Riemannian Manifolds and their Curvature

by

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Abstract

In this dissertation we study some of the fundamental concepts and results of Riemannian geometry. In particular we look at Riemannian manifolds and their curvature with an emphasis on the Riemannian manifolds that have constant sectional curvature. Starting with the fundamental concepts of Riemannian geometry: the Riemannian metric, the Riemannian connection, geodesics and curvature, this dissertation goes on to cover deep results such as the fundamental theorem of Riemannian geometry, the Hopf-Rinow theorem, the Hadamard theorem and the classification theorem for Riemannian manifolds of constant sectional curvature. Along the way we also cover useful tools such as Jacobi fields. In [3: p. 159], M. P. do Carmo considers a diffeomorphism $h : P \to Q$ between Riemannian manifolds (P, g) and $(Q, \tilde{g})$ which preserves the corresponding $(0, 4)$-Riemannian curvature tensors $\mathcal{R}$ and $\widetilde{\mathcal{R}}$. Referring to R. S. Kulkarni [44] and S. T. Yau [46], he poses a problem of deciding whether h is an isometry. Accordingly, at the end of this dissertation, we look at the problem of deciding if a diffeomorphism between two Riemannian manifolds which preserves the sectional curvature is an isometry.

Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

Artur Correia

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This 1st day of September 2020, at Johannesburg, South Africa.

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Introduction

Riemannian geometry, named after the great German mathematician Bernhard Riemann (1826–1866), is the branch of differential geometry that studies n-dimensional differentiable manifolds endowed with *Riemannian metrics*, which are, broadly speaking, rules for computing lengths of tangent vectors and angles between them. It is the most "geometric" branch of differential geometry. A great part of the study of differential geometry is aimed at attempting to comprehend what curvature is and what it tells us about a space. Thus the notion of curvature is one of the focal ideas of differential geometry. One could contend that it is the focal one, distinguishing the geometrical core of the subject from those aspects that are analytic, algebraic, or topological. To quote Marcel Berger [47, p. 9], curvature is "the N° 1 Riemannian invariant and the most natural. Gauss and then Riemann saw it instantly." The word *curvature* corresponds to our everyday understanding of what it means for something to be curved: balloons, tubes, cans, soap films, the surface of our planet earth are all physical models of spaces that have a lot of curvature, where as floors, ceilings, and cardboard boxes are all physical models of spaces that do not have a lot of curvature.

Being able to measure curvature of non-flat objects has many real world applications. Engineers frequently use curvature to add strength to structures and save on material used to build the structures. For example, the hyperbolic shape of a nuclear power plant cooling tower incorporates curvature not only to strengthen the structure but also to minimize the amount of material required to build it. Curvature also plays an important role in physics. The magnitude of a force required to move an object at constant speed along a curved path is, according to Newton's laws, a constant multiple of the curvature of the trajectory. The motion of a body in a gravitational field is determined, according to Einstein, by the curvature of space-time. All sorts of shapes, from soap bubbles to red blood cells, seem to be determined by various curvatures.

Given the vastness of the subject and the limited time and space available, we decided to restrict ourselves mainly to the core concepts and results of Riemannian geometry and to the study of manifolds (spaces) of constant curvature. The manifolds of constant curvature are on the one hand the simplest Riemannian manifolds and yet on the other hand are sufficiently complicated to present a fascinating object of study. They provide the simplest examples of curvature in its purest form and they can also provide us with a launching point in our quest to understand what types of spaces we should expect from positive curvature, from negative curvature and so on. Constant curvature has also played an important role in the history of geometry. Spaces with constant curvature supplied the first examples of non-Euclidean geometries, putting an end to the quest to prove Euclid's parallel axiom from the other geometric axioms. So, these are the spaces that represent the first level of study beyond Euclidean space, and it is by reference to these spaces that the general Riemannian manifolds are studied.

The study of spaces of constant curvature is made very manageable because of a few simplifying properties these spaces have. The first such property is the fact that normal neighborhoods in spaces of the same dimension and the same constant curvature are isometric. So any space with constant curvature is locally the same as any other space with the same curvature. The second distinguishing property is that every complete, simply connected manifold with constant sectional curvature is isometric to one of the frame-homogeneous model spaces $\mathbb{S}^n$, $\mathbb{R}^n$ or $\mathbb{H}^n$; the next theorem, due to W. Killing and H. Hopf, illustrates this fact.

Theorem (A). *[2, Theorem 4.3] Let (M, g) be a complete, simply connected, n-dimensional Riemannian manifold with constant sectional curvature $\mathcal{K}$, $n \geq 2$. Then M is isometric to $\mathbb{R}^n$ if $\mathcal{K} = 0$, to $\mathbb{S}^n\left(\frac{1}{\sqrt{\mathcal{K}}}\right)$ if $\mathcal{K} > 0$, and to $\mathbb{H}^n\left(\frac{1}{\sqrt{-\mathcal{K}}}\right)$ if $\mathcal{K} < 0$.*

Overview of the Chapters

The main reference that was used for this dissertation is the book by Manfredo Perdigão do Carmo titled *Riemannian Geometry* [3]. We follow closely the style of presentation from this book. The only exception is Section 7 which is an exposition of the work done by Ravindra Shripad Kulkarni in the paper titled *Curvature and Metric* [44]. Most of the books listed in the reference section were used as a supplement to the book by do Carmo-to help with a better understating of some of the concepts and proofs of this dissertation.

We now briefly describe the contents: The first section is a brief review of the theory on n-dimensional differentiable manifolds which we make use of throughout this dissertation. The next four sections present in some detail the basic but fundamental concepts of Riemannian Geometry (Riemannian metrics, Riemannian connections, geodesics and curvature). In particular, in Section 2 we prove the fundamental theorem of Riemannian Geometry and in Section 3 we prove the Hopf-Rinow Theorem which states the fact that given any two points on a complete Riemannian manifold there exists a minimizing geodesic joining these two points. Section 4 addresses the notion of curvature on a Riemannian manifold. Specifically, Section 4 addresses the various characterizations of curvature such as the Riemannian curvature tensor, sectional curvature, the Ricci curvature tensor and the scalar curvature. A good part of the study of Riemannian Geometry consists of understanding the relationship between geodesics and curvature. Jacobi fields, an essential tool for this understanding, are introduced in Section 5. In Section 5 we also prove one of the main theorems of this dissertation, namely the Theorem of Hadamard which states that there is a homeomorphism of complete simply connected manifolds of dimension n, whose sectional curvature $\mathcal{K} \leq 0$, onto $\mathbb{R}^n$. In Section 6 is where we begin to deal with Riemannian manifolds of constant sectional curvature; we describe, in some detail, the model spaces ($\mathbb{S}^n$, $\mathbb{R}^n$ and $\mathbb{H}^n$) of constant sectional curvature and prove one of the main theorems of this dissertation which is Theorem A stated in the introduction. Finally, in Section 7, we question if a sectional curvature preserving diffeomorphism is an isometry or not? We answer this question by proving the following theorem, due to R. S. Kulkarni [44], which shows that the only exceptional case is a diffeomorphism between two manifolds of the same constant curvature.

Theorem (B). *If P is an analytic Riemannian manifold with dimension ≥ 4, then either P has constant sectional curvature or a sectional curvature preserving diffeomorphism between P and another analytic Riemannian manifold Q $(h : P \to Q)$ is an isometry.*

0 Differentiable Manifolds

0.1 Introduction

The main purpose of this section is to introduce the necessary tools needed for an adequate study of Riemannian geometry. We define the fundamental concepts which we deal with throughout this dissertation. However, we will not be going into a lot of detail and we will not prove any of the results so that we do not detour from our main objectives. Much of the theory and the proofs for this section can be found in any standard differential geometry text. We begin with the important notion of differentiable manifolds. Riemannian geometry is usually developed on differentiable manifolds.

0.2 Differentiable Manifolds

A manifold, roughly, is a topological space that locally looks like the Euclidean space $\mathbb{R}^n$. A differentiable manifold M is a manifold for which this resemblance with the Euclidean space is sharp enough to allow the establishment of partial differentiation-in fact, all the essential features of calculus-on M. We may also say, at least intuitively, that a differentiable manifold is a generalization of curves in $\mathbb{R}^2(\mathbb{R}^3)$ and surfaces in $\mathbb{R}^3$ to arbitrary dimensions.

We now give the precise definitions.

Definition 0.2.1. *An n-dimensional topological manifold M (or simply an n-manifold) is a topological space with the following properties:*

1. *M is Hausdorff, that is, for every pair of distinct points y, x in M there are neighborhoods $\widetilde{U}$ of y and $\overline{U}$ of x such that $\widetilde{U} \cap \overline{U} = \varnothing$, in other words, $\widetilde{U}$ and $\overline{U}$ do not intersect.*

2. *Every point $y \in M$ possesses a neighborhood U homeomorphic to an open subset A of $\mathbb{R}^n$.*

3. *M satisfies the second countability axiom, that is, M has a countable basis for its topology.*

Differentiable manifolds are topological manifolds equipped with an extra structure that permits us to differentiate functions and maps.

Definition 0.2.2. *An n-dimensional differentiable manifold M is a is a topological manifold of dimension n together with a collection of homeomorphisms $f_\alpha : A_\alpha \subset \mathbb{R}^n \to M$ defined on open sets A_α of $\mathbb{R}^n$ into M (for α in an at most numerable set of indices I) satisfying the following three conditions:*

1. $\bigcup_{\alpha \in I} f_\alpha(A_\alpha) = M$.

2. *For every pair of indices* α, β *such that* $\mathcal{O} := f_\alpha(A_\alpha) \cap f_\beta(A_\beta) \neq \varnothing$, *the sets* $f_\alpha^{-1}(\mathcal{O})$, $f_\beta^{-1}(\mathcal{O})$ *are open sets in* $\mathbb{R}^n$ *and the overlap (or transition) maps* $f_\beta^{-1} \circ f_\alpha : f_\alpha^{-1}(\mathcal{O}) \to f_\beta^{-1}(\mathcal{O})$, $f_\alpha^{-1} \circ f_\beta : f_\beta^{-1}(\mathcal{O}) \to f_\alpha^{-1}(\mathcal{O})$ *are differentiable.*

3. *The family* $\mathfrak{F} = \{(A_\alpha, f_\alpha) \mid \alpha \in I\}$ *is maximal with respect to conditions 1 and 2, meaning that if* $f_0 : A_0 \to M$ *is a homeomorphism such that* $f_0^{-1} \circ f$ *and* $f^{-1} \circ f_0$ *are* C^∞ *for all* f *in* $\mathfrak{F}$, *then* (A_0, f_0) *is in* $\mathfrak{F}$.

The pair (A_α, f_α) (or the map $f_\alpha : A_\alpha \subset \mathbb{R}^n \to M$) with the point y in $f_\alpha(A_\alpha)$ is called a local coordinate system (or a parametrization) of M at y. $f_\alpha(A_\alpha)$ is then referred to as a coordinate neighborhood at y. A family $\{(A_\alpha, f_\alpha)\}$ which satisfies conditions 1 and 2 is called a differentiable structure on M.

Remark 0.2.1. *We have placed the Hausdorff and the countable basis restrictions on the topology of* M *to avoid it from being too peculiar. The natural topology of a differentiable manifold can be fairly peculiar because it can occur that the Hausdorff condition or the countable basis condition or indeed both these conditions not be satisfied by* M. *However, such examples are bizarre and useless. We need both these conditions to be satisfied by* M. *Specifically, we need the Hausdorff condition to guarantee that the limit of a convergent sequence is unique. This, along with the countable basis condition, ensures the existence of partitions of unity (Cf. Subsection 0.7), which is an indispensable tool in differential geometry. Differentiable manifolds that satisfy the Hausdorff and the countable basis conditions are said to be sufficiently topologically well-behaved. We are making the class of objects of study (differentiable manifolds) as general as it can be, while still staying significant for geometry.*

Remark 0.2.2. *We will use* n *or* d *to denote the dimension of* M *whenever either one is more convenient.*

0.3 Differentiable Maps between Manifolds

We will now broaden the notion of differentiability to mappings among differentiable manifolds.

Definition 0.3.1. *Let* P *be a differentiable manifold of dimension* n *and let* Q *be another differentiable manifold of dimension* d. *We say that a continuous map* $g : P \to Q$ *is differentiable at a point* $y \in P$ *if there are local coordinate systems* (A, f) *of* P *around* y *(i.e.* $y \in f(A)$*) and* $(B, \bar{f})$ *of* Q *around*

$g(y)$, with $g(f(A)) \subset \bar{f}(B)$, such that the map $\tilde{g} := \bar{f}^{-1} \circ g \circ f : A \subset \mathbb{R}^n \to \mathbb{R}^d$ is differentiable at $f^{-1}(y)$.

The map g is said to be differentiable on an open subset of P if it is differentiable at each and every point of this open subset. We obtain, from Definition 0.2.1 part 2, that the definition above is not dependent on the choice of the local coordinate systems. The mapping $\tilde{g} := \bar{f}^{-1} \circ g \circ f : A \subset \mathbb{R}^n \to \mathbb{R}^d$ is referred to as a local representation of g and is the expression of g in the local coordinate systems f and $\bar{f}$.

Remark 0.3.1. *In this dissertation, unless otherwise stated, either explicitly or by unequivocal context, the words differentiable and smooth will be used to mean infinitely differentiable (C^∞).*

Definition 0.3.2. *Let P, Q be a pair of differentiable manifolds. A mapping h between P and Q is referred to as a diffeomorphism if it is differentiable, bijective, and its inverse $h^{-1} : Q \to P$ is also differentiable. h is named a local diffeomorphism at a point y in M if there are open neighborhoods A of y and B of $h(y)$ such that $h|_A : A \to B$ is a diffeomorphism.*

Two differentiable manifolds are considered to be the same if they are diffeomorphic, that is, if there is a diffeomorphism h between them. So, Definition 0.3.2 is the equivalent notion of an isomorphism but for manifolds.

0.4 Tangent Spaces and Differentials

We wish to expand the concept of tangent vectors to differentiable manifolds. But first, we recall, from classical differential geometry of regular surfaces in $\mathbb{R}^3$, that a vector $r \in \mathbb{R}^3$ is said to be tangent to a surface $Z \subset \mathbb{R}^3$ at a point $y \in Z$ if there exists a differentiable curve $\psi : (-\epsilon, \epsilon) \to Z \subset \mathbb{R}^3$ such that $\psi(0) = y$ and $\frac{d\psi}{dt}\big|_{t=0} = r$. The set T_yZ of all these vectors is a 2-dimensional vector space, called the tangent plane to Z at y and can be identified with the plane in $\mathbb{R}^3$ which is tangent to Z at y. To generalize this to an abstract n-dimensional manifold we need to find a description of r which does not involve the ambient Euclidean space $\mathbb{R}^3$. Without going into detail on how this is achieved we have the following definition:

Definition 0.4.1. *Let $\psi : (-\epsilon, \epsilon) \to M$ be a differentiable curve on a differentiable manifold M and let $\mathcal{D}$ be the set of all functions $g : M \to \mathbb{R}$ that are differentiable at $\psi(0) = y \in M$. The tangent vector to the curve ψ at y is the operator $\frac{d\psi}{dt}\big|_{t=0} : \mathcal{D} \to \mathbb{R}$ defined by*

$$\frac{d\psi}{dt}\bigg|_{t=0}(g) = \frac{d(g \circ \psi)}{dt}\bigg|_{t=0}.$$

So, a tangent vector to a differentiable manifold M at a point $y \in M$ is a tangent vector to some differentiable curve $\psi : (-\epsilon, \epsilon) \to M$ with $\psi(0) = y$.

Definition 0.4.2. *The set of all tangent vectors at the point y in M is referred to as the tangent space at y, and is denoted by T_yM.*

In a local coordinate system $f : A \subset \mathbb{R}^n \to M$ around the point $y = f(0)$, the curve ψ is given in local coordinates by the curve in A:

$$\widetilde{\psi}(t) := (f^{-1} \circ \psi)(t) = (x_1(t), \ldots, x_n(t)),$$

and

$$\left.\frac{d\psi}{dt}\right|_{t=0}(g) = \left(\sum_{\eta=1}^{n} \dot{x}_\eta(0) \left(\frac{\partial}{\partial x_\eta}\right)_{t=0}\right)(g).$$

In other words, the vector $\left.\frac{d\psi}{dt}\right|_0$ in the local coordinate system f can be given by

$$\left.\frac{d\psi}{dt}\right|_{t=0} = \sum_{\eta=1}^{n} \dot{x}_\eta(0) \left(\frac{\partial}{\partial x_\eta}\right)_{t=0}.$$

Proposition 0.4.1. *The tangent space to M at y is an n-dimensional vector space.*

With the concept of a tangent space at hand, we are now able to also extend the concept of the differential of a differentiable mapping to differentiable manifolds .

Proposition 0.4.2. *Let P be a differentiable manifold of dimension n, let Q be another differentiable manifold of dimension d and let ρ be a differentiable mapping between P and Q. For each point y in P and for each tangent vector r in T_yP, select a differentiable curve $\psi : (-\epsilon, \epsilon) \to P$ such that $\psi(0) = y$, $\left.\frac{d\psi}{dt}\right|_{t=0} = r$. Take $\phi = \rho \circ \psi$. The mapping $d\rho_y : T_yP \to T_{\rho(y)}Q$ defined by $d\rho_y(r) = \left.\frac{d\phi}{dt}\right|_{t=0}$ is a linear mapping that is independent of the choice of ψ.*

Definition 0.4.3. *The linear mapping $d\rho_y$ as defined by Proposition 0.4.2 is referred to as the differential of ρ at y.*

0.5 Vector Fields and Lie Brackets

Classically, a vector field J on $\mathbb{R}^n$ is a differentiable map $J : \mathbb{R}^n \to \mathbb{R}^n$. In this subsection, we take a look at vector fields over differentiable manifolds.

Definition 0.5.1. *Let M be a differentiable manifold. If we examine the disjoint union of all tangent spaces T_yM at every point of M, we acquire a space denoted by TM,*

$$TM = \bigcup_{y \in M} T_yM = \{(y, J_y) \mid y \in M, J_y \in T_yM\}.$$

TM admits (we won't verify this here) a differentiable structure naturally induced by the one on M. With this differentiable structure, TM is a differentiable manifold of dimension $2 \times \dim M$ which is called the tangent bundle of M. We shall denote by $\pi : TM \to M$ the projection of this bundle, with $\pi(v) = y$, where $y \in M$ and $v \in T_yM$.

Definition 0.5.2. *Let M be a differentiable manifold. A vector field J on M is a correspondence that assigns to every point $y \in M$ a tangent vector $J_y \in T_yM$. In terms of mappings, J is a mapping of M into the tangent bundle TM satisfying the property $\pi \circ J = \mathrm{id}_M$. The vector field is called differentiable if the map $J : M \to TM$ is differentiable. The set of all differentiable vector fields on M is denoted by $\mathfrak{X}(M)$.*

In a local coordinate system $f : A \subset \mathbb{R}^n \to M$ we can express J by

$$\begin{aligned} J_y &= a_1(y)\left(\frac{\partial}{\partial x_1}\right)_y + \cdots + a_n(y)\left(\frac{\partial}{\partial x_n}\right)_y \\ &= \sum_{\eta=1}^{n} a_\eta(y)\left(\frac{\partial}{\partial x_\eta}\right)_y, \end{aligned}$$

where every $a_\eta : A \to \mathbb{R}$ is a well-defined differentiable function on A and $\left\{\frac{\partial}{\partial x_\eta}\right\}_{1 \le \eta \le n}$ is the basis associated to f. And so we notice that J is differentiable if and only if the functions a_η are differentiable for some and, thus, for any local coordinate system.

We now take a pair vector fields $J, K \in \mathfrak{X}(M)$. Generally, the operators JK and KJ will contain derivatives of the second order, and thus will not correspond to vector fields. Nonetheless, the commutator $JK - KJ$ does give us a vector field.

Proposition 0.5.1. *Let $J, K \in \mathfrak{X}(M)$ be a pair of differentiable vector fields on a differentiable manifold M, then there is a unique differentiable vector field $L \in \mathfrak{X}(M)$ satisfying*

$$Lg = (JK - KJ)g,$$

for each differentiable function $g \in \mathcal{D}(M)$, where $\mathcal{D}(M)$ is the set of differentiable functions defined on M.

The vector field L is referred to as the Lie bracket of J and K, and is denoted by $[J, K]$. In a local coordinate system, L is expressed by

$$[J, K] = \sum_{\eta=1}^{n} (JK_\eta - KJ_\eta) \frac{\partial}{\partial x_\eta}.$$

If $[J, K] = 0$ then we state that the vector fields $J, K \in \mathfrak{X}(M)$ commute. We list the properties of the Lie bracket in the next proposition.

Proposition 0.5.2. *If $J, K, L \in \mathfrak{X}(M)$, then we obtain:*

1. *Bilinearity: for all $b, c \in \mathbb{R}$,*

$$\begin{aligned} [bJ + cK, L] &= b[J, L] + c[K, L] \\ [J, bK + cL] &= b[J, K] + c[J, L]; \end{aligned}$$

2. *Antisymmetry:*

$$[J, K] = -[K, J];$$

3. *Jacobi identity:*

$$[[J, K], L] + [[K, L], J] + [[L, J], K] = 0;$$

4. *Leibniz rule: for all $g, h \in \mathcal{D}(M)$,*

$$[gJ, hK] = gh[J, K] + g(Jh)K - h(Kg)J.$$

0.6 Orientability

Definition 0.6.1. *Let M be a differentiable manifold. We call M an orientable manifold if M is equipped with a differentiable structure $\{(A_\alpha, f_\alpha)\}$ such that:*

(a) *for each pair α, β, with $\mathcal{O} = f_\alpha(A_\alpha) \cap f_\beta(A_\beta) \neq \varnothing$, the differential of the change of coordinates (or the Jacobian of the overlap maps) $f_{\beta\alpha} = f_\beta^{-1} \circ f_\alpha$ has positive determinant at every point in the domain.*

M is referred to as non-orientable if the above does not hold. A selection of a differentiable structure which meets condition (a) from the definition above is called an orientation of M. We say that M is oriented if M is orientable together with a choice of an orientation. If M is equipped with two separate differentiable structures that meet condition (a), then they give the same orientation to M if condition (a) is again met by their union. There is exactly two different orientations on M if M is orientable and connected.

Now, we let h be a diffeomorphism between two differentiable manifolds P and Q. It is not difficult to check that P is orientable if and only if Q is orientable. Furthermore, if P and Q are connected and are oriented, h induces an orientation on Q which may or may not be the same as the original orientation of Q. If it is the same then we call h orientation preserving and if it is not the same we call h orientation reversing.

0.7 Partitions of Unity

Partitions of unity are a fundamental tool in differential geometry. They can assist in breaking down a global object into a sum of local objects or they can also assist in piecing together locally defined objects into a global object. For example, we will use partitions of unity to prove existence theorems which we will come across in the upcoming sections of this dissertation.

We need some preliminary definitions before we can define what a partition of unity is. Let M be a differentiable manifold. We call a collection of open sets $\{U_\alpha\}_{\alpha \in I} \subset M$ with $\bigcup_\alpha U_\alpha = M$ (i.e. U_α covers M) locally finite if all points y in M have a neighborhood $\mathcal{O}$ satisfying $\mathcal{O} \cap U_\alpha \neq \varnothing$ for only finitely many indices α. The support of a real-valued function $g : M \to \mathbb{R}$, denoted supp g, is the closure of the set of points where the value of g is not equal to zero, that is, $\operatorname{supp} g = \overline{\{y \in M \mid g(y) \neq 0\}}$.

Definition 0.7.1. *A collection $\{g_\alpha\}_{\alpha \in I}$ of differentiable functions $g_\alpha : M \to \mathbb{R}$ is called a differentiable partition of unity subordinate to the covering $\{U_\alpha\}$ if the following three conditions are satisfied:*

(i) *For each $\alpha \in I$ and $y \in M$, $g_\alpha(y) \geq 0$ and the support of g_α is contained in a coordinate neighborhood $U_\alpha = f_\alpha(A_\alpha)$ of a differentiable structure $\{(A_\beta, f_\beta)\}$ of M.*

(ii) *The collection $\{U_\alpha\}$ is locally finite.*

(iii) *$\sum_\alpha g_\alpha(y) = 1$, for every point y in M.*

Theorem 0.7.1 (Existence of Partitions of Unity). *A differentiable manifold M has a differentiable partition of unity if and only if every connected component of M is Hausdorff and has a countable basis.*

1 Riemannian Metrics

1.1 Introduction

The study of Riemannian geometry begins in this section. Whilst in the previous section we looked at manifolds from the point of view of differentiability, here we start to address questions of a geometric sort, for example, the length of a vector (and of a curve), the angle between two vectors (and between two intersecting curves) and area of a region on an n-dimensional differentiable manifold. In this section and in many of the sections to come, we will draw intuition and give most examples from surfaces in Euclidean space; for these-together with curves on the plane and in space-were the original objects of study in classical differential geometry and are the source of much of the current theory.

In the theory of regular surfaces $Z \subset \mathbb{R}^3$, the mathematical tool that allows for the calculation of the length of a vector tangent to Z and the angle between two vectors tangent to Z, is the first fundamental form. The first fundamental form on Z is defined as the restriction of the usual Euclidean dot product in $\mathbb{R}^3$ to the tangent space T_yZ at every given point $y \in Z$. More accurately, there is an induced inner product on T_yZ coming from the the standard inner product (the dot product) in $\mathbb{R}^3$. This induced inner product is what is referred to as the first fundamental form and is denoted by $\mathrm{I}_y(\ ,\ )$ or $\langle\ ,\ \rangle_y$. So for tangent vectors r, $s \in T_yZ$, $\mathrm{I}_y(r,s)$ is equal to the standard inner product of r, s thought of as vectors in $\mathbb{R}^3$, that is, $\mathrm{I}_y(r,s) = r \cdot s$. It is this definition of the first fundamental form along with integration that permits for geometric measurements such as the length of a curve on Z, the angle between two intersecting curves on Z, the area of a region on Z, and all other metric properties of Z.

Everything discussed in the paragraph above is to do with surfaces which are embedded in $\mathbb{R}^3$, that is, with reference to the ambient space in which the surface sits in. In general, an n-dimensional differentiable manifold M is defined without reference to a specific embedding (without reference to the ambient space), so we don't instantly have a first fundamental form as defined above. Moreover, it is not clear from the definition of M whether it has a metric or not. But we would still like to calculate lengths, angles, areas and volumes on M. To achieve this we must add an extra structure on M, namely, a Riemannian metric.

1.2 Riemannian Metrics

Before we define what a Riemannian metric is, we first recall the definition of an inner product on a real vector space W. Inner products are a natural generalization of the dot product to arbitrary vector spaces.

Definition 1.2.1. *An inner product or scalar product on W is a map*

$$\langle\ ,\ \rangle : W \times W \to \mathbb{R}$$

which satisfies the following three properties:

(i) $\langle\ ,\ \rangle$ *is Symmetric, that is,*

$$\langle r, s\rangle = \langle s, r\rangle,$$

for all r, $s \in W$.

(ii) $\langle\ ,\ \rangle$ *is Positive Definite, that is,*

$$\langle r, r\rangle \geq 0,$$

for all $r \in W$. The equality holds only when $r = 0$.

(iii) $\langle\ ,\ \rangle$ *is Bilinear (linear in each argument), that is,*

$$\langle ar + bs, t\rangle = a\langle r, t\rangle + b\langle s, t\rangle$$

$$\langle r, cs + dt\rangle = c\langle r, s\rangle + d\langle r, t\rangle,$$

for all r, s, $t \in W$ and $a, b, c, d \in \mathbb{R}$.

We call such a map a *symmetric, positive definite, bilinear form.*

Definition 1.2.2. *A Riemannian metric on an n-dimensional differentiable manifold M is a mapping that assigns to every point $y \in M$ an inner product $g_y(\ ,\ ) : T_yM \times T_yM \to \mathbb{R}$ on the tangent space T_yM. Furthermore, $g_y(\ ,\ )$ is required to vary differentiably, that is, if J and K are any two differentiable vector fields on M, then the function $g(J, K) : M \to \mathbb{R}$ defined by $y \mapsto g_y(J_y, K_y)$, should be differentiable on M.*

Hence, the extra structure that we have added to M, is an inner product assigned to each tangent space at each point that varies differentiably from point to point. We can look at it as a natural generalization of the inner product between two vectors in $\mathbb{R}^n$ to an n-dimensional differentiable manifold. This concept was presented by Riemann in his 1854 habilitation lecture [48].

Remark 1.2.1. *We will often use the same notation as the first fundamental form and write* $\langle\ ,\ \rangle_y$ *instead of* $g_y(\ ,\ )$. *The subscript* y *is dropped whenever the point* y *is implied by context, hence we write* $\langle\ ,\ \rangle$ *or* $g(\ ,\ )$ *with the understanding that it is to be calculated at every point* y.

Definition 1.2.3. *A differentiable manifold* M *of dimension* n *equipped with a Riemannian metric* $\langle\ ,\ \rangle$ *is called a Riemannian manifold, and is denoted by the pair* $(M, \langle\ ,\ \rangle)$.

In order to perform computations and to better comprehend the notion of a Riemannian metric, it is helpful to express it in terms of a local coordinate system.

Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold, let $f : A \subset \mathbb{R}^n \to M$ be a local coordinate system on M, let y be any point on the coordinate neighborhood $f(A)$, let J, K be vector fields on M and let $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\}$ be the standard basis vector fields of T_yM induced by f. As seen in subsection 0.5, with respect to this basis, J and K can be expressed as:

$$J = \sum_{\eta=1}^{n} J_\eta \frac{\partial}{\partial x_\eta}, \quad K = \sum_{\omega=1}^{n} K_\omega \frac{\partial}{\partial x_\omega},$$

where the J_η and K_ω are differentiable functions on A.

Then, in this local coordinate system f, we can write the Riemannian metric $\langle\ ,\ \rangle$ as

$$\begin{aligned}\langle J, K\rangle &= \left\langle \sum_{\eta=1}^{n} J_\eta \frac{\partial}{\partial x_\eta}, \sum_{\omega=1}^{n} K_\omega \frac{\partial}{\partial x_\omega} \right\rangle \\ &= \sum_{\eta,\omega=1}^{n} J_\eta K_\omega \left\langle \frac{\partial}{\partial x_\eta}, \frac{\partial}{\partial x_\omega} \right\rangle \\ &= \sum_{\eta,\omega=1}^{n} g_{\eta\omega} J_\eta K_\omega,\end{aligned}$$

where we have set

$$g_{\eta\omega} = \left\langle \frac{\partial}{\partial x_\eta}, \frac{\partial}{\partial x_\omega} \right\rangle.$$

It can be shown that $\langle\ ,\ \rangle$ is symmetric and positive definite iff the matrix $g_{\eta\omega}$ has the same properties, that is, $\langle\ ,\ \rangle$ is symmetric iff $g_{\eta\omega} = g_{\omega\eta}$ and

$\langle\ ,\ \rangle$ is positive definite iff $g_{\eta\omega}$ is a positive definite matrix. Therefore, in a local coordinate system, the Riemannian metric is represented by a symmetric, positive definite matrix $g_{\eta\omega}$.

Remark 1.2.2. *In a local coordinate system, for $g_y(\ ,\ )$ to vary differentiably as in Definition 1.2.2, means that, if $f : A \subset \mathbb{R}^n \to M$ is a system of coordinates for M near y, such that $f(x_1, x_2, ..., x_n) = u \in f(A)$ and $\frac{\partial}{\partial x_\eta}(u) = df_u(0, ...1, ..., 0)$, then $g_u(\frac{\partial}{\partial x_\eta}(u), \frac{\partial}{\partial x_\omega}(u)) = g_{\eta\omega}(x_1, ..., x_n)$ is a differentiable function on A.*

Example 1.2.1. *The manifold $M = \mathbb{R}^n$, where $T_yM = \mathbb{R}^n$ for every point y, is naturally equipped with a Riemannian metric: the usual dot product. Specifically,*

$$g_{\eta\omega} = \left\langle \frac{\partial}{\partial x_\eta}, \frac{\partial}{\partial x_\omega} \right\rangle = \begin{cases} 1, & \text{if } \eta = \omega, \\ 0, & \text{if } \eta \neq \omega. \end{cases}$$

Such a metric is called the Euclidean metric.

$M = \mathbb{R}^n$ together with

$$\langle r, s\rangle = r \cdot s = \sum_{\eta=1}^{n} r_\eta s_\eta$$

is a Riemannian manifold.

The Riemannian metric enables us to define numerous common notions in geometry on a differentiable manifold, just as the dot product does in Euclidean geometry of $\mathbb{R}^n$.

Definition 1.2.4. *Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold and let r, s be vectors in T_yM.*

1. *The length of r, $\|r\|$, is given by $\|r\| = \sqrt{\langle r, r\rangle}$.*
2. *The angle between r and s is given by $\cos\theta = \dfrac{\langle r, s\rangle}{\|r\|\|s\|}$.*
3. *r and s are orthogonal if $\langle r, s\rangle = 0$.*

We now give the definition(s) of when we can regard two Riemannian manifolds as being the same.

Definition 1.2.5. *Let P and Q be a pair of Riemannian manifolds. A diffeomorphism $h : P \to Q$ is called an isometry if for every $q \in P$ and every $r, s \in T_qP$,*

$$\langle r, s\rangle_q = \langle dh_q(r), dh_q(s)\rangle_{h(q)}.$$

Two Riemannian manifolds are said to be isometric if there is an isometry between them.

Definition 1.2.6. *Let P and Q be a pair of Riemannian manifolds. A differentiable map $h : P \to Q$ is a local isometry if for every point $y \in P$ there is a neighborhood $A \subset P$ of y such that $h|_A : A \to h(A)$ is an isometry onto an open subset of Q.*

Intuitively, an isometry is a transformation that bends one manifold into another without stretching, shrinking or cutting it. In other words, isometries are a class of maps (diffeomorphisms) that preserve the metric (the first fundamental form). Thus, geometric quantities such as the distance between two points, are preserved by an isometry. Riemannian geometry, is concerned primarily with properties of Riemannian manifolds that are preserved by isometries.

The next proposition shows that any differentiable manifold can be equipped with at least one Riemannian metric.

Proposition 1.2.1. *Every differentiable manifold M has a Riemannian metric.*

Proof. Let M be a differentiable manifold with a differentiable structure $\{(A_\alpha, f_\alpha)\}_{\alpha\in I}$. For every $\alpha \in I$, we can define a Riemannian metric $\langle\ ,\ \rangle^\alpha$ by $\langle J, K\rangle^\alpha = \sum_\eta J_\eta K_\eta$ (the usual dot product) with respect to the coordinate basis over $f_\alpha(A_\alpha)$. This Riemannian metric is the metric induced by the local coordinate system. To construct a Riemannian metric on M one needs to piece together the Riemannian metrics from the various coordinate neighborhoods of a differentiable structure. The standard tool to do this is partitions of unity. And so we let $\{g_\alpha\}_{\alpha\in I}$ be a partition of unity on M that is subordinate to the covering $\{f_\alpha(A_\alpha)\}_{\alpha\in I}$. Now, for each $y \in M$ and for each $J, K \in T_yM$, we define the bilinear form $\langle\ ,\ \rangle_y$ on T_yM by

$$\langle J, K\rangle_y \overset{\text{def}}{=} \sum_{\alpha\in I} g_\alpha(y)\langle J, K\rangle^\alpha_y.$$

We have to verify that this construction defines a Riemannian metric on M. Since for each $y \in M$, only a finite number of $\alpha \in I$ have $g_\alpha(y) \neq 0$, then the sum in the equation above is finite. It is clear by construction that $\langle J, K\rangle_y$ is symmetric. To prove that $\langle\ ,\ \rangle_y$ is positive definite, we note that each $\langle J, K\rangle_y^\alpha$ is. Let $\tilde{I}$ be the set of all indices $\alpha \in I$ such that $g_\alpha(y) \neq 0$. By definition of a partition of unity, $0 < g_\alpha(y) \leq 1$. Thus, for all $J \in T_yM$, clearly $\langle J, J\rangle_y \geq 0$. Furthermore, if $\langle J, J\rangle_y = 0$, then at least one summand in

$$\sum_{\alpha\in\tilde{I}} g_\alpha(y)\langle J, J\rangle_y^\alpha$$

is 0. (In fact, all summands are 0). Thus, there exists an $\alpha \in I$ with $\langle J, J\rangle_y^\alpha = 0$. Since $\langle\ ,\ \rangle_y^\alpha$ is positive definite, then $J = 0$. Hence, $\langle J, J\rangle_y$ itself is positive definite. Therefore the construction above does indeed define a Riemannian metric. ■

1.3 Lengths and Volume on Riemannian Manifolds

In this subsection we present some of the most key ideas from classical geometry into the Riemannian setting. In particular, we will look at the length of a curve, the distance between a pair of points and the volume of a region on a Riemannian manifold M. Using integration, the Riemannian metric allows for formulas that measure the length of a curve and volume of a region on a manifold. We start with the length of a curve on M.

Let $\psi : [b, c] \to M$ be a differentiable curve on M, where $[b, c]$ is a closed interval in $\mathbb{R}$. At every point $\psi(t)$ $(t \in [b, c])$ in M, the vector $\psi'(t) = \frac{d\psi}{dt}$ is a tangent vector to M at $\psi(t)$, referred to as the velocity vector. We can therefore use the Riemannian metric $g(\ ,\ )$ to compute its length $\left\|\frac{d\psi}{dt}\right\| = \sqrt{g_{\psi(t)}\left(\frac{d\psi}{dt}, \frac{d\psi}{dt}\right)}$ (Definition 1.2.4, part 1), which we call the speed. The length of ψ is then obtained by integrating the speed.

Definition 1.3.1. *Let M be a Riemannian manifold and let $\psi : [b, c] \to M$ be a differentiable (of class C^∞) curve on M. Then the length $\ell(\psi)$ of the curve ψ is defined as:*

$$\ell(\psi) = \int_b^c \left\|\frac{d\psi}{dt}\right\| dt = \int_b^c \sqrt{g_{\psi(t)}\left(\frac{d\psi}{dt}, \frac{d\psi}{dt}\right)}\, dt.$$

In terms of a local coordinate system (or a parametrization) $f : A \subset \mathbb{R}^n \to M$, $\ell(\psi)$ is given by

$$\ell(\psi) = \int_b^c \sqrt{\sum_{\eta,\omega=1}^n g_{\eta\omega}\frac{du^\eta}{dt}\frac{du^\omega}{dt}}\, dt,$$

where $\hat{\psi} = f^{-1} \circ \psi : [b, c] \to A \subset \mathbb{R}^n$, $\hat{\psi} = (u^1(t), ..., u^n(t))$. Then $\psi = f \circ \hat{\psi}$, hence, $\psi(t) = f(u^1(t), ..., u^n(t))$ and thus $\frac{du^\eta}{dt}, \frac{du^\omega}{dt}$ are the components of the velocity vector with respect to the basis $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\}$.

If ψ is a piecewise differentiable curve, we define its length as follows:

Definition 1.3.2. *Let $\psi : [b, c] \to M$ be a curve and let $b = t_0 < t_1 < ... < t_{n-1} < t_n = c$ be a finite subdivision of the closed interval $[b, c]$ such that the restrictions $\psi\big|_{[t_i, t_{i+1}]}$ are differentiable for $i = 0, ..., n-1$. The length of ψ from b to c is given by*

$$\ell(\psi) = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left\| \frac{d\psi}{dt} \right\| dt = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \sqrt{g_{\psi(t)}\left(\frac{d\psi}{dt}, \frac{d\psi}{dt}\right)} dt.$$

Example 1.3.1. *We consider $M = \mathbb{R}^2$ with Cartesian coordinates (x, y) and with the standard Euclidean metric on $\mathbb{R}^2$. For a regular parametrized curve $\psi : [b, c] \to \mathbb{R}^2$ represented by $\psi(t) = (x^1(t), x^2(t))$, we get*

$$\begin{aligned} g_{\psi(t)}\left(\frac{d\psi}{dt}, \frac{d\psi}{dt}\right) &= g_{\psi(t)}\left(\left(\frac{dx^1}{dt}, \frac{dx^2}{dt}\right), \left(\frac{dx^1}{dt}, \frac{dx^2}{dt}\right)\right) \\ &= \left(\frac{dx^1}{dt}\right)^2 + \left(\frac{dx^2}{dt}\right)^2, \end{aligned}$$

thus

$$\ell(\psi) = \int_b^c \sqrt{\left(\frac{dx^1}{dt}\right)^2 + \left(\frac{dx^2}{dt}\right)^2} dt.$$

As expected, this is the standard calculus formula for the arc length of a parameterized curve.

One of the key properties of the length of a curve is that it does not dependent on the choice of a parametrization.

Definition 1.3.3. *A reparametrization of a differentiable curve $\psi : [b, c] \to M$ is a curve of the form $\widetilde{\psi} = \psi \circ \phi : [a, d] \to M$, where $\phi : [a, d] \to [b, c]$ (the reparametrization) is a diffeomorphism (differentiable map with differentiable inverse).*

Proposition 1.3.1. *Let (M, g) be a Riemannian manifold and let $\psi : [b, c] \to M$ be a differentiable curve on M. If $\widetilde{\psi} : [a, d] \to M$ is any reparametrization of ψ, then $\ell(\psi) = \ell(\widetilde{\psi})$.*

Proof. Because ψ and $\widetilde{\psi}$ are two paths with an identical image curve, we have a diffeomorphism $\phi : [a, d] \to [b, c]$ such that $\widetilde{\psi} = \psi \circ \phi$. Since a diffeomorphism is one-to-one and continuous, we have that either $\phi' > 0$ (ϕ is increasing) or $\phi' < 0$ (ϕ is decreasing), and also endpoints must map to endpoints. We start with the case for when $\phi' > 0$. Using the bilinearity of g, the chain rule for curves and the change of variables formula for integrals, we obtain:

$$\begin{aligned}
\ell(\widetilde{\psi}) &= \int_a^d \big\|\widetilde{\psi}'(s)\big\| ds = \int_a^d \big\|(\psi \circ \phi)'(s)\big\| ds \\
&= \int_a^d \big\|\psi'(\phi(s))\phi'(s)\big\| ds = \int_a^d \big\|\psi'(\phi(s))\big\|\big|\phi'(s)\big| ds \\
&= \int_a^d \big\|\psi'(\phi(s))\big\|\phi'(s) ds = \int_{\phi(a)}^{\phi(d)} \big\|\psi'(t)\big\| dt \\
&= \int_b^c \big\|\psi'(t)\big\| dt = \ell(\psi).
\end{aligned}$$

For $\phi' < 0$, the computation is the same except for two sign changes: (1) $|\phi'(s)| = -\phi'(s)$ since $\phi' < 0$, and (2) after we change the variable, the sign changes again because ϕ reverses the limits of integration. So these two sign changes cancel each other out:

$$\begin{aligned}
\ell(\widetilde{\psi}) &= \int_a^d \big\|\widetilde{\psi}'(s)\big\| ds = \int_a^d \big\|(\psi \circ \phi)'(s)\big\| ds \\
&= \int_a^d \big\|\psi'(\phi(s))\phi'(s)\big\| ds = \int_a^d \big\|\psi'(\phi(s))\big\|\big|\phi'(s)\big| ds \\
&= \int_a^d \big\|\psi'(\phi(s))\big\|(-\phi'(s)) ds = -\int_c^b \big\|\psi'(t)\big\| dt \\
&= \int_b^c \big\|\psi'(t)\big\| dt = \ell(\psi).
\end{aligned}$$

If ψ turns out to be a piecewise differentiable curve, then on each subinterval on which ψ is differentiable we apply the same procedure as above. ∎

Now that we have covered the notion of length of a curve, we can define the distance between two points on a Riemannian manifold.

Definition 1.3.4. *Let u, y be any pair of points on a connected Riemannian manifold M. The distance between u and y, denoted by $d(u, y)$, is defined as*

$$\begin{aligned}
d(u, y) = \inf\{\ell(\psi) \mid \psi : [b, c] \to M &\textit{ is a piecewise differentiable} \\
&\textit{curve with } \psi(b) = u \textit{ and } \psi(c) = y\}.
\end{aligned}$$

So to obtain the distance between two given points on a Riemannian manifold, we consider all the piecewise differentiable curves joining these two points and take the infimum of all its lengths. $d(\ ,\ )$ is also referred to as the Riemannian distance.

Remark 1.3.1. *Since M is connected, any two points in M can be joined by a piecewise differentiable curve. Therefore $d(\ ,\ )$ is always defined.*

The proofs of the next two propositions shall be presented in Section 3 as we need some concepts from there.

Proposition 1.3.2. *With the distance $d(\ ,\ )$, M is a metric space, that is, $d(\ ,\ )$ satisfies the three defining properties of a metric on a metric space: for all $u, y, w \in M$,*

- (i) Symmetry: $d(u, y) = d(y, u)$;
- (ii) Positive Definiteness: $d(u, y) \geq 0$, *the equality holds iff* $u = y$;
- (iii) Triangle Inequality: $d(u, w) \leq d(u, y) + d(y, w)$.

Proposition 1.3.3. *[3, Proposition 2.6] Let (M, g) be a connected Riemannian manifold. With the Riemanian distance function, M is a metric space whose metric topology is the same as the original manifold topology.*

To end this section, we are going to demonstrate how a Riemannian metric allows us to define the concept of volume on an orientable n-dimensianal manifold M.

First we require a few preparatory facts. Let y be a point in M, let $g_{\eta\omega} = \left\langle \frac{\partial}{\partial x_\eta}, \frac{\partial}{\partial x_\omega} \right\rangle$ and let $f : A \subset \mathbb{R}^n \to M$ be a parametrization around y which is a member of a group of parametrizations with the same orientation as that of M. Such parametrizations are considered to be positive. Next we write $J_\eta(y) = \frac{\partial}{\partial x_\eta}(y)$ in a positively orientated orthonormal basis $\{e_1, ..., e_n\}$ of the tangent space T_yM: $J_\eta(y) = \sum_{\eta\omega} a_{\eta\omega} e_\omega$. Then

$$g_{\eta\nu}(y) = \langle J_\eta, J_\nu \rangle(y) = \sum_{\omega\xi} a_{\eta\omega} a_{\nu\xi} \langle e_\omega, e_\xi \rangle = \sum_\omega a_{\eta\omega} a_{\nu\omega}.$$

The parallelepiped formed by the vectors $J_1(y), ..., J_n(y)$ in T_yM has volume equal to the the determinant of the matrix $(a_{\eta\omega})$ multiplied by the volume of $(e_1, ..., e_n)$. Thus we obtain

$$\mathrm{vol}(J_1(y), ..., J_n(y)) = \mathrm{vol}(e_1, ..., e_n) \cdot \det(a_{\eta\omega}) = \sqrt{\det(g_{\eta\omega})}(y),$$

where in the last equality we used the fact that $\mathrm{vol}(e_1, ..., e_n) = 1$.

Now if $\bar{f} : B \subset \mathbb{R}^n \to M$ is another positive parametrization around y with $K_\eta(y) = \frac{\partial}{\partial y_\eta}(y)$, $h_{\eta\omega}(y) = \langle K_\eta, K_\omega\rangle(y)$ (which is the metric for this parametrization $\bar{f}$) and $X = \det(\frac{\partial x_\eta}{\partial y_\omega}) = \det(d\bar{f}^{-1} \circ df)(y) > 0$, the determinant of the derivative of change of coordinates, we get

$$\begin{aligned}\sqrt{\det(g_{\eta\omega})}(y) &= \mathrm{vol}(J_1(y), ..., J_n(y)) \\ &= X\mathrm{vol}(K_1(y), ..., K_n(y)) = X\sqrt{\det(h_{\eta\omega})}(y).\end{aligned}$$

We then define the volume of a region R (an open connected subset of M) contained in a coordinate neighborhood $f(A)$ of a positive parametrization $f : A \subset \mathbb{R}^n \to M$ by the integral in $\mathbb{R}^n$:

$$\mathrm{vol}(R) = \int_{f^{-1}(R)} \sqrt{\det(g_{\eta\omega})}dx_1 \ . \ . \ . \ dx_n.$$

This definition is well-defined, in other words, it is independent of the parametrization. To show this we suppose that R is contained in another coordinate neighborhood $\bar{f}(B)$ with a positive parametrization $\bar{f} : B \subset \mathbb{R}^n \to M$ and then use the change of variable theorem for multiple integrals:

$$\begin{aligned}\int_{f^{-1}(R)} \sqrt{\det(g_{\eta\omega})}dx_1 \ . \ . \ . \ dx_n &= \int_{f^{-1}(R)} X\sqrt{\det(h_{\eta\omega})}dx_1 \ . \ . \ . \ dx_n \\ &= \int_{\bar{f}^{-1}(R)} \sqrt{\det(h_{\eta\omega})}dy_1 \ . \ . \ . \ dy_n \\ &= \mathrm{vol}(R),\end{aligned}$$

and therefore this shows that the definition given above is independent on the choice of the coordinate system.

2 Connections

2.1 Introduction

Let J and K be vector fields on an n-dimensional differentiable manifold M. The goal of this section is to show how to define a new vector field $\nabla_J K$ on M whose value at each point y is the vector rate of change of K in the J_y direction. In other words we would like to define differentiation of vector fields with respect to other vector fields on M. $\nabla_J K$ is commonly known as the directional (or covariant) derivative of K in the direction of J.

Given an n-dimensional differentiable manifold M, there is a natural way of differentiating functions on the manifold, but we do not have a naturally determined method of differentiation of vector fields on the manifold. So in order to get an idea of how to define $\nabla_J K$ on M we start by looking at differentiation of vector fields on submanifolds of $\mathbb{R}^n$, for which we do have a naturally determined method. This is not the most direct way of approaching the subject of differentiation of vector fields on manifolds but it helps our geometric understanding.

Suppose that M is a submanifold of $\mathbb{R}^n$. Let $J_y \in T_y M$ be a tangent vector for any point $y \in M$. Let $\psi : (-\epsilon, \epsilon) \to M$ be a curve on M of class C^1 or higher with initial point y and initial vector J_y, that is, $\psi(0) = y$ and $\psi'(0) = J_y$. Let K be a differentiable tangent vector field on M, that is, $K_y \in T_y M$ for every $y \in M$, which is defined everywhere on M and let $K(t)$, $t \in (-\epsilon, \epsilon)$, be the restriction of the vector field K to the curve ψ. Since every vector tangent to M is also a vector in $\mathbb{R}^n$ we can ignore M and differentiate $K(t)$ as a vector field along a curve in $\mathbb{R}^n$ obtaining $\frac{dK}{dt}$, another vector field along the curve. Even though both J_y and K are tangent to M, in general, $\frac{dK}{dt}$, will not be tangent to M. This would leave the realm of intrinsic geometry of the submanifold M and thus will not have meaning for M. To fix this we consider, instead of the usual derivative $\frac{dK}{dt}$, the orthogonal projection of $\frac{dK}{dt}$ on $T_{\psi(t)}M$. This orthogonally projected vector we call the directional (covariant) derivative at y of the vector field K in the direction of (relative to the vector) J_y and denote it by $\nabla_{J_y} K$ or $\frac{DK}{dt}$. The covariant derivative of K is the derivative of K as seen from the viewpoint of M. The intuition here is that all the geometry defined on M should be defined relative to the tangent space $T_y M$, in this way it only depends on the first fundamental form of M and therefore the concept of the covariant derivative is a concept which can be considered within Riemannian geometry.

It can be shown, although we will not show it here, that $\nabla_{J_y}K$ being defined as above, satisfies the following properties:

1. $\nabla_{J_y}K$ is linear over $\mathbb{R}$ in K:
$$\nabla_{J_y}(aK + bL) = a\nabla_{J_y}K + b\nabla_{J_y}L.$$

2. $\nabla_{J_y}K$ is linear over $C^\infty(M)$ in J:
$$\nabla_{mJ_y+n\bar{J}_y}K = m\nabla_{J_y}K + n\nabla_{\bar{J}_y}K.$$

3. ∇ obeys the following product rule:
$$\nabla_{J_y}(mK) = (J_y m)K(y) + m(y)\nabla_{J_y}K.$$

4. Agreement with the metric:
$$\nabla_{J_y}\langle K, L\rangle = \langle \nabla_{J_y}K, L(y)\rangle + \langle K(y), \nabla_{J_y}L\rangle.$$

5. Symmetric property:
$$[K, L] = \nabla_K L - \nabla_L K.$$

Where K, L are differentiable vector fields on M; $J_y, \overline{J}_y$ are vectors in T_yM; m, n are differentiable functions on M and $a, b \in \mathbb{R}$.

Both K and $\nabla_{J_y}K$ are tangent vector fields and thus have meaning for an arbitrary manifold M. However, the way by which $\nabla_{J_y}K$ is obtained from K and ψ makes use of the embedding of M in $\mathbb{R}^n$. In other words, it was possible to define $\nabla_{J_y}K$ because M was a submanifold in $\mathbb{R}^n$ and one already has a notion of derivative in $\mathbb{R}^n$ and one also has the notion of orthogonal projection into the tangent space T_yM in $\mathbb{R}^n$. On an arbitrary manifold M, which is not embedded in a Euclidean space in which to differentiate, we cannot define the directional derivative of a vector field on M in the same way as above. Our ultimate aim is to obtain for an arbitrary manifold M a definition of the directional derivative of a vector field along a curve which does not make use of an embedding but is intrinsic to the Riemannian and differentiable structure of M itself.

This is where, a connection comes in: it will be a new extra structure on a manifold that will help us overcome the difficulty mentioned above. Intuitively, a connection on a manifold will be a rule for computing directional derivatives of vector fields on the manifold.

2.2 Affine Connections

We will now study the general case. Let M be an n-dimensional differentiable manifold. As we mentioned in the introduction, there is no natural rules for the derivation of vector fields on M. So we shall introduce such rules axiomatically, as operations satisfying some of the key properties $(1-5)$ of the directional derivative of vector fields on submanifolds of $\mathbb{R}^n$. We do this because we are interested in having as much similarity to the standard differential calculus in $\mathbb{R}^n$ as possible and by having an axiomatic definition we obtain a more general mathematical structure on a domain which is independent of the metric.

In all that follows we shall denote $\mathfrak{X}(M)$ to be the space of all differentiable vector fields on M, $\mathcal{D}(M)$ to be the space of real-valued differentiable functions on M and $\Gamma(T_{\psi(t)}M)$ to be the space of all differentiable vector fields along the curve $\psi(t)$.

Definition 2.2.1. *Suppose M is a differentiable manifold. An affine connection on M is an $\mathbb{R}$-bilinear map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ denoted by $(J, K) \mapsto \nabla_J K$ that satisfies for all $J, K, L \in \mathfrak{X}(M)$, all $m, n \in \mathcal{D}(M)$ and all $a, b \in \mathbb{R}$:*

(∇1) *$\nabla_J K$ is linear over $\mathcal{D}(M)$ in J:*

$$\nabla_{mJ+nK} L = m\nabla_J L + n\nabla_K L.$$

(∇2) *$\nabla_J K$ is linear over $\mathbb{R}$ in K:*

$$\nabla_J (aK + bL) = a\nabla_J K + b\nabla_J L.$$

(∇3) *∇ satisfies the following product rule:*

$$\nabla_J (mK) = J(m)K + m\nabla_J K.$$

$\nabla_J K$ is called the *covariant derivative* of K in the direction of J.

Remark 2.2.1. *We defined $\nabla_J K$ in Definition 2.2.1 for a pair of vector fields J, K instead of $\nabla_{J_y} K$ for a single tangent vector J_y and a field K as done in the introduction of this section. This is fine since as y varies over M we have that $\nabla_{J_y} K = (\nabla_J K)_y$. Thus $\nabla_J K$ consists of all the covariant derivatives of K with respect to the vectors of J.*

Now we can address the question that originally motivated us to define a connection: How can we make sense of the directional derivative of vector fields along a curve on an arbitrary manifold?

Proposition 2.2.1. *Let M be a differentiable manifold, let $\psi : I \to M$ be a differentiable curve, let R, S be differentiable vector fields along ψ that are tangent to M along ψ, let $m : I \to \mathbb{R}$ be a differentiable function, and let ∇ be an affine connection on M. Then there exists a unique map $\frac{D}{dt} : \Gamma(T_{\psi(t)}M) \to \Gamma(T_{\psi(t)}M)$ which associates to a vector field R along the differentiable curve $\psi : I \to M$ another vector field $\frac{DR}{dt}$ along ψ, called the covariant derivative of R along ψ, characterized by the following three properties:*

(i) $\dfrac{D(R+S)}{dt} = \dfrac{DR}{dt} + \dfrac{DS}{dt}$;

(ii) $\dfrac{D(mR)}{dt} = \dfrac{dm}{dt}R + m\dfrac{DR}{dt}$;

(iii) *If R is induced by a vector field $K \in \mathfrak{X}(M)$, that is if $R(t) = K(\psi(t))$ for all t, then $\frac{DR}{dt} = \nabla_{\frac{d\psi}{dt}} K$ (= the covariant derivative of K in the direction of the velocity vector of ψ).*

Proof. We prove uniqueness first and then existence. Suppose that there exists a map $\frac{D}{dt}$ which satisfies the properties (i)-(iii). Let $f : A \subset \mathbb{R}^n \to M$ be a local coordinate system for M and let $(x_1(t), x_2(t), ..., x_n(t))$ denote the coordinate functions of $\psi(t)$ over $f(A)$, $t \in I$. Then the vector field $R(t)$ along ψ can locally be written uniquely as a linear combination:

$$R(t) = \sum_{\omega=1}^{n} r_\omega \frac{\partial}{\partial x_\omega},$$

where $r_1, ..., r_n$ are unique C^∞ real valued functions on $\mathbb{R}$ (or an appropriate subset of $\mathbb{R}$ such as I), and $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}$ are the basis vector fields on the coordinate neighborhood $f(A)$.

If (i)-(iii) are to hold, then we must have:

$$\begin{aligned}
\frac{DR}{dt} &= \sum_{\omega=1}^{n} \frac{D}{dt}\left(r_\omega \frac{\partial}{\partial x_\omega}\right) && \text{by (i)} \\
&= \sum_{\omega=1}^{n} \left\{ \frac{dr_\omega}{dt}\frac{\partial}{\partial x_\omega} + r_\omega \frac{D}{dt}\frac{\partial}{\partial x_\omega} \right\} && \text{by (ii)} \\
&= \sum_{\omega=1}^{n} \left\{ \frac{dr_\omega}{dt}\frac{\partial}{\partial x_\omega} + r_\omega \nabla_{\frac{d\psi}{dt}} \frac{\partial}{\partial x_\omega} \right\} && \text{by (iii)} \qquad (1)
\end{aligned}$$

So (1) proves that if there does exist a map $\frac{D}{dt}$ satisfying properties (i)-(iii), then the map is unique.

To prove existence, we define $\frac{DR}{dt}$ for a curve ψ in a coordinate neighborhood $f(A)$ by (1). It is easy to verify that (1) satisfies the three properties (i)-(iii). Thus, for curves in $f(A)$, $\frac{D}{dt}$ does exist. If $\bar{f}(\bar{A})$ is another coordinate neighborhood which overlaps with $f(A)$, that is, $\bar{f}(\bar{A}) \cap f(A) \neq \emptyset$, we define $R(t) = \sum_{\omega=1}^{n} \bar{r}_\omega \frac{\partial}{\partial \bar{x}_\omega}$ and we define $\frac{\bar{D}R}{dt}$ in $\bar{f}(\bar{A})$ by (1):

$$\frac{\bar{D}R}{dt} = \sum_{\omega=1}^{n} \left\{ \frac{d\bar{r}_\omega}{dt} \frac{\partial}{\partial \bar{x}_\omega} + \bar{r}_\omega \nabla_{\frac{d\psi}{dt}} \frac{\partial}{\partial \bar{x}_\omega} \right\}.$$

By the uniqueness of $\frac{DR}{dt}$ in $f(A)$ we have that $\frac{DR}{dt} = \frac{\bar{D}R}{dt}$ in $\bar{f}(\bar{A}) \cap f(A)$. Thus when we define $\frac{DR}{dt}$ by the formula in (1) we can extend it over all of M, and this concludes the proof. ■

Remark 2.2.2. *Proposition 2.2.1, in particular property (iii), illustrates the fact that we do not need to know the vector field K everywhere on M to compute $(\nabla_J K)_y$. It is enough to know $J_y \in T_yM$ and K at the points of a curve $\psi : (-\epsilon, \epsilon) \to M$ such that $\psi(0) = y$ and $\psi'(0) = J_y$. In other words, one way to compute $(\nabla_J K)_y$ is to take a curve curve $\psi : (-\epsilon, \epsilon) \to M$ with $\psi(0) = y$ and $\psi'(0) = J_y$, and to calculate $R'(0)$, where $R(t) = K(\psi(t))$. So the notion of affine connection is actually a local notion.*

We shall now describe affine connections in local coordinates. Let $(x_1, x_2, ..., x_n)$ be a local coordinate system defined on a coordinate neighborhood $f(A)$ of M, let $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}$ be the corresponding standard basis vector fields on $f(A)$ and let $J, K \in \mathfrak{X}(M)$. Then J and K on $f(A)$ can be expressed as:

$$J = \sum_{\eta=1}^{n} J_\eta \frac{\partial}{\partial x_\eta}, \quad K = \sum_{\omega=1}^{n} K_\omega \frac{\partial}{\partial x_\omega},$$

and by using the properties listed in Definition 2.2.1, $\nabla_J K$ on $f(A)$ can be expressed as:

$$\begin{aligned}
\nabla_J K &= \nabla_{\sum_{\eta=1}^{n} J_\eta \frac{\partial}{\partial x_\eta}} \left(\sum_{\omega=1}^{n} K_\omega \frac{\partial}{\partial x_\omega} \right) \\
&= \sum_{\eta=1}^{n} J_\eta \nabla_{\frac{\partial}{\partial x_\eta}} \left(\sum_{\omega=1}^{n} K_\omega \frac{\partial}{\partial x_\omega} \right) \\
&= \sum_{\eta=1}^{n} J_\eta \left(\sum_{\omega=1}^{n} K_\omega \nabla_{\frac{\partial}{\partial x_\eta}} \left(\frac{\partial}{\partial x_\omega} \right) + \frac{\partial}{\partial x_\eta} \left(\sum_{\omega=1}^{n} K_\omega \right) \frac{\partial}{\partial x_\omega} \right) \\
&= \sum_{\eta,\omega=1}^{n} J_\eta K_\omega \nabla_{\frac{\partial}{\partial x_\eta}} \left(\frac{\partial}{\partial x_\omega} \right) + \sum_{\eta,\omega=1}^{n} J_\eta \frac{\partial}{\partial x_\eta} \left(K_\omega \right) \frac{\partial}{\partial x_\omega}.
\end{aligned}$$

Now, there exists n^3 differentiable functions $\Gamma^{\nu}_{\eta\omega} : f(A) \to \mathbb{R}$, for $\eta, \omega, \nu = 1, .., n$, called the *Christoffel symbols* of the connection ∇ or the *coefficients of the connection*, such that:

$$\nabla_{\frac{\partial}{\partial x_\eta}} \left(\frac{\partial}{\partial x_\omega} \right) = \sum_{\nu=1}^{n} \Gamma^{\nu}_{\eta\omega} \frac{\partial}{\partial x_\nu}$$

on $f(A)$. We then have:

$$\nabla_J K = \sum_{\eta,\omega=1}^{n} J_\eta K_\omega \sum_{\nu=1}^{n} \Gamma^{\nu}_{\eta\omega} \frac{\partial}{\partial x_\nu} + \sum_{\eta,\omega=1}^{n} J_\eta \frac{\partial}{\partial x_\eta} \left(K_\omega \right) \frac{\partial}{\partial x_\omega}.$$

By changing the the variable of summation in the last term from ω to ν we obtain the following formula:

$$\nabla_J K = \sum_{\nu=1}^{n} \left(\sum_{\eta,\omega=1}^{n} J_\eta K_\omega \Gamma^{\nu}_{\eta\omega} + J(K_\nu) \right) \frac{\partial}{\partial x_\nu}.$$

In particular,

$$(\nabla_J K)_y = \sum_{\nu=1}^{n} \left(\sum_{\eta,\omega=1}^{n} J_\eta(y) K_\omega(y) \Gamma^{\nu}_{\eta\omega}(y) + J_y(K_\nu)(y) \right) \frac{\partial}{\partial x_\nu}\Big|_y.$$

This formula proves that $(\nabla_J K)_y$ depends only on the values of the J_η, K_ν at y, and the directional derivatives of the K_ν in the direction of J_y. Moreover, $J_\eta(y)$ and $K_\nu(y)$ depend only on J_y and K_y, and $(JK_\nu)(y) = \frac{d}{dt} K_\nu(\psi(t))|_{t=0}$ depends only on the values of K_ν (or K) along a curve ψ whose tangent vector at $y = \psi(0)$ is J_y, so what we wrote in Remark 2.2.2 is indeed true.

Definition 2.2.2. *Let M be a differentiable manifold with an affine connection ∇ and let R be a vector field defined at each point of a curve $\psi : I \to M$. Then we shall say that R is a constant or parallel vector field if $\frac{DR}{dt}(t) = 0$, for all $t \in I$.*

2.3 Riemannian Connections

Using a partition of unity, it can be shown that every differentiable manifold admits an affine connection and that, in fact, there are an infinite number of distinct affine connections on any differentiable manifold. To narrow down the number of affine connections on a manifold, we impose additional properties on a connection and define such a connection not just on any manifold but on a Riemannian manifold.

Definition 2.3.1. *An affine connection* ∇ *on a Riemannian manifold* M *is said to be compatible (consistent) with the metric* $\langle\ ,\ \rangle$ *iff for every* $J, K, L \in \mathfrak{X}(M)$:

$$J\langle K, L\rangle = \langle \nabla_J K, L\rangle + \langle K, \nabla_J L\rangle.$$

Definition 2.3.2. *An affine connection* ∇ *on a differentiable manifold* M *is said to be symmetric iff for any* $J, K \in \mathfrak{X}(M)$:

$$[J, K] = \nabla_J K - \nabla_K J.$$

It turns out that the properties to be compatible with metric and to be symmetric uniquely define a connection on a Riemannian manifold as the following theorem shows.

Theorem 2.3.1 (**Fundamental Theorem of Riemannian Geometry**). *[3, Theorem 3.6] For a given Riemannian manifold* M*, there is a unique affine connection* ∇ *on* M *which satisfies for all* $J, K, L \in \mathfrak{X}(M)$ *the following two conditions:*

(∇4) ∇ *is compatible with the Riemannian metric:*

$$J\langle K, L\rangle = \langle \nabla_J K, L\rangle + \langle K, \nabla_J L\rangle;$$

(∇5) ∇ *is symmetric :*

$$[J, K] = \nabla_J K - \nabla_K J.$$

This connection is called the *Riemannian connection* or the *Levi-Civita connection* with respect to the metric $\langle\ ,\ \rangle$ on M.

Proof. First, we show uniqueness. Let J, K, $L \in \mathfrak{X}(M)$. If such a connection ∇ exists then by cyclically permuting J, K, L in (∇4) we must have the following compatibility conditions:

$$J\langle K, L\rangle = \langle \nabla_J K, L\rangle + \langle K, \nabla_J L\rangle; \tag{2}$$
$$K\langle L, J\rangle = \langle \nabla_K L, J\rangle + \langle L, \nabla_K J\rangle; \tag{3}$$
$$L\langle J, K\rangle = \langle \nabla_L J, K\rangle + \langle J, \nabla_L K\rangle, \tag{4}$$

and by (∇5) we must have the following symmetry conditions:

$$\begin{aligned} [J, K] &= \nabla_J K - \nabla_K J; \\ [K, L] &= \nabla_K L - \nabla_L K; \\ [L, J] &= \nabla_L J - \nabla_J L. \end{aligned} \tag{5}$$

Adding equations (2) and (3) and subtracting equation (4), using the bilinearity and symmetric properties of $\langle\ ,\ \rangle$, we obtain:

$$\begin{aligned} J\langle K,L\rangle + K\langle L,J\rangle - L\langle J,K\rangle =& \langle \nabla_K L - \nabla_L K, J\rangle + \langle \nabla_J L - \nabla_L J, K\rangle \\ &- \langle \nabla_J K - \nabla_K J, L\rangle + 2\langle \nabla_J K, L\rangle. \end{aligned} \tag{6}$$

Applying the symmetry conditions from (5) to (6) yields:

$$\begin{aligned} J\langle K,L\rangle + K\langle L,J\rangle - L\langle J,K\rangle =& \langle [K,L], J\rangle + \langle [J,L], K\rangle \\ &- \langle [J,K], L\rangle + 2\langle \nabla_J K, L\rangle. \end{aligned} \tag{7}$$

Finally, solving for $\langle \nabla_J K, L\rangle$ in (7), we get:

$$\begin{aligned} 2\langle \nabla_J K, L\rangle =& J\langle K,L\rangle + K\langle L,J\rangle - L\langle J,K\rangle \\ &- \langle [K,L], J\rangle - \langle [J,L], K\rangle + \langle [J,K], L\rangle. \end{aligned} \tag{8}$$

Thus if we are given the restrictions (∇4) and (∇5), the formula in (8) proves that a Riemmanian connection is unique (because of the non-degeneracy of the metric), if it exists. This completes the proof of uniqueness.

To show that such a connection ∇ exists, we define ∇ by (8) and verify that ∇ satisfies conditions (∇1) – (∇5). We begin by verifying that ∇ satisfies conditions (∇1) – (∇3), this will show whether ∇ is a well-defined connection to begin with.

We note that functions can be factored out of the metric $\langle\ ,\ \rangle$ and from the properties of Lie brackets we have, for example, $[gK, L] = -(Lg)K + g[K, L]$, for every $g \in \mathcal{D}(M)$.

(∇1) Let $J, \overline{J} \in \mathfrak{X}(M)$. $\nabla_{J+\overline{J}}K = \nabla_J K + \nabla_{\overline{J}} K$ follows from the fact that $2\langle \nabla_{J+\overline{J}}K, L\rangle = 2\langle \nabla_J K, L\rangle + 2\langle \nabla_{\overline{J}} K, L\rangle$. So to prove ($\nabla$1) it suffices to show that $\nabla_{gJ} K = g\nabla_J K$, $g \in \mathcal{D}(M)$. By (8) we have:

$$\begin{aligned} 2\langle \nabla_{gJ} K, L\rangle =& gJ\langle K,L\rangle + K\langle L, gJ\rangle - L\langle gJ, K\rangle \\ &- \langle [K,L], gJ\rangle - \langle [gJ, L], K\rangle + \langle [gJ, K], L\rangle \\ =& gJ\langle K,L\rangle + (Kg)\langle L,J\rangle + gK\langle L,J\rangle - (Lg)\langle J,K\rangle - gL\langle J,K\rangle \\ &- g\langle [K,L], J\rangle - \langle g[J,L] - (Lg)J, K\rangle + \langle g[J,K] - (Kg)J, L\rangle \\ =& gJ\langle K,L\rangle + (Kg)\langle L,J\rangle + gK\langle L,J\rangle - (Lg)\langle J,K\rangle - gL\langle J,K\rangle \\ &- g\langle [K,L], J\rangle - g\langle [J,L], K\rangle + (Lg)\langle J,K\rangle + g\langle [J,K], L\rangle \\ &- (Kg)\langle J,L\rangle \\ =& 2g\langle \nabla_J K, L\rangle. \end{aligned}$$

Thus ∇ is linear over differentiable functions in J.

($\nabla 2$) Let L be arbitrary, $V, H \in \mathfrak{X}(M)$ and $a, b \in \mathbb{R}$. From the bilinearity of both the Lie bracket and the metric we have by (8):

$$\begin{aligned}
2\langle \nabla_J(aV+bH), L\rangle =& J\langle aV+bH, L\rangle + (aV+bH)\langle L, J\rangle - L\langle J, aV+bH\rangle \\
& - \langle [aV+bH, L], J\rangle - \langle [J,L], aV+bH\rangle \\
& + \langle [J, aV+bH], L\rangle \\
=& aJ\langle V, L\rangle + bJ\langle H, L\rangle + aV\langle L, J\rangle + bH\langle L, J\rangle \\
& - aL\langle J, V\rangle - bL\langle J, H\rangle - a\langle [V,L], J\rangle - b\langle [H,L], J\rangle \\
& - a\langle [J,L], V\rangle - b\langle [J,L], H\rangle + a\langle [J,V], L\rangle \\
& + b\langle [J,H], L\rangle \\
=& 2a\langle \nabla_J V, L\rangle + 2b\langle \nabla_J H, L\rangle \\
=& 2\langle a\nabla_J V + b\nabla_J H, L\rangle.
\end{aligned}$$

Thus ∇ is linear over the reals in K, that is, $\nabla_J(aV+bH) = a\nabla_J V + b\nabla_J H$.

($\nabla 3$) For an arbitrary L and $g \in \mathcal{D}(M)$, we have:

$$\begin{aligned}
2\langle \nabla_J(gK), L\rangle =& J\langle gK, L\rangle + gK\langle L, J\rangle - L\langle J, gK\rangle \\
& - \langle [gK, L], J\rangle - \langle [J,L], gK\rangle + \langle [J, gK], L\rangle \\
=& (Jg)\langle K, L\rangle + gJ\langle K, L\rangle + gK\langle L, J\rangle - (Lg)\langle J, K\rangle - gL\langle J, K\rangle \\
& - g\langle [K,L], J\rangle + (Lg)\langle K, J\rangle - g\langle [J,L], K\rangle + g\langle [J,K], L\rangle \\
& + (Jg)\langle K, L\rangle \\
=& (Jg)\langle K, L\rangle + g2\langle \nabla_J K, L\rangle + (Jg)\langle L, K\rangle \\
=& 2\langle JgK + g\nabla_J K, L\rangle.
\end{aligned}$$

Thus we have shown that $\nabla_J(gK) = JgK + g\nabla_J K$ which is ($\nabla 3$) as required.

So ∇ does satisfy conditions ($\nabla 1$) – ($\nabla 3$) and thus defines a connection.

($\nabla 4$) We compute:

$$2\langle \nabla_J K, L\rangle + 2\langle K, \nabla_J L\rangle = J\langle K, L\rangle + J\langle L, K\rangle = 2J\langle K, L\rangle,$$

Therefore ∇ is compatible with the metric.

($\nabla 5$) Finally, we compute:

$$\begin{aligned}
2\langle \nabla_J K - \nabla_K J, L\rangle &= 2\langle \nabla_J K, L\rangle - 2\langle \nabla_K J, L\rangle \\
&= \langle [J,K], L\rangle - (-\langle [J,K], L\rangle) = 2\langle [J,K], L\rangle,
\end{aligned}$$

Hence ∇ is symmetric. ■

We conclude this section by obtaining the formula which defines the Christoffel symbols for the Riemannian connection in terms of local coordinates. Let (A, f) be any local coordinate system on the Riemannian manifold M and let J, K, L be given respectively by the tangent vector fields $\frac{\partial}{\partial x_\eta}, \frac{\partial}{\partial x_\omega}, \frac{\partial}{\partial x_\nu}$. In a local coordinate system, the fact that ∇ is symmetric implies that for all $\eta, \omega, \nu = 1, ..., n$:

$$\nabla_{\frac{\partial}{\partial x_\eta}} \frac{\partial}{\partial x_\omega} - \nabla_{\frac{\partial}{\partial x_\omega}} \frac{\partial}{\partial x_\eta} = \left[\frac{\partial}{\partial x_\eta}, \frac{\partial}{\partial x_\omega}\right] = 0 \quad \Longleftrightarrow \quad \Gamma^\nu_{\eta\omega} = \Gamma^\nu_{\omega\eta}.$$

We recall that the Christoffel symbols and the metric coefficients are defined respectively in local coordinates by:

$$\nabla_{\frac{\partial}{\partial x_\eta}}\left(\frac{\partial}{\partial x_\omega}\right) = \sum_{\nu=1}^{n} \Gamma^\nu_{\eta\omega} \frac{\partial}{\partial x_\nu}, \quad g_{\eta\omega} = \left\langle \frac{\partial}{\partial x_\eta}, \frac{\partial}{\partial x_\omega} \right\rangle.$$

Therefore (8) yields:

$$\begin{aligned}
2\left\langle \nabla_{\frac{\partial}{\partial x_\eta}} \frac{\partial}{\partial x_\omega}, \frac{\partial}{\partial x_\varrho} \right\rangle &= \frac{\partial}{\partial x_\eta} g_{\omega\varrho} + \frac{\partial}{\partial x_\omega} g_{\varrho\eta} - \frac{\partial}{\partial x_\varrho} g_{\eta\omega} \\
\Leftrightarrow \left\langle \sum_{\nu=1}^{n} \Gamma^\nu_{\eta\omega} \frac{\partial}{\partial x_\nu}, \frac{\partial}{\partial x_\varrho} \right\rangle &= \frac{1}{2}\left(\frac{\partial}{\partial x_\eta} g_{\omega\varrho} + \frac{\partial}{\partial x_\omega} g_{\varrho\eta} - \frac{\partial}{\partial x_\varrho} g_{\eta\omega} \right) \\
\Leftrightarrow \sum_{\nu=1}^{n} \Gamma^\nu_{\eta\omega} g_{\nu\varrho} &= \frac{1}{2}\left(\frac{\partial}{\partial x_\eta} g_{\omega\varrho} + \frac{\partial}{\partial x_\omega} g_{\varrho\eta} - \frac{\partial}{\partial x_\varrho} g_{\eta\omega} \right).
\end{aligned}$$

Multiplying both sides by $g^{\nu\varrho}$ (the inverse of the matrix $g_{\nu\varrho}$), we get:

$$\Gamma^\nu_{\eta\omega} = \frac{1}{2} \sum_{\varrho=1}^{n} g^{\nu\varrho} \left(\frac{\partial}{\partial x_\eta} g_{\omega\varrho} + \frac{\partial}{\partial x_\omega} g_{\varrho\eta} - \frac{\partial}{\partial x_\varrho} g_{\eta\omega} \right).$$

The last formula defines the Christoffel symbols for the Riemannian connection. In the latter sections of this dissertation we will see that this formula is very useful for computations and makes some results easier to prove.

3 Geodesics

3.1 Introduction

Having presented two key thoughts of Riemannian geometry, the Riemannian metric and the Riemannian connection, we can now analyze some geometric ideas past those of length and angle introduced in section 1. Specifically, in this section, we will take a look at geodesics. Geodesics are curves in arbitrary manifolds that play a role analogous to that of straight lines in the plane. They are, in a sense, the straightest possible curves on a manifold. In classical Euclidean geometry, straight lines assume an essential role as the reason for most constructions and for the formation of most figures studied. The goal of this section is to look for curves on arbitrary manifolds, in particular on Riemannian manifolds, that assume an analogous role. This notion will depend massively on the concept of the covariant derivative presented in the previous section.

In classical Euclidean geometry, straight lines on a plane have three imperative properties:

1. For every pair of distinct points, there is exactly one straight line containing both of them.
2. The shortest distance between two points is a straight line.
3. It is a curve where its tangent vectors are all parallel, or equivalently, the second derivative always vanishes. This could be restated less precisely by saying that a straight line "does not bend to the left or right as we travel along it".

In order to generalize the notion of a straight line on a plane to the setting of Riemannian manifolds, we first need to have a sense of what properties of a straight line we hope to generalize, in light of the fact that, albeit each of the three of these properties hold at the same time for straight lines in the plane, things are not all that straightforward on Riemannian manifolds.

To help our intuition and geometric understanding we first deal with how to generalize the straight line in a plane to a submanifold of $\mathbb{R}^n$ such as the surface of the Earth, which to a good approximation is a perfect sphere. From the point of view of a resident (that is, from the perspective of a creature who lives on Earth and has no impression of the third dimension) driving a car (which is very small compared to the size of the Earth) on a road that is part of a great circle on Earth, that is, a circle with the biggest conceivable diameter, such as the equator or a line of longitude, it would feel as though you were traveling in a straight line, since with respect to Earth you would not be turning the wheels of the car to the left or the right. Consequently, great circles intuitively satisfy property 3. Between antipodal points (for example, the North and South poles) there are infinitely many lines of longitudes between them, therefore great circles do not satisfy property 1. Now, if we take two neighboring points on a sphere, and join them by the portion of the great circle that goes around 'the long way', such a great circle still doesn't veer to the left or to the right, but it is unquestionably not the shortest distance between the two points. Therefore great circles do not satisfy property 2 as well. Hence, properties 1-3 will not always hold simultaneously. The idea is that whatever curve is analogous to a straight line is one where you never have to steer.

So from the discussion above we see that, intuitively, the most useful property needed for the generalization of a straight line in the plane to the setting of a Riemannian manifold is property 3, in other words, we will generalize property 3 thus forming the basis for our definition of geodesics on a Riemannian manifold.

3.2 The Definition of a Geodesic

We now turn to a more precise look at property 3. What does "a curve where its tangent vectors are all parallel" actually mean? From the viewpoint of a Riemannian manifold M, a tangent vector field $\frac{d\phi}{dt}$ appears to be parallel to itself along a curve $\phi(t)$ on M if it has covariant derivative equal to zero. This motivates the following definition:

Definition 3.2.1. *Let M be a Riemannian manifold together with its Riemannian connection ∇ and I an open, closed, or half-open interval in $\mathbb{R}$. A parametrized curve $\phi : I \to M$ is a geodesic if the covariant derivative $\frac{D}{dt}(\frac{d\phi}{dt})$ of its tangent vector field $\frac{d\phi}{dt}$ is identically zero. That is, if*

$$\frac{D}{dt}\left(\frac{d\phi}{dt}\right) = \nabla_{\frac{d\phi}{dt}} \frac{d\phi}{dt} = 0, \tag{9}$$

for all $t \in I$.

We can think of $\phi(t)$ as the trajectory of a particle moving in M. $\frac{d\phi}{dt}$ is then the velocity and $\frac{d^2\phi}{dt^2}$ the acceleration of ϕ. The covariant derivative $\frac{D}{dt}(\frac{d\phi}{dt})$ of the velocity vector field $\frac{d\phi}{dt}$ is the tangential component of the acceleration

$\frac{d^2\phi}{dt^2}$. Intuitively $\frac{D}{dt}(\frac{d\phi}{dt})$ is the acceleration of the point $\phi(t)$ as seen from M. Then ϕ is a geodesic if the acceleration of the particle is zero, that is, if $\frac{d^2\phi}{dt^2} = 0$.

In particular, if ϕ is a geodesic, then the speed (or the length of the tangent vector $\frac{d\phi}{dt}$) $\|\frac{d\phi}{dt}\| = \sqrt{\langle \frac{d\phi}{dt}, \frac{d\phi}{dt} \rangle}$ is constant, since

$$\frac{d}{dt}\left\langle \frac{d\phi}{dt}, \frac{d\phi}{dt} \right\rangle = 2\left\langle \frac{D}{dt}\frac{d\phi}{dt}, \frac{d\phi}{dt} \right\rangle = 0.$$

Therefore, $\|\frac{d\phi}{dt}\| = a \neq 0$, for some constant a.

So geodesics are curves that have zero acceleration, or equivalently, a geodesic is a constant-speed curve.

3.3 Existence of Geodesics

To analyze properties of geodesics it is helpful to write (9) in a local coordinate system. Suppose M is a Riemannian manifold with a Riemannian connection ∇. Let $f : A \subset \mathbb{R}^n \to M$ be a local coordinate system, let $\phi : [a, b] \to M$ be a differentiable curve and let $\Gamma^{\nu}_{\eta\omega}$, $\eta, \omega, \nu = 1, .., n$ be given by $\nabla_{\frac{\partial}{\partial x_\eta}} \left(\frac{\partial}{\partial x_\omega} \right) = \sum_{\nu=1}^{n} \Gamma^{\nu}_{\eta\omega} \frac{\partial}{\partial x_\nu}$, as it was in section 2. In the local coordinate system (A, f) the curve ϕ has coordinate functions $\phi(t) = (\phi_1(t), ..., \phi_n(t))$. Then the tangential vector field $\frac{d\phi}{dt}$ on $f(A)$ can be expressed uniquely as:

$$\begin{aligned} \frac{d\phi}{dt} &= \frac{d\phi_1}{dt}\frac{\partial}{\partial x_1} + ... + \frac{d\phi_n}{dt}\frac{\partial}{\partial x_n} \\ &= \sum_{\eta=1}^{n} \frac{d\phi_\eta}{dt}\frac{\partial}{\partial x_\eta}. \end{aligned}$$

Therefore, we obtain:

$$\begin{aligned} \nabla_{\frac{d\phi}{dt}} \frac{d\phi}{dt} &= \nabla_{\sum_{\eta=1}^{n} \frac{d\phi_\eta}{dt}\frac{\partial}{\partial x_\eta}} \left(\sum_{\omega=1}^{n} \frac{d\phi_\omega}{dt}\frac{\partial}{\partial x_\omega} \right) \\ &= \sum_{\nu=1}^{n} \left(\sum_{\eta,\omega=1}^{n} \frac{d\phi_\eta}{dt}\frac{d\phi_\omega}{dt}\Gamma^{\nu}_{\eta\omega}(\phi(t)) + \frac{d^2\phi_\nu}{dt^2} \right) \frac{\partial}{\partial x_\nu}. \end{aligned}$$

So, by (9), $\phi(t)$ is a geodesic iff

$$\frac{d^2\phi_\nu}{dt^2} + \sum_{\eta,\omega=1}^{n} \frac{d\phi_\eta}{dt}\frac{d\phi_\omega}{dt}\Gamma^{\nu}_{\eta\omega}(\phi(t)) = 0, \quad \nu = 1, ..., n.$$

We sum up the discussion above in the subsequent theorem.

Theorem 3.3.1. *[3, pp.61-62] Let M be a Riemannian manifold equipped with its Riemannian connection, let $f : A \subset \mathbb{R}^n \to M$ be a local coordinate system for M and let $\phi : [a,b] \to f(A)$ be a parameterized curve on M. Then ϕ is a geodesic of M iff relative to any local coordinate system, the coordinate components of $\phi(t) = (\phi_1(t), ..., \phi_n(t))$ satisfy the following system of differential equations:*

$$\frac{d^2\phi_\nu}{dt^2} + \sum_{\eta,\omega=1}^{n} \frac{d\phi_\eta}{dt}\frac{d\phi_\omega}{dt}\Gamma^\nu_{\eta\omega}(\phi(t)) = 0, \quad \nu = 1, ..., n. \tag{10}$$

Equation (10) is a system of ordinary, non-linear, second order differential equations on the real line, called the geodesics equations. Among other things, we will use this equation to prove the existence and uniqueness of geodesics.

Example 3.3.1. *Let $(\mathbb{R}^3, g_{euclid})$ be the 3-dimensional Euclidean space together with its standard metric. In local coordinates we have*

$$g_{\eta\omega} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and by using the formula which defines the Christoffel symbols for the Riemannian connection in terms of local coordinates

$$\Gamma^\nu_{\eta\omega} = \frac{1}{2}\sum_{\varrho=1}^{n} g^{\nu\varrho}\left(\frac{\partial}{\partial x_\eta}g_{\omega\varrho} + \frac{\partial}{\partial x_\omega}g_{\varrho\eta} - \frac{\partial}{\partial x_l}g_{\eta\omega}\right)$$

we see that $\Gamma^\nu_{\eta\omega} = 0 \ \forall \ \eta, \omega, \nu$. Therefore (10) yields

$$\frac{d^2\phi_\nu}{dt^2} = 0, \quad \nu = 1, ..., n$$

whose solutions all have the form

$$\phi(t) = a + bt$$

for arbitrary constants a, b. This can obliviously by generalized to n-dimensional Euclidean space. Hence in the Euclidean case, geodesics are straight lines with the usual linear parametrization.

Standard theorems on ordinary differential equations give rise to the following result.

Theorem 3.3.2 (**Existence and Uniqueness of Geodesics**). *Let M be a Riemannian manifold with Riemannian connection ∇. For any point $y \in M$, any tangent vector $r \in T_yM$, and any $t_0 \in \mathbb{R}$, there exits an open interval $I \subset \mathbb{R}$ with $t_0 \in I$ and a geodesic $\phi : I \to M$ with initial conditions $\phi(t_0) = y$ and $\frac{d\phi}{dt}\big|_{t_0} = r$. The geodesic ϕ is unique in the following sense: if the curve $\bar{\phi} : \bar{I} \to M$ is another geodesic satisfying the same initial conditions $\bar{\phi}(t_0) = y$ and $\frac{d\bar{\phi}}{dt}\big|_{t_0} = r$, then $\bar{\phi}(t) = \phi(t)$ for all t in the intersection of the domains of the two curves.*

In short, Theorem 3.3.2 says that there exists a unique geodesic through any given point of a Riemannian manifold in any given tangent direction.

Proof. To prove this we will apply the existence and uniqueness theorem for the solutions of ordinary differential equations with initial conditions to Equation (10). Let $f : A \subset \mathbb{R}^n \to M$ be a local coordinate system such that $y \in f(A)$. If we introduce new variables $\frac{d\phi_\eta}{dt} = r_\eta$ we can rewrite the second-order system of differential equations (10) as the following first-order system of differential equations:

$$\begin{cases} \dfrac{d\phi_\nu}{dt} = r_\nu(t) \\ \dfrac{dr_\nu}{dt} = -\displaystyle\sum_{\eta,\omega=1}^{n} r_\eta(t) r_\omega(t) \Gamma^\nu_{\eta\omega}(\phi(t)), \quad \nu = 1, ..., n. \end{cases}$$

By the existence and uniqueness theorem for first-order ordinary differential equations, there exists a number $\epsilon > 0$ and a differentiable function $C : (t_0 - \epsilon, t_0 + \epsilon) \to A \times \mathbb{R}^n$, written $C(t) = (\phi_\eta(t), r_\eta(t))$ such that the functions $\phi_\eta(t), r_\eta(t)$ are solutions of (10) with initial conditions $C(t_0) = y$ and $C'(t_0) = r$. Thus it easily follows that the curve $\phi(t) = (\phi_1(t), ..., \phi_n(t))$ in $f(A)$ will be a geodesic with $\phi(t_0) = y$ and $\phi'(t_0) = r$. Existence is proved.

To prove the uniqueness condition, we suppose that there exists $t_1 \in I \cap \bar{I}$ such that $\phi(t_1) \neq \bar{\phi}(t_1)$. Without loss of generality we may assume $t_1 > t_0$. Thus the set $U = \{t \in I \cap \bar{I} : t > t_0 \text{ and } \phi(t) \neq \bar{\phi}(t)\}$ is non-empty and bounded below and so it has a greatest lower bound b, that is, $b = \inf U$. We want to show that $\phi(b) = \bar{\phi}(b)$ and $\frac{d\phi}{dt}\big|_b = \frac{d\bar{\phi}}{dt}\big|_b$. This is trivial for $b = t_0$. If $b > t_0$ then $\phi = \bar{\phi}$ on the interval (t_0, b) then by continuity $\phi(b) = \bar{\phi}(b)$ and $\frac{d\phi}{dt}\big|_b = \frac{d\bar{\phi}}{dt}\big|_b$. Now, since $t \mapsto \phi(t + b)$ and $t \mapsto \bar{\phi}(t + b)$ are also geodesics and since they have the same initial conditions, the existence part of Theorem 3.3.2 implies that $\phi = \bar{\phi}$ on some interval J around b, but this contradicts the definition of b. The argument for $t_0 > t_1$ is analogous, therefore $\bar{\phi}(t) = \phi(t)$ for all $t \in I \cap \bar{I}$. ∎

The preceding theorem shows the existence of the geodesic ϕ by solving the system of differential equations (10) over a coordinate neighborhood. For this situation, the interval I might be restricted by virtue of the fact that $\phi(I) \subset f(A)$, that is, equation (10) is for geodesics with image in a fixed coordinate neighborhood $f(A)$. By covering arguments we can extend geodesics over other coordinate neighborhoods. We just let I be the union of all the intervals on which such a geodesic is defined, then the various geodesics agree where they overlap. So, if $\phi(t_1)$ for some $t_1 \in I$ is in another coordinate neighborhood $\bar{f}(\bar{A})$, then we can uniquely extend the geodesic over $\bar{f}(\bar{A})$ as going through the point $\phi(t_1)$ with tangent vector $\frac{d\bar{\phi}}{dt}\big|_{t_1}$. The geodesic is said to be maximal or geodesically in-extendible if its domain interval I cannot be extended to a larger interval.

3.4 The Exponential Map

For some fixed point y of a Riemannian manifold we would like to look at the set of every geodesic going through y in all directions, in other words, we would like to study the collective behaviour of geodesics. In order to achieve this, one is lead to the exponential map. The exponential map depicts the behavior, at least for a small quantity of time, of every geodesic starting at a solitary point of a manifold. The definition and properties of the exponential map are used to prove many fundamental results in Riemannian geometry, some of which we will prove in this dissertation.

The existence and uniqueness of geodesics (Theorem 3.3.2) allows for the next definition.

Definition 3.4.1. *Let M be a Riemannian manifold, let y be a point on M, and let $\mathfrak{D}_y \subset T_yM$ be the set of all tangent vectors $n \in T_yM$ such that the unique maximal geodesic ϕ_n, with $\phi_n(0) = y$ and $\frac{d\phi_n}{dt}\big|_0 = n$, is defined at least over the interval $[0, 1]$. Then, the exponential map of M at y is the function $\exp_y : \mathfrak{D}_y \to M$ such that*

$$\exp_y(n) = \phi_n(1), \quad \forall\ n \in \mathfrak{D}_y.$$

In words, the function $\exp_y : \mathfrak{D}_y \to M$ assigns to every tangent vector $n \in T_yM$ a point on M and $\exp_y(n) = \phi_n(1)$ says that the image of n under the exponential mapping is defined to be that point on the unique geodesic determined by n at which the parameter takes the value of $+1$.

Since the length of the tangent vector $\frac{d\phi}{dt}$ along the geodesic $\phi(t)$ is constant, we have that the length ℓ of $\frac{d\phi}{dt}$ from $\phi(0)$ to $\phi(1)$ is:

$$\ell = \int_0^1 \left\| \frac{d\phi}{dt} \right\| dt = \int_0^1 \|n\| dt = \|n\|.$$

Therefore, $\exp_y(n)$ is the point on the unique geodesic $\phi(t)$ that starts at y, has tangent vector n at y and whose distance from y along the geodesic is the length of n.

The next lemma shows that we can increase the initial vector of a geodesic by decreasing its interval of definition, or, we can increase the interval of definition of the geodesic to the detriment of shortening the initial vector.

Lemma 3.4.1. *If ϕ_n is the maximal geodesic with $\phi_n(0) = y$ and $\frac{d\phi_n}{dt}\big|_0 = n \in T_yM$, then for any real number a, we have that,*

$$\phi_{an}(t) = \phi_n(at),$$

as soon as either side is defined.

Proof. If $a = 0$ we have nothing to show. If $a \neq 0$ we begin by showing that $\phi_{an}(t) = \phi_n(at)$ holds whenever $\phi_n(at)$ is defined. First, we recall from the previous subsection that for a point y in M and a tangent vector n in T_yM, there is a unique geodesic $\phi_n : I \to M$, defined on a maximal open interval $I \subset \mathbb{R}$, with $0 \in I$, $\phi_n(0) = y$ and $\frac{d\phi_n}{dt}\big|_0 = n$. We now set a new curve $\hat{\phi} : J \to M$ given by $\hat{\phi}(t) = \phi_n(at)$, where $J = \{t : at \in I\}$, that is, J is the inverse image of I by the map $t \mapsto at$. From the chain rule we obtain:

$$\frac{d\hat{\phi}}{dt} = a\frac{d\phi_n}{dt}(at),$$

and as a result we have,

$$\nabla_{\frac{d\hat{\phi}}{dt}} \frac{d\hat{\phi}}{dt} = \nabla_{a\frac{d\phi_n}{dt}} \left(a\frac{d\phi_n}{dt} \right) = a^2 \nabla_{\frac{d\phi_n}{dt}} \frac{d\phi_n}{dt} = 0.$$

Thus $\hat{\phi}$ is a geodesic with initial conditions $\hat{\phi}(0) = \phi_n(0) = y$ and $\frac{d\hat{\phi}}{dt}\big|_0 = a\frac{d\phi_n}{dt}\big|_0 = an$. So $\hat{\phi}$ is the unique geodesic with initial velocity $an \in T_yM$. Then it follows by uniqueness that $\hat{\phi}(t) = \phi_{an}(t)$, for every t in I. Therefore, we see that if $\phi_n(at)$ is defined then so is $\phi_{an}(t)$ and we have equality.

Conversely, we assume that $\phi_{an}(t)$ is defined and let $u = an$, $s = at$, $b = \frac{1}{a}$. Then $\phi_{an}(t) = \phi_u(bs)$ is defined and it is identical to $\phi_{bu}(s) = \phi_n(at)$. And so we deduce that if either side of $\phi_{an}(t) = \phi_n(at)$ is defined then so is the other. ∎

Lemma 3.4.1 gives us a different and somewhat easier characterization of the exponential map:

$$\begin{aligned}\exp_y(tn) &= \phi_{tn}(1) && \text{(by the definition of } \exp_y\text{)}\\ &= \phi_n(t) && \text{(by Lemma 3.4.1).}\end{aligned}$$

Geometrically we interpret this as follows: geodesics $t \mapsto \phi_n(t)$ in M through y are the images under the exponential map $\exp_y$ of straight lines $t \mapsto tn$ through the origin in T_yM and, as the next proposition demonstrates, this is done in a diffeomorphic way.

Proposition 3.4.1. *Let y be a point in a Riemannian manifold M. There is a neighborhood $\mathcal{V}$ of $0 \in T_yM$ and a neighborhood $\mathcal{U}$ of $y \in M$ such that the exponential map $\exp_y : \mathcal{V} \to \mathcal{U}$ is a diffeomorphism.*

Proof. The exponential map $\exp_y$ is well-defined and differentiable on some neighborhood of $0 \in T_yM$. One can verify this by utilizing results concerning the smooth dependence of the solutions of an ordinary differential equation on its initial conditions. Now, as T_yM is a vector space, the tangent space of T_yM at $0 \in T_yM$, denoted by $T_0(T_yM)$, can be canonically identified with T_yM, that is, $T_0(T_yM) = T_yM$. The differential at the origin of $\exp_y$ then turns into a map from T_yM onto itself:

$$d(\exp_y)_0 : T_yM \to T_yM.$$

To calculate $d(\exp_y)_0(n)$ for an arbitrary vector $n \in T_yM$, we pick a curve $\psi(t)$ in T_yM which satisfies $\psi(0) = 0$ and $\frac{d\psi}{dt}\big|_0 = n$. A clear choice for such a curve is $\psi(t) = tn$. Consequently,

$$\begin{aligned}d(\exp_y)_0(n) &= \frac{d}{dt}\exp_y(\psi(t))\Big|_{t=0}\\ &= \frac{d}{dt}\exp_y(tn)\Big|_{t=0}\\ &= \frac{d}{dt}\phi_n(t)\Big|_{t=0}\\ &= n,\end{aligned}$$

where $\phi_n(t)$ is the unique geodesic such that $\phi_n(0) = y$ and $\frac{d\phi_n}{dt}\big|_0 = n$. Hence the differential map $d(\exp_y)_0 : T_yM \to T_yM$ of $\exp_y$ at 0 is just the identity map $id_{|T_yM} : T_yM \to T_yM$ and so it is invertible and by the inverse function theorem there is a neighborhood of $0 \in T_yM$ which is mapped diffeomorphically onto a neighborhood of $y \in M$. ∎

Definition 3.4.2. *If $\mathcal{V}$ is a neighborhood of $0 \in T_yM$ and $\mathcal{U}$ is a neighborhood of $y \in M$ such that $\exp_y : \mathcal{V} \to \mathcal{U}$ is a diffeomorphism as in the proposition above, then the image $\exp_y(\mathcal{V}) = \mathcal{U}$ is called a normal neighborhood of y.*

We can transfer coordinates from the tangent space T_yM to the Riemannian manifold M by using the exponential map. Let $(e_1, ..., e_d)$ be an orthonormal basis for T_yM, where $d = \dim M$, and let $y \in M$ with $\exp_y(\mathcal{V}) = \mathcal{U}$ a normal neighborhood of y. An orthonormal basis for T_yM induces an isomorphism $\varphi : T_yM \to \mathbb{R}^d$ by $\varphi(x^1, ..., x^d) = \sum x^\eta e_\eta$. We can merge the exponential map with this isomorphism to set up a system of coordinates $\rho : \mathcal{V} \subset \mathbb{R}^d \to M$ around the point y on M defined by $\rho(x^1, ..., x^d) = \exp_y(x^1e_1 + ... + x^de_d) = \exp_y(\sum x^\eta e_\eta)$. This local coordinates $(x^1, ..., x^d)$ are referred to as the normal coordinates on $\mathcal{U}$ centered at y. The normal coordinate system $(x^1, ..., x^d)$ at y with respect to $(e_1, ..., e_d)$ associates to every point $w \in \mathcal{U}$ the coordinates of the vector relative to $(e_1, ..., e_d)$ of the corresponding point:

$$\exp_y^{-1}(w) \in \mathcal{V} \subset T_yM.$$

In summary,

$$\exp_y^{-1}(w) = \sum x^\eta(w)e_\eta.$$

Therefore, if $(f^1, \dots, f^d)$ is the dual basis to $(e_1, \dots, e_d)$ then $x^\eta = f^\eta(\exp_y^{-1})$ on $\mathcal{V}$.

Theorem 3.4.2. *If $(x^1, ..., x^d)$ is a normal coordinate system defined on $\mathcal{U}$ and centered at $y \in M$, then the components of the Riemannian metric $g_{\eta\omega}$ reduce to $\delta_{\eta\omega}$, all the first partial derivatives of $g_{\eta\omega}$ and all the Christoffel symbols $\Gamma^\nu_{\eta\omega}$ vanish at y. That is, for all η, ω, ν we have:*

1. $g_{\eta\omega}(y) = \delta_{\eta\omega}$;
2. $\Gamma^\nu_{\eta\omega}(y) = 0$;
3. $\dfrac{\partial g_{\eta\omega}}{\partial x^\nu}(y) = 0$.

Proof. 1. Let $n \in \mathcal{V} \subset T_yM$. In normal coordinates, the geodesics $\phi_n(t) = \exp_y(tn)$ that go through y for $n = \sum a^\eta e_\eta$ have a notably elementary coordinate expression:

$$x^\eta(\phi_n(t)) = f^\eta(\exp_y^{-1}(\phi_n(t))) = f^\eta(tn) = tf^\eta(n) = ta^\eta.$$

We may write this expression as $(x^1, \dots, x^d) = (a^1t, \dots, a^dt)$. Then for the initial velocity we get:

$$n = \left.\frac{d\phi_n}{dt}\right|_0 = \sum \left.\frac{d(x^\eta(\phi_n))}{dt}\right|_0 \left.\frac{\partial}{\partial x^\eta}\right|_y = \sum a^\eta \left.\frac{\partial}{\partial x^\eta}\right|_y.$$

For fixed index η, if we let $a^\eta = \delta_{\eta\omega}$, then $e_\eta = \frac{\partial}{\partial x^\eta}\big|_y$, this is due to the fact that $n = \sum a^\eta e_\eta$ was arbitrary. Thus:

$$g_{\eta\omega}(y) = \left\langle \frac{\partial}{\partial x^\eta}, \frac{\partial}{\partial x^\omega} \right\rangle_y = \langle e_\eta, e_\omega \rangle = \delta_{\eta\omega}.$$

2. Let $(x^1, \ldots, x^d) = (a^1 t, \ldots, a^d t)$ be a geodesic that goes through the point y. Then, it satisfies the geodesic differential equations:

$$\ddot{x}^\nu + \sum_{\eta,\omega} \Gamma^\nu_{\eta\omega} \dot{x}^\eta \dot{x}^\omega = 0, \quad \nu = 1, \ldots, d.$$

Since $x^\eta(\phi_n(t)) = ta^\eta$, the geodesic equation above for $\phi_n(t)$ reduces to:

$$\sum_{\eta,\omega} \Gamma^\nu_{\eta\omega}(\phi_n(t)) a^\eta a^\omega = 0, \quad \forall\ \nu.$$

Plugging in $t = 0$ we see that $\sum_{\eta,\omega} \Gamma^\nu_{\eta\omega}(y) a^\eta a^\omega = 0$ for every $(a^1, \ldots, a^d)$ in $\mathbb{R}^d$. Now, letting $(a^1, \ldots, a^d) = (0, \ldots, 1, 0, \ldots, 0, 1, \ldots, 0)$ such that $a^\eta = a^\omega = 1$ and all the other entries are equal to zero, we obtain

$$\Gamma^\nu_{\eta\omega} + \Gamma^\nu_{\omega\eta} = 0.$$

And so $\Gamma^\nu_{\eta\omega}$ is identically zero at the point y by the symmetry of the connection ∇.

3. By the compatibility of the connection ∇ on a Riemannian manifold M with the metric we obtain:

$$\frac{\partial g_{\eta\omega}}{\partial x^\nu} = \frac{\partial}{\partial x^\nu}\left\langle \frac{\partial}{\partial x^\eta}, \frac{\partial}{\partial x^\omega} \right\rangle = \left\langle \nabla_{\frac{\partial}{\partial x^\nu}} \frac{\partial}{\partial x^\eta}, \frac{\partial}{\partial x^\omega} \right\rangle + \left\langle \frac{\partial}{\partial x^\eta}, \nabla_{\frac{\partial}{\partial x^\nu}} \frac{\partial}{\partial x^\omega} \right\rangle.$$

From property 2 we have that $\nabla_{\frac{\partial}{\partial x^\nu}} \frac{\partial}{\partial x^\eta} = 0$ at y and $\nabla_{\frac{\partial}{\partial x^\nu}} \frac{\partial}{\partial x^\omega} = 0$ at y. Therefore, $\frac{\partial g_{\eta\omega}}{\partial x^\nu}(y) = 0$. ■

Remark 3.4.1. *As a result of properties $1-3$ listed in the preceding theorem, a number of computations in Riemannian geometry may be simplified by working in normal coordinate systems. So normal coordinate systems are very useful. Although, we should take note that properties $1-3$ are only guaranteed to hold at the point in which the coordinates are centered. In general properties $1-3$ fail to hold even for points arbitrarily close to y.*

3.5 Minimizing Properties of Geodesics

The way in which we defined geodesics in subsection 3.2 is not unique. A different way to defining geodesics on a Riemannian manifold is to define it as a curve of shortest length between a pair of distinct points. However, as we discussed in the introduction of this section, this definition poses a problem. To recap, we used the example of great circles on the sphere to show that in general we can't find a unique geodesic joining any two distinct points, this is because on a sphere you can have two curves which are geodesics joining two distinct points but one geodesic is shorter than the other. But what we hope is that points which are sufficiently 'close to each other' would have a unique 'short' geodesic joining them. The objective of this subsection is to show that the shortest curves between any pair of points on a Riemannian manifold are geodesics, and that the converse is also true but only locally.

We begin with some preliminary definitions which are needed for what is to come:

Definition 3.5.1. *We call a continuous curve $\phi : [b, c] \to M$ piecewise differentiable if there exists a finite subdivision $b = t_0 < t_1 < ... < t_{d-1} < t_d = c$ of the closed interval $[b, c]$ such that the restrictions $\phi\big|_{[t_{i-1},t_i]}$ are differentiable for $i = 1, ..., d$. We say that the point $y \in M$ is joined to the point $x \in M$ by ϕ if $\phi(b) = y$ and $\phi(c) = x$.*

For most applications, it is useful to restrict the exponential map, $\exp_y$, to an open subset of the tangent space T_yM, that is,

$$\exp_y : \mathcal{B}(0, \epsilon) \subset T_yM \to M,$$

where $\mathcal{B}(0, \epsilon)$ is an open ball with radius ϵ and center $0 \in T_yM$.

Definition 3.5.2. *For $\epsilon > 0$ such that the closed ball $\overline{\mathcal{B}(0, \epsilon)}$ is contained in $\mathcal{V}$, that is, $\overline{\mathcal{B}(0, \epsilon)} \subset \mathcal{V} = \exp_y^{-1}(\mathcal{U})$, we define the normal ball (or the geodesic ball) centered at the point y and with radius ϵ as the open set $\exp_y(\mathcal{B}(0, \epsilon)) = \mathcal{B}(y, \epsilon)$. The geodesics that start at y and lying in $\mathcal{B}(y, \epsilon)$ are called radial geodesics.*

Definition 3.5.3. *A normal neighborhood $\mathcal{W}$ of a point $x \in M$ is totally normal if there exists some $\epsilon > 0$ such that for all $y \in \mathcal{W}$, $\mathcal{W}$ is contained in a normal ball of radius ϵ and center y, that is, $\mathcal{W} \subset \mathcal{B}(y, \epsilon)$ $\forall y \in \mathcal{W}$.*

We now state and prove the theorem which says that geodesics are locally the shortest curves.

Theorem 3.5.1. *[2, Theorem 4.5] Let M be a Riemannian manifold, let $y \in M$ be a point, let $\mathcal{U}$ be a normal neighborhood of y and let $\mathcal{B}(y,\epsilon) \subset \mathcal{U}$ be a normal ball of center y. Then, there exists a geodesic $\phi : I \to M$ joining y to x for every point $x \in \mathcal{B}(y,\epsilon)$. Furthermore, we have that $\ell(\phi) \leq \ell(\psi)$, where $\psi : J \to M$ is any other piecewise differentiable curve joining y to x, and $\ell(\phi) = \ell(\psi)$ iff ψ is a reparameterization of ϕ.*

To prove this theorem we need Gauss' lemma. The proof and details surrounding this lemma can be found in M. do Carmo [3, p. 67].

Lemma 3.5.2 (Gauss)**.** *Let $y \in M$ and let $n \in T_yM$ such that $\exp_y n$ is defined. Let $m \in T_yM \approx T_n(T_yM)$. Then*

$$\langle (d\exp_y)_n(n), (d\exp_y)_n(m)\rangle = \langle n, m\rangle.$$

Proof of Theorem 3.5.1. Without any loss of generality we may suppose that $I, J = [0,1]$ and that, initially, $\psi([0,1]) \subset \mathcal{B}$. Since, on $\mathcal{U}$, the exponential map $\exp_y$ is a diffeomorphism, we may express the curve $\psi(t)$, for $t \neq 0$, as $\exp_y(s(t)\cdot n(t)) = q(s(t),t)$, where $s : (0,1] \to \mathbb{R}$ is a positive piecewise differentiable function, $t \to n(t)$ is a curve in T_yM satisfying $|n(t)| = 1$, and q is a parametrized surface. It follows that, with the exception of a limited number of points (the singularities),

$$\frac{d\psi}{dt} = \frac{\partial q}{\partial s}\frac{ds}{dt} + \frac{\partial q}{\partial t}.$$

Now, $\left\langle \frac{\partial q}{\partial s}, \frac{\partial q}{\partial t}\right\rangle = 0$, this follows from the Gauss lemma. As $\left|\frac{\partial q}{\partial s}\right| = 1$, we have that,

$$\left|\frac{d\psi}{dt}\right|^2 = \left|\frac{ds}{dt}\right|^2 + \left|\frac{\partial q}{\partial t}\right|^2 \geq \left|\frac{ds}{dt}\right|^2,$$

and so,

$$\int_\epsilon^1 \left|\frac{d\psi}{dt}\right| dt \geq \int_\epsilon^1 \left|\frac{ds}{dt}\right| dt \geq \int_\epsilon^1 \frac{ds}{dt} dt = s(1) - s(\epsilon).$$

We get that $\ell(\psi) \geq \ell(\phi)$ by taking $\epsilon \to 0$. If either of the inequalities in the two equations above is strict then $\ell(\psi) > \ell(\phi)$. If $\ell(\psi) = \ell(\phi)$, then $\left|\frac{\partial q}{\partial t}\right| = 0$, that is, $n(t)$ is equal to a constant and $\left|\frac{ds}{dt}\right| = \frac{ds}{dt} > 0$. It follows that ψ is a monotonic reparameterization of ϕ, and therefore $\psi([0,1]) = \phi([0,1])$.

If $\psi([0,1]) \not\subset \mathcal{B}$, we take the first point $\tilde{t}$ in the open interval $(0,1)$ for which $\psi(\tilde{t})$ resides in the boundary of $\mathcal{B}$. If φ is the radius of $\mathcal{B}$, we obtain,

$$\ell(\psi) \geq \ell_{[0,\tilde{t}]}(\psi) \geq \varphi > \ell(\phi).$$

■

Next we turn to the converse of Theorem 3.5.1 which shows that although a geodesic may not be the shortest path between two points but the shortest path between two points must be a geodesic. Intuitively, this implies that there is no turning in the shortest road between two points, since doing as such could probably increase the length of the road.

Theorem 3.5.3. *[2, Theorem 4.6] Let M be a Riemannian manifold and let $y, x \in M$ be points. If a piecewise differentiable curve $\phi : I \to M$ joining y to x, has length less or equal to the length of any other piecewise differentiable curve $\psi : J \to M$ joining y to x, that is, $\ell(\phi) \leq \ell(\psi)$, then ϕ is a reparameterized geodesic.*

To prove this theorem we use the following result which shows that totally normal neighborhoods always exist. The proof can be found in M. do Carmo [3, p. 72].

Theorem 3.5.4. *For each point $x \in M$ there is a neighborhood $\mathcal{W}$ of x such that for each point $y \in \mathcal{W}$, $\mathcal{W}$ is a normal neighborhood of y as well.*

Proof of Theorem 3.5.3. Let $I = [b, c]$, let $t_o \in [b, c]$ and let $\mathcal{W}$ be a totally normal neighborhood of $\phi(t_o)$. Then, there is an interval $E \subset [b, c]$ which is also closed and which its interior is not empty, such that $\phi(E) \subset \mathcal{W}$, for $t_o \in E$. The restriction of the curve ϕ to this closed interval E, $\phi_E : E \to \mathcal{W}$, is then a piecewise differentiable curve connecting a pair of points of a normal ball. From Theorem 3.5.1 and from the hypothesis of Theorem 3.5.3, the length of the curve ϕ_E, $\ell(\phi_E)$, is the same as the length of a radial geodesic connecting these pair of points, this is due to the fact that $\ell(\phi_E)$ lies in a normal ball. Again, from Theorem 3.5.1, ϕ_E must be a reparameterized geodesic in each totally normal neighborhood it intersects, in other words, ϕ_E is a geodesic on E and hence at t_o, since t_o is arbitrary. ∎

3.6 The Hopf-Rinow Theorem

In the previous subsection we saw that an essential consequence of Gauss' lemma is that points which are close enough together can always be joined by a unique minimizing geodesic. We also saw from the existence of totally normal neighborhoods that if a piecewise differentiable curve ϕ is minimizing, then ϕ is geodesic. These are both local properties. A local property is a property which relies upon the behavior of the manifold in the neighborhood of a point. The goal of this subsection is to study these results globally. A global property is a property that relies upon the behavior of the manifold taken as a whole.

To start with, we define a special kind of manifold which is the natural setting for global properties, viz., a complete Riemannian manifold.

Definition 3.6.1. *A Riemannian manifold M is said to be (geodesically) complete if for each point $y \in M$, the exponential map, $\exp_y$, is defined on all of T_yM (i.e., $\mathfrak{D}_y = T_yM$), or, in other words, if the domain of all the geodesics in M can be extended to the entire real line $\mathbb{R}$, that is, if any geodesic $\phi(t)$ in M such that $\phi(0) = y$ is defined for every t in $\mathbb{R}$.*

Intuitively, a complete Riemannian manifold M, is a manifold which does not have any boundaries or holes, thus you can walk forever along a geodesic and not worry about falling off the manifold. In general, given any pair of distinct points on an arbitrary manifold which are not sufficiently close to one another, there does not necessarily exist a minimizing geodesic connecting these two points. What ends up being valuable on complete manifolds is that there does exist a minimizing geodesic connecting any two distinct points in M. This is the essence of the Hopf-Rinow Theorem. But first we look at some examples.

Example 3.6.1. *The Euclidean space $\mathbb{R}^n$ is geodesically complete.*

Example 3.6.2. *The sphere $\mathbb{S}^2$ is geodesically complete.*

Example 3.6.3. *The plane with the origin removed or, in other words, the punctured plane $\mathbb{R}^2 \setminus \{(0,0)\}$ is not geodesically complete, since the domain of the geodesic $\phi(t) = (t,t)$, $1 < t < \infty$, cannot be extended to $-\infty < t < \infty$.*

Theorem 3.6.1 (Hopf-Rinow). *[3, Theorem 2.8] Let M be a Riemannian manifold and let $y \in M$ be a point. Then the statements* (a), (b), (c) *and* (d)*, stated below, are equivalent:*

(a) *M is complete as a metric space.*

(b) *M is geodesically complete, i.e., for every point $y \in M$, the exponential map, $\exp_y$, is defined on all of T_yM.*

(c) *M is geodesically complete at y, i.e., there exists some point $y \in M$ such that, the exponential map, $\exp_y$, is defined on all of T_yM.*

(d) *The closed and bounded subsets of M are compact.*

Moreover, any of the statements (a) – (d) also imply (but are not equivalent to):

(e) *For any two points $y, x \in M$ there exists a geodesic ϕ joining y to x with length $\ell(\phi) = d(y, x)$, i.e., any two distinct points of M can be joined by a length minimizing geodesic.*

Proof. This theorem was first proved by W. Rinow and H. Hopf. However, the proof that we are going to present here follows a simpler proof that was provided by G. de Rham. The main point of the proof lies in the implication that if $\exp_y$ is defined on all of T_yM, then any $x \in M$ can be connected with y by a shortest geodesic. In particular, this is the implication $(b) \Rightarrow (e)$. To prove this implication, we first need to find a candidate for the tangent vector $r \in T_yM$ giving the minimal geodesic, that is, we need to produce a minimizing geodesic from y to a given point $x \in M$. Roughly speaking, the idea of the proof is to start from y with a geodesic in the "right direction", and then prove that this geodesic eventually reaches x. To this end, let $d(y, x) = v$, let $\delta > 0$ be such that $\mathcal{B}(y, \delta)$ is a normal ball centered at y, let $\mathcal{S} = \partial\mathcal{B}(y, \delta)$ denote the boundary of $\mathcal{B}(y, \delta)$ and let $p_0 \in \mathcal{S}$ be a point such that $d(p_0, x) = \min\{d(p, x)|p \in \mathcal{S}\}$ which exists by the continuity of the distance function and by the compactness of $\mathcal{S}$. Then, by the definition of the exponential map, we have that $\exp_y(\delta r) = p_0$, for some unique vector $r \in T_yM$ with $\|r\| = 1$. We consider the geodesic $\phi : \mathbb{R} \to M$ defined by $\phi(s) = \exp_y(sr)$, which by assumption is defined on the whole of $\mathbb{R}$. We want to show that $\phi(v) = x$. For that, we consider the set

$$A := \{s \in [0, v] \mid d(\phi(s), x) = v - s\},$$

and show that $v \in A$. We will do this by showing that A is non-empty, closed, and open, and thus that it contains all of $[0, v]$. First, $0 \in A$, since $d(\phi(0), x) = d(y, x) = v - 0$, hence A is non-empty. Also, it's closed, since both sides of $d(\phi(s), x) = v - s$ are continuous in s (so the condition commutes with limits). It's more challenging to show that it's open. We want to prove that if $s_0 \in A$ with $s_0 < v$ then $s_0 + \delta' \in A$, where $\delta' > 0$ is sufficiently small. Similar to the above, at a given s_0, we consider a normal ball $\mathcal{B}_{\delta'}(\phi(s_0))$ around $\phi(s_0)$ with $\mathcal{S}' = \partial B_{\delta'}(\phi(s_0))$ as its boundary. Thus, the distance function again has a minimum at p_0' on $\mathcal{S}'$, so It suffices to show that $p_0' = \phi(s_0 + \delta')$, because if so, then $d(\phi(s_0), x) = v - s_0$ and

$$d(\phi(s_0), x) = \delta' + d(p_0', x) = \delta' + d(\phi(s_0 + \delta'), x),$$

so $d(\phi(s_0 + \delta'), x) = v - (s_0 + \delta')$, and so $s_0 + \delta' \in A$. Thus, all we need to do is show $p_0' = \phi(s_0 + \delta')$. To show this we observe that, by the triangle inequality and by the first equality of the equation above,

$$d(y, p_0') \geq d(y, x) - d(x, p_0') = v - (v - s_0 - \delta') = s_0 + \delta',$$

and, since the piecewise differentiable curve which connects and y to p_0' via $\phi(s_0)$ has length $s_0 + \delta'$, we conclude that this is a minimizing curve, hence a (reparameterized) geodesic. Thus, as promised, $p_0' = \phi(s_0 + \delta')$.

Now, the rest of the proof of the Hopf-Rinow theorem is straight forward:

$(c) \Rightarrow (d)$ For a closed and bounded set $K \subset M$ we can always find a metric ball $B(y, \varepsilon)$ containing K, that is, $K \subset B(y, \varepsilon)$, for some $\varepsilon > 0$. By what we have shown in the beginning, any point in $B(y, \varepsilon)$ can be connected with y by a geodesic (of length $\leq \varepsilon$). Hence, $B(y, \varepsilon)$ is the image of the compact ball in T_yM of radius ε under the continuous map $\exp_y$. Hence, $B(y, \varepsilon)$ is compact itself. Since K is assumed to be closed and shown to be contained in a compact set, it must be compact itself.

$(d) \Rightarrow (a)$ Let $(p_n)_{n \in \mathbb{N}} \subset M$ be a Cauchy sequence. It then is bounded, and, by (d), its closure is compact. It therefore contains a convergent subsequence, and being Cauchy, it has to converge itself. This shows completeness of M.

$(a) \Rightarrow (b)$ We suppose M is not geodesically complete. Then there exists a normalized geodesic ϕ that is defined for values $s < s_0$ but not for s_0. Let $(s_n)_n$ be a sequence such that $s_n \to s_0$ with $s_n < s_0$. For sufficiently large indices we have $|s_n - s_m| < \varepsilon$. This tells us that

$$d(\phi(s_n), \phi(s_m)) \leq |s_n - s_m| < \varepsilon,$$

and hence that $\{\phi(s_n)\}$ is Cauchy. Because M is complete there is $y_0 \in M :$ $\phi(s_n) \longrightarrow y_0$. Let $(\mathcal{W}, \delta)$ be a totally normal neighborhood of y_0. Let N be big enough such that $|s_m - s_n| < \delta$, $\phi(s_n)$, $\phi(s_m) \in \mathcal{W}$ whenever $m, n > N$. Then by definition there is a unique geodesic g connecting $\phi(s_n)$ and $\phi(s_m)$ with length less than δ. Wherever ϕ is defined it is obvious that $g = \phi$. Since we know that $\exp_{\phi(s_n)} : B(0, \delta) \longrightarrow M$ is a diffeomorphism on its image which contains $\mathcal{W}$, we see that g extends ϕ for values beyond s_0.

$(b) \Rightarrow (c)$ Follows from definition, so it is trivial. ■

Remark 3.6.1. *The length minimizing geodesic from the theorem above is not unique. For example, there are an infinite number of length minimizing geodesics joining the antipodal points of the sphere.*

Remark 3.6.2. *The converse of the Hopf-Rinow theorem is false. As an example, if we take the open canonical half-sphere, we observe that any pair of* *distinct points in it can be connected by a unique length minimizing geodesic, but this manifold is not geodesically complete.*

4 Curvature

4.1 Introduction

The aim of this section is to define the curvature of Riemannian manifolds. The definition of curvature that we are going to present in this section has many different geometric interpretations. We will describe at least two of them. The first interpretation is that curvature measures to which extent a Riemannian manifold is different from a plane, or, in other words, curvature measures the amount by which a Riemannian manifold deviates from being flat (having no curvature). For the second interpretation we take a look at the parallel transport of a vector along a curve. If we parallel transport a vector along a closed curve in the plane ($\mathbb{R}^2$) then when it comes back to the starting position we wind up with an indistinguishable vector from the one we began with, that is, in a non-curved space the orientation of the vector is preserved along the transport. However, for a curved surface such as the sphere, this not the case. If we parallel transport a vector along a closed curve on a sphere it might alter its orientation. Hence, we can decide if a Riemannian manifold is curved or not by transporting a vector around a closed curve and measuring the difference of the orientation at start and the end of the transport. So, intuitively, curvature also measures the failure of a vector to return to its starting position with the same orientation after it got parallel transported around a closed curve.

4.2 Curvature

Definition 4.2.1. *Let M be a Riemannian manifold with Riemannian connection ∇. The curvature $\mathcal{R}$ of M is a correspondence that associates to each pair of vector fields $J, K \in \mathfrak{X}(M)$ the map*

$$\mathcal{R}(J,K) : \mathfrak{X}(M) \to \mathfrak{X}(M)$$

defined by the formula

$$\mathcal{R}(J,K)L = \nabla_K \nabla_J L - \nabla_J \nabla_K L + \nabla_{[J,K]} L, \quad L \in \mathfrak{X}(M).$$

Now, if $M = \mathbb{R}^n$, then for each $J, K, L \in \mathfrak{X}(\mathbb{R}^n)$, we have that $\mathcal{R}(J,K)L = 0$. To check this we plug the vector field $L = (\ell_1, ..., \ell_n) \in \mathbb{R}^n$ into the formula given in the above Definition. Therefore, we see that, as mentioned in the introduction of this section, $\mathcal{R}$ adequately measures the amount M varies from being flat (Euclidean).

Another way of writing the formula in Definition 4.2.1 is to use local coordinates. Let $(x_1, x_2, ..., x_n)$ be a local coordinate system of M at y and let $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}$ be the corresponding standard basis vector fields. Then

$$\begin{aligned}\mathcal{R}\left(\frac{\partial}{\partial x_\eta}, \frac{\partial}{\partial x_\omega}\right)\frac{\partial}{\partial x_\nu} &= \nabla_{\frac{\partial}{\partial x_\omega}}\nabla_{\frac{\partial}{\partial x_\eta}}\frac{\partial}{\partial x_\nu} - \nabla_{\frac{\partial}{\partial x_\eta}}\nabla_{\frac{\partial}{\partial x_\omega}}\frac{\partial}{\partial x_\nu} + \nabla_{\left[\frac{\partial}{\partial x_\eta}, \frac{\partial}{\partial x_\omega}\right]}\frac{\partial}{\partial x_\nu} \\ &= \left(\nabla_{\frac{\partial}{\partial x_\omega}}\nabla_{\frac{\partial}{\partial x_\eta}} - \nabla_{\frac{\partial}{\partial x_\eta}}\nabla_{\frac{\partial}{\partial x_\omega}} + \nabla_{\left[\frac{\partial}{\partial x_\eta}, \frac{\partial}{\partial x_\omega}\right]}\right)\frac{\partial}{\partial x_\nu} \\ &= \left(\nabla_{\frac{\partial}{\partial x_\omega}}\nabla_{\frac{\partial}{\partial x_\eta}} - \nabla_{\frac{\partial}{\partial x_\eta}}\nabla_{\frac{\partial}{\partial x_\omega}}\right)\frac{\partial}{\partial x_\nu},\end{aligned}$$

where the last equality follows from the fact that $[\frac{\partial}{\partial x_\eta}, \frac{\partial}{\partial x_\omega}] = 0$.

Again, if we let $M = \mathbb{R}^n$, we get that

$$\nabla_{\frac{\partial}{\partial x_\omega}}\nabla_{\frac{\partial}{\partial x_\eta}} = \nabla_{\frac{\partial}{\partial x_\eta}}\nabla_{\frac{\partial}{\partial x_\omega}}.$$

So, in a sense, curvature measures the non-commutativity (the noninterchangeability of order of differentiation) of second-order covariant derivatives. In general, $\nabla_K\nabla_J L \neq \nabla_J\nabla_K L$. We can consider this as another interpretation of curvature, but this one is more of a formal interpretation and less of a geometric one.

In the next four propositions we study some useful algebraic properties of $\mathcal{R}$.

Proposition 4.2.1. *Let M be a Riemannian manifold, let $J, J_1, J_2, K, K_1, K_2, L, L_1, L_2 \in \mathfrak{X}(M)$ and let $g_1, g_2 \in \mathcal{D}(M)$. The curvature $\mathcal{R}$ of M satisfies the following properties:*

(a) *Bilinearity in $\mathfrak{X}(M) \times \mathfrak{X}(M)$:*

  (i) $\mathcal{R}(g_1J_1 + g_2J_2, K)L = g_1\mathcal{R}(J_1, K)L + g_2\mathcal{R}(J_2, K)L,$

  (ii) $\mathcal{R}(J, g_1K_1 + g_2K_2)L = g_1\mathcal{R}(J, K_1)L + g_2\mathcal{R}(J, K_2)L.$

(b) *Linearity in $\mathcal{R}(J, K) : \mathfrak{X}(M) \to \mathfrak{X}(M)$:*

  (iii) $\mathcal{R}(J, K)(g_1L_1 + g_2L_2) = g_1\mathcal{R}(J, K)L_1 + g_2\mathcal{R}(J, K)L_2.$

Proof. We prove these properties in the same manner in which we proved Theorem 2.3.1.

(i)

$$\begin{aligned}\mathcal{R}\left(g_1J_1+g_2J_2,K\right)L =&\nabla_K\nabla_{g_1J_1+g_2J_2}L-\nabla_{g_1J_1+g_2J_2}\nabla_KL+\nabla_{[g_1J_1+g_2J_2,K]}L\\ =&\nabla_K\left(g_1\nabla_{J_1}+g_2\nabla_{J_2}\right)L-\left(g_1\nabla_{J_1}+g_2\nabla_{J_2}\right)\nabla_KL\\ &+\nabla_{[g_1J_1,K]+[g_2J_2,K]}L\\ =&g_1\nabla_K\nabla_{J_1}L+\left(Kg_1\right)\nabla_{J_1}L+g_2\nabla_K\nabla_{J_2}L+\left(Kg_2\right)\nabla_{J_2}L\\ &-g_1\nabla_{J_1}\nabla_KL-g_2\nabla_{J_2}\nabla_KL+\nabla_{g_1[J_1,K]-(Kg_1)J_1}L\\ &+\nabla_{g_2[J_2,K]-(Kg_2)J_2}L\\ =&g_1\nabla_K\nabla_{J_1}L-g_1\nabla_{J_1}\nabla_KL+g_1\nabla_{[J_1,K]}L+g_2\nabla_K\nabla_{J_2}L\\ &-g_2\nabla_{J_2}\nabla_KL+g_2\nabla_{[J_2,K]}L\\ =&g_1\mathcal{R}\left(J_1,K\right)L+g_2\mathcal{R}\left(J_2,K\right)L.\end{aligned}$$

(ii) Similar to (i).

(iii) It is enough to show that $\mathcal{R}$ satisfies

$$\mathcal{R}(J,K)(L_1+L_2)=\mathcal{R}(J,K)L_1+\mathcal{R}(J,K)L_2$$

and

$$\mathcal{R}(J,K)(gL)=g\mathcal{R}(J,K)L.$$

$$\begin{aligned}\mathcal{R}(J,K)\left(L_1+L_2\right)=&\nabla_K\nabla_J\left(L_1+L_2\right)-\nabla_J\nabla_K\left(L_1+L_2\right)+\nabla_{[J,K]}\left(L_1+L_2\right)\\ =&\nabla_K\left(\nabla_JL_1+\nabla_JL_2\right)-\nabla_J\left(\nabla_KL_1+\nabla_KL_2\right)\\ &+\nabla_{[J,K]}L_1+\nabla_{[J,K]}L_2\\ =&\nabla_K\nabla_JL_1-\nabla_J\nabla_KL_1+\nabla_{[J,K]}L_1+\nabla_K\nabla_JL_2-\nabla_J\nabla_KL_2\\ &+\nabla_{[J,K]}L_2\\ =&\mathcal{R}(J,K)L_1+\mathcal{R}(J,K)L_2.\end{aligned}$$

$$\begin{aligned}\mathcal{R}(J,K)(gL)=&\nabla_K\nabla_J(gL)-\nabla_J\nabla_K(gL)+\nabla_{[J,K]}(gL)\\ =&\nabla_K\left(g\nabla_JL\right)+\nabla_K((Jg)L)-\nabla_J\left(g\nabla_KL\right)-\nabla_J((Kg)L)\\ &+g\nabla_{[J,K]}L+[J,K](g)L\\ =&g\nabla_K\nabla_JL+(Kg)\nabla_JL+(Jg)\nabla_KL+(K(Jg))L\\ &-g\nabla_J\nabla_KL-(Jg)\nabla_KL-(Kg)\nabla_JL-(J(Kg))L\\ &+g\nabla_{[J,K]}L+[J,K](g)L\\ =&g\nabla_K\nabla_JL-g\nabla_J\nabla_KL+((KJ-JK)g)L+g\nabla_{[J,K]}L\\ &+[J,K](g)L\\ =&g\nabla_K\nabla_JL-g\nabla_J\nabla_KL+[K,J](g)L+g\nabla_{[J,K]}L+[J,K](g)L\\ =&g\mathcal{R}(J,K)L.\end{aligned}$$

$\blacksquare$

The proposition above shows that $\mathcal{R}(J, K)L$ is multilinear over $C^\infty(M)$ in each of the three vector fields. Hence, $\mathcal{R}$ is a tensor field of type $(1, 3)$.

$\mathcal{R}$ satisfies the first Bianchi identity when the connection is symmetric as in the case of the Riemannian connection. In short, the Bianchi identity says that a certain combination of derivatives of the curvature vanishes.

Proposition 4.2.2 (First Bianchi Identity)**.** *The curvature $\mathcal{R}$ of a Riemannian manifold M with Riemannian connection ∇ satisfies:*

$$\mathcal{R}(J, K)L + \mathcal{R}(K, L)J + \mathcal{R}(L, J)K = 0.$$

Proof.

$$\begin{aligned}
&\mathcal{R}(J, K)L + \mathcal{R}(K, L)J + \mathcal{R}(L, J)K \\
&=\nabla_K\nabla_J L - \nabla_J\nabla_K L + \nabla_{[J,K]}L + \nabla_L\nabla_K J - \nabla_K\nabla_L J + \nabla_{[K,L]}J \\
&\quad + \nabla_J\nabla_L K - \nabla_L\nabla_J K + \nabla_{[L,J]}K \\
&=\nabla_K (\nabla_J L - \nabla_L J) + \nabla_J (\nabla_L K - \nabla_K L) + \nabla_L (\nabla_K J - \nabla_J K) \\
&\quad - \nabla_{[K,J]}L - \nabla_{[L,K]}J - \nabla_{[J,L]}K.
\end{aligned}$$

Applying the symmetric property of the Riemannian connection ,we obtain:

$$\begin{aligned}
&\mathcal{R}(J, K)L + \mathcal{R}(K, L)J + \mathcal{R}(L, J)K \\
&=\nabla_K[J, L] + \nabla_J[L, K] + \nabla_L[K, J] - \nabla_{[K,J]}L - \nabla_{[L,K]}J - \nabla_{[J,L]}K \\
&=[K, [J, L]] + [J, [L, K]] + [L, [K, J]] \\
&=0
\end{aligned}$$

The last equality comes from one of the properties of Lie brackets, namely, the Jacobi identity. $\blacksquare$

By using the Riemannian metric, it will be useful to change the tensor field of type $(1, 3)$ from Definition 4.2.1 to a tensor field of type (0.4) as follows:

Definition 4.2.2. *The Riemannian curvature tensor, also denoted by $\mathcal{R}$, is the tensor of type* $(0, 4)$ *defined by*

$$\mathcal{R}(J, K, L, H) = \langle \mathcal{R}(J, K)L, H\rangle.$$

Proposition 4.2.3. *Let M be a Riemannian manifold equipped with Riemannian connection ∇ and let $J, K, L, H \in \mathfrak{X}(M)$. Then the Riemannian curvature tensor satisfies the following symmetry properties:*

1. $\mathcal{R}(J,K,L,H)+\mathcal{R}(K,L,J,H)+\mathcal{R}(L,J,K,H)=0$;

2. $\mathcal{R}(J,K,L,H)=-\mathcal{R}(K,J,L,H)=-\mathcal{R}(J,K,H,L)$;

3. $\mathcal{R}(H,J,K,L)=\mathcal{R}(K,L,H,J)$.

Proof. 1. It is simply the Bianchi Identity once more, which we have already proved.

2. From Definition 4.2.1 we have that $\mathcal{R}(J,K)L = -\mathcal{R}(K,J)L$, hence $\mathcal{R}(J,K,L,H)=-\mathcal{R}(K,J,L,H)$. To prove $\mathcal{R}(J,K,L,H)=-\mathcal{R}(J,K,H,L)$ it is equivalent to proving that $\mathcal{R}(J,K,L,L)=0$. Using (∇4) we get:

$$\begin{aligned}
\mathcal{R}(J,K,L,L) =& \left\langle \nabla_K\nabla_J L - \nabla_J\nabla_K L + \nabla_{[J,K]}L, L\right\rangle\\
=& \left\langle \nabla_K\nabla_J L, L\right\rangle - \left\langle \nabla_J\nabla_K L, L\right\rangle + \left\langle \nabla_{[J,K]}L, L\right\rangle\\
=& K\left\langle \nabla_J L, L\right\rangle - \left\langle \nabla_J L, \nabla_K L\right\rangle - J\left\langle \nabla_K L, L\right\rangle + \left\langle \nabla_K L, \nabla_J L\right\rangle\\
&+\frac{1}{2}[J,K]\langle L,L\rangle\\
=& K\left\langle \nabla_J L, L\right\rangle - J\left\langle \nabla_K L, L\right\rangle + \frac{1}{2}[J,K]\langle L,L\rangle\\
=& K\left(\frac{1}{2}J\langle L,L\rangle\right) - J\left(\frac{1}{2}K\langle L,L\rangle\right) + \frac{1}{2}[J,K]\langle L,L\rangle\\
=& -\frac{1}{2}[J,K]\langle L,L\rangle + \frac{1}{2}[J,K]\langle L,L\rangle\\
=& 0.
\end{aligned}$$

3. First we write property 1 four times with indices cyclically permuted:

$$\begin{aligned}
\mathcal{R}(H,J,K,L)+\mathcal{R}(J,K,H,L)+\mathcal{R}(K,H,J,L)=0;\\
\mathcal{R}(J,K,L,H)+\mathcal{R}(K,L,J,H)+\mathcal{R}(L,J,K,H)=0;\\
\mathcal{R}(K,L,H,J)+\mathcal{R}(L,H,K,J)+\mathcal{R}(H,K,L,J)=0;\\
\mathcal{R}(L,H,J,K)+\mathcal{R}(H,J,L,K)+\mathcal{R}(J,L,H,K)=0.
\end{aligned}$$

Adding up all four equations and using the second equality of property 2, $\mathcal{R}(J,K,L,H)=-\mathcal{R}(J,K,H,L)$, we obtain:

$$\mathcal{R}(K,H,J,L)+\mathcal{R}(L,J,K,H)+\mathcal{R}(H,K,L,J)+\mathcal{R}(J,L,H,K)=0.$$

Applying both equalities of property 2 to the equation above we get:

$$2\mathcal{R}(K,H,J,L)-2\mathcal{R}(J,L,H,K)=0,$$

and, thus,

$$\mathcal{R}(K,H,J,L)=\mathcal{R}(J,L,H,K).$$

■

There is one more identity that is satisfied by the covariant derivatives of the Riemannian curvature tensor on every Riemannian manifold. It is called the second Bianchi identity.

Proposition 4.2.4 (Second Bianchi Identity)**.** *Let M be a Riemannian manifold of dimension n. The total covariant derivative of the Riemannian curvature tensor $\mathcal{R}$ satisfies the following identity:*

$$\nabla\mathcal{R}(J,K,L,H,T)+\nabla\mathcal{R}(J,K,H,T,L)+\nabla\mathcal{R}(J,K,T,L,H)=0,$$

for all $J,K,L,H,T\in\mathfrak{X}(M)$.

Proof. Since the objects involved are all tensors, it is enough to show the equality at a point $y\in M$. To show this we will use a geodesic frame at y. Let $U\subset M$ be a neighborhood of y. A geodesic frame at y is a family of vector fields $e_1,\dots,e_n\in\mathfrak{X}(U)$ which are orthonormal at every point of U and, at y, $\nabla_{e_\eta}e_\omega(y)=0$. So by choosing a geodesic frame $\{e_\eta\}$ based at y, and cyclically permuting the vector fields e_ν, e_ϱ, e_κ, we obtain

$$\begin{aligned}\nabla\mathcal{R}(e_\eta,e_\omega,e_\nu,e_\varrho,e_\kappa)(y)&=e_\kappa\langle\mathcal{R}(e_\eta,e_\omega)e_\nu,e_\varrho\rangle(y)=e_\kappa\langle\mathcal{R}(e_\nu,e_\varrho)e_\eta,e_\omega\rangle(y)\\&=\langle\nabla_{e_\kappa}\nabla_{e_\varrho}\nabla_{e_\nu}e_\eta-\nabla_{e_\kappa}\nabla_{e_\nu}\nabla_{e_\varrho}e_\eta+\nabla_{e_\kappa}\nabla_{[e_\nu,e_\varrho]}e_\eta,e_\omega\rangle(y),\end{aligned}$$

$$\begin{aligned}\nabla\mathcal{R}(e_\eta,e_\omega,e_\varrho,e_\kappa,e_\nu)(y)&=e_\nu\langle\mathcal{R}(e_\eta,e_\omega)e_\varrho,e_\kappa\rangle(y)=e_\nu\langle\mathcal{R}(e_\varrho,e_\kappa)e_\eta,e_\omega\rangle(y)\\&=\langle\nabla_{e_\nu}\nabla_{e_\kappa}\nabla_{e_\varrho}e_\eta-\nabla_{e_\nu}\nabla_{e_\varrho}\nabla_{e_\kappa}e_\eta+\nabla_{e_\nu}\nabla_{[e_\varrho,e_\kappa]}e_\eta,e_\omega\rangle(y),\end{aligned}$$

$$\begin{aligned}\nabla\mathcal{R}(e_\eta,e_\omega,e_\kappa,e_\nu,e_\varrho)(y)&=e_\varrho\langle\mathcal{R}(e_\eta,e_\omega)e_\kappa,e_\nu\rangle(y)=e_\varrho\langle\mathcal{R}(e_\kappa,e_\nu)e_\eta,e_\omega\rangle(y)\\&=\langle\nabla_{e_\varrho}\nabla_{e_\nu}\nabla_{e_\kappa}e_\eta-\nabla_{e_\varrho}\nabla_{e_\kappa}\nabla_{e_\nu}e_\eta+\nabla_{e_\varrho}\nabla_{[e_\kappa,e_\nu]}e_\eta,e_\omega\rangle(y).\end{aligned}$$

Adding the three equalities above, we find that

$$\begin{aligned}&\nabla\mathcal{R}(e_\eta,e_\omega,e_\nu,e_\varrho,e_\kappa)(y)+\nabla\mathcal{R}(e_\eta,e_\omega,e_\varrho,e_\kappa,e_\nu)(y)+\nabla\mathcal{R}(e_\eta,e_\omega,e_\kappa,e_\nu,e_\varrho)(y)\\&\quad=\langle\mathcal{R}(e_\varrho,e_\kappa)\nabla_{e_\nu}e_\eta-\nabla_{[e_\varrho,e_\kappa]}\nabla_{e_\nu}e_\eta+\mathcal{R}(e_\kappa,e_\nu)\nabla_{e_\varrho}e_\eta-\nabla_{[e_\kappa,e_\nu]}\nabla_{e_\varrho}e_\eta\\&\qquad+\mathcal{R}(e_\nu,e_\varrho)\nabla_{e_\kappa}e_\eta-\nabla_{[e_\nu,e_\varrho]}\nabla_{e_\kappa}e_\eta+\nabla_{e_\kappa}\nabla_{[e_\nu,e_\varrho]}e_\eta+\nabla_{e_\nu}\nabla_{[e_\varrho,e_\kappa]}e_\eta\\&\qquad+\nabla_{e_\varrho}\nabla_{[e_\kappa,e_\nu]}e_\eta,e_\omega\rangle(y)\\&\quad=\langle\mathcal{R}(e_\varrho,e_\kappa)\nabla_{e_\nu}e_\eta+\mathcal{R}(e_\kappa,e_\nu)\nabla_{e_\varrho}e_\eta+\mathcal{R}(e_\nu,e_\varrho)\nabla_{e_\nu}e_\eta+\mathcal{R}([e_\nu,e_\varrho],e_\kappa)e_\eta\\&\qquad-\nabla_{[[e_\nu,e_\varrho],e_\kappa]]}e_\eta+\mathcal{R}([e_\varrho,e_\kappa],e_\nu)e_\eta-\nabla_{[[e_\varrho,e_\kappa],e_\nu]]}e_\eta+\mathcal{R}([e_\kappa,e_\nu],e_\varrho)e_\eta\\&\qquad-\nabla_{[[e_\kappa,e_\nu],e_\varrho]]}e_\eta,e_\omega\rangle(y)\\&\quad=\mathcal{R}(e_\varrho,e_\kappa,\nabla_{e_\nu}e_\eta,e_\omega)(y)+\mathcal{R}(e_\kappa,e_\nu,\nabla_{e_\varrho}e_\eta,e_\omega)(y)+\mathcal{R}(e_\nu,e_\varrho,\nabla_{e_\kappa}e_\eta,e_\omega)(y)\\&\qquad+\mathcal{R}([e_\nu,e_\varrho],e_\kappa,e_\eta,e_\omega)(y)+\mathcal{R}([e_\varrho,e_\kappa],e_\nu,e_\eta,e_\omega)(y)+\mathcal{R}([e_\kappa,e_\nu],e_\varrho,e_\eta,e_\omega)(y)\\&\qquad-\langle\nabla_{([[e_\nu,e_\varrho],e_\kappa]+[[e_\varrho,e_\kappa],e_\nu]+[[e_\kappa,e_\nu],e_\varrho])}e_\eta,e_\omega\rangle(y).\end{aligned}$$

Since $\nabla_{e_\eta} e_\omega(y) = 0$ and $\mathcal{R}$ is a tensor, the first six terms all vanish at y (we note that $[e_\nu, e_\varrho](y) = \nabla_{e_\nu} e_\varrho(y) - \nabla_{e_\varrho} e_\nu(y) = 0 - 0 = 0$, etc.). The last term vanishes identically because of the Jacobi Identity:

$$[[e_\nu, e_\varrho], e_\kappa] + [[e_\varrho, e_\kappa], e_\nu] + [[e_\kappa, e_\nu], e_\varrho] = 0,$$

identically in the chosen neighborhood of y where the geodesic frame at y is defined.

By linearity, for any $J, K, L, H, T \in \mathfrak{X}(M)$ we have

$$\nabla\mathcal{R}(J, K, L, H, T)(y) + \nabla\mathcal{R}(J, K, H, T, L)(y) + \nabla\mathcal{R}(J, K, T, L, H)(y) = 0.$$

Since $\nabla\mathcal{R} : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{D}(M)$ is a tensor, the identity above is preserved under change of coordinates (the invariance under change of frames follows from linearity). Finally by the arbitrariness of $y \in M$, we conclude the second Bianchi identity

$$\nabla\mathcal{R}(J, K, L, H, T) + \nabla\mathcal{R}(J, K, H, T, L) + \nabla\mathcal{R}(J, K, T, L, H) = 0,$$

for all $J, K, L, H, T \in \mathfrak{X}(M)$. ■

We shall now express the Riemannian curvature tensor in local coordinates. Let (M, g) be a Riemannian manifold with associated Riemannian connection ∇, let (A, f) be a local coordinate system of M at y and let $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ be the corresponding standard basis vector fields. If $J = \sum_{\eta=1}^n \alpha^\eta \frac{\partial}{\partial x_\eta}$, $K = \sum_{\omega=1}^n \beta^\omega \frac{\partial}{\partial x_\omega}$ and $L = \sum_{\nu=1}^n \gamma^\nu \frac{\partial}{\partial x_\nu}$ are vector fields, we can then write, by the linearity of $\mathcal{R}$:

$$\begin{aligned}
\mathcal{R}(J, K)L &= \mathcal{R}\left(\sum_{\eta=1}^n \alpha^\eta \frac{\partial}{\partial x_\eta}, \sum_{\omega=1}^n \beta^\omega \frac{\partial}{\partial x_\omega}\right) \sum_{\nu=1}^n \gamma^\nu \frac{\partial}{\partial x_\nu} \\
&= \sum_{\eta,\omega,\nu=1}^n \alpha^\eta \beta^\omega \gamma^\nu \left(\mathcal{R}\left(\frac{\partial}{\partial x_\eta}, \frac{\partial}{\partial x_\omega}\right) \frac{\partial}{\partial x_\nu}\right) \\
&= \sum_{\eta,\omega,\nu,\kappa=1}^n \alpha^\eta \beta^\omega \gamma^\nu \mathcal{R}^\kappa_{\eta\omega\nu} \frac{\partial}{\partial x_\kappa},
\end{aligned}$$

where by $\mathcal{R}^\kappa_{\eta\omega\nu}$ we mean the components of the vector field $\mathcal{R}\left(\frac{\partial}{\partial x_\eta}, \frac{\partial}{\partial x_\omega}\right) \frac{\partial}{\partial x_\nu}$ expressed in the local coordinate system (A, f), i.e.,

$$\mathcal{R}\left(\frac{\partial}{\partial x_\eta}, \frac{\partial}{\partial x_\omega}\right) \frac{\partial}{\partial x_\nu} = \sum_{\kappa=1}^n \mathcal{R}^\kappa_{\eta\omega\nu} \frac{\partial}{\partial x_\kappa}.$$

Our aim is now to express these components in terms of the Christoffel symbols $\Gamma^{\nu}_{\eta\omega}$ of the Riemannian connection. Using the fact that $[\frac{\partial}{\partial x_\eta}, \frac{\partial}{\partial x_\omega}] = 0$, $\nabla_{\frac{\partial}{\partial x_\eta}}\left(\frac{\partial}{\partial x_\omega}\right) = \sum_{\nu=1}^{n} \Gamma^{\nu}_{\eta\omega}\frac{\partial}{\partial x_\nu}$, the definition of $\mathcal{R}$ and the properties listed in Definition 2.2.1, we obtain:

$$\begin{aligned}
\mathcal{R}\left(\frac{\partial}{\partial x_\eta}, \frac{\partial}{\partial x_\omega}\right)\frac{\partial}{\partial x_\nu} &= \nabla_{\frac{\partial}{\partial x_\omega}}\nabla_{\frac{\partial}{\partial x_\eta}}\frac{\partial}{\partial x_\nu} - \nabla_{\frac{\partial}{\partial x_\eta}}\nabla_{\frac{\partial}{\partial x_\omega}}\frac{\partial}{\partial x_\nu} \\
&= \nabla_{\frac{\partial}{\partial x_\omega}}\left(\sum_{\varrho=1}^{n}\Gamma^{\varrho}_{\eta\nu}\frac{\partial}{\partial x_\varrho}\right) - \nabla_{\frac{\partial}{\partial x_\eta}}\left(\sum_{\varrho=1}^{n}\Gamma^{\varrho}_{\omega\nu}\frac{\partial}{\partial x_\varrho}\right) \\
&= \sum_{\varrho=1}^{n}\left(\frac{\partial}{\partial x_\omega}\Gamma^{\varrho}_{\eta\nu} - \frac{\partial}{\partial x_\eta}\Gamma^{\varrho}_{\omega\nu}\right)\frac{\partial}{\partial x_\varrho} - \sum_{\kappa,\varrho=1}^{n}\left(\Gamma^{\varrho}_{\eta\nu}\Gamma^{\kappa}_{\omega\varrho} - \Gamma^{\varrho}_{\omega\nu}\Gamma^{\kappa}_{\eta\varrho}\right)\frac{\partial}{\partial x_\kappa} \\
&= \sum_{\kappa=1}^{n}\left(\frac{\partial}{\partial x_\omega}\Gamma^{\kappa}_{\eta\nu} - \frac{\partial}{\partial x_\eta}\Gamma^{\kappa}_{\omega\nu} + \sum_{\varrho=1}^{n}\Gamma^{\varrho}_{\eta\nu}\Gamma^{\kappa}_{\omega\varrho} - \sum_{\varrho=1}^{n}\Gamma^{\varrho}_{\omega\nu}\Gamma^{\kappa}_{\eta\varrho}\right)\frac{\partial}{\partial x_\kappa}.
\end{aligned}$$

Therefore,

$$\mathcal{R}^{\kappa}_{\eta\omega\nu} = \frac{\partial}{\partial x_\omega}\Gamma^{\kappa}_{\eta\nu} - \frac{\partial}{\partial x_\eta}\Gamma^{\kappa}_{\omega\nu} + \sum_{l=1}^{n}\Gamma^{\varrho}_{\eta\nu}\Gamma^{\kappa}_{\omega\varrho} - \sum_{l=1}^{n}\Gamma^{\varrho}_{\omega\nu}\Gamma^{\kappa}_{\eta\varrho}.$$

Using the fact that in local coordinates $g_{\eta\omega} = \langle\frac{\partial}{\partial x_\eta}, \frac{\partial}{\partial x_\omega}\rangle$ and using the bilinearity of $\langle\ ,\ \rangle$, we get the components of the Riemannian curvature tensor:

$$\begin{aligned}
\left\langle\mathcal{R}\left(\frac{\partial}{\partial x_\eta}, \frac{\partial}{\partial x_\omega}\right)\frac{\partial}{\partial x_\nu}, \frac{\partial}{\partial x_\kappa}\right\rangle &= \left\langle\sum_{\mu=1}^{n}\mathcal{R}^{\mu}_{\eta\omega\nu}\frac{\partial}{\partial x_\mu}, \frac{\partial}{\partial x_\kappa}\right\rangle \\
&= \sum_{\mu=1}^{n}\mathcal{R}^{\mu}_{\eta\omega\nu}\left\langle\frac{\partial}{\partial x_\mu}, \frac{\partial}{\partial x_\kappa}\right\rangle \\
&= \sum_{\mu=1}^{n}\mathcal{R}^{\mu}_{\eta\omega\nu}g_{\mu\kappa} \\
&= \mathcal{R}_{\eta\omega\nu\kappa}.
\end{aligned}$$

We can then rewrite the symmetry properties from Proposition 4.2.3 using local coordinates:

$$\mathcal{R}_{\eta\omega\nu\kappa} + \mathcal{R}_{\omega\nu\eta\kappa} + \mathcal{R}_{\nu\eta\omega\kappa} = 0;$$

$$\mathcal{R}_{\eta\omega\nu\kappa} = -\mathcal{R}_{\omega\eta\nu\kappa} = -\mathcal{R}_{\eta\omega\kappa\nu};$$

$$\mathcal{R}_{\eta\omega\nu\kappa} = \mathcal{R}_{\nu\kappa\eta\omega}.$$

4.3 Sectional Curvature

The Riemannian curvature tensor is an intricate object which is difficult to work with directly and whose full geometric interpretation is still not easily understood. Thus, we look for notions of curvature of Riemannian manifolds which are simpler and have a more explicit geometric interpretation. One of the simpler notions is sectional curvature. Sectional curvature is an important one because it is the natural generalization of Gauss curvature for surfaces in $\mathbb{R}^3$ and we can recover the Riemannian curvature tensor from it, in other words, none of the information carried by the Riemannian curvature tensor is lost by considering only sectional curvatures.

Sectional curvature is defined on plane sections (tangent planes).

Definition 4.3.1. *Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold and let $y \in M$ be a point. A plane section or tangent plane at y, denoted by $\mathcal{P}$, is defined to be the 2-dimensional subspace of the tangent space T_yM. The set*

$$G_2(T_yM) = \{\mathcal{P} \mid \mathcal{P} \text{ is a section of } T_yM\}$$

of sections is called the Grassmannian of 2-planes at y.

Definition 4.3.2. *Let $(M, \langle\ ,\ \rangle)$ be a Riemannian manifold, let y be any point on M and let $\mathcal{P} \subset T_yM$ be a 2-dimensional subspace of the tangent space T_yM spanned by the linearly independent tangent vectors $J_y, K_y \in T_yM$. Then, the sectional curvature of $\mathcal{P}$, denoted $\mathcal{K}(\mathcal{P})$, is given by*

$$\mathcal{K}(\mathcal{P}) = \mathcal{K}(J_y, K_y) = \frac{\mathcal{R}(J_y, K_y, J_y, K_y)}{\langle J_y, J_y\rangle\langle K_y, K_y\rangle - \langle J_y, K_y\rangle^2}. \tag{11}$$

The term in the denominator of equation (11) represents the square of the area of the parallelogram in T_yM generated by the vectors $J_y, K_y \in T_yM$. It is needed as a normalization factor so that the definition of sectional curvature above is independent of the choice of the linearly independent vectors J_y, K_y. It must only depend on the plane section in T_yM spanned by J_y, K_y. The following proposition demonstrates all of this.

Proposition 4.3.1. *Let $\{J_y, K_y\}$ and $\left\{\bar{J}_y, \bar{K}_y\right\}$ be two distinct pairs of linearly independent vectors on the tangent space T_yM such that the two plane sections span(J_y, K_y) and span$(\bar{J}_y, \bar{K}_y)$ are equal. Then*

$$\mathcal{K}(J_y, K_y) = \mathcal{K}(\bar{J}_y, \bar{K}_y).$$

Hence the sectional curvature is not dependent on the the choice of spanning vectors.

Proof. First we write $\{\bar{J}_y, \bar{K}_y\}$ in terms of $\{J_y, K_y\}$:

$$\bar{J}_y = aJ_y + bK_y, \quad \bar{K}_y = cJ_y + dK_y, \text{ where } a, b, c, d \in \mathbb{R} \text{ with } ad - bc \neq 0.$$

Then we must compute $\langle \bar{J}_y, \bar{J}_y\rangle\langle \bar{K}_y, \bar{K}_y\rangle - \langle \bar{J}_y, \bar{K}_y\rangle^2$ and $\mathcal{R}(\bar{J}_y, \bar{K}_y, \bar{J}_y, \bar{K}_y)$. These computations are very easy to do but are very lengthy and so we will skip it right to the very last steps. Using the bilinearity of $\langle\ ,\ \rangle$ we get:

$$\langle \bar{J}_y, \bar{J}_y\rangle\langle \bar{K}_y, \bar{K}_y\rangle - \langle \bar{J}_y, \bar{K}_y\rangle^2 = (ad - bc)^2(\langle J_y, J_y\rangle\langle K_y, K_y\rangle - \langle J_y, K_y\rangle^2).$$

Using the symmetric properties of $\mathcal{R}$ we get:

$$\mathcal{R}(\bar{J}_y, \bar{K}_y, \bar{J}_y, \bar{K}_y) = (ad - bc)^2\mathcal{R}(J_y, K_y, J_y, K_y).$$

Therefore,

$$\mathcal{K}(\bar{J}_y, \bar{K}_y) = \frac{\mathcal{R}(J_y, K_y, J_y, K_y)}{\langle J_y, J_y\rangle\langle K_y, K_y\rangle - \langle J_y, K_y\rangle^2} = \mathcal{K}(J_y, K_y).$$

■

Remark 4.3.1. *Since $J_y, K_y \in T_yM$ are linearly independent vectors we have that $\langle J_y, J_y\rangle\langle K_y, K_y\rangle - \langle J_y, K_y\rangle^2 \neq 0$ and so we are dealing with a non-degenerate plane section. We also note that if (J_y, K_y) is an orthonormal basis then $\langle J_y, J_y\rangle\langle K_y, K_y\rangle - \langle J_y, K_y\rangle^2 = 1$.*

As referenced in the introduction of this subsection, the significance of sectional curvature originates from the fact that knowing $\mathcal{K}(\mathcal{P})$, fol all plane sections $\mathcal{P}$ of T_yM, completely determines the Riemannian curvature tensor $\mathcal{R}$ at y, that is, the knowledge of $\mathcal{K}$ is equivalent to that of $\mathcal{R}$. This is a purely algebraic fact:

Lemma 4.3.1. *Let W be a vector space of dimension ≥ 2 with an inner product $\langle\ ,\ \rangle$. Let $\mathcal{R}_1 : W \times W \times W \to W$ and $\mathcal{R}_2 : W \times W \times W \to W$ be tri-linear mappings such that properties 1, 2 and 3 of Proposition 4.2.3 are satisfied by*

$$\mathcal{R}_1(J_y, K_y, L_y, T_y) = \langle \mathcal{R}_1(J_y, K_y)L_y, T_y\rangle,$$
$$\mathcal{R}_2(J_y, K_y, L_y, T_y) = \langle \mathcal{R}_2(J_y, K_y)L_y, T_y\rangle.$$

If for every pair of linearly independent vectors $J_y, K_y \in W$,

$$\frac{\mathcal{R}_1(J_y, K_y, J_y, K_y)}{\langle J_y, J_y\rangle\langle K_y, K_y\rangle - \langle J_y, K_y\rangle^2} = \frac{\mathcal{R}_2(J_y, K_y, J_y, K_y)}{\langle J_y, J_y\rangle\langle K_y, K_y\rangle - \langle J_y, K_y\rangle^2},$$

then $\mathcal{R}_1 = \mathcal{R}_2$.

Proof. It is enough to show that $\mathcal{R}_1(J_y, K_y, L_y, T_y) = \mathcal{R}_2(J_y, K_y, L_y, T_y)$ for all $J_y, K_y, L_y, T_y \in W$. Now, since $\mathcal{K}_1(J_y, K_y) = \mathcal{K}_2(J_y, K_y)$ by hypothesis, we must have that the first line holds, that is, for all $J_y, K_y \in W$, $\mathcal{R}_1(J_y, K_y, J_y, K_y) = \mathcal{R}_2(J_y, K_y, J_y, K_y)$. Then

$$\mathcal{R}_1(J_y + L_y, K_y, J_y + L_y, K_y) = \mathcal{R}_2(J_y + L_y, K_y, J_y + L_y, K_y).$$

Using linearity of $\mathcal{R}_1$ and $\mathcal{R}_2$ to expand the above equation and then using property 3 of proposition 4.2.3, we get:

$$\begin{aligned}&\mathcal{R}_1(J_y, K_y, J_y, K_y) + 2\mathcal{R}_1(J_y, K_y, L_y, K_y) + \mathcal{R}_1(L_y, K_y, L_y, K_y)\\ &= \mathcal{R}_2(J_y, K_y, J_y, K_y) + 2\mathcal{R}_2(J_y, K_y, L_y, K_y) + \mathcal{R}_2(L_y, K_y, L_y, K_y),\end{aligned}$$

therefore, for all $J_y, K_y, L_y \in W$,

$$\mathcal{R}_1(J_y, K_y, L_y, K_y) = \mathcal{R}_2(J_y, K_y, L_y, K_y).$$

Using what we have just proved and using properties 2 and 3 from proposition 4.2.3, we apply the same procedure as above but now in the second and forth argument:

$$\begin{aligned}&\mathcal{R}_1(J_y, K_y + T_y, L_y, K_y + T_y) = \mathcal{R}_2(J_y, K_y + T_y, L_y, K_y + T_y)\\ \Rightarrow\ &\mathcal{R}_1(J_y, K_y, L_y, T_y) + \mathcal{R}_1(J_y, T_y, L_y, K_y) = \mathcal{R}_2(J_y, K_y, L_y, T_y) + \mathcal{R}_2(J_y, T_y, L_y, K_y)\\ \Rightarrow\ &\mathcal{R}_1(J_y, K_y, L_y, T_y) - \mathcal{R}_2(J_y, K_y, L_y, T_y) = \mathcal{R}_1(K_y, L_y, J_y, T_y) - \mathcal{R}_2(K_y, L_y, J_y, T_y).\end{aligned}$$

So we see that, the expression $\mathcal{R}_1(J_y, K_y, L_y, T_y) - \mathcal{R}_2(J_y, K_y, L_y, T_y)$ is invariant under cyclic permutations of the first three arguments, namely J_y, K_y, and L_y. But, by property 1 of Proposition 4.2.3, the cyclic sum is zero, thus we have

$$3[\mathcal{R}_1(J_y, K_y, L_y, T_y) - \mathcal{R}_2(J_y, K_y, L_y, T_y)] = 0,$$

hence, for all $J_y, K_y, L_y, T_y \in W$,

$$\mathcal{R}_1(J_y, K_y, L_y, T_y) = \mathcal{R}_2(J_y, K_y, L_y, T_y).$$

■

In the case of a 2-dimensional manifold, that is, a surface Z in $\mathbb{R}^3$, Definition 4.3.2 takes a special form. The tangent space T_yZ of Z is 2-dimensional. This implies that in this situation there is just a single section $\mathcal{P}$ at $y \in Z$ namely the entire 2-dimensional tangent plane T_yZ. Hence, for surfaces in $\mathbb{R}^3$, the number $\mathcal{K}(\mathcal{P})$ is equal to its Gaussian curvature.

4.4 Ricci Curvature and Scalar Curvature

Albeit sectional curvature is simpler to work it than the Riemannian curvature tensor, it is still reasonably complex since it determines the latter and thus it has an identical number of components. In this subsection we introduce, two easier and weaker notions of curvature, namely Ricci curvature and scalar curvature. We will see that, in some sense, Ricci and scalar curvature, curtail the information encrypted in the Riemannian curvature tensor, in other words, Ricci and scalar curvature has less information in general than the Riemannian curvature tensor but they are more manageable.

Definition 4.4.1. *The Ricci curvature tensor of a Riemannian manifold M at a point $y \in M$, denoted by* Ric *or* Rc*, is the bilinear map* $\mathrm{Ric}_y : T_yM \times T_yM \to \mathbb{R}$ *given by*

$$\mathrm{Ric}_y(J_y, L_y) = \mathrm{trace}(K_y \mapsto \mathcal{R}_y(J_y, K_y)L_y),$$

where J, L is any pair of tangent vectors in T_yM.

So the Ricci curvature tensor associates to any pair of tangent vectors J_y, L_y in T_yM at the point y, the trace of the linear endomorphism $K_y \mapsto \mathcal{R}_y(J_y, K_y)L_y$. The components of the Ricci curvature tensor relative to a local coordinate system are described by: $\mathrm{Ric}_{\eta\omega} = \mathcal{R}^{\nu}_{\nu\eta\omega} = g^{\nu\kappa}\mathcal{R}_{\nu\eta\omega\kappa}$, where $g_{\eta\omega} = \langle \frac{\partial}{\partial x_\eta}, \frac{\partial}{\partial x_\omega} \rangle$ and $g^{\nu\kappa}$ is the inverse matrix of $g_{\nu\kappa}$. Choosing any orthonormal basis $\{e_1, \ldots, e_n\}$ of T_yM, we obtain:

$$\mathrm{Ric}_y(J_y, L_y) = \sum_{\eta=1}^{n} \langle \mathcal{R}_y(J_y, e_\eta)L_y, e_\eta \rangle.$$

Lemma 4.4.1. *The Ricci curvature tensor is a symmetric tensor of type (0,2).*

Proof. Using the definition of Ric and the symmetric properties of $\mathcal{R}$ we get the desired result:

$$\begin{aligned}
\mathrm{Ric}_y(J_y, L_y) &= \sum_{\eta=1}^{n} \langle \mathcal{R}_y(e_\eta, J_y)L_y, e_\eta \rangle \\
&= \sum_{\eta=1}^{n} \langle \mathcal{R}_y(L_y, e_\eta)e_\eta, J_y \rangle \\
&= \sum_{\eta=1}^{n} \langle \mathcal{R}_y(e_\eta, L_y)J_y, e_\eta \rangle \\
&= \mathrm{Ric}_y(L_y, J_y).
\end{aligned}$$

■

Therefore we observe that the Ricci curvature tensor is a symmetric bi-linear form. Ric can likewise be defined as the symmetric tensor of type $(1,1)$:

$$\mathrm{Ric}(J) = \sum_{\eta=1}^{n} \mathcal{R}(J, e_\eta)e_\eta.$$

Definition 4.4.2. *The scalar curvature of a Riemannian manifold M, denoted by* Sc *or just* S*, is defined as the trace of the Ricci curvature tensor with respect to Riemannian metric g:*

$$\mathrm{Sc} = \mathrm{trace}(\mathrm{Ric}).$$

Scalar curvature is a function $\mathrm{Sc} : M \to \mathbb{R}$, since it depends only on the point y in M. So in summary we see that sectional curvature depends on a point and a 2-dimensional subspace of that point's tangent space, Ricci curvature depends on a point and on a tangent vector and scalar curvature just relies upon a point.

The components of the scalar curvature relative to a local coordinate system are given by: $\mathrm{Sc} = g^{\eta\omega}\,\mathrm{Ric}_{\eta\omega} = g^{\eta\omega}\mathcal{R}_{\eta\omega}$. Choosing any orthonormal basis $\{e_1, \ldots, e_n\}$ of T_yM, we obtain:

$$\begin{aligned}
\mathrm{Sc} &= \mathrm{trace}(\mathrm{Ric}) \\
&= \sum_{\omega=1}^{n} \langle \mathrm{Ric}(e_\omega), e_\omega \rangle \\
&= \sum_{\eta,\omega=1}^{n} \langle \mathcal{R}(e_\eta, e_\omega)e_\omega, e_\eta \rangle.
\end{aligned}$$

Corollary 4.4.1. *Let M be an n-dimensional Riemannian manifold of constant sectional curvature k. Then the Ricci curvature tensor and the scalar curvature are given by the following formulas:*

1. $\mathrm{Ric} = (n-1)k$;

2. $\mathrm{Sc} = n(n-1)k$.

We shall prove the above Corollary in Section 6, where Riemannian manifolds with constant sectional curvature are dealt with, in thorough detail.

5 Jacobi Fields

5.1 Introduction

In this section we study how curvature influences the conduct of nearby geodesics. Generally speaking, a manifold with positive curvature makes nearby geodesics converge with time, while manifolds with negative curvature makes nearby geodesics move away from each other with time. Jacobi fields supply tools to study this impact of curvature on the behavior of nearby geodesics. They can likewise be utilized to identify points where the exponential map $\exp_y$ fails to be a local diffeomorphism. At the end of this section, we prove one of the main results of this dissertation, namely, Hadamard's theorem.

5.2 The Jacobi Equation

Definition 5.2.1. *Let M be a Riemannian manifold and let $\phi : [0, b] \to M$ be a geodesic in M. A vector field $\mathcal{J}$ along ϕ is said to be a Jacobi field if, for all $t \in [0, b]$, it satisfies the following equation:*

$$\frac{D^2}{dt^2}\mathcal{J}(t) + \mathcal{R}(\phi'(t), \mathcal{J}(t))\phi'(t) = 0. \tag{12}$$

Equation (12) is called the Jacobi equation. The existence and uniqueness of Jacobi fields is assured by the next proposition.

Proposition 5.2.1. *Let M be a Riemannian manifold, let y be a point in M and let $\phi : [0, b] \to M$ be a geodesic in M with $\phi(0) = y$. Then, for any given pair of tangent vectors $J_y, K_y \in T_yM$, there exists a unique Jacobi field $\mathcal{J}$ along ϕ satisfying the initial conditions $\mathcal{J}(0) = J_y$ and $\frac{D\mathcal{J}}{dt}(0) = K_y$.*

Proof. First, we pick any orthonormal basis $(e_\eta)_{1\leq\eta\leq n}$ of T_yM such that $e_1 = \phi'(0)/\|\phi'(0)\|$. Next we parallel transport the vectors e_η along ϕ to obtain parallel orthonormal vector fields $e_1(t), \ldots, e_n(t)$ along ϕ. Then, we can express any arbitrary vector field $\mathcal{J}$ along ϕ as a linear combination of the $e_\eta(t)$:

$$\mathcal{J}(t) = \sum_{\eta=1}^{n} q_\eta(t)e_\eta(t),$$

for some C^∞ real-valued functions q_η. By definition, $\mathcal{J}$ is a Jacobi field if and only if it satisfies equation (12). Since the $e_\eta(t)$ are parallel along ϕ, we have

$$\frac{D^2}{dt^2}\mathcal{J}(t) = \sum_{\eta=1}^{n} q''_\eta(t)e_\eta(t). \tag{A}$$

For the Riemannian curvature tensor we have

$$\mathcal{R}(\phi', \mathcal{J})\phi' = \sum_{\eta=1}^{n} \langle \mathcal{R}(\phi', \mathcal{J})\phi', e_\eta \rangle e_\eta = \sum_{\eta=1}^{n} \left(\sum_{\omega=1}^{n} q_\omega \langle \mathcal{R}\left(\phi', e_\omega\right) \phi', e_\eta \rangle \right) e_\eta. \tag{B}$$

Substituting (A) and (B) into (12), we conclude that $\mathcal{J}$ is a Jacobi field if and only if

$$q_\eta'' + \sum_{\omega=1}^{n} q_\omega \langle \mathcal{R}(\phi', e_\omega)\phi', e_\eta \rangle = 0 \text{ for all } \eta = 1, \ldots, n. \tag{C}$$

And so we see that if we localize the Jacobi equation by using a local coordinate system, the Jacobi equation is locally equivalent to a linear system of second-order ordinary differential equations (C). By the existence and uniqueness theorem for solutions of second-order ordinary differential equations, for each pair of vectors, J_y, K_y in T_yM, there exists a unique Jacobi field such that $\mathcal{J}(0) = J_y$ and $\frac{D\mathcal{J}}{dt}(0) = K_y$. ∎

5.3 The Form of Jacobi Fields

One systematic method of constructing a Jacobi field along a geodesic is to use the exponential map in the following way: Let M be Riemannian manifold, let $y \in M$, let $n \in T_yM$ so that $\exp_y$ is defined at $n \in T_yM$ and let $m \in T_n(T_yM)$. We can construct a Jacobi field along the geodesic $\phi : [0,1] \to M$, defined by $\phi(t) = \exp_y tn$. For that, we define a parametrized surface $q : [0,1] \times [-\epsilon, \epsilon] \to M$ by $q(t,u) = \exp_y tn(u)$, where $n(u)$ is a curve in T_yM such that $n(0) = n$, $n'(0) = m$, and take $\mathcal{J}(t) = \frac{\partial q}{\partial u}(t,0)$. More details on this can be found in M. do Carmo [3, p. 110]. In this subsection, we are going to prove that this is fundamentally the only method of building Jacobi fields along the geodesic ϕ such that $\mathcal{J}(0) = 0$. More accurately, we have the subsequent proposition which essentially says that $\mathcal{J}$ is a Jacobi field satisfying $\mathcal{J}(0) = 0$ if and only if $\mathcal{J}(t) = \frac{\partial}{\partial u} \exp_y(tn(u))|_{u=0}$ where $n : (-\epsilon, \epsilon) \to T_yM$ such that $n(0) = \phi'(0)$.

Proposition 5.3.1. *Let $\mathcal{J}$ be a Jacobi field along a geodesic $\phi : [0,b] \to M$ such that $\mathcal{J}(0) = 0$. Set $\phi'(0) = n$ and $\frac{D\mathcal{J}}{dt}(0) = m$. Regard m as a constituent of $T_{bn}(T_{\phi(0)}M)$ and build up a curve $n(u)$ in $T_{\phi(0)}M$ satisfying $n(0) = bn$, $n'(0) = m$. Set $q(t,u) = \exp_y(\frac{t}{b}n(u))$ with $y = \phi(0)$, and define a Jacobi field $\widetilde{\mathcal{J}}$ by $\widetilde{\mathcal{J}}(t) = \frac{\partial q}{\partial u}(t,0)$. Then $\widetilde{\mathcal{J}} = \mathcal{J}$ on $[0,b]$.*

Proof. Proposition 5.2.1 suggests that it will suffice to prove that $\mathcal{J}(0) = \widetilde{\mathcal{J}}(0) = 0$ and that $\frac{D\mathcal{J}}{dt}(0) = \frac{D\widetilde{\mathcal{J}}}{dt}(0) = m$. From the hypothesis of the

Proposition, we have,

$$\widetilde{\mathcal{J}}(t) = \frac{\partial q}{\partial u}(t,u)\bigg|_{u=0},$$

so,

$$\begin{aligned}
\widetilde{\mathcal{J}}(t) &= \frac{\partial q(t,u)}{\partial u}\bigg|_{u=0} \\
&= \frac{\partial}{\partial u}\left(\exp_t(tn(u))\right)\bigg|_{u=0} \\
&= (d\exp_y)_{tn(u)}\left(t\frac{Dn}{du}\right)\bigg|_{u=0} \\
&= (d\exp_y)_{tn}\left(t\frac{Dn}{du}\right)\bigg|_{u=0} \\
&= (d\exp_y)_{tn}(tm).
\end{aligned}$$

For $t=0$, this yields $\widetilde{\mathcal{J}}(0) = (d\exp_y)_0(0) = 0 = \mathcal{J}(0)$. Where we have used the fact that $(d\exp_y)_0(m) = m$ for all m. We obtained this fact in the proof of Proposition 3.4.1. Moreover, again for $u=0$, we have

$$\begin{aligned}
\frac{D}{dt}\frac{\partial q}{\partial u}(t,0) = \frac{D}{dt}((d\exp_y)_{tn}(tm)) &= \frac{D}{dt}(t(d\exp_y)_{tn}(m)) \\
&= (d\exp_y)_{tn}(m) + t\frac{D}{dt}((d\exp_y)_{tn}(m)),
\end{aligned}$$

thus, for $t=0$,

$$\frac{D\widetilde{\mathcal{J}}}{dt}(0) = \frac{D}{dt}\frac{\partial q}{\partial u}(0,0) = (d\exp_y)_0(m) = m = \frac{D\mathcal{J}}{dt}(0).$$

And so we have shown that $\mathcal{J}(0) = \widetilde{\mathcal{J}}(0) = 0$ and that $\frac{D\mathcal{J}}{dt}(0) = \frac{D\widetilde{\mathcal{J}}}{dt}(0) = m$, this concludes the proof. ■

Corollary 5.3.1. *Let $\mathcal{J}$ be a Jacobi field along the geodesic $\phi : [0,b] \to M$ such that $\mathcal{J}(0) = 0$. Then, for $t \in [0,b]$, we have*

$$\mathcal{J}(t) = (d\exp_y)_{t\phi'(0)}(t\mathcal{J}'(0)).$$

Proof. Follows immediately from the Proposition above. ■

5.4 Effect of Curvature on Geodesics

As mentioned in the introduction of this section, Jacobi fields may be used as a tool to acquire a relationship among the spreading of geodesics that start from the same point and the curvature at this point. In this subsection we see how this is done.

Proposition 5.4.1. *Let y be a point in the Riemannian manifold M, let $\phi : [0,b] \to M$ be a geodesic satisfying $\phi(0) = y$, $\phi'(0) = n$, let $m \in T_n(T_yM)$ such that $|m| = 1$ and let $\mathcal{J}$ be a Jacobi field along ϕ given by*

$$\mathcal{J}(t) = (d\exp_y)_{tn}(tm), \quad t \in [0,b]. \tag{D}$$

Then the Taylor expansion of $|\mathcal{J}(t)|^2$ about $t = 0$ is given by

$$|\mathcal{J}(t)|^2 = t^2 - \frac{1}{3}\langle \mathcal{R}(n,m)n, m\rangle t^4 + O(t), \tag{E}$$

where $\lim_{t\to 0} \frac{O(t)}{t^4} = 0$.

Proof. To show (E) we need to compute the first four coefficients of the Taylor expansion. We start by computing the first three coefficients using the fact that $\mathcal{J}(0) = (d\exp_y)_0(0) = 0$ and that $\frac{D\mathcal{J}}{dt}(0) = m$.

$$\begin{aligned}
\langle \mathcal{J},\mathcal{J}\rangle(0) &= \langle \mathcal{J}(0),\mathcal{J}(0)\rangle = \langle 0,0\rangle = 0,\\
\langle \mathcal{J},\mathcal{J}\rangle'(0) &= \left(\left\langle \frac{D\mathcal{J}}{dt},\mathcal{J}\right\rangle + \left\langle \mathcal{J},\frac{D\mathcal{J}}{dt}\right\rangle\right)(0)\\
&= 2\left\langle \mathcal{J},\frac{D\mathcal{J}}{dt}\right\rangle(0) = 2\left\langle \mathcal{J}(0),\frac{D\mathcal{J}}{dt}(0)\right\rangle = 2\langle 0,m\rangle = 0,\\
\langle \mathcal{J},\mathcal{J}\rangle''(0) &= 2\left\langle \frac{D\mathcal{J}}{dt},\frac{D\mathcal{J}}{dt}\right\rangle(0) + 2\underbrace{\left\langle \frac{D^2\mathcal{J}}{dt^2},\mathcal{J}\right\rangle(0)}_{=0} = 2\underbrace{\langle m,m\rangle}_{=1} = 2,\\
\langle \mathcal{J},\mathcal{J}\rangle'''(0) &= 6\underbrace{\left\langle \frac{D\mathcal{J}}{dt},\frac{D^2\mathcal{J}}{dt^2}\right\rangle(0)}_{=0} + 2\underbrace{\left\langle \frac{D^3\mathcal{J}}{dt^3},\mathcal{J}\right\rangle(0)}_{=0} = 0,
\end{aligned}$$

where for the third coefficient we have also used the fact that $\mathcal{J}$ is a Jacobi field and so $\frac{D^2\mathcal{J}}{dt^2}(0) = -\mathcal{R}(\phi'(0),\mathcal{J}(0))\phi'(0) = 0$. To compute the fourth coefficient we must first prove the following:

$$\frac{D}{dt}[\mathcal{R}(\phi',\mathcal{J})\phi'](0) = \mathcal{R}\left(\phi',\frac{D\mathcal{J}}{dt}\right)\phi'(0). \tag{F}$$

Since $\frac{d}{dt}\langle K,L\rangle = \langle \frac{DK}{dt},L\rangle + \langle K,\frac{DL}{dt}\rangle$, for any H we have

$$\begin{aligned}
\frac{d}{dt}\langle \mathcal{R}(\phi',H)\phi',\mathcal{J}\rangle &= \frac{d}{dt}\langle \mathcal{R}(\phi',\mathcal{J})\phi',H\rangle\\
&= \left\langle \frac{D}{dt}\mathcal{R}(\phi',\mathcal{J})\phi',H\right\rangle + \left\langle \mathcal{R}(\phi',\mathcal{J})\phi',\frac{DH}{dt}\right\rangle,
\end{aligned}$$

so we get

$$
\begin{aligned}
\left\langle \frac{D}{dt}(\mathcal{R}(\phi',\mathcal{J})\phi'), H \right\rangle &= \frac{d}{dt}\left\langle \mathcal{R}(\phi',H)\phi',\mathcal{J} \right\rangle - \left\langle \mathcal{R}(\phi',\mathcal{J})\phi', \frac{DH}{dt} \right\rangle \\
&= \left\langle \frac{D}{dt}\mathcal{R}(\phi',H)\phi',\mathcal{J} \right\rangle + \left\langle \mathcal{R}(\phi',H)\phi', \frac{D\mathcal{J}}{dt} \right\rangle \\
&\quad - \left\langle \mathcal{R}(\phi',\mathcal{J})\phi', \frac{DH}{dt} \right\rangle \\
&= \left\langle \mathcal{R}\left(\phi', \frac{D\mathcal{J}}{dt}\right)\phi', H \right\rangle + \left\langle \frac{D}{dt}\mathcal{R}(\phi',H)\phi',\mathcal{J} \right\rangle \\
&\quad - \left\langle \mathcal{R}(\phi',\mathcal{J})\phi', \frac{DH}{dt} \right\rangle,
\end{aligned}
$$

which for $t = 0$ gives us (F), since $\mathcal{J}(0) = 0$ and the equality above is true for every H. Differentiating the Jacobi equation $\mathcal{J}'' + \mathcal{R}(\phi',\mathcal{J})\phi' = 0$ we obtain

$$
\begin{aligned}
0 &= \mathcal{J}''' + \frac{D}{dt}[\mathcal{R}(\phi',\mathcal{J})\phi'] \\
&= \mathcal{J}''' + \mathcal{R}(\phi',\mathcal{J}')\phi',
\end{aligned}
$$

at $t = 0$ this yields $\mathcal{J}'''(0) = -\mathcal{R}(\phi'(0), \frac{D\mathcal{J}}{dt}(0))\phi'(0) = -\mathcal{R}(n,m)n$. Finally we are able to compute the fourth coefficient:

$$
\begin{aligned}
\langle \mathcal{J},\mathcal{J}\rangle''''(0) &= 8\left\langle \frac{D\mathcal{J}}{dt}, \frac{D^3\mathcal{J}}{dt^3} \right\rangle(0) + 6\underbrace{\left\langle \frac{D^2\mathcal{J}}{dt^2}, \frac{D^2\mathcal{J}}{dt^2} \right\rangle(0)}_{=0} + 2\underbrace{\left\langle \frac{D^4\mathcal{J}}{dt^4}, \mathcal{J} \right\rangle(0)}_{=0} \\
&= -8\left\langle \frac{D\mathcal{J}}{dt}, \mathcal{R}\left(\gamma', \frac{D\mathcal{J}}{dt}\right)\phi' \right\rangle(0) \\
&= -8\langle \mathcal{R}(n,m)n, m\rangle.
\end{aligned}
$$

Putting these coefficients together, we obtain (E):

$$
\begin{aligned}
|\mathcal{J}(t)|^2 &= 2\frac{t^2}{2!} - \frac{8}{4!}\langle \mathcal{R}(n,m)n,m\rangle t^4 + O(t) \\
&= t^2 - \frac{1}{3}\langle \mathcal{R}(n,m)n,m\rangle t^4 + O(t).
\end{aligned}
$$

■

Corollary 5.4.1. *Let $\phi : [0,c] \to M$ be a geodesic on the Riemannian manifold M which is parametrized by arc length and let $\langle m,n\rangle = 0$, then the term*

$\langle \mathcal{R}(n,m)n,m\rangle$ is the sectional curvature at y with respect to the plane section π spanned by n and m. Thus, in this case, we have that

$$|\mathcal{J}(t)|^2 = t^2 - \frac{1}{3}\mathcal{K}(y,\pi)t^4 + O(t) \tag{G}$$

and that

$$|\mathcal{J}(t)| = t - \frac{1}{6}\mathcal{K}(y,\pi)t^3 + \overline{O}(t), \quad \lim_{t\to 0}\frac{\overline{O}(t)}{t^3} = 0. \tag{H}$$

Proof. From the hypothesis, vectors $n, m \in T_yM$ are orthonormal, thus $\langle \mathcal{R}(n,m)m,n\rangle = \mathcal{K}(n,m)$, and hence (G) follows. So (G) is an immediate application of Proposition 5.4.1. For (H) we simply need to correlate the coefficients of the Taylor expansion with the coefficients of the Taylor expansion raised to the power of two. ■

It is equation (H) which basically accommodates the relation between curvature and geodesics. To see this we need first to make some observations. Let q be a parametrized surface given by

$$q(t,u) = \exp_y tn(u), \quad 0 \le t \le \lambda, \quad -\epsilon < u < \epsilon,$$

where λ is picked to be really small so that $\exp_y tn(u)$ is indeed defined and $n(u)$ is a curve in T_yM such that $|n(u)| = 1$, $n(0) = n$, $n'(0) = m$ and $|m| = 1$.

Firstly, we observe that the rays $t \mapsto tn(u)$, $0 \le t \le \lambda$, originating from the origin $0 \in T_yM$, diverge from the ray $t \mapsto tn(0)$ with velocity:

$$\left|\left(\frac{\partial}{\partial u}tn(u)\right)(0)\right| = \left|t\left(\frac{\partial}{\partial u}n(u)\right)(0)\right| = |tn'(0)| = |tm| = t.$$

Secondly, we observe from Equation (H) that the geodesics $t \mapsto \exp_y(tn(u))$ diverge from the geodesic $\phi(t) = \exp_y tn(0)$ with a velocity that is distinct from t by a term of the third order of t, given by $-\frac{1}{6}\mathcal{K}(y,\pi)t^3$.

And so, specifically, we see that, locally, the geodesics spread apart more than the rays in T_yM for $\mathcal{K}_y(\pi) < 0$ (negative curvature) and less apart for $\mathcal{K}_y(\pi) > 0$ (positive curvature) and that for small t the value $\mathcal{K}(y,\pi)t^3$ provides an approximation for the amount of this spread with an error of order t^3.

5.5 Conjugate Points

As also mentioned in the introduction of this section, Jacobi fields can also be used to determine the singularities of the exponential mapping. In this subsection we look at this relation between Jacobi fields and the singularities of the exponential map and then we will derive some additional properties of Jacobi fields. We begin with the key definition of this subsection:

Definition 5.5.1. *Let $\phi : [0, b] \to M$ be a geodesic on the Riemannian manifold M. Along ϕ, the point $\phi(t_0)$, $0 < t_0 \leq b$, is said to be conjugate to the point $\phi(0)$, if there is, along ϕ, a non-zero Jacobi field $\mathcal{J}$, satisfying $\mathcal{J}(0) = 0 = \mathcal{J}(t_0)$. The maximum number of such linearly independent fields is called the multiplicity of the conjugate point $\phi(t_0)$.*

Since M is a Riemannian manifold of dimension d we have that the dimension of T_yM is also d for every point y in M. Therefore, along any geodesic $\phi : [0, b] \to M$ there exists exactly d linearly independent Jacobi fields along ϕ which are equal to 0 at $\phi(0)$. This follows from the next lemma which we present without proof.

Lemma 5.5.1. *Let $\phi : [0, b] \to M$ be a geodesic and let $\mathcal{J}_1, \ldots, \mathcal{J}_s$ be Jacobi fields along ϕ such that $\mathcal{J}_\eta(0) = 0$, $\eta = 1, \ldots, s$. Then $\mathcal{J}_1, \ldots, \mathcal{J}_s$ are linearly independent if and only if $\frac{D\mathcal{J}_1}{dt}(0), \ldots, \frac{D\mathcal{J}_s}{dt}(0)$ are linearly independent.*

5.5.1 Conjugate Points and the Singularities of the Exponential Map

The exponential map, $\exp_y$, is defined on all of T_yM and near to the point 0 it is a local diffeomorphism if M is complete Riemannian manifold. Nevertheless, at points that are not near to 0, the exponential map may fail to be a local diffeomorphism. The next proposition is a significant result in that it associates the singularities of the exponential map with conjugate points:

Proposition 5.5.1. *Let $\phi : [0, b] \to M$ be a geodesic on the Riemannian manifold M. Set $\phi(0) = y$. Along ϕ, $x = \phi(t_0)$, $0 < t_0 \leq b$, is a conjugate point of y if and only if $n_0 = t_0\phi'(0)$ is a critical value of the exponential map* exp. *Furthermore, the multiplicity of x as a conjugate point of y is equal to the dimension of the kernel of the linear map $(d\exp_y)_{n_0}$.*

Proof. From Definition 5.5.1, the point $x = \phi(t_0)$ is conjugate to the point y if and only if, along ϕ, there is a Jacobi field $\mathcal{J}$ that does not vanish identically, but satisfies $\mathcal{J}(0) = 0 = \mathcal{J}(t_0)$. Let $n = \phi'(0)$ and let $m = \frac{D\mathcal{J}}{dt}(0)$. From Corollary 5.3.1, we have that $\mathcal{J}$ is of the form

$$\mathcal{J}(t) = (d\exp_y)_{tn}(tm), \quad 0 \leq t \leq b,$$

specifically, we obtain that $\mathcal{J}$ does not vanish if and only if $m \neq 0$. Thus, x is a conjugate point of y if and only if

$$0 = \mathcal{J}(t_0) = (d\exp_y)_{t_0n}(t_0m), \quad m \neq 0,$$

i.e., if and only if $t_0 n$ is a critical value of the exponential mapping $\exp_y$. This proves the first part of the proposition.

For the second part, we know that the multiplicity of x is the same as the number of linearly independent Jacobi fields $\mathcal{J}_1, \ldots, \mathcal{J}_s$ which are 0 at t_0 and at 0. From Lemma 5.5.1 and the construction above we have that the multiplicity of x is identical to the dimension of the kernel of $(d\exp_y)t_0 n$. ■

5.5.2 Properties of Jacobi Fields

We conclude this subsection by giving some additional properties of Jacobi fields which we will make use of in the sections to come.

Proposition 5.5.2. *Let $\phi : [0, b] \to M$ be a geodesic on the Riemannian manifold M and let $\mathcal{J}$ be a Jacobi field along ϕ. Then, for $0 \leq t \leq b$,*

$$\langle \mathcal{J}(t), \phi'(t) \rangle = \langle \mathcal{J}'(0), \phi'(0) \rangle t + \langle \mathcal{J}(0), \phi'(0) \rangle.$$

Proof. From the Jacobi equation, we have, $\langle \mathcal{J}', \phi' \rangle' = \langle \mathcal{J}'', \phi' \rangle = -\langle \mathcal{R}(\phi', \mathcal{J})\phi', \phi' \rangle = 0$. And so we get that $\langle \mathcal{J}', \phi' \rangle = \langle \mathcal{J}'(0), \phi'(0) \rangle$. Furthermore, $\langle \mathcal{J}, \phi' \rangle' = \langle \mathcal{J}', \phi' \rangle = \langle \mathcal{J}'(0), \phi'(0) \rangle$. We now integrate the last equation with respect to t to get the desired result: $\langle \mathcal{J}, \phi' \rangle = \langle \mathcal{J}'(0), \phi'(0) \rangle t + \langle \mathcal{J}(0), \phi'(0) \rangle$. ■

From the proposition above we can get two immediate corollaries:

Corollary 5.5.1. *$\langle \mathcal{J}, \phi' \rangle$ is independent of t if $\langle \mathcal{J}, \phi' \rangle(t_1) = \langle \mathcal{J}, \phi' \rangle(t_2)$, where $t_1, t_2 \in [0, b]$, such that $t_1 \neq t_2$. Specifically, if $\mathcal{J}(0) = 0 = \mathcal{J}(b)$, then $\langle \mathcal{J}, \phi' \rangle(t) \equiv 0$.*

Proof. $\langle \mathcal{J}(t), \phi'(t) \rangle = \langle \mathcal{J}(0), \phi'(0) \rangle$ for every t in $[0, b]$. This follows from Proposition 5.5.2. And if $\mathcal{J}(0) = 0 = \mathcal{J}(b)$ we get $\langle \mathcal{J}(0), \phi'(0) \rangle = 0$. ■

Corollary 5.5.2. *Let $\mathcal{J}(0) = 0$. Then $\langle \mathcal{J}'(0), \phi'(0) \rangle = 0$ if and only if $\langle \mathcal{J}, \phi' \rangle(t) \equiv 0$. Specifically, the space of Jacobi fields $\mathcal{J}$ satisfying $\mathcal{J}(0) = 0$ and $\langle \mathcal{J}, \phi' \rangle \equiv 0$ has dimension equal to $d - 1$.*

Proof. The first affirmation follows right away from Proposition 5.5.2. Moreover, when $\mathcal{J}(0) = 0$ and $\langle \mathcal{J}, \phi' \rangle \equiv 0$ we have $d - 1$ degrees of freedom for $\mathcal{J}'(0)$ and therefore we have that the dimension of the space of such Jacobi fields is $d - 1$, by Lemma 5.5.1. ■

Proposition 5.5.3. *Let M be a Riemannian manifold, let $\phi : [0, b] \to M$ be a geodesic on M, let $K_1 \in T_{\phi(0)}M$ and let $K_2 \in T_{\phi(b)}M$. Then, along ϕ, there is a unique Jacobi field $\mathcal{J}$ satisfying $\mathcal{J}(0) = K_1$ and $\mathcal{J}(b) = K_2$, if $\phi(b)$ is not conjugate to $\phi(0)$.*

Proof. Let $\mathfrak{J}$ be the space of Jacobi fields such that $\mathcal{J}(0) = K_1$ and let $\Upsilon : \mathfrak{J} \to T_{\phi(b)}M$ be a map given by

$$\Upsilon(\mathcal{J}) = \mathcal{J}(b), \quad \mathcal{J} \in \mathfrak{J}.$$

Υ is an injection (one-to-one), this is due to the fact that $\phi(b)$ and $\phi(0)$ are not conjugate. Certainly, if $\mathcal{J}_1 \neq \mathcal{J}_2$ such that $\mathcal{J}_1(b) = \mathcal{J}_2(b)$, then $\mathcal{J}_1 - \mathcal{J}_2$ would be a non vanishing Jacobi field satisfying $(\mathcal{J}_1 - \mathcal{J}_2)(0) = K_1 - K_1 = 0 = \mathcal{J}_1(b) - \mathcal{J}_2(b) = (\mathcal{J}_1 - \mathcal{J}_2)(b)$, which would be a contradiction.

As Υ is a linear injection and $\dim T_{\phi(b)}M = \dim \mathfrak{J}$ we also have that Υ is an isomorphism. Thus, there is $\widetilde{\mathcal{J}} \in \mathfrak{J}$ such that $\widetilde{\mathcal{J}}(0) = K_1$ and $\widetilde{\mathcal{J}}(b) = K_2$. Uniqueness is straightforward since Υ is an isomorphism. ■

5.6 The Theorem of Hadamard

As an application of Hopf-Rinows theorem, we will prove the following local-global theorem which is a characterization of simply connected Riemannian manifolds of non-positive sectional curvature ($\mathcal{K} \leq 0$).

Theorem 5.6.1 (Hadamard). *[3, Theorem 3.1] Let M be a simply connected, complete Riemannian manifold whose sectional curvature $\mathcal{K}$, at every point y of M and at every plane section $\mathcal{P} \subset T_yM$, is ≤ 0. Then $\exp_y : T_yM \to M$ is a diffeomorphism, that is, M is diffeomorphic to $\mathbb{R}^n$, where $n =$ dimension of M.*

Requiring a Riemannian manifold to have non-positive sectional curvature is a restriction on the infinitesimal geometry of the space. Much of the power and elegance of the theory of such manifolds stems from the fact that one can use this infinitesimal condition to make deductions about the global geometry and topology of the manifold. One of the results which underpins this passage from the local to the global context is a fundamental theorem that is due to Hadamard which we have stated above. Hadamard's theorem is an example of the relation between local and global properties, in which a local condition ($\mathcal{K} \leq 0$) together with weak global restrictions (complete and simply connected) imply a strong global restriction (homeomorphic to $\mathbb{R}^n$).

For the proof of Hadamard's Theorem we will need the following two lemmas:

Lemma 5.6.2. *Let M be a complete Riemannian manifold whose sectional curvature $\mathcal{K}$, at every point y of M and every plane section $\mathcal{P} \subset T_yM$, is ≤ 0. Then y does not have a conjugate point along any geodesic ϕ starting from y, for each y in M. Specifically, $\exp_y : T_yM \to M$ is a local diffeomorphism.*

Proof. Let $\phi : [0, \infty) \to M$ be a geodesic with $\phi(0) = y$ and let $\mathcal{J}$ be a not identically zero Jacobi field along ϕ with $\mathcal{J}(0) = 0$. By our assumption that the curvature is non-positive, we obtain, from the Jacobi equation

$$\begin{aligned}\frac{d^2}{dt^2}\langle \mathcal{J}, \mathcal{J}\rangle &= 2\left\langle \frac{D\mathcal{J}}{dt}, \frac{D\mathcal{J}}{dt}\right\rangle + 2\left\langle \frac{D^2\mathcal{J}}{dt^2}, \mathcal{J}\right\rangle \\ &= 2\left\langle \frac{D\mathcal{J}}{dt}, \frac{D\mathcal{J}}{dt}\right\rangle - 2\left\langle \mathcal{R}\left(\frac{d\phi}{dt}, \mathcal{J}\right)\frac{d\phi}{dt}, \mathcal{J}\right\rangle \\ &= 2\left|\frac{D\mathcal{J}}{dt}\right|^2 - 2\mathcal{K}\left(\frac{d\phi}{dt}, \mathcal{J}\right)\left|\frac{d\phi}{dt} \wedge \mathcal{J}\right|^2 \geq 0.\end{aligned}$$

Hence we see that $\frac{d}{dt}\langle \mathcal{J}, \mathcal{J}\rangle$ is an increasing function, that is, if $t_2 > t_1$ then $\frac{d}{dt}\langle \mathcal{J}, \mathcal{J}\rangle(t_2) \geq \frac{d}{dt}\langle \mathcal{J}, \mathcal{J}\rangle(t_1)$. Recalling that $\frac{D\mathcal{J}}{dt}\big|_0 \neq 0$ and $\frac{d}{dt}\langle \mathcal{J}, \mathcal{J}\rangle\big|_0 = 0$, it follows, furthermore, that for a sufficiently small $t > 0$, $\langle \mathcal{J}, \mathcal{J}\rangle(t) > \langle \mathcal{J}, \mathcal{J}\rangle(0)$. Thus for every $t > 0$, $\langle \mathcal{J}, \mathcal{J}\rangle(t) > 0$. Therefore $\phi(t)$ is not conjugate to $\phi(0)$ along ϕ which implies that $\exp_y$ must be a local diffeomorphism by the inverse function theorem. ■

Lemma 5.6.3. *Let $h : M \to Q$ be a local diffeomorphism between a complete Riemannian manifold M and a Riemannian manifold Q. Suppose that h has the following property: for every $y \in M$ and for every $u \in T_yM$, we have $|dh_y(u)| \geq |u|$, that is, h is a local isometry. Then h is a covering map.*

Proof. Since we already have a local diffeomorphism we only need to check that h has the path-lifting property, that is, we need to check that given a differentiable curve $\psi : [0, 1] \to Q$ and a point $x \in M$ with $h(x) = \psi(0)$, there exists a curve $\hat{\psi} : [0, 1] \to M$ with $\hat{\psi}(0) = x$ and $h \circ \hat{\psi} = \psi$. To show this, we first observe that, since h is a local diffeomorphism around an open neighborhood of x, ψ can be lifted to a small interval starting from x, that is, we can uniquely define for some small enough $\epsilon > 0$, $\hat{\psi} : [0, \epsilon] \to M$ such that $\hat{\psi}(0) = x$ and $h \circ \hat{\psi} = \psi$. Because h is a local diffeomorphism over all of M, the set of values $B \subset [0, 1]$, such that ψ can be lifted on B starting from x, is an open interval on the right, that is, $B = [0, t_o)$. If we can show that $t_o \in B$, we shall have B open and closed in $[0, 1]$, therefore $B = [0, 1]$ and ψ can be lifted on the entire interval.

To prove that $t_o \in B$, we consider an increasing sequence $\{t_n\}$ in B with $\lim t_n = t_o$, that is, $\{t_n\}$ is an increasing sequence converging to t_0. Then the sequence $\{\hat{\psi}(t_n)\}$ must be contained in a compact set $K \subset M$. Otherwise, if this were not true, since M is complete, the distance from $\hat{\psi}(t_n)$ to $\hat{\psi}(0)$ would be arbitrarily large, that is, $d(\hat{\psi}(t_n), \hat{\psi}(0)) \to \infty$ (This follows from a result in topology). However, by the hypothesis on $|dh|$ we have

$$\begin{aligned}
\ell(\psi|_{[0,t_n]}) = \int_0^{t_n} \left|\frac{d\psi}{dt}\right| dt = \int_0^{t_n} \left| dh_{\hat{\psi}(t)}\left(\frac{d\hat{\psi}}{dt}\right)\right| dt \\
\geq \int_0^{t_n} \left|\frac{d\hat{\psi}}{dt}\right| dt \\
\geq d(\hat{\psi}(t_n), \hat{\psi}(0)),
\end{aligned}$$

this implies that $\ell(\psi|_{[0,t_o]}) \to \infty$, that is, the length of ψ between 0 and t_o is arbitrarily large. This is nonsensical and thus proves the affirmation we made above.

And so, since $\{\hat{\psi}(t_n)\} \subset K \subset M$, there is an accumulation point (limit point) $w \in M$ of $\{\hat{\psi}(t_n)\}$. Let W be an open neighborhood of w with $h|_W$ a diffeomorphism. Then $\psi(t_o) \in h(W)$ and, by continuity, there exists an interval $I \subset [0,1]$, $t_o \in I$, such that $\psi(I) \subset h(W)$. Next, we select an index n such that $\hat{\psi}(t_n) \in W$ and study the lifting β of ψ on I going through the accumulation point w. Since $h|_W$ is bijective, the liftings β and $\hat{\psi}$ concur on their common domain, $[0, t_n) \cap I$. Hence, β is an extension of $\hat{\psi}$ to I, therefore $\hat{\psi}$ can be defined at $t_o \in I$ and $t_o \in B$. ∎

The first lemma demonstrates that under the assumption of Riemannian manifolds with non-positive sectional curvature, the exponential map is a local diffeomorphism. The second lemma is an important one which we will also make use of in the latter sections of this dissertation.

Proof of Theorem 5.6.1. For every point y in M, $\exp_y : T_yM \to M$ is defined and is surjective. This is due to the fact that M is complete. We also have that $\exp_y$ is a local diffeomorphism everywhere, by Lemma 5.6.2. This permits us to add a Riemannian metric $\hat{g} = (\exp_y)^*g$ on T_yM in such a way that $\exp_y : (T_yM, \hat{g}) \to (M, g)$ is a local isometry. Since the geodesics of $(T_yM, \hat{g})$ going through the origin ($0 \in T_yM$) are straight lines, it follows that $\exp_0 : T_0T_yM \to T_yM$ is defined for every $J_0 \in T_0T_yM$. $(T_yM, \hat{g})$ is then a complete manifold (Cf. Hopf-Rinows Theorem, $(c) \Rightarrow (b)$). And so, from Lemma 5.6.3, $\exp_y$ is a covering map. As M is simply connected, $\exp_y$ is a diffeomorphism. ∎

6 Spaces of Constant Sectional Curvature

6.1 Introduction

In this section, we characterize an important class of Riemannian manifolds, those with constant sectional curvature. They assumed an essential role in the advancement of Riemannian geometry. We will study these manifolds in some detail in this section.

6.2 Definitions and Important Results

We begin with the notion of an isotrpic Riemannian manifold M. Isotropy is a property which says that at every point of M, all directions are geometrically identical. In other words, the geometry of M does not depend on directions.

Definition 6.2.1. *A Riemannian manifold M is said to be isotropic at a point $y \in M$ if its sectional curvature is the same constant k, $k \in \mathbb{R}$, for every plane section $\mathcal{P} \subset T_yM$, that is, if $\mathcal{K}(\mathcal{P})$ is not dependent on the particular choice of plane section $\mathcal{P}$ at y. Additionally, M is said to be isotropic if it is isotropic at all points $y \in M$.*

Example 6.2.1. *Every 2-dimensional manifold is trivially isotropic as there is only a single plane section through any point.*

Since sectional curvature completely determines the Riemannian curvature tensor $\mathcal{R}$, we have an explicit formula for $\mathcal{R}$ when M is isotropic:

Proposition 6.2.1. *Let M be a Riemannian manifold and let $f : A \subset \mathbb{R}^n \to M$ be a local coordinate system around any point $y \in M$. Then M is isotropic at y if and only if the coefficients of the Riemannian curvature tensor at y are given by*

$$\mathcal{R}_{\eta\omega\nu\xi} = k(g_{\eta\nu}g_{\omega\xi} - g_{\eta\xi}g_{\omega\nu}). \tag{13}$$

Proof. One can easily verify by a direct computation that M is of constant sectional curvature k at the point y if (13) holds. Conversely, suppose that M is of constant sectional curvature k at y, that is, $\mathcal{K}(\mathcal{P}) = k$ for all $\mathcal{P} \subset T_yM$. We define a tri-linear mapping $\widetilde{\mathcal{R}} : T_yM \times T_yM \times T_yM \to T_yM$ by

$$\widetilde{\mathcal{R}}(J, K, H, L) = \langle J, H\rangle\langle K, L\rangle - \langle K, H\rangle\langle J, L\rangle,$$

for all $J, K, H, L \in T_yM$. Then $\widetilde{\mathcal{R}}$ is a tensor of type $(0, 4)$ which satisfies properties 1, 2, and 3 of Proposition 4.2.3. We now wish to show that $\mathcal{R} = k\widetilde{\mathcal{R}}$, which will complete the proof. Since

$$\widetilde{\mathcal{R}}(J, K, J, K) = \langle J, J\rangle\langle K, K\rangle - \langle J, K\rangle^2,$$

we have from Definition 4.3.2:

$$\mathcal{R}(J,K,J,K) = k(\langle J,J\rangle\langle K,K\rangle - \langle J,K\rangle^2) = k\widetilde{\mathcal{R}}(J,K,J,K),$$

for all pairs of vectors $J, K \in T_yM$. Lemma 4.3.1 implies that,

$$\mathcal{R}(J,K,H,L) = k\widetilde{\mathcal{R}}(J,K,H,L), \quad \text{for all} \quad J,K,H,L.$$

Hence $\mathcal{R} = k\widetilde{\mathcal{R}}$. ∎

Definition 6.2.2. *A Riemannian manifold M is called a space of constant sectional curvature or a space form if $\mathcal{K}(J_y, K_y) = const. = k$ for all linearly independent vectors $J_y, K_y \in T_yM$ and all $y \in M$, that is, if it is isotropic and k is the same constant value at all points of M.*

It is clear that if a Riemannian manifold has constant sectional curvature then it is isotropic. The converse is true in dimensions greater or equal to 3. This is a result due to Friedrich Schur.

Theorem 6.2.1 (F. Schur, 1886). *Let $(M, \langle\ ,\ \rangle)$ be a connected Riemannian manifold of dimension ≥ 3. If the sectional curvature $\mathcal{K}(\mathcal{P})$, for each $y \in M$, does not depend on the plane section $\mathcal{P} \subset T_yM$, but only on the point y at which it is calculated, then M has constant sectional curvature, that is, $\mathcal{K}(\mathcal{P})$ also does not depend on the point y.*

Proof. First, we define a tensor $\widetilde{\mathcal{R}}$ by

$$\widetilde{\mathcal{R}}(H,L,J,K) = \langle H,J\rangle\langle L,K\rangle - \langle L,J\rangle\langle H,K\rangle.$$

By hypothesis, $\mathcal{K}(\mathcal{P}) = k$ does not depend on $\mathcal{P}$ for any point $y \in M$, this means that by Proposition 6.2.1, we have $\mathcal{R} = k\widetilde{\mathcal{R}}$. By taking derivatives and observing the fact that $\nabla_V\widetilde{\mathcal{R}} = 0$, we obtain,

$$\nabla_V\mathcal{R} = (Vk)\widetilde{\mathcal{R}}, \quad \text{for all} \quad V \in \mathfrak{X}(M).$$

We now want to prove that $Jk = 0$ for all J. By cyclically permuting the arguments and using the second Bianchi identity:

$$\nabla\mathcal{R}(H,L,J,K,V) + \nabla\mathcal{R}(H,L,K,V,J) + \nabla\mathcal{R}(H,L,V,J,K) = 0,$$

we get, for all $J,K,H,L,V \in \mathfrak{X}(M)$,

$$\begin{aligned} 0 \quad = \quad & (Vk)(\langle H,J\rangle\langle L,K\rangle - \langle L,J\rangle\langle H,K\rangle) \\ & + (Jk)(\langle H,K\rangle\langle L,V\rangle - \langle L,K\rangle\langle H,V\rangle) \\ & + (Kk)(\langle H,V\rangle\langle L,J\rangle - \langle L,V\rangle\langle H,J\rangle). \end{aligned}$$

Fix $y \in M$. Because M is of dimension ≥ 3, it is feasible, fixing J at y, to pick K and L at y such that $\langle J, K\rangle = \langle K, L\rangle = \langle L, J\rangle = 0$, $\langle L, L\rangle = 1$, that is, J, K, L are linearly independent (orthogonal) at y. Put $V = L$ at y. The relation above gives us:

$$\langle (Jk)K - (Kk)J, H\rangle = 0, \quad \text{for all} \quad H.$$

Since J and K are linearly independent at y, we may conclude that $Jk = 0$ for all $J \in T_yM$. Therefore k is locally constant, and since M is connected, k is globally constant. ∎

Schur's theorem tells us that the isotropy of a Riemannian manifold, implies the homogeneity, that is, that all points are geometrically identical. Specifically, a pointwise property infers a global one. In higher dimensional manifolds isotropy forces the sectional curvature to be constant.

With the concept of Riemannian manifolds with constant sectional curvature in hand, we are now able to prove Corollary 4.4.1 from Section 4.

Proof of Corollary 4.4.1. 1. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis. Using Proposition 6.2.1 we obtain:

$$\begin{aligned}
\operatorname{Ric}(e_\omega, e_\omega) &= \sum_{\eta=1}^{n} \langle \mathcal{R}(e_\omega, e_\eta)e_\eta, e_\omega\rangle \\
&= \sum_{\eta=1}^{n} \langle k(\langle e_\eta, e_\eta\rangle e_\omega - \langle e_\omega, e_\eta\rangle e_\eta), e_\omega\rangle \\
&= k\left(\sum_{\eta=1}^{n} \langle e_\eta, e_\eta\rangle\langle e_\omega, e_\omega\rangle - \sum_{\eta=1}^{n} \langle e_\eta, e_\omega\rangle\langle e_\eta, e_\omega\rangle\right) \\
&= k\left(\sum_{\eta=1}^{n} 1 - \sum_{\eta=1}^{n} \delta_{\eta\omega}\right) = (n-1)k.
\end{aligned}$$

2. To acquire the formula for the scalar curvature we just have to multiply $\operatorname{Ric}(e_\omega, e_\omega)$ by n. ∎

6.3 Jacobi Fields on Spaces of Constant Curvature

Before we move on to deal with the most important aspects of this section, we detour slightly to take a look at Jacobi fields on Riemannian manifolds of constant sectional curvature curvature. Let $\phi : [0, d] \to M$ be a normalized (unit speed) geodesic on a Riemannian manifold M of constant sectional curvature k and let $\mathcal{J}$ be a Jacobi field along ϕ, normal to ϕ'. Using Proposition 6.2.1 and using the fact that $\|\phi'\| = 1$ as well as that $\langle \mathcal{J}, \phi' \rangle = 0$ ($\mathcal{J}$ is normal to ϕ'), we obtain, for each vector field T along ϕ:

$$\langle \mathcal{R}(\phi', \mathcal{J})\phi', T \rangle = k\{\langle \phi', \phi' \rangle \langle \mathcal{J}, T \rangle - \langle \phi', T \rangle \langle \mathcal{J}, \phi' \rangle\} = k\langle \mathcal{J}, T \rangle = \langle k\mathcal{J}, T \rangle,$$

which then implies that $\mathcal{R}(\phi', \mathcal{J})\phi' = k\mathcal{J}$. And so the Jacobi equation for manifolds of constant curvature is of the form:

$$\frac{D^2}{dt^2}\mathcal{J} + k\mathcal{J} = 0. \tag{$*$}$$

Since $(*)$ asserts that the second covariant derivative of $\mathcal{J}$ is a multiple of $\mathcal{J}$ itself, it is sensible to attempt to construct a solution by picking a parallel normal vector field along the geodesic ϕ. So, next we let $s(t)$ be this parallel vector field along ϕ such that $\|s(t)\| = 1$ and $\langle \phi'(t), s(t) \rangle = 0$ for every $t \in [0, d]$. Then

$$\mathcal{J}(t) = \begin{cases} \frac{\sin(t\sqrt{k})}{\sqrt{k}} s(t), & \text{if } k > 0, \\ ts(t), & \text{if } k = 0, \\ \frac{\sinh(t\sqrt{-k})}{\sqrt{-k}} s(t), & \text{if } k < 0, \end{cases}$$

is a solution for Equation $(*)$ with initial conditions $\mathcal{J}(0) = 0$ and $\frac{D\mathcal{J}}{dt}(0) = s(0)$. One can verify this without difficulty. For instance, in the case $k > 0$ we get

$$\begin{aligned} \frac{D^2}{dt^2}\mathcal{J}(t) + k\mathcal{J}(t) &= \frac{D}{dt}\left(\cos(t\sqrt{k})s(t) + \frac{\sin(t\sqrt{k})}{\sqrt{k}} \underbrace{\frac{D}{dt}(s(t))}_{=0}\right) + k\frac{\sin(t\sqrt{k})}{\sqrt{k}}s(t) \\ &= -\sqrt{k}\sin(t\sqrt{k})s(t) + \sqrt{k}\sin(t\sqrt{k})s(t) \\ &= 0, \end{aligned}$$

for every $t \in [0, d]$.

6.4 Spaces of Constant Sectional Curvature

In this subsection we look at examples of Riemannian manifolds of constant sectional curvature. In particular, we will look at the three model spaces of Riemannian geometry, Euclidean $\mathbb{R}^n$, Spherical $\mathbb{S}^n$ and Hyperbolic space $\mathbb{H}^n$. We will show that they have constant sectional curvature $\mathcal{K} = 0$, $\mathcal{K} = +1$ and $\mathcal{K} = -1$ respectively. In actual fact, these are the only possible values of the constant sectional curvature of a Riemannian manifold. To verify this we rescale the Riemannian metric. By scaling one means the process of multiplying a Riemannian metric by a positive constant λ. So we replace $\langle\ ,\ \rangle$ by $\lambda\langle\ ,\ \rangle$. By doing this, it is easy to check that the Christoffel symbols, the Riemannian connection and the Riemannian curvature tensor all remain unchanged, but the sectional curvature is changed, with $\widetilde{\mathcal{K}} = \frac{1}{\lambda}\mathcal{K}$, that is, the sectional curvature is multiplied by $\frac{1}{\lambda}$. As a result, If M is a Riemannian manifold of constant sectional curvature, by rescaling the metric, we may assume that either $\mathcal{K} = -1$, or $\mathcal{K} = 0$, or $\mathcal{K} = +1$.

6.4.1 Euclidean Space

Let $M = \mathbb{R}^n$ with its standard inner product $\langle r, s\rangle = r \cdot s = \sum_{i=1}^{n} r_i s_i$, $r, s \in \mathbb{R}^n$. As we showed in section 4, the Riemannian curvature tensor $\mathcal{R}(J, K)L = 0$, for all $J, K, L \in \mathfrak{X}(\mathbb{R}^n)$. Therefore, by Definition 4.3.2, the sectional curvature $\mathcal{K}(\mathcal{P}) = 0$ for any plane $\mathcal{P} \subset T_y\mathbb{R}^n$ at any point $y \in \mathbb{R}^n$. Thus, $\mathbb{R}^n$ with its standard Rimeannian metric is flat, that is, it has constant sectional curvature equal to 0.

6.4.2 Spherical Space

Let $M = \mathbb{S}^n \subset \mathbb{R}^{n+1}$ be the n-dimensional unit sphere. Let $N(x) = -x \in \mathbb{R}^n$, $|x| = 1$, that is, the orientation of $\mathbb{S}^n$ is given by the inward pointing unit normal. Then all of the eigenvalues of the self adjoint operator associated to H_N (the second fundamental along the normal vector N) are equal to 1, this is since the Gauss mapping (M. do Carmo [3, p.129]) is equal to $(-i)$, where i is the identity of $\mathbb{S}^n$. And this means that for all $y \in \mathbb{S}^n$, each $r \in T_y\mathbb{S}^n$ is an eigenvector. Using the following expression from [3, p.131],

$$\mathcal{K}(e_\eta, e_\omega) - \overline{\mathcal{K}}(e_\eta, e_\omega) = \lambda_\eta\lambda_\omega,$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of T_yM and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues, we conclude the sectional curvature of the unit sphere is a constant equal to $+1$.

6.4.3 Hyperbolic Space

Let $M = \mathbb{H}^n$ be the n-dimensional hyperbolic space. There are several models of hyperbolic space, here we will use the upper half-space of $\mathbb{R}^n$, also referred to as the Poincaré half-plane model:

$$\mathbb{H}^n = \{(y_1, ..., y_n) \in \mathbb{R}^n : y_n > 0\},$$

equipped with the metric:

$$g_{\eta\omega}(y_1, ..., y_n) = \frac{\delta_{\eta\omega}}{y_n^2}.$$

Before we can begin with the calculations of the curvature of $\mathbb{H}^n$, we need the concept of when two metrics are conformal. This concept will be dealt with in much more detail in Section 7.

Definition 6.4.1. *Two metrics* $\langle\ ,\ \rangle$ *and* $\langle\langle\ ,\ \rangle\rangle$ *on a differentiable manifold* M *are conformal if there exists a positive differentiable function* $d : M \to \mathbb{R}$*, such that for all* $y \in M$ *and all* $r, s \in T_yM$

$$\langle r, s\rangle_y = d(y)\langle\langle r, s\rangle\rangle_y.$$

With the above definition in mind, we consider on $\mathbb{H}^n$, the following metric which is conformal to the usual metric of Euclidean space $\mathbb{R}^n$:

$$g_{\eta\omega}(y_1, ..., y_n) = \frac{\delta_{\eta\omega}}{D^2},$$

where D is a positive differentiable function on $\mathbb{H}^n$. To get the sectional curvature of $\mathbb{H}^n$ we need to compute the coefficients of the Riemannian curvature tensor $\mathcal{R}$. But before we can compute $\mathcal{R}$ we need to compute the Christoffel symbols. We put $logD = d$ and let $g^{\eta\omega} = D^2\delta_{\eta\omega}$ be the inverse metric of $g_{\eta\omega}$. Under these conditions, we have:

$$\frac{\partial g_{\eta\nu}}{\partial y_\omega} = -\delta_{\eta\nu}\frac{2}{D^3}D_\omega = -2\frac{\delta_{\eta\nu}}{D^2}\frac{\partial d}{\partial y_\omega}.$$

The Christoffel symbols are then calculated as follows:

$$\begin{aligned}
\Gamma^\nu_{\eta\omega} &= \frac{1}{2}\sum_{\zeta=1}^{n} g^{\zeta\nu}\left(\frac{\partial}{\partial y_\eta}g_{\omega\zeta} + \frac{\partial}{\partial y_\omega}g_{\zeta\eta} - \frac{\partial}{\partial y_\zeta}g_{\eta\omega}\right) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial y_\eta}g_{\omega\nu} + \frac{\partial}{\partial y_\omega}g_{\nu\eta} - \frac{\partial}{\partial y_\nu}g_{\eta\omega}\right)D^2 \\
&= -\delta_{\omega\nu}\frac{\partial d}{\partial y_\eta} - \delta_{\nu\eta}\frac{\partial d}{\partial y_\omega} + \delta_{\eta\omega}\frac{\partial d}{\partial y_\nu},
\end{aligned}$$

hence if two indices are equal, we have

$$\Gamma^\eta_{\eta\omega} = -\frac{\partial d}{\partial y_\omega}, \quad \Gamma^\omega_{\eta\eta} = \frac{\partial d}{\partial y_\omega}, \quad \Gamma^\omega_{\eta\omega} = -\frac{\partial d}{\partial y_\eta}, \quad \Gamma^\eta_{\eta\eta} = -\frac{\partial d}{\partial y_\eta},$$

while if all three indices are different, we have:

$$\Gamma^{\nu}_{\eta\omega} = 0.$$

The coefficients of the Riemannian curvature tensor are calculated as follows:

$$\begin{aligned}
\mathcal{R}_{\eta\omega\eta\omega} &= \sum_{\xi=1}^{n} \mathcal{R}^{\xi}_{\eta\omega\eta} g_{\xi\omega} = \mathcal{R}^{\omega}_{\eta\omega\eta} g_{\omega\omega} = \mathcal{R}^{\omega}_{\eta\omega\eta} \frac{1}{D^2} \\
&= \frac{1}{D^2}\Bigg(\frac{\partial}{\partial y_\omega} \Gamma^{\omega}_{\eta\eta} - \frac{\partial}{\partial y_\eta} \Gamma^{\omega}_{\omega\eta} + \sum_{\xi=1}^{n} \Gamma^{\xi}_{\eta\eta} \Gamma^{\omega}_{\omega\xi} - \sum_{\xi=1}^{n} \Gamma^{\xi}_{\omega\eta} \Gamma^{\omega}_{\eta\xi} \Bigg) \\
&= \frac{1}{D^2}\Bigg(- \sum_{\substack{\xi=1 \\ \xi\neq\eta,\ \xi\neq\omega}}^{n} \frac{\partial}{\partial y_\omega} \frac{\partial d}{\partial y_\omega} + \frac{\partial}{\partial y_\eta} \frac{\partial d}{\partial y_\eta} + \frac{\partial d}{\partial y_\xi} \frac{\partial d}{\partial y_\xi} \\
&\qquad + \left(\frac{\partial d}{\partial y_\eta}\right)^2 - \left(\frac{\partial d}{\partial y_\omega}\right)^2 - \left(\frac{\partial d}{\partial y_\eta}\right)^2 + \left(\frac{\partial d}{\partial y_\omega}\right)^2 \Bigg) \\
&= \frac{1}{D^2}\Bigg(- \sum_{\xi=1}^{n} \frac{\partial}{\partial y_\omega} \frac{\partial d}{\partial y_\omega} + \frac{\partial}{\partial y_\eta} \frac{\partial d}{\partial y_\eta} + \left(\frac{\partial d}{\partial y_\xi}\right)^2 + \left(\frac{\partial d}{\partial y_\eta}\right)^2 + \left(\frac{\partial d}{\partial y_\omega}\right)^2 \Bigg),
\end{aligned}$$

hence if any three indices are different, we have:

$$\mathcal{R}^{\eta}_{\eta\omega\nu} = -\frac{\partial d}{\partial y_\nu} \frac{\partial d}{\partial y_\omega} - \frac{\partial}{\partial y_\nu} \frac{\partial d}{\partial y_\omega}, \quad \mathcal{R}^{\omega}_{\eta\omega\nu} = \frac{\partial d}{\partial y_\eta} \frac{\partial d}{\partial y_\nu} + \frac{\partial}{\partial y_\nu} \frac{\partial d}{\partial y_\eta}, \quad \mathcal{R}^{\nu}_{\eta\omega\nu} = 0,$$

while if all four indices are different, we have:

$$\mathcal{R}_{\eta\omega\nu\xi} = 0.$$

Using Definition 4.3.2, we can finally calculate the sectional curvature of $\mathbb{H}^n$ with respect to the plane section spanned by the orthogonal vectors $\frac{\partial}{\partial y_\eta}$, $\frac{\partial}{\partial y_\omega}$:

$$\begin{aligned}
\mathcal{K}_{\eta\omega} &= \frac{\mathcal{R}_{\eta\omega\eta\omega}}{g_{\eta\eta} g_{\omega\omega}} = \mathcal{R}_{\eta\omega\eta\omega} D^4 \\
&= \Bigg(- \sum_{\xi=1}^{n} \frac{\partial}{\partial y_\omega} \frac{\partial d}{\partial y_\omega} + \frac{\partial}{\partial y_\eta} \frac{\partial d}{\partial y_\eta} + \left(\frac{\partial d}{\partial y_\xi}\right)^2 + \left(\frac{\partial d}{\partial y_\eta}\right)^2 + \left(\frac{\partial d}{\partial y_\omega}\right)^2 \Bigg) D^2.
\end{aligned}$$

For the case $D^2 = y_n^2$ ($\Rightarrow d = log\ y_n$), we have:

$$\mathcal{K}_{\eta\omega} = \left(- \frac{1}{y_n^2} \right) y_n^2 = -1, \quad \text{if} \quad \eta \neq n \quad \text{and} \quad \omega \neq n;$$

$$\mathcal{K}_{n\omega} = \left(\frac{\partial}{\partial y_n} \frac{\partial d}{\partial y_n} + \left(\frac{\partial d}{\partial y_n}\right)^2 - \left(\frac{\partial d}{\partial y_n}\right)^2 \right) D^2 = \left(- \frac{1}{y_n^2} \right) y_n^2 = -1, \quad \text{if} \quad \eta = n \quad \text{and} \quad \omega \neq n;$$

$$\mathcal{K}_{\eta n} = \left(\frac{\partial}{\partial y_n} \frac{\partial d}{\partial y_n} + \left(\frac{\partial d}{\partial y_n}\right)^2 - \left(\frac{\partial d}{\partial y_n}\right)^2 \right) D^2 = \left(- \frac{1}{y_n^2} \right) y_n^2 = -1, \quad \text{if} \quad \eta \neq \omega \quad \text{and} \quad \omega = n.$$

Using the expressions we obtained on the previous page for the coefficients of the Riemannian curvature tensor and Proposition 6.2.1, we may conclude that the sectional curvature of $\mathbb{H}^n$ is a constant equal to -1.

Remark 6.4.1. *$\mathbb{R}^n$, $\mathbb{S}^n$ and $\mathbb{H}^n$ are, although we will not show it here, complete and simply connected. The importance of these facts is made clear in the next subsection.*

6.5 Classification Theorem

We now prove the main theorem of this section, that $\mathbb{R}^n$, $\mathbb{S}^n$ and $\mathbb{H}^n$ are essentially the only simply connected, complete Riemannian manifolds, with constant sectional curvature. But before we can prove it we need a few results. We begin with a theorem by Cartan, which basically implies that the metric is, in a certain sense, determined locally by the curvature.

Theorem 6.5.1 (É. Cartan). *Let P, Q be two n-dimensional Riemannian manifolds, let $\mathcal{R}$, $\widetilde{\mathcal{R}}$ be the curvatures of P and Q respectively and let u, w be points on P and Q respectively. We pick a linear isometry $\pi : T_uP \to T_wQ$ and let $V \subset P$ be a normal neighborhood of u such that $\exp_w$ is defined at $\pi \circ \exp_u^{-1}(V)$. Define a mapping $h : V \to Q$ by $h(y) = \exp_w \circ \pi \circ \exp_u^{-1}(y)$, $y \in V$. Since V is a normal neighborhood of u there exist a normalized geodesic $\phi : [0,t] \to P$ joining u and y, that is, with $\phi(0) = u$, $\phi(t) = y$. We also consider the geodesic $\widetilde{\phi} : [0,t] \to Q$ with $\widetilde{\phi}(0) = w$, $\widetilde{\phi}'(0) = \pi(\phi'(0))$. Denote by T_t and $\widetilde{T}_t$ the parallel transport along ϕ from $\phi(0)$ to $\phi(t)$ and the parallel transport along $\widetilde{\phi}$ from $\widetilde{\phi}(0)$ to $\widetilde{\phi}'(0)$ respectively. With all these machinery we define $\gamma_t : T_yP \to T_{h(y)}Q$ by $\gamma_t(J) = \widetilde{T}_t \circ \pi \circ T_t^{-1}(J)$, $J \in T_yP$. If for all $y \in V$ and for all $J, K, L, H \in T_yM$ we have*

$$\langle \mathcal{R}(J,K)L, H\rangle = \langle \widetilde{\mathcal{R}}(\gamma_t(J), \gamma_t(K))\gamma_t(L), \gamma_t(H)\rangle,$$

then $h : V \to h(V) \subset Q$ is a local isometry and $dh_u = \pi$.

Proof. To show that h is an isometry in V it is sufficient to prove that for every $y \in V$ and every $J \in T_yP$, $|dh_y(J)| = |J|$, that is, we need to prove that h preserves lengths. We pick any point y in V and any J in T_yP. Let $\phi : [0,b] \to P$ be a normalized geodesic joining u and y. From Proposition 5.5.3 in Section 5, we can write:

$$\mathcal{J}(t) = \sum_{\eta=1}^{n} q_\eta(t) e_\eta(t),$$

where $\mathcal{J}$ is the unique Jacobi field along ϕ such that $\mathcal{J}(0) = 0$ and $\mathcal{J}(b) = J$, $e_\eta(t)$ is the parallel transported orthonormal basis $\{e_1, ..., e_n\}$ of T_uP along ϕ with $e_n = \phi'(0)$, q_η is some differentiable function. From the Jacobi equation at $t = 0$ (Cf. Section 5), we get:

$$q''_\omega + \sum_{\eta=1}^{n} \langle \mathcal{R}(e_n, e_\eta)e_n, e_\omega \rangle q_\eta = 0, \quad \omega = 1, ..., n.$$

Now we consider Q. Let $\widetilde{\mathcal{J}}(t) = \gamma_t(\mathcal{J}(t))$, $t \in [0, b]$, where $\widetilde{\mathcal{J}}(t)$ is a vector field along the normalized geodesic $\widetilde{\phi} : [0, b] \to Q$ satisfying $\widetilde{\phi}(0) = w$, $\widetilde{\phi}'(0) = \pi(\phi'(0))$. Then from the linearity of γ_t, we obtain:

$$\widetilde{\mathcal{J}}(t) = \sum_{\eta=1}^{n} q_\eta(t)\tilde{e}_\eta(t),$$

where we set $\tilde{e}_\omega(t) = \gamma_t(e_\omega(t))$. As consequence of the hypothesis of the theorem we have $\langle \mathcal{R}(e_n, e_\eta)e_n, e_\omega \rangle = \langle \widetilde{\mathcal{R}}(\tilde{e}_n, \tilde{e}_\eta)\tilde{e}_n, \tilde{e}_\omega \rangle$. And so it turns out that $\widetilde{\mathcal{J}}$ is a Jacobi field along $\widetilde{\phi}$ with $\widetilde{\mathcal{J}}(0) = \gamma_t(\mathcal{J}(0)) = 0$ since it satisfies the Jacobi equation:

$$q''_\omega + \sum_{\eta=1}^{n} \langle \widetilde{\mathcal{R}}(\tilde{e}_n, \tilde{e}_\eta)\tilde{e}_n, \tilde{e}_\omega \rangle q_\eta = 0, \quad \omega = 1, ..., n.$$

Since $\widetilde{\mathcal{J}}$ and $\mathcal{J}$ are Jacobi fields along ϕ and $\widetilde{\phi}$ respectively and $\mathcal{J}(0) = 0 = \widetilde{\mathcal{J}}(0)$, we have from Corollary 5.3.1 of Section 5 for $t \in [0, b]$:

$$\begin{aligned} \mathcal{J}(t) &= (d\ \exp_u)_{t\phi'(0)}(t\mathcal{J}'(0)), \\ \widetilde{\mathcal{J}}(t) &= (d\ \exp_w)_{t\widetilde{\phi}'(0)}(t\widetilde{\mathcal{J}}'(0)). \end{aligned}$$

Computing $\widetilde{\mathcal{J}}(t) = \gamma_t(\mathcal{J}(t))$ at $t = 0$ we see that $\gamma_0 = \pi$ since there is no parallel transport. And so we have that $\widetilde{\mathcal{J}}'(0) = \pi(\mathcal{J}'(0))$. Therefore:

$$\begin{aligned} \widetilde{\mathcal{J}}(b) &= (d\ \exp_w)_{b\widetilde{\phi}'(0)} b\pi(\mathcal{J}'(0)) \\ &= (d\ \exp_w)_{b\widetilde{\phi}'(0)} \circ \pi \circ ((d\ \exp_u)_{b\phi'(0)})^{-1}(\mathcal{J}(b)) \\ &= dh_y(\mathcal{J}(b)) \\ &= dh_y(J). \end{aligned}$$

But as γ_t (parallel transport) is an isometry, we have that $|\widetilde{\mathcal{J}}(b)| = |\mathcal{J}(b)| = |J|$. This proves that h is a local isometry. Finally:

$$dh_u(J) = \widetilde{\mathcal{J}}(b) = \gamma_0(\mathcal{J}(b)) = \widetilde{T}_0 \circ \pi \circ T_0^{-1} \circ \mathcal{J}(b) = \widetilde{T}_0 \circ \pi \circ T_0^{-1}(J) = \pi(J).$$

■

We are particularly interested in the following two corollaries.

Corollary 6.5.1. *Let P, Q be two n-dimensional Riemannian manifolds with the same constant sectional curvature. Let u, w be points on P and Q respectively. Let $\{e_\eta\}$, $\{\hat{e}_\eta\}$ be the orthonormal bases of T_uP and T_wQ respectively. Then there exist a neighborhood $I \subset P$ of u, a neighborhood $\hat{I} \subset Q$ of w, and an isometry $h : I \to \hat{I}$ such that $dh_u(e_\eta) = \hat{e}_\eta$.*

Proof. We choose an isometry $\pi : T_uP \to T_wQ$ such that $\pi(e_\eta) = \hat{e}_\eta$. The condition on the curvature is easily verified. $dh_u = (d \exp_u)_0 \circ \pi \circ (d \exp_w)_0^{-1}$ and $(d \exp_u)_0$ is identity for all u, therefore $dh_u = \pi$. ■

The next Corollary shows that a Riemannian manifold with constant sectional curvature has a lot of isometries.

Corollary 6.5.2. *Let M be Riemannian manifold with constant sectional curvature. Let u, y be points on M. Let $\{e_\eta\}$ be the orthonormal basis of T_uM and let $\{\hat{e}_\eta\}$ be the orthonormal basis of T_yM. Then there exist a neighborhood $I \subset M$ of u, a neighborhood $\hat{I} \subset M$ of y, and an isometry $h : I \to \hat{I}$ such that $dh_u(e_\eta) = \hat{e}_\eta$.*

Lemma 6.5.2. *Let P, Q be two connected Riemannian manifolds and let h_1, h_2 be two local isometries between P and Q. If for some point y in P we have $h_1(y) = h_2(y)$ and $(dh_1)_y = (dh_2)_y$, then $h_1 = h_2$.*

Proof. Let V be a normal neighborhood of y such that h_1 and h_2 are diffeomorphisms on V. We define $\tau = h_1^{-1} \circ h_2 : V \to V$. Then by the hypothesis $\tau(y) = y$ and $d\tau_y = id$ (the identity). Pick a point u in V, then there exists a unique vector $J \in T_yM$ with $\exp_y(J) = u$ as V is normal neighborhood of y. Now, if we have that $\tau(u) = u$, then $h_1 = h_2$ on V. To see that $\tau(u) = u$, we write ϕ_J the geodesic satisfying $\phi_J(0) = y$ and $\phi'_J(0) = J$. Since τ is an isometry, $\tau \circ \phi_J$ is a geodesic starting at $\tau(\phi_J(0)) = \tau(y) = y$. Since $(\tau \circ \phi_J)'(0) = d\tau_y(\phi'_J(0)) = id_y(\phi'_J(0)) = J$, then $\tau \circ \phi_J = \phi_J$. Finally, $\tau(u) = \tau(\exp_y(J)) = \tau(\phi_J(1)) = \phi_J(1) = \exp_y(J) = u$. Any point w in M can be joined to y by a path $\psi : [0,1] \to M$ such that $\psi(0) = y$, $\psi(1) = w$. This is possible since M is connected. Next we consider:

$$G = \{t \in [0,1]; h_1(\psi(t)) = h_2(\psi(t)) \text{ and } (dh_1)_{\psi(t)} = (dh_2)_{\psi(t)}\}.$$

$\sup G > 0$ since $h_1 = h_2$ on V. If $\tilde{t} = \sup G \neq 1$, we repeat the argument above for the point $\psi(\tilde{t})$ but get a contradiction because then there is no normal neighborhood around y. Therefore $\sup G = 1$, hence $h_1(w) = h_2(w)$, for every point w in M. ■

We are now ready to state and prove the main theorem of this section.

Theorem 6.5.3. *[3, Theorem 4.1] Let M be a complete Riemannian Manifold with constant sectional curvature $\mathcal{K}$. Then the universal covering $\overline{M}$ of M, with the covering metric, is isometric to:*

(i) $\mathbb{H}^n$, if $\mathcal{K} = -1$,

(ii) $\mathbb{R}^n$, if $\mathcal{K} = 0$,

(iii) $\mathbb{S}^n$, if $\mathcal{K} = +1$.

Proof. We start with case (iii). Select points u, w in M and $\overline{M}$ respectively and select a linear isometry $\pi : T_u\mathbb{S}^n \to T_w\overline{M}$. Define a map $h : \mathbb{S}^n\backslash\{y\} \to \overline{M}$ by $h = \exp_w \circ \pi \circ \exp_u^{-1}$, where $y \in \mathbb{S}^n$ is the antipodal point of u. It is clear that, from Cartans Theorem, h is a local isometry. Next, we select a point u' in $\mathbb{S}^n$ such that $u' \neq u$, $u' \neq y$ and let $w' = h(u')$, $\pi' = dh_{u'}$. Using the linear isometry $\pi' : T_{u'}\mathbb{S}^n \to T_{w'}\overline{M}$, we are capable of defining an analogous map $h' : \mathbb{S}^n\backslash\{y'\} \to \overline{M}$ by $h' = \exp_{w'} \circ \pi' \circ \exp_{u'}^{-1}$, where y' is the antipodal point of u'. Since $\mathbb{S}^n\backslash(\{y\} \cup \{y'\})$ is connected and $h(u') = w' = h'(u')$, $dh_{u'} = \pi' = dh'_{u'}$, we have, from Lemma 6.5.2, that $h = h'$ on $\mathbb{S}^n\backslash(\{y\}\cup\{y'\})$, that is, where h and h' overlap, they must agree. Thus, we are able to define a map $m : \mathbb{S}^n \to \overline{M}$ by

$$m(p) = \begin{cases} h(p), & \text{if } p \in \mathbb{S}^n\backslash\{y\} \\ h'(p), & \text{if } p \in \mathbb{S}^n\backslash\{y'\}. \end{cases}$$

m is local diffeomorphism since, by Cartans Theorem, m is a local isometry. m is also a covering map since it is a local diffeomorphism between compact, connected manifolds $\mathbb{S}^n$ and $\overline{M}$. As $\overline{M}$ is simply connected, we have from Algebraic Topology, that m is a diffeomorphism and therefore m is an isometry. In other words, we obtain a globally defined local isometry, by putting h and h' together.

Now we consider the first two cases, (i) and (ii). These two cases can be treated simultaneously and so to avoid repetition we will denote $\mathbb{R}^n$ as well as $\mathbb{H}^n$ by $\diamondsuit$. Pick points u, w on $\diamondsuit$ and $\overline{M}$ respectively and pick a linear isometry $\pi : T_u\diamondsuit \to T_w\overline{M}$. Next, we define a map $h : \diamondsuit \to \overline{M}$ by $h = \exp_w \circ \pi \circ \exp_u^{-1}$. h is well defined. We verify this by first observing that since $\diamondsuit$ is a simply connected, complete Riemannian manifold with non-positive sectional curvature, we have by Hadamards Theorem, that $\exp_u : T_u\diamondsuit \to \diamondsuit$ is a diffeomorphism and so $\exp_u^{-1}$ is well defined on $\diamondsuit$. We also observe that $\exp_w$ is defined on all of $T_w\overline{M}$, this is due to the fact $\overline{M}$ is complete. Hence it follows, that h is well defined. Applying Cartans Theorem, we see that h is a local isometry and, for all $J \in T_u\diamondsuit$, $dh_u(J) = \pi(J)$. As π is an isometry, we obtain $|dh_u(J)| = |J|$. Therefore, by Lemma 5.6.3 of Section 5, we can conclude that h is diffeomorphism, and this completes the proof for case (i) and case (ii). ■

7 Curvature Preserving Diffeomorphisms

7.1 Introduction

In this section we ask the following question: Let (P, g), $(Q, \tilde{g})$ be two Riemannian manifolds with sectional curvatures $\mathcal{K}$, $\widetilde{\mathcal{K}}$ respectively. We will refer a diffeomorphism $h : P \to Q$ as sectional curvature preserving if and only if for each point y in P and for each plane section Π of the tangent space T_yP, we have

$$\mathcal{K}(\Pi) = \widetilde{\mathcal{K}}(h_*\Pi).$$

What are we able to state regarding the metrics of P and Q? In other words, we question if a sectional curvature preserving diffeomorphism is an isometry? In general, the response to this inquiry is that it is not true. For instance, a sectional curvature preserving diffeomorphism between two Riemannian manifolds of the same constant sectional curvature does not need to be an isometry, as any map will conserve their curvatures. However, in R. S. Kulkarnis thesis in 1970 ([44], [45]), he demonstrated that Riemannian manifolds of constant sectional curvature are very nearly the only exceptions and proved the following theorem:

Theorem 7.1.1 (**Kulkarni, 1970**). *If P is an analytic Riemannian manifold with dimension ≥ 4, then either P has constant sectional curvature or a sectional curvature preserving diffeomorphism $h : P \to Q$ is an isometry.*

Essentially the theorem is saying that, when the sectional curvature is not constant and the dimension is larger than 3, diffeomorphisms preserving the sectional curvature are isometries. A different and simpler proof of the above theorem was given in 1974 by S. T. Yau [46]. In this dissertation we are only going to present Kulkarni's proof, as Yau used techniques which are beyond the scope of this dissertation. For 2-dimensional Riemannian manifolds (surfaces), Theorem 7.1.1 is a kind of conserve to the Theorema Egregium of Gauss, and is completely false, since the notion of a non-isotropic point does not make sense when the dimension is 2. Recalling Definition 6.2.1, a point $y \in M$ is said to be isotropic if $\mathcal{K}(\Pi) = const.$ for every 2-plane section $\Pi \in T_yM$, and is said to be non-isotropic otherwise. There are known examples of 2-dimensional manifolds where there exists sectional curvature preserving diffeomorphisms between them but they are non-locally isometric. One can find some of these examples in Kreyszig [41, p.164] and in do Carmo [42, p. 237]. For 3-dimensional Riemannian manifolds, Yau in [46], constructed a counter example for open 3-dimensional Riemannian manifolds and proved the following theorem:

Theorem 7.1.2 (Yau, 1974). *Let $h : P \to Q$ be a sectional curvature preserving diffeomorphism between two compact Riemannian manifolds of dimension ≥ 3. Suppose that P is nowhere constantly curved, that is, the sectional curvature function of P is not constant at each point of P. Then h is an isometry.*

For the 3-dimensional case, Kulkarni [44], had only limited results. He assumed compactness and confined the sign of curvature (he assumed that the sectional curvature of P is everywhere negative). So, under the dimension restriction (≥ 4), Theorem 7.1.1 is the best possible. And so we observe that the different behaviours of curvature preserving diffeomorphisms $h : P \to Q$ obviously depends on the dimension n of P, namely $n = 2$, $n = 3$ or $n \geq 4$. To the reason for this appearance it should be mentioned that the higher the dimension the more conditions the diffeomorphism has to satisfy. Kulkarni in [44], also looked at stronger affirmations for Theorem 7.1.1, such as the case when P has only a smooth metric. However, we will not go through that here.

In order to prove Theorem 7.1.1, some preliminary work will have to done. In Subsections 7.2 and 7.3 we will go trough definitions and results needed for an adequate proof of the theorem as done by Kulkarni [44]. The theorem is then proved by considering independently the conformally non-flat case (Subsection 7.4) and the conformally flat case (Subsection 7.5).

7.2 A General Theorem

First, we need to prove that if the dimension of P is ≥ 3, then a curvature preserving diffeomorphism is conformal on the closure of the set of points which are not isotropic. Essentially, we are proving that the curvature, in general, determines the conformal class of a metric. We begin with a few definitions.

Definition 7.2.1. *Let (X, τ) be a topological space and let A be a subset of X. We define the closure of A in (X, τ), which we denoted with $\overline{A}$ or* $\mathrm{cl}(A)$, *by:*

$y \in \overline{A}$ if and only if every open set U containing y contains a point of A.

Or in symbols:

$$\overline{A} = \{y \in X \mid \forall U \in \tau \text{ such that } y \in U,\ U \cap A \neq \varnothing\}.$$

Definition 7.2.2. *Let (X, τ) be a topological space. The set $A \subseteq X$ is said to be dense in X if the intersection of every nonempty open set with A is nonempty, that is, $A \cap U \neq \varnothing$ for all $U \in \tau \backslash \{\varnothing\}$.*

Intuitively, the closure can be thought of as all the points that are either in A or "near/close" to A. The closure allows for "points on the edge of A". A dense set in one that is close to everything, in other words, for every point in X, the point is either in A or arbitrarily "close" to A. So, A is dense in X if and only if $\overline{A} = X$ (i.e. the closure is the entire space X).

Definition 7.2.3. *Let W be real vector space equipped with an inner product g. A "curvature tensor" with respect to g is a bilinear map $\mathcal{R} : W \times W \to \operatorname{End} W$, which satisfies the same algebraic properties of a Riemannian curvature tensor:*

(i) $\mathcal{R}(J, K) = -\mathcal{R}(K, J)$;

(ii) $\mathcal{R}(J, K)L + \mathcal{R}(K, L)J + \mathcal{R}(L, J)K = 0$;

(iii) $\langle \mathcal{R}(J, K)L, H\rangle = \langle \mathcal{R}(L, H)J, K\rangle$.

Remark 7.2.1. *This "curvature tensor" above is also referred to as the algebraic curvature tensor. Algebraic curvature tensors are introduced to put things in an abstract setting. Essentially what we doing is developing the notion of a curvature structure in a more general set up by replacing the tangent bundle of P by an arbitrary vector space with an inner product.*

Definition 7.2.4. *We define the "sectional curvature" the same way we did in Section 4: Let $\{J, K\}$ be a basis for a non-degenerate 2-plane section $\Pi \subseteq W$. The "sectional curvature" $\mathcal{K}(\Pi)$ is defined to be*

$$\mathcal{K}(J, K) = \frac{\langle \mathcal{R}(J, K)J, K\rangle}{\langle J, J\rangle\langle K, K\rangle - \langle J, K\rangle^2}.$$

Theorem 7.2.1. *Let $\mathcal{R} : W \times W \to \operatorname{End} W$, $\widetilde{\mathcal{R}} : \widetilde{W} \times \widetilde{W} \to \operatorname{End} \widetilde{W}$ be curvature tensors with respect to inner products $g, \tilde{g}$ respectively, where W, $\widetilde{W}$ are real vector spaces of dimension ≥ 3 equipped with $g, \tilde{g}$ respectively. Let $\mathcal{K}$, $\widetilde{\mathcal{K}}$ be the corresponding sectional curvatures. Suppose that $\mathcal{K}$ is not a constant, and that $h : W \to \widetilde{W}$ is a sectional curvature preserving linear isomorphism. Then, $g(h(u), h(w)) = k \cdot g(u, w)$, where k is a constant. That is, h is a homothety.*

As an immediate consequence we have the following two theorems which are essentially the same,

Theorem 7.2.2. *Let $h : (P, g) \to (Q, \tilde{g})$ be a sectional curvature preserving diffeomorphism between two Riemannian manifolds of dimension ≥ 3. Then h is conformal on the closure of the set of non-isotropic points.*

Theorem 7.2.3. *Let $h : (P, g) \to (Q, \tilde{g})$ be a curvature-preserving diffeomorphism of two Riemannian manifolds with dimension ≥ 3. Suppose that the set $\{y \in P \mid y \text{ is non-isotropic}\}$ is dense in P. Then h is conformal.*

It is these theorems which generically reduces considerations about curvature-preserving maps to those about conformal maps. In particular, we attempt to answer the classical problem: when is a conformal map of Riemannian manifold into itself necessarily an isometry (or a homothety)? We will deal with the concept of conformal maps in detail in the next subsection.

Before we can prove Theorem 7.2.1, we need the following two lemmas. The first Lemma (Cf. Eisenhart [40, p. 83]) is a result which is essentially equivalent to Proposition 6.2.1:

Lemma 7.2.4. *Let W be a vector space endowed with an inner product g and curvature tensor $\mathcal{R}$. The following are equivalent:*

(i) *The sectional curvature is constant.*

(ii) *$\mathcal{R}$ is given by*

$$\mathcal{R}(J, K)L = k \cdot \{\langle J, L\rangle K - \langle K, L\rangle J\}, \quad \textit{where } k \textit{ is a constant.}$$

Lemma 7.2.5. *Under the hypothesis of Theorem 7.2.1, W admits an acceptable frame.*

A frame is an orthonormal basis of the vector space W. If for each triple of different indices η, ω, ν, we have that the values of $\mathcal{K}(e_\eta, e_\omega)$, $\mathcal{K}(e_\eta, e_\nu)$ and of $\mathcal{K}(e_\omega, e_\nu)$ are all different, then an orthonormal set of vectors $\{e_\eta\}$ is referred to as acceptable.

Proof of Lemma 7.2.5. (A) We begin the proof by first supposing that the dimension of the vector space W is equal to 3. Let $\{e_\eta, e_\omega, e_\nu\}$ be any frame. If $\mathcal{K}(e_\eta, e_\omega)$, $\mathcal{K}(e_\eta, e_\nu)$, $\mathcal{K}(e_\omega, e_\nu)$ are not all distinct, we have two cases:

(1) All equal to a constant $\mathcal{K}_0$. Using Lemma 7.2.4, we can observe that the curvature would be constant if also the mixed components $\mathcal{R}_{\eta\nu\omega\nu} \cdots$ are zero. So we are going to suppose that $\mathcal{R}_{\eta\nu\omega\nu} \neq 0$. Let g be an orthogonal transformation which rotates the $\{e_\eta, e_\omega\}$-plane through an angle θ, and leaves e_ν fixed. Then we obtain a new frame:

$$\begin{aligned} ge_\eta &= ue_\eta + we_\omega \\ ge_\omega &= -we_\eta + ue_\omega, \end{aligned}$$

where $u = \cos\theta, w = \sin\theta$. Using the formula in Definition 7.2.4, the properties of the Riemannian curvature tensor from Section 4 and some basic trigonometric identities, we obtain the following

$$\begin{aligned}
\mathcal{K}(ge_\eta, ge_\omega) &= \mathcal{K}(e_\eta, e_\omega) = \mathcal{K}_0;\\
\mathcal{K}(ge_\eta, ge_\nu) &= u^2\mathcal{R}_{\eta\nu\eta\nu} + 2uw\mathcal{R}_{\eta\nu\omega\nu} + w^2\mathcal{R}_{\omega\nu\omega\nu}\\
&= \mathcal{K}_0 + \sin 2\theta\mathcal{R}_{\eta\nu\omega\nu};\\
\mathcal{K}(ge_\omega, ge_\nu) &= u^2\mathcal{R}_{\omega\nu\omega\nu} + 2uw\mathcal{R}_{\eta\nu\omega\nu} + w^2\mathcal{R}_{\eta\nu\eta\nu}\\
&= \mathcal{K}_0 - \sin 2\theta\mathcal{R}_{\eta\nu\omega\nu}.
\end{aligned}$$

And so we see that, since we supposed that the mixed components of $\mathcal{R}$ are not equal to zero, the new frame is acceptable when $\sin 2\theta$ is non-zero.

(2) For the second case we suppose that $\mathcal{K}(e_\eta, e_\omega) = \mathcal{K}(e_\eta, e_\nu) \neq \mathcal{K}(e_\omega, e_\nu)$. Let $|\mathcal{K}(e_\eta, e_\omega) - \mathcal{K}(e_\omega, e_\nu)| = \varepsilon$. Using the same orthogonal transformation g as in case (1) above, we obtain

$$\begin{aligned}
&\mathcal{K}(ge_\eta, ge_\omega) = \mathcal{K}(e_\eta, e_\omega);\\
&\mathcal{K}(ge_\eta, ge_\nu) - \mathcal{K}(e_\eta, e_\nu) = \sin^2\theta(\mathcal{K}(e_\omega, e_\nu) - \mathcal{K}(e_\eta, e_\nu)) + \sin 2\theta\mathcal{R}_{\eta\nu\omega\nu};\\
&\mathcal{K}(ge_\omega, ge_\nu) - \mathcal{K}(e_\omega, e_\nu) = \sin^2\theta(\mathcal{K}(e_\eta, e_\nu) - \mathcal{K}(e_\omega, e_\nu)) - \sin 2\theta\mathcal{R}_{\eta\nu\omega\nu}.
\end{aligned}$$

We pick θ to be very small that each of the differences in the last two equations above do not vanish (using small angle approximations), but have absolute value less than $\varepsilon/2$. It is easily seen that the new frame is acceptable.

(B) We now finally prove the general case. To every frame $a = \{e_1, \ldots, e_n\}$ we assign an integer $p(a)$. This integer will reflect the number of distinct triples $\{\eta, \omega, \nu\}$ such that $\mathcal{K}(e_\eta, e_\omega)$, $\mathcal{K}(e_\eta, e_\nu)$ and $\mathcal{K}(e_\omega, e_\nu)$ are all distinct. Now, $a \mapsto p(a)$ is a lower semi-continuous function. This fact follows from the continuity of curvature. $a \mapsto p(a)$ being a lower semi-continuous function means that, there is a neighbourhood of the identity in the orthogonal group $O(n)$, such that by a variation via g belonging to this neighbourhood, the acceptable triplets remain acceptable and so we have the inequality $p(a) \leq p(ga)$. Next, we select a a such that $p(a)$ is maximum. If a is not acceptable, then there will be a triplet $\{e_\eta, e_\omega, e_\nu\}$ such that $\mathcal{K}(e_\eta, e_\omega)$, $\mathcal{K}(e_\eta, e_\nu)$ and $\mathcal{K}(e_\omega, e_\nu)$ are not all distinct. Now, in a neighbourhood of the identity in $O(n)$, variations of the triplet $\{e_\eta, e_\omega, e_\nu\}$ by all elements do not always give a 3-dimensional space of constant curvature. This is because otherwise, curvature itself would be constant. Therefore, there exists amply small variations of $\{e_\eta, e_\omega, e_\nu\}$ such that the resulting 3-dimensional space is not of constant curvature. So, by part (A) of this proof, there is amply small variations of $\{e_\eta, e_\omega, e_\nu\}$ such that the triplet $\{e_\eta, e_\omega, e_\nu\}$ becomes acceptable, while the other acceptable triplets remain acceptable. This is a contradiction on the choice we made for a, and thus thereby proving the lemma. ∎

Proof of Theorem 7.2.1. Let $\{e_1, \ldots, e_n\}$ be an acceptable frame. Set $he_\eta = \tilde{e}_\eta$ and $\tilde{g}(\tilde{e}_\eta, \tilde{e}_\omega) = b_{\eta\omega}$. Since h is a sectional curvature preserving linear isomorphism, we have that $\mathcal{K}(e_\eta, e_\omega) = \widetilde{\mathcal{K}}(\tilde{e}_\eta, \tilde{e}_\omega)$. Using Definition 7.2.4, the components of $\widetilde{\mathcal{R}}$ with respect to the basis $\{\tilde{e}_\eta\}$ then becomes

$$\widetilde{\mathcal{R}}_{\eta\omega\eta\omega} = \left(b_{\eta\eta}b_{\omega\omega} - b_{\eta\omega}^2\right)\mathcal{R}_{\eta\omega\eta\omega}. \tag{14}$$

Let u, $w \in \mathbb{R}$ such that not both of them are allowed to be zero and let $\{\eta, \omega, \nu\}$ be distinct. Then, by using Definition 7.2.4, the properties from Definition 1.2.1 and the properties from Proposition 4.2.1, we get the following two equations

$$\mathcal{K}(ue_\eta + we_\omega, e_\nu) = \frac{u^2\mathcal{R}_{\eta\nu\eta\nu} + 2uw\mathcal{R}_{\eta\nu\omega\nu} + w^2\mathcal{R}_{\omega\nu\omega\nu}}{u^2 + w^2} \tag{15}$$

$$\widetilde{\mathcal{K}}(u\tilde{e}_\eta + w\tilde{e}_\omega, \tilde{e}_\nu) = \frac{u^2\widetilde{\mathcal{R}}_{\eta\nu\eta\nu} + 2uw\widetilde{\mathcal{R}}_{\eta\nu\omega\nu} + w^2\widetilde{\mathcal{R}}_{\omega\nu\omega\nu}}{(u^2b_{\eta\eta} + 2uwb_{\eta\omega} + w^2b_{\omega\omega})b_{\nu\nu} - (ub_{\eta\nu} + wb_{\omega\nu})^2}. \tag{16}$$

We obtain an identity in u and w by cross multiplying equations (15) and (16). The next three equations are obtained by comparing the coefficients of u^3w, uw^3 and u^2w^2 respectively.

$$\widetilde{\mathcal{R}}_{\eta\nu\omega\nu} = \mathcal{R}_{\eta\nu\omega\nu}(b_{\eta\eta}b_{\nu\nu} - b_{\eta\nu}^2) + \mathcal{R}_{\eta\nu\eta\nu}(b_{\eta\omega}b_{\nu\nu} - b_{\eta\nu}b_{\omega\nu}); \tag{17}$$

$$\widetilde{\mathcal{R}}_{\eta\nu\omega\nu} = \mathcal{R}_{\eta\nu\omega\nu}(b_{\omega\omega}b_{\nu\nu} - b_{\omega\nu}^2) + \mathcal{R}_{\omega\nu\omega\nu}(b_{\eta\omega}b_{\nu\nu} - b_{\eta\nu}b_{\omega\nu}); \tag{18}$$

$$\begin{aligned}\widetilde{\mathcal{R}}_{\eta\nu\eta\nu} + \widetilde{\mathcal{R}}_{\omega\nu\omega\nu} =& \mathcal{R}_{\eta\nu\eta\nu}(b_{\omega\omega}b_{\nu\nu} - b_{\omega\nu}^2) \\ &+ \mathcal{R}_{\omega\nu\omega\nu}(b_{\eta\eta}b_{\nu\nu} - b_{\eta\nu}^2) \\ &+ 4\mathcal{R}_{\eta\nu\omega\nu}(b_{\eta\omega}b_{\nu\nu} - b_{\eta\nu}b_{\omega\nu}).\end{aligned} \tag{19}$$

Next, to avoid the tedious exercise of re-writing the expressions from the equations above, we let

$$\begin{aligned}M &= (b_{\eta\eta}b_{\nu\nu} - b_{\eta\nu}^2) - (b_{\omega\omega}b_{\nu\nu} - b_{\omega\nu}^2),\\ N &= b_{\eta\omega}b_{\nu\nu} - b_{\eta\nu}b_{\omega\nu},\\ x &= \mathcal{R}_{\eta\nu\eta\nu} - \mathcal{R}_{\omega\nu\omega\nu},\\ y &= \mathcal{R}_{\eta\nu\omega\nu}.\end{aligned}$$

Equation (17) $-$ Equation (18) $= Nx + My = 0$. Simplifying equation (19) using equation (14) we get $-Mx + 4Ny = 0$. Combining these two equations we get $M(y+x) - N(4y-x) = 0$. Now, $x \neq 0$, this is due to the fact that the frame is acceptable, and so we have that $M = N = 0$. Using property 2 from Definition 1.2.4, $N = 0$ amounts to

$$\begin{aligned} 0 &= b_{\eta\omega} b_{\nu\nu} - b_{\eta\nu} b_{\omega\nu} \\ &= \tilde{g}(\tilde{e}_\eta, \tilde{e}_\omega)\tilde{g}(\tilde{e}_\nu, \tilde{e}_\nu) - \tilde{g}(\tilde{e}_\eta, \tilde{e}_\nu)\tilde{g}(\tilde{e}_\omega, \tilde{e}_\nu) \\ &= \|\tilde{e}_\eta\|\|\tilde{e}_\omega\| \cos\theta \|\tilde{e}_\nu\|\|\tilde{e}_\nu\| \cos\gamma - \|\tilde{e}_\eta\|\|\tilde{e}_\nu\| \cos\varphi \|\tilde{e}_\omega\|\|\tilde{e}_\nu\| \cos\psi \\ &= \cos\theta - \cos\varphi\cos\psi, \end{aligned} \tag{$\boldsymbol{a}$}$$

where we let $\alpha = \angle(\tilde{e}_\eta, \tilde{e}_\eta)$, $\beta = \angle(\tilde{e}_\omega, \tilde{e}_\omega)$, $\gamma = \angle(\tilde{e}_\nu, \tilde{e}_\nu)$, $\theta = \angle(\tilde{e}_\eta, \tilde{e}_\omega)$, $\varphi = \angle(\tilde{e}_\omega, \tilde{e}_\nu)$, $\psi = \angle(\tilde{e}_\nu, \tilde{e}_\eta)$. By symmetry, we also have,

$$\cos\varphi - \cos\theta\cos\psi = 0, \tag{$\boldsymbol{b}$}$$

$$\cos\psi - \cos\theta\cos\varphi = 0. \tag{$\boldsymbol{c}$}$$

Combining equations ($\boldsymbol{a}$) and ($\boldsymbol{b}$) we get

$$\begin{aligned} 0 &= \cos\theta - \cos\theta\cos\psi\cos\psi \\ &= \cos\theta(1 - \cos^2\psi) \\ \therefore\ \cos\theta &= 0 \quad \text{or} \quad \cos\psi = \pm 1. \end{aligned}$$

But, since $\cos\psi$ can't be equal to ± 1, we just have that $\cos\theta = 0$. This obviously means that $\tilde{e}_\eta$ and $\tilde{e}_\omega$ are orthogonal. Following the same method above of combining equations ($\boldsymbol{a}$), ($\boldsymbol{b}$), ($\boldsymbol{c}$) we also get $\cos\varphi = 0$, $\cos\psi = 0$, which then implies that $\{\tilde{e}_\eta, \tilde{e}_\omega, \tilde{e}_\nu\}$ are all orthogonal to each other. Now, for $M = 0$, we obtain

$$\begin{aligned} 0 &= (b_{\eta\eta} b_{\nu\nu} - b_{\eta\nu}^2) - (b_{\omega\omega} b_{\nu\nu} - b_{\omega\nu}^2) \\ &= (\tilde{g}(\tilde{e}_\eta, \tilde{e}_\eta)\tilde{g}(\tilde{e}_\nu, \tilde{e}_\nu) - \tilde{g}(\tilde{e}_\eta, \tilde{e}_\nu)\tilde{g}(\tilde{e}_\eta, \tilde{e}_\nu)) \\ &\quad - (\tilde{g}(\tilde{e}_\omega, \tilde{e}_\omega)\tilde{g}(\tilde{e}_\nu, \tilde{e}_\nu) - \tilde{g}(\tilde{e}_\omega, \tilde{e}_\nu)\tilde{g}(\tilde{e}_\omega, \tilde{e}_\nu)) \\ &= (\|\tilde{e}_\eta\|^2 \cos\alpha \|\tilde{e}_\nu\|^2 \cos\gamma - \|\tilde{e}_\eta\|^2 \|\tilde{e}_\nu\|^2 \cos^2\psi) \\ &\quad - (\|\tilde{e}_\omega\|^2 \cos\beta \|\tilde{e}_\nu\|^2 \cos\gamma - \|\tilde{e}_\omega\|^2 \|\tilde{e}_\nu\|^2 \cos^2\varphi) \\ &= (\|\tilde{e}_\eta\|^2 \|\tilde{e}_\nu\|^2) - (\|\tilde{e}_\omega\|^2 \|\tilde{e}_\nu\|^2) \\ &= \|\tilde{e}_\eta\|^2 - \|\tilde{e}_\omega\|^2, \end{aligned}$$

therefore

$$\|\tilde{e}_\eta\| = \|\tilde{e}_\omega\|.$$

By using the same procedure as above, expressions similar to M obtained by cyclic permutations of $\{\eta, \omega, \nu\}$, demonstrate that $\{\tilde{e}_\eta, \tilde{e}_\omega, \tilde{e}_\nu\}$ all have equal length. As this is true for each triple $\{\eta, \omega, \nu\}$, we can observe that $\{\tilde{e}_\eta\}$ is an orthogonal frame whose vectors are of equal length. In other words, h is a homothety. ■

7.3 Conformal Changes of Riemannian Metrics

In light of Theorem 7.2.1, we need to do a comprehensive study of a conformal change of a metric. Let P be an n-dimensional differentiable connected Riemannian manifold with metric tensor g. Since we consider several Riemannian metrics on the same manifold P, in order to distinguish g from other metrics on P, we shall denote by (P, g) the Riemannian manifold P with metric tensor g. If $\tilde{g}$ is another metric on P, and there exists a positive real-valued function $h \in C^\infty(P)$ on P (i.e. $h : P \to \mathbb{R}$) such that $\tilde{g}(J, K) = hg(J, K)$, for all $J, K \in \mathfrak{X}(P)$, then g and $\tilde{g}$ are said to be conformally related, or to be conformal to each other, and such a change of a metric $g \to \tilde{g}$ is referred to as a conformal change of a Riemannian metric.

In the above discussion, what has been changed is the Riemannian metric g at each point of the manifold P. We are now going to consider point transformations which induce a conformal change of metric of the manifold. Let $\tau : P \to Q$ be a diffeomorphism between two Riemannian manifolds (P, g) and (Q, g'). Then $\tilde{g} = \tau^{-1} g'$ is a Riemannian metric on P which is referred to as the induced metric. When g and $\tilde{g}$ are conformally related, that is, when there exists a function h on P such that $\tilde{g} = hg$, we call $\tau : (P, g) \to (Q, g')$ a conformal transformation (or a conformal diffeomorphism). Another way of saying this is, a diffeomorphism $\tau : P \to Q$ is called a conformal transformation if it pulls g' back to a metric that is conformal to g:

$$\tau^* g' = hg \text{ for some positive } h \in C^\infty(P).$$

Specifically, if h is a constant function ($h = \text{constant} \neq 0$), then τ is said to be a homothetic transformation or a homothety (Cf. Subsection 7.2) and if h is identically equal to 1 ($h = 1$), then τ is said to be an isometric transformation or an isometry.

Remark 7.3.1. *If (P, g) is a Riemannian manifold with Riemannian metrics g and $\tilde{g}$ conformally related on P, then the angles between two vectors with respect to g and $\tilde{g}$ are always equal at each point of P, that is, $g(J, K)/[\sqrt{g(J, J)} \cdot \sqrt{g(K, K)}] = \tilde{g}(J, K)/[\sqrt{\tilde{g}(J, J)} \cdot \sqrt{\tilde{g}(K, K)}]$, for all pairs of vectors J, K.*

If g and $\tilde{g}$ are conformal metrics on a differentiable manifold P, there is no reason to expect that the curvature tensor, Ricci curvature, scalar curvature of g and $\tilde{g}$ should be closely related to each other. In this subsection, we look at how the curvature and these related quantities change under a conformal transformation. We will omit most the proofs in these subsection as they are tedious and are straightforward calculations. Most of the proofs can be found in Eisenhart [40, p. 89]. First, we need to determine how the Riemannian connection changes when a metric is changed conformally. We will denote the covariant derivative of a vector field K with respect to a vector field J and with respect to the metric g, just as we did in Section 2, that is, by $\nabla_J K$. The corresponding quantities such as covariant derivatives, curvature tensors and others with respect to the metric $\tilde{g}$ are denoted by a tilde overhead. We will we abbreviate $g(J, K) = \langle J, K\rangle$, $\tilde{g}(J, K) = \widetilde{\langle X, Y\rangle}$ whenever convenient.

Proposition 7.3.1. *Let P be a differentiable manifold, let g and $\tilde{g}$ be two conformally related metrics on P and let ∇ and $\widetilde{\nabla}$ be the Riemannian connections of g and $\tilde{g}$ respectively. Then, for any two vector fields J, K, we have*

$$\widetilde{\nabla}_J K = \nabla_J K + S(J, K),$$

where $S(J, K) = \frac{1}{2h}\{(Jh)K + (Kh)J - g(J, K)\operatorname{grad} h\}$ and grad h is the gradient of h with respect to the metric g, that is, $J(h) = g(J, \operatorname{grad} h)$.

Remark 7.3.2. *For each $y \in P$ and for any $J \in T_yP$, $\operatorname{grad} h$ is a vector in T_yP such that the inner product with J is the derivation of h by J. This definition is compatible with the one in standard multi-variable calculus. The interpretation of the gradient in a Riemannian manifold is analogous to the one in Euclidean space: its direction is the direction of steepest ascent of h and it is orthogonal to the level sets of h; and its length is the maximum directional derivative of h in any direction.*

Proof of Proposition 7.3.1. Since $S(J, K) = S(K, J)$ by its definition, one has

$$\begin{aligned}
\widetilde{\nabla}_J K - \widetilde{\nabla}_K J &= \nabla_J K - \nabla_K J + S(J, K) - S(K, J) \\
&= \nabla_J K - \nabla_K J \\
&= [J, K], \quad \text{for all } J, K \in \mathfrak{X}(P).
\end{aligned}$$

Hence, $\widetilde{\nabla}$ is symmetric. To see that $\widetilde{\nabla}$ is metric-compatible, we note that

$$\begin{aligned}
h[g(S(J,K),L)+g(K,S(J,L))] &= h\bigg[g\bigg(\frac{1}{2h}\{(Jh)K+(Kh)J-g(J,K)\,\mathrm{grad}\,h\},L\bigg)\\
&\quad + g\bigg(K,\frac{1}{2h}\{(Jh)L+(Lh)J-g(J,L)\,\mathrm{grad}\,h\}\bigg)\bigg]\\
&= \frac{1}{2}(Jh)g(K,L)+\frac{1}{2}(Kh)g(J,L)\\
&\quad - \frac{1}{2}g(J,K)g(\mathrm{grad}\,h,L)+\frac{1}{2}(Jh)g(K,L)\\
&\quad + \frac{1}{2}(Lh)g(K,J)-\frac{1}{2}g(J,L)g(K,\mathrm{grad}\,h)\\
&= (Jh)g(K,L)+\frac{1}{2}(Kh)g(J,L)+\frac{1}{2}(Lh)g(K,L)\\
&\quad - \frac{1}{2}g(J,K)(Lh)-\frac{1}{2}g(J,L)(Kh)\\
&= (Jh)g(K,L),
\end{aligned}$$

and hence

$$\begin{aligned}
J(\tilde{g}(K,L)) &= J(h)g(K,L)+hg(\nabla_J K,L)+hg(K,\nabla_J L)\\
&= hg(S(J,K),L)+hg(K,S(J,L))+hg(\nabla_J K,L)+hg(K,\nabla_J L)\\
&= hg(\nabla_J K+S(J,K),L)+hg(K,\nabla_J L+S(J,L))\\
&= \tilde{g}(\widetilde{\nabla}_J K,L)+\tilde{g}(K,\widetilde{\nabla}_J L).
\end{aligned}$$

Therefore we have shown that $\widetilde{\nabla}$ is compatible with $\tilde{g}$. By the uniqueness of Riemannian connections on a Riemannian manifold (Cf. Theorem 2.3.1 in Section 2), we know that $\widetilde{\nabla}$ defined as $\widetilde{\nabla}_J K = \nabla_J K + S(J,K)$ is the Riemannian connection of $(P,\tilde{g})$. ■

Now, as h is a positive real-valued function we may represent it as $h = e^{2\varphi}$ for $\varphi = \frac{1}{2}\ln h$. φ is also a positive C^∞ real-valued function. We simply choose to work with the conformal factor in this form ($e^{2\varphi}$) to make some computations simpler. We now substitute $h = e^{2\varphi}$ into the formula for S above. Since

$$\frac{Jh}{2h} = J\varphi, \quad \frac{\mathrm{grad}\,h}{2h} = \mathrm{grad}\,\varphi,$$

we have

$$S(J,K) = \{J\varphi K + K\varphi J - \langle J,K\rangle G\},$$

where we have denoted $\mathrm{grad}\,\varphi$, with respect to the metric g, by G and we have abbreviated $g(J,K)$ to $\langle J,K\rangle$.

The Riemannian metric enables us to define a hessian of a function $\varphi : P \to \mathbb{R}$,

Definition 7.3.1. *Let (P, g) be a Riemannian manifold equipped by the Riemannian connection ∇. Given a scalar field $\varphi : P \to \mathbb{R}$ and any two vector fields $J, K \in \mathfrak{X}(P)$, the Riemannian Hessian of φ is the covariant 2 -tensor field* $\mathrm{hess}_\varphi := \nabla^2\varphi := \nabla\nabla\varphi$*, defined by*

$$\begin{aligned} \mathrm{hess}_\varphi(J, K) &= (\nabla_J d\varphi)\, K \\ &= JK\varphi - (\nabla_J K)\,\varphi \\ &= J\langle K, \mathrm{grad}\,\varphi\rangle - (\nabla_J K)\,\varphi \\ &= \langle K, \nabla_J(\mathrm{grad}\,\varphi)\rangle\,. \end{aligned}$$

The Hessian is a $(0, 2)$-tensor field which is bilinear and symmetric.

For convenience, we use the function φ to define another $(0, 2)$-tensor field which is also bilinear and symmetric:

$$V(J, K) = \mathrm{hess}_\varphi(J, K) - J\varphi K\varphi.$$

Next, we let $\mathcal{T}(P)$ be the tangent bundle of P and $\mathcal{T}^*(P)$ be the cotangent bundle of P. Since V is a symmetric $(0, 2)$-tensor field, it defines a bundle map $V_0 : \mathcal{T}(P) \to \mathcal{T}^*(P)$ given by $(V_0(J))K = V(J, K)$, for any two vector fields J, K. Now, V_0 defines an endomorphism of the tangent bundle $\mathcal{T}(P)$. This is due to fact that, we are able to canonically identify $\mathcal{T}^*(P)$ with $\mathcal{T}(P)$, thanks to the Riemannian metric. We will again use V_0 to denote this endomorphism. If we let $G = \mathrm{grad}\,\varphi$, using the expression for the hessian and using the expression for the tensor field V, we obtain the following expression for V_0:

$$V_0(J) = \nabla_J G - J\varphi G.$$

The next few propositions illustrates the deformation of the Riemannian curvature tensor, the Ricci tensor, the Scalar curvature and other tensors under the conformal change of metric. All the formulas in these propositions are classical and most of the proofs can be found in Eisenhart [40, p. 89], where calculations are made in local coordinates.

Proposition 7.3.2. *Let P be a differentiable manifold with Riemannian metric g and let $\tilde{g} = e^{2\varphi}g$. Then, for any three vector fields J, K, L, the Riemannian curvature tensor of $\tilde{g}$ is related to that of g by the following formula:*

$$\widetilde{\mathcal{R}}(J, K)L = \mathcal{R}(J, K)L + T(J, K)L$$

where

$$T(J,K)L = \left\{V(K,L) + \langle K,L\rangle \|G\|^2\right\} J \\ - \left\{V(J,L) + \langle J,L\rangle \|G\|^2\right\} K \\ + \langle K,L\rangle V_0(J) - \langle J,L\rangle V_0(K), \quad \|G\| = \langle G,G\rangle^{\frac{1}{2}}.$$

In view of the proposition above, we are able to see why it was useful to define the tensor field V.

Proof. We start by using the definition of the Riemannian curvature tensor,

$$\begin{aligned}\widetilde{\mathcal{R}}(J,K)L =& \widetilde{\nabla}_J \widetilde{\nabla}_K L - \widetilde{\nabla}_K \widetilde{\nabla}_J L - \widetilde{\nabla}_{[J,K]} L \\ =& \widetilde{\nabla}_J [\nabla_K L + S(K,L)] - \widetilde{\nabla}_K [\nabla_J L + S(J,L)] \\ &- \{\nabla_{[J,K]} L + S([J,K],L)\} \\ =& \nabla_J \{\nabla_K L + S(K,L)\} + S(J, \nabla_K L + S(J,L)) \\ &- \nabla_K \{\nabla_J L + S(J,L)\} - S(K, \nabla_J L + S(J,L)) \\ &- \{\nabla_{[J,K]} L + S([J,K],L)\},\end{aligned}$$

where by definition

$$\nabla_J S(K,L) := \nabla_J (S(K,L)) - S(\nabla_J K, L) - S(K, \nabla_J L).$$

Substituting

$$S(J,K) = \{J\varphi K + K\varphi J - \langle J,K\rangle G\}$$

into the expression for $\widetilde{\mathcal{R}}(J,K)L$ above, we get the desired result after a straightforward but tedious calculation. ■

Using the expression for the conformal transformation of the curvature from Proposition 7.3.2 we can find expressions for the Ricci curvature, the scalar curvature etc. upon taking traces.

Recalling from section 4, the Ricci tensor is a symmetric bilinear $(0,2)$ tensor field defined by: for $y \in P$ and $J, L \in T_y(P)$, the tangent space at y,

$$\mathrm{Ric}(J,L) = \mathrm{Trace}\{K \longrightarrow \mathcal{R}(J,K)L\}.$$

Proposition 7.3.3. *For any two vector fields J, L,*

$$\widetilde{\mathrm{Ric}}(J,L) = \mathrm{Ric}(J,L) + \mathcal{T}(J,L),$$

where

$$\mathcal{T}(J,L) = -\left\{(n-2)V(J,L) + (n-1)\langle J,L\rangle \|G\|^2 + \langle J,L\rangle \,\mathrm{Trace}\, V_0\right\}.$$

Ric defines an endomorphism of the tangent bundle:

$$\mathrm{Ric}_0 = \mathcal{T}(P) \xrightarrow{c} \mathcal{T}^*(P) \xrightarrow{i} \mathcal{T}(P)$$

where c is the canonical map defined by Ric and i is the identification $\mathcal{T}^*(P)$ with $\mathcal{T}(P)$ via the Riemannian metric. Ric_0 is the corresponding linear transformation defined by $\langle \mathrm{Ric}_0 J, K\rangle = \mathrm{Ric}(J, L)$.

Proposition 7.3.4. *For any vector filed J,*

$$\begin{aligned}\widetilde{\mathrm{Ric}_0} J =& \frac{\mathrm{Ric}_0 J}{h} - \frac{n-2}{h} V_0(J)\\ &- \frac{1}{h}\left\{(n-1)\|G\|^2 + \operatorname{Trace} V_0\right\} J.\end{aligned}$$

Again, recalling from section 4, Scalar curvature is a function $P \xrightarrow{\mathrm{Sc}} \mathbb{R}$ defined by $\mathrm{Sc}(y) = \operatorname{Trace}\mathrm{Ric}_0(y)$.

Proposition 7.3.5.

$$\widetilde{\mathrm{Sc}} = \frac{\mathrm{Sc}}{h} - \frac{n(n-1)}{h}\|G\|^2 - \frac{2n-2}{h}\operatorname{Trace} V_0.$$

Proposition 7.3.6.

$$\begin{aligned}T(J,K)L =& \frac{1}{n-2}\{\mathcal{T}(J,L)K - \mathcal{T}(K,L)J\}\\ &+ \frac{1}{n-2}\{(\widetilde{\langle J, L\rangle}\widetilde{\mathrm{Ric}_0}K - \langle J, L\rangle \mathrm{Ric}_0 K)\\ &\quad - (\widetilde{\langle K, L\rangle}\widetilde{\mathrm{Ric}_0}J - \langle K, L\rangle \mathrm{Ric}_0 J))\}\\ &+ \frac{1}{(n-1)(n-2)}\{(\widetilde{\langle K, L\rangle}\widetilde{\mathrm{Sc}} - \langle K, L\rangle \mathrm{Sc})J\\ &\quad - (\widetilde{\langle J, L\rangle}\,\mathrm{Sc} - \langle J, L\rangle \mathrm{Sc})K\}.\end{aligned}$$

From the proposition above we obtain a second order conformal invariant (we explain how in the next paragraph) which is referred to as the Weyl conformal curvature tensor, named after the German mathematician Hermann Weyl, who was the first to consider it:

$$\begin{aligned}C(J,K)L =& \mathcal{R}(J,K)L\\ &+ \frac{1}{n-2}\{\mathrm{Ric}(K,L)J - \mathrm{Ric}(J,L)K + \langle K, L\rangle \mathrm{Ric}_0 J - \langle J, L\rangle \mathrm{Ric}_0 K\}\\ &- \frac{\mathrm{Sc}}{(n-1)(n-2)}\{\langle K, L\rangle J - \langle J, L\rangle K\}. \qquad (\dagger)\end{aligned}$$

In mathematics, invariant means a function, quantity or property which remains unchanged when a specified transformation is applied. So, by C being a conformal invariant, we mean a quantity which remains invariant under the conformal changes of metrics, that is, under changes of metrics which stretch the length of vectors but preserve the angles between any pair of vectors. To see that C is a conformal invariant we perform the simple computation $\widetilde{C}(J,K)L - C(J,K)L = 0$, where we use $\widetilde{\mathcal{R}}(J,K)L - \mathcal{R}(J,K)L = T(J,K)L$ (Cf. Proposition 7.3.2) and for T we use the expression from Proposition 7.3.6. And so $\widetilde{C}(J,K)L = C(J,K)L$. It is very easy to use C under the sectional curvature preserving hypothesis as we will see in the next subsection. As noted in Section 4, the Ricci and scalar curvatures contain only part of the information encoded into the Riemannian curvature tensor. The Weyl tensor, encodes all the rest.

There is a well known and very important result due to Weyl which we will make use of in the next subsection. But before we can state this result, we first need to look at the the notion of conformally flat Riemannian manifolds. Roughly speaking, a Riemannian manifold (P, g) is conformally flat if it is taken to a flat manifold by a conformal transformation.

Definition 7.3.2. *A Riemannian manifold (P, g) is said to be conformally flat if every point of y of M, there exists a neighborhood U of y and a C^∞ real-valued function φ defined on U such that $(U, e^{2\varphi}g)$ is flat (i.e. the curvature of $e^{2\varphi}g$ vanishes on U, $\mathcal{R}(e^{2\varphi}g|_U) = 0$). More specifically, (P, g) is called conformally flat if it can be covered by neighborhoods $\{U_\alpha\}$ such that there exists a conformal map $\tau_\alpha : U_\alpha \to \mathbb{R}^n$.*

In a conformally flat Riemannian manifold P, each point of P is contained in a neighborhood on which the metric g can be written as $g = f \cdot g_0$ where f is a positive real-valued function defined on the neighborhood, and g_0 is the flat (i.e., Euclidean) metric. We note that we do not require f to be defined on the whole of P.

We can now state the result due to Weyl. This result gives us an obvious necessary condition to detect conformal flatness. For a detailed proof one can check Eisenhart [40, p. 92].

Theorem 7.3.1 (Weyl). *Let (P, g) be a Riemannian manifold whose dimension is ≥ 4. Then P is conformally flat if and only if $C \equiv 0$.*

This theorem essentially says that a necessary and sufficient condition for a metric to be conformally flat is that its Weyl conformal curvature tensor vanishes everywhere. So, for $n \geq 4$, the Weyl conformal curvature tensor is the only obstruction to conformal flatness.

7.4 Conformally Non-Flat Case

The following proposition demonstrates, under the curvature preserving condition, the relations between second order tensors associated to metrics g, $\tilde{g}$ respectively.

Proposition 7.4.1. *Let W be a real vector space endowed with inner products $g, \tilde{g}$ and curvature tensors $\mathcal{R}, \widetilde{\mathcal{R}}$ such that*

1. $\tilde{g} = \delta g$ *where* $\delta \in \mathbb{R} > 0$.
2. $\widetilde{\mathcal{K}} = \mathcal{K}$ *(equality of corresponding sectional curvatures).*

Then

(a) $\widetilde{\mathcal{R}} = \delta \mathcal{R}$,

(b) $\widetilde{\mathrm{Ric}} = \delta\, \mathrm{Ric}$,

(c) $\widetilde{\mathrm{Ric}}_0 = \mathrm{Ric}_0$,

(d) $\widetilde{\mathrm{Sc}} = \mathrm{Sc}$,

(e) $\widetilde{C} = \delta C$.

Proof. Using Definition 7.2.4, condition 2 means that for any two linearly independent vectors J, K we have:

$$\frac{\widetilde{\langle \widetilde{\mathcal{R}}(J,K)J,K \rangle}}{\widetilde{\langle J,J \rangle}\widetilde{\langle K,K \rangle} - \widetilde{\langle J,K \rangle}^2} = \frac{\langle \mathcal{R}(J,K)J,K \rangle}{\langle J,J \rangle \langle K,K \rangle - \langle J,K \rangle^2},$$

by using condition 1, the above equation becomes

$$\frac{\delta \langle \widetilde{\mathcal{R}}(J,K)J,K \rangle}{\delta \langle J,J \rangle \delta \langle K,K \rangle - (\delta \langle J,K \rangle)^2} = \frac{\langle \mathcal{R}(J,K)J,K \rangle}{\langle J,J \rangle \langle K,K \rangle - \langle J,K \rangle^2},$$

this reduces to

$$\langle \widetilde{\mathcal{R}}(J,K)J,K \rangle = \delta \langle \mathcal{R}(J,K)J,K \rangle. \qquad (\boldsymbol{A})$$

Replacing K by $K+L$ in ($\boldsymbol{A}$) and then using property (iii) from Definition 1.2.1, we see that

$$\langle \widetilde{\mathcal{R}}(J,K+L)J,K+L \rangle = \delta \langle \mathcal{R}(J,K+L)J,K+L \rangle$$
$$\langle \widetilde{\mathcal{R}}(J,K+L)J,K \rangle + \langle \widetilde{\mathcal{R}}(J,K+L)J,L \rangle = \delta \langle \mathcal{R}(J,K+L)J,K \rangle + \delta \langle \mathcal{R}(J,K+L)J,L \rangle$$

Using property (a) from Proposition 4.2.1 on the equation above, we get

$$\begin{aligned}
\text{LHS} &= \langle \widetilde{\mathcal{R}}(J,K)J,K\rangle + \langle \widetilde{\mathcal{R}}(J,L)J,K\rangle + \langle \widetilde{\mathcal{R}}(J,K)J,L\rangle + \langle \widetilde{\mathcal{R}}(J,L)J,L\rangle,\\
\text{RHS} &= \delta\langle \mathcal{R}(J,K)J,K\rangle + \delta\langle \mathcal{R}(J,L)J,K\rangle + \delta\langle \mathcal{R}(J,K)J,L\rangle + \delta\langle \mathcal{R}(J,L)J,L\rangle.
\end{aligned}$$

Now, using ($\boldsymbol{A}$) and property (iii) from Definition 7.2.3, the above reduces to

$$\langle \widetilde{\mathcal{R}}(J,K)J,K\rangle = \delta\langle \mathcal{R}(J,K)J,L\rangle.$$

Since this holds for every L, we have

$$\widetilde{\mathcal{R}}(J,K)J = \delta\mathcal{R}(J,K)J. \tag{$\boldsymbol{B}$}$$

Next, we replace J by $J+L$ in ($\boldsymbol{B}$) and again use property (a) from Proposition 4.2.1 to get

$$\begin{gathered}
\widetilde{\mathcal{R}}(J+L,K)(J+L) = \delta\mathcal{R}(J+L,K)(J+L)\\
\widetilde{\mathcal{R}}(J,K)(J+L) + \widetilde{\mathcal{R}}(L,K)(J+L) = \delta\mathcal{R}(J,K)(J+L) + \delta\mathcal{R}(L,K)(J+L)
\end{gathered}$$

Then, by using property (b) from Proposition 4.2.1, we obtain

$$\begin{aligned}
\text{LHS} &= \widetilde{\mathcal{R}}(J,K)J + \widetilde{\mathcal{R}}(J,K)L + \widetilde{\mathcal{R}}(L,K)J + \widetilde{\mathcal{R}}(L,K)L,\\
\text{RHS} &= \delta\mathcal{R}(J,K)J + \delta\mathcal{R}(J,K)L + \delta\mathcal{R}(L,K)J + \delta\mathcal{R}(L,K)L.
\end{aligned}$$

Using property (i) from Definition 7.2.3 and equation ($\boldsymbol{B}$), we finally get

$$\widetilde{\mathcal{R}}(J,K)L - \widetilde{\mathcal{R}}(K,L)J = \delta\{\mathcal{R}(J,K)L - \mathcal{R}(K,L)J\}. \tag{$\boldsymbol{C1}$}$$

By symmetry we also have

$$\widetilde{\mathcal{R}}(K,L)J - \widetilde{\mathcal{R}}(L,J)K = \delta\{\mathcal{R}(K,L)J - \mathcal{R}(L,J)K\}, \tag{$\boldsymbol{C2}$}$$

$$\widetilde{\mathcal{R}}(L,J)K - \widetilde{\mathcal{R}}(J,K)L = \delta\{\mathcal{R}(L,J)K - \mathcal{R}(J,K)L\}. \tag{$\boldsymbol{C3}$}$$

And also, by property (ii) of Definition 7.2.3, we have

$$\widetilde{\mathcal{R}}(J,K)L + \widetilde{\mathcal{R}}(K,L)J + \widetilde{\mathcal{R}}(L,J)K = 0 = \mathcal{R}(J,K)L + \mathcal{R}(K,L)J + \mathcal{R}(L,J)K. \tag{$\boldsymbol{D}$}$$

Hence, by combining equations ($\boldsymbol{C1}$), ($\boldsymbol{C2}$), ($\boldsymbol{C3}$), ($\boldsymbol{D}$) we get

$$\widetilde{\mathcal{R}}(J,K)L = \delta\mathcal{R}(J,K)L,\ \ \widetilde{\mathcal{R}}(K,L)J = \delta\mathcal{R}(K,L)J,\ \ \widetilde{\mathcal{R}}(L,J)K = \delta\mathcal{R}(L,J)K.$$

This proves (a). The rest now obviously follows immediately. However, we will show one more which we will make use of in the next theorem, namely (e). So, using equation (†) with condition 1 and parts (a) to (d), we get

$$
\begin{aligned}
\widetilde{C}(J,K)L &= \widetilde{\mathcal{R}}(J,K)L \\
&\quad + \frac{1}{n-2}\left\{\widetilde{\mathrm{Ric}}(K,L)J - \widetilde{\mathrm{Ric}}(J,L)K + \widetilde{\langle K,L\rangle}\widetilde{\mathrm{Ric}}_0 J - \widetilde{\langle J,L\rangle}\widetilde{\mathrm{Ric}}_0 K\right\} \\
&\quad - \frac{\widetilde{\mathrm{Sc}}}{(n-1)(n-2)}\left\{\widetilde{\langle K,L\rangle}J - \widetilde{\langle J,L\rangle}K\right\}. \\
&= \delta\mathcal{R}(J,K)L \\
&\quad + \frac{1}{n-2}\{\delta\,\mathrm{Ric}(K,L)J - \delta\,\mathrm{Ric}(J,L)K + \delta\langle K,L\rangle\,\mathrm{Ric}_0\,J - \delta\langle J,L\rangle\,\mathrm{Ric}_0\,K\} \\
&\quad - \frac{\mathrm{Sc}}{(n-1)(n-2)}\{\delta\langle K,L\rangle J - \delta\langle J,L\rangle K\} \\
&= \delta C(J,K)L.
\end{aligned}
$$

This proves (e). ■

Definition 7.4.1. *We shall call a Riemannian manifold (P, g) nowhere conformally flat, if there is no nonempty open subset $\subseteq P$ which is conformally flat (in the inherited Riemannian metric).*

We will also need the following definition which is an equivalent definition of a dense subset (Cf. Definition 7.2.2):

Definition 7.4.2. *A is dense in X if, for any point $y \in X$, there is a sequence $\{a_n\}$ such that*

$$\lim_{n\to\infty} a_n = y, \text{ with } a_n \in A \text{ for all } n.$$

Theorem 7.4.1. *Let $(P, g), (Q, \tilde{g})$ be Riemannian manifolds such that there exists a sectional curvature preserving diffeomorphism between them. Suppose that P is nowhere conformally flat and that its dimension is ≥ 4. Then a sectional curvature preserving diffeomorphism is an isometry.*

Proof. Let $h : P \to Q$ be a sectional curvature preserving diffeomorphism. We identify P with Q via h and by doing so we cut down to the following scenario: P has two metrics $g, h^*\tilde{g}$ such that the identity map of

$$(P, g) \xrightarrow{1_P} (P, h^*\tilde{g})$$

is curvature preserving. Now, $C \neq 0$ (Cf. Theorem 7.3.1) on an open dense subset of P, this is due to the fact that, from the hypothesis of the theorem,

P is not conformally flat and its dimension is ≥ 4. $C = 0$ when a point is isotropic, this is easily verified by substituting the formula from Lemma 7.2.4 part (ii) and the formulas from Corollary 4.4.1 into the expression (†) for C. Therefore, this means that the non-isotropic points are dense. So we may apply Theorem 7.2.1 and we observe, from continuity studies (using Definition 7.4.2 from above), that there is a positive real-valued function ρ on P such that $h^*\tilde{g} = \rho \cdot g$. By Proposition 7.4.1 part (e) we then have that $\widetilde{C} = \rho C$. But since C is invariant under conformal changes to the metric we also have that $\widetilde{C} = C$. As $C \neq 0$, on a dense subset, we must conclude that $\rho \equiv 1$. Hence h is indeed an isometry. ■

7.5 Conformally Flat Case

The conformally flat case necessitates a totally different reasoning which we go through in this subsection. We wish to prove the following theorem:

Theorem 7.5.1. *Let $(P, g), (Q, \tilde{g})$ be Riemannian manifolds such that there exists a sectional curvature preserving diffeomorphism between them. Suppose that P is conformally flat and that its dimension is ≥ 4. Furthermore, we suppose that the set of non-isotropic points is dense. Then the sectional curvature preserving diffeomorphism is an isometry.*

Before we can prove the theorem above, we need to make some observations. Since (P, g) is conformally flat, then given a point $y \in P$, there exists a neighborhood U of y such that $g|_U = e^{2\alpha} g_0$, where $\alpha : U \to \mathbb{R}$ and g_0 is the flat metric. Also, since it is supposed that the set of non-isotropic points are dense, then the sectional curvature preserving diffeomorphism h between P and Q is conformal (C.f. Theorem 7.2.1), and so, as in the proof of Theorem 7.4.1 we come down to the following scenario: P is endowed with metrics g, $h^*\tilde{g}$, such that $h^*\tilde{g} = e^{2\varphi} g$ where $\varphi : P \to \mathbb{R}$, and the identity map $(P, g) \xrightarrow{1_P} (P, h^*\tilde{g})$ is curvature preserving. Our task will be to show that $\varphi \equiv 0$.

We are now going to derive some basic formulas for the curvature-preserving diffeomorphism h. These formulas will be necessary for us prove Theorem 7.5.1 later on in this subsection. We have 3 metrics, namely g, $h^*\tilde{g}$ (which from now on, for convenience, we will write simply as $\tilde{g}$) and g_0. The quantities associated to the metric $\tilde{g}$ will be designated by a tilde overhead. The quantities associated to the metric g_0 will be designated by a subscript$_0$. As $\tilde{g} = e^{2\varphi} g$, from Proposition 7.4.1 part (a), we obtain

$$\widetilde{\mathcal{R}} = e^{2\varphi} \mathcal{R}. \tag{20}$$

Since g_0 is the flat metric, its curvature tensor vanishes, and since $g = e^{2\alpha} g_0$, we get, for any 3 vector fields J, K, L, by Proposition 7.3.2:

$$\begin{aligned} \mathcal{R}(J,K)L = & \left\{V(K,L) + \langle K,L\rangle_0 \|\mathrm{grad}_0\, \alpha\|_0^2\right\} J \\ & - \left\{V(J,L) + \langle J,L\rangle_0 \|\mathrm{grad}_0\, \alpha\|_0^2\right\} K \\ & + \langle K,L\rangle_0 V_0(J) - \langle J,L\rangle_0 V_0(K), \end{aligned} \tag{21}$$

where V and V_0 are defined with respect to the metric g_0.

Specifically, if J, K, L are orthonormal vector fields with respect to g_0, then

$$\mathcal{R}(J,L)L = V(K,L)J - V(J,L)K \tag{22}$$

$$\mathcal{R}(J,K)J = V(J,K)J - \left\{V(J,J) + \|\mathrm{grad}_0\, \alpha\|_0^2\right\} K - V(K) \tag{23}$$

Recalling the Bianchi Identity from Section 4, for any 3 vector fields J, K, L:

$$\left(\nabla_J \mathcal{R}\right)(K,L) + \left(\nabla_K \mathcal{R}\right)(L,J) + \left(\nabla_L \mathcal{R}\right)(J,K) = 0. \tag{24}$$

For any 4 vector fields J, K, L, H we have a $(1,3)$ tensor field F:

$$\begin{aligned} \left(\nabla_J F\right)(K,L,H) = & \nabla_J(F(K,L,H)) - F\left(\nabla_J K, L, H\right) - F\left(K, \nabla_J L, H\right) \\ & - F\left(K, L, \nabla_J H\right). \end{aligned} \tag{25}$$

Using the formula for change of connection under conformal change (Cf. Subsection 7.3) and using (25), we obtain

$$\begin{aligned} & (\nabla_J \widetilde{\mathcal{R}})(K,L)H - (\widetilde{\nabla}_J \widetilde{\mathcal{R}})(K,L)H \\ & \quad = -S(J, \widetilde{\mathcal{R}}(K,L)H) + \widetilde{\mathcal{R}}(S(J,K),L)H \\ & \qquad - \widetilde{\mathcal{R}}(S(L,J),K)H + \widetilde{\mathcal{R}}(K,L)S(J,H) \\ & \quad = -\{J\varphi \widetilde{\mathcal{R}}(K,L)H + \langle \widetilde{\mathcal{R}}(K,L)H, G\rangle J - \langle J, \widetilde{\mathcal{R}}(K,L)H\rangle G\} \\ & \qquad + \widetilde{\mathcal{R}}(S(J,K),L)H - \widetilde{\mathcal{R}}(S(L,J),K)H \\ & \qquad + \{J\varphi \widetilde{\mathcal{R}}(K,L)H + H\varphi \widetilde{\mathcal{R}}(K,L)J - \langle J,H\rangle \widetilde{\mathcal{R}}(K,L)G\} \\ & \quad = -\langle \widetilde{\mathcal{R}}(K,L)H, G\rangle J - \{\langle \widetilde{\mathcal{R}}(K,L)J, H\rangle G\} \\ & \qquad + \{\widetilde{\mathcal{R}}(S(J,K),L)H - \widetilde{\mathcal{R}}(S(L,J),K)H\} \\ & \qquad + \{H\varphi \widetilde{\mathcal{R}}(K,L)J\} - \langle J,H\rangle \widetilde{\mathcal{R}}(K,L)G, \end{aligned} \tag{26}$$

where we have used the exact same notation as in Proposition 7.3.1 and G is again $\mathrm{grad}\,\varphi$ with respect to the metric g.

Next, we perform a cyclic sum over J, K and L while fixing H. Using (20) and (24), the left hand side of (26) becomes

$$\begin{aligned}(\nabla_J\widetilde{\mathcal{R}})(K,L)H + (\nabla_K\widetilde{\mathcal{R}})(L,J)H + (\nabla_L\widetilde{\mathcal{R}})(J,K)H \\ = 2e^{2\varphi}\{J\varphi\mathcal{R}(K,L)H + K\varphi\mathcal{R}(L,J)H + L\varphi\mathcal{R}(J,K)H\}.\end{aligned}$$

In a cyclic sum over the vector fields J, K, L, the terms in curly brackets on the utmost right hand side of (26) vanish and we remain with

$$\begin{aligned}-e^{2\varphi}\{\langle\mathcal{R}(K,L)H,G\rangle J + \langle\mathcal{R}(L,J)H,G\rangle K + \langle\mathcal{R}(J,K)H,G\rangle L \\ + \langle J,H\rangle\mathcal{R}(K,L)G + \langle K,H\rangle\mathcal{R}(L,J)G + \langle L,H\rangle\mathcal{R}(J,K)G\}.\end{aligned}$$

Cancelling $e^{2\varphi}$ from the left and right side, we obtain the following relation

$$\begin{aligned}&2\{J\varphi\mathcal{R}(K,L)H + K\varphi\mathcal{R}(L,J)H + L\varphi\mathcal{R}(J,K)H\} \\ &\quad + \{\langle\mathcal{R}(K,L)H,G\rangle J + \langle\mathcal{R}(L,J)H,G\rangle K + \langle\mathcal{R}(J,K)H,G\rangle L \\ &\quad + \langle J,H\rangle\mathcal{R}(K,L)G + \langle K,H\rangle\mathcal{R}(L,J)G + \langle L,H\rangle\mathcal{R}(J,K)G\} = 0.\end{aligned} \tag{27}$$

Since the dimension of the Riemannian manifold P is ≥ 4, it is feasible to let the vector fields J, K, L, H be orthonormal with respect to the metric g_0. Using (22) and taking the inner product of the left hand side of (27) with J, we get

$$\begin{aligned}\langle\text{first bracket}, J\rangle_0 &= 2\{V(K,H)L\varphi - V(L,H)K\varphi\} \\ \langle\text{second bracket}, J\rangle_0 &= \langle\mathcal{R}(K,L)H,G\rangle \\ &= V(L,H)K\varphi - V(K,H)L\varphi.\end{aligned}$$

Thus, for any 3 orthonormal vector fields K, L, H,

$$V(K,H)L\varphi - V(L,H)K\varphi = 0. \tag{28}$$

Let J, K, H be orthonormal vector fields with respect to the metric g_0 and let $H = J$ in (27), then (27) becomes

$$\begin{aligned}&2\{J\varphi\mathcal{R}(K,L)J + K\varphi\mathcal{R}(L,J)J + L\varphi\mathcal{R}(J,K)J\} \\ &\quad + \{\langle\mathcal{R}(K,L)J,G\rangle J + \langle\mathcal{R}(L,J)J,G\rangle K + \langle\mathcal{R}(J,K)J,G\rangle L \\ &\quad + \|J\|^2\mathcal{R}(K,L)G\} = 0.\end{aligned} \tag{29}$$

Taking inner product of the left hand side of (29) with K and using (22), (23) we obtain the following two expressions

$$\begin{aligned}\langle\text{first bracket}, K\rangle_0 = 2[&V(J,L)J\varphi + V(K,L)K\varphi \\ &- \left\{V(J,J) + V(K,K) + \|\mathrm{grad}_0\,\alpha\|_0^2\right\} L\varphi].\end{aligned}$$

$$\begin{aligned}\langle \text{second bracket}, K\rangle_0 &= -\langle \mathcal{R}(J,L)J, G\rangle + \|J\|^2 \langle \mathcal{R}(K,L)G, K\rangle_0 \\ &= -\langle \mathcal{R}(J,L)J, G\rangle - \langle \mathcal{R}(K,L)K, G\rangle \\ &= -\big[V(J,L)J\varphi - \big\{V(J,J) + \|\mathrm{grad}_0\, \alpha\|_0^2\big\}L\varphi \\ &\quad - \langle V_0(L), G\rangle\big] - \big[V(K,L)K\varphi - \big\{V(K,K) + \|\mathrm{grad}_0\, \alpha\|_0^2\big\}L\varphi \\ &\quad - \langle V_0(L), G\rangle\big].\end{aligned}$$

But, since $G = \mathrm{grad}\,\varphi = e^{-2\alpha}\,\mathrm{grad}_0\,\varphi$, we have

$$\begin{aligned}\langle V_0(L), G\rangle &= \langle V_0(L), \mathrm{grad}_0\,\varphi\rangle_0 \\ &= V(L, \mathrm{grad}_0\,\varphi),\end{aligned}$$

and so,

$$\begin{aligned}\langle \text{second bracket}, K\rangle_0 = -[&V(J,L)J\varphi + V(K,L)K\varphi \\ &- \left\{V(J,J) + V(K,K) + 2\,\|\mathrm{grad}_0\,\alpha\|_0^2\right\} L\varphi \\ &- 2V(L, \mathrm{grad}_0\,\varphi)].\end{aligned}$$

Now, with respect to the metric g_0, we let $\{E_1, E_2, \ldots, E_n\}$ be orthonormal vector fields, so that $\mathrm{grad}_0\,\varphi = \sum_y (E_y\varphi)E_y$. So, finally we have

$$\begin{aligned}V(J,L)J\varphi + V(K,L)K\varphi - \{V(J,J) + V(K,K)\}L\varphi \\ + 2\sum_y V(L, E_y)E_y\varphi = 0.\end{aligned} \tag{30}$$

The proof of Theorem 7.5.1 will follow by the following two lemmas. The first lemma describes a space which is conformally flat among a space which is of constant curvature. The second lemma is an observation which will instantly give us Theorem 7.5.1.

Lemma 7.5.2. *Let (P, g) be a conformally flat Riemannian manifold whose dimension is ≥ 3. Let the curvature tensor $\mathcal{R}$ be expressed as in formula* (21)*. Then the following are equivalent:*

(i) *P is of constant curvature.*

(ii) *$V(J,K) = 0$, for all pairs of orthogonal vector fields $\{J, K\}$.*

(iii) *$V = \gamma g_0$, where $\gamma : P \to \mathbb{R}$.*

(iv) *$V(E_\eta, E_\omega) = \gamma\delta_{\eta\omega}$, where $\gamma : P \to \mathbb{R}$ and $\{E_1, \ldots, E_n\}$ are orthonormal vector fields.*

The proof of the lemma above can be found in Kulkarni [44, p. 324].

Lemma 7.5.3. *Suppose that* $(\mathrm{grad}_0\,\varphi)y \neq 0$*, where* y *is a point on the Riemannian manifold* P*. Then* y *is isotropic.*

Proof. Let

$$E_1 = \frac{\mathrm{grad}_0\,\varphi}{\|\mathrm{grad}_0\,\varphi\|_0},$$

and with respect to g_0, we let $\{E_2, \ldots, E_n\}$ be orthonormal vector fields tangent to the level surface of φ in a neighbourhood of y. Then $\{E_\eta\}$ are orthonormal, $(E_1\varphi)(y) = \|\mathrm{grad}_0\,\varphi\|_0 \neq 0$, and for $\eta > 1$, $E_\eta\varphi = 0$. In equation (28) we let $K = E_1$, $L = E_\eta$, $H = E_\omega$, where $\{1, \eta, \omega\}$ are distinct and $\eta \neq \omega$, $\eta > 1$, $\omega > 1$. We are then left with $V(E_\eta, E_\omega) = 0$. Next, in equation (30) we let $J = E_1$, $K = E_\eta$, $L = E_\omega$, where $\{1, \eta, \omega\}$ are distinct and $\eta > 1$. We then get $3V(E_1, E_\eta) = 0$ or $V(E_1, E_\eta) = 0$. Finally, again in equation (30) we let $J = E_\eta$, $K = E_1$, $L = E_1$ where $\{1, \eta, \omega\}$ are again distinct. Then $V(E_\eta, E_\eta) + V(E_\omega, E_\omega) = 2V(E_1, E_1)$. Since the dimension of the Riemannian manifold P is ≥ 4, this obviously implies that $V(E_\eta, E_\eta) = V(E_1, E_1)$. And so it follows that y is isotropic from condition (iv) of Lemma 7.5.2. ■

Proof of Theorem 7.5.1. First we observe that $\mathrm{grad}\,\varphi \equiv 0$ or $\varphi \equiv$ constant. This is due to the fact that, if $\mathrm{grad}\,\varphi \not\equiv 0$, then the set of isotropic points has non-empty interior, this follows from Lemma 7.5.3. But then we have a contradiction on one of our hypothesis of the theorem which says that non-isotropic points are dense. From Definition 7.2.2 we see that if the set of non-isotropic points are dense then the set of isotropic points cannot have an interior, so indeed $\mathrm{grad}\,\varphi \equiv 0$ or $\varphi \equiv$ constant. Using Definition 7.2.4 and the fact that $\tilde{g} = e^{2\varphi}g$ (Cf. Theorem 7.2.1), we now compute the corresponding sectional curvatures $\mathcal{K}$ and $\widetilde{\mathcal{K}}$:

$$\mathcal{K}(J, K) = \frac{\langle \mathcal{R}(J, K)J, K\rangle}{\langle J, J\rangle\langle K, K\rangle - \langle J, K\rangle^2},$$

$$\begin{aligned}
\widetilde{\mathcal{K}}(J, K) &= \frac{\widetilde{\langle \widetilde{\mathcal{R}}(J, K)J, K\rangle}}{\widetilde{\langle J, J\rangle}\widetilde{\langle K, K\rangle} - \widetilde{\langle J, K\rangle}^2} \\
&= \frac{e^{2\varphi}\langle \mathcal{R}(J, K)J, K\rangle}{e^{2\varphi}\langle J, J\rangle e^{2\varphi}\langle K, K\rangle - (e^{2\varphi}\langle J, K\rangle)^2}.
\end{aligned}$$

Therefore the corresponding sectional curvatures are associated by

$$\widetilde{\mathcal{K}} = \frac{\mathcal{K}}{e^{2\varphi}}.$$

Since the value of the constant curvature of P is not zero, the curvature preserving hypothesis gives us that $\varphi \equiv 0$. This completes the proof. ■

7.6 A Proof of the Theorem

We are now ready to present the proof of the fundamental theorem of this section, Theorem 7.1.1. But before taking the proof of the theorem proper, we need to recall a few concepts and make some observations which we will make use of in the proof. First we recall Schur's theorem from Section 6, which said that, if the sectional curvature is a constant at every point of a connected Riemannian manifold of dimension at least 3, this constant is the same at all points. We also need to make a comment on the connected components of a topological space.

Definition 7.6.1. *A subset of a topological space X is said to be connected if it is connected under its subspace topology. Given a point $y \in X$, we define*

$$\mathcal{C}(y) = \bigcup \{\mathcal{C} \subset X \mid \mathcal{C} \text{ is connected and } y \in \mathcal{C}\}$$

as the union of all connected subspaces of X that contain the point y. This set $\mathcal{C}(y)$ is called the connected component of y. It is the maximal connected subspace of X containing the point y.

And so, the maximal connected subsets of a topological space are called the connected components of the topological space. Then, we may say that, the sectional curvature $\mathcal{K}$ has no dependence on the 2-planes if and only if it is a constant on each connected component of our Riemannian manifold.

Proof of Theorem 7.1.1. Simultaneously, subsections 7.4 and 7.5, demonstrate that, if the dimension of the Riemannian manifold P is ≥ 4 and $h : (P, g) \to (Q, \tilde{g})$ is a curvature preserving diffeomorphism, then h itself is an isometry on the closure of the set of points which are not isotropic. Therefore, a sufficient requirement for global isometry is that the set of points which are not isotropic must be dense. When the metric is analytic, this is equivalent to saying that there is one point which is not isotropic. In the event that there is an interior on the set of isotropic points, that is, the set of non-isotropic is not dense, then it follows from a Schur's theorem type argument (Cf. Theorem 6.2.1), that if every point of an open subset of P is isotropic, then the sectional curvature would be constant on every one of the connected components of this open set of isotropic points. Specifically, it is constant on an open subset of $G_2(P)$ (the Grassman bundle of 2-plane sections of P), and thus, by means of analyticity, constant everywhere. This obviously contradicts the hypothesis of the theorem, that if dim ≥ 4, then sectional curvature preserving diffeomorphisms between two manifolds with analytic metric are globally isometric except in the case of diffeomorphic but not globally isometric manifolds of the same constant curvature. ■

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