Dynamical Systems in Walrasian General Equilibrium

Steven Jon Davey

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I declare that this research report is my own, unaided work. It has been submitted for the degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any other degree or examination in any other University

(Signature of candidate)

20th day of November 1997

Abstract

The role of dynamical systems in general equilibrium analysis is chronicled. Starting with the introduction of a suitable phase space and evolution model, fundamental existence and stability results are proved. From here, many negative statements regarding the convergence of the price mechanism and attempts to rectify these conclusions are examined.

The conclusion is that either the traditional economic hypotheses behind the model need to be reexamined or the economist's dream of a universal pricing mechanism must be abandoned.

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1. Introduction

The central issue addressed here is the behaviour of a competitive economy as envisaged and described by Leon Walras between 1874 and 1877. This model was conceived as an attempt to explain the state of an equilibrium reached by a large number of small agents interacting through markets. The equilibrium positions were perceived as solutions to a system of (non-linear) equations of the type supply equals demand.

Despite the fact that Walras himself realised that his theory would be vacuous without a mathematical argument in support of the existence of at least one equilibrium state, for several decades no rigorous account of his theory was developed. Indeed, for many years, the belief in the existence of market equilibria was founded on the bold assumption that a system with equal numbers of equations and unknowns must be solvable.

The history of general equilibrium analysis is both long and central to all areas of economics. Most procedures in mathematical economics and many theories in economics in general have origins in equilibrium analysis. However, despite its significance, general equilibrium analysis is not well understood. The reason for this is simply that the traditional mathematical tools applied in this area are ill suited to answer the types of questions which naturally arise.

In a standard approach to general equilibrium theory, the focus is on a static or comparative static analysis of general equilibrium. Briefly, these can be described as follows :

> Clearly, if a state of equilibrium is reached it will persist as long as the knowledge of external events, relevant to the decision of the individuals, corresponds to the common expectation. The observed equilibrium magnitudes can be considered as possible solutions to a set of postulated or hypothesised relationships imposed on economic variables. The specification of these hypothetical functional relationships, the explicit definition of the particular conditions and environment in which these relations hold (that is, the parameters or external events) and the elucidation of the restrictions upon the empirically observable values that are implied thereof are the main ingredients of static analysis.
>
> Comparative statics is the study of changes in the system from one position of equilibrium to another, without regard to the transitional process involved in the adjustment. It is the study of the responses of equilibrium unknowns to designated changes in parameters. - *Antoniou* [1]

To justify this approach to equilibrium theory, conditions and forces which restore equilibrium once there is a change in the parameters must be both fully understood and justifiable. Several alternative stability conditions have been proposed but two have come to dominate economic literature. They are the Marshallian and Walrasian stability conditions. The choice between the two depends on the underlying market. If we have a good theoretical or empirical reason to believe that in a particular market, prices are more flexible than quantities, we would favour Walrasian stability.

If on the other hand we suspect quantities more flexible than prices we would use Marshallian stability. Because our model will not include production it makes sense to examine Walrasian stability.

1.1. **Why Smooth Dynamical Systems ?** Smooth dynamical systems have developed as a pure mathematical subject : an attempt to study all ordinary differential equations on a manifold. One of its main achievements has been the removal of the fear of making a global study of differential equations on a manifold of dimension greater than 2. Since a differentiable manifold is the principal candidate for the state space of a physical system with its progression through time associated with the evolution of a dynamical system, the use of methods from smooth dynamical systems should be natural in economics. The truth, however, is that such methods are rare in this area.

The most important advantage associated with the dynamic approach is the ability to address fundamental questions in equilibrium theory which could not be satisfactorily handled using traditional methods. Questions such as *"How is equilibrium reached?"* or a dual and more frequently posed question *"Why is economic equilibrium stable?"*, while difficult to handle in a static theory, seem ideally posed for a dynamic approach.

To quote Smale

> A successful introduction of dynamical systems into economic theory would give greater validity to equilibrium theory. On the other hand, it may be that the resolution of these problems will require the recasting of the foundations of equilibrium theory. We may well keep in mind some historical perspectives from physics, drawing an analogy between Walrasian equilibrium theory and Newtonian mechanics. - *Smale* [30]

Remark 1.1. From a dynamical perspective, it makes sense to classify goods into two ideal classes, completely perishable or completely durable, since the theory seems different for the two kinds. Walrasian theory seems suited to a perishable, continually endowed class of goods where the equilibrium reached depends on the agents' initial endowment.

The chief example of such a perishable goods is labour. Each day an economic agent begins with a fixed quantity. Since it cannot be used the following day, all unused labour is wasted. Both the endowment and consumption bundles in the commodity space are interpreted as the rates of endowment (fixed over time) and consumption respectively. The goal of Walrasian equilibrium is to construct and analyse paths over time in the state of spaces.

> These paths would obey economically justifiable axioms of exchange and price adjustment, and probably should lead at least under some economic conditions to a Walrasian equilibrium, starting from any endowment allocation and any price system. At the most satisfactory level, these paths should be given interpretations in terms of individual agent's actions in price offerings and purchases. - *Smale* [28]

At present, a completely satisfactory dynamic for this problem is not available.

During the real passage of time an actual exchange of durable goods takes place and initial conditions become irrelevant in the shuffle. Thus if we allow this passage of time, (non-tatônnement) we must replace Walrasian price equilibrium by a different notion of price equilibrium more commonly found in the fundamental theorem of welfare economics. For the purposes of this paper, we shall not examine such commodities.

1.2. **Some Economics.** The model we shall consider throughout is considered the starting point in a standard development of the theory of general equilibrium in most courses in microeconomics. We shall see, however, that this simplified economy is still rich enough to allow for some very interesting behaviour - the kind of behaviour which is definitely not presented in such courses !

We shall use the following notational conventions : superscripts will denote components of vectors while subscripts will be reserved for vectors, that is $x^\alpha \in \mathbb{R}$ while $x_i \in \mathbb{R}^n$ for some n.

An *economy* will be a simple consumption only world in which $l \geq 2$ commodities are traded. m agents come together and exchange goods at certain prices, assumed for now to be positive (economically, such commodities are called *scarce*). We shall not be overly concerned with *free* commodities (with zero price), although these will be discussed, or *noxious* commodities (with negative price).

No theory of money is considered here and it is assumed throughout that the economy works without the help of a good serving as a medium of exchange. The procedure of exchange is described as follows : *To each commodity α, $\alpha \in 1, \ldots, l$ is associated a number p^α called the* price . *An agent's account is debited by an amount p^α when he accepts delivery of one unit of commodity α and credited with an amount p^β when he makes a delivery of one unit of commodity β. The balance of his account will then influence his decisions about his future consumption.*

Let $P = \{x \in \mathbb{R}^n | x^\alpha > 0, \alpha \in 1, \ldots, l\}$ and $\bar{P} = P \cup \partial P$. A *commodity bundle*, $x \in \bar{P}$, is a vector whose elements represent quantities of the various commodities. A *price system* is a non-trivial linear function p on $\bar{P} \subset \mathbb{R}^l$ which associates a commodity bundle $x \in \bar{P}$ with its value $p(x) \in \mathbb{R}$. Identifying $\mathbb{R}^l$ with its dual space, we write $p = (p^1, \ldots, p^l)$ and then $p(x) = \sum p^\alpha x^\alpha$. Further, we use the customary abuse of notation and talk about the price system p when we actually mean the equivalence class $[p]$ of price systems where

$$[p] = \{q \in \bar{P} \mid q = \lambda p; \lambda \in \mathbb{R}, \lambda > 0\}.$$

This equivalence is justified economically on the grounds that an economy is assumed to be self contained, that is, prices are not relative to some commodity outside the system. The assumption of homogeneity of degree zero allows us to work on a mathematically more convenient space. Let

$$S^{l-1} = \{p \in \mathbb{R}^l \mid \|p\| = 1\}$$

where $\|\cdot\|$ represents the Euclidean norm. Now define

$$S = S^{l-1} \cap P$$

as the *space of price systems*.

In an economy without production, the amount the i-th agent can purchase depends on the amount he can sell. This amount is in turn determined by his *initial endowment* or *resources* $r_i = (r_i^1, \ldots, r_i^l)$. These resources will determine the boundary conditions for the price dynamic. In economics they are called exogenous variables because their value is not determined by the price dynamic but rather fixed "outside" the system. Define the i-th agent's *wealth* as $p(r_i)$. The *budget set* for the i-th agent,

$$\{x \in \bar{P} \mid p(x) \leq p(r_i)\},$$

is then the set of all commodity bundles the agent can afford. The plane passing through r_i with normal p is called the *budget plane* .

Since no natural order exists on $\bar{P}$, we introduce one via an agent's *preferences*, $\succcurlyeq$. Preferences determine the agent's consumption at a price system p. They are assumed to be a total preordering on $\bar{P}$ (that is, they are transitive and reflexive). Mathematically, a more convenient concept is that of a *utility function* which is a C^r, $r \geq 0$, function $u_i : \bar{P} \to \mathbb{R}$ with the property $u_i(x) \geq u_i(y)$ if and only if $x \succcurlyeq_i y$. The necessary conditions for the existence of such functions are presented in the following theorem by Debreu [2] (chapter 4.6)

Theorem 1.2. Let $\succcurlyeq$ be a total preordering on $\bar{P}$ such that, for all $y \in \bar{P}$ sets of the form $\{x \in \bar{P} | y \succcurlyeq x\}$ and $\{x \in \bar{P} | x \succcurlyeq y\}$ are closed. Then there exist a continuous utility function $u : \bar{P} \to \mathbb{R}$.

For the most part we assume that preferences are *realised* as C^r utility functions where $r \geq 1$ and we shall deal directly with utility functions[1]. We shall also assume the utility functions satisfy the following desirability condition :

D: $Du_i \in P$.

That is, an agent prefers more of a commodity to less of the commodity (no good is noxious !). The *indifference surfaces* for the i-th agent are $u_i^{-1}(c) \subset \bar{P}$ as c varies over $\mathbb{R}$.

For now we shall assume utility functions are, in addition, strictly convex :

C: for any commodity bundle x, the set $\{y \in \bar{P} | u_i(y) \geq u_i(x)\}$ is strictly convex

and therefore we can define the i-th agent's *demand function* $\mathcal{D}_i$ by

$$\mathcal{D}_i(p) = \text{ maximum } x \text{ of } u_i \text{ on the set } \{y \in \bar{P} | p(y) = p(r)\}$$

r a given initial endowment. Using elementary Lagrange multiplier techniques it is easy to verify that the demand at a price p is the point of tangency between the utility function and the budget plane.

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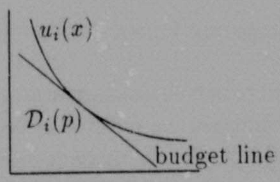
[1]By C^r on $\partial \bar{P}$, we mean that there is an open set Q of $\mathbb{R}^l$ containing $\bar{P}$ and an extension of u to $u^* : Q \to \mathbb{R}$ such that u^* is C^r. Furthermore, the derivatives of order $i \leq r$ of u at $x \in \bar{P}$ will be defined to be those of u^*. Such derivatives are independent of the choice of u^*.



As this requires the matrix of first derivatives Du_i to be orthogonal to the budget plane at $\mathcal{D}_i(p)$, there is a positive scalar λ such that

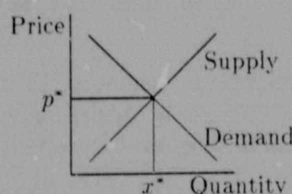
$$\lambda p = Du_i(\mathcal{D}_i(p)).$$

The i-th agents *excess demand function*, $f_i(p) = \mathcal{D}_i(p) - r_i$, is the difference between what is demanded, $\mathcal{D}_i(p)$, and what is "supplied" r_i. The *aggregate* or *market excess demand function* $f(p) = \sum_{i=1}^n f_i(p)$ is then easily seen to have the following properties :

C: $f(p)$ is single valued and smooth (because of u_i's convexity and smoothness):
H: $f(p)$ is homogeneous of degree zero (because each $f_i(p)$ is defined by the tangency between the utility function and the budget plane; and for positive scalar μ, both p and μp define the same budget plane); and
W: $f(p)$ is orthogonal to p (because both r_i and $\mathcal{D}_i(p)$ are in the budget plane).

Condition (W) is known as *Walras' law*. It states that if all prices are normalised then $f(p)$ is a smooth tangent vector field on S.

For the simple case of one market where prices are measured in terms of some extra market standard, the following familiar diagram gives some justification for the existence of the equilibrium price p^*.



General equilibrium treats this problem for several markets simultaneously. Even within the confines of a competitive economy this can be a very cumbersome agenda.

Remark 1.3. As it is stated and as we shall examine it, equilibrium does not correspond to the special case in which it is regarded as a sort of optimum position of some objective function. Even though many economists have examined equilibrium states in terms of their optimality or non-optimality, this aspect will not be addressed here.

This section is dedicated to the mathematical background for the economic applications which follow. It is designed as a reference section to be read in conjunction with, but not necessarily before, the application which follow.

2.1. Manifold Theory. We begin with some introductory definitions from differential topology.

Definition 2.1. A topological space X is called *locally compact* if every point has a neighbourhood with a compact closure. It is called *paracompact* if every open cover $\{U_\lambda\}_{\lambda\in\Lambda}$ has a locally finite open refinement $\{V_\gamma\}_{\gamma\in\Gamma}$, that is, $\{V_\gamma\}_{\gamma\in\Gamma}$ is an open cover of X, each V_γ is contained in some U_λ and for each $x \in X$, the set $\gamma \in \Gamma$ for which $x \in V_\gamma$ is finite.

Let M be a Hausdorff topological space with a countable basis. A *local chart* in M is a pair (ϕ, U), where $U \subset M$ is open and $\phi : U \to U_0 \subset \mathbb{R}^m$ is a homeomorphism onto an open set U_0 of $\mathbb{R}^m$. We say that U is a *parametrised neighbourhood* in M. If (ϕ, U) and (φ, V) are local chart in M, with $U \cap V \neq \emptyset$, the change of co-ordinates

$$\varphi\phi^{-1} : \phi(U \cap V) \to \varphi(U \cap V)$$

is a homeomorphism. A *differentiable manifold of class* C^r ($r \in \mathbb{N} \cup \infty$ or real analytic) is a topological space together with a family of local charts which satisfy

a. the parametrised neighbourhoods cover M; and
b. the changes of co-ordinate are C^r diffeomorphisms.

Such a family of charts is called a C^r *atlas* for M. A maximal atlas is called a differentiable structure.

All the manifolds considered in the economic applications sections will be subsets of Euclidean space $\mathbb{R}^m$ where differentiable of class C^r, $1 \leq r \leq \infty$, is the same as differentiable of class C^∞. We do however use general manifolds to develop some of the required results.

To define differentiable maps between manifolds, let M, N be manifolds of dimension m, n and $f : M \to N$ be a map. A pair of charts (ϕ, U) for M and (ψ, V) for N is *adapted to* f if $f(U) \subset V$. We say that f is *of class* C^r if for each point $x \in M$ there are local charts $(\phi, U), x \in U$ and (ψ, V) adapted to f and

$$\psi f \phi^{-1} : \phi(U) \to \psi(V)$$

is of class C^r. This choice is independent of the charts if M, N are C^r. In particular, a curve $\alpha : (-\epsilon, \epsilon) \to M$ is differentiable if $\phi\alpha : (-\epsilon, \epsilon) \to \mathbb{R}^m$ is differentiable where (ϕ, U) is a local chart with $\alpha(-\epsilon, \epsilon) \in U$.

Definition 2.2. Let $\psi : M \to N$ be a C^∞ map of manifolds then :

- ψ is called an *immersion* if $d\psi_m$ is non-singular for all $m \in M$.
- ψ is called an *embedding* if ψ is a one-to-one immersion which is also a homeomorphism into; that is ψ is open as a map into $\psi(M)$ with the relative topology.

The *tangent vector to α at $x = \alpha(0)$*, denoted by $\dot\alpha(0)$ or $\frac{d}{dt}\alpha|_0$, is defined as the set of differentiable curves $\beta : (-\epsilon, \epsilon) \to M$ such that $\beta(0) = x$ and $d(\phi\beta)(0) = d(\phi\alpha)(0)$. This definition does not depend on the choice of local chart (ϕ, U). The *tangent space* to M at x, denoted M_x or T_xM, is the set of tangent vectors to differentiable curves passing through x. It follows that M_x has natural m-dimensional vector space structure. If $f : M \to N$ is a differentiable map with $f(x) = y$, we define $T_xf(x) : M_x \to N_y$ as the map which takes the tangent vector at x to the curve $\alpha : (-\epsilon, \epsilon) \to M$ to the tangent vector at y to the curve $f\alpha : (-\epsilon, \epsilon) \to N$. This definition does not depend on the choice of the curve and T_xf is a linear map.

Given a specific chart (ϕ, U), we define the standard basis of T_xM by taking a canonical basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ and setting

$$\frac{\partial}{\partial x_i} := \dot\alpha(0)$$

where $\alpha_i(t) = \phi^{-1}(\phi(x) + te_i)$. Define the *tangent bundle* of M to be the disjoint union

$$TM = \bigcup_{x\in M} T_xM$$

of the tangent spaces with the canonical projection $\pi : TM \to M$ such that $\pi(T_xM) = x$. Any chart (ϕ, U) of M then induces a chart $(\varphi, U \times \bigcup_{x\in U} T_xU)$ by taking

$$\varphi := (\phi(x), (v_1, \ldots v_n)) \in \mathbb{R}^n \times \mathbb{R}^n$$

where v_i are the coefficients of $v \in T_xM$ with respect to the basis $\{\frac{\partial}{\partial x_1}; \ldots; \frac{\partial}{\partial x_n}\}$ of T_xM in such a way that TM is a differentiable manifold.

Definition 2.3. A *vector field* is a map $X : M \to TM$ such that $\pi \circ X = 1_M$, that is X assigns each $m \in M$ a tangent vector at m denoted X_m. It is called *smooth* if $X \in C^\infty(M, TM)$. If f is a function on M, then $X(f)$ or Xf is the function on M whose value at m is $X_m(f)$.

Definition 2.4. If X, Y are smooth vector fields on M define a vector field $[X, Y]$, called the *Lie bracket* of X and Y, by setting

$$[X, Y]_m(f) = X_m(Yf) - Y_m(Xf).$$

It is not difficult to show

Proposition 2.5. 1. $[X, Y]$ is indeed a smooth vector field on M.
2. If $f, g \in C^\infty$, then $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$.
3. $[X, Y] = -[Y, X]$.
4. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ for all smooth vector fields X, Y and Z on M.

A subset S of a manifold M is called a *submanifold of class C^r* of dimension s if, for each $x \in S$, there are open sets $U \subset M$ containing x, $V \subset \mathbb{R}^s$ containing 0 and $W \subset \mathbb{R}^{m-s}$ containing 0 and a diffeomorphism $\phi : U \to V \times W$ of class C^r such that $\phi(S \cap U) = V \times \{0\}$. The *codimension* of S is $m - s$. We call such a (ϕ, U) a *submanifold chart* for (M, S).

2.1.1. *Manifolds with Boundary.* A *halfspace* of $\mathbb{R}^m$ is a subset of the form

$$H = \{x \in \mathbb{R}^m \mid \lambda(x) \geq 0\}$$

where $\lambda : \mathbb{R}^m \to \mathbb{R}$ is a linear map. If $\lambda \equiv 0$ the $H = \mathbb{R}^m$ otherwise H is called a *proper* halfspace. If H is proper its boundary is $\partial H = \text{kernel}(\lambda)$; this is a linear subspace of dimension $m-1$. If $H = \mathbb{R}^m$ we set $\partial H = \emptyset$.

We extend the definition of a chart on a space M to mean a map $\phi : U \to \mathbb{R}^m$ which takes open sets $U \in M$ homeomorphically onto open sets of a halfspace in $\mathbb{R}^m$. Since $\mathbb{R}^m$ is itself a halfspace, this definition subsumes the previous one. Using this definition of a chart, we can extend the meaning of C^r atlas and C^r manifold .

Let (M, Φ) be a C^r manifold (in the new sense). Suppose $(\phi, U) \in \Phi$ and $\phi(U)$ is an open subset of the proper halfspace $H \subset \mathbb{R}^m$. If $x \in \phi^{-1}(\partial H)$ we say that x is a *boundary point* for the chart (ϕ, U). This condition is independent of the chart (for $r \geq 1$, the case we shall consider, this follows from the inverse function theorem).

The *boundary* of the C^r manifold (M, Φ) is defined to be the set of points $x \in M$ which are boundary points for some (and hence any) chart. These points are denoted ∂M. By restricting charts to ∂M, it can be shown that ∂M is a C^r manifold of dimension $m-1$.

The definition of a C^r map between C^r manifolds remains unchanged as does the definition of tangent vectors.

A subset $S \subset M$ is a C^r submanifold if, for each $x \in S$, there is an open set $U \subset M$ containing x, a C^r embedding $g : U \to \mathbb{R}^m$ and a halfspace H of $\mathbb{R}^k \subset \mathbb{R}^m$ such that

$$S \cap U = g^{-1}(H).$$

We call $S \subset M$ a *neat submanifold* if $\partial S = S \cap \partial M$ and S is covered by charts (ϕ, U) of M such that

$$S \cap U = \phi^{-1}(\mathbb{R}^s)$$

where $s = \dim S$.

If S is a submanifold of M and $\partial S = \emptyset$, then S is neat if and only if $S \cap \partial M = \emptyset$. In general, S is neat if and only if $\partial S = S \cap \partial M$ and (for $r \geq 1$) S is not tangent to ∂M at any point $x \in \partial S$, that is, we have $S_x \not\subset (\partial M)_x$.

Remark 2.6. We shall now make the implicit assumption that, unless there is an explicit statement to the contrary, all manifolds will be assumed to be paracompact, with a countable base, and Hausdorff.

2.2. **Weak and Strong Topologies in $C^r(M, N)$.** Since these topologies arise frequently in the sections which follow[2] we spend some time illuminating the most apparent and useful properties. The strong and weak topologies are introduced here via jets. This is done not only because the approach is co-ordinate free, but also to introduce concepts which we have cause to revisit in §2.3.

[2] For example, in §3, an economy will be defined in terms of utility functions $u \in C^2(P, \mathbb{R})$ and we will prove some denseness results about relevant subsets.

Let M, N be C^r manifolds and let $C^r(M, N)$ denote the set of C^r maps from M to N. $C^r(M, N)$ will be identified with a subset of continuous maps from M to $J^r(M, N)$, where $J^r(M, N)$ is the manifold of r-jets of maps from M to N. We begin therefore by defining these topologies on the space $C(X, Y)$ of continuous maps from a space X to a space Y.

The *weak* or *compact open* topology on $C(X, Y)$, denoted $C_w(X, Y)$, is generated by the subbase comprising all sets of the form

$$\{f \in C(X, Y) \mid f(K) \subset V\}$$

where $K \subset X$ is compact and $V \subset Y$ is open.

The weak topology is most useful when X is locally compact. When Y is a metric space, the topology is the same as that of uniform convergence on compact sets. If X is compact and Y is a metric, $C_w(X, Y)$ has the metric

$$d(f, g) = \sup_{x \in X} d(f(x), g(x)).$$

This metric is complete provided Y is a complete metric space. More generally,

Theorem 2.7. Let each component of X be locally compact with a countable base; let Y be a complete metric. Then $C_w(X, Y)$ has a complete metric.

Proof. It suffices to construct a complete metric on $C_w(X_\alpha, Y)$ for each component X_α of X, therefore, assume X is locally compact with a countable base. Then X has a countable covering by compact sets $\{X_n\}$. Each space $C_w(X_n, Y)$ has a complete metric. Define a map

$$\rho : C_w(X, Y) \to \prod_n C_w(X_n, Y)$$
$$\rho_n(f) = f \mid X_n.$$

Then ρ is a homeomorphism onto a closed subspace. Since the product of a countable number of complete metric spaces has a complete metric, $C_w(X, Y)$ is homeomorphic to a closed subspace of a complete metric space and thus has a complete metric. $\square$

Let X, Y be arbitrary spaces. The space $C_s(X, Y)$ is the set $C(X, Y)$ with the following *strong* topology (this topology is also called the *fine* or *Whitney* topology). Let $\Gamma_f \subset X \times Y$ denote the graph of f. If $W \subset X \times Y$ is an open set containing Γ_f, let

$$\mathcal{N}(f, W) = \{g \in C(X, Y) \mid \Gamma_g \subset W\}.$$

These sets, for all f and W, form a base for the strong topology. The induced topology on a subset of $C(X, Y)$ is also called *strong*. When X is paracompact, Y is metric, $C_s(X, Y)$ has a base consisting of all sets of the form

$$\mathcal{N}(f, \epsilon) = \{g \in C(X, Y) \mid d(g(x), f(x)) < \epsilon(x) \text{ for all } x \in X\} \tag{2.1}$$

where $f \in C(X, Y)$ and $\epsilon \in C(X, \mathbb{R}^+)$ are arbitrary.

If X is compact, the weak and strong topologies are the same.

We cannot expect the strong topology to have a complete metric, since it may not have any metric. We shall, however, see that in many cases it is a *Baire space*, that is, a space in which the intersection of a countable family of open dense sets is dense.

Let Y be a metric space. A subset of $C(X,Y)$ is *uniformly closed* if it contains the limit of every uniformly convergent sequence in it. Obviously this concept depends on the metric in Y. We note here that a subset which is closed under pointwise convergence is uniformly closed as is a subset which is closed in the weak topology.

Theorem 2.8. Let X be paracompact and Y a complete metric space. Then every uniformly closed subset $Q \subset C(X,Y)$ is a Baire space in the strong topology.

Proof. Let $\{A_n\}_{n\in N}$ be a sequence of open dense subsets of Q (working here and throughout this theorem in the strong topology) and let $U \subset Q$ be a nonempty open set. Then $A_0 \cap U$ is a nonempty subset of Q. Therefore there is a $f_0 \in A_0 \cap U$ and $\epsilon_0 \in C(X,\mathbb{R}^+)$ such that

$$Q \cap \bar{\mathcal{N}}(f_0, \epsilon_0) \subset A_0 \cap U.$$

where $\bar{\mathcal{N}}(f_0,\epsilon_0) = \{g \mid d(f_0(x), g(x)) \le \epsilon_0(x)\}$. We may assume $\epsilon_0 < 1$.

By recursion, there are sequences $\{f_n\} \in Q$ and $\{\epsilon_n\} \in C(X,\mathbb{R}^+)$ such that, for all $n \in N$

$$Q \cap \bar{\mathcal{N}}(f_{n+1}, \epsilon_{n+1}) \subset A_{n+1} \cap \mathcal{N}(f_n, \epsilon_n)$$

and $\epsilon_{n+1} \le \epsilon_n/2$. The sequence $\{f_n\}$ satisfies

$$d(f_{n+1}(x), f_n(x)) \le 2^{-n}$$

and so is uniformly convergent. The limit f is in Q since Q is uniformly closed. Also, f belongs to every $\mathcal{N}(f_n,\epsilon_n)$, so $f \in U$ and $f \in \bigcap A_n$ □

Corollary 2.9. If M and N are C^0 manifolds, every weakly closed subset $Q \subset C(M,N)$ is a Baire space in the strong topology.

2.2.1. *Jets and the Baire Property.* We now define the jets of finite order r, treating firstly manifolds without boundary. Let M, N be C^s manifolds $0 \le s \le \infty$. An *r-jet*, r finite and $r \le s$, from M to N is an equivalence class $[x,f,U]_r$ of triples (x,f,U) where $U \subset M$ is open, $x \in U$ and $f : U \to N$ is a C^r map. The equivalence relation is $[x,f,U]_r = [x',f',U']_r$ if $x = x'$ in some (and hence any) pair of charts adapted to f at x, f and f' have the same derivatives up to order r at x. We use the notation

$$[x,f,U]_r = j^r_x f = j^r f(x)$$

to denote the r-jet of f at x. We call x the *source* and $f(x)$ the *target* of $[x,f,U]_r$.

The set of all r-jets from M to N is denoted $J^r(M,N)$. There are well defined source and target maps :

$$\sigma : J^r(M,N) \to M \quad \sigma[x,f,U]_r = x$$
$$\tau : J^r(M,N) \to N \quad \tau[x,f,U]_r = f(x).$$

We define $J^r_x(M,N) = \sigma^{-1}(x), J^r(M,N)_y = \tau^{-1}(y)$ and $J^r_{x,y}(M,N) = J^r_x(M,N) \cap J^r(M,N)_y$ which is the set of all r-jets from M to N with source x and target y. In the special case $M = \mathbb{R}^m, N = \mathbb{R}^n$, we write $J^r(\mathbb{R}^m,\mathbb{R}^n) = J^r(m,n)$. Suppose