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MASTERS DISSERTATION

Two-dimensional and axisymmetric turbulent thermal jets

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Abstract

Faculty of Science
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Master of Science

Two-dimensional and axisymmetric turbulent thermal jets

by Erick MUBAI

In this dissertation, the two-dimensional and axisymmetric turbulent thermal jets are studied. The closure problem in the Reynolds averaged Navier-Stokes equations is resolved by introducing the eddy viscosity while the closure problem in the averaged energy balance equation is resolved by introducing the turbulent thermal conductivity. The kinematic eddy viscosity and turbulent thermal conductivity are modelled using the mixing lengths formulation. The governing equations are then written in terms of the stream function to give a non-linear system of a third order partial differential equation coupled with a second order partial differential equation for the temperature difference. The governing system of two partial differential equations is reduced to two ordinary differential equations using the associated Lie point symmetry approach.

The associated Lie-point symmetry approach is very powerful when dealing with partial differential equations. If a system of partial differential equations has a conserved vector, the Lie point symmetry associated with that conserved vector can be used to reduce the system of partial differential equations to a system of ordinary differential equations. This system of ordinary differential equations can be integrated at least once, according to the double reduction theorem [10,11].

The associated Lie point symmetry is derived using conserved vectors from conservation laws of the governing partial differential equations. The conserved vectors are obtained from the multiplier method. The conservation laws are used to obtain conserved quantities by integrating the conservation laws along the axis perpendicular to the centreline of the jet from the centreline to the boundary of the jet.

The reduced ordinary differential equations for the invariant solutions can only be solved analytically when the kinematic viscosity and the thermal conductivity are both zero for the two-dimensional jet and for other cases they are solved numerically. The conserved quantities are used to supplement the homogeneous boundary conditions to fully solve the jet problem. The shooting method is used to obtain the numerical solution.

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Nomenclature

$B(m,n)$	Beta function
$y_B(x), r_B(z)$	Boundary of the jet
$\underline{\nabla}$	Del vector operator, $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \dots\right)^T$
ρ	Density of the fluid
μ_T	Eddy viscosity of the flow
E_ψ, E_S	Euler operators
$S(x,y)$	Difference between the temperature at position (x,y) and the temperature at $(x, y_B(x))$, where $y_B(x)$ is the boundary of the jet.
$S(z,r)$	Difference between the temperature at position (x,y) and the temperature at $(x, y_B(x))$, where $y_B(x)$ is the boundary of the jet.
F'	Fluctuations of the variable F
$\Gamma(n)$	Gamma function
F'	Fluctuations of the variable F
T^i	i-th component of the conserved vector
v_i	i-th component of fluid velocity
x_i	i-th independent variable
Λ_i	i-th multiplier
ν_T	Kinematic eddy viscosity of the flow
ν	Kinematic viscosity of the fluid
X	Lie point symmetry generator
$X^{[\infty]}$	Lie point symmetry generator extension containing all derivatives.
$X^{[n]}$	Lie point symmetry generator extension containing up to n-th order derivatives.
$\overline{(\cdot)}$	Mean operator
β_1	Momentum boundary constant
J	Momentum conserved quantity
l	Momentum mixing length
l_0	Momentum mixing length constant
$\frac{\partial}{\partial x_i}$	Partial differentiation with respect to the i-th independent variable.
F_i	Partial differentiation with respect to the i-th independent variable applied to the dependent variable F.
Pr	Prandtl's number
p	Pressure
ψ	Stream function
c_v	Specific heat capacity of the fluid
T	Temperature
Re	The Reynolds number
β_2	Thermal boundary constant
K	Thermal conserved quantity
κ	Thermal conductivity
l_θ	Thermal mixing length
l_1	Thermal mixing length constant
D_i	Total derivative with respect to the i-th independent variable.
κ_T	Turbulent thermal conductivity
$\underline{x}$	Vector containing all independent variables.

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Dedicated to Rosa Angelina Masehle Mubai.

Chapter 1

Introduction

In this dissertation, we study the two-dimensional and axisymmetric turbulent thermal jets with the surrounding fluid at rest and constant temperature. The conduit for the two-dimensional jet is rectangular with infinite width and the conduit for the axisymmetric jet is small and circular and the fluid flow is axially symmetric. In practical terms, infinite width means that the width of the conduit is very large compared to the height. Some real-world jet examples include a wastewater discharging from pipes into rivers and gas exiting an aircraft nozzle. For both the two-dimensional and axisymmetric free jets, the flow is laminar close to the conduit and transitions from laminar to turbulence as the distance from the conduit increases [1]. The analytical solution for the laminar region of the jet is not easily accessible [1]. A numerical solution in [2] and an analytical solution in [3] have been presented for laminar jets. However, the solution for the laminar jets is not stable farther downstream.

The Navier-Stokes and the energy balance equations are averaged by splitting every variable into the mean and fluctuations. The turbulence closure problem is resolved by relating the Reynolds stresses and covariances between the velocity components and temperature to the eddy viscosity and the turbulent thermal conductivity. The coupled system of governing equations is further simplified using the boundary layer theory. The eddy viscosity and turbulent thermal conductivity are modelled using mixing length hypothesis. The associated Lie point symmetry theory is used to reduce these coupled system of governing equations to ordinary differential equations.

In recent decades, there has been a growing interest in Lie point symmetries due to their powerful ability to reduce non-linear partial differential equations to ordinary differential equations. Lie point symmetries were first discovered by Sophus Lie and pioneered by Emmy Noether who applied them to differential equations. Noether then uncovered a relationship between the conservation laws and the Noether symmetries giving rise to the Noether theorem [4]. The formula relating the conserved vector of the partial differential equation and the Lie-Bäcklund symmetry associated to the conserved vector was provided in [5].

Conserved quantities are essential when determining constants or arbitrary functions which cannot be found by the homogeneous boundary conditions of the jet. A direct method for obtaining these conserved quantities is by integrating the coupled system of governing equations along the axis perpendicular to the centreline of the jet from the centreline to the boundary of the jet. This method is however not systematic. A systematic method for finding conserved quantities is by using conservation laws.

Conservation laws are obtained using various methods, which include: the Noether's approach [3], the direct method [6] and the multiplier method [7]. Naz et al [8] compared all these methods and concluded that the multiplier method was more systematic and easier compared to its counterparts. Hence, in this work, we use the multiplier method. This method was first introduced by Steudel et al [9] and invoked by Anco and Blumann [7] and Naz, Mohammed and Mason [9]. From the conserved vectors, we obtain the associated Lie point symmetries, which are then used to reduce the coupled system of governing partial differential equations to ordinary differential equations. Therefore by the double reduction theorem [10,11], we can integrate the ordinary differential equations at least once. Obtaining the associated Lie point symmetries has advantages over the direct methods of obtaining the Lie point symmetries. By using the conserved vectors to obtain associated Lie point symmetries, the calculations become less tedious.

Schlichting [1] assumed a similarity solution for the two-dimensional laminar jet and successfully reduced the governing equations to ordinary differential equations. The assumed similarity solution can be shown to be a scaling transformation. Hence, scaling transformations work for two-dimensional laminar jets. It is shown in this work that the coupled system of governing equations for the turbulent thermal free jets admits a scaling transformation.

The ordinary differential equations obtained by reducing the coupled system of governing partial differential equations using the Lie point symmetry method can be solved analytically under special conditions. For the two dimensional free jets, when the kinematic viscosity and the thermal conductivity are zero, the reduced ordinary differential equation for the mean momentum is a second order autonomous differential equation. However, when the kinematic viscosity and the thermal conductivity are non-zero, the reduced ordinary differential equations are highly nonlinear and hence have to be solved numerically. For the axisymmetric jet, there are no special cases where the reduced ordinary differential equations can be solved analytically. Hence, we resort to numerical techniques. The ordinary differential equations combined with the boundary conditions and conserved quantities form a complete system. This system will be solved using the shooting method adapted from [13].

The analytical results obtained for the special cases can be used to validate the numerical techniques.

This dissertation is structured in the following manner: In Chapter 2 we give a review of the methods that will be used in this dissertation. In Chapter 3 we derive the coupled system of governing equations for the two-dimensional and axisymmetric turbulent thermal jets. In Chapter 4 we solve the coupled system of governing equations for the two-dimensional jet. In Chapter 5 we find solutions to the coupled system of governing equations for the axisymmetric jet. In Chapter 6 we give the conclusions.

Chapter 2

Methods: Lie point symmetries, conservation laws and invariant solutions

2.1 Introduction

Sophus Lie first introduced Lie point symmetries at the end of the nineteenth century. He realised that a symmetry generator admitted by a differential equation may be used to reduce a partial differential equation to an ordinary differential equation [14], to reduce the order of an ordinary differential equation by one, and to linearize and transform an ordinary differential equation to a simpler simpler differential equation [15]. Emmy Noether took the subject further by relating Noether symmetries to conservation laws giving rise to the Noether theorem [4]. Kara et al [5] later gave a formula relating conserved vectors of a differential equation and the Lie-Bäcklund symmetry associated to the conserved vector.

Conservation laws are essential when dealing with free jet problems since they provide conserved quantities. There are various methods for obtaining conservation laws. Naz, Mahomed and Mason [8] comprehensively compared these methods and showed that the multiplier method is simpler and more systematic than its counterparts. Conserved quantities allow for the governing partial differential equations for the free jet problem to be fully solved after they have been reduced to ordinary differential equations using a Lie point symmetry. In this chapter, we review the methods that will be used in this dissertation to solve the two-dimensional and axisymmetric turbulent thermal jets.

This chapter is organized in the following manner. In Section 2.2 we discuss conserved vectors. Then, in Section 2.3 we discuss associated Lie point symmetries. Finally, we provide conclusions in Section 2.4.

2.2 Conserved vectors

Conserved vectors give a less tedious process for obtaining Lie point symmetries than the standard procedure. In this section we show how these conserved vectors are found.

Consider system of two partial differential equations

$$F^i(\underline{x}, \psi, S, \psi_{(1)}, S_{(1)}, \psi_{(2)}, ...) = 0, \; i = 1, 2 \tag{2.2.1}$$

for the two unknown functions $\psi(\underline{x})$ and $S(\underline{x})$ where $\psi_{(k)}$ and $S_{(k)}$ are all k-th order derivatives with respect to $\underline{x}$. The conserved vector $(T^1, T^2, ...)$, written in Einstein's summation notation, is defined by

$$D_i T^i \Big|_{F^j=0 \text{ for all } j} = 0, \tag{2.2.2}$$

where

$$D_i = \frac{\partial}{\partial x_i} + \psi_i \frac{\partial}{\partial \psi} + S_i \frac{\partial}{\partial S} + \psi_{ij} \frac{\partial}{\partial \psi_j} + S_{ij} \frac{\partial}{\partial S_j} + ... \tag{2.2.3}$$

and (2.2.2) is called the conservation law for the system with components (2.2.1). The multiplier method assumes that the divergence of the conserved vector can be written as

$$\Lambda_i F^i = D_i T^i, \tag{2.2.4}$$

where Λ_i is a multiplier and

$$\Lambda_i = \Lambda_i(\underline{x}, \psi, S, \psi_{(1)}, S_{(1)}, \psi_{(2)}, ...). \tag{2.2.5}$$

The standard Euler operators defined by

$$E_\psi = \frac{\partial}{\partial \psi} + \sum_{s\geq 1} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial \psi_{i_1 \cdots i_s}} \tag{2.2.6}$$

and

$$E_S = \frac{\partial}{\partial S} + \sum_{s\geq 1} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial S_{i_1 \cdots i_s}} \tag{2.2.7}$$

annihilate the right hand side of equation (2.2.4) to give

$$E_\psi \left(\Lambda_i F^i \right) = 0, \tag{2.2.8}$$

$$E_S \left(\Lambda_i F^i \right) = 0. \tag{2.2.9}$$

Equations (2.2.8) and (2.2.9) are used to find determining equations for multipliers Λ_i. The multipliers are then substituted into (2.2.4) to obtain the conserved vector, $(T^1, T^2, ...)$.

Example

Consider the partial differential equation

$$\frac{\partial u}{\partial x} + u \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2}. \tag{2.2.10}$$

When dealing with conservation laws and Lie point symmetries the partial derivatives are written using the subscript notation in order to indicate that u and its partial derivatives are independent variables. Using the multiplier method we need to find scalar $\Lambda(x, y, u)$ such that

$$\Lambda(x, y, u) \left(u_x + u u_y - u_{yy} \right) = D_x T^1 + D_y T^2, \tag{2.2.11}$$

for all functions $u(x, y)$. Applying the Euler operator

$$E_u = \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} - D_y \frac{\partial}{\partial u_y} + D_y^2 \frac{\partial}{\partial u_{yy}} + ... \tag{2.2.12}$$

to (2.2.11) we obtain

$$-2u_{yy}\frac{\partial \Lambda}{\partial u}-\frac{\partial \Lambda}{\partial x}-u\frac{\partial \Lambda}{\partial y}-\frac{\partial^2 \Lambda}{\partial y^2}-2u_y\frac{\partial^2 \Lambda}{\partial y \partial u}-\left(u_y\right)^2\frac{\partial^2 \Lambda}{\partial u^2}-uu_y\frac{\partial \Lambda}{\partial u}=0. \quad (2.2.13)$$

Splitting equation (2.2.13) into coefficients of independent partial derivatives of u we obtain

$$u_{yy}: \frac{\partial \Lambda}{\partial u}=0, \quad (2.2.14)$$

$$u_y: \frac{\partial^2 \Lambda}{\partial y \partial u}+u\frac{\partial \Lambda}{\partial u}=0, \quad (2.2.15)$$

$$\left(u_y\right)^2: \frac{\partial^2 \Lambda}{\partial u^2}=0, \quad (2.2.16)$$

$$\text{Remainder}: \frac{\partial \Lambda}{\partial x}+u\frac{\partial \Lambda}{\partial y}+\frac{\partial^2 \Lambda}{\partial y^2}=0. \quad (2.2.17)$$

Equations (2.2.14) to (2.2.17) reduce to

$$\Lambda=c, \quad (2.2.18)$$

where c is a constant. Hence equation (2.2.11) can be written as

$$c\left(u_x+uu_y-u_{yy}\right)=D_x\left(cu\right)+D_y\left(\frac{1}{2}cu^2-cu_y\right), \quad (2.2.19)$$

for arbitrary $u(x,y)$. When $u(x,y)$ is a solution of (2.2.10) then (2.2.19) becomes

$$D_x\left(u\right)+D_y\left(\frac{1}{2}u^2-cu_y\right)\Big|_{(2.2.10)}=0. \quad (2.2.20)$$

Thus the conserved vector is

$$\left(u,\frac{1}{2}u^2-u_y\right). \quad (2.2.21)$$

2.3 Associated Lie point symmetries

A Lie point symmetry generator of the system (2.2.1) is of the form

$$X=\xi^i(\underline{x},\psi,S)\frac{\partial}{\partial x_i}+\eta^1(\underline{x},\psi,S)\frac{\partial}{\partial \psi}+\eta^2(\underline{x},\psi,S)\frac{\partial}{\partial S}. \quad (2.3.1)$$

It is associated with the conserved vector $T=(T^1,T^2,...)$ provided [5]

$$X^{[\infty]}(T^i)+T^iD_k(\xi^k)-T^kD_k(\xi^i)=0, \quad (2.3.2)$$

where the prolongation $X^{[\infty]}$ is defined by

$$X^{[\infty]}=\xi^i\frac{\partial}{\partial x_i}+\eta^1\frac{\partial}{\partial \psi}+\eta^2\frac{\partial}{\partial S}+\sum_{s\geq 1}\zeta^\alpha_{i_1\cdots i_s}\frac{\partial}{\partial u_{i_1\cdots i_s}},\ \alpha=1,2 \quad (2.3.3)$$

and

$$\zeta^\alpha_i=D_i\left(\eta^\alpha-\xi^ju^\alpha_j\right)+\xi^ju^\alpha_{ij}, \quad (2.3.4)$$

$$\zeta^{\alpha}_{i_1\cdots i_s} = D_{i_1}\cdots D_{i_s}\left(\eta^{\alpha} - \xi^{j}u^{\alpha}_{j}\right) + \xi^{j}u^{\alpha}_{ji_1\cdots i_s} \tag{2.3.5}$$

and

$$u^1 = \psi, \qquad\qquad u^2 = S. \tag{2.3.6}$$

Example continued

Continuing with the example in the previous section, we find the Lie point symmetry

$$X = \xi^1(x,y,u)\frac{\partial}{\partial x} + \xi^2(x,y,u)\frac{\partial}{\partial y} + \eta(x,y,u)\frac{\partial}{\partial u} \tag{2.3.7}$$

associated with the conserved vector (2.2.21) in order demonstrate the methods described above. The associated Lie point symmetry condition (2.3.2), with $i = 1$ and using the conserved vector (2.2.21), becomes

$$X(T^1) + T^1 D_y\xi^2 - T^2 D_y\xi^1 = 0, \tag{2.3.8}$$

which expands to

$$\eta + u\frac{\partial\xi^2}{\partial y} - \frac{1}{2}u^2\frac{\partial\xi^1}{\partial y} + \left(u\frac{\partial\xi^2}{\partial u} + \frac{\partial\xi^1}{\partial y} - \frac{1}{2}u^2\frac{\partial\xi^1}{\partial u}\right)u_y + \left(u_y\right)^2\frac{\partial\xi^1}{\partial u} = 0. \tag{2.3.9}$$

Splitting equation (2.3.9) into coefficients of derivatives of u we obtain

$$u_y : u\frac{\partial\xi^2}{\partial u} + \frac{\partial\xi^1}{\partial y} - \frac{1}{2}u^2\frac{\partial\xi^1}{\partial u} = 0, \tag{2.3.10}$$

$$\left(u_y\right)^2 : \frac{\partial\xi^1}{\partial u} = 0. \tag{2.3.11}$$

$$\text{Remainder: } \eta + u\frac{\partial\xi^2}{\partial y} - \frac{1}{2}u^2\frac{\partial\xi^1}{\partial y} = 0. \tag{2.3.12}$$

Equation (2.3.11) implies

$$\xi^1 = \xi^1(x,y). \tag{2.3.13}$$

Substituting (2.3.13) into (2.3.10) we obtain

$$u\frac{\partial\xi^2}{\partial u} + \frac{\partial\xi^1}{\partial y} = 0. \tag{2.3.14}$$

The second term of equation (2.3.14) is independent of u. Hence, the first term should also be independent of u, that is

$$u\frac{\partial\xi^2}{\partial u} = A(x,y), \qquad\qquad \frac{\partial\xi^1}{\partial y} = -A(x,y), \tag{2.3.15}$$

where $A(x,y)$ is an arbitrary function of x and y. Hence integrating the first equation of (2.3.15) we obtain

$$\xi^2 = A(x,y)\ln u + B(x,y), \tag{2.3.16}$$

where $B(x,y)$ is an arbitrary function of x and y. Substituting equation (2.3.16) and the second equation of (2.3.15) into equation (2.3.12) we obtain

$$\eta = -u\ln u\frac{\partial A}{\partial y} - u\frac{\partial B}{\partial y} - \frac{1}{2}u^2 A(x,y). \tag{2.3.17}$$

The associated Lie point symmetry equation (2.3.2), with $i = 2$ and using the conserved vector (2.2.21), becomes

$$X^{[1]}(T^2) + T^2 D_x\xi^1 - T^1 D_x\xi^2 = 0, \tag{2.3.18}$$

which expands to

$$u\eta - \zeta_y + \frac{1}{2}u^2\frac{\partial\xi^1}{\partial x} - u_y\frac{\partial\xi^1}{\partial x} - u\frac{\partial\xi^2}{\partial x} - uu_x\frac{\partial\xi^2}{\partial u} = 0, \tag{2.3.19}$$

where the prolongation ζ_y, using (2.3.4), is

$$\zeta_y = D_y\eta - u_x D_y\xi^1 - u_y D_y\xi^2 \tag{2.3.20}$$

The prolongation ζ_y can be further expanded to

$$\zeta_y = \frac{\partial\eta}{\partial y} + u_y\frac{\partial\eta}{\partial u} - u_x\frac{\partial\xi^1}{\partial y} - u_x u_y\frac{\partial\xi^1}{\partial u} - u_y\frac{\partial\xi^2}{\partial y} - (u_y)^2\frac{\partial\xi^2}{\partial u}. \tag{2.3.21}$$

Substituting the prolongation (2.3.21) into (2.3.19) we obtain

$$\begin{gathered} u\eta - \frac{\partial\eta}{\partial y} + \frac{1}{2}u^2\frac{\partial\xi^1}{\partial x} - u\frac{\partial\xi^2}{\partial x} + u_x\left(\frac{\partial\xi^1}{\partial y} - u\frac{\partial\xi^2}{\partial u}\right) + \\ u_y\left(\frac{\partial\xi^2}{\partial y} - \frac{\partial\eta}{\partial u} - \frac{\partial\xi^1}{\partial x}\right) + (u_y)^2\frac{\partial\xi^2}{\partial u} = 0. \end{gathered} \tag{2.3.22}$$

Substituting equations (2.3.15), (2.3.16) and (2.3.17) into (2.3.22) we obtain

$$\begin{gathered} -u^2\ln u\frac{\partial A}{\partial y} - u^2\frac{\partial B}{\partial y} - \frac{1}{2}u^3 A + u\ln u\frac{\partial^2 A}{\partial y^2} + u\frac{\partial^2 B}{\partial y^2} + \frac{1}{2}u^2\frac{\partial A}{\partial y} + \\ \frac{1}{2}u^2\frac{\partial\xi^1}{\partial x} - u\ln u\frac{\partial A}{\partial x} - u\frac{\partial B}{\partial x} - 2u_x A(x,y) + \\ u_y\left(2\ln u\frac{\partial A}{\partial y} + 2\frac{\partial B}{\partial y} + uA(x,y) - \frac{\partial\xi^1}{\partial x} + \frac{\partial A}{\partial y}(x,y)\right) + (u_y)^2\frac{A(x,y)}{u} = 0. \end{gathered} \tag{2.3.23}$$

Splitting equation (2.3.23) into coefficients of derivatives of u we obtain

$$u_x : A(x,y) = 0, \tag{2.3.24}$$

$$u_y : 2\ln u\frac{\partial A}{\partial y} + 2\frac{\partial B}{\partial y} + uA(x,y) - \frac{\partial\xi^1}{\partial x} + \frac{\partial A}{\partial y} = 0, \tag{2.3.25}$$

$$(u_y)^2 : A(x,y) = 0, \tag{2.3.26}$$

$$\begin{gathered} \text{Remainder} : -u^2\ln u\frac{\partial A}{\partial y} - u^2\frac{\partial B}{\partial y} - \frac{1}{2}u^3 A + u\ln u\frac{\partial^2 A}{\partial y^2} + u\frac{\partial^2 B}{\partial y^2} + \frac{1}{2}u^2\frac{\partial A}{\partial y} \\ + \frac{1}{2}u^2\frac{\partial\xi^1}{\partial x} - u\ln u\frac{\partial A}{\partial x} - u\frac{\partial B}{\partial x} = 0. \end{gathered} \tag{2.3.27}$$

It can be shown that equations (2.3.24) to (2.3.27) reduce to

$$\xi^1 = \xi^1(x), \tag{2.3.28}$$

$$A = 0, \tag{2.3.29}$$

$$2\frac{\partial B}{\partial y} - \frac{d\xi^1}{dx} = 0, \tag{2.3.30}$$

$$\frac{\partial^2 B}{\partial y^2} - \frac{\partial B}{\partial x} = 0, \tag{2.3.31}$$

Differentiating equation (2.3.30) with respect to y we obtain

$$\frac{\partial^2 B}{\partial y^2} = 0. \tag{2.3.32}$$

Therefore, the first term of equation (2.3.31) vanishes to give

$$B = B(y). \tag{2.3.33}$$

Solving equation (2.3.32) we obtain

$$B = c_1 y + c_2, \tag{2.3.34}$$

where c_1 and c_2 are constants. From equation (2.3.30) we obtain

$$\xi^1 = 2c_1 x + c_3, \tag{2.3.35}$$

where c_3 is a constant. From equations (2.3.16) and (2.3.17) we obtain

$$\xi^2 = c_1 y + c_2 \tag{2.3.36}$$

and

$$\eta = -c_1 u, \tag{2.3.37}$$

respectively. The symmetry generator then becomes

$$X = (2c_1 x + c_3)\frac{\partial}{\partial x} + (c_1 y + c_2)\frac{\partial}{\partial y} - c_1 u\frac{\partial}{\partial u}. \tag{2.3.38}$$

We consider only the general case when $c_1 \neq 0$. The case when $c_1 = 0$ may lead to another reduction, but it is not considered in this example. Hence, the Lie point symmetry generator (2.3.38) may be divided by c_1 to give

$$X = (2x + c_3^*)\frac{\partial}{\partial x} + (y + c_2^*)\frac{\partial}{\partial y} - u\frac{\partial}{\partial u}, \tag{2.3.39}$$

where $c_2^* = c_2/c_1$ and $c_3^* = c_3/c_1$. For further calculations, we will omit $*$ to simplify notation.

2.4 Invariant solutions of differential equations

Once the symmetry generator X is found using (2.3.2), the invariant solutions $\psi = f(\underline{x})$ and $S = g(\underline{x})$ can be obtained using the following invariance conditions:

$$X(\psi - f(\underline{x}))\Big|_{\psi=f(\underline{x})} = 0, \tag{2.4.1}$$

$$X(S - g(\underline{x}))\Big|_{S=g(\underline{x})} = 0. \tag{2.4.2}$$

This expands to first order linear partial differential equations for ψ and S and they are solved for ψ and S by integrating the differential equations of the characteristic curves of the partial differential equations.

Example continued

We continue with the example in the previous two sections by finding the invariant solution. For this example with one dependent variable, the invariant solution $u = f(x,y)$ satisfies

$$X\left(u - f(x,y)\right)\Big|_{u=f(x,y)} = 0. \tag{2.4.3}$$

Equation (2.4.3) expands to

$$(2x + c_3)\frac{\partial f}{\partial x} + (y + c_2)\frac{\partial f}{\partial y} = -f. \tag{2.4.4}$$

The characteristic equations for the linear partial differential equation (2.4.4) are

$$\frac{dx}{2x + c_3} = \frac{dy}{y + c_2} = \frac{df}{-f}. \tag{2.4.5}$$

The equation consisting of the first and second terms of (2.4.5) gives

$$\frac{y + c_2}{\left(x + \frac{c_3}{2}\right)^{1/2}} = I_1, \tag{2.4.6}$$

where I_1 is a constant. The equation consisting of the first and third terms of (2.4.5) gives

$$\left(x + \frac{c_3}{2}\right)^{1/2} f = I_2, \tag{2.4.7}$$

where I_2 is a constant. Therefore the general solution of the partial differential equation (2.4.4) is

$$I_2 = F(I_1), \tag{2.4.8}$$

where F is an arbitrary function. Since $f = u(x,y)$ it follows that the general solution for u is

$$u(x,y) = \left(x + \frac{c_3}{2}\right)^{-1/2} F(\xi), \qquad \xi = \frac{y + c_2}{\left(x + \frac{c_3}{2}\right)^{1/2}}. \tag{2.4.9}$$

Substituting (2.4.9) into the partial differential equation (2.2.10) gives the ordinary differential equation

$$-\frac{1}{2}F - \frac{1}{2}\xi\frac{dF}{d\xi} + F\frac{dF}{d\xi} = \frac{d^2F}{d\xi^2}, \tag{2.4.10}$$

which may be rewritten as

$$\frac{d^2F}{d\xi^2} + \frac{1}{2}\frac{d}{d\xi}\left(\xi F - F^2\right) = 0. \tag{2.4.11}$$

Therefore, we have reduced the partial differential equation (2.2.10) to the ordinary differential equation (2.4.11) using the method of associated Lie point symmetry. The ordinary differential equation (2.4.11) can be integrated to give

$$\frac{dF}{d\xi} + \frac{1}{2}\xi F - \frac{1}{2}F^2 = k, \tag{2.4.12}$$

where k is a constant of integration. This demonstrates the double reduction theorem of Sjoberg [10,11] which states that if a partial differential equation is reduced to an ordinary differential equation using a Lie point symmetry associated with a conserved vector of the partial differential equation then the ordinary differential equation can be integrated at least once.

2.5 Conclusions

In this chapter, we have demonstrated the associated Lie point symmetry method through an example. The solution using an associated Lie point symmetry will be derived for the two-dimensional and axisymmetric turbulent thermal jets in Chapters 4 and 5, respectively.

Chapter 3

Reynolds averaged Navier-Stokes and averaged energy balance equations applied to turbulent thermal jets

3.1 Introduction

When a fluid with a larger momentum than the ambient intrudes through a conduit, a jet is formed. Examples of such flow include gas exiting an aircraft nozzle. For this dissertation, the intruding and the ambient fluids have the same intrinsic properties. The line containing the center of the conduit and perpendicular to the conduit is called the centreline of the jet. The mean velocity along the flow changes rapidly with the axis perpendicular to the centreline. Hence the Prandtl boundary layer theory apply for mean momentum. The mean temperature also changes rapidly with the axis perpendicular to the centreline. Therefore boundary layer theory can also be applied to simplify the energy balance equation.

The jet can be separated into three stages, the initial stage, the transitional stage and the fully-developed turbulent stage [16]. The flow is laminar in the initial stage and turbulent in the fully developed stage. The analytical solutions to the initial and transitional stage are not easily accessible. The velocity profiles in these regions depend on the geometry of the nozzle. In the fully developed stage, the jet can be simply modelled by a fictitious point source ejecting heat and momentum into the ambient fluid [16]. This point is close to the conduit. This dissertation considers the fully-developed turbulent stage.

The irregular nature of turbulence makes it difficult to model mathematically. Reynolds [17] proposed the idea of splitting every variable in the Navier-Stokes equations into mean and fluctuations. The fluctuations are supposed to model the irregular nature of turbulence. The problem with this approach is that the averaged Navier-Stokes equations now contains more unknowns than the number of equations. The additional quantities are called the Reynolds stresses which are proportional to the covariances of velocity components. The Boussinesq approximation writes these Reynolds stresses in terms of the mean velocity gradients by introducing the eddy viscosity. The eddy viscosity can be interpreted as an additional drag to the mean flow. The eddy viscosity is the property of the flow not of the fluid. The averaged Navier-Stokes equations with the Boussinesq approximation are simplified using boundary layer theory. The eddy viscosity is modelled using Prandtl's mixing length hypothesis.

The energy balance equation is also averaged by splitting the temperature and the velocity components into mean and fluctuations. We also encounter the closure problem which is resolved by introducing the turbulent thermal conductivity. Similar to eddy viscosity, the turbulent thermal conductivity is a property of flow, not of the fluid. The averaged energy balance equations with the turbulent thermal conductivity are simplified using boundary layer theory. The turbulent thermal conductivity is modelled using the thermal mixing length.

In this chapter, the governing equations for the two-dimensional turbulent thermal jet are derived in Section 3.2 and the governing equations for the axisymmetric turbulent jet are derived in Section 3.3. The derivations for both the two-dimensional and the axisymmetric jet follows the six steps:
(1) Derive the average the Navier-Stokes equations by splitting every variable except for density into mean and fluctuations (subsections 3.2.1 and 3.3.1),
(2) Average the energy balance equation by splitting every variable in the energy balance equation into mean and fluctuations (subsections 3.2.2 and 3.3.2),
(3) Introduce the eddy viscosity and the turbulent thermal conductivity in order to resolve the closure problem (subsections 3.2.3 and 3.3.2),
(4) Apply the boundary layer approximation on both the averaged Navier-Stokes equations and the energy balance equation (subsections 3.2.4 and 3.3.3),
(5) Derive the formula for the eddy viscosity and turbulent thermal conductivity in terms of the momentum mixing length and thermal mixing length (subsections 3.2.5 and 3.3.4),
(6) Derive jet boundary conditions (subsections 3.2.6 and 3.3.5). A summary of the governing equations is given in Section 3.4 and the conclusions are given in Section 3.5.

3.2 Two-dimensional thermal jets

In this section, we derive the governing equations for the two-dimensional turbulent thermal jet. The two-dimensional jet assumes that the width is very large when compared with height of the conduit. Therefore the jet flow is symmetrical along the width of the conduit and the only axes considered are the axis perpendicular to the cross-sectional area of the conduit and the axis along the height of the conduit.

3.2.1 The Reynolds averaged Navier-Stokes equation

The cartesian coordinate axes are constructed in a such a way that the point source resembling the conduit has the coordinates $(0, 0)$ and the jet flows parallel to the positive side of the x-axis. Hence, for the region we are studying, x is positive. Time is represented by t. The velocity components v_x, v_y and v_z represent velocities along the x, y and z axes, respectively. Pressure and density of the fluid are represented by p and ρ, respectively.

The Navier-Stokes equations in cartesian coordinates are

$$\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right), \tag{3.2.1}$$

$$\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right), \tag{3.2.2}$$

$$\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right), \tag{3.2.3}$$

where ν is the kinematic viscosity. For an incompressible fluid the continuity equation is

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0. \tag{3.2.4}$$

But for a two-dimensional jet, v_z and $\frac{\partial}{\partial z}$ are zero. Thus, equations (3.2.1) to (3.2.4) become

$$\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right), \tag{3.2.5}$$

$$\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} \right), \tag{3.2.6}$$

$$0 = \frac{\partial p}{\partial z}, \tag{3.2.7}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0. \tag{3.2.8}$$

We introduce an averaging operator, defined when applied to F as

$$\overline{F}(x,y,z) = \lim_{N\to\infty} \frac{1}{N} \int_0^N F(x,y,z,t)dt, \tag{3.2.9}$$

The averaging operator, $\overline{(\cdot)}$, gives the time average of F. The averaging operator is linear, that is

$$\overline{F+G} = \overline{F} + \overline{G}, \tag{3.2.10}$$

$$\overline{kF} = k\overline{F}, \tag{3.2.11}$$

where k is a constant. The averaging operator is idempotent, that is

$$\overline{\overline{F}} = \overline{F}, \tag{3.2.12}$$

In general

$$\overline{FG} \neq \overline{F}\,\overline{G}. \tag{3.2.13}$$

The averaging operator commutes with spatial differentiation:

$$\overline{\frac{\partial P}{\partial x_i}} = \frac{\partial \overline{P}}{\partial x_i}. \tag{3.2.14}$$

We can write the velocity components and the pressure as the sum of a time average and fluctuations. Thus

$$v_i(x,y,t) = \overline{v}_i(x,y) + v_i'(x,y,t), \tag{3.2.15}$$

$$p(x,y,t) = \overline{p}(x,y) + p'(x,y,t), \tag{3.2.16}$$

where the prime represents the fluctuations. Applying the averaging operator, $\overline{(\cdot)}$, on both sides of the system of the partial differential equations (3.2.5), (3.2.6) and

(3.2.7) we obtain

$$\overline{\frac{\partial v_x}{\partial t}} + \overline{v_x \frac{\partial v_x}{\partial x}} + \overline{v_y \frac{\partial v_x}{\partial y}} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + \nu \left(\frac{\partial^2 \overline{v}_x}{\partial x^2} + \frac{\partial^2 \overline{v}_x}{\partial y^2} \right), \tag{3.2.17}$$

$$\overline{\frac{\partial v_y}{\partial t}} + \overline{v_x \frac{\partial v_y}{\partial x}} + \overline{v_y \frac{\partial v_y}{\partial y}} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial y} + \nu \left(\frac{\partial^2 \overline{v}_y}{\partial x^2} + \frac{\partial^2 \overline{v}_y}{\partial y^2} \right), \tag{3.2.18}$$

$$\frac{\partial \overline{v}_x}{\partial x} + \frac{\partial \overline{v}_y}{\partial y} = 0. \tag{3.2.19}$$

In order to show that the averaged time derivatives in (3.2.17) and (3.2.18) vanish we make the following assumptions:

1. v_i and $\frac{\partial v_i}{\partial t}$ are continuous with respect to t,
2. $\lim_{N\to\infty} v_i(N,x,y)/N = 0$ for all (x,y) in the jet,

where $i = x, y$. Assumption 2 is reasonable since the velocity of the jet does not grow to infinity as time increases. Instead, we expect the velocity to fluctuate about the steady state mean velocity. Hence

$$\overline{\frac{\partial v_i}{\partial t}}(x,y) = \lim_{N\to\infty} \frac{1}{N} \int_0^N \frac{\partial v_i}{\partial t}(t,x,y)dt \tag{3.2.20}$$

$$= \lim_{N\to\infty} \frac{1}{N} v_i(t,x,y)\Big|_{t=0}^{t=N} \tag{3.2.21}$$

$$= \lim_{N\to\infty} \left[\frac{1}{N} v_i(N,x,y) - \frac{1}{N} v_i(0,x,y) \right] \tag{3.2.22}$$

$$= 0 + 0 = 0. \tag{3.2.23}$$

Substituting equations (3.2.15) and (3.2.16) into (3.2.17) and (3.2.18) and using the fact that averaged time derivatives vanish we obtain

$$\overline{(\overline{v}_x + v'_x)\frac{\partial}{\partial x}(\overline{v}_x + v'_x)} + \overline{(\overline{v}_y + v'_y)\frac{\partial}{\partial y}(\overline{v}_x + v'_x)} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + \nu \left(\frac{\partial^2 \overline{v}_x}{\partial x^2} + \frac{\partial^2 \overline{v}_x}{\partial y^2} \right), \tag{3.2.24}$$

$$\overline{(\overline{v}_x + v'_x)\frac{\partial}{\partial x}\left(\overline{v}_y + v'_y\right)} + \overline{(\overline{v}_y + v'_y)\frac{\partial}{\partial y}\left(\overline{v}_y + v'_y\right)} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial y} + \nu \left(\frac{\partial^2 \overline{v}_y}{\partial x^2} + \frac{\partial^2 \overline{v}_y}{\partial y^2} \right). \tag{3.2.25}$$

Using the averaging operator properties (3.2.10) to (3.2.14) to simplify (3.2.24) and (3.2.25), we obtain

$$\overline{v}_x \frac{\partial \overline{v}_x}{\partial x} + \overline{v}_y \frac{\partial \overline{v}_x}{\partial y} + \overline{v'_x \frac{\partial v'_x}{\partial x}} + \overline{v'_y \frac{\partial v'_x}{\partial y}} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + \nu \left(\frac{\partial^2 \overline{v}_x}{\partial x^2} + \frac{\partial^2 \overline{v}_x}{\partial y^2} \right), \tag{3.2.26}$$

$$\overline{v}_x \frac{\partial \overline{v}_y}{\partial x} + \overline{v}_y \frac{\partial \overline{v}_y}{\partial y} + \overline{v'_x \frac{\partial v'_y}{\partial x}} + \overline{v'_y \frac{\partial v'_y}{\partial y}} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial y} + \nu \left(\frac{\partial^2 \overline{v}_y}{\partial x^2} + \frac{\partial^2 \overline{v}_y}{\partial y^2} \right). \tag{3.2.27}$$

Using the product rule

$$v_i \frac{\partial v_j}{\partial x_i} = \frac{\partial}{\partial x_i}(v_i v_j) - v_j \frac{\partial v_i}{\partial x_i}, \tag{3.2.28}$$

equations (3.2.26) and (3.2.27) become

$$\overline{v}_x \frac{\partial \overline{v}_x}{\partial x} + \overline{v}_y \frac{\partial \overline{v}_x}{\partial y} - \overline{v'_x \left(\frac{\partial v'_x}{\partial x} + \frac{\partial v'_y}{\partial y} \right)} + \overline{\frac{\partial}{\partial y}(v'_x v'_y)} + \overline{\frac{\partial}{\partial x}(v'_x v'_x)} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + \nu \left(\frac{\partial^2 \overline{v}_x}{\partial x^2} + \frac{\partial^2 \overline{v}_x}{\partial y^2} \right), \tag{3.2.29}$$

$$\overline{v}_x \frac{\partial \overline{v}_y}{\partial x} + \overline{v}_y \frac{\partial \overline{v}_y}{\partial y} - \overline{v'_y \left(\frac{\partial v'_x}{\partial x} + \frac{\partial v'_y}{\partial y} \right)} + \overline{\frac{\partial}{\partial x}(v'_x v'_y)} + \overline{\frac{\partial}{\partial y}(v'_y v'_y)} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial y} + \nu \left(\frac{\partial^2 \overline{v}_y}{\partial x^2} + \frac{\partial^2 \overline{v}_y}{\partial y^2} \right). \tag{3.2.30}$$

Subtracting equation (3.2.19) from (3.2.8) we see that

$$\frac{\partial v'_x}{\partial x} + \frac{\partial v'_y}{\partial y} = 0. \tag{3.2.31}$$

Hence the third terms under the operator $\overline{(\cdot)}$ in equations (3.2.29) and (3.2.30) vanish to give

$$\overline{v}_x \frac{\partial \overline{v}_x}{\partial x} + \overline{v}_y \frac{\partial \overline{v}_x}{\partial y} + \frac{\partial}{\partial y}\overline{(v'_x v'_y)} + \frac{\partial}{\partial x}\overline{(v'_x v'_x)} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + \nu \left(\frac{\partial^2 \overline{v}_x}{\partial x^2} + \frac{\partial^2 \overline{v}_x}{\partial y^2} \right), \tag{3.2.32}$$

$$\overline{v}_x \frac{\partial \overline{v}_y}{\partial x} + \overline{v}_y \frac{\partial \overline{v}_y}{\partial y} + \frac{\partial}{\partial x}\overline{(v'_x v'_y)} + \frac{\partial}{\partial y}\overline{(v'_y v'_y)} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial y} + \nu \left(\frac{\partial^2 \overline{v}_y}{\partial x^2} + \frac{\partial^2 \overline{v}_y}{\partial y^2} \right). \tag{3.2.33}$$

From above, we see that we have a system of three partial differential equations (3.2.19), (3.2.32) and (3.2.33) with six unknowns p, $\overline{v}_x$, $\overline{v}_y$, $\overline{v'_x v'_y}$, $\overline{v'_x v'_x}$ and $\overline{v'_y v'_y}$. This problem is called the turbulence closure problem. The covariances $\rho\overline{v'_x v'_y}$, $\rho\overline{v'_x v'_x}$ and $\rho\overline{v'_y v'_y}$ are called the Reynolds stresses and will be related to the mean velocity gradients $\frac{\partial \overline{v}_x}{\partial y}$, $\frac{\partial \overline{v}_x}{\partial x}$ and $\frac{\partial \overline{v}_y}{\partial y}$ in Subsection 3.2.3.

In summary:

The Reynolds averaged Navier-Stokes equations are

$$\overline{v}_x \frac{\partial \overline{v}_x}{\partial x} + \overline{v}_y \frac{\partial \overline{v}_x}{\partial y} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + \nu \left(\frac{\partial^2 \overline{v}_x}{\partial x^2} + \frac{\partial^2 \overline{v}_x}{\partial y^2} \right) - \frac{\partial}{\partial x}\overline{(v'_x v'_x)} - \frac{\partial}{\partial y}\overline{(v'_x v'_y)}, \tag{3.2.34}$$

$$\overline{v}_x \frac{\partial \overline{v}_y}{\partial x} + \overline{v}_y \frac{\partial \overline{v}_y}{\partial y} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial y} + \nu \left(\frac{\partial^2 \overline{v}_y}{\partial x^2} + \frac{\partial^2 \overline{v}_y}{\partial y^2} \right) - \frac{\partial}{\partial x}\overline{(v'_x v'_y)} - \frac{\partial}{\partial y}\overline{(v'_y v'_y)}, \tag{3.2.35}$$

$$\frac{\partial \overline{v}_x}{\partial x} + \frac{\partial \overline{v}_y}{\partial y} = 0. \tag{3.2.36}$$

3.2.2 The averaged energy balance equation

The energy balance equation, with the viscous terms neglected, in cartesian coordinates is

$$\rho c_v \left[\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y}\right] = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right), \tag{3.2.37}$$

where T represents the temperature of fluid at time t and position (x, y), κ is the thermal conductivity of the fluid and c_v is the specific heat capacity of the fluid at a constant volume. The thermal conductivity and the specific capacity of the fluid are assumed to be constant. Taking the averaging operator, $\overline{(\cdot)}$, on both sides of (3.2.37) we obtain

$$\rho c_v \left[\overline{\frac{\partial T}{\partial t}} + \overline{v_x \frac{\partial T}{\partial x}} + \overline{v_y \frac{\partial T}{\partial y}}\right] = \kappa \left(\frac{\partial^2 \overline{T}}{\partial x^2} + \frac{\partial^2 \overline{T}}{\partial y^2}\right). \tag{3.2.38}$$

As we have seen above for velocity, the averaged time derivative vanishes. We also write the temperature as the sum of a time average and fluctuations:

$$T(x, y, t) = \overline{T}(x, y) + T'(x, y, t). \tag{3.2.39}$$

Substituting equations (3.2.15) and (3.2.39) into (3.2.38) we obtain

$$\rho c_v \left[\overline{(\overline{v}_x + v_x') \frac{\partial}{\partial x}(\overline{T} + T')} + \overline{(\overline{v}_y + v_y') \frac{\partial}{\partial y}(\overline{T} + T')}\right] = \kappa \left(\frac{\partial^2 \overline{T}}{\partial x^2} + \frac{\partial^2 \overline{T}}{\partial y^2}\right). \tag{3.2.40}$$

Using the averaging properties (3.2.10) to (3.2.14) to simplify (3.2.40) we obtain

$$\rho c_v \left(\overline{v}_x \frac{\partial \overline{T}}{\partial x} + \overline{v}_y \frac{\partial \overline{T}}{\partial y} + \overline{v_x' \frac{\partial T'}{\partial x}} + \overline{v_y' \frac{\partial T'}{\partial y}}\right) = \kappa \left(\frac{\partial^2 \overline{T}}{\partial x^2} + \frac{\partial^2 \overline{T}}{\partial y^2}\right). \tag{3.2.41}$$

Using the product rule

$$v_i' \frac{\partial T'}{\partial x_i} = \frac{\partial}{\partial x_i}(v_i' T') - T' \frac{\partial v_i}{\partial x_i}, \tag{3.2.42}$$

equation (3.2.41) becomes

$$\begin{gathered}\rho c_v \left(\overline{v}_x \frac{\partial \overline{T}}{\partial x} + \overline{v}_y \frac{\partial \overline{T}}{\partial y} + \overline{T' \left(\frac{\partial v_x'}{\partial x} + \frac{\partial v_y'}{\partial y}\right)} + \frac{\partial}{\partial x}\overline{(v_x' T')} + \frac{\partial}{\partial y}\overline{(v_y' T')}\right) = \\ \kappa \left(\frac{\partial^2 \overline{T}}{\partial x^2} + \frac{\partial^2 \overline{T}}{\partial y^2}\right).\end{gathered} \tag{3.2.43}$$

Again, using (3.2.31) the third term under the averaging operator $\overline{(\cdot)}$ in equation (3.2.43) vanishes to give

$$\rho c_v \left(\overline{v}_x \frac{\partial \overline{T}}{\partial x} + \overline{v}_y \frac{\partial \overline{T}}{\partial y} + \frac{\partial}{\partial x}\overline{(v_x' T')} + \frac{\partial}{\partial y}\overline{(v_y' T')}\right) = \kappa \left(\frac{\partial^2 \overline{T}}{\partial x^2} + \frac{\partial^2 \overline{T}}{\partial y^2}\right). \tag{3.2.44}$$

Hence, in summary:

$$\rho c_v \left(\overline{v}_x \frac{\partial \overline{T}}{\partial x} + \overline{v}_y \frac{\partial \overline{T}}{\partial y}\right) = \frac{\partial}{\partial x}\left(\kappa \frac{\partial \overline{T}}{\partial x} - \rho c_v \overline{(v_x' T')}\right) + \frac{\partial}{\partial y}\left(\kappa \frac{\partial \overline{T}}{\partial y} - \rho c_v \overline{(v_y' T')}\right). \tag{3.2.45}$$

3.2.3 Eddy viscosity and turbulent thermal conductivity

Joseph Valentin Boussinesq in 1877 related the covariances $\overline{v_i' v_j'}$ with the mean velocity gradients by introducing the eddy viscosity, μ_T, defined by

$$-\rho\overline{v_i' v_j'} = \mu_T\left(\frac{\partial \overline{v}_i}{\partial x_j} + \frac{\partial \overline{v}_j}{\partial x_i}\right). \tag{3.2.46}$$

Dividing equation (3.2.46) by ρ gives

$$-\overline{v_i' v_j'} = \nu_T\left(\frac{\partial \overline{v}_i}{\partial x_j} + \frac{\partial \overline{v}_j}{\partial x_i}\right), \tag{3.2.47}$$

where

$$\nu_T = \frac{\mu_T}{\rho} \tag{3.2.48}$$

is called the kinematic eddy viscosity. The Reynolds averaged Navier-Stokes equations (3.2.34) and (3.2.35), written in terms of (3.2.47), become

$$\begin{aligned}\overline{v}_x\frac{\partial \overline{v}_x}{\partial x} + \overline{v}_y\frac{\partial \overline{v}_x}{\partial y} &= -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + \frac{\partial}{\partial x}\left((\nu+\nu_T)\frac{\partial \overline{v}_x}{\partial x}\right) + \frac{\partial}{\partial y}\left((\nu+\nu_T)\frac{\partial \overline{v}_x}{\partial y}\right)\\ &\quad + \frac{\partial}{\partial x}\left(\nu_T\left(\frac{\partial \overline{v}_x}{\partial x} + \frac{\partial \overline{v}_y}{\partial y}\right)\right),\end{aligned} \tag{3.2.49}$$

$$\begin{aligned}\overline{v}_x\frac{\partial \overline{v}_y}{\partial x} + \overline{v}_y\frac{\partial \overline{v}_y}{\partial y} &= -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial y} + \frac{\partial}{\partial x}\left((\nu+\nu_T)\frac{\partial \overline{v}_y}{\partial x}\right) + \frac{\partial}{\partial y}\left((\nu+\nu_T)\frac{\partial \overline{v}_y}{\partial y}\right)\\ &\quad + \frac{\partial}{\partial y}\left(\nu_T\left(\frac{\partial \overline{v}_x}{\partial x} + \frac{\partial \overline{v}_y}{\partial y}\right)\right).\end{aligned} \tag{3.2.50}$$

Using the continuity equation (3.2.36) for the mean velocity the last terms of (3.2.49) and (3.2.50) vanish to give

$$\overline{v}_x\frac{\partial \overline{v}_x}{\partial x} + \overline{v}_y\frac{\partial \overline{v}_x}{\partial y} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + \frac{\partial}{\partial x}\left((\nu+\nu_T)\frac{\partial \overline{v}_x}{\partial x}\right) + \frac{\partial}{\partial y}\left((\nu+\nu_T)\frac{\partial \overline{v}_x}{\partial y}\right), \tag{3.2.51}$$

$$\overline{v}_x\frac{\partial \overline{v}_y}{\partial x} + \overline{v}_y\frac{\partial \overline{v}_y}{\partial y} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial y} + \frac{\partial}{\partial x}\left((\nu+\nu_T)\frac{\partial \overline{v}_y}{\partial x}\right) + \frac{\partial}{\partial y}\left((\nu+\nu_T)\frac{\partial \overline{v}_y}{\partial y}\right), \tag{3.2.52}$$

where $\nu + \nu_T$ is the effective kinematic viscosity.

The covariances $\overline{T' v_i'}$ are related to the mean temperature gradient by

$$\rho c_v \overline{T' v_i'} = -\kappa_T\frac{\partial \overline{T}}{\partial x_i}, \tag{3.2.53}$$

where κ_T is turbulent thermal conductivity. Substituting (3.2.53) into (3.2.45) gives

$$\rho c_v\left(\overline{v}_x\frac{\partial \overline{T}}{\partial x} + \overline{v}_y\frac{\partial \overline{T}}{\partial y}\right) = \frac{\partial}{\partial x}\left((\kappa+\kappa_T)\frac{\partial \overline{T}}{\partial x}\right) + \frac{\partial}{\partial y}\left((\kappa+\kappa_T)\frac{\partial \overline{T}}{\partial y}\right), \tag{3.2.54}$$

where $\kappa + \kappa_T$ is the effective conductivity.

3.2.4 Boundary layer approximation

Boundary layer theory does not require a solid boundary to be applicable. It applies to jets, wakes and plumes, as well as boundary layers, in which there is a region of rapid change. In the two-dimensional jet $\overline{v}_x(x, y)$ changes rapidly with y in the region of the centreline of the jet.

Let

- characteristic thermal conductivity be κ_0,
- characteristic dynamic viscosity be ν_0,
- characteristic pressure be P,
- characteristic distance in the x-direction be L,
- characteristic temperature be T_0,
- characteristic velocity in the x-direction be U,
- thickness of velocity boundary layer be δ_V,
- thickness of thermal boundary layer be δ_T,
- characteristic velocity in the y-direction be V.

The boundary layer approximation must still satisfy the continuity equation. Hence the continuity equation

$$\frac{\partial \overline{v}_x}{\partial x} + \frac{\partial \overline{v}_y}{\partial y} = 0 \tag{3.2.55}$$

gives

$$\frac{[\overline{v}_x]}{[x]} \sim \frac{[\overline{v}_y]}{[y]}, \tag{3.2.56}$$

which after substituting characteristic lengths and velocities becomes

$$V = \frac{\delta_V}{L} U. \tag{3.2.57}$$

We note the following:

- when differentiating $\overline{T}$ with respect to y, y is scaled with δ_T.
- when differentiating $\overline{v}_x$ and $\overline{v}_y$ with respect to y, y is scaled with δ_V.

Hence we define dimensionless variables

$$\overline{T}^* = \frac{\overline{T}}{T_0}, \qquad \overline{p}^* = \frac{\overline{p}}{P}, \qquad \overline{v}_x^* = \frac{\overline{v}_x}{U}, \qquad \overline{v}_y^* = \frac{L}{\delta_V} \frac{\overline{v}_y}{U}, \tag{3.2.58}$$

$$\kappa^* = \frac{\kappa + \kappa_T}{\kappa_0}, \qquad \nu^* = \frac{\nu + \nu_T}{\nu_0}, \qquad x^* = \frac{x}{L}, \qquad y^* = \frac{y}{\delta_V} \text{ or } \frac{y}{\delta_T}, \tag{3.2.59}$$

which when substituted into the momentum equations (3.2.51) and (3.2.52) gives

$$\overline{v}_x^* \frac{\partial \overline{v}_x^*}{\partial x^*} + \overline{v}_y^* \frac{\partial \overline{v}_x^*}{\partial y^*} = -\frac{P}{\rho U^2} \frac{\partial \overline{p}^*}{\partial x^*} + \frac{\nu_0}{UL} \frac{\partial}{\partial x^*} \left(\nu^* \frac{\partial \overline{v}_x^*}{\partial x^*} \right) + \frac{L^2}{\delta_V^2} \frac{\nu_0}{UL} \frac{\partial}{\partial y^*} \left(\nu^* \frac{\partial \overline{v}_x^*}{\partial y^*} \right), \tag{3.2.60}$$

$$\frac{\rho\delta_V^2 U^2}{PL^2}\left(\overline{v}_x^*\frac{\partial\overline{v}_y^*}{\partial x^*}+\overline{v}_y^*\frac{\partial\overline{v}_y^*}{\partial y^*}\right)=-\frac{\partial\overline{p^*}}{\partial y^*}+\frac{\nu_0\delta_V^2\rho U}{PL^3}\frac{\partial}{\partial x^*}\left(\nu^*\frac{\partial\overline{v}_y^*}{\partial x^*}\right)$$
$$+\frac{\nu_0\rho U}{LP}\frac{\partial}{\partial y^*}\left(\nu^*\frac{\partial\overline{v}_y^*}{\partial y^*}\right). \tag{3.2.61}$$

For the pressure gradient to balance with the inertia term in (3.2.60) we take

$$P=\rho U^2. \tag{3.2.62}$$

The Reynolds number,

$$Re=\frac{UL}{\nu_0}, \tag{3.2.63}$$

which is the ratio between inertial and viscous forces, is an important dimensionless number for predicting the flow patterns of the jet. For turbulent flow the Reynolds number is very large. Writing equations (3.2.60) and (3.2.61) in terms of the Reynolds number (3.2.63) we obtain

$$\overline{v}_x^*\frac{\partial\overline{v}_x^*}{\partial x^*}+\overline{v}_y^*\frac{\partial\overline{v}_x^*}{\partial y^*}=-\frac{\partial\overline{p}^*}{\partial x^*}+\frac{1}{Re}\frac{\partial}{\partial x^*}\left(\nu^*\frac{\partial\overline{v}_x^*}{\partial x^*}\right)+\frac{1}{Re}\left(\frac{L}{\delta_V}\right)^2\frac{\partial}{\partial y^*}\left(\nu^*\frac{\partial\overline{v}_x^*}{\partial y^*}\right), \tag{3.2.64}$$

$$\left(\frac{\delta_V}{L}\right)^2\left(\overline{v}_x^*\frac{\partial\overline{v}_y^*}{\partial x^*}+\overline{v}_y^*\frac{\partial\overline{v}_y^*}{\partial y^*}\right)=-\frac{\partial\overline{p}^*}{\partial y^*}+\frac{1}{Re}\left(\frac{\delta_V}{L}\right)^2\frac{\partial}{\partial y^*}\left(\nu^*\frac{\partial\overline{v}_y^*}{\partial y^*}\right)$$
$$+\frac{1}{Re}\frac{\partial}{\partial y^*}\left(\nu^*\frac{\partial\overline{v}_y^*}{\partial y^*}\right). \tag{3.2.65}$$

The rapid change of $\overline{v}_x$ with y in the region of the centreline should be just sufficient to prevent the viscous term in (3.2.64) to be the same order of magnitude as the pressure gradient and inertial terms we require that

$$\frac{1}{Re}\left(\frac{L}{\delta_V}\right)^2\sim 1. \tag{3.2.66}$$

We take

$$\frac{L}{\delta_V}=\frac{1}{\sqrt{Re}}. \tag{3.2.67}$$

Equations (3.2.64) and (3.2.65) become

$$\overline{v}_x^*\frac{\partial\overline{v}_x^*}{\partial x^*}+\overline{v}_y^*\frac{\partial\overline{v}_x^*}{\partial y^*}=-\frac{\partial\overline{p}^*}{\partial x^*}+\frac{1}{Re}\frac{\partial}{\partial x^*}\left(\nu^*\frac{\partial\overline{v}_x^*}{\partial x^*}\right)+\frac{\partial}{\partial y^*}\left(\nu^*\frac{\partial\overline{v}_x^*}{\partial y^*}\right), \tag{3.2.68}$$

$$\frac{1}{Re}\left(\overline{v}_x^*\frac{\partial\overline{v}_y^*}{\partial x^*}+\overline{v}_y^*\frac{\partial\overline{v}_y^*}{\partial y^*}\right)=-\frac{\partial\overline{p}^*}{\partial y^*}+\frac{1}{Re^2}\frac{\partial}{\partial y^*}\left(\nu^*\frac{\partial\overline{v}_y^*}{\partial y^*}\right)+\frac{1}{Re}\frac{\partial}{\partial y^*}\left(\nu^*\frac{\partial\overline{v}_y^*}{\partial y^*}\right). \tag{3.2.69}$$

In boundary layer theory, ν is small and Re is large. In turbulent flow Re is large. We assume that

$$\sqrt{Re}>>1, \tag{3.2.70}$$

which reduces equations (3.2.68) and (3.2.69) to

$$\overline{v}_x^* \frac{\partial \overline{v}_x^*}{\partial x^*} + \overline{v}_y^* \frac{\partial \overline{v}_x^*}{\partial y^*} = -\frac{\partial \overline{p}^*}{\partial x^*} + \frac{\partial}{\partial y^*}\left(\nu^* \frac{\partial \overline{v}_x^*}{\partial y^*}\right), \tag{3.2.71}$$

$$\frac{\partial \overline{p}^*}{\partial y^*} = 0. \tag{3.2.72}$$

Writing equations (3.2.71) and (3.2.72) in terms of original variables we obtain

$$\overline{v}_x \frac{\partial \overline{v}_x}{\partial x} + \overline{v}_y \frac{\partial \overline{v}_x}{\partial y} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + \frac{\partial}{\partial y}\left((\nu + \nu_T) \frac{\partial \overline{v}_x}{\partial y}\right), \tag{3.2.73}$$

$$\frac{\partial \overline{p}}{\partial y} = 0. \tag{3.2.74}$$

This implies that the mean pressure is constant along the y-direction. The main-

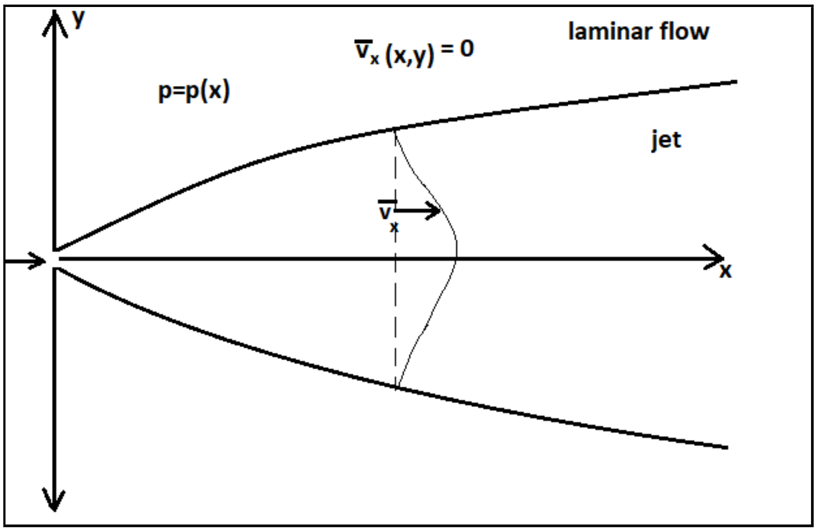

FIGURE 3.1: Schematic diagram of the turbulent jet

stream flow is steady and laminar and the Euler equation applies. Thus

$$\rho U \frac{dU}{dx} = -\frac{dp}{dx}. \tag{3.2.75}$$

But for jet flow $\frac{dU}{dx}$. Therefore equation (3.2.73) reduces to

$$\frac{d\overline{p}}{dx}(x) = 0 \tag{3.2.76}$$

at the boundary of the jet. However, equation (3.2.76) is applicable inside the boundary layer since the mean pressure was shown to be the same along the y direction. Hence, equation (3.2.73) becomes

$$\overline{v}_x \frac{\partial \overline{v}_x}{\partial x} + \overline{v}_y \frac{\partial \overline{v}_x}{\partial y} = \frac{\partial}{\partial y}\left((\nu + \nu_T) \frac{\partial \overline{v}_x}{\partial y}\right). \tag{3.2.77}$$

Writing the energy balance equation (3.2.54) in terms of dimensionless variables we obtain

$$\overline{v}_x^* \frac{\partial \overline{T}^*}{\partial x^*} + \frac{\delta_V}{\delta_T} \overline{v}_y^* \frac{\partial \overline{T}^*}{\partial y^*} = \frac{\kappa_0}{\rho c_v \nu_0} \left[\frac{\nu_0}{UL} \frac{\partial}{\partial x^*} \left(\kappa^* \frac{\partial \overline{T}^*}{\partial x^*} \right) + \frac{\nu_0}{UL} \left(\frac{L}{\delta_T} \right)^2 \frac{\partial}{\partial y^*} \left(\kappa^* \frac{\partial \overline{T}^*}{\partial y^*} \right) \right]. \tag{3.2.78}$$

Now the turbulent Prandtl number is

$$Pr = \frac{\text{viscous diffusion rate}}{\text{thermal diffusion rate}} = \frac{\nu_0}{(\kappa_0 / (\rho c_v))}. \tag{3.2.79}$$

In (3.2.79) the characteristic ν_0 and κ_0 for $\nu + \nu_T$ and $\kappa + \kappa_T$ were used. Equation (3.2.78) becomes

$$\overline{v}_x^* \frac{\partial \overline{T}^*}{\partial x^*} + \frac{\delta_V}{\delta_T} \overline{v}_y^* \frac{\partial \overline{T}^*}{\partial y^*} = \frac{1}{PrRe} \frac{\partial}{\partial x^*} \left(\kappa^* \frac{\partial \overline{T}^*}{\partial x^*} \right) + \frac{1}{PrRe} \left(\frac{L}{\delta_T} \right)^2 \frac{\partial}{\partial y^*} \left(\kappa^* \frac{\partial \overline{T}^*}{\partial y^*} \right). \tag{3.2.80}$$

Now we will consider $PrRe$ to be large. The rapid change of T with y in the region of the centreline should be just sufficient to prevent the thermal conduction term on the right hand side of (3.2.80) from being negligible, even though $PrRe$ is large. For the thermal conduction to be the same order of magnitude as the conduction term on the left hand side we require that

$$\frac{1}{PrRe} \left(\frac{\delta_T}{L} \right)^2 \sim 1. \tag{3.2.81}$$

We take

$$\frac{\delta_T}{L} = \frac{1}{\sqrt{PrRe}}. \tag{3.2.82}$$

From (3.2.67) and (3.2.82)

$$\frac{\delta_V}{\delta_T} = \sqrt{Pr}. \tag{3.2.83}$$

Equation (3.2.80) becomes

$$\overline{v}_x^* \frac{\partial \overline{T}^*}{\partial x^*} + \sqrt{Pr} \overline{v}_y^* \frac{\partial \overline{T}^*}{\partial y^*} = \left[\frac{Pr}{Re} \frac{\partial}{\partial x^*} \left(\kappa^* \frac{\partial \overline{T}^*}{\partial x^*} \right) + \frac{\partial}{\partial y^*} \left(\kappa^* \frac{\partial \overline{T}^*}{\partial y^*} \right) \right]. \tag{3.2.84}$$

We consider

$$Pr \sim 1. \tag{3.2.85}$$

Then from (3.2.83), $\delta_T \sim \delta_V$ and the thickness of the thermal boundary layer is the same order of magnitude as the magnitude of the momentum boundary layer. Since $\sqrt{Re} >> 1$, equation (3.2.84) reduces to

$$\overline{v}_x^* \frac{\partial \overline{T}^*}{\partial x^*} + \sqrt{Pr} \overline{v}_y^* \frac{\partial \overline{T}^*}{\partial y^*} = \frac{\partial}{\partial y^*} \left(\kappa^* \frac{\partial \overline{T}^*}{\partial y^*} \right), \tag{3.2.86}$$

which, when written in original variables, gives

$$\rho c_v \left(\overline{v}_x \frac{\partial \overline{T}}{\partial x} + \overline{v}_y \frac{\partial \overline{T}}{\partial y} \right) = \frac{\partial}{\partial y} \left((\kappa + \kappa_T) \frac{\partial \overline{T}}{\partial y} \right). \tag{3.2.87}$$

3.2.5 Mixing lengths for the momentum and heat transfer

The mixing length for the momentum transfer and heat transfer has been considered by Schlichting and Gerstern [1].

Consider first the mixing length for the momentum transfer. Consider mean flow particles parallel to the x-axis illustrated in Figure 3.2:

$$\overline{v}_x = \overline{v}_x(x, y), \qquad \overline{v}_y = 0, \qquad \overline{v}_z = 0. \tag{3.2.88}$$

Thus

$$\tau_T = \tau_{xy} = -\rho\overline{v_x' v_y'} = \mu_T\left(\frac{\partial \overline{v}_x}{\partial y} + \frac{\partial \overline{v}_y}{\partial x}\right) \mu_T = \frac{\partial \overline{v}_x}{\partial y}. \tag{3.2.89}$$

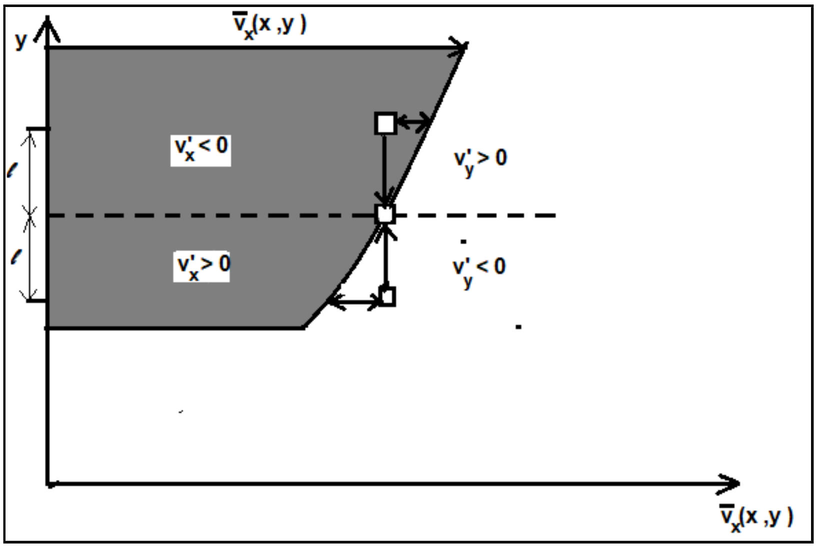

FIGURE 3.2: Demonstration of the mixing length for the momentum transfer.

Ludwig Prandtl approximated the Reynolds stresses in terms of the characteristic length l called the momentum mixing length, which is the average distance travelled by a fluid lump before mixing with the mean flow. As the fluid lump shown in Figure 3.2 travels from level $y + l'$ to y it retains its momentum in the x-direction and therefore retains its velocity in the x-direction. Thus

$$v_x' = -\overline{v}_x(x, y + l') + \overline{v}_x(x, y). \tag{3.2.90}$$

Expanding $\overline{v}_x(x, y + l')$ using the Taylor series expansion to first order equation (3.2.90) becomes

$$v_x' = -l'\frac{\partial \overline{v}_x}{\partial y}. \tag{3.2.91}$$

Prandtl assumed that v_y' and v_x' are of the same order of magnitude. Fluid lumps which arrive at level y downwards from above have $v_y' > 0$ and $v_x' < 0$ while the fluid lumps which arrive at level y upwards from below have $v_y' < 0$ and $v_x' > 0$. Hence v_x' and v_y' always have opposite sign and

$$v_y'(x, y) = -cl'\frac{\partial \overline{v}_x}{\partial y}, \; c > 0 \tag{3.2.92}$$

where c is a positive constant. Thus

$$\tau_T = -\rho\overline{v'_x v'_y} = \rho l^2 \left(\frac{\partial \overline{v}_x}{\partial y}\right)^2 \tag{3.2.93}$$

where

$$l^2 = c\overline{l'l'}. \tag{3.2.94}$$

But the sign of τ_T must change with the sign of $\frac{\partial \overline{v}_x}{\partial y}$ and it is therefore correct to write

$$\tau_T = \rho l^2 \left|\frac{\partial \overline{v}_x}{\partial y}\right| \frac{\partial \overline{v}_x}{\partial y} \tag{3.2.95}$$

where l is the momentum mixing length and

$$\nu_T = \frac{\mu_T}{\rho} = l^2 \left|\frac{\partial \overline{v}_x}{\partial y}\right|. \tag{3.2.96}$$

The constant c is absorbed into the mixing length l. Substituting equation (3.2.96) into (3.2.77) we obtain

$$\overline{v}_x \frac{\partial \overline{v}_x}{\partial x} + \overline{v}_y \frac{\partial \overline{v}_x}{\partial y} = \frac{\partial}{\partial y}\left(\left(\nu + l^2(x)\left|\frac{\partial \overline{v}_x}{\partial y}\right|\right)\frac{\partial \overline{v}_x}{\partial y}\right). \tag{3.2.97}$$

The mixing length l will be assumed to be a function of the x only.

Consider next the mixing length for the heat transfer. We require an expression for the turbulent thermal conductivity, κ_T.

From (3.2.87) the y-component of the turbulent heat flux vector $\underline{h}_T$ satisfies

$$h_T = -\kappa_T \frac{\partial \overline{T}}{\partial y} = \rho c_v \overline{T' v'_y}. \tag{3.2.98}$$

We use dimensional analysis to obtain $\frac{h_T}{\rho c_v}$. The heat flux, $\frac{h_T}{\rho c_v}$, is assumed to depend on $\frac{|\tau_T|}{\rho}$, $\left|\frac{\partial \overline{T}}{\partial y}\right|$ and l_θ to give the relation

$$\frac{h_T}{\rho c_v} = \left(\frac{|\tau_T|}{\rho}\right)^a \left(\left|\frac{\partial \overline{T}}{\partial y}\right|\right)^b l_\theta^c, \tag{3.2.99}$$

where the exponents a, b and c will be determined using dimensional analysis. The notation used for dimensions is:

- dimension of temperature is θ
- dimension of time is T
- dimension of length is L

Square brackets indicate the dimension of the quantity enclosed by them. Therefore

$$\left[\frac{h_T}{\rho c_v}\right] = \left[\overline{v'_y T'}\right] = \frac{L\theta}{T}, \tag{3.2.100}$$

$$\left[\frac{\tau_T}{\rho}\right] = \left[-\overline{v'_x v'_y}\right] = \frac{L^2}{T^2}, \tag{3.2.101}$$

$$\left[\frac{\partial \overline{T}}{\partial y}\right] = \frac{\theta}{L}, \tag{3.2.102}$$

$$[l_\theta] = L. \tag{3.2.103}$$

Substituting these dimensions into (3.2.99) we obtain

$$L\theta T^{-1} = \left(\frac{L^2}{T^2}\right)^a \left(\frac{\theta}{L}\right)^b L^c. \tag{3.2.104}$$

Equating the exponents of L, T and θ we obtain

$$L : 1 = 2a - b + c, \tag{3.2.105}$$

$$T : -1 = -2a, \tag{3.2.106}$$

$$\theta : 1 = b, \tag{3.2.107}$$

which gives

$$a = \frac{1}{2}, \qquad b = 1, \qquad c = 1. \tag{3.2.108}$$

Substituting equations (3.2.108) into (3.2.99) we obtain

$$\frac{h_T}{\rho c_v} = \left(\frac{|\tau_T|}{\rho}\right)^{\frac{1}{2}} \left|\frac{\partial \overline{T}}{\partial y}\right| l_\theta. \tag{3.2.109}$$

The temperature is the highest at the centreline of the jet and decreases as we move away from the centreline. Hence

$$\frac{\partial \overline{T}}{\partial y} < 0 \tag{3.2.110}$$

in the upper half, $y > 0$, of the jet. Therefore for the upper half of the jet

$$\frac{h_T}{\rho c_v} = -l_\theta \left(\frac{|\tau_T|}{\rho}\right)^{\frac{1}{2}} \frac{\partial \overline{T}}{\partial y}. \tag{3.2.111}$$

But from (3.2.95)

$$\frac{|\tau_T|}{\rho} = l^2 \left|\frac{\partial \overline{v}_x}{\partial y}\right|^2 \tag{3.2.112}$$

and therefore

$$\frac{h_T}{\rho c_v} = -l_\theta l \left|\frac{\partial \overline{v}_x}{\partial y}\right| \frac{\partial \overline{T}}{\partial y}. \tag{3.2.113}$$

Hence from (3.2.98)

$$\kappa_T = \rho c_v l_\theta l \left|\frac{\partial \overline{v}_x}{\partial y}\right|. \tag{3.2.114}$$

The thermal mixing length l_θ is assume to be a function of x only. Substituting equation (3.2.114) into (3.2.87) we obtain

$$\overline{v}_x \frac{\partial \overline{T}}{\partial x} + \overline{v}_y \frac{\partial \overline{T}}{\partial y} = \frac{\partial}{\partial y}\left[\left(\frac{\kappa}{\rho c_v} + \rho c_v l_\theta(x) l(x)\left|\frac{\partial \overline{v}_x}{\partial y}\right|\right)\frac{\partial \overline{T}}{\partial y}\right]. \qquad (3.2.115)$$

3.2.6 Boundary conditions

The jet is symmetrical about the centreline, $y = 0$. Hence

$$\overline{v}_y(x,0) = 0. \qquad (3.2.116)$$

The x-component of the mean velocity is maximum at $y = 0$. Hence

$$\frac{\partial \overline{v}_x}{\partial y}(x,0) = 0. \qquad (3.2.117)$$

The temperature at the boundary of the jet is ambient. i.e.

$$T(x, y_{Tb}(x)) = T_B, \qquad (3.2.118)$$

where $y_{Tb}(x)$ is the thermal boundary of the jet in the region $y > 0$ and T_B is the ambient temperature. The temperature is maximum at the centreline of the jet, $y = 0$. Hence

$$\frac{\partial \overline{T}}{\partial y}(x,0) = 0. \qquad (3.2.119)$$

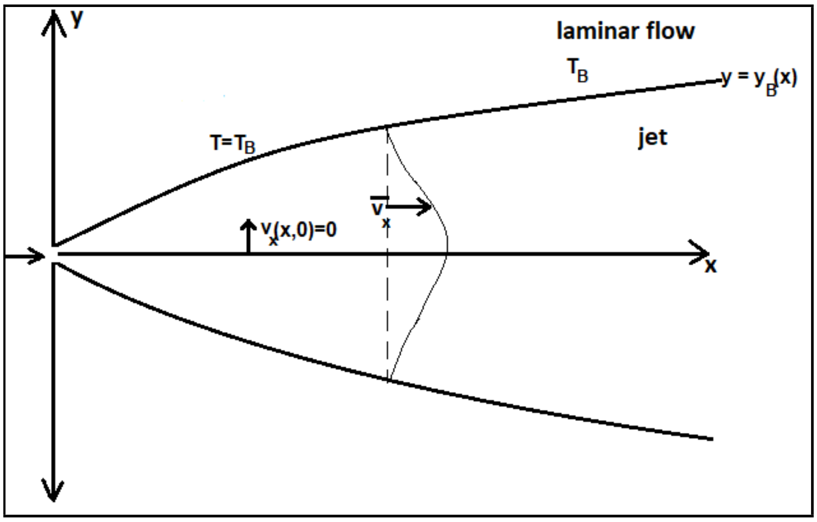

FIGURE 3.3: Schematic diagram for the mean velocity profile

The mean velocity flux through the jet boundary with normal $\underline{n}$ is zero (see Figure 3.4). Thus

$$y = y_B(x) : \overline{v}_y \cos\alpha - \overline{v}_x \sin\alpha = 0, \qquad (3.2.120)$$

where α is the angle between the tangent of the jet boundary and the x-component of the mean velocity. Writing this equation in terms of dimensionless variables gives

$$y = y_B(x) : \frac{\delta_V}{L}\overline{v}_y^* \cot\alpha - \overline{v}_x^* = 0, \qquad (3.2.121)$$

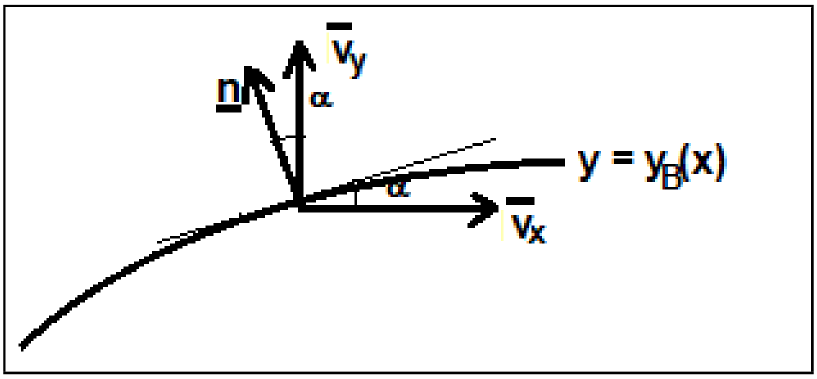

FIGURE 3.4: Schematic diagram of the boundary of the jet.

which, since $\frac{\delta_V}{L}$ is very small, reduces to

$$\overline{v}_x(x, y_B(x)) = 0. \tag{3.2.122}$$

The flow is laminar at the boundary. Hence

$$\nu_T(x, y_B(x)) = l^2 \left| \frac{\partial \overline{v}_x}{\partial y} \right| (x, y_B(x)) = 0, \tag{3.2.123}$$

which gives

$$\frac{\partial \overline{v}_x}{\partial y}(x, y_B(x)) = 0. \tag{3.2.124}$$

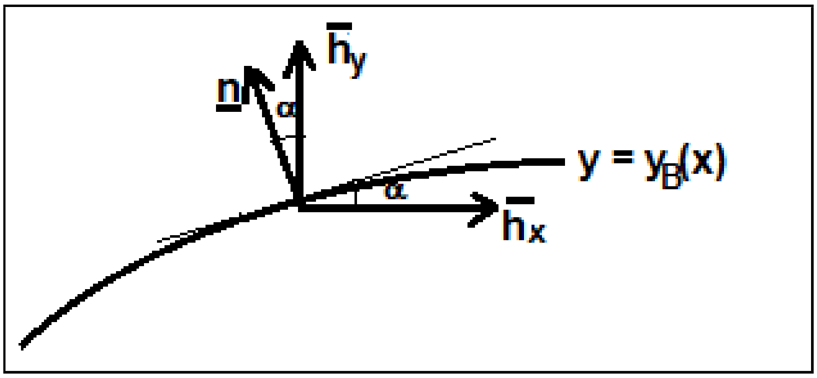

FIGURE 3.5: Schematic diagram of the boundary of the jet.

Let

$$h_x = -(\kappa + \kappa_T)\frac{\partial \overline{T}}{\partial x}, \tag{3.2.125}$$

$$h_y = -(\kappa + \kappa_T)\frac{\partial \overline{T}}{\partial y}. \tag{3.2.126}$$

The heat flux through the jet boundary with normal $\underline{n}$ is zero (see Figure 3.5). Therefore

$$y = y_{Tb}(x) : h_x \sin\alpha - h_y \cos\alpha = 0, \tag{3.2.127}$$

which, after substituting (3.2.125) and (3.2.126), becomes

$$y = y_{Tb}(x) : \sin\alpha \frac{\partial \overline{T}}{\partial x} - \cos\alpha \frac{\partial \overline{T}}{\partial y} = 0. \tag{3.2.128}$$

Writing (3.2.128) in terms of dimensionless variables gives

$$y = y_{Tb}(x) : \frac{\delta_V}{L} \sin\alpha \frac{\partial \overline{T}^*}{\partial x^*} - \cos\alpha \frac{\partial \overline{T}^*}{\partial y^*} = 0, \tag{3.2.129}$$

which, since

$$\frac{\delta_V}{L} << 1, \tag{3.2.130}$$

reduces to

$$y = y_{Tb}(x) : \frac{\partial \overline{T}}{\partial y} = 0. \tag{3.2.131}$$

3.3 Axisymmetric thermal jets

In this section, we derive the governing equations for the axisymmetric thermal jet. The conduit for the axisymmetric jet is circular and the flow is axially symmetric.

3.3.1 The Reynolds averaged Navier-Stokes equations with eddy viscosity

The cylindrical polar coordinate axes are constructed in a such a way that the point source resembling the conduit is at the origin and the jet flows in the positive side of the z-axis. Hence, for the region we are studying, z is positive. Time is represented by t. The velocity components v_r, v_ϕ and v_z represent velocities along the r, ϕ and z base vectors, respectively.

The Navier-Stokes equations written in cylindrical polar coordinates are

$$\frac{\partial v_r}{\partial t} + (\underline{v} \cdot \underline{\nabla}) v_r - \frac{v_\phi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu \left[\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\phi}{\partial \phi} \right], \tag{3.3.1}$$

$$\frac{\partial v_\phi}{\partial t} + (\underline{v} \cdot \underline{\nabla}) v_\phi + \frac{v_\phi v_r}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \phi} + \nu \left[\nabla^2 v_\phi - \frac{v_\phi}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \phi} \right], \tag{3.3.2}$$

$$\frac{\partial v_z}{\partial t} + (\underline{v} \cdot \underline{\nabla}) v_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \nabla^2 v_z \tag{3.3.3}$$

and the continuity equation for incompressible fluids is

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} = 0, \tag{3.3.4}$$

where

$$\underline{v} \cdot \underline{\nabla} = v_r \frac{\partial}{\partial r} + \frac{v_\phi}{r}\frac{\partial}{\partial \phi} + v_z \frac{\partial}{\partial z}, \tag{3.3.5}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}. \tag{3.3.6}$$

For the axisymmetric jet with vanishing azimuthal velocity

$$v_\phi = 0, \tag{3.3.7}$$

$$\frac{\partial}{\partial \phi} = 0, \tag{3.3.8}$$

which reduces equations (3.3.1) to (3.3.4) to

$$\frac{\partial v_r}{\partial t} + (\underline{v} \cdot \underline{\nabla}) v_r = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu \nabla^2 v_r, \tag{3.3.9}$$

$$\frac{\partial v_z}{\partial t} + (\underline{v} \cdot \underline{\nabla}) v_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \nabla^2 v_z, \tag{3.3.10}$$

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0, \tag{3.3.11}$$

where

$$\underline{v} \cdot \underline{\nabla} = v_r \frac{\partial}{\partial r} + v_z \frac{\partial}{\partial z}, \tag{3.3.12}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{3.3.13}$$

The averaging operator $\overline{(\cdot)}$ defined by (3.2.9) also apply for cylindrical coordinates. The averaged time derivative vanishes. Applying the averaging operator $\overline{(\cdot)}$ on the system of partial differential equations (3.3.9) to (3.3.11) we obtain

$$\overline{v_r \frac{\partial v_r}{\partial r}} + \overline{v_z \frac{\partial v_r}{\partial z}} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial r} + \nu \nabla^2 \overline{v}_r, \tag{3.3.14}$$

$$\overline{v_r \frac{\partial v_z}{\partial r}} + \overline{v_z \frac{\partial v_z}{\partial z}} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial z} + \nu \nabla^2 \overline{v}_z, \tag{3.3.15}$$

$$\frac{\partial \overline{v_r}}{\partial r} + \frac{\overline{v_r}}{r} + \frac{\partial \overline{v_z}}{\partial z} = 0. \tag{3.3.16}$$

We write velocity components and pressure as the sum of a time average and fluctuations. Thus

$$v_i(z,r,t) = \overline{v}_i(z,r) + v_i'(z,r,t), \tag{3.3.17}$$

$$p(z,r,t) = \overline{p}(z,r) + p'(z,r,t). \tag{3.3.18}$$

Substituting equations (3.3.17) and (3.3.18) into (3.3.14) and (3.3.15) we obtain

$$\overline{(\overline{v}_r + v_r')\frac{\partial}{\partial r}(\overline{v}_r + v_r')} + \overline{(\overline{v}_z + v_z')\frac{\partial}{\partial z}(\overline{v}_r + v_r')} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial r} + \nu \nabla^2 \overline{v}_r, \tag{3.3.19}$$

$$\overline{(\overline{v}_r + v_r')\frac{\partial}{\partial r}(\overline{v}_z + v_z')} + \overline{(\overline{v}_z + v_z')\frac{\partial}{\partial z}(\overline{v}_z + v_z')} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial z} + \nu \nabla^2 \overline{v}_z. \tag{3.3.20}$$

The averaging operator properties listed in (3.2.10) to (3.2.14) apply to cylindrical polar coordinates as well. Using the averaging operator properties (3.2.10) to (3.2.14) to simplify (3.3.19) and (3.3.20) we obtain

$$\overline{v}_r \frac{\partial \overline{v}_r}{\partial r} + \overline{v}_z \frac{\partial \overline{v}_r}{\partial z} + \overline{v_r' \frac{\partial v_r'}{\partial r}} + \overline{v_z' \frac{\partial v_r'}{\partial z}} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial r} + \nu \nabla^2 \overline{v}_r, \tag{3.3.21}$$

$$\overline{v}_r \frac{\partial \overline{v}_z}{\partial r} + \overline{v}_z \frac{\partial \overline{v}_z}{\partial z} + \overline{v_r' \frac{\partial v_z'}{\partial r}} + \overline{v_z' \frac{\partial v_z'}{\partial z}} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial z} + \nu \nabla^2 \overline{v}_z. \tag{3.3.22}$$

Using the product rule

$$v_i \frac{\partial v_j}{\partial x_i} = \frac{\partial}{\partial x_i}\left(v_i v_j\right) - v_j \frac{\partial v_i}{\partial x_i}, \tag{3.3.23}$$

equations (3.3.21) and (3.3.22) become

$$\overline{v}_r \frac{\partial \overline{v}_r}{\partial r} + \overline{v}_z \frac{\partial \overline{v}_r}{\partial z} - \overline{v_r' \left(\frac{\partial v_r'}{\partial r} + \frac{\partial v_z'}{\partial z}\right)} + \frac{\partial}{\partial r}\overline{(v_r' v_r')} + \frac{\partial}{\partial z}\overline{(v_z' v_r')} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial r} + \nu \nabla^2 \overline{v}_r, \tag{3.3.24}$$

$$\overline{v}_r \frac{\partial \overline{v}_z}{\partial r} + \overline{v}_z \frac{\partial \overline{v}_z}{\partial z} - \overline{v_z' \left(\frac{\partial v_r'}{\partial r} + \frac{\partial v_z'}{\partial z}\right)} + \frac{\partial}{\partial z}\overline{(v_r' v_z')} + \frac{\partial}{\partial r}\overline{(v_z' v_z')} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial z} + \nu \nabla^2 \overline{v}_z. \tag{3.3.25}$$

Subtracting equation (3.3.16) from (3.3.11) and using (3.3.17) we obtain

$$\frac{\partial v_r'}{\partial r} + \frac{\partial v_z'}{\partial z} = -\frac{v_r'}{r}. \tag{3.3.26}$$

Substituting equation (3.3.26) into (3.3.24) and (3.3.25) for the third terms of the left hand sides we obtain

$$\overline{v}_r \frac{\partial \overline{v}_r}{\partial r} + \overline{v}_z \frac{\partial \overline{v}_r}{\partial z} + \frac{\overline{v_r' v_r'}}{r} + \frac{\partial}{\partial r}\overline{(v_r' v_r')} + \frac{\partial}{\partial z}\overline{(v_r' v_z')} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial r} + \nu \nabla^2 \overline{v}_r, \tag{3.3.27}$$

$$\overline{v}_r \frac{\partial \overline{v}_z}{\partial r} + \overline{v}_z \frac{\partial \overline{v}_z}{\partial z} + \frac{\overline{v_r' v_z'}}{r} + \frac{\partial}{\partial z}\overline{(v_r' v_z')} + \frac{\partial}{\partial r}\overline{(v_z' v_z')} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial z} + \nu \nabla^2 \overline{v}_z. \tag{3.3.28}$$

Hence in summary:

$$\overline{v}_r \frac{\partial \overline{v}_r}{\partial r} + \overline{v}_z \frac{\partial \overline{v}_r}{\partial z} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial r} + \nu \nabla^2 \overline{v}_r - \frac{\overline{v_r' v_z'}}{r} - \frac{\partial}{\partial r}\overline{(v_r' v_r')} - \frac{\partial}{\partial z}\overline{(v_z' v_r')}, \tag{3.3.29}$$

$$\overline{v}_r \frac{\partial \overline{v}_z}{\partial r} + \overline{v}_z \frac{\partial \overline{v}_z}{\partial z} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial z} + \nu \nabla^2 \overline{v}_z - \frac{\overline{v_r' v_z'}}{r} - \frac{\partial}{\partial z}\overline{(v_r' v_z')} - \frac{\partial}{\partial r}\overline{(v_z' v_z')}, \tag{3.3.30}$$

$$\frac{\partial \overline{v_r}}{\partial r} + \frac{\overline{v_r}}{r} + \frac{\partial \overline{v_z}}{\partial z} = 0, \tag{3.3.31}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{3.3.32}$$

The Reynolds stresses are $\rho\overline{v_r' v_r'}$, $\rho\overline{v_z' v_r'}$ and $\rho\overline{v_z' v_z'}$, which can be written in terms of the mean velocity gradients using the Boussinesq closure model (3.2.47) as

$$\overline{v_r' v_r'} = -2\nu_T \frac{\partial \overline{v}}{\partial r}, \qquad \overline{v_z' v_r'} = -\nu_T \left(\frac{\partial \overline{v}_z}{\partial r} + \frac{\partial \overline{v}_r}{\partial z}\right), \qquad \overline{v_z' v_z'} = -2\nu_T \frac{\partial \overline{v}_z}{\partial z}. \tag{3.3.33}$$

Hence

$$\overline{v}_r \frac{\partial \overline{v}_r}{\partial r} + \overline{v}_z \frac{\partial \overline{v}_r}{\partial z} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial r} + \frac{\partial}{\partial r}\left((\nu + \nu_T)\frac{\partial \overline{v}_r}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left((\nu + \nu_T)\overline{v}_r\right) + \frac{\partial}{\partial z}\left((\nu + \nu_T)\frac{\partial \overline{v}_r}{\partial z}\right)$$
$$+ \frac{\partial}{\partial r}\left(\nu_T \left(\frac{\partial \overline{v}_r}{\partial r} + \frac{\overline{v}_r}{r} + \frac{\partial \overline{v}_z}{\partial z}\right)\right), \tag{3.3.34}$$

$$\overline{v}_r\frac{\partial\overline{v}_z}{\partial r}+\overline{v}_z\frac{\partial\overline{v}_z}{\partial z}=-\frac{1}{\rho}\frac{\partial\overline{p}}{\partial z}+\frac{\partial}{\partial r}\left((\nu+\nu_T)\frac{\partial\overline{v}_z}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left((\nu+\nu_T)\overline{v}_z\right)+\frac{\partial}{\partial z}\left((\nu+\nu_T)\frac{\partial\overline{v}_z}{\partial z}\right)$$
$$+\frac{\partial}{\partial z}\left(\nu_T\left(\frac{\partial\overline{v}_r}{\partial r}+\frac{\overline{v}_r}{r}+\frac{\partial\overline{v}_z}{\partial z}\right)\right). \tag{3.3.35}$$

Using equation (3.3.31) the last terms of (3.3.34) and (3.3.35) vanish to give

$$\overline{v}_r\frac{\partial\overline{v}_r}{\partial r}+\overline{v}_z\frac{\partial\overline{v}_r}{\partial z}=-\frac{1}{\rho}\frac{\partial\overline{p}}{\partial r}+\frac{\partial}{\partial r}\left((\nu+\nu_T)\frac{\partial\overline{v}_r}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left((\nu+\nu_T)\overline{v}_r\right)$$
$$+\frac{\partial}{\partial z}\left((\nu+\nu_T)\frac{\partial\overline{v}_r}{\partial z}\right), \tag{3.3.36}$$

$$\overline{v}_r\frac{\partial\overline{v}_z}{\partial r}+\overline{v}_z\frac{\partial\overline{v}_z}{\partial z}=-\frac{1}{\rho}\frac{\partial\overline{p}}{\partial z}+\frac{\partial}{\partial r}\left((\nu+\nu_T)\frac{\partial\overline{v}_z}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left((\nu+\nu_T)\overline{v}_z\right)$$
$$+\frac{\partial}{\partial z}\left((\nu+\nu_T)\frac{\partial\overline{v}_z}{\partial z}\right). \tag{3.3.37}$$

3.3.2 Energy balance equation with turbulent thermal conductivity

The energy balance equation written in cylindrical polar coordinates is

$$\rho c_v\left[\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+v_z\frac{\partial T}{\partial z}\right]=\kappa\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right). \tag{3.3.38}$$

Writing equation (3.3.38) in terms averages and the fluctuations (3.3.17) and (3.3.18) and using the fact that the averaged time derivative vanishes we obtain

$$\rho c_v\left[\overline{(\overline{v}_r+v_r')\frac{\partial}{\partial r}(\overline{T}+T')}+\overline{(\overline{v}_z+v_z')\frac{\partial}{\partial z}(\overline{T}+T')}\right]=\kappa\left(\frac{\partial^2\overline{T}}{\partial r^2}+\frac{1}{r}\frac{\partial\overline{T}}{\partial r}+\frac{\partial^2\overline{T}}{\partial z^2}\right). \tag{3.3.39}$$

Using the averaging properties (3.2.16) to (3.2.14) to expand the equation (3.3.39) we obtain

$$\rho c_v\left[\overline{v}_r\frac{\partial\overline{T}}{\partial r}+\overline{v}_z\frac{\partial\overline{T}}{\partial z}+\overline{v_r'\frac{\partial T'}{\partial r}}+\overline{v_z'\frac{\partial T'}{\partial z}}\right]=\kappa\left(\frac{\partial^2\overline{T}}{\partial r^2}+\frac{1}{r}\frac{\partial\overline{T}}{\partial r}+\frac{\partial^2\overline{T}}{\partial z^2}\right). \tag{3.3.40}$$

Using the product rule

$$v_i'\frac{\partial T'}{\partial x_i}=\frac{\partial}{\partial x_i}(v_i'T')-T'\frac{\partial v_i}{\partial x_i}, \tag{3.3.41}$$

equation (3.3.40) becomes

$$\rho c_v\left[\overline{v}_r\frac{\partial\overline{T}}{\partial r}+\overline{v}_z\frac{\partial\overline{T}}{\partial z}-\overline{T'\left(\frac{\partial v_r'}{\partial r}+\frac{\partial v_z'}{\partial z}\right)}+\frac{\partial}{\partial r}\overline{(v_r'T')}+\frac{\partial}{\partial z}\overline{(v_z'T')}\right]=$$
$$\kappa\left(\frac{\partial^2\overline{T}}{\partial r^2}+\frac{1}{r}\frac{\partial\overline{T}}{\partial r}+\frac{\partial^2\overline{T}}{\partial z^2}\right). \tag{3.3.42}$$

Substituting equation (3.3.26) into (3.3.42) for the third term of the left hand side we obtain

$$\rho c_v \left[\overline{v}_r \frac{\partial \overline{T}}{\partial r} + \overline{v}_z \frac{\partial \overline{T}}{\partial z} + \frac{\overline{T' v_r'}}{r} + \frac{\partial}{\partial r} \overline{(v_r' T')} + \frac{\partial}{\partial z} \overline{(v_z' T')} \right] = \kappa \left(\frac{\partial^2 \overline{T}}{\partial r^2} + \frac{1}{r} \frac{\partial \overline{T}}{\partial r} + \frac{\partial^2 \overline{T}}{\partial z^2} \right). \tag{3.3.43}$$

Hence in summary:

$$\rho c_v \left[\overline{v}_r \frac{\partial \overline{T}}{\partial r} + \overline{v}_z \frac{\partial \overline{T}}{\partial z} \right] = \frac{\partial}{\partial r} \left(\kappa \frac{\partial \overline{T}}{\partial r} - \rho c_v \overline{(v_r' T')} \right) + \frac{1}{r} \left(\kappa \frac{\partial \overline{T}}{\partial r} - \rho c_v \overline{(v_r' T')} \right) + \frac{\partial}{\partial z} \left(\kappa \frac{\partial \overline{T}}{\partial z} - \rho c_v \overline{(v_z' T')} \right). \tag{3.3.44}$$

But equation (3.2.53), which is expressed in cartesian coordinates, becomes in cylindrical polar coordinates

$$\rho c_v \overline{T' v_r'} = -\kappa_T \frac{\partial \overline{T}}{\partial r}, \qquad \rho c_v \overline{T' v_z'} = -\kappa_T \frac{\partial \overline{T}}{\partial z}. \tag{3.3.45}$$

Thus equation (3.3.44) becomes

$$\rho c_v \left[\overline{v}_r \frac{\partial \overline{T}}{\partial r} + \overline{v}_z \frac{\partial \overline{T}}{\partial z} \right] = \frac{\partial}{\partial r} \left((\kappa + \kappa_T) \frac{\partial \overline{T}}{\partial r} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left((\kappa + \kappa_T) \overline{T} \right) + \frac{\partial}{\partial z} \left((\kappa + \kappa_T) \frac{\partial \overline{T}}{\partial z} \right). \tag{3.3.46}$$

3.3.3 Boundary layer approximation

Let

- characteristic thermal conductivity be κ_0
- characteristic dynamic viscosity be ν_0
- characteristic pressure be P
- characteristic distance in the z-direction be L
- characteristic temperature be T_0
- characteristic velocity in the z-direction be U
- thickness of velocity boundary layer be δ_V
- thickness of thermal boundary layer be δ_T
- characteristic velocity in the r-direction be V.

From the continuity equation for the mean velocity,

$$\frac{\partial \overline{v}_r}{\partial r} + \frac{\overline{v}_r}{r} + \frac{\partial \overline{v}_z}{\partial z} = 0, \tag{3.3.47}$$

we have

$$\frac{[\overline{v}_z]}{[z]} \sim \frac{[\overline{v}_r]}{[r]}, \tag{3.3.48}$$

which gives

$$V = \frac{\delta_V}{L} U. \tag{3.3.49}$$

As with the two-dimensional boundary layer we make the following scaling:

- when differentiating $\overline{T}$ with respect to r, r is scaled with δ_T.
- when differentiating $\overline{v}_z$ and $\overline{v}_r$ with respect to r, r is scaled with δ_V.

We define dimensionless variables

$$\overline{T}^* = \frac{\overline{T}}{T_0}, \qquad \overline{p}^* = \frac{\overline{p}}{P}, \qquad \overline{v}_z^* = \frac{\overline{v}_z}{U}, \qquad \overline{v}_r^* = \frac{L}{\delta_V} \frac{\overline{v}_r}{U}, \tag{3.3.50}$$

$$\kappa^* = \frac{\kappa + \kappa_T}{\kappa_0}, \qquad \nu^* = \frac{\nu + \nu_T}{\nu_0}, \qquad z^* = \frac{z}{L}, \qquad r^* = \frac{r}{\delta_V} \text{ or } \frac{r}{\delta_T}. \tag{3.3.51}$$

Substituting these dimensionless variables into the momentum equations (3.3.36) and (3.3.37) we obtain

$$\begin{aligned} \frac{1}{Re}\left(\overline{v}_r^* \frac{\partial \overline{v}_r^*}{\partial r^*} + \overline{v}_z^* \frac{\partial \overline{v}_r^*}{\partial z^*}\right) = -\frac{\partial \overline{p}^*}{\partial r^*} + \frac{1}{Re}\frac{\partial}{\partial r^*}\left(\nu^* \frac{\partial \overline{v}_r^*}{\partial r^*}\right) + \frac{1}{Re}\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(\nu^* \overline{v}_r^*\right) + \\ \frac{1}{Re^2}\frac{\partial}{\partial z^*}\left(\nu^* \frac{\partial \overline{v}_r^*}{\partial z^*}\right), \end{aligned} \tag{3.3.52}$$

$$\overline{v}_r^* \frac{\partial \overline{v}_z^*}{\partial r^*} + \overline{v}_z^* \frac{\partial \overline{v}_z^*}{\partial z^*} = -\frac{\partial \overline{p}}{\partial z} + \frac{\partial}{\partial r*}\left(\nu^* \frac{\partial \overline{v}_z^*}{\partial r^*}\right) + \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(\nu^* \overline{v}_z^*\right) + \frac{1}{Re}\frac{\partial}{\partial z^*}\left(\nu^* \frac{\partial \overline{v}_z^*}{\partial z}\right), \tag{3.3.53}$$

where

$$P = \rho U^2, \tag{3.3.54}$$

$$Re = \frac{UL}{\nu} \tag{3.3.55}$$

and

$$\delta_V = \frac{L}{\sqrt{Re}}, \tag{3.3.56}$$

which, for large Reynolds number, reduce to

$$\frac{\partial \overline{p}^*}{\partial r^*} = 0, \tag{3.3.57}$$

$$\overline{v}_r^* \frac{\partial \overline{v}_z^*}{\partial r^*} + \overline{v}_z^* \frac{\partial \overline{v}_z^*}{\partial z^*} = -\frac{\partial \overline{p}^*}{\partial z} + \frac{\partial}{\partial r^*}\left(\nu^* \frac{\partial \overline{v}_z^*}{\partial r^*}\right) + \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(\nu^* \overline{v}_z^*\right). \tag{3.3.58}$$

Writing equations (3.3.57) and (3.3.58) in terms of original variables we obtain

$$\frac{\partial \overline{p}}{\partial r} = 0, \tag{3.3.59}$$

$$\overline{v}_r \frac{\partial \overline{v}_z}{\partial r} + \overline{v}_z \frac{\partial \overline{v}_z}{\partial z} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial z} + \frac{\partial}{\partial r}\left((\nu + \nu_T)\frac{\partial \overline{v}_z}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left((\nu + \nu_T)\overline{v}_z\right). \tag{3.3.60}$$

It follows from (3.3.59) that $\overline{p} = \overline{p}(z)$. The flow outside the jet is laminar with $\frac{dU}{dx} = 0$ for the mainstream velocity U. The Euler equation outside the jet gives

$$0 = -\frac{\partial \overline{p}}{\partial z} \tag{3.3.61}$$

and since $\overline{p} = \overline{p}(z)$, equation (3.3.61) applies inside the jet. Therefore equation (3.3.60) becomes

$$\overline{v}_r \frac{\partial \overline{v}_z}{\partial r} + \overline{v}_z \frac{\partial \overline{v}_z}{\partial z} = \frac{\partial}{\partial r}\left((\nu + \nu_T)\frac{\partial \overline{v}_z}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left((\nu+\nu_T)\overline{v}_z\right). \tag{3.3.62}$$

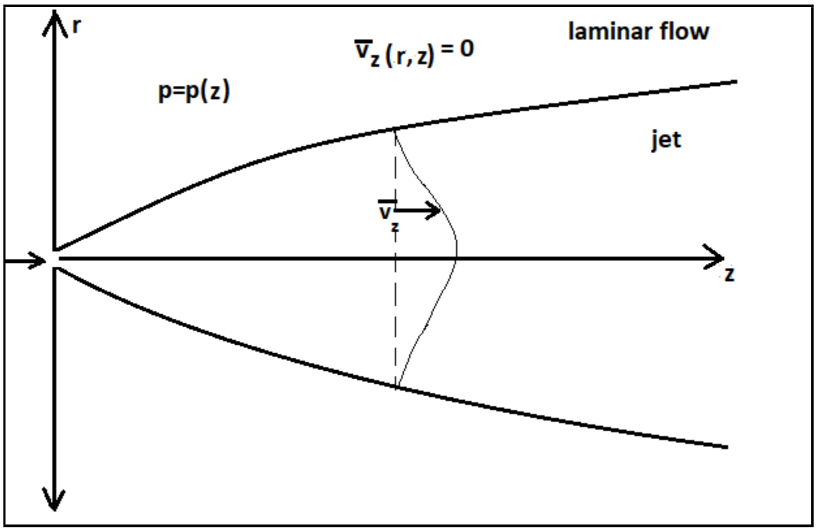

FIGURE 3.6: Schematic diagram for the mean velocity profile of the axisymmetric jet

Writing the energy balance equation (3.3.44) in terms of dimensionless variables we obtain

$$\sqrt{Pr}\,\overline{v}_r^* \frac{\partial \overline{T}^*}{\partial r^*} + \overline{v}_z^* \frac{\partial \overline{T}^*}{\partial z^*} = \frac{\partial}{\partial r^*}\left(\kappa^* \frac{\partial \overline{T}^*}{\partial r^*}\right) + \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(\kappa^* \overline{T}^*\right) + \frac{1}{PrRe}\frac{\partial}{\partial z^*}\left(\kappa^* \frac{\partial \overline{T}^*}{\partial z^*}\right), \tag{3.3.63}$$

where

$$\delta_T = \delta_V \sqrt{Pr}. \tag{3.3.64}$$

We again consider $Pr \sim 1$. Since $\sqrt{Re} >> 1$, equation (3.3.63) reduces to

$$\sqrt{Pr}\,\overline{v}_r^* \frac{\partial \overline{T}^*}{\partial r^*} + \overline{v}_z^* \frac{\partial \overline{T}^*}{\partial z^*} = \frac{\partial}{\partial r^*}\left(\kappa^* \frac{\partial \overline{T}^*}{\partial r^*}\right) + \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(\kappa^* \overline{T}^*\right). \tag{3.3.65}$$

Writing (3.3.65) in terms of the original variables we obtain

$$\rho c_v \left[\overline{v}_r \frac{\partial \overline{T}}{\partial r} + \overline{v}_z \frac{\partial \overline{T}}{\partial z}\right] = \frac{\partial}{\partial r}\left((\kappa + \kappa_T)\frac{\partial \overline{T}}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left((\kappa+\kappa_T)\overline{T}\right). \tag{3.3.66}$$

3.3.4 Mixing lengths for the momentum and heat transfer

In Section 3.2.5 we found that the kinematic eddy viscosity and the turbulent thermal conductivity are related to momentum mixing length l and the thermal mixing length l_θ. The same calculation for the axisymmetric jet gives

$$\nu_T = l^2 \left| \frac{\partial \overline{v}_z}{\partial r} \right|, \tag{3.3.67}$$

$$\kappa_T = \rho c_v l_\theta l \left| \frac{\partial \overline{v}_z}{\partial r} \right|. \tag{3.3.68}$$

Substituting equations (3.3.67) and (3.3.68) into (3.3.62) and (3.3.66) we obtain

$$\overline{v}_r \frac{\partial \overline{v}_z}{\partial r} + \overline{v}_z \frac{\partial \overline{v}_z}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r} \left[r \left(\nu + l^2 \left| \frac{\partial \overline{v}_z}{\partial r} \right| \right) \frac{\partial \overline{v}_z}{\partial r} \right], \tag{3.3.69}$$

$$\rho c_v \left[\overline{v}_r \frac{\partial \overline{T}}{\partial r} + \overline{v}_z \frac{\partial \overline{T}}{\partial z} \right] = \frac{1}{r} \frac{\partial}{\partial r} \left[r \left(\kappa + \rho c_v l_\theta l \left| \frac{\partial \overline{v}_z}{\partial r} \right| \right) \frac{\partial \overline{T}}{\partial r} \right]. \tag{3.3.70}$$

3.3.5 Boundary conditions

The axisymmetric free jet is symmetrical about the centreline. Hence it follows that

$$\overline{v}_r(z,0) = 0. \tag{3.3.71}$$

The mean velocity along the jet is maximum at the centreline of the jet. Hence

$$\frac{\partial \overline{v}_z}{\partial r}(z,0) = 0. \tag{3.3.72}$$

The temperature at the boundary of the jet is ambient, that is

$$T(z, r_{Tb}(z)) = T_B \tag{3.3.73}$$

where $r = r_{Tb}(z)$ is the thermal boundary of the jet and T_B is the ambient temperature. The temperature is maximum at the centreline of the jet. Hence

$$\frac{\partial \overline{T}}{\partial r}(z,0) = 0. \tag{3.3.74}$$

The fluid flux through the jet boundary with normal $\underline{n}$ is zero (see Figure 3.9). Thus the component of the velocity normal to the boundary is zero:

$$r = r_B(z) : \overline{v}_r \cos\alpha - \overline{v}_z \sin\alpha = 0, \tag{3.3.75}$$

where α is the angle between the z-component of the fluid velocity and the tangent of the boundary of the jet. Writing (3.3.75) in terms of dimensionless variables we obtain

$$r = r_B(z) : \frac{\delta_V}{L} \overline{v}_r^* \cot\alpha - \overline{v}_z^* = 0, \tag{3.3.76}$$

which, since $\frac{\delta_V}{L} << 1$, gives

$$\overline{v}_z(z, r_B(z)) = 0. \tag{3.3.77}$$

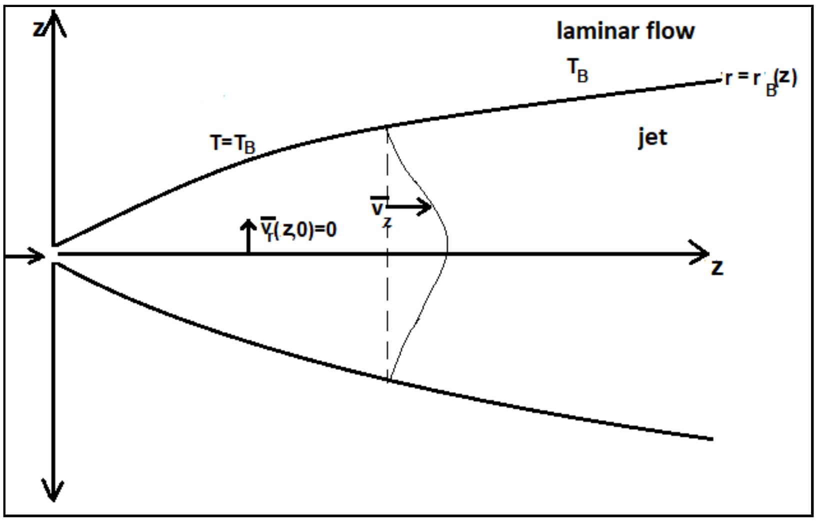

FIGURE 3.7: Schematic of the mean velocity profile of the axisymmetric jet.

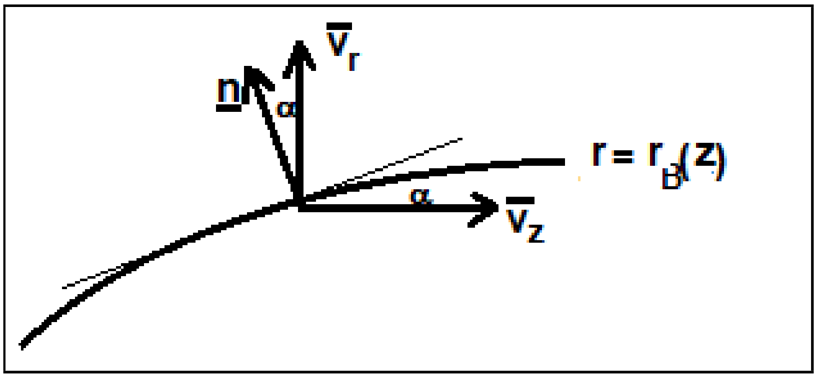

FIGURE 3.8: Schematic of the boundary of the jet showing the velocity components.

Since the flow is laminar at the boundary we have

$$\nu_T = l^2 \left| \frac{\partial \overline{v}_z}{\partial r} \right| = 0, \tag{3.3.78}$$

which gives

$$\frac{\partial \overline{v}_z}{\partial r}(z, r_B(z)) = 0. \tag{3.3.79}$$

The z and r components of the heat flux vector are

$$h_z = -(\kappa + \kappa_T)\frac{\partial \overline{T}}{\partial z}, \tag{3.3.80}$$

$$h_r = -(\kappa + \kappa_T)\frac{\partial \overline{T}}{\partial r}. \tag{3.3.81}$$

The heat flux through the jet boundary with normal $\underline{n}$ is zero (see Figure 3.10). Therefore

$$r = r_{Tb}(z) : h_z \sin\alpha - h_r \cos\alpha = 0. \tag{3.3.82}$$

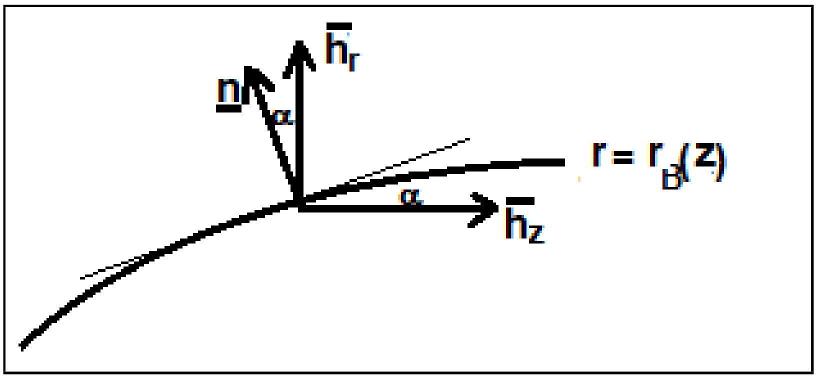

FIGURE 3.9: Schematic diagram for the showing the components of the heat flux vector.

Substituting equations (3.3.80) and (3.3.81) into this equation we obtain

$$r = r_{Tb}(z) : \sin\alpha \frac{\partial \overline{T}}{\partial z} - \cos\alpha \frac{\partial \overline{T}}{\partial r} = 0. \tag{3.3.83}$$

Writing (3.3.83) in terms of dimensionless variables we obtain

$$r = r_{Tb}(z) : -\frac{\delta_T}{L} \sin\alpha \frac{\partial \overline{T}^*}{\partial z^*} + \cos\alpha \frac{\partial \overline{T}^*}{\partial r^*} = 0, \tag{3.3.84}$$

which, since $\frac{\delta_T}{L} << 1$, gives

$$r = r_{Tb}(z) : \frac{\partial \overline{T}}{\partial r} = 0. \tag{3.3.85}$$

3.4 Summary

The coupled system of governing equations for the two-dimensional turbulent thermal jet are

$$\overline{v}_x \frac{\partial \overline{v}_x}{\partial x} + \overline{v}_y \frac{\partial \overline{v}_x}{\partial y} = \frac{\partial}{\partial y}\left[\left(\nu + l^2(x)\left|\frac{\partial \overline{v}_x}{\partial y}\right|\right)\frac{\partial \overline{v}_x}{\partial y}\right], \tag{3.4.1}$$

$$\overline{v}_x \frac{\partial \overline{T}}{\partial x} + \overline{v}_y \frac{\partial \overline{T}}{\partial y} = \frac{\partial}{\partial y}\left[\left(\frac{\kappa}{\rho c_v} + l(x) l_\theta(x)\left|\frac{\partial \overline{v}_x}{\partial y}\right|\right)\frac{\partial \overline{T}}{\partial y}\right], \tag{3.4.2}$$

$$\frac{\partial \overline{v}_x}{\partial x} + \frac{\partial \overline{v}_y}{\partial y} = 0 \tag{3.4.3}$$

with boundary conditions

$$\frac{\partial \overline{v}_x}{\partial y}(x,0) = 0, \quad \overline{v}_x(x, y_B(x)) = 0, \quad \frac{\partial \overline{v}_x}{\partial y}(x, y_B(x)) = 0, \quad \overline{v}_y(x,0) = 0 \tag{3.4.4}$$

$$\frac{\partial \overline{T}}{\partial y}(x,0) = 0, \qquad \frac{\partial \overline{T}}{\partial y}(x, y_{Tb}(x)) = 0, \qquad \overline{T}(x, y_{Tb}(x)) = T_B. \tag{3.4.5}$$

The governing equations for the axisymmetric turbulent thermal jet are

$$\overline{v}_r \frac{\partial \overline{v}_z}{\partial r} + \overline{v}_z \frac{\partial \overline{v}_z}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r} \left[r \left(\nu + l^2(z) \left| \frac{\partial \overline{v}_z}{\partial r} \right| \right) \frac{\partial \overline{v}_z}{\partial r} \right], \tag{3.4.6}$$

$$\overline{v}_r \frac{\partial \overline{T}}{\partial r} + \overline{v}_z \frac{\partial \overline{T}}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r} \left[r \left(\frac{\kappa}{\rho c_v} + l_\theta(z) l(z) \left| \frac{\partial \overline{v}_z}{\partial r} \right| \right) \frac{\partial \overline{T}}{\partial r} \right], \tag{3.4.7}$$

$$\frac{\partial \overline{v}_z}{\partial z} + \frac{1}{r} \overline{v}_r + \frac{\partial \overline{v}_r}{\partial r} = 0, \tag{3.4.8}$$

with boundary conditions

$$\frac{\partial \overline{v}_z}{\partial r}(z,0) = 0, \quad \overline{v}_z(z, r_B(z)) = 0, \quad \frac{\partial \overline{v}_z}{\partial r}(z, r_B(z)) = 0, \quad \overline{v}_r(z,0) = 0, \tag{3.4.9}$$

$$\frac{\partial \overline{T}}{\partial r}(z,0) = 0, \qquad \frac{\partial \overline{T}}{\partial r}(z, r_{Tb}(z)) = 0, \qquad \overline{T}(z, r_{Tb}(z)) = T_B. \tag{3.4.10}$$

3.5 Conclusions

In this chapter, we derived the governing equations for both the axisymmetric and the two-dimensional turbulent thermal jets which will be analysed further in Chapter 4 and 5.

Chapter 4

Two-dimensional turbulent jet: conservation laws and Lie point symmetries and group invariant solutions

4.1 Introduction

In this chapter, we apply the Lie point symmetry method to reduce the coupled system of governing partial differential equations derived in Chapter 3 to ordinary differential equations. The conserved vectors of the coupled system of governing partial differential equations are used to find the associated Lie point symmetry. Hence, we first derive the conserved vectors and conservation laws for the system of partial differential equations. The conservation laws are used to find conserved quantities which are necessary to solve for the undetermined constants or arbitrary functions contained in the Lie point symmetry and to completely solve the jet problem since the homogeneous boundary conditions of the jet are not sufficient.

Naz et al [8] compared different methods for obtaining conservation laws and found that the multiplier method, first introduced by Steudel et al [9] and invoked by Anco et al [4], to be simpler and more systematic compared to its counter parts. Hence, in this chapter, the multiplier method is used to obtain conserved vectors and conservation laws.

When $\nu = 0$ and $\kappa = 0$ the associated Lie point symmetry contains an arbitrary function. The invariant solution for this case can thus be obtained using two methods: (1) letting $\nu \to 0$ and $\kappa \to 0$ in the invariant solution for $\nu \neq 0$ and $\kappa \neq 0$; (2) using Prandtl's hypothesis [16] which states that the momentum mixing length is proportional to the half-width of the jet and the conserved quantity for the momentum to find the arbitrary function in the associated Lie point symmetry. The results for these two methods will be compared.

The differential equations for the invariant solution are solved analytically only when $\nu = 0$ and $\kappa = 0$. Otherwise they are solved numerically using the shooting method. The numerical solution for $\nu = 0$ and $\kappa = 0$ will be compared with the analytical solution. This is done to validate the numerical solution.

This chapter is structured in the following manner: In Section 4.2, we derive the conservation laws which are then used to determine conserved quantities in Section 4.3. In Section 4.4, we find the associated Lie point symmetries to the conservation laws derived in Section 4.2, which are then used to reduce the model partial differential equations to ordinary differential equations in Section 4.5. In Section 4.6 we give the analytical solution for the invariant solution for when $\nu \to 0$ and $\kappa \to 0$. In

Section 4.7 we give the analytical solution for when $\nu = 0$ and $\kappa = 0$ using Prandtl's hypothesis [18]. In Section 4.8 we give the numerical solution for the invariant solutions. We conclude the chapter by providing a general discussion on the results in Section 4.9 and concluding remarks in Section 4.10.

4.2 Conservation laws

The coupled system of governing partial differential equations for the two-dimensional turbulent thermal jet is (3.4.1), (3.4.2) and (3.4.3);

$$\overline{v}_x \frac{\partial \overline{v}_x}{\partial x} + \overline{v}_y \frac{\partial \overline{v}_x}{\partial y} = \frac{\partial}{\partial y}\left[\left(\nu + l^2(x)\left|\frac{\partial \overline{v}_x}{\partial y}\right|\right)\frac{\partial \overline{v}_x}{\partial y}\right], \tag{4.2.1}$$

$$\overline{v}_x \frac{\partial \overline{T}}{\partial x} + \overline{v}_y \frac{\partial \overline{T}}{\partial y} = \frac{\partial}{\partial y}\left[\left(\frac{\kappa}{\rho c_v} + l(x) l_\theta(x)\left|\frac{\partial \overline{v}_x}{\partial y}\right|\right)\frac{\partial \overline{T}}{\partial y}\right] \tag{4.2.2}$$

and

$$\frac{\partial \overline{v}_x}{\partial x} + \frac{\partial \overline{v}_y}{\partial y} = 0. \tag{4.2.3}$$

The jet is symmetrical about the centreline, $y = 0$. Therefore, it suffices to find the solution for the coupled system of governing partial differential equations (4.2.1), (4.2.2) and (4.2.3) for the $y > 0$ region. For this region (see Figure 4.1) the mean velocity gradient is negative, that is

$$\frac{\partial \overline{v}_x}{\partial y} < 0, \text{ when } y > 0. \tag{4.2.4}$$

The solution obtained for the $y > 0$ region can be reflected about the centreline, $y = 0$, in order to obtain the solution for the $y < 0$ region.

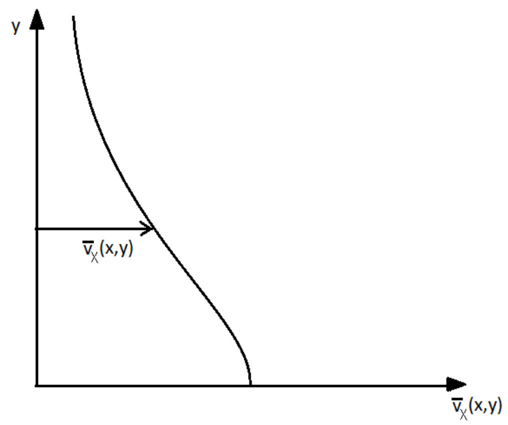

FIGURE 4.1: Schematic diagram of the mean velocity profile of the free jet.

We introduce the stream function ψ defined by

$$\overline{v}_x = \frac{\partial \psi}{\partial y}, \qquad \overline{v}_y = -\frac{\partial \psi}{\partial x}, \tag{4.2.5}$$

which identically satisfies the continuity equation (4.2.3).

The momentum boundary conditions for the two-dimensional turbulent thermal jet are (3.4.4),

$$\frac{\partial \overline{v}_x}{\partial y}(x,0) = 0, \quad \overline{v}_x(x, y_B(x)) = 0, \quad \frac{\partial \overline{v}_x}{\partial y}(x, y_B(x)) = 0, \quad \overline{v}_y(x,0) = 0, \tag{4.2.6}$$

where $y_B(x)$ is the boundary of the jet in the region $y > 0$. Rewriting the boundary conditions (4.2.6) in terms of the stream function (4.2.5) we obtain

$$\frac{\partial^2 \psi}{\partial y^2}(x,0) = 0, \quad \frac{\partial \psi}{\partial y}(x, y_B(x)) = 0, \quad \frac{\partial^2 \psi}{\partial y^2}(x, y_B(x)) = 0, \quad \frac{\partial \psi}{\partial x}(x,0) = 0. \tag{4.2.7}$$

We let

$$S = \overline{T}(x,y) - \overline{T}(x, y_{Tb}(x)), \tag{4.2.8}$$

where $y_{Tb}(x)$ is the boundary of jet in the $y > 0$ region. Writing the thermal boundary conditions (3.4.5) for the two-dimensional turbulent thermal jet in terms of S we obtain

$$\frac{\partial S}{\partial y}(x,0) = 0, \qquad \frac{\partial S}{\partial y}(x, y_{Tb}(x)) = 0, \qquad S(x, y_{Tb}(x)) = 0. \tag{4.2.9}$$

Substituting equations (4.2.5) and (4.2.8) into (4.2.1) and (4.2.2) we obtain the third-order partial differential equation

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial}{\partial y}\left[\left(\nu - l^2(x)\frac{\partial^2 \psi}{\partial y^2}\right)\frac{\partial^2 \psi}{\partial y^2}\right] \tag{4.2.10}$$

and

$$\frac{\partial \psi}{\partial y}\frac{\partial S}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial S}{\partial y} = \frac{\partial}{\partial y}\left[\left(\frac{\kappa}{\rho c_v} - l(x) l_\theta(x)\frac{\partial^2 \psi}{\partial y^2}\right)\frac{\partial S}{\partial y}\right] \tag{4.2.11}$$

as the governing equations in terms of the stream function.

When working with Lie point symmetries and conservation laws, x, y, ψ, S and all the partial derivatives of ψ and S with respect to x and y are regarded as independent variables and the suffix notation is used for partial differentiation. Thus,

$$x, y, \psi, S, \psi_x, \psi_y, S_x, S_y, \psi_{xx}, \psi_{xy}, \psi_{yy}, S_{xx}, S_{xy}, S_{yy}, ...$$

are independent variables. The notation $\frac{\partial \psi}{\partial x}$, $\frac{\partial^2 S}{\partial x \partial y}$,... is used when ψ and S and the partial derivatives of ψ and S are regarded as functions of independent variables x and y. Expanding equations (4.2.10) and (4.2.11) and writing them in terms of partial derivatives with the subscript notation, we obtain

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} - \nu \psi_{yyy} + 2l^2(x)\psi_{yy}\psi_{yyy} = 0 \tag{4.2.12}$$

and

$$\psi_y S_x - \psi_x S_y - \frac{\kappa}{\rho c_v} S_{yy} + l(x) l_\theta(x) \psi_{yy} S_{yy} + l(x) l_\theta(x) \psi_{yyy} S_y = 0. \tag{4.2.13}$$

In this section, we seek conservation laws for the system (4.2.10) and (4.2.11), which are defined by

$$D_x T^1 + D_y T^2 \Big|_{(4.2.10),(4.2.11)} = 0, \tag{4.2.14}$$

where

$$D_x = \frac{\partial}{\partial x} + \psi_x \frac{\partial}{\partial \psi} + S_x \frac{\partial}{\partial S} + \psi_{xx} \frac{\partial}{\partial \psi_x} + S_{xx} \frac{\partial}{\partial S_x} + \psi_{xy} \frac{\partial}{\partial \psi_y} + S_{xy} \frac{\partial}{\partial S_y} + ..., \tag{4.2.15}$$

$$D_y = \frac{\partial}{\partial y} + \psi_y \frac{\partial}{\partial \psi} + S_y \frac{\partial}{\partial S} + \psi_{yy} \frac{\partial}{\partial \psi_y} + S_{yy} \frac{\partial}{\partial S_y} + \psi_{xy} \frac{\partial}{\partial \psi_x} + S_{xy} \frac{\partial}{\partial S_x} + ... \tag{4.2.16}$$

are total derivatives with respect to x and y, respectively, and T^1, T^2 are the components of the conserved vector (T^1, T^2). Using the multiplier method, we need to find scalars Λ_1 and Λ_2 such that

$$\begin{aligned} &\Lambda_1 \left(\psi_y \psi_{xy} - \psi_x \psi_{yy} - \nu \psi_{yyy} + 2l^2(x) \psi_{yy} \psi_{yyy} \right) + \\ &\Lambda_2 (\psi_y S_x - \psi_x S_y - \frac{\kappa}{\rho c_v} S_{yy} + l(x) l_\theta(x) \psi_{yy} S_{yy} + l(x) l_\theta(x) \psi_{yyy} S_y) = D_x T^1 + D_y T^2, \end{aligned} \tag{4.2.17}$$

for all functions $\psi(x,y)$ and $S(x,y)$ where

$$\Lambda_1 = \Lambda_1(x, y, \psi, S), \tag{4.2.18}$$

$$\Lambda_2 = \Lambda_2(x, y, \psi, S). \tag{4.2.19}$$

When the standard Euler operators defined by

$$E_\psi = \frac{\partial}{\partial \psi} - D_x \frac{\partial}{\partial \psi_x} - D_y \frac{\partial}{\partial \psi_y} + D_x^2 \frac{\partial}{\partial \psi_{xx}} + D_x D_y \frac{\partial}{\partial \psi_{xy}} + D_y^2 \frac{\partial}{\partial \psi_{yy}} - D_y^3 \frac{\partial}{\partial \psi_{yyy}} - ... \tag{4.2.20}$$

and

$$E_S = \frac{\partial}{\partial S} - D_x \frac{\partial}{\partial S_x} - D_y \frac{\partial}{\partial S_y} + D_x^2 \frac{\partial}{\partial S_{xx}} + D_x D_y \frac{\partial}{\partial S_{xy}} + D_y^2 \frac{\partial}{\partial S_{yy}} - D_y^3 \frac{\partial}{\partial S_{yyy}} - ..., \tag{4.2.21}$$

are applied to equation (4.2.17) the right hand side vanishes. Hence, by applying the Euler operators (4.2.20) and (4.2.21) on equation (4.2.17), we obtain

$$\begin{aligned} &E_\psi(\Lambda_1 \left(\psi_y \psi_{xy} - \psi_x \psi_{yy} - \nu \psi_{yyy} + 2l^2(x) \psi_{yy} \psi_{yyy} \right) + \\ &\Lambda_2 (\psi_y S_x - \psi_x S_y - \frac{\kappa}{\rho c_v} S_{yy} + l(x) l_\theta(x) \psi_{yy} S_{yy} + l(x) l_\theta(x) \psi_{yyy} S_y)) = 0 \end{aligned} \tag{4.2.22}$$

and

$$\begin{aligned} &E_S(\Lambda_1 \left(\psi_y \psi_{xy} - \psi_x \psi_{yy} - \nu \psi_{yyy} + 2l^2(x) \psi_{yy} \psi_{yyy} \right) + \\ &\Lambda_2 (\psi_y S_x - \psi_x S_y - \frac{\kappa}{\rho c_v} S_{yy} + l(x) l_\theta(x) \psi_{yy} S_{yy} + l(x) l_\theta(x) \psi_{yyy} S_y)) = 0. \end{aligned} \tag{4.2.23}$$

Expanding equation (4.2.23) we obtain

$$\begin{gathered}
\frac{\partial \Lambda_1}{\partial S}\psi_y\psi_{xy} - \frac{\partial \Lambda_1}{\partial S}\psi_x\psi_{yy} - \nu\frac{\partial \Lambda_1}{\partial S}\psi_{yyy} + 2l^2(x)\frac{\partial \Lambda_1}{\partial S}\psi_{yy}\psi_{yyy} - \frac{\kappa}{\rho c_v}\frac{\partial \Lambda_2}{\partial S}S_{yy} + \\
l(x)l_\theta(x)\frac{\partial \Lambda_2}{\partial S}\psi_{yy}S_{yy} + 2l(x)l_\theta(x)\frac{\partial \Lambda_2}{\partial S}\psi_{yyy}S_y - \frac{\partial \Lambda_2}{\partial x}\psi_y + \frac{\partial \Lambda_2}{\partial y}\psi_x + l(x)l_\theta(x)\frac{\partial \Lambda_2}{\partial y}\psi_{yyy} \\
+l(x)l_\theta(x)\frac{\partial \Lambda_2}{\partial \psi}\psi_y\psi_{yyy} - \frac{\kappa}{\rho c_v}\frac{\partial^2 \Lambda_2}{\partial y^2} - 2\frac{\kappa}{\rho c_v}\frac{\partial^2 \Lambda_2}{\partial y \partial S}S_y - 2\frac{\kappa}{\rho c_v}\frac{\partial^2 \Lambda_2}{\partial y \partial \psi}\psi_y - \frac{\kappa}{\rho c_v}\frac{\partial^2 \Lambda_2}{\partial S^2}(S_y)^2 \\
-\frac{\kappa}{\rho c_v}\frac{\partial^2 \Lambda_2}{\partial S \partial \psi}S_y\psi_y - \frac{\kappa}{\rho c_v}\frac{\partial^2 \Lambda_2}{\partial \psi^2}(\psi_y)^2 + l(x)l_\theta(x)\frac{\partial^2 \Lambda_2}{\partial y^2}\psi_{yy} + 2l(x)l_\theta(x)\frac{\partial^2 \Lambda_2}{\partial S \partial y}S_y\psi_{yy} \\
+2l(x)l_\theta(x)\frac{\partial^2 \Lambda_2}{\partial \psi \partial y}\psi_y\psi_{yy} + 2l(x)l_\theta(x)\frac{\partial^2 \Lambda_2}{\partial S \partial \psi}S_y\psi_y\psi_{yy} + \frac{\partial^2 \Lambda_2}{\partial S^2}(S_y)^2\psi_{yy} \\
+\frac{\partial^2 \Lambda_2}{\partial \psi^2}(\psi_y)^2\psi_{yy} = 0.
\end{gathered} \tag{4.2.24}$$

The multipliers Λ_1 and Λ_2 are independent of all partial derivatives of ψ and S. Equation (4.2.24) is a polynomial in partial derivatives of ψ and S, which means this equation may be split into coefficient equations. This leads to

$$\psi_y\psi_{xy} : \frac{\partial \Lambda_1}{\partial S} = 0, \tag{4.2.25}$$

$$\psi_y\psi_{yyy} : \frac{\partial \Lambda_2}{\partial \psi} = 0, \tag{4.2.26}$$

$$\psi_y : \frac{\partial \Lambda_2}{\partial x} = 0, \tag{4.2.27}$$

$$\psi_x : \frac{\partial \Lambda_2}{\partial y} = 0, \tag{4.2.28}$$

$$\psi_{yy}S_{yy} : \frac{\partial \Lambda_2}{\partial S} = 0. \tag{4.2.29}$$

The system of partial differential equations (4.2.25) to (4.2.29) gives

$$\Lambda_1 = \Lambda_1(x, y, \psi), \tag{4.2.30}$$

$$\Lambda_2 = c_2, \tag{4.2.31}$$

where c_2 is a constant. It can be verified that the coefficients of the other partial derivatives and the remainder in (4.2.24) are zero when (4.2.30) and (4.2.31) are satisfied.

Expanding equation (4.2.22) we obtain

$$\begin{gathered}
\frac{\partial \Lambda_1}{\partial \psi}\psi_y\psi_{xy} - \frac{\partial \Lambda_1}{\partial \psi}\psi_x\psi_{yy} - \nu\frac{\partial \Lambda_1}{\partial \psi}\psi_{yyy} + 2l^2(x)\frac{\partial \Lambda_1}{\partial \psi}\psi_{yy}\psi_{yyy} \\
-2\psi_{xy}D_y\Lambda_1 - \psi_x D_y^2\Lambda_1 + \psi_y D_x D_y\Lambda_1 + \psi_{yy}D_x\Lambda_1 + \nu D_y^3\Lambda_1 - 2l^2(x)\psi_{yy}D_y^3\Lambda_1 \\
-4l^2(x)D_y^2\Lambda_1\psi_{yyy} - 2l^2(x)\psi_{yy}D_y^3\Lambda_1 - 2l^2(x)D_y\Lambda_1\psi_{yyyy} = 0.
\end{gathered} \tag{4.2.32}$$

The last term of equation (4.2.32) is the only term which depend on ψ_{yyyy}. Hence, by taking $\frac{\partial}{\partial \psi_{yyyy}}$ on equation (4.2.32) we obtain

$$D_y \Lambda_1 = 0, \tag{4.2.33}$$

which expands to

$$\frac{\partial \Lambda_1}{\partial y}(x, y, \psi) + \frac{\partial \Lambda_1}{\partial \psi}(x, y, \psi)\, \psi_y = 0. \tag{4.2.34}$$

Splitting equation (4.2.34) into coefficients of ψ_y and the remainder we obtain

$$\frac{\partial \Lambda_1}{\partial y} = \frac{\partial \Lambda_1}{\partial \psi} = 0, \tag{4.2.35}$$

which gives

$$\Lambda_1 = \Lambda_1(x). \tag{4.2.36}$$

Substituting (4.2.36) into (4.2.32) equation (4.2.32) reduces to

$$\frac{\partial \Lambda_1}{\partial x} = 0, \tag{4.2.37}$$

which gives

$$\Lambda_1 = c_1, \tag{4.2.38}$$

where c_1 is a constant.

Substituting equations (4.2.31) and (4.2.38) into (4.2.17) we obtain

$$\begin{aligned}
&c_1 \left(\psi_y \psi_{xy} - \psi_x \psi_{yy} - \nu \psi_{yyy} + 2l^2(x)\psi_{yy}\psi_{yyy}\right) + \\
&c_2 \left(\psi_y S_x - \psi_x S_y - \frac{\kappa}{\rho c_v} S_{yy} + l(x) l_\theta(x) \psi_{yy} S_{yy} + l(x) l_\theta(x) \psi_{yyy} S_y\right) \\
&= D_x \left[c_1 \psi_y^2 + c_2 \psi_y S\right] \\
&+ D_y \left[c_1 \left(-\psi_x \psi_y - \nu \psi_{yy} + l^2(x)\psi_{yy}^2\right) + c_2 \left(-\frac{\kappa}{\rho c_v} S_y - \psi_x S + l(x) l_\theta(x) \psi_{yy} S_y\right)\right]
\end{aligned} \tag{4.2.39}$$

for arbitrary $\psi(x, y)$ and $S(x, y)$. When $\psi(x, y)$ and $S(x, y)$ are solutions of the system of equations (4.2.10) and (4.2.11) then (4.2.39) reduces to

$$\begin{aligned}
&D_x \left[c_1 \psi_y^2 + c_2 \psi_y S\right] + D_y \left[c_1 \left(-\psi_x \psi_y - \nu \psi_{yy} + l^2(x)\psi_{yy}^2\right)\right. \\
&\left.\left. + c_2 \left(-\frac{\kappa}{\rho c_v} S_y - \psi_x S + l(x) l_\theta(x) \psi_{yy} S_y\right)\right]\right|_{(4.2.10),(4.2.11)} = 0.
\end{aligned} \tag{4.2.40}$$

Thus any conserved vectors of the system of partial differential equations (4.2.10) and (4.2.11) with multipliers $\Lambda_1(x, y, \psi, S)$ and $\Lambda_2(x, y, \psi, S)$ is a linear combination of the following two conserved vectors:

$c_1 = 1$ **and** $c_2 = 0$**:**

$$T^1 = \left(\psi_y\right)^2, \tag{4.2.41}$$

$$T^2 = -\psi_x \psi_y - \nu \psi_{yy} + l^2(x) \left(\psi_{yy}\right)^2 \tag{4.2.42}$$

$c_1 = 0$ **and** $c_2 = 1$**:**

$$T^1 = \psi_y S, \tag{4.2.43}$$

$$T^2 = -\frac{\kappa}{\rho c_v} S_y - \psi_x S + l(x) l_\theta(x) \psi_{yy} S_y. \tag{4.2.44}$$

4.3 Conserved quantities

Having derived conservation laws in the previous section, we are now able to find conserved quantities by integrating the conservation laws with respect to y from $y = 0$ to $y = y_B(x)$ or $y_{Tb}(x)$.

4.3.1 Conserved quantity derived from the momentum flux conserved vector (4.2.41) and (4.2.42)

Firstly, we deal with the case when $c_1 = 1$ and $c_2 = 0$. The conserved vector components for this case are

$$T^1 = \left(\psi_y\right)^2, \tag{4.3.1}$$

$$T^2 = -\psi_x \psi_y - \nu \psi_{yy} + l^2(x) \left(\psi_{yy}\right)^2. \tag{4.3.2}$$

We now treat T^1 and T^2 as functions of the independent variables x and y. Then

$$D_x T^1 + D_y T^2 = \frac{\partial T^1}{\partial x}(x, y) + \frac{\partial T^2}{\partial y}(x, y). \tag{4.3.3}$$

Therefore conserved vectors with x and y regarded as independent variables satisfy

$$\frac{\partial T^1}{\partial x}(x, y) + \frac{\partial T^2}{\partial y}(x, y) = 0. \tag{4.3.4}$$

Hence the conserved vector with components (4.3.1) and (4.3.2) satisfies

$$\frac{\partial}{\partial x}\left[\left(\frac{\partial \psi}{\partial y}\right)^2\right] + \frac{\partial}{\partial y}\left[-\frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial y} - \nu \frac{\partial^2 \psi}{\partial y^2} + l^2(x)\left(\frac{\partial^2 \psi}{\partial y^2}\right)^2\right] = 0. \tag{4.3.5}$$

Integrating equation (4.3.5) with respect to y from the centreline of the jet to the boundary, $y_B(x)$, we obtain

$$\int_0^{y_B(x)} \frac{\partial}{\partial x}\left[\left(\frac{\partial \psi}{\partial y}\right)^2\right] dy + \left(-\frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial y} - \nu \frac{\partial^2 \psi}{\partial y^2} + l^2(x)\left(\frac{\partial^2 \psi}{\partial y^2}\right)^2\right)\Bigg|_0^{y_B(x)} = 0. \tag{4.3.6}$$

Using boundary conditions (4.2.7) of the jet all terms except the first of equation (4.3.6) vanish. Hence, we obtain

$$\int_0^{y_B(x)} \frac{\partial}{\partial x}\left[\left(\frac{\partial \psi}{\partial y}\right)^2\right] dy = 0. \tag{4.3.7}$$

Using differentiation under the integral sign [19], equation (4.3.7) expands to

$$\frac{d}{dx}\int_0^{y_B(x)} \left(\frac{\partial \psi}{\partial y}\right)^2 dy - \frac{dy_B}{dx}\left(\frac{\partial \psi}{\partial y}(x, y_B(x))\right)^2 = 0, \tag{4.3.8}$$

which, when imposing boundary conditions (4.2.7), becomes

$$\frac{d}{dx}\int_0^{y_B(x)}\left(\frac{\partial\psi}{\partial y}\right)^2 dy = 0 \tag{4.3.9}$$

and yields the conserved quantity

$$J = 2\rho\int_0^{y_B(x)}\left(\frac{\partial\psi}{\partial y}(x,y)\right)^2 dy = \text{constant independent of } x, \tag{4.3.10}$$

which is the total momentum flux along the jet.

4.3.2 Conserved quantity derived from thermal conserved vector (4.2.43) and (4.2.44)

Now, we turn to the case when $c_1 = 0$ and $c_2 = 1$. The conserved vector components for this case are

$$T^1 = \psi_y S, \tag{4.3.11}$$

$$T^2 = l_\theta(x)l(x)\psi_{yy}S_y - \frac{\kappa}{\rho c_v}S_y - \psi_x S. \tag{4.3.12}$$

The conservation law (4.3.4) is then

$$\frac{\partial}{\partial x}\left(S\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(l_\theta(x)l(x)\frac{\partial^2\psi}{\partial y^2}\frac{\partial S}{\partial y} - \frac{\kappa}{\rho c_v}\frac{\partial S}{\partial y} - S\frac{\partial\psi}{\partial x}\right) = 0. \tag{4.3.13}$$

Integrating equation (4.3.13) with respect to y from zero to the thermal boundary $y_{Tb}(x)$ we obtain

$$\int_0^{y_{Tb}(x)}\frac{\partial}{\partial x}\left(S\frac{\partial\psi}{\partial y}\right)dy + \left(l_\theta(x)l(x)\frac{\partial^2\psi}{\partial y^2}\frac{\partial S}{\partial y} - \frac{\kappa}{\rho c_v}\frac{\partial S}{\partial y} - S\frac{\partial\psi}{\partial x}\right)\Bigg|_{y=0}^{y=y_{Tb}(x)} = 0. \tag{4.3.14}$$

Using boundary conditions (4.2.7) and (4.2.9) the terms outside the integral sign in equation (4.3.14) vanish. Hence,

$$\int_0^{y_{Tb}(x)}\frac{\partial}{\partial x}\left(S\frac{\partial\psi}{\partial y}\right)dy = 0. \tag{4.3.15}$$

The partial derivative with respect to x inside the integral sign can be moved outside by using the differentiation under the integral sign formula [19] to obtain

$$\frac{d}{dx}\int_0^{y_{Tb}(x)} S\frac{\partial\psi}{\partial y}dy - \frac{dy_{Tb}}{dx}\frac{\partial\psi}{\partial y}(x,y_{Tb}(x))S(x,y_{Tb}(x)) = 0. \tag{4.3.16}$$

Using boundary conditions (4.2.7) the second term of equation (4.3.16) vanishes. Hence,

$$\frac{d}{dx}\int_0^{y_{Tb}(x)} S\frac{\partial\psi}{\partial y}dy = 0. \tag{4.3.17}$$

Therefore

$$K = 2\rho c_v\int_0^{y_{Tb}(x)}\frac{\partial\psi}{\partial y}(x,y)S(x,y)dy = \text{constant independent of } x. \tag{4.3.18}$$

The constant K is the power per unit length of the jet.

4.4 Associated Lie point symmetry

In this section, we find the Lie point symmetry of the coupled system of governing partial differential equations (4.2.10) and (4.2.11), which is associated with the conserved vectors derived in Section 4.2.

A Lie point symmetry of the system (4.2.10) and (4.2.11) is of the form

$$X = \xi^1(x,y,\psi,S)\frac{\partial}{\partial x} + \xi^2(x,y,\psi,S)\frac{\partial}{\partial y} + \eta^1(x,y,\psi,S)\frac{\partial}{\partial \psi} + \eta^2(x,y,\psi,S)\frac{\partial}{\partial S}. \quad (4.4.1)$$

It is associated with the conserved vector $T = (T^1, T^2)$ provided

$$X^{[\infty]}(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i) = 0, \quad (4.4.2)$$

where the prolongation $X^{[\infty]}$ is defined by

$$\begin{aligned} X^{[\infty]} = \xi^i\frac{\partial}{\partial x_i} + \eta^1\frac{\partial}{\partial \psi} + \eta^2\frac{\partial}{\partial S} + \zeta_i^1\frac{\partial}{\partial \psi_{x_i}} + \zeta_i^2\frac{\partial}{\partial S_{x_i}} + \zeta_{11}^1\frac{\partial}{\partial \psi_{x_1x_1}} + \zeta_{12}^1\frac{\partial}{\partial \psi_{x_1x_2}} \\ +\zeta_{22}^1\frac{\partial}{\partial \psi_{x_2x_2}} + \zeta_{11}^2\frac{\partial}{\partial S_{x_1x_1}} + \zeta_{12}^2\frac{\partial}{\partial S_{x_1x_2}} + \zeta_{22}^2\frac{\partial}{\partial S_{x_2x_2}}\dots \end{aligned} \quad (4.4.3)$$

where

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \quad (4.4.4)$$

$$\zeta_{ij}^\alpha = D_i(\zeta_j^\alpha) - u_{il}^\alpha D_j(\xi^l) \quad (4.4.5)$$

and

$$u^1 = \psi, \qquad u^2 = S, \qquad x_1 = x, \qquad x_2 = y. \quad (4.4.6)$$

4.4.1 Associated with the momentum flux conserved vector (4.2.41) and (4.2.42)

We begin by considering the conserved vector (T^1, T^2), from (4.2.41) and (4.2.42), given by

$$T^1 = \psi_y^2, \quad (4.4.7)$$

$$T^2 = -\psi_x\psi_y - \nu\psi_{yy} + l^2(x)\psi_{yy}^2. \quad (4.4.8)$$

Equation (4.4.2), with $i = 1$, becomes

$$X^{[\infty]}(T^1) + T^1 D_y\xi^2 - T^2 D_y\xi^1 = 0. \quad (4.4.9)$$

Expanding equation (4.4.9) we obtain

$$2\psi_y\zeta_y^1 + (\psi_y)^2 D_y\xi^2 + \left(\psi_x\psi_y + \nu\psi_{yy} - l^2(x)(\psi_{yy})^2\right) D_y\xi^1 = 0. \quad (4.4.10)$$

Substituting the prolongation term ζ_y^1 defined by (4.4.4) into equation (4.4.10) we obtain

$$\begin{aligned} 2\psi_y\left(\frac{\partial\eta^1}{\partial y} + \psi_y\frac{\partial\eta^1}{\partial\psi} + S_y\frac{\partial\eta^1}{\partial S} - \psi_x D_y\xi^1 - \psi_y D_y\xi^2\right) + \\ (\psi_y)^2 D_y\xi^2 + (\psi_x\psi_y + \nu\psi_{yy} - l^2(\psi_{yy})^2) D_y\xi^1 = 0, \end{aligned} \quad (4.4.11)$$

which expands to the determining equation

$$2\frac{\partial\eta^1}{\partial y}\psi_y + \left(2\frac{\partial\eta^1}{\partial\psi} - \frac{\partial\xi^2}{\partial y}\right)\psi_y^2 + 2\frac{\partial\eta^1}{\partial S}\psi_y S_y - \frac{\partial\xi^1}{\partial y}\psi_x\psi_y - \frac{\partial\xi^1}{\partial\psi}\psi_x(\psi_y)^2 - \frac{\partial\xi^1}{\partial S}\psi_x\psi_y S_y - \frac{\partial\xi^2}{\partial\psi}(\psi_y)^3 - \frac{\partial\xi^2}{\partial S}(\psi_y)^2 S_y + \nu\frac{\partial\xi^1}{\partial y}\psi_{yy} + \nu\frac{\partial\xi^1}{\partial\psi}\psi_y\psi_{yy} + \nu\frac{\partial\xi^1}{\partial S}S_y\psi_{yy} - l^2(x)\frac{\partial\xi^1}{\partial y}(\psi_{yy})^2 - l^2(x)\frac{\partial\xi^1}{\partial\psi}\psi_y(\psi_{yy})^2 - l^2(x)\frac{\partial\xi^1}{\partial S}S_y(\psi_{yy})^2 = 0. \tag{4.4.12}$$

Splitting equation (4.4.12) into the coefficients of the independent partial derivatives of ψ and S we obtain

$$\psi_y : \frac{\partial\eta^1}{\partial y} = 0, \tag{4.4.13}$$

$$(\psi_y)^2 : 2\frac{\partial\eta^1}{\partial\psi} - \frac{\partial\xi^2}{\partial y} = 0, \tag{4.4.14}$$

$$S_y : \frac{\partial\eta^1}{\partial S} = 0, \tag{4.4.15}$$

$$\psi_x\psi_y : \frac{\partial\xi^1}{\partial y} = 0, \tag{4.4.16}$$

$$\psi_x(\psi_y)^2 : \frac{\partial\xi^1}{\partial\psi} = 0, \tag{4.4.17}$$

$$\psi_x\psi_y S_y : \frac{\partial\xi^1}{\partial S} = 0, \tag{4.4.18}$$

$$\psi_y^3 : \frac{\partial\xi^2}{\partial\psi} = 0, \tag{4.4.19}$$

$$S_y\psi_y^2 : \frac{\partial\xi^2}{\partial S} = 0. \tag{4.4.20}$$

When (4.4.13) to (4.4.20) are satisfied the coefficients of the remaining partial differential derivatives are zero. From equations (4.4.13), (4.4.15) to (4.4.20) we obtain

$$\xi^1 = \xi^1(x), \tag{4.4.21}$$

$$\xi^2 = \xi^2(x,y), \tag{4.4.22}$$

$$\eta^1 = \eta^1(x,\psi). \tag{4.4.23}$$

Differentiating (4.4.14) with respect to y and using the fact that η^1 is independent of y by (4.4.23) we obtain

$$\frac{\partial^2\xi^2}{\partial y^2} = 0, \tag{4.4.24}$$

which gives

$$\xi^2 = a_1(x)y + a_2(x), \tag{4.4.25}$$

where $a_1(x)$ and $a_2(x)$ are arbitrary functions of x. Differentiating equation (4.4.14) with respect to ψ and using the fact that ξ^2 is independent of ψ by (4.4.22) we obtain

$$\frac{\partial^2\eta^1}{\partial\psi^2} = 0, \tag{4.4.26}$$

which gives

$$\eta^1 = a_3(x)\psi + a_4(x), \tag{4.4.27}$$

where $a_3(x)$ and $a_4(x)$ are arbitrary functions of x. Substituting equations (4.4.25) and (4.4.27) into (4.4.14) we obtain

$$a_3(x) = \frac{1}{2}a_1(x). \tag{4.4.28}$$

Therefore,

$$\eta^1 = \frac{1}{2}a_1(x)\psi + a_4(x). \tag{4.4.29}$$

In summary, the first determining equations therefore gives, for $\nu \neq 0$ and $\nu = 0$,

$$\xi^1 = \xi^1(x), \qquad \xi^2(x,y) = a_1(x)y + a_2(x), \tag{4.4.30}$$

$$\eta^1(x,\psi) = \frac{1}{2}a_1(x)\psi + a_4(x), \tag{4.4.31}$$

$$\eta^2 = \eta^2(x,y,\psi,S). \tag{4.4.32}$$

Equation (4.4.2), with $i = 2$, becomes

$$X^{[\infty]}(T^2) + T^2 D_x\xi^1 - T^1 D_x\xi^2 = 0. \tag{4.4.33}$$

Now expanding equation (4.4.33) we obtain

$$\begin{aligned} &-\frac{1}{2}a_1'(x)\psi\psi_y - a_4'(x)\psi_y - \nu\left(-\frac{3}{2}a_1(x) + \frac{d\xi^1}{dx}\right)\psi_{yy} + \\ &\left(-3l^2(x)a_1(x) + l^2(x)\frac{d\xi^1}{dx} + 2l(x)\frac{dl}{dx}(x)\xi^1\right)(\psi_{yy})^2 = 0. \end{aligned} \tag{4.4.34}$$

Splitting equation (4.4.34) into coefficients of derivatives of ψ we obtain

$$\psi_y : \frac{1}{2}a_1'(x)\psi + a_4'(x) = 0, \tag{4.4.35}$$

$$\psi_{yy} : \nu\left(-\frac{3}{2}a_1(x) + \frac{d\xi^1}{dx}\right) = 0, \tag{4.4.36}$$

$$(\psi_{yy})^2 : -3l^2(x)a_1(x) + l^2(x)\frac{d\xi^1}{dx} + 2l(x)\frac{dl}{dx}(x)\xi^1 = 0. \tag{4.4.37}$$

Splitting equation (4.4.35) into coefficients of ψ and the remainder we obtain

$$\psi : a_1'(x) = 0, \tag{4.4.38}$$

$$\text{Remainder} : a_4'(x) = 0. \tag{4.4.39}$$

Therefore a_1 and a_4 are constants. Thus for $\nu \neq 0$ and $\nu = 0$,

$$\xi^1 = \xi^1(x), \tag{4.4.40}$$

$$\xi^2 = a_1 y + a_2(x), \tag{4.4.41}$$

$$\eta^1 = \frac{1}{2}a_1\psi + a_4, \tag{4.4.42}$$

$$\eta^2 = \eta^2(x, y, \psi, S), \tag{4.4.43}$$

$$\frac{d\xi^1}{dx} + \frac{2}{l(x)}\frac{dl}{dx}(x)\xi^1(x) = 3a_1, \tag{4.4.44}$$

$$\nu\left(-\frac{3}{2}a_1 + \frac{d\xi^1}{dx}\right) = 0. \tag{4.4.45}$$

Case (i): $\nu \neq 0$

When $\nu \neq 0$ equation (4.4.45) becomes

$$-\frac{3}{2}a_1 + \frac{d\xi^1}{dx} = 0, \tag{4.4.46}$$

which gives

$$\xi^1 = \frac{3}{2}a_1 x + a_5, \tag{4.4.47}$$

where a_5 is a constant. Substituting equation (4.4.47) into (4.4.44) we obtain

$$2\frac{dl}{dx}\left(\frac{3}{2}a_1 x + a_5\right) = \frac{3}{2}a_1 l(x), \tag{4.4.48}$$

which gives

$$l(x) = l_0\sqrt{x + \frac{2}{3}\frac{a_5}{a_1}}, \tag{4.4.49}$$

where l_0 is a constant. The flow is laminar close to the nozzle, meaning that the mixing length at $x = 0$ is zero. Hence

$$a_5 = 0, \qquad l(x) = l_0\sqrt{x} \tag{4.4.50}$$

and the Lie point symmetry associated with the conserved vector described by (4.2.41) and (4.2.42), with $\nu \neq 0$, is

$$X = \frac{3}{2}a_1 x\frac{\partial}{\partial x} + (a_1 y + a_2(x))\frac{\partial}{\partial y} + \left(\frac{1}{2}\psi + a_4\right)\frac{\partial}{\partial \psi} + \eta^2(x, y, \psi, S)\frac{\partial}{\partial S}. \tag{4.4.51}$$

Case (ii): $\nu = 0$

When $\nu = 0$, equation (4.4.45) is identically satisfied and ξ^1 remains an arbitrary function of x. Hence, the Lie point symmetry associated with the conserved vector described by (4.2.41) and (4.2.42), with $\nu = 0$, becomes

$$X = \xi^1(x)\frac{\partial}{\partial x} + (a_1 y + a_2(x))\frac{\partial}{\partial y} + \left(\frac{1}{2}a_1\psi + a_4\right)\frac{\partial}{\partial \psi} + \eta^2(x, y, \psi, S)\frac{\partial}{\partial S} \tag{4.4.52}$$

and $\xi^1(x)$ and $l(x)$ are related by (4.4.44).

4.4.2 Association with thermal conserved vector (4.2.43) and (4.2.44)

In the previous subsection we saw that ξ^1 remained an arbitrary function of x or $\frac{3}{2}a_1 x$ depending on whether ν is zero or non-zero. To accommodate both cases, we leave ξ^1 as an arbitrary function of x.

Expanding equation (4.4.2), with $i = 1$, using the conserved vector described by (4.2.43) and (4.2.44) and the associated Lie point symmetry (4.4.51) we obtain

$$\psi_y \eta^2 + S\zeta_y^1 + a_1 S\psi_y = 0, \tag{4.4.53}$$

which simplifies to

$$\left(\eta^2 + \frac{1}{2}a_1 S\right)\psi_y = 0. \tag{4.4.54}$$

Therefore

$$\eta^2 = -\frac{1}{2}a_1 S. \tag{4.4.55}$$

The associated Lie point symmetry then becomes

$$X = \xi^1(x)\frac{\partial}{\partial x} + (a_1 y + a_2(x))\frac{\partial}{\partial y} + \left(\frac{1}{2}a_1\psi + a_4\right)\frac{\partial}{\partial \psi} - \frac{1}{2}a_1 S\frac{\partial}{\partial S}. \tag{4.4.56}$$

Expanding equation (4.4.2), with $i = 2$, using the conserved vector described by (4.2.43) and (4.2.44) and the associated Lie point symmetry (4.4.56) we obtain

$$\begin{aligned}
&\xi^1 \frac{d}{dx}\left(l(x)l_\theta(x)\right)\psi_{yy}S_y + \frac{1}{2}a_1 S\psi_x - S\zeta_x^1 + \left(-\frac{\kappa}{\rho c_v} + l(x)l_\theta(x)\psi_{yy}\right)\zeta_y^2 \\
&+l(x)l_\theta(x)S_y\zeta_{yy}^1 + \frac{d\xi^1}{dx}\left(-\frac{\kappa}{\rho c_v}S_y - S\psi_x + l(x)l_\theta(x)\psi_{yy}S_y\right) - \frac{da_2}{dx}S\psi_y = 0,
\end{aligned} \tag{4.4.57}$$

which simplifies to

$$\begin{aligned}
&\frac{\kappa}{\rho c_v}\left(\frac{3}{2}a_1 - \frac{d\xi^1}{dx}\right)S_y \\
&+\left(\frac{d\xi^1}{dx}l(x)l_\theta(x) - 3a_1 l(x)l_\theta(x) + \xi^1(x)\frac{d}{dx}\left(l(x)l_\theta(x)\right)\right)\psi_{yy}S_y = 0.
\end{aligned} \tag{4.4.58}$$

Splitting equation (4.4.58) into coefficients of S_y and $\psi_{yy}S_y$ we obtain

$$S_y : \frac{\kappa}{\rho c_v}\left(\frac{3}{2}a_1 - \frac{d\xi^1}{dx}\right) = 0, \tag{4.4.59}$$

$$\psi_{yy}S_y : \frac{d\xi^1}{dx}l(x)l_\theta(x) - 3a_1 l(x)l_\theta(x) + \xi^1(x)\frac{d}{dx}\left(l(x)l_\theta(x)\right) = 0. \tag{4.4.60}$$

Case (i): $\nu \neq 0$ **and** $\kappa \neq 0$

From equation (4.4.51) for $\nu \neq 0$, we have

$$\xi^1 = \frac{3}{2}a_1 x \tag{4.4.61}$$

and (4.4.59) is therefore identically satisfied and from (4.4.50)

$$l(x) = l_0\sqrt{x}. \tag{4.4.62}$$

We assume that $a_1 \neq 0$. When $a_1 = 0$, we might have another reduction, but this is not considered in this dissertation. Substituting equations (4.4.61) and (4.4.62) into

(4.4.60) we obtain

$$\frac{d}{dx}\left(\sqrt{x}l_\theta(x)\right) = \frac{l_\theta(x)}{\sqrt{x}}, \tag{4.4.63}$$

which gives

$$l_\theta(x) = l_1\sqrt{x}, \tag{4.4.64}$$

where l_1 is a constant. The associated Lie point symmetry generator (4.4.51) then becomes

$$X = \frac{3}{2}a_1 x\frac{\partial}{\partial x} + (a_1 y + a_2(x))\frac{\partial}{\partial y} + \left(\frac{1}{2}a_1\psi + a_4\right)\frac{\partial}{\partial \psi} - \frac{1}{2}a_1 S\frac{\partial}{\partial S}. \tag{4.4.65}$$

Case (ii): $\nu \neq 0$ **and** $\kappa = 0$

From equation (4.4.51) we have

$$\xi^1 = \frac{3}{2}a_1 x \tag{4.4.66}$$

and from (4.4.50)

$$l(x) = l_0\sqrt{x}. \tag{4.4.67}$$

We assume that $a_1 \neq 0$. When $a_1 = 0$, we might have another reduction, but this is not considered in this dissertation. Since $\kappa = 0$, equation (4.4.59) is identically satisfied. The mixing length $l(x)$ and ξ^1 are the same as in Case (i). Hence, the calculation for $l_\theta(x)$ remains valid and

$$l_\theta(x) = l_1\sqrt{x}, \tag{4.4.68}$$

where l_1 is a constant.

Case (iii): $\nu = 0$ **and** $\kappa \neq 0$

When $\kappa \neq 0$, equation (4.4.59) becomes

$$\frac{3}{2}a_1 - \frac{d\xi^1}{dx} = 0. \tag{4.4.69}$$

Solving the ordinary differential equation (4.4.69) we obtain

$$\xi^1 = \frac{3}{2}a_1 x + a_5, \tag{4.4.70}$$

where a_5 is a constant. We assume that $a_1 \neq 0$. When $a_1 = 0$, we might have another reduction, but this is not considered in this dissertation. Substituting equation (4.4.70) into the ordinary differential equation (4.4.37) we obtain

$$2\frac{dl}{dx}\left(\frac{3}{2}a_1 x + a_5\right) = \frac{3}{2}a_1 l(x), \tag{4.4.71}$$

which gives

$$l(x) = l_0\sqrt{x + \frac{2}{3}\frac{a_5}{a_1}}, \tag{4.4.72}$$

where l_0 is a constant. The mixing length must be zero at $x = 0$, since the jet flow is laminar close to the nozzle. Hence

$$l(x) = l_0\sqrt{x}, \tag{4.4.73}$$

$$a_5 = 0 \tag{4.4.74}$$

and

$$\xi^1 = \frac{3}{2}a_1 x. \tag{4.4.75}$$

The mixing length $l(x)$ and ξ^1 are the same as in Case (i). Hence, the calculation for l_θ remains valid and

$$l_\theta(x) = l_1\sqrt{x}, \tag{4.4.76}$$

where l_1 is a constant.

Case (iv): $\nu = 0$ **and** $\kappa = 0$

When $\kappa = 0$ equation (4.4.59) is identically satisfied and ξ^1 remains an arbitrary function of x. Hence, the Lie point symmetry, with $\nu = 0$ and $\kappa = 0$, remains (4.4.51):

$$X = \xi^1(x)\frac{\partial}{\partial x} + (a_1 y + a_2(x))\frac{\partial}{\partial y} + \left(\frac{1}{2}a_1\psi + a_4\right)\frac{\partial}{\partial \psi} - \frac{1}{2}a_1 S\frac{\partial}{\partial S} \tag{4.4.77}$$

and $l(x)$ and $l_\theta(x)$ satisfy (4.4.44) and (4.4.60),

$$\frac{d\xi^1}{dz} + \frac{2}{l(x)}\frac{dl}{dx}\xi^1(x) = 3a_1, \tag{4.4.78}$$

$$\xi^1(x)\frac{d}{dx}\left(l(x)l_\theta(x)\right) + \left(\frac{d\xi^1}{dx} - 3a_1\right)l(x)l_\theta(x) = 0. \tag{4.4.79}$$

We assume that $a_1 \neq 0$. When $a_1 = 0$, we might have another reduction, but this is not considered in this dissertation. Now replacing $\frac{d\xi^1}{dz}$ in (4.4.79) using (4.4.78), equation (4.4.79) reduces to

$$\frac{1}{l_\theta(x)}\frac{dl_\theta}{dx} = \frac{1}{l(x)}\frac{dl}{dx} \tag{4.4.80}$$

and therefore

$$l_\theta(x) = Dl(x), \tag{4.4.81}$$

where D is a constant.

4.5 Invariant solution

In this section we derive the invariant solution for the coupled system of governing partial differential equations in terms of the stream function (4.2.10) and (4.2.11) using the Lie point symmetry generators (4.4.65) and (4.4.77).

Case (i): ν **and/or** κ **non-zero**

The solutions $\psi = f(x,y)$ and $S = g(x,y)$ are invariant if

$$X(\psi - f(x,y))\Big|_{\psi=f(x,y)} = 0 \tag{4.5.1}$$

and

$$X(S-g(x,y))\Big|_{S=g(x,y)} = 0. \tag{4.5.2}$$

Expanding equation (4.5.1) using the Lie point symmetry generator (4.4.65) we obtain

$$\frac{3}{2}a_1 x\frac{\partial f}{\partial x} + (a_1 y + a_2(x))\frac{\partial f}{\partial y} = (\frac{1}{2}a_1 f + a_4). \tag{4.5.3}$$

The characteristic equations for the differential equation (4.5.3) are

$$\frac{dx}{\frac{3}{2}a_1 x} = \frac{dy}{(a_1 y + a_2(x))} = \frac{df}{\frac{1}{2}a_1 f + a_4}. \tag{4.5.4}$$

The equation consisting of the first and second terms of (4.5.4) gives

$$\frac{dy}{dx} - \frac{2}{3x}y = \frac{2}{3a_1}\frac{a_2(x)}{x}. \tag{4.5.5}$$

The integrating factor for this differential equation is

$$I = \exp\left[-\frac{2}{3}\int^x \frac{dx}{x}\right] = x^{-2/3}. \tag{4.5.6}$$

Thus

$$\frac{d}{dx}\left(x^{-2/3}y\right) = \frac{2}{3a_1}a_2(x)x^{-5/3} \tag{4.5.7}$$

and

$$\frac{y - \frac{2}{3}a_1 x^{2/3}\int^x a_2(x)x^{-5/3}dx}{x^{2/3}} = k_1, \tag{4.5.8}$$

where k_1 is a constant. Let

$$H(x) = \frac{2}{3}a_1 x^{2/3}\int^x a_2(x)x^{-5/3}dx. \tag{4.5.9}$$

Thus

$$\frac{y - H(x)}{x^{2/3}} = k_1. \tag{4.5.10}$$

The pair equation consisting of the first and third terms of equation (4.5.4) gives

$$\frac{f + 2\frac{a_4}{a_1}}{x^{1/3}} = k_2, \tag{4.5.11}$$

where k_2 is a constant. The general solution for the differential equation (4.5.3) is then

$$k_2 = F(k_1), \tag{4.5.12}$$

where F is an arbitrary function. Since $f = \psi(x,y)$

$$\psi(x,y) = x^{1/3}F(\xi) - 2\frac{a_4}{a_1}, \tag{4.5.13}$$

where

$$\xi = \frac{y - H(x)}{x^{2/3}}. \tag{4.5.14}$$

Substituting the invariant solution (4.5.13) and (4.5.14) into the conserved quantity (4.3.10) we obtain

$$J = 2\rho \int_{-\frac{H(x)}{x^{2/3}}}^{\frac{y-H(x)}{x^{2/3}}} \left(\frac{dF}{d\xi}\right)^2 d\xi \tag{4.5.15}$$

and J is independent of x if

$$\text{lower limit: } \frac{H(x)}{x^{2/3}} = \frac{2}{3}a_1 \int^x a_2(x)x^{-5/3}dx = k, \tag{4.5.16}$$

$$\text{upper limit: } \frac{y_B(x)}{x^{2/3}} - k = \beta_1, \tag{4.5.17}$$

But (4.5.16) is satisfied for all values of x provided $a_2(x) = 0$. Thus $H(x) = 0$ and $k = 0$. Hence

$$J = 2\rho \int_0^{\beta_1} \left(\frac{dF}{d\xi}\right)^2 d\xi \tag{4.5.18}$$

and the boundary of the jet and the similarity variable are

$$y_B(x) = \beta_1 x^{2/3}, \qquad \xi = \frac{y}{x^{2/3}}. \tag{4.5.19}$$

Now substituting the invariant solution (4.5.13) with ξ given by (4.5.19) and the mixing length (4.4.50) into equation (4.2.10) we obtain

$$\frac{d}{d\xi}\left[\left(\nu - l_0^2\frac{d^2F}{d\xi^2}\right)\frac{d^2F}{d\xi^2}\right] + \frac{1}{3}\frac{d}{d\xi}\left(F\frac{dF}{d\xi}\right) = 0. \tag{4.5.20}$$

Expanding equation (4.5.2) using the Lie point symmetry generator (4.4.65) with $a_2(x) = 0$ we obtain

$$\frac{3}{2}x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = -\frac{1}{2}g. \tag{4.5.21}$$

The differential equations of the characteristic curves for this equation are

$$\frac{dx}{\frac{3}{2}x} = \frac{dy}{y} = \frac{dg}{-\frac{1}{2}g}. \tag{4.5.22}$$

The equation consisting of the first and second terms of (4.5.22) gives

$$\frac{y}{x^{\frac{3}{2}}} = k_3, \tag{4.5.23}$$

where k_1 is a constant. The equation consisting of the first and third terms of (4.5.22) gives

$$x^{1/3}g = k_4, \tag{4.5.24}$$

where k_4 is a constant. Therefore the general solution of the partial differential equation (4.5.21) is, since $g = S(x,y)$,

$$S(x,y) = x^{-\frac{1}{3}}G(\xi), \qquad \xi = \frac{y}{x^{2/3}}, \tag{4.5.25}$$

where G is an arbitrary function. Substituting invariant solutions (4.5.25) and (4.5.13) into equation (4.2.11) with (4.4.62) for $l(x)$ and (4.4.64) for $l_\theta(x)$ we obtain

$$\frac{d}{d\xi}\left[\left(\frac{\kappa}{\rho c_v} - l_0 l_1 \frac{d^2F}{d\xi^2}\right)\frac{dG}{d\xi}\right] + \frac{1}{3}\frac{d}{d\xi}\left(FG\right) = 0. \tag{4.5.26}$$

Expressed in terms of the invariant solution the conserved quantity (4.3.18) becomes

$$K = 2\rho c_v \int_0^{\frac{y_{Tb}(x)}{x^{2/3}}} \frac{dF}{d\xi} G(\xi) d\xi = \text{ constant independent of } x. \tag{4.5.27}$$

Hence

$$\frac{y_{Tb}(x)}{x^{2/3}} = \beta_2 \tag{4.5.28}$$

where β_2 is a constant and therefore

$$K = 2\rho c_v \int_0^{\beta_2} \frac{dF}{d\xi} G(\xi) d\xi. \tag{4.5.29}$$

Writing boundary conditions (4.2.7) and (4.2.9) in terms of the invariant solutions (4.5.25) and (4.5.13) gives

$$\frac{d^2F}{d\xi^2}(0) = 0, \qquad F(0) = 0, \qquad \frac{d^2F}{d\xi^2}(\beta_1) = 0, \qquad \frac{dF}{d\xi}(\beta_1) = 0 \tag{4.5.30}$$

$$\frac{dG}{d\xi}(0) = 0, \qquad \frac{dG}{d\xi}(\beta_2) = 0, \qquad G(\beta_2) = 0, \tag{4.5.31}$$

where β_2 is a constant defined by (4.5.28).

Case (ii): ν and κ both zero

When $\nu = 0$ and $\kappa = 0$ the associated Lie point symmetry is (4.4.77) and $\xi^1(x)$, $l(x)$ and $l_\theta(x)$ are related by (4.4.78) and (4.4.81).

The solution $\psi = f(x, y)$ is an invariant solution provided

$$X\left(\psi - f(x,y)\right)\Bigg|_{\psi = f(x,y)} = 0. \tag{4.5.32}$$

Expanding this equation using the Lie point symmetry generator (4.4.77) we obtain

$$\xi^1(x)\frac{\partial f}{\partial x} + \left(a_1 y + a_2(x)\right)\frac{\partial f}{\partial y} = \frac{1}{2}a_1 f + a_4. \tag{4.5.33}$$

The differential equations of the characteristic curves are

$$\frac{dx}{\xi^1(x)} = \frac{dy}{a_1 y + a_2(x)} = \frac{df}{\frac{1}{2}a_1 f + a_4}. \tag{4.5.34}$$

Solving the equation consisting of the first and second terms of (4.5.34) we obtain

$$\frac{dy}{dx} - \frac{a_1}{\xi^1(x)} y = \frac{a_2(x)}{\xi^1(x)}, \tag{4.5.35}$$

which gives

$$\frac{y}{\exp\left(a_1 \int^x \frac{dx}{\xi^1(x)}\right)} - H(x) = k_1, \tag{4.5.36}$$

where k_1 is a constant and

$$H(x) = \int^x \frac{a_2(x)}{\xi^1(x)} \exp\left(-a_1 \int^x \frac{dx}{\xi^1(x)}\right) dx. \tag{4.5.37}$$

The equation consisting of the first and third terms of (4.5.34) gives

$$\frac{f + 2\frac{a_4}{a_1}}{\exp\left(\frac{1}{2} a_1 \int^x \frac{dx}{\xi^1(x)}\right)} = k_2, \tag{4.5.38}$$

where k_2 is a constant. The general solution is

$$k_2 = F(k_1), \tag{4.5.39}$$

where F is an arbitrary function, which gives, since $f = \psi(x, y)$,

$$\psi(x, y) = \exp\left(\frac{1}{2} a_1 \int^x \frac{dx}{\xi^1(x)}\right) F(\xi) - 2\frac{a_4}{a_1}, \tag{4.5.40}$$

where

$$\xi = \frac{y}{\exp\left(a_1 \int^x \frac{dx}{\xi^1(x)}\right)} - H(x). \tag{4.5.41}$$

Substituting the invariant solution (4.5.40) into the conserved quantity (4.3.10) we obtain

$$J = 2\rho \int_{-H(x)}^{\beta_1(x) - H(x)} \left(\frac{dF}{d\xi}\right)^2 d\xi, \tag{4.5.42}$$

where

$$\beta_1(x) = \frac{y_B(x)}{\exp\left(a_1 \int^x \frac{dx}{\xi^1(x)}\right)}. \tag{4.5.43}$$

Now J is independent of x provided the lower and upper limits in the integral in (4.5.42) are constant, that is provided

$$H(x) = H_0, \tag{4.5.44}$$

$$\beta_1(x) = \beta_1, \tag{4.5.45}$$

where H_0 and β_1 are constants. Differentiating (4.5.37) with respect to x gives

$$\frac{dH}{dx} = \frac{a_2(x)}{\xi^1(x)} \exp\left(-a_1 \int^x \frac{dx}{\xi^1(x)}\right) = 0, \tag{4.5.46}$$

provided

$$a_2(x) = 0. \tag{4.5.47}$$

Hence $H(x) = 0$ and from (4.5.42)

$$J = 2\rho \int_0^{\beta_1} \left(\frac{dF}{d\xi}\right)^2 d\xi \tag{4.5.48}$$

and from (4.5.43)

$$y_B(x) = \beta_1 \exp\left(a_1 \int^x \frac{dx}{\xi^1(x)}\right). \tag{4.5.49}$$

Also, $S = g(x, y)$ is an invariant solutions generated by the Lie point symmetry X provided that

$$X\left(S - g(x, y)\right)\bigg|_{S=g(x,y)} = 0, \tag{4.5.50}$$

which, using (4.4.77) with $a_2(x) = 0$, expands to

$$\xi^1(x)\frac{\partial g}{\partial x} + a_1 y\frac{\partial g}{\partial y} = -\frac{1}{2}g. \tag{4.5.51}$$

The differential equations of the characteristic curves are

$$\frac{dx}{\xi^1(x)} = \frac{dy}{a_1 y} = \frac{dg}{-a_1\frac{1}{2}g}. \tag{4.5.52}$$

The first pair of terms gives

$$\frac{y}{\exp\left(a_1 \int^x \frac{dx}{\xi^1(x)}\right)} = k_3, \tag{4.5.53}$$

where k_3 is a constant. The first and last terms give

$$g\exp\left(\frac{1}{2}a_1 \int^x \frac{dx}{\xi^1(x)}\right) = k_4. \tag{4.5.54}$$

The general solution is

$$k_4 = G(k_3), \tag{4.5.55}$$

where G is an arbitrary function. Thus since $g = S(x, y)$

$$S(x, y) = \left(-\frac{1}{2}a_1 \int^x \frac{dx}{\xi^1(x)}\right) G(\xi). \tag{4.5.56}$$

The invariant solution still contains the arbitrary function $\xi^1(x)$ and $l(x)$ and $l_\theta(x)$ which are still to be determined. The three functions are related by the two equations (4.4.78) and (4.4.81). In order to obtain a third equation we use Prandtl's hypothesis [18], which relates the boundary of the jet and the momentum length by

$$y_B(x) = Cl(x), \tag{4.5.57}$$

where C is a constant. Therefore,

$$\beta_1 \exp\left(a_1 \int^x \frac{dx}{\xi^1(x)}\right) = Cl(x). \tag{4.5.58}$$

Differentiating (4.5.58) with respect to x gives

$$\beta_1 \frac{a_1}{\xi^1(x)} \exp\left(a_1 \int^x \frac{dx}{\xi^1(x)}\right) = C\frac{dl}{dx} \tag{4.5.59}$$

Eliminating C from (4.5.59) using (4.5.58) gives

$$\frac{dl}{dx} = a_1 \frac{l(x)}{\xi^1(x)}. \tag{4.5.60}$$

Substituting (4.5.60) into (4.4.78) gives

$$a_1 = \frac{\partial \xi^1}{\partial x} \tag{4.5.61}$$

and therefore

$$\xi^1(x) = a_1 x + a_5, \tag{4.5.62}$$

where a_5 is a constant. Substituting (4.5.62) into (4.5.58) gives

$$l(x) = l_0 \left(x + \frac{a_5}{a_1} \right), \tag{4.5.63}$$

where $l_0 = \frac{\beta_1 a_1}{C}$ is a constant. The mixing length is zero at $x = 0$, since the jet flow is laminar close to the nozzle. Therefore

$$l(x) = l_0 x, \qquad \xi^1(x) = a_1 x \tag{4.5.64}$$

and, by (4.4.81),

$$l_\theta = l_1 x, \tag{4.5.65}$$

where $l_1 = D l_0$ is a constant. The Lie point symmetry generator (4.4.77) on dividing through by a_1 becomes

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \left(\frac{1}{2}\psi + a_4 \right) \frac{\partial}{\partial \psi} - \frac{1}{2} S \frac{\partial}{\partial S}. \tag{4.5.66}$$

The invariant solutions (4.5.40) and (4.5.56) become

$$\psi(x, y) = x^{1/2} F(\xi) - 2\frac{a_4}{a_1}, \tag{4.5.67}$$

$$S(x, y) = x^{-1/2} G(\xi) \tag{4.5.68}$$

where from (4.5.41),

$$\xi = \frac{y}{x}. \tag{4.5.69}$$

From (4.2.5) we see that the fluid velocity depends only on the spatial derivative of $\psi(x, y)$. We can therefore take $a_4 = 0$. Substituting the invariant solutions (4.5.67) and (4.5.68) into the partial differential equations (4.2.10) and (4.2.11), with $\nu = 0$ and $\kappa = 0$, gives

$$l_0^2 \frac{d}{d\xi} \left[\left(\frac{d^2 F}{d\xi^2} \right)^2 \right] - \frac{1}{2} \frac{d}{d\xi} \left(F \frac{dF}{d\xi} \right) = 0, \tag{4.5.70}$$

$$l_0 l_1 \frac{d}{d\xi} \left[\frac{d^2 F}{d\xi^2} \frac{dG}{d\xi} \right] - \frac{1}{2} \frac{d}{d\xi} (FG) = 0. \tag{4.5.71}$$

Expressed in terms of the invariant solutions the thermal conserved quantity (4.3.18) becomes

$$K = 2\rho c_v \int_0^{\frac{y_{Tb}(x)}{x}} G(\xi)\frac{dF}{d\xi}d\xi = \text{ constant independent of x} \tag{4.5.72}$$

and therefore

$$\frac{y_{Tb}(x)}{x} = \beta_2 \tag{4.5.73}$$

where β_2 is a constant. Hence

$$y_{Tb}(x) = \beta_2 x, \tag{4.5.74}$$

$$K = 2\rho c_v \int_0^{\beta_2} G(\xi)\frac{dF}{d\xi}d\xi. \tag{4.5.75}$$

Writing boundary conditions (4.2.7) and (4.2.9) in terms of the invariant solution (4.5.67) and (4.5.68) gives (4.5.30) and (4.5.31).

4.6 Analytical solution with $\kappa \to 0$ and $\nu \to 0$

Integrating (4.5.20) and utilising boundary conditions (4.5.30) at either $\xi = 0$ or $\xi = \beta_1$ in order to obtain the constant of integration gives

$$\left(\nu - l_0^2 \frac{d^2F}{d\xi^2}\right)\frac{d^2F}{d\xi^2} + \frac{1}{3}F\frac{dF}{d\xi} = 0 \tag{4.6.1}$$

and integrating (4.5.26) and using the boundary conditions (4.5.30) at $\xi = \beta_1$ to obtain the constant of integration we find that

$$\left(\frac{\kappa}{\rho c_v} - l_0 l_1 \frac{d^2F}{d\xi^2}\right)\frac{dG}{d\xi} + \frac{1}{3}FG = 0. \tag{4.6.2}$$

This is an example of the double reduction theorem of Sjoberg [10,11] which states that if a partial differential equation is reduced to an ordinary differential equation using an associated Lie point symmetry with a conserved vector of the partial differential equation then the ordinary differential equation can be integrated at least once.

Letting $\nu \to 0$ in (4.6.1) gives

$$\left(\frac{d^2F}{d\xi^2}\right)^2 = \frac{1}{3l_0^2}F\frac{dF}{d\xi}. \tag{4.6.3}$$

Now

$$\overline{v}_x = \frac{\partial\psi}{\partial y} = x^{-1/3}\frac{dF}{d\xi} > 0 \tag{4.6.4}$$

and taking the square root on both sides of equation (4.6.3), we obtain

$$\frac{d^2F}{d\xi^2} = \pm\frac{1}{\sqrt{3}l_0}F^{1/2}\left(\frac{dF}{d\xi}\right)^{1/2}. \tag{4.6.5}$$

But since we are considering the upper half of the jet

$$\frac{\partial \overline{v}_x}{\partial y} = \frac{1}{x}\frac{d^2F}{d\xi^2} < 0 \tag{4.6.6}$$

and the negative sign is taken in (4.6.5). Thus,

$$\frac{d^2F}{d\xi^2} = -\frac{1}{\sqrt{3}l_0}F^{1/2}\left(\frac{dF}{d\xi}\right)^{1/2}. \tag{4.6.7}$$

Equation (4.6.7) is an autonomous second order differential equation. Multiplying both sides by $\left(\frac{dF}{d\xi}\right)^{1/2}$ and integrating with respect to ξ gives

$$\int^{\xi}\left(\frac{dF}{d\xi}\right)^{1/2}\frac{d^2F}{d\xi^2}d\xi = -\frac{1}{\sqrt{3}l_0}\int^{\xi}F^{1/2}\frac{dF}{d\xi}d\xi. \tag{4.6.8}$$

But

$$\frac{d^2F}{d\xi^2}d\xi = \frac{d}{d\xi}\frac{dF}{d\xi}d\xi = dF' \tag{4.6.9}$$

and (4.6.8) becomes

$$\int^{F'}(F')^{1/2}dF' = -\frac{1}{\sqrt{3}l_0}\int^{F}(F)^{1/2}dF. \tag{4.6.10}$$

Thus

$$\left(\frac{dF}{d\xi}\right)^{3/2} = -\frac{1}{\sqrt{3}l_0}F^{3/2} + b^*, \tag{4.6.11}$$

where b^* is the constant. Writing

$$b^* = \frac{b^{3/2}}{\sqrt{3}l_0}, \tag{4.6.12}$$

equation (4.6.11) becomes

$$\frac{dF}{d\xi} = \left(\frac{1}{\sqrt{3}l_0}\right)^{2/3}\left(b^{3/2} - F^{3/2}\right)^{2/3}, 0 \le F \le b. \tag{4.6.13}$$

Substituting the boundary condition $F'(\beta_1) = 0$ from (4.5.30) into equation (4.6.11), one verifies that

$$F(\beta_1) = b. \tag{4.6.14}$$

Equation (4.6.13) is variable separable. Thus

$$d\xi = \left(\sqrt{3}l_0\right)^{2/3}\frac{dF}{(b^{3/2} - F^{3/2})^{2/3}} \tag{4.6.15}$$

and therefore

$$\int_0^{\xi}d\xi = \left(\sqrt{3}l_0\right)^{2/3}\int_{F(0)}^{F}\frac{dF}{(b^{3/2} - F^{3/2})^{2/3}} \tag{4.6.16}$$

and using the boundary condition $F(0) = 0$ from (4.5.30) we find that

$$\xi = \left(\sqrt{3}l_0\right)^{2/3} \int_0^F \frac{dF}{(b^{3/2} - F^{3/2})^{2/3}},\ 0 \leq F \leq b, \tag{4.6.17}$$

which becomes

$$\xi = (\sqrt{3}l_0)^{2/3} \int_0^f \frac{df}{(1 - f^{3/2})^{2/3}}, \tag{4.6.18}$$

where

$$f = \frac{F(\xi)}{b},\ 0 \leq f \leq 1, \tag{4.6.19}$$

which can be rewritten as

$$F(\xi) = bf,\ 0 \leq f \leq 1. \tag{4.6.20}$$

Equations (4.6.18) and (4.6.20) form a parametric solution for $F(\xi)$ where the parameter is f with $0 \leq f \leq 1$. For each value of f in the range, $0 \leq f \leq 1$, there is a value for F and ξ.

The constant b is obtained from the conserved quantity J given by equation (4.5.18) which can be written as

$$J = 2\rho \int_{F(0)}^{F(\beta_1)} \frac{dF}{d\xi} dF, \tag{4.6.21}$$

where $F(\beta_1) = b$ and, from the boundary condition (4.5.30), $F(0) = 0$. Substituting $\frac{dF}{d\xi}$ from equation (4.6.13) into (4.6.21) we obtain

$$J = \frac{2\rho}{(\sqrt{3}l_0)^{2/3}} \int_0^b (b^{3/2} - F^{3/2})^{2/3} dF \tag{4.6.22}$$

and writing equation (4.6.22) in terms of the variable f, we obtain

$$J = \frac{2\rho}{(\sqrt{3}l_0)^{2/3}} b^2 \int_0^1 (1 - f^{3/2})^{2/3} df. \tag{4.6.23}$$

But

$$\int_0^1 (1 - f^{3/2})^{2/3} df = \frac{2}{3} \int_0^1 x^{-1/3} (1 - x)^{2/3} dx, \tag{4.6.24}$$

where $x = f^{3/2}$. Equation (4.6.24) can be expressed in terms of beta and gamma functions defined by [19]

$$B(m, n) = \int_0^1 x^m (1 - x)^n dx,\ n > 0, m > 0 \tag{4.6.25}$$

and

$$\Gamma(n) = \int_0^1 x^{n-1} e^{-x} dx,\ n > 0. \tag{4.6.26}$$

The gamma function satisfies

$$\Gamma(n) = (n - 1)\Gamma(n - 1),\ n > 1. \tag{4.6.27}$$

Beta and gamma functions are related by

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}. \tag{4.6.28}$$

Using these results it can be shown that

$$\int_0^1 (1 - f^{3/2})^{2/3} df = \frac{(\Gamma\left(\frac{2}{3}\right))^2}{\Gamma\left(\frac{1}{3}\right)} \tag{4.6.29}$$

and hence from (4.6.23),

$$b = \left(\frac{J\Gamma\left(\frac{1}{3}\right)}{2\rho}\right)^{1/2} \frac{1}{\Gamma\left(\frac{2}{3}\right)} \left(\sqrt{3}l_0\right)^{1/3} \tag{4.6.30}$$

The constant β_1 which determines the boundary of the jet,

$$y_B(x) = \beta_1 x^{2/3}, \tag{4.6.31}$$

is obtained with $f = 1$ and $\xi = \beta_1$. Equation (4.6.18) becomes

$$\beta_1 = (\sqrt{3}l_0)^{2/3} \int_0^1 \frac{df}{(1 - f^{3/2})^{2/3}}. \tag{4.6.32}$$

Making the transformation $x = f^{3/2}$, we obtain

$$\beta_1 = \frac{2}{3}(\sqrt{3}l_0)^{2/3} \int_0^1 x^{-1/3}(1-x)^{-2/3} dx = \frac{2}{3}(\sqrt{3}l_0)^{2/3} B\left(\frac{1}{3}, \frac{2}{3}\right), \tag{4.6.33}$$

where $B(\frac{1}{3}, \frac{2}{3})$ is a beta function. The integral in (4.6.33) is convergent. Hence the jet is bounded in the y-direction. But

$$B\left(\frac{1}{3}, \frac{2}{3}\right) = \frac{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}{\Gamma(1)} = \frac{2\pi}{\sqrt{3}}, \tag{4.6.34}$$

where we used

$$\Gamma(1) = 1, \qquad \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(p\pi)}, \qquad 0 < p < 1. \tag{4.6.35}$$

Thus,

$$\beta_1 = \frac{4\sqrt{3}}{9}\pi(\sqrt{3}l_0)^{2/3}. \tag{4.6.36}$$

We now turn our attention to G which satisfies equation (4.6.2). We let $\kappa \to 0$ in (4.6.2) which reduces to

$$\frac{d^2F}{d\xi^2}\frac{dG}{d\xi} = \frac{1}{3l_0 l_1} FG \tag{4.6.37}$$

and using (4.6.7) for $F''(\xi)$ we obtain

$$\frac{1}{G}\frac{dG}{d\xi} = -\frac{1}{\sqrt{3}l_1} \frac{F^{1/2}}{\left(\frac{dF}{d\xi}\right)^{1/2}}. \tag{4.6.38}$$

Equation (4.6.38) is written in variable separate form by first writing it as

$$\frac{dG}{G} = -\frac{1}{\sqrt{3}l_1}\frac{F^{1/2}}{\left(\frac{dF}{d\xi}\right)^{3/2}}dF \tag{4.6.39}$$

and using (4.6.13),

$$\frac{dG}{G} = -\frac{l_0}{l_1}\frac{F^{1/2}}{(b^{3/2}-F^{3/2})}dF. \tag{4.6.40}$$

Hence

$$\int_{G(0)}^{G}\frac{dG}{G} = -\frac{l_0}{l_1}\int_{F(0)}^{F}\frac{\sqrt{F}}{(b^{3/2}-F^{3/2})}dF. \tag{4.6.41}$$

Now $G(0)$ is unknown and is obtained later from the conserved quantity (4.5.75). Using the boundary condition (4.5.30), $F(0) = 0$, the integral on the right hand side of (4.6.41) can be calculated by making the substitution

$$w = b^{3/2} - F^{3/2}. \tag{4.6.42}$$

Thus

$$G = G(0)\left(1 - f^{3/2}\right)^{\frac{2}{3}\frac{l_0}{l_1}}, \tag{4.6.43}$$

where f is defined by (4.6.20). Therefore $G(0) > G(\xi) \geq 0$ for all values of ξ satisfying $0 \leq \xi < \beta_1$ and $G(\beta_1) = 0$. Thus $\beta_1 = \beta_2$. That is, the thermal and momentum boundaries are equal when $\nu = 0$ and $\kappa = 0$.

The constant $G(0)$ can be found by utilising the conserved quantity (4.5.75). Substituting equation (4.6.43) into (4.5.75) we obtain

$$K = 2\rho c_v G(0)\int_0^b G dF = 2\rho c_v b G(0)\int_0^1\left(1 - f^{3/2}\right)^{\frac{2}{3}\frac{l_0}{l_1}} df \tag{4.6.44}$$

since $F(0) = 0$ and $F(\beta_1) = b$. But

$$\int_0^1 (1 - f^{3/2})^{\frac{2}{3}\frac{l_0}{l_1}} df = \frac{2}{3}\int_0^1 x^{-1/3}(1-x)^{\frac{2}{3}\frac{l_0}{l_1}} dx = \frac{2}{3}B\left(\frac{2}{3}, 1 + \frac{2}{3}\frac{l_0}{l_1}\right) \tag{4.6.45}$$

and using (4.6.25) and (4.6.26) for beta and gamma functions we find that

$$\int_0^1 (1 - f^{3/2})^{\frac{2}{3}\frac{l_0}{l_1}} df = \frac{2}{3}\frac{\frac{l_0}{l_1}}{\left(1 + \frac{l_0}{l_1}\right)}\frac{\Gamma(\frac{2}{3})\Gamma(\frac{2}{3}\frac{l_0}{l_1})}{\Gamma\left(\frac{2}{3}(1 + \frac{l_0}{l_1})\right)} \tag{4.6.46}$$

and therefore

$$G(0) = \frac{3K}{4\rho c_v b}\frac{(1 + \frac{l_0}{l_1})}{\frac{l_0}{l_1}}\frac{\Gamma(\frac{2}{3}(1 + \frac{l_0}{l_1}))}{\Gamma(\frac{2}{3})\Gamma(\frac{2}{3}\frac{l_0}{l_1})}, \tag{4.6.47}$$

where b is given by (4.6.30).

Equations (4.6.20) and (4.6.43) form a parametric solution for $G(\xi)$ where the parameter is f with $0 \leq f \leq 1$.

We now summarise the analytical solution for $\nu \to 0$ and $\kappa \to 0$. The parametric solution for $F(\xi)$ and $G(\xi)$ is

$$F = bf, \tag{4.6.48}$$

$$G = G(0)\left(1 - f^{3/2}\right)^{\frac{2}{3}\frac{l_0}{l_1}}, \tag{4.6.49}$$

$$\xi = (\sqrt{3}l_0)^{2/3}\int_0^f \frac{dg}{(1-g^{3/2})^{2/3}}, \tag{4.6.50}$$

where the parameter, f, satisfies

$$0 \le f \le 1. \tag{4.6.51}$$

The physical variables at a given value of

$$x > 0 \qquad \text{and} \qquad 0 \le y \le \beta_1 x^{2/3} \tag{4.6.52}$$

are

$$\psi(x,y) = bx^{1/3}f, \tag{4.6.53}$$

$$S(x,y) = G(0)x^{-1/3}\left[1 - f^{3/2}\right]^{\frac{2}{3}\frac{l_0}{l_1}}, \tag{4.6.54}$$

$$\overline{v}_x(x,y) = \frac{b}{(\sqrt{3}l_0)^{2/3}}x^{-1/3}\left[1 - f^{3/2}\right]^{2/3}, \tag{4.6.55}$$

$$\overline{v}_y(x,y) = \frac{b}{3}x^{-2/3}\left[2(1-f^{3/2})^{2/3}\int_0^f \frac{dg}{(1-g^{3/2})^{2/3}} - f\right], \tag{4.6.56}$$

$$y = x^{2/3}\xi = (\sqrt{3}l_0)^{2/3}x^{2/3}\int_0^f \frac{dg}{(1-g^{3/2})^{2/3}}, \tag{4.6.57}$$

where

$$0 \le f \le 1. \tag{4.6.58}$$

The boundary of the upper half of the jet and the mixing lengths are

$$\xi = \frac{y}{x^{2/3}}, \qquad l(x) = l_0 x^{1/2}, \qquad l_\theta(x) = l_1 x^{1/2} \tag{4.6.59}$$

and

$$\beta_1 = \frac{4\sqrt{3}}{9}\pi(\sqrt{3}l_0)^{2/3}, \tag{4.6.60}$$

$$b = \left(\frac{J\Gamma\left(\frac{1}{3}\right)}{2\rho}\right)^{1/2}\frac{1}{\Gamma\left(\frac{2}{3}\right)}\left(\sqrt{3}l_0\right)^{1/3}, \tag{4.6.61}$$

$$G(0) = \frac{3K}{4\rho c_v}\left(\frac{2\rho}{J\Gamma\left(\frac{1}{3}\right)}\right)^{1/2}\left(\frac{1}{\sqrt{3}l_0}\right)^{1/3}\frac{(1+\frac{l_0}{l_1})}{\frac{l_0}{l_1}}\frac{\Gamma(\frac{2}{3}(1+\frac{l_0}{l_1}))}{\Gamma(\frac{2}{3}\frac{l_0}{l_1})}. \tag{4.6.62}$$

4.7 Analytical solution with $\nu = 0$ and $\kappa = 0$ using Prandtl's hypothesis

Inequalities (4.6.3) and (4.6.4), since $\frac{\partial \overline{v}_x}{\partial y} < 0$ and $\overline{v}_x(x,y) > 0$ for $0 < y < y_B(x)$:

$$\frac{d^2F}{d\xi^2} < 0, \; 0 < \xi < \beta_1, \tag{4.7.1}$$

$$\frac{dF}{d\xi} > 0,\ 0 < \xi < \beta_1. \tag{4.7.2}$$

remain valid even when $\nu = 0$ and $\kappa = 0$. The differential equations (4.5.70) and (4.5.71) obtained using Prandtl's hypothesis and (4.5.26) and (4.5.20) with $\nu = 0$ and $\kappa = 0$ differ only by the coefficient of $\frac{d}{d\xi}\left(F\frac{dF}{d\xi}\right)$ and $\frac{d}{d\xi}(FG)$. The boundary conditions and the two conserved quantities for the two problems are the same. Hence the two problem can be solved using the same method. Using the same approach as in Section 4.6 we find the solution of (4.5.70) to be

$$\xi = \left(\sqrt{2}l_0\right)^{2/3}\int_0^f \frac{dg}{(1-g^{3/2})^{2/3}}, \tag{4.7.3}$$

with f defined by

$$f = \frac{F}{b} \tag{4.7.4}$$

and satisfying

$$0 \le f \le 1, \tag{4.7.5}$$

where

$$b = F(\beta_1) = \left(\frac{J\Gamma\left(\frac{1}{3}\right)}{2\rho}\right)^{1/2}\frac{1}{\Gamma\left(\frac{2}{3}\right)}\left(\sqrt{2}l_0\right)^{1/3} \tag{4.7.6}$$

and

$$\beta_1 = \frac{4\sqrt{3}}{9}\pi\left(\sqrt{2}l_0\right)^{2/3}. \tag{4.7.7}$$

We also find that the solution for (4.5.71) to be

$$G(\xi) = G(0)\left(1 - f^{3/2}\right)^{\frac{2}{3}\frac{l_0}{l_1}}, \tag{4.7.8}$$

where

$$G(0) = \frac{3K}{4\rho c_v b}\frac{1+\frac{l_0}{l_1}}{\frac{l_0}{l_1}}\frac{\Gamma\left(\frac{2}{3}\left(1+\frac{l_0}{l_1}\right)\right)}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{2}{3}\frac{l_0}{l_1}\right)}. \tag{4.7.9}$$

The physical variables at a given value of

$$x > 0 \qquad \text{and} \qquad 0 \le y_B(x) \le \beta_1 x \tag{4.7.10}$$

are

$$\psi(x,y) = bx^{1/2}f, \tag{4.7.11}$$

$$S(x,y) = G(0)x^{-1/2}\left(1 - f^{3/2}\right)^{\frac{2}{3}\frac{l_0}{l_1}}, \tag{4.7.12}$$

$$\overline{v}_x(x,y) = \frac{b}{\left(\sqrt{2}l_0\right)^{2/3}}x^{-\frac{1}{2}}\left(1 - f^{3/2}\right)^{2/3}, \tag{4.7.13}$$

$$\overline{v}_y(x,y) = \frac{b}{2}x^{-1/2}\left[2\left(1-f^{3/2}\right)^{2/3}\int_0^f \frac{dg}{(1-g^{3/2})^{2/3}} - f\right]. \tag{4.7.14}$$

The boundary constant β_1 was shown to be finite when ν and κ are zero. Hence, the boundary is finite in this case. However, it will be argued in Section 4.8 that the boundary constant β_1 is infinite when $\nu \neq 0$. In this section, we derived the analytical solution by assuming Prandtl's hypothesis [18] and obtained a different solution by letting ν and κ tend to zero.

4.8 Numerical solution

In this section we give a numerical solution to the differential equations (4.6.1) and (4.6.2) subject to the boundary conditions (4.5.30) and (4.5.31) and the conserved quantities (4.5.18) and (4.5.29).

Expanding the differential equation (4.6.1) we obtain the quadratic equation

$$\left(\frac{d^2F}{d\xi^2}\right)^2 - \frac{\nu}{l_0^2}\frac{d^2F}{d\xi^2} - \frac{1}{3l_0^2}F\frac{dF}{d\xi} = 0. \tag{4.8.1}$$

Hence

$$\frac{d^2F}{d\xi^2} = \frac{\nu}{2l_0^2} \pm \sqrt{\left(\frac{\nu}{2l_0^2}\right)^2 + \frac{1}{3l_0^2}F(\xi)\frac{dF}{d\xi}}. \tag{4.8.2}$$

The is discriminant in (4.8.2) is real because

$$\overline{v}_x(x,y) = \frac{\partial\psi}{\partial y} = x^{-1/3}\frac{dF}{d\xi} > 0. \tag{4.8.3}$$

We will consider the numerical solution for the upper half of the jet. Then

$$\frac{\partial\overline{v}_x}{\partial y} = \frac{1}{x}\frac{d^2F}{d\xi^2} < 0. \tag{4.8.4}$$

We therefore take the negative sign in (4.8.2), which becomes

$$\frac{d^2F}{d\xi^2} = \frac{\nu}{2l_0^2} - \sqrt{\left(\frac{\nu}{2l_0^2}\right)^2 + \frac{1}{3l_0^2}F(\xi)\frac{dF}{d\xi}}. \tag{4.8.5}$$

From the differential equation (4.6.2) we obtain

$$\frac{dG}{d\xi} = -\frac{1}{3}\frac{F(\xi)G(\xi)}{\frac{\kappa}{\rho c_v} - l_0 l_1\frac{d^2F}{d\xi^2}}. \tag{4.8.6}$$

The denominator in (4.8.6) is non-zero because of (4.8.4).

The boundary conditions (4.5.30) and (4.5.31) are not sufficient to obtain a unique solution for the system of the ordinary equations (4.8.5) and (4.8.6). Hence, we impose conserved quantities (4.3.10) and (4.3.18), written in terms of the invariant solutions (4.5.13) and (4.5.25):

$$J = 2\rho\int_0^{\beta_1}\left(\frac{dF}{d\xi}\right)^2 d\xi, \tag{4.8.7}$$

$$K = 2\rho c_v\int_0^{\beta_2} G(\xi)\frac{dF}{d\xi}d\xi. \tag{4.8.8}$$

In order to convert this problem to an initial value problem (IVP), consider

$$F(\xi) = A\xi^n \text{ as } \xi \to 0 \tag{4.8.9}$$

where A is a constant. Now

$$\overline{v}_x(x,y) \text{ is positive and finite as } y \to 0. \tag{4.8.10}$$

Now since

$$\psi(x,y) = x^{1/3}F(\xi), \qquad \xi = \frac{y}{x^{2/3}} \tag{4.8.11}$$

we have

$$\overline{v}_x(x,y) = \frac{1}{x^{1/3}}\frac{dF}{d\xi} \qquad = \frac{nA}{x^{1/3}}\xi^{n-1} \text{ as } \xi \to 0. \tag{4.8.12}$$

Thus $n = 1$ and

$$F(\xi) = A\xi \qquad \text{as } \xi \to 0. \tag{4.8.13}$$

The boundary condition $\frac{d^2F}{d\xi^2}(0) = 0$ in (4.5.30) is satisfied.

From (4.5.31)

$$\frac{dG}{d\xi}(0) = 0, \tag{4.8.14}$$

but the boundary condition on $G(0)$ is not specified. Consider

$$G(\xi) = B\xi^m \text{ as } \xi \to 0, \tag{4.8.15}$$

where B is a constant. Thus

$$\frac{dG}{d\xi} = mB\xi^{m-1} \text{ as } \xi \to 0 \tag{4.8.16}$$

and (4.8.14) is satisfied by $m = 1$ and any $m > 1$. We choose $m = 0$ so that

$$\frac{dG}{d\xi} = B, \text{ as } \xi \to 0. \tag{4.8.17}$$

The initial value problem in the shooting method can be summarised as follows. Initial conditions

$$F(0) = 0, \qquad \frac{dF}{d\xi}(0) = A, \qquad G(0) = B. \tag{4.8.18}$$

Differential equations

$$\frac{d^2F}{d\xi^2} = \frac{\nu}{2l_0^2} - \sqrt{\left(\frac{\nu}{2l_0^2}\right)^2 + \frac{1}{3l_0^2}F(\xi)\frac{dF}{d\xi}}, \tag{4.8.19}$$

$$\frac{dG}{d\xi} = -\frac{1}{3}\frac{F(\xi)G(\xi)}{\frac{\kappa}{\rho c_v} - l_0 l_1 \frac{d^2F}{d\xi^2}}. \tag{4.8.20}$$

The targets

$$J = 2\rho \int_0^{\beta_1} \left(\frac{dF}{d\xi} \right)^2 d\xi, \tag{4.8.21}$$

$$K = 2\rho c_v \int_0^{\beta_2} \frac{dF}{d\xi} G(\xi) d\xi. \tag{4.8.22}$$

The conserved quantities J and K are prescribed and also ρ and c_v and the mixing lengths l_0 and l_1. The shooting method determines the constants A and B, the upper limits β_1 and β_2 and the solution for $F(\xi)$ and $G(\xi)$.

Initially, we guess values for constants A and B and use the solution corresponding to these constants to calculate the conserved quantities. If equations (4.8.21) and (4.8.22) for the conserved quantities are not satisfied then we modify the values of A and B and try again. This process is repeated until equations (4.8.21) and (4.8.22) for the conserved quantities are satisfied.

Let

$$F(\xi, A, B), \qquad\qquad G(\xi, A, B)$$

be the solution of the system of ordinary differential equations (4.8.19) and (4.8.20) with the initial conditions (4.8.18) and corresponding constants A and B. Assume that these solutions yield conserved quantities

$$\overline{J}(A, B) = 2\rho \int_0^{\beta_1} \left(\frac{dF}{d\xi}(\xi, A, B) \right)^2 d\xi, \tag{4.8.23}$$

$$\overline{K}(A, B) = 2\rho c_v \int_0^{\beta_2} G(\xi, A, B) \frac{dF}{d\xi}(\xi, A, B) d\xi. \tag{4.8.24}$$

We require that

$$J - \overline{J}(A, B) = 0, \tag{4.8.25}$$

$$K - \overline{K}(A, B) = 0, \tag{4.8.26}$$

that is, equations (4.8.21) and (4.8.22) for the conserved quantities are satisfied.

Suppose (A_k, B_k) is the current best guess of A and B that satisfy (4.8.25) and (4.8.26). To update (A_k, B_k) we need steps δ_1 and δ_2 which get us closer to the solution, that is

$$J - \overline{J}(A_k + \delta_1, B_k + \delta_2) = 0, \tag{4.8.27}$$

$$K - \overline{K}(A_k + \delta_1, B_k + \delta_2) = 0. \tag{4.8.28}$$

Expanding equations (4.8.27) and (4.8.28) using the Taylor series expansion we obtain

$$\begin{pmatrix} \frac{\partial \overline{J}}{\partial A}(A_k, B_k) & \frac{\partial \overline{J}}{\partial B}(A_k, B_k) \\ \\ \frac{\partial \overline{K}}{\partial A}(A_k, B_k) & \frac{\partial \overline{K}}{\partial B}(A_k, B_k) \end{pmatrix} \begin{pmatrix} \delta_1 \\ \\ \delta_2 \end{pmatrix} = \begin{pmatrix} J - \overline{J}(A_k, B_k) \\ \\ K - \overline{K}(A_k, B_k) \end{pmatrix}. \tag{4.8.29}$$

Hence, the updated guess for constants A and B become

$$A_{k+1} = A_k + \delta_1, \tag{4.8.30}$$

$$B_{k+1} = B_k + \delta_2. \tag{4.8.31}$$

We use the secant method to compute the partial derivatives in equation (4.8.29). For example:

$$\frac{\partial \overline{J}}{\partial A}(A_k, B_k) = \frac{\overline{J}(A_k, B_k) - \overline{J}(A_{k-1}, B_k)}{A_k - A_{k-1}}. \tag{4.8.32}$$

This suggests the numerical algorithm to calculate A and B:

1. Choose (A_0, B_0) and (A_1, B_1) (Initial guess).
2. Set $k = 1$.
3. Compute $F(\xi, A_k, B_k)$, $G(\xi, A_k, B_k)$, $F(\xi, A_{k-1}, B_{k-1})$ and $G(\xi, A_{k-1}, B_{k-1})$ using the system of differential equations (4.8.19) and (4.8.20) with initial conditions (4.8.18). For this we use the Runge-Kutta scheme.
4. Compute derivatives $\frac{\partial \overline{J}}{\partial A}$, $\frac{\partial \overline{J}}{\partial B}$, $\frac{\partial \overline{K}}{\partial A}$ and $\frac{\partial \overline{K}}{\partial B}$ using the secant method (4.8.32).
5. Compute steps δ_1 and δ_2 using equation (4.8.29).
6. Update $A_{k+1} = A_k + \delta_1$, $B_{k+1} = B_k + \delta_2$, $A_k = A_{k-1}$ and $B_k = B_{k-1}$.
7. If $|J - J(A_{k+1}, B_{k+1})| < \epsilon$ and $|K - K(A_{k+1}, B_{k+1})| < \epsilon$ then stop, else continue to step 3 with $k \to k+1$ (increment k), where ϵ is the tolerance of the numerical method.

Once the values of A and B are found we use the Runge-Kutta scheme to solve the resulting IVP.

Table 4.1 shows the results of the shooting method when ν and κ are equal to 0.1. It can be seen from this table that the method converges just after six iterations.

A_k	B_k	$\lvert J - \overline{J}(A_k, B_k)\rvert$	$\lvert K - \overline{K}(A_k, B_k)\rvert$
0.2	0.2	0.885520838931	0.885520768116
1.32877480037	1.99327115373	2.74136184438	4.61235055957
0.475595792714	-5.4722581264	0.464300111173	7.16382495535
0.599168119926	-3.1011766543	0.178009505226	5.25446298219
0.676002805976	2.30537871804	0.03040319706	2.51399478108
0.664794179652	0.422622679809	0.00145233503741	0.365203495493
0.665305195468	0.660684694495	$1.07933631134e-05$	0.00695534388078
0.665309021635	0.665313156217	$3.8893215315e-09$	$6.52955836777e-06$

TABLE 4.1: Shooting method results with $\nu = 0.1$ and $\kappa = 0.1$.

For the numerical results presented in Figures 4.2 to 4.8 $J = 1$, $K = 1$, $\rho = 1$, $l_0 = 1$, $l_1 = 1$, $c_v = 1$ and the values of ν and κ are varied from zero to one.

Case (i): $\nu \neq 0$ **and** $\kappa \neq 0$

It can be shown that

$$\overline{v}_x = x^{-1/3}\frac{dF}{d\xi}. \tag{4.8.33}$$

Figure 4.2 shows the numerical results for $\frac{dF}{d\xi}$ plotted against ξ for different values of kinematic viscosity, ν. As the kinematic viscosity increases the mean velocity along the centreline of the jet decreases. It can be seen that $\frac{dF}{d\xi}$ gets closer and closer to zero as ξ becomes large. When $\frac{dF}{d\xi}$ is sufficiently small we cannot reliably determine whether $\frac{dF}{d\xi}$ is zero or non-zero since we limited by the accuracy of the numerical scheme. When $\nu = 0$ and $\kappa = 0$, Figure 4.7, the value of $F(\xi)$ becomes a non-real number when ξ is greater than some value we assume to be the boundary β_1, but when $\nu \neq 0$ and $\kappa \neq 0$ $\frac{dF}{d\xi}$ remains real for all considered values of ξ. This suggests that $\frac{dF}{d\xi}$ remains positive for all values of ξ, which leads us to conclude that the momentum boundary β_1 is infinite when $\nu \neq 0$ and $\kappa \neq 0$.

It can also be shown that

$$\frac{\partial \overline{v}_x}{\partial y} = \frac{1}{x}\frac{dF}{d\xi}. \tag{4.8.34}$$

Figure 4.3 shows the numerical results for $\frac{d^2F}{d\xi^2}$ plotted against ξ for different values of ν. The region where $\frac{d^2F}{d\xi^2}$ is most negative is called the shear layer of the jet. As the kinematic viscosity increase the shear layer region becomes wider.

The temperature difference is given by

$$S(x,y) = x^{-1/3}G(\xi). \tag{4.8.35}$$

Figure 4.4 shows the numerical results for $G(\xi)$ plotted against ξ for different values of the thermal conductivity, κ, and kinematic viscosity, ν. Similar to the numerical solution for $\frac{dF}{d\xi}$, Figure 4.2, $G(\xi)$ gets closer and closer to zero as ξ increase but never becomes non-real for all considered values of ξ. This leads us to conclude that β_2 is infinite.

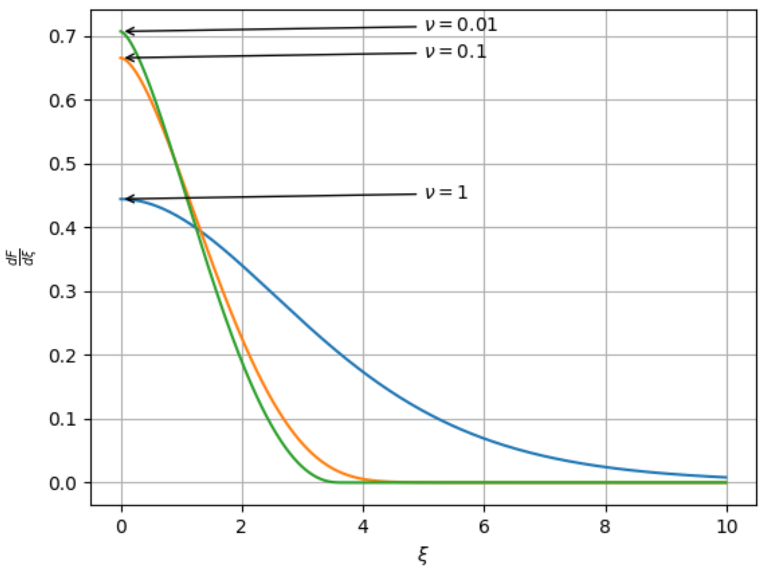

FIGURE 4.2: Graphs of $\frac{dF}{d\xi}$ versus ξ for $\nu = 0.01$, $\nu = 0.1$ and $\nu = 1$.

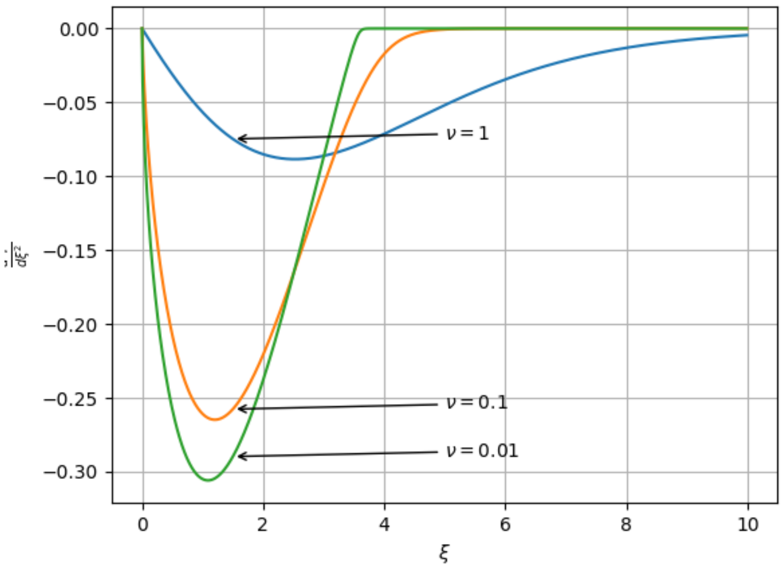

FIGURE 4.3: Graphs of $\frac{d^2F}{d\xi^2}$ versus ξ for $\nu = 0.01$, $\nu = 0.1$ and $\nu = 1$.

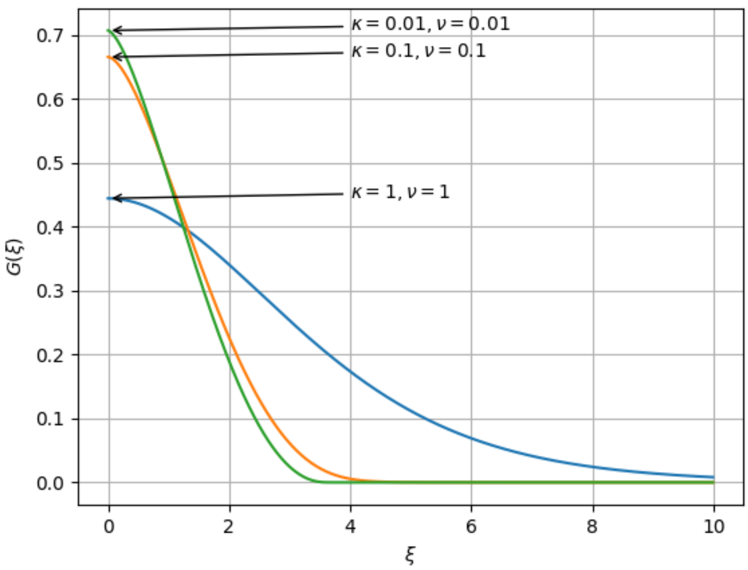

FIGURE 4.4: Graphs of $G(\xi)$ versus ξ for varying values of ν and κ.

Case (ii): $\nu \neq 0$ **and** $\kappa = 0$

Figure 4.5 shows the numerical results for $G(\xi)$ plotted against ξ when $\nu = 0.01, \kappa = 0$ and $\nu = 0.1, \kappa = 0$. The numerical results for $\frac{dF}{d\xi}$ for this case are the same as in Figure 4.2 since F is independent of G. The numerical results for $G(\xi)$ are approximately flat when ξ is between 3.2 and 4.8 which then become unstable when ξ is larger than 4.8.

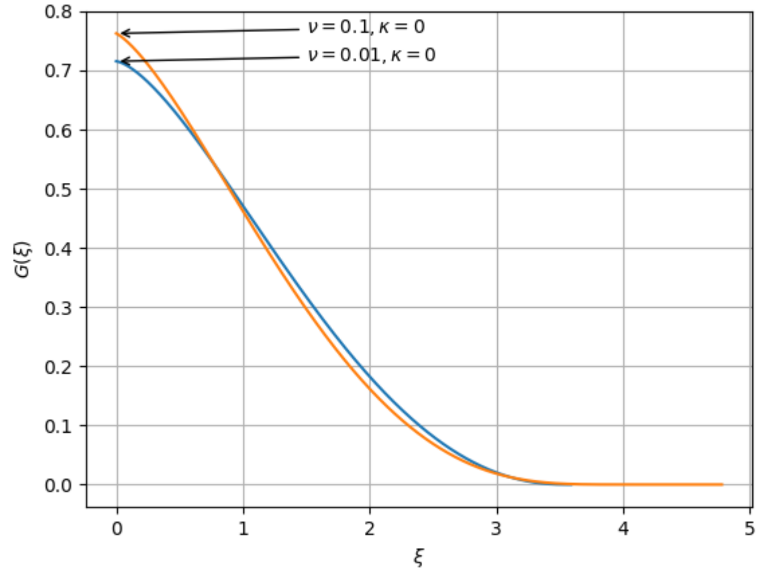

FIGURE 4.5: Graphs of $G(\xi)$ versus ξ for $\nu = 0.01$ and $\nu = 0.1$ with $\kappa = 0$.

Case (iii): $\nu = 0$ **and** $\kappa \neq 0$

When $\nu = 0$ and $\kappa \neq 0$ the numerical algorithm described above cannot accurately determine F and G. Suppose the initial guesses for A and B are 0.71 and 0.9, respectively, giving the numerical results for $\frac{dF}{d\xi}$ and $G(\xi)$ plotted against ξ shown in Figure 4.6. It can be seen that β_1 for this initial guesses is about 3.5 and $G(\beta_1)$ is positive. Therefore $\beta_2 > \beta_1$. The value of F cannot be determined when $\xi > \beta_1$, because F is required when calculating G using (4.8.6). Thus G cannot be calculated when $\xi > \beta_1$ which also means that the conserved quantity (4.8.8) cannot be computed accurately. This conserved quantity is needed when updating the values of A and B. In order to obtain values for G for $\beta_1 < \xi < \beta_2$ we thus need a different model. This is outside the scope of this dissertation.

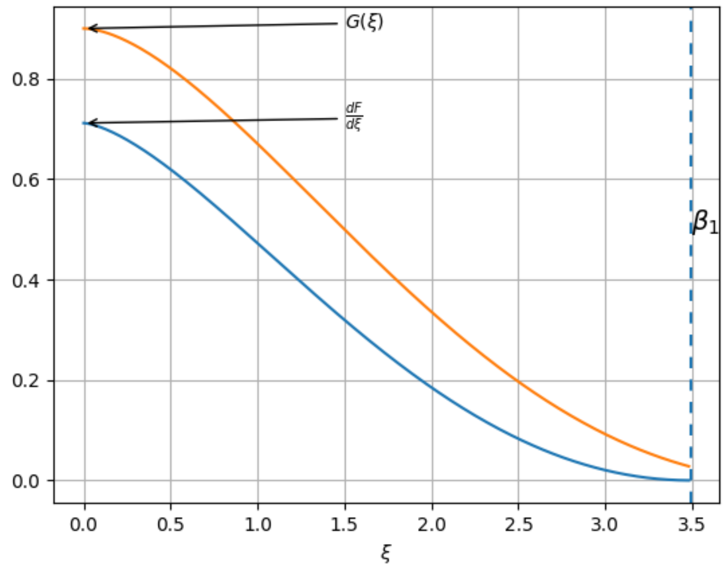

FIGURE 4.6: The initial guesses $A = 0.71$ and $B = 0.9$ of $\frac{dF}{d\xi}$ and $G(\xi)$ for $\nu = 0$ and $\kappa = 1$.

Case (iv): $\nu = 0$ **and** $\kappa = 0$

It can be seen in Figures 4.7 and 4.8 that the numerical solution perfectly matches the analytical solution, for the limit $\nu \to 0$ and $\kappa \to 0$, that was obtained in Section 4.6. Calculating the boundary β_1 from equation (4.6.60) we obtain

$$\beta_1 = \frac{4\sqrt{3}}{9}\pi\left(\sqrt{3}l_0\right)^{2/3} = 3.4879351$$

which agrees with the numerical result

$$\beta_1 = 3.49$$

seen in Figure 4.7. It can also be seen that the boundaries β_1 and β_2 are finite and equal when $\nu = 0$ and $\kappa = 0$. This was proven analytically in Section 4.6. In Figure 4.8 the numerical result for β_2 is

$$\beta_2 = 3.49.$$

Figures 4.9 to 4.11 show numerical results when $\nu = 0$, $\kappa = 0$, $\rho = 1$, $c_v = 1$, $l_0 = 1$, $l_1 = 1$ and the values of conserved quantities J and K are varied. As the conserved quantity J increases the mean flow velocity increases at each point except the boundary of the jet while the mean temperature difference decrease. The conserved quantity K has no effect on the mean velocity but as K increases the mean temperature difference at each point except the boundary increases. The analytical solution, $\nu \to 0$, $\kappa \to 0$ gives the boundary

$$\beta_1 = \frac{4\sqrt{3}}{9}\pi\left(\sqrt{3}l_0\right)^{2/3} = 3.4879351$$

which verifies that the boundary of the jet does not depend on J or K and agrees well with the numerical result

$$\beta_1 = 3.49.$$

Figures 4.12 to 4.14 show the numerical results when $\nu = 0$, $\kappa = 0$, $\rho = 0$, $c_v = 1$, $J = 1$, $K = 1$ and the values of mixing lengths l_0 and l_1 are varied. As the momentum mixing length l_0 increase the boundary of the jet increase, but the mean velocity at the centreline decrease. This is because an increase in l_0 means an increase in the effective viscosity which slows the mean flow at the centreline and entrains the fluid farther from the centreline. The thermal mixing length of l_1 has no effect on the boundary of the jet. This fact was shown analytically, (4.6.60), in Section 4.6. As the thermal mixing length, l_1 increase the mean temperature difference at the centreline decrease while the temperature difference increases close to the boundary of the jet. An increase in the thermal mixing length means an increase in the effective conductivity which means the mean temperature difference can then be easily transferred from the centreline to the boundary of the jet.

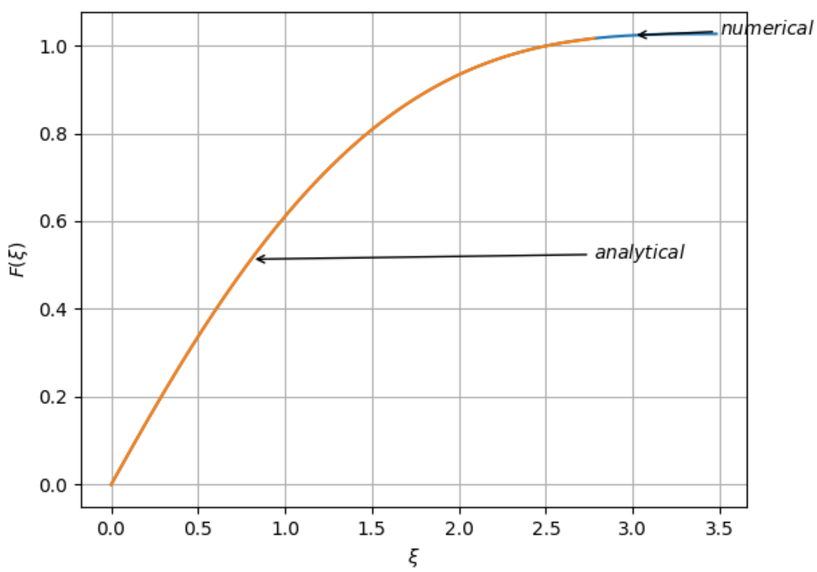

FIGURE 4.7: Comparison of the analytical solution for $\nu \to 0$ and $\kappa \to 0$ with the numerical solution for $F(\xi)$ for $\nu = 0$.

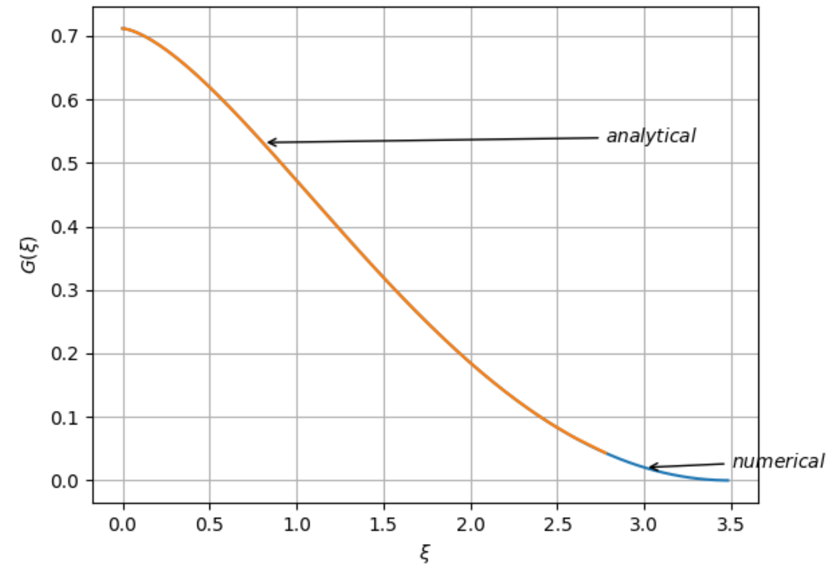

FIGURE 4.8: Comparison of the analytical solution for $\nu \to 0$ and $\kappa \to 0$ with the numerical solution for $G(\xi)$ for $\nu = 0$.

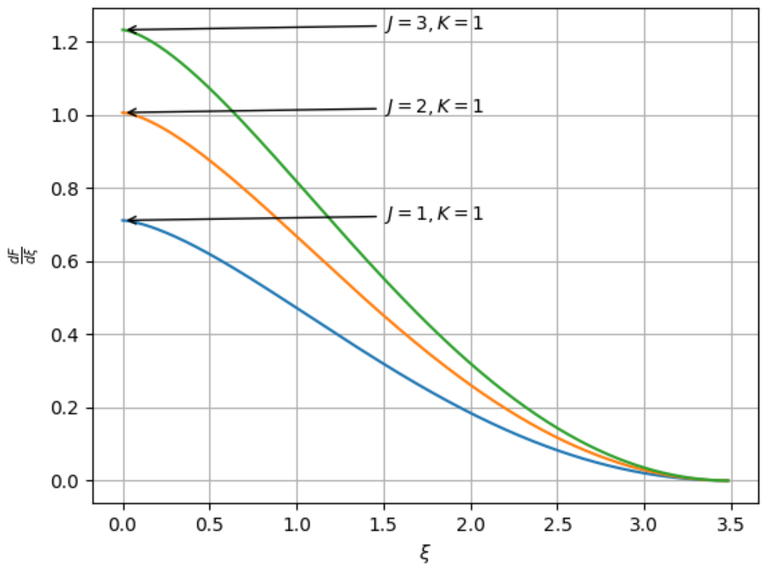

FIGURE 4.9: Graphs of the numerical solution for $\nu = 0$ and $\kappa = 0$ for $\frac{dF}{d\xi}$ versus ξ for varying values of the conserved quantity J.

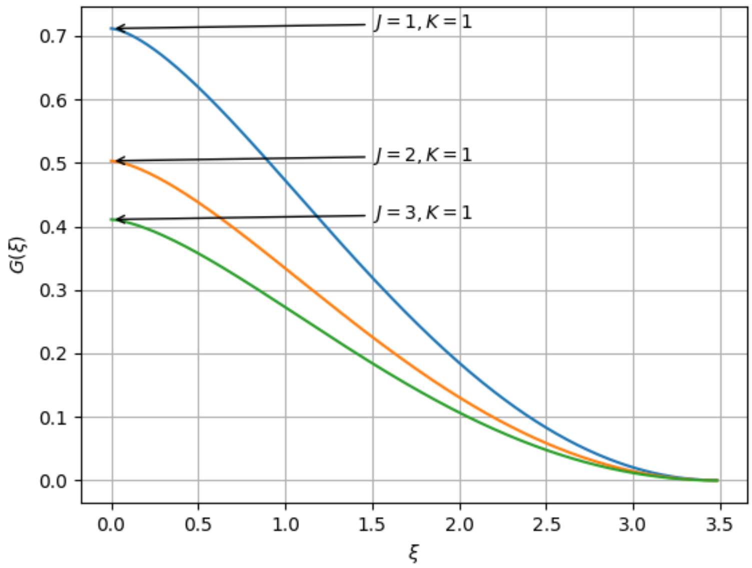

FIGURE 4.10: Graphs of the numerical solution for $\nu = 0$ and $\kappa = 0$ for $G(\xi)$ versus ξ for varying values of the conserved quantity J.

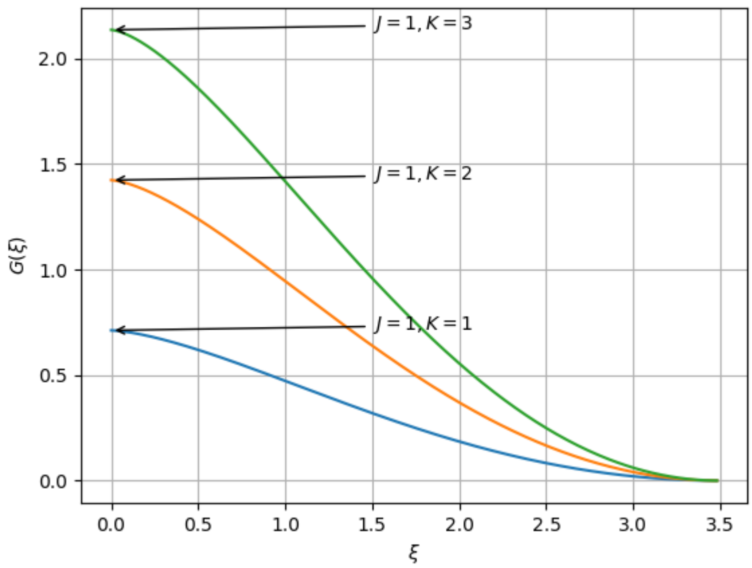

FIGURE 4.11: Graphs of the numerical solution for $\nu = 0$ and $\kappa = 0$ for $G(\xi)$ versus ξ for varying values of the conserved quantity K.

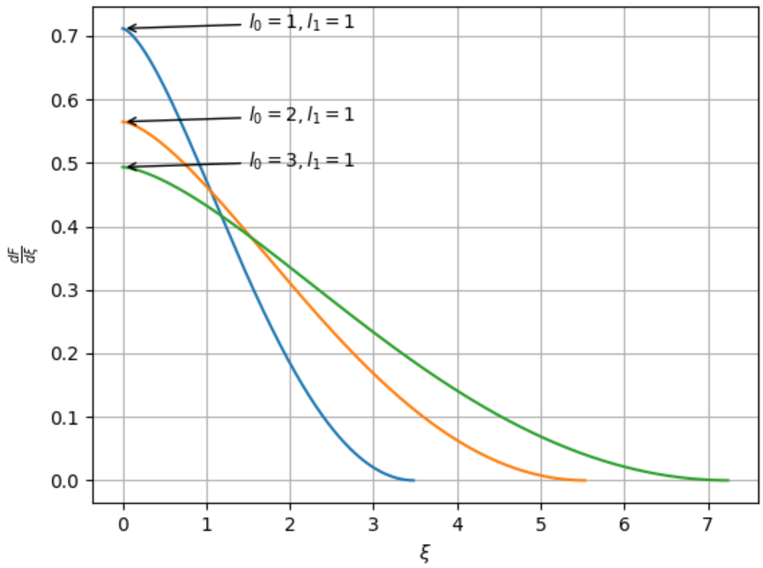

FIGURE 4.12: Graphs of the numerical solution for $\nu = 0$ and $\kappa = 0$ for $\frac{dF}{d\xi}$ versus ξ for varying values of the mixing length l_0.

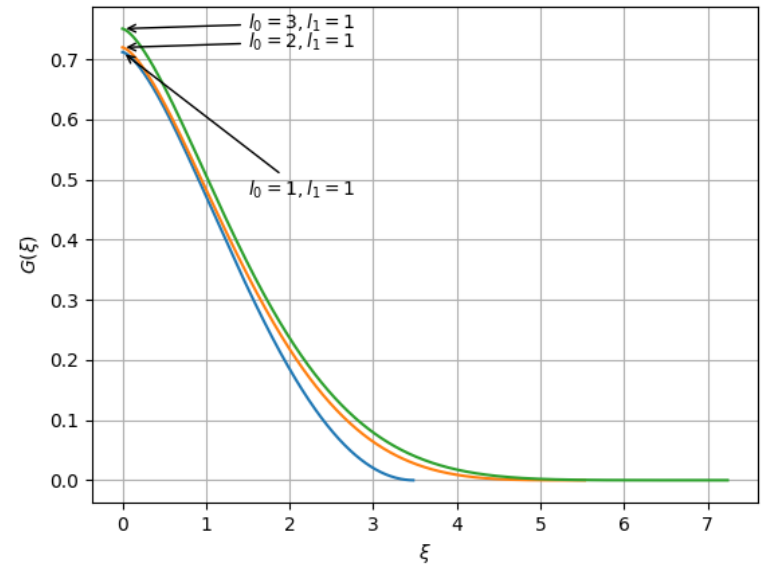

FIGURE 4.13: Graphs of the numerical solution for $\nu = 0$ and $\kappa = 0$ for $G(\xi)$ versus ξ for varying values of the mixing length l_0.

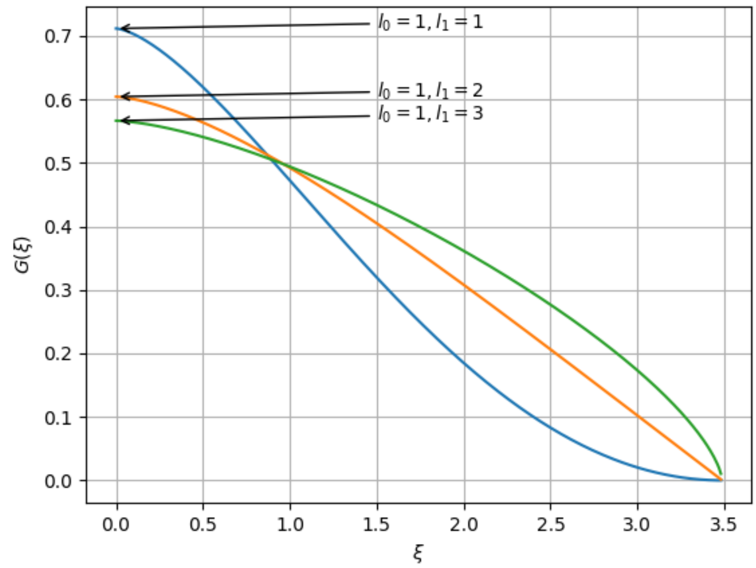

FIGURE 4.14: Graphs of the numerical solution for $\nu = 0$ and $\kappa = 0$ for $G(\xi)$ versus ξ for varying values of the mixing length l_1.

4.9 Discussions of the results

In this chapter, we reduced the coupled system of governing partial differential equations in terms of the stream function, (4.2.10) and (4.2.11), to ordinary differential equations. We achieved this by first obtaining conservation laws to the model partial differential equations (4.2.10) and (4.2.11). We found that the conserved vector space for multipliers of the form $\Lambda_1(x, y, \psi, S)$ and $\Lambda_2(x, y, \psi, S)$ is spanned by the conserved vector with components (4.2.41) and (4.2.42) and the conserved vector with components (4.2.43) and (4.2.44). The conserved quantities were obtained by integrating the conservation laws with respect to y from the centreline to the boundary of the jet. From the basis of conserved vectors, we obtained the associated Lie point symmetry. In the associated Lie point symmetry $\xi^1(x)$ was an arbitrary function of x when $\nu = 0$ and $\kappa = 0$. To solve for this arbitrary function, we invoked Prandtl's hypothesis [18].

For the analytical solution for $\nu = 0$ and $\kappa = 0$ and Prandtl's hypothesis that the motion perpendicular to the flow are mechanically and geometrically similar, it was found that

$$l(x) = l_0 x, \qquad l_\theta(x) = l_1 x, \qquad y_B(x) = \beta_1 x,$$

while for the analytical solution obtained by letting $\nu \to 0$ and $\kappa \to 0$,

$$l(x) = l_0 x^{1/2}, \qquad l_\theta(x) = l_1 x^{1/2}, \qquad y_B(x) = \beta_1 x^{2/3}.$$

For both solutions the momentum and thermal mixing lengths are proportional to each other. Prandtl's hypothesis is not satisfied for the analytical solution for $\nu \to 0$ and $\kappa \to 0$, but the mixing lengths lie within the boundary for sufficiently large values of x. The functional dependence for the two solutions of $l(x)$, $l_\theta(x)$ and $y_B(x)$ are different which could be tested experimentally.

4.10 Conclusions

For a two-dimensional turbulent jet, Prandtl's hypothesis that the mixing length and the half-width of the jet are proportional is not satisfied by the analytical solution obtained by letting $\nu \to 0$ and $\kappa \to 0$. The mixing length lies within the boundary of the jet for sufficiently large x.

The numerical results match perfectly with the analytical solution when $\nu \to 0$ and $\kappa \to 0$. It was shown both analytically and numerically that the momentum and thermal boundaries are finite and equal when $\nu = 0$ and $\kappa = 0$ and are infinite when $\nu \neq 0$ and $\kappa \neq 0$. The thermal mixing length does not affect the boundary of the jet when $\nu = 0$ and $\kappa = 0$. As the momentum mixing length increased the entrainment of the mean flow also increased. As the momentum mixing length constant l_0 increased the boundary β_1 also increased. The shooting method presented in this Chapter could not accurately determine the invariant solution when $\nu = 0$ and $\kappa \neq 0$. The values of G when $\beta_1 < \xi < \beta_2$ were required but could not be found since F is only determined when $0 < \xi < \beta_1$ and is needed when calculating G. This means we need a different model to calculate G when $\beta_1 < \xi < \beta_2$ perhaps matching the to an exterior flow which is outside the scope of this dissertation. The numerical solution was unstable for G when $\xi > 4.8$ when $\nu \neq 0$ and $\kappa = 0$. This may indicate that there is no solution for $G(\xi)$ for $\xi > 4.8$ and that there is a finite boundary for $G(\xi)$ in the $y-$direction.

Chapter 5

Axisymmetric turbulent jet: conservation laws, Lie point symmetry and group invariant solutions

5.1 Introdution

In this chapter, the governing equations (3.4.6), (3.4.7) and (3.4.8) for the axisymmetric turbulent thermal jet, written in terms of the stream function, are investigated. These equations are reduced to ordinary differential equations using the associated Lie point symmetry. The associated Lie point symmetry is found using conserved vectors and depends on whether ν and κ are zero or non-zero. The formula for finding associated Lie point symmetries from conserved vectors was discovered in [5].

There are various methods for finding conservation laws and were compared in Naz et al [8], which concluded that the multiplier method is more systematic and simpler than its counterparts. Hence in this chapter, the multiplier method is used to obtain conservation laws and are also used to find conserved quantities. This is done by integrating the conservation law with respect to the radial coordinate r from the centreline to the boundary of the jet. This method for obtaining conserved quantities is more systematic than directly integrating the governing equations.

For the special case, $\nu = 0$ and $\kappa = 0$, the associated Lie point symmetry generator contains an arbitrary function of z. The conserved quantities do not assist in solving for this arbitrary function. Hence Prandtl's hypothesis [18] which hypothesise that the mixing length is proportional to the half-width of the jet is used to determine this arbitrary function.

This chapter will be structured in the following manner: In Section 5.2 we derive the conservation laws which are then used to determine conserved quantities in Section 5.3. In Section 5.4, we find the associated Lie point symmetry to the conserved vector derived in Section 5.2 for ν and κ non-zero and ν and κ zero, which is then used to reduce the governing partial differential equations to ordinary differential equations for the invariant solution in Section 5.5. The numerical solution for the invariant solution is given in Section 5.6. We end the chapter by giving concluding remarks in Section 5.7.

5.2 Conservation laws

The governing equations for the axisymmetric turbulent thermal jet are (3.4.6), (3.4.7) and (3.4.8);

$$\overline{v}_z\frac{\partial \overline{v}_r}{\partial z}+\overline{v}_r\frac{\partial \overline{v}_r}{\partial r}=\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\nu+l^2(z)\left|\frac{\partial \overline{v}_z}{\partial r}\right|\right)\frac{\partial \overline{v}_z}{\partial r}\right], \tag{5.2.1}$$

$$\overline{v}_z\frac{\partial \overline{T}}{\partial z}+\overline{v}_r\frac{\partial \overline{T}}{\partial r}=\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\kappa}{\rho c_v}+\rho c_v l(z) l_\theta(z)\left|\frac{\partial \overline{v}_z}{\partial r}\right|\right)\frac{\partial \overline{T}}{\partial r}\right] \tag{5.2.2}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}(r\overline{v}_r)+\frac{\partial \overline{v}_z}{\partial z}=0, \tag{5.2.3}$$

with boundary conditions (3.4.9) and (3.4.10),

$$\frac{\partial \overline{v}_z}{\partial r}(z,0)=0,\qquad \overline{v}_z(z,r_B(z))=0,\qquad \frac{\partial \overline{v}_z}{\partial r}(z,r_B(z))=0,\qquad \overline{v}_r(z,0)=0 \tag{5.2.4}$$

and

$$\frac{\partial \overline{T}}{\partial r}(z,0)=0,\qquad \frac{\partial \overline{T}}{\partial r}(z,r_{Tb}(z))=0,\qquad \overline{T}(z,r_{Tb}(z))=T_B, \tag{5.2.5}$$

where $r_B(z)$ is the boundary of the jet. For a free jet, the mean velocity gradient is negative (see Figure 5.1). i.e.

$$\frac{\partial \overline{v}_z}{\partial r}<0. \tag{5.2.6}$$

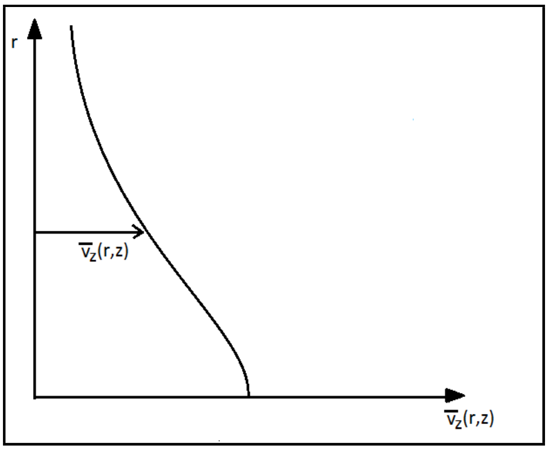

FIGURE 5.1: Schematic diagram for the mean velocity profile

The stream function ψ is defined by

$$\overline{v}_z=\frac{1}{r}\frac{\partial \psi}{\partial r},\qquad\qquad \overline{v}_r=-\frac{1}{r}\frac{\partial \psi}{\partial z}. \tag{5.2.7}$$

Let

$$\overline{S}=\overline{T}-\overline{T}(z,r_B(z)), \tag{5.2.8}$$

where $r_{Tb}(z)$ is the boundary of the jet. In order to simplify notation we omit bars on S for the rest of the chapter. The boundary conditions (5.2.4) and (5.2.5), written in terms of the stream function ψ and S, become

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)(z,0)=0,\qquad \frac{\partial\psi}{\partial r}(z,r_B(z))=0,\qquad \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)(z,r_B(z))=0, \tag{5.2.9}$$

$$\frac{1}{r}\frac{\partial\psi}{\partial z}(z,0)=0 \tag{5.2.10}$$

and

$$\frac{\partial S}{\partial r}(z,0)=0,\qquad \frac{\partial S}{\partial r}(z,r_{Tb}(z))=0,\qquad S(z,r_{Tb}(z))=0. \tag{5.2.11}$$

Substituting the stream function (5.2.7) and equation (5.2.8) into the governing equations (5.2.1) and (5.2.2) we obtain

$$\begin{aligned}-\frac{1}{r}\frac{\partial\psi}{\partial z}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial\psi}{\partial r}\frac{\partial\psi}{\partial z\partial r}=\\ \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\nu-l^2(z)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right],\end{aligned} \tag{5.2.12}$$

$$\frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial S}{\partial z}-\frac{1}{r}\frac{\partial\psi}{\partial z}\frac{\partial S}{\partial r}=\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\kappa}{\rho c_v}-\rho c_v l(z)l_\theta(z)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right)\frac{\partial S}{\partial r}\right]. \tag{5.2.13}$$

When working with Lie point symmetries and conservation laws, z, r, ψ, S and all the partial derivatives of ψ and S with respect to z and r are regarded as independent variables and the suffix notation is used for partial differentiation. Thus,

$$z,r,\psi,S,\psi_z,\psi_r,S_z,S_r,\psi_{zz},\psi_{zr},\psi_{rr},S_{zz},S_{zr},S_{rr},...$$

are independent variables. The notation $\frac{\partial\psi}{\partial z}$, $\frac{\partial^2 S}{\partial z\partial r}$,... is used when ψ and S and the partial derivatives of ψ and S are regarded as functions of independent variables z and r. Expanding equations (5.2.12) and (5.2.13) and writing them in terms of partial derivatives with the subscript notation we obtain

$$\begin{aligned}-\frac{1}{r}\psi_z\psi_{rr}+\frac{1}{r}\psi_r\psi_{zr}+\nu\left[\frac{1}{r^2}\psi_r\psi_z+\frac{1}{r}\psi_{rr}-\frac{1}{r^2}\psi_r-\psi_{rrr}\right]-3\frac{l^2(z)}{r^2}(\psi_{rr})^2\\ +6\frac{l^2(z)}{r^3}\psi_r\psi_{rr}-3\frac{l^2(z)}{r^4}(\psi_r)^2+2\frac{l^2(z)}{r}\psi_{rr}\psi_{rrr}-2\frac{l^2(z)}{r^2}\psi_r\psi_{rrr}=0,\end{aligned} \tag{5.2.14}$$

$$\begin{aligned}\psi_rS_z-\psi_zS_r-\frac{\kappa}{\rho c_v}S_r-\frac{\kappa}{\rho c_v}rS_{rr}-l(z)l_\theta(z)\frac{1}{r}\psi_{rr}S_r+l(z)l_\theta(z)\frac{1}{r^2}\psi_rS_r\\ +l(z)l_\theta(z)\psi_{rrr}S_r+l(z)l_\theta(z)\psi_{rr}S_{rr}-l(z)l_\theta(z)\frac{1}{r}\psi_rS_{rr}=0.\end{aligned} \tag{5.2.15}$$

We introduce the standard Euler operators which are defined by

$$E_\psi=\frac{\partial}{\partial\psi}-D_z\frac{\partial}{\partial\psi_z}-D_r\frac{\partial}{\partial\psi_r}+D_z^2\frac{\partial}{\partial\psi_{zz}}+D_zD_r\frac{\partial}{\partial\psi_{rz}}+D_r^2\frac{\partial}{\partial\psi_{rr}}-D_r^3\frac{\partial}{\partial\psi_{rrr}}-...,\tag{5.2.16}$$

$$E_S=\frac{\partial}{\partial S}-D_z\frac{\partial}{\partial S_z}-D_r\frac{\partial}{\partial S_r}+D_z^2\frac{\partial}{\partial S_{zz}}+D_zD_r\frac{\partial}{\partial S_{rz}}+D_r^2\frac{\partial}{\partial S_{rr}}-D_r^3\frac{\partial}{\partial S_{rrr}}-...,\tag{5.2.17}$$

where

$$D_z = \frac{\partial}{\partial z} + \psi_z \frac{\partial}{\partial \psi} + S_z \frac{\partial}{\partial S} + \psi_{zz} \frac{\partial}{\partial \psi_z} + S_{zz} \frac{\partial}{\partial S_z} + \psi_{zr} \frac{\partial}{\partial \psi_r} + S_{zr} \frac{\partial}{\partial S_r} + ..., \tag{5.2.18}$$

$$D_r = \frac{\partial}{\partial r} + \psi_r \frac{\partial}{\partial \psi} + S_r \frac{\partial}{\partial S} + \psi_{rr} \frac{\partial}{\partial \psi_r} + S_{rr} \frac{\partial}{\partial S_r} + \psi_{zr} \frac{\partial}{\partial \psi_z} + S_{zr} \frac{\partial}{\partial S_z} + ... \tag{5.2.19}$$

A conservation law for the system of partial differential equations (5.2.14) and (5.2.15) is defined by

$$D_z T^1 + D_r T^2 \Big|_{(5.2.14),(5.2.15)} = 0 \tag{5.2.20}$$

where T^1 and T^2 are the components of the conserved vector (T^1, T^2). The conserved vector can be found by the multiplier method. We need to find multipliers Λ_1 and Λ_2 such that

$$\Lambda_1 L_1 + \Lambda_2 L_2 = D_z T^1 + D_r T^2 \tag{5.2.21}$$

for all functions $\psi(z,r)$ and $S(z,r)$ where

$$\begin{aligned} L_1 = & -\frac{1}{r}\psi_z\psi_{rr} + \frac{1}{r}\psi_r\psi_{zr} + \nu\left[\frac{1}{r^2}\psi_r\psi_z + \frac{1}{r}\psi_{rr} - \frac{1}{r^2}\psi_r - \psi_{rrr}\right] - 3\frac{l^2(z)}{r^2}(\psi_{rr})^2 \\ & +6\frac{l^2(z)}{r^3}\psi_r\psi_{rr} - 3\frac{l^2(z)}{r^4}(\psi_r)^2 + 2\frac{l^2(z)}{r}\psi_{rr}\psi_{rrr} - 2\frac{l^2(z)}{r^2}\psi_r\psi_{rrr}, \end{aligned} \tag{5.2.22}$$

$$\begin{aligned} L_2 = & \psi_r S_z - \psi_z S_r - \frac{\kappa}{\rho c_v} S_r - \frac{\kappa}{\rho c_v} r S_{rr} - l(z) l_\theta(z) \frac{1}{r}\psi_{rr} S_r + l(z) l_\theta(z) \frac{1}{r^2}\psi_r S_r \\ & + l(z) l_\theta(z) \psi_{rrr} S_r + l(z) l_\theta(z) \psi_{rr} S_{rr} - l(z) l_\theta(z) \frac{1}{r}\psi_r S_{rr} \end{aligned} \tag{5.2.23}$$

are the left hand sides of equations (5.2.14) and (5.2.15) respectively. we consider multiplier of the form

$$\Lambda_1 = \Lambda_1(z, r, \psi, S), \tag{5.2.24}$$

$$\Lambda_2 = \Lambda_2(z, r, \psi, S). \tag{5.2.25}$$

We find that multipliers of the form (5.2.24) and (5.2.25) are sufficient to solve the problem of the turbulent thermal jet. Further conservation laws could be investigated by considering multipliers which depend on the partial derivatives ψ_r, ψ_z, S_r, S_z, ψ_{rr}, ..., but calculations become complicated and are best done using Software.

It can be shown that the Euler operators (5.2.16) and (5.2.17) annihilate the right hand side of (5.2.21). Hence,

$$E_\psi(\Lambda_1 L_1 + \Lambda_2 L_2) = 0, \tag{5.2.26}$$

$$E_S(\Lambda_1 L_1 + \Lambda_2 L_2) = 0. \tag{5.2.27}$$

Expanding equation (5.2.27) using the Euler operator definition (5.2.16) we obtain

$$\begin{aligned} & \frac{\partial \Lambda_1}{\partial S}(L_1) + \frac{\partial \Lambda_2}{\partial S}(L_2) - D_z(\Lambda_2 \psi_r) \\ & -D_r\left[\Lambda_2\left(-\psi_z - \frac{\kappa}{\rho c_v} - l(z) l_\theta(z) \frac{1}{r}\psi_{rr} + l(z) l_\theta(z) \frac{1}{r^2}\psi_r + l(z) l_\theta(z) \psi_{rrr}\right)\right] \\ & +D_r^2\left[\Lambda_2\left(-\frac{\kappa}{\rho c_v} r + l(z) l_\theta(z) \psi_{rr} - l(z) l_\theta(z) \frac{1}{r}\psi_r\right)\right] = 0. \end{aligned} \tag{5.2.28}$$

Equation (5.2.28) can be split into coefficients of powers and products of the derivatives of ψ and S to give

$$\psi_r\psi_{rz} : \frac{\partial \Lambda_1}{\partial S} = 0, \tag{5.2.29}$$

$$\psi_{rrr} : -\nu\frac{\partial \Lambda_1}{\partial S} + l(z)l_\theta(z)\frac{\partial \Lambda_2}{\partial r} = 0, \tag{5.2.30}$$

$$(\psi_{rr})^2 : -3\frac{l^2(z)}{r^2}\frac{\partial \Lambda_1}{\partial S} + l(z)l_\theta(z)\frac{\partial \Lambda_2}{\partial \psi} = 0, \tag{5.2.31}$$

$$\psi_r S_{rr} : \frac{\partial \Lambda_2}{\partial S} = 0, \tag{5.2.32}$$

$$\begin{aligned}\psi_r : -\frac{\nu}{r}\frac{\partial \Lambda_1}{\partial S} - \frac{\partial \Lambda_2}{\partial z} - \frac{\kappa}{\rho c_v}\frac{\partial \Lambda_2}{\partial \psi} + l(z)l_\theta(z)\frac{1}{r^2}\frac{\partial \Lambda_2}{\partial r}\\ -l(z)l_\theta(z)\frac{1}{r}\frac{\partial^2 \Lambda_2}{\partial r^2} - 2\frac{\kappa}{\rho c_v}r\frac{\partial^2 \Lambda_2}{\partial r\partial \psi} = 0.\end{aligned} \tag{5.2.33}$$

From the system of partial differential equations (5.2.29) to (5.2.33), it can be concluded that

$$\Lambda_1 = \Lambda_1(z, r, \psi), \tag{5.2.34}$$

$$\Lambda_2 = c_2, \tag{5.2.35}$$

where c_2 is a constant. Equations (5.2.34) and (5.2.35) are satisfied for $\nu \neq 0$, $\kappa \neq 0$ and $\nu = 0$, $\kappa = 0$. It can be verified that the coefficients of the other partial derivatives and the remainder in (5.2.28) are zero when (5.2.34) and (5.2.35) are satisfied.

Now expanding equation (5.2.26) using the Euler definition (5.2.16) we find that the second term, $E_\psi\left(c_2L_2\right)$, of (5.2.26) vanishes. Hence we are left with just expanding the first term, $E_\psi\left(\Lambda_1(z, r, \psi)\right)$, of (5.2.26) to obtain

$$\begin{aligned}&\frac{\partial \Lambda_1}{\partial \psi}(L_1)\\ &-D_r\left[\Lambda_1\left(\frac{1}{r}\psi_{rz} + \frac{1}{r^2}\psi_z - \frac{\nu}{r^2} + 6\frac{l^2(z)}{r^3}\psi_{rr} - 6\frac{l^2(z)}{r^4}\psi_r - 2\frac{l^2(z)}{r^2}\psi_{rrr}\right)\right]\\ &-D_z\left[\Lambda_1\left(-\frac{1}{r}\psi_{rr} + \frac{1}{r^2}\psi_r\right)\right] + D_r^2\left[\Lambda_1\left(\frac{\nu}{r} - \frac{1}{r}\psi_z - 6\frac{l^2(z)}{r^2}\psi_{rr} + 6\frac{l^2(z)}{r^3}\psi_r + 2\frac{l^2(z)}{r}\psi_{rrr}\right)\right]\\ &+D_rD_z\left[\Lambda_1\left(\frac{1}{r}\psi_r\right)\right] - D_r^3\left[\Lambda_1\left(-\nu r + 2\frac{l^2(z)}{r}\psi_{rr} - 2\frac{l^2(z)}{r^2}\psi_r\right)\right] = 0.\end{aligned} \tag{5.2.36}$$

Equation (5.2.36) is a polynomial in the powers which are independent and products of the derivatives of ψ and S and hence can be split into coefficients of the powers and products of the derivatives of ψ and S to give

$$\psi_{rrrr} : \frac{\partial \Lambda_1}{\partial r} = 0, \tag{5.2.37}$$

$$\begin{aligned}\psi_{rr} : 2\frac{\nu}{r}\frac{\partial \Lambda_1}{\partial \psi} - 24\frac{l^2(z)}{r^3}\frac{\partial \Lambda_1}{\partial r} + \frac{1}{r}\frac{\partial \Lambda_1}{\partial r} + \frac{1}{r}\frac{\partial \Lambda_1}{\partial z} + 6\frac{l^2(z)}{r^2}\frac{\partial^2 \Lambda_1}{\partial r^2}\\ -2\frac{l^2(z)}{r}\frac{\partial^3 \Lambda_1}{\partial r^3} + 3\nu\frac{\partial^2 \Lambda_1}{\partial r\partial \psi} = 0,\end{aligned} \tag{5.2.38}$$

$$\psi_{rrr}\psi_{rr} : \frac{\partial \Lambda_1}{\partial \psi} = 0. \tag{5.2.39}$$

From equations (5.2.37) to (5.2.39) we obtain

$$\Lambda_1 = c_1, \tag{5.2.40}$$

where c_1 is a constant. This results is satisfied for both $\nu \neq 0, \kappa \neq 0$ and $\nu = 0, \kappa = 0$. Using (5.2.40) it can be verified that all the coefficients of the powers and products of the other partial derivatives of ψ and also the remainder in (5.2.36) vanish identically.

Returning to the definition of multipliers

$$\Lambda_1 L_1 + \Lambda_2 L_2 = D_z T^1 + D_r T^2, \tag{5.2.41}$$

and substituting equations (5.2.35) and (5.2.40) into (5.2.41) we obtain

$$\begin{aligned} c_1 L_1 + c_2 L_2 = {} & D_z\left[c_1 \frac{1}{r}(\psi_r)^2 - c_2 \psi S_r\right] \\ & + D_r\left[c_1\left(-\frac{1}{r}\psi_r\psi_z + \nu\frac{1}{r}\psi_r - \nu\psi_{rr} - 2\frac{l^2(z)}{r^2}\psi_r\psi_{rr} + \frac{l^2(z)}{r}(\psi_{rr})^2 + \frac{l^2(z)}{r^3}(\psi_r)^2\right)\right] \\ & + D_r\left[c_2\left(\psi S_z - \frac{\kappa}{\rho c_v} r S_r - l(z) l_\theta(z)\frac{1}{r}\psi_r S_r + l(z) l_\theta(z)\psi_{rr} S_r\right)\right] \end{aligned} \tag{5.2.42}$$

for arbitrary functions $\psi(z,r)$ and $S(z,r)$. When $\psi(z,r)$ and $S(z,r)$ are solutions of the system of partial differential equations (5.2.14) and (5.2.15) then equation (5.2.43) becomes

$$\begin{aligned} & D_z\left[-c_1 \frac{1}{r}(\psi_r)^2 - c_2 \psi S_r\right] + \\ & D_r\left[c_1\left(-\frac{1}{r}\psi_r\psi_z + \nu\frac{1}{r}\psi_r - \nu\psi_{rr} - 2\frac{l^2(z)}{r^2}\psi_r\psi_{rr} + \frac{l^2(z)}{r}(\psi_{rr})^2 + \frac{l^2(z)}{r^3}(\psi_r)^2\right)\right] \\ & + D_r\left[c_2\left(\psi S_z - \frac{\kappa}{\rho c_v} r S_r - l(z) l_\theta(z)\frac{1}{r}\psi_r S_r + l(z) l_\theta(z)\psi_{rr} S_r\right)\right]\Bigg|_{(5.2.14),(5.2.15)} = 0. \end{aligned} \tag{5.2.43}$$

Therefore any conserved vector of the system of partial differential equations (5.2.14) and (5.2.15) with multipliers $\Lambda_1(z,r,\psi,S)$ and $\Lambda_2(z,r,\psi,S)$ is a linear combination of the following two vectors

Momentum flux conserved vector: $c_1 = 1$ **and** $c_2 = 0$**:**

$$T^1 = \frac{1}{r}(\psi_r)^2, \tag{5.2.44}$$

$$T^2 = -\frac{1}{r}\psi_z\psi_r + \nu\left(\frac{1}{r}\psi_r - \psi_{rr}\right) + l^2(z)\left(\frac{1}{r^3}(\psi_r)^2 + \frac{1}{r}(\psi_{rr})^2 - 2\frac{1}{r^2}\psi_r\psi_{rr}\right) \tag{5.2.45}$$

and

Thermal flux conserved vector: $c_1 = 0$ **and** $c_2 = 1$**:**

$$T^1 = -\psi S_r, \tag{5.2.46}$$

$$T^2 = \psi S_z - \frac{\kappa}{\rho c_v} r S_r + l(z) l_\theta(z) \left(\psi_{rr} S_r - \frac{1}{r} \psi_r S_r \right). \tag{5.2.47}$$

5.3 Conserved quantities

Conserved quantities are useful in solving for constants or arbitrary functions contained in the associated Lie point symmetry generators obtained in the next section. The conserved quantities are obtained by integrating the conservation law with respect r from the centreline to the boundary of the jet.

The conserved vector components T^1 and T^2 are now treated as functions of the independent variables z and r. Then

$$D_z T^1 + D_r T^2 = 0 \tag{5.3.1}$$

can be written as

$$\frac{\partial T^1}{\partial z}(z, r) + \frac{\partial T^2}{\partial r}(z, r) = 0. \tag{5.3.2}$$

5.3.1 Momentum flux conserved quantity

Substituting the momentum conserved vector (5.2.44) and (5.2.45) into equation (5.3.2) we obtain

$$\begin{aligned} &\frac{\partial}{\partial z}\left[\frac{1}{r}\left(\frac{\partial \psi}{\partial r}\right)^2\right] + \frac{\partial}{\partial r}\left[-\frac{1}{r}\frac{\partial \psi}{\partial z}\frac{\partial \psi}{\partial r}\right. \\ &\left. +\nu\left(\frac{1}{r}\frac{\partial \psi}{\partial r} - \frac{\partial^2 \psi}{\partial r^2}\right) + l^2(z)\left(\frac{1}{r^3}\left(\frac{\partial \psi}{\partial r}\right)^2 - \frac{2}{r^2}\frac{\partial \psi}{\partial r}\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\left(\frac{\partial^2 \psi}{\partial r^2}\right)^2\right)\right] = 0. \end{aligned} \tag{5.3.3}$$

Integrating equation (5.3.3) with respect to $r = 0$ from the centreline of the jet, 0, to the boundary of the jet, $r = r_B(z)$, we obtain

$$\begin{aligned} &\int_0^{r_B(z)} \frac{\partial}{\partial z}\left(-\frac{1}{r}\left(\frac{\partial \psi}{\partial r}\right)^2\right) dr + \left[-\frac{1}{r}\frac{\partial \psi}{\partial z}\frac{\partial \psi}{\partial r} + \nu\left(\frac{1}{r}\frac{\partial \psi}{\partial r} - \frac{\partial^2 \psi}{\partial r^2}\right)\right. \\ &\left. + l^2(z)\left(\frac{1}{r^3}\left(\frac{\partial \psi}{\partial r}\right)^2 - \frac{2}{r^2}\frac{\partial \psi}{\partial r}\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\left(\frac{\partial^2 \psi}{\partial r^2}\right)^2\right)\right]\Bigg|_0^{r_B(z)} = 0. \end{aligned} \tag{5.3.4}$$

Using the formula for differentiation under the integral sign [19]

$$\int_0^{r_B(z)} \frac{\partial}{\partial z}\left(-\frac{1}{r}\left(\frac{\partial \psi}{\partial r}\right)^2\right) dr = \frac{d}{dz}\int_0^{r_B(z)}\left(-\frac{1}{r}\left(\frac{\partial \psi}{\partial r}\right)^2\right) dr + \frac{1}{r_B(z)}\psi_r(z, r_B(z))\frac{dr_B}{dz}, \tag{5.3.5}$$

and simplifying the second term in equation (5.3.4) we obtain

$$\begin{aligned} &\frac{d}{dz}\int_0^{r_B(z)} \frac{1}{r}\left(\frac{\partial \psi}{\partial r}\right)^2 dr - \frac{1}{r_B(z)}\frac{\partial \psi}{\partial r}(z, r_B(z))\frac{dr_B}{dz} + \left[-\frac{1}{r}\frac{\partial \psi}{\partial z}\frac{\partial \psi}{\partial r} + \nu\left(\frac{1}{r}\frac{\partial \psi}{\partial r} - \frac{\partial^2 \psi}{\partial r^2}\right) + \right. \\ &\left. l^2(z)\left(\frac{1}{r^3}\left(\frac{\partial \psi}{\partial r}\right)^2 - \frac{2}{r^2}\frac{\partial \psi}{\partial r}\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\left(\frac{\partial^2 \psi}{\partial r^2}\right)^2\right)\right]\Bigg|_0^{r_B(z)} = 0. \end{aligned} \tag{5.3.6}$$

All terms outside the integral sign in equation (5.3.6) vanish when boundary conditions (5.2.9), (5.2.10) are imposed. Hence

$$\frac{d}{dz}\int_0^{r_B(z)} \frac{1}{r}\left(\frac{\partial\psi}{\partial r}(z,r)\right)^2 dr = 0, \tag{5.3.7}$$

which gives

$$\int_0^{r_B(z)} \frac{1}{r}\left(\frac{\partial\psi}{\partial r}(z,r)\right)^2 dr = \text{constant independent of } z. \tag{5.3.8}$$

Equation (5.3.8) can be multiplied by $2\pi\rho$ to give the total momentum flux in the direction of the jet. Thus the conserved quantity becomes

$$J = 2\pi\rho\int_0^{r_B(z)} \frac{1}{r}\left(\frac{\partial\psi}{\partial r}(z,r)\right)^2 dr = \text{ constant independent of } z. \tag{5.3.9}$$

5.3.2 Thermal flux conserved quantity

Substituting the thermal flux conserved vector (5.2.46) and (5.2.47) into (5.3.2) we obtain

$$\begin{aligned} &\frac{\partial}{\partial z}\left[-\psi\frac{\partial S}{\partial r}\right] \\ &+\frac{\partial}{\partial r}\left[\psi\frac{\partial S}{\partial z} - \frac{\kappa}{\rho c_v}r\frac{\partial S}{\partial r} + l(z)l_\theta(z)\left(\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r}\right)\frac{\partial S}{\partial r}\right] = 0 \end{aligned} \tag{5.3.10}$$

Integrating equation (5.3.10) with respect to $r = 0$ from the centreline of the jet, 0, to the boundary of the jet, $r = r_{Tb}(z)$, we obtain

$$\begin{aligned} &\int_0^{r_{Tb}(z)} \frac{\partial}{\partial z}\left(-\psi\frac{\partial}{\partial r}\right) dr + \int_0^{r_{Tb}(z)} \frac{\partial}{\partial r}\left(\psi\frac{\partial S}{\partial z}\right) dr \\ &+\left[-\frac{\kappa}{\rho c_v}r\frac{\partial S}{\partial r} + l(z)l_\theta(z)\left(\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r}\right)\frac{\partial S}{\partial r}\right]\Bigg|_0^{r_{Tb}(z)} = 0 \end{aligned} \tag{5.3.11}$$

Using boundary conditions (5.2.11) equation (5.3.11) becomes

$$\int_0^{r_{Tb}(z)} \frac{\partial}{\partial z}\left(-\psi\frac{\partial S}{\partial r}\right) dr + \int_0^{r_{Tb}(z)} \frac{\partial}{\partial r}\left(\psi\frac{\partial S}{\partial z}\right) dr = 0, \tag{5.3.12}$$

which simplifies to

$$\int_0^{r_{Tb}(z)} \left(-\frac{\partial\psi}{\partial z}\frac{\partial S}{\partial r} + \frac{\partial\psi}{\partial r}\frac{\partial S}{\partial z}\right) dr = 0. \tag{5.3.13}$$

Using integration by parts on the first term inside the integral sign of (5.3.13), equation (5.3.13) becomes

$$-\frac{\partial\psi}{\partial z}S\Bigg|_{r=0}^{r=r_{Tb}(z)} + \int_0^{r_{Tb}(z)} \left(\frac{\partial^2\psi}{\partial z\partial r}S + \frac{\partial\psi}{\partial r}\frac{\partial S}{\partial z}\right) dr = 0. \tag{5.3.14}$$

The first term of equation (5.3.14) vanish when boundary conditions (5.2.9), (5.2.10) and (5.2.11) are imposed. Hence equation (5.3.14) simplifies to

$$\int_0^{r_{Tb}(z)} \frac{\partial}{\partial z}\left(\frac{\partial \psi}{\partial r} S\right) dr = 0. \tag{5.3.15}$$

Using the formula for differentiation under the integral sign [19] we obtain

$$\int_0^{r_{Tb}(z)} \frac{\partial}{\partial z}\left(\frac{\partial \psi}{\partial r} S\right) dr = \frac{d}{dz}\int_0^{r_{Tb}(z)} \left(\frac{\partial \psi}{\partial r} S\right) dr - \frac{\partial \psi}{\partial r}(z, r_{Tb}(z)) S(z, r_{Tb}(z)) \frac{dr_{Tb}}{dz}, \tag{5.3.16}$$

The second term on the right hand side of (5.3.16) vanishes when the third boundary condition (5.2.11) is imposed. Hence

$$\frac{d}{dz}\int_0^{r_{Tb}(z)} \left(\frac{\partial \psi}{\partial r} S\right) dr = 0, \tag{5.3.17}$$

which gives that

$$K = 2\pi\rho c_v \int_0^{r_{Tb}(z)} \frac{\partial \psi}{\partial r}(z,r)S(z,r)dr = \text{constant independent of } z. \tag{5.3.18}$$

5.4 Associated Lie point symmetry

In this section, we obtain associated Lie point symmetries for conserved quantities obtained in Section 4.3.

An associated Lie point symmetry to the conserved vector (T^1, T^2) is defined by

$$X = \xi^1(z,r,\psi,S)\frac{\partial}{\partial z} + \xi^2(z,r,\psi,S)\frac{\partial}{\partial r} + \eta^1(z,r,\psi,S)\frac{\partial}{\partial \psi} + \eta^2(z,r,\psi,S)\frac{\partial}{\partial S} \tag{5.4.1}$$

which can prolonged to contain all derivatives and satisfy

$$X^{[\infty]}(T^1) + T^1 D_r \xi^2 - T^2 D_r \xi^1 = 0, \tag{5.4.2}$$

$$X^{[\infty]}(T^2) + T^2 D_z \xi^1 - T^1 D_z \xi^2 = 0. \tag{5.4.3}$$

The prolongation $X^{[\infty]}$ is defined by

$$\begin{aligned} X^{[\infty]} = {} & \xi^i \frac{\partial}{\partial x_i} + \eta^1 \frac{\partial}{\partial \psi} + \eta^2 \frac{\partial}{\partial S} + \zeta_i^1 \frac{\partial}{\partial \psi_{x_i}} + \zeta_i^2 \frac{\partial}{\partial S_{x_i}} + \zeta_{11}^1 \frac{\partial}{\partial \psi_{x_1 x_1}} + \zeta_{12}^1 \frac{\partial}{\partial \psi_{x_1 x_2}} \\ & + \zeta_{22}^1 \frac{\partial}{\partial \psi_{x_2 x_2}} + \zeta_{11}^2 \frac{\partial}{\partial S_{x_1 x_1}} + \zeta_{12}^2 \frac{\partial}{\partial S_{x_1 x_2}} + \zeta_{22}^2 \frac{\partial}{\partial S_{x_2 x_2}} \ldots \end{aligned} \tag{5.4.4}$$

where

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j), \tag{5.4.5}$$

$$\zeta_{ij}^\alpha = D_j(\zeta_i^\alpha) - u_{il}^\alpha D_j(\xi^l) \tag{5.4.6}$$

and

$$u^1 = \psi, \qquad u^2 = S, \qquad x_1 = z, \qquad x_2 = r. \tag{5.4.7}$$

5.4.1 Associated with the momentum flux conserved vector, (5.2.44) and (5.2.45)

Substituting the momentum flux conserved vector, (5.2.44) and (5.2.45), into (5.4.2) we obtain

$$
\begin{aligned}
&-\frac{1}{r^2}\psi_r^2\xi^2+\frac{2}{r}\psi\zeta_r^1+\frac{1}{r}\psi_r^2 D_r\xi^2\\
-&\left[-\frac{1}{r}\psi_z\psi_r-\nu\psi_{rr}+\frac{\nu}{r}\psi_r-2\frac{l^2(z)}{r^2}\psi_r\psi_{rr}+\frac{l^2(z)}{r^3}\left(\psi_r\right)^2+\frac{l^(z)}{r}\left(\psi_{rr}\right)^2\right]D_r\xi^1=0.
\end{aligned}
\tag{5.4.8}
$$

Splitting equation (5.4.8) into coefficients of derivatives of ψ and S we obtain

$$\psi_z\left(\psi_r\right)^2:\frac{\partial\xi^1}{\partial\psi}=0, \tag{5.4.9}$$

$$\psi_z\psi_r S_r:\frac{\partial\xi^1}{\partial S}=0, \tag{5.4.10}$$

$$\left(\psi_{rr}\right)^2:\frac{\partial\xi^1}{\partial r}=0, \tag{5.4.11}$$

$$\left(\psi_r\right)^3:\frac{1}{r}\frac{\partial\xi^2}{\partial\psi}-\frac{l^2(z)}{r^3}\frac{\partial\xi^1}{\partial\psi}-\frac{2}{r}\frac{\partial\xi^2}{\partial\psi}=0, \tag{5.4.12}$$

$$\left(\psi_r\right)^2 S_r:-\frac{1}{r}\frac{\partial\xi^2}{\partial S}-\frac{l^2(z)}{r^3}\frac{\partial\xi^1}{\partial S}=0, \tag{5.4.13}$$

$$\psi_r:\frac{2}{r}\frac{\partial\eta^1}{\partial r}-\frac{\nu}{r}\frac{\partial\xi^1}{\partial r}=0, \tag{5.4.14}$$

$$\psi_r S_r:\frac{2}{r}\frac{\partial\eta^1}{\partial S}-\nu\frac{\partial\xi^1}{\partial S}=0, \tag{5.4.15}$$

$$\left(\psi_r\right)^2:\frac{2}{r}\frac{\partial\eta^1}{\partial\psi}-\frac{1}{r}\frac{\partial\xi^2}{\partial r}-\frac{1}{r^2}\xi^2-\frac{l^2(z)}{r^3}\frac{\partial\xi^1}{\partial r}-\frac{\nu}{r}\frac{\partial\xi^1}{\partial\psi}=0. \tag{5.4.16}$$

The system of partial differential equations (5.4.9) to (5.4.16) gives

$$\xi^1=\xi^1(z),\qquad \xi^2=\xi^2(z,r),\qquad \eta^1=\eta^1(z,\psi) \tag{5.4.17}$$

and equation (5.4.16) reduces to

$$2\frac{\partial\eta^1}{\partial\psi}-\frac{\partial\xi^2}{\partial r}-\frac{1}{r}\xi^2=0. \tag{5.4.18}$$

Differentiating equation (5.4.18) with respect to ψ we obtain

$$\frac{\partial^2\eta^1}{\partial\psi^2}=0 \tag{5.4.19}$$

giving

$$\eta^1=a_1(z)\psi+a_2(z), \tag{5.4.20}$$

where $a_1(z)$ and $a_2(z)$ are arbitrary functions of z. Again, differentiating equation (5.4.18) with respect to r gives the Euler equation

$$\frac{\partial^2 \xi^2}{\partial r^2} + \frac{1}{r}\frac{\partial \xi^2}{\partial r} - \frac{1}{r^2}\xi^2 = 0 \tag{5.4.21}$$

with the solution

$$\xi^2 = a_3(z)r + a_4(z)\frac{1}{r}, \tag{5.4.22}$$

where $a_3(z)$ and $a_4(z)$ are arbitrary functions of z. Substituting (5.4.20) and (5.4.22) back into (5.4.18) gives

$$a_3(z) = a_1(z). \tag{5.4.23}$$

The Lie point symmetry generator (5.4.1) becomes

$$X = \xi^1(z)\frac{\partial}{\partial z} + \left(a_1(z)r + a_4(z)\frac{1}{r}\right)\frac{\partial}{\partial r} + (a_1(z)\psi + a_2(z))\frac{\partial}{\partial \psi} + \eta^2(z,r,\psi,S)\frac{\partial}{\partial S}. \tag{5.4.24}$$

Substituting the momentum flux conserved vector (5.2.44) and (5.2.45) and the Lie point symmetry generator (5.4.24) into (5.4.3) then the following prolongations are needed:

$$\zeta_1^1 = \left(a_1(z) - \frac{d\xi^1}{dz}\right)\psi_z - \left(r\frac{da_1}{dz} + \frac{1}{r}\frac{da_4}{dz}\right)\psi_r + \psi\frac{da_1}{dz} + \frac{da_2}{dz}, \tag{5.4.25}$$

$$\zeta_2^1 = \frac{1}{r^2}a_4(z)\psi_r, \tag{5.4.26}$$

$$\zeta_{22}^1 = -\frac{2}{r^3}a_4(z)\psi_r + \left(\frac{2}{r^2}a_4(z) - a_1(z)\right)\psi_{rr}. \tag{5.4.27}$$

Separating (5.4.3) by powers and products of the partial derivatives of ψ we obtain

$$\psi_{rr} : \nu\left[\frac{d\xi^1}{dz} + \frac{2}{r^2}a_4(z) - a_1(z)\right] = 0, \tag{5.4.28}$$

$$\psi_r : -\frac{da_1}{dz}\psi - \frac{da_2}{dz} + \nu\left[\frac{d\xi^1}{dz} - \frac{2}{r^2}a_4(z) - a_1(z)\right] = 0, \tag{5.4.29}$$

$$(\psi_r)^2 : -3l^2(z)a_1(z) + 3\frac{1}{r^2}a_4(z) + \frac{d}{dz}\left(l^2(z)\right)\xi^1(z) + l^2(z)\frac{d\xi^1}{dz} = 0, \tag{5.4.30}$$

$$\psi_r\psi_{rr} : -3l^2(z)a_1(z) + 3\frac{1}{r^2}a_4(z) + \frac{d}{dz}\left(l^2(z)\right)\xi^1(z) + l^2(z)\frac{d\xi^1}{dz} = 0, \tag{5.4.31}$$

$$(\psi_{rr})^2 : -3l^2(z)a_1(z) + 3\frac{1}{r^2}a_4(z) + \frac{d}{dz}\left(l^2(z)\right)\xi^1(z) + l^2(z)\frac{d\xi^1}{dz} = 0. \tag{5.4.32}$$

There two cases to consider, $\nu = 0$ and $\nu \neq 0$

Case (i) $\nu = 0$

Equations (5.4.32) can be separated in powers of r and (5.4.29) in powers of ψ. We find that a_1 and a_2 are constants,

$$a_4 = 0 \tag{5.4.33}$$

and $\xi^1(z)$ and $l(z)$ satisfy

$$\frac{d\xi^1}{dz} + \frac{2}{l(z)}\frac{dl}{dz}\xi^1(z) = 3a_1. \tag{5.4.34}$$

Thus

$$\xi^1 = \xi^1(z), \quad \xi^2(z) = a_1 r, \quad \eta^1(\psi) = a_1\psi + a_2, \quad \eta^2 = \eta^2(z, r, \psi, S) \tag{5.4.35}$$

and therefore

$$X = \xi^1(z)\frac{\partial}{\partial z} + a_1 r\frac{\partial}{\partial r} + (a_1\psi + a_2) + \eta^2(z, r, \psi, S)\frac{\partial}{\partial S}. \tag{5.4.36}$$

Case (ii): $\nu \neq 0$

From (5.4.28), (5.4.29) and (5.4.30) we again have a_1 and a_2 are constants, $a_4 = 0$ and (5.4.34) but in addition

$$\frac{d\xi^1}{dz} = a_1 \tag{5.4.37}$$

we find that

$$\xi^1 = a_1 z + a_5, \quad \xi^2(z) = a_1 r, \quad \eta^1(\psi) = a_1\psi + a_2, \quad \eta^2 = \eta^2(z, r, \psi, S), \tag{5.4.38}$$

$$l(z) = A(a_1 z + a_5) \tag{5.4.39}$$

where a_5 and A are constants. Thus

$$X = (a_1 z + a_5)\frac{\partial}{\partial z} + a_1 r\frac{\partial}{\partial r} + (a_1\psi + a_2)\frac{\partial}{\partial \psi} + \eta^2(z, r, \psi, S)\frac{\partial}{\partial S}. \tag{5.4.40}$$

5.4.2 Association with thermal conserved vector

It has been shown so far that the associated Lie point symmetry is of the form

$$X = \xi^1(z)\frac{\partial}{\partial z} + a_1 r\frac{\partial}{\partial r} + (a_1\psi + a_2)\frac{\partial}{\partial \psi} + \eta^2(z, r, \psi, S)\frac{\partial}{\partial S} \tag{5.4.41}$$

where

$$\xi^1(z) = \begin{cases} a_1 z + a_5, & \nu \neq 0 \\ \xi^1(z), & \nu = 0 \end{cases} \tag{5.4.42}$$

and

$$l(z) = \begin{cases} A(a_1 z + a_5), & \nu \neq 0 \\ l(z), & \nu = 0 \end{cases} \tag{5.4.43}$$

and when $\nu = 0$, $\xi^1(z)$ and $l(z)$ satisfy

$$\frac{d\xi^1}{dz} + \frac{2}{l(z)}\frac{dl}{dz}\xi^1(z) = 3a_1. \tag{5.4.44}$$

Substituting the thermal flux conserved vector (5.2.46) and (5.2.47) and the symmetry generator (5.4.41) into (5.4.2) we obtain for $\nu = 0$ and $\nu \neq 0$

$$-S_r\eta^1 - \psi\zeta_r^2 - a_1\psi S_r = 0 \tag{5.4.45}$$

which expands to

$$-S_r\left(\psi\left(\frac{\partial\eta^2}{\partial S} + a_1\right) + a_2\right) - \psi\frac{\partial\eta^2}{\partial r} - \psi\psi_r\frac{\partial\eta^2}{\partial\psi} = 0. \tag{5.4.46}$$

Splitting this equation into coefficients of S_r and ψ_r and the remainder we obtain

$$\text{Remainder}: \frac{\partial\eta^2}{\partial r} = 0, \tag{5.4.47}$$

$$S_r : \psi\left(a_1 + \frac{\partial\eta^2}{\partial S}\right) + a_2 = 0, \tag{5.4.48}$$

$$\psi_r : \frac{\partial\eta^2}{\partial\psi} = 0. \tag{5.4.49}$$

From equations (5.4.47) and (5.4.49) we obtain

$$\eta^2 = \eta^2(z, S). \tag{5.4.50}$$

Splitting equation (5.4.48) into coefficients of ψ and the remainder we obtain

$$\psi : a_1 + \frac{\partial\eta^2}{\partial S} = 0, \tag{5.4.51}$$

$$\text{Remainder}: a_2 = 0. \tag{5.4.52}$$

Equation (5.4.51) gives

$$\eta^2 = -a_1 S + a_5(z) \tag{5.4.53}$$

where $a_5(z)$ is an arbitrary function z. Hence the Lie point symmetry generator (5.4.41) becomes

$$X = \xi^1(z)\frac{\partial}{\partial z} + a_1 r\frac{\partial}{\partial r} + a_1\psi\frac{\partial}{\partial\psi} + (-a_1 S + a_5(z))\frac{\partial}{\partial S}, \tag{5.4.54}$$

where $\xi^1(z)$ and $l(z)$ are given by (5.4.42) and (5.4.43) and when $\nu = 0$, (5.4.44) is satisfied.

Substituting the thermal flux conserved vector (5.2.46) and (5.2.47) and the Lie point symmetry generator (5.4.54) into (5.4.3) then the following prolongation coefficients are needed

$$\zeta_1^2 = \frac{da_6}{dz} - \left(a_1 + \frac{d\xi^1}{dz}\right)S_z, \tag{5.4.55}$$

$$\zeta_2^2 = -2a_1 S_r, \tag{5.4.56}$$

$$\zeta_2^1 = 0, \tag{5.4.57}$$

$$\zeta_{22}^1 = -a_1\psi_{rr}. \tag{5.4.58}$$

Splitting equation (5.4.3) into coefficients of powers and products of the partial derivatives of ψ and S we obtain

$$S_r : \frac{\kappa}{\rho c_v}\left(\frac{d\xi^1}{dz} - a_1\right) = 0, \tag{5.4.59}$$

$$\psi_r S_r : l(z)l_\theta(z)\frac{d\xi^1}{dz} + \frac{d}{dz}\left(l(z)l_\theta(z)\right) = 3a_1 l(z)l_\theta(z), \tag{5.4.60}$$

$$\psi_{rr} S_r : l(z)l_\theta(z)\frac{d\xi^1}{dz} + \frac{d}{dz}\left(l(z)l_\theta(z)\right) = 3a_1 l(z)l_\theta(z), \tag{5.4.61}$$

$$\text{Remainder} : \frac{da_5}{dz} = 0. \tag{5.4.62}$$

Therefore, from equation (5.4.62), a_5 is a constant. The Lie point symmetry generator (5.4.54) becomes

$$X = \xi^1(z)\frac{\partial}{\partial z} + a_1 r\frac{\partial}{\partial r} + a_1\psi\frac{\partial}{\partial \psi} + \left(-a_1 S + a_5\right)\frac{\partial}{\partial S}, \tag{5.4.63}$$

where

$$\frac{\kappa}{\rho c_v}\left(\frac{d\xi^1}{dz} - a_1\right) = 0 \tag{5.4.64}$$

and ξ^1 is related to the momentum mixing length $l(z)$ and the thermal mixing length $l_\theta(z)$ by

$$l(z)l_\theta(z)\frac{d\xi^1}{dz} + \frac{d}{dz}\left(l(z)l_\theta(z)\right) = 3a_1 l(z)l_\theta(z). \tag{5.4.65}$$

Also $\xi^1(z)$ and $l(z)$ are given by (5.4.42) and (5.4.43) and depend on whether $\nu = 0$ or $\nu \neq 0$ and when $\nu = 0$, $\xi^1(z)$ and $l(z)$ satisfy a further condition, (5.4.44).

Case (i): $\nu \neq 0$ **and** $\kappa \neq 0$

When $\nu \neq 0$ and $\kappa \neq 0$,

$$l(z)l_\theta(z)\frac{d\xi^1}{dz} + \frac{d}{dz}\left(l(z)l_\theta(z)\right) = 3a_1 l(z)l_\theta(z), \tag{5.4.66}$$

$$\frac{d\xi^1}{dz} - a_1 = 0 \tag{5.4.67}$$

and

$$\xi^1(z) = a_1 z + a_2, l(z) = A(a_1 z + a_2). \tag{5.4.68}$$

Equation (5.4.67) is identically satisfied. Substituting (5.4.68) into (5.4.65) gives

$$\left(a_1 z + a_2\right)\frac{dl_\theta}{dz} = a_1 l_\theta(z) \tag{5.4.69}$$

and therefore

$$l_\theta(z) = B\left(a_1 z + a_2\right), \tag{5.4.70}$$

where B is a constant. But the mixing lengths $l(z)$ and $l_\theta(z)$ must be zero at $z = 0$ since turbulent effects are absent close to the nozzle of the jet. Thus from (5.4.66) and

(5.4.70),

$$a_2 = 0, \qquad l(z) = l_0 z, \qquad l_\theta(z) = l_1 z, \tag{5.4.71}$$

where $l_0 = Aa_1$ and $l_1 = Ba_1$ are positive constants. We assume that $a_1 \neq 0$. When $a_1 = 0$, we might have another reduction, but this is not considered in this dissertation. The Lie point symmetry generator (5.4.63) becomes

$$X = z\frac{\partial}{\partial z} + r\frac{\partial}{\partial r} + \psi\frac{\partial}{\partial \psi} + (-S + a_5^*)\frac{\partial}{\partial S}, \tag{5.4.72}$$

where $a_5 = a_5/a_1$. The star is suppressed.

Case (ii): $\nu \neq 0$ **and** $\kappa = 0$

We have seen in Case (i) that when $\nu \neq 0$,

$$\xi^1 = a_1 z \tag{5.4.73}$$

and

$$l(z) = l_0 z. \tag{5.4.74}$$

Equation (5.4.64) is now identically satisfied. Substituting equation (5.4.74) into (5.4.65) we obtain

$$z\frac{dl_\theta}{dz} = l_\theta(z) \tag{5.4.75}$$

which gives

$$l_\theta(z) = l_1 z, \tag{5.4.76}$$

where l_1 is a constant. We assume that $a_1 \neq 0$. When $a_1 = 0$, we might have another reduction, but this is not considered in this dissertation. The Lie point symmetry generator is given by (5.4.72).

Case (iii): $\kappa \neq 0$ **and** $\nu = 0$

Since $\nu = 0$,

$$\xi^1 = \xi^1(z), \qquad l = l(z) \tag{5.4.77}$$

and from (5.4.44)

$$\frac{d\xi^1}{dz} + \frac{2}{l(z)}\frac{dl}{dz}\xi^1(z) = 3a_1. \tag{5.4.78}$$

Also from (5.4.59), since $\kappa \neq 0$

$$\frac{d\xi^1}{dz} - a_1 = 0 \tag{5.4.79}$$

and from (5.4.65)

$$l(z)l_\theta(z)\frac{d\xi^1}{dz} + \frac{d}{dz}\left(l(z)l_\theta(z)\right) = 3a_1 l(z)l_\theta(z). \tag{5.4.80}$$

From (5.4.79)

$$\xi^1 = a_1 z + a_6, \tag{5.4.81}$$

where a_6 is a constant. Substituting (5.4.81) into (5.4.78) we obtain

$$\frac{1}{l(z)}\frac{dl}{dz} = \frac{a_1}{a_1 z + a_6}, \tag{5.4.82}$$

which gives

$$l(z) = l_0\left(z + \frac{a_6}{a_1}\right), \tag{5.4.83}$$

where l_0 is a constant. But the mixing length is zero at $z = 0$ since the jet flow is laminar close to the conduit. Therefore $a_6 = 0$ and equations (5.4.81) and (5.4.83) become

$$l(z) = l_0 z, \tag{5.4.84}$$

$$\xi^1(z) = a_1 z. \tag{5.4.85}$$

Substituting (5.4.84) and (5.4.85) into (5.4.65) we obtain

$$z\frac{dl_\theta}{dz} = l_\theta(z) \tag{5.4.86}$$

which gives

$$l_\theta(z) = l_1 z, \tag{5.4.87}$$

where l_1 is a constant. We assume that $a_1 \neq 0$. When $a_1 = 0$, we might have another reduction, but this is not considered in this dissertation. The Lie point symmetry generator is again given by (5.4.72).

Case (iv): $\nu = 0$ **and** $\kappa = 0$

When $\nu = 0$ and $\kappa = 0$,

$$\xi^1 = \xi^1(z), \qquad\qquad l = l(z) \tag{5.4.88}$$

and from (5.4.44),

$$\frac{d\xi^1}{dz} + \frac{2}{l(z)}\frac{dl}{dz}\xi^1(z) = 3a_1. \tag{5.4.89}$$

Equations (5.4.59) is identically satisfied since $\kappa = 0$ but from (5.4.65)

$$l(z)l_\theta(z)\frac{d\xi^1}{dz} + \frac{d}{dz}\left(l(z)l_\theta(z)\right) = 3a_1 l(z)l_\theta(z). \tag{5.4.90}$$

Now replacing $\frac{d\xi^1}{dz}$ in (5.4.90) using (5.4.90), equation (5.4.90) reduces to

$$\frac{1}{l_\theta}\frac{dl_\theta}{dz} = \frac{1}{l}\frac{dl}{dz} \tag{5.4.91}$$

and therefore

$$l_\theta(z) = Dl(z), \tag{5.4.92}$$

where D is a constant. We assume that $a_1 \neq 0$. When $a_1 = 0$, we might have another reduction, but this is not considered in this dissertation. Substituting (5.4.92) back into (5.4.90) we find that (5.4.90) reduces to (5.4.89).

Thus when $\nu = 0$ and $\kappa = 0$, the Lie point symmetry generator is

$$X = \xi^{*1}(z)\frac{\partial}{\partial z} + r\frac{\partial}{\partial r} + \psi\frac{\partial}{\partial \psi} + (-S + a_5^*)\frac{\partial}{\partial S}, \tag{5.4.93}$$

where

$$\xi^{*1}(z) = \frac{\xi^1(z)}{a_1}, \qquad a_5^* = a_5/a_1 \tag{5.4.94}$$

and

$$l_\theta(z) = Dl(z), \tag{5.4.95}$$

$$\frac{d\xi^{*1}}{dz} + \frac{2}{l(z)}\frac{dl}{dz}\xi^{*1}(z) = 3. \tag{5.4.96}$$

5.5 Invariant solutions

In this section we reduce the governing partial equations (5.2.12) and (5.2.13) in terms of the stream function to ordinary differential equations using the Lie point symmetry generators obtained in the previous section. These are the two cases:

Case A $\nu \neq 0, \kappa \neq 0; \nu \neq 0, \kappa = 0; \nu = 0, \kappa \neq 0$

$$X = z\frac{\partial}{\partial z} + r\frac{\partial}{\partial r} + \psi\frac{\partial}{\partial \psi} + (-S + a_5)\frac{\partial}{\partial S}, \tag{5.5.1}$$

$$l(z) = l_0 z, \qquad l_\theta(z) = l_1 z. \tag{5.5.2}$$

Case B $\nu = 0, \kappa = 0$

$$X = \xi^1(z)\frac{\partial}{\partial z} + r\frac{\partial}{\partial r} + \psi\frac{\partial}{\partial \psi} + (-S + a_5)\frac{\partial}{\partial S}, \tag{5.5.3}$$

$$l_\theta(z) = Dl(z), \tag{5.5.4}$$

$$\frac{d\xi^1}{dz} + \frac{2}{l(z)}\frac{dl}{dz}\xi^1(z) = 3. \tag{5.5.5}$$

In Case B there are two equations for the three unknowns $\xi^1(z)$, $l(z)$, $l_\theta(z)$. On further equation is required. This remaining equation is obtained using the conserved quantity and Prandtl's hypothesis [18].

Case A : $\nu \neq 0, \kappa \neq 0$**;** $\nu \neq 0, \kappa = 0$**;** $\nu = 0, \kappa \neq 0$

We first deal with the case when ν or/and κ are non zero. In Section 7.4.2 the Lie point symmetry generator for this case was shown to be

$$X = z\frac{\partial}{\partial z} + r\frac{\partial}{\partial r} + \psi\frac{\partial}{\partial \psi} + (-S + a_5)\frac{\partial}{\partial S}. \tag{5.5.6}$$

The Lie point symmetry generator (5.5.6) will be used to find invariant solutions for the system of governing partial differential equations (5.2.12) and (5.2.13).

The invariant solutions $f(z,r)$ and $g(z,r)$ are defined by

$$X(\psi - f(z,r))\Big|_{\psi=f(z,r)} = 0 \tag{5.5.7}$$

and

$$X(S - g(z,r))\Big|_{S=g(z,r)} = 0. \tag{5.5.8}$$

From equation (5.5.7) we obtain

$$f = z\frac{\partial f}{\partial z} + r\frac{\partial f}{\partial r}. \tag{5.5.9}$$

The characteristic equations for the differential equation (5.5.9) are

$$\frac{df}{f} = \frac{dz}{z} = \frac{dr}{r}. \tag{5.5.10}$$

The equation consisting of the second and third terms of (5.5.10) gives

$$\frac{r}{z} = k_1, \tag{5.5.11}$$

where k_1 is a constant. The equation consisting of the first and second terms of (5.5.10) gives

$$\frac{f}{z} = k_2, \tag{5.5.12}$$

where k_2 is a constant. The general solution is

$$k_2 = F(k_1), \tag{5.5.13}$$

where F is an arbitrary function. Thus since $f = \psi(z,r)$

$$\psi(r,z) = zF(\xi), \tag{5.5.14}$$

where

$$\xi = \frac{r}{z}. \tag{5.5.15}$$

Substituting invariant solution (5.5.14) into the governing partial differential equation (5.2.12) we obtain

$$\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right) - \frac{d}{d\xi}\left[\xi\left(\nu - l_0^2\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right)\right)\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right)\right] = 0. \tag{5.5.16}$$

From equation (5.5.8) we obtain

$$-(g - a_5) = z\frac{\partial g}{\partial z} + r\frac{\partial g}{\partial r}. \tag{5.5.17}$$

The characteristic equations for the differential equation (5.5.17) are

$$\frac{dg}{-(g - a_5)} = \frac{dz}{z} = \frac{dr}{r}. \tag{5.5.18}$$

The equation consisting of the second and third terms of (5.5.29) gives (5.5.11). The equation consisting of the first and second terms of equation (5.5.29) gives

$$z\,(g - a_5) = k_3 \tag{5.5.19}$$

where k_4 is a constant. The general solution is

$$k_4 = G(k_1). \tag{5.5.20}$$

Thus since $g = S(z, r)$

$$S(z, r) = z^{-1}G(\xi) + a_5. \tag{5.5.21}$$

Substituting the invariant solutions (5.5.21) and (5.5.14) into the thermal partial differential equation (5.2.13) we obtain

$$\frac{d}{d\xi}\left[\xi\left(\frac{\kappa}{\rho c_v} - l_0 l_1 \frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right)\right)\frac{dG}{d\xi}\right] + \frac{d}{d\xi}\,(FG) = 0. \tag{5.5.22}$$

Writing the boundary conditions (5.2.9), (5.2.10) and (5.2.11) in terms of the invariant solutions (5.5.14) and (5.5.21) gives

$$\lim_{\xi\to 0}\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right) = 0, \quad \lim_{\xi\to 0}\frac{1}{\xi}\left(-F + \xi\frac{dF}{d\xi}\right) = 0, \quad \frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right)(\beta_1) = 0, \tag{5.5.23}$$

$$\frac{1}{\xi}\frac{dF}{d\xi}(\beta_1) = 0, \tag{5.5.24}$$

$$\frac{dG}{d\xi}(0) = 0, \qquad \frac{dG}{d\xi}(\beta_2) = 0, \qquad G(\beta_2) = 0, \tag{5.5.25}$$

where

$$\beta_2 = \frac{r_{Tb}(z)}{z} \tag{5.5.26}$$

is a constant.

Case B: $\nu = 0$ **and** $\kappa = 0$

We now consider the case when ν and κ are both zero. For this case ξ^1 is an arbitrary function of z and the associated Lie point symmetry is

$$X = \xi^1(z)\frac{\partial}{\partial z} + r\frac{\partial}{\partial r} + \psi\frac{\partial}{\partial \psi} + (-S + a_5)\,\frac{\partial}{\partial S}. \tag{5.5.27}$$

Now $\psi = f(z, r)$ is an invariant solution generated by X provided (5.5.7) is satisfied which expands to

$$\xi^1(z)\frac{\partial f}{\partial z} + r\frac{\partial f}{\partial r} = 0. \tag{5.5.28}$$

The differential equations of the characteristic curves are

$$\frac{dz}{\xi^1(z)} = \frac{dr}{r} = \frac{df}{f}. \tag{5.5.29}$$

Solving the differential equation consisting of the first and second terms of equation (5.5.29) we obtain

$$\frac{r}{\exp\left(\int^z \frac{dz}{\xi^1(z)}\right)} = k_1, \tag{5.5.30}$$

where k_1 is a constant. Solving the differential equation consisting of the first and third terms of equation (5.5.29) we obtain

$$\frac{f}{\exp\left[\int^z \frac{dz}{\xi^1(z)}\right]} = k_2, \tag{5.5.31}$$

where k_2 is a constant. The general solution is

$$k_2 = F(k_1), \tag{5.5.32}$$

where F is an arbitrary function, which gives

$$\psi(z,r) = \exp\left(\int^z \frac{dz}{\xi^1(z)}\right) F(\xi) \tag{5.5.33}$$

where

$$\xi = \frac{r}{\exp\left(\int^z \frac{dz}{\xi^1(z)}\right)}. \tag{5.5.34}$$

Also, $S = g(z,r)$ is an invariant solution generated by the Lie point symmetry X provided (5.5.8) is satisfied. Using (5.5.27) this gives

$$\xi^1(z)\frac{\partial g}{\partial z} + r\frac{\partial g}{\partial r} = -\left(g - a_5\right). \tag{5.5.35}$$

The differential equation of the characteristic curves are

$$\frac{dz}{\xi^1(z)} = \frac{dr}{r} = \frac{dg}{-\left(g - a_5\right)} \tag{5.5.36}$$

The first pair of terms gives again (5.5.30). The first and last terms give

$$\frac{g - a_5}{\exp\left[-\int^z \frac{dz}{\xi^1(z)}\right]} = k_3. \tag{5.5.37}$$

The general solution is

$$k_3 = G(k_1) \tag{5.5.38}$$

where G is an arbitrary function. Thus since $g = S(z,r)$

$$S(z,r) = \exp\left[-\int^z \frac{dz}{\xi^1(z)}\right] G(\xi) + a_5. \tag{5.5.39}$$

Substituting the invariant solution (5.5.33) and (5.5.34) into the conserved quantity (5.3.9) we obtain

$$J = 2\rho \int_0^{\beta_1} \frac{1}{\xi}\left(\frac{dF}{d\xi}\right)^2 d\xi \tag{5.5.40}$$

where

$$\beta_1 = \frac{r_B(z)}{\exp\left(\int^z \frac{dz}{\xi^1(z)}\right)}. \tag{5.5.41}$$

Therefore,

$$r_B(z) = \beta_1 \exp\left(\int^z \frac{dz}{\xi^1(z)}\right). \tag{5.5.42}$$

Using Prandtl's hypothesis [18] which relates the momentum mixing lengths with the height of the boundary of the jet by

$$r_B(z) = Cl(z), \tag{5.5.43}$$

where C is a positive constant, results in the integral equation

$$\beta_1 \exp\left(\int^z \frac{dz}{\xi^1(z)}\right) = Cl(z). \tag{5.5.44}$$

Differentiating equation (5.5.44) with respect to z gives

$$\beta_1 \frac{\exp\left(\int^z \frac{dz}{\xi^1(z)}\right)}{\xi^1(z)} = C\frac{dl}{dz}. \tag{5.5.45}$$

Dividing equation (5.5.45) by (5.5.44) gives

$$\frac{dl}{dz} = \frac{l(z)}{\xi^1(z)}. \tag{5.5.46}$$

Substituting (5.5.46) into (5.5.5) we obtain

$$1 = \frac{\partial \xi^1}{\partial z} \tag{5.5.47}$$

which gives

$$\xi^1(z) = z + a_6, \tag{5.5.48}$$

where a_6 is a constant. Now substituting equation (5.5.48) into (5.5.44) we obtain

$$l(z) = l_0\left(z + \frac{a_6}{a_1}\right), \tag{5.5.49}$$

where $l_0 = \frac{\beta_1}{C}$ is a positive constant. However, the mixing length is zero at $z = 0$, since turbulent effects are absent close to the conduit. Hence the momentum mixing length becomes

$$l(z) = l_0 z \tag{5.5.50}$$

and, by (5.5.4) and (5.5.48),

$$l_\theta(z) = l_1 z, \qquad \xi^1(z) = z, \tag{5.5.51}$$

where $l_1 = Dl_0$ is a constant. The associated Lie point symmetry generator (5.5.1) becomes

$$X = z\frac{\partial}{\partial z} + r\frac{\partial}{\partial r} + (\psi + a_4)\frac{\partial}{\partial \psi} + (-S + a_5)\frac{\partial}{\partial S}. \tag{5.5.52}$$

The invariant solutions (5.5.33) and (5.5.39) with ξ defined by (5.5.34) reduce

$$\psi(z,r) = zF(\xi) \tag{5.5.53}$$

and

$$S(z,r) = z^{-1}G(\xi), \tag{5.5.54}$$

where

$$\xi = \frac{r}{z}. \tag{5.5.55}$$

Substituting invariant solutions (5.5.53) to (5.5.55) with the mixing lengths (5.5.50) and (5.5.51) into the governing equations (5.2.12) and (5.2.13), with $\nu = 0$ and $\kappa = 0$, we obtain

$$\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right) = -\frac{d}{d\xi}\left[\xi l_0^2\left(\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right)\right)^2\right], \tag{5.5.56}$$

$$\frac{d}{d\xi}(FG) = \frac{d}{d\xi}\left[\xi l_0 l_1 \frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right)\frac{dG}{d\xi}\right]. \tag{5.5.57}$$

Writing the boundary conditions (5.2.9), (5.2.10) and (5.2.11) in terms of the invariant solutions (5.5.53) to (5.5.55) gives (5.5.23), (5.5.24) and (5.5.25) with

$$\beta_1 = \frac{r_B(z)}{z}, \qquad \beta_2 = \frac{r_{Tb}(z)}{z}. \tag{5.5.58}$$

5.6 Numerical solution

Integrating both equations (5.5.16) and (5.5.22) and imposing boundary conditions (5.5.23), (5.5.24) and (5.5.25) at $\xi = \beta_1$ to determine the constant of integration we obtain

$$\xi\left(\nu - l_0^2\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right)\right)\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right) + \frac{F(\xi)}{\xi}\frac{dF}{d\xi} = 0, \tag{5.6.1}$$

$$\xi\left(\frac{\kappa}{\rho c_v} - l_0 l_1\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right)\right)\frac{dG}{d\xi} + F(\xi)G(\xi) = 0. \tag{5.6.2}$$

This is another example of the double reduction theorem of Sjoberg [10,11] that if a reduction from a partial differential equation to an ordinary differential equation is performed by an associated Lie point symmetry with a conserved vector of the partial differential equation then the ordinary differential equation can be integrated at least once.

Expanding equation (5.6.1) we obtain

$$\left(\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right)\right)^2 - \frac{\nu}{l_0^2}\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right) - \frac{1}{l_0^2\xi^2}F\frac{dF}{d\xi} = 0, \tag{5.6.3}$$

which is a quadratic equation for $\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right)$. Hence

$$\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right) = \frac{\nu}{2l_0^2} \pm \sqrt{\left(\frac{\nu}{2l_0^2}\right)^2 + \left(\frac{1}{l_0^2\xi}\right)^2 F\frac{dF}{d\xi}}. \tag{5.6.4}$$

The discriminant in (5.6.4) is real because

$$\overline{v}_z(z,r) = \frac{1}{r}\frac{\partial\psi}{\partial r} = \frac{1}{z\xi}\frac{dF}{d\xi} > 0. \tag{5.6.5}$$

Writing the inequality (5.2.6) in terms of the invariant solution (5.5.21) we obtain

$$\frac{\partial\overline{v}_z}{\partial r}(z,r) = \frac{1}{z^2}\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right) < 0. \tag{5.6.6}$$

Hence, since also $\frac{1}{\xi}\frac{dF}{d\xi} > 0$, we take the negative sign in (5.6.4) and obtain

$$\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right) = \frac{\nu}{2l_0^2} - \sqrt{\left(\frac{\nu}{2l_0^2}\right)^2 + \left(\frac{1}{l_0^2\xi}\right)^2 F\frac{dF}{d\xi}}, \tag{5.6.7}$$

which expands to

$$\frac{d^2F}{d\xi^2} = \frac{1}{\xi}\frac{dF}{d\xi} + \frac{\nu\xi}{2l_0^2} - \sqrt{\left(\frac{\nu}{2l_0^2}\right)^2 + \left(\frac{1}{l_0^2\xi}\right)^2 F\frac{dF}{d\xi}}. \tag{5.6.8}$$

Making $\frac{dG}{d\xi}$ the subject of (5.6.2) gives

$$\frac{dG}{d\xi} = -\frac{F(\xi)G(\xi)}{\xi\left(\frac{\kappa}{\rho c_v} - l_0 l_1 \frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right)\right)}. \tag{5.6.9}$$

The system of ordinary differential equations (5.6.8) and (5.6.9) cannot be solved analytically and therefore is solved numerically.

Because of the factor $\frac{1}{\xi}$ in (5.6.8) and (5.6.9) the numerical solution cannot start at $\xi = 0$. The initial values $F(\delta)$ and $\frac{dF}{d\xi}(\delta)$ have therefore to be obtained.

Consider

$$F(\xi) = A\xi^n \text{ as } \xi \to 0, \tag{5.6.10}$$

where A is a constant. From the schematic mean velocity profile of the jet (Figure 5.1), it can be seen that

$$\overline{v}_z \text{ is positive and finite as } r \to 0. \tag{5.6.11}$$

Writing (5.6.11) in terms of the similarity solution (5.5.21) we obtain

$$\overline{v}_z = \frac{1}{z\xi}\frac{dF}{d\xi} = \frac{An(n-1)}{z}\xi^{n-2} \text{ is positive and finite as } r \to 0. \tag{5.6.12}$$

Therefore,

$$n = 2 \tag{5.6.13}$$

and (5.6.10) becomes

$$F(\xi) = A\xi^2 \text{ as } \xi \to 0. \tag{5.6.14}$$

The initial conditions at $\xi = \delta$ become

$$F(\delta) = A\delta^2, \tag{5.6.15}$$

$$F'(\delta) = 2A\delta. \tag{5.6.16}$$

The value of A is found by the shooting method, with the target being the conserved quantity (5.3.9) written in terms of the invariant solution (5.5.21):

$$J = 2\pi\rho \int_0^{\beta_1} \frac{1}{\xi} \left(\frac{dF}{d\xi} \right)^2 d\xi. \tag{5.6.17}$$

The upper limit β_1 is not known and is to be determined during the numerical solution. We can now check that the boundary conditions (5.5.23), (5.5.24) are satisfied at $\xi = 0$ when (5.6.15) holds:

$$\overline{v}_z(z,r) = \frac{1}{z}\frac{1}{\xi}\left[-F + \xi\frac{dF}{d\xi}\right] = \frac{A\xi}{z} = 0, \text{ as } \xi \to 0, \tag{5.6.18}$$

$$\frac{\partial \overline{v}_z}{\partial r}(z,r) = \frac{1}{z}\frac{d}{d\xi}\left[\frac{1}{\xi}\frac{dF}{d\xi}\right] = \frac{1}{z}\frac{d}{d\xi}\left[2A\right] = 0, \text{ as } \xi \to 0. \tag{5.6.19}$$

The boundary conditions (5.5.23), (5.5.24) are therefore satisfied.

From (5.5.25) the boundary condition on $G(\xi)$ at $\xi = 0$ is

$$\frac{dG}{d\xi}(0) = 0. \tag{5.6.20}$$

The boundary condition on $G(0)$ is not known. Consider

$$G(\xi) = B\xi^m, \text{ as } \xi \to 0. \tag{5.6.21}$$

Then

$$\frac{dG}{d\xi} = mB\xi^{m-1}, \text{ as } \xi \to 0 \tag{5.6.22}$$

and (5.6.20) is satisfied by $m = 0$ or any $m > 1$. We take $m = 0$ so that

$$G(\xi) = B, \text{ as } \xi \to 0. \tag{5.6.23}$$

The initial condition for G at $\xi = \delta$ is therefore

$$G(\delta) = B, \tag{5.6.24}$$

where B is a constant. Similarly, the value of B is found by the shooting method with the target being the conserved quantity (5.3.18), written in terms of the invariant solution (5.5.14) and (5.5.21),

$$K = 2\pi\rho c_v \int_0^{\beta_2} G(\xi)\frac{dF}{d\xi}d\xi. \tag{5.6.25}$$

The initial value problem in the shooting method can be summarised as follows: Initial conditions

$$F(\delta) = A\delta^2, \tag{5.6.26}$$

$$\frac{dF}{d\xi}(\delta) = 2A\delta, \tag{5.6.27}$$

$$G(\delta) = B. \tag{5.6.28}$$

Differential equations

$$\frac{d^2F}{d\xi^2} = \frac{1}{\xi}\frac{dF}{d\xi} + \frac{\nu\xi}{2l_0^2} - \sqrt{\left(\frac{\nu}{2l_0^2}\right)^2 + \left(\frac{1}{l_0^2\xi}\right)^2 F\frac{dF}{d\xi}}, \tag{5.6.29}$$

$$\frac{dG}{d\xi} = -\frac{F(\xi)G(\xi)}{\xi\left(\frac{\kappa}{\rho c_v} - l_0 l_1 \frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right)\right)}. \tag{5.6.30}$$

The targets

$$J = 2\pi\rho\int_0^{\beta_1}\frac{1}{\xi}\left(\frac{dF}{d\xi}\right)^2 d\xi, \tag{5.6.31}$$

$$K = 2\pi\rho c_v\int_0^{\beta_2} G(\xi)\frac{dF}{d\xi}d\xi. \tag{5.6.32}$$

The conserved quantities J and K are prescribed as well as ρ, c_v and the mixing lengths l_0 and l_1. The shooting method determines the constants A and B, the upper limits β_1 and β_2 as well as $F(\xi)$ and $G(\xi)$.

Initially, we guess values of A and B and find the corresponding solution in order to calculate the conserved quantities (5.6.31) and (5.6.32). If equations (5.6.31) and (5.6.32) for conserved quantities are not satisfied then we modify values of A and B and try again.

Let

$$F(\xi, A, B), \qquad\qquad G(\xi, A, B)$$

be the solution for the system of ordinary differential equations (5.6.29) and (5.6.30), with corresponding constants A and B. Assume these solutions yield conserved quantities

$$\overline{J}(A,B) = 2\pi\rho\int_0^{\beta_1}\frac{1}{\xi}\left(\frac{dF}{d\xi}(\xi,A,B)\right)^2 d\xi, \tag{5.6.33}$$

$$\overline{K}(A,B) = 2\pi\rho c_v\int_0^{\beta_2} G(\xi,A,B)\frac{dF}{d\xi}(\xi,A,B)d\xi. \tag{5.6.34}$$

We require that

$$J - \overline{J}(A,B) = 0, \tag{5.6.35}$$

$$K - \overline{K}(A,B) = 0. \tag{5.6.36}$$

Suppose (A_k, B_k) is the current best guess of A and B which satisfy (5.6.35) and (5.6.36). To update (A_k, B_k) we need steps δ_1 and δ_2 which get us closer to the solution, i.e

$$J - \overline{J}(A_k+\delta_1, B_k+\delta_2) = 0, \tag{5.6.37}$$

$$K - \overline{K}(A_k+\delta_1, B_k+\delta_2) = 0. \tag{5.6.38}$$

Expanding equations (5.6.37) and (5.6.38) using the Taylor series expansion we obtain

$$\begin{pmatrix} \frac{\partial\overline{J}}{\partial A}(A_k,B_k) & \frac{\partial\overline{J}}{\partial B}(A_k,B_k) \\ \frac{\partial\overline{K}}{\partial A}(A_k,B_k) & \frac{\partial\overline{K}}{\partial B}(A_k,B_k) \end{pmatrix}\begin{pmatrix}\delta_1 \\ \delta_2\end{pmatrix} = \begin{pmatrix} J - \overline{J}(A_k,B_k) \\ K - \overline{K}(A_k,B_k)\end{pmatrix}. \tag{5.6.39}$$

Then, the updated guess of constants A and B becomes

$$A_{k+1} = A_k + \delta_1, \tag{5.6.40}$$

$$B_{k+1} = B_k + \delta_2. \tag{5.6.41}$$

We use the secant method to compute the partial derivatives in equation (5.6.39). For example:

$$\frac{\partial \overline{J}}{\partial A}(A_k, B_k) = \frac{\overline{J}(A_k, B_k) - \overline{J}(A_{k-1}, B_k)}{A_k - A_{k-1}}. \tag{5.6.42}$$

This suggests the numerical algorithm to calculate A and B:

1. Choose (A_0, B_0) and (A_1, B_1).
2. Set $k = 1$.
3. Compute $F(\xi, A_k, B_k)$, $G(\xi, A_k, B_k)$, $F(\xi, A_{k-1}, B_{k-1})$ and $G(\xi, A_{k-1}, B_{k-1})$ using the system of differential equations (5.6.8) and (5.6.9) with initial conditions (5.6.15), (5.6.16) and (5.6.24). For this we used the Runge-Kutta scheme.
4. Compute derivatives $\frac{\partial \overline{J}}{\partial A}$, $\frac{\partial \overline{J}}{\partial B}$, $\frac{\partial \overline{K}}{\partial A}$ and $\frac{\partial \overline{K}}{\partial B}$ using the secant method (5.6.42).
5. Compute steps δ_1 and δ_2 using equation (5.6.39).
6. Update $A_{k+1} = A_k + \delta_1$, $B_{k+1} = B_k + \delta_2$, $A_k = A_{k-1}$ and $B_k = B_{k-1}$.
7. If $\left|J - \overline{J}(A_{k+1}, B_{k+1})\right| < \epsilon$ and $\left|K - \overline{K}(A_{k+1}, B_{k+1})\right| < \epsilon$ then stop, else continue to step 3 with $k \to k+1$ (increment k), where ϵ is the tolerance of the numerical method.

Once the values of A and B are found, we use the Runge-Kutta scheme to solve the resulting IVP.

Table 5.1 shows the results for the shooting method when ν and κ are 0.1. It can be seen that the shooting method converges just after four iterations.

A_k	B_k	$\left\|J - \overline{J}(A_k, B_k)\right\|$	$\left\|K - \overline{K}(A_k, B_k)\right\|$
0.2	0.2	0.208729225733	0.604365803881
0.239783187898	0.634127641884	0.07501439799	0.421480030115
0.229265552819	0.428104169707	0.00416678339063	0.0702507669816
0.229819026591	0.458403513182	7.46862339684e-05	0.00276368176197
0.229829128226	0.459662841124	7.7089337136e-08	6.7417674583e-06

TABLE 5.1: Shooting method results for $\nu = 0.1$ and $\kappa = 0.1$.

For Figures 5.2 to 5.8 $J = 1$, $K = 1$, $\rho = 1$, $l_0 = 1$, $l_1 = 1$, $c_v = 1$ and the values of ν and κ are varied from zero to one.

Case (i): $\nu \neq 0$ **and** $\kappa \neq 0$

It can be shown that

$$\overline{v}_z(z, r) = \frac{1}{z}\frac{1}{\xi}\frac{dF}{d\xi}. \tag{5.6.43}$$

Figure 5.2 shows the numerical results for $\frac{1}{\xi}\frac{dF}{d\xi}$ plotted against ξ for different values of the dynamic viscosity, ν. It can be seen that $\frac{1}{\xi}\frac{dF}{d\xi}$ gets closer and closer to zero as ξ increases. When $\frac{1}{\xi}\frac{dF}{d\xi}$ is sufficiently small we cannot reliably determine whether $\frac{1}{\xi}\frac{dF}{d\xi}$ is zero or not since we are restricted by the accuracy of the numerical scheme. When $\nu = 0$ and $\kappa = 0$ the value of $\frac{1}{\xi}\frac{dF}{d\xi}$ becomes a non-real number when ξ is greater than some value which we assume to be β_1, but when $\nu \neq 0$ and $\kappa \neq 0$ the value of $\frac{1}{\xi}\frac{dF}{d\xi}$ is real for all values of ξ considered. This suggests that $\frac{1}{\xi}\frac{dF}{d\xi}$ remains a real number close to zero for all values ξ sufficiently large, which leads us to conclude that the momentum boundary β_1 is infinite when $\nu \neq 0$ and $\kappa \neq 0$.

It can be also shown that

$$\frac{\partial \overline{v}_z}{\partial r}(z,r) = \frac{1}{z}\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right). \tag{5.6.44}$$

Figure 5.3 shows the numerical results for $\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right)$ plotted against ξ for different values of ν. The region where $\frac{\partial}{\partial r}\overline{v}_z$ is largest is called the shear layer of the jet. As the kinematic viscosity increases the shear layer region increases in width due to the increase in viscous diffusion.

The temperature difference is given by

$$S(z,r) = \frac{1}{z}G(\xi). \tag{5.6.45}$$

Figure 5.4 shows the numerical results for $G(\xi)$ for different values of thermal conductivity, κ, and dynamic viscosity, ν. Similar to the numerical results for $\frac{1}{\xi}\frac{dF}{d\xi}$ the value of $G(\xi)$ gets closer and closer to zero when ξ gets sufficiently large but never becomes a non-real value. Hence we conclude that β_2 is infinite.

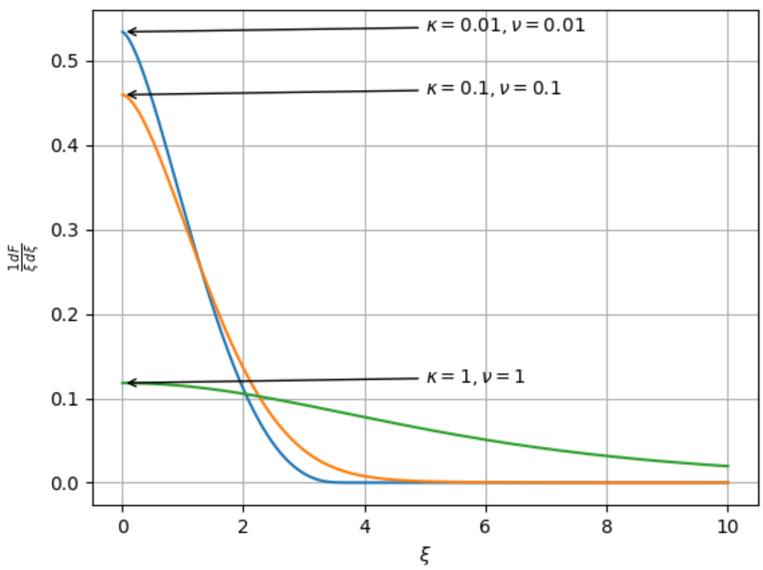

FIGURE 5.2: Graphs of $\frac{1}{\xi}\frac{dF}{d\xi}$ versus ξ for $\nu = 0.01$, $\nu = 0.1$ and $\nu = 1$.

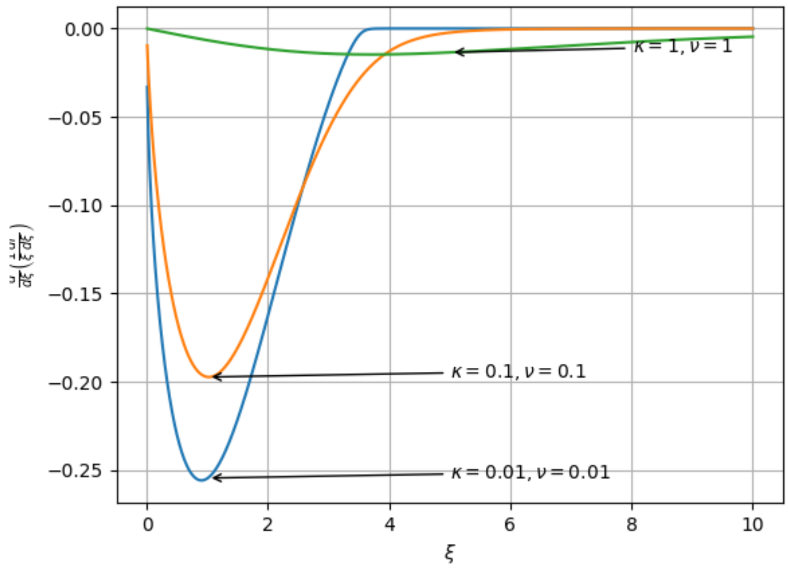

FIGURE 5.3: Graphs of $\frac{d}{d\xi}\left(\frac{1}{\xi}\frac{dF}{d\xi}\right)$ versus ξ for $\nu = 0.01$, $\nu = 0.1$ and $\nu = 1$.

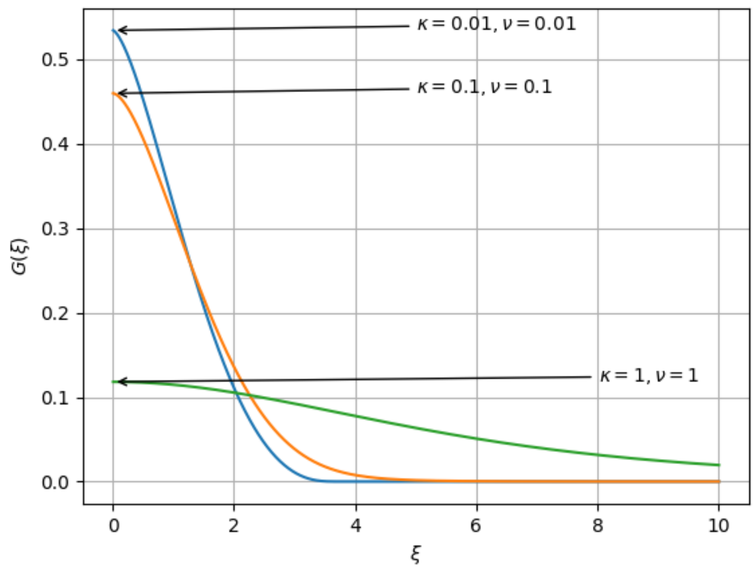

FIGURE 5.4: $G(\xi)$ versus ξ for varying values of ν and κ.

Case (ii): $\nu \neq 0$ **and** $\kappa = 0$

Figure 5.5 shows the graphs of the numerical solution for $G(\xi)$ plotted against ξ when $\nu = 0.01, \kappa = 0$ and $\nu = 0.1, \kappa = 0$. These numerical results for $G(\xi)$ are stable when $\xi < 6$ but unstable when $\xi > 6$. This suggests that there is a boundary for $G(\xi)$ at $\xi = 6$.

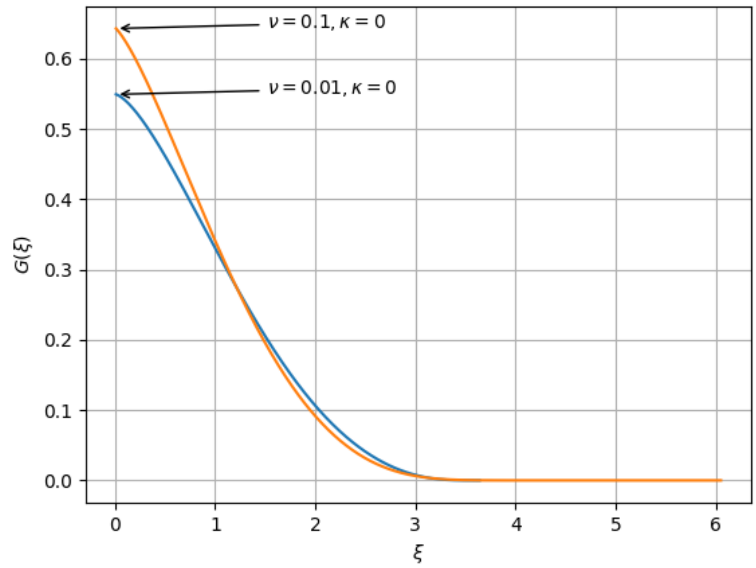

FIGURE 5.5: Graph of $G(\xi)$ versus ξ for varying values of ν with $\kappa = 0$.

Case (iii): $\nu = 0$ **and** $\kappa \neq 0$

We have seen in Chapter 4 that when $\nu = 0$ and $\kappa \neq 0$ the numerical scheme presented for the two-dimensional jet could not accurately determine F and G. Figure 5.6 shows the graphs of the numerical solution of $\frac{1}{\xi}\frac{dF}{d\xi}$ and $G(\xi)$ plotted against ξ for when $\nu = 0$, $\kappa = 1$ with A and B set to 0.27 and 0.6, respectively. In order to update guesses of A and B we must compute conserved quantities (5.6.31) and (5.6.32). It can be seen that β_1 is finite and $\beta_2 > \beta_1$. Therefore G cannot be determined when $\beta_1 < \xi < \beta_2$ and hence we cannot compute the conserved quantity (5.6.32). Thus the guesses for A and B cannot be updated using the above numerical scheme. Therefore the shooting method presented in this chapter cannot accurately determine the invariant solution when $\nu = 0$ and $\kappa \neq 0$. In order to obtain an accurate solution a different model must be applied to find G when $\beta_1 < \xi < \beta_2$. This model may require matching to an outer solution in the region $\xi > \beta_1$ and is beyond the scope of this dissertation.

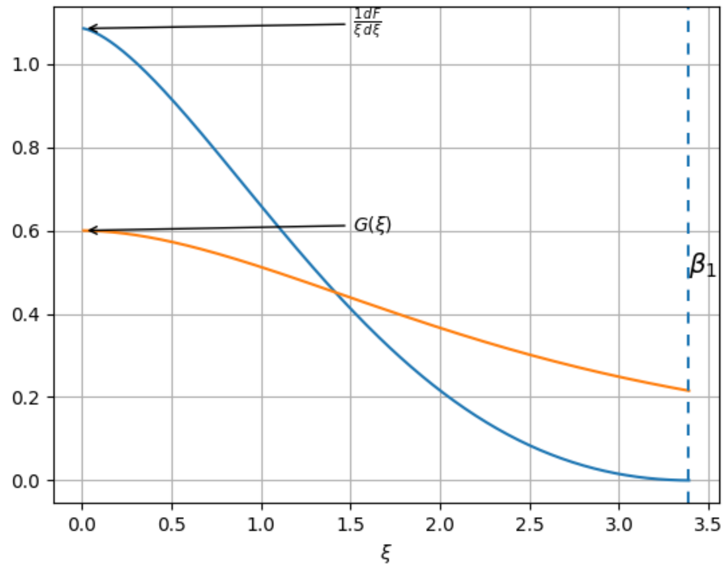

FIGURE 5.6: The initial guesses of $\frac{1}{\xi}\frac{dF}{d\xi}$ and $G(\xi)$.

Case (iv): $\nu = 0$ **and** $\kappa = 0$

Figures 5.7 and 5.8 show the graphs of the numerical solution for $\frac{1}{\xi}\frac{dF}{d\xi}$ and $G(\xi)$ plotted against ξ for $\nu = 0$ and $\kappa = 0$. It can be seen in these figures that the boundaries β_1 and β_2 are finite and equal. This was also shown to be the case for the two-dimensional jet.

Figures 5.9 to 5.11 show numerical results when $\nu = 0$, $\kappa = 0$, $\rho = 1$, $c_v = 1$, $l_0 = 1$, $l_1 = 1$ and varying values of the conserved quantities J and K. As J increases the mean velocity increases while the mean temperatures decrease at every point except the boundary of the jet. As K increases the mean temperature increases at every point except the boundary. The conserved quantity K has no effect on the mean velocity.

Figures 5.12 to 5.14 show numerical results when $\nu = 0$, $\kappa = 0$, $\rho = 1$, $c_v = 1$, $J = 1$, $K = 1$ and varying values of the mixing lengths l_0 and l_1. As the momentum mixing length increases the boundary of the jet increases and the centreline mean velocity decrease. This is because an increase in l_0 means an increase in the effective viscosity which slows the fluid at the centreline of the jet and entrains the fluid farther from the centreline. It can be seen from Figure 5.14 that, unlike the momentum mixing length, the thermal mixing length does not affect jet boundary width. This fact was also shown to be true for two-dimensional jets. As the thermal mixing length increases the mean temperature at the centreline decreases while close to the boundary of the jet the mean temperature difference increases. An increase in the thermal mixing length means an increase in the effective conductivity which increases the transfer of heat from the centreline to the boundary of the jet.

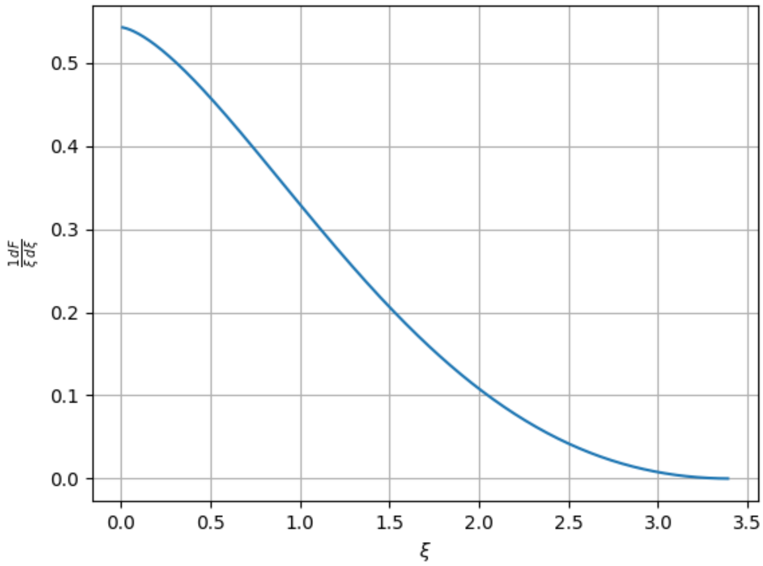

FIGURE 5.7: The graph of the numerical solution for $\frac{1}{\xi}\frac{dF}{d\xi}$ for $\nu = 0$ and $\kappa = 0$.

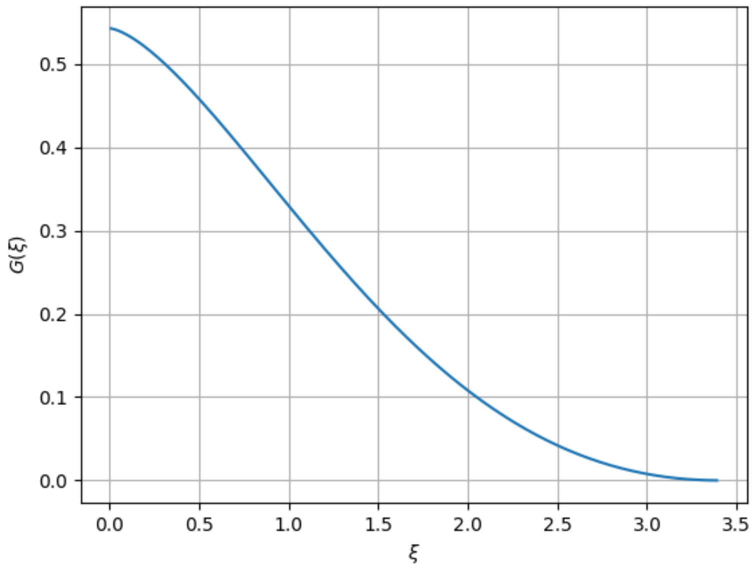

FIGURE 5.8: The graph of the numerical solution for $G(\xi)$ for $\nu = 0$ and $\kappa = 0$.

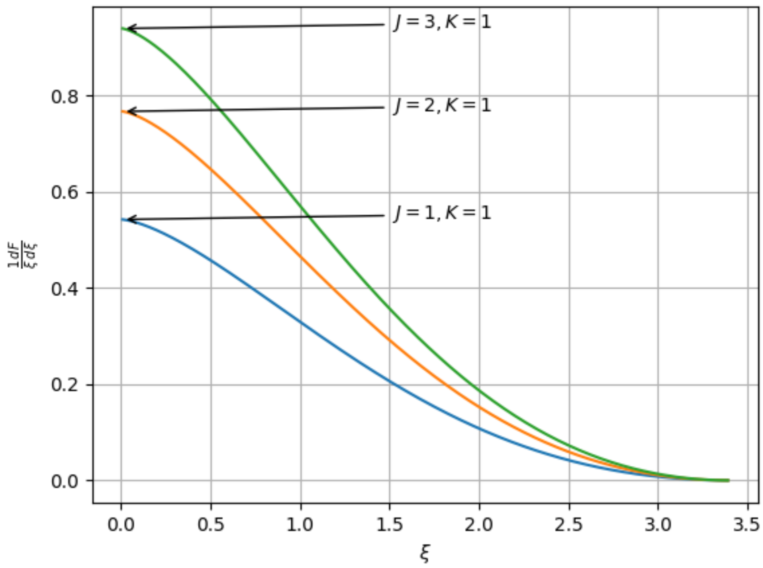

FIGURE 5.9: Graphs of $\frac{1}{\xi}\frac{dF}{d\xi}$ versus ξ for varying values of the conserved quantity J with $\nu = 0$ and $\kappa = 0$.

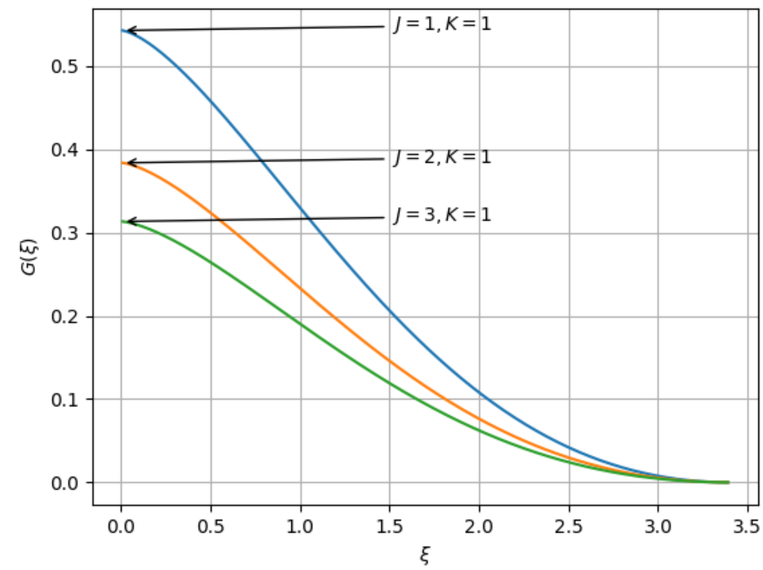

FIGURE 5.10: Graph of $G(\xi)$ versus ξ for varying values of the conserved quantity J with $\nu = 0$ and $\kappa = 0$.

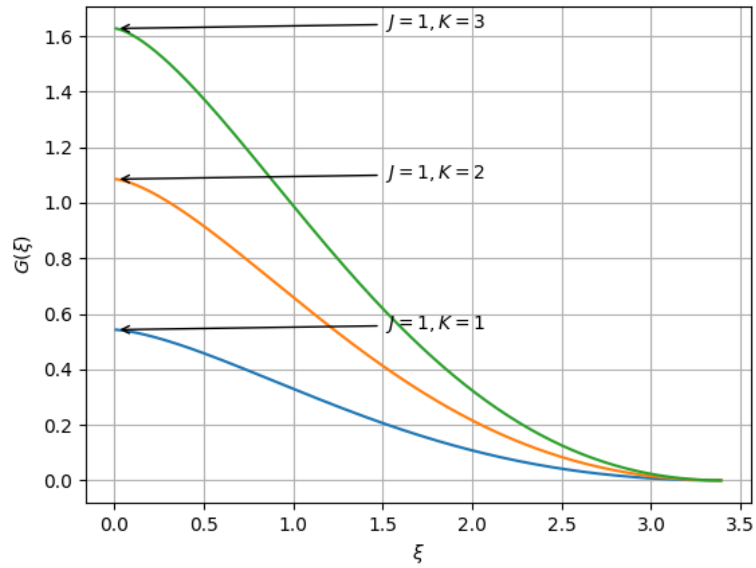

FIGURE 5.11: Graphs of $G(\xi)$ versus ξ for varying values of the conserved quantity K with $\nu = 0$ and $\kappa = 0$.

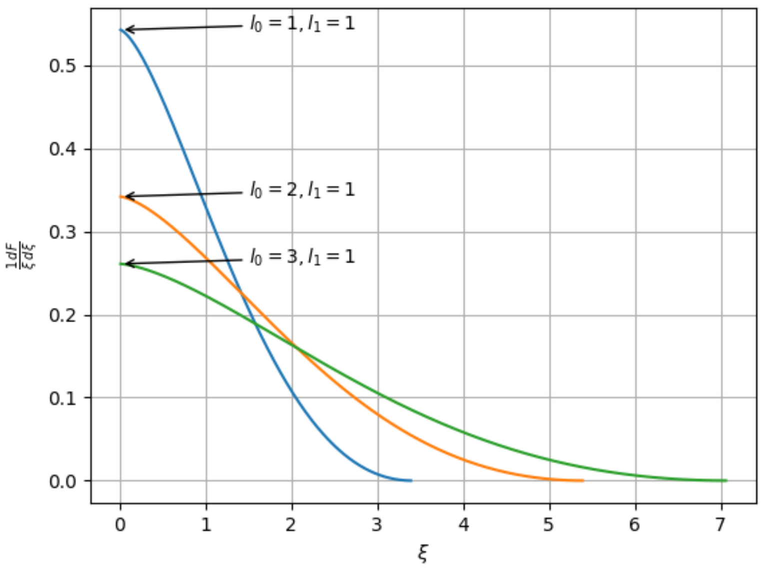

FIGURE 5.12: Graphs of $\frac{1}{\xi}\frac{dF}{d\xi}$ versus ξ for varying values of the mixing length l_0 with $\nu = 0$ and $\kappa = 0$.

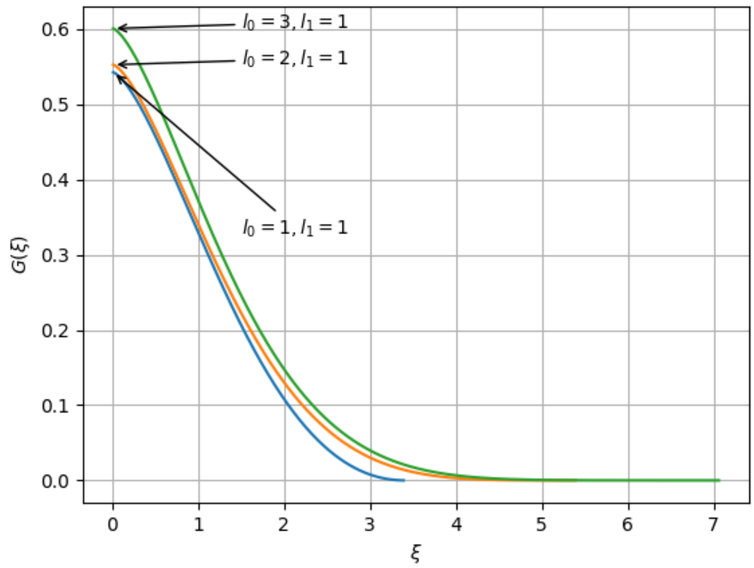

FIGURE 5.13: Graphs of $G(\xi)$ versus ξ for varying values of the mixing length l_0 with $\nu = 0$ and $\kappa = 0$.

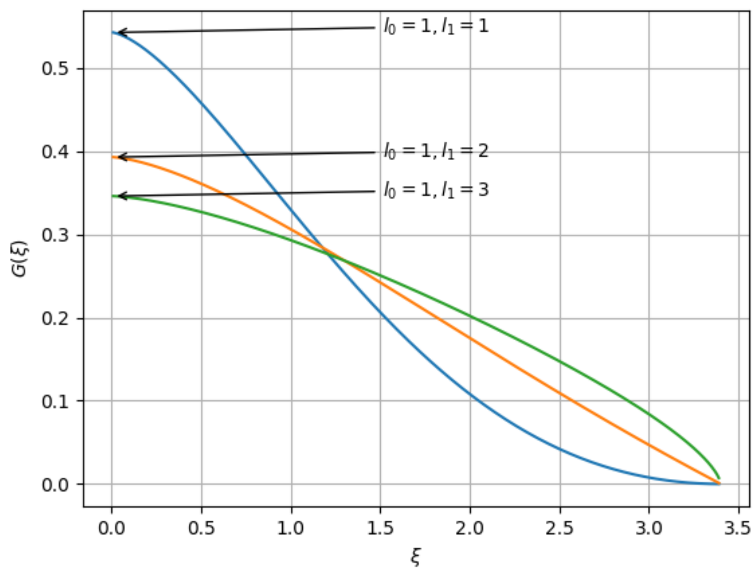

FIGURE 5.14: Graphs of $G(\xi)$ versus ξ for varying values of the mixing length l_1 with $\nu = 0$ and $\kappa = 0$.

5.7 Conclusions

In this chapter, we reduced the governing partial differential equations (5.2.12) and (5.2.13) to ordinary differential equations (5.5.16) and (5.5.22) using the associated Lie point symmetry which was derived using the conserved vectors for the partial differential equations. These ordinary differential equations were integrated once, further illustrating the double reduction theorem of Sjoberg [10,11].

We found the form for the momentum and thermal mixing lengths for an invariant solution to exist. When $\nu = 0$ and $\kappa = 0$ the jet is bounded in the radial direction and the mixing lengths were obtained from Prandtl's hypothesis that the length is proportional to the half-width of the jet. The numerical solution showed that the width of the momentum and thermal jets were equal. The invariant solution for $\nu \to 0$, $\kappa \to 0$ agreed with the invariant solution for $\nu = 0$, $\kappa = 0$ which was not the case for the two-dimensional jet.

From the numerical results, we can conclude that when $\nu \neq 0$ and $\kappa \neq 0$ the widths of the jet are infinite. The numerical results also showed that the thermal mixing length does not affect the boundary thickness when $\nu = 0$ and $\kappa = 0$. Hence l_1 can be increased without affecting the boundary of the jet, but keeping in mind that $l_1 < \beta_1$ since the fluid flow outside the boundary is laminar. The momentum mixing length does, however, affect the boundary of the jet. An increase in the momentum mixing length constant l_0 resulted in an increase in the width of the jet. An increase in l_0 also resulted in a decrease in the mean velocity along the centreline but an increase in the mean velocity farther from the centreline. This means that the drag on the fluid on the centreline increased and the entrainment of the mean flow also increased.

We were not able to find the numerical solution when $\nu = 0$ and $\kappa \neq 0$ since $G(\xi)$ could not be determined for $\beta_1 < \xi < \beta_2$. A different model is therefore needed for the laminar flow and mean temperature when $r_B(z) < r < r_{Tb}(x)$. This is beyond the scope of this dissertation. An interesting fact to note is that there the flow is laminar in this region and the boundary conditions on $T(z, r_B(z))$ can be determined from the current model and $T(z, r_{Tb}(z)) = T_B$.

Chapter 6

Conclusions

In this dissertation, we investigated the two-dimensional and axisymmetric turbulent thermal jets using the associated Lie point symmetry approach to reduce the coupled system of governing partial differential equations to ordinary differential equations which were then integrated once, further demonstrating the double reduction theorem [10,11]. The ordinary differential equations for the invariant solutions could only be solved analytically for the two-dimensional jet when $\nu = 0$ and $\kappa = 0$.

The associated Lie point symmetries were obtained from conserved vectors which were derived from conservation laws. The conservation laws were again used to obtain conserved quantities. The conserved quantities were essential in ensuring that the system of ordinary differential equations for the invariant solution had a unique solution. The conserved quantities were also used to find arbitrary functions in the Lie point symmetry when $\nu = 0$ and $\kappa = 0$.

For the two-dimensional free jet and when $\nu = 0$ and $\kappa = 0$, the solution obtained using Prandtl's hypothesis [18] was inconsistent with the solution obtained by letting ν and κ tend to zero in the solution for $\nu \neq 0$ and $\kappa \neq 0$. The correct solution can only be determined through experiments. The solution for the axisymmetric jet when $\nu = 0$ and $\kappa = 0$ obtained using Prandtl's hypothesis was consistent with the solution obtained by letting ν and κ tend to zero in the solution for $\nu \neq 0$ and $\kappa \neq 0$.

The solution obtained numerically perfectly matched the analytical solution when $\nu \to 0$ and $\kappa \to 0$ for the two-dimensional jet. The momentum boundary β_1 and the thermal boundary β_2 were shown to be equal in the two-dimensional and axisymmetric jets when $\nu = 0$ and $\kappa = 0$. The thermal mixing length was shown not to affect the width of the jet for both the two-dimensional and axisymmetric jets. This means the thermal mixing length can be increased without affecting the width of the boundary, but cannot be greater than the width of the boundary since the jet is laminar outside of the boundary. However, as the momentum mixing length increases the width of both the momentum and thermal boundaries, β_1 and β_2, increase.

The numerical solution for $G(\xi)$ when $\nu \neq 0$ and $\kappa = 0$ was shown to be unstable when ξ is greater than some value for both the two-dimensional and axisymmetric jets. This may indicate a finite boundary for the turbulent thermal effects and could be investigated in future work.

The numerical solution when $\nu = 0$ and $\kappa \neq 0$ could not be accurately determined since $G(\xi)$ was undefined when $\beta_1 < \xi < \beta_2$. Hence a different model is required for $G(\xi)$ when $\beta_1 < \xi < \beta_2$ which is outside the scope of this dissertation. This may require matching with a laminar flow in the region $\beta_1 < \xi < \beta_2$ and could also be considered in future work.

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