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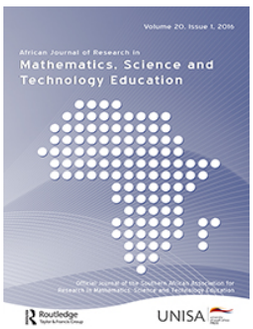
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Students' Difficulties with Definitions in the Context of Proofs in Elementary Set Theory

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In this paper we explore first-year students' difficulties with the use and interpretation of definitions of mathematical objects as they attempt proof construction exercises in the area of elementary set theory. The participants are students at a historically disadvantaged university in South Africa. In this study the activities and utterances of 10 students who took part in consultative group sessions were observed and analysed. Consultative sessions were organised so as to encourage and develop students' active participation while engaging in the task of proof construction. The framework that was used to analyse students' proof comprehension and construction actions and contributions, particularly their interpretation and use of definitions, is described in the paper. The findings of the study resonate closely with those of researchers in the developed world. Students' difficulties with definitions of mathematical objects include their misinterpretation of definitions of objects such as the union of sets and the Cartesian product and their association of mathematical objects with a word or symbol contained in their definitions.

Keywords: *Conceptual difficulties*; *mathematical definitions*; *proof construction*; *union of sets*; *Cartesian product*; *elementary set theory*

Introduction

Formal mathematical proof construction is a crucial component of undergraduate mathematics major courses at universities. At university level students are expected to read and produce mathematical proofs that obey well-defined conventions in line with the acceptable practices of the mathematical community (Weber & Alcock, 2011). For an argument to be considered a mathematical proof it must be based on accepted axioms and definitions (Tall, 1989). Furthermore most proofs need to proceed using deductive reasoning while employing the appropriate mathematical notation and proof techniques. These stringent requirements coupled with the newly met mathematical objects, often introduced through definitions, are often overwhelming for first-year university students.

Research carried out at college and university level by researchers such as Moore (1994), Stylianou, Blanton and Knuth (2011) and Solow (2002) on students' challenges in proof construction has found three major areas of difficulty. Firstly students are challenged by the mathematical language and notation contained within the mathematical area of proof construction including the appropriate use of definitions. Secondly, the lack of knowledge of logical reasoning processes and proof methods involved in the proof construction process, which may be likened to essential road maps on the journey through proof construction, is a great hindrance. The third major challenge is students' limited deductive

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reasoning abilities and a lack of an appreciation of the need for justification of each deduction in the proof-construction process.

Clark and Lovric (2008) explored the many challenges students face as they make the transition to proof construction in university-level mathematics. They propose that this transition requires students to change the kinds of reasoning used, that is, to shift from informal to formal language; to reason from mathematical definitions; to understand and apply theorems; and make connections between mathematical objects. These challenges are exacerbated for first-year students at the university in which this study takes place as the majority of these students do not have English as their first language. The Third International Mathematics and Science Study (TIMSS), which took place in 1994, highlighted the importance of English language proficiency as a foundation for the development of mathematical fluency and skill (Howie et al., 2000). Students who do not have English as their first language often have difficulty with their understanding of logical connectives, which are particularly crucial for proof comprehension and construction. The lack of English language proficiency may further impede students' understanding of mathematical definitions as the vocabulary used, although sometimes also used in everyday language, may have different or more precise meanings when used in the context of mathematical definitions (The National Council of Teachers of Mathematics, 2000).

In general South African students' introduction to formal proof construction is rather sudden. At high school level students are engaged with mainly algorithmic mathematics. Proof construction is confined to the proof of a few theorems in Euclidean geometry and proof of trigonometric identities. In first-year undergraduate courses mathematics students may have met a few other proof constructions in their Calculus course but there is generally no introductory course on methods of proof. Moore (1994) reports that similarly the transition to formal proof construction in the United States is abrupt and that 'at many colleges and universities students are expected to write proofs in real analysis, abstract algebra and other advanced courses with no explicit instruction on how to write proofs' (p. 249). In the context of Algebra, one of the areas in which first-year students majoring in mathematics at this university meet formal proofs is in the area of elementary set theory. These proofs have a rigid axiomatic structure and require students to use and interpret definitions of mathematical objects and link these definitions to the steps required in the theorem (Selden, 2012). Selden argues that, when one compares the typical proofs in geometry that students have come across in high school with proofs at tertiary level, 'one sees that the objects in geometry are idealisations of real things (points, lines, planes), whereas objects in real analysis, linear algebra, abstract algebra or topology (functions, vector spaces, groups, topological spaces) are abstract reifications' (p. 393).

Researchers such as Stylianou et al. (2011) and Moore (1994) have found that students' problems with grasping mathematical language, signs and symbols lead to difficulties with understanding, remembering and applying definitions. Similarly Weber (2001) found that students lacked real understanding of definitions and were thus unable to apply them correctly. Definitions also play a crucial role in providing the structural framework of a proof. In fact Moore (1994) emphasised that definitions not only provide the mathematical language and notation necessary for proof construction but they also reveal the overall logical structure of the proof providing the justification for each step or deduction. Tall and Vinner (1981) refer to the concept definition as the words used to define the mathematical notion, while an individual's concept image describes their personal mental understanding of the notion encompassing all the mental pictures and the properties and processes associated with the notion. They investigate several students' concept images of the notions of limit and continuity and demonstrate that, when these differ from the formal concept definitions, this may cause cognitive conflict. Edwards and Ward (2004) report on the difficulties that undergraduate students had with their understanding and use of definitions in real analysis and introductory abstract algebra. They recommend that the concept of mathematical definitions needs to be engaged with more directly in mathematics courses and especially those where students are being introduced to formal proof construction. This engagement could take place by introducing a discussion on the notions of concept definition and concept image (Vinner, 1991). This could be augmented with readings on the centrality of rigor and proof in mathematics (Stewart, 1995) and material on methods for teaching the use of mathematical definitions (Solow, 2002). Harel (2013) stresses the importance of an intellectual need in students for what they are taught. He reasons that, when students do not realise such a

need, they become intellectually aimless: this makes their learning problematic. Harel has identified five categories of intellectual need: the need for certainty; the need for causality; the need for computation; the need for communication; and the need for structure. His categories of needs for certainty and causality directly address formal proof construction: in proving an assertion one sets out to achieve certainty about its truth while the need for causality refers to the mental act of understanding why the assertion is true or false. In the context of this paper in particular, students should be nurtured to develop a need for a correct and accurate interpretation and application of the definitions of the mathematical objects that they encounter.

The aim of this paper is to examine some of the difficulties students have with the interpretation and application of definitions in the context of formal proof construction in the area of elementary set theory. This paper looks at the difficulties students experience in constructing proofs at a historically disadvantaged university in South Africa. We report specifically on our observations of students' challenges with their interpretation of definitions of mathematical objects such as the union, the Cartesian product and the intersection of two sets while they were engaged in proof construction attempts. Significant forms of scaffolding that seem to resolve these difficulties were also explored in a broader study and these are the subject of a future publication.

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Setting of the Study

The study took place in the second semester of 2010. Students at the university mainly come from previously disadvantaged schools, most of which are rural. Even more than 15 years after apartheid, many of these feeder schools are still disadvantaged owing to the fact that mathematics teachers at these schools are often not as qualified as those teaching at urban schools.

The study took place in the context of a first-year mathematics major course. The course was taught in two parts: a Calculus portion which took place in the first semester and an Algebra portion which took place in the second semester. The Algebra portion included the topics simultaneous linear equations and matrices, elementary set theory, relations and functions, mathematical induction, permutations and combinations, the binomial theorem, complex numbers and mathematical logic. The lectures were taught in a standard lecture format where large numbers of students (roughly 500 students in each class) sat quietly, listening to the lecturer and taking notes as he/she talked and wrote on the blackboard. The students also attended weekly tutorials where a large group of students would work on the exercises assigned for that week and had the opportunity of asking questions from the lecturer and several student assistants.

The actual research took place in the context of consultative group sessions in the area of elementary set theory. These sessions were offered to the chosen participants in addition to the normal lectures and tutorials. Consultative group sessions (a form of inquiry based collaborative session) are geared to getting learners to interact with one another and more knowing others while making functional use (Vygotsky, 1986) of newly met terms and symbols as well as logical reasoning processes, proof methods and justification methods. We explain the term 'functional use' later in the paper.

Ten students participated in the consultative group sessions which the first author led. In these sessions the aim was to create a warm and tolerant environment where every contribution would be welcome. Students were encouraged to take ownership of the proof construction process from the very beginning of the sessions. A volunteer would come up to the board to attempt the proof of a proposition or theorem posed by the lecturer while other students (and lecturer whenever necessary) made contributions. These contributions questioned points of confusion and critiqued incorrect proof construction steps, provided guidance towards proof construction strategy and clarified mathematical terms, definitions and proof methods. The lecturer (the first author) and the student's peers would offer advice on the way in which the proof construction should proceed. The practice of using logical and deductive reasoning and justification was encouraged by both the first author and peers. Students were encouraged to critique and question proof construction steps that did not make sense. Guidance was given by the lecturer whenever necessary, such as in instances where incorrect ideas and proof methods persisted.

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Methods of the Study

The 10 students were selected from different strata of mathematical ability (according to their first semester results). This stratified selection was made in order to investigate the effectiveness of the sessions for students of varied mathematical abilities. Since the first author was taking an active role in organising and participating in the group sessions and thus was not able to observe the students systematically or make detailed field notes during the course of the sessions, the sessions were video-recorded. There were four consultative group sessions in which the proofs of propositions and theorems relating to the topic of Elementary Set Theory were covered in detail. The video-recordings of these sessions enabled us to do more detailed observation after the completion of the sessions and transcripts for analysis were generated from these video-recordings.

These transcriptions went through several iterations of review, revision and correction until they were accurate and contained no incongruences. After some preliminary coding, particular events (on which more detailed coding and analysis would be carried out) were selected to form the basis for our study. To this end, the transcripts were examined for events that best illustrated the challenges and difficulties of students. The selection of these events was based not only on the transcripts and video records but also on the first author's field notes and memos written during and after each session. The two vignettes discussed in this paper from the analysis of the consultative sessions are taken from the first and second consultative sessions. These sessions were the main arena where students' difficulties with respect to the use and interpretation of definitions arose and were engaged with.

Analytical Framework used in the Study

The analytical framework for analysis of students' proof construction actions and contributions is based on a comprehensive assessment model for assessing proof comprehension in advanced mathematics at the undergraduate level developed by Mejia-Ramos, Fuller, Weber, Rhoads and Samkoff (2012). Since the model is aimed at assessment, we adapted it to enable its use in the analysis of students' attempts at proof construction. We used the model in combination with a grounded approach, allowing sub-categories to emerge as we worked with the data.

The Mejia-Ramos et al. model was expanded using the Vygotskian (1986) notion of the functional use of the sign and the theory of concept formation. Vygotsky's experiments led him to believe that the (functional) use of a word (or sign) for communication purposes (and even before full understanding) is a necessary stage for the child in their advancement to true concept-level thinking. With concept-level thinking the word becomes personally meaningful and is used in a way that is appropriate to the relevant community, for example, the community of mathematicians (Berger, 2004a, 2006). Vygotsky elaborated that the use of words and verbal thinking are the main processes that lead to concept formation and its generative cause is a specific use of words as functional tools to communicate with others. In terms of this theory it is important that, while students are struggling to develop the various abilities needed for proof construction, the various mathematical objects and processes that are in the process of formation should be discussed with peers and more knowing others so that through this discussion and communication these mathematical objects and processes are gradually made more personally meaningful and their usage becomes congruent with that of the wider mathematical community (Berger, 2004a). We use this notion of the functional use of a sign to interpret students' evolving usage of terms, signs and symbols (and their definitions) and to interpret their reasoning processes, proof methods and processes of justification.

The adapted Meija-Ramos et al. (2012) model considers two aspects of students' understanding of proof in advanced mathematics: the *local comprehension*/*construction* of a proof and students' *holistic comprehension*/*construction*. The first aspect focuses on students' understanding of the local characteristics of proof construction such as the meaning of specific terms, symbols and definitions, the logical reasoning and proof methods employed, and whether each statement and conclusion has been made with the necessary justification. The second aspect focuses on students' holistic

understanding of proof construction. In this paper we focus mostly on the local aspects of proof construction and will describe this in more detail below.

The first category of the students' *local comprehension/construction* of a proof is concerned with the *meaning of terms, symbols and signs* (L1) and measures students' understanding of key terms, symbols and definitions in the proof. Here we are concerned with students' use of new and unfamiliar terminology, signs and symbols and also students' knowledge of definitions, that is, students' ability to explain definitions in their own words and in more formal language.

The next category in the model is concerned with the *logical status of statements and proof framework* (L2). This criterion is aimed at assessing student's mastery of proof methodologies and techniques.

The last category in the local aspect of proof comprehension is the *justification of claims* (L3). This category explores whether the student is able to justify and provide reasons for new assertions based on previous assumptions or statements.

In this paper we focus on the first category, L1. Indicators of students' actions in each of the sub-categories relating to L1 were developed and these are described in Table 1.

In all the analyses given below (in Tables 3–5), the student's ability is indicated by the code in the Table 1, while in contrast, the students' *inability* to perform these actions is indicated by an 'x' attached to the code. For example, L1b indicates that the student has correctly described a definition while L1bx indicates that the student has incorrectly described or interpreted a definition. In the analyses done in this study each proof construction action and contribution is coded according to the local proof comprehension codes and categories such as those found in Table 1.

As discussed above, when investigating how students interpret and apply the newly met mathematical terms, symbols, signs, logical reasoning processes, proof methods and justification we used the Vygotskian notion of the functional use of the object or word. The indicators for the various categories (Berger 2004b, c) are outlined in Table 2. Note that the term 'object' includes mathematical terms, definitions, symbols and signs.

Table 1. Sub-categories and indicators for category L1 of the local comprehension/construction aspect of proof model

Category	Sub-category	Indicators
L1: Meaning of terms, symbols and signs *Purpose:* In this category we consider whether the student can identify the definition of key terms in the proof or specify what is meant by signs, symbols or terms that are met in the proof.	L1a: Using newly met terms, symbols and signs	Students correctly use newly met terms and symbols in the proof construction process (written or spoken). This is interpreted using the functional use of the object for which indicators are given in Table 2.
	L1b: Using mathematical definitions	Students • describe or explain the meaning of terms or symbols; • provide definitions of symbols or terms used in the proof using formal language or in their own words; • make reference to or call to mind definitions appropriate to the proof construction; • question the meaning of terms, symbols and signs.
	L1c: Illustrating mathematical objects and definitions with examples	Students illustrate a mathematical object or definition with simple examples.

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Findings: Students' Difficulties with the use and Interpretation of Definitions

In this paper we offer two vignettes from the analysis of the consultative sessions illustrating how the use of definitions often hinders rather than supports the students' journey in proof construction.

In the tables that follow, the column with the heading 'analysis of proof construction actions and contributions' uses codes from the sub-categories and indicators outlined in Table 1 while the 'interpretation' column uses categories and indicators from Table 2 and also includes general comments. In the paragraph after the analyses presented in the tables, the analysis and interpretation of categories is further explained or justified if required.

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The Union of Two Sets

In the first session while Maria (a pseudonym) was attempting the proof of the equivalence of $A \subseteq$ B and $A \cap B$ = A, Laura asked a question about the difference between the notions of the union and intersection. This sparked an interesting discussion where it became evident that most students, for example Gary (lines 99 and 101), did not have an adequate understanding of the mathematical object, 'union of sets'. The analysed transcript is found in Table 3.

As shown in Table 3, at the request of the lecturer, Gary went to the board to illustrate his interpretation of the union of two sets using a Venn diagram (line 101). His explanation and depiction of the union of two sets (given codes L1bx and L1cx from Table 1) clearly showed what seemed to be a commonly held misconception: that the union contains all the elements in both sets except elements contained in the intersection of the two sets. This could perhaps have been brought about as a result of students' consideration of the definition of the union of two sets: $A \cup B = \{x: x \in A \text{ or } x \in B\}$. Gary seems to have interpreted this as 'x may be in A or in B but not in both (exclusive "or" vs inclusive "or")'. This is an example of complex thinking (refer to Table 2) whereby the learner associates an object or word with a more familiar object or word (exclusive 'or' as in 'You or I should fetch the bread from the shop').

Table 2. Categories and indicators for the functional use of objects

Categories for functional use of object or process	Indicators
Heap level thinking	Students associate the newly met object with a more familiar object having a vague or chance connection and based on non-mathematical criteria.
Complex level thinking	Students associate the newly met object with: • an object which shares a similar attribute; • an object previously met in an example; • a more familiar object which reminds the student of the newly met object in some way; • a more familiar object having a similarity of templates.
Pseudoconcept-level thinking	Students might be able to use or apply the newly met object correctly (thus giving the appearance of concept-level knowledge) but reveal their incomplete or contradictory knowledge (revealing complex-level knowledge) in earlier or later activities.
Concept-level thinking	Students are able to: • correctly and logically explain or describe the properties and attributes of the newly met object; • correctly identify and appreciate differences in properties of the newly met object as distinguished from other newly met or more familiar objects; • correctly and logically use or apply the object.

Table 3. Analysis of an excerpt of transcript concerning the notions of the union and intersection

	Speech and actions	Analysis of proof construction actions and contributions	Interpretation according to theoretical framework and general comments
99	Gary: OK, ah, can I say something about the union and intersection. At the union it's either let's say for example *x*, let's say *x* is in *A*, right? If we say union it's either in *A* or in *B*, it cannot be in both *A* and *B*. But then if you say intersection it means *A* and *B*, all of them, they contain *x*.	L1bx: incorrectly describing the meaning of the terms intersection and union in own words.	Gary's thinking of the notion of union seems to be at complex level. See paragraph below for justification of this inference.
100	T: Would you go up and draw a Venn diagram and just show us as a picture what does union and intersection mean? I think that will clarify it.		The lecturer encourages the use of examples (in this case Venn diagrams) to clarify students' interpretation of mathematical objects.
101	Gary: [goes to the board and draws two intersecting circles A and B. The part that intersects he colours in black. Writes: 'intersection'.] This one will be *A*. Right. This one is for intersection. Right? Then for union. [Writes: 'Union' and draws the same two intersecting circles. The part that intersects he leaves white, the part on the left that does not intersect he fills with dots, and the part on the right that does not intersect he fills with lines and he points to the intersection diagram.] Here *A* and *B* they have a common number, let's say for *x*, that's why we say it's an intersection—it means *x* is in both *A* and *B*. This is where they intersect [points to the black part]. [Points to the union diagram.] Then here it's … Let's say this *A* [left-hand circle] and this B [right-hand circle]. The value that we find in *A* is not in *B* but	L1cx: illustrating the notions of intersection and union with incorrect examples. L1bx: incorrect description and example for union. The union is shown as not containing the intersection.	Using Venn diagrams and shading Gary depicts the intersection and union of two sets. The depiction of the union of two sets clearly shows what seems to be a commonly held misconception; that the union does not contain elements in the intersection of the two sets. This is an example of complex thinking. See below.

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The Cartesian Product and the Intersection of Two Sets

In the second session it became evident that even students who had been struggling in the first session such as Maria, had greatly improved their proof construction abilities. This was apparent in Maria's proof construction attempt of the proposition: $(A \cap B) \times C = (A \times C) \cap (B \times C)$. Maria's correct attempt (in lines 1–7) to prove the first component of the proof (without the accompanying lengthy analysis) is given below:

Maria [1–7]: We want to show: $(A \cap B) \times C \subseteq (A \times C) \cap (B \times C)$ and
$(A \times C) \cap (B \times C) \subseteq (A \cap B) \times C$,
Let $(x, y) \in (A \cap B) \times C$,
$\Rightarrow x \in (A \cap B)$ and $y \in C$,
$\Rightarrow x \in A$ and $x \in B$ and $y \in C$,
$\Rightarrow x \in A$ and $y \in C$ and $x \in B$ and $y \in C$,
$\Rightarrow (x, y) \in (A \times C)$ and $(x, y) \in (B \times C)$,
$\Rightarrow (x, y) \in (A \times C) \cap (B \times C)$.

However, difficulties with interpretation of mathematical definitions became apparent once again when one of the students (Christine) seemed to associate the mathematical notions of intersection and Cartesian product with a word contained in their definitions. The notions of the Cartesian product of two sets (whose definition is: $A \times B = \{(x, y): x \in A \text{ and } y \in B\}$) as well as the intersection of two sets (whose definition is: $A \cap B = \{x: x \in A \text{ and } x \in B\}$) both contain the word 'and' in their definitions. This seemed to lead Christine to link the notion of the Cartesian product with the notion of the intersection. Christine asked the questions (in lines 10 and 12) shown in Table 4.

Christine (in line 10) clearly associated the word 'and' with intersection when she asked: 'Because "and" means intersection … ' and she went on to ask whether the symbol of the Cartesian product could be replaced by the symbol of the intersection. In line 12 she explicitly displayed her linking of the Cartesian product with the notion of the intersection (as a result of associating both these notions with the word 'and') when she said 'that cross stands for an intersection, right?' and repeated her request, to replace the symbol of the Cartesian product by the symbol of the intersection.

This request was made notwithstanding the fact that the Cartesian product is a binary operation acting on two sets (for example A and B) to create a new set ($A \times B$), whose elements are ordered pairs (x, y), x being an element of A and y an element of B; in contrast the intersection of two sets comprises single elements that are common to both sets. Students' association of these two mathematical objects with the word 'and' reveals how difficult it is for them to understand and process the full mathematical definition. Instead they seem to focus on one word that is common to both definitions (but in very different contexts) and use this as a pivot on which to base their reasoning.

Although Maria showed an able functional use of terms and symbols in her proof construction attempt (in lines 1–7 shown above), and was able to use the correct proof method and logical reasoning processes to bring the proof to completion, her response in line 18 (see Table 5), to Christine's questions (lines 10 and 12 in Table 4) implies that her understanding of the notion of the Cartesian product was confused. She used terms and symbols incorrectly in her explanation. Indeed Maria

Table 4. Analysis of an excerpt of transcript concerning the Cartesian product

	Speech and actions	Analysis of proof construction actions and contributions	Interpretation according to theoretical framework and general comments
10	Christine: Because 'and' means intersection can we say, in the bracket say A intersection C. Can you say that?	L1bx: suggesting incorrect use of terms, symbols and signs	Christine seems to be associating both the intersection and Cartesian product with the word 'and'. She now wants to know if she can substitute an intersection sign for the symbol denoting the Cartesian product. This implies complex thinking.
11	Maria: Hmm?		
12	Christine: That cross stands for an intersection, right? Can we put intersections in the bracket?	L1bx: suggesting incorrect use of terms, symbols and signs	As in above, Christine seems to associate the cross (Cartesian product) with an intersection.

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Table 5. Analysed transcript of an excerpt concerning the Cartesian product

	Speech and actions	Analysis of proof construction actions and contributions	Interpretation according to theoretical framework and general comments
18	Maria: I think that here because we're speaking of a multiplication … [points to: Proposition: $(A \cap B) \times C = (A \times C) \cap (B \times C)$]. Here we started with a multiplication sign and we want to prove that you see this side here [points to: $(A \cap B) \times C$]. We've got an intersection and here we've got a multiplication sign. And here we've got [points to: $(A \times C) \cap (B \times C)$] two multiplication signs. So if we prove this [points to the lower part of the board] we must prove this also looking at this side that what this side contains [points to: $(A \times C) \cap (B \times C)$]	L1bx: not able to identify the correct meaning and use of terms, symbols and signs.	Despite doing the proof correctly, Maria's explanation of the proof is confused and does not meet the criterion of (conceptual thinking) of being meaningful to the mathematical community. Thus her use and interpretation of the definition of the Cartesian product is classified as pseudoconceptual.

primarily seemed to be intent on keeping the same number of each sign or symbol on both sides of the equality. She did not seem to be considering the interpretation of the definitions of the Cartesian product and the intersection. Her correct construction of the proof is thus classified as communication at a pseudoconcept level (refer to Table 2).

Concluding Summary

The findings of this study resonate closely with those of researchers in the developed world such as Moore (1994), Solow (2002), Weber (2001) and Blanton, Stylianou and David (2011). Students' inaccurate interpretation, use and application of definitions would result in their inability to use definitions to provide the overall structure and proof method for a particular proof and also to use definitions to provide the logical and deductive reasoning needed to proceed in the proof construction as found by Moore (1994). Stylianou et al. (2011) have noted that students' difficulty in grasping mathematical language, signs and symbols hinders their understanding of definitions. Similarly Weber (2001) found that students lacked real understanding of definitions and were thus unable to apply them correctly. These findings are echoed in this study which is set in the context of students from poorly resourced schools (in terms of well-qualified teachers and other resources) and whose mother tongue is not English. The findings seem to suggest that students have similar conceptual difficulties regarding the interpretation and application of definitions irrespective of whether or not the language of learning and teaching is their mother tongue.

To summarise, among the difficulties that students had with definitions of mathematical objects were their misinterpretation of definitions such as the union and the association of mathematical objects and symbols with a word contained in their definitions. With respect to the misinterpretation of the notion of the union, students' interpretation of the word 'or' contained in its definition seemed to be taken as exclusive and not inclusive. Thus their depiction of the union of two sets contained all elements in both sets except those contained in the intersection of the two sets. An example of linking mathematical objects and symbols with a word contained in their definitions was observed when students associated the notions of the intersection (∩) and the Cartesian product (×) with the word 'and'. Students'

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association of both notions with the word 'and' led to their subsequent tendency to want to interchange the symbol of the Cartesian product with the symbol of intersection. This observation alerted us to the realisation of the great difficulty students have in understanding and interpreting the full mathematical definition. Students instead seemed to focus on one word common to both definitions and based all their thinking on this limited focus.

Edwards and Ward (2004) have recommended that the concept of mathematical definitions needs to be engaged with more directly in mathematics courses where students are being introduced to formal proof construction. In our broader study we explored the development of students' proof construction abilities, which include their use, interpretation and application of definitions, in the context of group consultative sessions. A separate paper will report on how, through further functional use of the mathematical objects, words and symbols, students' sometimes incomplete and confused way of working with definitions evolved into ways that are consistent, meaningful and commensurate with those of the mathematical community.

Harel (2013) stresses cultivating an intellectual need in students for what they are taught. In this context this translates to a need for grasping an accurate interpretation of mathematical definitions. This need could be stimulated and developed as students realise the usefulness of definitions in revealing the logical structure of the proof and providing the justification for their deductions when working together in the consultative sessions. The effectiveness of such an approach needs further research.

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Disclosure Statement

No potential conflict of interest was reported by the authors.

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