University of the Witwatersrand

Master's Dissertation

Bipartite Ramsey Number for Different Types of Trees

Student:
Mr. Lesego R. E. Mabusela
(1043480)

Supervisor:
Mr. Ronald J. Maartens

Co-Supervisor:
Prof. Elizabeth Jonck

A Master's dissertation submitted to the Faculty of Science in fulfillment of the requirements for the degree of Master's of Science in Mathematics in the

School of Mathematics

May 4, 2022

Declaration of Authorship

I, Lesego Mabusela, student number 1043480, declare that this dissertation titled "Bipartite Ramsey Number for Different Types of Trees" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a masters degree at this University.
- Where any part of this dissertation has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the dissertation is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed:

Date:

UNIVERSITY OF THE WITWATERSRAND

Abstract

Faculty of Science

School of Mathematics

Master's of Science in Mathematics

Bipartite Ramsey Number for Different Types of Trees

by Ronald J. Maartens, Elizabeth Jonck and Lesego Mabusela.

For any two graphs F and H, the *Ramsey number* $R(F, H)$ is the smallest positive integer n such that for any red-blue coloring of the edges of the complete graph K_n there is a subgraph of K_n isomorphic to F whose edges are all colored red, or a subgraph of K_n isomorphic to H whose edges are all colored blue.

Let F and H now be two bipartite graphs, and s a positive integer. The *s-bipartite Ramsey number* $b_s(F, H)$ is the smallest positive integer t, with $t \geq s$, such that for any red-blue coloring of the edges of $K_{s,\,t}$ there is a subgraph of $K_{s,\,t}$ isomorphic to F whose edges are all colored red, or a subgraph of $K_{s,\,t}$ isomorphic to H whose edges are all colored blue. The case where $s = t$ is known as the *bipartite Ramsey number*, denoted by $b(F, H)$.

Finally, let T_n denote a *tree* of order $n \geq 5$ with maximum degree $n - 2$. In this dissertation we determine the Ramsey numbers $b(K_{1,\,n}, K_{1,\,m})$, $b(K_{1,\,m}, T_n)$, $b(T_m, T_n)$, $b_s(K_{1,m}, T_n)$ and $b_s(T_m, T_n)$.

Acknowledgements

I would like to express my deep and sincere gratitude to my research supervisors Mr. Ronald J. Maartens and Prof. Elizabeth Jonck for providing me with an opportunity to carry out my research under their invaluable guidance and patience. Their sincerity and motivation kept me inspired throughout my research. They taught me the methodology to carry out this research and present it in the best way possible. Their background knowledge in this area as well as their assistance in the different phases of this dissertation have been of great help and are much appreciated.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Main Research Questions

In this dissertation our main research questions are:

- *What is the bipartite Ramsey number of two trees with various structures?*
- *What is the s-bipartite Ramsey number of two trees with various structures?*

1.2 Basic Definitions and Terminology

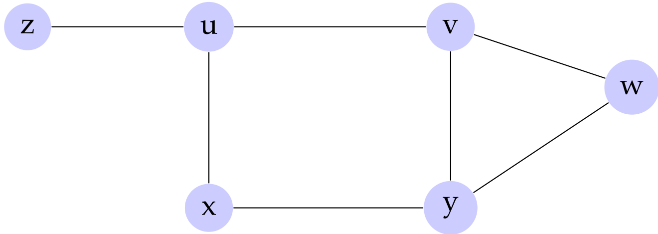
We give some basic Graph Theory definitions in this section. The reader is referred to the textbook *Graphs and Digraphs* by *Chartrand, Lesniak and Zhang* [1] for any undefined terms and concepts in Graph Theory. We advise the reader that we use colored diagrams in this dissertation to better visualise concepts.

A *graph* G is a finite nonempty set V of objects called *vertices* (singular called *vertex*) together with a (possibly empty) set E of 2-element subsets of V called *edges*. In this dissertation we only consider finite *simple graphs* which are graphs with no loops or multiple edges between the same two vertices, unless otherwise stated. A *loop* is defined to be an edge which connects a vertex to itself. We consider graphs that are *undirected* which are graphs with edges that have no direction attached to them.

The number of vertices in the graph G is called the *order* of G and the number of edges is called the *size* of G, that is, the cardinality of the vertex set and the edge set respectively. In Figure 1.2.1 the graph G has order six and size seven.

To explicitly show that a graph G has vertex set V and edge set E it is sometimes denoted by $G = (V, E)$. To stress that V is the vertex set of G we write V as $V(G)$ and for a similar explanation we write the edge set E as $E(G)$.

Figure 1.2.1: A simple graph G.



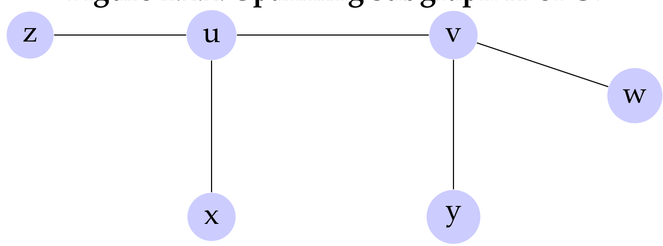
If uv is an edge of G, then vertices u and v are said to be *adjacent* to each other. Two adjacent vertices are called *neighbours* of each other. If v is a vertex and uv is an edge, then v and uv are said to be *incident* to each other. The *degree* of a vertex v in the graph G is the number of vertices in G that are adjacent to v. Equivalently the *degree* of vertex v is the number of edges incident to v.

In this dissertation the degree of a vertex v is denoted by $deg_G\ v$ or, more simply, by $deg\ v$ if the graph G under consideration is clear. In graph G of Figure 1.1 we have that $deg\ v = 3$ and $deg\ w = 2$. $\Delta(G)$ denotes the *maximum degree* of G which is defined as the largest degree among the vertices of G . Similarly, the *minimum degree* of G is denoted by $\delta(G)$. In Figure 1.2.1 graph G has $\Delta(G) = 3$ and $\delta(G) = 1$.

Two graphs that have the exact same structure are said to be *isomorphic*. If G and H are isomorphic, then there exists a bijective function $\phi : V(G) \to V(H)$ such that any two vertices u and v are adjacent in G if and only if $\phi(u)$ and $\phi(v)$ are adjacent in H. Isomorphic graphs G and H have the same number of vertices, same number of edges and the degree of the vertices in G is the same as the degree of the corresponding vertices in H.

A graph H is said to be a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(H) = V(G)$ and $E(H) \subseteq E(G)$, then H is called a *spanning subgraph* of G. The graph in Figure 1.2.2 is the spanning subgraph H of graph G given in Figure 1.2.1. We see in Figure 1.2.1 and Figure 1.2.2 that G and H have the same number of vertices but graph H has less edges than graph G.

Figure 1.2.2: Spanning subgraph H of G.

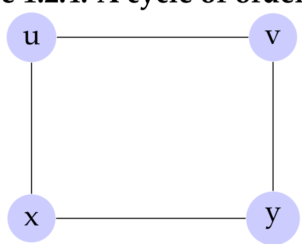


For any two vertices u and v, not necessarily distinct, a $u - v$ *walk* in G is a sequence of vertices and edges in G, beginning at u and ending at v. In a walk we repeat edges and vertices. A walk whose initial and terminal vertices are distinct is called an *open walk*, otherwise it is called a *closed walk*. A $u - v$ open walk with no repeated edges and vertices is called a *path*. A path of order m is denoted by P_m. The graph in Figure 1.2.3 gives the graph of P_4.

Figure 1.2.3: A $z - w$ path of order four and size three.

A $u - v$ closed walk with no repeated edges and no repeated vertices apart from the initial vertex and terminal vertex is called a *cycle*. A cycle of order m is denoted by C_m where $m \geq 3$. The graph in Figure 1.2.4 is the graph of C_4.

Figure 1.2.4: A cycle of order four.

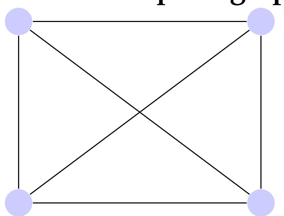


A graph is said to be *complete* if every two distinct vertices of the n vertices in the graph are adjacent to each other and is denoted by K_n. Therefore, K_n is a graph of order n and size

$$\binom{n}{2} = \frac{n(n-1)}{2}.$$

The graph in Figure 1.2.5 gives an example of K_4. A *trivial graph* is a graph of order one and a *nontrivial graph* is a graph with two or more vertices. A graph of size zero is called an *empty graph* and so a *nonempty graph* has one or more edges.

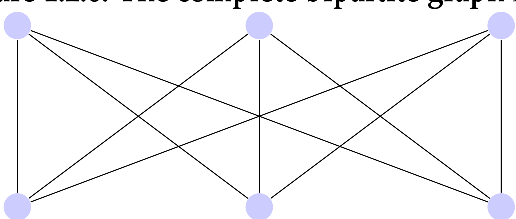
Figure 1.2.5: Complete graph K_4.



If all the vertices of a graph G have the same degree, then G is called a *regular* graph. That is, if every vertex of G has degree r, then we say that G is *r-regular*. Note that if we only consider a cycle graph, then every vertex on the cycle has degree two and thus the cycle graph is $2 - regular$.

A nontrivial graph G is a complete *bipartite* graph if it is possible to partition $V(G)$ into two subsets U and W, called *partite sets*, such that every vertex in U is a neighbour of a vertex in W with no edge between any two vertices in the same partite set. More generally, a graph G is a *k-partite* graph if it is possible to partition $V(G)$ into k partite sets, say $V_1, V_2, \cdots, V_k$, such that no vertices in the same partite set are adjacent. A graph G is said to be *balanced* if the number of vertices between any two partite sets in G differ by at most one.

Figure 1.2.6: The complete bipartite graph $K_{3,3}$.

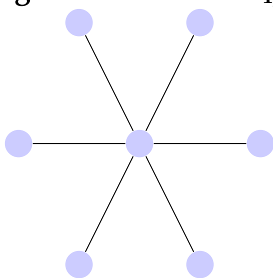


A k-partite graph having partite sets $V_1, V_2, \cdots, V_k$ is called a complete k-partite graph if all of the vertices in V_i are adjacent to all of the vertices in V_j, this is when

$1 \le i, j \le k$ for $i \ne j$. The complete bipartite graph is denoted by $K_{n,\,m}$ if it has two partites U and W, where U has n vertices and W has m vertices. In Figure 1.2.6 we have an example of the complete bipartite graph $K_{3,3}$.

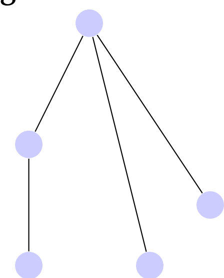
A special case of the complete bipartite graphs is the graph $K_{1,n}$ which we call a *star*. The graph in Figure 1.2.7 gives an example of a $K_{1,6}$. A *tree* is a connected graph without cycles, and a tree on n vertices is denoted by T_n. Stars and paths are all special types of trees.

Figure 1.2.7: A star $K_{1,6}$



We use T'_n to denote the tree of order $n \ge 5$ whose maximum degree is $n - 2$. We highlight that T'_n is not a star since the maximum degree of T'_n is $n - 2$. The graph in Figure 1.2.8 is a tree T'_5.

Figure 1.2.8: A tree T'_5.



For the tree T'_n we consider $n \ge 5$ because

- if $n = 4$, then the maximum degree of T'_4 is

$$4 - 2 = 2$$

  implying that T'_4 is isomorphic to P_4,

- if $n = 3$, then the maximum degree of T_3' is

$$3 - 2 = 1$$

implying that T_3' a disconnected graph and we are only considering connected graphs,

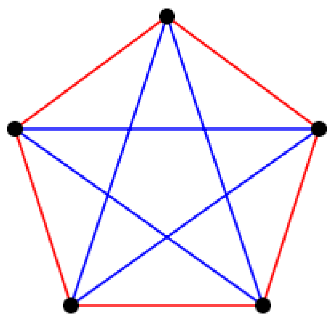
- if $n = 2$, then T_2' is an empty graph since the maximum degree is zero and we are done, and
- if $n = 1$, then the graph T_1' does not exist as it has a negative maximum degree.

There are numerous ways of producing a new graph from at least one graph. The *complement* $\overline{G}$ of a graph G is a graph whose vertex set is $V(G)$ and where uv is an edge in $\overline{G}$ if and only if uv is not an edge in graph G. Obviously the empty graph is the complement of the complete graph.

The *coloring of edges* is a process where a color from a group of possible colors is assigned to every edge given in a graph. In this dissertation we only assign one color to an edge and we only make use of two colors, namely red and blue.

Since an edge is assigned either a red color or a blue color, we have that in the complete graph all of the edges colored red form a subgraph which we denote by G_R and all of the edges colored blue form a subgraph which we denote by G_B. Note that the two subgraphs G_R and G_B are complementary graphs of each other. The graph in Figure 1.2.9 gives an example of this.

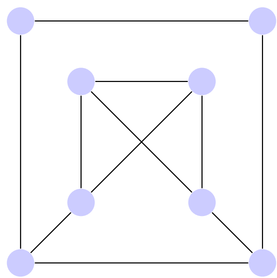
Figure 1.2.9: A red-blue coloring of edges of K_5.



We mostly make use of the complements when we talk of the red-blue coloring of edges of the graph G since all the edges that are not colored blue in G is colored red producing a red subgraph of G.

Let G be a graph. A *Hamiltonian path* of graph G is a path in G that contains every vertex of G, while a cycle in G that contains every vertex of G is called a *Hamiltonian cycle*. Figure 1.2.10 gives an example of a graph containing a Hamiltonian cycle. A graph that contains a Hamiltonian cycle is in short called *Hamiltonian*. Necessarily, the order of a Hamiltonian graph is at least three and every Hamiltonian graph contains a Hamiltonian path. However, a graph with a Hamiltonian path need not be Hamiltonian.

Figure 1.2.10: A Hamiltonian graph.



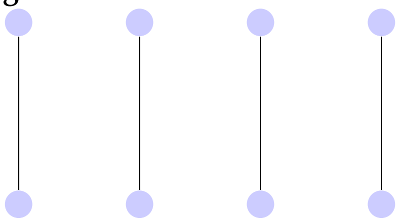
The following results helps us later on in determining whether a graph is Hamiltonian or not.

Theorem 1.2.1. *[2] All graphs G of order $n \geq 3$ with minimum degree $\delta(G) \geq n/2$ are Hamiltonian.*

Theorem 1.2.2. *[2] If G is the balanced complete k-partite graph with $k \geq 3$, then G is Hamiltonian.*

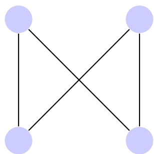
In a graph G, a set M of edges, with no two edges incident to the same vertex, is called a *matching*. A matching M in G is a *perfect* matching if every vertex of G is incident with some edge in M. If M is a perfect matching in G, then G has order $n = 2k$ for some positive integer k where $|M| = k$. Thus, only graphs of even order have a perfect matching. The graph in Figure 1.2.11 gives an example of a perfect matching when $n = 8$.

Figure 1.2.11: A Perfect Matching.



We also refer to factors of a graph G. A *factor* of a graph G simply means a spanning regular subgraph of G. A spanning k-regular subgraph of G is called a $k - factor$. The edges of the graph are partitioned into $disjoint$ k-factors. The $disjoint\ union$ of a graph is an operation that combines two or more graphs to from a larger graph. Figure 1.2.12 gives us an example of a 2-factor graph. The edge set of a 1-factor in a graph G is a perfect matching in G.

Figure 1.2.12: A 2-factor graph.



The following results are needed for our results later on.

Theorem 1.2.3. *[3], [4] The complete graph K_{2n} is 1-factorable.*

Theorem 1.2.4. *[5] A graph G is 2-factorable if and only if G is $2p$ regular, where p is a positive integer.*

Theorem 1.2.5. *[6] For every positive integer n the graph $K_{n,\,n}$ is 1-factorable.*

The notation $n \mid a$ means that a is *divisible* by n. In mathematics, divisibility means that a number divides evenly into another number or itself with no remainder. For example two divides evenly into eight and so we write $2 \mid 8$. However, three would leave us with a remainder when we divide eight by three so we write $3 \nmid 8$ to denote that eight is not divisible by three.

Modulo is a mathematics operation that finds the remainder when one integer is divided by another. Modulo is often represented with *mod*. For integer a, b and r, if a is the dividend, b the divisor and r the remainder we write $a \ (mod\ b) \equiv r$.

1.3 Ramsey Theory

Ramsey Theory is named after the British Mathematician Frank Plumpton Ramsey (1903 - 1930). Frank was the first to mention a lemma on what is named today as Ramsey's Theorem in his second paper *On a problem of formal logic* [7] in 1930.

Ramsey Theory is a branch in mathematics that focuses on the appearance of order in a structure given a substructure of a particular property. Ramsey Theory mostly answers the question:

"How many vertices in a structure is needed to ensure that a particular substructure of some given property is contained in that structure?".

Ramsey Theory has many interesting applications and often quickly become complicated especially if we consider very large structures. We focus on a few variations of the original Ramsey number in this dissertation. Each following chapter in this dissertation focus on a particular type of Ramsey number for two trees.

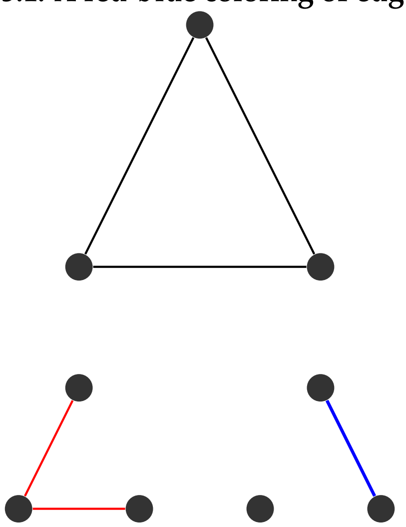
In Chapter 2 we begin by confirming the Ramsey number for each pair $K_{1,\,n}$ and $K_{1,\,m}$ of stars of order $n+1 \geq 3$ and $m+1 \geq 3$ respectively. This result was first mentioned by S. Burr and J. Roberts in their article *On the Ramsey numbers for stars* [8] in 1973, their version of a Ramsey number was as follows.

Given any two graphs F and H, the Ramsey number $R(F,\,H)$ is defined to be the smallest positive integer n such that for any red-blue coloring of the edges of the complete graph of order n, K_n, there is either a subgraph of K_n isomorphic to F whose edges are all colored red, or a subgraph of K_n isomorphic to H whose edges are all colored blue.

If $R(F,\,H) = n$ in the definition of the Ramsey numbers of two graphs F and H, then for every red-blue coloring of the edges of the complete graph K_n there exist either a red F or a blue H. Also, we have that there exists a red-blue coloring of edges of the complete graph K_{n-1} where there is neither a red F nor a blue H.

We emphasize that in a complete graphs where the red subgraph is the complement of the blue subgraph and *vice versa*. The graph in Figure 1.3.1 illustrates this statement again. In Figure 1.3.1 we first have a K_3 with no edge coloring and then we assign a red-blue edge coloring in the graphs below the K_3 with no edge coloring. We see that the red subgraph is the complement of the blue subgraph and *vice versa*.

Figure 1.3.1: A red-blue coloring of edges of K_3.



In Section 2.4 we prove the Ramsey number for a star of order $m \geq 2$ and a tree T'_n of order $n \geq 5$ whose maximum degree is $n-2$. We prove the Ramsey number for two trees T'_m and T'_n where $m \geq 5$ and $n \geq 5$ in Section 2.5. The Ramsey numbers $R(K_{1,m}, T'_n)$ and $R(T'_m, T'_n)$ was studied by Guo and Volkmann in their article *Tree Ramsey numbers* [9] in 1995.

We then analyse the k-Ramsey number for each pair $K_{1,n}$ and $K_{1,m}$ of stars of order $n+1$ and $m+1$ respectively, where $n \geq 2$ and $m \geq 2$ in Chapter 3. Andrews, Chartrand, Lumduanhom and Zhang introduced the concept of k-Ramsey numbers in their article *Stars and Their k-Ramsey numbers* [10] in 2017. Their version of a Ramsey number is as follows.

For any two graphs F and H, and an integer k with $2 \leq k < R(F, H)$*, the k-Ramsey number* $R_k(F,\ H)$ *is the smallest positive order of the balanced complete k-partite graph G such that for any red-blue coloring of the edges of G results in a subgraph of G isomorphic to F whose edges are all colored red, or a subgraph of G isomorphic to H whose edges are all colored blue.*

The authors of *Stars and Their k-Ramsey numbers* [10] presented a formula for the k-Ramsey number of any two stars $K_{1,\,n}$ and $K_{1,\,m}$ of order $n + 1 \geq 3$ and $m + 1 \geq 3$ respectively. They also considered the k-Ramsey number for two cycles of order four in the article.

We stress that k is defined strictly between two and the Ramsey number $R(F,\ H)$. Since if $k = 1$, then the graph considered is an empty graph as all the vertices are in one partite set and if $k = R(F,\ H)$, then the graph considered has one vertex in each partite set, resulting in the balanced complete k-partite graph to be a complete graph of order k and so $R_k(F,\ H)= R(F,\ H)$.

The goal of Chapter 4 is to present a formula for the bipartite Ramsey number of two trees. We begin by determining the bipartite Ramsey number for the pair of stars $K_{1,\,n}$ and $K_{1,\,m}$ of order $n + 1 \geq 3$ and $m + 1 \geq 3$ in Section 4.3. We use the following lemma from the article *Another view on bipartite Ramsey numbers* [11] to assist us in estimating the bipartite Ramsey number from Corollary 3.3.1 in Chapter 3.

Lemma 1. *If F and H are bipartite graphs, then* $R_2(F, H)$ *is either* $2b(F, H)$ *or* $2b(F, H) - 1$.

The conditions for $R_2(F, H) = 2b(F, H)$ or $R_2(F, H) = 2b(F, H) - 1$ is still unknown.

In Section 4.4 we investigate the bipartite Ramsey number for a star of order $n \geq 2$ and a tree T'_n of order $n \geq 5$ whose maximum degree is $n - 2$. Christou, Iliopoulos and Miller studied the bipartite Ramsey number of simple graphs such as trees in their article *Bipartite Ramsey numbers involving stars, stripes and trees* [12] in 2013. We furthermore investigate the biparite Ramsey number for two trees T'_m and T'_n of order $m, n \geq 5$ in Section 4.5

In 1998, Hattingh and Henning introduced the concept of bipartite Ramsey numbers in their article *Star-path bipartite Ramsey number* [13]. Their version of a bipartite Ramsey number is as follows.

For bipartite graphs F and H, the bipartite Ramsey number b(F, H) is the smallest positive integer n such that for any red-blue coloring of edges of the complete bipartite graph $K_{n,\,n}$ there is either a subgraph of $K_{n,\,n}$ isomorphic to F whose edges are all colored red or a subgraph of $K_{n,\,n}$ isomorphic to H whose edges are all colored blue.

In the article *Star-path bipartite Ramsey number* [13] the authors also considered the multicoloring of edges but we only consider a two edge coloring in this dissertation. The authors of *Star-path bipartite Ramsey number* [13] established the number $b(P_m, K_{1,\,n})$ for all integers $n, m \geq 2$, where P_m denotes a path of order m and $K_{1,\,n}$ a star of order $n + 1 \geq 3$. In our dissertation we focus on the bipartite Ramsey number $b(K_{1,\,n}, K_{1,\,m})$ for the pair $K_{1,\,n}$ and $K_{1,\,m}$ of stars of order $n + 1 \geq 3$ and $m + 1 \geq 3$ respectively.

In our study of bipartite Ramsey numbers for two stars that we only find the number $b(K_{1,\,n}, K_{1,\,m})$ when the number of vertices is divisible by the number of partite sets. When the number of vertices is not divisible by the number of partite sets, then k is defined between three and $R(K_{1,\,n}, K_{1,\,m})$.

From our definition of the bipartite Ramsey number, if $b(F, H) = b$ for bipartite graphs F and H, then there exist a complete bipartite graph $K_{b,\,b}$ such that for every red-blue edge coloring of $K_{b,\,b}$ there is either a red F or a blue H. That is there exists a red-blue edge coloring of the complete graph $K_{b-1,\,b-1}$ where there is neither a red F nor a blue H. We have used the formula we established in Corollary 3.3.1 of Chapter 3 to estimate the bipartite Ramsey results by also using the result from Lemma 1. The case where both F and H are stars is proved in Chapter 4.

The main goal of Chapter 5 is to present the formula for the s-bipartite Ramsey number for the pair of bipartite graphs F and H.

We begin by first proving the s-bipartite Ramsey number for the pair of stars $K_{1,\,n}$ and $K_{1,\,m}$ of size $n \geq 2$ and $m \geq 2$, respectively, which was studied by Olejniczak in his PhD dissertation *Variations in Ramsey Theory* [14] in 2019. We then investigate the s-bipartite Ramsey number for a star $K_{1,\,m}$ of order $m + 1 \geq 3$ and a tree T'_n of order $n \geq 5$ in Section 5.3, followed by the s-bipartite Ramsey number for two trees T'_m and T'_n of order $m, n \geq 5$.

In Chapter 3 we focus on the number $R_k(K_{1,\,n},\, K_{1,\,m})$ of the balanced complete k-partite graphs. A graph is said to be *balanced* if the number of vertices in any of the two partite sets differ by at most one. In Chapter 5 we determine the formula of the s-bipartite Ramsey number for the pair of bipartite graphs F and H of the complete bipartite graph which is not necessarily balanced.

Bi, Chartrand and Zhang first mentioned the concept of s-bipartite Ramsey numbers in their article *Another view of bipartite Ramsey numbers* [11] in 2018. This version of the Ramsey number is as follows:

For bipartite graphs F and H, and a positive integer s, the s-bipartite Ramsey number $b_s(F, H)$ is the smallest positive integer t, with $t \geq s$, such that for any red-blue edge coloring of $K_{s,\,t}$ results in a red F or a blue H.

In the article *Another view of bipartite Ramsey numbers* [11] the authors only considered the s-bipartite Ramsey number of two 4-cycles and some complete bipartite graphs. We aim to further their study by finding the full results of the number $b_s(K_{1,\,m},\, T'_n)$ and the number $b_s(T'_n,\, T'_m)$.

Chapter 2

The Ramsey Number for Two Trees

2.1 Purpose

The purpose of this chapter is to prove the general formula for the Ramsey number of two trees F and H. For purposes of our dissertation, the trees we consider are stars and trees T_n' on $n \geq 5$ vertices with $\Delta(T_n') = n - 2$.

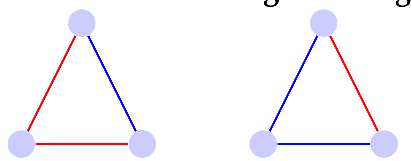
2.2 The Ramsey Number for Two Stars $K_{1,2}$ and $K_{1,m}$

In this section we look at $R(K_{1,2}, K_{1,m})$ where $m \geq 2$. For the sake of completeness we give the Ramsey result of two stars $K_{1,1}$ and $K_{1,1}$. We notice that the star $K_{1,1}$ is isomorphic to the path P_2. It is trivial from Figure 2.2.1 that in every red-blue coloring of the edges of K_2 we always either have a red $K_{1,1}$ or a blue $K_{1,1}$. In the article *On Ramsey-type problems* [15] we have that $R(P_2, P_2) = 2$ and consequently we have that $R(K_{1,1}, K_{1,1}) = 2$. Otherwise, we either have an empty graph or multiple copies of $K_{1,1}$.

Figure 2.2.1: A red-blue edge coloring of the complete graph of order two.

We also have it necessary to show that $R(K_{1,2}, K_{1,2})$= 3. Notice that $K_{1,2}$ is isomorphic to P_3. In Figure 2.2.2 we have that in every red-blue coloring of the edges of K_3 there is always either a red $K_{1,2}$ or a blue $K_{1,2}$.

Figure 2.2.2: A red-blue edge coloring of K_3.



We see in Figure 2.2.1 that there is a red-blue coloring of the edges of K_2 where there is neither a red $K_{1,2}$ nor a blue $K_{1,2}$. Therefore we have $R(K_{1,2}, K_{1,2})= 3$.

In order to determine $R(K_{1,2}, K_{1,m})$ where $m \geq 2$, we reflect consideration on two cases according to the parity of m. The statement of the theorem below was given by Burr and Roberts in 1973 in their article *On the Ramsey number of stars* [8]. The proof is original and not from the paper.

Theorem 2.2.1. *For $m \geq 2$,*

$$R(K_{1,2}, K_{1,m}) = \begin{cases} m+1 & \text{if } m \text{ is even,} \\ m+2 & \text{if } m \text{ is odd.} \end{cases}$$

Proof

We reflect consideration on two cases as indicated by whether m is even or m is odd.

Case 1: m is odd.

We first show that

$$R(K_{1,2}, K_{1,m}) \leq m+2.$$

Let $G = K_{m+2}$ be the complete graph of order $m+2$. Then every vertex v of G has degree $m+1$. Also, consider a red-blue coloring of the edges of the complete graph G resulting in a red subgraph G_R of G and a blue subgraph G_B of G. Assume that there is neither a red $K_{1,m}$ nor a blue $K_{1,2}$.

Let v be any arbitrary vertex of the complete graph G. In Table 2.2.1 we give all of the different ways to choose red and blue edges incident to vertex v which has degree $m+1$ in G. We see that if $deg_{G_R}\, v \geq m$, then G always has a red $K_{1,m}$, and if $deg_{G_B}\, v \geq 2$, then G always has a blue $K_{1,2}$ as indicated in the last column.

Table 2.2.1: The number of red and blue edges incident to v in K_{m+2}.

$deg_{G_R}\ v$	$deg_{G_B}\ v$	Contradiction
$m+1$	0	red $K_{1,m}$
m	1	red $K_{1,m}$
$m-1$	2	blue $K_{1,2}$
$m-2$	3	blue $K_{1,2}$
$\vdots$	$\vdots$	$\vdots$
2	$m-1$	blue $K_{1,2}$
1	m	blue $K_{1,2}$
0	$m+1$	blue $K_{1,2}$

As a result we have that there is always either a blue $K_{1,2}$ or a red $K_{1,m}$ in every red-blue coloring of the edges of the complete graph G. Therefore, we have

$$R(K_{1,2}, K_{1,m}) \leq m+2.$$

We now show that

$$R(K_{1,2}, K_{1,m}) > m+1.$$

Let H be a complete graph of order $m+1$. Since m is odd, $m+1$ is even. Let v be an arbitrary vertex of H. Let C be a cycle that passes through the vertices of H exactly once. We have that C is a Hamiltonian cycle in H by Theorem 1.2.1. Let $F = H - E(C)$. Then F is a regular graph of degree $m-2$.

Reflect into consideration the red-blue coloring of the edges of H. Color the edges of graph F red and alternate the colors red and blue on the edges of C in the complete graph H. Then the blue subgraph H_B of H is 1-regular and the red subgraph H_R of H is $(m-1)$-regular.

As a result, we have that there is a red-blue coloring of the edges of the complete graph G where there is neither a blue $K_{1,2}$ nor a red $K_{1,m}$. Therefore we conclude that

$$R(K_{1,2}, K_{1,m}) > m+1$$

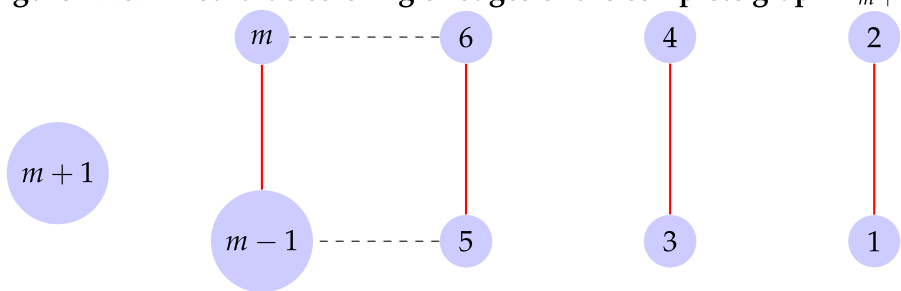
and so, when m is odd, we have that $R(K_{1,2}, K_{1,m}) = m + 2$.

Case 2: m is even.

If m is even, then $m+1$ is odd. We begin by showing that $R(K_{1,2}, K_{1,m}) \leq m+1$. Let $G = K_{m+1}$ be the complete graph of order $m+1$. Let v be an arbitrary vertex of graph G and so $deg\ v = m$. Consequently, G is an m-regular graph. Assume, to the contrary, that in any red-blue coloring of the edges of G there is neither a red $K_{1,2}$ nor a blue $K_{1,m}$.

We consider the red subgraph G_R of G. Let G_R be a perfect matching graph with no two adjacent edges as in Figure 2.2.3. Otherwise, if G_R has no perfect matching, then G contains a red $K_{1,2}$ which is a contradiction to our assumption.

Figure 2.2.3: A red-blue coloring of edges of the complete graph K_{m+1}.



Further we have the complement of G_R as the blue subgraph G_B of graph G. Then the vertices in the blue subgraph G_B of G are either of degree $m-1$ or m. If the vertex with degree m has all blue edges incident to it, then we have a blue $K_{1,m}$, a contradiction to our assumption. If the vertex with degree m has one or more red edges incident to it, then we have a red $K_{1,2}$, again a contradiction to our assumption. Therefore,

$$R(K_{1,2}, K_{1,m}) \leq m+1.$$

We now show that

$$R(K_{1,2}, K_{1,m}) > m.$$

Let H be a complete graph of order m. We have that m is even. If v is an arbitrary vertex of H, then $deg\ v = m-1$. Let C be the cycle that passes through every vertex of H exactly once. We use Theorem 1.2.1 to conclude that C is a Hamiltonian cycle in H, that is, C is a cycle that goes through each vertex of H exactly once. Let $F = H - E(C)$. Then F is a regular graph of degree $m - 2$.

Let there be a red-blue coloring of the edges of graph H. Color the edges of graph F blue and alternate the colors red and blue on the edges of C in the complete graph H. Then the red subgraph H_R of H is 1-regular and the blue subgraph H_B of H is $m - 1$ regular.

As a result we have that there is a red-blue coloring of the edges of the complete graph H where there is neither a red $K_{1,m}$ nor a blue $K_{1,2}$. Therefore, we conclude that

$$R(K_{1,2}, K_{1,m}) > m.$$

Consequently, we have that

$$R(K_{1,2}, K_{1,m}) = m + 1.$$

□

2.3 The Ramsey Number for Two Stars $K_{1,n}$ and $K_{1,m}$

In this section we prove the Ramsey number for two stars $K_{1,n}$ and $K_{1,m}$ of order $n + 1 \geq 3$ and $m + 1 \geq 3$ respectively. This results appeared in two articles: *Recent results on generalized Ramsey Theory for graphs* [16] and *On Ramsey numbers for stars* [8]. Harary in his article *Recent results on generalized Ramsey Theory for graphs* [16] only considered the Ramsey number for two stars of equal order, that is $n = m$. Burr and Roberts in their article *On Ramsey numbers for stars* [8] considered stars of all orders $n + 1 \geq 3$ and $m + 1 \geq 3$.

We begin by proving the Ramsey number of two stars $K_{1,n}$ and $K_{1,m}$ where both $n \geq 2$ and $m \geq 2$ are even integers. Our proof is original.

Proposition 2.3.1. *Let n and m be even integers with $n \geq 2$ and $m \geq 2$. Then*

$$R(K_{1,\,n}, K_{1,\,m}) = n + m - 1.$$

Proof

Since n and m are even integers, let $n = 2x$ and $m = 2y$ for some positive integers x and y. We show that

$$R(K_{1,\,n}, K_{1,\,m}) > n + m - 2.$$

In this regard, let H be the complete graph of order $n + m - 2$. Since the order $n + m - 2$ of H is even, we have that H is 1-factorable by Theorem 1.2.3. If u is an arbitrary vertex in H, then

$$\begin{aligned} deg\ u &= (n + m - 2) - 1 \\ &= n + m - 3 \\ &= (n - 2) + (m - 1) \\ &= (n - 1) + (m - 2). \end{aligned}$$

That is, the graph H is 1-factorable into either

$$(n - 2) + (m - 1)$$

or

$$(n - 1) + (m - 2)$$

1-factors. We now consider these two possibilities.

Case 1: H is 1-factorable into $(n - 2) + (m - 1)$ factors.

Let there be a red-blue coloring of edges of the complete graph H. Color $n - 2$ edges of these 1-factors red and the remaining $m - 1$ edges of these 1-factors blue in H. The resulting red subgraph H_R of H is a regular graph of degree $n - 2$ and the resulting blue subgraph is a regular graph of degree $m - 1$.

Thus, there is a red-blue coloring of the edges of the complete graph H of order $n + m - 2$ where there is neither a red $K_{1,\,n}$ nor a blue $K_{1,\,m}$.

Case 2: H is 1 - factorable into $(n-1)+(m-2)$ factors.

We reflect consideration on the complete graph H with a red-blue coloring of edges. Color $n-1$ edges of these 1-factors red and the remaining $m-2$ edges of these 1-factors blue in H. The resulting red subgraph H_R of H is a regular graph of degree $n-1$ and the resulting blue subgraph H_B of H is a regular graph of degree $m-2$. Thus, there is a red-blue coloring of the edges of the complete graph H of order $n+m-2$ where there is neither a red $K_{1,\,n}$ nor a blue $K_{1,\,m}$.

From the two cases above we may conclude that

$$R(K_{1,\,n}, K_{1,\,m}) > n+m-2.$$

We now show that $R(K_{1,\,n}, K_{1,\,m}) \leq n+m-1$. In this regard, let G be the complete graph of order $n+m-1$. If v is an arbitrary vertex of G, then

$$\begin{aligned} deg\ v &= (n+m-1)-1 \\ &= n+m-2 \\ &= 2x+2y-2 \\ &= 2(x-1)+2y \\ &= 2x+2(y-1). \end{aligned}$$

Consequently, we see that the graph G is $2(x+y-1)$-regular, and so, by Theorem 1.2.4, we have that G is 2-factorable. The graph G is 2-factorable into $(x-1+y)$ 2-factors. Assume to the contrary that in any red-blue coloring of the edges of the graph G there is neither a red $K_{1,\,n}$ nor a blue $K_{1,\,m}$.

We now consider two cases according to the grouping of the factors of G when we assign colors to the edges.

Case 1: G is 2-factorable into $(x-1)+y$ factors.

Consider a red-blue coloring of the edges of the graph G by coloring $x-1$ edges of these 2-factors red and coloring the remaining y edges of these 2-factors blue. The resulting red subgraph G_R of G is a regular graph of degree

$$2(x-1) = 2x-2 = n-2$$

and the resulting blue subgraph G_B of G is a regular graph of degree

$$2y = m.$$

So we have that in any red-blue coloring of the edges of G there is always a blue $K_{1,m}$ since the blue subgraph G_B of G is m-regular, a contradiction.

Case 2: G is 2-factorable into $[x+(y-1)]$ 2-factors.
Let there be a red-blue coloring of the edges of the graph G by coloring x of these 2-factors red and coloring the remaining $y-1$ edges of these 2-factors blue. The resulting red subgraph G_R of G is a regular graph of degree

$$2x = n$$

and the resulting blue subgraph G_B of G is a regular graph of degree

$$2(y-1) = 2y-2 = m-2.$$

So we have that in any red-blue coloring of the edges of G there is always a red $K_{1,n}$ since the red subgraph G_R of G is n-regular, a contradiction.

In both cases we get a contradiction, and so

$$R(K_{1,n}, K_{1,m}) = n+m-1.$$

□

We now prove the Ramsey number for two stars $K_{1,n}$ and $K_{1,m}$ where both $n \geq 3$ and $m \geq 3$ are odd. When $n = 1$, then star $K_{1,n}$ is isomorphic to the path P_2. From the article *On Ramsey-type problems* [15] authored by L. Gerencsér we have that $R(P_2, P_2) = 2$. Consequently, we have that $R(K_{1,1}, K_{1,1}) = 2$.

Proposition 2.3.2. *Let n and m be odd integers with $n \geq 3$ and $m \geq 3$. Then*

$$R(K_{1,n}, K_{1,m}) = n + m.$$

Proof

Since n and m are odd integers, let $n = 2x + 1$ and $m = 2y + 1$ for some positive integers x and y. We show that

$$R(K_{1,n}, K_{1,m}) > n + m - 1.$$

With that said, let H be the complete graph of order $n + m - 1$. If u is an arbitrary vertex of H, then

$$\begin{aligned} deg\ u &= (n + m - 1) - 1 \\ &= n + m - 2 \\ &= (2x + 1) + (2y + 1) - 2 \\ &= 2x + 2y. \end{aligned}$$

We see that the graph H is $2(x + y)$-regular, and so by Theorem 1.2.4 we have that H is 2-factorable. The graph H is 2-factorable into $(x + y)$ factors. Let there be a red-blue coloring of the edges of the complete graph H. Color x edges of these 2-factors red and the remaining y edges of these 2-factors blue in H. The red subgraph H_R of H is a regular graph of degree

$$2x = n - 1$$

and the blue subgraph H_R of H is regular graph of degree

$$2y = m - 1.$$

Thus, we have that there is a red-blue coloring of the edges of H where there is a neither a red $K_{1,n}$ nor a blue $K_{1,m}$, and so

$$R(K_{1,n}, K_{1,m}) > n + m - 1.$$

We now show that

$$R(K_{1,n}, K_{1,m}) \leq n + m.$$

Let G be the complete graph of order $n + m$. If v is an arbitrary vertex in G, then

$$deg\ v = (n + m) - 1.$$

Hence G is a regular graph of degree $n + m - 1$. Consider a red-blue coloring of the edges of the complete graph G resulting in a red subgraph G_R of G and a blue subgraph G_B of G. Assume there is neither a red $K_{1,n}$ nor a blue $K_{1,m}$ in G. Table 2.3.1 gives all of the different ways to choose red and blue edges incident to vertex v. We conclude that if $deg_{G_R}\ v \geq m$, then G has a red $K_{1,m}$, and if $deg_{G_B}\ v \geq n$, then G has a blue $K_{1,n}$ as indicated in the last column.

Table 2.3.1: The number of red and blue edges incident to v in K_{n+m}.

$deg_{G_B}\ v$	$deg_{G_R}\ v$	Contradiction
0	$m+n-1$	red $K_{1,m}$
1	$m+n-2$	red $K_{1,m}$
2	$m+n-3$	red $K_{1,m}$
$\vdots$	$\vdots$	$\vdots$
$n-1$	m	red $K_{1,m}$
n	$m-1$	blue $K_{1,n}$
$n+1$	$m-2$	blue $K_{1,n}$
$\vdots$	$\vdots$	$\vdots$
$m+n-3$	2	blue $K_{1,n}$
$m+n-2$	1	blue $K_{1,n}$
$m+n-1$	0	blue $K_{1,n}$

Consequently, in any red-blue coloring of edges of the complete graph G there is always either a red $K_{1,m}$ or a blue $K_{1,n}$. Thus,

$$R(K_{1,m}, K_{1,n}) \leq n + m$$

and so

$$R(K_{1,m}, K_{1,n}) = n + m.$$

□

We finally confirm the Ramsey number of two stars $K_{1,n}$ and $K_{1,m}$ where $n \geq 2$ and $m \geq 2$ are of opposite parity.

Proposition 2.3.3. *Let n and m be integers of opposite parity with $n \geq 2$ and $m \geq 2$. Then*

$$R(K_{1,n}, K_{1,m}) = n + m.$$

Proof

Since n and m are integers of opposite parity, let $n = 2x$ and $m = 2y + 1$ for some positive integers x and y. We show that

$$R(K_{1,n}, K_{1,m}) > n + m - 1.$$

Let H be the complete graph of order $n + m - 1$. Since the order $n + m - 1$ of H is even, we have that H is 1-factorable by Theorem 1.2.3. If u is an arbitrary vertex in H, then

$$\begin{aligned} deg\ u &= (n + m - 1) - 1 \\ &= n + m - 2 \\ &= (n - 1) + (m - 1). \end{aligned}$$

The graph H is 1-factorable into $(n - 1) + (m - 1)$ factors. Consider a red-blue coloring of the edges of the complete graph H by coloring $n - 1$ edges of these 1-factors red and coloring the remaining $m - 1$ edges of these 1-factors blue. The resulting red subgraph H_R of H is a regular graph of degree $n - 1$ and the resulting blue subgraph H_B of H is a regular graph of degree $m - 1$.

Consequently, there is a red-blue coloring of the edges of the complete graph H where there is neither a red $K_{1,n}$ nor a blue $K_{1,m}$, and so

$$R(K_{1,n}, K_{1,m}) > n + m - 1.$$

We now show that $R(K_{1,n}, K_{1,m}) \leq n + m$.

Let G be the complete graph of order $n + m$. If v is an arbitrary vertex in G, then

$$deg\ v = n + m - 1.$$

We have that G is a regular graph of degree $n + m - 1$. Consider a red-blue coloring of the edges of the complete graph G resulting in a red subgraph G_R of G and a blue subgraph G_B of G. Assume there is neither a red $K_{1,n}$ nor a blue $K_{1,m}$ in G. In Table 2.3.2 we give all of the different ways to choose red and blue edges incident to vertex v. We see that if $deg_{G_R}\ v \geq m$, then G has a red $K_{1,m}$, and if $deg_{G_B}\ v \geq n$, then G has a blue $K_{1,n}$ as indicated in the last column of Table 2.3.2.

Table 2.3.2: The number of red and blue edges incident to v in K_{n+m}.

$deg_{G_B}\ v$	$deg_{G_R}\ v$	Contradiction
0	$m+n-1$	red $K_{1,m}$
1	$m+n-2$	red $K_{1,m}$
2	$m+n-3$	red $K_{1,m}$
$\vdots$	$\vdots$	$\vdots$
$n-1$	m	red $K_{1,m}$
n	$m-1$	blue $K_{1,n}$
$n+1$	$m-2$	blue $K_{1,n}$
$\vdots$	$\vdots$	$\vdots$
$m+n-3$	2	blue $K_{1,n}$
$m+n-2$	1	blue $K_{1,n}$
$m+n-1$	0	blue $K_{1,n}$

Consequently, we have that in any red-blue coloring of the edges of the complete graph G there is always either a red $K_{1,m}$ or a blue $K_{1,n}$. So we conclude that

$$R(K_{1,m}, K_{1,n}) \leq n + m$$

and so

$$R(K_{1,m}, K_{1,n}) = n + m.$$

□

By combining Proposition 2.3.1, Proposition 2.3.2 and Proposition 2.3.3, we obtain Theorem 2.3.1. This result is used as the upper bound for the number of partite sets when we confirm the results in Chapter 3.

Theorem 2.3.1. *Let n and m be integers with $n \geq 2$ and $m \geq 2$. Then*

$$R(K_{1,n}, K_{1,m}) = \begin{cases} n+m-1 & \text{if } n \text{ and } m \text{ are both even,} \\ n+m & \text{otherwise.} \end{cases}$$

2.4 The Ramsey Number for a Star $K_{1,m}$ and a Tree T'_n

In this section we look at the Ramsey number of $K_{1,m}$ and T'_n where T'_n is a tree of order $n \geq 5$ with a maximum degree of $n-2$. The statement of Theorem 2.4.1 was given by Guo and Volkmann in their article *Tree Ramsey numbers* [9] in 1995, however, for the sole purpose of maintaining similarity in the structure of the proof we use an original proof.

We prove Theorem 2.4.1 in two propositions according to whether $2 \mid (m+1)(n-1)$ or $2 \nmid (m+1)(n-1)$. We begin by proving the Ramsey number for a star $K_{1,m}$ and a tree T'_n when $2 \mid (m+1)(n-1)$.

Proposition 2.4.1. *If $n > m+1 \geq 4$ and $2 \mid (m+1)(n-1)$, then*

$$R(K_{1,m}, T'_n) = m+n-2.$$

Proof

We show that

$$R(K_{1,m}, T'_n) \leq m+n-2.$$

In this regard, let G be the complete graph of order $m+n-2$. If v is an arbitrary vertex of G, then

$$deg\ v = (m+n-2)-1 = m+n-3.$$

Assume that in any red-blue coloring of the edges of the complete graph G there is neither a red $K_{1,m}$ nor a blue T'_n. Consider a red-blue coloring of the edges of the complete graph G resulting in a red subgraph G_R of G and a blue subgraph G_B of G. Table 2.4.1 gives all of the different ways to choose red and blue edges incident to vertex v. We see that if $deg_{G_R}\, v \geq m$, then we have a red $K_{1,m}$, and if $deg_{G_B}\, v \geq n-2$, then we have a blue T'_n as indicated in the last column of Table 2.4.1.

Table 2.4.1: The number of red and blue edges incident to v in K_{n+m-2}.

$deg_{G_R}\, v$	$deg_{G_B}\, v$	Contradiction
$m+n-3$	0	red $K_{1,m}$
$m+n-4$	1	red $K_{1,m}$
$m+n-5$	2	red $K_{1,m}$
$\vdots$	$\vdots$	$\vdots$
m	$n-3$	red $K_{1,m}$
$m-1$	$n-2$	blue T'_n
$m-2$	$n-1$	blue T'_n
$\vdots$	$\vdots$	$\vdots$
2	$m+n-5$	blue T'_n
1	$m+n-4$	blue T'_n
0	$m+n-3$	blue T'_n

Consequently, every red-blue coloring of the edges of the complete graph G has a red $K_{1,m}$ or a blue T'_n. We conclude that

$$R(K_{1,m}, T'_n) \leq m+n-2.$$

We now consider the lower bound,

$$R(K_{1,m}, T'_n) > m+n-3.$$

Let H be the complete graph of order $m+n-3$. If u is an arbitrary vertex in H, then

$$deg\, u = (m+n-3)-1 = m+n-4.$$

Since $2 \mid (m+1)(n-1)$ we have that $(m+1)(n-1) = 2q$ is even for some positive integer q. This implies that both $m+1$ and $n-1$ are even or one of $m+1$ and $n-1$

is even. We now consider these three cases.

Case 1: Both $m+1$ and $n-1$ are even.

Since both $m+1$ and $n-1$ are even, we have that both m and n are odd. That is $m = 2x+1$ and $n = 2y+1$ for some positive integers x and y. If u is an arbitrary vertex of H, then

$$\begin{aligned} deg\ u &= m+n-4 \\ &= (2x+1)+(2y+1)-4 \\ &= 2x+2y-2 \\ &= 2x+2(y-1) \end{aligned}$$

We find that the graph H is $2[x+(y-1)]$-regular and by Theorem 1.2.4, we have that H is 2-factorable into $x+(y-1)$ factors. Let there be a red-blue coloring of the edges of the complete graph H. Color x edges of these 2-factors red and the remaining $y-1$ edges of these 2-factors blue. The resulting red subgraph H_R of H is a regular graph of degree

$$2x = m-1$$

and the resulting blue subgraph H_B of H is a regular graph of degree

$$\begin{aligned} 2(y-1) &= 2y-2 \\ &= (n-1)-2 \\ &= n-3. \end{aligned}$$

Thus, we have that there is a red-blue coloring of the edges of H where there is neither a red $K_{1,m}$ nor a blue T'_n. For Case 1 we conclude that

$$R(K_{1,m}, T'_n) > m+n-3.$$

Case 2: $m+1$ is odd and $n-1$ is even.

Since $m+1$ is odd, we have that m is even and since $n-1$ is even, we have that n is odd. That is, $m = 2x$ and $n = 2y+1$ for some positive integers x and y. Therefore, the order $m+n-3$ of H is even and, by Theorem 1.2.3, we have that H is

1-factorable. Consequently, if u is an arbitrary vertex of H, then

$$deg\ u = m + n - 4 = (m - 1) + (n - 3).$$

The graph H is 1-factorable into $(m-1)+(n-3)$ factors. Let there be a red-blue coloring of the edges of the complete graph H by coloring $m-1$ edges of these 1-factors red and coloring the remaining $n-3$ edges of these 1-factors blue. The resulting red subgraph of H is a regular graph of degree $m-1$ and the resulting blue subgraph of H is a regular graph of degree $n-3$.

As a result we have that there is a red-blue coloring of edges of the complete graph H where there is neither a red $K_{1,\,m}$ nor a blue T_n'. For Case 2, we conclude that

$$R(K_{1,\,m}, T_n') > m + n - 3.$$

Case 3: $m+1$ is even and $n-1$ is odd.
Since $m+1$ is even and $n-1$ is odd, let that $m = 2x+1$ and $n = 2y$ for some positive integers x and y. Consequently, we have

$$\begin{aligned} m + n - 3 &= (2x+1) + (2y) - 3 \\ &= 2x + 2y - 2 \\ &= 2(x + y - 1) \end{aligned}$$

and so the order $m+n-3$ of H is even. By Theorem 1.2.3 we have that H is 1-factorable. If u is an arbitrary vertex of H, then

$$deg\ u = m + n - 4 = (m - 1) + (n - 3).$$

Thus, the graph H is 1-factorable into $(m-1)+(n-3)$ factors. Let there be a red-blue coloring of edges of the complete graph H. Color $m-1$ edges of these 1-factors red and color remaining $n-3$ edges of these 1-factors blue. The resulting red subgraph of H is a regular graph of degree $m-1$ and the resulting blue subgraph of H is a regular graph of degree $n-3$.
As a result we have that there is a red-blue coloring of the edges of the complete graph H where there is neither a red $K_{1,\,m}$ nor a blue T_n'. For Case 3 we conclude that

$$R(K_{1,\,m}, T'_n) > m + n - 3.$$

Therefore, $R(K_{1,\,m}, T'_n) = m + n - 2$ in all three cases above. □

We now confirm the Ramsey number for a star $K_{1,\,m}$ and a tree T'_n when

$$2 \nmid (m+1)(n-1).$$

Proposition 2.4.2. *If $n > m + 1 \geq 4$ and $2 \nmid (m+1)(n-1)$, then*

$$R(K_{1,\,m}, T'_n) = m + n - 3.$$

Proof

Since $2 \nmid (m+1)(n-1)$ we have that $(m+1)(n-1) = 2a+1$ is odd for some positive integer a. For $(m+1)(n-1)$ to be odd we have that both $m+1$ and $n-1$ are odd, implying that both m and n are even. Thus, $m = 2x$ and $n = 2y$ for some positive integers x and y.

We begin by proving the upper bound

$$R(K_{1,\,m}, T'_n) \leq m + n - 3.$$

Let G be the complete graph of order $m + n - 3$. If v is an arbitrary vertex of G, then

$$\begin{aligned} deg\ v &= (m + n - 3) - 1 \\ &= m + n - 4 \\ &= 2x + 2y - 4 \\ &= 2x + 2(y - 2) \\ &= 2(x - 1) + 2(y - 1). \end{aligned}$$

We find that the graph G is a $2(x + y - 2)$ regular graph which is 2-factorable into $x + (y - 2)$ or $(x - 1) + (y - 1)$ factors by Theorem 1.2.4. Assume that in any red-blue coloring of the edges of the complete graph G there is neither a red $K_{1,\,m}$ nor a blue T'_n. We now consider these two cases.

Case 1: G is 2-factorable into $x + (y - 2)$ factors.

Let there be a red-blue coloring of the edges of the complete graph G by assigning x edges of these 2-factors the color red and the remaining $y - 2$ edges of these 2-factors the color blue. The resulting red subgraph of G is a regular graph of degree $2x = m$. Consequently, we have a red $K_{1,\,m}$ which is a contradiction to our assumption. For Case 1 we conclude that

$$R(K_{1,\,m}, T_n') \leq m + n - 3.$$

Case 2: G is 2-factorable into $(x - 1) + (y - 1)$ factors.

Let there be a red-blue coloring of edges of the complete graph G by assigning $x - 1$ edges of these 2-factors the color red and the remaining $y - 1$ edges of these 2-factors the color blue. The resulting blue subgraph of G is a regular graph of degree

$$2(y - 1) = 2y - 2 = n - 2.$$

As a result, we have a blue T_n' which is a contradiction to our assumption. For Case 2 we conclude that in any red-blue edge coloring of G there is always a blue T_n'. Consequently,

$$R(K_{1,\,m}, T_n') \leq m + n - 3.$$

We now proof the lower bound

$$R(K_{1,\,m}, T_n') > m + n - 4.$$

Let H be the complete graph of order $m + n - 4$. Since m and n are even, we have that the order of H is even and we have that H is 1-factorable by Theorem 1.2.3. Let u be an arbitrary vertex of H. Then

$$\begin{aligned} deg\ u &= (m + n - 4) - 1 \\ &= m + n - 5 \\ &= (m - 2) + (n - 3). \end{aligned}$$

The graph H is 1-factorable into $(m - 2) + (n - 3)$ factors. Consider a red-blue coloring of the edges of the complete graph H by assigning the color red to $m - 2$ edges of these 1-factors and the color blue to the remaining $n - 3$ edges of these 1-factors. The resulting red subgraph H_R of H is a regular graph of degree $m-2$ and the resulting blue subgraph H_B of H is a regular graph of degree $n-3$.

Therefore, there exist a red-blue coloring of the edges of the complete graph H where there is neither a red $K_{1,\,m}$ nor a blue T_n'. Consequently,

$$R(K_{1,\,m}, T_n') > m+n-4,$$

and so

$$R(K_{1,\,m}, T_n') = m+n-3.$$

□

By combining Proposition 2.4.1 and Proposition 2.4.2 we have Theorem 2.4.1.

Theorem 2.4.1. *For $m \geq 2$ and $n \geq 5$,*

$$R(K_{1,\,m}, T_n') = \begin{cases} m+n-2 & \text{if } 2 \mid (m+1)(n-1), \\ m+n-3 & \text{if } 2 \nmid (m+1)(n-1). \end{cases}$$

2.5 The Ramsey Number for Two Trees

In this section we prove the number $R(T_m', T_n')$ of two trees of order $m \geq 5$ and $n \geq 5$ with maximum degree $m-2$ and $n-2$ respectively. In 1995 Guo and Volkmann did study the Ramsey number for two trees T_m' and T_n' in their article *Tree-Ramsey numbers* [9] and proved the result in Theorem 2.5.1 below.

In the article *Ramsey Number for Trees* [17] Zhi-Hong Sun studied the Ramsey number for a tree T_n' and G_m where G_m is a connected graph of order m which need not be complete. Zhi-Hong Sun also considered the Ramsey number of two trees T_n'' where T_n'' is a tree on n vertices with

$$V = \{v_0, v_1, \cdots, v_{n-2}, v_{n-1}\}$$

and

$$E = \{v_0v_i | 1 \leq i \leq n-3\} \bigcup \{v_{n-3}v_{n-2}, v_{n-2}v_{n-1}\}.$$

Theorem 2.5.1. *[17] If n and m are positive integers where $n \geq 5$ and $m \geq 5$, then*

$$R(T'_m, T'_n) = \begin{cases} m+n-5 & \textit{if } m = n \equiv 0(mod\ 2), \\ m+n-3 & \textit{if } (m\text{ - }1) \mid (n-3), \\ m+n-4 & \textit{otherwise.} \end{cases}$$

We changed the statement in Theorem 2.5.1 to the statement in Theorem 2.5.2 below.

In our analysis of the proof of Theorem 2.5.1 we observed that where we consider the complete graph on $m+n-3$ vertices is covered by considering the complete graphs on $m+n-5$ vertices and $m-n-4$ vertices respectively. When considering the complete graph on $m+n-3$ vertices Guo and Volkmann considered the condition $(m-1) \mid (n-3)$ implying that $n-3 = k(m-1)$ for some positive integer k. This condition is covered in Theorem 2.5.2 with the following reasoning.

Consider when m odd. Since m is odd, we have that $m = 2a+1$ for some positive integer a. Therefore

$$\begin{aligned} n-3 &= k(m-1) \\ &= k[(2a+1)-1] \\ &= k(2a) \\ &= 2ak. \end{aligned}$$

As a result, we have that $n-3 = 2ak$ which implies that

$$\begin{aligned} n &= 2ak+3 \\ &= 2ak+(2+1) \\ &= 2(ak+1)+1 \end{aligned}$$

where $ak+1$ is an integer. Therefore, we have that n is odd. The instance where both m and n are odd is covered when considering the complete graph on $m+n-4$ vertices in Proposition 2.5.2.

Consider m is even. Since m is even, we have that $m = 2a$ for some positive integer a. Therefore,

$$\begin{aligned} n - 3 &= k(m-1) \\ &= k(2a-1). \end{aligned}$$

We now consider two instances according to the parity of k.

If k is even, then we have that $k = 2b$ for some positive integer b. Therefore, we have that

$$\begin{aligned} n - 3 &= k(2a-1) \\ &= 2b(2a-1) \\ &= 4ab - 2b \end{aligned}$$

and

$$\begin{aligned} n &= 4ab - 2b - 3 \\ &= 4ab - 2b - (2+1) \\ &= 4ab - 2b - 2 - 1 \\ &= 2(2ab - b - 1) - 1 \end{aligned}$$

where $2(ab) - b - 1$ is some positive integer. Thus, n is odd. The instance where m is even and n is odd is considered in Proposition 2.5.2. If k is odd, then we have that $k = 2b+1$ for some positive integer b. Therefore,

$$\begin{aligned} n - 3 &= k(2a-1) \\ &= (2b+1)(2a-1) \\ &= 4ab - 2b + 2a - 1 \end{aligned}$$

and

$$\begin{aligned} n &= 4ab - 2b + 2a - 1 + 3 \\ &= 4ab - 2b + 2a + 2 \\ &= 2(2ab - b + a + 1) \end{aligned}$$

where $2ab - b + a + 1$ is a positive integer and so n is an even integer. The instance where both n and m are even is covered in Proposition 2.5.1.

By the reasoning given above, we managed to simplify the statement of Theorem 2.5.1 to the statement given in Theorem 2.5.2.

Theorem 2.5.2. *If n and m are positive integers where $n \geq 5$ and $m \geq 5$, then*

$$R(T'_m, T'_n) = \begin{cases} m + n - 5 & \text{if } m \text{ and } n \text{ are both even,} \\ m + n - 4 & \text{otherwise.} \end{cases}$$

We prove Theorem 2.5.2 by considering the following two propositions according to the different parities of m and n.

Proposition 2.5.1. *Let m and n be positive integers where $n \geq 5$ and $m \geq 5$. If both m and n are even, then*

$$R(T'_m, T'_n) = m + n - 5.$$

Proof

We begin by proving the upper bound

$$R(T'_m, T'_n) \leq m + n - 5.$$

Let G be the complete graph of order $m + n - 5$ and let v be an arbitrary vertex of G. Then

$$deg\ v = (m + n - 5) - 1 = m + n - 6.$$

Since we have that both m and n are even, let $m = 2x$ and $n = 2y$ for some positive integers x and y. Then,

$$\begin{aligned} deg\ v &= m + n - 6 \\ &= 2x + 2y - 6 \\ &= 2(x-2) + 2(y-1) \\ &= 2(x-1) + 2(y-2). \end{aligned}$$

We find that the graph G is a $2(x+y-3)-$regular graph and we have that G is 2-factorable into either $(x-2)+(y-1)$ or $(x-1)+(y-2)$ factors by Theorem 1.2.4. Assume that in any red-blue coloring of the edges of the complete graph G there is neither a red T'_m nor a blue T'_n. We consider two cases according to the factors of G.

Case 1: G is 2-factorable into $(x-2)+(y-1)$ factors.
Let there be a red-blue coloring of the edges of the complete graph G. Color $(x-2)$ edges of these 2-factors red the remaining $(y-1)$ edges of these 2-factors blue in G. The resulting blue subgraph of G is a regular graph of degree

$$2(y-1) = 2y - 2 = n - 2.$$

Consequently, we have a blue T'_n, which is a contradiction to our assumption. Thus, in any red-blue coloring of the edges of G there is always a blue T'_n.

Case 2: G is 2-factorable into $(x-1)+(y-2)$ factors.
Let there be a red-blue coloring of the edges of the complete graph G. Color $(x-1)$ edges in these 2-factors red and the remaining $(y-2)$ edges of these 2-factors blue. The resulting red subgraph of G is a regular graph of degree

$$2(x-1) = 2x - 2 = m - 2.$$

Consequently, we have a red T'_m which is a contradiction. Thus, in any red-blue coloring of the edges of G there is always a red T'_m.

From the two cases above we may conclude that

$$R(T'_m, T'_n) \leq m + n - 5.$$

We now consider the lower bound, where we want to show that

$$R(T'_m, T'_n) > m + n - 6.$$

Let H be the complete graph of order $m + n - 6$. Since m and n are even we have that the order of H is even and by Theorem 1.2.3 we have that H is 1-factorable. If u is an arbitrary vertex of H, then

$$\begin{aligned} deg\ u &= (m + n - 6) - 1 \\ &= m + n - 7 \\ &= (m - 4) + (n - 3). \end{aligned}$$

The graph H is 1-factorable into $(m - 4) + (n - 3)$ factors. Let there be a red-blue coloring of the edges of the complete graph H by coloring $(m - 4)$ edges of these 1-factors red and remaining $(n - 3)$ edges of the remaining 1-factors blue. The resulting red subgraph of H is a regular graph of degree $m - 4$ and the resulting blue subgraph of H is a regular graph of degree $n - 3$.

Therefore, there is a red-blue coloring of edges of the complete graph H where there is neither a red T_m nor a blue T'_n. As a result we get that

$$R(T'_m, T'_n) > m + n - 6$$

and so

$$R(T'_m, T'_n) = m + n - 5.$$

□

Lastly, we need to consider when m and n are not both even.

Proposition 2.5.2. *Let m and n be positive integers where $m \geq 5$ and $n \geq 5$. If m and n are not both even, then*

$$R(T'_m, T'_n) = m + n - 4.$$

Proof

We begin by proving the upper bound

$$R(T'_m, T'_n) \leq m + n - 4.$$

Let G be the complete graph of order $m + n - 4$. If v is an arbitrary vertex of G, then

$$deg\ v = (m + n - 4) - 1 = m + n - 5.$$

Assume to the contrary that in any red-blue coloring of the edges of the complete graph G there is neither a red T'_m nor a blue T'_n. Consider a red-blue coloring of the edges of G resulting in a red subgraph G_R of G and a blue subgraph G_B of G. In Table 2.5.1 we give all the possible colorings of the edges incident to vertex v. We see that if $deg_{G_R}\ v \geq m - 2$, then G has a red T'_m, and if $deg_{G_B}\ v \geq n - 2$, then G has a blue T'_n as indicated in the last column of Table 2.5.1. As a result in every red-blue coloring of the edges of the complete bipartite graph G there is always a blue T'_n or a red T'_m. As such we conclude that

$$R(T'_m, T'_n) \leq m + n - 4.$$

We now consider the lower bound

$$R(T'_m, T'_n) > m + n - 5.$$

Let H be the complete graph of order $m + n - 5$. If u is an arbitrary vertex of H, then

$$deg\ u = (m + n - 5) - 1 = m + n - 6.$$

Since m and n are not both even, we have that either m and n are both odd or m and n are of opposite parity. We now consider these two cases.

Table 2.5.1: The number of red and blue edges incident to v in K_{n+m-4}.

$deg_{G_R}\ v$	$deg_{G_B}\ v$	Contradiction
$m+n-5$	0	red T'_m
$m+n-6$	1	red T'_m
$m+n-7$	2	red T'_m
$\vdots$	$\vdots$	$\vdots$
$m-1$	$n-4$	red T'_m
$m-2$	$n-3$	red T'_m
$m-3$	$n-2$	blue T'_n
$m-4$	$n-1$	blue T'_n
$\vdots$	$\vdots$	$\vdots$
2	$m+n-7$	blue T'_n
1	$m+n-6$	blue T'_n
0	$m+n-5$	blue T'_n

Case 1: Both m and n are odd.

If both m and n are odd, then we let $m = 2x+1$ and $n = 2y+1$ for some positive integers x and y. Therefore,

$$\begin{aligned} deg\ u &= m+n-6 \\ &= (2x+1)+(2y+1)-6 \\ &= 2x+2y-4 \\ &= 2(x-1)+2(y-1). \end{aligned}$$

We have that H is a $2(x+y-2)$-regular graph. By theorem 1.2.4, we have that H is 2-factorable into $(x-1)+(y-1)$ factors.

Let there be a red-blue coloring of the edges of the complete graph H. Color $(x-1)$ of these 2-factors red and the remaining $(y-1)$ edges of these 2-factors blue. The resulting red subgraph of G is a regular graph of degree

$$\begin{aligned} 2(x-1) &= 2x-2 \\ &= (m-1)-2 \\ &= m-3 \end{aligned}$$

and the resulting blue subgraph of G is a regular graph of degree

$$\begin{aligned} 2(y-1) &= 2y-2 \\ &= (n-1)-2 \\ &= n-3. \end{aligned}$$

Consequently, we have that there is a red-blue coloring of the edges of the complete graph H where there is neither a red T'_m nor a blue T'_n.

Case 2: m and n are of opposite parity.
Without loss of generality, assume that m is even and n is odd. Thus, $m = 2x$ and $n = 2y + 1$ for positive integers x and y. We have that the order of H is

$$\begin{aligned} m+n-5 &= 2x+2y+1-5 \\ &= 2x+2y-4 \\ &= 2(x+y-2) \end{aligned}$$

which is even and, by Theorem 1.2.3, we have that H is 1-factorable. Thus,

$$\begin{aligned} deg\ u &= m+n-6 \\ &= (m-3)+(n-3). \end{aligned}$$

and so H is 1-factorable into $[(m-3)+(n-3)]$.

Assume, to the contrary, that in every red-blue coloring of the edges of the complete graph H there is always either a red T'_m or a blue T'_n. Reflect into consideration a red-blue coloring of the edges of the complete graph H by coloring $(m-3)$ edges of these 1-factors red and the remaining $(n-3)$ edges of these 1-factors blue. The resulting red subgraph of G is a regular graph of degree $m-3$ and the resulting blue subgraph of G is a regular graph of degree $n-3$.

Consequently, we have that there is a red-blue coloring of the edges of the complete graph H where there is neither a red T'_m nor a blue T'_n. Consequently,

$$R(T'_m, T'_n) > m+n-5$$

and so

$$R(T'_m, T'_n) = m+n-4.$$

□

2.6 Conclusion

This chapter mainly focuses on confirming the Ramsey number $R(F, H)$ of two trees F and H. The result in Theorem 2.3.1 is used as an upper bound for the number of partite sets when dealing with k-Ramsey numbers in Chapter 3.

Theorem 2.3.1 *Let n and m be integers with $n \geq 2$ and $m \geq 2$. Then*

$$R(K_{1,n}, K_{1,m}) = \begin{cases} n+m-1 & \text{if } n \text{ and } m \text{ are both even,} \\ n+m & \text{otherwise.} \end{cases}$$

We focus on confirming the Ramsey number $R(K_{1,m}, T'_n)$ for a star of order $m+1 \geq 3$ and a tree T'_n of order $n \geq 5$.

Theorem 2.4.1 *For $m \geq 2$ and $n \geq 5$,*

$$R(K_{1,\,m}, T'_n) = \begin{cases} m+n-2 & \text{if } 2 \mid (m+1)(n-1), \\ m+n-3 & \text{if } 2 \nmid (m+1)(n-1). \end{cases}$$

We furthermore prove the Ramsey number $R(T'_m, T'_n)$ for two trees of order $m \geq 5$ and $n \geq 5$, respectively.

Theorem 2.5.2 *If n and m are positive integers where $n \geq 5$ and $m \geq 5$, then*

$$R(T'_m, T'_n) = \begin{cases} m+n-5 & \text{if } m \text{ and } n \text{ are both even}, \\ m+n-4 & \text{otherwise}. \end{cases}$$

Chapter 3

The k-Ramsey Number for Two Stars

3.1 Purpose

The main aim of this chapter is to determine the k-Ramsey number $R_k(K_{1,\,m},\, K_{1,\,n})$ for the pair of stars $K_{1,\,m}$ and $K_{1,\,n}$ of order $m+1 \geq 3$ and $n+1 \geq 3$ respectively. We are particularly interested in the case where $k = 2$ as this result is used to estimate the bipartite Ramsey number $b(K_{1,\,m},\, K_{1,\,n})$ in Chapter 4.

3.2 The 2-Ramsey Number for Two Stars $K_{1,\,2}$ and $K_{1,\,m}$

The main objective of this section is to present a few background results. We notice that $K_{1,\,2}$ is isomorphic to P_3 and so we have

$$R_2(K_{1,\,2}, K_{1,\,2}) = 2b(P_3, P_3) - 1.[18]$$

The bipartite Ramsey number of two paths was studied by Faudree and Schelp in their article *Path-path Ramsey-type Numbers for the Complete Bipartite graph* [18]. From the article we have that $R_2(K_{1,\,2},\, K_{1,\,2}) = 5$. We now consider $R_k(K_{1,\,n},\, K_{1,\,m})$ where variables k and n are fixed with $k = 2$ and $n = 2$.

Theorem 3.2.1. *[10] For each integer $m \geq 2$, $R_2(K_{\,2},\, K_{1,\,m}) = 2m+1$.*

Proof

We begin by proving the upper bound

$$R_2(K_{1,\,2}, K_{1,\,\mathrm{m}}) \leq 2m + 1.$$

Consider the balanced complete bipartite graph G of order $2m + 1$. Let U be the partite set of G with m vertices and V be the partite set of G with $m + 1$ vertices. If $u \in U$, then

$$deg\ u = m + 1.$$

Consider a red-blue coloring of the edges of graph G resulting in a red subgraph G_R of G and a blue subgraph G_B of G. In Table 3.2.1 we give all of the different ways to choose red and blue of the edges incident to vertex u. We find that if $deg_{G_R}\ u \geq m$, then G has a red $K_{1,\,m}$, and if $deg_{G_B}\ u \geq 2$, then G has a blue $K_{1,\,2}$ as indicated in the last column of Table 3.2.1. Therefore, we have a contradiction to our assumption and so

$$R_2(K_{1,\,2}, K_{1,\,m}) \leq 2m + 1.$$

Table 3.2.1: The number of red and blue edges incident to u in K_{2m+1}.

$deg_{G_R}\ u$	$deg_{G_B}\ u$	Contradiction
$m + 1$	0	red $K_{1,\,m}$
m	1	red $K_{1,\,m}$
$m - 1$	2	blue $K_{1,\,2}$
$m - 2$	3	blue $K_{1,\,2}$
$\vdots$	$\vdots$	$\vdots$
2	$m - 1$	blue $K_{1,\,2}$
1	m	blue $K_{1,\,2}$
0	$m + 1$	blue $K_{1,\,2}$

We now consider the lower bound

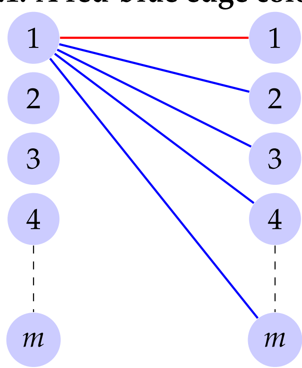
$$b(K_{1,\,2}, K_{1,\,m}) > m.$$

In this regard, let H be the complete bipartite graph of order $2m$ where there is an equal number of vertices in the partite sets. If v is an arbitrary vertex in graph H, then $deg\ v = m$. By Theorem 1.2.5, we have that the graph H is 1-factorable. Thus,

H is 1-factorable into m factors otherwise we have a red $K_{1,2}$ or a blue $K_{1,\,\mathrm{m}}$ in H.

Consider a red-blue coloring of the edges of graph H by assigning the color red to one of these 1-factors and the color blue to the remaining $m-1$ of these 1-factors as indicated in Figure 3.2.1. In Figure 3.2.1 we only colored the edges incident to one vertex in a partite set to avoid confusion. However, this coloring is repeated for all of the vertices in the partite set.

Figure 3.2.1: A red-blue edge coloring of H.



From the graph in Figure 3.2.1 we have that

$$deg_{H_R}\, v = 1$$

and

$$deg_{H_B}\, v = m - 1.$$

We find that the resulting red subgraph of H is one regular and the resulting blue subgraph of H is $m-1$ regular. Therefore, there exists a red-blue coloring of the edges of graph H where there is neither a red $K_{1,\,2}$ nor a blue $K_{1,\,m}$. It follows that

$$R_2(K_{1,\,2}, K_{1,\,m}) > 2m.$$

and so,

$$R_2(K_{1,\,2}, K_{1,\,m}) = 2m + 1.$$

□

3.3 The k-Ramsey Number for Two Stars $K_{1,\,n}$ and $K_{1,\,m}$

This section focuses on the k-Ramsey number for two stars $K_{1,\,n}$ and $K_{1,\,m}$ where $m, n \geq 3$. We are mainly interested with the instance where $k = 2$, however, we give the proof for the case where the number of vertices is divisible by the number of partite sets, that is $(m + n - 2) \mid (k - 1)$. This is because for the case where $(m + n - 2) \nmid (k - 1)$ we have that $k \geq 3$. For the other cases please refer to the article *Stars and their k-Ramsey numbers* [10].

We begin by confirming the upper bound for the k-Ramsey number of two stars $K_{1,\,m}$ and $K_{1,\,n}$ where $m \geq 2$ and $n \geq 3$.

Proposition 3.3.1. *Let k, n and m be positive integers with $2 \leq k < R(K_{1,\,n}, K_{1,\,m})$, $m \geq 3$ and $n \geq 3$. If $m + n - 2 = (k-1)q$ for some positive integer q, then*

$$R_k(K_{1,\,m}, K_{1,\,n}) \leq kq + 1.$$

Proof

Consider the balanced complete k-partite graph $G = K_{(k-1)q,\,q+1}$ of order $kq + 1$. Then G has $k - 1$ partite sets of order q and one partite set of order $q + 1$. If v is an arbitrary vertex from one of the partite sets of G with q vertices, then

$$\begin{aligned} deg\ v &= (k-2)q + q + 1 \\ &= qk - 2q + q + 1 \\ &= qk - q + 1 \\ &= (k-1)q + 1. \end{aligned}$$

This is the maximal degree of the balanced k-partite graph G and from the statement of the theorem we have that

$$m + n - 2 = (k-1)q.$$

Consequently, we have

$$\begin{aligned} deg\ v &= (k-1)q+1 \\ &= (m+n-2)+1 \\ &= m+n-1 \\ &= \Delta G. \end{aligned}$$

Consider a red-blue coloring of the edges of graph G resulting in a red subgraph G_R of G and a blue subgraph G_B of G. In Table 3.3.1 we give all the different ways to choose red and blue edges incident to vertex v, the last column gives the graph we are trying to avoid. We find that if $deg_{G_R}\ v \geq m$, then G has a red $K_{1,\,m}$, and if $deg_{G_B}\ v \geq n$, then G has a blue $K_{1,\,n}$.

Table 3.3.1: The number of red and blue edges incident to v in $K_{(k-1)q,\,q+1}$.

$deg_{G_B}\ v$	$deg_{G_R}\ v$	Contradiction
0	$m+n-1$	red $K_{1,\,m}$
1	$m+n-2$	red $K_{1,\,m}$
2	$m+n-3$	red $K_{1,\,m}$
$\vdots$	$\vdots$	$\vdots$
$n-1$	m	red $K_{1,\,m}$
n	$m-1$	blue $K_{1,\,n}$
$n+1$	$m-2$	blue $K_{1,\,n}$
$\vdots$	$\vdots$	$\vdots$
$m+n-3$	2	blue $K_{1,\,n}$
$m+n-2$	1	blue $K_{1,\,n}$
$m+n-1$	0	blue $K_{1,\,n}$

Thus, in every red-blue coloring of the edges of the balanced complete k-partite graph G there is always either a red $K_{1,\,m}$ or a blue $K_{1,\,n}$ which is a contradiction to our assumption. Hence, we conclude that

$$R_k(K_{1,\,m}, K_{1,\,n}) \leq kq+1.$$

$\square$

We now confirm the k-Ramsey numbers of two stars $K_{1,\,m}$ and $K_{1,\,n}$ where $m, n \geq 3$ and $(k-1)q$ is odd.

Proposition 3.3.2. *Let k, n and m be positive integers with $2 \leq k < R(K_{1,\,n}, K_{1,\,m})$, $m \geq 3$ and $n \geq 3$. If $m+n-2 = (k-1)q$ for some positive integer q where $(k-1)q$ is odd, then*

$$R_k(K_{1,\,m}, K_{1,\,n}) = kq + 1.$$

Proof

Let $(k-1)q$ be odd. Due to the fact that the product of two integers is odd if and only if both integers are odd we have that $k-1$ and q are both odd. Thus, k is even since $k-1$ is odd. We show that $R_k(K_{1,\,n}, K_{1,\,m}) = kq+1$. From Proposition 3.3.1 we only need to show that $R_k(K_{1,\,n}, K_{1,\,m}) > kq$.

Consider the balanced complete k-partite graph H of order kq. Then H has k partite sets each of order q, that is $H = K_{kq}$. If v is an arbitrary vertex from one of the partite sets of H, then

$$deg\ v = (k-1)q.$$

Hence graph H is a $(k-1)q$-regular graph of degree $n+m-2$. Since

$$(k-1)q = n+m-2$$

is odd, it follows that n and m are of opposite parity. This is due to the fact that the sum of two integers is odd only if the integers are of opposite parity. Without loss of generality, let n be even and m be odd, that is $n = 2x$ and $m = 2y+1$ for some positive integers x and y.

Let $V_1, V_2, \cdots, V_k$ denote the partite sets of $V(H)$. Since k is even, we divide the partite sets V_i for $1 \leq i \leq k$ into $k/2$ pairs : $\{V_1, V_2\}, \{V_3, V_4\}, \cdots, \{V_{k-1}, V_k\}$. These $k/2$ pairs of partite sets bring about $k/2$ induced subgraphs $H_1, H_2, \cdots, H_{k/2}$ isomorphic to $K_{q,\,q}$. Let M_i be a perfect matching between V_{2i-1} and V_{2i} where $1 \leq i \leq k/2$.

Consider a subgraph F obtained by removing all of edges in the perfect matching M_i from H for $1 \leq i \leq k/2$. Then F is a regular spanning subgraph of degree $(k-1)q - 1$. If u is an arbitrary vertex of F, then

$$\begin{aligned} deg\ u &= (k-1)q - 1 \\ &= (n+m-2) - 1 \\ &= n+m-3 \\ &= 2x + (2y+1) - 3 \\ &= 2x + 2y - 2 \\ &= 2x - 2 + 2y \\ &= 2(x-1) + 2y. \end{aligned}$$

Thus, the subgraph F is 2-factorbale into $(x-1) + y$ factors.

Reflect into consideration a red-blue coloring of the edges of graph H by coloring $(x-1)$ edges in these 2-factors and each M_i red, where $1 \leq i \leq k/2$, and the remaining y edges in these 2-factors blue. The resulting red subgraph of H is a regular graph of degree

$$\begin{aligned} 2(x-1) + 1 &= 2x - 2 + 1 \\ &= 2x - 1 \\ &= n - 1 \end{aligned}$$

and the resulting blue subgraph of H is a regular graph of degree

$$2y = m - 1.$$

Thus, there is a red-blue coloring of the edges of graph H where there is neither a red $K_{1,\,n}$ nor a blue $K_{1,\,m}$, and so $R_k(K_{1,\,n}, K_{1,\,m}) > kq$. Consequently,

$$R_k(K_{1,\,n}, K_{1,\,m}) = kq + 1.$$

The same result is attained if n is odd and m is even. □

We finally confirm the k-Ramsey number of two stars $K_{1,m}$ and $K_{1,n}$ where $m \geq 3$, $n \geq 3$ and $(k-1)q$ is even.

Proposition 3.3.3. *Let k, n and m be positive integers with $2 \leq k < R(K_{1,n}, K_{1,m})$, $m \geq 3$ and $n \geq 3$. If $m+n-2 = (k-1)q$ for some positive integer q where $(k-1)q$ is even, then*

$$R_k(K_{1,m}, K_{1,n}) = \begin{cases} kq & \text{if } k, q \text{ are both odd and } n, m \text{ are both even,} \\ kq+1 & \text{otherwise.} \end{cases}$$

Proof

Let $(k-1)q$ be even. Then we have that either both $k-1$ and q are even, or one of $k-1$ and q is even. This is due to the fact that the product of any number by an even number is even. If $(k-1)q+2 = n+m$ is even, then it follows that n and m are of the same parity. The sum of any two numbers of the same parity is even. We consider two cases; the first case we assume that both n and m are odd, and in the second case we assume that both are even.

We show that

$$R_k(K_{1,n}, K_{1,m}) = kq+1,$$

but from Proposition 3.3.1 we are only left with showing that

$$R_k(K_{1,n}, K_{1,m}) > kq.$$

Case 1: n and m are both odd.

Let $n = 2x+1$ and $m = 2y+1$ for some positive integers n and m. Also let H be the balanced complete k-partite graph of order kq. Then $H = K_{kq}$ has k partite sets each of order q. If v is an arbitrary vertex from one of the partite sets of H, then

$$\begin{aligned} deg\ v &= (k-1)q \\ &= n+m-2 \\ &= (2x+1)+(2y+1)-2 \\ &= 2x+2y+2-2 \\ &= 2x+2y. \end{aligned}$$

This implies that H is a regular graph of degree

$$(k-1)q = 2x + 2y = 2(x+y).$$

By Theorem 1.2.4 we have that H is 2-factorable into $(x+y)$ factors.

Reflect into consideration a red-blue coloring of the edges of the balanced complete k-partite graph H by coloring x edges of these 2-factors red and the remaining y edegs of these 2-factors blue. The resulting red subgraph of H is a regular graph of degree $2x = n-1$ and the resulting blue subgraph of H is a regular graph of degree $2y = m-1$.

Therefor, there exists a red-blue coloring of the edges of graph H where there is neither a red $K_{1,\,n}$ nor a blue $K_{1,\,m}$ and so for Case 1 we have

$$R_k(K_{1,\,n}, K_{1,\,m}) > kq.$$

Case 2: n and m are both even.
Let $n = 2x$ and $m = 2y$ for some positive integers x and y. Further, let $(k-1)q$ be even and so we have that either both $k-1$ and q are even or one of $k-1$ and q is even. We consider the three different subcases.

Subcase 2.1: $k-1$ is odd and q is even.
Let H be the balanced complete k-partite graph of order kq. Then H has k-partite sets each of order q. If v is an arbitrary vertex from one of the partite sets of H, then

$$\begin{aligned} deg\ v &= (k-1)q \\ &= n + m - 2 \\ &= 2x + 2y - 2 \\ &= 2(x-1) + 2y. \end{aligned}$$

By Theorem 1.2.1, we have now that H is Hamiltonian. Let C be a Hamiltonian cycle that passes through all the vertices of H exactly once and let F be the spanning subgraph obtianed by removing the edges of C from H, that is, $F = H - E(C)$. If u

is an arbitrary vertex of F, then

$$\begin{aligned} deg\ u &= (k-1)q-2 \\ &= (n+m-2)-2 \\ &= 2x+2y-2-2 \\ &= 2x-2+2y-2 \\ &= 2(x-1)+2(y-1). \end{aligned}$$

Thus, F is a regular spanning subgraph of H of degree $2(x-1)+2(y-1)$. Consequently, we see that the graph F is 2-factorable into $(x-1)+(y-1)$ factors.

Consider a red-blue coloring of the edges of graph H by coloring $(x-1)$ edges of these 2-factors red, coloring the remaining $(y-1)$ of these 2-factors blue, and alternating the colors red and blue to the edges of C. The resulting red subgraph of H is a regular graph of degree

$$\begin{aligned} 2(x-1)+1 &= 2x-2+1 \\ &= 2x-1 \\ &= n-1 \end{aligned}$$

and the resulting blue subgraph of H is a regular graph of degree

$$\begin{aligned} 2(y-1)+1 &= 2y-2+1 \\ &= 2y-1 \\ &= m-1. \end{aligned}$$

Therefor, there exists a red-blue coloring of the edges of graph H where there is neither a red $K_{1,\,n}$ nor a blue $K_{1,\,m}$. So we have here

$$R_k(K_{1,\,n}, K_{1,\,m}) > kq$$

and so

$$R_k(K_{1,\,n}, K_{1,\,m}) = kq+1.$$

Subcase 2.2: Both $k-1$ and q are even.

Let $H = K_{kq}$ be the balanced complete k-partite graph of order kq. Then H has k partite sets each of order q. If v is an arbitrary vertex from one of the partite sets of H, then we find that

$$\begin{aligned} deg\ v &= (k-1)q \\ &= n+m-2 \\ &= 2x+2y-2 \\ &= 2x+2y-2 \\ &= 2(x-1)+2y. \end{aligned}$$

Thus, H is a regular graph of degree $(k-1)q = 2(x-1)+2y$. Since k is odd, and q is even, let $k = 2a+1$ and $q = 2b$ for some positive integers a and b. Hence,

$$\begin{aligned} n+m-2 &= (k-1)q \\ &= (2a+1-1)(2b) \\ &= (2a)(2b) \\ &= 4ab. \end{aligned}$$

Since $4 \mid (n+m-2)$, we write that $n+m \equiv 2\ (mod\ 4)$. Since both n and m are even, we have that one of n and m is congruent to 0 modulo 4 or 2 modulo 4, say $n \equiv 0\ (mod\ 4)$ and $m \equiv 2\ (mod\ 4)$. Then, since $n \equiv 0\ (mod\ 4)$, we have that $4 \mid n$ and so we have $n = 4c$ for some positive integer c. Similarly, since $m \equiv 2\ (mod 4)$ we have that $4 \mid (m-2)$ and so we may write $m-2 = 4d$ for some positive integer d.

By Theorem 1.2.1 we now have that H is Hamiltonian. Let C be a Hamiltonian cycle that goes through all the vertices of H once. Let F be the spanning subgraph obtained by removing the edges of C from H, that is $F = H - E(C)$. If u is an arbitrary vertex of F, then we have that

$$\begin{aligned} deg_F\ u &= (k-1)q - 2 \\ &= (n+m-2) - 2 \\ &= 4c + (4d+2) - 2 - 2 \\ &= 4c + 4d - 2 \\ &= 2(2c-1) + 4d. \end{aligned}$$

As a result, F is a regular spanning subgraph of H of degree $2(2c-1)+4d$. Consequently, the graph F is 2-factorable into $(2c-1)+2d$ factors.

Consider a red-blue coloring of the edges of the balanced complete k-partite graph H by assigning the color red to the edges in $(2c-1)$ of these 2-factors, the color blue to the edges in the remaining $2d$ of these 2-factors, and alternate the colors red and blue to the edges of C. The resulting red subgraph of H is a regular graph of degree

$$\begin{aligned} 2(2c-1)+1 &= 4c - 2 + 1 \\ &= 4c - 1 \\ &= n - 1 \end{aligned}$$

and the resulting blue subgraph of H is a regular graph of degree

$$\begin{aligned} 2(2d)+1 &= 4d + 1 \\ &= m - 1. \end{aligned}$$

That is, there is a red-blue coloring of the edges of graph H where there is neither a red $K_{1,\,n}$ nor a blue $K_{1,\,m}$, and so

$$R_k(K_{1,\,n}, K_{1,\,m}) > kq$$

so in Subcase 2.2 we have

$$R_k(K_{1,\,n}, K_{1,\,m}) = kq + 1.$$

Subcase 2.3: $k-1$ is even and q is odd.

If $k-1$ is even, then we have that $(k-1)q$ is even. This is due to the fact that the product of any number with an even number is even. So since

$$(k-1)q = n + m - 2$$

is even, it follows that both n and m is even. We want to show that

$$R_k(K_{1,\mathrm{n}}, K_{1,\mathrm{m}}) > kq.$$

Consider the balanced complete k-partite graph H of order kq. Then H has k partite sets each of order q, that is $H = K_{kq}$. If v is an arbitrary vertex of H, then

$$deg\ v = (k-1)q.$$

Hence H is a regular graph of degree $(k-1)q$. Consider a red-blue coloring of the edges of graph H resulting in a red subgraph of H and a blue subgraph of H. We claim that there is either a red $K_{1,n}$ or a blue $K_{1,m}$ in H. Note that H is a regular graph of degree

$$\begin{aligned}(k-1)q &= n + m - 2\\ &= (n-1) + (m-1).\end{aligned}$$

Thus, the graph H is 1-factorable into $(n-1)+(m-1)$ factors. Assigning the color red to the edges in $(n-1)$ of these 1-factors results in a red subgraph of H of degree $(n-1)$ and assigning the color blue to the edges in $(m-1)$ of these 1-factors results in a blue subgraph of H of degree $(m-1)$.

But $(n-1)$, $(m-1)$ and kq are odd. Thus our claim is true, and in any red-blue coloring of edges of the balanced complete k-partite graph H there is always either a red $K_{1,n}$ or a blue $K_{1,m}$.

Now we show that

$$R_k(K_{1,n}, K_{1,m}) > kq - 1.$$

Let $F = K_{q-1,\,(k-1)q}$ be the balanced complete k-partite graph of order $kq-1$. Then, F has one partite set of order $q-1$ and $k-1$ partite sets of order q. Let

$$V_1, V_2, \cdots, V_{k-1}$$

be the partite sets of F having q vertices each. If u is an arbitrary vertex in V_i where $1 \leq i \leq k-1$, then we have

$$\begin{aligned} deg\ u &= (q-1) + (k-2)q \\ &= kq - 2q + q - 1 \\ &= kq - q - 1 \\ &= (k-1)q - 1. \end{aligned}$$

Let V_k be the partite set of F having $q-1$ vertices. If v is an arbitrary vertex in V_k, then we find that

$$\begin{aligned} deg\ v &= (k-1)q \\ &= n + m - 2. \end{aligned}$$

Now, since both k and q are odd, let

$$k = 2a + 1$$

and

$$q = 2b + 1$$

for some positive integers a and b. We have that

$$\begin{aligned} degv &= n + m - 2 \\ &= (k-1)q \\ &= (2a + 1 - 1)(2b + 1) \\ &= 2a(2b + 1) \\ &= 4ab + 2a. \end{aligned}$$

We now reflect consideration on two subcases according to the parity of a.

Subcase 2.3.1: a is even.

Let $a = 2x$ for some positive integer x. Then we have that

$$\begin{aligned}(k-1)q &= n+m-2 \\ &= 4ab+2a \\ &= 4ab+2(2x) \\ &= 4ab+4x.\end{aligned}$$

We see that $4 \mid (n+m-2)$ and so we have $n+m \equiv 2 \ (mod\ 4)$. Since n and m are both even, we say without loss of generality, that one of n and m is congruent to 0 modulo 4 and the other is congruent to 2 modulo 4, that is, $n \equiv 0 \ (mod\ 4)$ and $m \equiv 2 \ (mod\ 4)$. Then, since $n \equiv 0 \ (mod\ 4)$, we have that $4 \mid n$ and so we write that $n = 4c$ for some positive integer c. Similarly, since $m \equiv 2 \ (mod\ 4)$ we have that $4 \mid (m-2)$ and so we write $m-2 = 4d$ for some positive integer d. Therefore,

$$\begin{aligned}(k-1)q-2 &= (n+m-2)-2 \\ & n+m-4 \\ &= 4c+(4d+2)-4 \\ &= 4c+4d-2 \\ &= 4c-2+4d \\ &= 2(2c-1)+4d.\end{aligned}$$

By Theorem 1.2.1 we have that F is a Hamiltonian graph. Let C be a Hamiltonian cycle in F that goes through all the vertices in F once. Note that the order of C is $kq-1$. Let F' be the spanning subgraph of F obtained by removing the edges of C from F, that is, $F' = F - E(C)$. If w is an arbitrary vertex of F' from the partite set of F' with q vertices, then

$$\begin{aligned}deg\ w &= (k-2)q+q-1-2 \\ &= kq-2q+q-3 \\ &= kq-q-3 \\ &= (k-1)q-3.\end{aligned}$$

If z is an arbitrary vertex of F' from the partite set of F' with $q-1$ vertices, then

$$deg\, z = (k-1)q - 2.$$

Thus, an arbitrary vertex in F' is either of degree $(k-1)q-2$ or $(k-1)q-3$.

Since $k-1$ is even, we divide the partite sets $V_1, V_2, \cdots, V_{k-1}$ into $(k-1)/2$ pairs, namely $\{V_1, V_2\}, \{V_3, V_4\}, \cdots, \{V_{k-2}, V_{k-1}\}$. Let M_i be the perfect matching between V_{2i-1} and V_{2i} where $1 \le i \le (k-1)/2$.

Let F'' be the regular multigraph produced from adding these $((k-1)/2)$ perfect matchings to the graph F'. The multigraph F'' is a regular graph of degree

$$\begin{aligned}(k-1)q - 3 + 1 &= (k-1)q - 2 \\ &= 2(2c-1) + 4d.\end{aligned}$$

As a result the multigraph F'' is 2-factorable into $(2c-1)+2d$ factors.

Consider a red-blue coloring of the edges of the multigraph F'' by assigning the color red to the edges in $(2c-1)$ of these 2-factors, the color blue to the remaining edges in $2d$ of these 2-factors, and alternate the color red and blue to edges in C. The resulting red-subgraph F''_R of F is a regular graph of degree

$$\begin{aligned}2(2c-1) + 1 &= 4c - 2 + 1 \\ &= 4c - 1 \\ &= n - 1\end{aligned}$$

and the resulting blue-subgraph F''_B of F is a regular graph of degree

$$\begin{aligned}2(2d) + 1 &= 4d + 1 \\ &= m - 1.\end{aligned}$$

Since F is a subgraph of F'' we have that the maximum degree of F_R is less than or equal to $n-1$, and the maximum degree of F_B is less than or equal to $m-1$, that is,

$\Delta F_R \leq n-1$ and $\Delta F_B \leq m-1$. Thus there exists a red-blue coloring of the edges of graph F where there is neither a red $K_{1,\,n}$ nor a blue $K_{1,\,m}$. Consequently,

$$R_k(K_{1,\,n}, K_{1,\,m}) > kq-1$$

and so

$$R_k(K_{1,\,n}, K)_{1,\,m} = kq.$$

Subcase 2.3.2: a is odd.

Let $a = 2x+1$ for some positive integer x. Then we have

$$\begin{aligned} n+m-2 &= (k-1)q \\ &= 4ab+2a \\ &= 2a(2b+1) \\ &= 2(2x+1)(2b+1) \\ &= (4x+2)(2b+1) \\ &= 4xb+4x+4b+2 \\ &= 4(xb+x+b)+2. \end{aligned}$$

Thus,

$$n+m-2 = 4(xb+x+b)+2$$

and, therefore,

$$\begin{aligned} n+m &= 4(xb+x+b)+4 \\ &= 4(xb+x+b+1). \end{aligned}$$

We see that $4 \mid (n+m)$ and so $n+m \equiv 0 \ (mod\ 4)$. Since n and m are both even, we have that either both n and m are congruent to 0 modulo 4 or both are congruent to 2 modulo 4.

We first consider the case where both n and m are congruent to 0 modulo 4. Since $n \equiv 0 \ (mod\ 4)$ and $m \equiv 0 \ (mod\ 4)$, we have that $4 \mid n$ and $4 \mid m$, and so we write

$n = 4c$ and $m = 4d$ for some positive integers c and d. Thus,

$$\begin{aligned}(k-1)q-2 &= (n+m-2)-2\\ &= n+m-4\\ &= 4c+4d-4\\ &= 4c-2+4d-2\\ &= 2(2c-1)+2(2d-1).\end{aligned}$$

By Theorem 1.2.1 we have that F is a Hamiltonian graph. Let C be a Hamiltonian cycle in F that passes through all the vertices in F once. Note that the order of C is $kq-1$. Let F' be the spanning subgraph of F obtained by removing the edges of C from F, that is, $F' = F - E(C)$. If w is an arbitrary vertex of F' from the partite set of F' with q vertices, then,

$$\begin{aligned}deg\ w &= [(k-2)q+q]-1-2\\ &= kq-2q+q-3\\ &= kq-q-3\\ &= (k-1)q-3.\end{aligned}$$

Further, let z be an arbitrary vertex of F' from the partite set of F' with $q-1$ vertices. Then

$$deg\ z = (k-1)q-2.$$

As a result, we have that an arbitrary vertex in F' is either of degree $(k-1)q-2$ or $(k-1)q-3$. From the condition of Subcase 2.3 we have that $k-1$ is even and so we divide the partite sets $V_1, V_2, \cdots, V_{k-1}$ into $(k-1)/2$ pairs, namely $\{V_1, V_2\}$, $\{V_3, V_4\}, \cdots, \{V_{k-2}, V_{k-1}\}$. Let M_i be the perfect matching between V_{2i-1} and V_{2i} where $1 \leq i \leq (k-1)/2$.

Let F'' be the regular multigraph produced from adding these $((k-1)/2)$ perfect matchings to the graph F'. The multigraph F'' is a regular graph of degree

$$\begin{aligned}(k-1)q-3+1 &= (k-1)q-2\\ &= n+m-4\\ &= 4c+4d-4\\ &= 4c-2+4d-2\\ &= 2(2c-1)+2(2d-1).\end{aligned}$$

We have that the multigraph F'' is 2-factorable into $(2c-1)+(2d-1)$ factors. Then, consider a red-blue coloring of the edges of the multigraph F'' by assigning the color red to edges in $(2c-1)$ of these 2-factors, the color blue to the remaining edges in $(2d-1)$ of these 2-factors, and alternate the color red and blue to edges in C. The resulting red subgraph F''_R of F is a regular graph of degree

$$\begin{aligned}2(2c-1)+1 &= 4c-2+1\\ &= 4c-1\\ &= n-1,\end{aligned}$$

and the resulting blue subgraph F''_B of F is a regular graph of degree

$$\begin{aligned}2(2d-1)+1 &= 4d-2+1\\ &= 4d-1\\ &= m-1.\end{aligned}$$

Since F is a subgraph of F'', we have that the maximal degree of F_R is less than or equal to $(n-1)$ and the maximal degree of F_B is less than or equal to $m-1$. That is $\Delta F_R \leq n-1$ and $\Delta F_B \leq m-1$. Therefor,

$$R_k(K_{1,\,n}, K_{1,\,m}) > kq-1.$$

Therefor,

$$R_k(K_{1,\,n}, K)_{1,\,m} = kq.$$

Finally, consider the case where both n and m are congruent to 2 modulo 4. Since $n \equiv 2\ (mod\ 4)$ and $m \equiv 2\ (mod\ 4)$ we have that $4 \mid (n-2)$ and $4 \mid (m-2)$. So we write $n-2=4c$ and $m-2=4d$ for some positive integers c and d. Thus,

$$\begin{aligned}(k-1)q-2 &= (n+m-2)-2\\ &= n+m-4\\ &= (4c+2)+(4d+2)-4\\ &= 4c+4d\\ &= 2(2c)+2(2d).\end{aligned}$$

Let C be a cycle in F that goes through all the vertices in F exactly once. The order of C is $kq-1$. Then C is a Hamiltonian cycle and F is a Hamiltonian graph by Theorem 1.2.1. Let F' be the spanning subgraph of F obtained by removing edges of C from F, that is, $F' = F - E(C)$. Let w be an arbitrary vertex of F' from the partite set of F' with q vertices. Then,

$$\begin{aligned}deg\ w &= [(k-2)q+q]-1-2\\ &= kq-2q+q-3\\ &= kq-q-3\\ &= (k-1)q-3.\end{aligned}$$

Let z be an arbitrary vertex of F' from the partite set of F' with $q-1$ vertices. Then,

$$deg\ z = (k-1)q-2.$$

Therefore, an arbitrary vertex in F' is either of degree $(k-1)q-2$ or $(k-1)q-3$. Since $k-1$ is even, we divide the partite sets $V_1, V_2, \cdots, V_{k-1}$ into $(k-1)/2$ pairs, namely $\{V_1, V_2\}, \{V_3, V_4\}, \cdots, \{V_{k-2}, V_{k-1}\}$. Let M_i be the perfect matching between V_{2i-1} and V_{2i} where $1 \leq i \leq (k-1)/2$.

Let F'' be the regular multigraph produced from adding these $((k-1)/2)$ perfect matchings to the graph F'. The multigraph F'' is a regular graph of degree

$$\begin{aligned}(k-1)q-3+1 &= (k-1)q-2\\ &= (n+m-2)-2\\ &= n+m-4\\ &= (4c+2)+(4d+2)-4\\ &= 4c+4d\\ &= 2(2c+2d).\end{aligned}$$

As a result, the multigraph F'' is 2-factorable into $(2c+2d)$ factors.

Consider a red-blue coloring of the edges of the multigraph F'' by assigning the color red to edges in $2c$ of these 2-factors, the color blue to the remaining edges in $2d$ of these 2-factors, and alternate the colors red and blue to edges in C. The resulting red subgraph F''_R of F is a regular graph of degree

$$\begin{aligned}2(2c)+1 &= 4c+1\\ &= (n-2)+1\\ &= n-1,\end{aligned}$$

and the resulting blue subgraph F''_B of F is a regular graph of degree

$$\begin{aligned}2(2d)+1 &= 4d+1\\ &= (m-2)+1\\ &= m-1.\end{aligned}$$

Since F is a subgraph of F'', we have that the maximal degree of F_R is less than or equal to $n-1$ and the maximal degree of F_B is less than or equal to $m-1$, that is, $\Delta F_R \leq n-1$ and $\Delta F_B \leq m-1$. Thus, there exists a red-blue coloring of the edges of F where there is neither a red $K_{1,\,n}$ nor a blue $K_{1,\,m}$, and so

$$R_k(K_{1,\,n}, K_{1,\,m}) > kq-1.$$

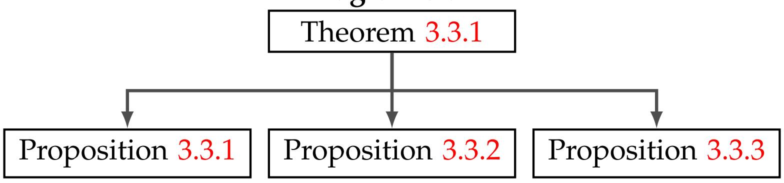
Consequently,

$$R_k(K_{1,\,n}, K_{1,\,m}) = kq.$$

$\square$

We combine the three propositions proved in Section 3.3 to give Theorem 3.3.1. The first proposition considered the upper bound and the last two propositions considered the lower bound. The difference between Proposition 3.3.2 and Proposition 3.3.3 is about the parity of $(k-1)q$.

Diagram 3.3.1



Theorem 3.3.1. *[10] Let k, n and m be positive integers with $2 \leq k < R(K_{1,\,n}, K_{1,\,m})$, $m \geq 3$ and $n \geq 3$. If $m + n - 2 = (k-1)q$ for some positive integer q, then*

$$R_k(K_{1,\,m}, K_{1,\,n}) = \begin{cases} kq & \text{if } k \text{ and } q \text{ are both odd and } n \text{ and } m \text{ are both even,} \\ kq+1 & \text{otherwise.} \end{cases}$$

When $k = 2$, we have the following corollary.

Corollary 3.3.1. *Let k, n and m be positive integers with $k = 2$, $m \geq 3$ and $n \geq 3$. If $m + n - 2 = q$ for some positive integer q, then*

$$R_2(K_{1,n}, K_{1,\,m}) = 2q + 1.$$

3.4 Conclusion

In this chapter we focused on confirming the k-Ramsey number,

$$R_k(K_{1,\,m}, K_{1,\,n})$$

for two stars of order $m, n \geq 2$. We began with a basic result where $k = 2$ and $n = 2$. That is we confirm the 2-Ramsey number

$$R_2(K_{1,2}, K_{1,m}).$$

We then considered the more general case where the number of vertices is divisible by the number of partite sets, that is $(m+n-2) \mid (k-1)$.

Theorem 3.3.1 *Let k, n and m be positive integers with $2 \leq k < R(K_{1,n}, K_{1,m})$, $m \geq 3$ and $n \geq 3$. If $m+n-2 = (k-1)q$ for some positive integer q, then*

$$R_k(K_{1,m}, K_{1,n}) = \begin{cases} kq & \text{if } k \text{ and } q \text{ are both odd and } n \text{ and } m \text{ are both even,} \\ kq+1 & \text{otherwise.} \end{cases}$$

Consequently, if $k = 2$ then

$$R_2(K_{1,n}, K_{1,m}) = 2q + 1.$$

Chapter 4

The Bipartite Ramsey Number for Two Trees

4.1 Purpose

The purpose of this chapter is to investigate the bipartite Ramsey number for a pair of bipartite graphs F and H, where F and H are different types of trees. We present a formula for the bipartite Ramsey number for a star and a tree T_n' with $n \geq 5$.

4.2 The Bipartite Ramsey Number for Two Stars $K_{1,2}$ and $K_{1,m}$

It is evident from Figure 4.2.1 that in every red-blue coloring of the edges of the complete bipartite graph of order two we always either have a red $K_{1,1}$ or a blue $K_{1,1}$. Trivially, we have that $b(K_{1,1}, K_{1,1}) = 1$.

Figure 4.2.1: A red-blue coloring of edges of $K_{1,1}$

In 1998 Hattingh and Henning studied the bipartite Ramsey numbers for stars and paths in their article *Star-path bipartite Ramsey numbers* [13]. Note that the star $K_{1,1}$ is isomorphic to the path P_2 and the star $K_{1,2}$ is isomorphic to the path P_3. In this regard we have that $b(K_{1,1}, K_{1,2}) = b(P_2, K_{1,2})$. From the article *Star-path bipartite Ramsey numbers* [13] we have that $b(P_2, K_{1,2}) = 2$ and so $b(K_{1,1}, K_{1,2}) = 2$.

Since we already mentioned that $K_{1,\,2}$ and P_3 are isomorphic graphs we have that

$$b(P_3, K_{1,\,m}) = b(K_{1,\,2}, K_{1,\,m}).$$

We make use of the above result from *Star-path bipartite Ramsey numbers* [13] and Theorem 3.2.1, to determine $b(K_{1,\,2}, K_{1,\,m})$, however, our proof is original.

Theorem 4.2.1. *For $m \geq 2$,*

$$b(K_{1,\,2}, K_{1,\,m}) = m + 1.$$

Proof

We show that

$$b(K_{1,\,2}, K_{1,\,m}) \leq m + 1.$$

Consider the complete bipartite graph G of order $2m + 2$, where each partite set is of order $m + 1$. If u is an arbitrary vertex of G, then $deg\ u = m + 1$.

Think about a red-blue coloring of the edges of the complete bipartite graph G resulting in a red subgraph G_R of G and a blue subgraph G_B of G. In Table 4.2.1 below we give all the different ways to choose red and blue edges incident to vertex u in graph G and the last column represent the graph that appears because of the specific coloring. We see that if $deg_{G_R}\ u \geq m$, then G has a red $K_{1,\,m}$, and if $deg_{G_B}\ u \geq 2$, then G has a blue $K_{1,\,2}$. As a result, in any red-blue coloring of the edges of the complete bipartite graph G there is always a blue $K_{1,\,2}$ or a red $K_{1,\,m}$, and so

$$b(K_{1,\,2}, K_{1,\,m}) \leq m + 1.$$

We now want to show that

$$b(K_{1,\,2}, K_{1,\,m}) > m.$$

Let H be the complete bipartite graph of order $2m$ where the order of each partite set is equal to m. If v is an arbitrary vertex of graph H, then $deg\ v = m$. Let M be a perfect matching between the two partite sets. Removing all the edges of M from H we have a spanning $m - 1$ regular subgraph F of H.

Table 4.2.1: The number of red and blue edges incident to u in $K_{m+1,\,m+1}$.

$deg_{G_R}\ u$	$deg_{G_B}\ u$	Contradiction
$m+1$	0	red $K_{1,\,m}$
m	1	red $K_{1,\,m}$
$m-1$	2	blue $K_{1,\,2}$
$m-2$	3	blue $K_{1,\,2}$
$\vdots$	$\vdots$	$\vdots$
2	$m-1$	blue $K_{1,\,2}$
1	m	blue $K_{1,\,2}$
0	$m+1$	blue $K_{1,\,2}$

Consider a red-blue coloring of the edges of the complete bipartite graph H by assigning the color blue to the edges of M and the color red to the edges of F. The resulting blue subgraph G is one regular and the resulting red subgraph of G is $m-1$ regular.

We have that there is a red-blue coloring of the edges of the complete bipartite graph H where there is neither a red $K_{1,\,m}$ nor a blue $K_{1,\,2}$. We conclude that

$$b(K_{1,\,2}, K_{1,\,m}) > m$$

and consequently we have that

$$b(K_{1,\,2}, K_{1,\,m}) = m+1.$$

□

4.3 The Bipartite Ramsey Number for Two Stars $K_{1,\,n}$ and $K_{1,\,m}$

This section focuses on finding the bipartite Ramsey number for two stars $K_{1,\,n}$ and $K_{1,\,m}$ of order $n+1 \geq 3$ and $m+1 \geq 3$ respectively. We consider $n \geq 2$ and $m \geq 2$ since, if $n = m = 1$, then both the stars $K_{1,\,n}$ and $K_{1,\,m}$ are isomorphic to the path P_2 and

$$b(K_{1,1}, K_{1,1}) = b(P_2, P_2) = 1$$

from the article *Path-path Ramsey-type numbers for complete bipartite graph* [18] by Faudree and Schelp.

Longani in 2003 considered the bipartite Ramsey number $b(K_{1,n}, K_{1,n})$ of stars where $n \geq 2$ in the article *Some Bipartite Ramsey numbers* [19]. In our dissertation we extend the study of Longani to the bipartite Ramsey number $b(K_{1,n}, K_{1,m})$ of stars where $n, m \geq 2$ and n and m are not necessarily equal to each other. The following is our own original work.

Theorem 4.3.1. *Let m, n be positive integers with $m \geq 2$ and $n \geq 2$. If $m + n - 2 = q$ for some positive integer q, then*

$$b(K_{1,n}, K_{1,m}) = q + 1.$$

Proof

We begin by proving that

$$b(K_{1,n}, K_{1,m}) \leq q + 1.$$

Let G be the complete bipartite graph of order $2q + 2 = 2(m + n - 1)$. If u is an arbitrary vertex of G, then

$$\begin{aligned} deg\ u &= q + 1 \\ &= (m + n - 2) + 1 \\ &= m + n - 1. \end{aligned}$$

Consider a red-blue coloring of the edges of G that results in a red subgraph G_R of G and a blue subgraph G_B of G. In Table 4.3.1 below we give all the different ways to choose red and blue edges incident to vertex u in G and in the last column we give the graph we are trying to avoid but exists due to a specific coloring. We see that if $deg_{G_R}\ u \geq m$, then G has a red $K_{1,m}$, and if $deg_{G_B}\ u \geq n$, then G has a blue $K_{1,n}$.

As a result in every red-blue coloring of the edges of the complete bipartite graph G there is always a blue $K_{1,n}$ or a red $K_{1,m}$. As such, we proved that

$$b(K_{1,\,n}, K_{1,\,m}) \leq q + 1.$$

Table 4.3.1: The number of red and blue edges incident to u in $K_{m+n-1,\,m+n-1}$.

$deg_{G_R}\ u$	$deg_{G_B}\ u$	Contradiction
$m+n-1$	0	red $K_{1,\,m}$
$m+n-2$	1	red $K_{1,\,m}$
$m+n-3$	2	red $K_{1,\,m}$
$\vdots$	$\vdots$	$\vdots$
m	$n-1$	red $K_{1,\,m}$
$m-1$	n	blue $K_{1,\,n}$
$m-2$	$n+1$	blue $K_{1,\,n}$
$\vdots$	$\vdots$	$\vdots$
2	$m+n-3$	blue $K_{1,\,n}$
1	$m+n-2$	blue $K_{1,\,n}$
0	$m+n-1$	blue $K_{1,\,n}$

We now consider the lower bound

$$b(K_{1,\,n}, K_{1,\,m}) > q.$$

Let H be the complete bipartite graph of order $2q$ where each partite set has q vertices. By Theorem 1.2.5 we have that for every positive integer q the graph H is 1-factorable. Thus, H is a q-regular bipartite graph and for any arbitrary vertex u of H we have that

$$\begin{aligned} deg\ u &= q \\ &= m+n-2 \\ &= (m-1)+(n-1). \end{aligned}$$

The graph H is 1-factorable into $(m-1)+(n-1)$ factors. Reflect into consideration a red-blue coloring of the edges of the complete bipartite graph H by coloring $(m-1)$ edges of the 1-factors red the remaining $(n-1)$ edges of these 1-factors blue. The resulting red subgraph of H is an $(m-1)$-regular graph and the resulting blue subgraph of H is an $(n-1)$-regular graph.

Thus, there is a red-blue coloring of the edges of the complete bipartite graph H where there is neither a red $K_{1,m}$ nor a blue $K_{1,n}$. We conclude that

$$b(K_{1,n}, K_{1,m}) > q$$

and so

$$\begin{aligned} b(K_{1,n}, K_{1,m}) &= q + 1 \\ &= (m + n - 2) + 1 \\ &= m + n - 1. \end{aligned}$$

□

From our study on bipartite Ramsey numbers for two stars we discovered the following result. We give a proof for this discovery below.

Proposition 4.3.1. *For $n \geq 2$ and $m \geq 2$,*

$$R_2(K_{1,n}, K_{1,m}) = 2b(K_{1,n}, K_{1,m}) - 1.$$

Proof

From Theorem 4.3.1 we have that

$$\begin{aligned} b(K_{1,n}, K_{1,m}) &= q + 1 \\ &= m + n - 1 \end{aligned}$$

where $q = m + n - 2$. In the proof of Theorem 4.3.1 we show that in a complete bipartite graph $K_{q,q}$ there is neither a red $K_{1,n}$ nor a blue $K_{1,m}$. We further show in the proof of Theorem 4.3.1 that in any red-blue coloring of the edges of the complete bipartite graph $K_{q+1,q+1}$ there is always either a red $K_{1,n}$ or a blue $K_{1,m}$. We now investigate the intermediate complete bipartite graph $K_{q,q+1}$.

Let G be the complete bipartite graph of order $2q + 1$ where V is the partite set of G

with q vertices and U is the partite set of G with $q+1$ vertices. If $v \in V$, then

$$\begin{aligned} deg\ v &= q+1 \\ &= (m+n-2)+1 \\ &= m+n-1. \end{aligned}$$

Consider a red-blue coloring of the edges of the complete bipartite graph G resulting in a red subgraph G_R of G and a blue subgraph G_B of G. In Table 4.3.2 we give all the different ways to choose red and blue edges incident to vertex v in G and the last column of the table gives us the graph we are trying to avoid, but which occurs according to how the colors are assigned. We have that if $deg_{G_R}\ v \geq n$, then G has a red $K_{1,\,n}$, and if $deg_{G_B}\ v \geq m$, then G has a blue $K_{1,\,m}$.

Table 4.3.2: The number of red and blue edges incident to v in $K_{q,\,q+1}$.

$deg_{\mathrm{G_B}}\ v$	$deg_{\mathrm{G_R}}\ v$	Contradiction
$m+n-1$	0	blue $K_{1,\,m}$
$m+n-2$	1	blue $K_{1,\,m}$
$m+n-3$	2	blue $K_{1,\,m}$
$\vdots$	$\vdots$	$\vdots$
$m+1$	$n-2$	blue $K_{1,\,m}$
m	$n-1$	blue $K_{1,\,m}$
$m-1$	n	red $K_{1,\,n}$
$m-2$	$n+1$	red $K_{1,\,n}$
$\vdots$	$\vdots$	$\vdots$
2	$m+n-3$	red $K_{1,\,n}$
1	$m+n-2$	red $K_{1,\,n}$
0	$m+n-1$	red $K_{1,\,n}$

As a result we have that in any red-blue coloring of edges of the complete bipartite graph G there is always either a blue $K_{1,\,m}$ or a red $K_{1,\,n}$ which is a contradiction to our assumption. Therefore,

$$
\begin{aligned}
R_2(K_{1,n}, K_{1,m}) &= 2q + 1 \\
&= 2(m + n - 2) + 1 \\
&= 2m + 2n - 4 + 1 \\
&= 2m + 2n - 2 - 2 + 1 \\
&= 2m + 2n - 2 - 1 \\
&= 2(m + n - 1) - 1 \\
&= 2b(K_{1,n}, K_{1,m}) - 1.
\end{aligned}
$$

□

4.4 The Bipartite Ramsey Number for a Star $K_{1,m}$ and a Tree T'_n

In 2013 Christou, Iliopoulos and Miller studied the bipartite Ramsey number of simpler monochromatic graphs such as stripes, stars and trees in their article *Bipartite Ramsey numbers involving stars, stripes and trees* [12]. Their bounds for the bipartite Ramsey number for trees were very complex which inspired us to do further research to come up with a much simpler proof. In this section we study the bipartite Ramsey number for a star $K_{1,m}$ of order $m + 1 \geq 3$ and a tree T'_n with $n \geq 5$ vertices whose maximum degree is $n - 2$.

We remind our reader that T'_n is defined as a tree of order $n \geq 5$ whose maximum degree is $n - 2$. In our study we consider $m \geq 2$ and $n \geq 5$ since

- if $m = 1$, then the star $K_{1,m}$ is isomorphic to path P_2,
- if $n = 4$, then T'_4 is isomorphic to P_4 which we consider in future studies,
- if $n = 3$, then T'_3 is a disconnected graph, we do not consider $n = 3$ because we are only considering connected graphs,
- if $n = 2$, then T'_2 is an empty graph and we are done because we have no edges to color, and

- the graph T_1' does not exist because it has a negative degree.

The following is our own original work.

Theorem 4.4.1. *Let m, n be positive integers with $m \geq 2$ and $n \geq 5$. Then,*

$$b(K_{1,\,m}, T_n') = m + n - 3.$$

Proof

We begin by proving that

$$b(K_{1,\,m}, T_n') \leq m + n - 3.$$

Let G be the complete bipartite graph of order $2(m+n-3)$. Then, G has two partite sets each of order $m+n-3$. If v is an arbitrary vertex of G, then

$$deg\ v = m + n - 3.$$

Consider a red-blue coloring of the edges of G result in a red subgraph G_R of G and a blue subgraph G_B of G. In Table 4.4.1 below we give all the different ways to choose red and blue edges of G incident to vertex v and the last column of the table gives us the graph that occurs because of the specific coloring. We see that if $deg_{G_R}\ v \geq m$, then G has a red $K_{1,\,m}$, and if $deg_{G_B}\ v \geq n-2$, then G has a blue T_n'. Thus, in every red-blue coloring of the edges of the complete bipartite graph G there is always a blue T_n' or a red $K_{1,\,m}$. Consequently,

$$b(K_{1,\,m}, T_n') \leq m + n - 3.$$

We now consider the lower bound

$$b(K_{1,\,m}, T_n') > m + n - 4.$$

Let H be the complete bipartite graph of order $2(m+n-4)$ where each partite set has $m+n-4$ vertices. Then H is a $(m+n-4)$-regular bipartite graph and for any arbitrary vertex u of H we have

$$\begin{aligned} deg\ u &= m + n - 4 \\ &= (m-1) + (n-3). \end{aligned}$$

Table 4.4.1: The number of red and blue edges incident to v in $K_{m+n-3,\, m+n-3}$.

$deg_{G_R}\ v$	$deg_{G_B}\ v$	Contradiction
$m+n-3$	0	red $K_{1,\, m}$
$m+n-4$	1	red $K_{1,\, m}$
$m+n-5$	2	red $K_{1,\, m}$
$\vdots$	$\vdots$	$\vdots$
m	$n-3$	red $K_{1,\, m}$
$m-1$	$n-2$	blue T'_n
$m-2$	$n-1$	blue T'_n
$\vdots$	$\vdots$	$\vdots$
2	$m+n-5$	blue T'_n
1	$m+n-4$	blue T'_n
0	$m+n-3$	blue T'_n

By Theorem 1.2.5 we have that the graph H is 1-factorable. The graph H is 1-factorable into $(m-1)+(n-3)$ factors. Reflect into consideration a red-blue coloring of the edges of the graph H by coloring $m-1$ edges of the 1-factors red and the remaining $n-3$ edges of these 1-factors blue. The resulting red subgraph of H is an $(m-1)$-regular graph and the resulting blue subgraph of H is an $(n-3)$-regular graph.

Consequently, there is a red-blue coloring of the edges of the complete bipartite graph H where there is neither a red $K_{1,\, m}$ nor a blue T'_n. We conclude that

$$b(K_{1,\, m}, T'_n) > m + n - 4$$

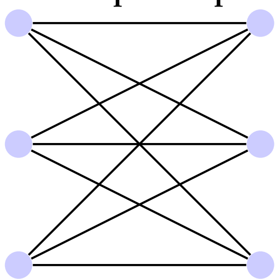
and so $b(K_{1,\, m}, T'_n) = m + n - 3$. $\square$

Consider the following example were $n = 5$ and $m = 2$ to illustrate the lower bound of Theorem 4.4.1. We show that

$$b(K_{1,2}, T_5') > 3.$$

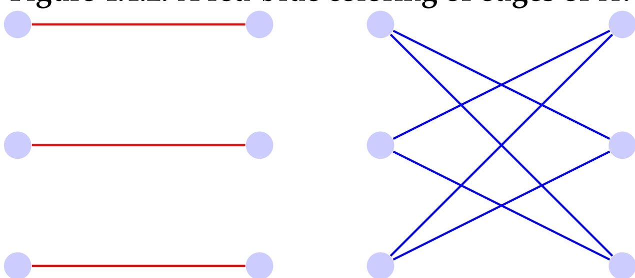
So we let H be the complete bipartite graph of order six where each partite set has three vertices as in the graph in Figure 4.4.1.

Figure 4.4.1: A complete bipartite graph H



We show in Figure 4.4.2 that there is a red-blue coloring of the edges of the complete graph H where there is neither a red $K_{1,2}$ nor a blue T_5', and so the result follows.

Figure 4.4.2: A red-blue coloring of edges of H.



From Lemma 1 in Chapter 1 we derive the following conjecture.

Conjecture 4.4.1. *If $n \geq 5$ and $m \geq 2$ are positive integers, then either*

$$R_2(K_{1,m}, T_n') = 2(m+n-3) - 1$$

or

$$R_2(K_{1,m}, T_n') = 2(m+n-3).$$

4.5 The Bipartite Ramsey Number for Two Trees

In this section we investigate the bipartite Ramsey number for two trees of order $m \geq 5$ and $n \geq 5$ whose maximum degree is $m-2$ and $n-2$ respectively. We consider the the instances when $m \geq 5$ since

- if $m = 4$, then T_4' is isomorphic to P_4 which we consider in future studies,
- if $m = 3$, then T_3' is a disconnected graph and is not consider $m = 3$ because we are only considering connected graphs,
- if $m = 2$, then T_2' is an empty graph and we are done because we have no edges to color, and
- if $m = 1$, then the graph T_1' does not exist because it has a negative degree.

The same reasoning hold for when $n \leq 4$.

The bipartite result when $m = n = 4$ was studied by Faudree and Schelp in their article *Path-path Ramsey-type numbers for complete bipartite graph* [18] published in 1975. We extend the work to when $m \neq n$. This is original work.

Theorem 4.5.1. *Let m, n be positive integers with $m \geq 5$ and $n \geq 5$. Then,*

$$b(T_m', T_n') = m + n - 5.$$

Proof

We begin by proving that

$$b(T_m', T_n') \leq m + n - 5.$$

Let G be the complete bipartite graph of order $2(m+n-5)$. Then G has two partite sets each of order $m+n-5$. If v is an arbitrary vertex of G, then

$$deg\ v = m + n - 5.$$

Consider a red-blue coloring of the edges of G results in a red subgraph G_R of G and a blue subgraph G_B of G. In Table 4.5.1 below we give all the different ways to choose red and blue edges incident to vertex v in graph G and the last column represents the graph we are trying to avoid but occurs because of the coloring assigned. We see that if $deg_{G_R}\ v \geq m-2$, then G has a red T'_m, and if $deg_{G_B}\ v \geq n-2$, then G has a blue T'_n.

Table 4.5.1: The number of red and blue edges incident to v in $K_{n+m-5,n+m-5}$.

$deg_{G_R}\ v$	$deg_{G_B}\ v$	Contradiction
$m+n-5$	0	red T'_m
$m+n-6$	1	red T'_m
$m+n-7$	2	red T'_m
$\vdots$	$\vdots$	$\vdots$
$m-1$	$n-4$	red T'_m
$m-2$	$n-3$	red T'_m
$m-3$	$n-2$	blue T'_n
$m-4$	$n-1$	blue T'_n
$\vdots$	$\vdots$	$\vdots$
2	$m+n-7$	blue T'_n
1	$m+n-6$	blue T'_n
0	$m+n-5$	blue T'_n

Thus, in every red-blue coloring of the edges of the complete bipartite graph G there is always a blue T'_n or a red T'_m. Consequently,

$$b(T'_m, T'_n) \leq m+n-5.$$

We now consider the lower bound

$$b(T'_m, T'_n) > m+n-6.$$

Let H be the complete bipartite graph of order $2(m+n-6)$ where each partite set has $m+n-6$ vertices. Then H is a $(m+n-6)$-regular bipartite graph, that is, for any arbitrary vertex u of H we have that

$$\begin{aligned} deg\ u &= m+n-6 \\ &= (m-3)+(n-3). \end{aligned}$$

By Theorem 1.2.5 we have that the graph H is 1-factorable. The graph H is 1-factorable into $(m-3)+(n-3)$ factors. Reflect into consideration a red-blue coloring of the edges of the complete bipartite graph H by assigning the color red to $m-3$ edges of the 1-factors and the color blue to the remaining $n-3$ edges of these 1-factors. The resulting red subgraph of H is an $(m-3)$-regular graph and the resulting blue subgraph of H is an $(n-3)$-regular graph.

Consequently, there is a red-blue coloring of the edges of the complete bipartite graph H where there is neither a red T'_m nor a blue T'_n. We conclude that

$$b(T'_m, T'_n) > m+n-6$$

and so

$$b(T'_m, T'_n) = m+n-5.$$

□

From Lemma 1 in Chapter 1 we derive the following conjecture.

Conjecture 4.5.1. *If n and m are positive integers where $n \geq 5$ and $m \geq 5$, then*

$$R_2(T'_m, T'_n) = 2(m+n-5)-1$$

or

$$R_2(T'_m, T'_n) = 2(m+n-5).$$

4.6 Conclusion

Longani studied the bipartite Ramsey number $b(K_{1,m}, K_{1,m})$ for two stars of equal order $m+1 \geq 2$. We extended this result by finding the general formula for the bipartite Ramsey number $b(K_{1,m}, K_{1,n})$ in Theorem 4.3.1.

The authors of *Bipartite Ramsey numbers involving stars, stripes and trees* [12] studied the bipartite Ramsey of different trees and gave very complex bounds which we simplified. We focused on the bipartite Ramsey number $b(T'_m, K_{1,n})$ and $b(T'_n, T'_m)$ in Theorem 4.4.1 and Theorem 4.5.1 respectively.

Summarizing Theorem 4.3.1, Theorem 4.4.1 and Theorem 4.5.1 we have the following main theorem.

Theorem 4.6.1. *Let T_n and T_m be any trees of order $n \geq 2$ and $m \geq 2$ respectively. Then,*

$$b(T_m, T_n) = \begin{cases} m+n-1 & \text{if } \Delta(T_m) = m-1 \text{ and } \Delta(T_n) = n-1, \\ m+n-3 & \text{if } \Delta(T_m) = m-1 \text{ and } \Delta(T_n) = n-2, \\ m+n-5 & \text{if } \Delta(T_m) = m-2 \text{ and } \Delta(T_n) = n-2. \end{cases}$$

Chapter 5

The s-Bipartite Ramsey Number for Two Trees

5.1 Purpose

The purpose of this chapter is to present a general formula for the s-bipartite Ramsey number for stars and trees T'_n for $n \geq 5$.

5.2 The s-Bipartite Ramsey Number for Two Stars $K_{1,\,n}$ and $K_{1,\,m}$

This section mainly focuses on finding the s-bipartite Ramsey number of two stars $K_{1,\,n}$ and $K_{1,\,m}$ of order $n+1 \geq 3$ and $m+1 \geq 3$ respectively. If $m = 1$, then $K_{1,\,m}$ is isomorphic to P_2 which we consider in future studies.

Bi, Chartrand and Zhang introduced the s-bipartite Ramsey number of cycles of length four and other small graph in their article *Another view of bipartite Ramsey numbers* [11] in 2018. In 2019 Olejniczak expanded on the topic by focusing on the s-bipartite Ramsey number for two stars in his dissertation *Variations in Ramsey Theory* [14].

Below the statement of the s-bipartite Ramsey number for two stars is given as in *Variations in Ramsey Theory* [14], however we came up with a simpler original proof.

Theorem 5.2.1. *Let s, n and m be positive integers with $n \geq 2$ and $m \geq 2$. If $t = n + m - 1$ where $t \geq s$, then*

$$b_s(K_{1,n}, K_{1,m}) = n + m - 1.$$

Proof

We begin by proving that

$$b_s(K_{1,n}, K_{1,m}) \leq n + m - 1.$$

In this regard, let U and V be the two disjoint partite sets of the vertices of the complete bipartite graph G where U is of order s and V is of order $t = n + m - 1$. If v is an arbitrary vertex in U, then

$$deg\ v = t = n + m - 1.$$

Consider a red-blue coloring of the edges of the graph G resulting in a red subgraph G_R of G and a blue subgraph G_B of G. In Table 5.2.1 below we give all the different ways to choose red and blue edges incident to vertex v in G and the last column represents the graph we are trying to avoid but which occurs because of the way we assigned the colors. We see that if $deg_{G_R}\ v \geq n$, then G has a red $K_{1,n}$, and if $deg_{G_B}\ v \geq m$, then G has a blue $K_{1,m}$.

As a result we have that in every red-blue coloring of the edges of the complete bipartite graph G there is always either a blue $K_{1,m}$ or a red $K_{1,n}$, which is a contradiction to our assumption. Therefore,

$$b_s(K_{1,n}, K_{1,m}) \leq n + m - 1.$$

We now want to show that

$$b_s(K_{1,n}, K_{1,m}) > n + m - 2.$$

Consider the complete bipartite graph H of order $s + t - 1$. Let U and V be the two partites sets of H where U is of order s and V is of order

$$\begin{aligned} t - 1 &= (n + m - 1) - 1 \\ &= n + m - 2. \end{aligned}$$

Table 5.2.1: The number of red and blue edges incident to v in $K_{s,\,t}$.

$deg_{G_R}\, v$	$deg_{G_B}\, v$	Contradiction
$n + m - 1$	0	red $K_{1,\,n}$
$n + m - 2$	1	red $K_{1,\,n}$
$n + m - 3$	2	red $K_{1,\,n}$
$\vdots$	$\vdots$	$\vdots$
n	$m - 1$	red $K_{1,\,n}$
$n - 1$	m	blue $K_{1,\,m}$
$n - 2$	$m + 1$	blue $K_{1,\,m}$
$\vdots$	$\vdots$	$\vdots$
3	$m + n - 3$	blue $K_{1,\,m}$
2	$m + n - 2$	blue $K_{1,\,m}$
1	$m + n - 1$	blue $K_{1,\,m}$

We know that $K_{s,\,t-1} \subseteq K_{t-1,\,t-1}$ since $s \leq t - 1$ and therefore H is 1-factorable by Theorem 1.2.5. If u is an arbitrary vertex in U, then

$$\begin{aligned} deg\ u &= t - 1 \\ &= n + m - 2 \\ &= (n - 1) + (m - 1). \end{aligned}$$

Thus, H is 1-factorable into $(n - 1) + (m - 1)$ factors. Reflect into consideration a red-blue coloring of edges of H by coloring $n - 1$ edges of these 1-factors red and the remaining $m - 1$ edges of these 1-factors blue. The resulting red subgraph of H is a regular graph of degree $n - 1$ and the resulting blue subgraph of H is a regular graph of degree $m - 1$. So we have that there is a red-blue coloring of edges of H where there is neither a red $K_{1,\,n}$ nor a blue $K_{1,\,m}$. We conclude that

$$b_s(K_{1,\,n}, K_{1,\,m}) > n + m - 2$$

and so

$$b_s(K_{1,\,n},\, K_{1,\,m}) = n + m - 1.$$

□

We conjecture the following result for two stars.

Corollary 5.2.1. *Let $n \geq 2$ and $m \geq 2$ be integers. If $t = n + m - 1$ where $t \geq s$, then*

$$b(K_{1,\,n}, K_{1,\,m}) = b_s(K_{1,\,n}, K_{1,\,m})$$

5.3 The s-Bipartite Ramsey Number for a Star $K_{1,\,m}$ and a Tree T'_n

In this section we determine the number $b_s(K_{1,\,m},\, T'_n)$ where $m \geq 2$ and $n \geq 5$ by beginning with the condition $2 \leq s \leq m + n - 4$. We mainly consider the case where $m \geq 2$ because, if $m = 1$, then the star $K_{1,\,m}$ is isomorphic to the path P_2 which we consider in future studies. Similarly we consider the case where $n \geq 5$ since, if

- if $n = 4$, then the maximum degree of T'_4 is

$$4 - 2 = 2$$

  implying that T'_4 is isomorphic to P_4 which is out of our study,

- if $n = 3$, then the maximum degree of T'_3 is

$$3 - 2 = 1$$

  implying that T'_3 a disconnected graph. We do not consider $n = 3$ because we are only considering connected graphs in our dissertation,

- if $n = 2$, then T'_2 is an empty graph since the maximum degree is zero and we are done because we have no edges to assign colors to, and

- if $n = 1$ the graph T'_1 does not exist as it has a negative maximum degree.

The following is original work.

Proposition 5.3.1. *Let $t = m + n - 3$ where $t \geq s$ for some positive integers s, $n \geq 5$ and $m \geq 2$. If $2 \leq s \leq m+n-4$, then*

$$b_s(K_{1,\,m}, T'_n) = m + n - 3.$$

Proof

We begin by proving the upper bound,

$$b_s(K_{1,\,m}, T'_n) \leq m + n - 3.$$

Let G be the complete bipartite graph of order $s+t$ where V is the partite set of G with s vertices and U is the partite set of G with $t = m+n-3$ vertices. If v is an arbitrary vertex in V, then

$$deg\ v = t = m + n - 3.$$

Take into consideration a red-blue coloring of the edges of G resulting in a red subgraph G_R of G and a blue subgraph G_B of G. In Table 5.3.1 we give all the different ways to choose red and blue edges incident to vertex v in G. The last column represents the graph we are trying to avoid, but occurs because of the way we assigned the colors. We see that if $deg_{G_R}\ v \geq m$, then G has a red $K_{1,\,m}$, and if $deg_{G_B}\ v \geq n-2$, then G has a blue T'_n.

Thus, in every red-blue coloring of the edges of the complete bipartite graph G there is always a blue T'_n or a red $K_{1,\,m}$. Consequently we have

$$b_s(K_{1,\,m}, T'_n) \leq m + n - 3.$$

We now want to show that

$$b_s(K_{1,\,m}, T'_n) > m + n - 4.$$

Let H be the complete bipartite graph of order $s+t-1$ where one partite set is of order s and the other partite set is of order $t-1 = m+n-4$.

Table 5.3.1: The number of red and blue edges incident to v in $K_{s,\,t}$.

$deg_{G_R}\, v$	$deg_{G_B}\, v$	Contradiction
$m+n-3$	0	red $K_{1,\,m}$
$m+n-4$	1	red $K_{1,\,m}$
$m+n-5$	2	red $K_{1,\,m}$
$\vdots$	$\vdots$	$\vdots$
m	$n-3$	red $K_{1,\,m}$
$m-1$	$n-2$	blue T_n'
$m-2$	$n-1$	blue T_n'
$\vdots$	$\vdots$	$\vdots$
2	$m+n-5$	blue T_n'
1	$m+n-4$	blue T_n'
0	$m+n-3$	blue T_n'

We know that $K_{s,\,m+n-4} \subseteq K_{m+n-4,\,m+n-4}$ since $2 \leq s \leq m+n-4$. So by Theorem 4.4.1 we have that there exists a red-blue coloring of the edges of the complete bipartite graph $K_{m+n-4,\,m+n-4}$ where there is neither a red $K_{1,\,m}$ nor a blue T_n'. Thus, we have that there exists a red-blue coloring of the edges of the bipartite graph $K_{s,\,m+n-4}$ where there is neither a red $K_{1,\,m}$ nor a blue T_n'. We conclude that

$$b_s(K_{1,\,m}, T_n') > m+n-4$$

and so

$$b_s(K_{1,\,m}, T_n') = m+n-3.$$

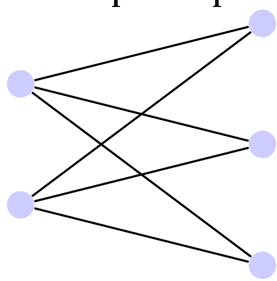
$\square$

Consider the following example when $n = 5$ and $m = 2$ to illustrate Proposition 5.3.1. We want to show that

$$b_s(K_{1,\,2}, T_5') > 3.$$

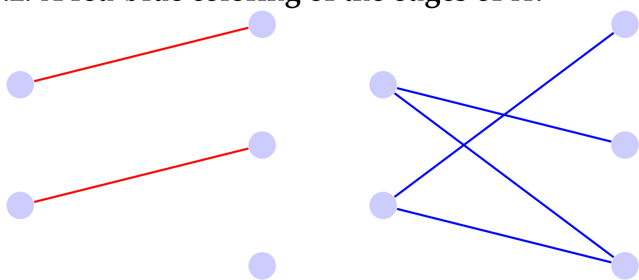
Let H be the complete bipartite graph of order five as in Figure 5.3.1. Let the partite set with three vertices be V and the other partite set with two vertices be U.

Figure 5.3.1: A complete bipartite graph H.



We show in Figure 5.3.2 that there is a red-blue coloring of the edges of the complete bipartite graph H where there is neither a red $K_{1,2}$ nor a blue T_5'.

Figure 5.3.2: A red-blue coloring of the edges of H.



Thus, we have that $b_2(K_{1,2}, T_5') > 3$. We make use of a proof by contradiction to show that $b_2(K_{1,2}, T_5') \leq 4$ and as a result we conclude that $b_2(K_{1,2}, T_5') = 4$.

We now determine the s-bipartite Ramsey number for $b_s(K_{1,m}, T_n')$ where $m \geq 2$, $n \geq 4$ and $s \geq m+n-3$. The following is original work.

Proposition 5.3.2. *Let t = m + n - 3 where t $\geq$ s for some positive integers s, n and m. If $m \geq 2$, $n \geq 4$ and $s \geq m+n-3$, then*

$$b_s(K_{1,m}, T_n') = s.$$

Proof

The lower bound was covered in Proposition 5.3.1.

$$b_s(K_{1,m}, T_n') = m+n-4.$$

was covered in Proposition 5.3.1. So it suffices to investigate the upper bound.

$$b_s(K_{1,\,m}, T_n') \leq m+n-3.$$

Let G be the complete bipartite graph of order $2(m+n-3)$. Then G has two partite sets each of order $m+n-3$. If v is an arbitrary vertex of G, then $deg\ v = m+n-3$.

Consider a red-blue coloring of the edges of G that result in a red subgraph G_R of G and a blue subgraph G_B of G. We remind our reader that T_n' is defined as a tree of order $n \geq 5$ whose maximum degree is $n-2$. In Table 5.3.2 below we give all the different ways to choose red and blue edges incident to vertex v in G and the last column represents the graph we are trying to avoid but occurs because of the way the colors are assigned. We see that if $deg_{G_R}\ v \geq m$, then G has a red $K_{1,\,m}$, and if $deg_{G_B}\ v \geq n-2$, then G has a blue T_n'.

Table 5.3.2: The number of red and blue edges incident to v in $K_{n+m-3,\,n+m-3}$.

$deg_{G_R}\ v$	$deg_{G_B}\ v$	Contradiction
$m+n-3$	0	red $K_{1,\,m}$
$m+n-4$	1	red $K_{1,\,m}$
$m+n-5$	2	red $K_{1,\,m}$
$\vdots$	$\vdots$	$\vdots$
m	$n-3$	red $K_{1,\,m}$
$m-1$	$n-2$	blue T_n'
$m-2$	$n-1$	blue T_n'
$\vdots$	$\vdots$	$\vdots$
2	$m+n-5$	blue T_n'
1	$m+n-4$	blue T_n'
0	$m+n-3$	blue T_n'

Thus, in every red-blue coloring of the edges of G there is always a blue T_n' or a red $K_{1,\,m}$. As such we conclude that $b_s(K_{1,\,m}, T_n') \leq m+n-3$ and so

$$b_s(K_{1,\,m}, T_n') = m+n-3.$$

□

We obtain Theorem 5.3.1 by combining Proposition 5.3.1 and Proposition 5.3.2.

Theorem 5.3.1. *Let s, n and m be positive integers with s, $m \geq 2$ and $n \geq 4$. If $t = m + n - 3$ where $t \geq s$, then*

$$b_s(K_{1,m}, T'_n) = \begin{cases} m+n-3 & \text{if } 2 \leq s \leq m+n-4, \\ s & \text{otherwise.} \end{cases}$$

5.4 The s-Bipartite Ramsey Number for Two Trees T'_m and T'_n

In this section we investigate the s-bipartite Ramsey number for all positive integers s of two bipartite graphs T'_m or T'_n where n and m are integers greater than or equal to five.

We study T'_m for $m \geq 5$ because

- if $m = 4$ then we have that T'_4 is isomorphic to P_4 which we consider in future studies,
- if $m = 3$ we have that T'_3 is disconnected and we only consider connected graphs,
- if $m = 2$ then T'_2 is an empty graph and we are done, and
- if $m = 1$ then T_1 does not exist.

Similarly, we study T'_n for $n \geq 5$ because of the same reasons provided above. The results from this section are our original work.

We begin by determining the number $b_s(T'_m,\ T'_n)$ where $m \geq 5$ and $n \geq 5$ for $2 \leq s \leq m + n - 6$.

Proposition 5.4.1. *Let t = m + n - 5 where $t \geq s$ for some positive integers s, n and m. If $m \geq 5$ and $n \geq 5$ and $2 \leq s \leq m + n - 6$, then*

$$b_s(T'_m, T'_n) = m + n - 5.$$

Proof

We begin by showing that

$$b_s(T'_m, T'_n) \leq m + n - 5.$$

Let G be the complete bipartite graph of order $s + t$ where V is the partite set of G with s vertices and U is the partite set of G with $t = m + n - 5$ vertices. If $v \in V$, then

$$deg\ v = m + n - 5.$$

Take into consideration a red-blue coloring of the edges of G resulting in a red subgraph G_R of G and a blue subgraph G_B of G. In Table 5.4.1 below we give all the different ways to choose red and blue edges incident to vertex v in G and the last column represent the graph we are trying to avoid but occurs because of the way we assigned the colors to the edges. We see that if $deg_{G_R}\ v \geq m - 2$, then G has a red T'_m, and if $deg_{G_B}\ v \geq n - 2$, then G has a blue T'_n.

Thus, in every red-blue coloring of the edges of G there is always a blue T'_n or a red T'_m. As such we conclude that

$$b_s(T'_m, T'_n) \leq m + n - 5.$$

We now consider the lower bound

$$b_s(T'_m, T'_n) > m + n - 6.$$

Let H be the bipartite graph of order $s + t - 1$ where one partite set is of order s and the other partite set is of order $t - 1 = m + n - 4$.

We know that $K_{s,\, m+n-6} \subseteq K_{m+n-6,\, m+n-6}$ since $2 \leq s \leq t - 1 = m + n - 6$ by definition. So by Theorem 4.5.1 we have that there exists a red-blue coloring of the edges of the complete bipartite graph $K_{m+n-4,\, m+n-4}$ where there is neither a red T'_m nor a blue T'_n. Therefore, we have that there exists a red-blue coloring of the edges of the bipartite graph $K_{s,\, m+n-6}$ where there is neither a red T'_m nor a blue T'_n. So we conclude that

$$b_s(T'_m, T'_n) > m + n - 6$$

and so

$$b_s(T'_m, T'_n) = m + n - 5.$$

Table 5.4.1: The number of red and blue edges incident to v in $K_{s,t}$.

$deg_{G_R}\ v$	$deg_{G_B}\ v$	Contradiction
$m+n-5$	0	red T'_m
$m+n-6$	1	red T'_m
$m+n-7$	2	red T'_m
$\vdots$	$\vdots$	$\vdots$
$m-1$	$n-4$	red T'_m
$m-2$	$n-3$	red T'_m
$m-3$	$n-2$	blue T'_n
$m-4$	$n-1$	blue T'_n
$\vdots$	$\vdots$	$\vdots$
2	$m+n-7$	blue T'_n
1	$m+n-6$	blue T'_n
0	$m+n-5$	blue T'_n

□

We now determine $b_s(T'_m, T'_n)$ where $m \geq 5$ and $n \geq 5$ for $s \geq m + n - 5$.

Proposition 5.4.2. *Let t = m + n - 3 where $t \geq s$ for some positive integers s, n and m. If $m \geq 5$ and $n \geq 5$ and $s \geq m + n - 5$, then*

$$b_s(T'_m, T'_n) = s.$$

Proof

The lower bound

$$b_s(T'_m, T'_n) > m + n - 6.$$

was studied in Proposition 5.4.1. So it suffices to investigate the upper bound

$$b_s(T'_m, T'_n) \leq m + n - 5.$$

Let G be the complete bipartite graph of order $2(m+n-5)$. Then G has two partite sets each of order $m+n-5$. If v is an arbitrary vertex of G, then

$$deg\ v = m+n-5.$$

Consider a red-blue coloring of the edges of G resulting in a red subgraph G_R of G and a blue subgraph G_B of G. In Table 5.4.2 below we give all of the different ways to choose red and blue edges incident to vertex v in G and the last column represent the graph we are trying to avoid but occurs because of the way we assigned the color to the edges. We find that if $deg_{G_R}\ v \geq m-2$, then G has a red T'_m, and if $deg_{G_B}\ v \geq n-2$, then G has a blue T'_n.

Table 5.4.2: The number of red and blue edges incident to v in $K_{n+m-5,\,n+m-5}$.

$deg_{G_R}\ v$	$deg_{G_B}\ v$	Contradiction
$m+n-5$	0	red T'_m
$m+n-6$	1	red T'_m
$m+n-7$	2	red T'_m
$\vdots$	$\vdots$	$\vdots$
$m-1$	$n-4$	red T'_m
$m-2$	$n-3$	red T'_m
$m-3$	$n-2$	blue T'_n
$m-4$	$n-1$	blue T'_n
$\vdots$	$\vdots$	$\vdots$
2	$m+n-7$	blue T'_n
1	$m+n-6$	blue T'_n
0	$m+n-5$	blue T'_n

Thus, in every red-blue coloring of the edges of G there is always a blue T'_n or a red T'_m, a contradiction to our assumption. As such we conclude that

$$b_s(T'_m, T'_n) \leq m+n-5$$

and so

$$b_s(T'_m, T'_n) = m+n-5.$$

□

We obtain Theorem 5.4.1 by combining Proposition 5.4.1 and Proposition 5.4.2.

Theorem 5.4.1. *Let s, n and m be positive integers with $s, m, n \geq 5$. If $t = m + n - 5$ where $t \geq s$, then*

$$b_s(T'_m, T'_n) = \begin{cases} m+n-5 & \text{if } 2 \leq s \leq m+n-6, \\ s & \text{otherwise.} \end{cases}$$

5.5 Conclusion

The authors of *Another view of bipartite Ramsey numbers* [11] introduced and studied the s-bipartite Ramsey number for 4-cycles and other small graphs. Olejniczak in his dissertation *Variations in Ramsey Theory* [14] studied the s-bipartite number of two stars. We, however, establish the following property for stars and with our own original proof:

$$b(K_{1,m}, K_{1,n}) = b_s(K_{1,m}, K_{1,n}).$$

We extend the theory of *Variations in Ramsey Theory* [14] by studying the s-bipartite Ramsey number of a star and a tree T'_n in Section 4.4 and proved the following theorem.

Theorem 5.3.1 Let s, n and m be positive integers with $s, m \geq 2$ and $n \geq 4$. If $t = m + n - 3$ where $t \geq s$, then

$$b_s(K_{1,m}, T'_n) = \begin{cases} m+n-3 & \text{if } 2 \leq s \leq m+n-4, \\ s & \text{otherwise.} \end{cases}$$

We furthermore studied the s-bipartite Ramsey number $b_s(T'_m, T'_n)$ in Section 5.4 and found the following theorem.

Theorem 5.4.1 Let s, n and m be positive integers with $s, m, n \geq 5$. If $t = m + n - 5$ where $t \geq s$, then

$$b_s(T'_m, T'_n) = \begin{cases} m+n-5 & \text{if } 2 \leq s \leq m+n-6, \\ s & \text{otherwise.} \end{cases}$$

Chapter 6

Closing Chapter

In this dissertation our main research questions are:

- *What is the bipartite Ramsey number of two trees with various structures?*
- *What is the s-bipartite Ramsey number of two trees with various structures?*

Ramsey Theory mostly focuses on how large a structure must be to ensure a particular substructure with a given property is contained. In this dissertation we focus on different variations of the Ramsey number for two trees.

We began by confirming the Ramsey number of two trees in Chapter 2. The result of the Ramsey number of two stars was used as the upper bound for the number of partite sets when we confirmed the k-partite Ramsey number of two stars in Chapter 3.

When confirming the k-Ramsey number of two stars in Chapter 3 we only looked at the instance were the number of partite sets is divisible by the number of vertices as given by Theorem 3.3.1.

Theorem 3.3.1 *Let k, n and m be positive integers with $2 \leq k < R(K_{1,n}, K_{1,m})$, $n \geq 3$ and $m \geq 3$. If $m + n - 2 = (k-1)q$ for some positive integer q, then*

$$R_k(K_{1,m}, K_{1,n}) = \begin{cases} kq & \text{if } k \text{ and } q \text{ are both odd and } n \text{ and } m \text{ are both even,} \\ kq + 1 & \text{otherwise.} \end{cases}$$

We are particularly interested in the instance when $k = 2$:

$$R_2(K_{1,m}, K_{1,n}) = 2q + 1 = 2(m + n - 2) + 1.$$

This result assisted us in estimating the bipartite Ramsey number for two stars. We further studied the bipartite Ramsey number of a star and a tree and the Ramsey number of two trees in Chapter 4. Combining the results in Chapter 4 enabled us to find Theorem 4.6.1. For $n \geq 2$, a tree T_n is a connected graph on n vertices which does not contain any cycles.

When studying the bipartite Ramsey number of a star and a tree and the Ramsey number of two trees, we discovered two conjectures which we intend to investigate further in future studies.

Theorem 4.6.1 *Let T_n and T_m be any trees where $n \geq 2$ and $m \geq 2$ respectively, then*

$$b(T_m, T_n) = \begin{cases} m + n - 1 & \text{if } \Delta(T_m) = m - 1 \text{ and } \Delta(T_n) = n - 1, \\ m + n - 3 & \text{if } \Delta(T_m) = m - 1 \text{ and } \Delta(T_n) = n - 2, \\ m + n - 5 & \text{if } \Delta(T_m) = m - 2 \text{ and } \Delta(T_n) = n - 2. \end{cases}$$

The above work formed partial building blocks to answer our research question on the s-bipartite Ramsey number of trees. We began by confirming the s-bipartite Ramsey number of two stars in Chapter 5. We then determined the s-bipartite Ramsey number of a star and a tree given by Theorem 5.3.1 below.

Theorem 5.3.1 Let s, n and m be positive integers with $s, m \geq 2$ and $n \geq 4$. If $t = m + n - 3$ where $t \geq s$, then

$$b_s(K_{1,m}, T'_n) = \begin{cases} m + n - 3 & \text{if } 2 \leq s \leq m + n - 4, \\ s & \text{otherwise.} \end{cases}$$

We furthermore studied the s-bipartite Ramsey number $b_s(T'_m, T'_n)$ in Section 5.4 and found the following theorem. We define T'_n to be a tree of order $n \geq 5$ whose maximum degree is $n - 2$.

Theorem 5.4.1 Let s, n and m be positive integers with $s, m \geq 5$ and $n \geq 5$. If $t = m + n - 5$ where $t \geq s$, then

$$b_s(T'_m, T'_n) = \begin{cases} m + n - 5 & \text{if } 2 \leq s \leq m + n - 6, \\ s & \text{otherwise.} \end{cases}$$

For completeness, one could study the s-bipartite numbers for trees in general. An idea we pursue in future studies.

Bibliography

[1] G. Chartrand, L. Lesniak, and P. Zhang, "Graphs & digraphs," vol. 39, 2010.

[2] G. A. Dirac, "Some theorems on abstract graphs," *Proceedings of the London Mathematical Society*, vol. 3, no. 1, pp. 69–81, 1952.

[3] T. P. Kirkman, "On a problem in combinations," *Cambridge and Dublin Mathematical Journal*, vol. 2, pp. 191–204, 1847.

[4] M. Reiss, "Ueber eine steinersche combinatorische aufgabe, welche im 45sten bande dieses journals, seite 181, gestellt worden ist.," *Journal für die reine und angewandte Mathematik*, vol. 1859, no. 56, pp. 326–344, 1859.

[5] J. Petersen, "Die theorie der regulären graphs," *Acta Mathematica*, vol. 15, no. 1, p. 193, 1891.

[6] D. König, "Über graphen und ihre anwendung auf determinantentheorie und mengenlehre," *Mathematische Annalen*, vol. 77, no. 4, pp. 453–465, 1916.

[7] F. Ramsey, "On a problem of formal logic. proceedings of the london mathematical society," *Sc, 2rid Set*, vol. 30, 1930.

[8] S. A. Burr and J. A. Roberts, "On ramsey numbers for stars," *Utilitas Mathematica*, vol. 4, pp. 217–220, 1973.

[9] Y. Guo and L. Volkmann, "Tree-ramsey numbers.," *Australasian Journal of Combinatorics*, vol. 11, pp. 169–176, 1995.

[10] E. Andrews, G. Chartrand, C. Lumduanhom, and P. Zhang, "Stars and their k-ramsey numbers," *Graphs and Combinatorics*, vol. 33, no. 2, pp. 257–274, 2017.

[11] Z. Bi, G. Chartrand, and P. Zhang, "Another view of bipartite ramsey numbers," *Discussiones Mathematicae Graph Theory*, vol. 38, no. 2, pp. 587–605, 2018.

[12] M. Christou, C. Iliopoulos, and M. Miller, "Bipartite ramsey numbers involving stars, stripes and trees," *Electronic Journal of Graph Theory and Applications (EJGTA)*, vol. 1, no. 2, pp. 89–99, 2013.

[13] J. H. Hattingh and M. A. Henning, "Star-path bipartite ramsey numbers," *Discrete Mathematics*, vol. 185, no. 1-3, pp. 255–258, 1998.

[14] D. Olejniczak, "Variations in ramsey theory," 2019.

[15] L. Gerencsér and A. Gyárfás, "On ramsey-type problems," *Annales Universitatis Scientiarum Budapest. Eötvös Sectio Mathematica*, vol. 10, pp. 167–170, 1967.

[16] F. Harary, "Recent results on generalized ramsey theory for graphs," in *Graph Theory and Applications*, Springer, 1972, pp. 125–138.

[17] Z. H. Sun, "Ramsey numbers for trees," *Bulletin of the Australian Mathematical Society*, vol. 86, no. 1, pp. 164–176, 2012.

[18] R. Faudree and R. Schelp, "Path-path ramsey-type numbers for the complete bipartite graph," *Journal of Combinatorial Theory, Series B*, vol. 19, no. 2, pp. 161–173, 1975.

[19] V. Longani, "Some bipartite ramsey numbers," *Southeast Asian Bulletin of Mathematics*, vol. 26, no. 4, pp. 583–592, 2003.