Commun Nonlinear Sci Numer Simulat 141 (2025) 108461

Contents lists available at ScienceDirect

Communications in Nonlinear Science and Numerical Simulation

journal homepage: www.elsevier.com/locate/cnsns

Research paper

Two-step inertial accelerated algorithms for solving split feasibility problem with multiple output sets

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ARTICLE INFO

MSC:
47H09
47H10
49J53
90C25

Keywords:
Split feasibility problem
Two-step inertial
Multiple output set

ABSTRACT

In this paper, we present and study two new two-step inertial accelerated algorithms for finding an approximate solution of split feasibility problems with multiple output sets. Our methods are extensions of CQ algorithms previously studied in the literature. In contrast to the related iterative methods for solving SFP, our methods incorporate a two-step inertial technique that speeds up the convergence rate of the generated sequences to the unique solution of SFP studied in this work. Furthermore, we present weak and strong convergence results where the strong convergence result of our method is obtained without the on-line rule, a feature absent in related algorithms in the literature. Finally, we demonstrate the applicability and performance of our algorithms through numerical experiments. Our numerical results reveal that our methods perform better than existing related methods in the literature.

1. Introduction

Let C and Q_i be nonempty, closed, and convex subsets of real Hilbert spaces H and H_i, $i = 1, 2, \ldots, N$, respectively. Let $\mathcal{T}_i : H \to H_i (i = 1, 2, \ldots, N)$ be bounded linear adjoint operators. The split feasibility problem (SFP) with multiple output sets is the problem of finding k^* such that

$$k^* \in \Gamma := C \cap \left(\bigcap_{i=1}^{N} \mathcal{T}_i^{-1}(Q_i) \right). \tag{1.1}$$

The problem (1.1) has been studied by Reich et al. in [1]. We note that when $N = 1$ in (1.1), we obtain the classical SFP defined as the problem of finding

$$k^* \in C \text{ such that } \mathcal{T} k^* \in Q. \tag{1.2}$$

Problem (1.2) was first presented and studied by Censor and Elfving [2] for modelling certain inverse problems emerging from areas such as signal processing and image reconstruction (see [3,4]). Other important applications of SFP (1.2) are in breast cancer prognosis treatment using Dantzig selectors (see [5]), intensity-modulated radiation therapy treatment (IMRT) planning (see [6]), dose-volume constraints in inverse planning of intensity-modulated photon or proton therapy (see [7]), gene regulatory network inference (see [8]), and in medical image reconstruction (see [4]). Some of these interesting applications have led to the introduction and analysis of several iterative algorithms for approximating a solution of (1.2). See, for example [2,3,9–17] and references therein. Other generalizations of (1.2) include the split common null point problem (SCNPP) [9,18–21], the split common fixed point

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https://doi.org/10.1016/j.cnsns.2024.108461
Received 14 June 2024; Received in revised form 11 October 2024; Accepted 12 November 2024
Available online 21 November 2024

problem(SCFPP) [22,23]and the split variational inequality (SVIP) [24,25] . Also, we note that when $H = H_1$, $C = C_1$, $Q_i = C_{i+1}$, $1 \leq i \leq N-1$, $\mathcal{T}_1 = \mathcal{A}_1$, $\mathcal{T}_2 = \mathcal{A}_2\mathcal{A}_1, \dots$, and $\mathcal{T}_N = \mathcal{A}_{N-1}\mathcal{A}_{N-2}\cdots\mathcal{A}_1$, where $\mathcal{A}_i, i = 1, 2, \dots, N-1$ are bounded linear operators, then (1.1) reduces to the generalized split feasibility problem studied by Reich and Tuyen in [26].

CQ algorithm, which was introduced by Byrne in [3], is a popular iterative method for finding an approximate solution of the SFPs. Since then, many scholars have extended and applied it in solving multiple-set split convex feasibility problem (MSSFP) (see [2,22,27]). Quite recently, Reich et al. in [1] introduced and studied two iterative CQ algorithms for solving (1.1). The authors in [1] showed that the iterative sequences generated by the two iterative CQ algorithms introduced and studied in their work converge weakly and strongly to the unique solution of (1.1). These results were achieved using the following condition on $\{\gamma_n\}$:

$$0 < a \leq \gamma_n \leq b < \frac{2}{N \max_{i=\overline{1,N-1}}\{\|\mathcal{T}_i\|^2\}}. \tag{1.3}$$

However, it is difficult to calculate operator norm when it is unknown. Very recently, Reich et al. in their work [28], modified the two methods studied in [1] by removing the rule governing the norms of the transfer operators $\mathcal{T}_i$ and also replacing the full set of indices $\{1, 2, \dots, N\}$ in the sums on the right-hand sides of the methods in [1] with a proper subset. This adjustment results to methods that are more cost-effective and efficient than related existing CQ algorithms in the literature.

Inertial extrapolation in algorithms have been justified by several authors as means of accelerating optimization algorithms. The inertial techniques are derived from a discretization of a second order dynamical system for minimizing a smooth convex function known as the heavy ball with friction (HBF):

$$\frac{d^2x}{dt^2}(t) + \gamma\frac{dx}{dt}(t) + \nabla f(x(t)) = 0,$$

where $\gamma > 0$ is a damping or friction parameter. The inertial extrapolation $x_n + \vartheta(x_n - x_{n-1})$ is the most popular means for achieving this purpose. The HBF was first traced to Polyak [29], Alvarez and Attouch [30] modified and applied it to a general maximal monotone operator.

Motivated by the work of Reich et al. in [28], Okeke et al. [31] modified the CQ algorithms studied in [28] by incorporating inertial extrapolation term. They aim to accelerate the convergence rate of the iterative sequences generated by their methods, which they justified through numerical experiments presented in their work. They authors in [31] obtained the weak and strong convergence results of their algorithms with similar conditions as in the methods studied in [28]. We remark that the strong convergence result of their method was achieved by the condition $\lim_{n\to\infty} \frac{\epsilon_n}{\sigma_n} = 0$. This condition is called the on-line rule and we remark that on-line rule is a stringent condition and not always desirable in practical problems since it leads to slow convergence, especially for large-scale problems.

However, it was shown in [32, Section 3] that the iterative sequences generated by the one-step inertial fixed point iteration studied in [32] may fail to provide acceleration. But, as shown in [33, Section 4], incorporating the multi-step inertial, like the two-step inertial term $x_n + \vartheta(x_n - x_{n-1}) + \beta(x_{n-1} - x_{n-2}), \vartheta > 0, \beta < 0$ could resolve the issue of not providing acceleration. Recently, several authors have studied multi-step inertial algorithms and showed that multi-step inertial term (for example, the two-step inertial term) improves the convergence rate of optimization algorithms (see [33–38]).

Motivated by ongoing research in this direction, we introduce and study two new two-step inertial accelerated algorithms for approximating a solution of (1.1) and show that the iterative sequences generated by these algorithms converge weakly and strongly to the unique solution of (1.1). We remark that the authors achieved strong convergence result without the application of on-line rule, a feature which is absent in related algorithms in the literature.

The rest of this paper is organized as follows: Section 2 contains fundamental results, concepts and lemmas that are instrumental in establishing the main results of this work. In Section 3, we introduce our first method and prove its weak convergence result; moreover, we present and study the strong convergence result of our second algorithm in Section 4. Finally, in Section 5, we show the performance of our methods in relation to other already established methods in the framework through numerical experiments.

2. Preliminaries

Throughout this work, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ represent the inner product and norm of a real Hilbert space H, respectively.

Definition 2.1. Let C be a closed, convex and nonempty subset of real Hilbert space H. For each $x \in H$, there exists a unique point $P_C x \in C$ such that

$$\|x - P_C x\| = \inf_{u\in C} \|x - u\|. \tag{2.1}$$

The operator $P_C : H \to C$ defined in (2.1) is the metric projection of H onto C which is characterized by (see [39,40])

$$\langle P_C x - x, y - P_C x\rangle \geq 0, \quad \forall y \in C, \quad \forall x \in H. \tag{2.2}$$

The metric projection P_C has the following well-known properties

Proposition 2.2 (*[1]*). *For any $x, y \in H$, it holds that:*

(i) $\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2$;

(ii) $\langle x-y,(I-P_C)x-(I-P_C)y\rangle \geq \|(I-P_C)x-(I-P_C)y\|^2$.

Definition 2.3. $\mathcal{A}: C \to C$ is called a nonexpansive operator if $\|\mathcal{A}x-\mathcal{A}y\| \leq \|x-y\|$, $\forall x,y \in C$.

We represent the set of fixed points of $\mathcal{A}$ by $F(\mathcal{A})$, that is $F(\mathcal{A}) = \{x \in C : \mathcal{A}x = x\}$.

Lemma 2.4 ([41]). *Let $\mathcal{A}$ be a nonexpansive operator on a real Hilbert space H. Then the operator $I-\mathcal{A}$ is demiclosed, that is, whenever $\{x_n\}$ is a sequence in H which weakly converges to some $x \in H$ and the sequence $\{(I-\mathcal{A})(x_n)\}$ strongly converges to some y, it follows that $(I-\mathcal{A})(x) = y$.*

Lemma 2.5. *The following identities are true*

(i) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x,y\rangle + \|y\|^2 \quad \forall x,y \in H$.
(ii) $2\langle x,y\rangle = \|x\|^2+\|y\|^2-\|x-y\|^2 = \|x+y\|^2-\|x\|^2-\|y\|^2 \quad \forall x,y \in H$.
(iii) $\|(1+r)x-(r-q)y-qz\|^2 = (1+r)\|x\|^2-(r-q)\|y\|^2-q\|z\|^2+(1+r)(r-q)\|x-y\|^2 + q(1+r)\|x-z\|^2-q(r-q)\|y-z\|^2 \quad \forall x,y,z \in H,\ r,q \in \mathbb{R}$.

Lemma 2.6. *If u and v are non-negative numbers, then*

(i) $uv \leq \frac{u^2}{2p}+\frac{pv^2}{2} \quad \forall p \geq 0$,
(ii) $(u+v)^2 \leq (2+\sqrt{2})u^2+\sqrt{2}v^2$.

Lemma 2.7 ([42]). *Let $\{u_n\}$ be a sequence of non-negative real numbers, $\{e_n\}$ be a sequence of numbers in (0,1) such that $\sum_{n=1}^{\infty} e_n = \infty$, and let $\{v_n\}$ be a sequence of real numbers. Assume that*

$$u_{n+1} \leq (1-e_n)u_n + e_n v_n \quad \forall\ n \geq 1.$$

If $\limsup_{i\to\infty} v_{n_i} \leq 0$ for every subsequence $\{u_{n_i}\}$ of $\{u_n\}$ satisfying $\liminf_{i\to\infty}(u_{n_i+1}-u_{n_i}) \geq 0$, then $\lim_{n\to\infty} u_n = 0$.

Lemma 2.8 ([43]). *Suppose that $\{u_n\}$, $\{r_n\} \subset \mathbb{R}_+$ such that*

$$u_{n+1} \leq (1-e_n)u_n + s_n + r_n, \quad \text{for each } n \in \mathbb{N},$$

where $\{e_n\}$ is a sequence in $(0,1)$ and s_n is a real sequence. Let $\sum_{n=1}^{\infty} r_n < \infty$ and $s_n \leq e_n N$ for some $N \geq 0$. Then, $\{u_n\}$ is bounded.

Proposition 2.9 ([28]). *An element $k^* \in H$ is a solution to problem (1.1) if and only if*

$$k^* = P_C\left[k^* - \gamma \sum_{i \in I(k^*)} \lambda_i \mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i k^*\right],$$

where $\lambda_i \geq 0$ for all $i \in I(k^*)$, $\sum_{i\in I(k^*)} \lambda_i = 1$, and γ is an arbitrary positive real number.

3. Weak convergence result

We establish our main results based on the following conditions:

Assumption 3.1.

(C1) Let $H, H_i, i = 1,2,\ldots,m$ be real Hilbert spaces.
(C2) Let $\mathcal{T}_i : H \to H_i, i = 1,2,\ldots,m$ be bounded linear operators with adjoint $\mathcal{T}_i^* : H_i \to H_i$.
(C3) Let C and Q_i be nonempty, closed, and convex subsets of H and $H_i, i = 1,2,\ldots,m$, respectively.
(C4) Suppose that the solution set Γ of the SFP with multiple outputs sets is nonempty. That is,

$$\Gamma := C \cap \left(\bigcap_{i=1}^{N} \mathcal{T}_i^{-1}(Q_i)\right) \neq \emptyset.$$

Assumption 3.2. Assume that the following conditions on β and ϑ hold:

(a) $0 \leq \vartheta < \min\left\{\frac{1}{2}, \frac{1-\rho}{1+\rho}\right\}$,
(b) $\max\left\{\frac{\vartheta(1+\rho)}{1-\rho} - 1, \vartheta\rho - \frac{(1-\rho)(1-\vartheta)^2}{1+\vartheta}\right\} < \beta \leq 0$,
(c) $\rho\vartheta(2\vartheta - 1) + 2\rho\beta^2 - (1-\vartheta)^2 - (1+\beta)^2 - \beta(2\vartheta - \rho) + (1+\rho) \leq 0$.

Algorithm 3.3. Weakly convergent two-step inertial algorithm

Step 0: Choose parameters ρ in $(0,1)$, β and ϑ such that Assumption 3.2 holds. For arbitrary starting points $x_{-1}, x_0, x_1 \in H$, set $n := 1$ and

Step 1: Compute

$$x_{n+1} = (1-\rho)y_n + \rho w_n \quad \forall n \geq 1. \tag{3.1}$$

Compute

$$\begin{cases} y_n = x_n + \vartheta(x_n - x_{n-1}) + \beta(x_{n-1} - x_{n-2}) \\ w_n = P_C\left[y_n - \gamma_n \sum_{i\in I(y_n)} \lambda_{i,n}\mathcal{T}_i^*(I - P_{Q_i})\mathcal{T}_i y_n\right], \end{cases} \tag{3.2}$$

where $I(y_n) = \{i : \|\mathcal{T}_i y_n - P_{Q_i}\mathcal{T}_i y_n\| = \max_{i=1,2,\cdots,N}\|\mathcal{T}_i y_n - P_{Q_i}\mathcal{T}_i y_n\|\}$, $\lambda_{i,n} \geq 0$ for each $i \in I(y_n)$, $\sum_{i\in I(y_n)}\lambda_{i,n} = 1$, and $\{\gamma_n\}$ is a sequence of nonnegative real numbers defined by

$$\gamma_n = \begin{cases} \frac{\chi_n d_n^2}{\|\sum_{i\in I(y_n)}\lambda_{i,n}\mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i y_n\|^2}, & \text{if} \quad \|\sum_{i\in I(y_n)}\lambda_{i,n}\mathcal{T}_i^*(I - P_{Q_i})\mathcal{T}_i y_n\|^2 > 0 \\ 0, & \text{otherwise;} \end{cases} \tag{3.3}$$

where $d_n = \max_{i=1,2,\cdots,N}\|\mathcal{T}_i y_n - P_{Q_i}\mathcal{T}_i y_n\|$ and $\{\chi_n\} \subset$ [a, b]$\subset (0,2)$.

Remark 3.4.

1. If $\vartheta = \beta = 0$, and $\rho = 1$, our method recovers the well-known CQ algorithm (Eq. (4.1)) studied by Reich et al. in [28], however, our method adopts two-step inertial technology, which accelerates the iteration efficiency; hence our method is an improvement of the method in [28]. Moreover, our method eliminates the need for operator norm in contrast to the method studied in [1] since it is difficult to calculate or even unknown.
2 If $\beta = 0$ and $\rho = 1$, our method recovers Algorithm 3.3 studied by Okeke et al. in [31]. The authors in [31] adopted the on-line rule; on-line rule is very restrictive and not always desirable in practical problems since it leads to slow convergence, especially for large-scale problems. Our method eliminates the need for this condition. In addition, our method provides acceleration using two-step inertial term unlike method in [31], hence our method improves their method since one-step inertial term may fail to provide acceleration for some practical problems as noted in [32, Section 3].

Lemma 3.5. *Suppose that Assumptions 3.1 and 3.2 hold. Let $\{\gamma_n\}$ be a sequence of nonnegative real numbers given by (3.3). Then Algorithm 3.3 generates a bounded sequence $\{x_n\}$.*

Proof. Let $k^* \in \Gamma$. Using the fact that $P_{Q_i}\mathcal{T}_i k^* = \mathcal{T}_i k^*$, Propositions 2.2(ii)and 2.9, and w_n in (3.2), we obtain

$$\begin{aligned} \|w_n - k^*\|^2 &= \left\| P_C\left[y_n - \gamma_n \sum_{i\in I(y_n)}\lambda_{i,n}\mathcal{T}_i^*(I - P_{Q_i})\mathcal{T}_i y_n\right] - P_C\left[k^* - \gamma\sum_{i\in I(y_n)}\lambda_{i,n}\mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i k^*\right]\right\|^2 \\ &\leq \left\| y_n - k^* - \gamma_n\sum_{i\in I(y_n)}\lambda_{i,n}\mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i y_n\right\|^2 \\ &= \|y_n - k^*\|^2 - 2\gamma_n\sum_{i\in I(y_n)}\lambda_{i,n}\langle y_n - k^*, \mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i y_n\rangle + \gamma_n^2\left\|\sum_{i\in I(y_n)}\lambda_{i,n}\mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i y_n\right\|^2 \\ &\leq \|y_n - k^*\|^2 - 2\gamma_n d_n^2 + \gamma_n^2\left\|\sum_{i\in I(y_n)}\lambda_{i,n}\mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i y_n\right\|^2 \end{aligned} \tag{3.4}$$

We now turn our attention to the following possibilities.
Case I: Suppose $\gamma_n = 0$. In this case, $w_n = P_C y_n$ and $\sum_{i\in I(y_n)}\lambda_{i,n}\mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i y_n = 0$. By Proposition 2.2(ii), we get

$$
\begin{aligned}
0 &= \left\langle \sum_{i\in I(y_n)} \lambda_{i,n}\mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i y_n, y_n - k^* \right\rangle \\
&= \sum_{i\in I(y_n)} \lambda_{i,n}\langle \mathcal{T}_i y_n - P_{Q_i}\mathcal{T}_i y_n, \mathcal{T}_i y_n - \mathcal{T}_i k^*\rangle \\
&\geq \sum_{i\in I(y_n)} \lambda_{i,n}\|\mathcal{T}_i y_n - P_{Q_i}\mathcal{T}_i y_n\|^2 \\
&\geq d_n^2 \geq 0.
\end{aligned}
$$

Therefore, $d_n = 0$. This further implies that

$$
\|\mathcal{T}_i y_n - P_{Q_i}\mathcal{T}_i y_n\| = 0, \quad \forall i \in I(y_n). \tag{3.5}
$$

It can be deduced from the definition of $I(y_n)$ that

$$
\|\mathcal{T}_i y_n - P_{Q_i}\mathcal{T}_i y_n\| = 0, \quad \forall i = 1,2,\dots,N. \tag{3.6}
$$

Additionally, from $w_n = P_C y_n$ we obtain

$$
\begin{aligned}
\|w_n - k^*\| &= \|P_C y_n - P_C k^*\| \\
&\leq \|y_n - k^*\|.
\end{aligned}
$$

So, from Case I, we obtain

$$
\|w_n - k^*\| \leq \|y_n - k^*\|. \tag{3.7}
$$

Case II: Suppose $\gamma_n = \chi_n \frac{d_n^2}{\|\sum_{i\in I(y_n)}\lambda_{i,n}\mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i y_n\|^2}$. It follows from (3.4) that

$$
\begin{aligned}
\|w_n - k^*\|^2 &\leq \|y_n - k^*\|^2 - \gamma_n\left[2d_n^2 - \gamma_n\left\|\sum_{i\in I(y_n)}\lambda_{i,n}\mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i y_n\right\|^2\right] \\
&= \|y_n - k^*\|^2 - 2\chi_n\frac{d_n^4}{\|\sum_{i\in I(y_n)}\lambda_{i,n}\mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i y_n\|^2} \\
&\quad +\chi_n^2\frac{d_n^4}{\|\sum_{i\in I(y_n)}\lambda_{i,n}\mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i y_n\|^4}\|\sum_{i\in I(y_n)}\lambda_{i,n}\mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i y_n\|^2 \\
&= \|y_n - k^*\|^2 - \chi_n(2-\chi_n)\frac{d_n^4}{\|\sum_{i\in I(y_n)}\lambda_{i,n}\mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i y_n\|^2}.
\end{aligned} \tag{3.8}
$$

Since $\chi_n \in (0,2)$, we obtain

$$
\|w_n - k^*\| \leq \|y_n - k^*\|. \tag{3.9}
$$

Combining (3.7) and (3.9), we obtain

$$
\|w_n - k^*\| \leq \|y_n - k^*\|. \tag{3.10}
$$

From (3.1), we obtain the following:

$$
\|x_{n+1} - k^*\|^2 = (1-\rho)\|y_n - k^*\|^2 + \rho\|w_n - k^*\|^2 - (1-\rho)\rho\|w_n - y_n\|^2. \tag{3.11}
$$

and

$$
\|w_n - y_n\|^2 = \frac{1}{\rho^2}\|x_{n+1} - y_n\|^2. \tag{3.12}
$$

Substitute (3.10) and (3.12) in (3.11), we get

$$
\begin{aligned}
\|x_{n+1} - k^*\|^2 &\leq (1-\rho)\|y_n - k^*\|^2 + \rho\|y_n - k^*\|^2 - \frac{(1-\rho)}{\rho}\|x_{n+1} - y_n\|^2 \\
&= \|y_n - k^*\|^2 - \frac{(1-\rho)}{\rho}\|x_{n+1} - y_n\|^2.
\end{aligned} \tag{3.13}
$$

By Lemma 2.5(iii), we obtain the following

$$
\begin{aligned}
\|y_n - k^*\|^2 &= \|x_n + \vartheta(x_n - x_{n-1}) + \beta(x_{n-1} - x_{n-2}) - k^*\|^2 \\
&= \|(1+\vartheta)(x_n - k^*) - (\vartheta-\beta)(x_{n-1} - k^*) - \beta(x_{n-2} - k^*)\|^2 \\
&= (1+\vartheta)\|x_n - k^*\|^2 - (\vartheta-\beta)\|x_{n-1} - k^*\|^2 - \beta\|x_{n-2} - k^*\|^2 \\
&\quad +(1+\vartheta)(\vartheta-\beta)\|x_n - x_{n-1}\|^2 + \beta(1+\vartheta)\|x_n - x_{n-2}\|^2 - \beta(\vartheta-\beta)\|x_{n-1} - x_{n-2}\|^2.
\end{aligned} \tag{3.14}
$$

Using Lemma 2.5(i), it follows that

$$\begin{aligned}\|x_{n+1}-y_n\|^2 &= \|(x_{n+1}-x_n)-(\vartheta(x_n-x_{n-1})+\beta(x_{n-1}-x_{n-2}))\|^2\\ &= \|x_{n+1}-x_n\|^2-2\langle x_{n+1}-x_n, \vartheta(x_n-x_{n-1})+\beta(x_{n-1}-x_{n-2})\rangle\\ &\quad +\|\vartheta(x_n-x_{n-1})+\beta(x_{n-1}-x_{n-2})\|^2\\ &= \|x_{n+1}-x_n\|^2-2\vartheta\langle x_{n+1}-x_n, x_n-x_{n-1}\rangle-2\beta\langle x_{n+1}-x_n, x_{n-1}-x_{n-2}\rangle\\ &\quad +\vartheta^2\|x_n-x_{n-1}\|^2+2\vartheta\beta\langle x_n-x_{n-1}, x_{n-1}-x_{n-2}\rangle+\beta^2\|x_{n-1}-x_{n-2}\|^2. \end{aligned} \tag{3.15}$$

Observe that

$$\begin{aligned}-2\vartheta\langle x_{n+1}-x_n, x_n-x_{n-1}\rangle &\geq -2\vartheta\|x_{n+1}-x_n\|\|x_n-x_{n-1}\|\\ &\geq -\vartheta\|x_{n+1}-x_n\|^2-\vartheta\|x_n-x_{n-1}\|^2, \end{aligned} \tag{3.16}$$

$$\begin{aligned}-2\beta\langle x_{n+1}-x_n, x_{n-1}-x_{n-2}\rangle &\geq -2|\beta|\|x_{n+1}-x_n\|\|x_{n-1}-x_{n-2}\|\\ &\geq -|\beta|\|x_{n+1}-x_n\|^2-|\beta|\|x_{n-1}-x_{n-2}\|^2, \end{aligned} \tag{3.17}$$

$$\begin{aligned}2\vartheta\beta\langle x_n-x_{n-1}, x_{n-1}-x_{n-2}\rangle &\geq -2|\beta|\vartheta\|x_n-x_{n-1}\|\|x_{n-1}-x_{n-2}\|\\ &\geq -|\beta|\vartheta\|x_n-x_{n-1}\|^2-|\beta|\vartheta\|x_{n-1}-x_{n-2}\|^2. \end{aligned} \tag{3.18}$$

Substituting (3.16), (3.17), (3.18) in (3.15), we get

$$\begin{aligned}\|x_{n+1}-y_n\|^2 &\geq \|x_{n+1}-x_n\|^2-\vartheta\|x_{n+1}-x_n\|^2-\vartheta\|x_n-x_{n-1}\|^2\\ &\quad -|\beta|\|x_{n+1}-x_n\|^2-|\beta|\|x_{n-1}-x_{n-2}\|^2+\vartheta^2\|x_n-x_{n-1}\|^2\\ &\quad -|\beta|\vartheta\|x_n-x_{n-1}\|^2-|\beta|\vartheta\|x_{n-1}-x_{n-2}\|^2+\beta^2\|x_{n-1}-x_{n-2}\|^2\\ &= (1-\vartheta-|\beta|)\|x_{n+1}-x_n\|^2+(\vartheta^2-\vartheta-|\beta|\vartheta)\|x_n-x_{n-1}\|^2\\ &\quad +(\beta^2-|\beta|\vartheta-|\beta|)\|x_{n-1}-x_{n-2}\|^2. \end{aligned} \tag{3.19}$$

Substituting (3.14) and (3.19) in (3.13) gives the following:

$$\begin{aligned}\|x_{n+1}-k^*\|^2 &\leq (1+\vartheta)\|x_n-k^*\|^2-(\vartheta-\beta)\|x_{n-1}-k^*\|^2-\beta\|x_{n-2}-k^*\|^2\\ &\quad +(1+\vartheta)(\vartheta-\beta)\|x_n-x_{n-1}\|^2+\beta(1+\vartheta)\|x_n-x_{n-2}\|^2\\ &\quad -\beta(\vartheta-\beta)\|x_{n-1}-x_{n-2}\|^2\\ &\quad -\frac{(1-\rho)}{\rho}\big[(1-\vartheta-|\beta|)\|x_{n+1}-x_n\|^2+(\vartheta^2-\vartheta-|\beta|\vartheta)\|x_n-x_{n-1}\|^2\\ &\quad +(\beta^2-|\beta|\vartheta-|\beta|)\|x_{n-1}-x_{n-2}\|^2\big]\\ &\leq (1+\vartheta)\|x_n-k^*\|^2-(\vartheta-\beta)\|x_{n-1}-k^*\|^2-\beta\|x_{n-2}-k^*\|^2\\ &\quad +\left[(1+\vartheta)(\vartheta-\beta)-\frac{(1-\rho)}{\rho}(\vartheta^2-\vartheta-|\beta|\vartheta)\right]\|x_n-x_{n-1}\|^2\\ &\quad -\left[\beta(\vartheta-\beta)+\frac{(1-\rho)}{\rho}(\beta^2-|\beta|\vartheta-|\beta|)\right]\|x_{n-1}-x_{n-2}\|^2\\ &\quad -\frac{(1-\rho)}{\rho}(1-\vartheta-|\beta|)\|x_{n+1}-x_n\|^2. \end{aligned} \tag{3.20}$$

From the inequality above, we obtain

$$\begin{aligned}&\|x_{n+1}-k^*\|^2-\vartheta\|x_n-k^*\|^2-\beta\|x_{n-1}-k^*\|^2+\frac{(1-\rho)}{\rho}(1-\vartheta-|\beta|)\|x_{n+1}-x_n\|^2\\ &\leq \|x_n-k^*\|^2-\vartheta\|x_{n-1}-k^*\|^2-\beta\|x_{n-2}-k^*\|^2+\frac{(1-\rho)}{\rho}(1-\vartheta-|\beta|)\|x_n-x_{n-1}\|^2\\ &\quad +\left[(1+\vartheta)(\vartheta-\beta)-\frac{(1-\rho)}{\rho}(\vartheta^2-2\vartheta-|\beta|-|\beta|\vartheta+1)\right]\|x_n-x_{n-1}\|^2\\ &\quad -\left[\beta(\vartheta-\beta)+\frac{(1-\rho)}{\rho}(\beta^2-|\beta|\vartheta-|\beta|)\right]\|x_{n-1}-x_{n-2}\|^2. \end{aligned} \tag{3.21}$$

Take $u_n=\|x_n-k^*\|^2-\vartheta\|x_{n-1}-k^*\|^2-\beta\|x_{n-2}-k^*\|^2+\frac{(1-\rho)}{\rho}(1-\vartheta-|\beta|)\|x_n-x_{n-1}\|^2$, we can rewrite (3.21) as follows:

$$\begin{aligned}u_{n+1} &\leq u_n+\left[(1+\vartheta)(\vartheta-\beta)-\frac{(1-\rho)}{\rho}(\vartheta^2-2\vartheta-|\beta|-|\beta|\vartheta+1)\right]\|x_n-x_{n-1}\|^2\\ &\quad -\left[\beta(\vartheta-\beta)+\frac{(1-\rho)}{\rho}(\beta^2-|\beta|\vartheta-|\beta|)\right]\|x_{n-1}-x_{n-2}\|^2. \end{aligned} \tag{3.22}$$

Next, we show that $u_n \geq 0$, for all $n \geq 1$.

$$\begin{aligned}
u_n &= \|x_n - k^*\|^2 - \vartheta\|x_{n-1} - k^*\|^2 - \beta\|x_{n-2} - k^*\|^2 + \frac{(1-\rho)}{\rho}(1 - \vartheta - |\beta|)\|x_n - x_{n-1}\|^2 \\
&\geq \|x_n - k^*\|^2 - 2\vartheta\|x_{n-1} - x_n\|^2 - 2\vartheta\|x_n - k^*\|^2 - \beta\|x_{n-2} - k^*\|^2 \\
&\quad + \frac{(1-\rho)}{\rho}(1 - \vartheta - |\beta|)\|x_n - x_{n-1}\|^2 \\
&= (1-2\vartheta)\|x_n - k^*\|^2 + \left[\frac{(1-\rho)}{\rho}(1 - \vartheta - |\beta|) - 2\vartheta\right]\|x_{n-1} - x_n\|^2 - \beta\|x_{n-2} - k^*\|^2 \\
&\geq 0,
\end{aligned} \tag{3.23}$$

since $\vartheta < \frac{1}{2}$, $\beta \leq 0$, $\frac{\vartheta(1+\rho)}{1-\rho} - 1 \leq \beta$ and $0 \leq \vartheta \leq \frac{1-\rho}{1+\rho}$.

We get from (3.22) that

$$\begin{aligned}
u_{n+1} - u_n &\leq \left[(1+\vartheta)(\vartheta - \beta) - \frac{(1-\rho)}{\rho}(\vartheta^2 - 2\vartheta - |\beta| - |\beta|\vartheta + 1)\right]\|x_n - x_{n-1}\|^2 \\
&\quad - \left[\beta(\vartheta - \beta) + \frac{(1-\rho)}{\rho}(\beta^2 - |\beta|\vartheta - |\beta|)\right]\|x_{n-1} - x_{n-2}\|^2 \\
&= -\left[(1+\vartheta)(\vartheta - \beta) - \frac{(1-\rho)}{\rho}(\vartheta^2 - 2\vartheta - |\beta| - |\beta|\vartheta + 1)\right]\left[\|x_{n-1} - x_{n-2}\|^2 - \|x_n - x_{n-1}\|^2\right] \\
&\quad + \left[(1+\vartheta)(\vartheta - \beta) - \frac{(1-\rho)}{\rho}(\vartheta^2 - 2\vartheta - |\beta| - |\beta|\vartheta + 1) - \beta(\vartheta - \beta)\right. \\
&\quad \left. - \frac{(1-\rho)}{\rho}(\beta^2 - |\beta|\vartheta - |\beta|)\right]\|x_{n-1} - x_{n-2}\|^2.
\end{aligned} \tag{3.24}$$

Take $c_1 = -\left[(1+\vartheta)(\vartheta - \beta) - \frac{(1-\rho)}{\rho}(\vartheta^2 - 2\vartheta - |\beta| - |\beta|\vartheta + 1)\right]$ and $c_2 = -\left[(1+\vartheta)(\vartheta - \beta) - \frac{(1-\rho)}{\rho}(\vartheta^2 - 2\vartheta - |\beta| - |\beta|\vartheta + 1) - \beta(\vartheta - \beta)\right.$ $\left.-\frac{(1-\rho)}{\rho}(\beta^2 - |\beta|\vartheta - |\beta|)\right]$. Since $\vartheta\rho - \frac{(1-\rho)(1-\vartheta)^2}{1+\vartheta} < \beta$, we have that $c_1 > 0$. Also, by Assumption 3.2(c) $c_2 > 0$. Moreover, we obtain from (3.24) that

$$u_{n+1} - u_n \leq c_1\left[\|x_{n-1} - x_{n-2}\|^2 - \|x_n - x_{n-1}\|^2\right] - c_2\|x_{n-1} - x_{n-2}\|^2.$$

The inequality above implies

$$u_{n+1} + c_1\|x_n - x_{n-1}\|^2 \leq u_n + c_1\|x_{n-1} - x_{n-2}\|^2 - c_2\|x_{n-1} - x_{n-2}\|^2. \tag{3.25}$$

Take $\bar{u}_n = u_n + c_1\|x_{n-1} - x_{n-2}\|^2$, then $\bar{u}_n \geq 0\ \ \forall n \geq 1$. So, we deduce from (3.25) that

$$\bar{u}_{n+1} \leq \bar{u}_n. \tag{3.26}$$

(3.26) implies that $\bar{u}_n$ is a decreasing sequence that is bounded from below. Therefore, $\lim n \to \infty \bar{u}_n$ exists. As a result, we obtain from (3.25) that

$$\lim_{n\to\infty} c_2\|x_{n-1} - x_{n-2}\|^2 = 0.$$

Therefore,

$$\lim_{n\to\infty} \|x_{n-1} - x_{n-2}\|^2 = 0. \tag{3.27}$$

Also,

$$\begin{aligned}
\|x_{n+1} - y_n\| &= \|x_{n+1} - x_n - \vartheta(x_n - x_{n-1}) - \beta(x_{n-1} - x_{n-2})\| \\
&\leq \|x_{n+1} - x_n\| + \vartheta\|x_n - x_{n-1}\| + |\beta|\|x_{n-1} - x_{n-2}\| \to 0, \quad n \to \infty,
\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}
\|x_n - y_n\| &= \|x_n - x_n - \vartheta(x_n - x_{n-1}) - \beta(x_{n-1} - x_{n-2})\| \\
&\leq \vartheta\|x_n - x_{n-1}\| + |\beta|\|x_{n-1} - x_{n-2}\| \to 0, \quad n \to \infty.
\end{aligned} \tag{3.29}$$

Combining (3.12) and (3.28), it ensues that

$$\|w_n - y_n\| \to 0, \quad n \to \infty. \tag{3.30}$$

And

$$\|w_n - x_n\| \leq \|w_n - y_n\| + \|y_n - x_n\| \to 0, \quad n \to \infty.$$

By (3.27) and the existence of $\lim n \to \infty \bar{u}_n$, it follows that $\lim_{n\to\infty} u_n$ exists and hence $\{u_n\}$ is bounded. Since $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, we obtain from the definition of u_n that

$$\lim_{n\to\infty}(\|x_n - k^*\|^2 - \vartheta\|x_{n-1} - k^*\|^2 - \beta\|x_{n-2} - k^*\|^2) \tag{3.31}$$

exists. Using the fact that $\{u_n\}$ is bounded, then, it ensues from (3.23) that the sequence $\{x_n\}$ is bounded. Hence, y_n and w_n are also bounded. This completes the proof. □

Lemma 3.6. *Suppose that Assumption 3.1 holds. Let $\{\gamma_n\}$ be a sequence of nonnegative real numbers given by (3.3). Then every sequential weak cluster points of sequence $\{x_n\}$ generated by Algorithm 3.3 is in Γ.*

Proof. Observe that (3.8) can be rewritten as follows:

$$\begin{aligned}
\chi_n(2-\chi_n)\frac{d_n^4}{\|\sum_{i\in I(y_n)}\lambda_{i,n}\mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i y_n\|^2} &\le \|y_n-k^*\|^2-\|w_n-k^*\|^2\\
&= \|x_n-k^*+\vartheta(x_n-x_{n-1})+\beta(x_{n-1}-x_{n-2})\|^2-\|w_n-k^*\|^2\\
&\le \left[\|x_n-k^*\|+\|\vartheta(x_n-x_{n-1})+\beta(x_{n-1}-x_{n-2})\|\right]^2\\
&\quad -\|w_n-k^*\|^2\\
&= \|x_n-k^*\|^2+2\|x_n-k^*\|\|\vartheta(x_n-x_{n-1})+\beta(x_{n-1}-x_{n-2})\|\\
&\quad +\|\vartheta(x_n-x_{n-1})+\beta(x_{n-1}-x_{n-2})\|^2-\|w_n-k^*\|^2\\
&= 2\|x_n-k^*\|\|\vartheta(x_n-x_{n-1})+\beta(x_{n-1}-x_{n-2})\|\\
&\quad +\|\vartheta(x_n-x_{n-1})+\beta(x_{n-1}-x_{n-2})\|^2\\
&\quad +\|x_n-k^*\|^2-\|w_n-k^*\|^2\\
&\le 2\vartheta\|x_n-k^*\|\|x_n-x_{n-1}\|+2|\beta|\|x_n-k^*\|\|x_{n-1}-x_{n-2}\|\\
&\quad +2\vartheta^2\|x_n-x_{n-1}\|^2+2|\beta|^2\|x_{n-1}-x_{n-2}\|^2\\
&\quad +\left[\|x_n-k^*\|-\|w_n-k^*\|\right]\left[\|x_n-k^*\|+\|w_n-k^*\|\right]\\
&\le \|x_n-x_{n-1}\|N_1+\|x_{n-1}-x_{n-2}\|N_2+2\vartheta^2\|x_n-x_{n-1}\|^2\\
&\quad +2|\beta|^2\|x_{n-1}-x_{n-2}\|^2+\|x_n-w_n\|N_3 \ \to 0, \quad n\to\infty,
\end{aligned}$$

where $N_1 := \sup_{n\ge1}(2\vartheta\|x_n-k^*\|)$, $N_2 := \sup_{n\ge1}(2|\beta|\|x_n-k^*\|)$ and $N_3 := \sup_{n\ge1}(\|x_n-k^*\|+\|w_n-k^*\|)$. We obtain from the condition $\{\chi_n\}\subset$ [a, b]$\subset(0,2)$ that

$$\frac{d_n^4}{\|\sum_{i\in I(y_n)}\lambda_{i,n}\mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i y_n\|^2}\to 0, \quad n\to\infty. \tag{3.32}$$

For each $i\in I(y_n)$, we get

$$\lambda_{i,n}d_n\|\mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i y_n\|\le \lambda_{i,n}d_n^2\|\mathcal{T}_i^*\|,$$

which implies that

$$d_n\sum_{i\in I(y_n)}\lambda_{i,n}\|\mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i y_n\|\le d_n^2\sum_{i\in I(y_n)}\lambda_{i,n}\|\mathcal{T}_i^*\|\le d_n^2\max_{i\in I(y_n)}\{\|\mathcal{T}_i^*\|\}. \tag{3.33}$$

Using the following inequality

$$\left\|\sum_{i\in I(y_n)}\lambda_{i,n}\mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i y_n\right\|\le\sum_{i\in I(y_n)}\lambda_{i,n}\left\|\mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i y_n\right\|$$

in (3.33) gives

$$d_n\left\|\sum_{i\in I(y_n)}\lambda_{i,n}\mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i y_n\right\|\le d_n^2\max_{i\in I(y_n)}\{\|\mathcal{T}_i^*\|\},$$

which implies

$$d_n\le\max_{i\in I(y_n)}\{\|\mathcal{T}_i^*\|\}\frac{d_n^2}{\|\sum_{i\in I(y_n)}\lambda_{i,n}\mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i y_n\|}. \tag{3.34}$$

Combining (3.32) and (3.34), show that $d_n\to 0$. Hence

$$\lim_{n\to\infty}\|\mathcal{T}_i y_n-P_{Q_i}\mathcal{T}_i y_n\|=0, \quad \forall i\in I(y_n). \tag{3.35}$$

From the definition $I(y_n)$, it follows that

$$\lim_{n\to\infty}\|\mathcal{T}_i y_n-P_{Q_i}\mathcal{T}_i y_n\|=0, \quad \forall i=1,2,\dots,N. \tag{3.36}$$

Combining (3.6) and (3.36), then

$$\lim_{n\to\infty}\|\mathcal{T}_i y_n-P_{Q_i}\mathcal{T}_i y_n\|=0, \quad \forall i=1,2,\dots,N \tag{3.37}$$

for the both Case I and Case II. Additionally, (3.5) and (3.35) imply that for all $i\in I(y_n)$,

$$\|\mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i y_n\|\le\max_{i\in I(y_n)}\{\|\mathcal{T}_i^*\|\}\|\mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i y_n\|\to 0, \quad n\to\infty.$$

Therefore,

$$\|\sum_{i\in I(y_n)} \lambda_{i,n}\mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i y_n\| \le \sum_{i\in I(y_n)} \lambda_{i,n}\|\mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i y_n\| \to 0, \quad n\to\infty. \tag{3.38}$$

Furthermore,

$$\begin{aligned}\|w_n - P_C(y_n)\| &= \left\| P_C\left[y_n - \gamma_n \sum_{i\in I(y_n)} \lambda_{i,n}\mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i y_n\right] - P_C(y_n)\right\| \\ &\le \gamma_n \|\sum_{i\in I(y_n)} \lambda_{i,n}\mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i y_n\| \to 0, \quad n\to\infty. \end{aligned} \tag{3.39}$$

Observe that

$$\begin{aligned}\|y_n - P_C(y_n)\| &= \|y_n - w_n + w_n - P_C(y_n)\| \\ &\le \|y_n - w_n\| + \|w_n - P_C(y_n)\|.\end{aligned}$$

By (3.30) and (3.39), we conclude that

$$\|y_n - P_C(y_n)\| \to 0. \quad n\to\infty. \tag{3.40}$$

By Lemma 3.5 we have that $\{x_n\}$ and $\{y_n\}$ are bounded, hence, there is a subsequence $\{x_{n_K}\}$ of $\{x_n\}$ such that x_{n_K} converges weakly to $q \in H$, $k\to\infty$. Also, by (3.29), there is a subsequence $\{y_{n_K}\}$ of $\{y_n\}$ so that y_{n_K} converges weakly to q. By Lemma 2.4 and (3.40) we have that $q \in F(P_C) = C$. Also, since $\mathcal{T}_i$ is a bounded linear operator, we have $\mathcal{T}_i y_{n_K}$ converges weakly to $\mathcal{T}_i q$ for each $i = 1,2,\dots,N$. Using Lemma 2.4 again and (3.37), we deduce that $\mathcal{T}_i q \in F(P_{Q_i})$. That is, $\mathcal{T}_i q \in Q_i$ for each $i = 1,2,\dots,N$. As a result, we have that $q \in \Gamma$ as required. □

We prove the main result of this paper as follows:

Theorem 3.7. *Suppose that Assumption 3.2 holds. Then the sequence $\{x_n\}$ iteratively generated by Algorithm 3.3 converges weakly to a unique point in Γ.*

Proof. We show that $\{x_n\}$ converges weakly to $k^* \in \Gamma$. We have already shown that every sequential weak cluster point of $\{x_n\}$ is in Γ. Since $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$ and $\lim_{n\to\infty} u_n$ exists, we obtain from definition of u_n that

$$\lim_{n\to\infty}(\|x_n - k^*\|^2 - \vartheta\|x_{n-1}-k^*\|^2 - \beta\|x_{n-2}-k^*\|^2)$$

exists for all $k^* \in \Gamma$. Suppose that there exists $\{x_{n_j}\} \subset \{x_n\}$ and $\{x_{n_m}\} \subset \{x_n\}$ such that $x_{n_j} \rightharpoonup q, \quad j\to\infty$ and $x_{n_m} \rightharpoonup k^*, \quad m\to\infty$. We show that $k^* = q$. Observe that

$$2\langle x_n, k^*-q\rangle = \|x_n - q\|^2 - \|x_n - k^*\|^2 - \|q\|^2 + \|k^*\|^2; \tag{3.41}$$

$$2\langle x_{n-1}, k^*-q\rangle = \|x_{n-1} - q\|^2 - \|x_{n-1} - k^*\|^2 - \|q\|^2 + \|k^*\|^2;$$

and

$$2\langle x_{n-2}, k^*-q\rangle = \|x_{n-2} - q\|^2 - \|x_{n-2} - k^*\|^2 - \|q\|^2 + \|k^*\|^2.$$

Therefore,

$$2\langle -\vartheta x_{n-1}, k^*-q\rangle = -\vartheta\|x_{n-1} - q\|^2 + \vartheta\|x_{n-1} - k^*\|^2 + \vartheta\|q\|^2 - \vartheta\|k^*\|^2. \tag{3.42}$$

and

$$2\langle -\beta x_{n-2}, k^*-q\rangle = -\beta\|x_{n-2} - q\|^2 + \beta\|x_{n-2} - k^*\|^2 + \beta\|q\|^2 - \beta\|k^*\|^2. \tag{3.43}$$

Addition of (3.41), (3.42) and (3.43) gives

$$\begin{aligned}2\langle x_n - \vartheta x_{n-1} - \beta x_{n-2}, k^*-q\rangle &= (\|x_n - q\|^2 - \vartheta\|x_{n-1}-q\|^2 - \beta\|x_{n-2}-q\|^2) - (\|x_n - k^*\|^2 \\ &\quad -\vartheta\|x_{n-1}-k^*\|^2 - \beta\|x_{n-2}-k^*\|^2) + (1-\vartheta-\beta)(\|k^*\|^2 - \|q\|^2).\end{aligned} \tag{3.44}$$

Observe that by (3.31), we know

$$\lim_{n\to\infty}(\|x_n - k^*\|^2 - \vartheta\|x_{n-1}-k^*\|^2 - \beta\|x_{n-2}-k^*\|^2)$$

exists and

$$\lim_{n\to\infty}(\|x_n - q\|^2 - \vartheta\|x_{n-1}-q\|^2 - \beta\|x_{n-2}-q\|^2)$$

also exists. Thus, from (3.44) it follows that $\lim_{n\to\infty}\langle x_n - \vartheta x_{n-1} - \beta x_{n-2}, k^*-q\rangle$ exists. Consequently,

$$\langle q - \vartheta q - \beta q, k^* - q\rangle = \lim_{j\to\infty}\langle x_{n_j} - \vartheta x_{n_j-1} - \beta x_{n_j-2}, k^*-q\rangle$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \langle x_n - \vartheta x_{n-1} - \beta x_{n-2}, k^* - q\rangle \\
&= \lim_{m\to\infty} \langle x_{n_m} - \vartheta x_{n_m-1} - \beta x_{n_m-2}, k^* - q\rangle \\
&= \langle k^* - \vartheta k^* - \beta k^*, k^* - q\rangle.
\end{aligned}
$$

Hence,

$$(1-\vartheta-\beta)\|k^*-q\|^2 = 0.$$

Since $\beta \le 0 < 1-\vartheta$, we obtain that $k^* = q$. Hence, $\{x_n\}$ converges weakly to a point in Γ. This complete this proof. □

4. Strong convergence result

Assumption 4.1. The following assumptions will be used in the convergence analysis.

(a) Choose $\vartheta \in \left(0, \frac{1}{3}\right]$, $\eta \in (0,1)$ and β such that $\max\{-\frac{\vartheta}{2}, \frac{2\vartheta-\frac{2}{3}-8\vartheta^2\tau}{3+5\vartheta}, \frac{3\vartheta-1-6\tau}{3}\} < \beta \le 0$, where $\tau = -\frac{1}{\eta}$;

(b) with the choices of ϑ, β and τ in (a) above, compute $\frac{3\beta\vartheta+2\beta+4\beta^2\tau}{\vartheta-2\vartheta\beta-\beta-4\vartheta^2\tau-\frac{2}{3}}$.

Now choose $\{\sigma_n\}$ in $\left(0, 1-\frac{3\beta\vartheta+2\beta+4|\beta|^2\tau}{2\vartheta-2\vartheta\beta-\beta-4\vartheta^2\tau-\frac{2}{3}}\right)$ such that $\lim_{n\to\infty}\sigma_n = 0$, $\sum_{n=1}^{\infty}\sigma_n = \infty$.

Remark 4.2. Observe that by the choices $\vartheta \in \left(0, \frac{1}{3}\right]$, and $\max\{-\frac{\vartheta}{2}, \frac{2\vartheta-\frac{2}{3}-8\vartheta^2\tau}{3+5\vartheta}, \frac{3\vartheta-1-6\tau}{3}\} < \beta \le 0$, we have

$$0 < \frac{3\beta\vartheta+2\beta+4|\beta|^2\tau}{2\vartheta-2\vartheta\beta-\beta-4\vartheta^2\tau-\frac{2}{3}} < 1.$$

Now, we introduce the following algorithm:

Algorithm 4.3. Strongly convergent two-step inertial accelerated algorithm

Step 0: Choose parameters $\{\sigma_n\}$, β, ϑ, and η such that Assumption 4.1 holds. For arbitrary x_{-1}, x_0, $y_0 \in H$, set $n := 1$ and

Step 1: Compute

$$x_{n+1} = P_C[\sigma_n x_0 + (1-\sigma_n)w_n] \quad \forall n \ge 1, \tag{4.1}$$

Compute

$$\begin{cases} y_n = x_n + \vartheta(x_n - x_{n-1}) + \beta(x_{n-1} - x_{n-2}) \\ w_n = y_n - \gamma_n \sum_{i\in I(y_n)} \lambda_{i,n} \mathcal{T}_i^*(I - P_{Q_i})\mathcal{T}_i y_n, \end{cases} \tag{4.2}$$

where $I(y_n) = \{i : \|\mathcal{T}_i y_n - P_{Q_i}\mathcal{T}_i y_n\| = \max_{i=1,2,\cdots,N}\|\mathcal{T}_i y_n - P_{Q_i}\mathcal{T}_i y_n\|\}$, $\lambda_{i,n} \ge 0$ $\forall i \in I(y_n)$, $\sum_{i\in I(y_n)}\lambda_{i,n} = 1$, and $\{\gamma_n\}$ is a sequence of nonnegative real numbers.

Step 2: Set n = n+ 1, and go to Step 1.

Remark 4.4.

1. If $\vartheta = \beta = 0$, and $\sigma_n = 0$, our method recovers the well-known CQ algorithm (Eq. (4.1)) studied by Reich et al. in [28], and if $\beta = 0$ and $\sigma_n = 0$, our method recovers Algorithm 3.3 studied in [31]. In contrast to the methods in [31], [1, Equation 1.5], [28] and our method generates iterative sequences that converge strongly to the unique solution of the problem studied in this paper. Another unique feature of our method is its ability to perform without prior knowledge of the operator norm in constraint to method in [1], since operator norm might be difficult to calculate or even unknown. Moreover, our method adopts two-step inertial technology without on-line rule unlike the methods in [1,28] with no inertial terms, and [31] with a single inertial term and performs with on-line rule. Hence, our method improves the methods in [1,28,31].

Lemma 4.5. *Suppose that Assumptions 3.1 and 4.1 hold. Let $\{\gamma_n\}$ be a sequence of nonnegative real numbers given by (3.3). Then the sequence $\{x_n\}$ generated by Algorithm 4.3 is bounded.*

Proof. Let $k^* \in \Gamma$ then, using (2.2) and (4.1), we have

$$\langle x_{n+1} - [\sigma_n x_0 + (1-\sigma_n)w_n], k^* - x_{n+1}\rangle \geq 0.$$

Thus,

$$0 \leq 2\langle x_{n+1} - [\sigma_n x_0 + (1-\sigma_n)w_n], k^* - x_{n+1}\rangle. \tag{4.3}$$

Take $b_n = \sigma_n x_0 + (1-\sigma_n)w_n$, so that (4.3) becomes

$$\begin{aligned} 0 &\leq 2\langle x_{n+1} - b_n, k^* - x_{n+1}\rangle \\ &= \|b_n - k^*\|^2 - \|x_{n+1} - b_n\|^2 - \|x_{n+1} - k^*\|^2. \end{aligned} \tag{4.4}$$

From (4.4), we have

$$\|x_{n+1} - k^*\|^2 \leq \|b_n - k^*\|^2 - \|x_{n+1} - b_n\|^2. \tag{4.5}$$

By Lemma 2.5(ii), we have

$$\begin{aligned} \|b_n - k^*\|^2 &= \|\sigma_n x_0 + (1-\sigma_n)w_n - k^*\|^2 \\ &= \|w_n - k^* - \sigma_n(w_n - x_0)\|^2 \\ &= \|w_n - k^*\|^2 + \sigma_n^2\|w_n - x_0\|^2 - 2\sigma_n\langle w_n - k^*, w_n - x_0\rangle \\ &= \|w_n - k^*\|^2 + \sigma_n^2\|w_n - x_0\|^2 - \sigma_n\|w_n - x_0\|^2 - \sigma_n\|w_n - k^*\|^2 + \sigma_n\|x_0 - k^*\|^2. \end{aligned} \tag{4.6}$$

Similarly,

$$\begin{aligned} \|x_{n+1} - b_n\|^2 &= \|b_n - x_{n+1}\|^2 \\ &= \|w_n - x_{n+1}\|^2 + \sigma_n^2\|w_n - x_0\|^2 - \sigma_n\|w_n - x_0\|^2 - \sigma_n\|w_n - x_{n+1}\|^2 + \sigma_n\|x_0 - x_{n+1}\|^2. \end{aligned} \tag{4.7}$$

Using (4.6) and (4.7) in (4.5), then

$$\begin{aligned} \|x_{n+1} - k^*\|^2 &\leq (1-\sigma_n)\|w_n - k^*\|^2 - (1-\sigma_n)\|w_n - x_{n+1}\|^2 + \sigma_n\|x_0 - k^*\|^2 - \sigma_n\|x_0 - x_{n+1}\|^2 \\ &\leq (1-\sigma_n)\|w_n - k^*\|^2 - (1-\sigma_n)\|w_n - x_{n+1}\|^2 + \sigma_n\|x_0 - k^*\|^2. \end{aligned} \tag{4.8}$$

We estimate $\|w_n - k^*\|^2$ as follows:

$$\begin{aligned} \|w_n - k^*\|^2 &= \left\| y_n - k^* - \gamma_n \sum_{i\in I(y_n)} \lambda_{i,n}\mathcal{T}_i^*(I - P_{Q_i})\mathcal{T}_i y_n \right\|^2 \\ &= \|y_n - k^*\|^2 - 2\gamma_n \sum_{i\in I(y_n)} \lambda_{i,n}\langle y_n - k^*, \mathcal{T}_i^*(I - P_{Q_i})\mathcal{T}_i y_n\rangle + \gamma_n^2 \left\| \sum_{i\in I(y_n)} \lambda_{i,n}\mathcal{T}_i^*(I - P_{Q_i})\mathcal{T}_i y_n \right\|^2 \\ &\leq \|y_n - k^*\|^2 - 2\gamma_n d_n^2 + \gamma_n^2 \left\| \sum_{i\in I(y_n)} \lambda_{i,n}\mathcal{T}_i^*(I - P_{Q_i})\mathcal{T}_i y_n \right\|^2. \end{aligned}$$

Following similar argument from (3.4) to (3.10), we obtain

$$\|w_n - k^*\| \leq \|y_n - k^*\|. \tag{4.9}$$

Also, the estimation of $\|w_n - x_{n+1}\|^2$ is as follows:

$$\begin{aligned} \|w_n - x_{n+1}\|^2 &= \left\| y_n - x_{n+1} - \gamma_n \sum_{i\in I(y_n)} \lambda_{i,n}\mathcal{T}_i^*(I - P_{Q_i})\mathcal{T}_i y_n \right\|^2 \\ &= \|y_n - x_{n+1}\|^2 - 2\langle \gamma_n \sum_{i\in I(y_n)} \lambda_{i,n}\mathcal{T}_i^*(I - P_{Q_i})\mathcal{T}_i y_n, y_n - x_{n+1}\rangle + \left\| \gamma_n \sum_{i\in I(y_n)} \lambda_{i,n}\mathcal{T}_i^*(I - P_{Q_i})\mathcal{T}_i y_n \right\|^2 \\ &\geq \|y_n - x_{n+1}\|^2 - 2\left[\|\gamma_n \sum_{i\in I(y_n)} \lambda_{i,n}\mathcal{T}_i^*(I - P_{Q_i})\mathcal{T}_i y_n\| \|y_n - x_{n+1}\| \right] \\ &\quad + \left\| \gamma_n \sum_{i\in I(y_n)} \lambda_{i,n}\mathcal{T}_i^*(I - P_{Q_i})\mathcal{T}_i y_n \right\|^2 \\ &\geq \|y_n - x_{n+1}\|^2 - \left[\eta\|\gamma_n \sum_{i\in I(y_n)} \lambda_{i,n}\mathcal{T}_i^*(I - P_{Q_i})\mathcal{T}_i y_n\|^2 + \frac{\|y_n - x_{n+1}\|^2}{\eta} \right] \\ &\quad + \left\| \gamma_n \sum_{i\in I(y_n)} \lambda_{i,n}\mathcal{T}_i^*(I - P_{Q_i})\mathcal{T}_i y_n \right\|^2. \end{aligned} \tag{4.10}$$

where the last inequality was obtained from Lemma 2.6(i). Now, since $\eta \in (0,1)$ it implies that $\eta < 1$, which further implies that $-\eta > -1$. Using this in (4.10), (4.10) reduces to the following inequality:

$$\|w_n - x_{n+1}\|^2 \geq (1+\tau)\|y_n - x_{n+1}\|^2, \tag{4.11}$$

where $\tau = -\frac{1}{\eta} < -1$. Now, using (4.9) and (4.11) in (4.8), we obtain

$$\begin{aligned}\|x_{n+1} - k^*\|^2 &\leq (1-\sigma_n)\|y_n - k^*\|^2 - (1-\sigma_n)(1+\tau)\|y_n - x_{n+1}\|^2 + \sigma_n\|x_0 - k^*\|^2 \\ &= (1-\sigma_n)\|y_n - k^*\|^2 - (1-\sigma_n)\|y_n - x_{n+1}\|^2 - \tau(1-\sigma_n)\|y_n - x_{n+1}\|^2 + \sigma_n\|x_0 - k^*\|^2.\end{aligned} \tag{4.12}$$

Now, we estimate $-\tau(1-\sigma_n)\|y_n - x_{n+1}\|^2$.

$$\begin{aligned}\|y_n - x_{n+1}\|^2 &= \|y_n - x_n + x_n - x_{n+1}\|^2 \\ &\leq 2\|y_n - x_n\|^2 + 2\|x_n - x_{n+1}\|^2.\end{aligned} \tag{4.13}$$

Observe that

$$\begin{aligned}2\|y_n - x_n\|^2 &= 2\|x_n + \vartheta(x_n - x_{n-1}) + \beta(x_{n-1} - x_{n-2}) - x_n\|^2 \\ &\leq 4\vartheta^2\|x_n - x_{n-1}\|^2 + 4|\beta|^2\|x_{n-1} - x_{n-2}\|^2.\end{aligned} \tag{4.14}$$

Therefore, using (4.13), (4.14) and the fact that $-\tau > 0$, we obtain the following

$$\begin{aligned}-\tau(1-\sigma_n)\|y_n - x_{n+1}\|^2 \leq\ & -4\vartheta^2\tau(1-\sigma_n)\|x_n - x_{n-1}\|^2 - 4|\beta|^2\tau(1-\sigma_n)\|x_{n-1} - x_{n-2}\|^2 \\ & -2\tau(1-\sigma_n)\|x_n - x_{n+1}\|^2.\end{aligned} \tag{4.15}$$

Substituting (3.14), (3.19) and (4.15) in (4.12), we have

$$\begin{aligned}\|x_{n+1} - k^*\|^2 \leq\ & (1-\sigma_n)\left[(1+\vartheta)\|x_n - k^*\|^2 - (\vartheta-\beta)\|x_{n-1} - k^*\|^2 - \beta\|x_{n-2} - k^*\|^2\right. \\ & +(1+\vartheta)(\vartheta-\beta)\|x_n - x_{n-1}\|^2 + \beta(1+\vartheta)\|x_n - x_{n-2}\|^2 \\ & \left.-\beta(\vartheta-\beta)\|x_{n-1} - x_{n-2}\|^2\right] \\ & -(1-\sigma_n)\left[(1-\vartheta-|\beta|)\|x_{n+1} - x_n\|^2 + (\vartheta^2 - \vartheta - |\beta|\vartheta)\|x_n - x_{n-1}\|^2\right. \\ & \left.+(\beta^2 - |\beta|\vartheta - |\beta|)\|x_{n-1} - x_{n-2}\|^2\right] \\ & -4\vartheta^2\tau(1-\sigma_n)\|x_n - x_{n-1}\|^2 - 4|\beta|^2\tau(1-\sigma_n)\|x_{n-1} - x_{n-2}\|^2 \\ & -2\tau(1-\sigma_n)\|x_n - x_{n+1}\|^2 + \sigma_n\|x_0 - k^*\|^2 \\ \leq\ & (1-\sigma_n)\left[(1+\vartheta)\|x_n - k^*\|^2 - (\vartheta-\beta)\|x_{n-1} - k^*\|^2 - \beta\|x_{n-2} - k^*\|^2\right] \\ & +(1-\sigma_n)\left[(1+\vartheta)(\vartheta-\beta) - (\vartheta^2 - \vartheta - |\beta|\vartheta) - 4\vartheta^2\tau\right]\|x_n - x_{n-1}\|^2 \\ & -(1-\sigma_n)\left[\beta(\vartheta-\beta) + (\beta^2 - |\beta|\vartheta - |\beta|) + 4|\beta|^2\tau\right]\|x_{n-1} - x_{n-2}\|^2 \\ & -(1-\sigma_n)\left[(1-\vartheta-|\beta|) + 2\tau\right]\|x_n - x_{n+1}\|^2 + \sigma_n\|x_0 - k^*\|^2 \\ & +(1-\sigma_n)\beta(1+\vartheta)\|x_n - x_{n-2}\|^2.\end{aligned} \tag{4.16}$$

Now, estimate $(1-\sigma_n)\beta(1+\vartheta)\|x_n - x_{n-2}\|^2$.

$$\begin{aligned}\|x_n - x_{n-2}\|^2 &= (\|x_n - x_{n-1} + x_{n-1} - x_{n-2}\|)^2 \\ &= (\|x_n - x_{n-1} - (x_{n-2} - x_{n-1})\|)^2 \\ &\geq (\|x_n - x_{n-1}\| - \|(x_{n-2} - x_{n-1})\|)^2 \\ &= \|x_n - x_{n-1}\|^2 - 2\|x_n - x_{n-1}\|\|(x_{n-2} - x_{n-1})\| + \|(x_{n-2} - x_{n-1})\|^2 \\ &\geq \|x_n - x_{n-1}\|^2 - \frac{\|x_n - x_{n-1}\|^2}{2} - 2\|x_{n-2} - x_{n-1}\|^2 + \|x_{n-2} - x_{n-1}\|^2 \\ &= \frac{1}{2}\|x_n - x_{n-1}\|^2 - \|x_{n-2} - x_{n-1}\|^2.\end{aligned} \tag{4.17}$$

From (4.17) and the fact that $\beta < 0$, we obtain

$$\begin{aligned}(1-\sigma_n)\beta(1+\vartheta)\|x_n - x_{n-2}\|^2 \leq\ & \frac{\beta(1+\vartheta)(1-\sigma_n)}{2}\|x_n - x_{n-1}\|^2 \\ & -(1-\sigma_n)\beta(1+\vartheta)\|x_{n-2} - x_{n-1}\|^2.\end{aligned} \tag{4.18}$$

Now, using (4.18) in (4.16), we obtain the following:

$$\begin{aligned}&\|x_{n+1} - k^*\|^2 - \vartheta\|x_n - k^*\|^2 - \beta\|x_{n-1} - k^*\|^2 \\ &\leq (1-\sigma_n)\left[\|x_n - k^*\|^2 - \vartheta\|x_{n-1} - k^*\|^2 - \beta\|x_{n-2} - k^*\|^2\right] \\ &\quad +(1-\sigma_n)\left[(1+\vartheta)(\vartheta-\beta) - (\vartheta^2 - \vartheta - |\beta|\vartheta) - 4\vartheta^2\tau\right]\|x_n - x_{n-1}\|^2 \\ &\quad -(1-\sigma_n)\left[\beta(\vartheta-\beta) + (\beta^2 - |\beta|\vartheta - |\beta|) + 4|\beta|^2\tau\right]\|x_{n-1} - x_{n-2}\|^2 + \sigma_n\|x_0 - k^*\|^2 \\ &\quad -(1-\sigma_n)\left[(1-\vartheta-|\beta|) + 2\tau\right]\|x_n - x_{n+1}\|^2 - \beta\sigma_n\|x_{n-1} - k^*\|^2 - \vartheta\sigma_n\|x_n - k^*\|^2\end{aligned}$$

$$
\begin{aligned}
&+\frac{\beta(1+\vartheta)(1-\sigma_n)}{2}\|x_n-x_{n-1}\|^2-(1-\sigma_n)\beta(1+\vartheta)\|x_{n-2}-x_{n-1}\|^2\\
&\le(1-\sigma_n)\left[\|x_n-k^*\|^2-\vartheta\|x_{n-1}-k^*\|^2-\beta\|x_{n-2}-k^*\|^2\right]\\
&+(1-\sigma_n)\left[(1+\vartheta)(\vartheta-\beta)-(\vartheta^2-\vartheta-|\beta|\vartheta)-4\vartheta^2\tau\right]\|x_n-x_{n-1}\|^2\\
&-(1-\sigma_n)\left[3\beta\vartheta+2\beta+4|\beta|^2\tau\right]\|x_{n-1}-x_{n-2}\|^2+\sigma_n\|x_0-k^*\|^2\\
&-(1-\sigma_n)\left[(1-\vartheta-|\beta|)+2\tau\right]\|x_n-x_{n+1}\|^2+(-2\beta\sigma_n-\vartheta\sigma_n)\|x_n-k^*\|^2\\
&+\left(-2\beta\sigma_n+\frac{\beta(1+\vartheta)(1-\sigma_n)}{2}\right)\|x_n-x_{n-1}\|^2. \qquad (4.19)
\end{aligned}
$$

Observe that

$$\lim_{n\to\infty}\left(-2\beta\sigma_n+\frac{\beta(1+\vartheta)(1-\sigma_n)}{2}\right)=\frac{\beta}{2}(1+\vartheta)\le 0.$$

Therefore, there exists $n_0\in\mathbb{N}$ such that

$$\left(-2\beta\sigma_n+\frac{\beta(1+\vartheta)(1-\sigma_n)}{2}\right)\le 0,\quad \forall n\ge n_0.$$

And, since $-\frac{\vartheta}{2}\le\beta$, we get

$$(-2\beta\sigma_n-\vartheta\sigma_n)\le 0.$$

So that (4.19) becomes

$$
\begin{aligned}
&\|x_{n+1}-k^*\|^2-\vartheta\|x_n-k^*\|^2-\beta\|x_{n-1}-k^*\|^2\le(1-\sigma_n)\left[\|x_n-k^*\|^2-\vartheta\|x_{n-1}-k^*\|^2\right.\\
&\left.-\beta\|x_{n-2}-k^*\|^2\right]+(1-\sigma_n)\left[(1+\vartheta)(\vartheta-\beta)-(\vartheta^2-\vartheta-|\beta|\vartheta)-4\vartheta^2\tau\right]\|x_n-x_{n-1}\|^2\\
&-(1-\sigma_n)\left[3\beta\vartheta+2\beta+4|\beta|^2\tau\right]\|x_{n-1}-x_{n-2}\|^2+\sigma_n\|x_0-k^*\|^2\\
&-(1-\sigma_n)\left[(1-\vartheta-|\beta|)+2\tau\right]\|x_n-x_{n+1}\|^2,\quad\forall n\ge n_0.
\end{aligned}
$$

The inequality above is equivalent to the following

$$
\begin{aligned}
&\|x_{n+1}-k^*\|^2-\vartheta\|x_n-k^*\|^2-\beta\|x_{n-1}-k^*\|^2+\frac{2}{3}\|x_n-x_{n+1}\|^2\\
&\le(1-\sigma_n)\left[\|x_n-k^*\|^2-\vartheta\|x_{n-1}-k^*\|^2-\beta\|x_{n-2}-k^*\|^2+\frac{2}{3}\|x_n-x_{n-1}\|^2\right]\\
&+(1-\sigma_n)(2\vartheta-2\vartheta\beta-\beta-4\vartheta^2\tau-\frac{2}{3})\|x_n-x_{n-1}\|^2+\sigma_n\|x_0-k^*\|^2\\
&-\left[(1-\sigma_n)[1-\vartheta-|\beta|+2\tau]-\frac{2}{3}\right]\|x_n-x_{n+1}\|^2\\
&-(1-\sigma_n)(3\beta\vartheta+2\beta+4|\beta|^2\tau)\|x_{n-1}-x_{n-2}\|^2,\quad\forall n\ge n_0. \qquad (4.20)
\end{aligned}
$$

Observe that

$$\liminf_{n\to\infty}\left[(1-\sigma_n)[1-\vartheta-|\beta|+2\tau]-\frac{2}{3}\right]=1-\vartheta-|\beta|+2\tau-\frac{2}{3}>0.$$

Since $-|\beta|=\beta$, and $\frac{3\vartheta-1-6\tau}{3}<\beta$. Therefore, there exists $n_1\in\mathbb{N}$ such that for all $n\ge n_1>n_0$, we have

$$(1-\sigma_n)[1-\vartheta-|\beta|+2\tau]-\frac{2}{3}>0.$$

Hence, (4.20) becomes

$$
\begin{aligned}
&\|x_{n+1}-k^*\|^2-\vartheta\|x_n-k^*\|^2-\beta\|x_{n-1}-k^*\|^2+\frac{2}{3}\|x_n-x_{n+1}\|^2\le\\
&(1-\sigma_n)\left[\|x_n-k^*\|^2-\vartheta\|x_{n-1}-k^*\|^2-\beta\|x_{n-2}-k^*\|^2+\frac{2}{3}\|x_n-x_{n-1}\|^2\right]\\
&+(1-\sigma_n)(2\vartheta-2\vartheta\beta-\beta-4\vartheta^2\tau-\frac{2}{3})\|x_n-x_{n-1}\|^2+\sigma_n\|x_0-k^*\|^2\\
&-(1-\sigma_n)(3\beta\vartheta+2\beta+4|\beta|^2\tau)\|x_{n-1}-x_{n-2}\|^2\quad\forall n\ge n_1, \qquad (4.21)
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\|x_{n+1}-k^*\|^2-\vartheta\|x_n-k^*\|^2-\beta\|x_{n-1}-k^*\|^2+\frac{2}{3}\|x_n-x_{n+1}\|^2\\
&-(1-\sigma_n)(2\vartheta-2\vartheta\beta-\beta-4\vartheta^2\tau-\frac{2}{3})\|x_n-x_{n-1}\|^2\le\\
&(1-\sigma_n)\left[\|x_n-k^*\|^2-\vartheta\|x_{n-1}-k^*\|^2-\beta\|x_{n-2}-k^*\|^2+\frac{2}{3}\|x_n-x_{n-1}\|^2\right]\\
&+\sigma_n\|x_0-k^*\|^2-(1-\sigma_n)(3\beta\vartheta+2\beta+4|\beta|^2\tau)\|x_{n-1}-x_{n-2}\|^2\quad\forall n\ge n_1. \qquad (4.22)
\end{aligned}
$$

Since $\sigma_n<1-\frac{3\beta\vartheta+2\beta+4|\beta|^2\tau}{2\vartheta-2\vartheta\beta-\beta-4\vartheta^2\tau-\frac{2}{3}}$, we have that

$$-(3\beta\vartheta+2\beta+4|\beta|^2\tau)<-(1-\sigma_n)(2\vartheta-2\vartheta\beta-\beta-4\vartheta^2\tau-\frac{2}{3}).$$

We get from (4.22) that

$$\begin{aligned}&\|x_{n+1}-k^*\|^2-\vartheta\|x_n-k^*\|^2-\beta\|x_{n-1}-k^*\|^2+\frac{2}{3}\|x_n-x_{n+1}\|^2\\&-(3\beta\vartheta+2\beta+4|\beta|^2\tau)\|x_n-x_{n-1}\|^2\le\\&(1-\sigma_n)\left[\|x_n-k^*\|^2-\vartheta\|x_{n-1}-k^*\|^2-\beta\|x_{n-2}-k^*\|^2+\frac{2}{3}\|x_n-x_{n-1}\|^2\right]\\&+\sigma_n\|x_0-k^*\|^2-(1-\sigma_n)(3\beta\vartheta+2\beta+4|\beta|^2\tau)\|x_{n-1}-x_{n-2}\|^2\quad\forall n\ge n_1.\end{aligned}\tag{4.23}$$

Define for each $n\ge n_1$,

$$\begin{aligned}s_n := &\Big[\|x_n-k^*\|^2-\vartheta\|x_{n-1}-k^*\|^2-\beta\|x_{n-2}-k^*\|^2\\&+\frac{2}{3}\|x_n-x_{n-1}\|^2-(3\beta\vartheta+2\beta+4|\beta|^2\tau)\|x_{n-1}-x_{n-2}\|^2\Big].\end{aligned}\tag{4.24}$$

Next, we prove that $s_n\ge 0,\quad\forall n\ge n_1$. Notice that

$$\|x_{n-1}-k^*\|^2\le 2\|x_n-x_{n-1}\|^2+2\|x_n-k^*\|^2.$$

So,

$$\begin{aligned}s_n &= \|x_n-k^*\|^2-\vartheta\|x_{n-1}-k^*\|^2-\beta\|x_{n-2}-k^*\|^2\\&\quad+\frac{2}{3}\|x_n-x_{n-1}\|^2-(3\beta\vartheta+2\beta+4|\beta|^2\tau)\|x_{n-1}-x_{n-2}\|^2\\&\ge \|x_n-k^*\|^2-2\vartheta\|x_n-x_{n-1}\|^2-2\vartheta\|x_n-k^*\|^2-\beta\|x_{n-2}-k^*\|^2\\&\quad+\frac{2}{3}\|x_n-x_{n-1}\|^2-(3\beta\vartheta+2\beta+4|\beta|^2\tau)\|x_{n-1}-x_{n-2}\|^2\\&= (1-2\vartheta)\|x_n-k^*\|^2+(\frac{2}{3}-2\vartheta)\|x_n-x_{n-1}\|^2-\beta\|x_{n-2}-k^*\|^2\\&\quad-\beta(3\vartheta+2+4\beta\tau)\|x_{n-1}-x_{n-2}\|^2\ge 0,\end{aligned}\tag{4.25}$$

since $0\le\vartheta<\frac{1}{3}$, $\beta\le 0$, and $\tau<-1$. We obtain from (4.23) that

$$s_{n+1}\le(1-\sigma_n)s_n+\sigma_n\|x_0-k^*\|^2.$$

Hence, using Lemma 2.8, we have that $\{s_n\}$ is a bounded sequence. Consequently, we deduce from (4.25) that $\{x_n\}$ is bounded. □

Theorem 4.6. *Suppose that Assumptions 3.1 and 4.1 hold. Let $\{\gamma_n\}$ be a sequence of nonnegative real numbers given by (3.3). Then the sequence $\{x_n\}$ generated by Algorithm 4.3 converges strongly to a unique point $P_\Gamma(x_0)$.*

Proof. Suppose $k^*=P_\Gamma(x_0)$. Then by Lemma 2.5, we get

$$\begin{aligned}\|b_n-k^*\|^2 &= \|\sigma_n(x_0-k^*)+(1-\sigma_n)(w_n-k^*)\|^2\\&= \sigma_n^2\|x_0-k^*\|^2+(1-\sigma_n)^2\|w_n-k^*\|^2+2\sigma_n(1-\sigma_n)\langle x_0-k^*,w_n-k^*\rangle.\end{aligned}\tag{4.26}$$

And,

$$\begin{aligned}\|x_{n+1}-b_n\|^2 &= \|\sigma_n(x_0-x_{n+1})+(1-\sigma_n)(w_n-x_{n+1})\|^2\\&= \sigma_n^2\|x_0-x_{n+1}\|^2+(1-\sigma_n)^2\|w_n-x_{n+1}\|^2-2\sigma_n(1-\sigma_n)\langle x_0-x_{n+1},w_n-x_{n+1}\rangle\\&\ge \sigma_n^2\|x_0-x_{n+1}\|^2+(1-\sigma_n)^2\|w_n-x_{n+1}\|^2-2\sigma_n(1-\sigma_n)\|x_0-x_{n+1}\|\|w_n-x_{n+1}\|\\&\ge \sigma_n^2\|x_0-x_{n+1}\|^2+(1-\sigma_n)^2\|w_n-x_{n+1}\|^2-2\sigma_n(1-\sigma_n)M_1\|w_n-x_{n+1}\|,\end{aligned}\tag{4.27}$$

where $M_1 := \sup_{n\ge n_1}\|x_0-x_{n+1}\|<\infty$, since $\{x_n\}$ is bounded. Using (4.26) and (4.27) in (4.4), we obtain

$$\begin{aligned}\|x_{n+1}-k^*\|^2 &\le \sigma_n^2\|x_0-k^*\|^2+(1-\sigma_n)^2\|w_n-k^*\|^2+2\sigma_n(1-\sigma_n)\langle x_0-k^*,w_n-k^*\rangle\\&\quad-\left[\sigma_n^2\|x_0-x_{n+1}\|^2+(1-\sigma_n)^2\|w_n-x_{n+1}\|^2-2\sigma_n(1-\sigma_n)M_1\|w_n-x_{n+1}\|\right]\\&\le (1-\sigma_n)\|w_n-k^*\|^2+\sigma_n\left(\sigma_n\|x_0-k^*\|^2+2(1-\sigma_n)\langle x_0-k^*,w_n-k^*\rangle\right.\\&\quad\left.+2(1-\sigma_n)M_1\|w_n-x_{n+1}\|\right)-(1-\sigma_n)^2\|w_n-x_{n+1}\|^2.\end{aligned}\tag{4.28}$$

Now, using (4.9) and (4.11) in (4.28), we obtain the following:

$$\begin{aligned}\|x_{n+1}-k^*\|^2 &\le (1-\sigma_n)\|y_n-k^*\|^2+\sigma_n\left(\sigma_n\|x_0-k^*\|^2+2(1-\sigma_n)\langle x_0-k^*,w_n-k^*\rangle\right.\\&\quad\left.+2(1-\sigma_n)M_1\|w_n-x_{n+1}\|\right)-(1-\sigma_n)^2(1+\tau)\|y_n-x_{n+1}\|^2\\&= (1-\sigma_n)\|y_n-k^*\|^2+\sigma_n\left(\sigma_n\|x_0-k^*\|^2+2(1-\sigma_n)\langle x_0-k^*,w_n-k^*\rangle\right.\\&\quad\left.+2(1-\sigma_n)M_1\|w_n-x_{n+1}\|\right)-(1-\sigma_n)^2\|y_n-x_{n+1}\|^2\\&\quad-(1-\sigma_n)^2\tau\|y_n-x_{n+1}\|^2.\end{aligned}\tag{4.29}$$

Using (3.14), (3.19) and (4.15) in (4.29), we get

$$\begin{aligned}\|x_{n+1}-k^*\|^2 \le\ & (1-\sigma_n)\left[(1+\vartheta)\|x_n-k^*\|^2-(\vartheta-\beta)\|x_{n-1}-k^*\|^2-\beta\|x_{n-2}-k^*\|^2\right.\\ & \left.+(1+\vartheta)(\vartheta-\beta)\|x_n-x_{n-1}\|^2+\beta(1+\vartheta)\|x_n-x_{n-2}\|^2-\beta(\vartheta-\beta)\|x_{n-1}-x_{n-2}\|^2\right]\\ & +\sigma_n\left(\sigma_n\|x_0-k^*\|^2+2(1-\sigma_n)\langle x_0-k^*,w_n-k^*\rangle\right.\\ & \left.+2(1-\sigma_n)M_1\|w_n-x_{n+1}\|\right)-(1-\sigma_n)^2\left[(1-\vartheta-|\beta|)\|x_{n+1}-x_n\|^2\right.\\ & \left.+(\vartheta^2-\vartheta-|\beta|\vartheta)\|x_n-x_{n-1}\|^2+(\beta^2-|\beta|\vartheta-|\beta|)\|x_{n-1}-x_{n-2}\|^2\right]\\ & -4\vartheta^2\tau(1-\sigma_n)^2\|x_n-x_{n-1}\|^2-4|\beta|^2\tau(1-\sigma_n)^2\|x_{n-1}-x_{n-2}\|^2\\ & -2\tau(1-\sigma_n)^2\|x_n-x_{n+1}\|^2,\end{aligned}$$

the inequality above implies the following:

$$\begin{aligned}&\|x_{n+1}-k^*\|^2-\vartheta\|x_n-k^*\|^2-\beta\|x_{n-1}-k^*\|^2+\frac{2}{3}\|x_n-x_{n+1}\|^2\\ &\ -(3\beta\vartheta+2\beta+4|\beta|^2\tau)\|x_n-x_{n-1}\|^2\le\\ &(1-\sigma_n)\left[\|x_n-k^*\|^2-\vartheta\|x_{n-1}-k^*\|^2-\beta\|x_{n-2}-k^*\|^2+\frac{2}{3}\|x_n-x_{n-1}\|^2\right.\\ &\ \left.-(3\beta\vartheta+2\beta+4|\beta|^2\tau)\|x_{n-1}-x_{n-2}\|^2\right]-\vartheta\sigma_n\|x_n-k^*\|^2-\beta\sigma_n\|x_{n-1}-k^*\|^2\\ &\ +\left[(1-\sigma_n)\left[(1+\vartheta)(\vartheta-\beta)-\frac{2}{3}\right]-(3\beta\vartheta+2\beta+4|\beta|^2\tau)-(1-\sigma_n)^2(\vartheta^2-\vartheta-|\beta|\vartheta+4\vartheta^2\tau)\right]\\ &\ \times\|x_n-x_{n-1}\|^2+(1-\sigma_n)\left[3\beta\vartheta+2\beta+4|\beta|^2\tau-\beta(\vartheta-\beta)-(1-\sigma_n)(\beta^2-|\beta|\vartheta-|\beta|+4|\beta|^2\tau)\right]\\ &\ \times\|x_{n-1}-x_{n-2}\|^2+\left[\frac{2}{3}-(1-\sigma_n)^2(1-\vartheta-|\beta|+2\tau)\right]\|x_{n+1}-x_n\|^2\\ &\ +\sigma_n\left(\sigma_n\|x_0-k^*\|^2+2(1-\sigma_n)\langle x_0-k^*,w_n-k^*\rangle+2(1-\sigma_n)M_1\|w_n-x_{n+1}\|\right).\end{aligned}\tag{4.30}$$

Define

$$r_n:=\sigma_n\|x_0-k^*\|^2+2(1-\sigma_n)\langle x_0-k^*,w_n-k^*\rangle+2(1-\sigma_n)M_1\|w_n-x_{n+1}\|,$$

then (4.30) reduces to

$$\begin{aligned}&s_{n+1}\le(1-\sigma_n)s_n+\sigma_nr_n+(-2\beta\sigma_n-\vartheta\sigma_n)\|x_n-k^*\|^2\\ &\ +\left[(1-\sigma_n)\left[(1+\vartheta)(\vartheta-\beta)-\frac{2}{3}\right]-(3\beta\vartheta+2\beta+4|\beta|^2\tau)-(1-\sigma_n)^2(\vartheta^2-\vartheta-|\beta|\vartheta+4\vartheta^2\tau)-2\beta\sigma_n\right]\\ &\ \times\|x_n-x_{n-1}\|^2+\left[\frac{2}{3}-(1-\sigma_n)^2(1-\vartheta-|\beta|+2\tau)\right]\|x_{n+1}-x_n\|^2\\ &\ +(1-\sigma_n)\left[3\beta\vartheta+2\beta+4|\beta|^2\tau-\beta(\vartheta-\beta)-(1-\sigma_n)(\beta^2-|\beta|\vartheta-|\beta|+4|\beta|^2\tau)\right]\|x_{n-1}-x_{n-2}\|^2,\end{aligned}\tag{4.31}$$

Since $-\frac{\vartheta}{2}\le\beta$, we have that

$$-2\beta\sigma_n-\vartheta\sigma_n\le 0\quad\forall n\ge n_1.\tag{4.32}$$

Observe that

$$\lim_{n\to\infty}\left[\frac{2}{3}-(1-\sigma_n)^2(1-\vartheta-|\beta|+2\tau)\right]=\frac{2}{3}-(1-\vartheta-|\beta|+2\tau)\le 0,$$

since $\frac{3\vartheta-1-6\tau}{3}<\beta$. Observe that

$$\begin{aligned}&\lim_{n\to\infty}\left[3\beta\vartheta+2\beta+4|\beta|^2\tau-\beta(\vartheta-\beta)-(1-\sigma_n)(\beta^2-|\beta|\vartheta-|\beta|+4|\beta|^2\tau)\right]\\ &=\beta(1+\vartheta)\le 0.\end{aligned}\tag{4.33}$$

Furthermore,

$$\begin{aligned}&\lim_{n\to\infty}\left[(1-\sigma_n)\left[(1+\vartheta)(\vartheta-\beta)-\frac{2}{3}\right]-(3\beta\vartheta+2\beta+4|\beta|^2\tau)-(1-\sigma_n)^2(\vartheta^2-\vartheta-|\beta|\vartheta+4\vartheta^2\tau)-2\beta\sigma_n\right]\\ &\quad=(1+\vartheta)(\vartheta-\beta)-\frac{2}{3}-(3\beta\vartheta+2\beta+4\beta^2\tau)-(\vartheta^2-\vartheta-|\beta|\vartheta+4\vartheta^2\tau)<0,\end{aligned}\tag{4.34}$$

since $\frac{2\vartheta-\frac{2}{3}-8\vartheta^2\tau}{3+5\vartheta}<\beta$, $\tau<-1$, and $-\vartheta\le\beta$, which implies $\beta^2\le\vartheta^2$. Using (4.32)–(4.34) in (4.31), there exists $n_2\ge n_1\in\mathbb{N}$ such that for all $n\ge n_2\ge n_1$,

$$\begin{aligned}&s_{n+1}\le(1-\sigma_n)s_n+\sigma_nr_n\\ &\ +\left[(1-\sigma_n)\left[(1+\vartheta)(\vartheta-\beta)-\frac{2}{3}\right]-(3\beta\vartheta+2\beta+4|\beta|^2\tau)-(1-\sigma_n)^2(\vartheta^2-\vartheta-|\beta|\vartheta+4\vartheta^2\tau)-2\beta\sigma_n\right]\\ &\ \times\|x_n-x_{n-1}\|^2.\end{aligned}\tag{4.35}$$

To conclude the proof, it is sufficient to prove in view of Lemma 2.7 that $\limsup_{i\to\infty}r_{n_i}\le 0$ for each subsequence $\{s_{n_i}\}\subset\{s_n\}$ such that $\liminf_{i\to\infty}(s_{n_i+1}-s_{n_i})\ge 0$. To this end, let $\{s_{n_i}\}$ be a subsequence of $\{s_n\}$ such that $\liminf_{i\to\infty}(s_{n_i+1}-s_{n_i})\ge 0$. From (4.35), we have

$$\limsup_{i\to\infty} -\Big[(1-\sigma_n)\left((1+\vartheta)(\vartheta-\beta)-\frac{2}{3}\right)-(3\beta\vartheta+2\beta+4|\beta|^2\tau)$$
$$-(1-\sigma_n)^2(\vartheta^2-\vartheta-|\beta|\vartheta+4\vartheta^2\tau)-2\beta\sigma_n\Big]\|x_n-x_{n-1}\|^2$$
$$\le \limsup_{i\to\infty}[(s_{n_i}-s_{n_i+1})+\sigma_{n_i}(r_{n_i}-s_{n_i})]$$
$$= -\liminf_{i\to\infty}(s_{n_i+1}-s_{n_i}) \le 0.$$

Since

$$\limsup_{i\to\infty} -\Big[(1-\sigma_n)\left((1+\vartheta)(\vartheta-\beta)-\frac{2}{3}\right)-(3\beta\vartheta+2\beta+4|\beta|^2\tau)$$
$$-(1-\sigma_n)^2(\vartheta^2-\vartheta-|\beta|\vartheta+4\vartheta^2\tau)-2\beta\sigma_n\Big] \ge 0.$$

We get

$$\lim_{i\to\infty}\|x_{n_i}-x_{n_i-1}\|=0.$$

From (3.38) and (4.2), we obtain

$$\|w_n-y_n\| = \|\gamma_n\sum_{i\in I(y_n)}\lambda_{i,n}\mathcal{T}_i^*(I-P_{Q_i})\mathcal{T}_i y_n\| \to 0, \quad n\to\infty.$$

And

$$\|w_n-x_{n+1}\| \le \|w_n-y_n\|+\|y_n-x_{n+1}\| \to 0, \quad n\to\infty. \tag{4.36}$$

Also,

$$\begin{aligned}\|x_{n+1}-P_C(y_n)\| &= \|P_C[\sigma_n x_0+(1-\sigma_n)w_n]-P_C(y_n)\|\\ &\le \|\sigma_n x_0+(1-\sigma_n)w_n-y_n\|\\ &= \|(\sigma_n(x_0-w_n))+(w_n-y_n)\|\\ &\le \sigma_n\|x_0-w_n\|+\|w_n-y_n\| \to 0, \quad n\to\infty.\end{aligned} \tag{4.37}$$

From (3.28) and (4.37), that we obtain

$$\|y_n-P_C(y_n)\| \le \|y_n-x_{n+1}\|+\|x_{n+1}-P_C(y_n)\| \to 0, \quad n\to\infty. \tag{4.38}$$

By Lemma 4.5 we have that $\{x_n\}$ is bounded. We have also shown that $\{y_n\}$ and $\{w_n\}$ are bounded, thus there exists a subsequence $\{x_{n_K}\}$ of $\{x_n\}$ such that x_{n_K} converges weakly to $q\in H$, $k\to\infty$. Also, by (3.29), there exists a subsequence $\{y_{n_K}\}$ of $\{y_n\}$ such that y_{n_K} converges weakly to q. By Lemma 2.4 and (4.38) we have that $q\in F(P_C)=C$. Also, since $\mathcal{T}_i$ is a bounded linear operator, we have $\mathcal{T}_i y_{n_K}$ converges weakly to $\mathcal{T}_i q$ for each $i=1,2,\dots,N$. Using Lemma 2.4 again and (3.37), we deduce that $\mathcal{T}_i q\in F(P_{Q_i})$. That is, $\mathcal{T}_i q\in Q_i$ for every $i=1,2,\dots,N$. Consequently, we have that $q\in\Gamma$.

Since $k^*=P_\Gamma(x_0)$, we have that

$$\limsup_{i\to\infty}\langle x_0-k^*, x_{n_i}-k^*\rangle = \langle x_0-k^*, q-k^*\rangle \le 0.$$

Also,

$$\limsup_{i\to\infty}\langle x_0-k^*, y_{n_i}-k^*\rangle \le 0,$$

therefore,

$$\limsup_{i\to\infty}\langle x_0-k^*, w_{n_i}-k^*\rangle \le 0, \tag{4.39}$$

By (4.39) and (4.36), we have that

$$\limsup_{i\to\infty} r_{n_i} \le 0.$$

Since $\sum_{n=1}^{\infty}\sigma_n=\infty$, we obtain by Lemma 2.7 in (4.31) that $\lim_{n\to\infty}s_n=0$. Using (4.25), it follows that $\{x_n\}$ converges strongly to $P_\Gamma(x_0)$. This completes the proof. □

5. Numerical examples

We now give some numerical examples to validate our results and illustrate the performance of our Algorithm 3.3 (Proposed Alg. (1)) and Algorithm 4.3 (Proposed Alg. (2)) with (Reich et al. Alg. (1)), (Reich et al. Alg. (2)) proposed by Reich et al. in [1], (Okeke et al. Alg. (1)) and (Okeke et al. Alg. (2)) Proposed by Okeke et al. [31]. We use $\|x_{n+1}-x_n\|<10^{-4}$ as the stopping criterion and the results are shown in Tables 2, 4 and Figs. 1–8.

The first example is in finite-dimensional spaces. Here, we compare the weak convergent algorithms, that is, (Proposed Alg. (1)) with (Reich et al. Alg. (1)) and (Okeke et al. Alg. (1)).

Table 1
Methods parameters for Example 5.1.

Reich *et al.* Alg. (1)	$\lambda_{i,n} = \frac{1}{N}$	$\chi_n = \frac{1}{n+3}$			
Okeke *et al.* Alg. (1)	$\lambda_{i,n} = \frac{1}{N}$	$\chi_n = \frac{1}{n+3}$	$\vartheta = 0.95$	$\epsilon_n = \frac{1}{(n+3)^2}$	
Proposed Alg. (1) 3.3	$\lambda_{i,n} = \frac{1}{N}$	$\chi_n = \frac{1}{n+3}$	$\rho = 0.20$	$\vartheta = 0.30$	$\beta = -0.20$

Table 2
Numerical results for Example 5.1.

	$m = 5$		$m = 10$		$m = 20$		$m = 40$	
	Iter.	CPU time	Iter.	CPU time	Iter.	CPU time	Iter.	CPU time
Reich *et al.* Alg. (1)	194	0.0091	100	10.0177	54	0.0179	31	0.0171
Okeke *et al.* Alg. (1)	106	0.0068	117	0.0064	104	0.0072	102	0.0079
Proposed Alg. (1) 3.3	28	0.0061	29	0.0060	28	0.0068	30	0.0071

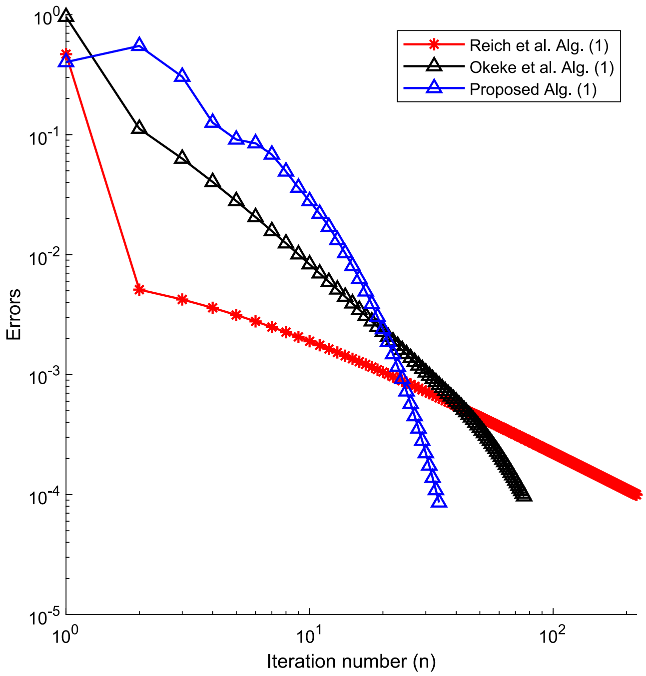

Fig. 1. Example 5.1 $m = 5$.

Example 5.1. Let $H = \mathbb{R}^m$ and $H_i = \mathbb{R}^m$ for $i = 1, 2, \ldots, N$. Suppose $C = \{x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : -5 \le x_k \le 5, k = 1, 2, \ldots, m\}$. For $i = 1, 2, \ldots, N$, let

$$Q_i = \{y \in \mathbb{R}^m : q_i(y) \le 0\},$$

where $q_i(y) = \frac{1}{2} y^T B_i y + b_i^T y + c_i$, B_i is a Hessian matrix, b_i and c_i are vectors generated randomly in $\mathbb{R}^m$ for $i = 1, 2, \ldots, N$. The projection onto Q_i can effectively be calculated using Optimization Toolbox solver "quadprog" in MATLAB. Let $T_i : \mathbb{R}^m \to \mathbb{R}^m$ be randomly generated matrices of which the elements of the representing matrices are randomly distributed in the interval $[-5, 5]$. We consider the following problem: Find $x^* \in C$ such that

$$x^* \in C \cap \left(\bigcap_{i=1}^{N} \mathcal{T}_i^{-1}(Q_i) \right).$$

The starting points x_{-1}, x_0 and x_1 are generated randomly in $\mathbb{R}^m$ for $m = 5, 10, 20, 40$ and $N = 6$ (see Table 1).

The second example is in infinite-dimensional spaces. Here, we compare the strong convergent algorithms, that is, (Proposed Alg. (2)) with (Reich et al. Alg. (2)) and (Okeke et al. Alg. (2)).

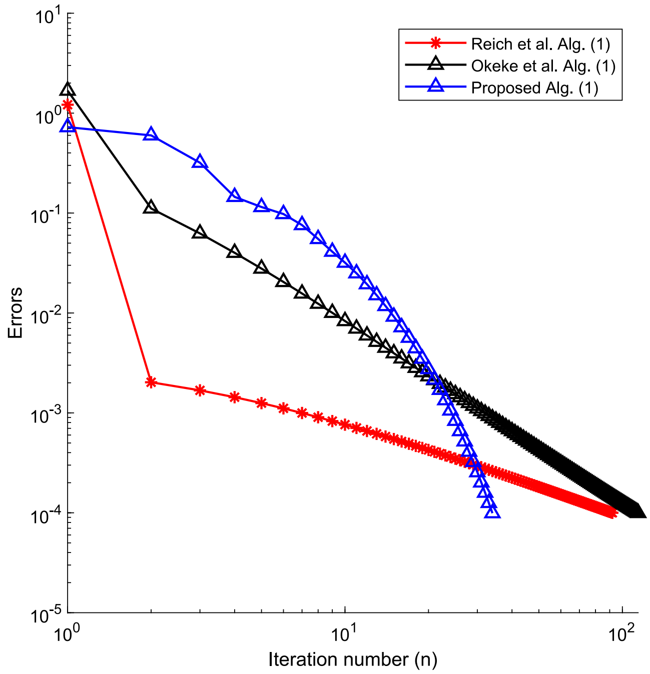

Fig. 2. Example 5.1 $m = 10$.

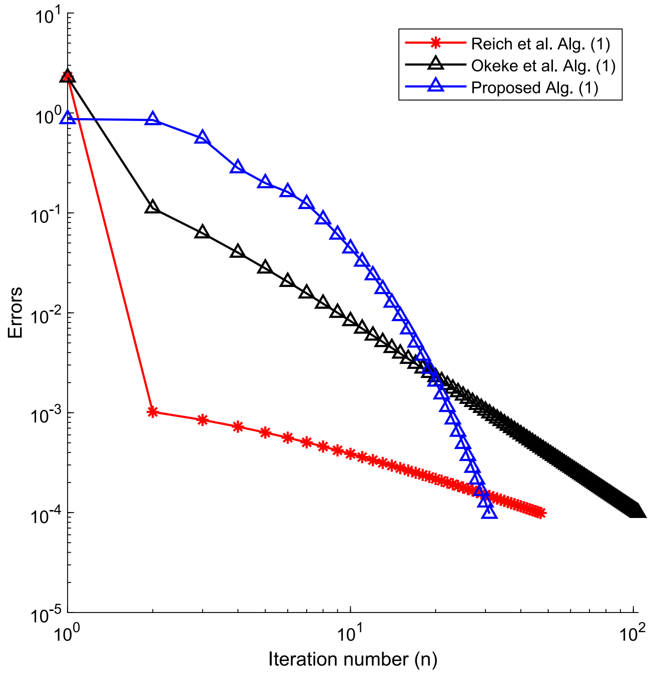

Fig. 3. Example 5.1 $m = 20$.

Example 5.2. Let $H = H_i = L^2([0,1])$ $(i = 1, 2, 3)$ with norm $\|x\| = \left(\int_0^1 |x(t)|^2 dt\right)^{1/2}$ and the inner product $\langle x, y\rangle = \int_0^1 x(t)y(t)dt$. Let C, Q_1, Q_2, Q_3 be defined by

$$C = \left\{ x \in L^2([0,1]) : \int_0^1 3t^2 x(t) = 0 \right\},$$

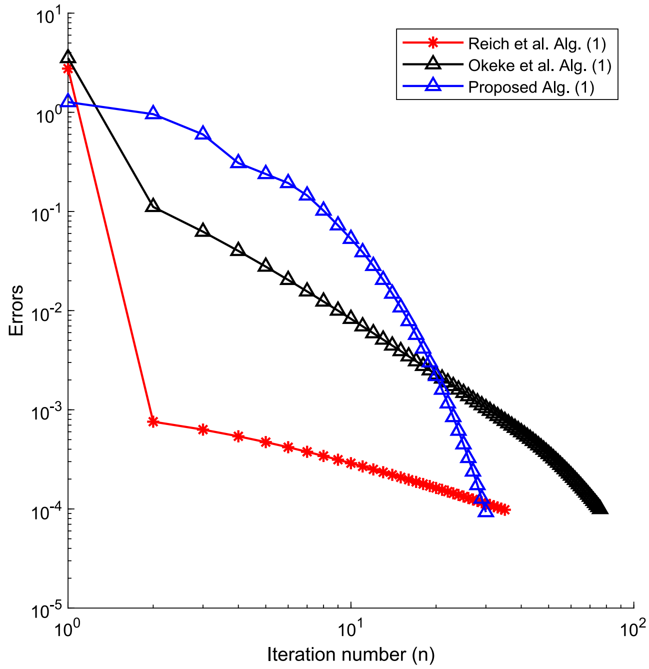

Fig. 4. Example 5.1 $m = 40$.

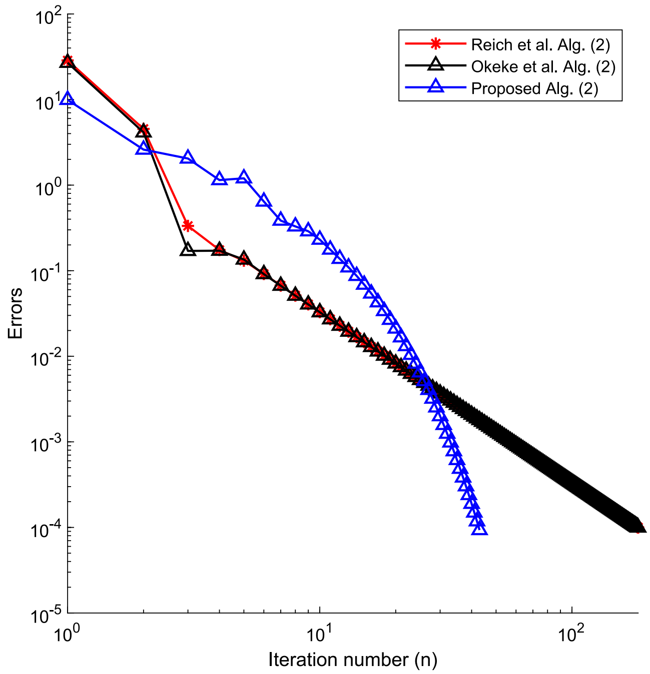

Fig. 5. Example 5.2 Case 1.

$$Q_1 = \left\{ x \in L^2([0,1]) : \int_0^1 \frac{tx(t)}{3} dt \geq -2 \right\},$$
$$Q_2 = \left\{ x \in L^2([0,1]) : \|x(t) - \sin(t)\|^2 \leq 16 \right\},$$

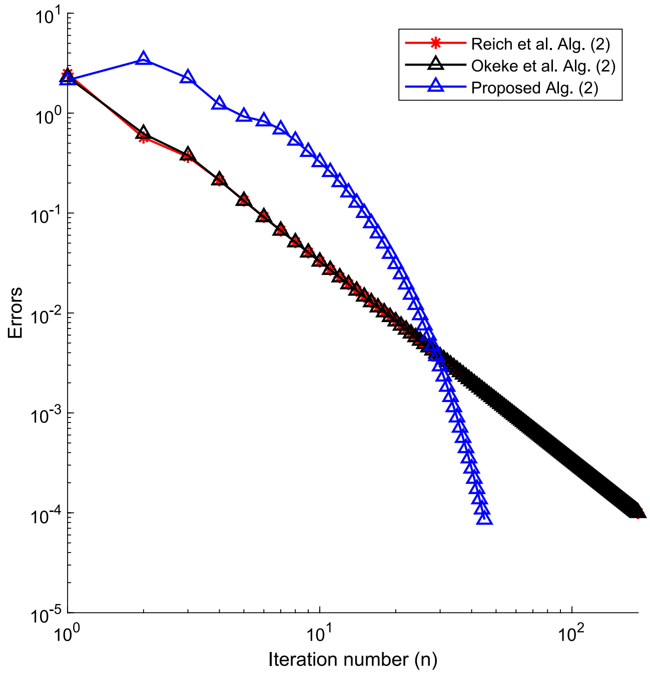

Fig. 6. Example 5.2 Case 1.

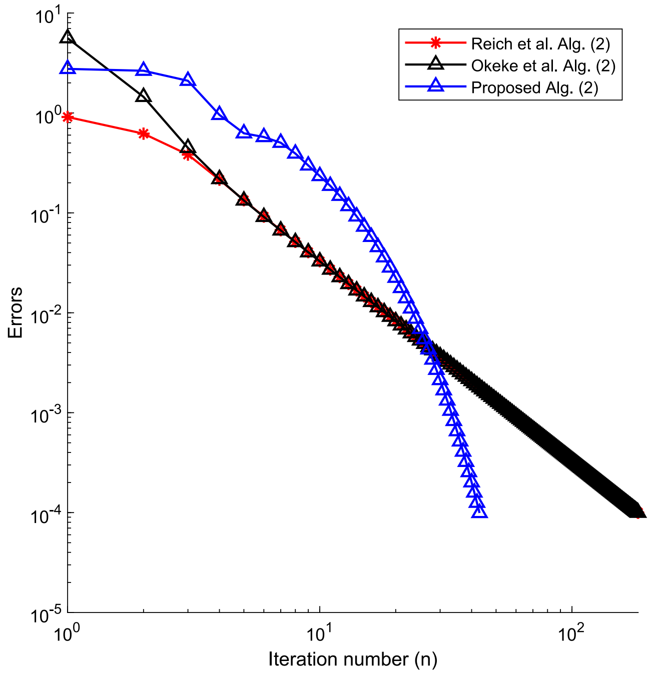

Fig. 7. Example 5.2 Case 3.

$$Q_3 = \left\{ x \in L^2([0,1]) : \|x\|^2 \leq 1 \right\}, \tag{5.1}$$

and $T_i : L^2([0,1]) \to L^2([0,1])$ is defined by $T_i x(t) = \frac{2ix(t)}{3}$ for $i = 1, 2, 3$. The projections onto C and Q_i are explicitly given as follows:

$$P_C x(t) = x(t) - 3t^2 \frac{\int_0^1 3t^2 x(t) dt}{\int_0^1 3t^2 dt},$$

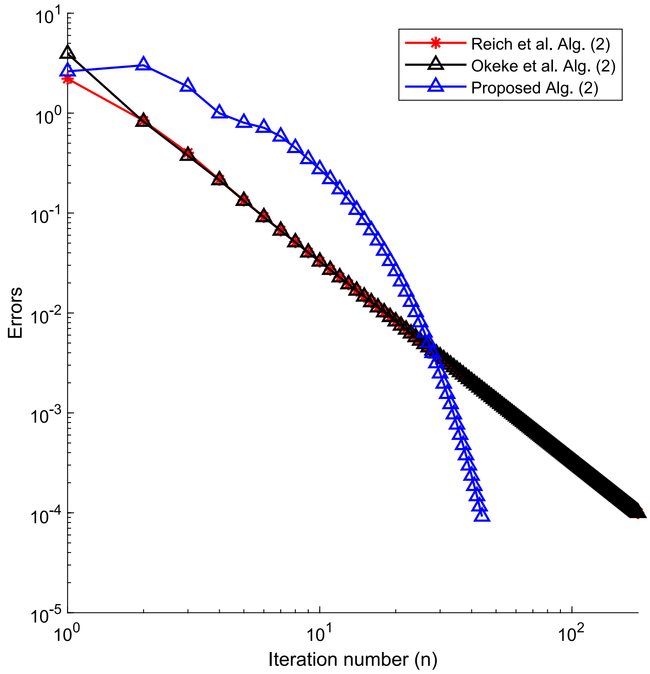

Fig. 8. Example 5.2 Case 4.

Table 3
Methods parameters for Example 5.2.

Reich *et al.* Alg. (2)	$\lambda_{i,n} = \frac{1}{N}$	$\chi_n = \frac{1}{n+3}$	$\sigma_n = \frac{1}{n+1}$	$f(x) = \frac{x}{3}$		
Okeke *et al.* Alg. (2)	$\lambda_{i,n} = \frac{1}{N}$	$\chi_n = \frac{1}{n+3}$	$\vartheta = 0.95$	$\epsilon_n = \frac{1}{(n+3)^2}$	$\sigma_n = \frac{1}{n+1}$	$f(x) = \frac{x}{3}$
Proposed Alg. (2) 4.3	$\lambda_{i,n} = \frac{1}{N}$	$\chi_n = \frac{1}{n+3}$	$\rho = 0.20$	$\vartheta = 0.30$	$\beta = -0.20$	$\sigma_n = \frac{1}{n+1}$

$$P_{Q_1}x(t) = \begin{cases} x(t) - t\dfrac{\int_0^1 \frac{tx(t)}{3}dt}{3\int_0^1 \frac{t}{3}dt} & \text{if} \quad \int_0^1 \frac{tx(t)dt}{3} < -2, \\ x(t) & \text{otherwise,} \end{cases}$$

$$P_{Q_2}x(t) = \begin{cases} x(t) & \text{if} \quad \|x(t) - \sin(t)\| \le 4, \\ \sin(t) + 4\frac{x(t)-\sin(t)}{\|x(t)-\sin(t)\|} & \text{otherwise,} \end{cases}$$

$$P_{Q_3}x(t) = \begin{cases} \frac{x(t)}{\|x(t)\|^2} & \text{if} \quad \|x(t)\|^2 > 1, \\ x(t) & \text{otherwise.} \end{cases}$$

For this example, take $\lambda_{i,n} = \frac{1}{N}$ with $N = 3$, 4 cases of the starting points x_{-1}, x_0 and x_1 are selected in $L_2([0,1])$ as follows:
Case 1: $x_{-1} = \exp(2t),\ x_0 = t + 3,\ x_1 = \sin(t)$;
Case 2: $x_{-1} = \sin(t),\ x_0 = \exp(-2t),\ x_1 = \cos(t)$;
Case 3: $x_{-1} = \frac{\exp(t)}{5},\ x_0 = \frac{9\cos(3t)}{10},\ x_1 = \frac{\sin(2t)}{2}$;
Case 4: $x_{-1} = \exp(-4t),\ x_0 = \cos(2t),\ x_1 = \sin(3t)$ (see Table 3).

Remark 5.3. We noted from the numerical Examples 5.1–5.2, Tables 2, 4 and Figs. 1–8, that in terms of number of iterations and CPU time, our proposed algorithms (Proposed Alg. (1)) and (Proposed Alg. (2)) perform better than the existing ones. Thus, our proposed methods are more efficient for solving the split feasibility problem with multiple output sets than the existing ones.

6. Conclusion

We presented and studied two new two-step inertial accelerated algorithms. Our algorithms are modification and extension of C-Q algorithms studied by Reich et al. in [28] and Okeke et al. in [31]. We adopted the two-step inertial technique and prove the

Table 4
Numerical results for Example 5.2.

	Case 1		Case 2		Case 3		Case 4	
	Iter.	CPU time	Iter.	CPU time	Iter.	CPU time	Iter.	CPU time
Reich *et al.* Alg. (2)	183	6.2595	183	6.2680	183	6.4824	183	6.2630
Okeke *et al.* Alg. (2)	183	5.5316	183	5.6436	183	5.6696	183	5.6371
Proposed Alg. (2) 4.3	43	1.2617	45	1.3157	43	1.2555	44	1.2870

weak and strong convergence of the introduced algorithms; moreover, we achieved strong convergence result without the application of on-line rule, which is in contrast to the method studied in [31]. Finally, we show the applicability and efficiency of our methods through numerical experiments.

CRediT authorship contribution statement

C.C. Okeke: Writing – review & editing, Supervision, Conceptualization. **K.O. Okorie:** Writing – review & editing, Writing – original draft. **C.E. Nwakpa:** Writing – review & editing. **O.T. Mewomo:** Writing – review & editing, Formal analysis.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The authors sincerely thank the anonymous referees for their careful reading, constructive comments and useful suggestions that improved the manuscript.

Data availability

No data was used for the research described in the article.

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