. Let $B \in \mathcal{A}$ be a minimal element and let $x \in B$. Now consider the subset $C = Bx \neq \emptyset$ as $x^2 \in C$. Also $C \subset B^2 \subset B$. Now

$$C^2 = (Bx)(Bx) \subset B^3x \subset Bx = C$$

and since right shifts are continuous C is a compact subset of S and thus $C \in \mathcal{A}$. However, B is a minimal element of $\mathcal{A}$ and thus $B = C$. Finally, define

$$D = \{y \in B : yx = x\} \neq \emptyset$$

as $C = Bx = B$. Also as right shifts are continuous we have that D is a compact subset of S. Now let $y, z \in C \subset B$ then $yz \in B$ and $yzx = zx = x$ and thus $D^2 \subset D$ i.e. $D \in \mathcal{A}$. As B is a minimal element of $\mathcal{A}$ we have that $x \in B = D$. But this implies that $x \in D$ and so $x^2 = x$. □

A semigroup with identity can have more then one idempotent, as the following example shows. Let X be an infinite Hausdorff, compact topological space and consider the semigroup $X^X = \{f : X \to X\}$ with the binary operation being composition of functions. We can endow X^X with the product topology and thus by Tychonoff's theorem we know that X^X is a compact topological space. A basic open neighbourhood of a function f is given as follows: Choose a finite number of points $\{x_i\}_{i=1..n} \subset X$ and choose open neighbourhoods $U_i \subset X$ of $f(x_i)$. A basic open neighbourhood of f is then of the form

$$V(\{x_i\}_{i=1..n}, \{U_i\}_{i=1..n}, f) = \{h : X \to X : h(x_i) \in U_i\}$$

Now as

$$V(\{x_i\}_{i=1..n}, \{U_i\}_{i=1..n}, f) \circ g \subset V(\{x_i\}_{i=1..n}, \{U_i\}_{i=1..n}, f)$$

the right shift $f \to f \circ g$ is continuous and thus X^X satisfies all the criteria of Theorem 3.4.2 and thus contains idempotents. In fact, every constant function on X is an idempotent and thus X^X contains at least $|X|$ many idempotents, one for each constant function.

Definition 3.4.3.

1. *P is a prime ideal of S if $B \cdot C \subset P$ where both B and C are ideals of S implies that $B \subset P$ or $C \subset P$.*

2. *M is a minimal ideal of S if $I \subset M$ with I an ideal on S implies that $I = M$.*

3. *M is a maximal ideal of S if $M \subset I$ with I an ideal on S implies that $I = M$.*

4. *M is a minimal left/right ideal of S if $I \subset M$ with I an left/right ideal on S implies that $I = M$.*

5. *M is a maximal left/right ideal of S if $M \subset I$ with I an left/right ideal on S implies that $I = M$.*

Lemma 3.4.4. *Let S be a semigroup and let $L \subset S$ be a left ideal of S. The following are equivalent:*

1. *L is a minimal left ideal of S.*

2. *$L \cdot x = L$ for all $x \in L$.*

3. *$S \cdot x = L$ for all $x \in L$.*

Proof.

- (1) $\implies$ (2): Let $x \in L$ then since L is a left ideal we have $L \cdot x \subset L$ but $S \cdot L \cdot x \subset L \cdot x$ and so $L \cdot x$ is a left ideal contained in L thus $L = L \cdot x$

- (2) $\implies$ (3): Let $x \in L$ then $L = L \cdot x \subset S \cdot x \subset L$ and so $S \cdot x = L$

- (3) $\implies$ (1): Suppose that $S \cdot x = L$ for all $x \in L$ and let I be a left ideal with $I \subset L$ then if $x \in L \subset I$ we have that $S \cdot x \subset I$. Then $L = S \cdot x \subset I$ so $I = L$

□

Lemma 3.4.5. *Let S be a semigroup and let $R \subset S$ be a right ideal of S. The following are equivalent:*

1. *R is a minimal right ideal of S.*

2. $x \cdot R = R$ *for all* $x \in R$.

3. $x \cdot S = R$ *for all* $x \in R$.

Proof. The proof mimics the previous one almost exactly □

Definition 3.4.6. *An idempotent* $p \in S$ *is minimal if it belongs to some minimal ideal of* S *.*

We first prove that the Stone-Čech compactification of a semigroup does contain minimal left ideals.

Lemma 3.4.7. *Let* S *be a compact, right shift continuous semigroup and let* T *be a right ideal of* S. *Then* T *contains a minimal left ideal. Furthermore, this left ideal is closed.*

Proof. Let T be a left ideal and let $x \in T$. Then Sx is a compact left ideal contained in T. The compactness follows from the continuity of right shifts in S while $S \cdot x \subset S \cdot T \subset T$. Let $\mathcal{H} = \{H : H$ is a compact left ideal $, H \subset T\} \neq \emptyset$. Let H_α be a chain in $\mathcal{H}$ then by the finite intersection property of S we know that $\cap H_\alpha$ is a lower bound for the chain. So by Zorn's Lemma we know that $\mathcal{H}$ contains a minimal element i.e. Any left ideal contains a minimal left ideal. □

We know show the existence of minimal ideals in a right shift invariant compact topological semigroup.

Lemma 3.4.8. *Let* S *be a compact, right shift continuous semigroup. Then* S *contains a minimal ideal.*

Proof. We proceed by constructing a minimal ideal as the largest minimal left ideal in the semigroup. So define

$$\mathcal{K} = \bigcup \{T : T \text{ is a minimal left ideal of } S\}$$

We can use this lemma to show the existence of minimal idempotents. By Lemma 3.4.7 we know that $\mathcal{K} \neq \emptyset$. Let $x \in S$ then $x \cdot \mathcal{K} = x \cdot T_\alpha \subset T_\alpha \subset \mathcal{K}$ for some α and thus $\mathcal{K}$ is a left ideal. To show that $\mathcal{K}$ is a right ideal, let $x \in S$ and $y \in \mathcal{T}$ which implies the existence of an α such that $y \in T_\alpha$. We know that $y \cdot \in T_\alpha \cdot x$ and that $T_\alpha \cdot x$ is trivially a right ideal. However, it can be shown that $T_\alpha \cdot x$ is in fact a minimal left ideal of S and so $y \cdot x \in T_\alpha \cdot x \in \mathcal{K}$ and thus $\mathcal{T}$ is in fact an ideal of the semigroup S. To show this claim, consider the set $A = \{z \in T_\alpha : z \cdot y \in H\}$ where H is a minimal left ideal contained in $T_\alpha \cdot x$. But then A is a trivially a left ideal which implies that $H = T_\alpha \cdot x$. Finally, let I denote any other ideal in S. We know that for any minimal right ideal T that $T \cap I \neq \emptyset$ and that $T \cap I$ is a left ideal contained in T and so $T = T \cap I$. But this implies that $T \subset I$ and thus $\mathcal{T} \subset I$ $\square$

Theorem 3.4.9. *Let S be a compact, right shift continuous semigroup. Then S contains a minimal idempotent.*

Proof. We know by Lemma 3.4.8 that S contains a minimal ideal, say I. By the proof of Lemma 3.4.8 we know that we may assume that there exists a minimal closed left ideal, say L, contained in this minimal ideal. The restriction of the extended semigroup operations to this minimal closed left ideal are well defined (since it is a right ideal) and are continuous (since it is closed). Also since the minimal left ideal is a closed subset of a compact space we know that it is compact too. So we can apply the Ellis theorem (theorem 3.4.2) to show the existence of an idempotent in $L \subset I$ $\square$

This shows that βS contains minimal idempotents and minimal ideals. Since we have mentioned minimal ideals, the idea of maximal ideals of a semigroup is natural. It can be shown that every maximal ideal of a semigroup is either a prime ideal or is of the form $S \setminus \{x\}$ where $x \in S^2 \setminus S$. (See [Gri69]).

Theorem 3.4.10. *Let S be a semigroup. S can contain at most one minimal ideal.*

Proof. Suppose K is the minimal ideal of S, let I be some other ideal of S then we have that $K \cdot I \subset K$ and $K \cdot I \subset I$ and so $K \cdot I \subset K \cap I \subset K$ but $K \cdot I$ is also an ideal of S and thus $K \subset K \cdot I$ which implies that $K = K \cap I$ i.e. $K \subset I$ $\square$

Definition 3.4.11. *Let S be an semigroup and denote by $K(S)$ the smallest ideal of S, if it exists*

Lemma 3.4.12. *Let S be a semigroup and let L be a minimal left ideal of S then:*

1. *$L \cdot a$ is a minimal left ideal of S for all $a \in S$.*
2. *Every left ideal of S contains a minimal left ideal.*
3. *The minimal left ideals form a decomposition of $K(S)$.*

Proof. 1. Let $\hat{L}$ be a left ideal contained in $L \cdot a$ and consider the set $L' = \left\{x \in L : x \cdot a \in \hat{L}\right\} \subset L$ and if $z \in y \cdot L'$there exists $x \in L$ such that $z = y \cdot x$ and $x \cdot a \in \hat{L}$. But since L is a left ideal we have that $z \in L$ and $z \cdot a = y \cdot x \cdot a \in \hat{L}$ and so L' is an ideal of S contained in L and thus $L = L'$ and thus $L \cdot a \subset \hat{L}$ which shows that $L \cdot a$ is a minimal ideal of S

2. Let A be a left ideal of S and let $a \in A$, since A is a left ideal we have that $L \cdot a \subset S \cdot A \subset A$ but $L \cdot a$ is a minimal ideal

3. The non-empty intersection of left ideals is still a left ideal and thus if $A, B \subset S$ are minimal left ideals then $C = A \cap B \neq \emptyset$ is a left ideal contained in both A and B and thus $A = C = B$. So consider the set $K = \bigcup_{a \in S} L \cdot a$ then
$$x \cdot K = \bigcup_{a \in S} x \cdot L \cdot a \subset \bigcup_{a \in S} L \cdot a = K$$
and
$$K \cdot x = \bigcup_{a \in S} L \cdot a \cdot x \subset \bigcup_{a \in S} L \cdot a = K$$
Thus K is an ideal of S. Let I be any other ideal of S then $I \cdot L \subset I$ and $I \cdot L \subset L$ thus $I \cdot L \subset I \cap L \subset L$ but $I \cdot L$ is left ideal and thus $L = I \cdot L \subset I$. This shows that
$$K = \bigcup_{a \in S} L \cdot a \subset \bigcup_{a \in S} I \cdot a \subset I$$
thus K is the smallest ideal of S

□

Definition 3.4.13. *Let S be a semigroup.*

1. *S is left [right] simple if it has no proper left [right] ideal.*

2. *S is simple if it contains no proper ideals.*

3. *S is completely simple if it is simple and contains a minimal left ideal which contains an idempotent.*

It is easy to see that S being left simple implies that $S \cdot x = S$ for all $x \in S$ i.e. the equation $a \cdot x = b$ has a solution for all $a, b \in S$

Corollary 3.4.14. *Let S be a compact, right shift continuous semigroup then S contains a smallest ideal which is a completely simple semigroup.*

Proof. We know that S contains a minimal left ideal by Lemma 3.4.7 and thus contains a smallest ideal $K(S)$ by Lemma 3.4.12. It is easy to see that $K(S)$ is a simple semigroup. But $K(S)$ is made up of minimal left ideals of S which are also minimal left ideals of $K(S)$ and by Theorem 3.4.9 one of these minimal left ideals contains an idempotent. $\square$

Definition 3.4.15. *Let S be a right shift continuous semigroup. The topological center is the set*

$$\Lambda(S) = \{x \in S : \lambda_x : S \to S \textit{ is continuous}\}$$

Definition 3.4.16. *Let S be a semigroup. The algebraic center is the set*

$$Z(S) = \{x \in S : \forall y \in s, x \cdot y = y \cdot x_x\}$$

Lemma 3.4.17. *If S is a right shift continuous semigroup then $Z(S) \subset \Lambda(S)$.*

Proof. Let $x \in Z(S)$ then $x \cdot y = y \cdot x$ for all $y \in S$ but then $\lambda_x(y) = x \cdot y = y \cdot x = \rho_x(y)$ and is thus continuous so $x \in \Lambda(S)$ $\square$

Theorem 3.4.18. *If S is a commutative semigroup then $Z(S)$ is dense in βS.*

Proof. Let $p \in S$ and $q \in \beta S$. Now

$$\begin{aligned} p \cdot q &= p\text{–}\lim_{s \in S}(q\text{–}\lim_{t \in S}(s \cdot t)) \\ &= p\text{–}\lim_{s \in S}(q\text{–}\lim_{t \in S}(t \cdot s)) \\ &= (q\text{–}\lim_{t \in S} t) \cdot p \\ &= q \cdot p \end{aligned}$$

Thus $p \in Z(S)$ and so $S \subset Z(S)$ but then $\beta S = \overline{S} \subset \overline{Z(S)} \subset \beta S$ $\square$

Lemma 3.4.19. *Let S be a compact right shift continuous semigroup. The algebraic center $Z(S)$ of S is dense in S if and only if $\Lambda(S)$ contains a dense commutative subset.*

Proof. If $Z(S)$ is dense in S and since $Z(S) \subset \Lambda(S)$ we have that $\Lambda(S)$ contains a dense commutative subset. So suppose that $\Lambda(S)$ contained a dense commutative subset $A \subset S$ then for all $x \in A$ we would have that both the left shifts λ_x and right shifts ρ_x would be continuous and since A is commutative we have that

$\lambda_x(y) = \rho_x(y)$ for all $y \in A$. But since A is in fact dense in S we have that $\lambda_x(y) = \rho_x(y)$ for all $y \in S$ and thus $x \in Z(S)$ and thus $S = \overline{A} \subset \overline{Z(S)} \subset S$ $\square$

Lemma 3.4.20. *If S is a right shift continuous semigroup then the closure of any right ideal is a right ideal.*

Proof. Let $R \subset S$ be a right ideal. Now the right shifts ρ_x are continuous and thus $\overline{R} \cdot x = \rho_x(\overline{R}) \subset \overline{\rho_x(R)} = \overline{R \cdot x} \subset \overline{R}$ for all $x \in S$. This implies that $\overline{R} \cdot S \subset S$ and so $\overline{R}$ is also a right ideal $\square$

It is not generally the case that the closure of an arbitrary left ideal in a right shift continuous semigroup is a left ideal.

Lemma 3.4.21. *If S is a right shift continuous semigroup such that $\overline{\Lambda(S)} = S$ then the closure of any left ideal is a left ideal.*

Proof. Let $x \in \overline{L}$ for some left ideal $L \subset S$ and let $y \in S$ with U an open neighbourhood of $y \cdot x$. We can choose a neighbourhood V of y such that $V \cdot x \subset U$ and since $\Lambda(S)$ is dense in S we can pick $z \in \Lambda(S) \cap V$. Now λ_z is a continuous function on S and so we can pick a neighbourhood W of x such that $z \cdot W \subset U$. Choose $w \in W \cap L$ then $z \cdot w \in z \cdot W \cap L \subset U \cap L \neq \emptyset$. Thus every neighbourhood of $y \cdot x$ has non-empty intersection with $\overline{L}$ i.e. $y \cdot x \in \overline{L}$ which implies that $S \cdot \overline{L} \subset \overline{L}$ $\square$

Definition 3.4.22. *Let S be a semigroup. Denote by $E(S)$ the set of all idempotents on S.*

Lemma 3.4.23. *Let S be a left simple semigroup with $E(S) \neq \emptyset$ and let $e \in E(S)$.*

1. *e is a right identity of S.*
2. *$e \cdot S$ is a group.*

Proof.

1. Let $x \in S$ then since S is left simple there must exist $y \in S$ such that $y \cdot e = x$. But then $x \cdot e = y \cdot e^2 = y \cdot e = x$
2. Let $x \in S$ then $e \cdot x \in e \cdot S$ and $e \cdot e \cdot x = e^2 \cdot x = e \cdot x$. So e is a left identity of $e \cdot S$ and since it is also a right identity it is the identity element of $e \cdot S$. Also for $x \in S$ and $y \in S$ we have that $(e \cdot x) \cdot (e \cdot y) = e \cdot (x \cdot e) \cdot y = e \cdot (x \cdot y) \in e \cdot S$ and thus $e \cdot S$ is a semigroup. Now let $x \in S$ then there must exist $z \in S$ such that $z \cdot x = e$. Let $y = e \cdot z$ then $y \cdot x = e \cdot z \cdot x = e^2 = e$. Now there

exists a $w \in S$ such that $w \cdot y = e$ then $w = w \cdot e = w \cdot y \cdot x = e \cdot x = x$ and thus $x \cdot y = e$

$\square$

Theorem 3.4.24. *Let S be a right shift continuous semigroup such that $\overline{\Lambda(S)} = S$ then:*

1. *If R is a right ideal of S then $\overline{R}$ is a two sided ideal of S.*

2. *If $e \in E(K(S))$ then $\overline{e \cdot S \cdot e} = S \cdot e$.*

Proof. 1. We know that the closure of R is a right ideal of S. Let $y \in \overline{R}$ then for any $x \in Z(S)$ we have that $\rho_y(x) = x \cdot y = y \cdot x \subset \overline{R} \cdot x \subset \overline{R}$. This implies that $\rho_y(S) = \rho_y(\overline{Z(S)}) \subset \overline{\rho_y(Z(S))} \subset \overline{R}$ i.e. $S \cdot \overline{R} \subset \overline{R}$ and thus $\overline{R}$ is a two sided ideal of S

2. Since the continuous image of a compact space is compact and any compact subset of a Hausdorff space is closed we know that $S \cdot e$ is closed. Thus $\overline{e \cdot S \cdot e} \subset S \cdot e$. Also

$$S \cdot e = \overline{Z(S)} \cdot e \subset \overline{Z(s) \cdot e} \subset \overline{e \cdot Z(s) \cdot e} \subset \overline{e \cdot S \cdot e}$$

$\square$

Lemma 3.4.25. *Let S be a left simple semigroup with $E(S) \neq \emptyset$ and let $e \in E(S)$.*

1. *e is a right identity of S.*

2. *$e \cdot S$ is a group.*

Proof. 1. Let $x \in S$ then since S is left simple there must exist $y \in S$ such that $y \cdot e = x$. But then $x \cdot e = y \cdot e^2 = y \cdot e = x$

2. Let $x \in S$ then $e \cdot x \in e \cdot S$ and $e \cdot e \cdot x = e^2 \cdot x = e \cdot x$. So e is a left identity of $e \cdot S$ and since it is also a right identity it is the identity element of $e \cdot S$. Also for $x \in S$ and $y \in S$ we have that $(e \cdot x) \cdot (e \cdot y) = e \cdot (x \cdot e) \cdot y = e \cdot (x \cdot y) \in e \cdot S$ and thus $e \cdot S$ is a semigroup. Now let $x \in S$ then there must exist $z \in S$ such that $z \cdot x = e$. Let $y = e \cdot z$ then $y \cdot x = e \cdot z \cdot x = e^2 = e$. Now there exists a $w \in S$ such that $w \cdot y = e$ then $w = w \cdot e = w \cdot y \cdot x = e \cdot x = x$ and thus $x \cdot y = e$

$\square$

Lemma 3.4.26. *Let S be a right simple semigroup with $E(S) \neq \emptyset$ and let $e \in E(S)$.*

1. *e is a left identity of S.*

2. *$S \cdot e$ is a group.*

Proof. 1. Let $x \in S$ then since S is right simple there must exist $y \in S$ such that $e \cdot y = x$. But then $e \cdot x = e^2 \cdot y = e \cdot y = x$

2. This proof is similar to the left simple semigroup case

□

Lemma 3.4.27. *Let S be a semigroup and let $e \in E(S)$. If $S \cdot e$ is a minimal left ideal then $e \cdot S$ is a minimal right ideal.*

Proof. Now $S \cdot e$ is a minimal left ideal and thus is a left simple semigroup containing the idempotent $e \cdot e = e \in E(S \cdot e)$. But then $e \cdot S \cdot e$ is a group with identity e. So let R be a right ideal with $R \subset e \cdot S$. Now if $y \in R \subset e \cdot S$ then $y \cdot e \in e \cdot S \cdot e$ but then there must exist $z \in e \cdot S \cdot e$ such that $y \cdot e \cdot z = e$ but since R is a right ideal we must have that $e \in R$. But then $e \cdot S \subset R$ and so $e \cdot S = R$ □

Lemma 3.4.28. *Let S be a semigroup and suppose that $R \subset S$ and $L \subset S$ are minimal left and right ideals respectively. Then $R \cdot L = R \cap L$ and $G = R \cdot L$ is a maximal subgroup of S.*

Proof.

- Since $R \cdot L \subset R$ and $R \cdot L \subset L$ we have that $R \cdot L \subset R \cap L$.

- Let $x \in R \cdot L \subset R$ then $x \cdot R \cdot L = (x \cdot R) \cdot L = R \cdot L$ and since $x \in R \cdot L \subset L)$ we have that $R \cdot L \cdot x = R \cdot (L \cdot x) = R \cdot L$ thus every equation $a \cdot x = b$ and $y \cdot a = b$ has a solution $x, y \in R \cdot L$ for any $a, b \in R \cdot L$ which implies that $R \cdot L$ is a group. Denote the identity of this group by $e \in R \cdot L \subset R \cap L$

- Let $x \in R \cap L$. Now suppose $x \cdot e = z$ then $z \in R \cdot L$ and $z = z \cdot e = x \cdot e^2 = x$ thus $x \in R \cdot L$ and so $R \cap L = R \cdot L$

□

Definition 3.4.29. *Let G be a group, let Λ and I be nonempty sets and let $P = (p_{\lambda i})_{\lambda \in \Lambda, i \in I}$ be a possibly infinite matrix with $p_{\lambda i} \in G$. The Rees matrix semigroup over the group G with $\Lambda \times I$ sandwich matrix P, $\mathcal{M}(G, I, \Lambda, P)$, is the set $R = I \times G \times \Lambda$ and the operation $\circ : R \times R \to R$ defined by*

$$(i, a, \lambda) \circ (j, b, \mu) = (i, a \cdot p_{\lambda j} \cdot b, \mu)$$

Lemma 3.4.30. *The Rees matrix semigroup $\mathcal{M}(G, I, \Lambda, P)$ is a completely simple semigroup.*

Proof.

- We have to show that $\circ : R \times R \to R$ is associative. Now

$$\begin{aligned}(i, a, \lambda) \circ ((j, b, \mu) \circ (k, c, \nu)) &= (i, a, \lambda) \circ (j, b \cdot p_{\mu k} \cdot c, \nu) \\ &= (i, a \cdot p_{\lambda j} \cdot b \cdot p_{\mu k} \cdot c, \nu)\end{aligned}$$

  and

$$\begin{aligned}((i, a, \lambda) \circ (j, b, \mu)) \circ (k, c, \nu) &= (i, a \cdot p_{\lambda j} \cdot b, \mu) \circ (k, c, \nu) \\ &= (i, a \cdot p_{\lambda j} \cdot b \cdot p_{\mu k} \cdot c, \nu)\end{aligned}$$

  thus $\mathcal{M}(G, I, \Lambda, P)$ is a semigroup

- Suppose $I \subset R$ is a proper ideal then $R \cdot I \cdot R \subset I$. Let $(j, b, \mu) \in I$ and $(i, a, \lambda) \in R$ then $(i, (p_{\lambda j} \cdot b \cdot p_{\mu i})^{-1}, \lambda) \in R$ and

$$\begin{aligned}(i, a, \lambda) \circ (j, b, \mu) \circ (i, (p_{\lambda j} \cdot b \cdot p_{\mu i})^{-1}, \lambda) &= (i, a \cdot p_{\lambda j} \cdot b \cdot p_{\mu i} \cdot (p_{\lambda j} \cdot b \cdot p_{\mu i})^{-1}, \lambda) \\ &= (i, a, \lambda) \in I\end{aligned}$$

  Thus $I = R$ and R is a simple semigroup

- Let $\lambda \in \Lambda$ and consider the set $J_\lambda = I \times G \times \{\lambda\}$ then if $(i, a, \lambda) \in J_\lambda$ and $(j, b, \mu) \in R$ then

$$(j, b, \mu) \circ (i, a, \lambda) = (j, b \cdot p_{\mu i} \cdot a, \lambda) \in J_\lambda$$

  and so J_λ is a left ideal of R. Now let $(i, a, \lambda) \in J_\lambda$ and $(j, b, \lambda) \in J_\lambda$ and consider the equation $x \circ (i, a, \lambda) = (j, b, \lambda)$. Now $x = (j, b \cdot (p_{\lambda i} \cdot a)^{-1}, \lambda) \in J_\lambda$ and

$$\begin{aligned}x \circ (i, a, \lambda) &= (j, b \cdot (p_{\lambda i} \cdot a)^{-1}, \lambda) \circ (i, a, \lambda) \\ &= (j, b \cdot (p_{\lambda i} \cdot a)^{-1} \cdot p_{\lambda i} \cdot a, \lambda) \\ &= (j, b, \lambda)\end{aligned}$$

  Thus $J_\lambda \circ (i, a, \lambda) = J_\lambda$ for all $(i, a, \lambda) \in J_\lambda$ and thus J_λ is a minimal left ideal of R. Now consider the general equation $x \circ (j, b, \mu) = (k, c, \nu)$ where

$(j, b, \mu), (k, c, \nu) \in J$ which is a minimal left ideal of R. Since

$$\begin{aligned}(i, a, \lambda) \circ (j, b, \mu) &= (i, a \cdot p_{\lambda j} \cdot b, \mu) \\ &= (k, c, \nu)\end{aligned}$$

implies that $\mu = \nu$ we have that the J_λ are the only minimal ideals of R

- Similarly it can be shown that the sets $K_i = \{i\} \times G \times \Lambda$ are the only minimal right ideals of R

- Now R contains both left minimal ideals and right minimal ideals and thus contains maximal groups of the form $J_\lambda \cap K_i = \{i\} \times G \times \{\lambda\}$ .Now there exists $j \in I$ and $\mu \in \Lambda$ such that $p_{\mu j} = e$ and thus (j, e, μ) is an idempotent since

$$\begin{aligned}(j, e, \mu) \circ (j, e, \mu) &= (j, e \cdot p_{\mu j} \cdot e, \mu) \\ &= (j, e^3, \mu) \\ &= (j, e, \mu)\end{aligned}$$

and since $(j, e, \mu) \in J_\mu$ we have that R is completely simple

□

Theorem 3.4.31. *Let S be a left simple semigroup. Let $e \in E(S)$. Define a function $f : E(S) \times e \cdot S \to S$ by $f(x, a) = x \cdot a$ then f is an isomorphism.*

Proof. Since every $x \in E(S)$ is a right identity on S we have that

$$\begin{aligned}f((x, a) \times (y, b)) &= f(x \cdot y, a \cdot b) \\ &= x \cdot y \cdot a \cdot b \\ &= x \cdot a \cdot b \\ &= x \cdot a \cdot y \cdot b \\ &= f(x, a) \cdot f(y, b)\end{aligned}$$

and thus we have that f is a homomorphism. Now let $x \in S$ then $S \cdot x = S$ we have that there must exist a $y \in S$ such that $y \cdot x = e$ but then $x \cdot y \cdot x \cdot y = x \cdot e \cdot y = x \cdot y$

and thus $x \cdot y \in E(S)$ thus

$$\begin{aligned} f(x \cdot y, e \cdot x) &= x \cdot y \cdot e \cdot x \\ &= x \cdot y \cdot x \\ &= x \cdot e \\ &= x \end{aligned}$$

Now let $f(x, a) = f(y, b)$ then $x \cdot a = y \cdot b$. Now

$$\begin{aligned} a &= e \cdot a \cdot e \\ &= e \cdot x \cdot a \cdot e \\ &= e \cdot y \cdot b \cdot e \\ &= e \cdot b \cdot e \\ &= b \end{aligned}$$

since $e \cdot S$ is a group and y is a right identity on S. Also we have that

$$\begin{aligned} x &= x \cdot e \\ &= x \cdot a \cdot a^{-1} \\ &= y \cdot b \cdot a^{-1} \\ &= y \cdot a \cdot a^{-1} \\ &= y \cdot e \\ &= y \end{aligned}$$

which shows that $(x, a) = (y, b)$ and thus f is one-to-one.

□

Theorem 3.4.32. *Let S be a completely simple semigroup. Let $e \in E(S)$ such that $S \cdot e$ is a minimal left ideal. Let $I = E(S \cdot e)$, $\Lambda = E(e \cdot S)$ and $G = e \cdot S \cdot e$ and $p_{\lambda i} = \lambda \cdot i$. Define a function $f : R \to S$ by $f(i, a, \lambda) = i \cdot a \cdot \lambda$ then f is an isomorphism.*

Proof. Since

$$
\begin{aligned}
f((i,a,\lambda)\circ(j,b,\mu)) &= f(i, a\cdot p_{\lambda j}\cdot b,\mu)\\
&= i\cdot a\cdot\lambda\cdot j\cdot b\cdot\mu\\
&= (i\cdot a\cdot\lambda)\cdot(j\cdot b\cdot\mu)\\
&= f((i,a,\lambda))\cdot f((j,b,\mu))
\end{aligned}
$$

we have that f is a homomorphism. Now since any idempotent $u \in E(S\cdot e)$ is a left identity for $S\cdot e$ and similarly every idempotent $v\in E(e\cdot S)$ is a right identity for $e\cdot S$ and since $e\cdot S\cdot e$ is a group we have that

$$
\begin{aligned}
E(S\cdot e)\cdot e\cdot S\cdot e\cdot E(e\cdot S) &= E(S\cdot e)\cdot e\cdot S\cdot e\cdot e\cdot S\cdot e\cdot E(e\cdot S)\\
&= S\cdot e^2\cdot S\\
&= S\cdot e\cdot S\\
&= S
\end{aligned}
$$

Since S contains no proper ideals. This implies that f is an onto function. To prove that f is also one-to-one requires a couple of steps. Suppose that $i{\cdot}a{\cdot}\lambda = j{\cdot}b{\cdot}\mu$then

- $a = e\cdot a\cdot e = e\cdot i\cdot a\cdot\lambda\cdot e = e\cdot j\cdot b\cdot\mu\cdot e = e\cdot b\cdot e = b$
- $i = i\cdot e = i\cdot a\cdot a^{-1} = i\cdot a\cdot\lambda\cdot a^{-1} = j\cdot b\cdot\mu\cdot b^{-1} = j\cdot b\cdot b^{-1} = j$
- $\lambda = e\cdot\lambda = a^{-1}\cdot a\cdot\lambda = a^{-1}\cdot i\cdot a\cdot\lambda = a^{-1}\cdot j\cdot b\cdot\mu = a^{-1}\cdot a\cdot\mu = \mu$

This implies that $f(i,a,\lambda) = f(j,b,\mu)$ implies that $(i,a,\lambda) = (j,b,\mu)$ and thus f is one-to-one

□

Since βS is a compact, right shift continuous semigroup we know that it contains a minimal left ideal which is a completely simple semigroup and thus βS contains a copy of the Rees Matrix semigroup.

3.5 Idempotent Orders and Green's Relations on Compact Semigroups

Definition 3.5.1. *Let X be a set with some partial order $\leq$ and let $x, y \in S$.*

- *x is a minimal element with respect to the partial order $\leq$ if $y \leq x$ implies that $x \leq y$.*
- *x is a maximal element with respect to the partial order $\leq$ if $x \leq y$ implies $y \leq x$.*

If a semigroup contains idempotents we can place a partial ordering on the set of idempotents $E(S)$ as follows.

Definition 3.5.2. *Let S be a semigroup that contains idempotents $x, y \in S$.*

- *$x \leq_L y$ if $x \cdot y = x$.*
- *$x \leq_R y$ if $y \cdot x = x$.*
- *$x \leq y$ if $x \cdot y = y \cdot x = x$.*

So $x \leq_L y$ if y is a right identity for x. Similarly $x \leq_R y$ is y is a left identity for x and $x \leq y$ if y is an identity for x.

Lemma 3.5.3. *The relation $\leq$ is a partial ordering on $E(S)$.*

Proof. It is easy to see that $x \leq y$ if and only if $x \leq_L y$ and $x \leq_R y$. Now since x is an idempotent we have that $x^2 = x$ and thus $x \leq_L x$ and $x \leq_R x$ and so $x \leq x$. Now let $x \leq y$ and $y \leq z$ then $x \cdot z = x \cdot y \cdot z = x \cdot y = x$ and $z \cdot x = z \cdot y \cdot x = y \cdot x = x$ and so $x \leq_L z$ and $y \leq_R z$ and so $x \leq z$. Finally let $x \leq y$ and $y \leq x$ then $x = y \cdot x = x \cdot y = y$ □

It should be noted that $\leq_L$ and $\leq_R$ are only pre-orders that Zorn's lemma does not directly apply to sets ordered by them. However, if $<$ is a given pre-order on a set X we can consider the equivalence relation $x \diamond y$ if and only if $x < y$ and $y < x$ and thus define a partial order $\leq$ on the equivalence classes of $\diamond$where $A \leq B$ if and only if there exists $x \in A$ and $y \in B$ such that $x < y$. It is easy to see that this is a partial ordering and thus we can pass to this generated partial order when wanting to invoke Zorn's lemma and thus find a maximal equivalence class. But then every element of that equivalence class is maximal under the original pre-order.

Theorem 3.5.4. *If S is a semigroup and $x \in S$ is an idempotent then the following statements are equivalent*

1. *x is a minimal element with respect to the pre-order $\leq_L$ i.e. $y \leq_L x \implies x_L \leq y$.*
2. *x is a minimal element with respect to the pre-order $\leq_R$ i.e. $y \leq_R x \implies x_R \leq y$.*
3. *x is a minimal element with respect to the partial-order$\leq$.*

Proof.

- (1) $\implies$ (2) : let x be a $\leq_L$ minimal element. Now let $z \leq_R x$ then

$$x \cdot z = z \implies (z \cdot x)^2 = z \cdot x \cdot z \cdot x = z \cdot z \cdot x = z \cdot x$$

and $(z \cdot x) \cdot x = z \cdot x$. But then $z \cdot x \leq_L x$ which implies that $x \cdot z \cdot x = x$. So

$$x = x^2 = x \cdot z \cdot x \cdot x \cdot z \cdot x = x \cdot z \cdot x \cdot z \cdot x = z^2 \cdot x = z \cdot x$$

and so $x \leq_R z$

- (2) $\implies$ (3) : let x be a $\leq_R$ minimal element. Now let $z \leq x$ then

$$z \cdot x = x \cdot z = z$$

but that implies that $z \leq_R x$ and so $z \cdot x = x$ then

$$x = x^2 = z \cdot x \cdot z \cdot x = x \cdot z$$

thus

$$x \cdot z = z \cdot x = x$$

and $x \leq z$

- (3) $\implies$ (1) : let x be a $\leq$ minimal element. Now let $z \leq_L x$ then $z \cdot x = z$ then

$$(x \cdot z)^2 = x \cdot z \cdot x \cdot z = x \cdot z^2 = x \cdot z$$

which implies $(x \cdot z) \cdot x = x \cdot z$ and

$$x \cdot (x \cdot z) = x^2 \cdot z = x \cdot z$$

so $x \cdot z \leq x$ and thus

$$x \cdot (x \cdot z) = (x \cdot z) \cdot x = x$$

i.e. $x \cdot z = x$ and $x \leq_L z$

□

Since any minimal element is minimal under all three partial orderings we can have a general catch all definition for the minimal elements of the semigroup S.

Definition 3.5.5. *An idempotent of a semigroup is a minimal idempotent with respect to $\leq$ if it is minimal under $\leq$ on $E(S)$.*

We have two competing ideas of minimal ideals, we now show that they actually coincide for certain types of semigroups.

Definition 3.5.6. *A semigroup S is abundant if every left ideal contains some minimal left ideal and every left ideal contains an idempotent.*

We have already proven that any compact right continuous topological group is abundant.

Lemma 3.5.7. *If S is an abundant semigroup with idempotent $e \in S$ and L is a left ideal contained in $S \cdot e$ then there exists an idempotent $f \in L$ such that $f \leq e$.*

Proof. Suppose that e is an idempotent of S and that there is some left ideal L of S such that $L \subset S \cdot e$. We can find another idempotent $f \in L$ since S is abundant. Now since $f \in L \subset S \cdot e$ we can find $s \in S$ such that $f = s \cdot e$. But then $f \cdot e = s \cdot e^2 = f$. Let $g = e \cdot f$ but then

$$g^2 = e \cdot f \cdot e \cdot f = e \cdot f^2 = e \cdot f = g$$

and so g is also an idempotent. Similarly $g \in L$. Now $e \cdot g = g$ and

$$g \cdot e = e \cdot f \cdot e = e \cdot f = g$$

and thus $g \leq e$. □

Theorem 3.5.8. *If S is an abundant semigroup then an idempotent is minimal with respect to $\leq$ if and only if it is contained in some minimal ideal of S.*

Proof. Now if e is an element of the minimal ideal $K(S)$ of S we know that it must be a member of some minimal left ideal L of S. Let x be another idempotent such that $x \leq e$ and consider the left ideal $S \cdot x$. Now $x \cdot e = e \cdot x = x$ and thus

$$S \cdot x = S \cdot x \cdot e \subset S \cdot e = L$$

but L is a minimal ideal and so $S \cdot x = L$. We can then find $s \in S$ such that $s \cdot x = e$ and thus

$$e = s \cdot x = s \cdot x \cdot x = e \cdot x = x$$

and $e \leq x$. So suppose now that e is minimal with respect to the partial order $\leq$and consider the left ideal $S \cdot e$. Since S is abundant we know that there exists another left ideal $L \subset S \cdot e$ but then there exists an idempotent $f \in L$ such that $f \leq e$ which implies that $f = e$ and that $L = S \cdot e$. But then $S \cdot e$ is a minimal left ideal and thus $e \in S \cdot e \subset K(S)$

□

The partial orders are well behaved under the semigroup operation on S

Lemma 3.5.9. *Let $x, y, z \in E(S)$.*

- *$x \leq_L y$ and $x \leq_L z$ implies that $x \leq_L y \cdot z$.*
- *$x \leq_R y$ and $x \leq_R z$ implies that $x \leq_R y \cdot z$.*
- *$x \leq z$ and $y \leq z$ implies that $x \cdot y \leq z$.*

Proof.

- Now $x \cdot y = x$ and $x \cdot z = x$ then $x \cdot y \cdot z = x \cdot z = x$
- Now $y \cdot x = x$ and $z \cdot x = x$ then $y \cdot z \cdot x = y \cdot x = x$
- Since it holds for both $\leq_L$ and $\leq_R$ it must hold for $\leq$

□

The existence of maximal left and right idempotents is an important area of study in βS. For example it can be shown that if p and q are idempotents of $\beta\mathbb{N}$ then $p \leq q$ or $q \leq p$ if and only if $p \cdot q = q \cdot p$.

Theorem 3.5.10. *If S is a right continuous, compact, Hausdorff semigroup then S contains a maximal right idempotent.*

Proof. Now since S is a right continuous, compact and Hausdorff semigroup we know that it contains some idempotent $x \in S$. Consider the pre-ordered set

$$A = \{y \in E(S) : x \leq_R y\} \ni x$$

and is thus $A \neq \emptyset$. Let C be some chain in A and consider the sets of the form

$$B_p = \{z \in X : z \cdot p = p\}$$

Now if $x \in B_p$and $y \in B_p$ we know that $p = x \cdot p$ and $p = y \cdot p$ and thus $p = x \cdot p = x \cdot y \cdot p$ and $x \cdot y \in B_p$. Also

$$\rho_p^{-1}(p) = \{z \in X : z \cdot p = p\} = B_p$$

and thus B_p is a closed semigroup. Now if $p \leq_R q$ and $x \in B_q$ we have that $x \cdot q = q$ and $q \cdot p = p$ but then $x \cdot p = x \cdot q \cdot p = q \cdot p = p$ and thus $x \in B_q$. This implies that

$$B = \bigcap_{p \in C} B_p$$

is non-empty and thus is a closed semigroup contained in S. But S is also compact and thus B is a compact right continuous semigroup and thus contains an idempotent $q \in B$. We claim that q is the upper limit of the chain C. For let $p \in C$ then $q \in B_p$ and so $q \cdot p = p$ which implies that $p \leq_R q$. But then by Zorn's lemma we know that A must contain an equivalence class of maximal element and thus S contains maximal right idempotent.

□

Now there is a canonical way of investigating the algebraic structure of any semigroup and that is by the Green's relationships.

Definition 3.5.11. *Let S be a semigroup with identity that contains idempotents $x, y \in S$.*

- $x \, L \, y$ *if* $S \cdot x = S \cdot y$.
- $x \, R \, y$ *if* $x \cdot S = y \cdot S$.
- $x \, J \, y$ *if* $S \cdot x \cdot S = S \cdot y \cdot S$.
- $x \, H \, y$ *if* $x \, L \, y$ *and* $x \, R \, y$.
- $x \, D \, y$ *if* $\exists z \in S$ *such that* $x \, L \, z$ *and* $z \, R \, y$.

There is a clear relationship between the Green's relations and the partial order on the idempotents. For example if $x \leq_L y$ then $S \cdot x = S \cdot x \cdot y \subset S \cdot y$ and if $x \leq_R y$ then x·S=y·x·S⊂y·S. Now we can use the partial ordering on the idempotents to discuss minimal and maximal ideals. Every Green's relation generates a subset of $S \times S$ and we will denote this subset by the operator of the relationship.

Theorem 3.5.12. *If S is a compact first countable topological semigroup with identity then L, R and J are closed subsets of $S \times S$.*

Proof. Now $S \times S$ is the compact and thus every sequence in $S \times S$ converges to some element of $S \times S$. Now suppose we have a sequence of points $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ converging to (x,y) and that for all $n \in \mathbb{N}$ we have that $x_n \, R \, y_n$ that is there exist a sequence of points $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ in S such that $x_n \cdot u_n = y_n$ and $y_n \cdot v_n = x_n$. Now since S is compact we may assume that both sequences $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ converging to $u \in S$ and $v \in S$ respectively. But since the operator

$$\cdot : S \times S \to S$$

is continuous we know that $x \cdot u = y$ and $y \cdot v = x$ which shows that $x \cdot S = y \cdot S$ and thus $x \, R \, y$. Similarly if $x_n \, L \, y_n$ for all $n \in \mathbb{N}$ we can find convergent sequences $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}$ converging to $u \in S$ and $v \in S$ respectively such that $u_n \cdot x_n = y_n$ and $v_n \cdot y_n = x_n$ but then $u \cdot x = y$ and $v \cdot y = x$ and thus $x \, L \, y$. Finally if $x_n \, J \, y_n$ we can find convergent sequences $\{t_n\}_{n\in\mathbb{N}}$, $\{u_n\}_{n\in\mathbb{N}}$, $\{v_n\}_{n\in\mathbb{N}}$ and $\{w_n\}_{n\in\mathbb{N}}$ converging to $t \in S$, $u \in S$, $v \in S$ and $w \in S$ respectively such that $t_n \cdot x_n \cdot u_n = y_n$ and $x_n = v_n \cdot y_n \cdot w_n$ but then $t \cdot x \cdot u = y$ and $x = v \cdot y \cdot w$ and thus $x \, J \, y$

$\square$

Now since H is the intersection of L and R we also know that H is a closed subset.

3.6 The Finite Sums and Products Theorem

Definition 3.6.1. *Let S be an infinite semigroup with $(x_s)_{s\in\mathbb{N}}$ a sequence in S. Let $\mathcal{P}_f(\mathbb{N})$ denote the set of all finite subsets of $\mathbb{N}$ and let $\prod_{n\in F} x_n$ denote the product in increasing order of indices $x_{n_1} \cdot x_{n_2} \cdot \ldots \cdot x_{n_m}$ for all $F \in \mathcal{P}_f(\mathbb{N})$. Then the set of all finite products is $FP(x_n)_{n\in\mathbb{N}} = \{\prod_{n\in F} x_n : F \in \mathcal{P}_f(\mathbb{N})$.*

Lemma 3.6.2. *Let S be a discrete semigroup with $p \in \beta S$ an idempotent and $A \in p$. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $FP(x_n)_{n\in\mathbb{N}} \subset A$.*

Proof. Now $p \in \beta S$ is an idempotent, so $p \cdot p = p$. But by Lemma 3.2.12 we know that $\{A \subset S : \Omega_p(A) \in p\} = p$ therefore $\Omega_p(A) = \{x \in S : x^{-1} \cdot A \in p\} \in p$. Let $B_1 = A$ and define a sequence of $B_n \in p$ such that $B_{n+1} = B_n \cap x_n^{-1} B_n$ where $x_n \in B_n \cap \Omega_p(B_n)$. Notice that this implies that $x_n^{-1} B_n \in p$ and so $B_{n+1} = B_n \cap x_n^{-1} B_n \in p$ as claimed. We claim that (x_n) is the required sequence. We proceed by induction on the length of the finite product. If $k = 1$ then

$$\prod_{i=1}^{k} = x_r \in B_r$$

for some $r \in \mathbb{N}$. So suppose the claim is true for all products $k < m$, then consider some finite product of elements of (x_n) of length $k = m$. Let F denote the set of all subscripts of that finite product with $r = \min(F)$ and $H = F \setminus \{r\}$ and $s = \min(H)$. Then

$$\prod_{n\in H} x_n \in B_s \subset B_{r+1} \subset x_r^{-1} B_r$$

But then

$$\prod_{n\in F} x_n = x_r \prod_{n\in H} x_n \subset x_r \cdot x_r^{-1} B_r \subset B_r \subset A$$

. □

Definition 3.6.3. *A collection of subsets $\mathcal{P}$ of a set S is called* Partition Regular *if for any finite partition $\{U_i\}$ of S there exists a n such that $U_n \in \mathcal{P}$.*

For example, any ultrafilter is partition regular. A partition regular collection is in some sense a large collection of subsets of S. We call a set an *IP* set if it contains any finite product of its elements. The Finite product theorem states that the family of all IP subsets of a semigroup are partition regular. We can now prove the original Finite Product Theorem.

Theorem 3.6.4 (Finite Product Theorem). *Let S be a semigroup and let A_i, $i = 1...r$ be a collection of finite disjoint subsets of S such that $S = \bigcup A_i$ i.e. The A_i form a finite partition of the semigroup. Then there exists a sequence (x_n) such that $FP(x_n) \subset A_m$ for some $m \in \{1...r\}$.*

Proof. Consider the Stone-Čech compactification of S when considered as a discrete topological space. Then by the Ellis theorem (theorem 3.4.2) we know that there exists an idempotent $p \in \beta S$ such that $A_m \in p$ for some $m \in \{1...r\}$. But then by Lemma 3.6.2 we know that there must exist a sequence (x_n) such that $FP(x_n)_{n\in\mathbb{N}} \subset A_m$ □

If the semigroup in question is Abelian then this theorem is also known as the Finite Sums Theorem (and was originally proved as such with $\mathbb{N}$ as the underlying semigroup see [Hin74]). We now give one application of the Finite Product theorem.

Theorem 3.6.5 (Van Der Warden). *[Ber03] Let S be a discrete Abelian semigroup with identity e and let $p \in \beta S$ a minimal idempotent. If $A \in p$ then A contains arbitrary long geometric sequences.*

Proof. Consider the compact, left right invariant semigroup given by the direct product of $k \in \mathbb{N}$ copies of βS. This is compact as a topological space since the product of compact spaces is compact and the continuity of the operator follows from the continuity of each projection onto the underlying space and the definition of the product topology. Let $p \in \beta S$ be any minimal idempotent of βS, whose existence is guaranteed by Theorem 3.4.9. Define $P = (p, p, ..., p) \in (\beta S)^k$ and consider the subsemigroup

$$E = cl(\{(a, a \cdot b, ..., a \cdot b^{k-1}) : a \in S \setminus \{e\}, b \in S\})$$

and the two sided ideal

$$I = cl(\{(a, a \cdot b, ..., a \cdot b^{k-1}) : a, b \in S \setminus \{e\}\})$$

Now $P \in E$ since if A_1, A_2, ..., $A_k \in p$ then $P \in \overline{A_1} \times \overline{A_2} \times ... \times \overline{A_k}$. But then if $a \in \bigcap A_i$ then

$$(a, a, ..., a) \in (\overline{A_1} \times \overline{A_2} \times ... \times \overline{A_k}) \cap \{(a, a \cdot b, ..., a \cdot b^{k-1}) : a \in S \setminus \{e\}, b \in S\}$$

which is our claim. Now let R be the minimal left ideal of βS such that $p \in R$. But $P \in E$ and so $E \cdot P$ is also a left ideal and thus their exists a minimal left ideal $\tilde{R}$

such that $\tilde{R} \subset E \cdot P$. Now since every minimal left ideal contains an idempotent, we can find a such an idempotent $q \in \tilde{R}$ such that $q = (q_1, q_2, ..., q_k)$. But then since $q \in E \cdot P$ there must exist $s \in E$ such that $q = s \cdot P$ where $s = (s_1, s_2, ..., s_k)$. But then $q_i \in R$ and since R is a minimal left ideal we know that $\beta S \cdot q_i = R$. This statement implies that $P = t_i \cdot q_i$ and thus

$$P \cdot q_i = t_i \cdot q_i \cdot q_i = t \cdot q_i = P$$

so $P = P + q$. But then $P \in R$ and so P is in the smallest two sided ideal in $(\beta S)^k$ and so $P \in I$. Now suppose $A \in p$ and so $P \in (\overline{A})^k$. But then $P \in (\overline{A})^k \cap I$ which shows that

$$A^k \cap \{(a, a \cdot b, ..., a \cdot b^{k-1}) : a, b \in S \setminus \{e\}\} \neq \emptyset$$

$\square$

Chapter 4

Decompositions of G^*

4.1 Maximal Principal Left Ideals

Definition 4.1.1. *If S is a discrete semigroup and $p \in \beta S$ then $(\beta S)p = \{(\beta S) \cdot p\} \cup \{p\}$ is the principal left ideal of βS generated by the point p.*

If G is a group or a semigroup with identity then trivially $(\beta S)p = (\beta S) \cdot p$. After studying minimal left ideals in the previous chapter, we move onto a class of maximal left ideal.

Definition 4.1.2. *A principal left ideal $(\beta S)\, p$ is maximal if and only if it cannot be properly contained in any other principal left ideal of βS.*

Since the union of arbitrary principal ideals need not be a principal ideal we cannot use the normal Zorn's lemma argument to prove that every principal ideal is contained in some maximal principal ideal. For example consider the semigroup $(\mathbb{Z}, \wedge)$ where $m \wedge n = min(m, n)$. Now the principal left ideals $\mathbb{Z}n = m : m \leq n$ satisfy $\mathbb{Z}n \subset \mathbb{Z}m$ for all $m \geq n$ and thus there exists no maximal principal left ideal in this semigroup.

Definition 4.1.3. *An element $p \in \beta S$ is called prime if $p \in S^* \setminus (S^*)^2$.*

By fluke of history, the literature has resorted to calling an element prime when its satisfies a irreducibility property. Now it can be shown that the algebraic center of βS is equal to the center of S for any weakly left cancellative semigroup which implies that βS can never be commutative for any weakly left cancellative semigroup. Thus we generally do not need to worry about the clash of nomenclature between the concept of a prime element on βS and the concept of a prime element on a commutaive semigroup with identity as the interesting algebra in βS only really occurs when S is at least weakly left cancellative.

Definition 4.1.4. *Denote by $\mathcal{P}$ the set of all prime elements of S^*.*

The principal left ideals generated by prime elements of the semigroup βS are the canonical examples of maximal principal ideals on βS.

Theorem 4.1.5. *Let G be a discrete group and let $p \in \beta G$ be a prime element. Then the principal ideal generated by p is maximal.*

Proof. Suppose there exists a principle left ideal I generated by some $q \in G^*$ such that $(\beta G)p \subset I$ but $r \in I \setminus (\beta G)p$. Then we know that there exists $x_1 \in \beta G$ such that $r = x_1 \cdot q$ and $x_2 \in \beta G$ such that $p = x_2 \cdot q$. However, since p is prime we know that $x_2 \in G$ and so its inverse exists and thus $q = x_2^{-1} \cdot p$. Then $r = x_1 \cdot x_2^{-1} \cdot p \in (\beta G)p$ which is a contradiction and thus $(\beta G)p$ is a maximal principal ideal. $\square$

We wish to characterize the subsets of βS which are in fact principal left ideals.

Definition 4.1.6. *Let S be a semigroup and let $p \in \beta S$. Define*

$$C(p) = \left\{A \subset S : \forall x \in S, x^{-1}A \in p\right\}$$

It is easy to see that $C(p)$ is a filter base on S since $x^{-1}(A \cap B) = (x^{-1}A) \cap (x^{-1}B)$ and thus $A, B \in C(p)$ implies that $A \cap B \in C(p)$.

Theorem 4.1.7. *If S is a semigroup and $p \in \beta S$ then*

$$\beta S \cdot p = \{q \in \beta S : C(p) \subset q\}$$

Proof. Let $r \in \beta S$ and suppose that $A \in C(p)$. Since $S = x \in S : x^{-1}A \in p = \Omega_p(A) \in r$ we have that $A \in r \cdot p$. Now let $q \in \beta S$ and suppose that $C(p) \subset q$. We know that the family of subsets $\mathcal{F}_A = \Omega_p(A)_{A \in q}$ form a base for some ultrafilter r on S. But then $r \in \beta S$ and by construction $q = r \cdot p$ $\square$

Recall that every closed subset of βS is of the form $q \in \beta S : \mathcal{F} \subset q$ for some filter $\mathcal{F}$ on S. In this case we can explicitly find the filter $\mathcal{F}$. Let $\mathcal{F}_p$ be the filter generated by $C(p)$ then trivially $q \in \beta S : \mathcal{F}_p \subset q = q \in \beta S : C(p) \subset q = \beta S \cdot p$ and thus $\beta S \cdot p$ is a closed subset of βS. Note that in general the right shift operator on βS is not a closed mapping and yet we have that the right shifts of the closed set βS are closed.

Lemma 4.1.8. *If S is a semigroup and $p \in \beta S$ then $\beta S \cdot p = \overline{S \cdot p}$.*

Proof. We know that $\overline{S \cdot p} \subset \beta S \cdot p$ since $S \cdot p \subset \beta S \cdot p$ and $\beta S \cdot p$ is closed. Now the function $\beta S \ni x \to x \cdot p$ is continuous and so $\overline{S} \cdot p \subset \overline{S \cdot p}$ but $\overline{S} = \beta S$ and so $\beta S \cdot p \subset \overline{S \cdot p}$ $\square$

If S is a semigroup with identity than the principal ideals of βS obviously cover the entire space of βS as $p \in \beta S \cdot p$ but they may be far from being pairwise disjoint. For example, it was shown in [HVMS92] that $\beta\mathbb{Z}$ contains a sequence $\{a_n\}_{n\in\mathbb{N}}$ such that $\beta\mathbb{Z} \cdot a_n \subset \beta\mathbb{Z} \cdot a_{n+1}$. However, if we limit our discussion to maximal principal left ideals of the Stone-Čech compactification of a discrete groups G which are countable than we do have a pairwise disjoint family of subsets of βG

Theorem 4.1.9. *Let G be a discrete group. If G is countable then any two distinct maximal principal left ideals of βG are disjoint.*

Proof. Let $p, q \in \beta G$ and suppose that the maximal principal ideals generated by p and q satisfy that

$$(\beta G)p \cap (\beta G q) \neq \emptyset$$

with $p \neq q$. Then $(\beta G)p$ and $(\beta G)q$ are closed subsets of a compact space and are thus σ-compact and so with out loss of generality we may assume

$$\overline{G \cdot p} \bigcap (G \cdot q) \neq \emptyset$$

However this implies $r \cdot p = s \cdot q$ for $r \in \beta S$ and $s \in S$ and thus $q = s^{-1} \cdot r \cdot p$ and so $(\beta G)q \subset (\beta G)p$ which implies that $(\beta G)q = (\beta G)p$. $\square$

Now every prime $p \in \mathcal{P}$ generates a maximal principal left ideal. So the number of maximal principal left ideals on a topological space is bounded below by $|\mathcal{P}|$. In this regard we investigate the cardinality of the set $\mathcal{P}$

Definition 4.1.10. *A subset A of a semigroup S is called* thin *if $x \cdot A \cap y \cdot A$ is finite, for all $x, y \in S$ with $x \neq y$.*

Lemma 4.1.11. *Let A be a subset of a group G. Then the following are equivalent*

1. *A is a thin set.*
2. *$x \cdot A \cap A$ is finite for all $x \in G \setminus \{e\}$.*
3. *if B is a finite subset G with $e \in B$ there exists a finite subset $C \subset G$ such that $B \cdot a \cap A = \{a\}$ for all $a \in A \setminus C$.*

Proof.

- (1) $\implies$ (2) : This follows trivially by considering $y = e$ and thus $x \cdot A \cap y \cdot A = x \cdot A \cap A$ is finite
- (2) $\implies$ (3): Let B be a finite subset of G with $e \in B$ and for all $b \in B \setminus \{e\}$ let $C_b = b^{-1}A \cap A$ which is a finite set and thus

  $$C = \bigcup_{b \in B \setminus \{e\}} C_b$$

  is a finite set. Then if $a \in A \setminus C$ and $b \in B \setminus \{e\}$ then $b \cdot a = x \in A$ implies that $a = b^{-1}x \in C_b$ which is a contradiction. So $B \cdot a \cap A = \{e\} \cdot a \cap A = \{a\}$
- (3) $\implies$ (1) : Suppose that A is not a thin set then there exists $x, y \in G$ with $x \neq y$ such that $x \cdot A \cap y \cdot A$ is infinite. Let $B = \{x\}$ then there exists finite C such that $B \cdot a \cap A = \{a\}$ for all $a \in A \setminus C$ but then $\{x \cdot a\} \cap A = \{a\}$which implies that $x = e$ similarly we can show that $y = e$ which is a contradiction

□

Thin sets are important structurally for a group. For example it can be shown that every infinite group is generated by some thin subset of itself (see [LP09]). Also, the cardinality of the intersection $x \cdot A \cap A$ can be made arbitrarily large and finite.

Lemma 4.1.12. *Let A be a thin subset of a group G. Then every element $p \in \overline{A} \setminus A$ is prime.*

Proof. Suppose that instead there exists $q, r \in G^*$ such that $p = qr$. But by the construction of the operation on βG we know that there must exist distinct $x, y \in G$ such that both $xr \in \overline{A}$ and $yr \in \overline{A}$ and so both $x^{-1}A$ and $y^{-1}A$ belong to r which is a free ultrafilter on G and thus contains no finite sets. Thus p must be an irreducible element. □

So every thin subset of a discrete group G generates a family of prime element in βG. So the question becomes one of determining how many thin subsets an infinite group G can have. It was shown in [Cho69] that every infinite subset of an obviously infinite group G contains a thin subset of the same cardinality as the original subset. So we do not have to remove too many points from any given subset to make it thin and thus thin subsets are in a sense very close to any subset in G.

Theorem 4.1.13. *Let S be an infinite subset of a group G then there exists a thin set $A \subset S$ with $|A| = |S|$.*

Proof. Let $x_1 \in S$ and define the subsets

$$X_\alpha = \{x_i^{\pm 1} \cdot x_j^{\pm 1} \cdot x_k^{\pm 1} : i, j, k < \alpha\}$$

for every ordinal $\alpha < |S|$ and where $x_\alpha \in S \setminus X_\alpha$ can be chosen by transfinite induction since $|X_\alpha| < \alpha < |S|$. Let $A = \{x_\alpha : \alpha < |S|\}$ then it is easy to see that $|A| = |S|$. Let $a, b \in G$, $a \neq b$ and suppose $aA \cap bA$ was countably infinite then $ax_{\alpha_n} = bx_{\beta_n}$ for some ordinal sequences $\alpha_n, \beta_n < |S|$ but then $x_{\alpha_n} \cdot x_{\beta_n}^{-1} = ba^{-1}$ for all $n < \omega$ and by taking subsequences we can assume that both (α_n) and (β_n) are increasing. But then

$$x_{\alpha_n} \cdot x_{\beta_n}^{-1} = x_{\alpha_{n+1}} \cdot x_{\beta_{n+1}}^{-1} \implies x_{\alpha_n} \cdot x_{\beta_n}^{-1} \cdot x_{\beta_{n+1}} = x_{\alpha_{n+1}}$$

and $x_{\alpha_{n+1}} \cdot x_{\beta_n} \cdot x_{\alpha_n}^{-1} = x_{\beta_{n+1}}$and since $a \neq b$ we must have that either $\alpha_{n+1} > \beta_{n+1}$ or $\alpha_{n+1} < \beta_{n+1}$ which in both cases contradicts the definition of the subsets X_α □

Theorem 4.1.14. *Let G be a discrete group then βG contains $2^{2^{|G|}}$maximal principle left ideals.*

Proof. We know that G contains a thin set A such that $|A| = |G|$. But then every $p \in \overline{A} \setminus A$ is a prime element of βG and generates a maximal principle left ideal. Furthermore $|\overline{A} \setminus A| = |\beta G \setminus G| = 2^{2^{|G|}}$ □

Now the space βG has $2^{2^{|G|}}$ members and $2^{2^{|G|}}$ maximal principal left ideals. Since almost nothing in mathematics is a coincidence this suggests an important and to date very difficult question to answer. Does every point in G^* lie in some maximal principal left ideal? This question was first asked to M. Rudin in the 1980's with regard to the maximal orbital closures of the shift function on $\beta\mathbb{Z} \setminus \mathbb{Z}$ and was asked in relation to dynamical system theory.

4.2 Upper Semicontinuous Left Ideal Decompositions

Definition 4.2.1. *The decomposition $\mathcal{F}$ of some space X is finer than some other decomposition $\mathcal{G}$ of X if for all $B \in \mathcal{G}$ and for all $b \in B$ there exists $A \in \mathcal{F}$ such that $b \in A \subset B$.*

Definition 4.2.2. *Let G be a countable, infinite group. Denote by $\mathcal{I}(G)$ the finest decomposition of G^* into closed left ideals of βG with the property that the corresponding quotient space of G^* is Hausdorff.*

Separation properties are in general not nice topological properties as in they are not preserved by many of the operations one would like to make on the spaces in question. For example the arbitrary product of normal spaces need not be normal (or worse even, subspaces of a normal space need not be normal) and the quotient spaces of a Hausdorff space need not be Hausdorff. The family of compact Hausdorff topological spaces is in fact the natural category in which to do topological analysis (at least according to the Category Theorists) , and it is in this space that we find a reasonable characterization of when a quotient topology inherits its separation properties from its parent space.

Definition 4.2.3. *A decomposition D of a topological space X into closed subsets is called upper semi-continuous if for every open subset $U \subset X$ we have that the set*

$$\bigcup\{A \in D : A \subset U\}$$

is open.

Upper semi-continuous decompositions come from the work of R.L. Moore (See [Moo25]) where he proved that if $\mathcal{D}$ is a decomposition of the plane $\mathbb{R}^2$ into compact connected subsets such that the decomposition is upper semicontinuous and no element $D \in \mathcal{D}$ separates $\mathbb{R}^2$ then $\mathbb{R}^2/\mathcal{D}$ is homeomorphic to $\mathbb{R}^2$. We can give two slightly easier equivalent definitions of an upper semicontinuous decompositions almost immediately.

Lemma 4.2.4. *A decomposition D of a topological space X is upper semicontinuous if and only if $\bigcup\{A \in D : A \cap C \neq \emptyset\}$ is closed for all closed $C \subset X$* .

Proof. It is easy to see that

$$X = \bigcup A \in D : A \cap C \neq \emptyset \cup \bigcup A \in D : A \subset X \setminus C$$

for any closed $C \subset X$ and that the two unions have to be disjoint. Now D being upper semicontinuous implies that $\bigcup A \in D : A \subset X \setminus C$ must be a open set but then

$$\bigcup A \in D : A \cap C \neq \emptyset = X \setminus \bigcup A \in D : A \subset X \setminus C$$

must be a closed set. Similarly if $U \subset X$ is some open subset then $C = X \setminus U$ implies that

$$\bigcup A \in D : A \subset U = \bigcup A \in D : A \subset X \setminus C = X \setminus \bigcup A \in D : A \cap C \neq \emptyset$$

must be open and thus D must be upper semicontinuous. □

Lemma 4.2.5. *A decomposition D of a topological space X is upper semicontinuous if and only if for all $A \in D$ and open $U \supset A$ there exists an open $V \supset A$ such that if $B \in D$ and $V \cap B \neq \emptyset$ then $B \subset U$.*

Proof. Let $A \in D$ and let $U \subset X$ be an open subset with $A \subset U$. We know that $V = \bigcup\{A \in D : A \subset U\}$ is an open non-empty subset of X and that $A \subset V$ and that if $B \in D$ and $V \cap B \neq \emptyset$ then since D is a decomposition we must have $B \in \mathcal{A}$ and thus $B \subset U$. So suppose for all $A \in D$ and open $U \supset A$ there exists an open $V \supset A$ such that if $B \in D$ and $V \cap B \neq \emptyset$ then $B \subset U$ but $\bigcup\{A \in D : A \subset U\}$ is not an open subset from some open $U \subset X$. We know then that there must exist a point

$$x \in \bigcup\{A \in D : A \subset U\} \neq \emptyset$$

(since the empty set is open) such that for all neighbourhoods $W \ni x$,

$$W \cap X \setminus \bigcup\{A \in D : A \subset U\} \neq \emptyset$$

Let $A \in D$ be the element of the decomposition that contains this point x then we know that there exists some $V \supset A \ni x$ such that if $B \in D$ and $V \cap B \neq \emptyset$ then $B \subset U$. But then since

$$V \cap X \setminus \bigcup\{A \in D : A \subset U\} \neq \emptyset$$

we can choose

$$y \in V \cap X \setminus \bigcup\{A \in D : A \subset U\}$$

and $B \in D$ such that $y \in B$. Then $V \cap B \neq \emptyset$ but $B \not\subset U$ which is a contradiction. □

A classical example of an upper semicontinuous decomposition is the following. Let $f : \mathbb{R} \to [0,1]$ be a function and consider the decomposition $\mathcal{D}$ of $\mathbb{R}\times[0,1]$ consisting of elements of the form $A_x = x \times [0, f(x)]$ and the singletons (x, y). It is easy to see that $\mathcal{D}$ is upper semicontinuous if and only if the function f is upper semicontinuous in the classic sense $(\forall\epsilon > 0, \exists\delta > 0, t \in [x, x+\delta) \implies |f(t) - f(x)| < \epsilon)$. Now consider the space $\mathbb{R}^2$ and let $\mathcal{D}$ be the set of all vertical parallel lines orthogonal to the x-axis line i.e. $\mathcal{D} = \{x \times \mathbb{R} : x \in \mathbb{R}\}$ then $\mathcal{D}$ is not upper semicontinuous.

Definition 4.2.6. *Let X and Y be spaces then a function $f : X \to Y$ is closed if $f(A)$ is closed for any closed $A \subset X$.*

There is a very clear relationship between the quotient map being closed and the resulting quotient space being Hausdorff. Any closed map preserves the T_1 nature of the preimage space and since any Hausdorff space is T_1 this property is required if we have any hope of the quotient space being Hausdorff.

Theorem 4.2.7. *Let X be a topological space and D be a decomposition of X. The natural quotient map $\pi : X \to X/D$ is closed if and only if D is upper semi-continuous .*

Proof. Let $\pi : X \to X/D$ be the natural quotient map (which is continuous by the definition of the quotient topology). We claim that for arbitrary open $U \subset X$ we have that

$$\begin{aligned}\pi^{-1}((X/D) \setminus \pi(X \setminus U)) &= \bigcup\{A \in D : \pi(A) \\ &\subset (X/D) \setminus \pi(X \setminus U)\} \\ &= \bigcup\{A \in D : A \subset U\}\end{aligned}$$

Since if $a \in A$ then $\pi(a) \in \pi(A) \subset (X/D) \setminus \pi(X \setminus U)$ and so

$$\pi(a) \notin \pi(X \setminus U) \implies a \notin X \setminus U \implies a \in U$$

If $C \subset X$ is a closed subset of X then $X \setminus C$ is an open subset and thus $\bigcup\{A \in D : A \subset X \setminus C\}$ is open. But thus set is exactly the set

$$\pi^{-1}((X/D) \setminus \pi(C)) = \pi^{-1}(\pi(X \setminus C))$$

and thus $(X/D) \setminus \pi(C)$ is open in X/D i.e. $\pi(C)$ is closed. Similarly if π is a closed map then we can reverse all the steps and see that $\bigcup\{A \in D : A \subset X \setminus C\}$

is open for arbitrary closed $C \subset X$ i.e. D is upper semi-continuous. $\square$

Theorem 4.2.8. *Let X be a compact space and Y be a Hausdorff space and $f : X \to Y$ be a continuous function then f is a closed map .*

Proof. Let A be a closed subset of X, then A is compact and thus $f(A)$ is a compact subset of the Hausdorff space Y. But any compact subspace of a Hausdorff space must be closed and thus $f(A)$ is closed $\square$

Theorem 4.2.9. *Let X be a compact, Hausdorff space and D be a decomposition of X. The quotient space X/D is Hausdorff if and only if D consists of closed subsets and the quotient map π is closed.*

Proof. Let $\pi : X \to X/D$ be the natural quotient map. If X/D is Hausdorff then points $y \in X/D$ are closed and thus D must consist of closed subsets of X as each D is the preimage of some point in X/D. Since the natural projection map is continuous, we have by Theorem 4.2.8 that the quotient map must be closed. So suppose that D only consists of closed subsets of X and that the quotient map is a closed map. Let x and y be points of X/D then there exist closed elements $D_1$and $D_2$of D such that $D_1 = \pi^{-1}(x)$ and $D_2 = \pi^{-1}(y)$. Since Hausdorff and compact implies normal there exist disjoint open subsets $U_1$and U_2 such that $D_1 \subset U_1$and $D_2 \subset U_2$. But then

$$V_1 = (X/D) \setminus \pi(X \setminus D_1)$$

and

$$V_2 = (X/D) \setminus \pi(X \setminus D_2)$$

are disjoint open subsets of X/D with $x \in V_1$and $y \in V_2$ $\square$

We now have sufficient technological machinery to investigate which decompositions of βS result in Hausdorff topologies. Note that Hausdorff decompositions are a careful balancing act between the number of new open sets generated and the size of the resulting set. It was shown in [Arh69] that X being Hausdorff implies that $|X| \leq 2^{L(X)\chi(X)}$ where $L(X)$ is the Lindelöf number and $\chi(X)$ is the character of the space. Since we are working in the quotient spaces of compact sets we have that $L(X) < \omega$ and so $|X| \leq 2^{\chi(X)}$. After our brief detour into the decomposition theory we are ready to consider the question for closed left ideals of βS.

Definition 4.2.10. *Let S be a discrete semigroup with identity e and let $p \in S^*$. Denote by $\mathcal{F}_p$ the filter on S generated by subsets of the form $\bigcup_{x \in S} x \cdot (A \setminus F_x)$ where $A \in p$ and F_x is a finite subset of S for each $x \in S$.*

Recall that $p \in S^*$ implies that p contains the Fréchet filter and thus $A \setminus F \in p$ for all finite F. The generating sets look very similar to the standard left ideals on S, in that we have the space S multiplied by the generator of the ideal on the right. We will show that this does generate the closed left ideals of βS. In fact we will later show that the left ideals of βG are exactly generated by the left shift invariant filters on G.

Theorem 4.2.11. *If S is a discrete semigroup with identity e then for all $p \in S^*$ we have that $\mathcal{F}_p \subset p$.*

Proof. The elements p of S^* are the non-principal ultrafilters on S and thus $A \setminus F \in p$ for any finite family of points F. But then $(A \setminus F_e) \subset \bigcup_{x \in S} x \cdot (A \setminus F_x)$ for any choice of F_x and thus $\mathcal{F}_p \subset p$ □

Definition 4.2.12. *Let $p \in S^*$ and denote by $I_p = \bigcap_{B \in \mathcal{F}_p} \overline{B} = \overline{\mathcal{F}_p}$.*

The method being used here is surprisingly powerful in the study of βS. If you can find a filter that encodes the behaviour you want the closed subsets of βS to exhibit then you can use that filter to generate the required closed subsets.

Theorem 4.2.13. *Let S be a discrete semigroup with identity e and let $p \in S^*$. Then I_p is a closed left ideal of βS .*

Proof. We know that $I_p = \{q \in \beta S : \mathcal{F}_p \subset q\}$ is a closed subset of βS. So we need to show that if $q \in \beta S$ with $\mathcal{F}_p \subset q$ and $r \in \beta S$ then $\mathcal{F}_p \subset r \cdot q$. Let $U \in \mathcal{F}_p$ then this implies the existence of a set $A \in P$ and a family of finite subsets F_x such that $\bigcup_{x \in S} x \cdot (A \setminus F_x) \subset U$. Let $V \in r$ and for all $y \in V$ define the set

$$B_y = \bigcup_{x \in S} x \cdot (A \setminus F_{yx}) \in \mathcal{F}_p \subset q$$

We have then that

$$y \cdot B_y = \bigcup_{x \in S} (y \cdot x) \cdot (A \setminus F_{yx}) \subset \bigcup_{x \in S} x \cdot (A \setminus F_x)$$

and so

$$\bigcup_{y \in V} y \cdot B_y \subset U$$

for $V \in r$ and $B_y \in q$. From this we see

$$U \in r \cdot q \implies \mathcal{F}_p \subset r \cdot q \implies r \cdot q \in I_p$$

for all $r \in \beta G$ $\square$

Now in general, a family of closed left ideals of a semigroup need not be a decomposition. A left ideal chain semigroup is one in which the left ideals are linearly ordered by set inclusion and is thus an example of a semigroup where attempts to decompose them by left ideals can fail spectacularly. Once again, the semigroup $\mathbb{Z}$ with the minimum operator and the discrete topology is the standard example of such a semigroup as $\mathbb{Z}n$ is a closed left ideal but $\mathbb{Z}n \subset \mathbb{Z}m$ if $n < m$. We require a technical lemma to show that we in fact do have a decomposition in the case of βG where G is a group.

Lemma 4.2.14. *Let G be a countably infinite discrete group, let $A \subset G$ be an infinite subset and let F_x be a finite subset of G for every $x \in G$. If*

$$B = G \setminus \bigcup_{x \in G} x \cdot (A \setminus F_x)$$

then there exists finite subsets H_x and K_x of G for every $x \in G$ such that

$$(\bigcup_{x \in G} x \cdot (A \setminus H_x)) \cap (\bigcup_{x \in G} x \cdot (B \setminus K_x)) = \emptyset$$

Proof. Since G is countably infinite we can enumerate it as $G = \{x_n : n \in \mathbb{N}\}$. Define

$$H_{x_n} = \bigcup_{i \leq n} F_{x_i^{-1} x_n}$$

and

$$K_{x_n} = \bigcup_{i \leq n} x_n^{-1} \cdot x_i F_{x_n^{-1} x_i}$$

thus

$$x_i^{-1} x_n (A \setminus H_{x_n}) \cap B = \emptyset$$

and

$$x_n^{-1} x_i A \cap (B \setminus K_{x_n}) = \emptyset$$

This implies that

$$x_n (A \setminus H_{x_n}) \cap x_i B = \emptyset$$

and

$$x_i A \cap x_n (B \setminus K_{x_n}) = \emptyset$$

and thus

$$x_n(A \setminus H_{x_n}) \cap x_m(B \setminus K_{x_m}) = \emptyset$$

which shows that

$$(\bigcup_{x\in G} x \cdot (A \setminus H_x)) \cap (\bigcup_{x\in G} x \cdot (B \setminus K_x)) = \emptyset$$

□

Note that the previous lemma can be extended to work for all groups where $|G|$ is a regular cardinal but this is not necessary for the application at hand. We can prove some basic results regarding the family $\{I_p : p \in G^*\}$

Lemma 4.2.15. *If G is a countably infinite discrete group, $p \in G^*$ and $q \in I_p$ then $I_q \subset I_p$.*

Proof. If $\mathcal{F}_p \subset \mathcal{F}_q$ then for all $x \in I_q$ we would have that $\mathcal{F}_p \subset \mathcal{F}_q \subset x$ i.e. $x \in I_p$. So let $C \in \mathcal{F}_p$ and suppose that $C \notin \mathcal{F}_q \subset q$. Since q is an ultrafilter we have that $G \setminus C \in q$. Since $C \in \mathcal{F}_p$ there must exist an $A \in p$ and a family of finite subsets F_x such that $\bigcup_{x\in G} x \cdot (A \setminus F_x) \subset C$ but this implies that $B = G \setminus \bigcup_{x\in G} x \cdot (A \setminus F_x) \in q$. Now there exist finite subsets H_x and K_x such that

$$(\bigcup_{x\in G} x \cdot (A \setminus H_x)) \cap (\bigcup_{x\in G} x \cdot (B \setminus K_x)) = \emptyset$$

However, $\bigcup_{x\in G} x \cdot (A \setminus H_x) \in \mathcal{F}_p \subset q$ since $q \in I_p$and $\bigcup_{x\in G} x \cdot (B \setminus K_x) \in \mathcal{F}_q \subset q$ since $q \in I_q$ and thus $\emptyset = (\bigcup_{x\in G} x \cdot (A \setminus H_x)) \cap (\bigcup_{x\in G} x \cdot (B \setminus K_x)) \in q$ which is a contradiction. Thus $C \in \mathcal{F}_q$ and so $\mathcal{F}_p \subset \mathcal{F}_q$ □

Lemma 4.2.16. *If G is a countably infinite discrete group, $p \in G^*$ and $q \in I_p$ then $I_p \subset I_q$.*

Proof. If $\mathcal{F}_q \subset \mathcal{F}_p$ then for all $x \in I_p$ we would have that $\mathcal{F}_q \subset \mathcal{F}_p \subset x$ i.e. $x \in I_q$. So let $C \in \mathcal{F}_q$ and suppose that $C \notin \mathcal{F}_p \subset q$ as $q \in \mathcal{F}_p$. Since q is an ultrafilter we have that $G \setminus C \in q$. Since $C \in \mathcal{F}_q$ there must exist an $A \in q$ and a family of finite subsets F_x such that $\bigcup_{x\in G} x \cdot (A \setminus F_x) \subset C$ but this implies that $G \setminus C \subset B = G \setminus \bigcup_{x\in G} x \cdot (A \setminus F_x) \in q$. However, $\bigcup_{x\in G} x \cdot (A \setminus H_x) \in \mathcal{F}_q \subset q$ and thus $\emptyset = (G \setminus \bigcup_{x\in G} x \cdot (A \setminus F_x)) \cap (\bigcup_{x\in G} x \cdot (B \setminus F_x)) \in q$ which is a contradiction. Thus $C \in \mathcal{F}_p$ and so $\mathcal{F}_q \subset \mathcal{F}_p$ □

For example, if $p \in \mathbb{Z}^* \cap \overline{\{2^n : n \in \mathbb{N}\}}$ then it is easy to see that $q = p + p \in \beta G \cdot p \subset I_p$ and also that $q \in I_q$. But $q \in I_p$ implies that $I_p \subset I_q$ and $I_q \subset I_p$and thus $I_p = I_q$. In fact since $\mathcal{F}_q \subset \mathcal{F}_p$ for all $q \in I_p$ we must have that $p \in I_q$ as $\mathcal{F}_q \subset \mathcal{F}_p \subset p$. Also if the point p is p-point then it can be shown that $I_p = \beta G \cdot p$, a principal left ideal. But since $p \notin \beta G \cdot q$ we know that $I_q \neq \beta G \cdot q$ i.e. if p is a p-point that for all $q \in \beta G \cdot p \setminus \{p\}$, q is not a p-point.

Theorem 4.2.17. *Let G be a countably infinite discrete group and let $p, q \in G^*$. Then $I_p \cap I_q \neq \emptyset$ implies that $I_p = I_q$.*

Proof. Let $q \in G^*$ and suppose that $\mathcal{F}_p \not\subset q$ (i.e. $q \notin I_p$) then there must exist $A \in p$ and F_x finite such that

$$B = G \setminus \bigcup_{x \in G} x \cdot (A \setminus F_x) \in q$$

but then we can find sets such that

$$(\bigcup_{x \in G} x \cdot (A \setminus H_x)) \cap (\bigcup_{x \in G} x \cdot (B \setminus K_x)) = \emptyset$$

But this implies that $\mathcal{F}_p \not\subset r$ for all $r \in I_q$since then $\mathcal{F}_p \subset r$ and $\mathcal{F}_q \subset r$ and so could not have empty intersections as subsets of an ultrafilter and so $I_p \cap I_q = \emptyset$ $\square$

So we have constructed a family of closed left ideals of βG which form a decomposition of the space if it is countable. We now proceed to investigate the relationship between this ideal decomposition and any other left ideal decomposition on βG.

Theorem 4.2.18. *If G is a countable discrete group then $\mathcal{I}(G) = \{I_p : p \in G^*\}$.*

Proof. We know that every I_p is a closed left ideal of βG and that $p \in I_p$. Further more we know that $\{I_p : p \in G^*\}$ forms a decomposition of G^*.

- We claim that $\mathcal{D}$ =$\{I_p : p \in G^*\}$ is upper semi-continuous. Let I_p be an arbitrary member of $\mathcal{D}$ and let $I_p \subset \overline{W}$ for some $W \subset G$. This implies that for all $r \in I_p$ we have that $W \subset r$ but each $r \in I_p$ is generated by sets of the form $\bigcup_{x \in G} x \cdot (A \setminus F_x)$ where $A \in p$ and F_x is a choice of finite subsets of G and thus with out loss of generality we can assume that $W = \bigcup_{x \in G} x \cdot (A \setminus F_x)$ for some $A \in p$ and choice F_x of finite subsets of G. But we know that there exists finite subsets H_x and K_x of G for every $x \in G$ such that

$$Q = \bigcup_{x \in G} x \cdot (A \setminus H_x)$$

and

$$R = \bigcup_{x \in G} x \cdot (B \setminus K_x)$$

and $Q \cap R = \emptyset$ where $B = G \setminus U$. Now suppose that $I_q \cap \overline{Q} \neq \emptyset$ then we know that for all $r \in I_q$, $Q \in r$. Now suppose $U \notin r$ then $B = G \setminus U \in r$. But his implies that

$$Q \cap (B \setminus K_e) \subset Q \cap \bigcup_{x \in G} x \cdot (B \setminus K_x) \subset Q \cap R = \emptyset$$

and so $Q \cap B \subset K_e$ for some finite subset $K_e \subset G$. But this contradicts the fact that $r \in G^*$ and thus $U \in r$ i.e. $I_r \subset \overline{U}$

- We claim that $\{I_p : p \in G^*\}$ is the finest decomposition of G^* into closed left ideals of βG such that the corresponding quotient space is Hausdorff. For suppose that $\mathcal{J}$ is some other decomposition of G^* into closed left ideals such that the quotient space is Hausdorff. We want to show that $I_p \subset J$ for all $p \in J \in \mathcal{J}$. We know that $\mathcal{J}$ must be upper semicontinuous thus for all $J \in \mathcal{J}$ and open $U \supset J$ there exists an open $V \supset J$ such that if $B \in \mathcal{J}$ and $V \cap B \neq \emptyset$ then $B \subset U$. Without loss of generality we may assume $V = U$. But then $G \cdot U \subset U$ since the decomposition $\mathcal{J}$ is made up of left ideals of βG and since $p \in U$ there must exist a $A \in p$ such that $\overline{A} \setminus A \subset U$ and so $G \cdot (\overline{A} \setminus A) \subset U$ and thus

$$\bigcap_{A \in p} \overline{G \cdot (\overline{A} \setminus A)} \subset \overline{U}$$

We claim that

$$I_p \subset \bigcap_{A \in p} \overline{G \cdot (\overline{A} \setminus A)}$$

Let $q \in I_p$ which implies that $\mathcal{F}_p \subset q$ and let $A \in p$ then we know that $\bigcup_{x \in G} x \cdot (A \setminus F_x) \in q$ for any finite choices F_x. So suppose

$$\bigcap_{A \in p} \overline{G \cdot (\overline{A} \setminus A)} \subset \overline{B}$$

for some $B \subset G$ then there must exist an $A \in p$ such that

$$\overline{G \cdot (\overline{A} \setminus A)} \subset \overline{B}.$$

But then we must have that $x \cdot (\overline{A} \setminus A) \subset \overline{B}$ for all $x \in G$ and thus we can choose finite subsets F_x such that $x \cdot (A \setminus F_x) \subset B$. This implies that $B \in \mathcal{F}_p$ and thus

$$q \in \bigcap_{A \in p} \overline{G \cdot (\overline{A} \setminus A)}$$

Thus for all neighbourhoods U of $J \in \mathcal{J}$ and points $p \in J$ we have that

$$I_p \subset \bigcap_{A \in p} \overline{G \cdot (\overline{A} \setminus A)} \subset \overline{U}$$

However, $\mathcal{J}$ consists of closed sets and thus $I_p \subset J$ and so

$$I(G) = \{I_p : p \in G^*\}$$

$\square$

4.3 Hausdorff topologies generated by Ideal Decompositions

Definition 4.3.1. *An ultrafilter $p \in \beta X$ is called uniform if for all $A \in p$ we have that $|A| = |X|$.*

Definition 4.3.2. *Denote by $U(X)$ the set of all uniform ultrafilters on the set X.*

If G is a group we know in fact that $U(X)$ is a two sided ideal of βX. For countable X it should be noted that $U(X) = X^*$ as every element of every ultrafilter in X^* is countable.

Theorem 4.3.3 (The left ideal theorem [DH91])**.** *If S is a cancellative semigroup with $|S| = \lambda \geq \omega$ then there exists a decomposition $\mathcal{I}$ of $U(S)$ into pairwise disjoint subsets such that*

- $|\mathcal{I}| = 2^{2^\lambda}$,
- *For each $I \in \mathcal{I}$, I is nowhere dense in $U(S)$ and*
- *For each $I \in \mathcal{I}$, I is a left ideal of βS.*

This theorem was strengthened in [Pro05], [FS07], [TZ12] and [Zel13] to show that for any group G, the set G^* can be decomposed into $2^{\mathfrak{c}}$ closed left ideals of βG so that the quotient space of G^* is Hausdorff. The original proof for regular cardinals was given in [Pro05] with the extension to all cardinals in [FS07].

Definition 4.3.4. *A topological group is maximally almost periodic if it can be continuously embedded into a compact topological group.*

Now any Abelian or free group can be algebraic embedded into a compact group and by endowing them with the discrete topology we can see that any Abelian or free group is maximally almost periodic.

Definition 4.3.5. *Let $(m_n)_{n<\omega}$ be a sequence of integers ≥ 2. Denote by $\oplus_{n<\omega}\mathbb{Z}_{m_n}$ the direct sum of the finite groups $\mathbb{Z}_{m_n}$ with the topology induced from the product topology on $\prod_{n<\omega} \mathbb{Z}_{m_n}$.*

Recall that the direct sum of an infinite family of groups is defined to be the subset of the product space $\prod_{\alpha\in\Lambda} G_\alpha$ where all but finitely many of the co-ordinates of any point in $\oplus_{\alpha\in\Lambda}G_\alpha$ are equal to the identities of the corresponding groups and with the group operation defined component wise.

Theorem 4.3.6. *Let G be a countable maximally almost periodic topological group. There exists a sequence $(m_n)_{n<\omega}$ of integers ≥ 2 and a bijection $h : G \to \oplus_{n<\omega}\mathbb{Z}_{m_n}$ with $h(e) = 0$ such that*

1. *$h(x \cdot y) = h(x) + h(y)$ for $r(x) + 2 \leq l(y)$ where $r(x) = \max supp(h(x))$ and $l(y) = \min supp(h(y))$.*

2. *$r(x^i y^j) \leq \max\{r(x), r(y)\} + 1$ for all $i, j \in \{1, -1\}$, and if $|r(x) - r(y)| \leq 2$, then $r(x^i y^j) \geq \max\{r(x), r(y)\} - 1$.*

3. *if $m \geq 1$ and $h(x)(n) = h(y)(n)$ for all $n \leq m$ then $l(x^{-1}y) \geq m - 1$ and $l(xy^{-1}) \geq m - 1$.*

Proof. See [Zel07] □

This shows that the countable maximally almost periodic topological groups are locally homeomorphic to the direct sum of additive group of the integers modulo m_n.

Lemma 4.3.7. *If G is a countably infinite group and $p \in G^*$ is a P-point then $I_p = (\beta G)p$.*

Proof. Choose $A \subset G$ and suppose that $(\beta G)p \subset \overline{A}$. For each $x \in G$ choose $A_x \in p$ such that $x \cdot A_x \subset A$. Since p is a P-point there must exist a $B \in p$ such that $|B \setminus A_x| < \infty$ for all $x \in G$. Then if

$$C = \bigcup_{x \in G} x \cdot (B \setminus (B \setminus A_x))$$

we have that $I_p \subset \overline{C}$. Since

$$C \subset \bigcup_{x \in G} x \cdot A_x \subset A$$

and $I_p \subset \overline{A}$ we have enough to show that $I_p = (\beta G)p$ as the smallest ideal containing p, $(\beta G)p$, must be a subset of I_p. □

So if p is a P-point then the decomposition ideal I_p is in fact a principal left ideal. We require a few technical lemmas before we can prove our first main theorem.

Lemma 4.3.8. *Let $x, y \in G$ and suppose that there is $n \in supp((h(y))$ with $n \geq r(x) + 2$. Then $l(xy) \leq n$.*

Proof. Let $z = x \cdot y$. Assume on the contrary that $l(z) > n$. Then $y = x^{-1}z$ and $r(x^{-1}) < n$ from 4.3.6. This implies that $h(y) = h(x^{-1}) + h(z)$ and thus $n \notin$supp$(h(y))$, which is a contradiction. □

Lemma 4.3.9. *For every $p \in G^*$, there is at most one $a \in G$ such that $l(a \cdot p) \in \omega^*$.*

Proof. Suppose that p can be written as $p = bp_0$, where $b \in G$, $p_0 \in G^*$, and $l(p_0) \in \omega^*$. Let $a \in G$. If $a = b^{-1}$, then $l(a \cdot p) = l(p_0) \in \omega*$. Otherwise $ab = 1$ and $l(a \cdot p) = l(a \cdot b) \in \omega$. Now suppose that we cannot write p this way. Then for every $m < \omega$ there are $n > m$ and $A \in p$ such that $n \in$supp$(h(b))$ for all $b \in A$. Let $a \in G$. Pick $n \geq r(a) + 2$ and $A \in p$ such that $n \in$ supp$(h(b))$ for all $b \in A$, then $l(ab) \leq n$. Thus, $\bar{l}(a \cdot p) \in \{0, 1, ..., n\}$. □

Lemma 4.3.10. *Let $(q_n)_{n<\omega}$ be a sequence of prime elements in G^* with $\bar{l}(q_n) \in \omega^*$ and let $q \in \overline{\{q_n : n < \omega\}}$. Then q is prime and $\bar{l}(q) \in \omega^*$.*

Proof. It is clear that $\bar{l}(q) \in \omega^*$. Suppose that $q = r \cdot s$ for some $r, s \in G^*$. Since

$$q \in (\overline{\{q_n : n < \omega\}}) \bigcap (G^s s)$$

we have that

$$(\overline{\{q_n : n < \omega\}}) \bigcap (G^s s) = \emptyset$$

So either $q_n = r's$ for some $n < \omega$ and $r \in G^s$ or $q = b \cdot s$ for some

$$q \in \overline{\{q_n : n < \omega\}}$$

and $b \in G^s$. Since q_n is prime, the first possibility implies that $q_n = c \cdot s$ for some $c \in G^s$. And since $\bar{l}(q_n), \bar{l}(q) \in \omega^*$ and $\bar{l}(b \cdot s), \bar{l}(c \cdot s) \in \omega$, we have a contradiction. □

Lemma 4.3.11. *Let $I \in I(G)$, let $p \in I$, and let $u = \bar{r}(p)$. Then either $\bar{r}(I) \subset \{u, u+1\}$ or $\bar{r}(I) \subseteq \{u-1, u\}$.*

Proof. We first show that $\bar{r}(I) \subset \{u-1, u, u+1\}$. Pick $U \in u$ such that $U - 1, U, U+1$ are pairwise disjoint, and pick $A \in p$ such that $\bar{r}(A) \subseteq U$. For every $x \in G$, let

$$F_x = \{y \in G : r(y) \leq r(x) + 1\}$$

and let

$$C = \bigcup_{x \in G} x \cdot (A \setminus F_x)$$

Then $I \subseteq C$ and

$$r(C) \subseteq (U-1) \bigcup U \bigcup (U+1)$$

Now if $r(I) \subseteq \{u, u+1\}$, we are done. Otherwise there is $p \in I$ such that $r(p) = u-1$, consequently $r(I) \subseteq \{u-2, u-1, u\}$, and so $r(I) \subseteq \{u-1, u\}$. $\square$

Lemma 4.3.12. *LetI* $\in \mathcal{I}(G)$, *let* $Q = \{p \in I : p \in \mathcal{P} \wedge \bar{l}(p) \in \omega^*\}$, *and let* $u \in \overline{r}(I)$. *If there is* $p \in Q$ *which is isolated in* I *and which is not a* P *-point in* G^*, *then* $|Q| = 2^{\mathfrak{c}}$.

Proof. Pick a decreasing sequence $(A_n)_{n<\omega}$ of members of p and a sequence $(a_n)_{n<\omega}$ in G such that

1. $\overline{A_0} \cap I = \{p\}$,
2. $\bigcap_{n<\omega} A_n = \emptyset$,
3. For all $A \in p$, there is $n < \omega$ such that $A \cap B_n$ is infinite, where $B_n = A_n \setminus A_n + 1$,
4. The sets an A_n,$n < \omega$, are pairwise disjoint, and
5. $l(a_n x) \geq$n for all $n < \omega$ and $x \in An$.

Define $f : G \to G$ by $f(x) = a_n x$ if $x \in B_n$ and f(x) $= 1$ if $x \in G \setminus A_0$. Let $q = \overline{f}(p)$, so q has a base consisting of subsets of the form $\bigcup_{n<\omega} a_n(A \cap Bn)$, where $A \in p$. Clearly $q \neq p$. Since $I \subset G^*$ has a neighbourhood base consisting of subsets of the form $\bigcup_{x \in G} x(A \setminus F_x)$, where $A \in p$ and $F_x \subset G$ is finite for each $x \in G$, and also that $q \in I$. But then $\theta(q) \in \omega^*$. Now q is prime since if that $q = r \cdot s$ for some $r, s \in G^*$ and since

$$q \in (\overline{G^s s}) \cap (\overline{\bigcup_{n<\omega} a_n B_n})$$

we must have that

$$(\overline{G^s s}) \cap (\overline{\bigcup_{n<\omega} a_n B_n}) = \emptyset$$

But then either $b \cdot s = q$ for some $b \in G^s$ and $q \in \overline{\bigcup_{n<\omega} a_n B_n}$ or $r \cdot s \in \overline{a_n B_n}$ for some $r \in \overline{G^s}$ and $n < \omega$. Since $\bar{l}(bs) \in \omega$ and $\bar{l}(q) \in \omega^*$, the first possibility gives us a contradiction. Also, since $s \in I$, the second possibility implies that $a_n^{-1} rs \in I \cap \overline{B_n}$. But $I \cap \overline{B_n} = \emptyset$ which is also a contradiction. Now ever subsequences of $(a_n)_{n<\omega}$, one can construct infinitely (in fact a continuum's worth) many prime $q \in I$ with $\bar{l}(q) \in \omega*$and thus $|Q| = 2^{\mathfrak{c}}$ $\square$

Lemma 4.3.13. *Let*$I \in \mathcal{I}(G)$, *let* $Q = \left\{p \in I : p \in \mathcal{P} \wedge \bar{l}(p) \in \omega^*\right\}$, *and let* $u \in \overline{r}(I)$. *If* $I \setminus Q$ *is dense in* I *then* u *is a P-point.*

Proof. Without loss of generality one may suppose that$\overline{r}(I) \subseteq \{u, u+1\}$. Pick $U \in u$ such that $U \bigcap (U+1) = \emptyset$, and let $\{U_n : 1 \leq n < \omega\}$ be a partition of U such that $U_n \notin u$ for all $n < \omega$. Define $\pi : \omega \to \omega$ by $\pi(m) = n$ if $m \in U_n \bigcup (U_{n+1})$ and $\pi(m) = 0$ if otherwise. For every $p \in I$, choose $A_p \in p$ such that

1. $r(A_p) \subseteq U \bigcup (U+1)$, and
2. either $\pi(r(x)) > l(x)$ for all $x \in A_p$ or $\pi(r(x)) \leq l(x)$ for all $x \in A_p$.

Consider two cases.

- Case 1: there is $p \in I$ such that $\pi(r(x)) \leq l(x)$ for all $x \in A_p$. Then for every $q \in \overline{A_p} \bigcap I$, one has $\bar{l}(q) \in \omega^*$. Indeed, assume on the contrary that $\bar{l}(q) = n$ for some $q \in \overline{A_p} \bigcap I$ and $n \in \omega$. Pick $B \in q$ such that $B \subseteq A_p$ and $l(B) = n$. Then for every $x \in B$, $\pi(r(x)) \leq l(x) = n$. Thus, $\overline{\pi}(\overline{r}(q)) \in \{0, 1, ..., n\}$. But $\overline{r}(q) \in \{u, u+1\}$ and $\overline{\pi}(u) = \overline{\pi}(u+1) \in \omega^*$, a contradiction. Since $I \setminus Q$ is dense in I and $\bar{l}(\overline{A_p} \bigcap I) \subseteq \omega^*$, there is a non prime $q \in \overline{A_p} \bigcap I$. i.e. $q = rs$ for some $r, s \in G^*$ and pick $x \in G^s$ such that $x \cdot s \in \overline{A_p}$. Then $\bar{l}(x \cdot s) \in \omega$. But since $q = r \cdot s$ that $s \in I$, so $x \cdot s \in \overline{A_p} \bigcap I$ and so $\bar{l}(x \cdot s) \in \omega^*$, another contradiction.

- Case 2: for every $p \in I$, $\pi(r(x)) > l(x)$ for all $x \in A_p$. Let $C = \bigcup_{p \in I} A_p$. Then $I \subseteq \overline{C}$. Pick $q \in I$ with $\overline{r}(q) = u$, and pick $B \in q$ and finite $F_x \subseteq F$ for each $x \in G$ such that $\bigcup_{x \in G} x(B \setminus F_x) \subseteq C$. Assume that there exists $n < \omega$ such that $r(B) \bigcap U_n$ is infinite. Then there must exist a sequence $(y_m)_{m<\omega} \subset B$ such that $r(y_m) < r(y_m + 1)$ and $r(y_m) \in U_n$, and thus $\pi(r(y_m)) = n$. By using a suitable subsequence, we also have that $h(y_m)(i) = h(y_k)(i)$ for all $m, k < \omega$ and $i \leq n+1$. Define $h(x)(i) = -h(y0)(i)$ if $i \leq n+1$ and $h(x)(i) = 0$ otherwise. Since F_x is finite, there exists a $m < \omega$ such that $y_m \in B \setminus F_x$ and $r(y_m) \geq n+3$. Then $|r(xy_m) - r(y_m)| \leq 1$, $xy_m \in C$, so $r(xy_m) \in U_n \bigcup U_{n+1}$, and $l(xy_m) \geq n$. Hence, $n = \pi(r(y_m)) = \pi(r(xy_m)) > l(xy_m) \geq n$, a contradiction

□

We are now ready to give a partial answer to the question of whether it can be shown in ZFC that every decomposition of G^* into closed left ideals such that the quotient space is Hausdorff contains a maximal principal left ideal.

Theorem 4.3.14. *[BZ13]It is consistent with ZFC that, if G is a countably infinite group which can be embedded algebraically into a compact group, then every $I \in \mathcal{I}(G)$ contains $2^{\mathfrak{c}}$ maximal principal left ideals of βG generated by prime elements.*

Proof. Any discrete group which can be algebraically embedded into a compact group can be continuously embedded into that group and thus is maximally almost periodic. This implies the existence of the bijection

$$h : G \to \oplus_{n<\omega}\mathbb{Z}_{m_n}$$

from 4.3.6. Let $I \in \mathcal{I}(G)$, let

$$Q = \left\{p \in I : p \in \mathcal{P} \wedge \bar{l}(p) \in \omega^*\right\}$$

and let $u \in \overline{r}(I)$. Now if u is not a P-point then by Lemma 4.3.13 we have that $I \setminus Q$ cannot be dense in I and thus interior V of Q cannot be empty. So let p be an element of the interior of Q. If p is an isolated point then by Lemma 4.3.12 we have that $|Q| = 2^{\mathfrak{c}}$ so suppose that p is not an isolated point then we can pick $A \in p$ such that $\overline{A} \cap I \subset V$ which implies that $F = \overline{A} \cap I$ is an infinite closed subset of $G^*$with $F \subset V \subset Q$ and thus $|Q| \geq |F| = 2^{\mathfrak{c}}$. However it is consistent with ZFC that there are no P-points in ω^* and thus it is consistent with ZFC that every $I \in \mathcal{I}(G)$ contains $2^{\mathfrak{c}}$ closed left ideals. But by construction each of those ideals contains a prime element and thus contains the principal ideal generated by that prime element. Finally since the principal ideal generated by a prime element is maximal we have that each $I \in \mathcal{I}(G)$ contains $2^{\mathfrak{c}}$ maximal principal left ideals of βG generated by prime elements. □

Any consistency result for ZFC raises the question if the result is actually independent of ZFC. The properties of βG and G^* are extremely set axiom dependant though, with J van Mill describing the space $\beta\mathbb{Z}$ in [vM84] as a three headed hydra with each head's properties dependant on what set theoretic assumptions you are prepared to make.

4.4 General Decompositions of Countable Groups by Left Ideals

To answer more questions about the decomposition structure of an the residual of some arbitrary topological group G we are required to expand our previous definitions.

Definition 4.4.1. *Let $X \subset G^*$ be a closed left ideal of βG. Denote by $\mathcal{I}'(X)$ the finest decomposition of X into closed left ideals of βG.*

We proceeded previously in a much more constructive manner to prove the existence of a family of decompositions on βG and thus arrived at an intrinsic definition of the finest decompositions in question. Now that we are dealing with families of decompositions are more extrinsic approach is required.

Lemma 4.4.2. *Let $X \subset G^*$ be a closed left ideal of βG. Then the family $\mathcal{I}'(X)$ is non-empty.*

Proof. Consider the family $\mathcal{A}$ of all decompositions of X into closed left ideals. Since $\{X\} \in \mathcal{A}$ we know that $\mathcal{A} \neq \emptyset$. Now let

$$h_\alpha : X \to X/\alpha$$

be the quotient map for every decomposition $\alpha \in \mathcal{A}$ and let

$$g : X \to \prod_{\alpha \in \mathcal{A}} X/\alpha$$

be the diagonal map which maps every point $x \in X$ to its corresponding equivalence class in each possible decomposition on X i.e.

$$g(x) = \prod_{\alpha \in \mathcal{A}} h_\alpha(x)$$

We claim that

$$\mathcal{I}'(X) = \left\{ g^{-1}(y) : y \in g(X) \right\}$$

So let $y \in g(X)$ then $y = \prod_{\alpha \in \mathcal{A}} D_\alpha$ where $D_\alpha \subset X$ is some closed left ideal. But then $g^{-1}(y) = \bigcap D_\alpha$ is the intersection of closed left ideals and is thus itself a closed left ideal. Also if $x \in g^{-1}(y) \cap g^{-1}(z)$ then $x \in (\bigcap D_\alpha) \cap (\bigcap E_\alpha)$ which implies that $x \in D_\alpha \cap E_\alpha$ and since we started of with decompositions we must have that $D_\alpha = E_\alpha$ which implies that $g^{-1}(y) = g^{-1}(z)$. Also for all $x \in X$ we know that

$$x \in g^{-1}(g(x)) \in \left\{g^{-1}(y) : y \in g(X)\right\}$$

Finally let $\alpha \in \mathcal{A}$ be any other decomposition and let $A \in \alpha$ and let $p \in A$. We claim that $g^{-1}(g(p)) \subset A$. Let $x \in g^{-1}(g(p))$ then $g(x) = g(p)$ but by the definition of the diagonal map we must have that $\pi_\alpha \circ g(x) = A$ where π_α is the α-coordinate projection function. But this implies that $x \in A$ and so $g^{-1}(g(p)) \subset A$. Thus $\{g^{-1}(y) : y \in g(X)\}$ is the finest decomposition of X into closed left ideals. □

Note that our definition of $\mathcal{I}'(X)$ makes sense when $X = G$ as G is a closed left ideal if G is a group and $\mathcal{I}' = \mathcal{I}'(G^*)$

Definition 4.4.3. *Let $X \subset G^*$ be a closed left ideal of βG. Denote by $\mathcal{I}(X)$ the finest decomposition of X into closed left ideals of βG with the property that the corresponding quotient space of X is Hausdorff.*

Lemma 4.4.4. *If $X \subset G^*$ is a closed left ideal of βG then the set $\mathcal{I}(X)$ is well-defined.*

Proof. We can proceed very similarly as the previous lemma. The set $\mathcal{A}$ is non-empty since any finite decomposition into closed subsets generates a Hausdorff quotient space (since it is $T_1$and finite and thus discrete and Hausdorff) and thus once again $\{X\} \in \mathcal{A}$. Similarly we know that $\{g^{-1}(y) : y \in g(X)\}$ is a finer decomposition of X into closed left ideals than any member of $\mathcal{A}$. So what is left to prove is that $D = \{g^{-1}(y) : y \in g(X)\}$ is in fact upper semicontinuous and thus since X is compact we know that the corresponding quotient space must be Hausdorff. So let $C \subset X$ be some closed subset and consider the set $\bigcup\{A \in D : A \cap C \neq \emptyset\}$. Now if $x \in A \cap D$ then $A = g^{-1}(g(x))$ and thus $A = \bigcap A_\alpha$ for some family of closed left ideals with $A_\alpha \in \alpha$. But then $A_\alpha \cap C \neq \emptyset$ and since each $\alpha \in \mathcal{A}$ is upper semicontinuous we must have that $\bigcup\{A_\alpha \in \alpha : A_\alpha \cap C \neq \emptyset\}$ is closed. But then

$$\bigcup\{A \in D : A \cap C \neq \emptyset\} = \bigcap \bigcup \{A_\alpha \in \alpha : A_\alpha \cap C \neq \emptyset\}$$

and is thus closed. Thus $\{g^{-1}(y) : y \in g(X)\}$ is the finest decomposition of X into closed left ideals such that the corresponding quotient space is Hausdorff □

Similarly here, our definition of $\mathcal{I}(X)$ makes sense when $X = G^*$ and $\mathcal{I}(G) = \mathcal{I}$.

Definition 4.4.5. *Let $X \subset G^*$ be a closed left ideal of βG and let $p \in X$. Denote by $\mathcal{I}(p, X) = \bigcap_{A \in p} \overline{G \cdot (\overline{A} \cap X)}$.*

The sets $\mathcal{I}(p, X)$ are in a sense the generators of the local ideal structure of βG. Also $\mathcal{I}(p, G^*) = \bigcap_{A \in p} \overline{G \cdot (\overline{A} \cap G^*)} = \bigcap_{A \in p} \overline{G \cdot (\overline{A} \setminus A)}$.

Lemma 4.4.6. *Let $X \subset G^*$ be a closed left ideal of βG and let $p \in X$. Then $\mathcal{I}(p, X)$ is a closed left ideal of βG contained in X.*

Proof. Being the intersection of closed subsets it is easy to see that $\mathcal{I}(p, X) = \bigcap_{A \in p} \overline{G \cdot (\overline{A} \cap X)}$ itself must be a closed subset. So let $x \in G$ then

$$
\begin{aligned}
x \cdot \mathcal{I}(p, X) &= \bigcap_{A \in p} x \cdot \overline{G \cdot (\overline{A} \cap X)} \\
&= \bigcap_{A \in p} \overline{x \cdot G \cdot (\overline{A} \cap X)} \\
&= \bigcap_{A \in p} \overline{G \cdot (\overline{A} \cap X)} \\
&= \mathcal{I}(p, X)
\end{aligned}
$$

Thus $\mathcal{I}(p, X)$ is a closed left ideal. Finally let $x \in \mathcal{I}(p, X)$ so

$$x \in \overline{G \cdot (\overline{A} \cap X)} \subset \overline{G \cdot X} \subset \overline{X} = X$$

since X is closed and thus $\mathcal{I}(p, X) \subset X$ □

Lemma 4.4.7. *Let $X \subset G^*$ be a closed left ideal of βG and let $\mathcal{D}$ be a decomposition of X into closed left ideals of βG such that the corresponding quotient space of X is Hausdorff. For all $D \in \mathcal{D}$ and $p \in D$ we have that $\mathcal{I}(p, X) \subset D$*

Proof. □

We know that $\mathcal{D}$ must be upper semicontinuous thus for all $D \in \mathcal{D}$ and open $X \supset U \supset D$ there exists an open $V \supset D$ such that if $B \in \mathcal{D}$ and $V \cap B \neq \emptyset$ then $B \subset U$. Without loss of generality we may assume $V = U$. But then$G \cdot U \subset U$ since the decomposition D is made up of left ideals of βG and since $p \in U$ there must exist a $A \in p$ such that $\overline{A} \cap X \subset U$ and so $G \cdot (\overline{A} \cap X) \subset U$ and thus

$$\mathcal{I}(p, X) = \bigcap_{A \in p} \overline{G \cdot (\overline{A} \cap X)} \subset \overline{U}$$

For arbitrary open neighbourhoods U of the closed subset D. But this implies that $\mathcal{I}(p, X) \subset D$

Definition 4.4.8. *Let $X \subset G^*$ be a closed left ideal of βG and let $\mathcal{F}$ be the smallest filter of G with $\overline{\mathcal{F}} = \{q \in \beta G : \mathcal{F} \subset q\} = X$. For all $p \in X$ denote by $\mathcal{F}_p$ the filter generated by subsets of the form $\bigcup_{x \in G} x \cdot (A \cap U_x)$ where $A \in p$ and $U_x \in \mathcal{F}$*

Lemma 4.4.9. *Let $X \subset G^*$ be a closed left ideal of βG, $B \subset G$ such that $\mathcal{I}(p, X) \subset \overline{B}$. Then there exists $A \in p$ such that*

$$\overline{G \cdot (\overline{A} \cap X)} \subset \overline{B}$$

Proof. Now since $\beta G \setminus \overline{B} = \overline{G \setminus B}$ is a closed subset of βG we have that it is compact and thus since all $y \in \overline{G \setminus B}$ there exist $A_y \in p$ such that

$$y \notin \overline{G \cdot (\overline{A_y} \cap X)}$$

(for else $\mathcal{I}(p, X) \subset \overline{G \setminus B}$). We can find some finite $F \subset G \setminus B$ such that

$$\bigcap_{y \in F} \overline{G \cdot (\overline{A_y} \cap X)} \subset \overline{B}$$

(since $\bigcup_{y \in F} \beta G \setminus \overline{G \cdot (\overline{A_y} \cap X)}$ form a finite subcover over $\overline{G \setminus B}$). Let

$$A = \bigcap_{y \in F} A_y$$

then

$$\overline{G \cdot (\overline{A} \cap X)} \subset \bigcap_{y \in F} \overline{G \cdot (\overline{A_y} \cap X)} \subset \overline{B}$$

□

Lemma 4.4.10. *Let $X \subset G^*$ be a closed left ideal of βG, $A, B \subset G$ and such that $G \cdot (\overline{A} \cap X) \subset \overline{B}$. Then for all $x \in G$ there exists $U_x \in \mathcal{F}$ such that $x \cdot (A \cap U_x) \subset B$.*

Proof. Once again since $\beta G \setminus \overline{B} = \overline{G \setminus B}$ is a closed subset of βG we have that it is compact and thus since all $y \in \overline{G \setminus B}$ there exist $V_y \in \mathcal{F}$ such that

$$y \notin x \cdot (\overline{A} \cap \overline{V_y}) = x \cdot \overline{A \cap V_y}$$

We can find some finite $F \subset G \setminus B$ such that

$$\bigcap_{y \in F} x \cdot (\overline{A \cap V_y}) \subset \overline{B}$$

Let $U_x = \bigcap_{y \in F} V_y$ then

$$x \cdot (A \cap U_x) \subset B$$

$\square$

Theorem 4.4.11. *If $X \subset G^*$ is a closed left ideal of βG then $I(p, X) = \overline{\mathcal{F}_p}$.*

Proof. Let

$$C = \bigcup_{x \in G} x \cdot (A \cap U_x)$$

where $A \in p$ and $U_x \in \mathcal{F}$ then for every $x \in G$ we have that

$$x \cdot (\overline{A} \cap X) \subset \overline{x \cdot (A \cap U_x)} \subset \overline{C}$$

and thus

$$\overline{G \cdot (\overline{A} \cap X)} \subset \overline{C}$$

and so

$$\mathcal{I}(p, X) = \bigcap_{A \in p} \overline{G \cdot (\overline{A} \cap X)} \subset \overline{C}$$

Which implies that $\mathcal{I}(p, X) \subset \overline{\mathcal{F}_p}$. Now let $B \subset G$ and suppose that $\mathcal{I}(p, X) \subset \overline{B}$ then by the previous lemma we know that there exists $A \in p$ such that

$$\overline{G \cdot (\overline{A} \cap X)} \subset \overline{B}$$

But then we also know that since $x \cdot (\overline{A} \cap X) \subset \overline{B}$ for all $x \in G$ there exists $U_x \in \mathcal{F}$ such that $x \cdot (A \cap U_x) \subset B$. Let

$$C = \bigcup_{x \in G} x \cdot (A \cap U_x)$$

then $C \in \mathcal{F}_p$ and $C \subset B$ and so $\overline{\mathcal{F}_p} \subset \overline{B}$ and thus $\overline{\mathcal{F}_p} \subset \mathcal{I}(p, X)$ $\square$

Now in the case that $X = G^*$ we have that $\mathcal{I}_p = \overline{\mathcal{F}_p}$ where $\mathcal{F}_p$ is the filter generated by subsets of the form $\bigcup_{x \in G} x \cdot (A \cap F_x)$ where $F_x \in \mathcal{F}$. Now $\mathcal{F} = \bigcap \{p : p \in G^*\}$ is the Fréchet filter and thus we can choose $F_x = A \setminus G_x$ where G_x is a finite subset. But then $\bigcup_{x \in G} x \cdot (A \cap F_x) = \bigcup_{x \in G} x \cdot (A \setminus G_x)$ and thus the previous definitions carry over smoothly to this new paradigm. We require a similar technical lemma to show that we in fact have a decomposition.

Lemma 4.4.12. *Let G be a countably infinite discrete group, let $A \subset G$ be an infinite subset and let $F_x \in \mathcal{F}$ for every $x \in G$ and some filter $\mathcal{F}$ such that $\overline{\mathcal{F}}$ is a* *closed left ideal on βG. If*

$$B = G \setminus \bigcup_{x \in G} x \cdot (A \cap F_x)$$

then there exists subsets H_x and K_x of G with $H_x \in \mathcal{F}$ and $K_x \in \mathcal{F}$ for every $x \in X$ such that

$$(\bigcup_{x \in G} x \cdot (A \cap H_x)) \cap (\bigcup_{x \in G} x \cdot (B \cap K_x)) = \emptyset$$

Proof. Since G is countably infinite we can enumerate it as $G = \{x_n : n \in \mathbb{N}\}$. Define

$$H_{x_n} = \bigcap_{i \leq n} F_{x_i^{-1} x_n}$$

which is an element of $\mathcal{F}$ by the finite intersection property and

$$K_{x_n} = \bigcap_{i \leq n} x_n^{-1} \cdot x_i F_{x_n^{-1} x_i}$$

which is an element of $\mathcal{F}$ since $\overline{\mathcal{F}}$ is a closed left ideal and thus $x \cdot \mathcal{F} = \mathcal{F}$ thus

$$x_i^{-1} x_n (A \cap H_{x_n}) \cap B = \emptyset$$

and

$$x_n^{-1} x_i A \cap (B \cap K_{x_n}) = \emptyset$$

This implies that

$$x_n (A \cap H_{x_n}) \cap x_i B = \emptyset$$

and

$$x_i A \cap x_n (B \cap K_{x_n}) = \emptyset$$

and thus

$$x_n (A \cap H_{x_n}) \cap x_m (B \cap K_{x_m}) = \emptyset$$

which shows that

$$(\bigcup_{x \in G} x \cdot (A \cap H_x)) \cap (\bigcup_{x \in G} x \cdot (B \cap K_x)) = \emptyset$$

□

We can know show that the closed left ideals $I(p, X)$ form a decomposition of X.

Theorem 4.4.13. *If $p, q \in X$ then $I(p,X) \cap I(q,X) \neq \emptyset$ implies that $I(p,X) = I(q,X)$.*

Proof. Let $q \in G^*$ and suppose that $\mathcal{F}_p \not\subset q$ (i.e. $q \notin I(p,X)$) then there must exist $A \in p$ and $F_x \in \mathcal{F}_p$ such that

$$B = G \setminus \bigcup_{x \in G} x \cdot (A \cap F_x) \in q$$

but then we can find sets such that

$$(\bigcup_{x \in G} x \cdot (A \cap H_x)) \cap (\bigcup_{x \in G} x \cdot (B \cap K_x)) = \emptyset$$

But this implies that $\mathcal{F}_p \not\subset r$ for all $r \in I_q$since then $\mathcal{F}_p \subset r$ and $\mathcal{F}_q \subset r$ and so could not have empty intersections as subsets of an ultrafilter and so $I_p \cap I_q = \emptyset$ $\square$

We can now show that the closed left ideals as defined generate a upper semicontinuous decomposition of X and thus the quotient topology on X, which is closed and thus compact, is Hausdorff.

Theorem 4.4.14. $\mathcal{I}(X) = \{I(p,X) : p \in X\}$.

Proof. We know that every $I(p,X)$ is a closed left ideal of βG and that $p \in I_p$. We also know that $\{I(p,X) : p \in X\}$ forms a decomposition of X and that for any other decomposition $\mathcal{D}$ of X into closed left ideals of βG such that the corresponding quotient space is Hausdorff we have that $\mathcal{I}(p,X) \subset D$ for $D \in \mathcal{D}$. We claim that $\mathcal{D}$ =$\{I(p,X) : p \in X\}$ is upper semi-continuous. Let $I(p,X)$ be an arbitrary member of $\mathcal{D}$ and let $I(p,X) \subset \overline{W}$ for some $W \subset G$. This implies that for all $r \in I(p,X)$ we have that $W \subset r$ but each $r \in I(p,X)$ is generated by sets of the form $\bigcup_{x \in G} x \cdot (A \cap F_x)$ where $A \in p$ and $F_x \in \mathcal{F}$ thus with out loss of generality we can assume that $W = \bigcup_{x \in G} x \cdot (A \cap F_x)$ for some $A \in p$ and choice $F_x \in \mathcal{F}$. But we know that there choices $H_x \in \mathcal{F}$ and $K_x \in \mathcal{F}$ for every $x \in G$ such that

$$Q = \bigcup_{x \in G} x \cdot (A \cap H_x)$$

and

$$R = \bigcup_{x \in G} x \cdot (B \cap K_x)$$

and $Q \cap R = \emptyset$ where $B = G \setminus U$. Now suppose that $I(q,X) \cap \overline{Q} \cap X \neq \emptyset$ then we know that for all $r \in I(q,X)$, $Q \in r$. Now suppose $U \notin r$ then $B = G \setminus U \in r$.

But his implies that

$$Q \cap (B \cap K_e) \subset Q \cap \bigcup_{x \in G} x \cdot (B \cap K_x) \subset Q \cap R = \emptyset$$

and so $Q \cap B \cap K_e = \emptyset$ and since $Q \cap B \neq \emptyset$ we have that $Q \cap K_e = \emptyset$. But this contradicts the fact that $\mathcal{F} \subset r$ and thus $U \in r$ thus $I(r, X) \subset \overline{U}$

$\square$

4.5 Closed left ideal decompositions

We now have all the pieces to prove the main theorem of this Thesis. We require a special kind of ultrafilter trying together the concept of a set being thin with the extended semigroup operations on βG.

Definition 4.5.1. *An ultrafilter $p \in \beta G$ is called uniformly discrete if there exists $A \in p$ such that the family of sets $\{x \cdot (\overline{A} \setminus A)\}_{x \in G}$ is pairwise disjoint.*

Definition 4.5.2. *Denote by $\mathcal{U}$ the set of all uniformly discrete ultrafilters on βG.*

Since the set of uniformly discrete ultrafilters is closely related to the concept of thins sets in G we know that every subset of G needs to be close to some thin subset. Recall that $\overline{A} \setminus A$ is just another way of saying that we want to look at the Fréchet filters on A and thus exclude finite subsets from the realm of discussion.

Lemma 4.5.3. *The set of uniformly discrete ultrafilters is open in G^*.*

Proof. Let $p \in \mathcal{U}$. There exists $A \in p$ such that the family of sets

$$\{x \cdot (\overline{A} \setminus A)\}_{x \in G}$$

is pairwise disjoint. Then for all

$$q \in \overline{A} = \{p \in \beta G : A \in p\}$$

we have that q is uniformly discrete. Thus $\overline{A} \subset \mathcal{U}$ and since $p \in \overline{A}$ we have that $\mathcal{U}$ contains an open neighbourhood of each of its points and is open. $\square$

Lemma 4.5.4. *The set of uniformly discrete ultrafilters is dense in G^*.*

Proof. Since every infinite subset $S \subset G$ contains a thin set $A \subset S$ we know that $xA \cap yA$ is finite for all $x \neq y \in G$ and so

$$(x \cdot (\overline{A} \setminus A)) \cap (y \cdot (\overline{A} \setminus A)) = \emptyset$$

Now we know that $\overline{S}$ is a basic open set in βG and thus $\overline{S} \cap G^*$ is a basic open subset of G^*. We claim that there exists some $r \in \overline{S} \cap G^*$ such that $A \in r$. In fact let r be any free ultrafilter on G such that $A \in r$ then since r is a filter and $A \subset S$ we have that $S \in r$ and so $r \in \overline{S} \cap G^*$ and thus $\mathcal{U}$ is dense in G^* $\square$

The previous two lemmas clarify what was meant by the informal statement that a family of subsets is close to all other subsets of a group, namely that we can find ultrafilters with an element that satisfies the property in question. These are called $\mathcal{I}$-ultrafilters (see [Bau95]) if the group g is countable and the study of which properties and thus sets $\mathcal{I}$ generate reasonable ultrafilters in ZFC is quite advanced. Now the left ideals of uniformly discrete ultrafilters can be used to separate the behaviour of the many different left ideals under investigation in this section.

Theorem 4.5.5. *Let $p \in G^*$ be a uniformly discrete ultrafilter, $X = I(p, G^*)$, and $Y = I(p, X)$. Then*

1. *p is isolated in X,*
2. *$Y = (\beta G)p$,*
3. *$X = (\beta G)p$ if and only if p is a P-point.*

Proof. 1. Let $A \in p$ be such that

$$x(\overline{A} \setminus A) \bigcap (\overline{A} \setminus A) = \emptyset$$

for every $x \in G \setminus e$. We claim that $\overline{A} \bigcap X = p$. Indeed, let $q \in X \setminus p$. Choose $B \in p$ such that $B \subseteq A$ and $q \notin \overline{B}$. Since

$$\overline{G \cdot (\overline{B} \setminus B)} = \overline{(G \setminus \{e\}) \cdot (\overline{B} \setminus B)} \bigcup (\overline{B} \setminus B)$$

and

$$\overline{(G \setminus \{e\}) \cdot (\overline{B} \setminus B)} \subseteq \beta G \setminus \overline{A}$$

thus

$$\overline{A} \bigcap \overline{G \cdot (\overline{B} \setminus B)} = \overline{A} \bigcap \overline{B} \setminus B = (\overline{B} \setminus B)$$

Hence,

$$q \notin \overline{A} \bigcap \overline{G \cdot (\overline{B} \setminus B)}$$

and so $q \notin \overline{A} \bigcap X$.

2. We know that there exists $A \in p$ such that $\overline{A} \bigcap X = p$. Then

$$\overline{G \cdot (\overline{A} \bigcap X)} = \overline{G \cdot p} = (\beta G)p$$

so $Y = (\beta G)p$.

3. Suppose that p is a P-point. Let $C \subseteq G$ be such that $(\beta G)p \subseteq C$. For every $x \in G$, pick $A_x \in p$ such that $x \cdot A_x \subseteq C$. Since p is a P-point, there is $A \in p$ such that
$$\overline{A} \setminus A \subseteq \overline{A_x} \setminus A_x$$
for all $x \in G$. Then $G \cdot (\overline{A} \setminus A) \subseteq \overline{C} \setminus C$, so $X \subseteq \overline{C} \setminus C$. Hence,$X = (\beta G)p$. Suppose that p is not a P-point. Choose a decreasing sequence $(A_n)_{n<\omega}$ of members of p such that
$$(x \cdot (\overline{A_0} \setminus A_0)) \bigcap (\overline{A_0} \setminus A_0) = \emptyset$$
for all $x \in G \setminus e$ and $\bigcap_{n<\omega}(\overline{A_n} \setminus A_n)$ is not a neighbourhood of $p \in G^*$. Enumerate G without repetitions as $x_n : n < \omega$. For every $n < \omega$, let
$$F_n = x_n^{-1} \cdot ((x_n A_0) \bigcap (\bigcup_{i<n} x_i A_0))$$
This implies that all F_n are finite, and the sets $x_n(A_0 \setminus F_n)$, $n < \omega$, are pairwise disjoint. Let
$$V = \bigcup_{n<\omega} x_n((A_0 \setminus A_n) \setminus F_n)$$
and
$$W = \bigcup_{n<\omega} x_n(A_n \setminus F_n)$$
Then $(\beta G)p \subseteq \overline{W}$ and $V \bigcap W = \emptyset$. Now for every$A \in p$, there is $n < \omega$ such that $A \bigcap (A_0 \setminus A_n)$ is infinite. Consequently, $(x_n A) \bigcap V$ is infinite, and so
$$(x_n \cdot (\overline{A} \setminus A)) \bigcap V = \emptyset$$
Thus, for every $A \in p$,
$$\overline{(G \cdot \overline{A} \setminus A))} \bigcap \overline{V} = \emptyset$$
Since
$$X = \bigcap_{A \in p} \overline{G \cdot (\overline{A} \setminus A)}$$
and for every finite $H \subseteq p$, one has
$$\bigcap_{A \in H} \overline{G \cdot (\overline{A} \setminus A)} \supseteq \overline{(G \cdot (\overline{(\cap_{A \in H} A)} \setminus (\cap_{A \in H} A)))}$$

It then follows that $X \bigcap \overline{V} = \emptyset$. Hence, $X \neq (\beta G)p$.

$\square$

We wish to investigate the relationship between the decompositions $\mathcal{I}'$and $\mathcal{I}$. But $\mathcal{I}'$ is a bit of a beast to work with so we weaken the decomposition slightly as follows.

Lemma 4.5.6. *The family $\mathcal{I}_2 = \bigcup_{X \in \mathcal{I}(G)} \mathcal{I}(X)$ is a decomposition of G^* into closed left ideals of βG.*

Proof. It is easy to see that every element of $\mathcal{I}_2$ is a closed left ideal of βG and that $\mathcal{I}_2$ covers G^*. Let $U, V \in \mathcal{I}_2$ and suppose $U \cap V \neq \emptyset$. There must exists $X \in \mathcal{I}(G)$ such that $U \in \mathcal{I}(X)$ and $V \in \mathcal{I}(X)$ (since the elements of $\mathcal{I}(G)$ are disjoint). But $\mathcal{I}(X)$ is also a decomposition and thus $U = V$ $\square$

We need to play off the availability of certain types of points, namely P-points and non P-points in the final argument. In fact, in ZFC we know that there are many non P-Points.

Theorem 4.5.7. *If G is a countable infinite discrete group then G^* contains a dense set of $2^{\mathfrak{c}}$ non-P-points.*

Proof. See [Wal74] $\square$

Theorem 4.5.8. *The set of all points $p \in G^*$ such that $(\beta G)p \in \mathcal{I}_2 \setminus \mathcal{I}(G)$ is dense in βG.*

Proof. Let $A \subset G$ be an infinite subset. We know that there exists an infinite set $B \subset A$ such that every point $p \in \overline{B} \setminus B$ is uniformly discrete and thus dense in G^*. Furthermore we can suppose that none of the $p \in \overline{B} \setminus B$ happen to be a P-point. But then $X = I(p, G^*) \in \mathcal{I}$ and $Y = I(p, X) \in \mathcal{I}_2$ but $Y = (\beta G)p$ and $X \neq (\beta G)p$ and so $(\beta G)p \in \mathcal{I}_2 \setminus \mathcal{I}(G)$ $\square$

We can now give a negative answer to the second question of this thesis, namely that $\mathcal{I} \neq \mathcal{I}'$

Theorem 4.5.9. *If G is a countable infinite discrete group then $\mathcal{I}'$ is finer then $\mathcal{I}(G)$.*

Proof. Let $D \in \mathcal{I}(G)$ and let $d \in D$. Let $U \subset D$ be open with $d \in U$. Now since the set of all points $p \in G^*$ such that

$$(\beta G)p \in \mathcal{I}_2 \setminus \mathcal{I}(G)$$

is dense in βG there must exist a $p \in U$ such that

$$d \in (\beta G)p \in \mathcal{I}_2 \setminus \mathcal{I}(G)$$

But since $\mathcal{I}_2$ is a decomposition of G^* into closed left ideals of βG we have that I' is necessarily finer than $\mathcal{I}_2$ and so we can find $V \in \mathcal{I}'$ such that $d \in V$ and $V \subset (\beta G)p \subset D$ which shows that $\mathcal{I}'$ is finer then $\mathcal{I}(G)$ $\square$

Chapter 5

Conclusion

We have shown that it is consistent with ZFC, the system of usual axioms of set theory, that if G is a group that can be algebraically embedded into a compact group then every $I \in \mathcal{I}$ contains $2^{\mathfrak{c}}$ maximal principal left ideals of βG [BZ13] i.e. $\mathcal{I}$ need not consist of maximal principal left ideals. We have also shown that there exists a dense subset of points $p \in G^*$ such that the principal left ideals $(\beta G)p \in \mathcal{I}' \setminus \mathcal{I}$, in particular, $\mathcal{I}'$ is finer than $\mathcal{I}$ [BZZ14]. We achieved these results by first considering the construction of βG and then using p-limits to extend the algebraic operations to the entire space. We then considered the topological and algebraic structure of the left ideals of the semigroup βG sufficiently to show the relationship between principal ideals of the space βG and uniformly discrete filters on G^* and P-points on βG. The main open question remains unsolved

Question 5.0.1. *Does every point of G^* lie in a maximal principal left ideal of βG?*

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