Telling and illustrating additive relations stories

A classroom-based design experiment on young children's use of narrative in mathematics

Nicky Roberts

Thesis submitted in fulfilment of the requirement for Doctor of Philosophy, University of Witwatersrand.

Supervisor: Prof Hamsa Venkat

DECLARATION

I hereby declare this PhD thesis, and the work presented in it, to by my own and to have been generated by me as result of my own original research. It has not been submitted for degree purposes to any other university.

N. Roberts

6 May 2016

Abstract

In South Africa, difficulties with learners solving word problems has been a recurrent problem identified through national standardised assessments extending from Foundation Phase into the Senior Phase. As is evident globally, particular difficulties have been identified with young children solving 'compare-type problems' where the numbers of objects in two disjoined sets are compared. This design experiment provides empirical data of young South African learners trying to make sense of compare-type problems. The task design from this design experiment suggested that engaging learners in narrative processes where they are expected to model the problem situations and then retell and vary the word problems, to become fluent in using the sematic schemata may assist them to become more experienced and better able to make sense of compare-type problems. This finding contradicts the advice offered in official South African government documentation.

The study was a three-cycle classroom-based design experiment which took place over 10 consecutive school days with Foundation Phase learners in a full service township school where the majority of learners were English Language Learners (ELLs), learning mathematics in English when their home language has not English. This study set out to research a 'narrative teaching approach' for a specific mathematics topic: additive relation word problems. At the heart of the study therefore, was a question relating to the efficacy of a teaching strategy: To what extent do young children's example space of additive relations expand to include compare type word problems?

This design experiment reveals that when adequately supported with careful task design and effort in monitoring and responding to learner activity, Grade 2 ELL children in a township school *can* improve their additive relations problem solving, in a relatively short time frame. The majority of the learners in this design experiment were able to solve compare-type problems at the end of the 10-day intervention. These learners were also able to produce evidence of movements towards more structured representations, and towards better learner explanation and problem posing using storytelling.

The design experiment intervention showed promise in expanding young children's example space for additive relations word problems. In both cycles the mean results improved from pre-test to post-test. The gains evident immediately after the intervention were retained in a delayed post-test administered for the third cycle which showed further improvements in the mean with a reduced standard deviation. The effect sizes of the shifts in means from pre-test to post-test was 0.7 (medium) in both cycles, while the effect size of shifts in the mean from pre-test to delayed post-test was 1.3 (large). T-tests established that these shifts in means were statistically significant. The core group showed the greatest learning gains, suggesting that the intervention was most successful in 'raising the middle' of the class.

Particular patterns of children's reasoning about additive relations word problems are documented from the South African ELL children in this design experiment. For example many ELLs in this design experiment initially responded to compare word problems like *'Mahlodi has 12 sweets. Moeketsi has 8 sweets. How many more sweets does Mahlodi have than Moeketsi?'* with: *'Mahlodi has 12 sweets'*. New actions and contrasts relating to additive relations are brought into focus. For example the empirical results indicated that inserting attention to 1:1 matching actions was found to be useful to helping learners to deal with static compare situations.

This study has helped to extend the theoretical foundations of what is meant by a 'narrative approach' as the theoretical features of the narrative approach are now situated within a broader theoretical framework of orienting theories, domain specific instructional theories, and related frameworks for action. The findings of this design experiment have been promising in the local context of the focal school. Should the intervention task design be found to yield similar results in other South African Foundation Phase contexts, when implemented by teachers other than the researcher, then it may be appropriate to use the research findings to improve the guidance provided to Foundation phase teachers (in curriculum documentation and through professional development offerings).

Acknowledgements

I would like to acknowledge Prof Hamsa Venkat who supervised this thesis. She provided valuable academic insight and rigour to my work. It was through the generous funding obtained from her Wits Connect project that I was able to attend research schools and two academic conferences: one in Durban South Africa; and another in Macau China. She was also a very important reference group member and mentor on overseeing the broader Focus on Primary Mathematics development project, on which this thesis is a small part. I am also grateful for the academic guidance and collegial encouragement provided by Dr Patrick Barmby, Dr Ulla Runnesson; Dr Audrey Msimanga; Prof Mike Askew, Prof Julia Anghileri, Dr Andreas Stylianides; Dr Cynthia Kros; Dr Sharon McAuliffe and Prof Mellony Graven who each provided collegial support and guidance on particular aspects of my thesis.

I was also the beneficiary of several University of Witwatersrand initiatives to support post-graduate students. I have been a recipient of a PhD completion grant, for which I am most grateful. I have also benefitted from the research weekends held as part of the research programme where I had regular opportunities to present my work and receive collegial feedback on my emerging ideas; while participating in the research initiatives of other post graduate students. In this regard I am most grateful to Corin Matthews, Thulelah Takane, Marie Weitz; Lawan Abdulhamid; Samantha Morrison; Herman Tshesane; Manono Mdluli; Anna-Lena Ekdahl; and Valerie Ramsingh who are all valuable members of our student support group and who each helped to keep me focused; motivated; thinking; reading more widely; and occasionally participating in very bad karaoke and dancing. I benefitted too from the on and off campus writing retreats organised by Prof Eric Worby in the Faculty of Humanities. I participated in three Southern African Association for Research in Mathematics Science and Technology Education (SAARMSTE) research schools, and each one moved my thesis forward considerably. I am grateful both to the organisers and readers made available at the SAARMSTE research school.

I thank the principal, deputy principal, teachers, children and their parents in the two primary schools in Cape Town for their enthusiastic participation and interest in this research. I deeply value their tacit knowledge of Foundation Phase teaching craft which continued to inspire me.

Numerous colleagues, friends and family members have supported me while I completed this thesis. I simply list them by name, as they each know whether they were inspirational colleagues and teachers; child minders, psychological therapists; proof readers, inspirational facilitators, continuous tea makers, pourers/suppliers of wine, patient sounding-boards, supportive colleagues and/or friends, or various combinations thereof: Anne Roberts, Anne Watson, Bongiwe Mlangeni; Carmel Marock, Catherine Sutherland, David Hornsby, Garth Spencer-Smith, Glynis Ponton, Heidi Jee, Ingrid Mostert, Jackie Johnson, John Mason, Katherine Malan; Lucy Allais; Helene Schoeman; Lynn Bowie, Marrianne Thamm, Melissa Mentoor, Mike Cameron, Owen Roberts; Pam Barmby; Sally Shield, Shauna Pamplin, Shireen Corker, Siddieka Hassen, Stephanie Allais, Sue

Fourie; Tanya Halls; Thelma Mort; Tracey-Lee Bester, Tshelitsi Marawala, Ursula Röntsch, and Vicky McDonald.

This thesis is dedicated to my husband Oscar, and our twins Tessa and Ayanda van Heerden. It was Tessa and Ayanda's formal introduction to schooling that inspired me to reflect deeply, as a parent, scholar and an educator, on mathematics in the early grades. It was Oscar who encouraged me to formalise this academically. I thank them all for their constant inspiration and insight. I am particularly grateful to Oscar for his support in this process, and for his sage advice to 'not do this thesis like a white person', I can only hope I have may have come close to meeting this criterion.

Contents

Figures and tables

List of figures

List of tables

CHAPTER 1: Introduction

This opening chapter provides a general overview of the scope of the thesis, including an outline of the research aims, and delineates the research questions. The way in which narrative is used in this study is defined as this was the main pedagogic strategy adopted for the content focus on additive relations word problems. An overview of the theoretical framework adopted which informed the design of the intervention is provided and this is followed by a synopsis of the research design. The chapter ends with an overview of the structure of this thesis.

Scope of this thesis

This thesis reports on a three-cycle classroom based design experiment and focuses attention on understanding the effect of the most recent (the third) instructional sequence cycle of young children's engagement with solving, posing and representing narratives relating to additive relations word problems.

All three cycles of this design experiment were located within Grades 2 and 3 in a disadvantaged 'township' school in the Western Cape in South Africa which was a 'full service' public school. South African government policy has allowed for the establishment of approximately 500 'full service schools' which are 'designated ordinary schools, which are intended to become examples of good inclusive practice' (Walton 2015, p. 213). The school was part of a broader development project Focus on Primary Mathematics involving two schools that I led. This broader work was focused on improving the teaching and learning of mathematics in the Foundation Phase. The broader project had the following aims: to support school-based teams of teachers to improve mathematics teaching and learning; to facilitate professional development interventions for these teachers; to identify areas of weakness as evident in the Annual National Assessment (ANA) and systemic assessment results from each school; to collaborate on classroom level interventions for these problem areas (of which this PhD study formed a part); and to share the lessons learnt with a wider audience. The broader project also meant that I had access to school assessment results from a range of different tests for the Foundation phase learners that participated in the design experiment. As is expected in design experiments, the focus is on the most recent research cycle, and comparison is made to the task design and learning gains evident in previous cycles.

Learning in this study was viewed in terms of two aspects of expansions in personal example spaces: firstly, I looked for expansions in the categories of additive relation word problems that the learners were able to solve, with these expansions linked to hierarchies of difficulty delineated in the literature; secondly, I looked for expansions in narratives – which were viewed in this study in terms of 'oral stories' and 'representations' that learners were able to bring into play when faced with an additive relation situation.

Watson and Mason (2005) within their body of work on example spaces, note that 'examples are usually not isolated; rather they are perceived as instances or classes of potential examples', thereby 'contributing to an example space' (p. 51). They distinguish several kinds of example spaces: a local personal example space which may be triggered by current tasks in the environment or recent experiences; a personal potential example space from which a local personal example space is drawn and consists of one person's past experience; a conventional example space which are understood by mathematicians, displayed in textbooks and into which a teacher hopes to induct her students; and a collective example space located in a particular classroom at a particular time (Watson and Mason 2005, p. 76). While the conventional example space of additive relations word problems was the focus of my attention as the teacher in planning the intervention, the learning objective was to expand learners' personal potential example space to include compare type problems. Compare type problem situations involve comparison of the numbers of objects in two disjoint sets. A typical compare problem is: 'I have 8 apples. You have 3 apples. How many more apples do I have than you?'

On the pedagogic side, the teaching strategies adopted were informed by a constructivist orientation, which drew on the concept of discernment from variation theory and adopted a 'narrative approach' to mathematics. Drawing on variation theory and defining example spaces as an object of study in mathematics, Watson and Mason (2005) refer to reference examples which are 'standard cases that are widely applicable and can be linked to several concepts and results' (p. 64). So the design of the teaching included the purposeful selection of reference examples of additive relation word problems from the conventional example space and sequencing of these examples and tasks, drawing on prior research in the mathematics education literature. The narrative approach employed in this study draws on literature on narrative and learning (Egan 1989; Egan 2002; Bruner 2003), a small body of literature on narrative and mathematics learning (Schiro 2004; Back, Piggott et al. 2010) and a prior design experiment which established that a narrative approach held promise in relation to the teaching of parity in Foundation Phase contexts in a public school in England and a South African private school (Roberts and Stylianides 2013). A narrative approach was seen as a potentially good 'fit' for addressing learning goals focused on expansion of personal potential example spaces in the context of the issues outlined in the last paragraph. The pedagogic approach for achieving this expansion was based on encouraging the telling and re-telling of 'narratives' (encompassing oral stories and representations) relating to a range of additive relations situations. During the intervention oral story telling was used as both a pedagogic and a cognitive strategy: when used as a pedagogic strategy story telling was used by the teacher for explaining; when used as a cognitive strategy learners were expected to re-tell, vary and create their own stories to support their sense making processes. Representations were used to interpret calculation strategies. Within representations, a sub-aspect of interest was attention to shifts over time towards more structured representations of number and additive relations.

So the intervention set out to induct Foundation Phase learners into a conventional example space of additive relations word problems, where the compare type word problems were brought into focus. This overarching learning objective was specified with three interrelated learning goals which took into account both *what* mathematics was to being learnt (additive relations word problems) and *how* this mathematics was approached (using narrative which encompass storytelling and representations). Learning goal 1 (LG 1) was for learners to solve a range of additive relations word problems. In support of the first goal, there were two further enabling learning goals which related to how learners were expected to make their thinking visible (to self and others): Learning goal 2 (LG 2) was for learners to flexibly use a range of representations to express and explain their solutions to additive relation word problems; and Learning goal 3 (LG 3) was for learners to tell stories to pose and explain additive relation word problems.

The design experiment approach generally begins with the identification of an experienced, practical learning problem and adopts an engineering approach to 'designing and systematically developing high-quality solutions' to this problem of practice (Burkhardt and Schoenfeld 2003, p.4). Literature and theory are then turned to in order to see if they can guide the design of a solution to the identified local problem. Given that this was the case in this study, I begin this thesis by detailing some of the 'local' problem related to children's working with additive relations word problems in the focal school. The assessment data in this focal school pointed towards an overarching learning objective of 'expanding learners' personal example spaces for additive relations to include compare type word problems'.

In a South African context of ongoing concerns about low performance in mathematics, the problems identified in the focal school were found to be more widespread, and have been linked to a range of issues identified in other classroom-based studies in the South African Foundation Phase mathematics education literature. These issues include challenging primary school contexts where schools and teachers frequently have limited capacity to support learning, and most learners do not speak English as their main language, yet many learn mathematics in English. As such the majority of learners are English Language Learners (ELLs). Mathematics in the majority of these schools is characterized by systemically poor mathematics attainment and very poor attainment in word problems in standardized assessments. This poor attainment has been linked to lack of sense making and number sense in mathematics, the language context in many classrooms, and lack of progression in the strategies and representations children use to solve word problems. All of the above are compounded by disruptive learner behaviour evident in many Foundation Phase classrooms (Marais and Meier 2010).

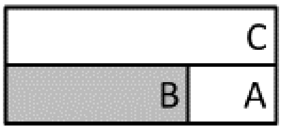
Appropriate use of comparative language phrases in English includes the use of 'more' or 'less' as descriptive adjectives ('He has more, I have less'); as determiners ('I need more stickers'); as comparative adjectives ('1 more than 6 is 7'); and as adverbs ('How many more is 7 than 6?'). The descriptive adjective and determiner uses of 'more' (or less) require an approximation. Cognitive scientists have identified the existence of an approximate number system (ANS) which is part of the 'core knowledge' which human are born with prior to their use of language (Feigenson, Dehaene et al. 2004). The ANS allows for the (non-exact) distinction between different quantities and it is seen as the primary foundation on which symbolic knowledge of number will develop. Innately, humans and other animals are able to distinguish between 'more' and 'fewer' in a set (Henning and Ragpot 2014). This discernment is not expressed linguistically, but both behavioural research and neuroscience research have confirmed that infants are able to make a distinction between quantities of many and few (in the ANS) (Henning and Ragpot 2014). The use of 'more' (and 'less') as comparative adjectives and as adverbs – which is the use required for compare type word problems - is not an approximation but requires some exact whole number arithmetic. This seems to be far more complex, where reasoning about the relationship between two sets is expected. This requires the use of symbolic knowledge where the child is expected to use language and symbols and reason about an additive relationship.

This thesis approaches additive relations primarily from a pedagogic perspective; and not from the developmental psychology and neuroscience perspectives. Suffice to note three important findings from the cognitive science perspective: Firstly pre-linguistic children are able to distinguish between more and fewer in a set (ANS); and have an object tracking system (OTS) which allows recognition of one, two and three objects. Secondly research coheres on the developmental process of learning to count where children first recite a 'counting list' from verbal memory without understanding the meaning of number words beyond three; gradually children begin to understand the concept of 'the next number' (in the counting list), and thereafter learn the principle of 'one more than' where their verbal knowing intersects with their conceptual knowing (Henning and Ragpot 2014). Later (and again gradually) children learn to use the cardinal values of numbers more than three which means that if they count out a number of objects, they will attribute the numerosity of the set to the last numeral used. Thirdly studies on learning and cognition suggest that the subsequent developmental process of constructing symbolic knowledge of number is not yet known. Butterworth asserts that 'the transition from approximations to exact whole number arithmetic is still mysterious' (Butterworth 2015, p. 24).

Some of the complexity of the shift from approximations to exact whole number arithmetic is evident when considering the linguistic demands of fluent use of comparative language in English. To illustrate, consider the linguistic demands on the following contrasting questions involving the term more: 'What number is 3 more than 8?' and 'How many more is 8 than 3?'.

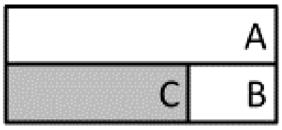
'What number is 3 more than 8?' directs attention to the number which is a result of increasing 8 by 3: '3 more than 8 is 11'. In this case 8 is the referent, and there is an implicit change increase action (of 3 more) in this problem situation. In this case the unknown (11) is the result of increasing a starting value (8) by a given quantity (3 more). This statement may be considered in its general form as 'A more than B is C' where the referent is B and depicted visually as follows:

Figure 1: Whole-part-part structure for 'A more than B is C'



In contrast, 'How many more is 8 than 3? directs attention to a comparison between 8 and 3: '8 is 5 more than 3'. Here the situation is static as no change is implicit in the problem statement, however there is an implicit 1:1 matching action which must be introduced into the situation to compare 8 and 3. In this case the unknown (5) is a relation (the difference) between two given quantities (3 and 8). In its general form this can be represented as 'A is B more than C' where the referent is C and depicted visually as follows:

Figure 2: Whole-part-part structure for 'A is B more than C'



Contrasting the general forms: 'A more than B is C' to 'A is B more than C' , reveals a very slight linguistic change, where the position of comparative clause 'more than' is switched with the position of the verb 'is'. To native speakers of English this shift in word order to change the meaning of the statement may appear obvious. However consider the situation of the ELL where a shift in word order may not commonly denote a shift in meaning. The language of comparison is complex in English as comparison requires a referent (something to compare to). The referent is not made explicit in the English language, but it inferred from the word order of the sentence. A slight change in word order results in significant change in meaning. Further it should be borne in mind that the above description of the linguistic challenges of comparative language limits attention to use of the word 'more', ignoring 'less', 'fewer', 'much' and 'many' as well as the comparative language of continuous measurement contexts where the even broader vocabulary of comparison for measurement (heavier, lighter, longer, shorter, broader, wider, later, slower, quicker, fuller, emptier etc.) is required.

I begin this introductory chapter by detailing the research aims and research questions. I follow this with a brief introduction to the ways in which narratives are defined and viewed in this study, and then go on to outline the additive relations literature base in which there is broad agreement about key categories of phenomenological situations that can be described in terms of additive relationships. Expanding the example space for stories related to additive relations could therefore be viewed in terms of expansions across these phenomenological categories. This literature base also points to a range of key representations that have been highlighted as potentially useful within learning to work with additive relation problems. These representations are also described briefly in this chapter, given that learning relating to representations was viewed in terms of expansions linked to this representational base. I then present a short overview of the research design – based on the literature on design experiments - and detail the analytical approaches that were used within the study. An outline of the structure of the broader thesis concludes this chapter.

Research aims and questions

In the context of a paucity of research in South African Foundation Phase mathematics, this study set out to research a teaching approach for a specific mathematics topic: additive relation word problems at Foundation Phase level. In particular the teaching approach made use of a narrative approach to mathematics with a topic focus of additive relations word problems and specifically 'compare' type word problems where the numbers of objects in two disjoint sets are compared.

Research aims

Firstly at the empirical level, this classroom-based design experiment aimed to develop and implement innovative approaches to teaching additive relation word problems in Foundation Phase South African township classrooms where the majority of learners are ELLs. An integral part of the design experiment was to research its outcomes.

Secondly, at the theoretical level, the aim was to deepen theoretical understanding of how narrative can be used as a pedagogic strategy in mathematics education. Stylianides and Stylianides, when writing about classroom-based design experiments, assert that:

> Research on classroom-based interventions in mathematics education has two core aims:
> (a) to improve classroom practice by engineering ways to act upon problems of practice; and
> (b) to deepen theoretical understanding of classroom phenomena that relate to these problems. (Stylianides and Stylianides 2013, p. 333) .

For this study, the problem of practice emerged in a particular school where expanding ELLs example space to include compare type word problems was found to be a necessary, but difficult to achieve, learning goal. As such an engineered response to a problem identified in a particular South African primary school, was planned and subjected to research analysis. It was expected that the theoretical understanding of the classroom phenomena, would be deepened. The broader objectives of so doing were to explore the efficacy of using 'narrative' as a pedagogic strategy in mathematics in a challenging school context, to address additive relation word problems and the inclusion of compare type word problems in particular.

If the classroom intervention was found to yield promising results, the research was further intended to inform possibilities for teaching practice in which colleagues could recognize aspects of the study as resonant with their own teaching and learning contexts, practicing teachers could replicate the intervention in their own classrooms and where a South African classroom experiment could be compared to other primary classroom experiments. The research was envisaged to potentially inform curriculum reform where refined curriculum guidelines for teachers on approaches to additive relation word problems were required, and to contribute to mathematics education theory by providing ontological innovations which delineated explanatory constructs and new categories relating to the teaching and learning of additive relation word problems.

Research questions

In the context of an intervention aimed at expanding Grade 2 English Language Learners' personal example spaces for additive relations to include compare type word problems, by making use of narrative as a pedagogic strategy my focus is on the following research question:

> To what extent do young children's example spaces of additive relations expand to include compare type word problems?

At the heart of the study therefore, is a question relating to the efficacy of a teaching strategy in a particular classroom-based design experiment. Efficacy of teaching was measured by looking for evidence of learning gains related to three interrelated learning goals: Learning goal 1: Solve a range of additive relation word problem types (LG1); Learning goal 2: Flexibly use a range of representations to pose and explain word problems (LG 2); and Learning goal 3: Tell stories to pose and explain word problems (LG 3).

Defining the use of 'narrative' in this study

Narratives were viewed as encompassing 'oral stories' and 'representations', and referred to for young children as 'telling and illustrating stories'. Narrative in its broadest sense refers to a spoken or written account of connected events, and thus encompasses representations in a range of forms. My use of narrative came to be defined thus in a previous design experiment:

> [Narrative was] used at times as a noun (a story) and at times as an adjective (describing the story-like quality of something). Narrative is considered to comprise both words (written or oral) and images (presented or imagined), as in a children's storybook. It should be noted that our use of 'narrative' is distinct from what others may term the 'narrative context' of a mathematical word problem. We view the 'narrative context' (or 'problem context') as the situation or milieu—comprising elements such as characters, a setting, numbers, and a question or conflict—on which a particular mathematical word problem is based. Our use of 'narrative' considers stories more generally, whereby several narratives (distinct stories) can be generated from a single narrative context, and the narratives need not be told in the genre of word problems (Roberts and Stylianides 2013, p. 454).

Narrative thus includes different forms of representation encompassing talk, gesture and writing/drawing with the key characteristic of this being an account of connected events. Prior writing in mathematics education has highlighted the fundamental importance of representations within mathematical learning while noting also, frequent difficulties for learners in moving between different representational forms (Duval 1999). South African evidence, outlined in the next chapter, points to particular learner difficulties with making sense of, representing and solving additive relation word problems. Taken in combination, these two bodies of evidence pointed to the usefulness of separating 'stories' (the term used for an oral or written narration using words, often involving natural language) from other representations (including: diagrammatic representations used for drawings or illustrations depicting mathematical situations; gestures, denoting movements or actions made by the narrator; and symbolic representations using number and operational notations).

From a pedagogic perspective, my focus on this narrative-based approach within the design experiment was aligned with Eisner's description of working with representations in terms of the transformation of the contents of consciousness into a public forum so that they can be stabilized, inspected, edited and shared by others (Eisner 1993). From a learning perspective following Cobb, Gravemeijer, Yackal, MacClain & Whitenack (1997) an emergent perspective is adopted in reflecting on the enculturation of mathematics symbols and notations with early grade learners. A reflexive relationship between individual thought and c ultural processes is assumed, and as such 'individual mathematical activity is seen to be necessarily socially and culturally situated' (p. 152).

Theoretical framework

The theoretical framework adopted for this study distinguishes the following types of theories: orienting theories; domain specific instructional theories, frameworks for action. A further type of theory - ontological innovation (which refers to the explanatory constructs and new categories) - relating to the teaching and learning additive relation word problems are put forward as the theoretical contribution of this study and are discussed in the concluding chapter. This section provides a broad synopsis of these major features of the theoretical framework.

My approach to the learning of mathematics falls within the constructivist tradition, where learners actively make sense of their environment in communities. The guiding principle relating to this is that learners make sense of problems for themselves while conforming to agreed social practices.

In this study, I follow the approach taken by John Mason across an extensive body of work in which variation is used as a key mechanism for 'educating awareness' – a notion that Mason (2004) links back to Gattegno (Gattegno 1964; Gattegno 1973). Following Mason, mathematics learning is viewed as supporting the development of mathematical thinking, where harnessing natural powers and educating awareness are valued. In particular I draw on the application of variation to the notions of discernment and as a means of educating awareness.

A conventional example space relating to additive relations word problem was articulated and then tasks designed to induct children into this conventional example space. Drawing on the work of Anghileri (2005) in relation to negotiating meanings of language and symbols and Gravemeijer (1997) in term of mathematical modelling processes, these tasks started with the natural language of the word problems and then shifted to encourage flexible movement between increasingly symbolic representations of the word problems. The construction of the conventional example space drew on mathematics education literature, and evolved as a dialectic as the relationship between theory and empirical data of learners' local engagements with particular tasks were reflected upon.

In this study the learning goals were defined in relation to an over-arching learning objective: 'to expand the learner's personal potential example space for additive relations to include compare type word problems' and related learning goals pertaining to the mathematics that was the intended focus. In this study variation in relation to a focal topic is not viewed as the only mechanism through which learning of the focal topic is made possible. Rather learning is seen as influenced by a range of other aspects of a classroom environment – the teacher role, the learning disposition of the learners, and the school culture. Thus, the theoretical ambit in this study extends to issues of emotion and behaviour within the social context of the classroom community, and these issues were consciously attended to by the teacher, and are reported on in the theoretical framework for action.

The mathematical context of the study is located in the content specific domain of additive relations. Additive relations problems refer to situations involving addition or subtraction that can be expressed in narrative forms. An extensive literature base has noted the range of problem types that feature within the remit of additive relations (Carpenter, Fennema et al. 1999; Clements and Sarama 2009; Askew 2012), while also noting the difficulties faced by some learners in moving from rudimentary counting based strategies to more abstract understandings of symbolic number relations. These difficulties are marked and extensive in the South African terrain, with evidence too of these issues being compounded by problems relating to fluency with number facts and generalised relationships and limited exposure to connections between natural language and mathematics.

One-step additive relations word problems have generated much research interest relating to early grade experiences of mathematics (Verschaffel and De Corte 1993). There are several major research traditions which have fed into this research domain. Initially the focus of the research endeavour relating to additive relation word problems was on the linguistic, presentational and computational demands of word problems. Dimensions of possible variation on the word problem example space included: linguistic and presentational considerations such as syntax, vocabulary, and key words; number of words in the word problem, inclusion of diagrams or just test; and computational considerations such as the number and nature of required operations, a nature and size of given numbers, and the type of open number sentences each word problem represented.

Research interest then shifted from the examination of externally observable performance of learners to the underlying cognitive schemes and thinking of students solving various kinds of word problems (Verschaffel, Depaepe et al. 2014). The research focused on cognitive processes of learners solving additive relations word problems resulted in three major research studies – all conducted in the early 1980s: Carpenter, Hiebert and Moser (1981) working on their Cognitively Guided Instruction (CGI) framework; Riley, Greeno and Heller (1983), (hereafter referred to as RGH analysis), working on information processing and computer programmes to support levels of problem solving; and Verschaffel and De Corte (1993)[1] (hereafter referred to as VDC analysis) working on semantic structures and children's invented solution strategies. All three studies organise and classify different additive relation word problems by using the types of actions or relationships described in the problems as their primary defining characteristic.

This cognitive research tradition coheres on the development of a classification framework for thinking about additive word problems type. Additive relations word problems are delineated into the following main classes of additive-relation word problems:[2]

> *Change problems* refer to word problems where there is an action of joining (change increase) or separating (change decrease) which changes (increases/decreases) the number in a set. The situation is dynamic. For example 'I have 8 apples. I eat three of them. How many are left?' There is a set of 8 apples, there is an action which changes/decreases this set (eating 3 apples), and the result is a decreased set of apples).
>
> *Collections problems* refer to word problems where two parts make a whole but there is no action. The situation is static. For example 'I have 8 apples. 3 are red. The rest are green. How many are green?' There is a whole (all the apples) and two parts (the red apples being one part, and the green apples being the other). There is no action or change. But a comparison is set up between the whole (8 apples in total) and a part (the 3 red apples).
>
> *Compare problems* where the numbers of objects in two disjoined sets are compared. For example 'I have 8 apples. You have 3 apples. How many more apples do I have than you?' There is a static situation comprising two disjoint sets (the apples I have and the apples you have), and a comparison is made between them, which attends to their difference in size.

While there have been some refinements to the classification framework with shifts in terminology over time (which are explained in more detail in Chapter 3 of this thesis), there was nevertheless broad agreement that these three phenomenological categories are distinct with empirical evidence of differing success rates for solving such problems being confirmed across these studies conducted in varying contexts.

The cognitive research however diverges around how the underlying cognitive processes, or calculation strategies, that learners use to solve word problems are construed. The RGH analysis adopted an information processing view on cognition and hypothesised different problem schemata which were conceived as organised knowledge structures of the essential components of the three problem types. They identified three levels of knowledge relating to increasingly sophisticated semantic networks used to represent propositional information in the problem texts: In level 1 processes sets are specified so models can be constructed directly; in level 2 processes sets cannot be constructed externally as the numbers are not specified and these processes represent relations between sets (so inferences are required); and in level 3 processes a whole-part relation is added into a representation that already has changes of comparisons of sets. (Riley, Greeno et al. 1983). RGH inferred these semantic networks from observing young children engaging in problem solving processes while manipulating blocks on sheets of coloured paper.

The CGI researchers adopted a different approach. They examined the way in which very young children went about solving additive relations word problems prior to any formal instruction. They did this in order to better understand the informal strategies that young learners adopt and which, it is then assumed, are brought with them into the formal schooling environment. They argued that building on children's informal strategies was an appropriate starting point for formal early grade instruction. These researchers identified three calculation strategies commonly used for solving additive relation problems and considered these to be in a hierarchy of mathematical sophistication: Direct modelling; counting; and calculating. Direct modelling refers to the use of concrete apparatus such as manipulatives (like counters, or actual objects) to enact a situation which closely resembles the problem situation. Counting refers to strategies making use of unit counting to calculate and includes varying levels of sophistication (count all, count on, and count back or up to reach a target). Calculating refers to using strategies which do not use unit counting but which may use counting in groups and/or build on known facts (often knowledge of bonds of five and ten) and the relationship between the numbers in the calculation to solve it. The instructional implication of this research base is that mathematical work with early grade learners should build on the informal strategies that young learners have been researched to adopt (direct modelling making use of unit counting), and shift them into more sophisticated strategies making use of counting and then calculating. The starting point in this tradition is the process or action of verbal counting. As a process is the starting point, the representation used to depict a counting action is commonly a number line. A number line allows learners to act on the discrete number objects using arcs to show movement forwards or backwards along the line. Distinctions can be made between actions on ones (which depict unit counting processes) and actions on groups (which denote a shift towards calculation processes using known facts).

The VDC researchers, working in a context where there is a 'heavy focus in the Flemish elementary schools on memorizing the basic addition and subtraction facts together with the early discouragement of verbal counting strategies' (Verschaffel and De Corte 1993, p. 247), adopted yet another approach. This work included longitudinal studies on first grade learners to test the hypothesise put forward by RGH with real children. Making use of individual interviews which were conducted the beginning, middle and end of the first grade, learners were asked not only to solve a set of eight word problems, but also to retell the problem, to explain and justify their solution strategy, to build a material representation of the problem with puppets and blocks, and to write down a matching number sentence. Re-telling the problem and building a material representation of the problem were included to gather empirical data about the learners' problem representations (which had previously only been inferred from their manipulation of cubes on coloured paper in the RGH study). The findings from this study supported the basic assumptions of the RGH analysis that the majority of young children's errors resulted from the inappropriate representation of the problem situation (the situation model). This contradicted the belief that the children's errors arose from not knowing which arithmetic operation was required to find the unknown (the mathematical model). Given the context of memorising basic addition and subtraction facts, the findings of this study differed from the CGI findings with far fewer learners using counting-based strategies.

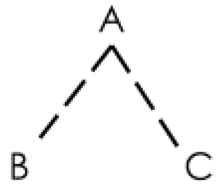
The VDC finding relating to few learners adopting counting-based strategies in the Flemish context is of particular relevance as it highlights a tension which has emerged more clearly in recent mathematics education literature from different parts of the world. In Sweden, the view that the development of early number concepts grows from direct modelling to counting and then to calculating as appropriate, has been challenged for some time. Neuman (1987) argued that the experiencing of structure – not counting – is the origin of arithmetic skills and made a distinction between learners using their fingers for counting (where fingers are used to keep track of a counting process), and using their fingers as 'finger numbers' (where the structure of hands having 5 fingers is used as a grouping structure). Reporting on this study Runesson and Kulberg (2010) explain that structure is evident when finger counting is used not for keeping track of counts, but rather :

> The 'sevenness' of seven becomes visible as a structure by visually seeing five (undivided!) fingers on one hand and two on the other. This offers an opportunity to develop a part-whole relationship. Structuring numbers promotes number sense whereas counting single objects could lead to math-difficulties (Runesson and Kullberg 2010, p. 4).

Runesson and Kulberg (2010) argue further that different ways of experiencing numbers are evident from the empirical data in this study. While some learners experience numbers only as ordinal (number as ordered names) other learners experience numbers only as cardinal (numbers as extents). For arithmetic skills simultaneous awareness of both cardinal and ordinal aspects as well as the part-whole relationship is necessary. Runesson and Kulberg explain that 'when experiencing numbers as 'finger numbers', the number is experienced as a structure' which 'makes it possible to visually see ordinal and cardinal features at the same time' (p. 5). I refer to this approach as a *structural approach* to early number development where simultaneous awareness of both the cardinal and ordinal aspects of numbers as well as their part-whole relationship is required.

A related argument challenging the progression from modelling to counting and then to calculating was put forward by Schmittau (2004) who adopted a cultural-historical perspective (in contrast to the constructivist perspective of the CGI researchers) drawing on the work of Vygotsky and Davydov. She highlighted concerns regarding taking children's naïve or spontaneous conceptions of number (experienced as unit counting) as a basis for number development and argued that pedagogic intervention is required for learners' conception of number to become 'scientific', arguing for an early introduction of number in which the measurement context of number was stressed and the discrete object unit counting context was backgrounded. She explained that a part-whole model where first grade learners write three equations derived from their actions with quantities (lengths, masses and volumes), before numbers are introduced is a particularly powerful (albeit deceptively simple) schematic:

Figure 3: Part-whole model (Schmittau)

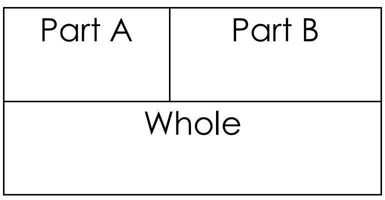


A = B + C
A – B = C
A – C = B

She goes on to explain that this model suggests putting together or taking apart a set of objects or quantities. Further as it represents the 'the essence of actions of composing and decomposing quantities, adding and subtracting are not perceived as formally separated operations, but as complementary actions' (Schmittau 2004, p. 235).

The part-whole structure has also been the focus of attention by Anghileri in England who argued for a focus on number triples and structural ways of representing these triples (Anghileri 2000, p. 54). In this regard the relationship between adding and subtracting, as well as equivalence and the commutative property of addition are foregrounded. Structures which visually show the relationship that adding and subtracting fit together (they are inverses) are in focus. Anghileri makes use of a particular whole number triple (8-5-3) to provide a structural image of the additive relation. I offer a general version of this structural imagery:

Figure 4: Structure of additive relations (Anghileri)



This gives rise to a family of equivalent number sentences:

Figure 5: Family of equivalent number sentences additive relations

Part A + Part B = Whole
Part B + Part A = Whole
Whole – Part A = Part B
Whole – Part B = Part A

This structural approach to additive relations puts forward teaching strategies for additive relations that encourage learners to work systematically to break up and combine numbers considering the whole, and each of the two parts, and to express these relationships in general terms to deepen their conceptual understanding of additive relations.

In the United States of America, Bass (2015) referring back to Davydov, distinguishes a 'counting pathway' which is an approach to early number development which takes verbal unit counting as its starting point (as in the CGI framework) from a 'measurement pathway' which is an approach to early number development which does not start with counting of discrete objects, but rather:

> Using the general context of quantity of various species of experiential objects, and addition as disjoint union[3] or concatenation[4]. This allows discussion of comparison of quantities (which one is more), and, implicitly that the larger quantity equals the smaller plus some other quantity. This can be done before any numerical values have been attached to the quantities, with the relations expressed symbolically (Bass 2015, p. 111).

In this approach, which overlaps with the views of Schmittau, the concept of number is introduced as 'choice of units' and the process of comparison is immediately invoked. Such an approach offers a strong connection between number work and spatial reasoning (shape and space, and measurement contexts) in early mathematics; as well as providing strong, explicit connections between the concepts of cardinality and ordinality. The structure of additive relations expressed as the whole-part-part diagram by Anghileri where rectangles are used to depict length invokes the measurement context of length, which may be absent from the whole-part model put forward by Davydov and adopted by Schmittau with first graders. The three authors – Anghileri, Schmittau and Bass – agree on the importance of invoking a measurement context of length (using paper strips, and or whole-part-part diagrams) to explore additive relations where a family of equivalent number sentences is made explicit to learners.

So the 'measurement pathway' offers a counter view to the 'counting pathway' which has held much currency in most English-speaking countries. Further a 'structural approach' to early number development is contrasted to the 'counting pathway'. Although there are overlaps between them I deliberately label a *structural approach* to additive relations word problems as distinct from a *measurement pathway* to early number development as while both are contrasted to a *counting pathway* they each make slightly different points:

- A *measurement pathway* considers the early number development as requiring reasoning about, and actions on, quantitative relationships in continuous measurement contexts as preceding work on discrete objects.
- *A counting pathway* advocates early number development which starts with learners informal strategies evident prior to formal instruction (direct modelling and their verbal counting of discrete objects), before progressing to more efficient counting strategies, and then to

calculation. A common representation used to express the calculation strategies is a number line where actions on numbers are visualised as hops of one, or bigger jumps.

- *A structural approach* foregrounds immediate recall of a family of number facts related to a particular number relation. A common representation used to express these number facts is a general whole-part structure (visualised diagrammatically using a context of length or as expressed by families of equivalent number sentences) which underlie all additive relations. In some cases, emphasis is placed on visualising the 5-wise and 10-wise structure of the number systems using finger numbers, and with apparatus using a 5-wise or 10-wise structure such as bead strings or abaci.

This tension fits within other broader developments in the mathematics education literature which are worth highlighting. Historically word problems at primary level were construed in terms of four basic operations, but definitions of algebra (traditionally the domain of secondary school mathematics) have been broadened to include consideration of algebraic reasoning at the primary school level. As a result, expression of general rules and structures underlying word problems (making use of words, actions and gestures in addition to formal symbolic notations) has been given greater attention at the early grade level. In this regard Kaput's (2008) definition of three strands for algebra: to generalise arithmetic patterns; to generalise towards the idea of a function; and to use language to model mathematical processes, may be useful. This lens of algebraic reasoning as including both generalised arithmetic and mathematical modelling has resulted in reframing how arithmetic is viewed and taught. As an example of this transition, the CGI work evolved into empirical studies on integrating arithmetic and algebra in early grade classrooms (Carpenter, Franke et al. 2003).

A final aspect of the mathematics education literature related to additive relations word problems which I have drawn on to inform this design experiment concerns the inclusion of what I term a 'partition problem' into the additive relation word problem conventional example space. A 'partition problem' is typified by the following example: 'There are 5 monkeys. They sleep in two trees. How many ways are there for the monkeys to sleep in the two trees?' (Cobb et al. 1997) This kind of problem has figured in research where a 'counting pathway' to early number development is developed, as well as in research where a structural approach is adopted. The use of this kind of problem was researched in South Africa in three Grade 3 classes in Gauteng at the same time as it was used in the cycle 3 intervention for this design experiment (Venkat, Ekdahl et al. 2014). The reason this problem type does not figure in the classification of one step additive relations word problem classes discussed above it that partition problems result in multiple possible solutions (whereas change, collection and compare problems result in single answer solutions). Partition problems do not include the typical one step process involving two givens with one unknown quantity to be found. Rather possible combinations of two unknowns which sum to a given total are explored.

This partition problem seemed to provide a bridge between the counting pathway and the structural approach. Composing and decomposing additive relations and depicting these using a continuous measurement context involving paper strips and whole-part-part diagrams draws from the literature on the structural approach; while opening work with a problem situation involving discrete (countable) objects draws on the counting-based approach. The partition problem makes use of discrete objects (the reference example in the conventional example space makes use of monkeys) and not measurement contexts. I felt that this partition problem could be used as a starting point to additive relations, as it provided opportunity to examine the structure of additive relations, which could be supported using a visual representation of a whole-part-part diagram, and related families of equivalent numbers sentences to express the relationship symbolically. In the partition problem, the 'whole' component of the additive relation is kept invariant (e.g. 5 monkeys) and the size of each part is changed by varying the position of the partition between part A and part B.

Story telling

The pedagogic approach to additive relation word problems centred on the use of narrative which was considered in relation to both storytelling and representations. Following Bruner (1996) and Chapman (2006) 'logical-scientific' knowing (which focuses on mathematical models and structures of the problem) was distinguished from 'narrative' knowing which focuses on the social context of the problem. Both forms of knowing were found to necessarily be invoked when engaging in word problem solving. Story-telling and representations were viewed as means by which children could communicate their thinking to self and others.

Storytelling was used as a pedagogic strategy to motivate learners and encourage sense making. This drew on literature relating to the role of storytelling in education in general (Egan 1989, 2002) and its application to mathematics learning (Schiro 2004 and Zazkis & Liljedahl 2009) where the use of storytelling by teachers is in focus. In addition storytelling was viewed as a cognitive strategy for children learning mathematics. This drew on the work of Bruner (1996) who saw storytelling as a vehicle of mind, used to make sense of the world; and on the work of Mason (2007) who described storytelling within mathematics learning as a fundamental human trait drawing on the powers of imaging and expressing.

Telling and re-telling of word problems had been used previously as means of gathering data on learners' problem solving strategies (Verschaffel 1994). In this study a narrative approach is adopted which expects students to engage in the social performances of telling and retelling word problem situations while simultaneously creating and elaborating symbolic models of their informal mathematical activity created when solving such problems. Problem solving is therefore seen as a process of mathematical modelling where the 'modelling activity might involve making drawings, diagrams or tables or it could involve developing informal notation or using conventional mathematical notations' (Cobb, Gravemeijer et al. 1997 p. 161).

In addition to examining learners' responses to problem solving (which is seen as a process of mathematical modelling), adopting a socio-cultural perspective, the word problem (traditionally presented as a written text given to learners through test-books and teachers) is viewed as a cultural artifact in itself into which the learner must be enculturated. As such the learners are expected to participate in the generation of word problem examples and to have some agency over both the form and content of word problem. In this way an interpretive stance, which draws on the traditions of Realistic Mathematics Education (RME) and the related 'emergent' perspective is adopted – to both the mathematical symbolisation (both formal and informal) in terms of representations created and then acted upon in the mathematical modelling process, as well as the construction and performance of the word problem (in written or oral form) to share with the classroom community. Cobb, Gravemeijer et al. (1997) explain that in an emergent approach ways of symbolising are not construed as means of bringing students into contact with established cultural meanings. Instead they are considered to support the emergence of mathematical meaning in the classroom. This has implications for the role of the teacher and instructional developer, as knowledge of the mathematical practices (such as additive relations word problems) institutionalised by society provides a direction to the process of emergence. However rather than bringing learners into contact with the conventional cultural meanings, a basic metaphor of building up towards participation in these practices underpins the teacher role which 'entails guiding both the development of individual students constructive activities and the evolution of classroom mathematical practices' (Cobb, Gravemeijer et al. 1997, p. 163).

A further rationale for adopting storytelling as a key pedagogic strategy and as a cognitive strategy for the intervention was the dominance of the oral language within the South African primary context (Hoadley 2012), and the dual need for ELLs to improve their English language proficiency and articulate and decode statements of comparison.

Representations

Representations were seen as entangled with storytelling as gestures, movement, actions, drawings and formal notations were involved as part of the narration of connected events. In this way story-telling and re-telling became a social act or performance. The representations were however viewed distinctly (separated from oral story telling), and figured from both a teaching perspective (how best to make use of representations through careful selection of reference examples making use of representations to express additive relations) and a learning perspective (representations, like story-telling, became a learning goal). Theoretical assumptions drawn from the mathematics education literature about the use of representations in the intervention included encouraging flexible movements between representations to facilitate sense-making. Modes of representation which distinguish various formats of representation (contrasting concrete, iconic, indexical, symbolic, and syntactical representations, as advocated by Ensor et al. 2009) were found to be useful when considering this flexible movement between representations. At the same time, secure use of particular representations was acknowledged to take time. The purpose and use of a representation was expected to change over time as they become increasingly reified to become objects upon which children could act. The representations foregrounded in the intervention were therefore carefully selected from a scan of the South African curriculum, and the literature on additive relations. Treffers' (2008) framework for using and interpreting representations which focused on teaching-learning trajectory which made use of increasingly structured representations was drawn on.

Reading of the literature led to the selection and sequencing of reference examples for additive relations stories, as well as the formal representations that were brought into focus during the three intervention cycles. Together – the stories and the representations – were used to define the local example space for additive relations word problems, into which the learners were to be enculturated.

The above domain-specific instructional theories codified the critical features of the mathematics: in relation to the object of learning, and how this object was to be approached. The framework for action codifies – from my perspective as the teacher of the intervention – the key features of my more general teacher roles which were not as intricately connected to the mathematics of the overarching learning objective. These are referred to as implementation features, as they were specific to the implementation of the third cycle intervention in the focal school. These features are not presented as an exhaustive list, and are constrained to those features of my teacher role to which I was consciously attending. I elaborate on these in detail in Chapter 3 focusing first on theoretical features relating to training behaviour, before reflecting on some more general features of my teacher role

Research design

In this section I briefly describe the methodology adopted for the study, offering a synopsis of the data gathering and data analysis processes.

Classroom-based design experiment

The study adopted a classroom based design experiment methodology (Brown 1992, Cobb, Confrey et al. 2003, Schoenfeld 2006) with teaching interventions undertaken over three iterative cycles, all of which were conducted in the same urban primary school in Cape Town South Africa. Each cycle comprised of 10 consecutive days' teaching with different Foundation Phase mathematics classes (of approximately 30 learners in a class).

In all three cycles, I was the lead teacher and worked alongside the normal classroom teacher. Two factors informed my decision to take on the lead teacher role, rather than work with the normal classroom teacher. First at the outset of the design experiment the detail of a theoretical framework in relation to additive relations was not yet established and as such I was not yet able to communicate this to a normal classroom teacher. Second the focal school had been identified as poor performing, and its teachers indicated that they required professional intervention to improve mathematics teaching and learning. It was therefore not expected that existing Foundation Phase teachers would be easily able to adopt the pedagogic approach to mathematics learning (a constructivist orientation, drawing on the concept of discernment from variation theory and adopting a 'narrative approach' to mathematics) as outlined in the theoretical framework. It was only when the theoretical approach and task design were securely defined, and if the intervention showed promise, that supporting a normal classroom teacher to adopt the approach would be appropriate.

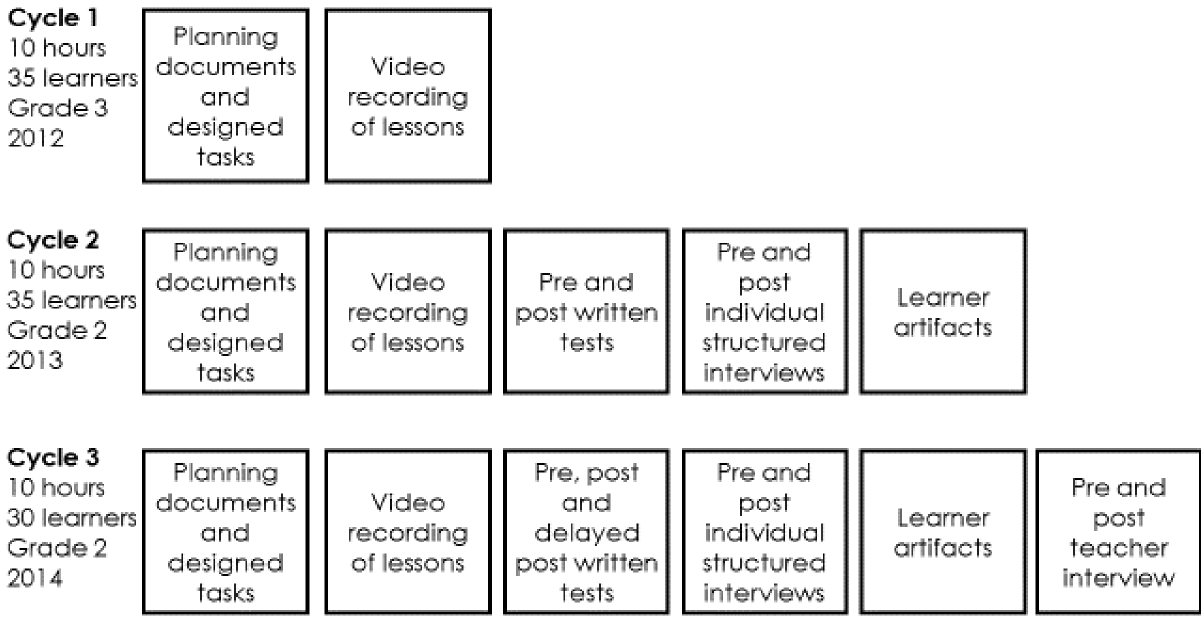
I refined the design of each iterative cycles based on mathematics education literature and empirical analysis of learner engagement in each intervention. This thesis answers the broad research question of the extent to which narrative can be used to support children to expand their personal potential example space of additive relation word problems to include compare type problems.

This study reports on a three-cycle design experiment paying most attention to its last research cycle. All learning cycles suggested promise in relation to learning gains, with statistically significant quantitative learning gains evident in both cycle 2 and cycle 3. In both Cycle 2 and Cycle 3, quantitative learning gains measured in shifts in attainment in written tests, were seen between pre- and post-tests, and maintained into the delayed post-test. I report in detail on the evidence obtained from the third intervention cycle which showed promise in relation to quantitative learning gains as measured in shifts in attainment in written tests, prior to and following the intervention. The third cycle intervention also showed promise when considering qualitative indicators of shifts in children's learning in relation to each of the learning goals, and I present examples of these from the examination of evidence of learning for three case study children. As part of this thesis I offer tasks designed to support children to make use of narrative in expanding understanding of additive relations, and corresponding coding frameworks for interpreting young children's representations and their spoken and written stories of additive relations. I discuss the limitations of this study and suggest possible further research in the light of its findings.

Data gathering

The data gathering for the overall design experiment sequence was as follows

Figure 6: Data gathering for the design experiment



Cycle 1 was experimental and exploratory and figured as a pilot for approaching the additive relations word problem topic and designing an intervention involving narrative. In Cycles 2 and 3, pre- and post-written tests were administered before and after the intervention. These tests included matched questions from pre- to post-test and most of the test items were identical from Cycle 2 to Cycle 3. In these latter cycles pre- and post-individual structured interviews were conducted with a selection of 12 learners from the intervention class in each cycle. The interviewed learners were selected for balance to include 4 learners from the lower attainment range, 4 learners from the middle attainment range, and 4 learners from the upper attainment range based on the written pre-test results. In Cycle 3 an additional delayed post-test was administered for the whole intervention class to explore retention of narratives beyond the end of the intervention.

While in Cycle 1 and Cycle 2 the intervention was approached using whole class teaching, in Cycle 3 the intervention included both whole class teaching, and small group teaching where learners were grouped according to their pre-test attainment, defined as support (blue), core (green) and extension (red) groups. In all three cycles the intervention lessons were video recorded, and the learner work (learning artifacts) completed during the intervention were collected.

Drawing on the Cycle 3 data, three case studies of individual learners were developed. These case studies include quantitative and qualitative analysis of their pre-, post- and delayed post written test responses, analysis of their learner artifacts from the intervention period, and their responses in the pre- and post interviews. These three case study learners were selected to include children drawn from each of the three attainment groups, where learning gains between pre and post written tests were in focus in the middle and lower attainment ranges, and absences of learning gains were in focus in the upper attainment range. The learners selected (pseudonyms used throughout) thus were:

- The child from the core group who made the most substantial shift in comparison to their peers (Retabile);
- The child from the lower attainment levels who made the most substantial shift in comparison to their peers (Mpho); and
- The child from the extension group who made the smallest shift in comparison to their peers (Gavril).

The three case studies are presented in detail in Annexure 1 (Mpho), Annexure 2 (Retabile), and Annexure 3 (Gavril). Extracts from the case studies were used to provide evidence of these particular children's experiences of the intervention. The children were not selected as representative either of the whole class or of their ability groups (shifts in attainment for which are described quantitatively). Rather Retabile and Mpho were purposively selected as cases where there was evidence of learning gains to allow for qualitative analysis of how such learning may have come about. Gavril was purposively selected as a case where there was minimal learning in the higher attainment ranges, as qualitative analysis of his learning was thought to be revealing of ceiling effects (in the intervention design and potentially in the assessment instruments and frameworks), and therefore of research interest.

Data analysis

The data is presented by focusing on the theoretical features and empirical data from the second and third interventions to answer the primary research question: 'To what extent do young children's example space of additive relations deepen to include compare type word problems?'. Data from cycle 2 is used as a benchmark to establish whether there was better or poorer learning; with the change in task design from this cycle to the last research cycle. This broad question is answered by responding to each of the following two subsidiary questions which break down the overall research question into component parts:

1. What was the teaching intervention that was undertaken for this design experiment:
   - How was the intervention designed?
   - How was the design refined over multiple research cycles?
   - How did this play out in the third cycle intervention in this particular local context?
2. What evidence of learning gains (in relation to the learning goals), if any, was seen as a result of the teaching intervention:
   - What evidence of learning to solve a range of additive relation word problems (LG 1) was seen during and following the intervention?
   - How did children make their thinking visible by using narrative in this intervention, particularly with regard to:
     - Flexibly using a range of representations to pose and explain word problems (LG 2); and
     - Telling stories to pose and explain word problems (LG 3).

The first set of subsidiary research questions relate to the teaching side of the intervention. The focus is on what was taught and how this content was approached from the perspective of the teacher. To answer 'How was the intervention designed?' a theoretical framework which codifies the theoretical features informing the design of the intervention is presented. The ways in which the conventional example space was defined is described in relation to constraints imposed on the focus of the intervention in terms of both mathematical and problem situation constraints. The design of the intervention is then described, with reference to the literature that informed it. In so doing attention is placed on the design features of the intervention in terms of the approach to additive relations word problems, the use of story telling as a pedagogic and cognitive strategy, and the use of representations to pose and explain additive relations word problems. This is referred to as the domain-specific instructional theories informing the design of the intervention. The way in which the design of the intervention was refined over multiple research cycles is then reflected upon. These refinements are commented on in relation changes made in the theoretical approach to the conventional example space of word problems. Attention is placed on the changes to the selection and sequencing of references examples for stories and representations in each iterative cycle. How the intervention played out in the third intervention cycle is provided as a summary description of the chronology of tasks from lesson one to lesson ten. This synopsis is drawn from a detailed description of the chronology of the lessons, which draws on the video transcripts. From the learning side examples of learning artifacts, as well as screen shots from the video recordings of the lessons are included. A detailed chronology of the third cycle intervention is presented in Annexure 4.

The second set of subsidiary questions focus attention on the learning side of the intervention. The focus is on what was learnt and evidence is drawn from learning gains evident in the pre- and post – assessments (written tests for the whole class, and individual interviews with the case study learners). The evidence of learning gains is considered in relation to the learning goals: Problem solving, and making thinking visible by using a range of representations and stories. In considering the learning goals several levels of analysis are offered. Firstly the learning gains in relation to each learning goal is reported for the whole class. In this case mean results, shifts in mean results and the significance of these shifts is in focus. Secondly learning gains are reported in relation to the each learning goal for the three ability groups within the class. Again shifts in the mean results are reported. Finally the learning gains of the particular case study learners are reported. While the case studies open with quantitative analysis of the pre and post-test assessment results, the focus shifts to qualitative analysis of their responses to tasks in the written tests and during pre and post interviews.

To answer the question 'what evidence of learning to solve a range of additive relations word problems is seen during and after the intervention?', quantitative analysis of pre- and post- and delayed post- written tests is presented for Cycle 3 and, where appropriate, this is compared to the Cycle 2 written tests.

To answer 'how did children make their thinking visible by flexibly using a range of representations to pose and explain word problems' the representations used by learners in the pre-, post- and delayed post- tests is analysed using a coding framework for interpreting young learners representations of additive relations. This coding framework is expounded upon in detail in Chapter 4. 'Typical' and 'telling' cases of learners' representations are selected to illustrate the coding framework distinctions (Mitchell 1984). The representations constructed by the three case study learners provide a further indication of the way in which particular learners were using the representations to pose and explain word problems.

To answer 'how did children make their thinking visible by telling stories to pose and explain word problems', evidence is presented by contrasting the way in which the three case study learners made use of stories during the intervention as well as in their pre and post structured interviews. Telling and typical examples of learners telling stories to pose and explain word problems are selected from the description of the learning intervention to support this.

Outline of chapters in this study

Chapter 1 provides a general overview of the scope of the thesis, makes clear the research aims and delineate the research questions. The way in which narrative is used in this study is defined, as this was the main pedagogic strategy adopted for the content focus on additive relations word problems. An overview of the theoretical framework adopted which informed the design of the intervention is provided and this is followed by a synopsis of the research design.

Motivation for the focus of the study is provided in Chapter 2. The identification of a practical problem in the focal school, which was found to have broader relevance in the South African primary school context lead to the specification of the research questions that guided the research methodology instituted in the context of the design experiment. The problem identified in the focal school, gave rise to a need for an engineered and experimental approach to the teaching of additive relations word problems to be trialled and implemented.

In Chapter 3 I discuss the literature bases that informed the design of the intervention from a mathematical (*what* was to be taught), mathematical pedagogy (*how* the mathematics teaching was to be approached) and general pedagogy perspective (*how* the teaching was to be approached). I provide a theoretical framework which codifies the critical theoretical features of the third cycle intervention. The critical theoretical features of each intervention evolved as a dialectic over the course of the design experiment drawing on the literature as well as reflections on the empirical data emerging from the implementation of each intervention cycle within this local context. This chapter answers the question: 'How was the third cycle intervention designed?'

The research design, which was undertaken as a classroom based design experiment, is discussed in Chapter 4. This outlines the features of a design experiment methodology and how this design experiment was undertaken. Attention is given to the sources of data as well as how they were analysed. The research design is presented after the theoretical framework as theoretical constructs set up in the framework are draw upon when explaining and justifying the research design. This includes a description of the intervention design, which answers this question: 'How was the intervention design refined over multiple research cycles?' This refers to the sequencing and selection of tasks, and how these evolved as a dialectic drawing on the literature and the experiences of the intervention cycles.

Chapter 5 responds to, 'How did the third cycle intervention play out in this local context?' This question is answered by reporting the intervention with reference to the theoretical features which informed its design. A detailed descriptive account of the last research third cycle is included in Annexure 4. What is meant by a narrative approach to additive relations word problems is exemplified by drawing on excerpts from the lesson transcripts, together with evidence from learner work during the intervention. The extent to which the hypothetical learning trajectory matched, or deviated from, the actual learning trajectory is analysed.

In Chapter 6 the focus of attention shifts from the implementation of the design experiment intervention to the findings relating to its impact on learning. In so doing, the question 'What evidence of learning gains (in relation to the learning goals), if any, is seen as a result of the teaching intervention?' is answered. This question is addressed in terms of the three interrelated learning goals (problem solving, representations and storytelling) and at differing levels of detail (whole class as well as smaller attainment groups).

In Chapter 7 data from both the teaching and the learning side of the intervention are brought together and examined in relation to three case study learners. In an attempt to account for the learning, examples of particular learning gains for these children are examined and interpreted in terms of the evidence of learner activity during the lesson intervention.

The thesis concludes with Chapter 8 where the implications for the above are discussed. The ontological innovations delineating explanatory constructs and new categories relating to the teaching and learning of additive relation word problems are discussed.

CHAPTER 2: Motivation, context and scope of the research

This chapter outlines the problem of practice that was identified in an urban township school. How this local problem was situated in the broader South African context and led to my decision to use a narrative approach in the mathematical context of additive relations is then explained. The identification of a practical problem in the focal school, which was found to have broader relevance in the South African primary school context led to the specification of the research questions that guided the research methodology instituted in the context of the design experiment.

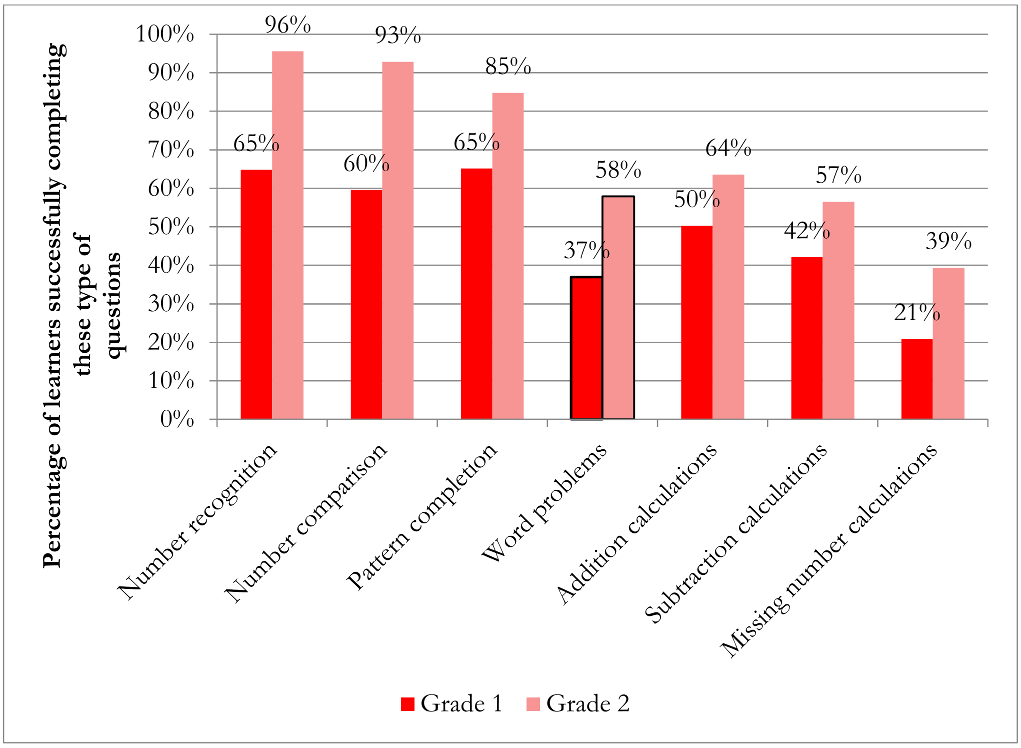
This study was motivated by a practical problem that was first identified within a particular local context. Poor attainment in word problems (and compare type additive relations problems in particular) was identified in the particular urban school servicing poor learners with which I had an established working relationship, and which is the context for this thesis.

The problem of practice

The 'local' learning problem: Grade 2's problems on additive relations

Empirical data from this focal school revealed learners' difficulties with conceptual understanding of word problems and compare type additive relation problems in particular. In June 2012 I conducted baseline number sense assessments with Foundation Phase learners in this school. These assessments, developed by Brombacher and Associates (Brombacher and Associates 2015), are based on the Early Grade Mathematics Assessment (EGMA) developed by RTI International. The tests were designed to assess basic number concepts in the early years and were administered using individual structured interviews. Analysis of Grade 1 and 2 baseline assessments with a sample of 95 learners from the approximately 240 learners in Grade 1 and 2 in this school, showed very poor performance in word problems (see Figure 7):

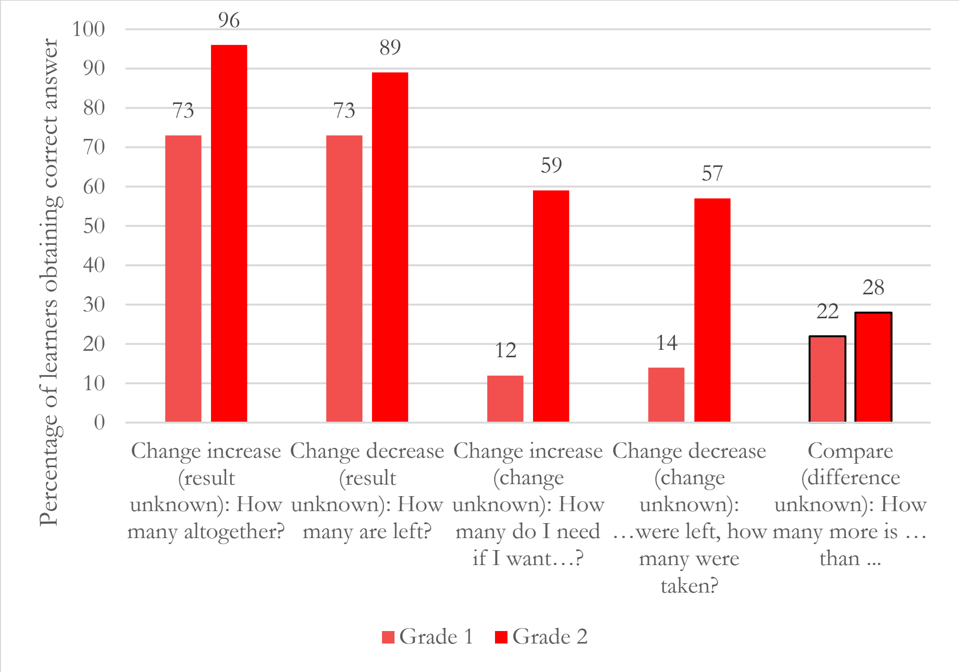
Figure 7: Learner attainment in number sense baseline assessment (Grade 1 and 2, 2012, n=95)



Word problems, subtraction calculations and missing number calculations (where there were either missing addends or missing subtrahends in a bare calculation) were the question types where Grade 2 learners were least successful in this assessment. Further analysis of learner performance in the word problem section, revealed that the compare type word problem was most poorly answered in relation to change increase and change decrease word problem types. The collection word problem type was not included in this assessment.

Few Grade 1 learners were able to solve 'change' word problems, however by Grade 2 the majority of learners could solve these. However, the 'compare' word problem was most poorly answered in both grades. Only 28% of Grade 2 learners were able to correctly respond to a question which required understanding of comparison: 'how many more is … than … ?' as depicted in Figure 8.

Figure 8: Learner attainment in number sense word problems (Grade 1 and 2, 2012, n=95)



Poor conceptual understanding of word problems in general and compare word problems in particular had been raised by Foundation Phase teachers in this school as an area of general concern. As such the choice to focus on additive relation word problems, with specific attention paid to compare word problems was informed by teacher identification of problem areas, and was supported by empirical data of learner assessments in this local school.

While there is international research on how to improve the learning of additive relation word problems, there is little South African research on this area specifically, although there are broader findings, reported in more detail below, pointing to problems with sense making and fluency, inefficient calculation strategies, interpretation of word problems, and predominance of concrete counting in ones representations. As such, supporting South African learners in a school servicing poor learners to interpret and solve additive relation problems was considered to be an important (and a difficult to achieve) learning goal which warranted attention and innovative classroom-based interventions.

The mathematics topic (additive relations word problems with a particular focus on compare word problems) was clearly identified empirically. What remained unclear was how this topic should be approached pedagogically. As word problems are expressed as narratives, I felt that an approach that made use of narratives as a means to support sense making might be appropriate. I had previously undertaken classroom-based design experiments on the use of narrative in mathematics learning in the early grades of primary schooling (Roberts and Stylianides 2013). The previous work had been undertaken in a state primary school in Cambridge, England, and in a private school in Knysna, South Africa and had shown promising results. These previous interventions had been located in well-resourced schools where English was the home language of the majority of learners.

I wished to examine the efficacy of these pedagogical approaches to mathematics learning in a disadvantaged context, where children came from poor home environments and where English was the home language of only a minority of learners. An engineering approach to research, which was concerned with practical impact on an identified problem, was therefore conceptualised. It was within this context that this classroom-based design experiment was undertaken.

Appropriateness of the focus on additive relations word problems in relation to the curriculum

Before proceeding with the design experiment on additive relations word problems I considered the contours of this topic focus in relation to the South African curriculum for Foundation Phase mathematics.

In the South African mathematics curriculum 'Numbers, Operations and Relationships' comprise the majority of the Foundation Phase curriculum (65% in Grade 1, 60% in Grade 2, and 55% in the Grade 3) (DBE 2011 p. 10). In this phase learners are expected to 'build an understanding of the basic operations of addition, subtraction, multiplication and division' (DBE p. 9) . Adding and subtracting are introduced before doubling and halving (which are the early introduction to multiplication and division) as additive relations are foundational to repeated addition and subtraction, which form the conceptual introduction to concepts of equal sharing and grouping for both multiplication and division. Additive relations are therefore a central concept to the Foundation Phase mathematics curriculum, and this topic focus was thus considered appropriate for this study.

The South African Curriculum and Assessment Policy Statement (CAPS) also identified solving problems as a key specific content focus for the 'Number Operations and Relationships' content area. The importance of solving problems as means of learners communicating mathematically is highlighted by stating: 'Solving problems in context enables learners to communicate their own thinking orally and in writing through drawings and symbols (DBE 2011, p. 9). The selection of the topic focus on additive relations word problems was therefore considered appropriate.

Situating the 'local' problem within the broader South African education landscape

The problem identified in this local context was then researched and found to be documented as an area of systemic concern in the South African landscape. As such literature on primary schooling, mathematics learning and the language contexts in which mathematics learning takes place in South African schools was of relevance.

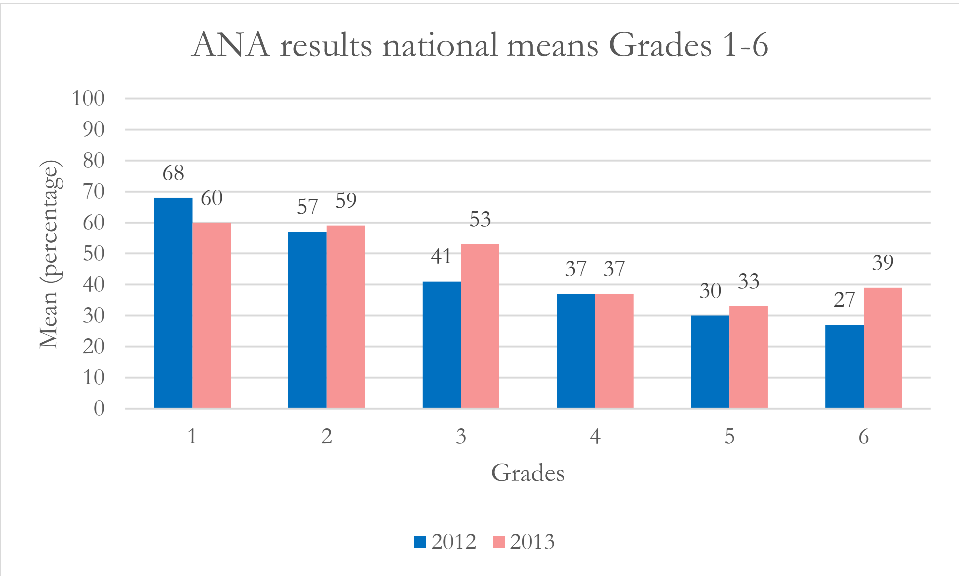
The section describes the South African primary schooling and Foundation Phase mathematics context. It opens with a brief discussion on system-wide poor attainment in mathematics, and then sketches the documented characteristics of learning and teaching in South African primary schools. Evidence of lack of sense making in mathematics is presented which is seen to be compounded by four factors: the South African language context; the over reliance on counting in ones; lack of fluencies required for number sense and disruptive learner behaviour.

Systemically poor attainment in mathematics

South Africa has well documented problems in its education system, which are particularly acute in the teaching and learning of mathematics, see for example (Taylor and Vinjevold 1999; Howie 2004; Fleisch 2008). Increasingly these challenges - seen most vividly at the endpoint of the system, with high drop-out and failure rates in national Grade 12 mathematics assessments - are thought to emanate from the primary school level, and with recognition that the early stages of learning mathematics impact on what is possible to learn in higher grades. Yet, a relatively recent review of South African research in mathematics education, revealed a paucity or gap with regard to South African mathematics education research focused at the primary school level (Venkat, Adler et al. 2009). While this review highlights limited mathematics education research overall at primary school level, this is compounded by further limitations in research that focuses at the Foundation Phase (Grades R-3).

With the introduction of Annual National Assessments (ANAs) in Grades 1 to 6 in 2010 more standardised and comparative assessment information is now available in South Africa on what primary learners are unable to do in mathematics assessments (DBE 2013). We know that there is significant under performance in mathematics in the system as a whole with a significant drop off in results from Grade 1 to Grade 6 level (DBE 2013).

Figure 9: South African ANA mean results Grades 1-6 for 2012 and 2013



Diagnostic reports are developed annually and areas of poor attainment across the system are reported. The Foundation Phase ANA diagnostic reports for 2012, 2013 and 2014 all identify word problems as a recurrent area of weakness (DBE 2012, DBE 2013, DBE 2015).

Challenging primary school contexts

This low achievement in mathematics takes place within a context of primary schooling which may be labelled challenging (in comparison to prior work undertaken in England and South African private schools). The majority of South African primary schools operate with limited resources, servicing communities living in poverty, where learners do not have English as their main language and where the school culture is frequently not supportive of learning (Carnoy, Chisholm et al. 2012).

This broad context of primary schooling in South Africa is sketched in a recent review of classroom-based studies on South African primary schools. Hoadley (2012) provides the following list of dominant descriptive features, organised by discourse, knowledge and time of the South African primary school context:

In terms of discourse:

- Oral discourse dominates – limited opportunities to read and write
- Classroom interaction patterns privilege the collective (chorusing)
- Limited feedback or evaluation of student responses
- Most learners learn in an additional language
- Learning is communalised rather than individualised

In terms of knowledge:

- Low levels of cognitive demand;
- Little use of textbooks, books or strong texts
- Concrete meanings dominate over abstract meanings
- Lack of focus on written text, reading and writing
- Focus on decoding text and neglect of reading extended text

In terms of time:

- Slow pacing
- The erosion of instructional time

(Hoadley 2012)

Together these dominant descriptive features provide a picture of what is known from classroom-based studies, sketching the context of a 'typical' South African primary school. Learning in an additional language in classroom contexts where oral language dominates, where there is lack of focus on written text, reading and writing has been referred to by other researchers as working with learners with 'low literacy levels' (Graven and Heyd-Metzuyanim 2014). This factor, together with a tendency to view meaning in relation to the concrete and not the 'abstract', and where learning is communalised rather than individualised are all particularly relevant to the context of learning to solve additive relations word problems. These characteristics are considered to be the challenges defining the South African primary school context and pertain to both teaching and learning.

Difficulties with inappropriate learner behaviour at Foundation Phase

In addition (perhaps unsurprisingly) within this challenging context where there is an absence of number sense, 'misbehaving learners and disciplinary problems' are identified as being 'a disproportionate and intractable part of every teacher's experience of teaching' (Marias and Meier

2010, p. 41). This is echoed by Moodley (forthcoming) who, writing about Foundation Phase teaching, states that 'in a South African context, learner behavioural difficulties continue to be a major problem in schools' (p. 94). Wasserman (2015) reporting on a mathematics intervention in a township primary school context, reported that the general learning culture and lack of classroom discipline were challenging, and this resulted in her opting to work with small groups rather than with the whole class. While this characteristic does not figure centrally within the literature on mathematics education, the issue is reported to be widespread in the broader primary education literature.

The South African language context

Hoadley (2012) documents that in South African primary schools most learners learn in an additional language and views this factor as compounding the lack of sense making in mathematics, which is particularly evident in poor attainment in word problems. As such some detail on this multi-lingual classroom context and its implications for mathematics teaching and learning is appropriate.

In South Africa, from Grade 4 upwards the majority of learners learn mathematics in English which is not their main language. Official policy since 1997 (Department of Education (DOE) 1997) dictates that in the first four years of primary schooling the language of teaching and learning (LoLT) for all learning areas (including mathematics) is to be in the home language of the learners. In practice the multiplicity of home languages in urban settings, combined with a 'press' for English due to its being seen as the language associated with socio-economic status, results in English being the language of teaching in many schools (Dalvit, Murray et al. 2009). The complexity of learning mathematics in English when this is not the main of language of the learners, impacts on all levels of the system (and was a contextual factor in the school in which this empirical research was set).

This contextual reality makes it difficult to separate the origin of poor attainment in word problems. The extent to which these difficulties arise either from 'mathematics' or from 'language' or from 'both language and mathematics' is in question. Schollar (2008) observed that the generally poor English reading levels brought into question whether learners' difficulties with mathematics problems were language or mathematical difficulties with conceptual difficulties with language and mathematics most acutely seen when considering learner performance in 'word problems' as opposed to 'bare calculations' in mathematics assessments. In support of the concern regarding the connection between mathematics and language learning, Ensor, Dunne et al. (2002) note that a mathematics assessment (even as high as Grade 7 level): 'cannot be seen in any simple way as a test of mathematical competence … Rather, it is a means of assessing learner's ability to answer mathematics questions, expressed in particular ways in English' (Ensor, Dunne et al. 2001, p. 30).

A more recent small empirical study (Sepeng 2014) involving 109 Grade 9 learners in six 'township schools in Grahamstown' contrasted learner's responses to word problems posed in the language of teaching and learning (English) with the same word problems posed in their home language (isiXhosa). This study found that language difficulties were evident in both languages: 'computation errors ... seem to stem from the inability to use language(s) (home *and/or* language of learning and teaching) effectively in order to solve problems in realistic settings (Sepeng 2014, p. 22). Sepeng suggests that learners should be encouraged to 'use their everyday knowledge and personal life experiences when making sense of word problems' facilitating 'a bond between learners' (informal) spoken language and formal (classroom) written mathematical language' (Sepeng 2014, p. 22). Sepeng (2014) views the home language and LoLT as having complementary roles, in supporting meaning making when solving word problems.

This notion of complementary roles for the home language and LoLT, aligns with the notion of the learners' main language being viewed as a resource rather than a problem (Adler 2002, p. 3). Setati (2002) points out that learning mathematics in multilingual classrooms is not simply a matter of managing the interaction between the LoLT and the main language of a learner. She argues that focusing on the meaning of individual words or the addition of new vocabulary is not adequate, as part of learning mathematics is acquiring control over the mathematics register (p. 9). As a result communicating mathematically in multilingual classrooms means managing not only the interaction between learners' main language and the LoLT, but also the interactions between ordinary English and mathematical English, between formal and informal mathematics language, and between procedural and conceptual discourses (Setati 2002).

It is important to bear in mind that these language issues are not unique to the South African multilingual classroom environment, as there is international documentation on the difficulties with flexibly moving between the semiotic systems of natural language and mathematical numbers for all learners, but developing symbol sense and transitioning between the symbolic and natural language is even more of a challenge for English Language Leaners (ELLs) (Askew and Brown 2003). In some of these international contexts ELLs are a minority of students in the class, whereas in South Africa, these learners are the majority of learners (and this was the case for the school in which the study was undertaken).

Within the constructivist paradigm for learning, language is an important consideration for learning mathematics, where meanings are negotiated as children are actively involved in constructing their own understanding:

> It is the nature of mathematics that concise communication is achieved through the use of symbols and formalised 'expressions' to convey information and to pose questions. However the route to understanding such mathematics is by successively broadening children's experiences of the language and meanings associated with formal mathematical expressions (Anghileri 2005, p. 90).

Rather than seeing word problems as the application of calculations recently learnt in a context, there is recognition that starting with a meaningful context (expressed in language) can support meaning making, and generate a need for a calculation. In this way the distinction between word problems and bare calculations, at least for young children, come to be blurred. For a bare calculation to be meaningful a context for a particular operation is sketched using language, and then symbolised using numbers and symbols. This also occurs in the reverse order where mathematical symbolisations must be interpreted as having a meaning within a context (commonly communicated using language). The 'words and symbols need to be *read* and *interpreted* with words or phrases being used to convey *meanings* of arithmetic expressions' (Anghileri 2005, p. 84). Language and mathematics are thus integrated conceptual structures that have a mutual inter-dependence.

Lack of number sense and fluency at Foundation Phase level

While above I sketched the general South African primary school and multilingual language context I now turn to mathematics learning in particular.

There is evidence of a lack of basic number sense, including a lack of mental strategies and fluency in young learners: 'the majority of South African Grade 3 learners have not developed foundational number sense before entering the intermediate phase' (Graven, Venkat et al. 2013, p. 134). Graven et al. (2013) assert that 'number sense is critical in the development of mathematical understanding throughout schooling and in everyday life' and cite Greeno (1991) when defining number sense as 'a set of capabilities for constructing and reasoning within mental models and includes flexible numerical computation, numerical estimation and quantitative judgement and inference' (Graven, Venkat et al. 2013, p. 132). This accords with the definition of number sense provided by Anghileri where flexibility in solving number problems as a result of an ability to 'make generalizations about the patterns and processes that they have met, and to link new information to their existing knowledge' is what characterises children with a 'feel' for numbers or number sense (Anghileri 2000, p. 1). This notion of number sense is present in the South African curriculum which links mental strategies with developing mental models and mental strategies for computation. However, number sense is described as frequently absent from the evidence gleaned from empirical work with learners:

> For example we have seen learners well into the intermediate phase across our work draw two groups of tally lines with 2 and 98 in each of the groups to answer the question 2 + 98. Not knowing a basic bond to 10 like 2 + 8 = 10, or that 2 + 98 is the same as 98 + 2 so one can count on 2 from 98, renders this simple computation tedious and highly error prone (Graven, Venkat et al. 2013, p. 132).

So a tendency to work procedurally, rather than to reflect on the relationships between numbers and build on known facts or generalised properties of number relations, is seen to contribute to the poor mathematics attainment. From the above description – and examples of – lack of number sense in general, one can infer that fluent and secure knowledge of basic number facts and generalised properties is not a common feature of the typical South African early grade classroom. There are two related implications of this assertion which are relevant to this study. One aspect of this empirical finding is the absence of connections between facts, and a general lack of fluency with known facts and generalised properties. Another aspect relates to over reliance on unit counting strategies seen in making use of ungrouped tally lines alluded to in the example above. Each aspect is now discussed in turn. Once again there is evidence of this from both teaching and learning perspectives in the South African context.

Lack of fluencies and negative mathematical identities

There is evidence of teachers not providing opportunities for learners to make use of known facts and to build on properties already established. By way of example, when examining an empirical episode from a Grade 2 teacher's classroom, Venkat and Naidoo (2012) identify the lack of coherence between the teacher talk/tasks, representations and examples. As such they identify that the teacher seems to treat examples 'ahistorically', in that 'each time a new example enters the scene, the past appears to vanish' (Venkat and Naidoo 2012, p. 32). These authors identify the absence of an identified base of known and established number facts and properties from which teaching proceeds.

There is also evidence of learners having negative mathematical identities within 'a passive, overly teacher dependent culture of learning mathematics' (Graven 2012, p. 60). Graven (2012) when working with young learners in South Africa, note that for learners 'current motivations for Mathematical participation are seemingly dominated by compliance with teacher instructions and getting answers right' (p. 60).

Together the above empirical findings from South African Foundation phase classrooms point to a passive teacher dependent culture, which is compounded by teachers not establishing and building on a base of known and established number facts and properties to support learning progression, which is over and above the concerns related to mathematical meaning making in the challenging schooling context. This lack of sense making relates to a lack of a 'productive disposition' towards mathematics which is defined by Kilpatrick, Swafford and Findell (2001) as: 'the tendency to see sense in mathematics, to perceive it as both useful and worthwhile, and to believe that steady effort in mathematics pays off and to see oneself as an effective learner and doer of mathematics' (p. 131). Considering the above, in this design experiment, the design of the intervention aimed to treat examples historically and to provide opportunities for learners to become fluent when working with basic number facts, and to identify, use and build on generalised properties of relationships between numbers. The details of this approach are described in the theoretical framework.

Over reliance on unit counting strategies for calculations

Concerns relating to sense-making in general, lack of basic number fluencies are compounded by documented concerns about calculation strategies. The tendency for South African Intermediate Phase (Grades 4 -6) learners to use unit counting strategies, even for very large numbers, was noted in a diagnostic report on the Annual National Assessment: 'Learners use drawings e.g. sticks, repeated addition and repeated subtraction for multiplication and division respectively even when working with large numbers (DBE 2012, p. 9). These concerns, which are now reported at the national level, have been identified in several smaller empirical research studies.

The heavy reliance in counting in ones strategies has been highlighted by Hoadley (2012) drawing on several earlier studies (Hoadley 2007, Schollar 2008, Davis 1996). In the Foundation Phase, Ensor et al. (2009) have identified that learners remain reliant on counting-based strategies for calculations, and do not shift to more 'abstract' calculation strategies. This was confirmed by Schollar (2008) who reported the prevalence of concrete counting strategies well into the Intermediate Phase. This concern is taken up by Weitz and Venkat (2013) who note that the significant proportions of South African learners that have difficulties with moving from highly concrete strategies relying on unit counting to more abstract strategies.

Scope of the Cycle 3 intervention

Historically design experiments have tended to be of long duration. For example a design experiment undertaken by Cobb et al. focused on Grade 1 learners in the United States of America, where 106 lessons were included in the design experiment (Cobb, Gravemeijer et al. 1997). The methodology has been criticised as being so data rich that it is lengthy and difficult to write up (Schoenfeld 2006). As a result of this criticism, more recent design experiments have attempted to design and reflect on lesson sequences which are of short duration. By way of contrast a design experiment by Stylianides and Stylianides (2014a) focused on a single lesson.

The cycle 3 intervention for this design experiment was deliberately designed to be of short duration (10 consecutive lessons). As a result several constraints were imposed which limited the scope of what was taught. Specifying these constraints helps to make clear the boundaries of the example space, and how it connects with other topics in mathematics education.

I took the general example space of mathematics word problems as my starting point. By identifying the dimensions of possible variation within this general word problem space, I am explicit in describing how each dimension of possible variation was constrained. These constraints defined the boundaries of the specific example space that was in focus: additive relations, one-step word problems using whole numbers less than twenty and making use of discrete object situations. Having identified the boundaries of the example space, and recognising the larger space of which this is a part, I discuss the dimensions of possible variation within the specific example space. In so doing I define the domain specific instructional design features underpinning the approach to additive relations word problems, the use of storytelling as a cognitive and pedagogic tool, and how this was supported by the use of representations to interpret calculation strategies.

This study was constrained to whole numbers (ignoring rational numbers) and the number range was limited from 0 to 20. This decision was based on prior intervention cycles where fluency with addition and subtraction calculation with answers up to 20 was found to be absent. This was supported by empirical research in other South African contexts where Grade 4 level, learners were found to be at least two years behind the curriculum expectations (Spaull 2013). As such the intervention focused on the number range specified for Grade 1 in the South African curriculum, even though the learners were in Grade 2.

Closely related to the constraints on the types and range of numbers, is distinguishing word problem contexts where the numbers involved are discrete, from those where the numbers are continuous. This is mathematically significant because the continuous measurement context expands the constraint on whole numbers to encompass rational numbers. Schmittau (2004) argues for young children to be introduced to numbers using measurement contexts rather than foregrounding discrete objects and she argues that failure to do this leads to over reliance on unit counting strategies. However, given the multi-lingual language context of the classes in this design experiment, with substantially greater English language requirements for continuous measurement contexts than for discrete objects (e.g. the continuous measurement comparative language vocabulary of heavier- lighter, fuller-emptier, longer-shorter, taller-wider, hotter-colder, more expensive-less expensive, cheaper) and the need for units of measurement vocabulary, I decided to restrict attention to discrete objects. This required the more limited comparative language requirements of 'more than' and 'less than', and 'how many more' and 'how many less'.

Another dimension of possible variation in the word problem example space is the extent of the action defined by the number of steps involved in obtaining a solution to a word problem. A word problem like this, can be solved in one step:

> 'Mother was cooking chicken feet for dinner. She counted 8 chicken feet in the pot. The mobile phone rang. She answered: 'Yes. I am fine. How are you? Bla bla bla' Oh no! What is that smell? Some feet got burnt. There are only 2 chicken feet to eat. We are hungry. Just pap and burnt gravy tonight. How many feet were burnt?'[5]

This problem can be solved in one step: '8 – 2 = … 'or '8 = 2 + …'. As only one step is required, this is referred to as a 1st degree word problem.

In contrast a word problem like this requires more than one step to solve:

> 'I have lots of money. I am rich. 500 moneys. I put it next to my bed. I go to sleep. Someone came in the night and stole my money. I woke up. It is gone! They stole 300 moneys. Oh no. I must now find my moneys. Who has taken my moneys? I find the thief. I beat him. I get only a 100 from him. Now I can buy that bicycle for 250. How much change do I get?'[6]

This problem can be solved following these steps:

Step 1: 500 – 300 = 200 (the money was stolen)
Step 2: 200 + 100 = 300 (the 100 was recovered)
Step 3: 300 – 250 = … (the change for the bicycle)

This is a three step word problem and is referred to as 3rd degree problem. Given that this intervention was designed for Foundation Phase learners the presented stories (narrated by the

[5] This was a learner generated example of an additive relation word problem, narrated in 2012 by a Grade 3 girl, in the first intervention cycle.
[6] This was a learner generated example of an additive relation story, narrated in 2012 by a Grade 3 boy, in the first intervention cycle.

teacher) were constrained to 1st degree problems. However if learners generated problems of higher degrees these were included in the class discussion.

Another dimension of possible variation in word problem situations relates to the role, value, and identified difficulties that children have with word problems in the mathematics curriculum. Word problems, as they commonly occur in mathematics classrooms, have been criticised as encouraging children to suspend sense-making (Schoenfeld 1991). Word problems tend to be stereotypical and are criticised as representing a game within the cultural norms of mathematics classrooms, which bear little or no resemblance to actual real world problem solving. This difficulty has been the subject of research using unsolvable problems such as 'There are 125 sheep and 5 dogs in a flock. How old is the shepherd?' (referred to by Reusser 1988, cited in Greer, 1997). Greer (1997) points to lack of sense making as learner do not consider their real-world situations, when answering word problems. So considering 'realism of the problem situation' as a dimension of possible variation, the range of permissible change varies from one extreme of stereotyped (at times unrealistic) problem situations, to the another of meaningfully realistic problem situation. While 'realism of the problem situation' has been discussed in the literature as a variable of consideration within word problems, this literature has usually been applied to older children that the Foundation Phase range. Thus, this dimension of possible variation was not emphasised in this study.

Several researchers recognise that there is value in structuring numbers to support a transition from counting to calculating, and that such structuring takes place using concrete materials, drawings of the concrete materials or objects and then as imagined/visualised objects of mind (Anghileri 2000; Treffers 2008; Wright, Stanger et al. 2012; Askew 2012). These authors refer to several possible ways in which numbers can be structured which denote a shift from counting in ones, to more efficient calculation strategies. These include structuring using 'doubles' and 'near doubles', using 'groups of 5' and 'groups of 10' aimed towards rapid recall of the doubles, bonds of five, bonds of ten, partitions involving 5s or 10s. Treffers (2008) refers to "smart' structuring using fives and tens, dividing into doubles and almost-doubles' (p. 46). The intervention design recognised this need for flexible and smart structuring of numbers to support a shift from counting in ones to calculating. However given the short duration of the intervention flexible partitioning using groups of five and ten was stressed and doubles and near doubles were not in focus. This was a pragmatic constraint on the design experiment intervention, as it was anticipated that most learners would be using a unit counting calculation strategy, and it was expected that neither doubles nor group by five nor group by tens partitions would be available as known facts to these children. This constraint was not intended to suggest that doubles and near doubles ought not to be available and rapidly recalled. It was simply a pragmatic decision for the groups by five and ten to be the focus of the short 10-day intervention.

Synopsis of the research context

So we are aware that the South African mathematics education system is performing poorly measured against annual national assessments. Additive relations is a central topic in the

Foundation Phase mathematics curriculum and there is evidence of systemically poor attainment in word problems, with explicit mention of difficulties with compare type problems. We know that many South African primary schools are challenging environments with low educational achievement. The challenging characteristics of the majority of South African primary school contexts include that the majority of learners are ELLs in classroom contexts where oral language dominates, where there is lack of focus on written text, reading and writing and a tendency to view meaning in relation to the concrete and not the 'abstract', and where learning is communalised rather than individualised. There seems to be very little sense making in mathematics learning where there are particular difficulties evident in poor attainment in word problems, lack of number sense and the over reliance on unit counting strategies. This context is compounded with difficulties with disruptive learner behaviour and the language context of these classrooms.

As a result a teaching approach that aimed to harness the oral discourse dominating primary mathematics classrooms, and to support this with a range of other representations (gesture, drawing and symbolic notation) to formalise the mathematical communication was considered appropriate. The need to develop fluencies required for flexible number sense and to facilitate shifts away from inefficient counting in ones strategies to more efficient symbolic calculation strategies was recognised. These challenges, documented in the South African landscape, refer to problems in both the teaching and the learning side of early grade mathematics. This context, together with promising findings relating to the use of narrative in other research contexts (Roberts and Stylianides 2013), suggested that a highly interventionist research design – a design experiment – where I took on the role of both teacher and researcher was appropriate.

The design experiment conducted by Cobb, Gravemeijer (1997) offers a synopsis of its research context which provides a useful juxtaposition to the research context of this design experiment. The Cobb Gravemeijer et al. (1997) design experiment took place over one year and involving 106 lessons with Mrs Smith's first grade class it took place in a private school in Nashville. There were 11 girls and 7 boys in the class. The majority of learners were from middle to upper income socio-economic families. Most parents held professional occupations. There were no minority children in the classroom, although a percentage of minority children attended the school. The students in the class were representative of the school's general population. Although not a Christian school, morals and values were part of the responsibility of schooling and children regularly participated in spiritual activities.

This design experiment took place over three cycles, each comprising of 10 consecutive days of in a full service public school in Cape Town where the language of teaching and learning was English. The focal school was a relatively well resourced school in the South African urban township context, and had access to an additional learning support staff member as a result of its full service status. It was classified as a poverty quintile 3[7] school and was a no fee school. It was supported

[7] South African schools have been classified by poverty quintile to track transformation targets and as a result of the pro-poor funding policy which aims to redress inequalities of the past. Poverty quintile 1 represents the poorest

by a philanthropic trust and its facilities included a school library and school hall. The teachers in this school had assistant teachers to support them in their classrooms. The teaching assistants included parents and caring adults from the community who received a small stipend for their role. The school day commenced with morning prayers and most children and teachers were Christian.

The third intervention cycle took place with Vanessa's mixed ability[8] Grade 2 class where mathematics lessons took place first thing in the morning. There were 11 girls and 19 boys in the class. All the learners were from poor socio economic families. Most parents were unemployed or had low-paying domestic or manual labour jobs. All the learners in the school were black children. The majority of children were South African with Afrikaans or isiXhosa as a home language. There were five children whose families were recent immigrants from central or north Africa who spoke a range of home languages including Portuguese and French. Only two learners had English as a home language. The students in the class were representative of the school's general population, although within the four Grade 2 classes, this class was considered to be particularly challenging. Vanessa (pseudonym for the normal classroom teacher of the class) with five years experience of teaching in this school described her Cycle 3 class as follows:

> This 2D class is, I think, been the most challenging class I've had yet. I'm finding that the level that the children are at is very low. [They are] definitely not at a grade 2 level, I would say they are maybe at middle of the year in grade 1 level. A lot of them can't read. They don't know the letters of the alphabet so when it comes to a story sum they can't do it because they don't know. They can't read it, at ALL. It's the largest number of age repeats I've had.[9] I think it's 14 out of the 30. And then also …the number of children that need to be assessed[10]…in comparison to what I've had in previous years…maybe two out of the whole class, whereas in this class there's already been…the three I think have already been assessed and at least, I think, 5 that need to be assessed.
> (Pre-interview with normal teacher Vanessa)

The class did have a higher proportion of learners who had repeated previous grades, and learners who were assessed as learners with special educational needs (LSEN) by the educational psychologist, compared to the other Grade 2 classes in this school.

schools in rural areas; whereas poverty quintile 5 represents affluent schools in suburban areas where parents pay school fees. This school – despite being in a very poor socio economic community - was classified as quintile 3 as it had facilities such as a library and school hall.

[8] A mixed ability model was in place for Grade 2 class selection at the time of the cycle 2 and 3 interventions. At the time of the cycle 1 intervention a streamed ability model was in place, and the intervention was conducted with top Grade 3 class.

[9] South African policy allows for a learner to repeat only one grade per phase. This is done when a child is not meeting the grade level curriculum expectations. A child who has already repeated a year in the Phase is referred to by teachers in this school (in collegial conversation) as an 'age repeat'. This means there were 14 children who had already repeated Grade 1 or Grade 2, and could not repeat another Grade before Grade 4.

[10] As a 'full service' school a number of children in each class had been formally identified as Learners with Special Educational Needs. Assessments by educational psychologists had been conducted for some learners, and other learners were waiting to be assessed to diagnose the extent of their developmental delays and the nature of the learning support required. Not all the teacher-identified learners had been assessed at the time of the cycle 3 intervention, as the school had only just been granted full service status.

CHAPTER 3: Literature review informing the theoretical framework

Chapter 3 draws on the mathematics education literature to outline the theoretical framework on which the design experiment was based. It has three main sections. The first section is the orienting theories describing my views of what mathematics is and how it ought to be taught and learnt. The second section is the domain-specific instructional theories that were the critical mathematical features of the intervention design (*what* was to be taught). Key elements of the international literature base on the nature of additive relations and word problems and suggested approaches within their teaching are presented. The third section of this chapter also extends the domain-specific instructional theories by focusing on the pedagogical features (*how* the teaching was to be approached). In the final section I discuss critical features of my general teacher role in this intervention cycle, which functioned as a framework for action. General pedagogical choices were made relating to the classroom management and general pedagogical style of the intervention.

This chapter defines the critical theoretical features of the design experiment intervention, and is part of the response to the research question: *'How was the intervention designed?*

Overall approach to the theoretical framework

I take 'theory' to refer broadly to what Mason (2014) describes as 'good theory': 'a collection of guiding principles used to inform practice' (p. 11). These guiding principles are viewed as the codified 'theoretical features' used to describe the design of the intervention, and which form the basis for interpreting the impact of the intervention. At the outset of the intervention these guiding principles were mainly implicit. It was through reflecting on, and communicating about, my behaviour, my emotions and my awareness in each iterative cycle that these guiding principles were brought to the surface (i.e. into my own awareness) and then weaved together to form a theoretical framework. I refer to a framework in the sense of 'a coherent set of ideas that together emphasise important aspects of a phenomenon, and that constitute an overt theory that can be put to some practical use' (Mason 2004, p. 12). The theoretical framework underpinned the design and implementation of the intervention and was informed by my theory of learning rooted in constructivism, which sees learners as active in making sense of the world, and as bringing with them natural powers of sense-making.

With this constructivist theory of learning as a guide I drew on a particular aspect of variation theory – the concept of discernment as supported through the use of variation in the midst of change, and the related ideas of dimensions of possible variation and the range of permissible change - to help to surface my own awareness of additive relations word problems and their related conventional, and personal example spaces.

I have found the distinctions that DiSessa and Cobb (2004) make between: grand theories, orienting frameworks, frameworks for action, domain-specific instructional theories, and ontological innovations, as characterising the nature of theory for design experiments to be helpful in delineating the various theoretical aspects of this study.

Grand theories are exemplified by Piaget's theory of intellectual development, or Skinner's behaviourist theory of learning and are argued to be difficult to apply or create in the field of design experiments as these are: 'simply too high-level to inform the vast majority of consequential decisions in creating good instruction' (DiSessa and Cobb 2004, p. 80). Rather *orientating frameworks* such as 'constructivist theory, or 'cultural-historical theory' are appealed to as they offer a basis for instructional design. These frameworks serve to suggest the core values of the person (or group) undertaking the design experiment. The detailed constraints placed on a particular design are the more detailed *frameworks for action* or pedagogic strategies that are in focus for the intervention. Frameworks for action 'are relatively inexplicit, complex, and often involve multiple very diverse elements that cannot plausibly be brought under a single umbrella' (DiSessa and Cobb 2004, p. 80). The framework for action cannot be specified for all aspects of the intervention, and certain elements of these frameworks are foregrounded at some points at the expense of others. In so doing, there is necessarily a gap between what is specified theoretically to be the pedagogic strategies in focus (the framework for action), and what takes place during the intervention (DiSessa and Cobb 2004, p. 80).

As a result 'theory' in design experiments is frequently taken to refer to *domain-specific instructional theories* which are specific to a subject domain and specify how to approach teaching or support learning in relation to a topic. These theories 'entail the conceptual analysis of a significant disciplinary idea' (DiSessa and Cobb 2004, p. 83). Domain-specific instructional theories also include 'specification of both successive patterns of reasoning and the means of supporting their emergence' (DiSessa and Cobb 2004, p. 83). In this design experiment conceptual analysis of a significant idea (additive relations word problems at Foundation Phase level) is conducted, to specify successive patterns of reasoning (evident in children's engagement in the intervention) and the means of supporting their emergence (the teacher's role in the intervention).

However, design experiments are situated in a particular learning context (in this case a poor, urban township school in South Africa where most learners were ELLs), and so the domain-specific instructional theories come about as an engineered response to a problem in a local context. The claims made relating to domain-specific instructional theories remain at the level of conjecture or hypothesis, when considering their generalisability to other contexts. This is not to suggest that they are not useful in resolving local problems, and that their approaches cannot be adapted for, and tested in, other contexts. Rather it serves to highlight that domain-specific instructional theories emphasise situated and pragmatic concerns over 'managing the gap between explanatory theory and pedagogic conjecture' (DiSessa and Cobb 2004, p. 83).

The final characteristic of theory in design experiments is *ontological innovation,* which is offered as an additional characteristic of design experiment theories, and intended as an adjunct to the other theoretical characteristics. Ontological innovation refers to 'hypothesizing and developing explanatory constructs, new categories of things in the world that help explain how it works' (DiSessa and Cobb 2004, p. 79). While domain-specific instructional theories were expected to be drawn on from the literature and to emerge and be documented in relation to the local context of the intervention, ontological innovation is the theoretical goal of the study given the empirical shift of the narrative-based approach from advantaged to a disadvantaged school context within this design experiment.

I use the above framework of theories characteristic of design experiments, to present the theoretical framework for this study. In presenting this theoretical framework I have first discussed those features which refer to orienting theories which underlay my choices relating to the intervention. These orienting theories of learning are the focus of this chapter. The theoretical features which related to the domain specific instructional theories were used to firstly impose constraints on the intervention design (which reflect theoretical choices in mathematics focus, which require some motivation or at least an explanation). Secondly I delineate guiding principles informing the design of the intervention. Thirdly theoretical features, which codify a framework for action in the implementation of the intervention in this particular school, are presented.

Overall theoretical framework for this study

The following table provides a synopsis of the overall theoretical framework.

Table 1: Summary of major features of the theoretical framework

Feature	Title		Brief description
Orienting theories			
1	Approach to learning mathematics	1.1	Learners make sense of problems for themselves while conforming to agreed social practices
		1.2	Discernment as awareness of contrast, with learning opportunity created by using invariance in the midst of change
		1.3	Learning as educating awareness and harnessing natural powers where learning transforms the human psyche (awareness, emotion and behaviour)
		1.4	Facilitating shifts in attention from the particular to the general (generalising) and from the general to the specific (specialising).
		1.5	Learning by engaging in 'tasks' where a learning trajectory is inferred from 'learner activity' with these tasks
		1.6	Varying tasks along prioritised dimensions of variation, while keeping the critical features invariant, and this should be experienced in rapid succession
		1.7	Seeking to gain insight into learners personal potential example spaces, through learners generating examples

Domain specific instructional theories: Mathematical design features of the intervention		
2	Approach to additive relations word problems	2.1 A counting-based conception (counting pathway) of early number development was adopted in that reference examples depicted discrete object problem situations 2.2 A 'take away' calculation strategy was contrasted to a 'difference' strategy, with consideration for efficiency in choice of strategy depending on the numbers involved. 2.3 The conventional example space for additive relations included: 'change increase', 'change decrease', 'collection' and 'compare' problem types 2.4 An equalise word problem and a 'compare (matching)' word problem were used as main tasks in the intervention design 2.5 A 'structural' approach to additive relations was foregrounded making use of a whole-part-diagram and families of equivalent number sentences for number triples for the reference examples. 2.6 A 'partition' word problem type was included as main task to introduce the additive relations structure.
3	A narrative approach to mathematics teaching: Story telling component	3.1 Paradigmatic knowing / logical-scientific thinking and narrative knowing/thinking were viewed as necessarily and simultaneously present in solving any mathematics word problem 3.2: Story telling was used as a pedagogic strategy to motivate learning and encourage sense making 3.3 Story telling was viewed as a cognitive strategy which draws on the human powers of imagining and expressing 3.4 Tasks that demanded story-telling and modelling in the LoLT were designed to support ELLs in their dual need to deepen conceptual understanding of additive relations, and to improve their English language proficiency 3.5 The word problem was viewed as a story-telling and retelling performance in the social community of the class
4	A narrative approach to mathematics teaching: Use of representations component	4.1 Flexible movement between representations where sense-making was primary, was encouraged 4.2 Secure use of a particular representations takes time and, over time, representations should be reified to become cognitive tools 4.3 In the process of being inducted into formal symbolic notation, shifts in attention are required in both directions – from concrete *objects* to symbolic *objects* (and vice versa), and both *processes* of specialising and generalising are necessary 4.4 A learning-teaching trajectory from counting to calculating which made reference to increasingly structured representations involving line, group and syntax models was adopted
Frameworks for action: Implementation features of the teacher role in the third cycle intervention		
5	Training behaviour	5.1 Teaching the whole class as well as in small groups 5.2 Encouraging a growing brains mindset. 5.3 Expecting learners to work independently, and ensuring that while they had similar tasks, that they each worked on unique problems 5.4 Providing immediate extrinsic recognition of appropriate effort which accumulated into a reward (and tied into the mathematics)
6	Principles of general pedagogic style	6.1 Adopting a mathematical thinking questioning style, listening to and exploring suggestions from learners 6.2 Providing specialised and explicit feedback and paying attention to learner errors 6.3 Facilitating opportunities for learners to practice (and receive feedback on) both main and enabling tasks

This chapter focuses on the underlying theory of learning that underpinned the intervention design and informed the choice of a design experiment methodology (the orientating theories). It opens with an overview of the basis features of variation theory which has its roots in constructivism. Two approaches to the use of variation within education are contrasted: a classic view on *variation* *theory* and a broader view on *variation*. This design experiment situates itself in the latter view. As this broader view on variation draws on, but is distinct from, the classic view on variation theory some time is spent discussing these differences. A description of three theoretical features which underpinned the design of the intervention: discernment, abstraction and examples spaces are expounded upon. This underlying theory of learning was then used to inform the major and interrelated theoretical features of the design and implementation of the intervention.

Orienting theory of this study

This design experiment draws on the epistemic paradigm of constructivism, and makes use of particular aspects of variation theory in its overall theory of learning. As such each concept requires some explanation, with reference to relevant background literature.

Feature 1.1 Learners make sense of problems for themselves while conforming to agreed social practices

The overarching guiding principle, which reflects my orienting theory of constructivism is that learners make sense of problems for themselves, while conforming to agreed social practices (Feature 1.1). This draws on both constructivism (how children make meaning individually) while recognising the social dimension where enculturation into a practice is social within a community of learners. As a result learners require opportunities to engage in problems that are meaningful to them, and what they bring to this problem is of interest to the classroom community. These ideas are discussed in more detail in relation to both problem solving and the narrative approach (encompassing both storytelling and representations) to mathematics learning that was adopted for this design experiment.

Variation Theory (a classic view)

Variation theory builds on the constructivist perspective on learning and is a view on the nature of learning which has been developed in relation to learning and awareness in general (Marton and Booth, 1997). In this view social constructivism is seen as the mirror image of individual constructivism, and the duality of person-world is brought together: 'the world is not constructed by the learner, not is it imposed on her, it is constituted as an internal relation between them' (Marton & Booth, 1997, p. 13). Importantly learning is described in terms of

> 'the experience of learning, learning as coming to experience the world in one way or another. Learning inevitably and inextricably involves a way of going about learning (learning's *how* aspect) and an object of learning (learning's *what* aspect). (Marton & Booth, 1997, p. 33)

In my reading of education research drawing on variation theory, variation is used in at least two ways, which I delineate as a 'classic view on variation theory' and a 'broader view on variation'. This study is situated in the broader view of the application of variation. I discuss this positioning in relation to my understanding of each view in order to make clear both what this study is claiming to build upon, and to make clear the variation theory landscape in which this design experiment is situated. As I expound on aspects of the broader view, I delineate the features of the theoretical framework adopted for this study.

In the classic view, variation theorists (like Marton, Tsui, Booth, Runesson, Pang, Kulberg) adopt variation theory as a theory of learning that has given rise to an approach to research which charts the variations in lived experiences of learners (phenomenography):

> The unit of phenomenographic research is *the way of experiencing something,* and the object of research is the *variation* in ways of experiencing phenomena… (Marton and Booth 1997, p. 111)

When considering an object of learning, these theorists identify the 'dimensions of possible of variation': those aspects of the object that can be changed. Working systematically with each dimension then they identify the 'range of permissible change' for that dimension of possible variation.

It is important to note that for classic theorists variation operates in two ways: firstly variation is viewed as a necessary condition for the possibility of learning; and secondly variation frameworks are used in a relatively tightly structured methodology referred to as learning studies. I discuss each of these in turn. Firstly, variation is viewed as a necessary condition for the possibility of learning: ' to learn the learner *must* discern what is to be learnt (the object of learning)…To discern an aspect the learner *must* experience potential alternatives, …against the background of invariance in other aspects of the same learning object' (Marton and Booth, p. 193 emphasis added). The central way in which the lived experience emerges is through discernment of invariance against a background of variation. In this view the focus of attention is explicitly on the object of learning, and other aspects of the learning environment are subordinate to this:

> The focus on what varies and what is invariant derives from the specific framework the teachers are making use of; and other factors such as whole class teaching, group work and form of representation, and the use of textual and other features are subordinated to the teachers' design of the pattern of variation and invariance (Marton and Booth 1997, p. 195).

The framework (sometimes termed the variation framework) which is used by the teachers again stresses the necessity of carefully planned variation and invariance to afford learning: 'According to this framework a particular pattern of variation and invariance is *necessary* to master *every* specific object of learning' (Marton and Booth 1997, p. 217). The object of learning is therefore structured in relation to its dimensions of possible variation, which are consciously constrained and relaxed to facilitate learner discernment of the focal features.

Recognising that there will be differences in the learners' experiences of the direct object of learning, classic variation theorists have developed a research methodology referred to as *learning study* which is a hybrid of a Japanese lesson study and a design experiment (Marton and Booth 1997, p. 194). Such learning studies have 'shown that exposure to variation is critical for the possibility to learn, and that what is learnt reflects the pattern of variation that was present in learning situation' (Runesson 2005, p. 72). It is again important to read this claim in terms of how the classic theorists understand variation: that variation and invariance are considered only in relation to the critical aspects of how they define 'the object of learning', which is in terms of direct objects of learning (the content) and the indirect objects of learning (the capability of using that content) (Marton and Booth 1997, p. 196).

In this classic view on variation theory the focus of attention (for teachers and researchers) is on two interrelated aspects (content and capability), and all other elements of learning (such as elements of affect and behaviour not related directly to the content) are bracketed out or delimited. While this is analytically possible, these broader phenomena were emphatically part of the terrain in the empirical space, where severe learning difficulties related to ADHD, foetal alcohol syndrome and pre-natal crystal-meth issues were all part of the classroom terrain. These issues needed to be considered within the design choices for this study. Thus, I opted to take a broader view of variation that draws on particular aspects of classic variation theory.

Variation Theory (a broader view)

In the broader view, certain elements of the classic view on variation theory are selected as useful, but are used in ways that are not tightly aligned to the learning study methodology. This view has been expressed by Askew who acknowledges drawing on the work of Mason, Watson and Davis in this regard (Askew 2012, p. xxii).

Feature 1.2 Discernment as awareness of contrast, with learning opportunity created by using invariance in the midst of change

In this broader view, aspects of variation theory (particularly learning as discernment of invariance in the midst of variation) is applied to particular research interests such as example spaces and related task design in mathematics, without being restricted to the confines of a rigorous methodological approach which constrains attention only to the object of learning. The design experiment in this study is situated within this broader view on variation. Below, how each concept figures within the theoretical framework of the design experiment is made clear (with reference to the features numbered in the summary table of the theoretical features above).

Feature 1.3: Learning is educating awareness through harnessing natural powers and learning is transforming the human psyche (awareness, behaviour and emotion)

This broader view adopts a definition of mathematics learning which privileges the broad idea of mathematical thinking. This approach has been succinctly defined by Carpenter, Franke et al.(2003) as follows:

> learning mathematics involves learning ways of thinking. It involves learning powerful mathematical ideas rather than a collection of disconnected procedures for carrying out calculations. But it also entails learning to generate those ideas, how to express them using words and symbols, and how to justify to oneself and others that those ideas are true. (Capenter, Franke et al. 2003, p. 1).

In this broader view discerning similarities and differences is considered a natural power that learners bring to bear on all that they experience. The power to distinguish, to discern, to make distinctions is central to education (Gravemeijer 1997, p. 125). Learning is considered to be 'educating awareness' - a notion introduced by Caleb Gattegno (1911 – 1988) where, in order to learn, people become aware of the actions they perform and turn these into objects of study. In so doing they become aware of their awareness. Such awareness of awareness relates to metacognition and mindfulness. The classic variation theorists share much of this overall approach to mathematics.

In this view, learning mathematics is seen as being part of human activity where humans' innate powers to discriminate, select and generalise are used naturally in their engagement with their world (Mason 2007). These powers include: imagining and expressing, focusing and de-focusing (discerning), specialising and generalising, conjecturing and convincing, classifying and characterising (Mason 2007). Notice the close alignment of these natural powers as elements of the Carpenter et al. (2003) definition of what mathematics learning entails: learning to generate those ideas (by the powers of discerning, specialising and generalising, characterising and classifying), how to express them using words and symbols (by the powers of imagining and expressing), and how to justify to oneself and others that those ideas are true (by the powers of conjecturing and convincing).

This view of mathematics which sees experiencing mathematics as a way of thinking means that the mathematics is explicitly not delimited or viewed in relation only to a specific object of learning. Mathematical thinking is a broader concept which may be exemplified by experiencing thinking about a particular topic, but the natural powers are considered to be broader ways of thinking, that are not tightly shackled to a particular mathematics topic (defined as the object of learning by the classic variation theorists). There are pedagogical consequences of the framing of mathematics learning as human activity, as the teaching focus shifts from the particular topic to reflect on the extent to which more general natural powers are being utilised. This informs the choice and design of tasks (explained in more detail later in this chapter).

When learning mathematics at primary school level, and particularly at the Foundation Phase, these powers are predominantly harnessed in relation to numbers and relationships. With the emphasis on mathematics as a way of thinking, major themes which cut across all mathematics learning are identified: 'freedoms and constraints', 'extending and restricting meaning'. 'invariance in the midst of change' and 'doing and undoing' (Mason 2004, p. 21). The natural powers are harnessed in relation to these major themes. In this way, the specific-content domain or mathematical topic in focus, is subordinated to the cross cutting mathematical themes used to invoke the natural powers. This idea is expressed in the theoretical framework for this design experiment as Feature 1.3: Learning is educating awareness through harnessing natural powers. In the broader view, variation is viewed as a <u>likely</u> condition for facilitating learning (and hence is valued), rather than as a <u>necessary</u> condition for all learning. Possibilities of other ways of bringing about learning, as well as considerations of affect are entertained.

Common to both views is that while aspects of the learning system are acknowledged to be important, variation theorists aim to attend to aspects that focus on the specific mathematics content (Askew 2012, p. 61). In this regard it is a theory of learning which is dependent on the actual content of learning. Thinking – and how to better support thinking – cannot be isolated from what is being thought about. As such, variation theorists spend time exploring and describing the objects of learning (the particular mathematics in focus) in detail. In the broader view, the definition of the objects of learning is broader than the definition espoused by the classic variation theorists. Again a distinction is made between direct and indirect objects of learning, but the indirect objects of learning are not as closely shackled to the direct objects, as Askew's broadening of the definition of indirect object of learning demonstrates:

> I would go further than Marton and his colleagues and suggest that every teaching activity embodies more than one indirect object of learning, additional objects that are *not necessarily mathematical*, for example learning that competition is good, or that some children are naturally 'better' at mathematics than others or that working together can be helpful (Askew 2012, p. 63, emphasis added).

Notice how in Askew's view the indirect object of learning is not necessarily mathematical, and need not relate explicitly to the capabilities related to the use of the content as defined by the classic variation theorists. Rather he argues that 'attending primarily to the cognitive in mathematics lessons should not make use blind to the moral values embedded in any lesson'. (Askew 2012, p. 63). As such, indirect objects of learning are considered broadly in relation to intellect, behaviour and emotion; these aspects are all acknowledged to be present (whether consciously intended or not) and attention is not strictly constrained only to the cognitive.

Feature 1.4: Facilitating shifts in attention from the particular to the general (generalising) and from the general to the specific (specialising)

In the 'broader view on variation' the thinking powers deliberately become objects of attention. One such power is 'generalising and specialising' which are interrelated and pervasive processes in all learning. The pervasiveness of these processes in learning is evident in Vygotsky's suggestion that language is a form of generalisation: At any age, a concept embodied in a word represents an act of generalisation (Vygotsky 1965). Mason explains that nouns such as 'cup' or 'chair' are inherently general. When a child first learns these new words the word (written or heard) refers to a specific cup or chair, but their subsequent experience or application of these words in a wide variety of situations and relation to different specific objects makes the words a generalisation of the class of all cups or chairs.

Generalising is the process of shifting from the specific object to the broader class of objects that the specific represents. Specialising is the reverse process of moving from the general class, to a particular case or object within the class. Whether one is specialising or generalising depends on what is being constrained and what is being allowed in terms of freedom (and recall that freedom and constraint were one of the four themes cutting across all mathematics). It is with the constraint (on the particular for specialising, and on the generic for generalising) against a background of freedom that the constraint is discerned.

Central to this conception of generalising and specialising is the distinction between process and object. In mathematics a learner frequently encounters a new idea as a process and later this becomes an object on which they can act. An example of this is initial encounters with number through counting of objects, while over time, number itself becomes a generalisation representing that number of counts, and applicable to the counting of any objects (Gelman and Gallistel 1978, 1986). At this point the number becomes a thinkable object, on which the child can act. Several researchers have made use of this notion of 'objectification' (the transition from process of object), (Thompson 1985), 'reification' (Sfard 2008) and the notion of a 'procept' (Gray and Tall 1994).

Abstracting is a closely related notion to specialising and generalising, which came into use when distinguishing the concrete (the material) from the abstract (the thinkable) (Dreyfus 2014). Mason notes that the word 'abstract' in mathematics is used differently by novice students of mathematics, who find the abstract nature of mathematics difficult, and experts, who find its abstract nature natural, however he contends that:

> The uses of the word abstract in mathematics by both novices and professionals refers to a common, root experience: an extremely brief moment which happens in the twinkling of an eye; a delicate shift of attention from seeing an expression as an expression of generality, to seeing the expression as an object or property. (Mason 1989, p. 3).

For Mason (1989) abstracting focuses on the processes of stressing and ignoring as facilitating generalising/specialising.[11] To see 'cup' as a specialisation, one has to stress the detail or the particular cup in question, and ignore all that is known about the general class of cups. To consider 'cup' in general terms one ignores the details of the particular and stresses the features of the general class. As such the connections with the classic variation theorists' notion of discernment is apparent: discernment is a process of stressing some features and ignoring others. In this view abstracting is a pervasive human activity which is the union of specialising and generalising (dualistic processes achieved through stressing and ignoring). This captures the notion that 'the abstraction' can be the concrete object which is attended to in the mind (a thinkable object): 'When forms become objects or components of thought, and when with familiarity they become mentally manipulable, becoming, as it were, concrete, mathematics finds its greatest power.' (Mason 1989). In the theoretical framework for this design experiment, these ideas are codified as Feature 1.4: Facilitating shifts in attention from the particular to the general (generalising) and from the general to the specific (specialising).

Variation applied to task design and examples in mathematics

The notion of variation (in the broader view on variation) has been applied to task design and examples in mathematics. These ideas are particularly pertinent to this design experiment as the design of the intervention focused attention on the selection of appropriate reference examples, and the design of tasks hypothesised to support the overall learning objective of the intervention.

[11] I recognise that there are differing definitions of the processes of generalising, specialising and abstracting in the mathematics literature, as well as in other places in Mason's own work. However it is this definition of abstraction which I have adopted for this study.

As such, I spend some time discussing the way in which variation is viewed when focused on the task design, and examples in mathematics. Where appropriate I again indicate how these concepts have found expression in the theoretical framework for the design experiment.

Feature 1.5 Learning by engaging in 'tasks' where a learning trajectory is inferred from 'learner activity' with these tasks

For Mason (2004) the aim of a mathematics lesson is for learners to learn something about a particular topic, and to do this they engage in tasks. The term 'tasks' refers to 'what the learners are asked to do: the calculations to be performed, the mental images and diagrams to be discussed, or the symbols to be manipulated' (Mason 2004, p. 4), rather than generic instructions such as 'listen to me' or assessment tasks.

In this approach (and within this design experiment), following Mason (2004), a distinction is drawn between task and activity: 'The purpose of a task is to initiate activity by a learners. In an activity learners construct and act upon objects whether physical, mental or symbolic that pertain to the mathematical topic' (Mason 2004, p. 4). Distinguishing tasks from activities in the design experiment allowed attention to be paid separately to the design of the task, and to the learner activity that resulted from these tasks. In so doing learner activity became a focus of research attention, which guided the design of tasks within each subsequent intervention cycle.

The distinction between task and learner activity goes to the heart of the complex relationship between activity and effect, and between teaching and learning. This is an important feature relating to how learning was construed in this design experiment, as the notion of children learning from tasks (and evident in their learning activity) informed the analysis of learning gains against each of the learning goals. While the notion of 'a task' is relatively fine grained, it is important to view tasks not as isolated but seen in relation to each other within the context of the intervention sequence. This process of viewing similarities and difference between tasks then follows a variation theory approach as consideration is given to that which is changing and that which is constant across a set of examples. From the teaching side, the instructional design of the intervention focused on the design and sequence of tasks, and the teacher roles of using storytelling and representations to mediate the mathematical tasks. From the learning side, the lived experiences of particular learners were inferred from their learner activity with these tasks (and as evident in their talk and actions during interviews and lessons recorded on video, and their writing and drawing in their learner books and written assessments).

Design experiments involve 'formulating, testing and revising a hypothetical learning trajectory' (DiSessa and Cobb 2004). As such both teaching and learning were considered in relation learning trajectories. In this study the *hypothetical learning trajectory*, refers to the teacher expectations of learning and is specified by a range and sequence of tasks and related expectations of learner activity (Simon and Tzur 2004). The *actual learning trajectories* of particular children refers to 'the learning routes that students seemed to have followed in the context of the implementation of [an instructional sequence]' (Stylianides and Stylianides, 2009). So the hypothesised learning trajectory specifies intended learning and is specified through attending to tasks and related teacher actions in mediating these tasks, while the actual learning trajectory specifies learning gains or absences of learning gains as evident in the particular learner's activity with the tasks. The iterative nature of design experiments means that over time, the hypothesised learning trajectory should increasingly closely match the actual learning trajectories. When such matching occurs, one can infer that the instructional sequence shows potential for facilitating the learning goals. When the instructional sequence is repeated with different cohorts of learners, and the same matching of hypothesised learning trajectory to actual learning trajectories is found, the inferred matching between effect and activity is strengthened. The above is described as Feature 1.5: Learning by engaging in 'tasks' where a learning trajectory is inferred from 'learner activity' with these tasks.

Feature 1.6: Varying tasks along prioritised dimensions of variation, while keeping critical features invariant, and this should be experienced in rapid succession

Examples and example spaces

The discernment component of classic variation theory, together with its attention to critical features and dimensions of possible variation has been applied by Watson and Mason (2005) to examples in mathematics. The term 'examples' is used in 'a very broad way to stand for anything from which a learner might generalise' (Watson and Mason 2005, p. 3).

This definition makes explicit the connection between 'an example' and its purpose which is 'to exemplify'. To be an example such an object must be an example of something – it represents a broader collection of similar objects. The definition of task is focused on the action (behaviour) which the learner is expected to perform; while the definition of an example focuses on the particular purpose to generalise (shift in awareness). There is a great deal of overlap between tasks and examples, as most of the things learners are asked to do lend themselves to generalisation (although the learners may not generalise them in the ways intended by teachers). Notice the connection between 'examples' where the focus is on the object and 'generalising and specialising' where the focus is on the process.

Working with this understanding of 'an example', Watson and Mason (2005) defined the phrase 'example space', as explained in the introductory chapter: 'examples are usually not isolated; rather they are perceived as instances or classes of potential examples', thereby 'contributing to an example space' (p. 51). The ideas of 'dimensions of possible variation' and the 'range of permissible change' were then applied to the mathematical object example spaces. The 'range of permissible change' defines the permissible change along an identified 'dimension of possible variation'. Example spaces then become thinkable objects, which are then compared to each other in relation to particular mathematics topics. With an 'example space' defined as an object worth of attention, different types of examples spaces, and different types of examples emerge which can be classified and characterised:

- Figural examples (classic examples of mathematical objects where their images are familiar to many although the details may be scant),
- Start-up examples (used at the beginning of learning some theory and from which basic problems, definitions and results can be conjectured),
- Reference examples (standard cases that are widely applicable and can be linked to several concepts and results);
- Model examples (generic cases summarising expectations and assumptions of concepts of theorems); and
- Counter examples (used to sharpen the distinction between concepts, also referred to as non-examples) (Watson and Mason 2005, p. 64).

The notion of a reference example is important as, when viewing example spaces, not all examples are equal. Reference examples can be introduced and foregrounded (and taken as shared within a classroom community). Familiarity with reference word problems allows learners openings to use the same process and model adopted for the reference problem to a new similar problem solving situation. The importance of viewing examples as members of a set, and not in isolation is a point that has been noted in the South African mathematics education literature (see Venkat and Naidoo 2012). As such the need to treat examples historically, acknowledging examples which had already been solved, and seeking to draw learners' attention to connections, rather than individual examples was adopted.

Mason explains that for all sets of mathematical tasks there is usually an invariant feature which is common to all of the tasks (Mason 2004, p. 22). So to learn from a set of similar examples learners need to use their power of focusing and defocusing to discern the invariance in the midst of change. Askew argues that engaging children in sets of mathematical examples can be beneficial 'if the children approach them mindfully' (Askew 2012, p. 61). This can be achieved when teachers are purposeful in directing children to look for, and think about, possible connection between examples

These kinds of variation in a task can be applied to compare type additive relation word problems for Foundation Phase learners as follows: 'I have 5 apples. You have 3 apples. How many more apples do I have than you?' This task could be varied in several ways without losing its purpose. Firstly the numbers involved in the problem could be varied: either '5' or '3' or both '5' and '3' could be changed (to other whole numbers, or to factions or to decimals, perhaps with a change in the context too) to create a new, but similar, task to the original. Secondly the characters in the problem could be changed, 'I' and 'you' could become 'Keletso' and 'Ayanda' for instance. Thirdly the objects in the problem context could be changed: 'apples' could be varied to be 'bananas', or 'toy cars' or 'tennis balls'. These changes would maintain a constraint on the problem of referring to discrete objects. However if this constraint were relaxed and measurement contexts were allowed, the problem could be varied to become: 'I am 5 years old, you are 3 years old. How much older am I than you?' Both word problems are compare type word problems. Relaxing the constraint on additive relations word problem type, could create a new problem type (while keeping the numbers invariant): 'I have 5 apples, you take 3 of my apples. How many apples do I have left?' With this variation the problem context keeps the characters, numbers and objects invariant (3,5, apples, you and me), but the imagined situation or action is changed, although the resulting solution remains the same.

Drawing on the classic variation theorists, such variation in a task is to be done deliberately moving along prioritised dimensions of variation, and in rapid succession (Mason 2004, p. 11). Attending to what is being constrained and what varies in a task involves considering 'freedoms and constraints' in the mathematical task (one of the four important themes that cut across all mathematics learning) (Mason 2004, p. 21).

These ideas informed the guiding principle of the theoretical framework of the design experiment that: Tasks should be carefully varied along prioritised dimensions of variation, while keeping the critical features invariant, and this should be experienced in rapid succession (Feature 1.6).

Feature 1: 7 Seeking to gain insight into learners personal potential example spaces, through learners generating examples

A fundamental belief within an approach to learning mathematics which puts learners' powers of sense making at the centre is that learners are able to exemplify for themselves and that to do so contributes to their learning' (Watson and Mason 2005). 'Imagining and expressing' and 'specialising and generalising' are considered to be human powers. It is these powers that are required to exemplify something. If example spaces are considered worthy of teacher and learner attention, it follows that initiating activity which requires learners to reflect on their own personal example spaces is a worthwhile undertaking. To do so may involve the teacher initiating tasks where the learner activity (and the main material of the lesson) is the 'generation of examples of questions, techniques, actions, notations and mathematical objects by learners' (Watson and Mason 2005, p. 24).

An example of a task to illustrate this idea is: 'Think of a word problem where we would need to use the calculation: 8 – 5 = …'. Generating the first instance of an example is usually easy (and highly personal). It is in applying variation to think about what you are constraining and what you are freeing that the depth of this process is revealed. By asking for another example, and another, and perhaps another example and then reflecting on what is remaining invariant, while other elements are changed helps to reveal the personal potential example space of word problems requiring '8 – 5 = ….' calculations. As mathematics is recognised to be a social activity, comparing personal example spaces with others in a community of mathematics practice further enriches this process.

The value in such tasks stems from the orienting framework, where attending to thinking and shifts in awareness (rather than attending solely to solutions – as the standardised assessments do) is in focus. Learner generated examples and problem posing processes support this. Watson and Mason (2005) claim that 'Making up examples that need particular techniques to solve them can focus learners on the mathematical structures that relate to those techniques' (p23). So setting tasks that expect learners to generate examples that require particular calculation techniques, has a dual benefit of firstly supporting them to think about their thinking (in relation to the topic in focus), and secondly the mathematical structure of the topic in focus is discerned as this is constrained (kept invariant), while other elements are varied.

This approach to engaging learners in mathematical thinking is however highly demanding on the teacher. This is a component acknowledged by Watson and Mason (2005) when they write:

> *If* learners are used to thinking mathematically *and* choose to do so for themselves instead of being dependent on some authority, they will learn mathematics more easily and more effectively as they will have developed a structured understanding of the subject and a network of meanings through which new experiences can be perceived and into which they can be assimilated (p24).

With this statement they acknowledge two of the conditions necessary for such an approach to be effective: Learners are accustomed to thinking mathematically (and this is not a socio-cultural practice which is developed quickly in a single or a few lessons), and the learners choose to do so for themselves (affect). These were both conditions which I anticipated would be challenging to establish in the South African primary mathematics context describe in the previous chapter; and in a teaching environment where I was not the normal classroom teacher.

Synopsis of my orienting theories and how these figure in the theoretical framework for this design experiment

The above sketches my orienting theories or the meta-theories which describe the general aspects that are hoped for in the intervention design. These theories required the systematic development of an argument grounded in the literature, with each term carefully defined, and as such has resulted in a somewhat dense text. As a support to the reader I present a synopsis of my orienting theories for this design experiment

My approach to the learning of mathematics falls within the constructivist tradition, where learners actively making sense of their environment in communities. The guiding principle relating to this is that learners make sense of problems for themselves while conforming to agreed social practices (Feature 1.1).

Classic variation theorists provide a useful construct of 'discernment' as awareness of contrast. In this view learning opportunities are created using invariance in the midst of change (Feature 1.2). This is supported by constructs such as 'dimensions of possible variation' and 'range of permissible change' which have been applied to mathematics learning in relation to particular mathematical thinking processes (such as specialising, generalising and abstraction), and to mathematical objects (such as example spaces) by theorists who take a broader view on variation.

Mathematics learning is viewed as supporting the development of mathematical thinking, where harnessing natural powers and educating awareness are valued. In particular I draw on the application of variation to notions of discernment (Feature 1.2) and as a means of educating awareness (Feature 1.3). This is done with regard to the interrelated processes of generalising, specialising and abstracting (Feature 1.4); and of task design (Feature 1.5). Tasks are designed by considering the conventional example space for a particular content domain and are varied along prioritised dimensions of variation (Feature 1.6). Gaining insight into learners' personal potential example spaces is done through learners generating examples (Feature 1.7).

In this study the learning goals were defined in relation to an over-arching learning objective, and related learning goals pertaining to the mathematics:

Table 2: Defining the object of learning for this study

Over-arching learning objective	Expand the learner's personal potential example space for additive relations to include compare type word problems.
Learning goal 1	Solve a range of additive relation word problem types
Learning goal 2	Flexibly use a range of diagrammatic representations to record and explain solutions to additive relation word problems
Learning goal 3	Tell stories to pose and explain additive relation word problems

These three interrelated learning goals were the direct object of learning, and were specific to the mathematics. In working with the intended learning goals depictions of the same concept, varied in some key way were presented in relation to one another in order to solicit a contrast that would facilitate learning. At the same time, the enacted learning goals were studied, via looking at which varying features different learners were attending to.

This broader view of variation acknowledges the other aspects of a classroom environment – the teacher role, the learning disposition of the learners, the school culture – as important features of the intervention, which could not be subordinated to the mathematical content and capabilities which are in focus for the classic variation theorists. The focus is on broader mathematical thinking goals – problem solving, using representations, and telling stories to pose and explain problems are the learning goals. These features of theoretical intent go beyond the content and capabilities related to the specific mathematical topic, to consider issues of emotion and behaviour in mathematics learning which were consciously attended to by the teacher, and which are reported on in the theoretical framework for action.

Domain-specific instructional theories relating to additive relations word problems

This section extends the theoretical framework to focus on the domain-specific (mathematical) instructional theories of the design of the intervention. The instructional theories relate to the over aching objective of learning: 'expanding the learners' examples spaces for additive relation word problems to include compare problems'. In this section, mathematics education literature on 'additive relations' and on 'word problems' was used to map out the conventional example space for this topic (*what* was to be learnt). The next section considers the pedagogic strategies (*how* this was to be taught), which were in focus for this learning goal. I codify these as theoretical features informing the domain-specific instructional theories underlying the intervention. These theoretical features are focused on the cognitive components of learning which concern *what* mathematics was in focus, and *how* the teaching of this mathematics was approached, both which were fundamental to the intervention design. As the design of the intervention, and its implementation, evolved dialectically drawing on the mathematics education literature as well as the empirical data of how children responded to the task design, the theoretical features of the design shifted with each design cycle. In this chapter I report on the literature informing the theoretical features underpinning the third design cycle.

The following table restates the theoretical features that were considered key to the '*what'* of the mathematics of the intervention. Together these features give the theoretical grounding for problem-solving 'strategies' and pedagogic task selections

Table 3: Summary of major features of the theoretical framework of what mathematics underpinned the last intervention

Feature	Title	Description
Mathematical design features of the intervention		
2	Approach to additive relations word problems	2.1 A counting-based conception of early number development was adopted 2.2 A 'take away' calculation strategy was contrasted to a 'difference' strategy, with consideration for efficiency in choice of strategy depending on the numbers involved. 2.3 The conventional example space for additive relations included: 'change increase', 'change decrease', 'collection' and 'compare' problem types 2.4 An equalise word problem and a 'compare (matching)' word problem were used as main tasks in the intervention design 2.5 A 'structural' approach to additive relations was foregrounded making use of a whole-part-diagram and families of equivalent number sentences for number triples. 2.6 A 'partition' word problem type was used to introduce the additive relations structure.

The literature on additive relations forms the heart of the mathematics education literature relevant to this study. The approach adopted to additive relations word problems is therefore described in detail.

Feature 2.1 A counting-based conception of early number development was adopted

Researching children's thinking in a project referred to as Cognitively Guided Instruction Carpenter et al. (1999) identified three calculation strategies commonly used for solving additive relation problems and also considered these in a hierarchy of mathematical sophistication: Direct modelling; counting; and calculating. Direct modelling refers to the use of concrete apparatus such as manipulatives (like counters, or actual objects) to enact a situation which closely resembles the problem situation. By counting, Carpenter et al. refer to strategies which make use of unit counting to calculate. These also vary in levels of sophistication: Count all, count on, and count back or up to reach a target. In this case counting strategies refer to unit counting, rather than counting in groups (such as twos, fives or tens). Calculating refers to more sophisticated strategies which do not use unit counting but which may use counting in groups and/or build on known facts (often knowledge of bonds of five and ten) and the relationship between the numbers in the calculation for solving.

Van den Heuvel-Panhuizen et al. (2008) provides a similar 'learning-teaching trajectory' of young children's likely progression for whole number calculation, but this trajectory makes specific reference to the expected progression towards more structured representations of number and additive relations. Importantly such a learning-teaching trajectory is not seen as a 'strictly linear, step-by-step regime' as it needs to do justice to 'the learning processes of individual children, discontinuities in the learning process...the fact that multiple skills can be learned simultaneously…differences in learning process at school as a result of difference in learning situations outside school' and 'different levels at which children master certain skills' (van den Heuvel-Panhuizen 2008, p. 13). Essentially, the learning-teaching trajectory is described as having a 'certain bandwidth' (van den Heuval-Panhuizen 2008, p. 13).

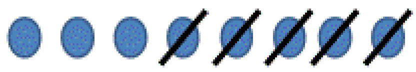
As part of this learning-teaching trajectory attainment targets are provided which are viewed as 'benchmarks', or reference points against which the development of students can be assessed. A series of levels are described from pre-school experiences to the upper primary grades (Treffers, 2008). While these cannot be used as fine grained shifts within a level which were the hoped for learning gains over my 10-day intervention, nevertheless this framework for counting to calculating provided the broader learning context for the intervention. Treffers includes extensive reference to the role of external representations in children's calculation strategies. I concur with this view of the importance of representations, but detail this role separately as part of *Feature 4: Representations.*

The grounding of number concepts in children's experiences of counting in ones provides a learning trajectory from counting to calculating which is succinctly described by Askew and Brown (2003) as a progression from 'count all, count on from the first number, count on from the larger number, use known facts and derive number facts' (p. 6). It is this counting-based approach to number concepts which is advocated for in the current South African curriculum where the mental strategies include putting the larger number first to count on or count back, and using a number line. There is also evidence in the curriculum statements of structured approach as the expected progression to a wider range of calculation strategies is anticipated. Building and breaking down numbers, and using the relations between addition and subtraction are explicitly stated as mental calculation strategies. Additional detail on how elements from the counting approach and the structural approach were incorporated into this intervention is provided in relation to *Feature 4: Representations*.

Feature 2.2 A 'take away' calculation strategy was contrasted to a 'difference' strategy, with consideration for efficiency in choice of strategy depending on the numbers involved.

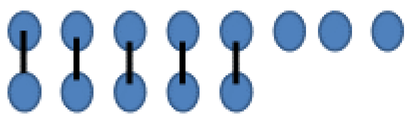
In addition to the counting-based approach to number development and an assumed learning trajectory from counting to calculating, attention has been paid to two ways in which to conceptualise subtraction and how these impact on calculation strategies. There are referred to as 'take away' and 'difference' models for subtraction, with CAPS for Foundation Phase emphasising the need for awareness of both of these (DBE 2011, p. 194). For the 'take-away' model, 8 minus 5 is conceptualised as 'remove 5 from 8'. This may be depicted diagrammatically using a 'take-away image'.

Figure 10: Take-away image for 8 minus 5



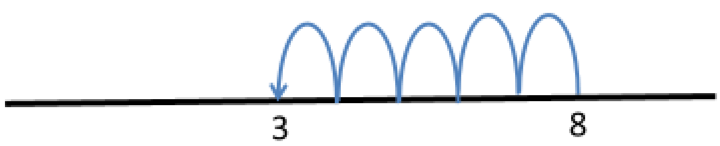
For the 'difference' model, 8 minus 5 is conceptualised as comparing 5 to 8: 'What is the difference between 5 and 8?' or 'How many more is 8 than 5?' or 'How many more must I add to 5 to get to 8?' This may be depicted diagrammatically using a 'difference image', where 8 is compared to 5:

Figure 11: Difference image for 8 minus 5



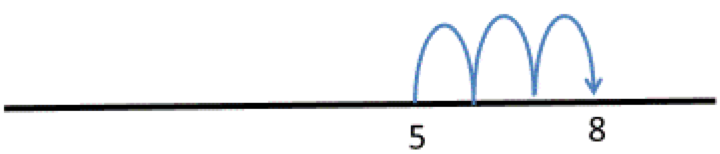
Askew (2012) argues for teachers to encourage learners to be flexible in their choice of strategy in ways that are informed by the relationship between the two numbers. He supports this with the different ways in which the two models are represented on an empty number line (and which can be used to clearly reveal which strategy children use for a subtraction calculation) (Askew 2012).

Figure 12: A take-away strategy for 8 minus 5 on an empty number line



Using a take-away strategy on an empty number line a child starts at 8 on the empty number line and hops back 5 to reach 3. The answer to 8 minus 5 is obtained by looking at the landing point after moving 5 hops back from 8.

Figure 13: A difference strategy for 8 minus 5 on an empty number line



Using a difference strategy on an empty number line a child compares 8 to 5 by placing both numbers on the empty number line. They find the difference between 8 and 5 by counting the hops from 5 to reach 8, or from 8 to reach 5.

The difference image (Figure 11) is appropriate for small numbers and problem situations involving discrete objects. The difference strategy on an empty number line (Figure 13) may be used to invoke a continuous measurement context where the length of 5 is compared to the length of 8. Using an empty number line with jumps (and not hops in ones) the difference is found by calculating the jump that is needed to move from 5 to 8.

For many young children their first introduction to subtraction tends to be through take-away contexts. Their dominant calculation strategy is one of removal or counting back from a number (9 minus 7 is solved by counting back 7 counts from 9 to reach 2). However, when the numbers are close together it is more efficient to use a difference strategy by counting up from a number to reach a target (9 minus 7 is solved by tracking how many counts are needed to count up from 7 to reach 9). Awareness of the two models, with both being relevant to subtraction situations, provides a choice of which model to use. One model may be implicit in the problem situation, while the numbers involved in the problem situation may be more efficiently worked with using the alternative model. Flexible movement between models is therefore encouraged.

Feature 2.3 Four different types of additive relations word problems were presented as part of the conventional example space: 'change increase', 'change decrease', 'collection' and 'compare' problem types

In this section I attempt to provide a historic overview of the development of additive relations word problem categorisations. Given the breadth of this literature, I refer only to key sources that have had influence on each other and are prominent in later citations, rather than including an exhaustive review of this topic.

The idea of 'additive relations' as combining the operations of both addition and subtraction is relatively new, and prior research considered word problems relating to addition as distinct from word problems relating to subtraction. As introduced earlier, three major research studies – all conducted in the early 1980s - have contributed to work focused on learners solving additive relations word problems: Carpenter, Hiebert and Moser (1981) working on their Cognitively Guided Instruction framework; Riley, Greeno and Heller (1983) working on information processing and computer programmes to support levels of problem solving; and Verschaffel and De Corte (1993) working on semantic structures and children's invented solution strategies. All three studies organise and classify different additive relation word problems by using the types of actions or relationships described in the problems as their primary defining characteristic. This research coheres on the development of a classification framework for thinking about additive word problems types. It is worth tracing the historic development of these categories, as the changes in classification reveal shifts in theoretical approaches.

The Riley et al. (1983) framework distinguishes action situations from static situations and suggests four categories of additive relations problem types: change, equalise, combine and compare.

Table 4: Types of additive relation word problems (Riley et al., 1983)

Action	**Static**
Change (result unknown) 1. Joe has 3 marbles. Then Tom gives him 5 more marbles. How many marbles does Joe have now? 2. Joe has 8 marbles. Then he gave 5 marbles to Tom. How many marbles does Joe have now?	Combine (combine value unknown) 1. Joe has 3 marbles. Tom has 5 marbles. How many marbles do they have altogether?
Change (change unknown) 3. Joe has 3 marbles. Then Tom gave him some more marbles. Now Joe has 8 marbles. How many marbles did Tom give him?	Combine (subset unknown) 2. Joe and Tom have 8 marbles altogether. Joe has 3 marbles. How many marbles does Tom have?

Action	**Static**
4. Joe has 8 marbles. Then he gave some marbles to Tom. Now Joe has 3 marbles. How many marbles did he give to Tom?	
Change (start unknown)	
5. Joe has some marbles. Then Tom gave him 5 more marbles. Now Joe has 8 marbles. How many marbles did Joe have in the beginning?	
6. Joe has some marbles. Then he gave 5 marbles to Tom. Now Joe has 3 marbles. How many marbles did Joe have in the beginning?	
Equalising	Compare (difference unknown)
1. Joe has 3 marbles. Tom has 8 marbles. What could Joe do to have as many marbles as Tom?	1. Joe has 8 marbles. Tom has 5 marbles. How many marbles does Joe have more than Tom?
2. Joe has 8 marbles. Tom has 3 marbles. What could Joe do to have as many marbles as Tom?	**2.** Joe has 8 marbles. Tom has 5 marbles. How many marbles does Tom have less than Joe?
	Compare (quantity unknown)
	3. Joe has 3 marbles. Tom has 5 marbles more than Joe. How many marbles does Tom have?
	4. Joe has 8 marbles. Tom has 5 marbles less than Joe. How many marbles does Tom have?
	Compare (referent unknown)
	5. Joe has 8 marbles. He has 5 more marbles than Joe. How many marbles does Tom have?
	6. Joe has 3 marbles. He has 5 marbles less than Tom. How many marbles does Tom have?

Verschaffel made use of the same four categories, but collapsed join and separate into a single category of ‘change’ while retaining the terminology of ‘combine’ and ‘compare’ categories (Verschaffel and De Corte 1993).

When considering the position of the unknown in each type, together with variation of the language of situations depicting increase and decrease, Riley et al. (1983) referred to a total of 16 sub-types. I draw on the vocabulary used in this framework to express in each word problem type using a general equation:

- Action: Change (start ± change = result; and change takes place in the present)
- Action: Equalise (start + change = result; and change takes place in the future)
- Static: Combine (subset A + subset B = combined set)
- Static: Compare (quantity ± difference = referent)

Varying the position of the unknown to generate sub-classes, was adopted by several other researchers (Carpenter, Fennema et al. 1999; Verschaffel and De Corte 1993; and Clements and Samar 1999). I therefore take the sub-categories as given, and attend only to the changes made over time in the classification and labelling of the four main problem types.

The Riley et al. (1983) framework was used and adapted by Carpenter et al. (1999), where they identified four classes of word problem types:

- *Class 1 Join* problems are where elements are added to a given set, for example "3 birds were sitting in a tree. 2 more birds flew into the tree. How many birds were in the tree then?";
- *Class 2 Separate* problems, where elements are removed from a given set, for example "Colleen had 8 guppies. She gave 3 guppies to Rodger. How many guppies does Colleen have left?";
- *Class 3 Whole-part-part* problems, which involve the relationship between a set and its two subsets: "10 children were playing soccer. 6 were boys and the rest were girls. How many girls were playing soccer?"; and
- *Class 4: Compare* problems, which involve comparison between two disjoint sets "Mark has 3 mice. Joy has 7 mice. Joy has how many more mice than Mark? (Carpenter, Franke et al. 1999, pp. 7-10).

How does this relate to the Riley et al. classification? The following table offers a comparison between the Riley and Carpenter framework:

Table 5: Comparing Riley and Carpenter et al.'s classifications of additive relations word problems

Riley et al. (1983)	**Carpenter et al. (1999)**	**Comment**
Action: Change	Level 1: Join Level 2: Separate	For Riley et al. actions of change could be either joining or separating. For Carpenter this distinction in action was significant, with 'separating' being more difficult than 'joining'
Action: Combine	Level 1: Join	While Rileyet al. considered 'combine' to be distinct from 'change', Carpenter considered 'combine' to reflect a joining action (if there was an action of bringing together) or to reflect a part-part-whole type

		if two attributes were contrasted within a static set. Riley et al. exemplified combine problems with Joe and Tom's marbles (ownership attribute of two sets). Carpenter exemplified combine problems with boys and girls on a playground (gender attribute contrasted within a set of children) (Carpenter and Moser 1984).
Not in this framework	Level 3: Part-part whole	Carpenter introduced a distinct static problem type, referred to as part-part-whole. This overlapped with the Riley et al. combine problems (in cases where two attributes where contrasted in a static set)
Static: Equalise	Level 1: Join	Riley et al.'s 'equalise' category is assumed to be considered as part of the Carpenter 'join' category, as the action of joining is still in focus, even though this action takes place in the future and not the present.
Static: Compare	Level 4: Compare	Both frameworks refer to compare problems.

It is notable that the RGH framework distinguished situations where there is action from situations that are static. Their framework also included an action word problem type referred to as equalise (see Table 5 above) -. The CGI framework considered this problem type to be part of the join category (and distinguished whether the change was a joining or separating action), or if a static set was being considered and two attributes within the set were contrasted this was categorised as a part-part-whole word problem.

The CGI framework feeds into the additive relation problem types described by Clements and Sarama (2009) who work with a simplified version of CGI with some revised labels for categories. The RGH concept of a 'change' problem type is reinserted:

- Change plus: An action of joining increases the number in a set;
- Change minus: An action of separating decreases the number in a set;
- Collection: Two parts make a whole but there is no action. The situation is static; and
- Compare: The numbers of objects in two sets are compared.(Clements and Samara 2009, p.62)

Askew (2012) built on this framework and introduced refinements by renaming 'change plus' to 'change increase', and 'change minus' to 'change decrease' in recognition of the fact that either operation (plus or minus) could be used to solve either problem type. For example a change decrease problem like: 'I have 8 apples. I eat 6 of them. How many are left?' could be solved using a subtraction calculation such as 8 – 6 = […], or an addition calculation such as 6 + […] = 8. Where the terminology of Clements and Samara draws attention to the likely operation (plus or minus) required to calculate a solution to the problem, the terminology introduced by Askew (2012) draws attention to the type of action or change in the problem situation (an increase or decrease to the set), and highlights that either operation could be used successfully.

It is notable that, since 1999, there has been relative stability in the four problem types identified by Carpenter et al., with slight refinements in labelling of these categories over time. Given the variety of terminology used to describe the additive relations problem types, for the purpose of this study I use a combination of Askew and Clements and Samara terminology to label the additive relation problem types as follows: Change (increase); Change (decrease); Collection; and Compare problems.

My motivation for using these terms is as follows. I prefer the terms 'change plus' and 'change minus' to 'join' and 'separate', as the emphasis on a common action of change is evident in their labels. I further prefer the 'change increase' and 'change decrease' to 'change plus' and 'change minus' terminology as attention is drawn to the problem situation and not the assumed likely operation. I prefer the term 'collection' to 'whole-part-part' as all additive relation problem types have a whole-part-part structure.

The slight modifications in the terminology used by these theorists are important and so I summarise them in this figure, before highlighting their differences:

Figure 14: Comparison of classification of additive relation problem types

Example	**Riley et al. (1983)**	**Verschaffel & DeCorte (1993)**	**Carpenter et al. (1999)**	**Clements and Samara (2009)**	**Askew (2012)**	**Roberts (this study)**
I have 5 apples. I get 3 more apples. How many apples do I have now?	Action: Change	Change	Join	Change plus	Change increase	Change increase
I have 5 apples. You have 3 apples. How many apples do we have altogether?	Action: Combine					
Joe has 5 marbles. Tom has 8 marbles. What could Joe do to have as many marbles as Tom?	Action: Equalise					
I have 8 apples. I eat three of them. How many are left?	Action: Change		Separate	Change minus	Change decrease	Change decrease
I have 8 apples. 3 are red. The rest are green. How many are red?	n/a	Combine	Whole-part-part	Collection	Part-part whole	Collection
I have 8 apples. You have 3 apples. How many more apples do I have than you?	Static: Compare	Compare	Compare	Compare	Compare	Compare

Feature 2.4 An equalise word problem and a 'compare (matching)' word problem were used as main tasks in the intervention design

I drew on research findings from the literature on additive relations word problems to inform my selection and sequencing of main tasks in the intervention design.

My decision to include a change increase (equalise) word problem and introduce a compare (matching) problem type, prior to learners working on a compare problem type was informed by three different sources. Firstly the CGI and VDC work both found that in the early stages of problem solving children's choice of calculation strategy is largely determined by the problem type and their calculation tends to follow the sequence of action presented in the problem text. Both studies concurred that 'the majority of young children's errors [in solving single solution additive relations word problems] result from inappropriate representations of the problem situation, which are, in turn, due to deficiencies or misconceptions in children's conceptual knowledge base' (Verschaffel and De Corte 1993, p. 256). This was in contrast to the formerly widespread belief that children's errors arose from difficulties with 'choosing the appropriate arithmetic operation to find the unknown element in the problem situation' (Verschaffel and De Corte 1993, p. 256). In static problems (where there is no action), learners to need to introduce an action into the problem situation, while in the dynamic problem types (change increase and change decrease) the action is explicit in the problem text. In static problem types a mental action of comparison is implied and this action has to be introduced into the problem situation.

Secondly, a 1:1 matching action was identified and labelled as a 'subtraction strategy' in the Carpenter Hiebert et al. (1981) study. This action was evident in the observed direct modelling of compare type problems, when concrete materials were available for learners to use. It was described as follows:

> Matching: Matching is only feasible when objects are available. The child puts out two sets of cubes, each set standing for one of the given numbers. The sets are then matched one-to-one. Counting the unmatched cubes gives the answer (Carpernter, Hiebert et al. 1981, p. 35).

In the CGI analysis of the empirical data, learners were found to use a matching action as their subtraction strategy more frequently when working on static problems (collection and compare type problems), while in the change problem situations, strategies of separating or joining (which modelled the change increase or decrease action explicit in the change problems) were used.

Thirdly, the VDC work supported the hypothesis that learners' choice of calculation strategy is influenced by the problem. However they also found that, within problem types, the wording of the problem can be adjusted to make the implied action more explicit, and this re-wording leads to more successful problem solving. Working with the finding that problem solving processes often break down at the point of constructing a situation model (when learners are working bottom-up, to first understand the problem situation), and not at the point of considering an appropriate mathematical model (where learners already have familiar mathematical models which can be imposed top-down onto the problem text), they conjectured that

> young and inexperienced children do not sufficiently master the semantic schemata for understanding the problems and for relating them to already available solution strategies. Therefore, they cannot rely on top-down processing of the problem text, but are instead committed largely to bottom-up processing. Rewording a problem by making the semantic relations more explicit compensates for the less developed schemata and facilitates appropriate bottom-up processing (Verschaffel and De Corte 1993, p. 249)

Thus this team found that word problems could be made more or less difficult to solve – within a problem type – depending not only the position of the unknown in problem text; but also on the extent to which the problem situation had to be inferred, or was explicit. They worked with common errors relating to misunderstanding the problem situation, and re-worded the problems to make the intended meaning of the problem situation more explicit. For example the collection problem: 'Pete and Ann have 9 apples altogether; Pete has 3 apples; how many apples does Ann have?' was reworded to 'Pete and Ann have 9 apples altogether; *3 of these apples belong to Pete and the rest belongs to Ann*; how many apples does Ann have?' and showed that learners had more success with the more explicit wording.

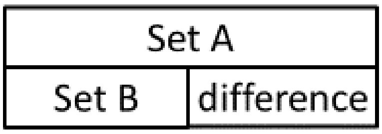
Subsequent writing by Carpenter, Fennema, et al. (1999) picked up on this finding, and noted that changing the wording of a compare problem, so that the implied 1:1 action is more explicit can make the compare problem easier to solve. They suggested changes in wording of the compare problem to make the action more explicit as follows: 'I have 11, you have 9; how many more must you get to have the same as me?'.

This change in wording changes the compare problem type into a change increase problem, which was termed an *Equalise* problem type of the RGH framework: 'Joe has 3 marbles. Tom has 8 marbles. What could Joe do to have as many marbles as Tom?' In this example an action of 'what could Joe do' is made explicit. The wording of the equalise problem type question introduces a change action of increasing (introducing a change increase action on the smaller set to equalise the two sets) or decreasing (introducing a change decrease action on the bigger set to equalise the two sets). Carpenter and Moser (1983) have noted that' equalize problems share characteristics of both Change and Compare problems. There is implied action on one of two given sets, but a comparison is also involved' (Capenter and Moser 1983, p. 56).

The equalise problem therefore remains an important problem within the change problem category. While classified as a change problem as the action of increasing or decreasing a set is explicit (making this a dynamic problem), the equalise problem type shares characteristics with the compare problem type, emphasised in the juxtaposition in Figure 15:

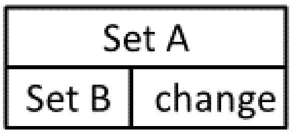
Figure 15: Juxtaposing compare and equalise problems

Compare problems:
Joe has 8. Toms has 5. How many more does Joe have than Tom?
Set A = Set B + difference



Equalise problems:
'Joe has 3 marbles. Tom has 8 marbles. What could Joe do to have as many marbles as Tom?'

Set A = Set B + change



Given that the wording of equalise problem made the action required for comparison explicit I chose to include an equalise problem as a reference example in the intervention design. The following equalise problem was included in the intervention design: 'I have 8 stickers. I get a surprise when I reach 10 stickers. How many more stickers do I need to reach 10?' The problem situation of stickers (and reaching a target of 10) was a deliberate design decision, as this situation was familiar to the children. A context of collecting stickers to reach a target of ten stickers was used as part of the framework for action (See *Feature 5.4: Providing immediate extrinsic recognition of effort which accumulated into a reward*). In addition the wording of the equalise problem deliberately made use of the phrase 'how many more', as in previous interventions cycles learners had shown difficulties in making sense of the 'how many more' phrase used in the compare type problems.

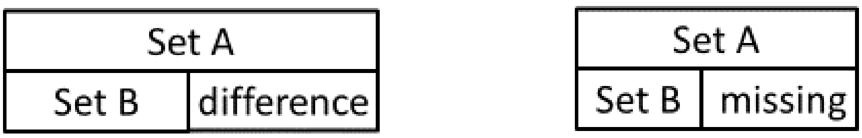
Taking these three findings into account, I also wanted to construct a reference example for the compare problem type that would be likely to invoke a 1:1 matching action – but where there was no need to introduce a change increase action – but which made a 1:1 matching action as explicit as possible. Drawing on the Realistic Mathematics Education (RME) tradition (van den Huevel-Panhuizen 2001), where the starting points of instruction should be experientially real to the learners so that they can immediately engage in personally meaningful activity, I sought to find a problem context that would be likely to invoke the representations and actions required to solve the compare problems. So in addition to changing the wording of the compare problem to be an equalise problem, I sought a compare problem context that would invoke a 1:1 matching action and draw attention to the difference.

I therefore introduced a particular type of compare problem, which was to be used as a starting point for engaging with the broader compare problem type. I referred to this as 'compare (matching)' word problem, and distinguished this from the conventional compare (disjoint set) problems. As with all compare problems, the compare (matching) problem type has two disjoint sets which have to be compared. However the wording of the compare (matching) problem explicitly draws attention to the absence of elements in a smaller set (the difference) by asking 'how many elements are missing?'. An example of a compare (matching) word problems is: 'There are 11 locks but only 9 keys. How many keys are missing?' The choice of locks and keys is deliberate, as in this problem context it is implicit that each key fits uniquely with a particular lock. This unique 1:1 matching of each element in one set to each element in another set is not explicitly implied in the compare (disjoint set) problem 'I have 11. You have 9. How many more do you have than me?' This juxtaposition is summarised in Figure 16.

Figure 16: Juxtaposing compare (disjoint) and compare (matching) problems

Compare (disjoint set) problems:
Joe has 8. Tom has 5. How many more does Joe have than Tom?
Set A = Set B + difference

Compare (matching) problems:
There are 8 locks and 5 keys. How many keys are missing?
Set A = Set B + missing



By including both a change increase (equalise) problem; as well as compare (matching) problem as main tasks in the intervention design, it was hoped that learners would have more experience of the possible actions required for comparison. It was hypothesised that the equalise problem would, simultaneously, give learners experience of introducing a change increase action (adding on the difference) to equalise two disjoint sets and draw learners' attention to a missing set (the difference), which may be mentally visualised when comparing two disjoint sets in a static compare problem situation.

Feature 2.5 A 'structural' approach to additive relations was foregrounded making use of a whole-part-diagram and families of equivalent number sentences for number triples.

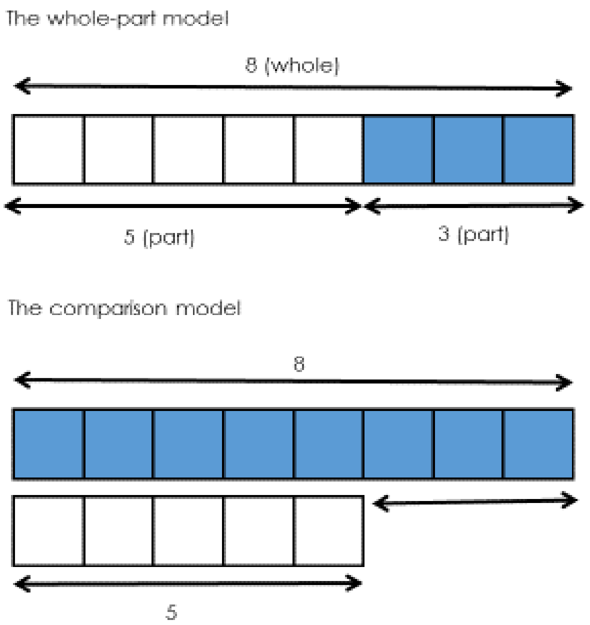
The research conducted by Runesson and Kulberg (2010) on additive relations complicates the CGI framework of change, compare and collection word problems that appear in the South African current curriculum. As noted already, in contrast to research work that advocates a 'counting pathway' for early number work, these researchers (and others such as Schmittau (2004) and Bass (2015)) background counting and foreground structure as pivotal to children's conceptual understanding of additive relations. This research puts forward teaching strategies for additive relations that encourage learners to work systematically to break up and combine numbers considering the whole, and each of the two parts, to deepen their conceptual understanding of additive relations. In this structural approach the relationship between adding and subtracting is foregrounded. The structure which shows this relationship is in focus. Adding and subtracting fit together – they are inverses. Anghileri explains that

> Associated with every addition fact are two subtraction facts that are immediately available without the need for any calculation. If 5 and 3 together make 8, this also means that "8 is 3 more than 5" and 8 is 5 more than 3 or that 8 subtract 3 is 5 and 8 subtract 5 is 3 (Anghileri 2000, p. 54).

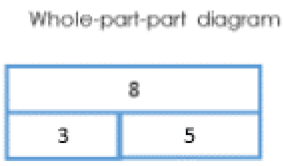
She argues that for young children these number triples (like 8, 5 and 3) should be identified and different ways of representing the triples should be explored. The flexible movement between carefully selected representations is discussed in detail in Feature 4.1: Flexible movement between representations where sense making was primary, was encouraged. However, as the representational images are inextricably bound to the theoretical approaches to additive relations, I focus on these representations in some detail in this section.

Diagrams depicting the structural relations of additive relations were referred to by Resnick et al. (1991) as proto-quantitative part-whole schemas that organise children's knowledge about the ways in which material around them comes apart and goes together. Anghileri described whole-part-part diagrams for particular number triples, and more recently the 'Singapore model method' with its 'whole-part model' and 'compare model' Kaur (2015) has captured much attention globally.

Figure 17: Whole-part-part diagrams



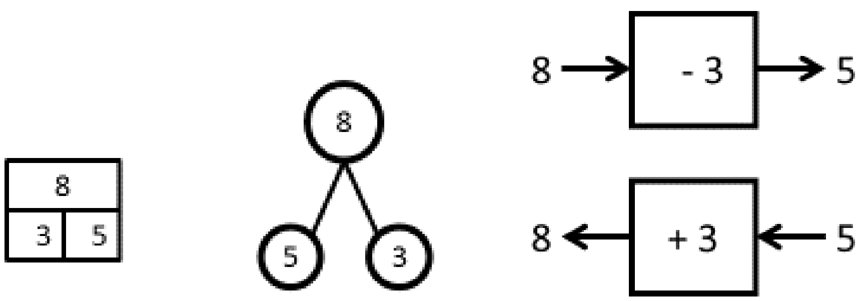
(Kaur 2015)



(Anghileri 2000)

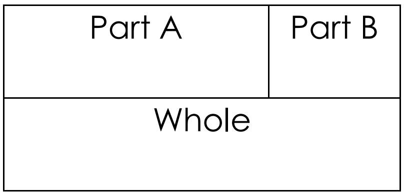
The structure of the whole-part-part diagram is important as the measurement context of length is invoked. The rectangle representing the number 3 is shorter than the rectangle representing 5, and together the two rectangles for 3 and 5, match the length of the whole rectangle of length 8. Although whole numbers are used as exemplars here, the continuous measurement context of length underlies the image. The significance of the distinction between discrete objects and continuous measurement contexts has been discussed in terms of the 'measurement pathway' and will not be repeated here. Suffice to note that the whole-part-part image making use of rectangles and attending to equivalent length was deliberate. This choice was in contrast to selecting an image which depicted a partition, but where there was no reference to (or possibility to extend to) continuous measurement contexts. Thus alternative diagrams which were considered and rejected included the following:

Figure 18: Alternative diagrammatic representations of a number triple (8,5 and 3)



The selected whole-part-part diagram supports the structural approach to additive relation word problems in three ways. Firstly, with this image, all three numbers are visible (the focus is on the three objects); secondly their structural relationship (how they fit together) is presented visually and the cardinal aspect of the numbers are visible (as their size is depicted by length); and thirdly as a generalised version of any additive relation this same imagery can be used for all of the additive relation word problem types.

Figure 19: Structure of additive relations (whole-part-part diagram)



In addition the 'structural approach' provides opportunity for keeping the 'whole' component of the structure invariant, while varying the position of the partition used to define the size of each part. In this way attention is drawn to the horizontal line in the whole-part-part diagram that separates Part A from Part B (and recognising that this can be slid horizontally left or right, to define new partitions). In this way the whole-part-part image becomes a dynamic diagram. The whole-part-part diagram provides a way of keeping all three quantities in the additive relation simultaneously visible, and thus, is in contrast with the 'counting pathway' and the CGI work on change word problems where values tend to be treated separately, with any one of the values being an unknown and so, initially out of view.

Because adding and subtracting are related – they fit together - there are families of number sentences which can be associated with the same diagrammatic representation and problem situation (involving the number triples to which Anghileri refers.)

Figure 20: Family of equivalent number sentences for any additive relation

Part A + Part B = Whole	Whole – Part A = Part B
Part B + Part A = Whole	Whole – Part B = Part A

Switching around the left and right hand sides of these equations gives rise to eight possible equivalent number sentences.

The notion of multiple equivalent number sentences modelling the same problem situation is labelled by Askew (2012) as 'meanings and symbols' and described as a big idea in primary mathematics, where the 'relationship between symbols and situations is not a simple one-to-one mapping but a many-to-many mapping'. The CGI researchers also considered the importance of syntax representations. The way that first and second grade children represented additive relations word problems using number sentences was examined using a quasi-experimental design with two groups of first grade learners. They distinguished between standard number sentences (in the form a + b = […] , or a – b = […]) from open number sentences (where the following six kinds of number sentences were used: a + b = […] , a – b = […]; a + […] = b; and a – […] = b, […] + a = b, […]- a = b. The Carpenter, Moser et al. (1988) study found that learners who were made aware of the alternative forms of number sentences were then more likely to use alternative number sentences where the action in a problem text matched the alternative form. Learners who had not been exposed to alternative number sentences used only the standard number sentence to represent the different word problem types (and tended to be less successful in solving the problems)(Carpenter, Moser et al. 1988).

Drawing on the structural approach together with the findings from CGI work that alternative forms of number sentences ought to be available to learners, in the intervention design I made use of whole-part-part diagrams and related families of equivalent numbers sentences to express the additive relation structure evident for any of the word problem types. I refer to the whole-part-part diagram, and its family of equivalent number sentences, together as a 'syntax model'. This idea is developed further when *Feature 4: Representations* is discussed in the next section.

This literature emphasises that it cannot be assumed that a single number sentence depicts a particular problem type or sub-type; rather there is a whole-part-part structure and related family of eight possible generic number sentences which is common to all word problem types. As such I wanted to ensure that a common whole-part-part structure and related family of number sentences (generalised version) would be experienced as the invariant attribute, while other attributes (such as number, the problem type, the problem context) would be varied.

Table 6: Analytical framework for additive relations word problems adopted in this study

<table>
<tr><th colspan="3">Conventional example space for single solution additive relations word problems</th><th colspan="2">Structural approach</th></tr>
<tr><th>Problem types</th><th>Problem type specific number sentence</th><th>Sub categories</th><th>Common whole-part-part structure (generalised versions)</th><th>Common family of equivalent number sentences (generalised version)</th></tr>
<tr><td>Change (increase)</td><td>start + change = result</td><td>Start unknown
Change unknown
Result unknown</td><td rowspan="4"><table><tr><td colspan="2">Whole</td></tr><tr><td>Part A</td><td>Part B</td></tr></table></td><td rowspan="4">Whole = Part A + Part B
Part A + Part B = Whole
Whole = Part B + Part A
Part A + Part B = Whole

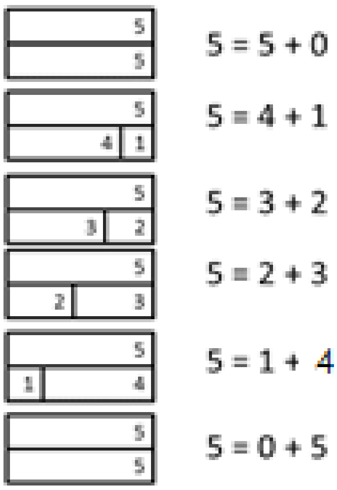
Whole – Part A = Part B
Part B = Whole – Part A
Whole – Part B = Part A
Part A = Whole – Part B</td></tr>
<tr><td>Change (decrease)</td><td>start - change = result</td><td>Start unknown
Change unknown
Result unknown</td></tr>
<tr><td>Collection</td><td>collection = subset + subset</td><td>Collection unknown
Subset unknown</td></tr>
<tr><td>Compare</td><td>whole = referent + difference</td><td>Whole unknown
Referent unknown
Difference unknown</td></tr>
</table>

Feature 2.5 A 'partition' word problem type was used to introduce the additive relations structure

The structural view introduces an additional problem type which is referred to in the literature, but which is distinct from the problem-types described above – partition problems: There are 5 monkeys in the monkey family. Each night they sleep in two trees (a big tree and a small tree). In how many ways (arrangements) could the monkeys be sleeping? (Cobb et al. 1997). I label this a 'partition problem' as the whole is kept invariant, and partitions vary.

Aligned with the broader approach, this problem type makes use of a discrete object problem situation (monkeys in trees). This partition problem was considered appropriate for introducing the structured approach focused on whole-part-part relations, and supporting learners to become familiar with the whole-part-part diagram. When working on the partition problem, whole-part-part diagrams, and families of number sentences to show the relationships between breaking down and recombining numbers (for the systematic development of number bond patterns) were used to support the problem solving process.

Figure 21: Whole-part-part diagrams and number sentences for partition problem



The partition problem was selected as it presented a problem situation where the structure of additive relations could come to be experienced as a dynamic structure. By keeping the whole invariant while varying the partitions, the dynamic nature of the whole-part-part diagram – and the significance of the lengths of each rectangle could be stressed. This dynamic interpretation of the whole-part-part diagram was considered as a prerequisite for meaningful work with a particular instance of a partition.

Feature 2.6 The process of solving a word problem was viewed as an exercise in mathematical modelling

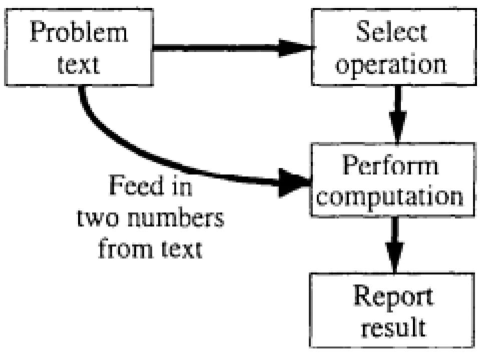
In this design experiment word problems were intended to present opportunities for mathematical problem solving. Literature reviews focused on mathematical problems solving often cite the seminar work of Polya on solving problems as a key source. In this work he provides a four phase guideline on how to approach any mathematics problem: starting with *understanding* the problem, considering how the various elements in the problem are connected in order to devise a *plan* for solving it, then proceeding to the third phase of *carrying out the plan*, and finally *looking back* to ensure that the solution makes sense (Polya 1945, pp. 5-6). Polya presents several guideline heuristics which teachers and students can adopt when approaching a problem where the route to the solution is not obvious (Polya 1945; Chapman 2006). These include some useful lines of questioning for teachers which can be applied even with early grade learners. In this list I present the questions recommended by Polya first, and follow each with the way in which these were adapted for early grade learners:

- *What is the unknown?* What are you trying to find? What is the problem or what is the question?
- *What is the data?* What information are you given in the story?
- *What is the condition?* How does the information fit together? What is happening in the story?
- *Introduce a suitable notation.* Can you write or draw something to help you?
- *Do you know any related problems?* Is there another problem you know that is like this one?
- *Can you check the result?* Does make sense? Does your answer work if you put it back in the story?
- *Can you derive the result differently*? Is there another way you could do this?
- *Can you change the data or the conditions?* Would it help to make the numbers easier?

These are high-level approaches and guideline teacher prompts to a generic problem solving process which were kept in mind when engaging children in solving problems.

A superficial solution to word problems can be contrasted to their problem solving *as* modelling approach. Greer (1997) presents the following flow diagram of the typical process for the superficial solution of word problems:

Figure 22: Superficial solution of word problems (Greer, 1997)



This superficial solution of word problems is frequently approached through a process of breaking down a problem text to identify key words that are associated with a particular operation. This invokes a linear process where, to solve and pose word problems, children are thought to move from the problem situation rooted in some real or imagined context, to select an appropriate mathematical model (commonly depicted using and open number sentence) and then calculating making use of an appropriate strategy. This conceptualisation is consistent with that of Greer, who built on Polya to see the modelling process as an act of translation from the problem situation to the mathematical calculation. In this 'translation' conceptualising of mathematical modelling, the two realms (problem situation, and mathematics/calculation) are seen as distinct from each other. Gravemeijer (1997) refers to this as a model-situation duality.

It was this conceptualisation of a superficial problem solving process which gave rise to a 'key word' approach to problem solving. This 'key word' approach was characteristic of the early research on mathematics word problems. Verschaffel, Dapaepe et al. (2014), referring back to Goldin and McClintock (1984) explain that this early research on word problems:

> focused mainly on the effects on performance of various kinds of linguistic, computational, and/or presentational task features (e.g., number of words, grammatical complexity, presence of particular key words, number and nature of the required operations, nature and size of the given numbers) and subject features (e.g., age, gender, general intelligence, linguistic, and mathematical ability of the problem solver) (Verschaffle, Depaepe et al. 2014, p. 357).

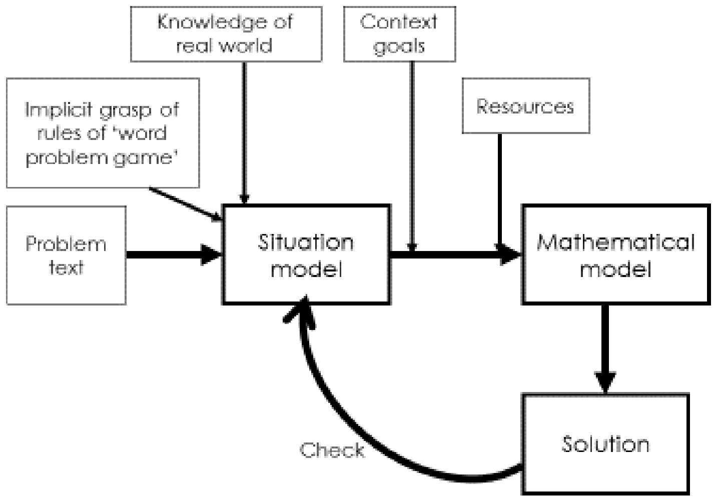
It is important to recognise that this dated approach remains part of the current South African discourse (at least in the government documentation on the diagnostic report on standardised assessments). This type of approach appears to be what is advocated for in South Africa, where having identified low attainment in word problems solving in the diagnostic report of the ANAs, the following 'remedial measures' were put forward by the South African government:

> Procedures of answering word problems should be taught. These include: reading with understanding, underlining the key words and information, translating certain words into mathematics language, identifying the appropriate mathematics operation, and constructing a mathematics sentence. (DBE 2012, p. 110)

Setati (2002) has argued that focusing on the meaning of individual words (key words) or the addition of new vocabulary is not adequate. Her claim is that part of learning mathematics is acquiring control over the mathematics register. This entails 'learning to speak, read, and write like a mathematician' and the mathematics register 'includes words; phrases; symbols; abbreviations; and ways of speaking, reading, writing and arguing that are specific to mathematics' (Setati 2002, p. 9). This suggests the need to go beyond the superficial solution of word problems, where translation of key words into operations is in focus. Thus, an alternative to the superficial solution of word problems was required for the intervention.

As an alternative to the superficial solution, Greer (1997) argues for mathematics teachers to 'treat word problems as exercises in modelling (or mathematization)' (p. 270). The modelling process is 'viewed as the link between the 'two faces' of mathematics, namely its grounding in aspects of reality, and the development of abstract formal structures'. This specifies the purpose of the word problem as to 'understand the situation, make reasoned and reasonable assumptions, and construct a model-or more than one model' (Greer 1997, p. 290). The following alterative flow diagram is offered for the process of solving a word problem as an exercise in mathematical modelling:

Figure 23: Schematic diagram of factors influencing the modelling of a word problem. (Greer, 1997)



In this approach to problem solving *as* modelling: the problem situation, the mathematical modelling process and the strategies used for calculations are interrelated, and constitute a dynamic (and not a linear) process. This complication is explained by Gravemeijer who refers to an organizing approach that emphasizes modelling as an activity which builds on Freudanthal's notion of 'organising the subject matter', rather than Polya's notion of translating from the situation into the mathematics (Gravemeijer 1997). Gravemeijer explains this notion of a dynamic modelling process as follows:

> To be more precise, at first a model is constituted as a context-specific model of acting in a situation, then the model is generalized over situations. Thus, the model changes character, it becomes an entity of its own, and in this new shape it can function as a model for more formal mathematical reasoning. (Gravemeijer 1997, p. 394)

To elaborate on mathematics as modelling, and argue against the superficial solution to word problems, Gravemeijer distinguished *models of* from *model for*. The *model of* refers to the situation sketched in the problem statement which is 'meaningful to the student because of this reference to a concrete situation'. The *model for* refers to a reified model, which results from a process of acting on the *model of*. With more experience of the *model of* in a variety of situations, attention shifts 'from the original situation to the mathematical relations involved'. The *model for* is used for reasoning about future problems, it is the generalised version of the *model of* (Gravemeijer 1997, p. 394). On the teaching side, attention is on ways to draw a learner's awareness to one specific realm (the calculation strategy, the process of modelling, or the problem situation) through the design of a task. Aspects can be stressed, while others are ignored and what is in focus is the shift in attention from the specific to the general and vice versa.

Domain-specific instructional theories relating to a narrative approach to mathematics

As explained in Chapter 1, I draw on narrative as comprising both stories and representations. The former refers to story-telling, the theoretical features of which are described here (Feature 3), and the latter refers to drawing, writing or illustrating the mathematics in these stories, which are elaborated upon in the next section (Feature 4). The following table provides a summary overview of the theoretical features that were considered to be key to the '*how*' of the mathematics of the intervention.

Table 7: Summary of major features of the theoretical framework of mathematics teaching underpinning the last design cycle

Feature	Title	Description
Mathematical design features of the intervention		
3	Using story telling as pedagogic and as a cognitive strategy	3.1 Paradigmatic knowing / logical-scientific thinking and narrative knowing/thinking were viewed as necessarily and simultaneously present in any mathematics word problem 3.2: Story telling was used as a pedagogic strategy to motivate learning and encourage sense making 3.3 Story telling was viewed as a cognitive strategy which draws on the human powers of imagining and expressing 3.4 Tasks that demanded story-telling and modelling in the LoLT were designed to support ELLs in their dual need to deepen conceptual understanding of additive relations, and to improve their English language proficiency 3.5 The word problem was viewed as a cultural artefact which required story-telling and retelling performance to build towards enculturation in the social community of the class
4	Using representations to interpret calculation strategies	4.1 Flexible movement between representations where sense-making was primary, was encouraged 4.2 Secure use of a particular representations takes time and, over time, representations should be reified to become cognitive tools 4.3 In the process of being inducted into formal symbolic notation, shifts in attention are required in both directions – from concrete *objects* to symbolic *objects* (and vice versa), and both *processes* of specialising and generalising are necessary. 4.3 A learning-teaching trajectory from counting to calculating which made reference to increasingly structured representations using line, group and syntax models was adopted

Feature 3: Using story telling as a cognitive strategy and a pedagogic strategy

Approaches which include teacher use of storytelling to motivate and encourage learners, are contrasted to approaches which envisage learner use of story telling as a cognitive strategy to support sense making. Both approaches to story telling (as a pedagogic strategy and as a cognitive strategy) relate to Feature1.3 Learning as educator awareness and harnessing natural powers (in this case of imagining and expressing), and both approaches figure within the design of the intervention. The choice to use oral story telling as a key teaching strategy within the intervention was motivated by the South African context of the dominance of oral expression, and the dual need of ELLs to deepen their understanding of additive relations and improve their English language proficiency.

Feature 3.1 Paradigmatic knowing / logical-scientific thinking and narrative knowing/thinking were viewed as necessarily and simultaneously present in any mathematics word problem

I introduced Chapman's distinction between narrative and paradigmatic ways of knowing in relation to word problems to explain the structuring of the constraints imposed on the intervention. Recall that narrative knowing in relation to word problems focuses on the social context of the problem, while the paradigmatic knowing focuses on the mathematical models and structures of the problem. A similar epistemological distinction has been made with regard to learning in general by Bruner (1996), who identified two broad ways by which human beings organise and manage their knowledge of the world—logical-scientific thinking and narrative thinking— and observed that schools have traditionally favoured the former. He criticised the limited attention to narrative as being problematic, for, as he said, "if narrative is to be made an instrument of mind on behalf of meaning making, it requires work on our part—reading it, making it, analysing it, understanding its craft, sensing its uses, discussing it" (p. 41). Bruner did not advocate for narrative thinking over logical-scientific thinking, but rather saw the two as mutually supportive: 'scientific explanations are adjuncts to narrative interpretation and vice versa' (p. 92) and noted that 'we may have erred in divorcing science from the narrative of culture' (p. 42).

The pedagogical approach adopted for this design experiment followed Bruner's argument giving rise to Feature 3.1 Paradigmatic knowing and narrative knowing were viewed as necessarily and simultaneously present in any mathematics word problem. Like Bruner I see logical-scientific thinking and narrative thinking as mutually supportive ways of thinking about the world, and solving problems. I recognise too that the narrative approach has been neglected in schools, particularly in the field of mathematics learning, where the social context is commonly viewed as background noise, to be stripped away to reveal the mathematical purpose – a point that has been made by Freudenthal (1991): 'context is not a mere garment clothing nude mathematics, and mathematising is quite another thing than simply unbuttoning this garment' (p. 75). It is within this awareness of duality and mutually supportive properties of narrative thinking and logic-scientific reasoning that I moved to focus explicitly on story telling.

Feature 3.2: Story telling was used as a pedagogic strategy to motivate learning and encourage sense making

Story telling has been selected as the focus of the pedagogic strategy as a result of growing attention in the field of cognitive psychology, which is starting to influence mathematics education, on the potential of narrative in supporting learning. Egan (1989) considered narrative in relation to education in general, and argued that story is the missing link that can bring learning and imagination together. According to him, stories can make whatever is to be learned into something meaningful, and can engage the imagination in the process of learning. Egan (2002) discussed further how educators 'might represent the world in narrative terms to children for whom this becomes a major tool for learning' (Egan 2002, p. 72). In relation to mathematics learning, Zazkis and Liljedahl (2009) refer to stories as used by teachers to explain a problem situation as 'stories for explaining'. A similar teacher centred approach to the use of storytelling in mathematics is advocated by Schiro (2004) where 'epic oral storytelling' by primary school teachers is described. From this research base – in education generally, and mathematics in particular storytelling was used as a pedagogical strategy to motivate learning and encourage sense-making (Feature 3.2).

Feature 3.3: Story telling was also viewed as a cognitive strategy which draws on the human powers of imagining and expressing

There is also another use of stories in learning, rooted in the literature on cognitive psychology, focused on providing an opportunity for children to tell and retell their own stories as a way of constructing meaning and make sense of ideas. This involves a shift from a focus on the teacher as story teller, to the child as narrator. Bruner (1996) reflected on this when referring to narrative as a vehicle of mind, and later elaborated on stories as a pervasive human habit which are used in multiple ways (Bruner 2003). Bruner (1996) proposed that humans are essentially narrative animals that tell stories to themselves and others as a way of making sense of the world. He broke down the components of narrative as follows:

> 'A 'story' (fictional or actual) involves an Agent who Acts to achieve a Goal in a recognizable Setting by the use of certain Means. What drives the story, what makes it worth telling is Trouble: some misfit between Agents, Acts, Goals, Setting and Means' (Bruner 1996, p. 94).

Mason (2008), focusing specifically on mathematics learning, identified 'imagining and expressing' as a key 'children's power', which should be harnessed to develop their mathematical thinking or sense-making in general, and to enable children to produce algebra, in particular. Mason (2007) described the relevance of stories as follows: 'human beings are narrative animals: they have a deep seated need to tell (portray, display, act out) stories that account for their experiences and their history, and a strong need to recount these to others as a basis for social interaction' (Mason 2007, p. 60) .

I therefore not only wanted children to share and understand teacher presented stories, but to give them ample opportunity to shape their own stories and recount these to each other.

Feature 3.4 Tasks that demand story-telling and modelling in English were designed to support ELLs in their dual need to deepen conceptual understanding of additive relations, and to improve their English language proficiency

A further rationale for experimenting with narrative in mathematics where children are supported to tell and retell stories orally, is the complex language context in South Africa.

There are few researched interventions where stories (that make extensive use of oral language) are used as the key pedagogic strategy in a mathematics classroom. A central hypothesis going into this study was that deliberate attention on language, through tasks that demand story-telling and modelling in English, may support learners in their dual need to deepen conceptual understanding of additive relations, and to improve their English language proficiency to articulate and decode statements of comparison (which are a critical feature required for solving many compare problems). The kind of tasks designed to engage children in a process of telling stories have been expounded upon in relation to Feature 1.7 Seeking to gain insight into learners personal potential example spaces, through learners generating examples.

Feature 4: Flexibly moving between representations to explain and interpret additive relations

I turn now to discuss the use of representations in mathematics learning in general with particular attention placed on their use in relation to additive relations concepts with young learners. There are two important findings reported in the mathematics education literature which have informed this theoretical feature.

Firstly, the progression described earlier in the 'counting pathway' from: count all, count on from the first number, count on from the larger number, and then use known and derived number facts has been associated with progression in children's representations of whole number arithmetic. Limited number sense is related to representations that are closer to concrete actions, and good number sense is related to more compressed symbolic representations (Askew and Brown 2003).

Secondly, there is growing consensus in South Africa that one of the major factors inhibiting learners' mathematical progression is continued using of counting in ones strategies for mathematical calculations. This has been highlighted by Hoadley (2012) drawing on several earlier studies. In the Foundation Phase, Ensor et al. (2009) have identified that learners remain reliant on counting-based strategies for calculations making use of concrete and iconic models of representation, and do not shift to more symbolic representation modes. This was confirmed by Schollar (2008) who reported the prevalence of concrete counting strategies well into the Intermediate Phase, and raised as a cause for concern by Weitz and Venkat (2013).

Taking these two findings together pointed to the importance of, and need to, focus on discussions of the external representations used to support calculation strategies and problem situations for additive relations. I considered these external representations of numbers and additive relations calculation strategies to comprise an 'example space'.

To provide some indication of the literature landscape relating to representations in mathematics learning, I first briefly revisit the use of the term 'representations' in this study, and refer to the sources of literature where flexible movement between representations is considered important for mathematics learning. The ways in which representations are used in mathematics, and how this takes time, and changes over time, is elaborated upon. I then narrow the scope from mathematics in general to focus attention on representations used to depict additive relations relating to whole numbers less than twenty. In so doing I provide an overview of the way in which additive relations are represented using formal mathematical notation of number symbols and operations; which is followed by examples of diagrammatic representations where additive relations are depicted drawn from the conventional example space. My source of data for the conventional example space of additive relations representations is two-fold: the mathematics education literature on early grade additive relations, and the South African curriculum for Foundation Phase (CAPS). Having defined, and exemplified what is being referred to in terms of formal mathematical notation and diagrammatic representations I then explain two conceptual frameworks which have been used to categorise the representations of young children. The first framework attends to different modes of representation. The second framework is an assumed learning trajectory in terms of representations used to support increasingly efficient calculation strategies from concrete modelling, to counting to calculating. This connects with Feature 2.1: A counting-based conception of early number development was adopted which encouraged a progression from concrete modelling to counting to calculating, but in this case the representations used and their increasing structure are in focus.

Feature 4.1 Flexible movement between representations where sense-making was primary, was encouraged

Several researchers have identified the importance of representations in supporting children's problem solving processes in mathematics, referring to these with various terms: picture and models (DeLoache, 1991), representations (Barmby, Bolden et al. 2014), models, images, and tools (Askew 2012). For the purpose of this study, I use the concept 'external representations' (hereafter 'representations') – to denote children's talk, markings, drawings and writings in mathematics. So external representations are taken to be a form of communication which makes children's internal modelling (thinking) visible to self and others. Further I have distinguished 'stories' (the term used for an oral or written narration using words, often involving natural language) from other representations (including: diagrammatic representations used for drawings or illustrations depicting mathematical situations; gestures, denoting movements or actions made by the narrator, teacher or child; and symbolic representations using written text, number symbols and operational notations).

Several early mathematics education researchers in primary mathematics explored the shifts from concrete, enacted experiences of processes to abstracting a general mathematical principle which is then symbolised using formal mathematical notation. This gave rise to much debate about 'abstract/symbol first' or 'procedure/concrete first' approaches to learning.

Dienes (1963) explored a procedure/concrete first approach and examined children's engagement with the Dienes blocks (blocks structured to define groups of tens, hundreds and thousands) with place value addition and subtraction. However through such experimentation with children, the expected progression from concrete to symbolic was found to be a complex process, which is closely related to abstraction (a shift in attention from the general to the specific and vice versa)

(Dienes 1963, p. 68). The criteria for establishing whether a symbolic structure had been learnt were found to be problematic, and the variability of embodiments enacted by different children was seen to become so 'noisy' that the envisaged structure could no longer be discerned.

The work by Dienes documents an early case of using concrete materials (often referred to as 'apparatus' or 'manipulables' in the literature) to support a mathematical process of addition and subtraction, where increasing symbolisation is envisaged. From these early stages, starting with concrete apparatus was thought to have some value in supporting the learning process. However this assumption was not universally accepted, with other theorists pointing out that embodiment (how the concrete materials are acted on and viewed) depends on the thinking of the child while they act. This was a point made by Holt (1964) and repeated by Gravemeijer (1994). A further point which brought into the question the supposed wisdom of 'begin with the concrete' highlighted the difficulties inherent in the concrete-abstract, and concrete-symbolic distinctions was made even earlier by Dewey:

> Instruction in number is not concrete merely because splits or beans or dots are employed. Whenever the use and bearing of number relations are clearly perceived, a number idea is concrete even if figures alone are used. Just what symbol is best to use at a given time – whether blocks or lines of figures – is entirely a matter of adjustment to a given case (Dewey 1933, p. 224).

Dewey refers to a 'number idea' as 'being concrete', which brings to mind Sfard's notion of a 'thinkable object'. The distinction between concrete and symbolic is not simply a matter of material and mind. The concrete can be in the mind, and the symbolic can be material (symbolic notation on a page is a concrete/materials artifact). The debate of 'abstract/symbol first' versus 'procedure/concrete first' has faded, and sense-making in all mathematical engagement (whatever the mode of representation) has been foregrounded:

> 'Neither the abstract-first nor procedure-first approach to learning fosters the intention to make sense…Often when manipulatives are used in teaching mathematics, the teacher demonstrates the way they are to be used and students are given very little freedom to meaning to the experiences in ways that make sense to them…' (Wheatley 1992, pp, 533-534)

Much research (Wheatley 1992, pp 533-534; Sasman, Linchevski et al. 1999; Tall 2004; Sfard 2008) has been conducted into children's thinking, and their conceptual and procedural development in mathematics. Across this body of work there appears to be growing consensus that central to mathematical thinking is the ability to move between the multiple representations, however these are defined -Bruner's (1994) enacted, icon, and symbolic representations, Tall's (2004) three worlds or any other categorisation of representational realms in mathematics. Boehster and Loeher (2008) refer to this as 'meta-representational competence': the ability to 'represent similarity in multiple forms,… to develop conceptual relationships among these different representational forms' (p. 212). Abstraction to, and manipulation of, the formal symbolic notations are no longer a destination, but rather one possible lens through which to view and think about a problem.

Feature 4.2 Secure use of a particular representation takes time and, over time, representations should be reified to become cognitive tools

While the above advocates for flexible movement between representations, there is simultaneously recognition that making effective use of a particular representation takes time, and the purpose of and way of using it evolves over time. Askew (2012) makes the point that introducing models such as the empty number line or the array takes time, and argues that children will only come to appreciate the power of these models through repeated exposure to them and then it takes them different times to take them on as models for working with and then as tool for thinking (Askew 2012). Askew acknowledges the time taken to introduce and encourage use of particular representations, and he refers back to the Gravemeijer's conception of mathematical modelling. At the point when a 'model for' becomes a thinkable object which can be brought to mind and used to reason about a new problem situation, it is referred to as 'a tool for thinking' (Askew 2012, p. 112).

Feature 4.3 In the process of being inducted into formal symbolic notation, shifts in attention are required in both directions – from concrete objects to symbolic objects (and vice versa), and both processes of specialising and generalising are necessary.

As flexible movement between representations (Feature 4.1) is valued for the learning of mathematics, the ways of describing different modes of representation are useful (although the direction of movement is no longer in focus). Various mathematics education theorists have focused on the formats or modes of representations and used different terminology to describe these. In this section I provide a short history of the development of these arguments, and then present and critique a framework for reflecting on the modes of representation which has featured in the South African Foundation Phase mathematics literature.

Piaget theorised four developmental stages: sensory-motor, to pre-operational, to concrete operational and to formal operational stages (Piaget 1970). These developmental stages were linked to developmental age-bands and informed curricula development for a generation or two (Gravemeijer 1997, p. 163). However this 'stage-age' mentality has been widely criticised and Mason advocates for the 'developmental stages' to be viewed as 'forms of attention', which all humans can utilize. He recognises that particular learners may have different dispositions with a tendency for one form of attention to be dominant, or triggered in different contexts. This marks an important shift away from stage-age approaches and aligns more comfortably with Feature 4.1 Flexible movement between representations where sense-making is primary.

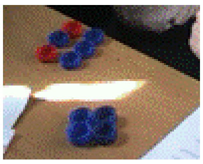
Responding to the shift away from 'stage-age' approaches, Bruner (1994) distinguished three representational realms: enactive, iconic, and symbolic representations, where he considered enactive representations consisting of material objects (such as 'a chair'), iconic representations to be drawings or pictures of the enacted representation (a picture of a chair), and symbolic representation to be notations agreed to signal the enacted object (the word 'chair'). Notice that he used 'realms' and not 'levels' to distinguish his categories, as he did not advocate for particular progression or direction of movements between these representational forms. Pierce included an additional category of representations between the iconic and symbolic realms, and referred to this as 'indexical'. An indexical representation refers to the object but the object is not recognisable in the marking. An example of an indexical representation is a tally mark, or dot used to depict a chair. This is contrasted to an iconic representation where chair-like properties (such as a seat and legs) are discernible in the representation. Dowling (1998) drew on the work of Peirce to develop three modes of signification: iconic, indexical and symbolic in an analysis of representations in a mathematics textbook scheme in England. Ensor et al. used Dowling's work to categorise the representations of number evident in early grade South African classrooms, describing each mode as follows:

- Concrete apparatus which entailed the manipulation of physical objects…This apparatus was used for counting and for calculation-by-counting strategies)
- Iconic (*images of everyday context realistic depictions*) apparatus including photographs, cartoons or drawings. This apparatus was used as concrete apparatus but could not be manipulated in the same way.
- Indexical (*images of everyday contexts – generic rather than realistic depiction of everyday contexts*) apparatus features drawing of sticks, tallies, dots, circles and other shapes represent everyday objects. This apparatus was used for counting and calculating-by-counting tasks
- Symbolic – number based *(use of numerals to represent numbers)* apparatus including number lines (structured or semi structured) number charts, number cards. This mode of representation supported calculation without counting but could also be used for calculation-by-counting tasks
- Symbolic-syntactical (*use of mathematical notation to produce mathematical statements*). This mode of representation is abstract and entails the deciphering and production of mathematical statements. It relies on known number facts, and facts which can be derived without counting.
- No representation used. This refers to tasks which learners are asked to carry out which did not entail the use of modes of representation (Ensor, Hoadley et al. 2009, p. 17).

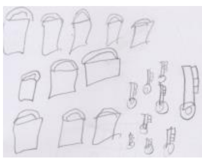
By referring to external representations that are drawn or written, the Ensor et al. (2009) framework is inherently visual. As such I exemplify their framework using examples from learners' work:

Figure 24: Exemplification of Ensor et al. framework using examples of learner work

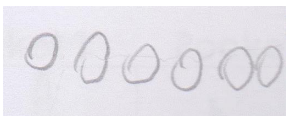
Concrete (manipulation of physical objects)



Iconic (images of everyday context realistic depictions). In this example 11 locks and 9 keys are depicted.

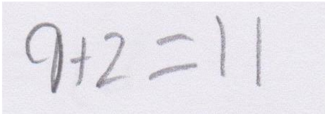


Indexical (indexes everyday contexts – generic rather than realistic depictions)



Symbolic – number based (use of numerals to represent numbers)

Symbolic syntactical (uses number sentence appropriately)



Ensor et al. (2009) acknowledge that their categories of strategies are neither full nor an ideal sequence in the learning of number, but they view them in relation to progression of increasing specialisation, stating that ‘the specialisation of content in the teaching of number entails a shift from counting, to calculating by counting to calculating without counting’ (Ensor, Hoadley et al. 2009, p. 15). The Ensor et al. (2009) study was limited as the only sources of data were children’s written work, and the artifacts in display in the classroom, and from this they inferred calculations strategies.

I concur with Askew and Venkat (2012) that there are gradations *within* Ensor et al.’s (2009) representational categories which include calculation strategies that are likely to involve some reified number facts and some counting. I think recognising that many South African children feel secure using indexical representations provides an opportunity to work *within* the indexical realm to make more group-wise and structured representations, which are more suited to larger numbers.

Feature 4.4 A learning-teaching trajectory from counting to calculating which made reference to increasingly structured representations using line, group and syntax models was adopted

The learning-teaching developed by Treffers (2008) was briefly introduced as part of Feature 2.1: A counting based conception of early number development was adopted. This trajectory outlined here takes the progression from counting to calculating as its starting point. While the teaching-learning trajectory has been refined for the South African context (van den Heuvel-Panhuizen, Kühne et al. 2012), I found that Treffers’ teaching-learning trajectory provided more detail on how shifts in representations are encouraged, and on the transition from object-based to number-based approaches where the shift from loose materials to structured materials is explained. Further, there was greater emphasis placed on real-world meanings and a central task in the early years of mathematics was to give real-world meanings to numbers and to structure them in ways which the various numerical operations can be visualised (Treffers and Buys 2008). It was this aspect of the trajectory which seemed to align more closely with the intervention content focus on additive relations word problems. As such a learning-teaching trajectory from counting to calculating which made reference to increasingly structured representations was adopted and guided the design of the intervention, and was used to interpret the children’s representations created during the study (Feature 4.4) – see Figure 25.

Treffers (2008) encourages processes of mathematization that incorporate shifts between the concrete and the symbolic rather than focusing on the mode of representation, and in so doing he attends to the structure of all modes of representations. The teaching-learning trajectory outlines 10 broad levels of development from counting to calculating spanning from very young pre-school children, to children in upper primary school. A synopsis description of the trajectory and the features of each level is presented below:

Figure 25: Summary of learning trajectory and progression in representations (Treffers 2008)

	Descriptor	Learning trajectory	Representations
1	Learning to count (approximately age 2)	The children know the counting sequence at least up to ten (p. 32),	Attention is paid to counting, singing and moving games and varied reciting of the counting sequence, for instance by counting whilst tagging or counting a row of items (p. 32).
2	Context-bound counting and calculating	Within what are, for them, meaningful context situations, children are able to count at least to ten, arrange numbers in the correct order, make reasonable estimates and compare quantities as being more, less or equal (p. 35).	Counting of up to four objects can be seen at a glance (subitizing) Arranging objects in a disordered pattern, in a row, or in a recognisable pattern affects reliability of counting (p. 37).
3	Object-bound counting-and-calculating	Children can order, compare, estimate and count up to ten objects (p. 37). They are able to select a suitable strategy for simple addition and subtraction situations in such things as concealment games for up to ten objects (p. 39).	Estimation, counting and calculation puzzles with concrete objects that are either visible or where some are covered up and therefore have to be imagined. Children gradually develop and understand various meanings and functions of small numbers and relationships between them. During this process they learn to recognise numerals (p. 40).
4	Towards pure counting-and-calculating via symbolization	Children can represent physical objects up to ten on their fingers and with lines and dots and are able to use such skills as adding up and taking away (p. 39).	Using fingers, drawing lines or dots, rapid recognition of number images up to twenty (doubles and images of groups of five and ten), positioning numbers on a number line, and related skill of ordering numbers, a string of twenty beads with a five structure is used to support this (p. 46).
5	Calculation by counting where necessary by counting materials	Children can recite the number sequence up to twenty and can count up and down from any number in this domain. They can also put numbers up to 20 into context by giving them a real-world meaning, can structure then by doubling and using groups of 5 and ten, and place them on an empty number line from zero to twenty (p. 46).	Children structure numbers by: - Using fingers in a way that shows structure (of 5 on one hand) - Doubling, - Using groups of 5 and 10 - Using a semi-structured number line from 0 to 20
6	From counting to structuring	Learn basic number images for all the number up to 10 using doubles, near doubles and grouping by 5.	Children structure numbers by: - Using fingers in a way that shows structure (of 5 on one hand) - Doubling, - Using groups of 5 and 10 Using a structured number line or number rack - Using semi-structured number line from 0 to 20

	Descriptor	Learning trajectory	Representations
7	Calculation by structuring with the help of suitable models	The children should be able to add and subtract quickly in the number area up to twenty by structuring the numbers and, in time, they should be able to perform formal calculations with the help of remembers number properties. They should be able to use this skill in elementary context situations and be able to both understand and use some conventional mathematical notation (p. 55).	Children make use of the following structured models: - Line model - Group model - Combination model
8	Formal calculation up to twenty using numbers as mental objects for smart and flexible calculation without the need for structured materials	Children should be able to perform formal calculations with the help of remembers number properties. They should be able to use this skill in elementary context situations and be able to both understand and use some conventional mathematical notation (p. 60).	Children in grade 1 need to give a 'real-world' meaning to numbers up to twenty (and beyond). That refers to the named numbers and measuring numbers from everyday experiences , such as anchor points for times, ages, measurements, prices and so on. They also need to develop a good grasp of the order and structure of the whole numbers up to twenty: they need to be able to position the numbers (approximately on the number line; order them by size, and split them into smaller numbers. In this way the numbers in the counting sequence acquire not only content but also structure, and they form and increasingly intricate network of relationships in the number system and beyond it in the real-world numerical information. Children learn to carry out, apply and write down the operations of addition and subtraction. They learn to see the underlying arithmetical structure of every manipulation and know their various 'real' manifestations (addition as 'with' or 'together' and subtraction was 'take-away' or 'difference' (p. 60).
9	Counting up to one hundred	Children can recite the counting sequence up to one hundred and can count up and down from any number in this domain. This applies to both small counting of ones (1, 2, 3, 4, ...) and the big counting sequence of tens (10, 20, 30, 40) (p. 65).	They should be able to place the numbers up to one hundred on an empty or almost empty number line, structure numbers into tens and ones and contextualise them in meaningful situations (p. 65).
10	Calculating up to one hundred (approximately age 9)	By then end of grade 2 the students have memorised additions and subtractions up to ten and have automized them up to twenty. They should be able to solve addition and subtraction problems up to one hundred both in context and in bare number format (p. 74).	The children may use an empty number line, write down intermediate steps or do it entirely in their heads (p. 74).

This Treffers framework is a coarse-grained delineation of development levels spans 7 years, and as such it was not appropriate to consider for the fine-grained learning gains hoped for in a short

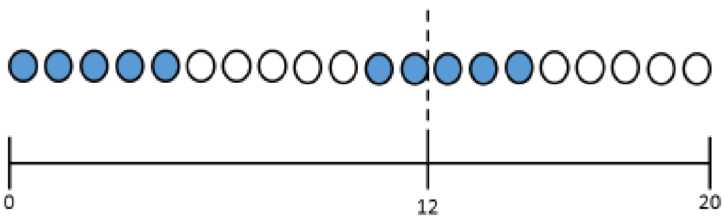
(10 day). Nevertheless, this trajectory provided the broader framework in which the short intervention was situated. It was used to map the assumed starting points for the learners in the intervention, which is described in detail in Chapter 4. Considering the starting points in this framework, it is important to notice that being able 'to count at least to ten, arrange numbers in the correct order, make reasonable estimates and compare quantities as being more, less or equal' is placed early on in the teaching-learning trajectory (located at level 2, and mapped to pre-kindergarten or 3 years of age, in the context of the Netherlands). This approximate age mapping coheres with literature review by Henning (2014) which asserts that 'Children begin to understand the concept of 'the next number' at around the age of 3 years (Huang, Spelke, & Snedeker, 2010; Spelke, 2011), and thereafter learn the principle of 'one more than'' (Henning and Ragpot 2014, p. 4).

What was taken from this trajectory, for the purpose of the short intervention, was the way in which representations can be used in increasingly structured ways. Treffers explains that in general, numbers up to twenty can be represented by means of three different structural models: a line model; a group model and a combination model. When describing these models he refers to different types of number lines, and specialised apparatus where the group-by-five structure of numbers is imposed.

Line model

The line model makes use of arranging or ordering along a horizontal or vertical line. When shown on a line the number 12 looks like this:

Figure 26: A line model for 12[12]



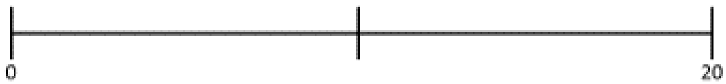
A structured number line includes all whole numbers, whereas a near empty or semi-structured number line includes pertinent reference numbers such as 0 and multiples of 5, 10 or 20, and an empty number line, refers only to the numbers which are in focus for a particular calculation. A bead string, in this case, refers to 20 beads on a string which are colour coded to depict groups of 5 (for example a string containing 5 red beads, 5 white beads, 5 red beads 5 white beads).

Notice how the line model combines several modes of representation: it could be a concrete depiction using counters or a bead string, it could be an iconic or indexical representation depicting

[12] Image source: Recreated from diagram in Treffers p. 45

drawings of the counters/beads, and it includes symbolic number symbols making use of the semi-structured number line. In elaborating on the learning trajectory for the use of the line model, Treffers provides two tasks which learners should be able to perform. Firstly 'the students should be able to select the numbers straight away using a string of 20 coloured beads divided into groups of 5' (Treffers 2008, p. 43). By using the phrase 'straight away', Treffers implies that children do not use a counting in ones approach to this task. Rather they make use of the 5-5-5-5 structure of the bead string and make use of known fact knowledge that 12 can be partitioned into '10 and 2', or '5; 5 and 2' to locate the 12 beads. Secondly the learners 'should be able to roughly locate numbers on a number line divided roughly into two or four equal parts' (Treffers 2008, p. 45).

Figure 27: Position a number on the number line (which is divided into 2 equal parts)[13]

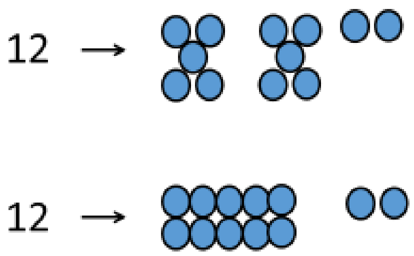


With this task, learners are not expected to use a count-by ones strategy to position the number onto the number line sketch. It is assumed that the midpoint of 20 (as 10) is a known fact. Learners then work with their knowledge of whether the number in question is bigger or smaller than 10, and estimate its position on the line.

Group model

When describing the group model, Treffers (2008) explains that numbers up to 20 can be grouped and split into ones, fives and tens, in a variety of representation. One example involves tallying, the same is possible using fingers (with one hand being a group of 5 ones).

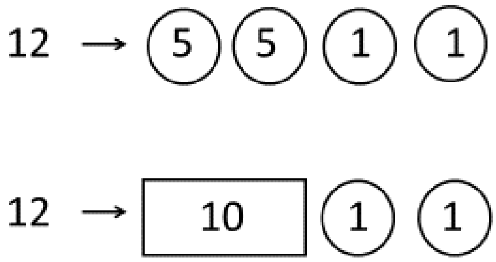
Figure 28: Group model depicting 12 as comprised of groups of ones, fives and tens (concrete/iconic/indexical mode of representation)[14]



[13] Image source: Recreated from diagram in Treffers, p45.
[14] Image source: Drawn from description provided by Treffers, p45

In these group models the objects remain countable within the groups, and they can be visualised and accessed by the children. With money using coins and notes (where number symbols are used), the units are no longer countable, only the value they express is visible.

Figure 29: Group model depicting 12 as comprised of groups of ones, fives and tens (symbolic mode of representation)

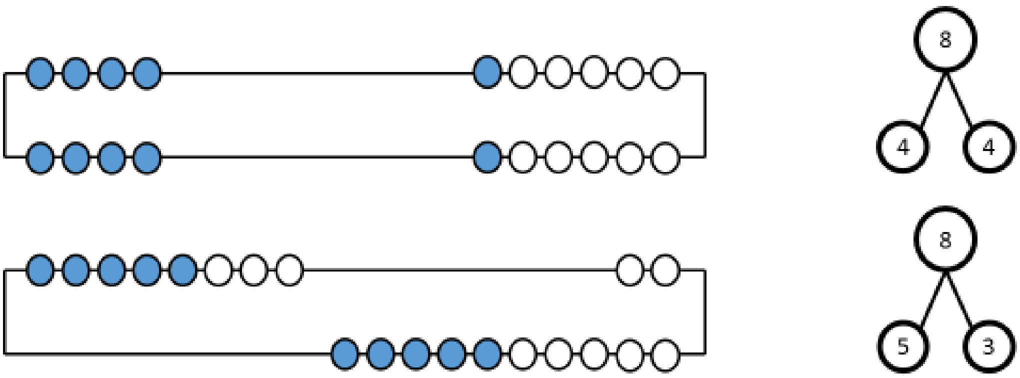


Using the group model, groups of fives, tens and ones are privileged. This is the same structure imposed on the line model. However in the group model the arrangements of these groups are not in a horizontal or vertical line. Learners are encouraged to impose the structure (by arranging single counters into groups (concrete mode of representations), or drawing arrangements of objects in groups (iconic/ indexical modes of representation) or partitioning number using symbolic representations (symbolic mode of representation) into groups of ones, fives and tens.

Combination model

To explain the combination model, Treffers (2008) uses an example of an arithmetic rack – a variation of the traditional abacus. This can be envisaged as a double bead string (where there are 10 beads on each string partitioned in groups of 5, and then used in parallel).

Figure 30: Combination model showing partitions of 8[15]



[15] Image source: Recreated from diagram or arithmetic rack in Treffers p. 45. The whole-part-part diagram was added into this to make the partitions of 8 explicit. Treffers makes use of an alternative diagrammatic

The combination model is used to work with both groups by five/ten (as in the line model) as well as groups by doubling. So Figure 8 shows 8 partitioned using halving/doubling (double 4 is 8) and 8 partitioned using the group of 5 structure (8 is 5 and 3). Although the constraint that options for structuring numbers, stressed groups of five and ten (doubles and near doubles were ignored) meant that portioning using doubles was not in focus, the idea of a structural relationship expressed as a partition of a whole into two parts was of relevance to the study. This aspect focused explicitly on the relationship between the objects in an additive relationship.

The ‘combination model’ referred to by Treffers, and exemplified by the arithmetic rack shows that structuring a number up to twenty can involve both a group and line. An arithmetic rack or double bead string is used to represent the possible partitions (shown either as groups or as lines).

Syntax model

Treffers refers to the partitioning of numbers in this flexible way as ‘splitting’ which he depicts using a number triple. Although the process of ‘splitting’ and representing these partitions visually with objects called ‘number triples’ is discussed in Treffers’ learning trajectory, these are omitted from his categorisaton of representations. He limits his categorisation of structural representations of numbers to line, group and combination models. In my view this suggests an omission or a gap in Treffers’ categorisation of representations which can be used to structure numbers. To fill this gap, I have introduced an additional model to the Treffers categorisation of structural models to represent numbers: a ‘syntax’ model.

The ‘syntax model’ makes use of the term syntax to mean ‘a system or orderly arrangement’ and is used to refer to the expression of the structural relationship of a particular additive relation. A syntax model could depict the structural relationship using a whole-part-part diagram, or the 8-5-3 number triple (as shown above by Treffers) using a triad of circles, or using number symbols and operations such as $8 = 5 + 3$ and the equivalent expressions of the same relationship $8 – 5 = 3$, $8 – 3 = 5$ etc. Note that syntax models, while related to a single representation and/or situation, lead to the construction of multiple examples rather than one example. This helps to explain why partition situations also fall out of the conventional example space related to additive relations, which have been constrained to single solution problems.

In this study, a ‘syntax model’ refers to the combined use of a whole-part-part diagram, and its family of equivalent number sentences:

Figure 31: Syntax model for additive relations

Part A + Part B = Whole

---

representations of the number triples 8,4,4 and 8,5,3 (which uses circles for each number, and not the rectangles of the whole-part-diagram).

Whole	Part B + Part A = Whole Whole – Part A = Part B Whole – Part B = Part A

The rationale for this selection of this representation, and its origins in the additive relations literature, were discussed earlier.

Of particular importance in the design of the teaching interventions for the second and third cycles, was a deliberate attempt to use multiple representations to depict additive relations and to encourage children to move between these. Although Feature 4.1 emphasises flexible movement between representations, the emphasis is not on a plethora of diagrammatic representations simply for the sake of facilitating movement between the representations. Rather a few key diagrammatic representations are selected which could support the hypothesised learning trajectory from counting in ones to more structured and approaches to working with numbers (Feature 4.4). The intention was to introduce diagrammatic representations that progressively got more structured and provide lasting imagery which can be built upon in Intermediate Phase. I therefore chose representations which were referred to in the literature with regard to additive relations and I included reference examples from the structural models for representing the numbers from 1-20 including group, line and syntax models.

Framework for action

The previous two sections outlined the literature informing the theoretical features of the intervention design that were focused on *what mathematics* (additive relations word problems) was in focus, and *how the teaching of this mathematics* (narrative) was approached (the means of supporting the learning of this mathematics). They ignore general behaviour and emotion, elements of learning which are referred to in Feature 1.3: Learning as educating awareness and harnessing natural powers where learning transforms the human psyche (awareness, emotion and behaviour).

Relaxing the constraint on the cognitive to encompass behaviour and emotion runs the risk of opening up the research focus to be so noisy, that the critical features of the intervention design can no longer be discerned. However failing to acknowledge these elements of learning – at least from the perspective of the teacher - runs the risk of being blind to the intended moral values of the intervention. Research points to teaching as a complex system of interactions where constraining one factor has influence on the other components of the system (Stigler and Hiebert 1998).

There are numerous other theoretical features of the teacher role which are not as tightly bound to the *mathematical* content focus. In this chapter I discuss my perspective on the key features of my general teacher role during the last cycle of the design experiment. These are referred to as 'implementation features' (as opposed to 'design features') as they concern the way in which the third cycle was implemented in this particular context. I acknowledge that this is in no way an exhaustive list of implementation features, and I constrain discussion to those features of my teacher role to which I was consciously attending. Table 9 provides a summary overview of the theoretical features that were considered to be key to my general teacher role.

Table 8: Summary of major features of the theoretical framework of the general teacher roles underpinning the third cycle

Feature	Title	Description
Implementation features of the teacher role		
5	Training behaviour	5.1 Teaching the whole class as well as in small groups. 5.2 Encouraging a growing brains mindset which emphasises learning goals over performance goals. 5.3 Expecting learners to work independently, and ensuring that while they had similar tasks, that they each worked on unique problems. 5.4 Providing immediate extrinsic recognition of appropriate behaviour which accumulated into a reward (and tied into the mathematics).
6	Principles of general pedagogic style	6.1 Adopting a mathematical thinking questioning style, listening to and exploring suggestions from learners 6.2 Providing specialised and explicit feedback and paying attention to learner errors. 6.3 Facilitating opportunities for learners to practice (and receive feedback on) both main and enabling tasks

Feature 5: Training behaviour

In this section I report on some of the classroom management decisions I made for the last design cycle. These features emerged as a dialectic between my reading of the literature on training behaviour and my experiences in the first and second cycles in this challenging school context, where learners discipline was an acknowledged problem. Learner behaviour is broader than 'disruptive learner behaviour' and is entangled with learner affect which includes consideration for learner motivation, belief, attitudes and dispositions towards mathematics.

Training behaviour relates to the way in which learners are expected to act or behave when engaging mathematically. Following from Feature 1.3: Learning as educating awareness and harnessing natural powers where learning transforms the human psyche (awareness, emotion and behaviour) it has long been recognised that these three elements of the human psyche are entangled and all come into play when focusing on a particular object of learning By way of example, Schoenfeld (1992) identified four categories of behaviours relevant for mathematics problem solving: resources (mathematical knowledge that can be brought to bear on the problem situation); heuristics (strategies and techniques for making progress on unfamiliar or non-standard problems); control (global decisions regarding the selection and implementation of resources and strategies); and belief systems (the mathematical world view relating to beliefs of self, environment, topic and mathematics). All four categories come into play when approaching a mathematical problem. Resources, heuristics and control have all been discussed in previous sections. My mathematical belief systems were explored to some extent in the orienting framework in this chapter, but up to now the teachers' role in inculcating or challenging the belief systems of learners has been ignored.

Disruptive learner behaviour was one of the context features outlined in the South African context for this study and presented in the previous chapter. My experiences in the second intervention cycle of this study included a significant number of incidents of 'disruptive learner behaviour'. It is therefore appropriate to reflect on what is meant by this phrase, and to attend to my shifts in classroom management from the second cycle to the third cycle, where learner behaviour was less disruptive. There is a vast global literature on learner behaviour in schools and in classrooms. In a South African context, disruptive behaviour is defined as 'inappropriate' to the schooling context (Mabeba and Prinsloo 2000) or learner behaviour that inhibits achievement of the teacher's purposes (Levin and Nolan 1996). My references to 'disruptive classroom behaviour' alludes similarly to learner behaviour that (in the teacher's view) disrupts lesson progress and impedes learning. Marais and Meyer (2010) distinguish 'surface behaviours' from 'more serious disruptive behaviours' (p. 44). The former include verbal interruptions, off-task behaviours, physical movement and disrespect and usually 'exist to some extent in all classrooms… and are usually not the result of deep-seated personal problems, but normal developmental behaviour of children' while the latter refers to violent physical contact (Marais and Meyer 2010, p. 44).

Feature 5.1 Teaching the whole class as well as in small groups

One of the ways in which I changed my teaching approach from the first two cycles to the third cycle, was to introduce small group work as part of each mathematics lesson. The previous cycles, (including those conducted in less challenging school contexts for previous studies focused on the use of narrative with young learners, and the first and second cycles of this design experiment in this school), had all been typified by a mental mathematics starter followed by whole-class activity.

In the third cycle I chose to work with the learners in small groups. The grouping of learners was done based on their attainment in the written pre-test on additive relations. As such the groups were 'attainment-based groups', which would be considered to be 'ability group's by Boaler's (2014) definition as 'a generic term to encompass any grouping, whether it be within class or between classes, flexible or inflexible, that involves students being separated according to perceptions of their ability' (p.1). Such group work is a common feature of Foundation Phase classrooms were a group or 10 learners engaging with the teacher on mat or in a circle, while the rest of the class is productive occupied, is thought to facilitate focused engagement in the mediated learning task (Treffers 2008).

Some sections of lessons were worked with in flexible and fluid attainment–based groups. The reasons for making this choice were as a result of pragmatic need to quickly develop a productive learning environment. This decision was intended to allow for more immediate responsive teaching, and 'trust-building' openings that were less possible in whole class environments where you had previously encountered multiple incidents of physical behaviour problems from several children. Attainment based grouping allowed many more learners to access group and individual level tasks without initial teacher input, and this too, was useful as a classroom management strategy. Some of the lesson sections included whole class discussion and activity. While I took care to ensure that the independent work which learners were engaged in was manageable to them (so the individual work cards were individually assigned to learners based on the evidence of their book-work on the previous day), the whole class engaged in similar word problem tasks when working in their groups.

Feature 5.2 Encouraging a growing brains mindset which emphasises learning goals over performance goals

Aware of the research evidence against ability grouping, I was mindful to ensure that I encouraged learners to think of themselves as having a growing, rather than a fixed, mind. This was drawn from the static brain versus the growing brain self-concept of children(Boaler 2010), drawing on the work of Dweck (1986) who distinguished performance goals from learning goals in her study of learner motivation. A *performance goal* focuses children on issues of ability and a strong orientation toward this goal can 'create a tendency to avoid challenge, to withdraw from challenge, or to show impaired performance in the face of challenge' (Dweck 1986, p. 1041). Performance goals are associated with ability to correctly and fluently perform tasks. This is contrasted with a learning goal which is associated with effort to learn a new task. A *learning goal* focuses children on effort where effort is seen to be 'a means of utilizing or activating their ability, of surmounting obstacles, and of increasing their ability', and encourage children to 'explore, initiate, and pursue tasks that promote intellectual growth' (Dweck 1986, p. 1041). This approach to motivation values challenge seeking and persistence in the face of failure in favour of confidence in ability. In the design of the third cycle intervention I aimed to emphasise effort, perseverance and hard work as means of improving learning.

Feature 5.3 Expecting learners to work independently on unique work cards

The research context sketched in Chapter 2 for this study included dominant descriptive features of the discourse in typical South African primary school classroom where classroom interaction patterns privilege the collective (chorusing) and learning is communalised rather than individualised (Hoadley 2012). My experiences in the first and second cycle interventions in this school, confirmed a culture of collective engagement with mathematical tasks in this focal school. Even in assessment contexts, learners were seen to routinely copy responses from each other and they did not seem to value doing their own work, or recording their own thinking.

As a response to this challenge, as well as a means to facilitate small group work, I designed a series of individual work cards (Annexure 5). Every card was unique, and was allocated to an individual learner to complete as part of their seatwork time. Seatwork tasks were conducted at the learners' desks, while I was working with a small group on the mat.[16] The work cards were designed to develop fluencies considered to be requirements for engagement with the additive relations word problems. By having each card unique, it was no longer possible for learners to copy from each other. Working individually was prioritised for seatwork tasks, whereas during group work social interaction between individuals and a more collective approach was encouraged.

Feature 5.4 Providing immediate extrinsic recognition of appropriate behaviour which accumulated into a reward (and tied into the mathematics)

I also noted in the discussion on the South African Foundation Phase mathematics classrooms in Chapter 2, that there seemed to be a lack of sense-making in mathematics. The lack of sense making is part of a broader lack of a 'productive disposition' towards mathematics. Kilpatrick, Swafford et al. (2001) defines a productive disposition as the tendency 'to see sense in mathematics, to perceive it as both useful and worthwhile, to believe that steady effort in mathematics pays off and see oneself as an effective learner and doer of mathematics' (p. 131).

Graven reports on evidence of dependence, and lack of agency in her work with South African learners, teachers and departmental officials (see Graven & Metzuyanim 2014; Graven 2014). The mathematical identities of young leaners are described as negative and taking shape within 'a passive, overly teacher dependent culture of learning mathematics' (Graven and Stott 2012).

A passive – largely teacher dependent culture of mathematics lessons had been observed in the broader development in this focal school, and changing the dispositions of both teachers and learners in relation to mathematics was an ongoing developmental aim. However changing a culture of collective engagement to a more individualised process, challenging beliefs about ability and shifting them to be concerned with effort, and interrupting a passive teacher-dominated culture where compliance with teacher instructions and getting answers right are not transformations that can be undertaken lightly. These are certainly not shifts which can be achieved within the scope of a ten day lesson intervention. These objectives were more in focus for the broader development study where working with teachers to shift their approaches to mathematics teaching and learning were in focus.

In working on a design experiment, which is highly interventionist, I needed to find ways in which to quickly encourage a productive disposition towards mathematics with the focal class. I was not the normal classroom teacher for these Grade 2 learners. While the normal class teacher was present and available to assist with classroom management, the cycle two experience had revealed

[16] In this context, there was no carpet or mat in the classroom. Nevertheless, small group work was undertaken with children sitting on a space on the floor. As this type of arrangement is commonly referred to as 'mat work', I refer to mat (despite its absence in this context).

that having a new lead teacher for a set period had not been a smooth process in this challenging school context. As such I set aside time in the first two lessons of the third intervention cycle to discuss my approach to managing behaviour and to establish some classroom rules. These were accompanied by agreeing some immediate rewards and punishments. It was agreed that learning effort would be acknowledged by a sticker (a small round dot stuck into a learners work book). Learners would work towards attaining 10 stickers, at which point they would be eligible for a reward or surprise. The rewards were gift-wrapped stationery (rulers, pencils, erasors, sharpeners) which were in short supply in this school context. The punishment – suggested and agreed by the learners in the class – was to 'stay in at break time and write lines' (this punishment was never applied as it was not required).

I am aware that extrinsic rewards and punishments are not a long term strategy for managing and supporting a productive mathematical disposition. I note the work internationally by Dweck (1986), and Boaler (2014) and locally by Graven and colleagues (Graven and Stott 2012; Graven and Heyd-Metzuyanim 2014) on the long term needs for developing productive mathematical dispositions where intrinsic motivation is in focus. However, in the context of experienced difficulties in managing disruptive learner behaviour in the second cycle, and the context of a short third cycle intervention where productive learning was required, the development of a process involving such extrinsic rewards and immediate agreed punishments proved useful for building the beginnings of more productive learning dispositions. Further, the experience of accumulating extrinsic acknowledgement (stickers and surprises) was linked to the mathematics which was the intervention focus. I did this by setting up 'a sticker story' as one of the additive relations word problem types which we were working on. The sticker story problem was a 'change increase' problem in the 'equalise' category as expounded by Riley et al.. A typical sticker problem was: 'Now I have 8 stickers in my book. I need ten stickers to get a surprise. How many more stickers do I need to get a surprise?' This sticker problem was worked on daily, and was unique to each learner. When a learner reached 10 stickers, their reward target shifted to other multiples of 10. As such, this was a personal story which was repeated with higher and higher numbers.

Feature 6: Principles of general pedagogical style

Deconstructing elements of a teacher's role is a complex undertaking, and is not the focus of this study. However, as my role in mediating the teaching intervention was clearly central to the outcomes, it must be reflected upon. While the orientating framework discussed in the previous chapter outlines the underlying theoretical features of my beliefs about learning and mathematics learning in particular, I turn now to give a brief account of my teaching behaviour. In my view the main features of my role as a teacher were: Adopting a mathematical thinking questioning style, listening to and exploring suggestions from learners; facilitating opportunities for learners to practice (and receive feedback on) fluencies in using particular notations and representations when recording their mathematics; and valuing specific feedback and paying attention to learner errors. These aspects of my pedagogic style may be viewed in relation to the (Kilpatrick, Swafford et al. 2001) definition of mathematical proficiency as including five interrelated strands: procedural fluency; conceptual understanding; adaptive reasoning; strategic competence and productive disposition.

Feature 6.1: Adopting a mathematical thinking questioning style, listening to learner's responses

This feature refers to the sensitive use of questions to focus attention on children's mathematical thinking as they make sense of problems. It relates to the orienting belief described as Feature 1.1 Learners make sense of problems for themselves, while conforming to agreed social practices in Chapter 3. This style of questioning stresses questions which are directed towards the development of mathematical thinking, and ignores questions which are geared mainly towards particular techniques, finding answers or checking correct use of technical terms (Watson and Mason 1998, p. 4). The intent of such a questioning style is for the teacher to model how mathematics can be questioned and discussed, and ultimately, for the learners to take over responsibility for asking the questions themselves (Watson and Mason 1998, p.4)

One way of thinking about the concept of a question, is to distinguish open questions (where there are multiple possible answers) from closed questions (where there is one definite answer). However, in line with the conceptualisation of Feature 1.4: Facilitating shifts in attention from the particular to the general (generalising) and from the particular to the specific (specialising), an open question could be increasingly constrained, to become closed, and the reverse process of identifying the constraints on a closed question, could result in the constraints being relaxed. So this dichotomous distinction between closed and open questions was not particularly useful. Rather questions focused on mathematical thinking which promoted thought about the structure of a concept, were distinguished from questions which had a different purpose to this (Watson and Mason 1998, p. 5). The purpose of asking the questions is for the teacher to listen to the response, and think: 'What does this reveal about understanding?' and adjust the lesson accordingly.

Typical questions and prompts which were used in the third cycle intervention included: Can you explain why you think that? Explain how you got that? Is that the same or different to …? What has changed? Give me an example of…? What if…? Is that always true or only sometimes true? Is this a new story, or an old one in disguise? Keep … the same, but change … , and Give me another one like …

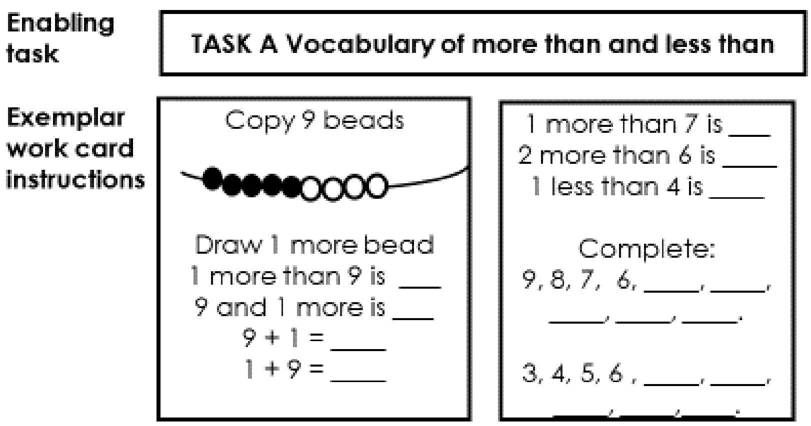
I was also aware of my option to use silence as a pedagogic strategy. During the third intervention cycle I was conscious of my choice to remain silent, which I then hoped would provoke children into deeper reasoning, and more communication with each other, rather than waiting for my approval. Part of this approach was in recognition that children were likely to view me as the arbitrator of the correctness of their reasoning. I knew that my feedback on work cards (see Feature: 6.2 Providing explicit and specialised feedback and paying attention to learner errors), and extrinsic rewards (Feature 5.4 Providing immediate extrinsic recognition of effort which accumulated into a reward) would feed into this mentality. During whole class and small group sessions I tried to listen carefully to learners' responses and to defer responsibility for accepting or rejecting an offer back to the group to judge its reasonableness.

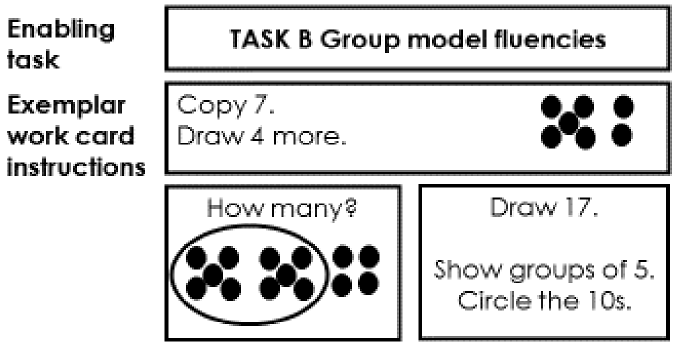
Feature 6.3 Facilitating opportunities for learners to practice (and receive feedback on) both main and enabling tasks

The South African context sketched in Chapter 2 included reference to the lack of sense making in mathematics where a tendency to work procedurally, rather than to reflect on the relationships between numbers and build on known facts or generalised properties of number relations, was seen to contribute to the poor mathematics attainment. In addition, one of the descriptive features of primary classrooms in South Africa was limited feedback or evaluation of student responses (Hoadley 2012). Taking these two contextual factors into account, this feature concerns the teachers' role in facilitating opportunity for routine practice and providing feedback on such tasks.

Treffers refers to the importance of a 'practice schedule' and distinguishes two forms of practice: reproductive practice and productive practice. The former 'focuses on the memorization and automatization of basic arithmetic skills up to twenty…the form and content of this kind of practice is determined and pre-structured by the teacher…and the tasks and questions are fixed' (Treffers 2008, p. 51). The latter is 'more indirect and problem-linked and requires learners to show more initiative' (Treffers 2008, p. 51).

The type of practice in focus in this design experiment was reproductive practice, as the individual work cards were designed to support the development of memorization and automation of basic arithmetic skills cards (see Feature 5.3: Expecting learners to work independently on unique work cards). Fluency with the basic facts and automatic recall of relationships between operations, were considered to be pre-requisites for shifting towards more structured approaches to calculations. I describe the kind of enabling task developed and the rationale for their selection in Chapter 4. To illustrate what is meant by pre-requisite fluency I illustrate the enabling tasks with this example:





These kinds of tasks were not focused on the learning goal of expanding the additive relations word problem example space of these Grade 2 learners. The number range was very low, with most tasks being pitched at level 4 on the (Treffers 2008) framework: 'towards pure counting-and-calculating via symbolization'. At this level 'children can represent physical objects up to ten on their fingers and with lines and dots and are able to use such skills as adding up and taking away' (p. 39)… 'they can order numbers' (p. 46).

I explicitly refer to opportunities to develop fluencies as part of the teacher's role. There has been some debate about the extent to which class time spent on 'automation' and 'practice' is appropriate use of time in teaching mathematics. For example Tahta (1972), referring to the writing of Mary Boole, notes education theorists who do not value mechanical repetition:

> Theorists in education sometimes imagine that a good teacher should not allow the work of his class to become mechanical at all. A year or two of practical work in a school (especially with examinations looming ahead) cures one of all such delusions. Education involves not only teaching, but also training (Tahta 1972, p. 15).

Note the distinction that Boole makes between teaching and training, and recall that training in behaviour is seen as part of learning (Feature 1.3 Learning is transforming educatoing awareness and harnessing natural powers where learning transforms the human psyche (awareness, emotion and behaviour)) Support for the inclusion of a 'practice schedule' where practice is part of the mathematics classroom routine, is found in the adage: 'practice makes perfect' which has been the focus of research in mathematics for theorists such as Li (1999). However the intensity of repetitive routine practice is in question: 'It is well know that repetition is an essential element to learning. Questions remain concerning the effects of the degree of variety in the set tasks' (Bell 1993, p. 15). So a balance between reproductive practice (training of behaviour leading to automation and memorization) and productive practice is required.

However questioning the appropriate intensity of reproductive practice raises the question of where the responsibility for the development of fluency lies (with the teacher or with the learner). It is my contention that for all learners, it is the teacher's role to provide opportunity for learners to develop procedural fluency. In certain contexts, such opportunities may be provided through the regular management and administration of homework tasks. However in the context of teaching young learners in challenging contexts, where opportunities for learning mathematics are constrained to the school environment (as a culture of supervised homework by a caring adult could not be assumed), more class time had to be devoted to routine practice. In the second cycle of the intervention, the empirical evidence of working with the Cycle 2 learners confirmed that the broader research finding (Spaull 2013) of below grade performance; was evident in this local context as most learners were not meeting Grade 2 curriculum expectations. As such the design and implementation of third cycle included tasks which demanded fluency with basic number facts (in the 0-20 range) as well as repetitive recall of particular phrases (such as '1 more than' and '1 less than') which were considered pre-requisite fluencies for the compare problem type.

Feature 6.3 Providing specialised and explicit feedback and paying attention to learner errors

It is recognised that children's previous knowledge may either help or hinder understanding new information in mathematics (Bransford, Brown et al. 2000). I therefore attended to the provision of immediate and specialised feedback to learners. This was administered through the daily collection of learners' workbooks which I marked and used to assign appropriate individual work cards for the next day's activities. Where recurrent errors were evident in a learner's work I assigned additional tasks which I thought would support them to address their errors. At times these tasks were written into their workbooks for completion, and at times these were individual work cards assigned to them. During the third cycle intervention I kept a journal where I made notes on each child, and used this as a reminder of which children required 1:1 discussion on particular tasks. I prioritised seeing these children individually at the beginning of a seatwork session, before I worked with a group of learners during each lesson.

Where errors were noticed as common across the work of several learners, these errors were addressed in whole class teaching. Given that misconceptions about compare type problems are common and arose amongst children in the first and second cycle of this research, it was a deliberate part of my teacher role to focus directly on these potential areas of difficulty. By their nature, common errors are relative to the mathematical object of learning. As such my discussion necessarily refers back to the theoretical features underpinning the additive relations word problems. The following common errors emerged during the first and second intervention cycles:

I was aware that children were likely to be using counting strategies where they would be working in ones and not groups (of five or tens). This relates to Feature 4.4 A learning trajectory from counting to calculating which made reference to increasingly structured representations was adopted. I therefore aimed to be deliberate in providing explicit feedback to learners that I expected to see them working in groups of fives or tens. Following Anghileri (2000) I deliberated distinguished the action of *hopping* in ones, from the action of *jumping* in groups by using different terms to describe these actions.

I expected that learners would have a dominant model for subtraction as 'take-away', and that shifting them to view subtraction as 'the difference' would take time. This related to Feature 2.2 A 'take-away' calculation strategy was contrasted to a 'difference strategy', with consideration for efficiency in choice of strategy depending on the numbers involved. Here I drew on the work of Anghileri (2000, 2007) and Askew (2012) by choosing numbers that were close to each other, to demonstrate the equivalence of the 'take away' and 'difference strategies'.

I expected that many learners would respond to the compare type problem as follows:

> Problem statement: Joe has 8 marbles. Tom has 5 marbles. How many marbles more does Joe have than Tom?
> Typical learner response: Joe has 8 marbles.

From this I inferred that the learners were not focusing attention on, or understanding, the use of the term 'more than'. It seemed as if the problem statement was understood to be: 'Joe has 8 marbles. Tom has 5 marbles. How many marbles does Joe have?' In this case, the learner response seemed reasonable. They seemed to respond to the problem statement as if it was an English comprehension question, where they ignored the phrase 'more than Tom'. I therefore aimed to direct attention to comparison, and to the person who had more, and then to the phrase 'how many more than?'. I intended to do this by posing three related questions about the problem:

> Revised problem statement (part 1): Joe has 8 marbles. Tom has 5 marbles. Who has more marbles?
> Typical learner response: 'Joe'
> Revised problem statement (part 2): How many marbles does Joe have?
> Typical learner response: 'Joe has 8 marbles'.
> Revised problem statement (part 3): How many marbles does Joe have more than Tom? Or How many more marbles does Joe have than Tom?
> Typical learner response: Some activity and thinking is required to figure out a response to this.

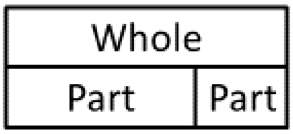
The rationale of changing the first part of the problem statement, was to enable the children to focus on the comparison, and to see that 'more' was a critical component of the problem. Posing a question 'who had more?' allowed them to demonstrate that they understood the context and the meaning of the term 'more'. The second part of the problem statement was introduced to acknowledge and anticipate their likely response to the original problem statement. They could then distinguish the question 'How many marbles does Joe have?' from the question 'How many *more* marbles does Joe have *than Tom*?'. By posing the three questions directly after each other, it was hoped that children would notice that the questions were distinct. They had already answered 'Joe has 8 marbles' to the second questions, and it was hoped that they would be less likely to respond with this answer to the third question. This approach to the misconception was seen as temporary scaffold, where the three questions would be removed/ faded, so that only the original compare problem statement would remain.

Finally many learners were observed not to notice the significance of the equivalence of the lengths of whole compared to the two parts in the whole-part-part diagrams. Diagrams such as the following were noticed in learner books (during the second intervention cycle):

Figure 32: Depiction of common learner drawings of whole-part-part diagrams

These diagrams were compared to the whole-part-part where the length of the whole was the same as the length of the two parts:

Figure 33: Whole-part-part diagrams



Some discussion about what was the same and what was different about these diagrams, and which diagram showed the whole-part-part diagram correctly, was facilitated during a whole class session.

In this chapter, all aspects of the theoretical framework that guided decisions about the structure, content and pedagogy within the design experiment have been detailed. In the next chapter, I describe the methodological decisions pertaining to the research.

CHAPTER 4: Methodology

This chapter is in two sections. The first section focuses on the research design and the second section focuses on the intervention design.

The research design was undertaken as a classroom based design experiment. This chapter outlines the features of a design experiment methodology and how this design experiment was undertaken. Attention is given to the sources of data as well as how they were analysed. The research design is presented after the theoretical framework (Chapters 3) as theoretical constructs set up in the framework are drawn upon when explaining and justifying the research design.

In design experiments intervention design is a key part of the research planning and methodology. The assumed social and intellectual starting points, and related endpoints are documented. In addition the hypothetical learning trajectory anticipated as a means of supporting learning is described. This trajectory describes the selection and sequencing of main and enabling tasks planned for the intervention. This chapter answers the questions: *'What was the design of the cycle 3 intervention?'* and *'How was the intervention design refined over multiple research cycles?'* which includes a reflection on the ways in which the intervention was refined over the three cycles. This refers to the sequencing and selection of main tasks, and how these evolved as a dialectic to form the cycle 3 theoretical framework which drew on the literature and the experiences of prior intervention cycles.

Design experiments

Design experiments are a relatively new methodological approach in the educational research arena. Design experiments were introduced in 1992 as an attempt to 'conduct research in (experimental) practice, and to contribute to both research and practice' (Burkhardt and Schoenfeld 2003, p.4). Design experiments adopt an *engineering* approach to research, where the 'research is directly concerned with practical impact—understanding how the world works and helping it "to work better" by designing and systematically developing high-quality solutions to practical problems. Internationally there has been growing recognition of the design experiment methods as part of evidence-based educational research (Design-Based Research Collaborative 2003; Kelly 2003). Design-based research methods incorporate both design research and empirical research, for the purpose of developing models and understanding learning in naturalistic intentional settings (Tabak 2004). According to Tabak (2004), design research involves 'designing an intervention that reifies a new form of learning' (p. 226) where the designs are based on principles informed by theory, literature in the field and prior research.

The naturalistic settings for design experiments in schools are therefore classrooms, with real classes of children. Design experiments are described by Cobb, Confrey, diSessa, Lehrer and Schauble (2003) as usually:

> conceptualised as cases of the process of supporting groups of students' learning in a particular content domain. The theoretical intent, therefore, is to identify and account for successive patterns in student thinking by relating these patterns to the means by which their development was supported and organised (p. 11).

Faced with a practical problem (poor learner attainment in additive relations word problems and compare-type problems in particular) in a focal school, this thesis was conceptualised as a classroom-based design experiment. The theoretical intent of this study was to identify (and attempt to account for) successive patterns in children's engagement with tasks designed to engage children in solving, posing and representing narratives relating to additive relations word problems.

Critical features of classroom-based design experiments

Cobb, Confrey et al. (2003) identified five cross cutting features that apply to design experiments. Other writers such as Turner and Meyer (2000), Gravemeijer and van Eerde (2009), Swan (2014) have offered similar categorisations of design-based research methods. I have chosen to adopt Cobb and DiSessa's cross cutting features, as in delineating the various theoretical features of this study I found the distinctions that DiSessa and Cobb (2004) make between: grand theories, orienting frameworks, frameworks for action, domain-specific instructional theories, and ontological innovations, for characterising the nature of theory for design experiments to be helpful. I used these to framework the theoretical features of the intervention design in the earlier chapters. I stay with this theoretical approach and I present each cross cutting feature of a design experiment in turn, and explain how each feature is exemplified in this study.

Critical feature 1: Theoretically focused

The first cross cutting feature of design experiments relates to the study purpose which has a theoretical intent and focuses attention both on the process of learning, and the means through which the learning is supported in particular communities (Cobb, Confrey et al. 2003, p.10). In this way design experiments expect that there is a dialectical relationship between theory development and improvement in practice. There is also a dialectical relationship between the process of learning and the means through which the learning is supported. As such it is expected that the design features of each iterative cycle relate to teaching-and-learning and are informed by theory. The empirical findings from the use of design features in authentic classroom contexts are used to further refine the theoretical basis for subsequent design.

In this study a hypothesised approach to expanding learners additive relations word problems example space was implemented, based on the additive relations literature and on prior research on the promising impact of narrative and diagrammatic representations in supporting mathematics learning in easier teaching contexts with young learners. The hypothesised approach was then empirically tested and refined based on analysis of empirical data from learners as well as reflection on efficacy. Contingencies were developed and refined to better match the approaches to realities of this new context.

The purpose of this study was theoretical in the sense that the initial design of the intervention - informed by theory – was interrogated empirically based on how children engaged with the tasks in the intervention. This informed re-design of subsequent intervention cycles (which were again examined empirically), with these analyses used to reflect back on theory and note any theoretical developments. A detailed theoretical framework guided the design of both the intervention and the research, while the theoretical intent of my study was for ontological innovation.

Critical feature 2: Highly interventionist

The second cross cutting feature of design experiments is that they take place in naturalistic settings, yet are 'highly interventionist…the intent is to investigate the possibilities for educational improvement by bringing about new forms of learning in order to study them' (Cobb, Confrey et al. 2003, p10). This study was highly interventionist as I played the role of teacher-researcher. I was not a full time teacher at the school, but I led a development project across this and another urban primary school. It was appropriate that I led the teaching intervention, as the use of narrative (storytelling and representations) as an approach to learning mathematics was expected to be discordant with the normative mathematics teaching in the intervention classes. These approaches were expected to be new forms of learning in this context that required further study with these communities of learners.

Critical feature 3: Iterative design

As a related characteristic to the prospective and reflective feature, the third cross cutting feature is that design experiments have an iterative design. Building on prior experiences of using a narrative approach to early grade mathematics learning in less challenging contexts (Roberts and Stylianides 2013), the intervention for this study was conducted in the same school with different cohorts of learners in three iterative design cycles.

The first exploratory cycle was conducted in November 2012 with Class 3A, where Vanessa (pseudonym, as are all names of teachers and learners in this study) was the normal class teacher. The second cycle was conducted with Class 2B in the same urban township school in November 2013, where Colleen was the normal class teacher. The third intervention cycle was conducted, again in the same urban township school in April 2014 with Class 2D where Vanessa was the normal class teacher. Concurrently the theory development and improvement in practice evolved as a dialectic over each intervention cycle. It is therefore appropriate to reflect on how this intervention evolved and to report on the changes made to the task design and theoretical features in each cycle.

In each intervention, children were introduced to an additive relation story and given opportunity to solve problems relating to a particular reference example. They were invited to generate similar stories to the one they had heard and to retell these stories in class activities. They were then introduced to the next story, which was compared to the first, and so on. Which additive relations word problems were selected for the main tasks, changed with each intervention cycle. In all three cycles, learners were expected to make use of representations to support them to solve the problems and to explain their reasoning. Which representations were in focus over the three intervention cycles however shifted from cycle 1 to cycle 3.

Critical feature 4: Prospective and reflective planning

The fourth cross cutting feature is that design experiments are both prospective and reflective. 'On the prospective side, designs are implemented with a hypothesized learning process and the means of supporting it in mind in order to expose the details of that process to scrutiny' (Cobb, Confrey et al. 2003, p.10). At the same time, potential pathways for learning and development may emerge by capitalizing on contingencies that arise as the design unfolds (Cobb, Confrey et al. 2003, p.10). In this regard, Tabak (2004) distinguishes between exogenous and endogenous education design: *Exogenous design* refers to the instructional materials, activity structures, or instructional strategies that have been developed for the purpose of the research; *endogenous design* refers to both the set of materials and practices that are already in place in the local setting and those devised by local participants 'in-action' as part of the enactment (Tabak 2004, p.227). Plans were adapted 'in action' within each cycle, and reflection on each cycle resulted in refined plans for subsequent intervention cycles. One of the orienting features of this design experiment was Feature 1.5 Learning by engaging in 'tasks' where a learning trajectory is inferred from 'learner activity' with these tasks, and as such the description of the hypothesised learning trajectory was closely related to the tasks designed to afford the intended learning.

The way in which tasks were seen to relate to learning was theoretically informed. The 'Mathematics Teaching Cycle' put forward by Simon (1995) (and see Simon (1997)) in which the notion of a hypothetical learning trajectory (HLT) is proposed offers a potential starting point for reflecting on the interrelationship between teaching and learning in the mathematics classroom. In the Mathematics Teaching Cycle an HLT consists of the goal for the students' learning, the mathematical tasks that will be used to promote student learning, and hypotheses about the process of the students' learning. In this conceptualisation there is an interdependence between the learning process and the tasks: 'The tasks are selected based on hypotheses about the learning process; the hypothesis of the learning process is based on the tasks involved' (Simon and tzur 2014, p.93). Building on the HLT concept, Leiken and Dinur (2007) recognise the inherent tension between teacher's attempts to be flexible and responsive to the thoughts of their learners and the need to proactively manage the students learning according the planned lesson agenda. They distinguish between a 'Planned Learning Trajectory' by which they refer to the HLT for a particular lesson, and the 'Actual Learning Trajectory' (ALT) which are the 'actual events and procedures' taking place in the lesson (Leikin and Dinur 2007).

Adopting an instructional engineering approach to mathematics teaching and learning Stylianides and Stylianides (2014b) use the concepts of HLT and ALT, and elaborate that (to them) the HLT is the 'learning milestones a classroom community is anticipated to achieve as the community goes through the implementation of the task sequence in an instructional engineering that aims to promote a particular learning goal' while the ALT is the 'learning milestones a classroom community seemed to have achieved during the implementation of the task sequence in an instructional engineering' (p. 380). As such the specification of the ALT is an evidence-based endeavour which is necessarily grounded in empirical data collected from the lesson intervention. This approach recognises that learning can be measured at different grain-sizes and that the learning of a community does not necessarily reflect the union of learning by each learner in the community. Stylianides and Stylianides (2014b) argue that efficacy of instructional engineering should be measured as degrees of success which are necessarily relational to the intended learning goal and its implementation in the classroom. For the learning goals targeted by the instructional engineering to be achieved, it is a necessary condition that there must be a close matching between the community's hypothetical and actual learning trajectories.

Design experiment methods recognise that the HLT reflects a teaching-learning duality, and that learning cannot be described in isolation to the means by which such learning is supported. As such it is expected that the HLT will not be clear in the initial intervention cycles, but is rather developed iteratively through repeated experimentation and dialectical engagement with theory and literature on the content domain which is the focus of the design experiment. In fact it is expected that intervention designs are inherently under specified, with recognition that the emergent ways in which local participants interpret novel tools may depart from the intended design (Hall 2001). As such it is expected that initially the HLT will not match the ALT closely, but that through careful reading of theory and literature, considered reflection and analysis of empirical data with a particular local context, the design evolves over time so that the HLT comes to more closely match the ALT. It is when this matching occurs in a particular local context, that a design experiment is considered mature enough to be tried outside of the particular local context, with the next intervention cycle potentially being less interventionist (involving the researcher/ research team working with a practicing teacher, rather than the intervention being led by one of the researchers).

At this point it is appropriate to offer a synopsis of the learning goals, starting and endpoints and task design sequence which constitute hypothetical learning trajectory for the intervention. The rationale for the design of the HLT adopted for the cycle 3 intervention of this design experiment is discussed in detail later in this chapter.

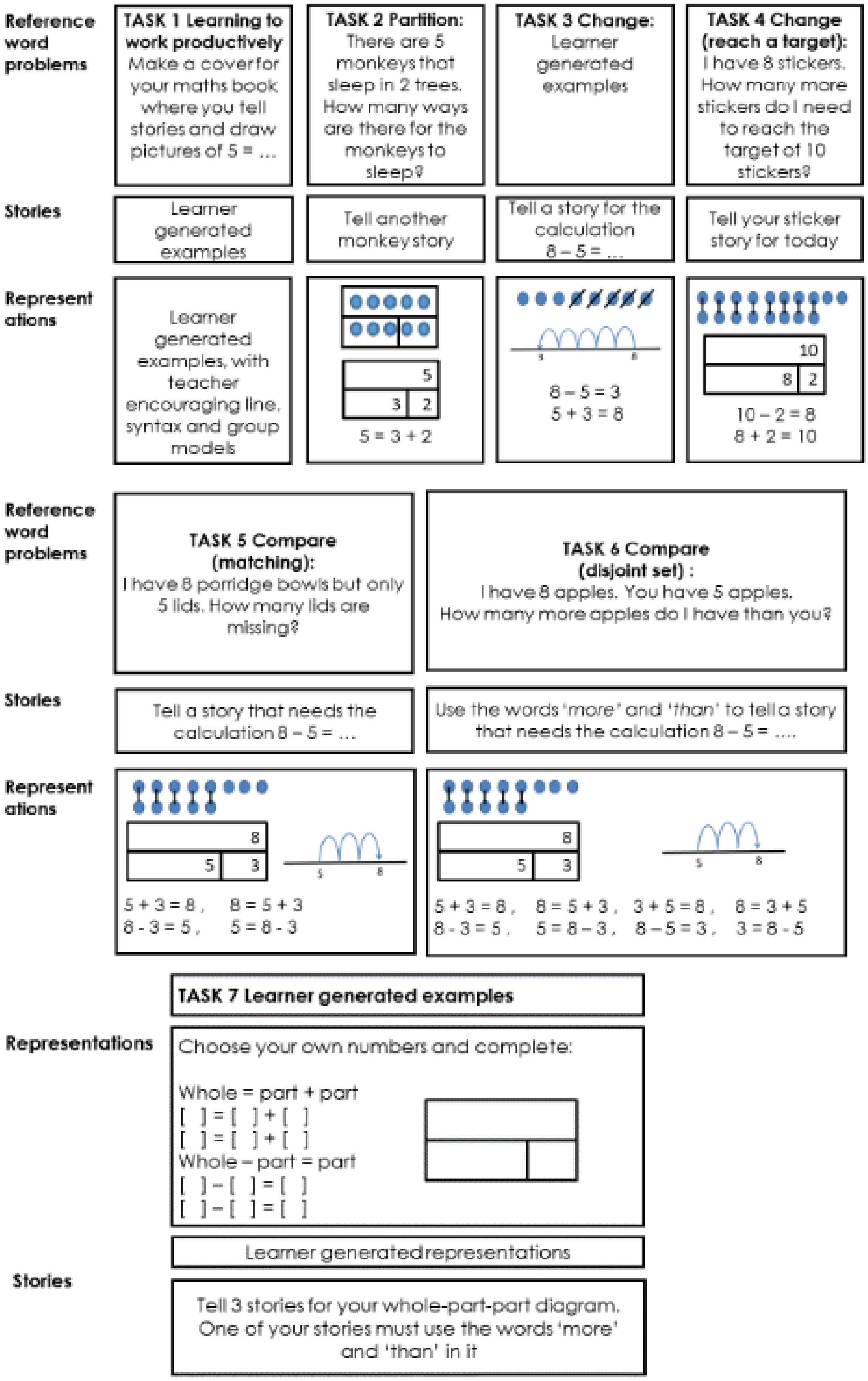
Table 9: Synopsis of learning goals in relation to start and end points

Overarching learning objective	To expand learners personal potential example space to include compare-type additive relation word problems	
Learning goal	**Assumed starting point**	**Prospective endpoint**
LG 1: Problem solving	1.1 Learners would be able to solve some of the word problems; with the easiest problem type being the change (decrease) problem. 1.2 Based on literature the compare (disjoint set) problem would be the most difficult	1.1 More learners would be able to solve a wider range of word problems: partition problem; change problem; change (reach a target) problem; collection problem; compare (matching) problem and a compare problem.

Overarching learning objective	To expand learners personal potential example space to include compare-type additive relation word problems	
	problem to solve. Based on prior intervention cycles, the compare (matching) problem would be easier to solve than the compare (disjoint set) 1.3 Many of the self-selected representations drawn to support their problem solving would be **incoherent**, and not depict the problem situation accurately.	1.2 Based on literature the compare (disjoint set) problem would be the most difficult problem to solve. Based on prior intervention cycles, the compare (matching) problem would be easier to solve than the compare (disjoint set) 1.3 More of the learners' self-selected representations would be **coherent** accurately depicting the problem situation.
LG 2: Representations	2.1 Learners would make use of group, line and syntax models; but group models would be the dominant coherent model. 2.2 Within group models: Learners would mostly use counting in ones strategies, with a few learners acting on groups of 5s or 10s. 2.3 Within line models: Learners would have used structured and semi structured number lines, but their use would not yet be secure. Most lines would depict hops in ones on the number line, with few learners using jumps of more than one. 2.4 Within syntax models: Learners would have used number sentences, mostly in the standard form They would not be familiar with alternative forms of number sentences of or with the whole-part-part diagram 2.5 Interpreting the learners' representation for their calculation action, most representations would make use of a 'take-away' action, with few using 'partition' or '1:1 matching' actions.	2.1 Learners would make use of group, line and syntax models with more line and syntax models and fewer group models. 2.2 Within group models, some learners would show evidence of acting on groups in indexical and iconic representations, and using jumps on a number line 2.3 Within line models, some learners would shift from structured to semi-structured or empty number lines, some would shift away from hops (actions in ones), to jumps (actions on groups). 2.4 Within syntax models, some learners would be more flexible in choosing to use alternative forms of number sentences. Some learners would make use of whole-part-part diagrams. Of these learners, some would show awareness of the measurement scale, while others would only attended to the whole and two part structure. 2.5 Interpreting the learners' representation for their calculation action, fewer representations would make use of a 'take-away' action, with more using 'partition' or '1:1 matching' actions.
LG 3: Story telling	3.1 When asked to pose an additive relation word problem, most learners would invoke a take-away action, or a change decrease problem type 3.2 Learners would not be familiar with explaining additive relations word problems to self and others 3.3 Learners would be able to explain a general change problem situation, but would not be able to explain a general compare problem situation	3.1 Learners would be able to pose a wider range of additive relations word problem examples, having shifted from a constraint of additive relations in 'change' contexts to include other additive relations contexts (including 'compare' contexts) 3.2 Learners would be able to explain a wider range of additive relations word problems to self and others 3.3 Learners would be able to explain a general change problem situation, and some learners would not be able to explain a general compare problem situation

Figure 34 provides the sequence of main tasks for the intervention.

Figure 34: Main tasks in cycle 3



The prospective and reflecting planning reified the task design so that by the third intervention cycle the main tasks, their major features and how these related the HLT were tightly defined:

Table 10: Major features of the main tasks in cycle 3

No.	Main task	Major features	Elements of hypothetical learning trajectory (assumed learner activity)
1	Learning to work productively	Establishes expectations regarding learner behaviour Establishes expectations regarding seatwork and group work tasks Offers diagnostic opportunity to review learner generated representations for 5 = …	Learners create a book cover with pictures or writing about '5 equals …' Learners experience using independent work cards in small groups.
2	Partition word problem	Models storytelling process Gives context for exploring whole-part-part additive relationship. Introduces dynamic version of whole-part-part diagram with reference to countable units and shifting to a symbolic image. Story retold and numbers varied.	Learners explore different partitions of 5 within a story context. Learners create part-part partitions by tearing 5 strips. Learners work randomly and then systematically Learners write number sentences in the form whole = part + part. Learners see a symbolic whole-part-part diagram. Learners retell and vary numbers in the partition problem.
3	Change word problems	Offers opportunity to start with symbolic and shifts to express additive relation context using natural language. Explores personal example space for additive relations word problems.	Learners generate their own examples and are likely to make use of change increase and change decrease contexts. Learners likely to use take-way or removal actions to model subtraction.
4	Change (reach a target) problems	Introduces comparative language of 'more' into a change increase (change unknown) context.	Learners count their stickers daily and calculate how many more stickers they need to reach the target of ten. Learners create images of comparison with 1:1 matching action. Learners draw whole-part-part diagrams and write number sentences in the form part + part = 10 10 – part = part. Learners see structured number line representation of counting on in ones to reach a target of 10.
5	Compare (matching)	Invokes a 1:1 matching context within a compare problem type.	Learners count out one set of objects, and then match the objects in the other set through a 1:1 matching action. Learners draw difference images for subtraction. Learners draw whole-part-part diagram (from the difference image, keeping the numbers invariant). Learners write number sentences in the form of part + part = whole, and whole – part = part.
6	Compare disjoint set problem	Building on the 1:1 matching experience, introduces a compare disjoint set problem.	Learners count out one set of objects on a bead string or using a 5-strip, and then match to a second bead string or using 5-strips. Learners draw a whole-part-part diagram. Learners write number sentences in the form of part + part = whole, and whole – part = part whole = part + part and part = whole – part.

No.	Main task	Major features	Elements of hypothetical learning trajectory (assumed learner activity)
7	Learner generated examples	Explores personal example spaces for additive relations word problems. Encourages use of terms 'more' and 'less' in problems contexts.	Learners generate their own examples, and are likely to make use of change increase and change decrease contexts. Learners likely to use take-away actions to model subtraction and write subtraction number sentences Learners encouraged to make use of difference contexts using comparative language of 'more than' and 'less than'

In examining the evidence of learning gains the starting point, end point and learner engagement with main tasks were examined for particular learners to infer the extent of matching of the HLT to the ALT for these learners.

Although close matching of the HLT to the ATL is a necessary condition, to claim a high degree of instructional engineering success it is not a sufficient condition. Stylianides and Stylianides (2014b) argue further that to claim a high degree of instructional engineering success, the implementation 'must be on the basis of the highly refined instructional plan' and also include 'low demands on the teacher for improvisation and in-the-moment decision making' (p.381). I did not expect that by the end of the third cycle of a design experiment on additive relations word problems in South Africa that claims relating to 'a high degree of success for the instructional engineering' could be made. Reporting only on three cycles also meant that there were not yet low demands on the teacher. What was however hoped for, was that progress towards the necessary condition of a close matching between the HLT and the ALT may be reported, coupled with proposed refinements to the instructional plan to further refine it (thereby decreasing demands on the teacher for future implementation cycles).

Critical feature 5: Theories developed are humble

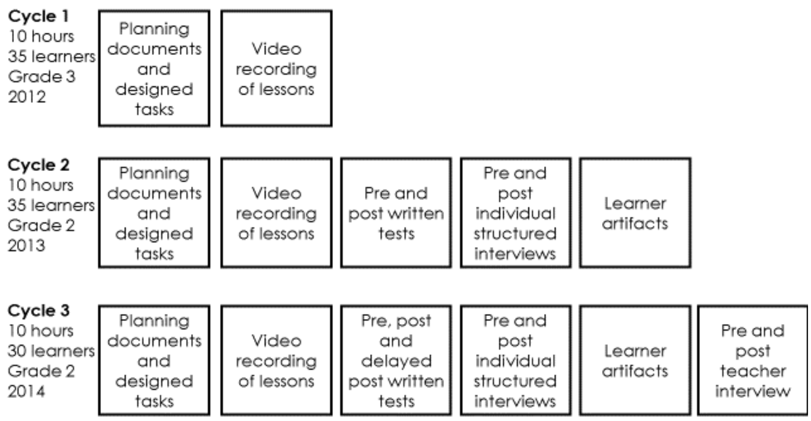
The fifth cross cutting feature of design experiments is that theories developed are humble: 'These theories are relatively humble in that they target domain-specific learning processes' (Cobb, Confrey et al. 2003, p.9). In this case the domain specific learning process was supporting learners to expand their additive relation word problem example space to include compare-type additive relations word problems at Foundation Phase level. The theories are also humble in that they are local. The theory 'provides reliable input to design of limited but known range' (Burkhardt and Schoenfeld 2003, p.10). Cobb, Confrey et al. (2003) elaborate that the theoretical products of design experiments have the potential for rapid pay-off because they are filtered in advance for instrumental effect and because design experiments directly investigate the types of problems that practitioners address in the course of their work.

This study reviewed the existing literature on additive relations, a HLT was developed, trialled and refined over three successes interventions. The implications of the study are discussed in relation to the efficacy of a 'narrative approach' to additive relations word problems and in relation to ontological innovations for teaching and learning additive relations word problems which emerge as promising features within this particular local context.

Methods adopted in this study

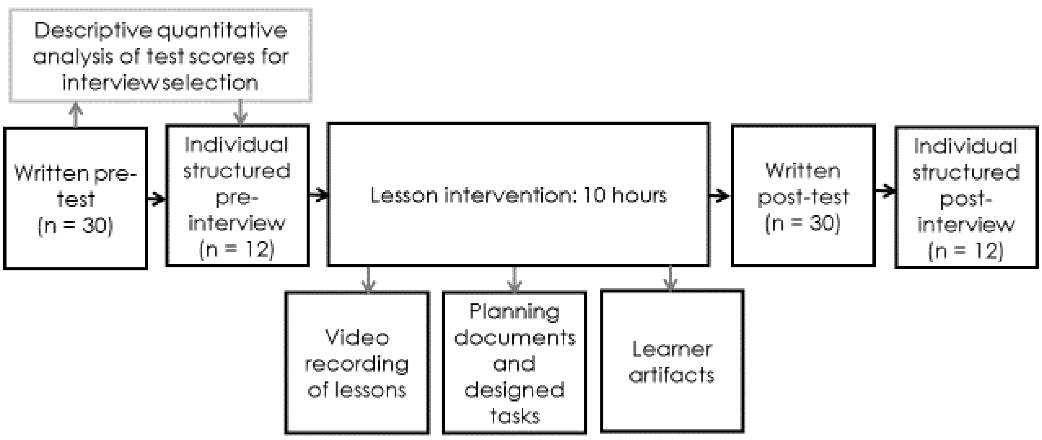
This study sought to examine whether an innovative pedagogical intervention impacts on children's learning of a topic in early grade mathematics in the local context of this focal school. The overall study in this local context consisted of a three-cycle design experiment with classes of early grade learners in this focal school, paying more attention to its last research cycle.

Figure 35: Three iterative intervention design cycle and data collated in each cycle



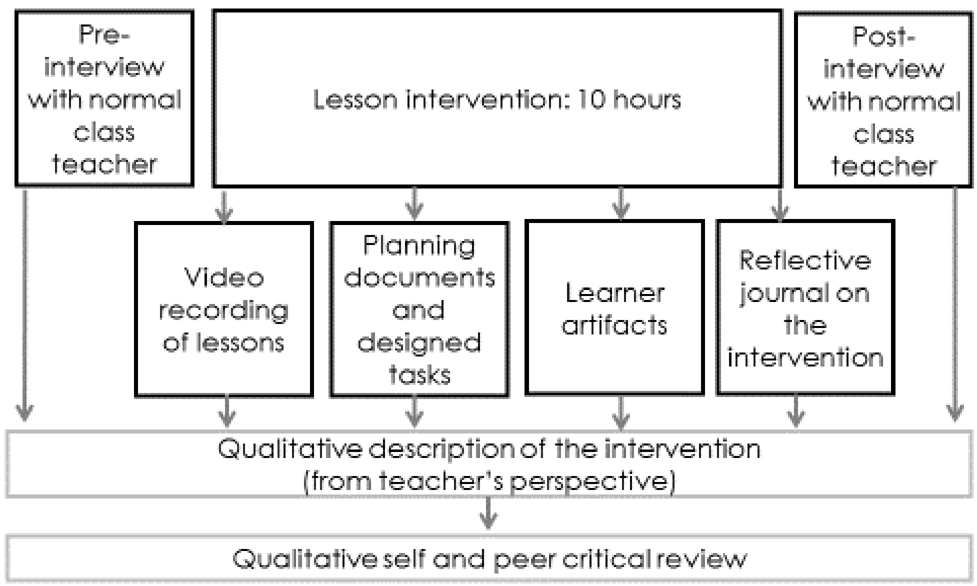
The data collected included data sources relevant to both the teaching and the learning sides of the intervention. Evidence of learning was collected via written tests for the whole class (administered prior to the intervention, just after the intervention and then 7 months after the intervention), as well as pre and post individual structured interviews with a selection of learners.

Figure 36: Data collected to measure learning



To gather empirical data relating to the teaching-side of the intervention the following data was collected:

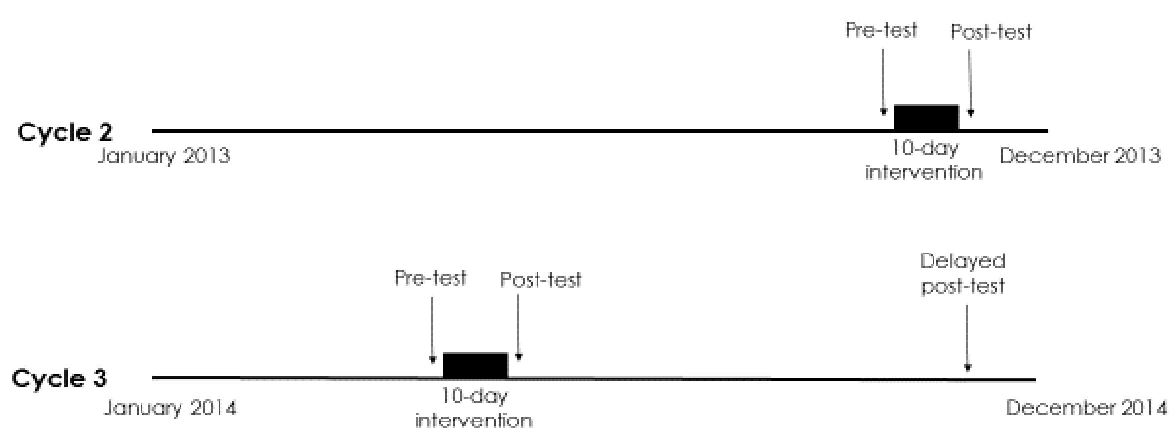
Figure 37: Data collection to describe teaching in the intervention



Quantitative analysis

The question of whether or not there was evidence of learning from the intervention was answered using quantitative analysis of pre-test, post-test and delayed post-test data from the learners in the intervention class (Cycle 3). Further the quantitative data from the Cycle 3 intervention class was compared to the same quantitative data from the previous Cycle 2 intervention. The same standardised written tests items were used and the same marking framework which made explicit the marking criteria, was applied to the learner test scripts in both cycles.

Figure 38: Administration of written tests in cycle 2 and cycle 3



The results of the marking of the written tests were recorded for each learner on a question-by-question and criterion by criterion basis. The marks for each criterion on each question were then summed. The total test scores were then tested using a Cronbach-alpha test for internal consistency (reliability) to establish the average correlation of items in the written test to gauge the reliability of using the summated total of the criteria (Cronbach 1951). The Cronbach-alpha test is widely used for this purpose, and was the test applied by Cobb, Wood et al. (1991) to establish the reliability of their Grade 2 standardised assessment in their year-long project adopting problem-centred approaches to mathematics learning.

The shifts in means and standard deviations of the marks in each written test were calculated. The same reporting on quantitative attainment in test scores was followed by Cobb, Wood et al. (1991). In this study mean results and standard deviations were calculated for five tests: the pre-test and post-test for Cycle 2, and the pre-test, post-test and delayed post-test from Cycle 3. For each cycle, the data was paired for each learner (resulting in pre-test- post-test pairs for each learner in each cycle, and pre-test-delayed post-test pairs for each learner in Cycle 3). For each individual the shift in test attainment was calculated (using the difference of the pre-test to the post-test score for each learner).

The means (and related standard deviation of the means) were tested for statistical significance making use a Student's t-test (repeated measures t-test). The Student's t-test was selected as the sample for each class was small (with less than 30 data pairs, as there were 26 learners with paired data in Cycle 3, and 22 learners with paired data for Cycle 2) and the test attainment data was on an interval scale. In each case a comparison between two small dependent samples was made. The degrees of freedom were calculated (df=25 for Cycle 3, and df=21 for Cycle 2). The null hypothesis for these tests was that the two samples were equal (no significant difference between the mean of the two tests), with the hypothesis being that the post-test mean was significantly greater than the pre-test mean (hence a one-directional or one-tailed t-test).

Given the small sample sizes (30 learners in a class) a parametric distribution of the written test attainment data was unlikely. A t-test assumes that a small sample is drawn from a normally distributed population. For this study it was assumed that with more data, the distribution of the written test attainment would come to more closely match the Gaussian distribution. This assumption (that the written test attainment data would be normally distributed with a bigger population of learners) was tested by creating a combined data set of all five of the tests (Cycle 2 pre-test, with n = 22, Cycle 2 Post-test n =22, Cycle 3 pre-test n = 26, Cycle 3 post-test n = 26 and Cycle 3 Delayed post-test, with n = 26; resulting in a test sample of n=148) and the distribution for the larger group was more closely matched to normal distribution. The assumed Gaussian distribution for a larger sample of learners writing the written test was therefore considered to be reasonable.

Further, the effect size (ES) of the shift in means (from each cycle) was calculated order to compare the attainment in written tests of the two cycles. Better attainment in the written test from one cycle to the next was inferred by comparing the effect sizes of the differences in means from the two tests in each cycle. The ES was calculated using Cohen's d value, making use of the pooled standard deviation (Cohen 1988). This provided a standardized difference taking into consideration the difference between two groups given their standard deviations (Mitchell and Jolley 2012). This ES was used as the purpose of calculating the ES was to detect differences in mean values (and not to account for amount of variance). Cohen's d provides a measure of the percentage of non-overlap between two independent groups (Cohen 1988). An ES of 0 indicated 0% non-overlap (complete overlap), while an ES of 0.8 indicated a 47% non-overlap, and an ES of 1.7 indicates a 75.4% non-overlap between the two samples. An ES of 0.7 to 1 may be considered to be 'large', while an ES of 0.3 - 0.5 may be considered to be medium, and an ES of 0 - 0.3 would be small (Cohen 1988).

Using the test attainment data for each learner and relating to each criterion and each item in the test, calculations of item difficulty defined by Crocker (2006) as the 'proportion of examinees who answered an item correctly' (p.374) were conducted and referred to as the 'facility' of items in the written test. Considering the mark allocations for each of the word problems, the total marks obtained by learners in a particular item in both cycles was summed and expressed as a proportion of the total available marks. This gives a facility score per question which ranges from 0 (all learners got all aspects of the question incorrect) to 1 (all learners got all aspects of the question correct), which was expressed as percentage. This offers a measure of the difficulty of a question in the test in relation to both groups. The same calculations were conducted on each cycle independently to offer a comparison of facility on each item between the two cycles.

I made use of a coding framework to classify the representations that learners were using this their written tests. The coding framework is elaborated on in detail below. For each question and each learner the representations they produced were categorised. Frequencies of types of representations were calculated by learner, as well as for the whole class. The work of the whole class was then collated to provide a total frequency of types of representations across the class in each written tests. Comparisons could then be made relating to types of representations created by the Cycle 3 class in the pre-test, post-test and delayed post-test. Changes in the distribution of representation categories were measured making use of contingency (two-way) tables of the representational categories. For each contingency table the actual shifts in number of representations was compared to the expected distribution of categories (where the null hypothesis assumes the same distribution between the two tests). A Chi-squared test was then applied to test whether the observed frequency of the occurrence of the categories corresponded to expected frequencies (assuming that the pre-test and post-test data would have the same frequencies) (Bless and Kathuria 1993). As such it measured whether the shifts in frequencies of categories from pre-test to post-test was significant or not.

Qualitative analysis

Qualitative approaches were adopted for the analysis of evidence of learning relating to Learning Goal 3: Story telling where learner activity on the whole class engagement with Main task 7:

Learners generating examples was the empirical base. Further qualitative approaches were adopted for analysing planning documents, and video recordings of lessons for the analysis of the means of supporting learning where the task design, changes in task design and the framework for action defining the teacher role were in focus.

If there was evidence of learning, then this study sought further to account for this learning through describing the means through which the learning was supported. Possible explanations for the observed differences in written test attainment and in the types of representations produced were then sought by reflecting on the changes made to the intervention task design evident in the qualitative data on the means of supporting learning. Analysis of changes in design made use of constant comparative method of qualitative analysis (Glaser 1965). This was done through offering a qualitative thick description of the intervention task design and the adaptations made to the task design over each cycle, and how the cycle 3 implementation played-out in this local context. This assumed causal link between changes in task design and changes in learning builds on the notion of hypothetical learning trajectories. The subsequent interventions cycles are considered to be reflective of better attainment (which implies better learner and better teaching) when the HLT for the class (or particular groups in the class) comes to closely match the actual empirically observed learning trajectory. While this does not assume that the ALT for a particular individual learners will exactly match the HLT for the class, this theoretical approach expects that general patterns of learning relating to the groups of learners in the class can be discerned. Recognising the inevitability of individual differences within the class, purposively selected individual learners were then examined in detail.

Sources of data for each research question

The following table outlines which data sources were used to answer each of the research questions

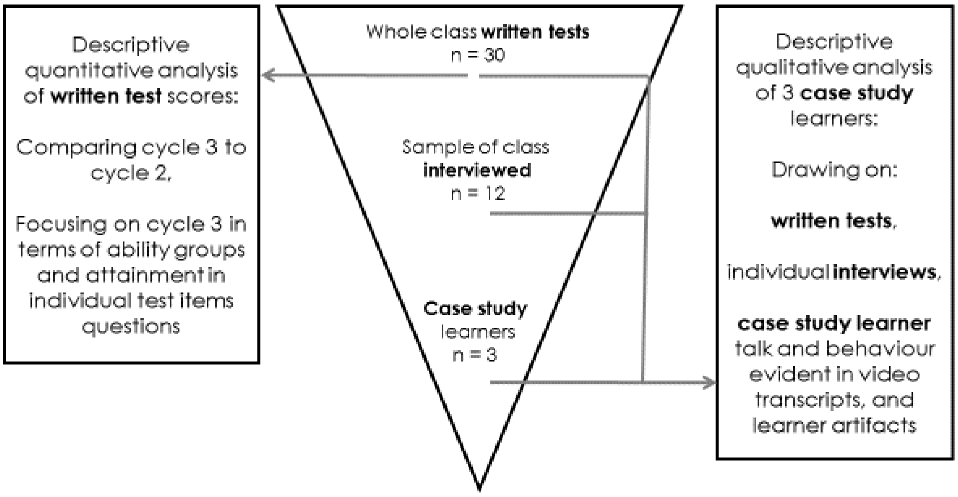
Table 11: Data sources against research questions

Research question	**Data sources for answering research question**
What evidence of learning gains (in relation to learning goals), if any, are seen as a result of the teaching intervention?	Evidence of learning was collated by analysing written pre and post-test results where LG 1: Problem solving and LG 2 Representations were in focus. The design of the tests, and the marking framework for this are presented below. The written test results were disaggregated by learning goal, by question, and by ability group. All the learner activity on Main task 7: Learner generated examples was analysed to provide an overview of learner activity on using storytelling to pose word problems. For three case study learners the video recordings of their individual structured interviews, learner artefacts from the interviews relating to using storytelling to pose and explain word problems were used. Together these two data sources were used to reflect on *LG* 3: Story telling

Research question	Data sources for answering research question
How was the intervention designed? How was the design refined over multiple research cycles? How did the third cycle intervention play out in this particular local context?	The planning documentation of the first, second and third intervention cycles, and the video recordings of the lessons in each cycle were analysed using the constant comparison qualitative method to reflect on the intervention design, and refinements of multiple cycles. Thick description of the third cycle teaching intervention to allow for it to be replicated was developed. This drew on planning documents, videos and selected transcripts of the lesson intervention, examples of learner activity in the lesson intervention. The description was subjected to critical self-reflection, as well as peer review.

The research questions were answered by considering the extent to which the evidence of learning gleaned from the third cycle intervention matched the HLT (with its assumed starting points and prospective end points). This evidence looked across learning from the whole class (compared to this evidence in the second cycle), together with the evidence of learning by ability group, and for each of the case study learners. From these sources of evidence, relationships between what was taught and what seemed to be learnt were inferred.

Figure 39: Types of analysis from each data source relating to evidence of learning



Below I provide more detail on the selection and ethical processes, the way in which the data was collected, the design of the written tests and interviews and the marking and coding frameworks adopted for analysing this data.

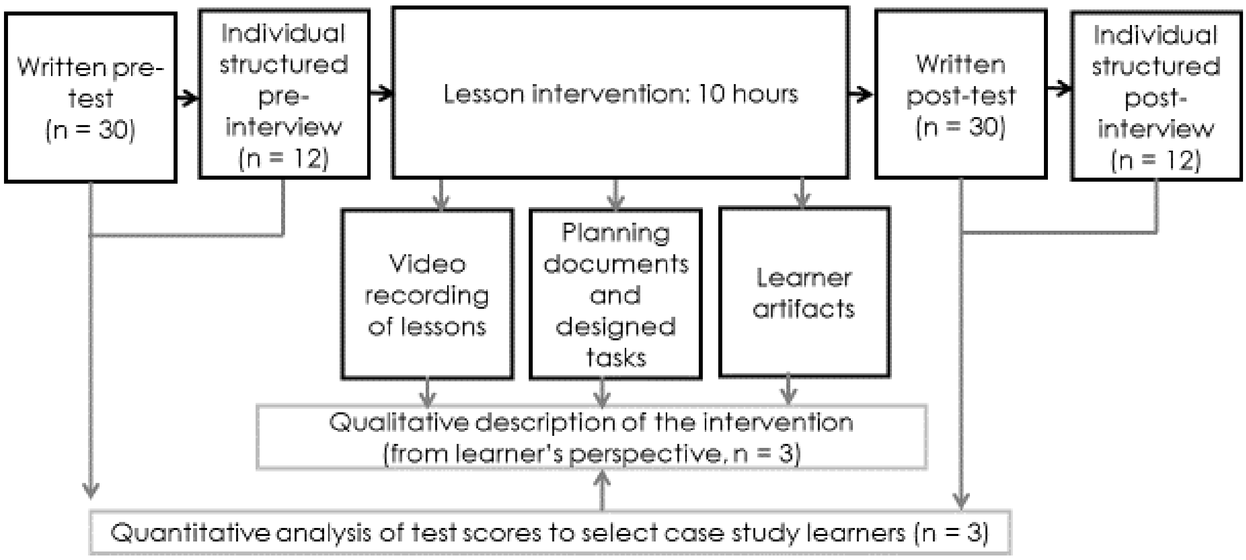
Selection of schools, classes and cases

I selected the schools and classes of children to be used in this study opportunistically. The first research cycle was conducted in 2012 with a Grade 3 class where I was providing professional development support to the Grade 3 teacher (Vanessa) as part of a development project. In the second research cycle I was not able to work with Vanessa's Grade 2 class as she was away on maternity leave. So the second research cycle was conducted in 2013 with a class in Grade 2 in the same school. The third research cycle was conducted in 2014 with Vanessa's Grade 2 class. I had no prior working knowledge of the children in this class.

When considering *whether* the intervention impacted on learning I used quantitative comparison of whole class and small group attainment in written tests. I made comparisons between the pre- post and delayed post-tests attainment; and made further comparisons with the attainment on the same test in the previous cycle. When considering *how* the intervention seemed to have impacted on learning, I shifted to a qualitative approach, adopting a case study method which is best applied 'when research addresses descriptive or explanatory questions and aims to produce first-hand understanding of people and events' (Yin 2006, p. 113). It is important to realise that the selection of case study learners was purposive. The cases were not intended to be representative of a bigger sample (for example their attainment group, or the whole class) where generalising from the case depends on statistical inference; rather the cases were selected to be reflective of 'a substantive topic or issue of interest, and the making of logical inferences (analytic generalisation)' (Yin 2006, p. 114).

I conducted quantitative analysis of the shifts in written test scores (from pre-test to post-test) for each learner in cycle 2 and cycle 3 making use a simple marking framework of correctness of solution (this marking framework is explained in more detail later in this chapter).

Figure 40: Selection of learners to be case study learners



I used the same criteria adopted in the earlier design experiment (Roberts and Stylianides 2013), to select the case study children for cycle 3:

- A child who made the most substantial shift in comparison to their peers,
- A child who made the most substantial shift in comparison to their peers, from the lower pre-attainment levels;
- A child made the smallest shift in comparison to their peers, in the upper pre-attainment levels;
- A child from each of the teaching groups (support, core and extension).

All cases were selected from the 12 learners where there was interview data. The first two cases - Retabile (from the core group) who made the most substantial shift in the class, and Gavril (from the support group) who made the most substantial shift in comparison to his peers in the support group - were deliberately chosen to examine cases where there was evidence of learning. Here, given that these were examples where there was evidence of learning (established through the quantitative analysis of their attainment in the written tests), I could analyse the more data from these children to make analytical inferences about how the observed learning seemed to have come about. I chose to select two cases of learning to ensure that analysis of learning did not exclude learners from the lower attainment reaches. The third case (Gavril from the extension group) was deliberately selected to focus on a child who could reveal ceiling effects in the instrumentation and the analytical methods. The prior design experiment revealed that by examining a case of lack of learning in the upper attainment levels, detail relating to ceiling effects in the written test design, and/or other kinds of learning (which were not in focus in the analysis of written tests) could be uncovered. So it was hypothesised that in this design experiment children from the upper attainment levels may not show shifts in attainment due to ceiling effects on the written test design (as they come into the intervention able to solve much of what is assessed in the test, and so there is little evidence of learning). As such, uncovering whether there was learning (which was not measured by the instrumentation), or inferring what could account for the lack of learning through detailed qualitative analysis of learner artifacts and interactions in the interviews was thought to be a worthwhile undertaking as these aspects could then be used to further improve both the task design and instrumentation of future design cycles

Research validity

From the outset of this study I recognised that my closeness to the study, as designer, teacher, researcher and analyst, was a research design limitation. While playing the teacher/researcher role was appropriate for the design experiment method (as an innovative approach unfamiliar to the normal teacher was required), this hindered my ability to step back from the intervention and examine its findings impartially. To mitigate this weakness in relation to the analysis of the means of supporting learning, I included multiple sources of data making use of video recordings, feedback from the normal teacher and data from the children before, during and after the intervention; as well as subjecting the qualitative descriptions of the intervention to critical self- and peer review.

The validity of the analysis of the data relating to learning gains were strengthened by making use of the quantitative and qualitative methods outlined above. For the quantitative data, the initial simple marking framework was deepened theoretically and the new marking re-applied to the written tests to ensure that similar results of learning gain could be inferred when approaching the written tests from a different perspective. I made use of statistical analysis (reliability tests, t-tests effect size) to establish the significance and effect size of observed shifts in the means on written test scores. Concerning the categorical data from the coding framework I tested for the significance of changes in categories using Chi-squared tests. Have established that significance of changes for the whole class using both interval (marks in the written tests) and categorical (representation types) data, I could exemplify the learning by explaining particular cases of learners who had shown the greatest shifts in attainment. Having three research cycles, from which there were comparable findings, also strengthened my confidence in the validity of the claims made in the third research cycle.

There are several parameters that are important to make explicit. As a design experiment, this study had a theoretical intent to deepen understanding of children's learning of additive relations word problems in this particular local context. Context is important and the claims made in this study relating to the patterns of reasoning evident amongst these young learners, draw on theories developed in other international contexts but remain limited to the context in which they have been examined. The claims made therefore pertain only to the findings from this local context. No attempt is made to generalise the findings from this design experiment to other local contexts. Further no attempt is made to generalise the learning of the three case study children (Mpho, Retabile and Gavril) to other learners (in other contexts, or even to other learners in their class). The case studies of individual learning were purposely selected to provide detailed information about how learning came about for Mpho and Retabile, and why there was little evidence of learning from Gavril. The insights from the cases, while suggestive of possible means of supporting learning with other children, have not had their efficacy established in this study. What is offered is a description of a theoretically informed instructional design for a hard-to-teach topic and findings of how this played within this particular local context. Thick description is offered so that where resonances with other spaces are recognized, possibilities exist for further trialling some of the tasks and pedagogical approaches designed and researched in this study

Ethical considerations

In all three research cycles the provincial department of education, principal and normal class teacher granted written permission for the research to be conducted. They saw value in the proposed teaching intervention and agreed that it should be undertaken as a normal part of the daily mathematics lessons. The letters of permission obtained from the Western Cape Department of Education is attached as Annexure 6, and the letter of permission from the University of Witwatersrand is attached as Annexure 7, with protocol number 2013ECE107D.

Parents of the children in the participating classes were sent a letter explaining that I was a qualified and experienced mathematics teacher, and informing them of the school decision for me to teach a series of numeracy lessons. The letter invited them to allow their children to participate in the research component of this. In all three cycles this letter included explicit parental permissions for the use of video-recordings and photographs of the children and their work. This letter is in included in Annexure 8. The parents of 26 of the 30 children in cycle three, agreed to participate in the research, with all giving permission to make use of video and photographs. Vanessa and I agreed that the four children, from whom no response had been obtained from their parents, would be asked to sit out of view of the video camera during the lessons. No data was recorded from these four children. All participants involved in the research were given pseudonyms.

It is difficult to obtain meaningful informed consent to participate in research from nine year olds. Nevertheless, I attempted to facilitate this in several ways. The purpose of the research and what was expected from the children, and that they could decide not to participate or to withdraw, was explained to the children, on the day that they took home the parental permission forms. This information was repeated when encouraging them to return the forms. I repeated this again in the first lesson. During the pre- and post-interviews, the purpose of the research and the reason for having a video camera were again explained to each child. They were told that they could ask to stop the video or the interview. Whenever a photograph of their work was taken in the interviews, they were asked permission for this. The learners were given a consent form which they completed (see Annexure 9).

Intervention design

Approach to intervention design in design experiments

This study was conducted as a design experiment, the methodology for which is presented in the next section. The design of each intervention cycle is a fundamental component of the methodology adopted in design experiments. Assumptions about the starting points for the envisioned forms of learning, the definition of the hypothetical learning trajectory (HLT), and the prospective endpoints are specified as part of the research planning (Cobb, Confrey et al. 2003). The design of the intervention informs the data collection and analytical methods used to analyse the data.

To describe the intervention design the theoretical features informing the intervention design must be specified in detail. Judgements on whether an intervention was successful relate to the particular learning goals, and the theoretical approach to how learning is viewed. The orienting framework for this design experiment makes clear the theoretical features relating to how learning in general, and learning of mathematics and additive relations word problems in particular was viewed. For the intervention in this study to be judged as resulting in the envisaged learning, it was expected that learning gains would be evident when comparing the learners' starting point (measured in pre-tests) before the intervention, to their endpoint (measured in post-tests) after the intervention. For these learning gains to be attributed to the means of supporting learning the aim was for the HLT to increasingly closely match actual learning trajectories. In this chapter I define the HLT and specify the trajectories of what I refer to as the 'enabling' and 'main' tasks included in the cycle 3 intervention. I also describe the way in which the intervention design was refined over each cycle with reference to both prior teaching experiences and evidence of learning, and the theoretical framework detailed in the previous chapters, which informed the intervention design for each cycle.

Intervention design for Cycle 3

Cycle 3 was conducted in April 2014 (in the first half of the academic year) in the same school as the Cycle 1 and Cycle 2 interventions. This Grade 2 class was a mixed ability class of 30 learners. The normal class teacher for this class was Vanessa (pseudonym). The majority (19) of learners were boys. This class also included a high proportion (30%) of Foundation Phase repeat learners (who had repeated either Grade 1 or Grade 2) and a high proportion (23%) of learners who were identified as having particular special needs as assessed by an educational psychologist.

Assumed starting points

Cobb, Confrey et al. (2003) refer to specifying the assumed 'social' and 'intellectual' starting points for any design experiment. I delineate assumed cognitive (intellectual) starting and ending points in relation to the mathematics. It is shifts in these cognitive starting points which were in focus for the overarching learning objective: 'expanding learners' personal potential example space for additive relations word problems to include compare type problems'. The cognitive end points provide targets or calibration points against which to assess attainment of learning goals, which were used to measure actual learning gains. However, and consistent with the orientating framework for this study it was expected from the outset that there would be variation in the lived experiences of each learner in the intervention. It was expected that the learners would come into the intervention with differing cognitive starting points, and that what they took from the intervention would differ (based on these starting points, but also on their unique lived experiences of the intervention). To take these assumed differences in account I refer to three ability groups: support, core and extension learners when reporting on start and end points.

In line with the orientating framework where learning is seen as educating awareness and harnessing natural powers where learning transforms the human psyche (awareness, emotion and behaviour) (Feature 1.3), I also make explicit the assumed social starting points in relation to behaviour and emotions. While these were not the focus of the intended learning, awareness of these starting points informed the framework for action in the theoretical framework.

Assumed social (behaviour and emotions) starting points

By the third cycle my assumptions about the social starting point drew on my experiences in the previous two cycles, and so were relatively clear. Prior to commencing the teaching intervention for the third cycle, prior evidence in classrooms informed me that the children would be:

1. From poor socio-economic contexts in terms of the home environments;

2. Accustomed to learning in English, but with the majority of learners having English as a second language (the home languages were expected to include isiXhosa, Afrikaans, Sesotho, French and other minority African languages);
3. Likely to include several children who exhibited disruptive behaviour and who struggled to focus on tasks and to work productively in groups;
4. Likely to include a few children who had developmental delays associated with foetal alcohol syndrome (FAS), and/or in-utero exposure to chrystalmeth (tic);
5. Likely to include a few children living in social contexts that included domestic abuse and lack of parental supervision;
6. Include a high proportion (about one third of the class) who were not obtaining the levels expected at Grade 2 defined in terms of the curriculum;
7. Accustomed to a flexible, approximately hour long daily numeracy lesson held in the morning;
8. Unfamiliar with differentiated mathematics lessons as the usual classroom teacher reported that she had not managed to differentiate the class and implement a system for group work in mathematics (although such an approach was used for reading groups).
9. At different cognitive levels for reading and writing, with some learners able to read and write short text fluently, and others not able to read independently, although able to copy writing.

Assumed intellectual starting points

Based on the prior intervention cycles and my broader experience in the school, I assumed that there would be a wide range of mathematical competences and estimated that learners would mostly be between level 4 – level 6 on the Treffers framework. The learner goals were focused on levels 6 and 7.

Figure 41: Learning trajectory (Treffers 2008)

	Descriptor	**Teacher expectations**	**Learning goals intention**
1	Learning to count		
2	Context-bound counting and calculating		
3	Object-bound counting-and-calculating		
4	Towards pure counting-and-calculating via symbolization	✓	
5	Calculation by counting where necessary by counting materials	✓	
6	From counting to structuring	✓	✓
7	Calculation by structuring with the help of suitable models		✓
8	Formal calculation up to twenty using numbers as mental objects for smart and flexible calculation without the need for structured materials		
9	Counting up to one hundred		
10	Calculating up to one hundred		

I specify the assumed intellectual starting points relating to the particular learning goals of the intervention.

Assumptions relating to Learning goal 1: Problem solving

To document my assumptions relating to Learning Goal 1: Problem solving, I made use of the cycle 2 intervention pre-test results (drawn from the 2013 Grade 2 cohort) and my cycle 1 experiences.

The cycle 2 written pre-test attainment revealed that the majority of learners could correctly answer bare calculations involving subtraction, but found solving additive relations word problems more difficult.

Table 12: Facility of each item cycle 2 (pre-test)

Question item	**Pre-test**
Question 1: Change decrease	79%
Question 2: Compare (matching)	68%
Question 3: Collection	57%
Question 4: Compare (disjoint set)	52%
Question 5: Partition	21%
Question 6: Bare calculation: 21 – 6 = […]	87%
Question 7: Bare calculation: 23 – 18 = […]	74%

The lowest attainment in terms of additive relations word problems was for the compare (disjoint set) problem, which accorded with the finding that this was the most difficult type of problem (Verschaffel 1994). The cycle 2 learners also performed poorly on the partition problem question. It was of interest that prior to the intervention, more learners were able to solve that compare (matching) word problem, than were able to solve the compare (disjoint set) problem. This suggested that the compare (matching) problem with its explicit focus on a difference (what is missing?), seemed to be a distinct problem to the compare (disjoint set problem), and the hypothesis (drawn from RGH and VDC research in other contexts) that the compare matching problem would be easier for learners to solve was found to be correct for the cycle 2 class in South Africa.

Given the attainment evident in cycle 2, it was expected that the learners in cycle 3 would experience the word problems in the written pre-test with the following order of increasingly difficulty: Change word problem; Compare (matching) word problem; Collection word problem; Compare (disjoint set) word problem, and partition word problem. As the cycle 3 intervention was conducted earlier in the academic year than the cycle 2 intervention; it was expected that the learners in cycle 3 would perform in a slightly lower range to cycle 2 learners.

Assumptions relating to Learning goal 2: Representations

In the cycle 2 intervention, I had seen that most learners were able to make use of semi structured number line representations, but that they tended to act on these using unit counting strategies. The same observation was made with regard to group models drawing iconic or indexical representations, where actions on single units dominated. In terms of the syntax models learners were familiar with the number sentences making use of +, - and = signs (in the standard form, where the unknown value is the 'result' of the addition or subtraction operation and on the right hand size of the equal sign). However they did not work with these sentences in relation to each other, and the whole-part-part diagram was not familiar to these learners.

Assumptions relating to Learning goal 3: Story telling

Based on my experiences in cycle 1 and cycle 2, I assumed that learners would be unfamiliar with using storytelling to explain and pose word problems.

General assumed starting point

The following were the assumed intellectual starting points for the cycle 3 intervention:

Table 13: General assumed starting points of the intervention

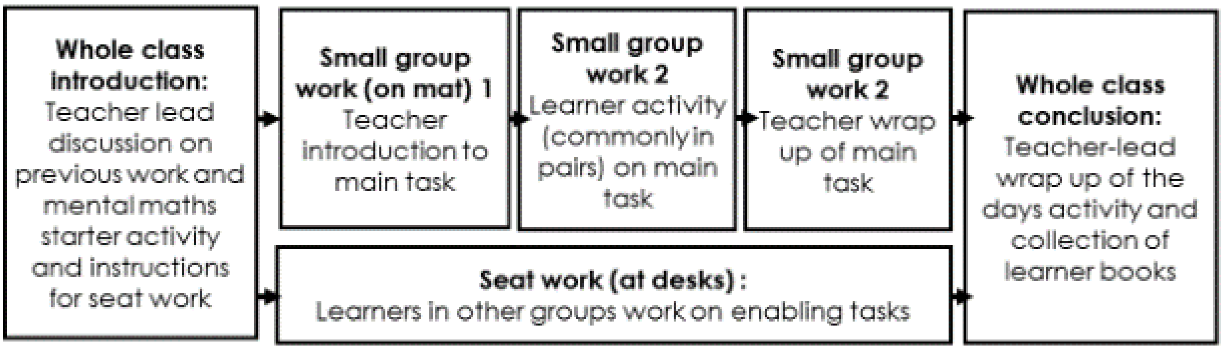
Overarching learning objective	To expand learners personal potential example space to include compare-type additive relation word problems
Learning goal	**Assumed starting point**
LG 1: Problem solving	• About half the learners would be able to solve change increase (result unknown) problems. • Solving the additive relation word problem types would be experienced in increasing difficulty as: change problem; compare (matching) problem; collection problem; and a compare (disjoint set) and problem partition problem;
LG 2: Representations	• Group models: Learners would mostly use counting in ones strategies, with few acting on groups of 5s or 10s. • Syntax models: Learners would have used number sentences, mostly in the form part + part = […], or whole – part = […]. They would not be familiar with families of equivalent number sentences or with the whole-part-part diagram • Line models: Learners would have used structured and semi structured number lines, but their use would be in ones and reflecting a take-away calculations strategy, wide range of abilities for number line use.
LG 3: Story telling	• Learners would not be familiar with posing additive relations word problem examples • When asked to pose an additive relation word problem, most learners would invoke a take-away action, associated with a change decrease problem type • Learners would not be familiar with explaining additive relations word problems to self and others

The hypothetical learning trajectory (HLT)

Characteristic of classroom-based design experiments the HLT was adapted with each refined design of the intervention. One of the orienting features of this design experiment was Feature 1.5: Learning by engaging in 'tasks' where a learning trajectory is inferred from 'learner activity' with these tasks, and as such the description of the HLT was closely related to the tasks designed to afford the intended learning. As explained in the theoretical framework for cycle 3, the design of the intervention was guided by Features 2: Approach to additive relations word problems, Feature 3: Narrative approach to mathematics teaching – story telling component and Feature 4: Narrative approach to mathematics teaching – Use of representations component.

There were three classroom formats in which it was expected that learning would take place: whole class teaching where oral mental maths fluencies were in focus; small group work sessions where particular reference examples of additive relations word problems were in focus; and individual seat work where independent learner activity on written tasks were in focus. The learning that was expected to take place was the union of these experiences and it was expected that there would be overlap and entanglement between the expected learning in each format.

Figure 42: Typical format in a lesson



To describe the tasks included the HLT I first discuss the enabling tasks, and then offer a description of the main tasks.

Enabling tasks

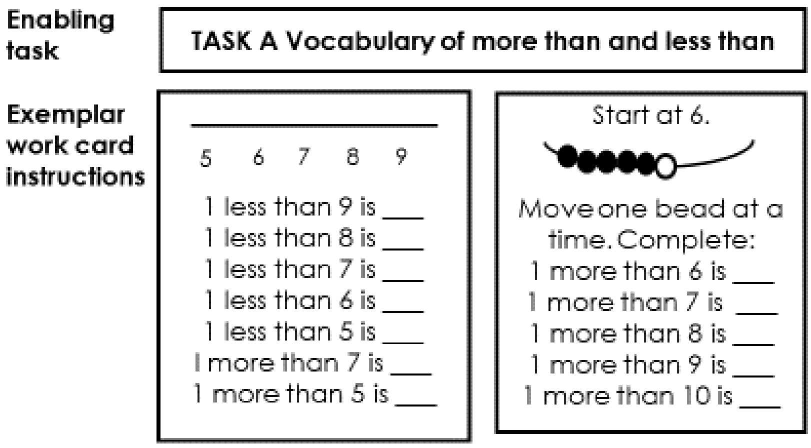
Enabling tasks contributed towards the attainment of the intended learning in the main tasks. All of the enabling tasks were posed using individual work cards which learners were expected to complete during individual seatwork sessions. The whole class teaching sessions were planned to respond to common learner errors (evident through the daily marking of learners' written work), and revisit main tasks, which had been introduced in the small group sessions. Each lesson was planned to open with some oral mental maths activities that introduced and/or built on the enabling tasks. Exactly what was planned for whole class sessions was not tightly specified before the intervention, as it was expected that these sessions would respond to observed learner activity on tasks from the previous day. The HLT for the individual work cards primarily made use of the Treffers framework starting at level 4 (towards pure counting-and-calculating via symbolization) where children were expected to represent physical objects up to ten on their fingers and will lines and dots to use such skills for adding and subtracting; and level 5 (calculation by counting where necessary be counting materials) where children were reciting and completing the number sequence up to twenty; putting number up to 20 into context; and structuring numbers using groups of 5 and 10 and structured or semi-structured number lines. In particular the enabling tasks focused on the vocabulary of 'more than' and 'less than' (although this is located at level 2 of the Treffers framework, prior intervention cycles revealed that many learners were not able to use these phrases appropriately); representations of number and number relationships using group models; line models and syntax models; and basic addition and subtraction number facts (using single digit numbers) and bridging tens calculations strategies for whole numbers between one and twenty.

The individual work cards for the enabling tasks were designed to support Feature 6.3: Providing opportunities for learners to practice (and receive feedback on pre-requisite fluencies and to keep learners productive occupied while teaching took place in small group sessions (Feature 5.1).

Examples of the individual work cards used in cycle 3 are presented in Annexure 5. In this section I provide a synopsis of each enabling task.

As part of the HLT it was anticipated that learners needed to experience using the terms 'more than' and 'less than' both orally and in written formats (Enabling task A: Vocabulary of more than and less than). Opportunities for learners to practice saying and writing '1 more than' and '1 less than' in a meaningful context were included for this purpose. A series of unique work cards – all using a similar problem format structure but varying the numbers was developed.

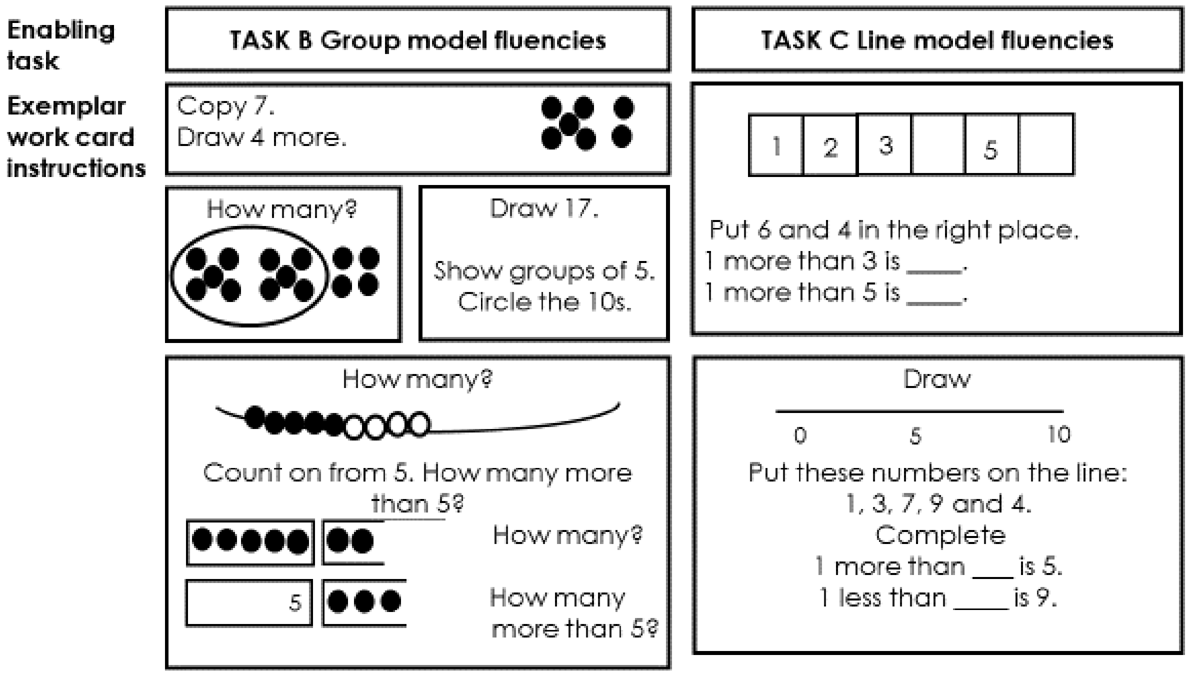
Figure 43 Enabling task A for cycle 3



The whole-class teaching sessions and individual work cards were also expected to provide opportunities for learners to see the teacher, and their peers making use of structured representations (line, group and syntax models). It was expected that learners would volunteer to complete representations, or create representations to show to their peers by coming up to the board to model a process. Learners were expected to progress from acting on countable units, to using structured materials where the reified objects could were acted on (Feature 4.4 A learning-teaching trajectory from counting to calculating which made use of increasingly structured representations was adopted). It was anticipated that not all learners would make this shift during the intervention, but that some learners would start to work with more structured materials and demonstrate some actions on groups using both line models and group models.

Enabling task B: Group models were intended to involve the use of the 5-5-5-5 bead string, the 5-strips and structured representations. Groups of 5 were to be shown using a dice pattern, and the groups of 10 were enclosed. These representations were contrasted to representations where unit counting was in focus, with no discernible groups. The teacher talk and writing or drawing on the board during whole class teaching was intended to support learners to notice the difference between ungrouped and grouped arrangements. Representations which showed groups of 5 and 10 would be praised.

Enabling task C: Line models were intended to include the 5-5-5-5 bead string, structured number lines, semi-structured number lines (depicting 0, 5, 10, 15 and 20) and empty number lines. Although the starting point assumed that learners would have had experience working with structured number lines, it was anticipated that amongst the learners there would be a wide range of familiarity with this line model (ranging from apparently completely foreign, to comfortable use of a an empty number line).



For Enabling task D: Syntax models involving families of equivalent number sentences, and the whole-part-part diagram were to be used. It was expected that learners would need to practice drawing whole-part-part diagrams and generating the families of equivalent number sentences that expressed this additive relation. These tasks aimed to help learners become aware of whole-part-part diagram and families of equivalent number sentences. Copying activities were included to provide the teacher with opportunities to see what children were noticing as they created their syntax representations.

Enabling task	**TASK D Syntax model fluencies**	**TASK E Basic number facts, and bridging tens calculations**	
Exemplar work card instructions	10 6 whole = part + part 10 = 6 + ___ 10 = ___ + 6 whole – part = part 10 – 6 = ___ 10 - ___ = 6	5 + 3 = ___ ___ = 7 + 1 ___ = 6 – 4 Replace * with + or – 8 * 1 = 7 2 * 5 = 7 9 = 3 * 6	3 + 2 + 1 = ___ 5 – 1 + 2 = ___ 6 = 8 – 1 - ___ 8 = 5 + 2 + ___ 3 + 2 = 1 + ___ 6 + 3 - __ = 7 8 – 2 = 3 + ___
	Draw a whole-part part diagram Write in the numbers. Write down number sentences for this diagram?	Draw 13 sticks. Show groups of 5. Circle the ten. 10 + 3 = ___ 10 + 5 = ___ 10 + 7 = ___ 10 + ___ = 12 8 + 2 + ___ = 14 9 + 1 + ___ = 16	Circle the ten 5 + 5 + 4 = ____ 8 + 2 + 3 = ____ 2 + 6 + 4 = ____ 3 + 3 + 7 = ____ 2 + 8 + 4 = ____ 1 + 9 – 2 = ____ 2 + 4 + 6 = ____ 7 + 7 + 3 = ____

Some of the individual work cards included tasks that were expected to already be familiar to learners (Enabling task E: Basic number facts and bridging tens calculations). These work cards were designed to facilitate practice with basic number facts involving single digit addition and subtraction. Success with these cards led to cards with a slightly higher number range being assigned to learners, where bridging the tens strategies were expected. Working in increasingly structured ways in their calculations, making use of structured groups of ten (with tasks such as circle the ten, or show groups of 5) were included.

Finally, some opportunities for learners to practice reading and solving word problems were designed for the cycle 3 intervention (Enabling task F: Word problems). These cards were only to be introduced on Day 6 when several word problems had been solved in small group work. The collection problem type was not included as a main task, but was included as part of the word problem fluency cards.

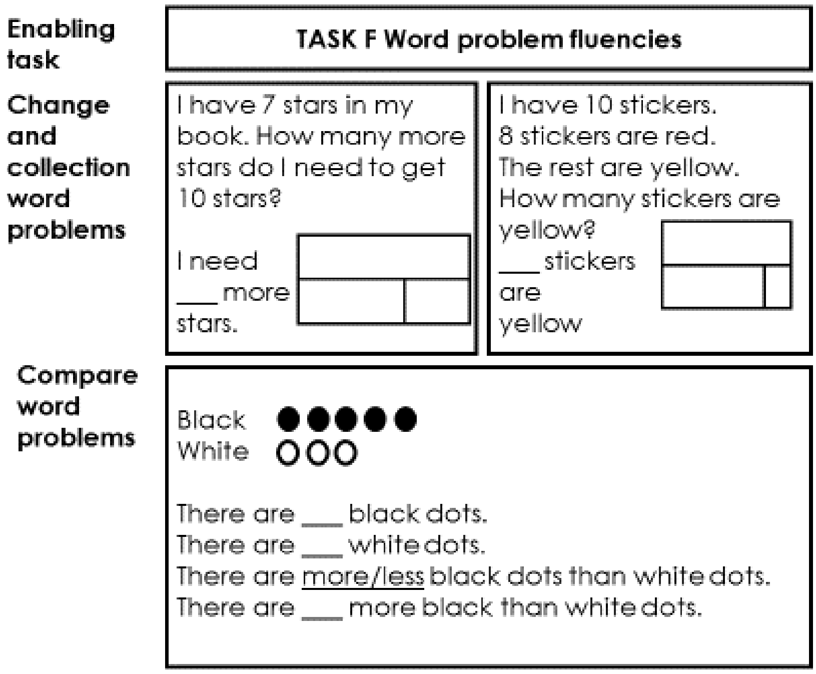


Table 15 provides a summary of the enabling tasks, their major features and relationship to the HLT

Table 14: Major features of the enabling tasks

	Enabling task	Major features	Elements of hypothetical learning trajectory (assumed learner activity)
A	Vocabulary of 'more than' and 'less than'	Provides contexts for the use of the terms 'more than' and 'less than'	Learners complete missing sections of sentences such as 1 more than 7 is ____. ____ more than 8 is 9. While moving bead along a string, or moving a finger along a structured number line, learners say the following sentences: '1 more than ___ is ___'. '2 more than ___ is ___'. ' 1 less than ___ is ___' .
B	Group model fluencies	Presents group model representations involving groups of 5, with each 10 enclosed. Expects learners to create group model representations – where groups of 5 and enclosed group of 10.	Learners say how many beads are on a string (using the 5-5-5-5) structure. Learners use 5-strips and count on from 5. Learners copy arrangements of a number of objects (using groups of 5 and enclosing the tens). Learners draw group model depictions of numbers less than 20 (showing groups of 5 and enclosing the 10s).
C	Line model fluencies	Offers opportunity to copy line models (structured number grids and number lines) Offers opportunity to fill in missing numbers in line models Offers opportunity to place numbers on a line model	Learners copy, complete and create line models using structured, then semi-structured then empty number lines. Learners depict hops in ones and jumps in groups.

	Enabling task	Major features	Elements of hypothetical learning trajectory (assumed learner activity)
D	Syntax model fluencies	Offers practice in copying whole-part-part image and related equivalent number sentences. Offer opportunity to fill in missing numbers in whole-part-part diagram and in families of equivalent number sentences.	Learners copy and complete whole-part-part diagram for a particular additive relation as well as equivalent number sentences. Learners fill in missing numbers in whole-part-diagram and related family of 4 equivalent number sentences. Learners fill in missing numbers in whole-part-diagram and related family of 4 equivalent number sentences.
E	Basic number facts and bridging the ten calculations	Offers opportunity to practice completing missing numbers and missing symbols in number sentences Exposes learners to numbers sentences in the form a = b + c or a = b – c Exposes learners to using enclosure action for finding a ten.	Learners fill in missing numbers and number symbols in number sentences involving single digits. Learners expected to use enclosing the ten action in calculations involving bonds of ten.
F	Word problem fluencies	Variations on the partition, change (reach a target), compare (matching) and compare (disjoint set) problems are posed. Change and collection problems are posed. At times these problems include scaffolds to make use of syntax models.	Learners solve problems of the partition, change (reach a target), compare (matching), compare (disjoint set) and collection problem types. Learners practice using syntax models for these problems.

Main tasks

It was expected that the main tasks would be introduced through small group work and then revisited during whole class teaching time. The learners were expected to engage with a sequence of tasks that drew on reference example word problems related to expanding learners' personal potential example space for additive relations identified from the literature and adapted based on the learning seen from cycles 1 and 2.

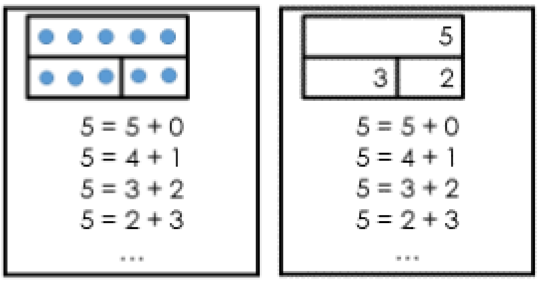
In each intervention, children were introduced to an additive relation story and given opportunity to solve problems relating to a particular reference example. They were invited to generate similar stories to the one they had heard and to retell these stories in class activities. They were then introduced to the next story, which was compared to the first, and so on.

While each task was numbered with the number depicting the order in which the tasks were introduced to the class, the intention was to revisit each task on several occasions. Learners were expected to recall and re-tell the problems that already been worked on, and to compare them to the new tasks being introduced.

The opening task Main task 1: Learning to work productively, was focused on introducing the intervention and encouraging a productive way of working while I was the lead teacher. As such the rules for learner behaviour were agreed and rewards and punishments established (Feature 5: Training behaviour). Learners were expected to make suggestions about what came to mind for them when trying to explain ‘5 equals’. The number sentence ‘5 =’ was to be written on the board in a deliberate attempt to challenge the likely dominant prior experience of seeing number sentences in the form ‘part + part = 5’, or ‘whole – part = 5’, where the equal sign is interpreted as an instruction to find the answer. The learners were to be encouraged to tell stories about five, or to show different ways of drawing 5. It was expected that the ideas of groups of five (for example 5 fingers on a hand, or the arrangement of 5 dots on a dice) would arise. This task was intended to allow for some diagnosis of the levels at which the children were working in terms of their representations and stories. Learners were encouraged to use their own informal representations to make any depictions of ‘five equals’.

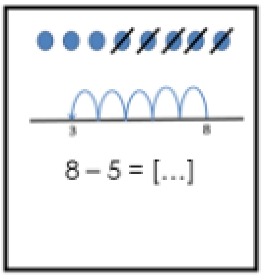
Having introduced expectations of working productively, it was then expected that learners would work on Main task 2: Partition problems in small groups. The partition problem was included in the intervention design as an example of a multiple-solution additive relation word problem (Feature 2.6 A partition problem type was used to introduce the additive relation structure). The problem context of 5 monkeys in 2 trees used a discrete object situation, which lent itself to learners concretely experiencing a partitioning action which could be used to introduce an indexical whole-part-part diagram. This was to be referred to as ‘the monkey story’. Over time the monkey story was to be varied to different numbers and different contexts, with the initial indexical representations being shifted towards more symbolic representations.

Figure 44: Representations for Main task 2: Partition problems



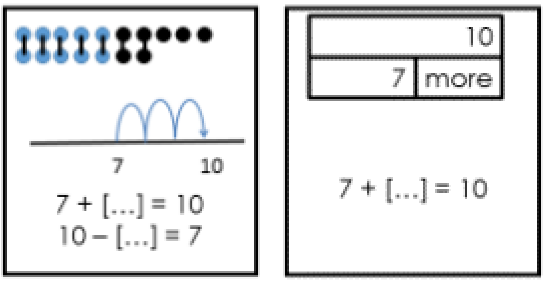
This was to be followed by Main task 3: Change which allowed for the introduction of learner generated examples in terms of story telling. The representations to be used included the line model with a take-away strategy, the take-away image and related action of removal. Learners were expected to write addition and subtraction number sentences to express the additive relation in their stories. As the previous cycle had found that the dominant stories narrated by learners made use of change actions (they were change problem types), learner generated examples of this problem type were thought to be appropriate. These stories would be referred to as ‘take-away/ change stories’.

Figure 45: Representations for Main task 3: Learner generated examples



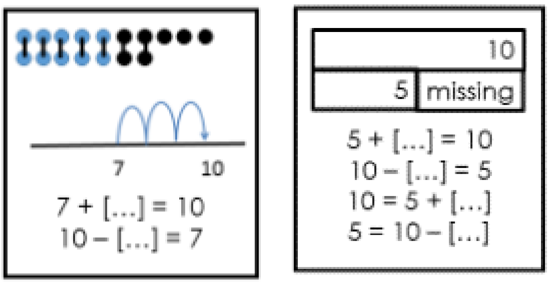
The take-away stories generated by learners were to be contrasted to Main task 4: Change increase (equalise), where the 1:1 matching action, and related difference image was to be introduced. The inclusion of this problem type was motivated by the theoretical framework (See Feature 2.4 A change increase (equalise) problem and a compare (matching) were included as a main task in the intervention design). Both problems had been identified as sharing characteristics of the compare problem type. The wording of the change increase (equalise) problem made explicit the change increase action of adding the difference on to a referent set. A context of learners accumulating stickers to reach a target of ten stickers before receiving a surpise was used (see Feature 5.4: Providing immediate extrinsic recognition for appropriate effort which accumulated into a reward – and tied into the mathematics) A whole-part-part image would be revisited for the change increase (equalise) problem contexts (again shifting from indexical representations with number sentences, towards a symbolic whole-part-part diagram and depicting the equalising action on a number line.

Figure 46: Representations for Main task 4: Change increase (equalise)



With both the partition problem (monkey story) and the change increase (equalise) problems in the context of sticker story having being introduced and repeatedly re-told and varied by the learners Main task 5: Compare (matching) problem was then to be introduced. The wording of the compare (matching) problem was intended to explicitly drew attention to the absence of elements in a smaller set by asking 'how many … are missing?' The compare (matching) story was to be told in the context of matching bowls to lids, or matching spoons to bowls. This context was chosen as eating porridge from plastic bowls with a lid and a spoon was a daily part of the classroom routine.

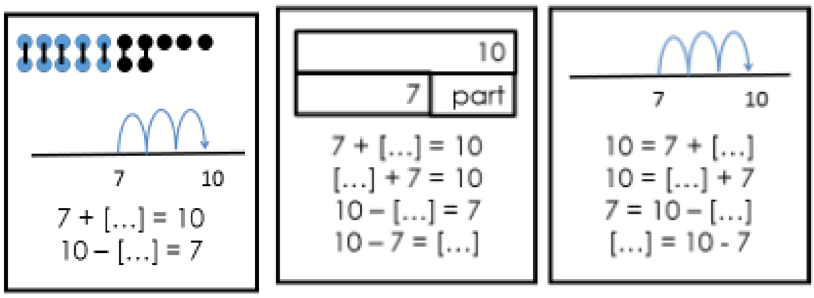
Figure 47: Representations for Main task 5: Compare (matching)



The matching problem context was expected to invoke a 1:1 matching action, and a related difference image. The difference strategy on the empty number line was to be introduced. This was to be supported by extending the syntax model to include four members of the family of equivalent number sentences.

Only once all the above stories: learners own stories (from Task 1 and Task 3), the monkey stories (Task 2), the sticker stories (Task 4), the bowls and lids (missing part) (Task 5) stories had been retold and varied by the teacher and the learners; would Main task 6: Compare (disjoint set problem) be introduced. Building on the conventional example space reference examples of Tom, and Joe with their marbles, the amounts that two people had of the same thing would be compared. This story was to be told using two learners from the class with a problem context chosen by the learners.

Figure 48: Representations for Main task 6: Compare (disjoint set)



By Main task 6 it was expected that eight members of the family of equivalent number sentences would be generated. The final task (Main task 7) for the third cycle intervention was designed to enable learners to show their application of the syntax model and to solicit learner generated examples of additive relations word problems.

Prospective endpoints

The prospective endpoints are described in the following table, against each of the learning goals:

Table 15: General prospective endpoints of the intervention

Overarching learning objective	To expand learners' personal potential example space to include compare-type additive relation word problems
Learning goal	**Prospective endpoint**
LG 1: Problem solving	• Learners would be able to solve a wider range of additive relation word problem types: partition problem; change problems; change (reach a target) problems; collection problems; compare (matching) problems and a compare (disjoint set problems) problems.
LG 2: Representations	• Learners would use a wider range of representation models (line, group and syntax models). • For group models, some learners would show evidence of acting on groups in indexical and iconic representations, when acting on bead string, and using jumps on a number line. • For line models, some learners would shift from structured to semi structured or empty number lines, some would shift away from hops (actions in ones), to jumps (actions on groups), some learners would sue both take-away and difference strategies on number lines. • For syntax models, learners would make use of whole-part-part diagrams and be more fluent with families of equivalent number sentences.
LG 3: Story telling	• Learners would be able to pose a wider range of additive relations word problem examples, having shifted from a constraint of additive relations in 'change' contexts to include other additive relations contexts (including 'compare' contexts). • Learners would be able to explain a wider range of additive relations word problems to self and others.

The above description details the starting points and the HLT for reaching the envisaged endpoints in the cycle 3 intervention. Below I report how the design of the intervention evolved over three intervention cycles.

The evolution of the intervention over multiple research cycles

The intervention was conducted in the same school with different cohorts of learners in three iterative design cycles. Concurrently the theory development and improvement in practice evolved as a dialectic over each intervention cycle. It is therefore appropriate to reflect on how this intervention evolved and to report on the changes made to the task design and theoretical features in each cycle. In this section I provide a brief account of the design changes from intervention over each intervention cycle. This has a two-fold purpose: firstly it is intended to highlight the key aspects of the third cycle by comparing it to the previous two cycles, secondly it exemplifies the iterative nature of the study which is a cross cutting feature of design experiments (Cobb Confrey et al. 2003).

The additive relations word problems selected for the main tasks changed with each intervention cycle, as did the representations that were in focus. To facilitate comparison across the three intervention cycles I refer back to the theoretical features and the related task design for the HLT for the cycle 3 intervention.

Intervention design for Cycle 1

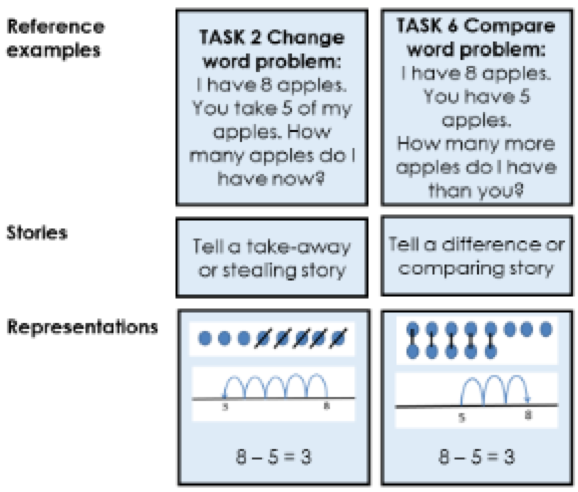
At the beginning of the design experiment I was focused on the shift in calculation strategy from a dominant ‘take-away’ strategy, to including a possible calculation strategy of ‘finding the difference’ (Feature 2.2). The difficulties children in this school had with compare type additive relations word problems were known (and are reported in Chapter 2). Given this evidence, I felt that making use of narratives as both a cognitive and a teaching strategy would support learners in moving towards two models of subtraction: take-away and the difference.

The cycle 1 intervention was undertaken with a Grade 3 class in this school where Vanessa (pseudonym, and the same teacher as for cycle 3) was the normal class teacher. This class was a high attaining Grade 3 mathematics class, as the school had opted to stream the grade into ability classes. The cycle 1 intervention took place near the end of the academic year (November 2012). There were 30 learners in the class including a balance of boys and girls, with the majority of learners speaking a language other than English at home, but learning in English at school. As this cycle was largely exploratory, and I was not yet clear on the detailed learning goals, I did not assess these learners.

Main tasks in cycle 1

My approach in cycle 1 was mainly informed by the counting pathway and neither the measurement pathway nor the structural approach were considered. During this first cycle intervention, there were only two main tasks. For ease of comparison with cycle 3, I use the numbering of main tasks adopted in cycle 3.

Figure 49: Main tasks in Cycle 1



Only two stories were contrasted, and only three representations were in focus: concrete counters (group model), number sentences in the standard form a – b = […] (syntax model), empty number lines (line model). In my planning processes there was little attention to the graduations and contrasts within each representation model.

Enabling tasks in cycle 1

I did not use any enabling tasks in cycle 1. All the teaching took place in whole class format and no individual work cards were used.

Theoretical features in cycle 1

The theoretical framework for the cycle 1 intervention was far less detailed than the framework for cycle 3. While the features of the orienting theories, and overarching learning objective were the same, the mathematics in cycle 1 was not tightly constrained.

Table 16: The theoretical framework for cycle 1, compared to cycle 3 (constraints)

Cycle 3 Feature	Title	Brief description	Present or absent for cycle 1
Domain specific instructional theories: Constraints imposed on the intervention design			
	Mathematical constraints imposed on the word problem example space	1 Possible mathematical operations were constrained to additive relations (ignoring multiplicative relations).	✓
		2 The degree of the word problem was constrained to 1-step (first degree) problem situations.	✓
		3. The types of numbers were constrained to whole numbers less than 20.	×
		4 The number context was constrained to discrete objects (and continuous measurement contexts were ignored).	×
		5 The options for structuring/partitioning numbers, stressed groups of five and ten (doubles and near doubles were ignored)	×

It is noteworthy that for the cycle 1 intervention my own awareness of the conventional example space for additive relations was limited, feeding into limited constraining of the aspects of the mathematics that I wanted to be in focus. I constrained attention to additive relations and to one step (first degree) word problems. However the number range was far bigger, and I was not yet constraining to whole numbers less than 20. My attention was drawn to the range of possible variations in terms of problem contexts (spanning both discrete and continuous measurement contexts) and the related vocabulary for these problem types. As such, constraining for discrete object contexts was not yet imposed on the intervention design.

The theoretical basis for the design of the cycle 1 intervention focused only on contrasting change type stories to compare type stories where the related calculations strategies (made visible on the empty number line) were take away and difference. In theoretical terms, in relation to selection of additive relations word problems, I only attended to counting-based conceptions of number and contrasting take-away and difference strategies. All theoretical elements relating to using story telling as a pedagogic and cognitive strategy were present though. On representations the take-away image (Figure 10 on page 62) was compared to the difference image (Figure 11 on page 62); and using an empty number line at take away calculation strategy (Figure 12 on page 63) was contrasted to the difference strategy (Figure 13 on page 63), however the whole-part-part diagram (Figure 19 on page 76) and family of equivalent number sentences was omitted.

Synopsis of learning from cycle 1

This first cycle intervention showed promise in using narrative in this school context, however several design changes were necessary.

From my experiences in cycle 1, I realised that the compare type problems were more difficult to teach than I had first assumed. I felt that learners needed substantially more support in seeing that adding and subtracting fitted together (the move from counting back to counting up had been difficult for the Grade 3 class). How adding and subtracting fitted together, and being able to solve compare (disjoint) set type problems, were learning goals that ought to have been attained in both Grade 1 and Grade 2 levels. I felt that focus of the intervention was foundational knowledge that ought to precede the Grade 3 level. I chose not to work with a Grade 1 class, as in the context of this school, at Grade 1 level, the English language skills of the learners were not yet sufficiently secure. So I chose Grade 2 as the appropriate grade level for the cycle 2 intervention.

However, learners had responded enthusiastically to most of the tasks intended to engage them in generating their own additive relation stories. There seemed to be potential in using storytelling as a cognitive and as a pedagogic tool (Feature 3) in this school context. But I found that while the children were able to make their thinking visible using stories and representations, they became fixated on the choice of problem context. This was consistent with the findings by Bastable and Shifter (2008). I had been aware of this research finding, which had informed my choice to ask for learner generated examples (Feature 1.7 seeking to gain insight into learners' personal example spaces though learners' generating examples). Even though I did not offer a teacher presented context of stealing, the learners generated their own change type word problem contexts and became fixated with stories about stealing.

Much time was spent in cycle 1 introducing and requesting use of a wider range of take-away actions – which included a wider vocabulary (other than stealing or robbing). Learners were encouraged to generate stories using verbs like: eating, hiding, throwing, giving, loosing. Some learners shifted away from the stealing problem context and were able to generate alterative stories using a wider range of vocabulary for removal/take away actions. However very few of the learners were able to make the shift to compare type stories, and their preferred and dominant narration was the change type problems, most commonly in the context of theft. I had not considered stealing to be a model for change type problems, but the children introduced this into the classroom, and 'stealing stories' became the prototype narrative for change type problems. The power of using tasks that stimulated learner generated examples – as means for finding contexts that are relevant for the particular children – was, somewhat sadly given this prototype, evident.

I found that most children still lacked the language of comparison, and struggled to explain or pose compare type problems. Many of the learners struggled to decode the compare problem: John has 5 and Kuhle has 8. How many more does Kuhle have than John? When posed this problem they commonly answered 'Kuhle has 8'. Their difficulties with this comparison context were clear when they were asked to pose problems similar to this. They did not seem to have the words for comparison, and used convoluted ways of articulating such problems, saying things like: 'I have 5 but I want to get to 8 like you. How many do I need?' I saw this as an opportunity, as this language structure supported the action of counting up (difference) strategy. However it was of concern that learners could not yet pose stories that made use of the phrases 'more than' or 'less than'.

Using representations to interpret calculation strategies (Feature 4) also showed promise. There were clear successes with most learners shifting their calculation strategies to include a difference calculation strategy by counting up or using an empty number line. Learners were also able to make use of both the take away image and the comparison image for subtraction. It was during the cycle 1 intervention that the importance of structuring these representations into arrangements that reflected groups of 10, was brought into focus. Although these were Grade 3 learners, whom I assumed would have a relatively secure sense of place value, the learners were not yet able to act on a group of ten, and could not arrange concrete counters using groups of 10 to quickly see how many there were.

Intervention design for Cycle 2

Cycle 2 was conducted with a Grade 2 class in the same school as cycle 1. In this cycle, the theoretical features of the intervention design were clearer, and learning gains were measured against each of the learning goals using both written tests and structured interviews.

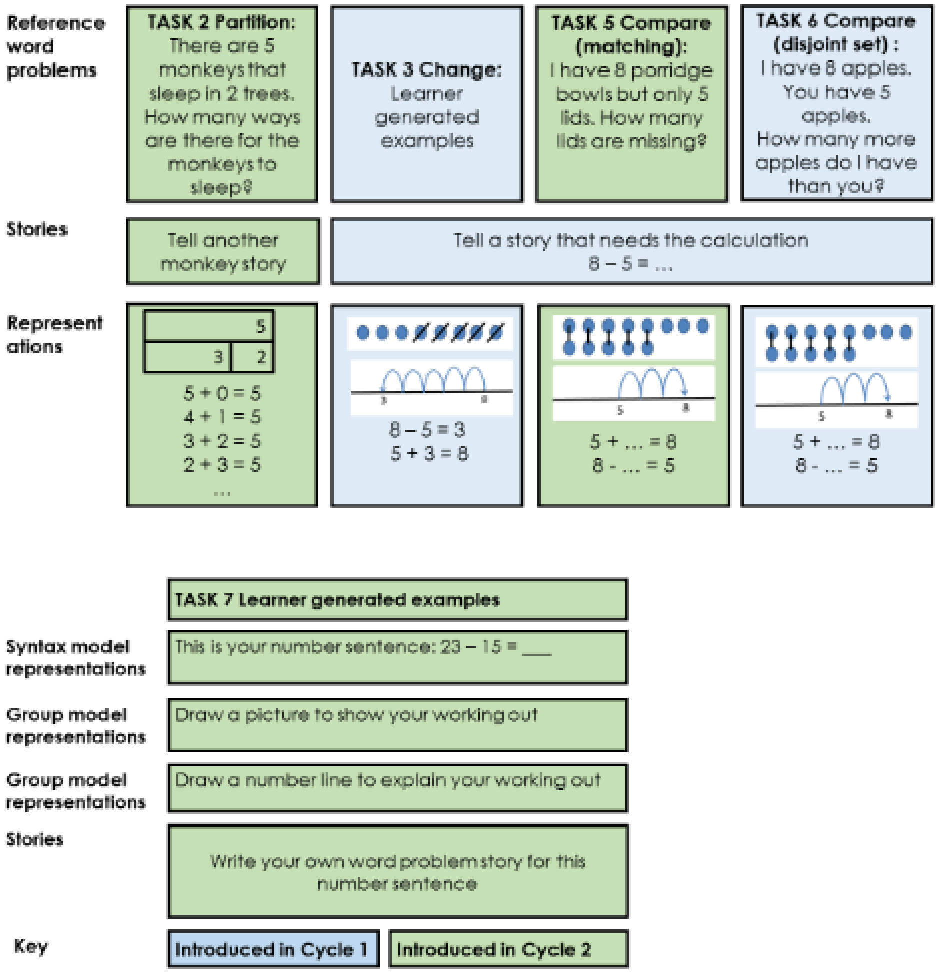
Cycle 2 was conducted towards the end of the Grade 2 academic year in November 2013. This was a mixed ability class of 30 learners. The normal class teacher was Colleen (pseudonym). Again there was a balance of boys and girls, and the majority of these learners did not speak English at home, but learnt in English at school. There were 5 learners (17%) who has repeated either Grade 1 or Grade 2 in previous years, and a similar number of learners considered to have special educational needs (as assessed by an educational psychologist).

I constrained the mathematics more tightly, considering only whole numbers less than 20, and focusing on discrete object contexts (ignoring the continuous measurement contexts). The constraints imposed on the mathematics in cycle 2, matched those imposed on cycle 3.

Main tasks in cycle 2

Based on the experiences in cycle 1, as well as the refinements to the theoretical framework drawn from the literature I inserted Main task 2 (Partition problem) and Main task 5 (Compare matching) into the task design for cycle 2. I also included a collection problem into the assessment tasks. With these insertions, the selection and sequencing of Main tasks was in cycle 2 was very similar to that adopted in cycle 3. The following provides an overview of the main tasks used during cycle 2:

Figure 50: Main tasks in Cycle 2



Enabling tasks in cycle 2

I did not use any enabling tasks in cycle 2. Most of the teaching took place in class format and towards the end of the cycle I experimented with some small group work in a separate venue, while the normal teacher worked with the rest of the class. No individual work cards were used.

Theoretical features of cycle 2

My experience of the first cycle intervention, coupled with deeper reading of the mathematics education literature on approaches to additive relations word problems (Feature 2), resulted in a more detailed theoretical framework relating to the domain specific instructional theories being used in the second cycle. This was closely aligned to the features of the domain-specific instructional theories used in the third cycle.

I chose to make several adjustments to Feature 2 of the domain-specific instructional theories used in cycle 1. Firstly I introduced Feature 2.3: The conventional example space for additive relations included change increase, change decrease, collection and compare type problems, which broadened the contrast between change and compare problem to include other problem types. Given the type of language which I had observed the learners using in trying to explain and pose compare problems, I introduced a compare (matching) problem context, before shifting to the compare (disjoint set) problem. I hoped that this would support learners in the move from the take away image to including the difference image, and to imaging actions of removal or 1:1 matching as both being relevant to subtraction contexts. This was the reason for the introduction of Main task 5: Compare (matching) word problems into the cycle 2 design.

Secondly I felt that the structural approach should be explored as a starting point. I therefore introduced a structural problem to open the cycle two intervention. As a result Feature 2.6 A partition word problem was introduced into the theoretical framework, and Main task 2: Partition word problem was introduced into the task design for cycle 2.

The planned approach to story telling (Feature 3) was much the same as for cycle 1. However the planned approach to representations (Feature 4) was more refined than in cycle 1, as some components of a structural approach to additive relations (Feature 2.5) were introduced. A whole-part-part diagram was used for the partition problem (Feature 2.6), but this was not seen as a cross cutting image for use across all the problem types. The whole-part-part diagram was also not accompanied by a family of equivalent number sentences. The representations were not introduced slowly with a view to develop, revisit, and act on them in different ways over time. As such planning for the development of representations to take time, and over time allowing representations to become reified (Feature 4.2) was not included in this intervention cycle.

Table 17: Summary of major features of the cycle 1 and cycle 2 theoretical framework (compared to cycle 3)

Cycle 3 Feature	Title	Brief description	Present or absent for cycle 1	Present or absent for cycle 2
Domain specific instructional theories: Mathematical design features of the intervention				
2	Approach to additive relations word problems	2.1 A counting-based conception of early number development was adopted	✓ Present	✓Present
		2.2 A 'take away' calculation strategy was contrasted to a 'difference' strategy, with consideration for efficiency in choice of strategy depending on the numbers involved.	✓ Present	✓Present
		2.3 The conventional examples space for additive relations word problems included: 'change increase', 'change decrease', 'collection' and 'compare' problem types	× Absent	✓Present
		2.4 2.4 An equalise word problem and a 'compare (matching)' word problem were used as main tasks in the intervention design	× Absent	Partially present[17]
		2.5 A 'structural' approach to additive relations was foregrounded making use of a whole-part-diagram and families of equivalent number sentences for number triples.	× Absent	Partially present[18]
		2.6 A 'partition' word problem type was used to introduce the additive relations structure.	× Absent	✓Present
3	Using story telling as pedagogic and	3.1 Paradigmatic knowing / logical-scientific thinking and narrative knowing/thinking were viewed as necessarily and simultaneously present in any mathematics word problem	✓ Present	✓ Present

[17] A compare (matching) word problem as included, but a change increase (equalise) problem was not included
[18] A symbolic whole-part-part diagram was included for the partition problem, but this representation was not used as common to all word problem types. The family of equivalent number sentences was not included.

Cycle 3 Feature	Title	Brief description	Present or absent for cycle 1	Present or absent for cycle 2
	as a cognitive strategy	3.2: Story telling was used as a pedagogic strategy to motivate learning and encourage sense making.	✓ Present	✓ Present
		3.3 Story telling was viewed as a cognitive strategy which draws on the human powers of imagining and expressing.	✓ Present	✓ Present
		3.4 Tasks that demanded story-telling and modelling in the LoLT were designed to support ELLs in their dual need to deepen conceptual understanding of additive relations, and to improve their English language proficiency.	✓ Present	✓ Present
		3.5 The word problem was viewed as a story-telling and re-telling performance in the social community of the class.	✓ Present	✓ Present
4	Using representations to interpret calculation strategies	4.1 Flexible movement between representations where sense-making was primary, was encouraged	✓Present	✓Present
		4.2 Secure use of a particular representations takes time and, over time, representations should be reified to become cognitive tools	× Absent	× Absent
		4.3 In the process of being inducted into formal symbolic notation, shifts in attention are required in both directions – from concrete objects to symbolic objects (and vice versa) and both processes of specialising and generalising are necessary	✓Present	✓Present
		4.4 A learning-teaching trajectory from counting to calculating which made reference to increasingly structured representations involving line, group and syntax models was	× Absent	× Absent

I had not experienced difficulties in managing learner behaviour with the cycle 1 intervention. So when designing the cycle 2 intervention I did not attend to the framework for action, which was included in cycle 3. Table 18 compares the theoretical features of the frameworks for action of cycle 2 to cycle 3:

Table 18: Summary of major features of the cycle 2 theoretical framework (compared to cycle 3)

No	Title	Brief description	Present or absent for cycle 2
Frameworks for action: Implementation features of the teacher role			
6	Training behaviour	6.1 Teaching the whole class as well as in small groups.	× Absent
		6.2 Encouraging a growing brains mindset.	× Absent
		6.3 Expecting learners to work independently, and ensuring that while they had similar tasks, that they each worked on unique problems.	× Absent
		6.4 Providing immediate extrinsic recognition of appropriate effort which accumulated into a reward (and tied into the mathematics).	× Absent
7	Principles of general pedagogic style	7.1 Adopting a mathematical thinking questioning style, listening to and exploring suggestions from learners.	✓Present
		7.2 Providing specialised and explicit feedback and paying attention to learner errors.	Partially present
		7.3 Facilitating opportunities for learners to practice (and receive feedback on) prerequisite fluencies	× Absent

The only aspect of the framework for action adopted for cycle 3, which was present for cycle 2 was my mathematical thinking questioning style where I aimed to listen to and explore suggestions from learners. I adopted a whole class teaching approach, and I did not expect learners to work independently on similar tasks where each learner had a unique problem. Much learner activity in cycle 2 reflected a culture of copying other learners' work and as such providing specialised and explicit feedback to learners was only partially present.

Synopsis of learning from cycle 2

The Cycle 2 intervention was difficult in terms of classroom management, and much of the teaching time was taken up in my attempts to manage disruptive learner behaviour. The learning gains in cycle 2 compared to cycle 3 are discussed in detail in the chapter that follows this. What is presented below is the main learning from cycle 2 which resulted in changes in the theoretical features of the intervention, and the adapted task design for cycle 3.

In cycle 2 I assessed learners using written tests and individual interviews, and there was evidence of learning gains. Learners had managed the compare (matching) problems, which was a step towards the compare (disjoint set) problems types, but only a minority of learners were able to solve compare (disjoint set) type problems. Once again (as was the case in cycle 1) learners in cycle 2 showed substantial difficulty with explaining or posing problems involving the terms 'more than' or 'less than'. I resolved to include the use of these terms (orally and in written form) as a prerequisite fluency that required practice in cycle 3.

I found that the time spent on the structural approach was productive, and provided a clearer mathematical basis for the shift from counting back to counting on, as there was more understanding of the structural relations underlying addition and subtraction. However, I noted that too much time was spent on the partitions problem in cycle 2, with insufficient attention to the compare (disjoint set) problem and the underlying structure of all additive relation contexts.

> The 5 monkeys story [Partitions problem] was enjoyed in cycle 2. However too much time was spent on these type of problems, without enough emphasis on the whole-part-part diagram and families of number sentences (Reflective journal, December 2013).

The story telling aspects of this intervention were dominated by teacher presentation of stories (Feature 3.2). There was little opportunity for learners' to tell stories (Feature 3.3). The disruptive learner behaviour in this intervention made it very difficult for learners to engage in oral story telling in a whole class context. Some experimentation with small group work, where learners were removed from the class in small groups, showed that these approaches were more successful in smaller groups.

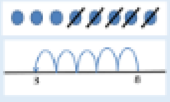
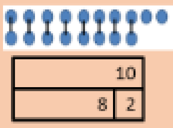
I felt that the connections between the stories would be strengthened if children constructed and engaged with the whole-part-part diagram in all of the additive relation story contexts. I felt that the whole-part-part image was a powerful representation of the additive relations structure. While this had been introduced in Cycle 2 as a possible representation, it was not in focus for many learners, and was not used as means of representing additive relations in their written test work. My reading of the literature deepened as I familiarised myself with the structural approach and the measurement pathway in early number work. The early algebra approaches of working with structure, suggested that a generalised approach to representations, where increasing structure was a deliberate teaching objective came into play. Also in this cycle I realised that children needed time and experiences to see the whole-part-part diagram as representative of a number triple (Feature 4.2). This is reflected as follows in my journal as part of my planned changes from Cycle 2 to Cycle 3:

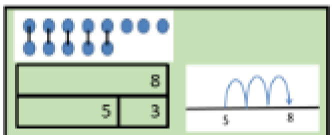
> The w-p-p diagram was just introduced [by the teacher as a complete artefact or representation]. Next time I want children to physically experience and build up the w-p-p diagram. I will give then 5 strips to represent the group of 5 monkeys. The action of partitioning the group of 5 will be to tear the 5 strip. This creates the partitions for the 2 parts in the whole-part-part diagram. Learners can then arrange the torn strips to be systematic, using the process of starting with all monkeys in the small tree, and moving them one at a time to the big tree. Once they are arranged systematically, they will stick the torn strips into their books and create a part-part wall for 5 (bonds of 5, with related number sentences). From the monkey story lesson I want to introduce the following vocabulary: group; whole; part; part; families of number sentences. So less time will be spent on completeness and working systematically [of the partitions problem]. The main idea here is to get the idea of a w-p-p diagram and partitioning options for "making 5". (Reflective journal, December 2013).

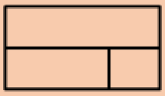
In my planning processes for cycle 3, I paid more attention to the graduations and contrasts within each representation model.

My experiences in cycle 2 resulted in the insertion of two more main tasks into the cycle 3 intervention: Main task 1: Learning to work productively and Main task 4: Compare (reach a target) involving stickers and rewards. These tasks were inserted into the cycle 3 design in response to the need to better manage learner behaviour. The cycle 2 intervention also resulted in my paying attention to 'frameworks for action' in an environment whether learner behaviour was challenging. Feature 5: Training behaviour and Feature 6: Principles of general pedagogic style came into focus for the design of cycle 3. Attending to these features in relation to the overall mathematical learning resulted in the identification of enabling tasks and related individual work cards to manage small group activity. How these aspects were approached in cycle 3 has been described in detail as part of the theoretical framework for cycle 3, and so will not be repeated here. I simply provide a summary of the seven main tasks used in cycle 3, with reference to their origins in cycle 1 and cycle 2:

Figure 51: Main tasks in cycle 3

Reference word problems	**TASK 1 Learning to work productively** Make a cover for your maths book where you tell stories and draw pictures of 5 = …	**TASK 2 Partition:** There are 5 monkeys that sleep in 2 trees. How many ways are there for the monkeys to sleep?	**TASK 3 Change:** Learner generated examples	**TASK 4 Change (reach a target):** I have 8 stickers. How many more stickers do I need to reach the target of 10 stickers?
Stories	Learner generated examples	Tell another monkey story	Tell a story for the calculation 8 – 5 = …	Tell your sticker story for today
Representations	Learner generated examples, with teacher encouraging line, syntax and group models	 5 = 3 + 2	 8 – 5 = 3 5 + 3 = 8	 10 – 2 = 8 8 + 2 = 10

Reference word problems	**TASK 5 Compare (matching):** I have 8 porridge bowls but only 5 lids. How many lids are missing?	**TASK 6 Compare (disjoint set) :** I have 8 apples. You have 5 apples. How many more apples do I have than you?
Stories	Tell a story that needs the calculation 8 – 5 = …	Use the words '*more*' and '*than*' to tell a story that needs the calculation 8 – 5 = ….
Representations	 5 + 3 = 8, 8 = 5 + 3 8 - 3 = 5, 5 = 8 - 3	 5 + 3 = 8, 8 = 5 + 3, 3 + 5 = 8, 8 = 3 + 5 8 - 3 = 5, 5 = 8 – 3, 8 – 5 = 3, 3 = 8 - 5

	TASK 7 Learner generated examples
Representations	Choose your own numbers and complete: Whole = part + part [ ] = [ ] + [ ] [ ] = [ ] + [ ] Whole – part = part [ ] – [ ] = [ ] [ ] – [ ] = [ ] 
	Learner generated representations
Stories	Tell 3 stories for your whole-part-part diagram. One of your stories must use the words 'more' and 'than' in it

Key Introduced in Cycle 1 | Introduced in Cycle 2 | Introduced in Cycle 3

The above offers a synopsis of changes made to the intervention design over the three intervention cycles. From an initial very loose experimental approach, the second cycle became far more detailed with more clearly theorised task design and selection. The experiences of the second cycle then further deepened the theoretical enquiry, as the theoretical approach had been trialled, empirical data emerging from this cycle then lead back into even more detailed theorising.

Analytical approaches

In this section I describe the design of the assessment instruments and analytical frameworks developed for analysing the data drawn from across the various data sources.

Design of written tests

The written tests were administered to assess both LG 1: Problem Solving and LG 2: Representations. LG 3: Stories was assessed through the inclusion of Main Task 7: Learners' generating examples in the cycle 3 intervention, and through structured interviews with the case study children. The purpose of marking the tests was to provide a quantitative overview of the broad evidence of learning gains (or absences of learning gains) for the whole class to allow for comparison between cycle 2 and cycle 3, as well as comparing the pre-test attainment to the post-test and delayed post-test attainment in cycle 2. Although the three learning goals were clearly related to each other, each learning goal suggested a different way to view the evidence of learning, and as a result analysis of learning was distinct for each goal. In this section I discuss how each learning goal was assessed and make explicit the criteria used for marking, categories applied for coding, and processes of comparison adopted to infer learning gains (or absences of learning).

Administration of written tests

Experiences in my broader development work with this school revealed that learner copying of each other's work in written test situations was a common practice in this school community. As a result, the written tests were administered in the small teaching groups, where 10 children were seated individually at desks spaced out in a classroom, to minimise copying. The written tests were administered in small groups over an approximately one hour time period.

The tests were administered in a similar way to how standardised assessments in mathematics (ANAs) are administered at this grade level. The test was given to learners in written format, but I read out the questions and then gave time for learners to complete their responses to each question. Each question item included a decorative picture which learners were encouraged to colour in when they had finished working on the problem. Only when all learners had completed a question and started colouring in the pictures was the next question posed by the teacher. As the learners wrote the tests in small groups, this meant that the time frames were different for different groups. The time frame was an estimate as the pace of completing the test depended on the group, and the intention was to allow sufficient time for every learner to complete the question.

Item selection and design for written tests

The same basic written test design (except for the additional prompt to draw a number line) was used for cycle 2 and for cycle 3, and the post-test (and delayed post-test in cycle 3) mirrored the pre-test format. All of the written tests are included in Annexure 10 for ease of reference. The same items were included in all five written tests. The format of the Cycle 3 Pre-test and Post-test included a prompt to draw a number line. Cycle 2 Pre-test, Cycle 2 Post-test and Cycle 3 Delayed post-test excluded this prompt. As a result of this change in format, Cycle 2 Pre-test and Post-test are compared to the Cycle 3 Pre-test and Post-test and Delayed post-test, excluding the prompted number line items.

The written tests were slightly refined from cycle 2 to cycle 3, to include prompts to draw a number line. Given that cycle 2 took place in the fourth academic term in the Grade 2 academic year (in November), whereas the Cycle 3 intervention took place in the second term of the Grade 2 academic year (in April), it was expected that children in Grade 2 should do better in the same test administered later in the year. Comparison between the Cycle 2 post-test, and the Cycle 3 delayed post-test offered a comparison where the time of year was less of a factor (as they were both written in November).

There were seven items used in the written test. From pre-test to post-test, and then to delayed post-test the sequencing of the questions within the written test was varied, and the problem contexts (objects and characters in the story) were adjusted slightly for each question. These changes were made as a result of the closeness of the pre-test to the post-test, as the intervention was conducted over a relatively short time period of 10 consecutive schools days. The pre-test was written in the few days prior to the intervention, and the post-test was written in the few days after the intervention (so depending on the grouping of learners[19] the post-test was written about 17-19 calendars days after the pre-test was written). The change in problem context resulted in different decorative pictures, which changed the colouring activity slightly.

Three types of problems were included in the written test: single solution word problems (change, collection and compare problems), a partition word problem where multiple solutions were expected, and bare calculations.

Following from Carpenter, Hiebert and Moser (1981) the word problems were 'constructed so as to provide a relatively simple example of the given type with respect to syntax, vocabulary, sentence length, familiarity of problem situations, and so on' (p. 30). The number range for all of the word problems (change, collection, compare) was carefully selected to be whole numbers less than 20. Working with first grade (grade 1) middle class children whose main language was English, Carpenter and Moser (1984) imposed the following constraints on their selection of number triples for additive relations word problem tasks:

[19] The learners wrote the written tests in small groups within 1-2 days of each other.

1. Doubles like 8 + 8 were not included, because there is evidence that children operate differently with these combinations (Groen & Parkman, 1972).
2. Consecutive addends like 6 + 7 were not included in the larger number set. It was not possible to impose this condition on the set of smaller numbers.
3. Number triples whose sum is 10 were not included.
4. Addends of 0 and 1 were not included. (Carpenter and Moser 1984, p.185)

Using these constraints, Carpenter and Moser defined 'small' number triples as 2-3-5; 2-4-6; 2-5-7; 2-6-8; 3-4-7; 3-6-9; and 'larger' number triples as 3-8-11; 4-7-11; 5-7-12; 4-9-13; 6-8-14; 6-9-15. Using this distinction, the number triples for this study were 'larger' number triples and included 6-8-14, 2-9-11, and 4-9-13. All number triples required bridging a ten, which allowed for comparisons between learners acting on groups and acting on ones (Feature 4.4: A learning teaching trajectory from counting to calculating which made use of increasingly structured representations using line, group and syntax models was adopted). The partition problems all involved partitions of five. The selection of 5 was deliberate, with fluency in making five seen as supportive of shifts towards fluent calculation making use of known facts relating to 5s and 10s (Treffers 2008 and Anghileri 2000). The number range for the bare calculations was slightly higher (21-15-6 and 23-5-18), where bridging the 20 was expected. These triples were selected to contrast an efficient take-away strategy (for 21 – 6 = 5) to an efficient difference strategy (for 23 – 18 = 5). Efficiency of subtraction strategy being dependent on the numbers involved in the calculation, is reported by several researchers including Carpenter and Moser (1984); Anghileri (2000), and Askew (2012).

The slight variations in the context of each word problems item from pre-test to post-test are presented in the table below. The delayed post-test made use of the pre-test questions, however the sequence of items was varied. I make use of the question sequence applied for the pre-test throughout this description.

Table 19: Item variation from pre-test to post-test and to delayed post-test

	Question	Pre-test questions	Post-test questions	Delayed post-test questions
1	Change (result unknown)	There are 14 people in a bus. 8 people get out of the bus. How many people are left in the bus? Space for working Number sentence: Number line:	There are 14 people in a taxi. 8 people get out of the taxi. How many people are left in the taxi?[20] Space for working Number sentence: Number line:	There are 14 people in a bus. 8 people get out of the bus. How many people are left in the bus? Number sentence Space for working:
2	Compare (matching) (difference unknown)	There are 11 bottles but only 9 lids. How many lids are missing? Space for working Number sentence: Number line:	There are 11 locks but only 9 keys. How many keys are missing? Space for working Number sentence: Number line:	There are 11 bottles but only 9 lids. How many lids are missing? Number sentence: Space for working
3		Sue has 13 stickers.	The teacher has 13 pens.	Sue has 13 stickers.

[20] Local context note: The children in this school commonly travel by 'mini-bus taxi' with a capacity of 16-20 people. This transport is commonly referred to as a 'taxi.'

	Collection (part unknown)	Some are gold and some are red. 4 of the stickers are gold. How many stickers are red? Space for working Number sentence: Number line:	Some are black and some are red. 4 of the pens are red. How many pens are black? Space for working Number sentence: Number line:	Some are gold and some are red. 4 of the stickers are gold. How many stickers are red? Number sentence: Space for working
4	Compare (disjoint set) (difference unknown)	Jarred has 9 sweets. Martha has 12 sweets. How many more sweets does Martha have than Jarred? Space for working Number sentence: Number line:	Jani has 9 stickers. Mpho has 12 stickers. How many more stickers does Mpho have than Jani? Space for working Number sentence: Number line:	Jarred has 9 sweets. Martha has 12 sweets. How many more sweets does Martha have than Jarred? Number sentence: Space for working
5	Partition problem	Sihle has 5 pencils. He keeps them in his two pockets. Show all the ways that the pencils can be kept in Sihle's pockets.[21] Space for working with blocks offered for partitions	Sue has 5 balls. She keeps them in two boxes. Show all the ways that the balls can be kept in her boxes. Space for working with blocks offered for partitions, and lines on which number sentences could be written	Sihle has 5 pencils. He keeps them in his two pockets. Show all the ways that the pencils can be kept in Sihle's pockets. Space for working with blocks offered for partitions, and lines on which number sentences could be written
6	Bare calculation (encouraging take-away)	21 – 6 = Space for working Number line:	21 – 6 = Space for working Number line:	21 – 6 = Space for working
7	Bare calculation (encouraging difference)	23 – 18 = Space for working Number line:	23 – 18 = Space for working Number line:	23 – 18 = Space for working

Four single-solution word problems (Questions 1-4) were posed. These problems were selected because they were representative of problem commonly included in Foundation Phase texts; and they spanned a variety of problem types that earlier studies indicated would elicit different solution strategies, and were of varying difficulty levels. These items included change decrease, collection and compare (matching) and compare (disjoint set) problem types, which were selected to offer a range of problem types from the conventional example space. All of the single-solution word problems made use of the easiest sub-type within a main problem type (change, collection and compare). It was expected that learning gains in solving the easier subtypes, would then suggest teaching moves to broaden learners' experiences to include the other sub-types.

The following figure illustrates the item design for the single answer word problems.

[21] Local context note: The children in this school were from poor socio-economic contexts. They do not have pencil cases, or lockers or storage space in their classroom. Keeping pencils in their pockets (of their school uniform shorts or skirt) was a common practice.

Figure 52: 'Single-answer' word problem (Cycle 2 pre- and post-tests and Cycle 3 delayed post-test)

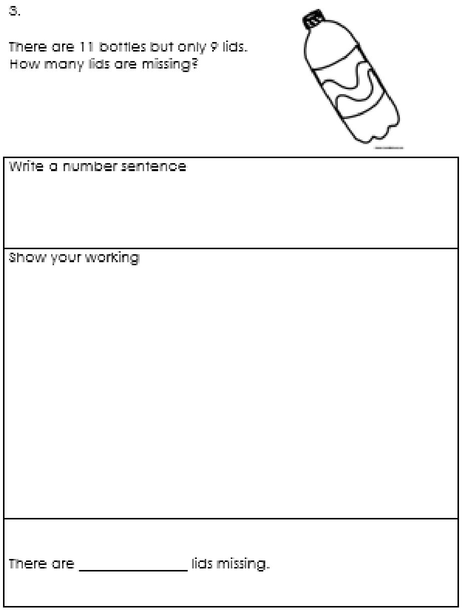


Figure 53: 'Single-answer' word problem (Cycle 3 pre- and post-tests)

The written test design for these word problems included several salient features. The word problem question was written in large font type for young learners. Each item included a decorative image (like a picture of a bottle, for a problem about bottles) which was included for two reasons. On the one hand, this decorative picture was thought to support ELL's need for sense making and entry into the word problem context (Chapman 2006, Clements and Samara 1999, and Bruner 1996); on the other hand the instruction 'if you have finished your work on the problem, then colour in the picture', could be used to keep all learners working at a similar pace through the written assessment. The questions were read one at a time for the class to work on. The question was repeated twice, and repeated again if any learner asked for further repetition.

There was a large space provided for learner working to allow for informal strategies to be used. This space was used to establish which representations were self-selected by the learner in their informal communication of mathematics when working on solving a problem. Then a written prompt was included for learners to complete an answer sentence 'There are ___ lids missing'. A prompt to write a number sentence (syntax model) was included.

There was a slight change in the format of the test questions from cycle 2 to cycle 3. The experiences of cycle 2 revealed that the children tended to first work in the space for working, frequently to directly model the problem situation. This empirical finding from cycle 2, aligns with the international literature on young children's solution strategies, which often start with direct modelling of the problem situation, and expressing this as a syntax model develops later as a cognitive tool for finding the solution (Carpenter, Fennema et al. 1999). It was observed in cycle 2, that once the solution to the problem was obtained, the learners then shifted to produce the prompted number sentence. Based on the experiences in cycles 1 and cycle 2, it was anticipated that most learners would most commonly self-select group models to sketch the problem situation and act on this through direct modelling to solve the problem. The number sentence and number line prompts were therefore included to encourage them to also produce more structured representations using syntax model and line model representations for the same problem situation. How learners responded to these prompts was considered as part of their production of mathematical representations.

Each test item was printed on a new page. This was a deliberate design decision to allow plenty of space to assess how leaners made their thinking visible (through first working informally with their self-selected representations, and then re-representing the problem both using number sentences and number lines) and also to ensure that the pacing of the test could be teacher directed. Learners were encouraged to work one page at a time and discouraged from turning over to the next page (and so to the next test item) until directed to do so by the teacher.

The fifth question in the written test was a partition problem type. It differed from single answer additive relation word problems described above, as its solution required multiple partitions. The rationale for the inclusion of this problem type was discussed in Feature 2.6 A partition problem was used to introduce the additive relations structure. The following figure provides the item design for this problem. A partition problem was included to assess learners' ability to create partitions from given whole, and attention was paid to the number of correct partitions and evidence of working systematically to produce completeness of solutions.

Figure 54 Partition problem written test design (Cycle 2 pre- and post-tests and Cycle 3 pre-test and delayed post-test)

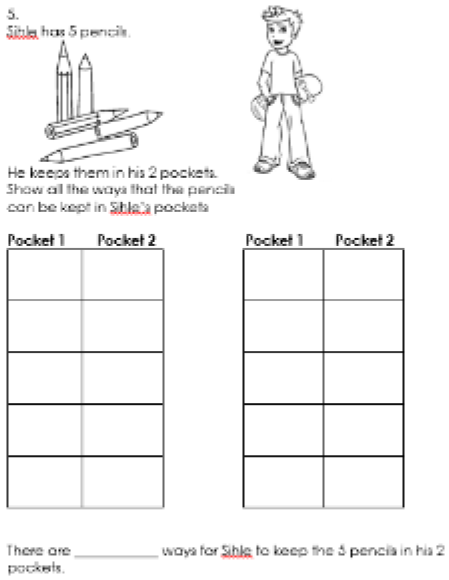
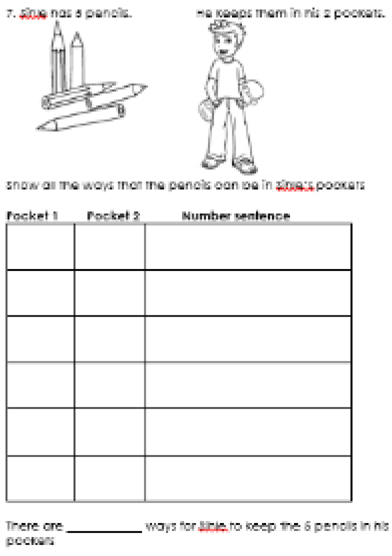


Figure 55: Partition problem written test design (Cycle 3 post-test)



The partition problem was more tightly structured to provide a layout in which children could quickly record the possible partitions using either number symbols (syntax model) and or indexical/ iconic representations (group model). As this test item neither included a blank section to show self- selected representations for working out the solution, nor prompts relating to particular representations, this item was excluded from the analysis of representations (Learning Goal 2: Representations).

Including space for more than six partitions was a deliberate design decision, as it was expected that if space was only provided for 6 partitions, that learners would not try and find more than 6 options. An answer sentence was included at the end of this item 'There are ____ ways for Sihle to keep the 5 pencils in his 2 pockets'. This was included as a teacher-directed prompt for learners to count how many unique partitions they had found.

Finally two bare subtraction calculations were included as question 6 and question 7 in the written test. These items were included to assess transfer of the representations used for word problems to bare calculation situations. As a result two contrasting questions were posed as bare calculations: a subtraction calculation which lent itself to an efficient take-away calculation strategy; and a subtraction calculation that lent itself to an efficient difference strategy. This contrast was a result of the inclusion of Feature 2.2 A 'take-away' calculation strategy was contrasted to a 'difference' strategy, with consideration for the efficiency in choice of strategy depending on the numbers involved. The calculation strategy included space for group models (a take-away image was contrasted to difference image) and line models (a take-away number line was contrasted to a difference number line). These two items were included on a single page as follows:

Figure 56: Bare calculations (Cycle 2 pre- and post-tests and Cycle 3 delayed post-test)

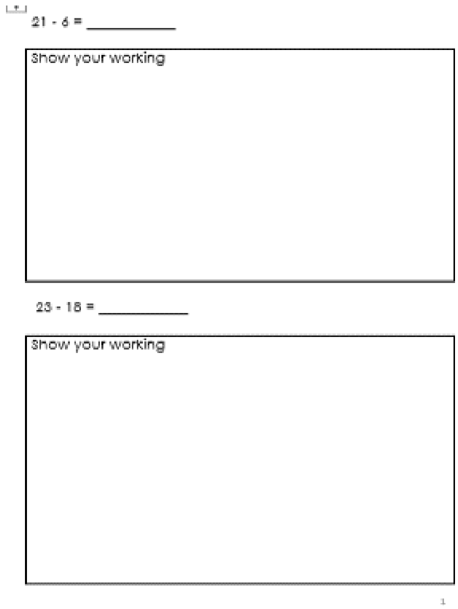
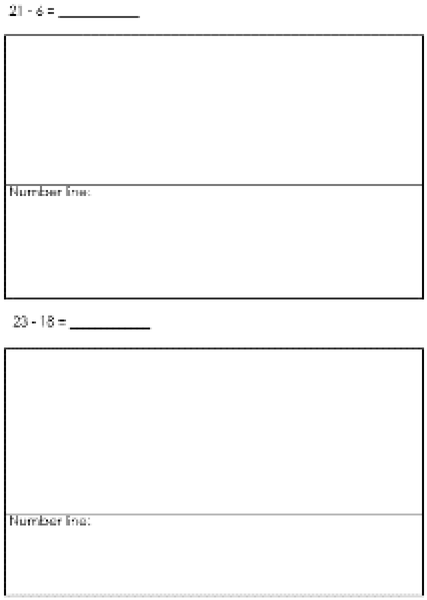


Figure 57: Bare calculations (Cycle 3 pre- and post-tests)



The layout of these questions in the written test was deliberate, and the decision to include a number line prompt in Cycle 3 drew on the empirical findings from cycle 2 (where number lines were not self-selected) and Feature 4.2 Secure use of a particular representation takes time, and over time, representations should be reified to become cognitive tools. The first space below each calculation was provided for learners to show their self-selected representations. As it was anticipated that many learners would not spontaneously initiate a solution working on a number line, a further prompt was provided to establish whether they could produce their calculation strategy on a number line. In the delayed post-test, in order for comparisons to be made with the cycle 2 post-test, the number line prompt was omitted.

Marking the written tests (marking framework)

The written test was marked in two ways: Firstly a simple marking framework for correctness of solution was applied; secondly the marking of the test was more deeply theorised in relation to the learning goals for the intervention.

For the first simple marking framework each item in the test was marked for a correct solution with one mark for a correct answer and 0 marks for no correct answer on each question. This included the five word problems and two bare calculation problems making a total of 7 items in the test, and total test score of 7 possible marks for the written test.

Table 20: Simple marking for written tests

	Question	Correct solution	Marking criterion: Correct answer
Q1.	**Change (increase):** There are 14 people in a bus. 8 people get out of the bus. How many people are left in the bus?	There are 6 people left in the bus.	1
Q2.	**Compare (matching):** There are 11 bottles but only 9 lids. How many lids are missing?	There are 2 lids missing.	1
Q3.	**Collection:** Sue has 13 stickers. Some are gold and some are red. 4 of the stickers are gold. How many stickers are red?	There are 9 red stickers.	1
Q4.	**Compare (disjoint set):** Jarred has 9 sweets. Martha has 12 sweets. How many more sweets does Martha have than Jarred?	Martha has 3 more sweets than Jarred.	1
Q5	**Partition:** Sihle has 5 pencils. He keeps them in his two pockets. Show all the ways that the pencils can be kept in Sihle's pockets.	There are 6 ways for Sihle to keep his pencils in his pockets.	1
Q6	21 – 6 =	15	1
Q7	23 – 18 =	5	1
	Total		**7**

The correct solution was awarded 1 mark awarded if children wrote the number symbol for the initial solution in their space for working, or into the answer sentence provided as a prompt. No learners wrote the answer in words, and where learners wrote a mirror image of the correct number symbol, they were awarded the mark. This simple marking framework was adopted early in the analysis stage to establish whether learners were better able to find the correct solution to each problem in the test, and to compare attainment in 'finding a correct solution' across the different items in the test. It was this simple marking framework which was applied to the written tests in both the second the third cycles. For the third cycle some of the teaching took place in ability groups where the pre-test attainment scores were used to create three small groups of learners. The simple marking framework was used to split the group into 10 support learners, 10 core learners and 10 extension learners, and to identify the three case study children for more detailed qualitative analysis.

Later in the analytical process, the written test was again analysed quantitatively, this time using a more deeply theorised marking framework developed against criteria relating to each of the two learning goals (Learning Goal 1: Problem solving and Learning Goal 2: Representations). As the overarching learning objective related to expanding example spaces for additive relations word problems, the two items on bare calculations were excluded from this marking framework. These tests items were included to measure transfer of the representation models developed for additive relations word problems to the bare calculation context. These questions were therefore analysed separately in relation to the representations used to solve these questions, where the calculation action evident in the representation was of interest. This built on Feature 2.2 Calculation actions were contrasted where actions depicting 'take away' were contrasted to 'difference' actions, with the likely choice of action likely to be influenced by the problem situation and consideration for efficiency in choice of strategy depending on the numbers involved.

The overall theoretical framing of this expanded marking framework two important distinctions were relevant.

The first distinction related to *coherence*, where children's representations were categorised as either 'coherent' or 'incoherent'. A representation was judged to be coherent if it could be interpreted as referring to the problem text in an appropriate additive relationship. This distinction was imposed in order to ensure that attainment considered only those representations which were connected to the problem text, and that marking was not allocated based on representation form. When analysing the representations produced in response to each item a record was kept of both coherent and incoherent models (this categorical coding of representations is explained in detail later in this chapter).

The particular *circumstance* that determines why a learner is using a particular representation, be it their own invention, self-selected from a range of teacher presented options, or prompted by a teacher is of relevance when interpreting children's representations (Roberts 2015). As such the second distinction in children's representations were categorised as either 'self-selected' or 'prompted'. The written test design included a space for working, where learners were encouraged to 'show their working' or 'use the space to show how they work out the problem'. Self-selected representations were any representation drawn into the space for working, or drawn in the margins of the paper. This drew on Feature 4.3 Secure use of representations takes time and over time representations should become reified to become cognitive tools. Considering an emergent approach to inducting learners into formal notations, what learners chose to represent informally (the representations they create themselves in the space for working) without any explicit prompting was therefore of interest. These self-selected representations were indicative of their cognitive tools as they were bringing these to mind on their own in response to a problem statement. At the same time however, as learners were in the process of being inducted into more formal use of structured representations, they were also prompted to depict the additive relation in the problem text using formal representations of number lines (line model) and number sentences (syntax model). The learners' responses to these prompts were referred to as 'prompted' representations. No prompt was included for a particular group model (iconic or indexical drawings), as findings from cycles 2 and 3 indicated that these were the most common self-selected models. Measuring attainment for Learning goal 1: Problem solving considered only the self-selected representations which led to solving the problem, while measuring attainment for Learning goal 2: Representations considered both self-selected and prompted representations. As a result the self-selected representations were valued both in relation to Learning Goal 1: Problem solving (where a coherent self-selected representation was marked as an indication of learner progress towards finding the solution) and Learning goal 2: Representations (where a coherent self-selected representation alongside coherent promoted number lines and coherent prompted number sentence was an indication of flexible use of multiple representations for the same problem situation). In comparison to the simple marking framework, the expanded marking framework gave a higher mark allocation to the ability to move between multiple representations for the same problem text.

Taking the five test items relating to additive relations word problems, criteria for the quantitative marking for each learning goal were developed, each of which is discussed in turn below.

Table 21: Expanded marking framework for written tests

		Learning goal 1: Problem solving		**Learning goal 2: Representations**			**Total**
	Question	**Criterion 1: Correct answer**	**Criterion 2: Any coherent self-selected model**	**Criterion 5: Coherent prompted number sentence**	**Criterion 6: Coherent prompted number line**	**Criterion 7: Other coherent self-selected model (not a number sentence or a number line)**	
Q1.	**Change (increase):** There are 14 people in a bus. 8 people get out of the bus. How many people are left in the bus?	1	1	1	1	1	5
Q2.	**Compare (matching):** There are 11 bottles but only 9 lids. How many lids are missing?	1	1	1	1	1	5
Q3.	**Collection:** Sue has 13 stickers. Some are gold and some are red. 4 of the stickers are gold. How many stickers are red?	1	1	1	1	1	5
Q4.	**Compare (disjoint set):** Jarred has 9 sweets. Martha has 12 sweets. How many more sweets does Martha have than Jarred?	1	1	1	1	1	5
.		**Criterion 3: Correct partitions**	**Criterion 4: Systematic working**				
Q5	**Partition:** Sihle has 5 pencils. He keeps them in his two pockets. Show all the ways that the pencils can be kept in Sihle's pockets.	1	1				2
	Total	5	5	4	4	4	22

Summing marks across criteria and items for each learner, resulted in a written test score of 22 marks which could be applied to the cycle 3 pre- and post-tests. The items in cycle 3 delayed-post were matched to the items in the cycle 2 post-test (which did not include the prompt for a number line). As a result when comparing cycle 3 to cycle 2, 'Criterion 6: Coherent prompted number line' was excluded from the marking framework, resulting in a written test score of a possible 18 marks.

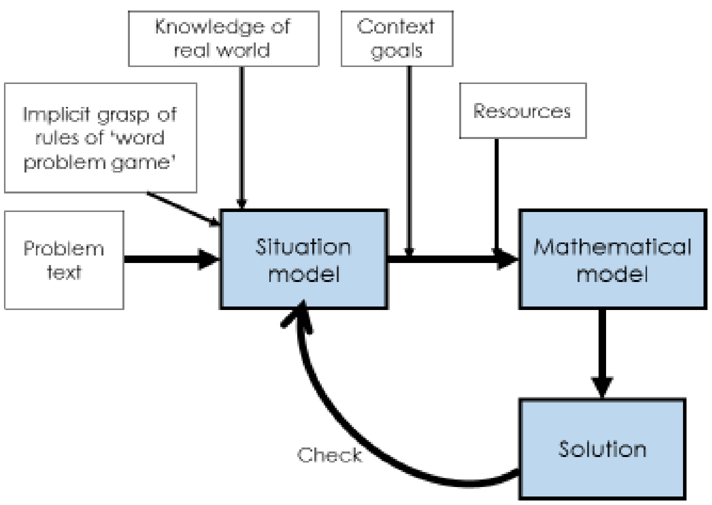
This marking framework provided the numerical data (in the form of test marks) for the written test which allowed for comparison of overall test score for each learner and for the class (considering the mean and standard deviation) from the overall test score. This allowed for quantitative comparisons from pre-test to post-test, and to delayed post-test where attainment in relation to Learning Goal 1: Problem Solving (10 marks) could be compared to attainment in relation to Learning Goal 2: Representations (12 marks).

I now elaborate on each of the marking criteria providing examples from learner work to exemplify how the marking framework was applied.

Criteria for marking Learning goal 1: Problem solving

The first two marking criteria were developed for problem solving and taken from Greer, where the process of solving a problem is seen to comprise of three related steps: understanding the situation model, selecting an appropriate mathematical model in order to solve the problem. The process of mathematization advocated by Greer (1997) views modelling as providing a link between the grounding aspects of reality described in the problem situation (understanding of which is a requirement towards finding the correct solution) and the more formal mathematical structures used for solving the problem.

Figure 58 Schematic diagram of factors influencing the modelling of a word problem. (Greer, 1997)



Criterion 1: Correct solution

'Criterion 1: Correct solution' related to the 'solution' step in Greer's model. If a child correctly identified the number to correctly answer the problem question they were awarded 1 mark. This number was commonly indicated by completing the answer sentence 'There are _6_ people left in the bus' by using the number symbol 6.

Criterion 2: Any self-selected coherent model

'Criterion 2: Any coherent model' was considered in relation to Learning Goal 1: Problem solving. This criterion was included in recognition of the related steps in Greer's model (situation model and mathematical model). Movement between the situation model, mathematical model and solution is expected, and this is not considered to be a linear sequence of steps. By way of example, a trial and error solution method could start with a possible solution and check this by relating the proposed solution to the problem situation. The process of modelling (working with a situation model or a mathematical model) was seen as part of the process of solving a problem. To meet this criterion only self-selected representations were considered. The prompted representations were ignored, as the prompts were intended to provide evidence relating to Learning Goal 2: Flexible use of representations, and were not expected to have been used by the learner during the problem solving process. Prompts were responded to only after the problem had been solved.

When attempting to distinguish the mathematical model from the situation model with the empirical data from these young children, it became clear that the distinction between the two was often blurred. For example, a child's drawing of 14 people, where 8 people are crossed out may be interpreted to reflect the situation model (the people are evident in their drawing) and the mathematical model (a calculation is conducted through direct modelling the subtraction operation evident in their action of crossing out 8 of the people). This observation is supported by the development pathway referred to by Carpenter et al. (1999), Carpenter et al. (1997) and Carpenter and Moser (1984) where direct modelling is seen to precede calculation strategies when solving additive relation problems. As such the criteria for a mathematical model and situation model (in the self-selected representations) were combined, and only 1 mark was awarded for any coherent model (where both situation models and mathematical models) could satisfy this criterion. The symbolic mathematical model in the form of the number sentence (which was prompted) was assessed as part of Learning Goal 2: Representations.

Recognising 'any coherent model' depended on the type of self-selected representations that the produced. I describe the expected evidence of mathematical modelling for each type of representation (group, line and syntax) in turn.

If, using group models, learners could demonstrate their concrete modelling action through the arrangement of iconic and indexical objects in their drawings and through actions introduced into their drawings: e.g., taking the examples of the 14 people in the bus, with 8 who get out, a coherent model would depict 14 indexical or iconic people and an action of drawing lines through (or encircling or colouring-in) 8 of the circles. This was interpreted as a concrete modelling action invoking a 'take-away action' to solve the problem. Evidence of having drawn 14 people in the bus, then acting on this drawing to erase 8 of these people, and then redrawing the 8 people outside of the bus, was taken as evidence of concrete modelling action of removal (which was appropriate for the take-away problem situation). Similarly for the test item 'There are 11 bottles but only 9 lids, how many lids are missing' a learner was considered to use a coherent situation model if 11 bottles and 9 lids featured in their representation. They were considered to be adopting an appropriate concrete modelling action to solve the problem if they acted on their situation model by arranging the 9 lids adjacent to /touching each bottles, or acting on their bottles and lids by drawing lines to connect each lid to each bottle. Such actions onto the problem situation were interpreted as invoking a '1:1 matching action' and as such as evidence of an appropriate model to solve this problem.

If using line models, learners could demonstrate the mathematical model they adopted to solve the problem using either hops (in ones) or jumps (in groups), shown with an arc drawn from one number to the next on a number line, a mark was awarded. Movement could either be forwards (to bigger numbers) denoting addition, or backwards (to smaller numbers) denoting subtraction. A coherent mathematical model of the 14 people, with 8 people getting out, would be depicted by showing 8 hops back from 14 to reach 6, or a jump of 4 back from 14 to reach 10, followed by another jumps of 4 backwards to reach 8 (or any other partitioning of 8). The direction of the jumps or hops was not considered significant (a learner could hop or jump forwards 8 from 6 to each 14) as the same number triple – 14-8-6 was being depicted.

If using syntax models, learners could demonstrate the mathematical model they adopted to solve the problem using formal mathematical notation, a mark was awarded. The relationship between numbers could either be expressed using a number sentence making use of number symbols (+ for addition, - for subtraction and = for equivalence), or using a whole-part-part diagram, where the relationship between number was inferred from the spatial arrangement of this diagram (the whole was the same length as the two parts, and one part was depicted smaller than the other). As these were young learners, awareness of the significance of the relative size of the two parts was not considered a substantive aspect of the whole-part-part diagram (and was therefore ignored) but the relative size of the two parts to the whole, was considered important. A whole-part-part diagram with the whole either smaller or bigger than the two parts was not considered coherent.

A few of the learners provided no written evidence of how they obtained the correct solution. They met 'criterion 1' as they wrote down the correct solution. As the oral instruction during the test was to show their working, a learner who gave no evidence of any working (either self-selected or prompted) was not awarded the mark for 'Criterion 2: Any coherent model'. There was only one case of this in the class, which impacted on only one question item. However, if a learner showed no evidence of working using self-selected models, but did show evidence of working by producing coherent prompted models, they were awarded the mark for criterion 2 as there was evidence of a coherent model to show their working, although this was through prompted models.

Criterion 3: Correct Partitioning

The final word problem was a partition word problem. It differed from the above word problems (which required a single solution) as multiple solutions were requested. There were 6 correct partitions for this word problem. To ensure that this word problem was given the same mark weighting for Learning Goal 1: Problem solving weighting as the previous four word problems, and total of 2 possible marks was adopted. As such half marks were introduced into this problem.

A full mark for criterion 3: Correct partitioning was awarded if a learner produced all six possible partitions: 5-0; 4-1; 3-2; 2-3; 1-4; 0-5. These partitions could either be shown using number symbols, or drawn using iconic or indexical markings. A partial mark allocation (1 half mark) was awarded if a learner produced three or more correct partitions.

Criterion 4: Systematic working

The final criterion adopted for Learning Goal 1: Problem solving was whether the learners adopted a systematic process to the problem solving process for the partition problem or not. Such systematic working suggested progress towards completeness, where the learner did not repeat partitions and evidence that all of the options had been exhausted. Learners were awarded a full mark if their solution was completely systematic making use of compensation (5-0; 4-1; 3-2; 2-3; 1-4; 0-5) or commutativity (5-0; 0-5; 4-1; 1-4; 3-2; 2-3). They were awarded a partial mark (1 half mark) if their working showed evidence of working systematically (3-4 partitions were systematic) and there was no evidence of repeating partitions.

Criteria for marking Learning goal 2: Representations

The second learning goal was for 'learners to flexibly use and produce a range of representations to solve, pose and explain word problems'. This was a cross cutting goal, which was relevant to all of the items the written tests. Two approaches were used to analyse learning gains for this learning goal: firstly a marking framework as detailed above was used to collate numerical data; secondly a categorical coding framework was developed to allow frequencies of the use and production of different types of representations to be tracked. In terms of the marking framework criteria 3, 4 and 5 together provided a measure of flexibility in using a range of representations in response to additive relation word problem texts. Question 5 was excluded from the marking for Learning goal 2: Representations, as the format of the question changed from pre-test to post-test making comparisons relating to using representations impossible.

Criterion 5: Coherent number sentence

A mark was awarded for this criterion if any one of the eight members of the family of equivalent number sentences depicting the additive relations in the problem text was written either in the space for working or in response to the prompt for a number sentence. Both open (9 + … = 11) and closed number sentences (11 = 9 + 2) were accepted. Mirror image reversals of number symbols were accepted as correct.

A number sentence was considered to be incoherent if the known values in the problem text did not appear in the number sentence, or the known values from the problem text appeared but the number sentence produced was not a true statement. For example for the problem statement: There are 11 bottles, but only 9 lids. How many lids are missing?, '9 + 3 = 11' and '11 + 9 = 2' where considered incoherent number sentences.

Criterion 6: Coherent number line

A mark was awarded for a number line accurately depicting the relationship between the two known values and unknown in the additive relation of the problem text. Number lines that only showed the numbers on the number line, and provided no evidence of a relationship between numbers (through drawing hops in ones, or jumps between numbers) were considered incomplete and categorised as incoherent. Number lines and/or number symbols that were reversed (mirror images of the conventional notation) were accepted. Similarly mirror image reversals of number symbols were accepted as correct.

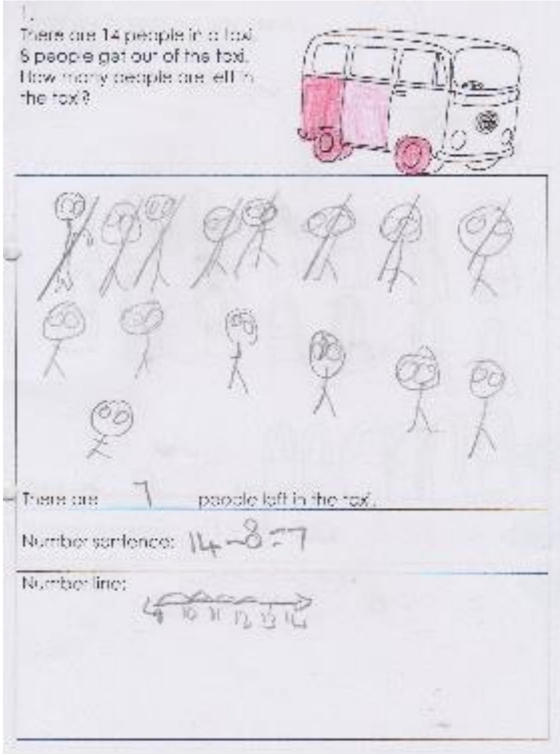
Criterion 7: Other coherent model (not a number line or a number sentence)

A mark was awarded for any coherent representation which was not a number line or a number sentence. This related to self-selected models and included group models (such as iconic and indexical drawings of the problem situation), as well as syntax models (such as whole-part-part diagrams).

Error carried principle

When applying the above criteria to the written tests, it was observed that in a few cases, a single calculation error was being penalised repeatedly. This was evident when a learner made a counting error in their self-selected representation (a drawing) which resulted in the incorrect answer. This error was potentially penalised repeatedly as Criterion 1: Correct solution, Criterion 2: Any coherent model; Criterion 5: Prompted number sentences and Criterion 6: Coherent self-selected model (not number line or number sentence) were all not met. As a result of this observation; the scripts were revisited to apply an 'error carried principle'. In so doing, half a mark was awarded for evidence of coherence between representation forms. So if a calculation error was evident in a drawing, which carried through into an incorrect solution; and which carried through to a number sentence then half marks were awarded. These half marks penalised the error (by not awarding the full mark) but recognised the coherence of the representation with other representations already created relating to the item. In cycle 3, with 26 learners, 4 items and 3 tests, there were a total of 312 learner responses to the test items. The 'error carried principle' was applied to a minority of responses, with 21 of the responses (4 in the pre-test scripts; 8 in the post-test scripts; and 9 in the delayed post-test) making use of this principle. I illustrate the application of this principle with this example:

Figure 59: Mark allocations and explanation for 'error carried' example



Learning goal 1: Problem solving		**Learning goal 2: Representations**			**Total**
Criterion 1: Correct answer	**Criterion 2: Any coherent self-selected model**	**Criterion 5: Coherent prompted number sentence**	**Criterion 6: Coherent prompted number line**	**Criterion 7: Coherent self-selected model (not a number sentence or a number line)**	
1	1	1	1	1	
0 There are <u>7</u> people left in the bus.	0 The drawing does not cohere with the problem text, as 15 people are shown. ×	0.5 Number sentence $14 - 8 = 7$. Although the number sentence is not true, it correctly represents the additive relation and coheres with the [incorrect] answer.	0 Number line does not cohere with the problem text. Numbers 9 to 14 are shown, with 4 hops from 9 to 13.×	0 The drawing does not cohere with the problem text, as 15 people are shown. ×	0.5

In this example it is important to note that a strict criterion relating to coherence of each representation with the problem text was applied. One may consider that this learner ought to allocated a half mark for Criterion 7. This was not awarded as the learner's self selected drawing did no cohere with the problem text. They drew 15 people, and the problem text referred to 14 people. This strict marking in relation to 'coherence' with the problem text was necessary to ensure that the criterion was only fulfilled if sense could be made of the learners' representation in relation to the problem text. The absence of this 'coherence' criterion would mean that if a learner drew anything (even if its relationship to the problem text was not apparent) they would be awarded a mark. This was a necessary constraint in the light the fact that there was more evidence of learners' attempting questions by the post test. Allocating a mark for a self-selected representation which did not cohere with the problem text, ran the risk of inappropriately inflating the post-test results. It should be born in mind that the expanded marking framework was applied in order to measure impact of the intervention and not as a means of communicating progress in mathematics learning to learners and/or parents where different marking criteria which include consideration for effort/attempt may apply.

The above marking framework, together with half mark allocations when applying the 'error carried' principle, was applied to all five sets of written tests (cycle 2 pre and post-tests and the cycle 3 pre-test; post-test and delayed post-tests).

Coding representations in written tests (coding framework)

In addition to applying the above marking framework to the written tests, I also coded the self-selected and promoted representations in each written test to provide categorical data relevant for the Learning Goal 2: Representations. This coding was developed in order to explore the collective example space of learners' representations in terms of the trajectory from counting to calculating, with attention to increasing structure (Feature 4.4). I considered all of the representations offered in response to items in the written test. For each question I separated self-selected representations from prompted representations. As before, representations that accurately depicted the situation model and/ or the mathematical model (coherent) were included, and representations that were not accurate depictions of the problem text (incoherent) were discarded.

The representations which emerged through the design experiment were conceptualised as defining and local collective examples space of representations of whole number additive relations in the early grades. Consistent with the notion of example spaces I considered 'dimensions of possible variation' and the 'range of permissible change' within the example space (Watson and Mason 2005, p.51).

There are several important dimensions of possible variation in children's representations which have been in focus in the expanded marking framework above, but which are placed in the background for the coding framework. For ease of reference, I list these dimensions of possible variation. Firstly, I consider the particular *circumstance* that determines why a learner is using a particular representation, be it their own invention, self-selected from a range of teacher presented options, or prompted by a teacher as a dimension of possible variation. Secondly mathematical representation occurs within a social context making the learners' representation dependent on the nature of the mathematical task (Feature 1.5). Thirdly central to mathematical thinking is the ability to *flexibly* move between multiple representations (Feature 4.1) and considering the extent to which a learner can coherently represent the same relationship in multiple ways becomes significant. I recognise that circumstance, task and flexibility are important dimensions of possible variation when interpreting learner representations as the backdrop to the coding framework.

For the coding framework the first dimension of possible variation when interpreting learners' representations was types (or modes) of representation which comprised of three main categories for structuring numbers less than twenty defined in the theoretical framework: line models, group models and syntax models (and related sub categories). Venkat and Askew (2013) note differences in modes of representation but suggest that there are gradations within these representational categories which relate to building calculating that is likely to involve some reified number facts and some counting. I take up Venkat and Askew's modification and argue that 'working within a particular mode of representation, makes it possible to discern shifts from counting to calculating when attention is focused on the structure of, and actions on, representations' (Roberts 2015). Three other dimensions of possible variation are dimensions for interpreting representations within a particular mode of representation. The second dimension is the arrangement of elements within a representation, referring to the spatial positioning of elements in relation to each other. The third dimension is the group-wise depictions evident in a child's representation. The way children group items, and act on their groups, suggests shifts of attention between considering each object as a member of a group and acting on the group itself. In the context of early grade learners' work on additive relations, these groups are commonly single objects (one's), pair-wise, 5-wise or 10-wise groups. Together the arrangement of objects, and the group-wise depictions may be considered to define the structure of the representation. The learners' actions (the fourth dimension) made visible by their gestures or markings are significant because they depict change or movement in their representations. In Roberts (2005) I exemplify these dimensions with reference to representations created by one of the learners in the cycle 3 intervention.

Table 23 offers a summary of the categories used for coding representations, which is described in more detail for each dimension of possible variation in the sections below.

Table 22: Categories for coding coherent representation

Group				Line				Syntax			
Iconic		Indexical		Structured number line		Empty number line		Whole-part-part diagram		Number sentence	
Ones	Groups	Ones	Groups	Hops in ones	Jumps in groups	Hops in ones	Jumps in groups	Unscaled	Scaled	Standard form	Alternative form

Frequencies for these representation categories were developed for each learner. This allowed for comparisons from pre-test to post- test to delayed post-test to be made. Increasing structure was inferred from shifts across representation types (from groups to line and then to syntax models) and shifts within each representation type that denoted movements from acting on ones towards acting on groups.

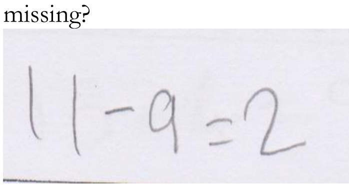
First dimension of possible variation: Representation type

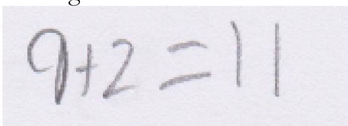
Drawing on Feature 4.4: A learning-teaching trajectory for counting to calculating which made reference to increasingly structured representations involving line, group and syntax models was adopted, the three main types of representations were line models, group models and syntax models. Sub-types for each main category where then defined. Considering group models 'iconic representations', which depicted the items in the situation model (a bottle was shaped like a bottle, a person was sketched as a stick figure), were distinguished from 'indexical representations' which symbolised the objects in the situation model using a circle or a tally mark. Considering line models structured number lines (showing all of the whole numbers from zero or one to the biggest number in the calculation) were distinguished from empty number lines (showing only the number involved in a calculation). Considering syntax models, the whole-part-diagrams were distinguished from number sentences.

Second dimension of possible variation: Arrangement

The categories relevant to the arrangement dimension depended on the representation type. Within group models, the representation was examined for possible arrangement where the iconic or indexical objects could be arranged horizontally, vertically or as an apparently random cluster (there were not arrangements depicting an array). Within line models all had a horizontal linear arrangement, and at times this was following the convention with smaller numbers on the left and bigger numbers on the right; and there were a few learners where this was reversed (bigger numbers on the left and smaller numbers on the right). Within syntax model representations I considered both number sentences and symbolic whole-part-part diagrams. All of the whole-part-part diagram were in a horizontal linear arrangement (no learner rotated the presented diagram to offer this as a top-bottom partition of vertical arrangement). All of the number sentences were also all written using the convention of a horizontal arrangement of number symbols and operation symbols, read from left to right. For the arrangement of number sentences I distinguished 'standard forms' which were in the form 'given value ± given value = unknown value', from 'alternative forms' which were in the form 'unknown value = given value ± given value' or given 'value ± unknown value = given value' or equivalent. This was considered part of the arrangement dimension as whether the unknown value was located on the right, or could be positioned on the left or the middle of the number sentence was of interest.

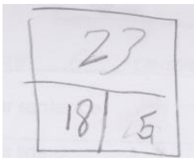
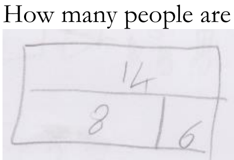
Table 23: Arrangement of number sentences: standard and alternative forms

Sub categories of type	**Structure**	**Examples**
Number sentence	Standard form	There are 11 bottles but only 9 lids. How many lids are missing?  11- 9 = unknown

	Alternative form	There are 11 bottles but only 9 lids. How many lids are missing?  9 + unknown = 11

As part of the arrangement dimension, for the whole-part-part diagrams I considered whether a learner was roughly drawing a 'scaled' version of the diagram which suggested awareness of the measurement component of the diagram (the larger part was drawn as rectangle that was longer than the rectangle for the smaller part), or an 'unscaled' version of the whole-part-part diagram, where the parts were drawn with similar rectangles, or the larger part was drawn smaller than the smaller part. There were no examples of whole-part-part diagrams where the whole was drawn either smaller or bigger than the two parts.

Table 24: Arrangement of whole-part-part diagrams: Unscaled and scaled diagrams

Sub categories of type	**Structure**	**Examples**
Whole-part-part diagram	Unscaled	23 – 18 = …  Parts 18 and 5 are drawn roughly equal to each other, or 5 part is slightly bigger than 18 part.
	Scaled	There are 14 people in a taxi. 8 people get out of the taxi. How many people are left in the taxi?  Parts 8 and 6 are drawn roughly to scale with part 6 clearly smaller than part 8.

Third dimension of possible variation: Group-wise depictions

This dimension was relevant to group and line models. This was not applied to syntax model representations was while use of number symbols in a syntax model may suggest a group-wise depiction but it is not possible to be certain that when a learner writes a number symbol that this is a group-wide depiction as they may have mentally (or using their fingers or other concrete representations) counted in ones to the number before writing it down. The following sub categories contrasting depictions of ones and of groups of more than one, were applied to the group and line models:

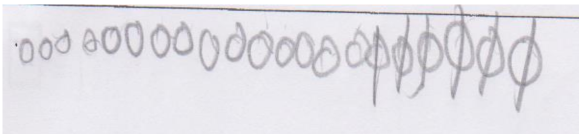
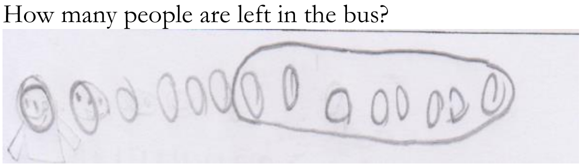
Table 25: Categories for coding group-wise depictions (applied to group and line models)

Group	Line

Iconic		Indexical		Structured number line		Empty number line	
Ones	Groups	Ones	Groups	Hops in ones	Jumps in groups	Hops in ones	Jumps in groups

For group models I considered whether the learners was arranging their objects using a 'ones' or 'groups' structure. Arrangements of indexical/iconic objects into groups were evident when learners encircled a group of objects, or arranged the objects in a structured pattern (such as dice pattern of 5s or a pair-wise pattern of 2s). When learners arranged their indexical or iconic objects singly (with no enclosure or arrangement to depict grouping) the group model was coded as having a 'ones' structure.

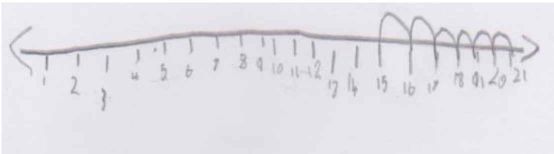
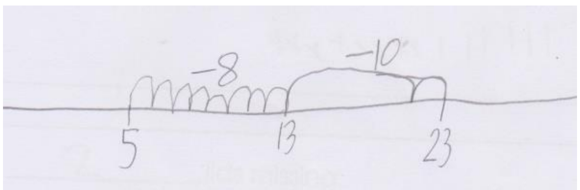
Table 26: Exemplifying the categorical coding of group model representations with examples of learner work

Sub categories of type	Structure	Examples
Iconic	Ones	There are 11 bottles but only 9 lids. How many lids are missing?  11 bottles are arranged singly in a horizontal linear arrangement. The learner matches 9 lids to each bottle, using a 1:1 matching action with each lid touching each bottle.
	Groups	No examples
Indexical	Ones	21 – 6 = 15  21 is arranged singly in a horizontal linear arrangement. The learner acts on 6 of the elements in ones, using a take-away action of crossing out.
	Groups	21 – 6 = 15  21 is arranged in a group-wise structure of 5 (using a dice pattern). The learner acts on 6 of the elements in ones (not on the whole group 5) using a take-away action of crossing out. There are 14 people in a bus. 8 people get out of the bus. How many people are left in the bus? 

		14 is arranged in ones using a horizontal linear arrangement. A group structure is imposed using a partitioning action of enclosing the 8 part of the set of 14.

For the line model representations I considered whether learners were arranging the numbers on their lines in relation to a reference point or 0 or 1, and requiring all of the whole numbers to be visible; or were arranging the numbers in their lines only in relation to the numbers given in the problem text. As such number lines were categorised as 'structured number lines' where all the numbers from 0 or 1 were drawn onto a line, or 'empty number lines' where only the numbers given or implied from the additive relation in the problem situation were drawn onto the number line. Group-wise depictions were then observable when a learner introduced a jump (of more than one at a time) between two numbers. This was contrasted to hops (or one at a time). Jumps and hops were evident on both structured and unstructured number lines.

Table 27: Exemplifying the categorical coding of line model representations with examples of learner work

Broad type	**Sub categories of type**	**Structure**	**Examples**
Line model	Structured	Hops in ones	21 – 6 = 15  The line includes all whole numbers from 1 to 21. The learner depicts 6 hops in ones between 15 and 21.
		Jumps in groups	No examples. All learners who used structured number lines, depicted hops in ones.
	Empty	Hops in ones	21-15 =  The line only includes the numbers 15 and 21. The learner depicts 6 hops in ones between 15 and 21.
		Jumps in groups	21 – 8 = 6  The line only includes the numbers 14 and 6. The learner depicts a jump of 8 (but does not label the size of this jump). 23 – 18 = 5  The line only includes a few numbers: 23, 13 and 5. The learner depicts a jump of 10 from 23, and labels this -10.

		They then depict 8 hops back from 13,and label this -8. The inclusion of group-wise jump, alongside hops in ones, resulted in this representation being coded as jumps in groups.

Fourth dimension of possible variation: Action

Each coherent group and line model representation was coded to contrast 'take-away' and 'difference' strategies as described in relation to Feature 2.2 A take-away calculation strategy was contrasted to a difference strategy. The HLT assumed that the take-away strategy would dominate the collective example space at the starting point and that pedagogic intervention (making use of Main Task 2: Partition problem, Main Task 5: Compare (matching) and Main Task 6: Compare (disjoint set) would be necessary for learners to bring to mind a difference strategy.

Take-away and difference labels refer to calculation strategies, and can be inferred from the learners representations but are essentially concerned with what a learner does mentally in order to find a solution. At the risk of being repetitive, but for the sake of clarity for the argument I put forward in this section, I will use a generic number sentence relevant to change decrease problem situation to describe the take-away calculation strategy: start – change = result

A 'take-away' strategy refers to starting with the 'start', removing the 'change' (through a backwards count in ones, or by partitioning the change into group for greater efficiency) to find the 'result'. I will use a generic number sentence relevant to a compare problem situation to describe the difference calculation strategy: referent – difference = whole

A difference calculation strategy refers to considering both the referent and the whole simultaneously at the start. A process of finding the difference involves calculating the jumps between the whole and the referent (and this may be done by counting up from the referent to reach a target of the whole; or counting back from the whole to reach a target of the referent; or using more efficient jumps between whole and referent.

It is therefore important to distinguish the mental strategy of a learner, from what is evident in a learners' representation (and which may be used to infer a particular calculation strategy). For this reason I introduced a notion of an *action* (what is evident within a learner representation which denotes the introduction of a change into a representation), that is distinct from a calculation *strategy* (the actual mental process the learner uses). It is not possible to categorise a learners calculation strategy based solely on their representations, but it is possible to categorise the actions evident in a learners' representations.

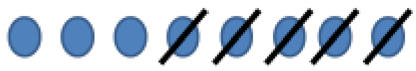
Working with the empirical data of all the learners examples it became clear that some learners used 'take-away' actions, and other learners used 'difference' actions in their representations. But there was a third possibility, which was distinct from either 'take-away' or 'difference' and which did not figure in the theoretical framework. I refer to these as 'partitioning actions' and acknowledge that such actions may be inferred from existing literature in relation to the join and separating labels used in the RGH problem types, partitioning problems (where a whole is kept invariant and different partitions are sought, as in the Cobb et al. work); and in work on syntax models (by Schmittau, Bass and Treffers work) and known fact calculation strategies relating to bridging the tens in the counting based conception of number. Partitioning actions are therefore not a new addition to mathematics education literature. My contribution in this regard is to draw these threads together to label a common action which is evident across modes of representations, and position this alongside the take-away and difference actions. I claim that the same partitioning action can be evident in group, line and syntax models. I now discuss each of the three calculation actions evident in the learners' representations in turn.

'Take-away' actions

This categorisation of calculation actions drew on Feature 2.2 A take-away calculation strategy was contrasted to a difference strategy with consideration for efficiency in choice of strategy depending on the numbers involved, which built on literature relating to 'take-away' and 'difference' actions (Angileri 2000 and Askew 2012).

A take-away action was inferred from acting on a group model from crossing out of objects ('take-away image'):

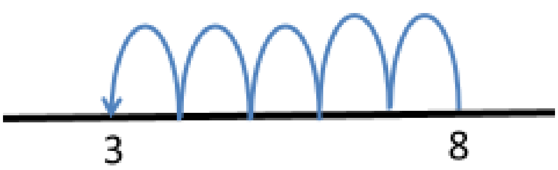
Figure 60: Take-away action on a group model for 8 – 5 = 3



Take-away actions can also be inferred by hiding or removing objects, so that they are no longer visible. So in instances of group model representations where learners drew, and then erased, objects this was coded as a take away action. It is important to notice that the all of the discrete objects are depicted in the same way (using the same indexical or iconic representation for each object) and that an action of removal is depicted by crossing out some objects or erasing them (in using concrete materials, the objects would be hidden or removed).

For line models, a take-away action was inferred from hops or jumps back from a starting number to produce a 'take-away number line'.

Figure 61: Take-away action on a line model for 8 – 5 = 3

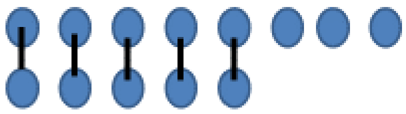


Difference or 1:1 matching actions

This categorisation of calculation actions also drew on Feature 2.2 A take-away calculation strategy was contrasted to a difference strategy with consideration for efficiency in choice of strategy depending on the numbers involved, which built on literature relating to 'take-away' and 'difference' actions (Angileri 2000 and Askew 2012).

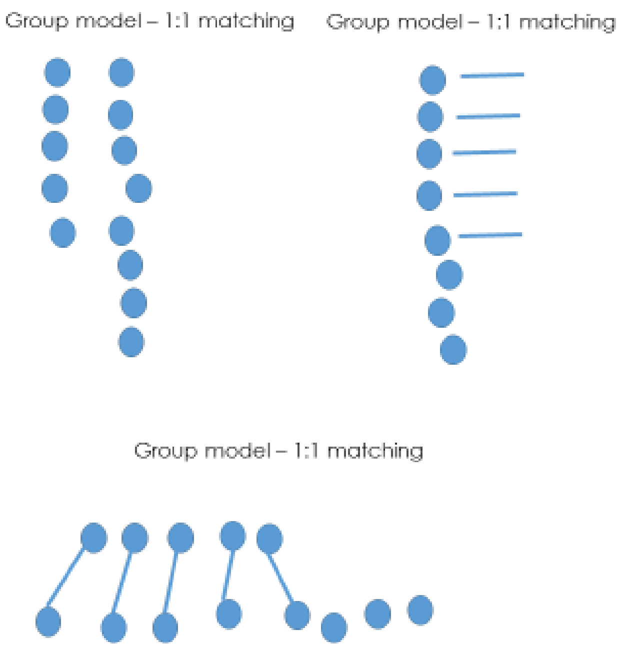
Examining children's direct modelling strategies Carpenter, Fennema et al. (1999) documented that a 1:1 matching action is required for comparison between two-disjoint sets (Carpenter, Fennema et al. 1999). Such an action can be pictured using a 'difference image' when working with group models:

Figure 62: Theorised version of a difference image for 8 minus 5



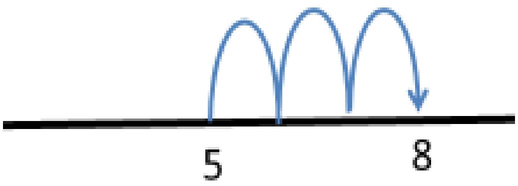
Working from the empirical examples of learner work, there were different ways in which this 1:1 matching action could be inferred. At times learners drew a line connecting each element of one set to each element of another; and at times learners used an iconic image where each element of one set was connected to each element in the other set by their arrangement as they touched each other, or drawn very close together.

Figure 63: Variations on the difference/ 1:1 matching image emerging from empirical data



The difference strategy for 8 - 5 = … , may be transformed to be an equivalent alternative form of number sentence of 5 + … = 8, which following (Askew 2012) and (Anghileri 2000) is depicted a on a number line as follows:

Figure 64: Difference action on a line model for 8 – 5 = 3

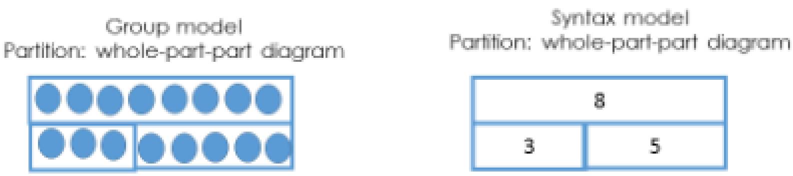


Both structured and empty number line versions of this image, as well as number lines using hops in ones or jumps in groups were included as difference actions on the line model.

Partitioning actions

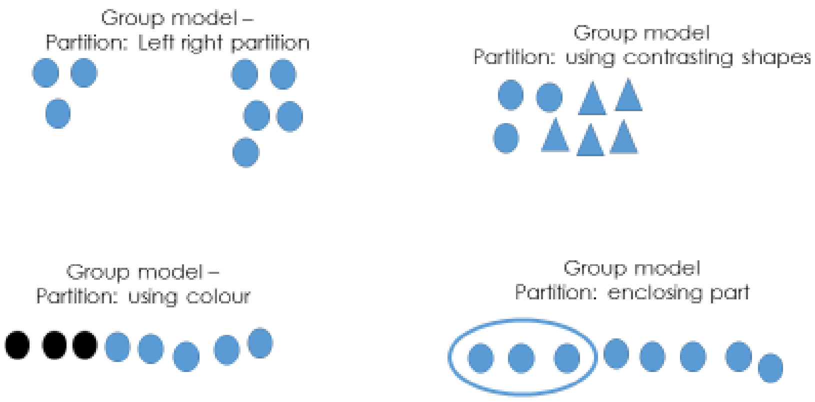
During the intervention, the whole-part-part diagram was introduced to learners in the intervention using an indexical representation during Main task 2: Partition Problem. Over the development of the main tasks, this image was shifted to become a symbolic whole-part-part diagram.

Figure 65: Indexical (group) whole-part diagram versus a symbolic (syntax) whole-part-part diagram



Drawing on the empirical data, it was evident that a partitioning action could be inferred from group model representations and from syntax model representations. The partitioning action on a group model was evident by learners arranging objects in a left-right group (or top-bottom group); by using colour to distinguish parts of a group, by enclosing a group of objects (drawing a rough circle around a part), or by using contrasting shapes (for example separating locks from keys using lines for locks and circles for keys). The partitioning action on a syntax model, was evident when learners drew a symbolic whole-part-part diagram.

Figure 66: Partition actions on group models



Notice that partitioning actions are distinct from the 1:1 matching action. For matching actions there is a discernible 1:1 relationship between each element of one set and each element of another, whereas for partitioning actions no such 1:1 relationship is evident but a set is separated into two parts, and the parts are represented as distinct from each other by their arrangement (left-right or top-bottom), by their contrasting shapes, by using colour or shading, or by enclosing one part. The partitioning actions are also distinct from the take-away action. For take-away actions part of a set is removed or hidden which is depicted by crossing out erasing the elements in the missing part; whereas for partitioning actions both parts remain visible they are just made visually distinguishable (through arrangement, shape or colour).

Within the collective example space of the representations developed in the design experiment, none of the syntax models made partitioning actions visible. There were no examples like those shown in Figure 68:

Figure 67: Examples of partitioning actions in syntax model representations

$$\begin{aligned} &14 - 6 \\ &= 14 - (4 + 2) \\ &= 14 - 4 - 2 \\ &= 10 - 2 \\ &= 8 \end{aligned}$$

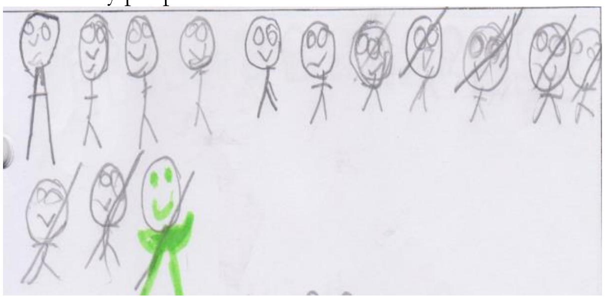
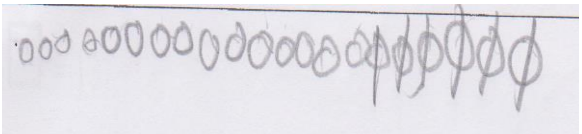
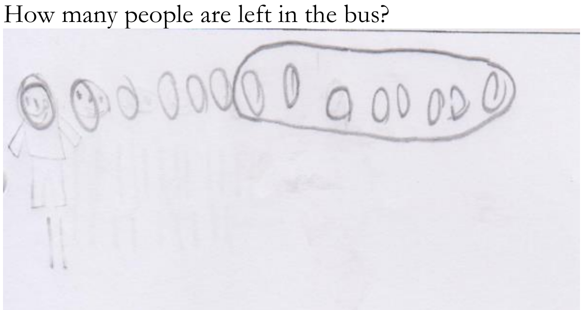
$$\begin{aligned} &14 - \underset{4\ \ 2}{6} \\ &= 10 - 2 \\ &= 8 \end{aligned}$$

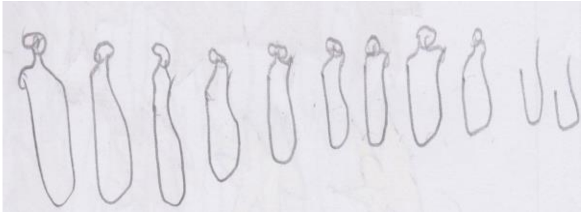
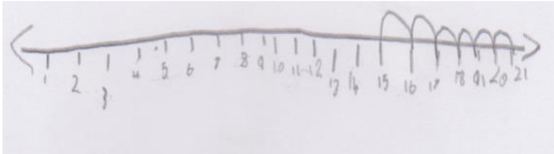
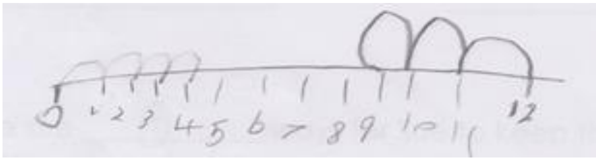
For this reason, I did not include partitioning actions in relation to the syntax models.

Exemplifying take-away, difference and partitioning actions

In the table below I exemplify the categorical coding of calculation actions as either take-away, 1:1 matching and partitioning actions, with reference to the empirical data of learner's representations. The coding of these representations was on coherent representations where the numbers and relationships depicted in the problem situation, were reflected in child's representation. As such I include the problem text to which the learner was responding for each example.

Table 28: Categorical coding of representations to distinguish calculation actions (take-away, 1:1 matching/difference and partitioning)

Broad type	**Sub categories of type**	**Calculation action**	**Examples**
Group model	Iconic/indexical	Take-away	There are 14 people in a bus. 8 people get out of the bus. How many people are left in the bus?  21 – 6 = … 
	Iconic/indexical	Partitioning action	There are 14 people in a bus. 8 people get out of the bus. How many people are left in the bus? 
	Iconic/ indexical	1:1 matching actions	There are 11 bottles but only 9 lids. How many lids are missing?  There are 11 bottles but only 9 lids. How many lids are missing?

			
Line model	Structured/ empty with hops/jumps	Take-away	21 – 6 = … 
	Structured/empty with hops/jumps	Difference	Jani has 9 stickers. Mpho has 12 stickers. How many more stickers does Mpho have than Jani? 

Analysing learning goal 3: Story telling

The third learning goal was explaining and telling additive relation stories (LG 2: Stories). As children of this age are still in the process of learning to write, storytelling was assessed in oral and not written form. Learning goal 3 was analysed by assessing the *Main task 7: Learner generated examples*, where the class was engaged in a 'storytelling task', and in relation to storytelling tasks posed in the structured interviews. The structured interviews were analysed for the case study learners.

Design of the whole-class story telling task (Main task 7)

The following was the final task for the intervention lesson, which was posed to the whole class:

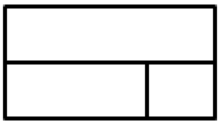
Figure 68: Main task 7: Learner generated examples

TASK Learner generated examples

Representations

Use your numbers to complete:

Whole = part + part
[ ] = [ ] + [ ]
[ ] = [ ] + [ ]
Whole – part = part
[ ] – [ ] = [ ]
[ ] – [ ] = [ ]



Learner generated representations

Stories

Make up 3 word problems (story problems) for your whole-part-part diagram.
One of your stories must use the word 'more' in it

I used Watson and Mason's (2005) concept of an example space, where any example in mathematics is seen as a member of a set of examples, as "examples are usually not isolated; rather they are perceived as instances or classes of potential examples" (p. 51). It followed then that initiating activity requiring learners to reflect on their own personal example spaces was a valuable undertaking.

The story telling task comprised three elements: *Element 1* the whole-part-part diagram; *Element 2* a family of related number sentence and *Element 3* the three stories to be told by learners. Learners were expected to use a number triple to draw a whole-part-part diagram and write the related family of number sentences. The story-telling aspect used 'Make up an example with some constraints'. Two constraints were imposed: learners were given a number triple with which to work and were expected to use this same triple in all of their stories; learners were expected to use the word 'more' in one of their stories. This task was individualised, as each learner in the class worked with a unique number triple provided either by the teacher, or self-selected.

Analysis of evidence from the story telling task

As the 'Make up a story' instruction (Element 3) made reference to the whole-part-part diagram (Element 1), whether learners were connecting their whole-part-part diagram to their stories by using the same number triple in both was of interest. Examining Element 1, the 28 learners were first categorised into learners who were able to draw a correct whole-part-part diagram (25 learners) and those who were not able to do this (3 learners). A correct whole-part-part diagram meant that they drew a whole-part-part diagram, positioned the biggest number as the whole with two smaller numbers as the parts thereby satisfying the relationship 'whole = part + part'.

The 25 learners with correct whole-part-part diagrams were then categorised into learners who had written at least one coherent story using the number triple from the whole-part-part diagram (22 learners), and those who had not (3 learners). A coherent story was defined as a story that made use of an additive relation where the same number triple used in the whole-part-part diagram satisfied the relationship 'whole = part + part'. Some learners generated stories where all three numbers in the number triple were used to express an additive relation in a narrative form such as 'I have 5 stickers. I found 3 stickers. I have 8 stickers' (for the number triple 8-5-3).

In most cases learners used two numbers from the number triple to express an additive relation, and pose a question in order to find the third number. For example 'I have 9 apples. I eat 2. How many I have?'[22] (for the number triple 9-7-2). Both versions of the word problem story – either with or without a question – were included for analysis. Across the 22 learners, 57 coherent stories were generated, and it was this set of 57 stories that constituted the empirical data for evidence relating to Learning Goal 3: Story telling.

Finally the learner activity on the story telling component of this task was considered for each individual learner. In this regard the extent to which a learner was generating stories that were all within the same problem type (or the same sub-type) or spanned across problem types was of interest. The theoretical assumption here was that learners who could bring to mind different types of word problems within their three stories showed a wider range of reference examples in their personal example space than learners who generated all three stories using the same problem type.

Structured interviews on story telling tasks

Design of the story telling tasks in structured interviews

As children of this age are still in the process of learning to write, storytelling was assessed in oral and not written form. As such this learning goal was only assessed in relation to the three case study learners, making use of their oral responses in the interviews.

During the interview, the learners were asked to pose problems for additive relation word problems: Can you tell an easy story to explain 10 – 7 = […]? And another harder one? And another harder/different one? This was intended to reveal the vocabulary spontaneously used to articulate their ideas, and the dimensions of possible variation in their example space for additive relations. I noted the type of word problems that children were posing, and their variation in problem context (the numbers remained invariant). Where children did not spontaneously vary the problem type, I provided prompts to establish whether they could recall and retell some of the other problem types.

Learners were also asked to explain a generalized problem situation contrasting change and compare situations. This task was included to gain some evidence relating to Feature 1.4

[22] All examples are transcribed directly from what learners wrote. They have not been edited for spelling or grammar.

Facilitating shifts in attention from the particular to the general (generalising) and from the general to the specific (specialising). Task design to solicit learner activity on generalising/specialising tasks was anticipated to be unfamiliar to learners. As such it was a deliberate design choice to include such tasks in the structured interviews, and exclude them from the written tests. It was expected that posing such tasks in the written tests would be too unfamiliar and provoke a need for teacher explanation, which would interrupt the learner's independent engagement with the written test items. So in the structured interviews, the selected learners were first set this task: I have some apples. You take some of my apples. How can you work out how many I have left? Then they were set the following task: I have some sweets. You have some sweets. You have more sweets than me. How can you work out how many more sweets you have? This task was intended to gain insight into the problem situations that they invoked to explain additive relation concepts. The generalized problem situation task made use of a presented story which the children were then encouraged to explain. This was intended to establish whether children could conceptualize a general case of the 'change decrease (result unknown)' problem, and compare this to a general case of the 'compare (disjoint set) difference unknown' problem type. In these tasks, by starting with a general problem situation, it was expected that learners would shift from the general to the particular (specialise) by introducing particular numbers of their own choosing into the problem situation. This built on Feature 1.4 Facilitation shifts in attention from the particular to the general (generalising) and from the general to the particular (specialising).

Analysing the story telling tasks in structured interviews

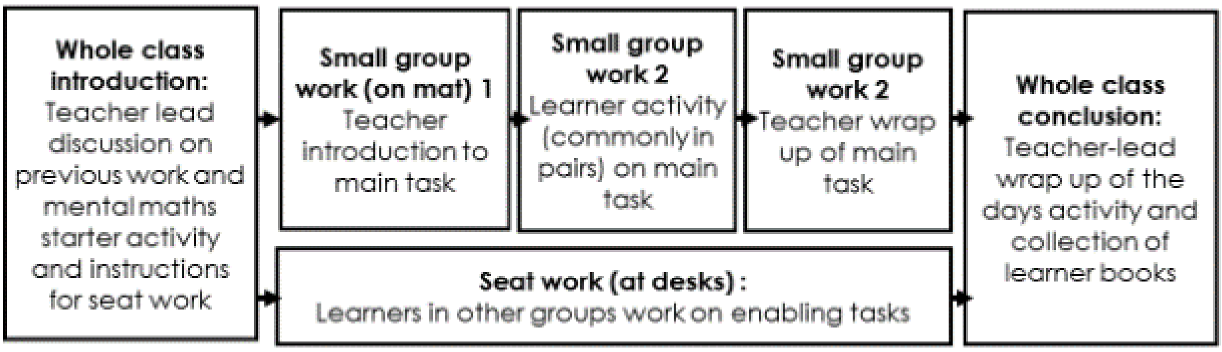
The story telling tasks for each learner were analysed by comparing the learner activity in the pre-interview to their activity on the same question in the post-interview. A constant comparison method (Glaser 1965) was adopted and learning gains or lack of learning gains were inferred from observed differences or similarities between the same learners responses at the two same interview items at the two points in time.

Evidence of the means of supporting learning

Data collection

Evidence of the means of supporting learning was collected via video recordings of the lessons, the collection of all learner artifacts (learner work in their books during seatwork, and on white-boards or their books in small group work).The intervention took place over ten consecutive school days with mathematics lessons taking place first thing each morning. Typically a lesson comprised of a whole class introduction and conclusion session, which was interspersed with small group activity. The whole class was set enabling tasks to be completed in their books (independent seat-work). The usual class teacher supported the independent seat work activity by keeping learners on task and responding to learner queries. Once the class was settled in this activity one of the ability groups was called to the mat at the front of the class for a small group activity with me. This focused on the main tasks of the intervention and involved a teacher introduction, learner activity on the problem solving tasks, and small group wrap up. Typically two ability groups worked with me each lesson.

Figure 69: Typical daily lesson structure



All lessons were video recorded, with the video camera focusing on teacher interactions with learners (during whole class and small group sessions). The learner work books for enabling tasks as well as any artifacts created by learners during small group activity were collected. The video recordings of lessons together with the tasks designed for each lesson, and the learner artifacts providing evidence of learner activity in response to these tasks were used to create a description of the lesson intervention. A detailed chronology of the lesson intervention which draws on these sources is presented in Annexure 4.

Analysis of the means of supporting learning

In this section I describe the process of analysing the various sources of data (video recordings of the lesson, the learner artifacts) and how the intervention description was refined through collegial engagement (with the usual classroom teacher, and through critical reflection and peer review).

Video recordings of the intervention lessons

I reviewed and transcribed each of the video recordings of the lessons to develop a detailed description of the teaching intervention. For whole class interactions the transcriptions were verbatim talk during whole class interactions capturing what was said as well as describing gestures made with artifacts (such as paper strips or bead strings). For small group interactions, I transcribed all teacher talk and learner interaction which took place during the introduction session in the small group. When small group work shifted to pair or individual work, the learner talk was not transcribed, but a record of learner activity obtained through video recording the artifacts that learners produced during these pair and individual work sessions. The concluding section of the small group interaction was again transcribed verbatim focusing on teacher and learner talk, gesture and drawings. The video recording of the independent seat work sessions of the rest of the class (which took place at the same time as small group work) were not video recorded. Evidence of learner activity during these sessions was obtained from the learner books as the seatwork tasks involved independent individual written work. During this review of the video lessons, I also captured key video screen shots which captured the type of activity which was being undertaken.

Learner artifacts

To describe the lesson intervention I worked from the video transcriptions, as well as video-stimulated recall of the lesson together with the evidence collected from the learners during that lesson. This included the learner activity on enabling tasks during seat-work sessions as well as their activity on the main tasks during the small group sessions. The learner artifacts for a particular lesson were reviewed to identify 'best' cases and 'telling' cases of learner activity on the tasks (Mitchell 1984). For best cases learner activity closely matched an appropriate solution to the task, while telling cases were revealing of common errors and misconceptions or not understanding the task instructions.

Pre and post interviews with normal classroom teacher

I interviewed the normal class teacher (Vanessa) before and after the intervention. The pre-interview was conducted in order to reflect on the assumed intellectual and social starting points for this class. Vanessa was present during the lesson interventions and helped with managing the individual seat work while I worked with the small groups. She shared the experience of the lesson intervention, and was available for informal discussions relating to learner activity on particular tasks. Her post-interview was structured to report her reflections on the third cycle intervention, and also her reflections on how this cycle compared to the first cycle (where she was also the normal class teacher). Vanessa also reviewed my description of the lesson intervention and offered critical comment on this.

Critical self-reflection and peer review

As I was both the researcher and the teacher for this intervention, I subjected my account of the intervention to self and peer scrutiny. I firstly reviewed the descriptive account of the intervention from a critical perspective, and noted episodes where I could have proceeded differently or felt that my teaching could have been improved. I marked these into the descriptive account making use of 'analytical/reflective note' text boxes. I then asked colleagues to read the descriptive account, attending to clarity in the description (identifying areas where they did not follow what was done), and suggestions for reflective notes (areas where my pedagogical decisions were not optimal, and in their view an alternative approach may have been more appropriate). These colleagues included: three academic reviewers (professors in education); three PhD students in mathematics education (with a focus on primary level mathematics) and two practicing teachers in Foundation Phase. Some reviewers discussed their observations with me in face-to-face discussions and others provided written annotations on the descriptions which were then discussed in a face-to-face context. This process allowed me to 'distance' myself from the teaching intervention and reflect critically on what was being observed by others in my teaching roles. The detailed account, including my reflective notes on the intervention, is included in Annexure 4.

Synopsis of the methods adopted in this study

In summary, the study examined both learning and teaching sides of the lesson interventions in a three-cycle design experiment, with a particular focus on the third cycle intervention. Learning was analysed by administering a written pre- test, post-test and a delayed post-test which was designed to assess learners' problem solving and use of representations in relation to five word problem items, and two bare calculation items. The overall attainment of learners in the third intervention cycle was compared to attainment of learners in the second intervention cycle (using the same test and same marking framework). The test was marked in two ways: firstly using a simple marking framework where only the correctness of solutions was examined; and secondly using an expanded marking framework in relation to the first two learning goals (Learning goal 1: Problem solving and Learning goal 2: representation). The more detailed marking framework developed criteria which drew on the theoretical features of the intervention design, as explained in the theoretical framework for the study. Although the second marking framework included the correctness of solution from the first marking framework I used the simple marking framework to assign learners to groups and to inform the design changes for cycle 3. It is for this reason that I report on both frameworks. The criteria were used for quantitative coding of marks on the test, which were then analysed quantitatively relating to the mean scores, standard deviations, significance and effect sizes of changes to these mean scores. For Learning goal 3: Story telling the evidence of learning by the whole class was analysed qualitatively making use of learner activity on the concluding task of the lesson intervention (Main task 7: Learners' generating examples). Further qualitative detail on learners' story telling process of progression from prior to the intervention to after the intervention was obtained through examining the pre- and post-interview responses of case study learners to story-telling items in the individual structured interviews.

Teaching was analysed qualitatively by developing a thick description of the task design for the HLT, and implementation of lesson intervention making use of multiple sources of data including planning documentations, video recordings and transcripts of the lessons, learner work, and subjecting this description to critical self-reflection and peer review. The HLT was compared to the ALT to establish whether or not there was close matching between the two trajectories. The HLT and ALT of particular case study learners were examined in detail making use of interview data, pre and post-test data; learner artifacts collected from the intervention, and video recordings of their interactions during the lessons. Two of the learners (Mpho and Retabile) were selected to provide insights into how learning came about for these children. The third case (Gavril) was selected to be revealing of possible ceiling effects, where absences of learning in the upper attainment levels could be examined.

The analytical frameworks used to interpret the various data sources were developed, drawing on the theoretical framework, the HLT and engagement with empirical data emerging from cycle 3.

CHAPTER 5: Findings on the implementation of the third cycle intervention

This chapter focuses attention on the implementation of the third cycle intervention. It provides evidence, drawn from a detailed account of the teaching-side of the intervention, to answer the question: '*How did the third cycle intervention play out in this local context?*' This necessarily precedes the findings relating to evidence of learning, interpretation of which requires evidence of how the third cycle was actually implemented.

This chapter aims to make a contribution towards the design-theory dualism which is characteristic of design experiment methods:

> 'theorizing how and why a task sequence in an instructional engineering "worked" in specific contexts can allow identification of specific aspects of the task sequence that are potentially core and cannot change, as well as others that may be adapted to accommodate different contexts'. (Stylianides and Stylianides 2014b, p.384)

I say 'make a contribution towards design-theory dualism' in recognition that reporting on a design experiment study after only three intervention cycles is relatively early in the research process. What is discussed in this chapter therefore draws on evidence (from lesson transcripts, teacher notes, learner activity on tasks, as well as the finding from the assessment processes reported in the previous chapter) to provide some evidence of how the teacher side of the ALT played out in the third cycle intervention against the HLT. Consistent with the iterative design of design experiments, further refinements of the HLT and related changes to the task design, which are conjectured to further support learning, and intended for subsequent design cycles are discussed.

While Chapter 4 focused on the design component of the research and the intervention, this chapter focuses on the empirical component relating to the teaching:

> Data from the classroom implementation of a task sequence are collected and analysed in order to document the classroom community's actual learning trajectory, examine the degree of matching between hypothetical and actual learning trajectories, and try to understand how different aspects of the task design sequence influenced the community's actual learning trajectory (Stylianides and Stylianides 2014b, p.384).

At the individual level, each Actual Learning Trajectory (ALT) is unique for each member of the classroom community, and cannot be exhaustively described. However, considering the classroom community as a collective; it is possible to put forward a general account of the ALT for the whole class. This is necessarily more focused on the teaching-side of the intervention as the descriptive accounts are written from my perspective as the task designer and lead teacher within the classroom community. By drawing on multiple sources of data (including video recordings of the lessons and related lesson transcripts; the learner activity resulting from the tasks undertaken; as well as critical self and peer- reflection on the lesson account) a descriptive account moves beyond the teacher- recollection of events to offer a thick description of the third cycle intervention.

This chapter contributes to satisfying several critical features of design experiments.

Firstly, in design experiment methods it is expected that the intervention in the particular local context is described in enough detail to be replicable to others. However the inclusion of such a detailed account of intervention can be both lengthy and disruptive to the flow of the research argument. As a result a detailed account of the intervention, which offers a description from the teaching-side, with reference to the task design and the features of its underlying theoretical framework, is presented in Annexure 4. In this chapter I simply give a synopsis account of the implementation of the Cycle 3 intervention from the teaching-side.

Secondly, as a narrative approach to learning mathematics is the cornerstone of this design experiment (this is described in detail in relation to Feature 3: Story telling and Feature 4: Use of representation), I draw from the descriptive account to exemplify what is meant by a narrative approach to mathematics learning, drawing on empirical data from the third cycle intervention. The claims made with regard to presence of particular theoretical features are exemplified through analysis of one main task (and this process is continued for other main tasks in the Annexure).

Thirdly, design experiment methods anticipate that within any intervention cycle, there are potentially mismatches between the HLT and ALT. These mismatches are examined to inform improved HLT designs for future intervention cycles. Exogenous design (reported as the HLT in Chapter 4) is compared to endogenous design (the in-action changes made when enacting the intervention, described as the teaching ALT in this chapter). In so doing the HLT is mapped to the ALT. Through developing the detailed account of the intervention (from the perspective of the teacher, but drawing on multiple sources of data to construct this thick description), and mapping the theoretical features to this description to the HLT, I was able to analyse the extent to which the HLT matched the ALT from the teacher planning perspective. I found that all of the theoretical features could be exemplified with empirical data drawn from the detailed accounts, which is presented in Annexure 4.

There were two ways in which the ALT differed from the HLT in the third cycle: firstly the ALT for the support group was not what was planned for in the HLT; and secondly there were endogenous design changes to Main task 3: Leaner generated examples. As these were fundamental deviations from the HLT I have drawn on the descriptive account in Annexure 4 to describe and then discuss these changes. This chapter therefore draws on the descriptive account of the intervention to examine the fidelity of the intervention by identifying, describing and accounting for mismatches between the HLT and the ALTs for the third intervention cycle. The implications of these mismatches for future design cycles are put forward.

Describing the third cycle intervention

As part of the analytical process relating to the teaching-side I developed two detailed accounts of the third cycle lesson intervention.

The first descriptive account was organised by chronology of events. As the main tasks were spread over several days and involved both whole class and small group work activities, the classroom management of these different learning formats, where episodes of learner engagement on a task were fragmented over time. This descriptions of the intervention drew together evidence from the lesson transcripts, and learner activity on tasks together with my personal reflective notes made as the teacher (during the intervention implementation) which were then extended when working on the descriptive account of the lesson intervention for the research, as well as the peer review comments obtained on this thick description.

The second descriptive account then drew on the chronological account of the intervention, as well as the theoretical framework and HLT, to re-organise the description in terms of each main task. This provided a clear overview of how each task was introduced by the teacher, with best case examples of learner activity on these tasks contrasted to telling case examples of common errors or evidence of emerging misconceptions emerging from learner activity on the tasks. Both of the descriptions offered detailed descriptive accounts – from my perspective, drawing on a range of data sources to support my recollection - of what transpired during the intervention. This description organised by main task with reference to each theoretical feature is presented in Annexure 4.

Synopsis of the lesson intervention

Cycle 3 was undertaken over ten consecutive school days. The class of 30 learners was divided into three attainment groups: support, core and extension groups, based on written pre-test results from the simple marking framework. Learning opportunities were offered in three formats. Firstly the whole class discussion format was a teacher-led discussions about the previous day's work and new concept where the teacher involved individual learners by asking them questions and asking them participate in creation of representations on the blackboard. Secondly learners were expected to engage in independent seat-work tasks. Each learner was allocated three to four unique work cards daily, which they were expected to complete in written form individually in their books. These were the *enabling tasks* as described in the intervention design. While the class was settling into this learning format I assisted learners individually where conceptual difficulties were evident in their work from the previous lesson. The independent seat-work tasks were marked daily to provide feedback to the learners, and new cards (based on the previous work) were assigned to each learner. Thirdly, learning opportunities were created through focused small group work where I worked with one of the groups of learners on *main tasks*. In general the small group work opened with a mental maths activity (related to the enabling tasks), and then teacher-sharing of a reference additive relation word problem. Finally the lesson was concluded with a whole-class discussion.

The mathematics lesson each day took place before the first break time. This included settling the children, as well as ensuring that they ordered and ate a hot porridge meal (which took place at some point each morning with learners eating in the classroom). The teaching time each day ranged from 1.5 to 2 hours for each lesson

The teacher sharing of additive relations word problems made use of narrative as a pedagogic strategy where story-telling was viewed as a process of involving the learners in co-creating a problem situation and simultaneously creating and acting on representations to support the problem solving process. Over time the stories were re-told and varied by the teacher and by the learners. New contexts and different numbers were introduced and the representations included in the story-telling performance became increasingly structured. While near the beginning of the intervention the representations used were mainly group models (making use of concrete materials such as bottle tops, bead strings, iconic or indexical drawings) as well as simple syntax models making use of addition number sentences, over time the representations used included line models (with shifts from structured to semi-structured number lines), and symbolic whole-part-part diagrams with related families of number sentences.

The intervention opened with providing learners opportunity to give their own informal representations and stories relating to '5 equals'. It closed with learners again creating their own stories and illustrating them with both their informal representations; but this was connected to a prompted whole-part-part diagram and family of number sentence that was particular to their number triple.

Between the opening and closing tasks, the following additive relations reference examples were structured as main tasks in Cycle 3:

Table 29: The reference example stories and presentations for Cycle 3

Taken as shared references/labels	**A monkey story**	**A take-away story**	**A sticker story**	**A matching story (bowls &lids story)**	**A 'How many more' story**
Task	Main task 2: Partition problem	Main task 3: Change problem	Main task 4: Change increase (equalise)	Main task 5: Compare (matching)	Main task 6: Compare (disjoint sets)
Narrative context	Monkeys in trees Golden books	Learner generated examples	Stickers	Bowls and lids Bowls and spoons Locks and keys	Compare numbers of things held by children Compare number of children in two families

Taken as shared references/labels	A monkey story	A take-away story	A sticker story	A matching story (bowls &lids story)	A 'How many more' story
Representations	Bead string Five frames paper strips Whole-part-part diagram Number sentences: Whole = […] + […]	Bead string Subtraction as take away Whole-part-part diagram Empty number line count back strategy Number sentence: Whole – part = […]	Subtraction as comparison Whole-part-part diagram Empty number line count on strategy Number sentence: Part + […] = whole	Subtraction as comparison Whole-part-part diagram Empty number line count on strategy Number sentences: part + […] = whole Whole – part = […]	Subtraction as comparison Whole-part-part diagram Five frames paper strips,bead strings Empty number line count on strategy Number sentences Part + […] = whole Whole – part = […]
Illustrative example of a story:	There are 5 monkeys. They sleep in two trees: a tall tree and a small tree. How many ways can the monkeys sleep in the two trees?	There are 5 birds. 3 fly away. How many birds are left?	I have 8 stickers. I need to get 10 stickers. How many more stickers do I need?	There are 6 porridge bowls but only 4 lids. How many lids are missing?	John has 5. Kuhle has 3. How many more does John have than Kuhle?

Exemplifying the narrative approach adopted in the third intervention cycle

In this section I draw on the descriptive accounts of the lesson intervention to illustrate the narrative approach to mathematics which was adopted for this intervention.

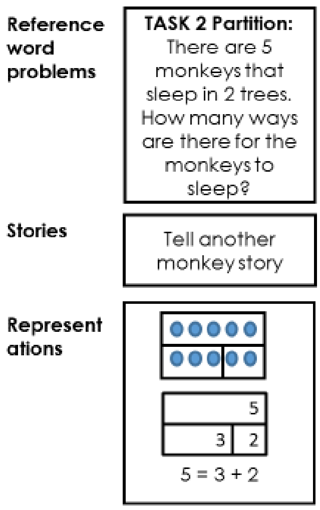
Each reference word problem (considered in terms of the stories presented in the table 30 above) was first introduced with a teacher-telling of the story. This teacher- telling made use of storytelling as performance, and as process. Frequently learners were invited to offer contexts or names of characters or to participate in the story by acting out parts of the story. Most of the stories were introduced through small group interaction first, and then revisited in the whole class setting. Typically, when working in a small group, children were introduced to an additive relation story and given opportunity to solve problems relating to a particular reference example. The group work tended to start off with collective engagement with the teacher-told story, where new representations were slowly introduced. Then learners worked in pairs or individually on similar kinds of word problems. Following the problem solving stage, learners were invited to generate similar stories to the one they had heard and to retell these stories in class activities. They were then introduced to the next story, which was compared to the first, and so on. As such the main tasks were revisited several times over the intervention period, and so each story was revisited and varied (sometimes by varying the numbers, sometimes by varying the contexts, and sometimes creating variation by learners telling 'another story like this one').

To exemplify what is meant by the narrative approach to mathematics teaching during the intervention I offer an account of the monkey story (Main Task 2: Partition problem). This includes descriptions of the use of storytelling as a pedagogic tools and as a cognitive strategy (Feature 3), as well as the use of representations (Feature 4).

Description of the monkey story

There were several episodes which related to the partition problem: the initial introduction to the word problem (teacher telling of the story), learner re-telling of the story, the teacher and learner use of representations relating to this story, and the subsequent re-telling and variations of their story from both teacher and learners. Although the use of representations has been defined as a distinct theoretical feature, both teacher and learners created or used a range of representation while telling their stories (story telling and use of representations are mutually constituted as part of the narrative). As such descriptions include both story telling and representations with reference to the partition problem.

Figure 70: Main task 2 Partition problems



The partition problem as introduced in small groups on Day 2 and 3 of the intervention. The problem context of the partition problem was monkeys in trees. The partition problem was revisited on Day 4. On Day 6 of the intervention the problem context was changed from monkeys to golden books.

Teacher telling of the monkey story

At the outset of the partitions problem I introduced a syntax model representation to introduce the vocabulary of a 'whole' and a 'part' which was the generalised vocabulary required for the structural approach (Feature 2.5). From previous intervention cycles I was aware that a common misconception was that the two parts were drawn as equal in size, or that the relative size of the parts was not noticed as being dependent on the numbers involved. I therefore showed learners two examples of generalised whole-part-part diagrams:

Figure 71: Whole-part-part posters for wall display

By varying the relative size of the two parts I hoped to draw attention to the partition between the two parts being dynamic (moving depending on the size of the two parts). I asked learners to repeat the words 'whole part part' back to me. I then provided examples which demonstrated the meaning of the words whole and part. Once I felt learners had some sense the vocabulary of 'whole' and 'part', I then introduced the partitions problem as a story:

> T: We are going to work on lots of stories while I am teaching you. And the story we are going to do today is a story about some monkeys. Monkeys
> L: Uh uh uh uh uh! [Learner makes a monkey noise and scratches under his arm]
> T: Who can make a monkey noise? I heard one there.
> [Some learners make monkey noises, and T encourages a quiet girl to try make a monkey noise. She declines. A boy bangs his chest, calls out loudly and T points out he is making a gorilla noise]
> (Day 2 Lesson transcript, extension group)

With the story context of monkeys grounded and enacted by some children, I drew a tall tree on the white board. I draw a shorter tree next to it. From prior lesson intervention cycles I was aware that being able to refer to each tree was useful (as when there were just two trees we struggled to discuss which tree was in focus). I labelled the trees 'tall' and 'short', saying the words and writing them. I continued the story as follows

> T: So I have two trees in my story….And there is a group. A group of monkeys. And there are five monkeys in the group. There are only five monkeys in the group. And these five monkeys sleep in these two trees. So when it is day time they are running around [monkey noises] running around doing all funny things. But at night time they climb up into the two trees. Now you are a scientist. And you want to know: where are the monkeys sleeping? So each night you come out into the place where the two trees are to come and see. How many monkeys are in this tree…?. And how many monkeys are in that tree? … And you want to find out how many different ways can the monkeys be in the two trees.
> (Day 2 lesson transcript, extension group)

The teacher telling of the monkey story was much the same for both the core group on day 2, and the support group on day 3.

Solving the monkey problem

I invited the extension group learners to draw their own trees on their white boards and put the five monkeys into their trees. Once all the learners had positioned their 5 monkeys in the 2 trees and shown their white boards I coached them through an example of how this could be represented using a whole-part-part diagram that was made of paper strips. I made use of the number triple 4-1-5 as this was the example I had drawn on my board. I then instructed learners to draw a whole-part-part diagram for the arrangement of monkeys in their trees. This is a best case example of a child's drawing of the five monkeys in the trees:

Figure 72: Best case example of learner drawing of 5 monkeys in two trees

Notice that children were first introduced to the general case of whole and two parts. They then had to specialise this relationship to an arrangement of their own choosing. In this way I aimed for them to shift attention from the general to the specific (specialising) (Feature 1.4).

I wanted to encourage flexible movement between representations (Feature 4.1) to ensure that symbolic whole-part-part diagram was fully understood. I handed out 5-strips to each learner. I modelled a process of tearing the whole 5 into two parts to match my 4-1 arrangement of monkeys drawn into my trees, and my 5-4-1 symbolic whole-part-part diagram. Learners then repeated this for their arrangement of monkeys. I encouraged the learners to find more ways and handed out more green 5-strips as required. Learners worked on tearing the cards and finding new arrangements of monkeys.

Table 30: Partitioning 5-strips to find how many ways

As they worked on finding new ways, I asked learners to check that they had not got repeats (the same way again) and at times pointed out repeats where I saw them.

I intervened with a learner when I saw he had torn his strip in two places making three partitions by drawing his focus back to the problem situation:

> T: You have got a problem as you made three trees. Are there three trees?
> Learner: No
> T: No I think you better give me this one back [T picks up the three pieces]. That would be three trees we only have two trees. [T gives him another strip to tear once to create two parts].
> (Day 2 lesson transcript, extension group)

This provides evidence of my paying attention to learner errors, and providing explicit feedback on such errors (Feature 6.2).

The children continued trying to find new ways and remove repeats. My intention was to encourage them to start working systematically. However noticing that the lesson was nearly over I opted to rather model a process of working systematically and recording the ways using number sentences. I drew a situation where all five monkeys are in the tall tree (iconic representation of 5-0 monkeys), and asked learners what would happen if the monkeys jumped across to the small tree one at a time. I changed my iconic drawing to 4-1 monkeys in the trees. By the third arrangement of a monkey jumping across to the tall tree, learners volunteered what numbers must be written into the whole-part-part diagram and said the corresponding number sentence, which I then wrote down. We now had a systematic set of six options which was drawn on the white board. I pointed out the pattern of what happening to the tall tree: 5, 4, 3, 2, 1, 0 and learners chorused the pattern, and then chorused the pattern for the small tree: 0, 1, 2, 3, 4, 5. I ended the lesson with learners counting how many ways were recorded. They concluded that there were six ways.

I then directed the learners to rearrange their two paper strip parts to make a systematic pattern. Once a learners' work was arranged systematically and they had checked that they had six ways, I gave them a worksheet to record their ways. This denoted a shift from using paper strips – where each 'monkey' was visible thereby forming an indexical whole-part-part diagram, to making use of a symbolic whole-part-part diagram which denoted increasingly structured representations (*Feature 4.4*).

Learners re-telling the monkey story

The next day (Day 3) I wanted to see if children could recall the partition problem:

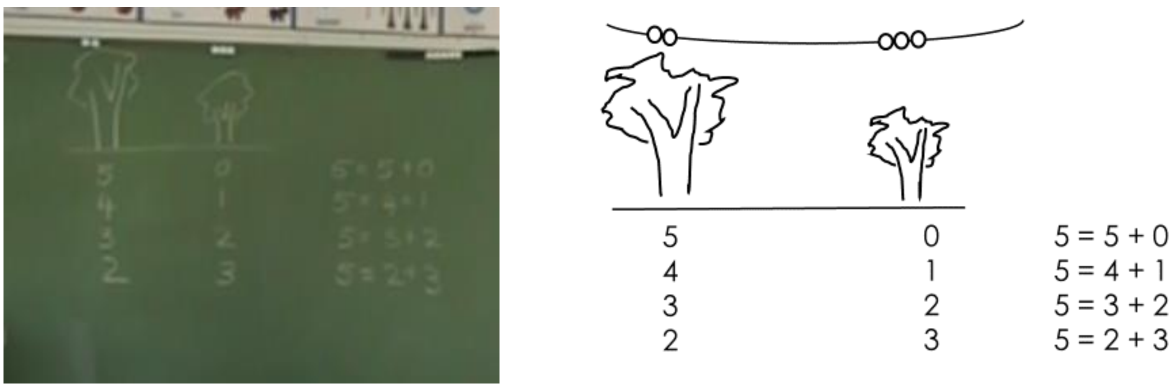
> T: Can someone remind me what that problem was by showing me you want to talk?
> [Core learner raises her hand].
> Learner: what was the problem we worked on?
> Learner: Monkeys sleeping in trees.
> T: How many monkeys were there? Gavril?
> Gavril: 5
> T: There were five monkeys...We had two trees. What was different about the two trees or were they exactly the same?
> Ls: No one was short and one was tall.
> T: Ok so we had a tall tree and we had a short tree.
> [T draws a short and tall trees on the board, just below the bead string].
> (Day 3 lesson transcript, whole class session)

Some of the children who had worked on the problem recalled the problem context and some of its contextual detail. My expectation was that children would retell stories previously told by the teacher, which would shift the story telling into being used as a cognitive strategy for the learners (Feature 3.3). I then re-told the monkey story (for the benefit of the blue group, and to remind the whole class). This denoted a shift back to story telling as pedagogic strategy (Feature 3.2).

Teacher use of representations to solve the monkey problem

I then modelled a systematic process of having all the monkeys starting in the short tree, and moving one monkey across the tall tree. At each stage I moved one bead from the bead string above the short tree, to be positioned above the tall tree. I wrote and said the number sentence for each one. As this progressed I paused and asked learners how many monkeys were in each tree, and asked the learners to tell me what to write for the number sentence.

Figure 73: Teacher modelling five monkeys in 2 trees



Once all six options were written down, I reminded the class of the need to find out how many ways there were. A learner volunteered that there were six ways and came up to the board to show how he counted the number sentences (touching each number sentence and saying the number).

I then repeated this process of modelling a systematic solution to this problem but this time I asked children to make their hands be the trees, and their fingers the monkeys. The learners were expected to move flexibly between the syntax models, and the concrete models using the bead string and their fingers (Feature 4.1).

Learners retelling and varying the monkey stories

I draw on empirical data from the core group session to exemplify how the monkey story was retold by learners. On the third day during the group work session with the core group took place in a separate venue. I allowed the boys to act out the 5 monkeys story, using two chairs. They worked systematically starting with all the boys in one tree and a boy moving one at a time to the other chairs.

Figure 74: Core group boys acting out 5 monkeys problem (4 and 1 option)

As the boys jumped one at a time from the one tree (chair) to the other, the girls had to provide the number sentence, which I wrote down on the board. At first the children offered the number sentences in the form 'part + part = whole', however I consistently reversed this and wrote 'whole = part + part' and talked about these two options being the same. By the end of story the girls were offering the number sentences in the form 'whole = part + part'. This task expected learners to tell stories, and model a narrative process of the problem situation using English (*Feature 3.4*).

This process was then repeated for the four girls. The slightly changed story (there were now only 4 monkeys) was retold, and the four girls acted this out again working systematically. The boys offered the number sentences with each jump. They were able to say the number sentences in the form 'whole = part + part'. They concluded that there were five ways.

The children seemed to enjoy using a bead string to model the monkey story, and the following extract provides a typical case of learners re-telling the monkey story with the bead string:

> There are 5 monkeys. One jumps to the other tree, then 4 monkeys are left.
> [She starts with 5 beads, and then moves 1 away from the group]
> Then the other one jumps to the short tree then there are 3 left
> [She keeps modelling with the bead string with each change]

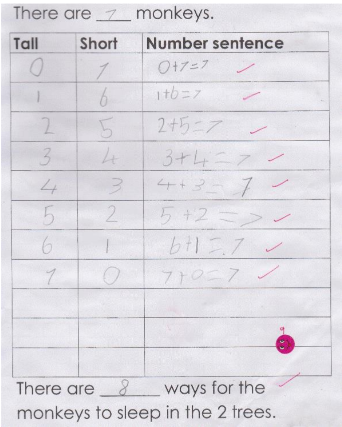
Then another one jumps to the short tree then there is 2 left.
Then another one jumps to the short tree and then there is 1 left.
And the last one jumps to the short tree. And then there is none left.
(Core learner re-telling the monkey story).

It was interesting that all the learners chose to re-tell the '5 monkeys' story, despite having worked on solving the problem for different numbers of monkeys.

Learner use of representations to solve the monkey story

The core group children then worked on a worksheet where each child was given a different number of monkeys for their story. While they had similar tasks they each worked on unique problems (Feature 5.3). Two learners were not able to work on their stories without concrete modelling support. For each of them I brought them two white boards and counters to support them to work concretely to directly model the situation using counters (each board was used to depict a tree, and the counters depicted the monkeys). They physically moved the counters systematically from one board to the next. I supported them in this process for the first one or two jumps. They then worked independently but continued to need to move the counters before recording each response.

Figure 75: Best case example of completion of a monkey's problem worksheet


There are 7 monkeys.

Tall	Short	Number sentence
0	7	0+7=7
1	6	1+6=7
2	5	2+5=7
3	4	3+4=7
4	3	4+3=7
5	2	5+2=7
6	1	6+1=7
7	0	7+0=7

There are 8 ways for the monkeys to sleep in the 2 trees.

Each learner successfully completed at least two versions of the monkey problem.

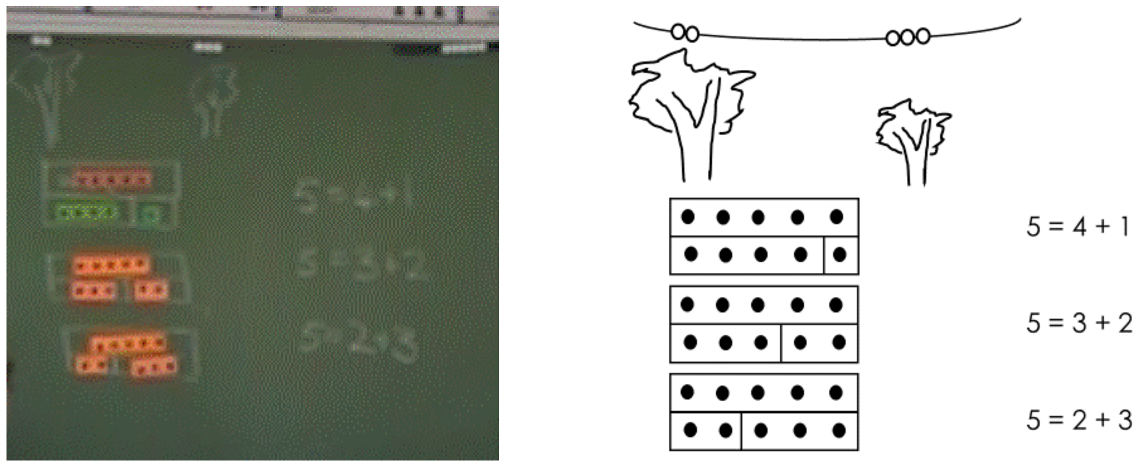
Collectively re-telling and representing the monkey story (whole class Day 4)

The partition problem was revisited in a whole class session on the fourth day. The learners recalled that the story was about monkeys and two trees. I then modelled a process of working with the 5 strips of card to create whole-part-part diagrams and number sentences for each options, and in so doing I modelled a process of flexibly moving between representations (*Feature 4.1*). I worked systematically starting with 5 monkeys in the tall tree. At each step I asked for learner engagement in the story and so learners were expected to participate in the story-telling using English (*Feature 3.4*). The following was a typical interaction:

T: What's the next part of the story?
L: One more monkeys jumped into the short tree,

T: One more monkey jumped into the short tree. So now we need the number sentence. Who can give me the number sentence?
L: 5 equals 2 plus 3
T: 5 equals 2 plus 3. We need to make that. Can you make that for me? [T hands volunteer learners a 5-strip]. I need a 2 and a 3. [T draws another whole-part-part diagram on the board] So we started off with the whole – all 5 monkeys – and we made two parts. [T takes 2 and 3 parts from learner] Who can help me? Must I put the 2 here? Or which way around must it go? The two must go there?
Ls: No
T: The two must go there?
Ls: Yes
T. Ok the two goes here. We can call it left. Two goes on the left. Think. What's the next step in the story?
(Day 4 lesson transcript, whole class)

Figure 76: Modelling the 5 monkeys problem making whole part part diagrams from 5-strips



Once I had worked through all the options of the monkey problem, with the interaction from children, telling me the next step in the story, making each partition and offering the number sentence, I returned to the main problem of how many ways were there. Some learners volunteered that there were six ways. A learner came up to board and showed how he knew it was six (he touched and counted each number sentence). As in the previous lesson I point out the systematic working and asked for descriptions of the patterns. The learners then modelled the monkeys in the tall tree with their fingers: '5; 4; 3; 2; 1; 0' and the monkeys in the small tree on their other hand: '0; 1; 2; 3; 4; 5'.

Teacher variation of the monkey story

In the opening of the lesson on the fifth day I asked volunteer children to retell the monkey story, and drew attention to the 2 trees and 5 monkeys. I then shifted to a 'new story' about golden books. I wanted a scenario where systematic working was required as part of the story. In so doing I varied the problem situation, while keeping critical features (the numbers and the mathematical situation) invariant. I therefore worked with golden books, which I pretended were very heavy and had to be moved from one box to another – one at a time. I chose heavy books as these required that only one book was moved at a time (moving too books was too heavy). In contrast to the monkey story, this story was intended to necessitate moving only one book at a time.

I narrated and acted out this story using 5 books, and with two boys holding a red and a blue box. After each movement of a book from the red to the blue tub, I asked a learner to make the appropriate partition, by tearing a 5-strip. This was stuck onto the board on the form of an indexical whole-part-part diagram. I selected another learner to provide a number sentence. It was noticible that by now, learners were offering the number sentences in the form whole = part + part. They were no longer using the form part + part = whole (which had previously been dominant). At the 5-3-2 partition I stopped the process, to ask whether this was really a new story:

> T: Is this story a new story?
> L: No…yes…[various answers called out by learners]
> T: Is it a new story? What's different about this story and the monkey story?
> L: There are golden books but in the other story there are monkeys
> T: Ok so on the one story there are monkeys and trees. And in the other story there are golden books and boxes. But is our picture looking the same?
> Ls: Yes…no… yes it's the same… It's the same…no
> (Day 5 lesson transcript, whole class)

I had deliberately varied the problem context, and kept the numbers invariant. I wanted to draw the learners' attention to the patterns in the systematic partitioning process. I pointed out the numbers and patterns in the golden books story and compared this to the monkey story. Learners agreed that this was the same as the monkey story and I concluded: 'So the things in the story have changed, but the numbers have stayed the same'.

Learners re-telling stories and teacher talk about story telling

The Day 6 lesson opened with revision and comparison of the three stories that had been introduced to this point in the intervention with the whole class: the partition problem about monkeys in trees; the partition problem about golden books in boxes; and the change increase (equalise) problem about stickers in the learner's books.

Volunteer learners told me about these stories and I wrote monkeys, stickers and golden books onto the blackboard. By learners being expected to re-tell the stories, they were expected to use story telling as a cognitive strategy and to tell stories using English (Feature 3.3). I asked the learners to re-tell the monkey story and used the bead string, and two sketches of trees to model the systematic process. After the second change in the plot (another monkey jumped to the short tree) I asked learners what numbers to put into the whole-part-part diagram. Learner volunteers offered that 5 monkeys were the whole, and that the two parts were 4 and 1. A learner then offered a number sentence: 5 = 4 + 1. This story was fluently retold by volunteer learners, with other learners supporting the recording of number sentences and symbolic whole-part-part diagrams

Discussion of the monkey story

The above description illustrates that the story-telling tasks were not approached as once off events which were concluded, before moving on to the next task. As is evident from revisiting the monkey problem repeatedly over several days the story slowly came to involve more of the learners, who took on different performance roles within the class as part of the story-telling process. The above description also makes clear the interconnected nature of the learning goals: problem solving, flexible use of representations and story-telling. While story-telling is the main element in focus for the vignette episodes relation to problem solving and use of representations are tightly connected with the story-telling. The narrative approach to mathematics learning therefore necessarily defines narrative as encompasses both story-telling and representations (and in a context of the overall mathematical goal of problem solving). The teacher and later the learners are seen to tell stories while creating representations and/or acting on and/or referring to a wide range of representations. The narrative is co-constituted through the use of talk, gesture and representations.

At first the use of story-telling was as a pedagogic strategy where the teacher was telling the story. The learner involvement in the narration was fairly superficial, their involvement was limited to demonstrating their understanding of the teacher presented story and context of 5 monkeys in 2 trees. Limited acting out of the story by learners involving monkey noises and a few monkey actions established the monkey context. Learner drawings of monkeys and trees created further visual cues relating to the situation model. Following this grounding of the situation model in the word problem, the learners' initial engagement with the word problem was fairly teacher directed, as the representations used were highly constrained. Learners were given 5 strips to use as their representation of the 5 monkeys. Their action of tearing the five strips allowed them to physically act on the group of 5 and break it into two. The torn five strips where then used to physically create indexical whole-part-part diagrams. There was direct teacher instruction – again constraining the representational options – by asking learners to write number sentences for each partition; or being directed to show the five monkeys using their fingers, with their hands as two trees.

As the story was revisited over time, learners were given a bigger part in its narration, and the constraints on the problem were slowly relaxed. As such shifts from story telling as pedagogic strategy to stimulate the imagination and create interest was shifted to story telling a cognitive strategies, where learners were expected to retell and vary the stories.

Reflecting back on how Main task 2: Partition problem unfolded over several days, it worth considering which dimensions of variation were being constrained (kept invariant) while other were allowed freedom. At first the focus was on varying the representations used to depict the problem situations. Multiple representations were used to support the telling of the same story, and the same problem solving process: bead strings, 5-strips, number sentences, indexical whole-part-part diagrams, symbolic whole-part-part diagrams, children and chairs, fingers on two hands (Feature 4.1). Familiarity with the story (the story stayed the same over four days), created familiarity which allowed more and more learners to be secure when participating in its narration. Gradually the representations became more structured (Feature 4.4). Initial use of indexical representations of the five strips, and using iconic representations involving drawings of monkeys and trees, shifted over time to indexical whole-part-part diagrams which were then replaced with the symbolic whole-part-part diagram. From the outset, symbolic number sentences were used alongside the iconic and indexical representations. The approach adopted recognised that secure use of particular representations takes time, and all representations were revisited over the intervention period (Feature 4.2).

The story was at first told orally and in detail making use of gesture, sound effects and inviting learner participation in its unfolding plot. In this regard story telling was used as pedagogical strategy to motivate learning and encourage sense making (Feature 3.2). The same story was also retold with different learners narrating different parts of the story, and other learners supporting the recording of each partition. This denoted a shift to learners as story-tellers, were story telling was viewed as a cognitive strategy (Feature 3.3). This collective and repeated narration the story gradually became part of taken-as-shared reference example.

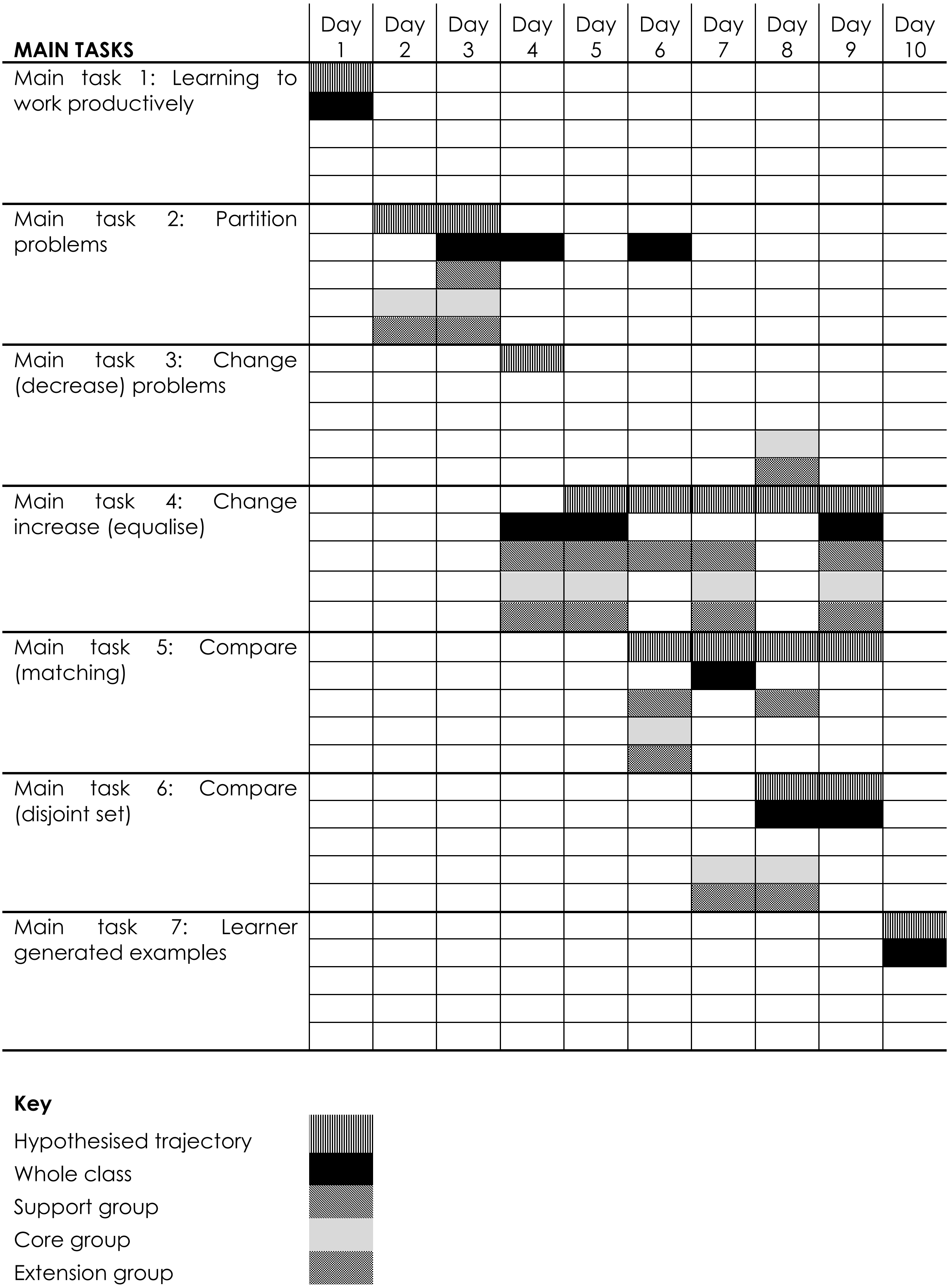
Besides the variation in representations discussed above, the partition problem story was varied in two additional ways. Firstly the core and extension learners retold and worked on telling and solving stories where the number of monkeys was varied. In this process there was no discussion relating to generalising this problem (to observe that if there are n monkeys, then there are n + 1 ways), as this was not the task focus during the intervention. Such generalisation was planned for work beyond the intervention. Secondly, the problem situation was varied: from monkeys in trees to heavy golden books, and was noted by learners as being 'the same' as the monkey story. This illustrates how tasks were varied along prioritised dimension of variation while keeping the critical features invariant, and this should be experienced in rapid succession (Feature 1.6).

Examining the fidelity of the implementation of the third cycle intervention

Drawing on the methodological approach adopted by Stylianides and Stylianides (2014a) in reporting on one of their design experiments, I discuss the fidelity of the implementation of the intervention, and identify endogenous design changes made to the task sequence or implementation as a result of the specific local context in the third cycle. While Stylianides and Stylianides (2014b) refer to and report on the hypothesised and actual learning trajectory of pre-defined stages in their design experiment study, I make use of the main tasks as defining the major stages of the lesson intervention in this design experiment.

I first provide a broad overview of how the main and enabling tasks transpired over the ten day intervention period. The lessons were structured using both whole class and small group interactions (Feature 5.1). The following table maps the main tasks to each day of the intervention, with reference to the teaching format (whole class, or group work).

Table 31: Mapping of main tasks to days in the third cycle intervention



In terms of the selection and sequencing of tasks, the ALT (from the teaching perspective) closely matched the HLT for the extension and core groups. The ALT for the support group did not following the HLT closely for the main tasks. The following table provides a synopsis of which tasks in the HLT were implemented as intended, and which tasks were omitted and or adapted.

Figure 77: Endogenous design changes to the HLT in the ALT for each group

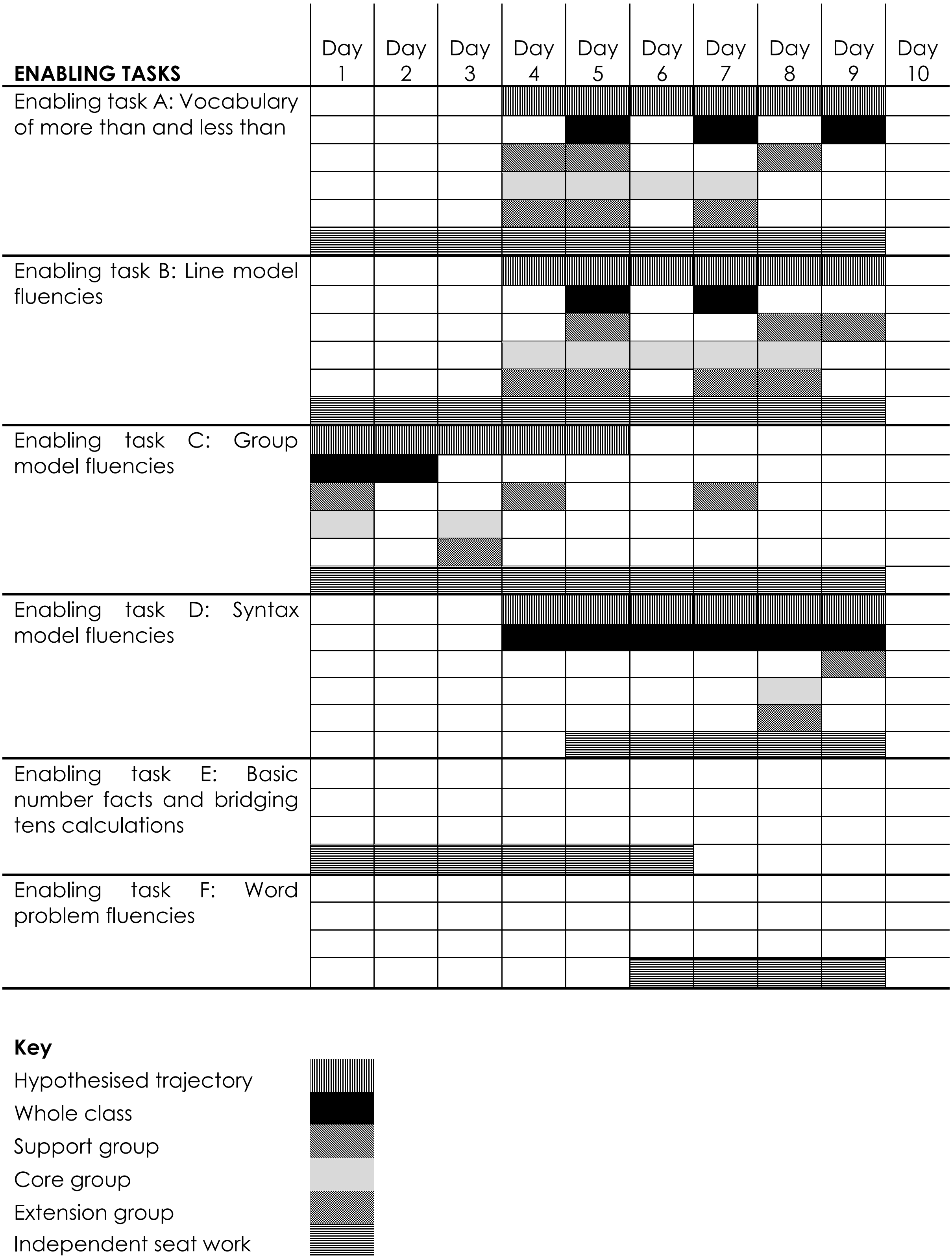
<table>
<tr><th>Main tasks in the HLT</th><th>Support ALT</th><th>Core ALT</th><th>Extension ALT</th></tr>
<tr><td>Task 1: Learning to work productively</td><td colspan="3">Task 1 was undertaken as a whole-class activity involving all three groups. It was implemented as planned.</td></tr>
<tr><td rowspan="2">Task 2: Partitition problem - sticker stories</td><td colspan="3">Task 2 was introduced in small group sessions and revisited during whole-class activity sessions across several days</td></tr>
<tr><td>The support group had one group-work session on Task 2.</td><td>The core group had two group-work sessions on Task 2.</td><td>The extension group had two group-work sessions on Task 2.</td></tr>
<tr><td rowspan="2">Task 3: Learner generated examples - change / take-away stories</td><td colspan="3">Task 3 was not implemented as planned.</td></tr>
<tr><td>Task 3 was omitted from the support group sessions.</td><td colspan="2">The core and extension groups worked on a component of Task 3, as they contrasted difference and take away calculation actions on a number line during a group session.</td></tr>
<tr><td>Task 4: Change increase (equalise) – sticker stories</td><td colspan="3">Task 4 was implemented as planned during group-work sessions for all three groups. It was revisited frequently in group-work sessions as well as during whole class teaching times.</td></tr>
<tr><td>Task 5: Compare (matching) – bowls and lids stories</td><td colspan="3">Task 5 was implemented as planned during group-work sessions for all three groups. It was revisited frequently in group-work sessions as well as during whole class teaching times.</td></tr>
<tr><td rowspan="2">Task 6: Compare (disjoint set) -</td><td>Task 6 was omitted as a group session task for the support group learners.</td><td colspan="2">Task 6 was included in small group sessions with the core and extension groups.</td></tr>
<tr><td colspan="3">Task 6 was also re-told and worked on during whole-class activity.</td></tr>
<tr><td>Task 7: Learner generated examples</td><td colspan="3">Task 7 was undertaken as planned and as a whole-class activity.</td></tr>
</table>

The core and extension group children were provided with opportunities to engage with all of the main tasks, except there was a change in the intended sequencing of Main task 3. As such their ALT in terms of opportunity for learner activity on the tasks followed the HLT as planned.

There were several changes in the ALT for the support group, implemented as contingent responses to the observed learner activity with the support group. As a result of these changes the support group children were not provided with the same opportunities to engage with the main tasks, and their ALT differed from the HLT. Main task 2 was done in a more compressed way. Main task 3: Change decrease and Main task 6: Compare (disjoint set) were omitted. While the support group learners were present in the class community when these main tasks were revisited during whole class sessions, without focused small group work on this (coupled with the lack of focus evident from most support groups learners) it was unlikely that their engagements with these whole class sessions resulted in learning. The reasons for these changes were based on a lower than expected starting point for the support group learners. The empirical evidence of their lack of foundational competencies, and the resulting endogenous design changes to their intervention design is presented below.

I turn now to the HLT and ALT relating to the learner activity on enabling tasks. The following table maps the enabling tasks to each day of the intervention, with reference to the teaching format (whole class, group work or independent seat work).

Table 32: Mapping of enabling tasks to days in the third cycle intervention



All of the enabling tasks were included for learners over the intervention period, with most learner activity taking place through independent seat work. While enabling tasks D, E, and F were not included in the HLT, these were introduced into the independent seat work sessions as a result of endogenous design changes. How the ALT differed from the HLT as well as the ways in which endogenous design changes were made to both main and enabling tasks, are discussed in the next section.

Fidelity of the implementation of the intervention

This section analyses the fidelity of the intervention in order to highlight particular aspects of the intervention where the HLT did not match the ALT from the teaching-side. By examining these in detail, implications for future design cycles are identified.

Considering the overall implementation of the intervention for whole class the HLT largely matched the ALT from a task inclusion and sequencing perspective (deviations are described in detail for particular groups below). The first lesson provided an introduction to the intervention where the focus was developing a productive way of working while engaging with Main Task 1: Learning to work productively. The word problem tasks, Main Task 2 and then Main Tasks 4-6 followed this same general structure.

I now discuss the fidelity of the task design for the core and extension groups. In light of the lack of fidelity for the support group I then provide evidence of the lower than expected starting point for the support group, and explain the changes made to the task design for the support group.

Fidelity of implementation for core and extension groups

For the core and extension groups the HLT generally matched the ALT, with only one endogenous design change. Main task 3: Change (learner generated examples) was not undertaken as originally planned, but aspects of this task were re-inserted later in the intervention.

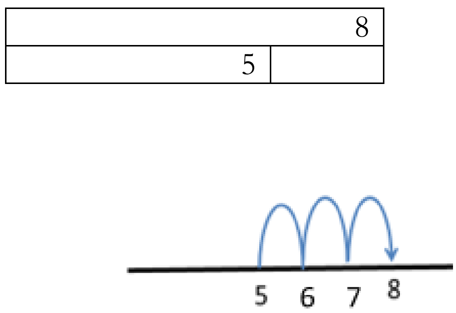
The reason for this change was that learner activity on Main task 2: Partition problems spanned over three days (Days 2-4). A slower introduction to this partitioning problem and the simultaneous introduction of the indexical and then symbolic whole-part-part diagram (in comparison to Cycle 2), was a result of a slower introduction to this symbolic notation, with an indexical whole-part-part diagram comprised of paper 5-strips which could be physically torn apart so that the partitioning action could be experienced. The process of creating the indexical whole-part-part diagrams by tearing five strips, as well as shifting from working randomly to working systematically on this task took longer than expected. But this task was considered to be a foundational task which set up the structural approach to additive relations. As such this additional time was through to be justified. In response to the need for additional time on this task, Main Task 3: Change word problems was omitted. Components of Main task 3 (contrasting take-away and difference calculation strategies on a number line) were reinserted on Day 8 for core and extension learners and on Day 10 with the whole class. For the core and extension groups, all of the enabling tasks featured in the whole class and small group sessions of the Cycle 3 intervention as planned in the hypothesised learning trajectory.

The intention of Main task 3: Learners generating examples, was for learners to tell their own stories. Prior intervention cycles revealed that learners were likely to tell change decrease or change increase problem types which would refer to a direct modelling action of joining or removing which could be visualised on a number line as a take-away image. Due to limitations in time, this learner generated examples task was omitted. It was used as planned in the last lesson (where the intention had been to compare take-away and difference problems). By not including the learner generated examples initially, opportunity to compare the structure of different story types was missed. The contrast between take-away and difference action was however the focus of attention on Day 8 of the intervention, when the core and extension groups were working on line model and syntax model fluencies. I provide a description of the extension group work, which was similar to that of the core group. The support group did not engage with this task.

I expected the extension group learners to write down a subtraction number sentence, draw the relevant whole-part-part diagram and show this as a 'count on' process on the number line. This was intended to draw attention to the equivalence of the number sentences: 'whole – part = part' and 'part + part = whole' and to connect this with the 'counting up' strategy on a number line. I started with 8 – 5 = … which I wrote on the white board. The learners were quickly able to work with this and identified 8 as the whole, 5 as a part, and 3 as the other part. I asked learners to help me draw a whole-part-part diagram for this number sentence. I sketched the diagram, and volunteer learners told me where to write the 5 and the 8

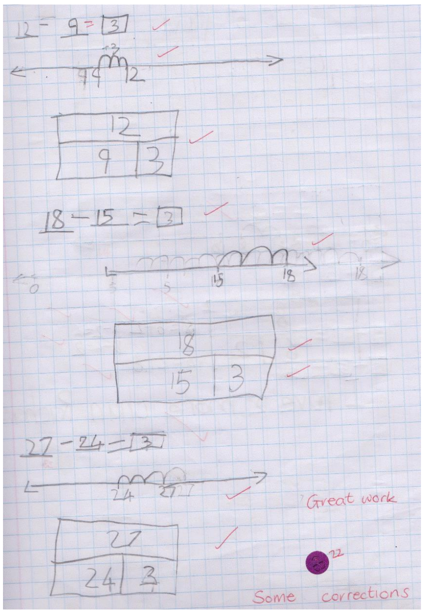
Learner and teacher talk
T: I want to say 8 minus 5
[T gestures to 8 – 5 = …].
I could also have my part plus my part is equal to eight.
[T gestures to 5 and the blank part and then to the whole 8]
5 plus what gets me to 8?
[no response from learners]
[T sketches empty number line and marks on 5]
T: I start at 5. If I add one 1 get to....
Ls: 6
[Teacher shows hop of 1 to the right and labels it 6]
T: If I add 1 I go to …
Ls: 7
[T sketches hop to 7]
T: If I add 1 I go to …
Ls: 8
[Teacher sketches hop to 8]
T: So how many have I jumped?
Ls: 8…3…3
T: So I have started with my part, and I have seen: How far must I jump to get to the 8?
Ls: 3
(Day 8 lesson transcript, extension group)

Black board
8 – 5 = …



In a similar way I then worked with a harder one of 18 – 14 = …Learners directed me to draw a whole-part-part diagram and show hops on a number line from 14 to reach 18. Learners then worked in their books to start with a subtraction number sentence, draw a whole-part-part diagram and then show this on a number line. I varied the number sentences given to each child, making sure that the numbers lent themselves to a different strategy (in that the number pair was close together). As learners finished I gave them more number sentences to work with (and each learner worked on their own number sentence). The following is a best case example of this work:

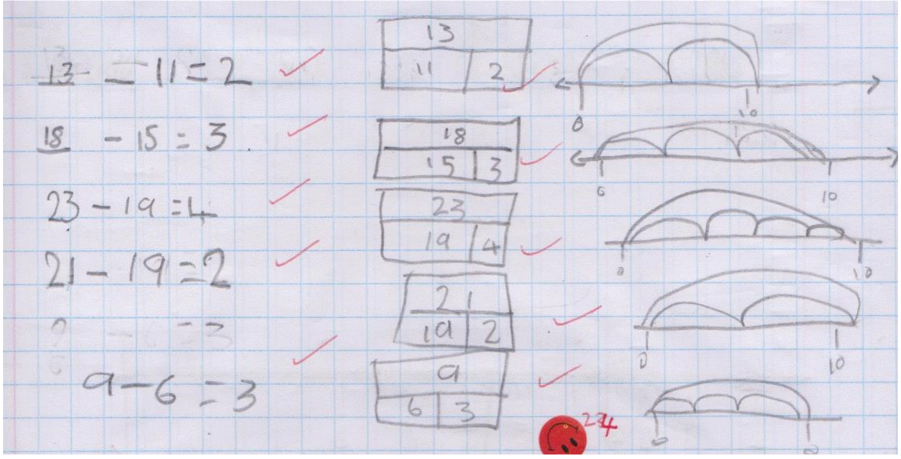
Figure 78: Best case example of learner activity



This learner worked on 12 – 9 = […]; 18 – 15 = […] and 27 – 24 = […]. The learners' activity reveals that for 18 – 15 = […] this child first attempted a take-away strategy showing counting back 15 from 18. However, following a teacher prompts to start with the part and count up to the whole, she erased this and depicted the more efficient counting up strategy. She repeated this efficient strategy for 27 – 24 = […].

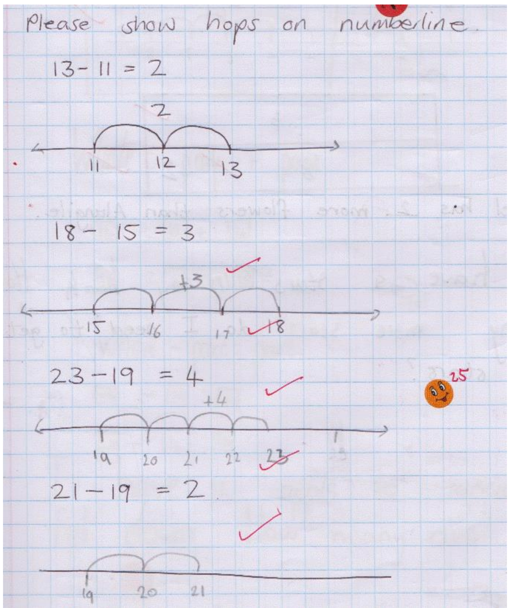
The following is a telling case example of a child who was able to solve the number sentences mentally, and depict a whole-part-part diagram accurately. However she was not yet clear on how to connect this with the number line representation.

Figure 79: Telling case example of learner activity in the extension group



This learner depicted a 0 – 10 number line and showed the correct number of hops between the two numbers. She showed 2 hops for 13 – 11, and 3 hops for 18 – 15, and 4 hops for 23 – 19. However she did not label the starting and ending points appropriately, using 0 and 10 for all her examples. I supported this learner to notice the significance of the labels on the number line by providing a task in her book which I wrote when I marked her work and noticed this error:

Figure 80: Telling case example of learner activity extension group



I repeated her number sentence examples. For the first case I provided the labels and the hops. In the second case I provided only the labels and she drew in the hops. In the third case I provided only the first label. In the final example I provided only a line, with no labels. She was then able to complete this task appropriately which seemed to be supported by the specialised and explicit feedback she received on her independent written work (*Feature 6.2*)

The above account drawn from the description of the lesson intervention was selected to illustrate the way in which Main task 3: Learner generated examples, was adapted from the original HLT of starting with a story-telling activity (where a take-away action would be likely which would lend itself to a take-away line model); to working directly with the number line representation to make use of a counting up action to create a difference image on a number line during the intervention.

My critical self-reflection on this endogenous change in task design was that the change meant that opportunity for a direct contrast between the two actions was lost, and the action was considered only in relation to the line model and not in terms of direct modelling of contrasting actions in a problem situation. By working directly with the line model image, the approach was procedural and introduced as a line model useful for a particular calculation. Learners were given opportunity to work on solving unique calculation problems. Some of them did first create a take-away image for a calculation, and then recreated a difference model for the same calculation. In this way the number were kept invariant and the two different actions could be visualised on the number line. However the task design did not include a deliberate instruction for the learners to use the two methods for the same calculation and then compare their efficiency.

Fidelity of implementation for the support group

The assumed starting point for some of the support group learners was below what had been expected, (where the assumed starting point had been based on the 2nd cycle intervention with the 2013 Grade 2 class). As noted already, the later timing of the Cycle 2 intervention and larger numbers of LSEN learners in Cycle 3 may well have contributed to the mismatch. The lower than expected starting point for the support group, resulted in several endogenous changes in the intervention design. In the section I briefly present the evidence of the lower than expected starting point for this group, before describing how the intervention was adapted based on this empirical data.

This lower than expected starting point for the support group was evident in results of the written pre-test. The mean result for the Cycle 3 pre-test was 8.08 (SD = 4.5) out of 18 marks; while the Cycle 2 pre-test mean had been 11.1 (SD = 4.3). The results of the support group in Cycle 3 were particularly low with only a pre-test mean of 3.7 (SD = 1.4).

Data collected from the video recording and related transcripts from the first few lessons, as well as evidence from support group learner activity on enabling tasks confirmed that the starting point for these learners was below what was expected. Several of the support group learners were not yet secure with some of the assumed starting points for the intervention such as reliable counting and basic addition and subtraction of single digits. These learners were not able to complete sentences like '1 more than 5 is …' or name the 'number after' a single digit number. They were not able to choose the correct phase when given a choice to select between 'more than' and 'less than' statements, for example '5 is more than / less than 9' could not be answered, or was answered incorrectly. To provide examples of the empirical data suggesting this lower than expected starting point I first provide a brief overview of the support group learner activity on the enabling tasks in the first lesson. The enabling tasks provided to learners during the first lesson were intended to be easily solved and completed, as the purpose of this lesson was to create a productive learning culture in the class community, where independent seat-word tasks could be completed independently. As such the kind of tasks assigned to learners as enabling tasks were below the grade level expectations.

In the first lesson, the support group work revealed that learners in this group used perceptual counting: counted all in ones, and needing to touch each object being counted. They were not able to count on from five, as the following extract from my lesson reflection reveals:

> For the support group I aimed to encourage counting on from five, for numbers that I displayed by holding up that number of beads on the bead string either horizontally or vertically on the bead string. If I asked: "how many are here?", while showing 7 beads, I discouraged counting in 1s and supported learners to "put the 5 in their heads" and count on 6,7 (in ones). Few of the members of the support could do this without my prompting, and most 'counted all' in ones and seemed to need to touch each bead to coordinate their verbal count with each object.
> (Teacher notes on lesson intervention description; Lesson 1 support group, teacher notes)

When this group started working on their enabling tasks, I found that several of them were not able to solve the enabling tasks which I had intended would be easily completed by them without mediation. In the first lesson, when I realised the several of the learners could not answer simple single digit addition calculations I brought out bottle top counters for several of them to work on adding or subtracting single digit numbers. Some were able to use the concrete materials adopting a counting all strategy to answer the addition questions. Very few were able to answer, or concretely model a single digit subtraction calculation.

Analysing the errors from the enabling tasks of the support group learners from in the second lesson revealed the kind of conceptual difficulties that these learners were having with basic addition and subtraction with the idea of '1 more than' , and '1 less than', and with the alternative form of the number sentence:

> Mpho: 2 more than 6 is [5]
> S2[23]: 4 + 3 = [1]
> S3: [11] = 6 – 5
> S4: 7 – 4 = [1]
> S5: 1 more than 3 is …(not complete); 2 more than 5 is [6]
> S6: 1 more than 6 is …(not complete); Complete: 6, [10], 8
> S7: 1 more than 2 is …(not complete), 4 + [8] = 4
> S8: 8 - 7 = [5], 9 + 7 = [8]
> (Example of support group learner errors in first few lessons, from learner books)

[23] S2 refers to the second learner in the support group. The first learner was Mpho (who obtained 0 in the pre-test), other learners have been ranked according to their pre-test attainment and labelled S (support), C (core) and E (extension).

Considering Treffers' ten level description of learner progression from counting to calculating, I had assumed that most learners in the focal class would be between level 4 – level 6, with a view to shifting into aspects of levels 6 and 7. It is of interest that while recognising that the learning-teaching trajectory has a bandwidth (which is not tightly mapped to ages or grade levels), the Treffers' framework situates 'compare quantities as being more, less or equal' to a level 2 skill and maps this to a pre-kindergarten (age 2-3 stage). However, the empirical data from this grade 2 class, where the learners are aged 8-9, suggested that some learners did not meet the level 2 skills and were unable to count reliably or compare single digit numbers as being more or less. As a result of this mismatch between the assumed starting point for the support group learners, and the actual starting point I made design changes to the group work sessions with the support group learners.

On the third day, the support group of eight learners were split into two groups of four, and additional group work time was arranged for these two groups. The small group teaching intervention sessions for the support group were adjusted to focus more on the following foundational competencies:

1. Reliable counting (coordinating verbal count with touching each object being counted, and announcing how many there are at the end of a verbal counting sequence);
2. Reading, writing and ordering single digit numbers less than ten with smallest on the left and biggest on the right;
3. Using comparative language of phrases using more and less
    a. 'more than' and 'less than' to compare two numbers such as '10 is <u>more/less</u> than 4'.
    b. '1 more than … is …' (where the variable was a single digit number less than 10) and associating the phrase more than with a hop to the right on a number line;
    c. '1 less than … is …' (where the variable was a single digit number less than 10) and associating the phrase 'less than' with a hop to the right on a number line;
4. Adding and subtracting single digit numbers.

The teaching time with the support group focused on developing the foundational fluencies when engaged with word problems like Task 4: Change increase and Task 5: Compare (matching). As a result of the additional time spent on these tasks working on the foundational skills, the support group did not work on Task 6 Compare (disjoint set) in a small group setting. They were present in the class when Task 6 Compare (disjoint set) was the focus a whole class sessions, with some of them participating in this communal whole class process.

The following extract of a support group work session on the fourth day is presented to illustrate how engagement with the Task 4: Change increase was used as an opportunity to consolidate the foundational competencies. Some of the support group learners were not able to count reliably, and did not know how to use the terms 'more' and 'less' correctly.

All learners had been given stickers for their efforts on the enabling tasks. To open the group-work session in Lesson 4, the support group learners were asked to count the stickers that were in their books. I watched learner S5 touch and count her six stickers, mouthing the words 1 to 6. When I asked her how many stickers she had, she said 'I don't know'. I asked her to count them again. This time, I noticed that she did not coordinate her touching with her verbal counting. She said she had 9 stickers. I asked her to touch and count each one, and to show me. She counted each one (with me slowing her down when her verbal count went ahead of her touching), and with each sticker touched and counted I moved a bead along the bead string. She counted 6 stickers, and when asked, said she had 6. Later in the session when I asked S5 how many stickers she had, she said '10'. We repeated the process of me watching her count and touch each sticker. I helped her to coordinate her verbal count with her touching each sticker. She again counted six stickers. When asked how many she had, this time she said she had 6 stickers.

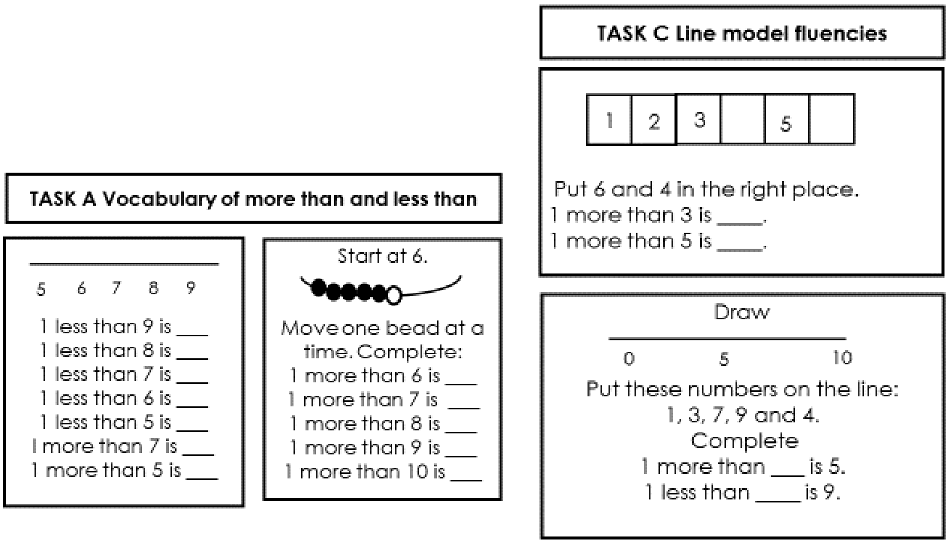
In the same session, learner S2 said he had 12 stickers. Mpho said 'You don't have 12!'. When S2 recounted he too did not keep his verbal count aligned with his touching action. He counted again and announced he had 11 stickers. The following except from the transcript on this lesson revealed that S2s counting was not reliable, and also indicated difficulties with ideas of 'more than' or 'less than' and 'smaller than' or 'bigger than':

> S2: I have 11
> Teacher: Let's see.
> S2: 1, 2, 3, 4 [S2 only touches 3 stickers on the page]
> Teacher: Don't count ones that are not there yet. Let's count [teacher touches each sticker, and waits for him to touch and count with her]
> S2: 1, 2, 3, 4, 5, 6, 7 [S2 coordinates verbal count with teacher and his touching the stickers]
> Teacher: Seven. Ok so how many do you have S2?
> S2: 7
> Teacher: Show me 7 on the bead string. [S2 moves a group of 5 beads and then moves 2 more]
> Teacher: Does S2 have more or less than 5?
> S2: Less
> Teacher: He has 7. Is 7 bigger than 5 or smaller than 5?
> S5: Smaller than 5.
> Teacher: Smaller than 5?
> S2: Big. Its big.
> S5: Bigger
> Teacher: Its bigger. 7 is bigger than 5.
> (Lesson 4 transcript, support group)

The support group was encouraged to start to use line, group and syntax models but the primary emphasis was on reliable counting and a shift from 'counting all' to 'counting on' using group actions. Unlike the core and extension groups, the support group did not work on syntax model fluencies where the difference strategy was compared to the take-away calculation strategy.

As a result of the lower than expected starting point I created additional enabling task cards to allow for further independent work on these foundational competencies:

Figure 81: Additional individual work cards added to Task A: Vocabulary and Task C: Line models



I made further endogenous design changes by inserting additional mental maths tasks into the small group work with the support group. For example I introduced a game called 'hands up', which was played with all three groups as a result of the difficulties with use of phrases '1 more than' and '1 less than'.

> Hands up: 1 more and 1 less than
> Learners sit in a circle. A volunteer learner chooses a starting number (whole number less than 5), for example 4.
> An ending number is agreed (whole number between 15 and 20), for example 20.
> The phrase which is to be used is agreed, for example '1 more than'. This phrase is written down on a white board.
> The learners each get a turn to use this phrase for a particular number, and the person to their left then continues from their stated number. For example using the phrase '1 more than' with a starting number of 4 the sequence of statements is as follows:
> Learner 1: 1 more than 4 is 5
> Learner 2: 1 more than 5 is 6
> Learner 3: 1 more than 6 is 7
> etc,
> (Description of hands up mental maths game, support group)

The first learner to say the number 20 (agreed ending number) was the winner, and they put their hand up. Then a new round commenced (where the winner chose the starting and ending numbers). This game was played repeatedly with the support group learners. At first they were given a bead string, where they could move a bead each time, to make the group bigger; then they were given structured 0-20 number lines and they moved their fingers along the structured number line with each hop of '1 more than'. This game was varied by using '1 less than', and using '2 more than' and '2 less than' phrases.

Positioning numbers with smaller numbers on the left, and bigger numbers on the right was also challenging for the support group. As a result, an embodied number line task was developed which built on the oral fluencies developed through the 'hands up' game, but shifted from the structured number line, to an imagined or empty number line on their bodies.

On Day 8 of the intervention I repeated '2 more/less than' oral activities using a structured number line for the support group. However for this group, the counting on and back in twos using comparative language was still not secure. I therefore switched to '1 more/less than' and used their bodies as number lines. I directed a child to hold out his arms with his back to the group, I encouraged the children to imagine 0 in the child's left hand and 10 in their right hand. When I asked what number would be on his head, the group established that 5 was in the middle. I modelled a process of moving along the child's arms from left to right saying: '1 more than 0 is 1', '1 more than 1 is 2', etc. The children joined in. The action of moving our hands in an arc (as we depicted hops on a number line) and landing further along the child's outstretched arms was used. Then in pairs (one child being the number line, the other doing the count and moving along the number line) counted '1 more than ___ is ___' moving right, and '1 less than ___ is ___' moving left. This task was repeated with the other support group and revisited on Day 9 in small groups. It was intended to support learners with three interrelated ideas: the arrangement of numbers in a line model with smaller on the left and bigger on the right, the direction of movement along this line for making bigger or 'more than', and for making smaller 'less than', and using the vocabulary of '1 more than' and ' 1 less then' in relation to particular numbers.

It was of interest to me that difficulties most acutely evident in the support group, were also visible for the core and extension group learners. Once the new tasks (enabling tasks for independent seat work) as well as new mental maths activities had been developed for the support group, they were also used with the core and extension group learners. In some cases the tasks were varied slightly for the groups. For example while the hands up game was played using '1 more than' with concrete materials (such as a bead string or counters) for the support group, for the core and extension groups this could be done fluently without concrete materials and only using a structured number line. Over time for the core and extension groups the structured line was removed, and the phrases were extended to '2 more than' (or at the request of an extension learner to try '3 more than'). The same basic task structure was used, however the tasks were varied slightly based on the demonstrated fluencies of the particular group.

In this way engagement with the support group learners was found to yield very useful information relating to foundational competencies, and suggested aspects that were thought to be possible pre-cursors for the main tasks to be accessible. At the same time it was clear engagement with the main tasks could be adjusted to suit the needs of a particular group. While the core and extensions groups used the sticker story (Main task 4: Change increase) to work on line and syntax model representations; the same main task could be used to focus on developing reliable counting with the support group.

Implications for intervention design in future cycle

Several issues emerge which are relevant to future improved design cycles.

The endogenous design changes as a result of the lower than expected starting points for the support learners were thought to be useful to the intervention design as foundational tasks were developed and introduced into the enabling task design, and suggestions for possible baseline assessment activities became clearer. In future design cycles, the enabling tasks designed for the support group should be integrated into the HLT. In addition some of the foundational competencies ought to be assessed in the assessment instruments. This would serve two purposes. Firstly including a few foundational competence items such as ' how many ?' (with an image of less than 10, or an image of 10-20 objects) or '1 more than 6 is ____', would help to diagnose the absence of foundational competences which would influence selection of enabling tasks, and the focus of small group tasks from earlier on in the intervention. Secondly including some foundational competences in the assessments may provide evidence of learning gains in relation to the foundational competencies for these learners.

The intervention should take place over a slightly longer period of 12 or 13 day so that all of the main tasks can be included as planned. The original HLT with Main task 3: Learner generated examples should be maintained, and learners should be given opportunity to tell stories (which are likely to invoke a change action), which can then be depicted in both group and line model representations. Time is important to establish both the change stories, and the take-away group model and take-away line models as taken-as-shared by the class community. It is against these images and stories, that the contrast with static problem situations (of the compare (matching), collection and compare (disjoint set) problem types) can be discussed. The calculation actions of take-away and difference should also be contrasted to a partitioning action (most closely related to the partition word problem and collection word problem type).

It may be further advantageous to include a classification task following Main Task 7: Learner generating examples. This final task would allow for learners to work with the word problem example space generated by the class community. Discussion classifying the word problems as movies (dynamic or change stories relating to the change increase and change decrease problem types), or as photographs (static stories relating to the collection, compare (matching) and compare (disjoint set) problem types). Further consideration of the contrasting actions explicit in the case of change problems) and implicit in the case of static problem situations, may then be facilitated. Learners could then be invited to depict the solution to these problems using line, group and syntax models.

Figure 82: Proposed main task 8 to consolidate contrast between take-away, difference and portioning actions

	TASK 8 Solving and classifying learner generated examples
Instruction	Work with a partner. Read all the stories. For each story: 1. Solve the problem 2. Draw a whole-part-part diagram 3. Write a number sentence Stick each story onto the wall under one of these labels: **MOVIE:** This story has action. There is a start, then something changes and there is an end. **PHOTO:** This story has no action. There are two parts and a whole. Nothing changes.

Concluding remarks on the third cycle intervention

The above analysis of how the third cycle intervention played out in this local context should be read in conjunction with the descriptive account in Annexure 4. It is this thick description of the the task-by-task analysis which provide sufficient detail for this intervention to be replicated in other contexts. What is offered in this chapter is a synopsis which illustrates first how the intervention panned out in terms of the extent to which the planned HLT matched the ALT from the teachers' perspective in terms of task design. It also then illustrates the analysis of the descriptive account of the intervention, where the description of the intervention was used to reflect on the extent to which there was evidence of the theoretical features specified in the theoretical framework. Inevitably, there was not perfect alignment with was hoped for in theory, and how actual lesson interventions played out in reality. While this chapter has provided qualitative data from the teaching-side, the next two chapters present the learning-side of the intervention firstly with quantitative analysis of the written tests and secondly with detailed qualitative analysis of learning in relation to particular children.

CHAPTER 6: Findings pertaining to learning gains for the whole class

> This chapter focuses on the findings relating to its effect on learning. In so doing, the question *'what evidence of learning gains (in relation to the learning goals), if any, is seen as a result of the teaching intervention*?' is answered. This question is addressed for Learning Goal 1: Problem solving, and Learning Goal 2: Representations firstly in relation to quantitative analysis of written tests using the simple (correctness of solutions) marking framework. Then this question is addressed in more detail using the expanded marking framework. Quantitative analysis of evidence of learning gains for the whole class and the smaller ability groups is presented. Learning Goal 3: Story telling is then analysed qualitatively using the whole class learning activity on the final main task in the intervention, with additional detail provided from the case study learners' activity on story telling tasks in the pre and post interviews.

This chapter opens with a quantitative analysis of the learning gains in relation to learning goals LG 1: Solving additive relation word problems, and LG 2: How children make their thinking visible by using diagrammatic representations drawing evidence from the results from the pre-post, and delayed post-test results from the written test. In addition, the whole class is disaggregated for the three ability groups (support, core and extension groups) where the quantitative results of each group is analysed. I compare the written test results in Cycle 2 to written test results in Cycle 3 and demonstrate that the Cycle 3 intervention showed more promise than Cycle 2. The third learning goal LG 3: How children make their thinking visible by using narrative to pose and explain additive relation word problems is assessed qualitatively drawing on the whole class learner activity on Main task 7, and examples of the use of narrative from the case study learners in Cycle 3.

The effect of the intervention was measured in relation to the learning goals and related learning trajectory from the assumed starting point to the end point. I looked for extent of match between the HLT and ALT, and for whether measured learning gains were found to be significant in relation to written test attainment for Learning goal 1: Problem solving and Learning goal 2: Representations. In addition it was expected that learning gains would be evident in whole class activity on Main Task 7: Learner generated examples, and for the case study children from pre- and post- individual interviews (particularly for Learning goal 3: Story telling, which could not be assessed in written format). If a smaller standard deviation was found, in addition to an improvement in the mean result, this would further strengthen evidence of learning.

For the more detailed analysis of learning relating to word problems, Student t-tests were expected to identify whether the shift in means was statistically significant; and the calculation of effect sizes was used as a basis for comparing learning gains in Cycle 3, to learning gains in Cycle 2. Categorical coding of representations in term of coherence, as well as representation type; and for calculation strategy was undertaken to measure shifts for the flexible use of representations. Chi-squared tests were undertaken on the contingency tables reporting frequency of representational category in the pre-test and the post-test to establish the probability of the observed shifts being a result of chance.

Learning relating to the third learning goal was reported on through qualitative analysis of whole class learner engagement with the final task of the intervention: Main task 7: Learners generating examples. Based on prior intervention cycles it was expected that learners would not be familiar with using story-telling to pose word problems; but that by the end of the intervention they would be able to do so; and that a variety of word problem types would figure in the examples generated by the class community. The learners' ability to use storytelling to explain problem situations is reported on with reference to the case study learners' engagement with story-telling tasks in the pre and post interviews. These tasks included learners telling stories for particular number sentences; explaining generalised word problems (where a general change-decrease problem type was contrasted to a general compare (disjoint set) problem type).

Quantitative analysis of written test attainment for problem solving and use of representations

In this section I report on the attainment in written tests. This is done with regard to the application of the 'simple marking framework' described in Chapter 4. The results from the application of the 'simple marking framework' were used to make decisions regarding changes in design of the intervention; and for the selection of learners to ability groups for the Cycle 3 intervention. Next I report on the application of the expanded marking framework to explore the effect of the intervention in relation to each of the learning goals. While the focus is on Cycle 3, for ease of reference I also include the comparative quantitative analysis from Cycle 2.

Before considering the detailed results, it is important to first reflect on the purpose of having a pre-test, post-test and delayed post-test. Results from the pre-test to post-test were used to establish whether there were learning gains for the class as a result of the intervention. A delayed post-test was included to test whether any gains evident in the post-test may have simply been a result of writing test soon after the intervention. The possibility that learning gain evident may have disappeared of dampened when there was a delay between the intervention and the written test. Therefore sustaining the learning gains in the delayed post-test would discount this hypothesis. However no claims relating to the impact of the intervention could be inferred by drawing on learning gains from the delayed post-test alone. Learners matured over the delay period, and the normal classroom teacher continue classroom teacher for this class, continued to participate in the broader development project focused on improving mathematics learning and teaching.

Written test attainment using the 'simple marking framework'

The simple marking framework concerned only the correctness of solutions in each of the test items (4 single-solution word problems, 1 multiple-solution word problem, and 2 bare calculations). This marking framework (explained in detail in Chapter 4) was used to provide an indication of the improvements, or lack of improvement in relation to finding the correct solution to each problem posed in the test. Using the simple marking framework, the test was marked out of the total of 7 marks (with 1 mark for each correct solution in each question).

Table 33: Results mean and standard deviation on written tests (simple marking framework)

	Mean	**(Standard deviation)**
Pre-test (marked out of 7)	3.5	(1.9)
Post-test (marked out of 7)	4.5	(2.2)
Delayed post-test (marked out of 7)	5.1	(1.7)

The above provided evidence that there were more learners obtaining correct solutions for the problems in the written test from pre-test to post-test, and that there were further improvements in correctly solving problems by the delayed post-test.

To gain a deeper understanding of the attainment of the class in relation to each item in the written test, and considering each learner in the three ability groups a visual presentation of this results of this simple marking framework was developed. For each learner and each item, their correct response to a question was colour-coded with red to indicate incorrect solution (0 marks), and with blue to indicate correct solution (1 mark). Each attainment group was then considered at each point in time. In addition, for each ability group the 'facility' of each item in the test was calculated. Facility was calculated by dividing the number of learners obtaining a correct solution by the number of learners in the group, which was then expressed as a percentage. This provided a '0 to 1' scale of difficulty with 0 indicating that no learner in the group had obtained the correct answer, and 1 indicating that all learners in the group had obtained the correct answer.

Table 34: Facility of each item (considering correct solutions)

	Cycle 3 (n=26)			
	Pre-test	**Post-test**	**Delayed post-test**	**Difference in facility (delayed post-test to pre-test, in percentage points)**
Question 1: Change decrease	69%	65%	85%	+16
Question 2: Compare (matching)	73%	81%	81%	+8
Question 3: Collection	54%	62%	69%	+15
Question 4: Compare (disjoint set)	34%	65%	58%	+24
Question 5: Partition	19%	42%	54%	+23
Question 6: Bare calculation 21 – 6 =…	42%	77%	81%	+39
Question 7: Bare calculation 23 – 18 = …	58%	54%	85%	+27

The calculation of facility per question showed that the greatest learning gains from pre-test to delayed post-test were for the bare calculation problems. In terms of the word problems, the greatest difference in facility was evident for the compare (disjoint set) word problem and the partition problem.

Analysis of learning in written tests involves considering the changes evident in the individual learners' activity on the same written test items at different points in time. Aggregated data (such as the mean result and standard deviation for the whole class; or the facility of questions for the whole class) does not maintain the relationships between individual learners and particular items. For this reason I then considered the individual learners each of the learning groups, and whether or not they obtained the correct solution for each item. I labelled the individual learners as S1, S2, S3 etc for the support group; as C1; C2; C3; etc for the core group and as E1, E2, E3, etc for the extension group. I introduced pseudonyms for the three case study children: Mpho; Retabile and Gavril, so that their attainment on the written test items as well as in relation to other learners in their groups would be visible.

Figure 83: Pre-test, post-test and delayed post-test correct solutions (support group)

	Q1 Change result unknown	Q2 Compare (matching)	Q3 Collection	Q4 Compare (disjoint set)	Q5: Partition problem	Q6: Bare calculation 21-6 =	Q7: Bare calculation 23-18 =	
Marks	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**7.0**
Mpho	0	0	0	0	0	0	0	0
S2	0	0	0	0	0	0	0	0
S3	0	0	0	0	0	0	0	0
S4	0	1	0	0	0	0	0	1
S5	0	1	1	0	0	0	0	2
S6	0	0	1	0	0	0	0	1
S7	1	1	0	0	0	1	1	4
S8	1	0	0	0	0	1	0	2
Pre-test (facility)	25%	38%	25%	0%	0%	25%	13%	**1.3 (mean)**
Mpho	1	1	0	0	0	1	0	3
S2	1	1	0	1	0	0	0	3
S3	0	0	0	0	0	0	0	0
S4	0	1	0	0	0	0	0	1
S5	0	0	0	0	0	0	0	0
S6	0	0	0	0	0	1	1	2
S7	1	1	0	1	0	1	1	5
S8	0	0	0	0	0	1	0	1
Post-test (facility)	38%	50%	0%	25%	0%	50%	25%	**1.9 (mean)**
Mpho	1	0	1	1	1	1	0	5
S2	0	1	0	1	0	1	1	4
S3	0	1	0	0	1	1	1	4
S4	1	0	0	0	0	0	0	1
S5	1	1	1	0	0	0	0	3
S6	1	0	0	0	0	0	0	1
S7	0	1	1	0	0	1	1	4
S8	1	1	1	0	1	1	1	6
Delayed post-test (facility)	63%	63%	50%	25%	38%	63%	50%	**3.5 (mean)**

For the support group, the mean result for correct solutions, improved slightly from pre-test to post-test, with further improvement evident by the delayed post-test. The facility of questions improved for all questions except the collection problem, and there was no improvement relating to the partition problem from pre-test to post-test. By the delayed post-test there were improved results in these questions. The compare (disjoint set) problem remained the most difficult question to solve, and more learners were able to solve the compare (matching problem). It is interesting to note that for this group, the improvement from pre-test to post-test was only slight; however the improvement from pre-test to delayed post-test was bigger. At both the pre-test and post-test stages the learners in this group struggled to solve almost all of the questions in the test. The test seemed to be pitched above the level at which most of this group were working (and evidence of their lower than expected starting point in the ALT compared to the HLT is discussed in some detail below). However by the delayed post-test, these learners were better able to solve some of the written test problems, although compare and partition problems remained unsolvable for the majority of this group. As mentioned above, the further improvements evident in the delayed post-test are not solely attributable to the intervention. The focal school, and the normal classroom teacher for this class, continued to participate in the broader development project focused on improving mathematics learning and teaching.

Figure 84: Pre-test, post-test and delayed post-test correct solutions (core group)

Marks	Q1 Change result unknown 1	Q2 Compare (matching) 1	Q3 Collection 1	Q4 Compare (disjoint set) 1	Q5: Partition problem 1	Q6: Bare calculation 21-6 = 1	Q7: Bare calculation 23-18 = 1	7
C1	1	1	1	0	0	0	1	4
C2	1	1	0	0	0	1	1	4
C3	1	1	1	1	0	0	0	4
C4	1	1	1	1	0	1	1	6
C5	1	1	0	0	0	0	1	3
C6	1	1	0	1	0	0	0	3
Retabile	1	1	1	0	0	0	0	3
C8	1	1	0	1	1	1	1	6
C9	1	1	1	0	1	1	0	5
Pre-test Facility	**100%**	**100%**	**56%**	**44%**	**22%**	**44%**	**56%**	**4.2 (mean)**
C1	1	1	1	1	1	1	0	6
C2	1	1	1	1	0	1	0	5
C3	1	1	1	1	1	1	1	7
C4	1	1	1	1	0	1	1	6
C5	0	1	1	1	0	1	0	4
C6	1	1	0	1	1	0	0	4
Retabile	1	1	1	1	1	1	1	7
C8	1	1	1	0	1	1	1	6
C9	1	1	1	1	1	1	0	6
Post-test facility	**89%**	**100%**	**89%**	**89%**	**67%**	**89%**	**44%**	**5.7 (mean)**
C1	1	1	1	1	0	1	1	6
C2	1	1	1	1	1	1	1	7
C3	1	1	1	1	1	1	1	7
C4	1	1	0	0	0	1	1	4
C5	1	1	1	0	1	1	1	6
C6	1	1	1	1	1	1	1	7
Retabile	1	1	1	1	1	1	1	7
C8	1	1	0	1	1	1	1	6
C9	1	1	0	1	0	1	1	5
Delayed post-test Facility	**100%**	**100%**	**67%**	**78%**	**67%**	**100%**	**100%**	**6.1 (mean)**

The core group showed greater learning gains in correct solutions than the support group. There were improvements in the mean result from pre-test to post-test, and further improvement by the delayed post-test. The core group were all able to solve the easier single answer word problems (change decrease and compare (matching)). Most of the learning gains in this group were in relation to the collection, compare (disjoint set) and partition problem. All core group learners could solve the compare (matching) word problem, which was found to be easier to solve than the compare (disjoint set) problem.

Figure 85: Pre-test, post-test and delayed post-test correct solutions (extension group)

	Q1 Change result unknown	Q2 Compare (matching)	Q3 Collection	Q4 Compare (disjoint set)	Q5: Partition problem	Q6: Bare calculation 21-6 =	Q7: Bare calculation 23-18 =	
Marks	1	1	1	1	1	1	1	
E1	1	1	1	0	1	0	1	5
E2	0	1	1	0	0	1	1	4
E3	0	0	0	1	0	1	1	3
E4	1	1	1	0	0	1	1	5
E5	1	0	1	1	0	0	1	4
E6	1	1	1	0	1	1	1	6
E7	1	1	1	1	0	0	1	5
E8	1	1	0	1	0	1	1	5
Gavril	1	1	1	1	1	0	1	6
Pre-test facility	**78%**	**78%**	**78%**	**56%**	**33%**	**56%**	**100%**	**4.8 (mean)**
E1	1	1	1	1	1	1	1	7
E2	0	1	0	1	1	1	1	5
E3	1	1	1	1	0	1	1	6
E4	1	1	1	1	1	1	1	7
E5	1	0	1	1	0	1	1	5
E6	0	1	1	0	0	1	1	4
E7	0	1	1	1	0	0	0	3
E8	1	1	1	0	1	1	1	6
Gavril	1	1	1	1	1	1	1	7
Post-test Facility	**67%**	**89%**	**89%**	**78%**	**56%**	**89%**	**89%**	**5.6 (mean)**
E1	1	1	0	1	1	1	1	6
E2	1	1	1	1	1	0	1	6
E3	1	1	1	1	0	1	1	6
E4	1	1	1	1	1	1	1	7
E5	0	0	1	0	0	1	1	3
E6	1	1	1	0	1	1	1	6
E7	1	1	1	0	0	1	1	5
E8	1	1	1	1	0	1	1	6
Gavril	1	0	1	1	1	0	1	5
Delayed post-test facility	**89%**	**78%**	**89%**	**67%**	**56%**	**78%**	**100%**	**5.6 (mean)**

The above analysis suggested that there had been overall learning gains in relation to the correct solutions when comparing the post-test and the delayed post-test to the pre-test. The biggest learning gains were for the core group, with bigger learning gains for the extension groups, than for the support group. While the support group attainment was the lowest across all three tests, it is interesting to note that the mean results for the core group (5.7 marks) were slightly better than the mean results for the extension group (5.6 marks) in the post-test, with greater improvements for the core group (a mean of 6.1 marks) than the extension group (a mean of 5.6 marks) for the delayed post-test.

For the whole class, the greatest learning gain was evident for the bare calculation problems. Considering the word problems, the greatest learning gains were evident for the compare (disjoint) set problem and the partition problem, although 40% of the learners could still not solve these problems correctly by the delayed post-test.

Written test attainment using the expanded marking framework

Analysis of the results obtained from applying the simple marking framework suggested that the intervention held promise in relation to learners getting more of the items in the written test correct. If the simple marking framework had not suggested learning gains, then the intervention would have been abandoned; and the approach this topic completely redesigned. The positive results made it appropriate to analyse the written test results in more detail (applying the expanded marking framework).

Reliability testing

All the written tests (both with and without the prompted number line item) were all found to be reliable with Cronbach α scores of above 0.7:

Table 35: Cronbach-alpha scores for reliability of the test (using the expanded marking framework)

Written test	**Cronbach α**		**Outcome**
Pre-test (with number lines) Cycle 3	0.85	> 0.7	Reliable
Post-test (with number lines) Cycle 3	0.92	> 0.7	Reliable
Pre-test (without number lines) Cycle 3	0.85	> 0.7	Reliable
Post-test (without number lines) Cycle 3	0.92	> 0.7	Reliable
Delayed-post-test (without number lines) Cycle 3	0.79	> 0.7	Reliable
Pre-test (without number lines) Cycles 2 and 3	0.86	> 0.7	Reliable
Cycle 2 post-test and Cycle 3 delayed post-test (without number lines) Cycles 2 and 3	0.84	> 0.7	Reliable

With all tests reliable, comparisons were made based on the consistency of the format of the test. When comparing Cycle 2 to Cycle 3 the prompted number line items were excluded. When comparing Cycle 3 over three points in time (pre-test, post-test and delayed post-test) the prompted number line items were excluded. When comparing Cycle 3 over two points in time (pre-test to post-test) the prompted number line items were included. The categorical coding of representations for cycle 3 included the representations drawn in response to the number line prompts.

Results of Cycle 2 attainment from the expanded marking framework (including the prompted number lines)

The expanded marking framework (presented in the Chapter 4) excluded learner activity on the bare calculations (which the simple marking framework showed were the items with the greatest learning gains), and focused only on the word problems (as the over- arching learning goal was expanding the learners example space for additive relations *word problems*). This marking framework included the criterion for the prompted number lines (which were introduced into the cycle 3 tests).

Table 36: Learning gains using expanded marking framework (including prompted number lines)

Written tests in cycle 3 (with prompted number line items)	N	Pre-test mean	(SD)	Post-test mean	(SD)	**Learning gain**
Total test marks (out of 22 marks)	26	8.9	(5.1)	12.5	(6.3)	3.6 marks (+16 percentage points)
Learning Goal 1: Problem solving (out of 10 marks)	26	4.7	(2.4)	6.4	(3.1)	1.7 marks (+17 percentage points)
Learning Goal 2: Representations (out of 12 marks)	26	4.3	(2.9)	6.1	(3.4)	1.8 marks (+15 percentage points)

There were positive learning gains for the whole test as well as for each learning goal in this written test. I calculated the facility of each criterion in the marking framework using a 0-1 scale with 0 being every learner failed to meet this criterion for every item, and 1 being every learner met this criterion for every item in the written test. This analysis revealed that the biggest learning gains were in relation to the multiple-solution word problem, and that the smallest learning gain relating to the prompted number lines.

Table 37: Changes in facility relating to the criteria of the expanded marking framework

Marking criteria	**Facility in pre-test**	**Facility in post-test**	**Difference in facility (percentage points)**
Criterion 1: Correct answer	57%	71%	+14
Criterion 2: Any coherent self-selected model	49%	63%	+15
Criterion 3: Correct partitions (multiple-solution problem)	25%	52%	+27
Criterion 4: Systematic working (multiple solution problem)	19%	48%	+29
Criterion 5: Coherent prompted number sentence	40%	60%	+20
Criterion 6: Coherent prompted number line	21%	27%	+7
Criterion 7: Coherent self-selected model (not a number sentence of number line)	49%	65%	+17

Comparing results of Cycle 3 to Cycle 2 attainment in the written tests (excluding the prompted number lines)

Comparisons between the cycle 3 intervention, and the cycle 2 intervention were appropriate as whether the changes in task design from cycle 2 to cycle 3 led to better learning gains (in terms of word problems) was of interest.

Comparisons between the cycle 2 and cycle 3 could be made by taking into account two factors: firstly the time of year at which the learners wrote the tests; and secondly the items in the written tests had to be marked in an identical manner, which meant that the prompted number line criterion was excluded from the marking framework. I explain each of these considerations in turn.

It is important to realise that cycle 2 took place in the fourth academic term, later in the Grade 2 academic year (in November), whereas the Cycle 3 intervention took place in the second term of the Grade 2 academic year (in April). As such it was expected that attainment in the written tests for Cycle 2 (where pre and post-tests were administered in November), would be higher than attainment in the written test for Cycle 3 (where pre and post-tests were administered in April).

The same items were included in all five written tests (the Cycle 2 pre-test and post-test; and the Cycle 3 pre-test, post-test and delayed post-test). However the format of the Cycle 3 Pre-test and Post-test included a prompt to draw a number line. Cycle 2 Pre-test, Cycle 2 Post-test and Cycle 3 Delayed post-test excluded this prompt. As a result of this change in format, Cycle 2 Pre-test and Post-test are compared to the Cycle 3 Pre-test and Post-test and Delayed post-test, excluding the prompted number line items.

Analysing the whole class attainment

In cycle 2 ($n = 23$), using the marking framework excluding prompted number lines with a possible mark allocation of 18, the mean result in the pre-test was 11.0 marks ($SD = 4.3$) and a mean of 13.9 marks ($SD = 4.1$) was seen in the post-test. In cycle 3 ($n = 26$), using the same marking framework (with 18 possible marks) the mean for the pre-test was 8.1 ($SD = 4.5$); the mean for the post-test was 11.4 ($SD = 5.5$); and the mean for the delayed post-test was 13.4 ($SD = 3.4$). T-tests established that the differences in means were significant.

There were improved mean results for the whole class in Cycle 2 and in Cycle 3 from pre-test to post-test. In Cycle 3 there was a further improvement in the mean result of the delayed post-test with a smaller standard deviations. This implies that there were positive learning gains in both cycles.

Paired samples t-tests comparing the shifts in means for Cycle 3 revealed that these learning gains were significant.

Table 38: Attainment in written tests in cycle 3

						Result of comparison and significance				
Written tests in cycle 3 (without prompted number line items)	N	Mean 1 (marks out of 18)	(SD)	Mean 2 (marks out of 18)	(SD)	**Learning gain**	**Paired samples t-test**	**Degrees of freedom (df)**	**Significance levels**	
Pre-test (mean 1) to post-test (mean 2)	26	8.1	(4.5)	11.4	(5.5)	3 marks (+17 percentage points)	-3.456	25	0.000	<0.05 and therefore significant
Pre-test (mean 1) to delayed post-test (mean 2)	26	8.1	(4.5)	13.4	(3.4)	5 marks (+30 percentage points)	-6.99		0.000	<0.05 and therefore significant
Post-test (mean 1) to delayed post-test (mean 2)	26	11.4	(5.5)	13.4	(3.4)	2 marks (+11 percentage points)	-2.786	25	0.010	<0.05 and therefore significant

Calculations of effect sizes of the shifts in means from pre-test to post-test, revealed that the effects were medium (just below cut off for of 0.7 for large effects (Cohen 1988)) for both cycles:

Table 39: Effect size in cycle 2 and cycle 3 (pre-test to post-test)

		Pre-test		**Post-test**				
	N	**Mean**	**(SD)**	**Mean**	**(SD)**	**Pooled SD**	**Effect size (r)**	
Cycle 3	26	8.08	4.5	11.4	5.5	5.0	0.65	< 0.7 medium
Cycle 2	23	11.1	4.3	13.9	4.1	4.0	0.68	< 0.7 medium

That there were positive shifts in the means in both cycles (coupled with a reduced standard deviation in cycle 3), suggests the intervention holds promise in shifting learning attainment on the written tests. It is encouraging that the positive outcome of the cycle 2 intervention, was replicated in the cycle 3 intervention, and makes the claims about the intervention holding promise, more robust.

Whether the shifts in learning attainment were retained over time was examined for the cycle 3 learners:

Table 40: Effect size in cycle 3 (pre-test to delayed post-test)

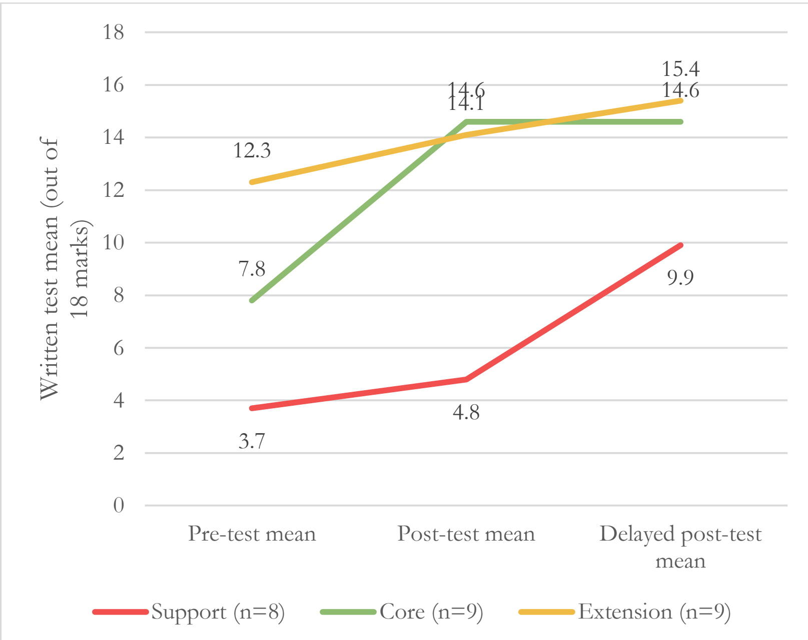
		Pre-test		**Delayed post-test**				
	N	**Mean**	**(SD)**	**Mean**	**(SD)**	**Pooled SD**	**Effect size (r)**	
Cycle 3	26	8.08	4.5	13.4	3.4	4	1.3	> 1 large

In cycle 3 there was better learning gains from pre-test to delayed post-test (compared to pre-test to post-test) based on bigger shift in mean (and reduced standard deviation) from pre-test to delayed post-test, providing evidence that the learning gains were retained and supported due to maturation of the learners and continued teaching and learning by the normal classroom teacher. The shifts from post-test to delayed post-test cannot be solely attributed to the cycle 3 intervention however. The role of continued teaching within the context of the broader development project, was most likely a contributing factor to the further improvements in delayed post-test attainment. The large effect sizes suggest that cycle 3 teaching intervention design holds promise in shifting overall learning attainment for the whole class in this local context, and that when teaching is supported within a broader development project, that such learning gains may be retained.

Learning gains for each learning group

As the cycle 3 whole class was broken into three learning group where small group teaching took place in these groups (as well as in whole class teaching sessions), it is of interest to compare the shifts in means for each learning group:

Figure 86: Shifts in mean test attainment (cycle 2) by ability group



The extension group mean improved by +2 marks (+11 percentage points) from pre-test (mean = 12.3, SD = 3.7) to post-test (mean = 14.1, SD = 3.05), and further improvements were evident by the delayed post-test (mean = 15.4, SD = 2.4). The declines in standard deviation were a positive indication of learning gains for this group (as the improved results were coupled with a tighter distribution). It is possible that ceiling effects on the assessment instruments, meant that learning gains for the extension group were dampened (this is explored in relation to a particular learner case from the extension group were minimal learning gains were evident).

The core group mean improved by +7 marks (+39 percentage points) from the pre-test (mean = 7.8, SD = 3.0) to the post-test (mean = 14.6, SD = 2.6) and retained this gain by the delayed post-test (mean = 14.6, SD = 2.3). Similarly the declining standard deviation was a further positive indication of learning gains for this group. As with using the simple marking framework, the mean results for the extension group (14.1 marks) was slightly lower than the mean results for the core group (14.6 marks) in the post-test. However, in contrast to the simple marking framework, when using the expanded marking framework the mean results for the extension group (15.4 marks) were higher than the mean results of the core group (14.6) in the delayed post-test. The expanded marking framework differentiated in greater detail between groups – with far more value placed on use of representations, than in the simple marking framework. The key gains are evident for the core group, suggesting that the intervention was most successful in 'raising the middle' of the class.

The support group mean only improved slightly by + 1 mark (+5 percentage points) from pre-test (mean = 3.7, SD = 1.4) to post-test (mean = 4.8, SD = 4.2), and then a larger improvement of + 6 marks (+18 percentage points) was evident by the delayed post-test (mean = 9.9, SD = 2.7). The slight increase in mean, coupled with the larger standard deviation supports the inference that the intervention was not as successful for the support group. Given the fairly substantial increase by the delayed post-test, it is conjectured at the time of the intervention the support group learners lacked several foundational competencies (which were assumed in the HLT starting point). Given the substantial increase in support group learner attainment by the delayed post-test, it is assumed that these foundational competencies were more secure for these learners, by the end of their Grade 2 year (when the delayed post-test was written). The improvements by the delayed post-test are attributed to the continued mathematical development that took place for these learners through the support of their normal classroom teacher (who in turn was further supported within the context of the broader development project).

Facility of items in the written test

Calculations of the 'facility' of items in the written test, were conducted to compare learner attainment of particular items and to compare test performance from cycle 2 to cycle 3. This made use of the expanded marking framework.

Facilities are given on an item by item basis for the written test across Cycle 2 and Cycle 3.

Table 41: Item level gains across cycle 2 to cycle 3 (expanded marking framework)

	Cycle 2 (n=23)			Cycle 3 (n=26)		
	Pre-test	**Post-test**	**Difference in facility (cycle 2)**	**Pre-test**	**Delayed post-test**	**Difference in facility (cycle 3)**
Question 1: Change decrease	79%	88%	+9	55%	85%	+30
Question 2: Compare (matching)	68%	88%	+20	55%	82%	+27
Question 3: Collection	57%	73%	+17	48%	70%	+22
Question 4: Compare (disjoint set)	52%	57%	+5	33%	72%	+38
Question 5: Partition	21%	41%	+20	11%	27%	+16

Question-by-question analysis of the changes in facility for each cycle is of interest.

The earlier timing starting point for cycle 3 is evident with the pre-test facility for cycle 3 on all questions being lower than the facility for cycle 2. A second possible contributing factor to this finding is the higher proportion of learners identified as having special educational needs (LSEN) by the school (4 learners in the Cycle 2 in comparison to 8 learners in the Cycle 3 class).

There were improvements in the difference in facility from the pre-test to the post-test for all of the single solution word problem in Cycle 3, in comparison to Cycle 2. For all of the word problem question items (except the partition problem) there as a higher difference in facility from pre-test to post-test in Cycle 3, than in Cycle 2. For example, the difference in facility for question 1: Change decrease was +30 in Cycle 3, and + 9 in Cycle 2.

Analysing representations in written tests

This section presents the findings from the categorical coding of the representations used by learners in their written tests.

Chi-squared testing

The coding of representations in the written tests in cycle 3 was conducted for pre-tests, post-tests (where a prompt for drawing a number line was included) and delayed post-test (where a prompt to include a number line representation was excluded). For each category the frequencies of these categories across the whole class (and/or for each attainment group) was calculated for the pre-test scripts and the post-test scripts. Each of these were presented in contingency tables of actual distributions of representations by category; which could then be compared to the expected distribution of representations by category (if there was the same probability of the categories being distributed identically in both pre-test and post-test). A Chi-squared test was then run to compare the actual distribution to the expected distribution. The Chi-squared test on the actual and expected values, using degrees of freedom (calculated using df = (rows- 1)(columns – 1), provide the probability that the change of distribution was due to change, (and $p<0.05$ was used as a threshold for the significance of this change). In the analysis below, I simply report on the p-value from the chi-squared test and whether or not this is significant.

Categorical coding of representations

Self-selected representations were distinguished from prompted representations. Then each representation was considered in relation to its problem text, and classified as coherent or incoherent. Additionally each representation was coded as being either a group, line or syntax model with the various sub categories related to each type of representation repeated in table 39.

Table 42: Categories for coding representations

Group				Line				Syntax			
Iconic		Indexical		Structured number line		Empty number line		Whole-part-part diagram		Number sentence	
Ones	Groups	Ones	Groups	Hops in ones	Jumps in groups	Hops in ones	Jumps in groups	Unscaled	Scaled	Standard form	Alternative form

Findings: Categorical coding of representations

Findings: Coherent vs incoherent models

There was an increase in coherent models from pre-test to post-test which was retained in the delayed post-test.

Table 43: Actual distribution of coherent models vs incoherent models (pre-test to post-test)

	Pre-test	Post-test	Total
Coherent models (total)	121	172	293
Incoherent models (total)	106	87	193
Total	227	259	486
Proportion of representations that were coherent	53%	66%	

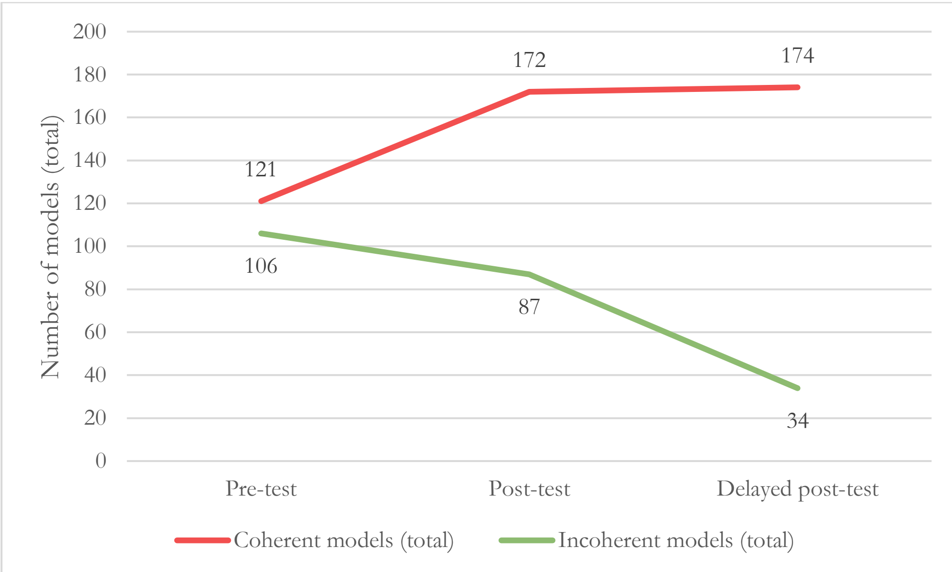
There was a decline in the proportion of incoherent models from pre-test to post-test. A Chi-squared test comparing the actual frequency of each category, to the expected frequency (assuming no difference in the distribution by category from pre-test to post-test), showed that the change was significant ($p = 1.27 \times 10^{-10} < 0.05$), and not a result of chance.

Table 44: Actual distribution of coherent models vs incoherent models (pre-test to delayed post-test)

	Pre-test	Delayed post-test	Total
Coherent models (total)	121	174	295
Incoherent models (total)	106	34	140
Total	227	208	435
Proportion of representations that were coherent	53%	84%	

There was a further increase in proportions of coherent models in the delayed post-test. A Chi-squared test on this change in categories was found to be significant ($p = 4.3 \times 10^{-4} < 0.05$).

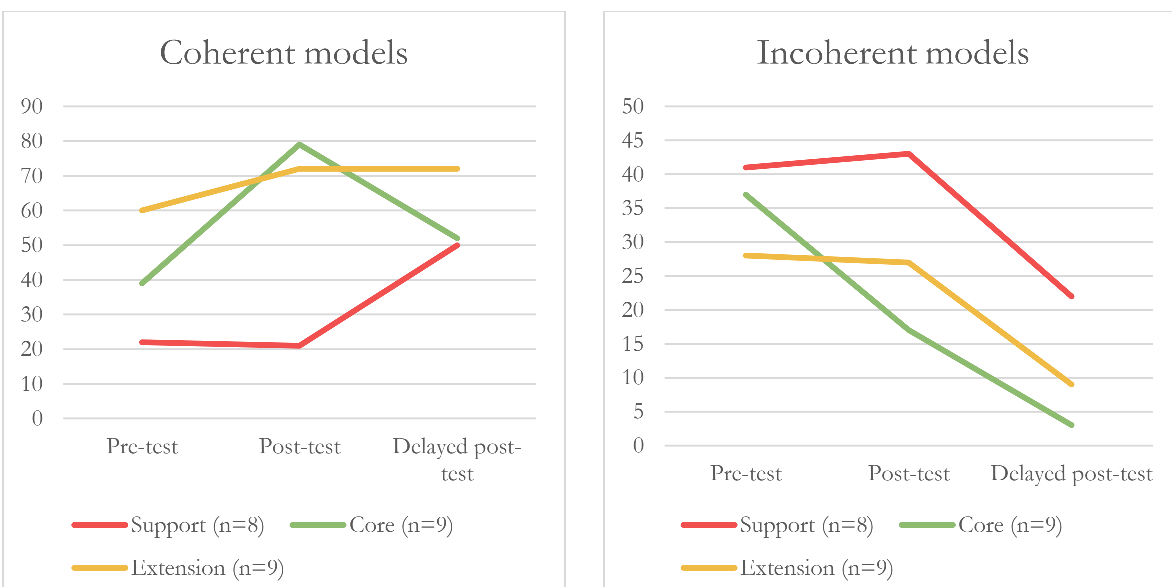
Figure 87: Coherent vs incoherent models in cycle 3



The further decline in incoherent representations in the delayed post-test may be a result of fewer prompts for representations (as the learners were not prompted to produce number lines in this test).

Considering the three learning groups, there was also evidence of increases in coherent models, and decreases in incoherent models over time (except for the support group, where there was a slight decline in coherent models from pre-test to post-test).

Figure 88: Coherent and incoherent models by ability group over time



The core group showed a steep increase in the number of coherent representations from pre-test to post-test. This declined in the delayed post-test (where they were not prompted to draw a number line). Similarly the extension group showed an increase in coherent representations and a decline in incoherent representations. The declines in the numbers of coherent models in the delayed post-test was a result of fewer prompts for representations (as this test excluded the number line prompts). As such it is informative to compare the proportions of coherent model (in relation to the total number of representations produced for each group):

Table 45: Percentage of representations which were coherent for each group

	Pre-test	Post-test	Delayed post-test
Support group	35%	33%	69%
Core group	51%	82%	95%
Extension group	68%	73%	89%

The support group was of interest as at the post-test stage this group were producing slightly fewer coherent models and more incoherent models (compared to the pre-test). This provides further evidence that the intervention held less promise for the support group learners, as it appeared that by the endpoint of the intervention they were more confused (and producing more incoherent representations) than prior to it. This however was corrected over time, with a substantial increase in coherent models, and decline in incoherent models evident for the support group in the delayed post-test. There were improvements in the proportion of coherent models produced over time for both the core and extension groups. However the core group showed greater improvements than the extension group over time, further supporting the inference that the intervention was most successful in 'raising the middle' of the class

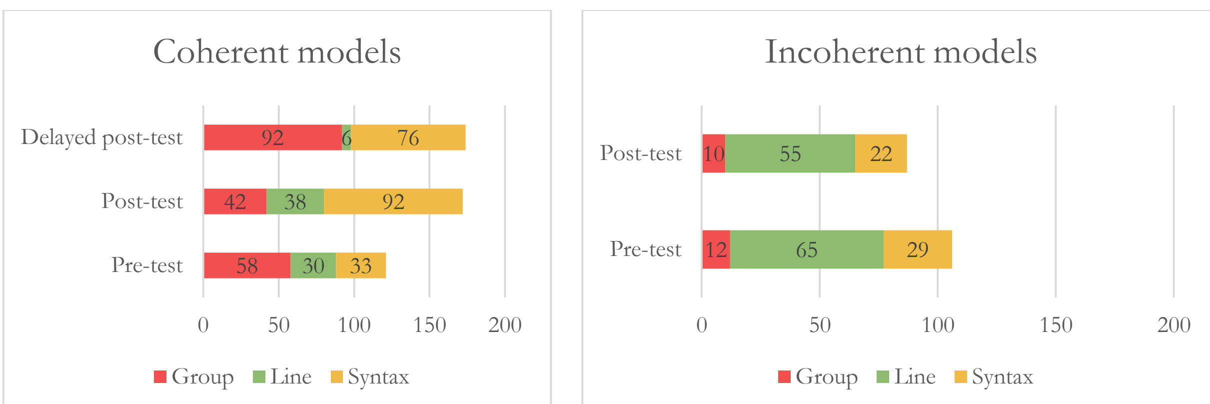
Findings: Types of representations (line, group and syntax models)

Examining the types of incoherent representations (considering group, line and syntax models) revealed that the core and extension groups were not yet secure in using the number line representations. The absence of the number line prompt in the delayed post-test means that their fluency in using this representation was not assessed at that endpoint. Of interest also is that the Grade 2 learners in this study did not self-select to use the line model in the post-test. This aligns with Feature 4.2 Secure use of a particular representations takes time and, over time, representations should be reified to become cognitive tools. Prior South African empirical findings relating to interventions of short duration at Grade 4 level reveal a reluctance to use the line model (Tshesane 2014; Wasserman 2015), which is consistent with the observation that secure use of the number line representations takes time to develop.

It is surprising that the support group showed an increase in incoherent representations from pre-test to post-test. However, this may be understood in terms of the analysis provided above that the learners in this group were below the assumed starting point for the HLT. It is conjectured that this group lacked the foundational competencies necessary to meaningfully use a range a representations, and while in the pre-test they did not offer solutions, in the pre-test they attempted to use a wider range of representations, which they were not yet able to use coherently. By the delayed post-test, it is conjectured that more of the foundational fluencies were secure for the support group, and therefore these support group learners were producing more coherent models at this point in time. This finding raises a further research question of whether improving foundational competencies are a sufficient condition for improving attainment in the written tests; and the extent to which the narrative approach was a necessary condition for this improvement. This design experiment has not investigated the efficacy of the narrative approach in comparison to other (or controlled) interventions. Such research would require a randomised control trial which is not part of the design experiment method. What this study has revealed is that the design intervention making use of a narrative approach holds promise for shifting the middle of a class, and is less effective for the support group learners who lacked the competencies assumed as the starting point.

Following on from Theoretical feature 4.4 A learning-teaching trajectory from counting to calculating which made reference to increasingly structured representations involving line, group and syntax models was adopted it was anticipated that there would be a shift away from group models towards more use of line and syntax models.

Figure 89: Coherent and incoherent models (group, line and syntax) by ability group



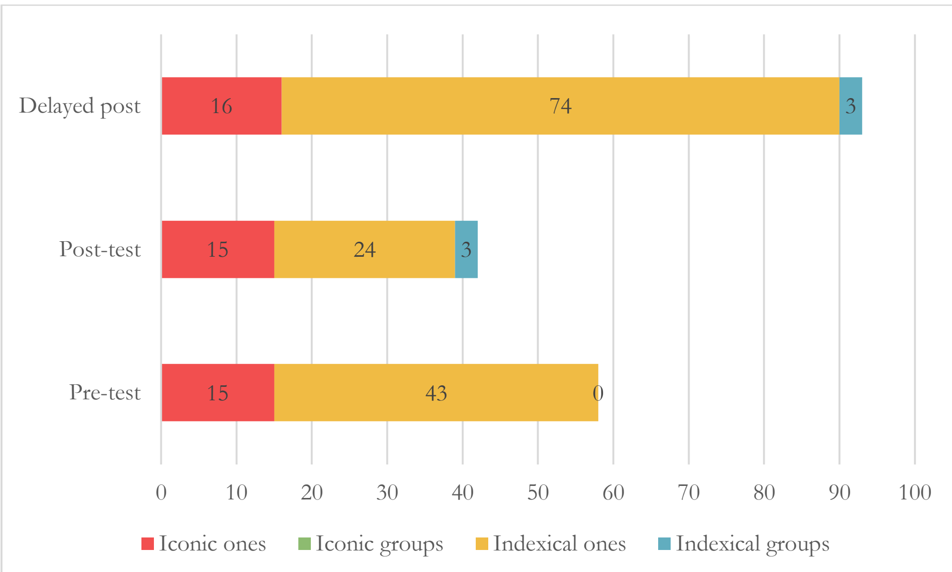
A Chi-squared test on the increase in coherent models by type of representation (group, line and syntax) from pre-test, to post-test was found to be significant ($p= 9.8\times 10^{-12} <0.05$). Similarly, the Chi-squared test on the decrease of incoherent models by type of representation was significant ($p = p= 1.0\times 10^{-6} <0.05$).

There was far more coherent use of syntax models (92 representations) in the post-test, compared to the pre-test (33 representations). The observation made above, that the line model was not yet secure for this class, was confirmed with this data, which showed that 65 incoherent line models were produced in the pre-test. This declined slightly with the post-test (with a modest increase in coherent line models and decrease in incoherent line models).

The sub-categories within each representation type were used to measure shifts <u>within</u> coherent representations, towards increased structure. For group models whether learners were acting on ones or acting on groups; for line models whether learners were hopping in ones, or jumping using more than one; and for syntax models whether the whole-part-part diagrams were scaled or unscaled and whether number sentences were in standard or alternative form (as defined by Carey (1991)) were of interest.

Findings: Within group model representations

Figure 90: Acting on ones versus acting on groups within coherent group models

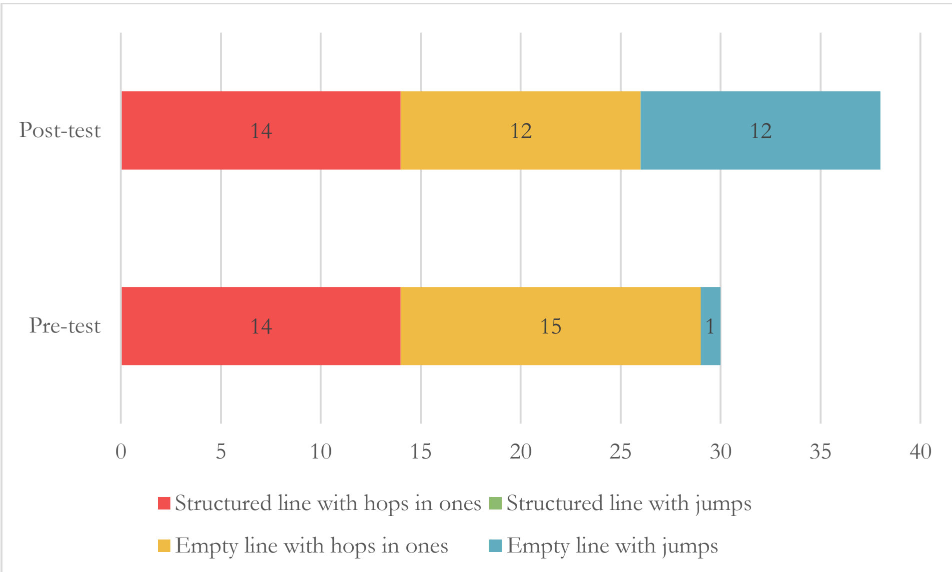


Considering the coherent group models, there was a decline in how many indexical representations were used, with a small number (3) of these indexical representations depicting actions on groups rather than actions on ones. There was not enough evidence of use of indexical group models in the post-test to say that the shift from ones to groups was significant (3 examples is below the threshold for the Chi-squared assumption), and is too small to claim any significance.

It is of theoretical interest that there were no empirical evidence of learners acting on groups within iconic representations, although there was some evidence of learners acting on groups for 3 indexical representations. This may suggest (as a very tentative conjecture) that indexical representations offer more opportunity to structure the representations to allow for group-wise actions. However the very small number of indexical representations making use of group-wise actions (only 3) also suggest that group-wise actions were not yet secure which aligns with Feature 4.2 Secure use of a particular representations takes time and, over time, representations should be reified to become cognitive tools. This class clearly needed more experience of shifting their actions from working with ones, to working with groups. Such shifts may be more evident in problem situations which make use of a bigger number range (which necessitates group-wise actions making use of the tens structure). In future intervention cycles, the HLT should be revised to include more examples making use of a bigger number range, so that group-wise actions may be brought more into focus.

Findings: Within line model representations

Figure 91: 'Hops in ones' versus 'jumps in groups' on coherent line models



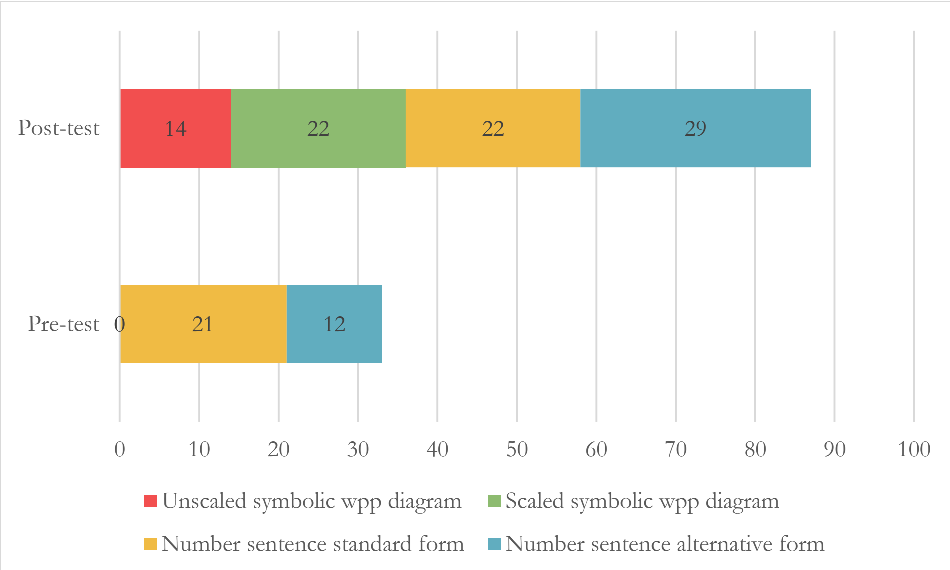
Considering the coherent line models, there was an increase in empty number line representations, with greater proportions of the empty number lines making use of jumps of more than one (rather than hops in ones). As the empty number line with jumps in the pre-test was so small that it violates assumption of the Chi-Squared test, I cannot claim that this shift was significant.

It is of interest that at this Grade 2 level, the line models continue to make use of structured number lines. Empty number lines feature more in the post-test (with 24 coherent empty number lines, compared to 16 in the pre-test). Compared to the group models (where there were only 3 examples of coherent group models using group actions), there is more evidence of group-wise actions being used on the line models (with 12 empty number lines using group-wise jumps). The absence of group-wise action on structured number lines is of interest, as this may suggest a developmental progression that learners who work on structured lines are more likely to continue to act in ones, and that with the shift to empty line models, there is a greater likelihood of group-wise jumps rather than hops in ones.

Findings: Within syntax model representations

Considering the coherent syntax models, there was an increase in the number of coherent syntax models.

Figure 92: 'Scaled' versus 'unscaled' whole-part-part diagrams, and 'standard' versus 'alternative' forms of number sentences in coherent syntax models



As anticipated in the HLT, the symbolic whole-part-part diagrams were completely absent in the pre-test (learners were not at all familiar with this diagram), but the whole-part-part diagram was adopted as a coherent model to express additive relations in 36 examples of learner work in the post-test. In 14 cases of the whole-part-part diagram the learners were not yet attending to the measurement scale of the two parts; but they were indicating the whole and two part structure. In 22 whole-part-part diagrams the diagrams were scaled suggesting awareness of this measurement aspect. As there were no examples of whole-part-part diagrams, the Chi-squared test could not be undertaken on this data. There were expansions in the range of syntax models and increased structuring within the syntax models.

As anticipated in terms of the HLT, where the starting point assumed the dominant use of number sentences in the standard form, by the post-test there was an increase in the coherent use of alternative forms for the number sentences. The Chi-squared test on this data revealed that the increase in use of alternative forms of number sentences was significant ($p= 8.1\times 10^{-8} <0.05$).

Findings: Calculation actions

As explained Chapter 4, each coherent representation was also coded for evidence of a calculation action where take-away, partition and 1:1 matching or difference actions were contrasted.

Table 46: Categorical coding of calculation actions

Take-away actions	Partitioning actions	1:1 matching / difference actions
Take-away: Crossing out on a group model	Partition: Arranging right left partition in a group model	Matching: Arranging 1:1 matching in a group model
Take away: Erasing and redrawing on a group model	Partition: Using colour to distinguish part in a group model	Difference: Hops or jumps between two numbers in a line model

Take away: Hops or jumps back from start on line model	Partition: Enclosing part in a group model	
	Partition: Drawing contrasting shapes in a group model	
	Partition: Symbolic whole-part-part syntax model	

It was expected that the assumed starting point would see take-away action dominating the learners calculation action across the test items; and that teaching intervention would be necessary for learners to bring to bring partitioning or 1:1 matching actions.

Table 47: Calculation actions comparing representations in pre-test to post-test

Calculation actions	Pre-test	Post-test
Take-away action	53	35
Partition action	12	26
Difference/ matching action	7	26
	72	87

As expected take-away calculation action dominated the pre-test representations. By the post-test there was more use of partitioning and difference/matching actions.

Looking within each type of calculation action, to consider changes within the calculation from pre-test to post-test was then conducted. As there were not enough examples of each sub-type of calculation action, the Chi-Squared test could not be conducted on this data.

Findings: Within take-away actions

As expected, take-away actions using group models where there was crossing out of objects, or movement back from a starting number on a number line dominated the pre-test responses. These take-away actions continued to be observed in the post-test (although this action was used less frequently):

Table 48: Types of take-away actions comparing representations in pre-test to post-test

Contrasting actions: Within take-away action	Pre-test	Post-test
Take-away: Crossing out on a group model	26	19
Take away: Erasing and redrawing on a group model	2	1
Take away: Hops or jumps back from start on line model	25	15
Total	53	35

Findings: Within partitioning actions

The main change in the types of partitioning actions evident in the written tests was the inclusion of the symbolic whole-part-part diagram. This representation was introduced to learners through Main Task 2: Partitioning problem, and extended as a representation to be used in the other main tasks. Most of the other group model representations from which partitioning actions were inferred continued to feature in the post-test data.

Table 49: Types of partition actions comparing representations in pre-test to post-test

Contrasting actions: Within partition action	Pre-test	Post-test
Partition: Arranging right left partition in a group model	5	4
Partition: Using colour to distinguish part in a group model	3	1
Partition: Enclosing part in a group model	3	0
Partition: Drawing contrasting shapes in a group model	1	1
Partition: Symbolic whole-part-part syntax model	0	20
Total	12	26

Findings: Within 1:1 matching / difference actions

Introducing a 1:1 matching action into the static problem situations for compare type problems was considered an important calculation action. The analysis of the types of matching/ difference actions reveals that these actions were used more frequently by the post-test.

Table 50: Types of 1:1 matching/ difference actions

Contrasting actions: Within matching/difference action	Pre-test	Post-test
Matching: Arranging 1:1 matching in a group model	3	10
Difference: Hops or jumps between two numbers in a line model	4	16
Total	7	26

By the post-test learners were no longer mostly using take-away actions, but also made use of partitioning and 1:1 matching/difference actions. The learning intervention seemed to be successful in supporting learners to be more flexible in their choice of calculation action.

Qualitative analysis of whole-class learner activity on the story-telling task

This section focuses on evidence of learning relating to Learning Goal 3: Story telling. It reports first on the last lesson in the third cycle intervention where a class of 27[24] Grade 2 learners were engaged in a 'storytelling task'.

[24] Two learners were absent on this day, so the class of 30 learners, was reduced to 28 learners. Parental permission was not obtained from one of the learners, and as such his data was excluded.

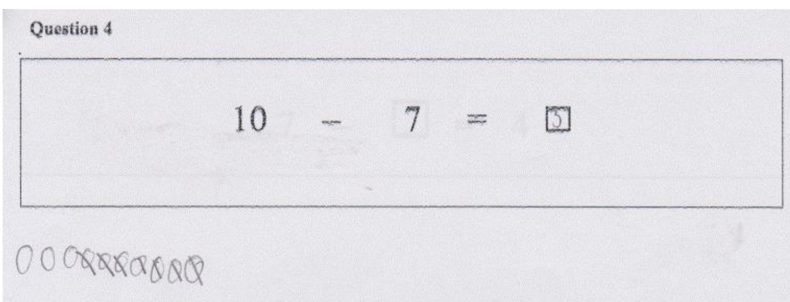
Evidence of learner activity on story-telling tasks prior to the intervention

In prior interventions (cycle 1 and cycle 2) I had observed that most learners were completely unfamiliar with using story-telling to pose questions or to explain solutions to word problems prior to the intervention. They required teacher modelling to understand what was expected of them for these tasks. When they told stories, they tended to be of the change increase or change decrease problem types. The case study learner from the support group's response to the story-telling task in the pre-interview was typical of the learners' unfamiliarity with these kinds of tasks. I present a vignette of Mpho's activity on a story-telling task in the pre-interview to illustrate the observation that most learners were not familiar with using story-telling to pose a word problem or using story-telling to explain a word problem situation.

Vignette 1: Mpho learning to tell stories during the pre-interview

In the pre-interview, it was clear that Mpho conceptualised subtraction using a change context, with a take-away model. When asked to tell a story for 10 – 7 = …:, he chose to first calculate the answer to this by drawing 10 lines, crossing out 7 of the lines, and writing down '3' as his solution.

Figure 93: Mpho's drawing to calculate 10 – 7 = …(pre-interview)



When prompted again to tell a story for this number sentence he offered: 'ten minus seven equals three'. His 'story' was simply a reading of the number sentence which he had just solved. I assumed that Mpho had never had to pose word problems before, and so offered him an example of a story, where he was able to choose one of two story contexts, before asking him to tell his own story:

> T: Let me give you an example of a story and then maybe you can also tell one. Aaaah, what do you like? Do you like balls, or cars?
> Mpho: Balls
> T: Balls, ok, then there were 10 balls. And something happened to them. 7 of the balls, went away. Okay, so shall we kick them over the fence? 7 of the balls were kicked over the fence. How many were left?
> Mpho: [Inaudible]
> T: So we had 10 balls, we've worked it out. There were 10 balls, 7 of them were kicked over the fence…
> Mpho: 3
> T: 3 are left.

Here is it apparent that for Mpho when faced with an open number sentence, his inclination was to solve the problem. While I was trying to bring to his attention the story structure: that there was a starting value (10), then some change (something happened) resulting in a new state; his attention seems to be on the remaining balls. However when asked to tell his own story he was able to keep the numbers invariant (working with 10 and 7), and choose a new context:

> T: Okay now can you tell me a story? I told you a story about balls.
> Mpho: Mmmm. There was 10 cars. 7 was broken. How much is left?

He selected the alternative context (cars not balls) which I had offered to him previously, and introduced an action or change in state of being 'broken'. He was able to pose a question, using a similar phrasing to what I had used, but changing this slightly to: 'how much is left' reflecting a common grammatical error for second language speakers of English (in the appropriate use of the verb to be (is/was/were) and the distinction between much/many). This revealed that he could change the problem context, although he mixed two problem types in his phrases. His story was a mixture of the collection type problem where two states are in focus: broken and not broken, however his question referred to a change type problem context where there had a change which meant some things remained or where left. Notwithstanding these errors, Mpho was on task in terms of creating his own story appropriate for the number sentence 10 – 7 =…

What is notable about this vignette of Mpho in the pre-interview, is that he required teacher support and modelling before he was able to tell a story given a number sentence. It was for this reason (together with the limited writing ability of Grade 2 learners, which would make telling a story in written form take a long time) that story telling tasks were excluded from the written test.

Findings: Whole class activity on story telling task

To examine the whole class activity on the story telling class I first focus on the *collective example space* (the example space located in a particular classroom at a particular time (Watson and Mason 2005), before quantifying the variety of additive word problem types which were brought to mind from the individual learners' *local personal example spaces* (examples which were triggered by current tasks in the environment or recent experiences, which in this case related to Main Task 7: Learners' generating examples). I provide two contrasting vignettes to illustrate the local personal example spaces of individual learners. I provide further details of personal example spaces relating to the case study learners and their story-telling activity before, during and after the intervention in Chapter 7.

Collective example space

Considering the collective example space of all the stories generated in response to Main Task 7: Learners' generating examples I focus first on the connection between the stories in relation to the whole-part-part diagram. I then reflect on the coherence of the stories generated and how many coherent stories each learner was able to write before discussing the nature and extent of two common misconceptions evident in the learners' stories. In so doing I consider the evidence of learning (inferred from coherent narration of stories), before turning to the evidence from learners where there was no evidence of learning (inferred from incoherent narratives).

Coherent story telling

The majority of learners (22) were able to draw a correct whole-part-part diagram, and generate at least one coherent word problem for the additive relation in this diagram. A minority of learners (5) were not able to do this (the remaining 2 learners in the class were absent). Amongst these 22 learners, 15 of these learners generated three coherent stories; 4 generated two coherent stories, and 3 learners generated one coherent story. The task design was judged to be appropriate for this class as the majority of learners completed it as expected.

The 57 connected and coherent stories were mapped as follows to the conventional problem types:

Table 51: Types of stories

Problem type	Frequency
Change (increase)	16
Change (decrease)	32
Collection	6
Compare	3
Total	57

The change problem types were the most commonly told stories, with the change (decrease) problem type being more common that the change (increase) problem type. It is notable that in the collective example space of the class activity on this task, all four problem types featured. Each connected and coherent story was then mapped further to the relevant sub classes. The following four tables present the frequency of stories for each sub-type. Up to four typical examples of the learner activity are included.

Table 52: Sub-types for change increase stories

Change increase	Frequency	Examples[25]
Result unknown	3	I have 1 flower. I pick 4 more. How many do I have? I have 7 frogs I found 3 more frogs How many frogs do I have altogether? I have 7 lino (sic)[26] and my mom give me 2 how many I have?
Change unknown	10	I have 2 cats. My mom bring more. I have 6 cats now. How many cats my mom bring? I have 6 lions how many more lions do I nid to get 10 lions? I have 2 car my dad bring more I have 6 car now. How many car my Dad bring? I have 10 pencils my mommy gave more pencil I have 15 pencil. How many more do I have?
Change increase situation but no question posed	3	I have 5 stickers. I found 3 stickers. I have 8 stickers. I have 6 books my mom bring 1 more. Do I have [incomplete] I have 6 books I need to have 4 more books

[26] Possibly a mis-spelling of 'lion'.

For the change (increase) problem type, examples of both result unknown and change unknown problems featured in the collective example space. No start unknown problems were told, which suggested that a next teaching move would be to include change increase (start unknown) problems into problem solving tasks. Difficulties with posing questions in English were evident as there were examples of change (increase) stories which did not include a question, or where there was evidence of difficulties when posing a question. This suggested that additional support was required for learners to fluently pose questions in English.

Table 53: Sub classes for change decrease stories

Change decrease	Frequency	Examples
Result unknown	22	I have 9 snakes 5 ran away How many Do I have? I have 9 apples. I eat 2. How many I have? I have 15 ducks. I gave 10 ducks to my friend. How may ducks do I have? I have 4 butterflies 3 butterflies fly a Way how many do I have?
Result unknown, but 'more' is incorrectly inserted into the question	8	I have 6 bows 3 bows get lost. How many more bows do I have now? I have 10 bols. 8 balls bounced away. How many more do I have. I have 10 puzzle. 8 puzzle get lost. How many more do I have? I have 13 sweets. I eat 3 sweets. How many more sweets are left?
Change unknown	1	I have 13 friends. Some of my friends ran away. I have 3 friends left. How many more friends ran away?
Change decrease situation but difficulty with posing question	1	I have 10 dog. 5 dog run away. dogs 5 left? Are left? How many are left?

The change (decrease) result unknown problem type dominated the collective example space. Only one learner told a story of the change (decrease) change unknown type. This is not surprising considering that the reference examples used in the intervention all were of the result unknown problem type. However this does suggest that in future design cycles, a possible new teaching move would be to include change (decrease) change unknown, and change (decrease) start unknown problems as reference examples.

Examining the stories of this problem type, it was clear that some learners attempted to fulfil the task constraint of using the word 'more' by inappropriately inserting the term into the change decrease situation. This suggested that learners were not yet secure with using 'more' in two ways: firstly to denote an action of increasing in change increase situations; and to secondly to denote a comparison in compare situations. How to use 'more' appropriately in these two ways, would require additional teacher intervention in future interventions.

Table 54: Collection stories

Collection	Frequency	Examples
Part unknown (contrasting two states)	2	I have 9 chalk. 5 brake. How many do I have? I have 14 pens 10 are red sum are Blue How many are Blue?
Part unknown (1 part missing)	4	I have 6 marbles and 4 marbles are missing. How many marbles are left? I have 6 bols (sic) and 4 bols were missing. How many are left? I have 18 tops. 8 are missing. How many do I have left? I have 6 cars. 4 cars were missing. How mane (sic) cars ore (sic) left?

There were very few stories of the collection problem type. This problem type is defined as relating to a static situation. Recall that the collection problem type was not included as a reference example in the intervention design. It is therefore not surprising that only a few learners made use of the problem type in their personal example spaces.

In classifying the stories, it was clear that there is potential overlap between 'change' problems and 'collection' problems. The change is expected to take place over time, and the duration of what is considered a reasonable time lapse is brought into question. The statement 'Something is missing from a set' describes a static situation in the present. However, if something is missing now, it may have been taken or lost or removed in the past. The significance of the change action in relation to a time frame was not obvious from definitions of the problem types. It was in considering whether to classify 'I have 6 cars. 4 cars were missing. How mane cars ore left?' as a change problem, or as a collection problem, that this aspect arose. For the purpose of this analysis, when an element of a set was described as missing, this was considered to be a statement relating to a static situation and two attributes were being contrasted: missing and present, as such this was classified as a collection problem. These 'missing part' problems were separated into a sub-class of their own within the collection stories.

This observation is distinct from the observation by Carpenter and Moser (1984) that there are overlaps between Change and Compare problems: 'Equalise problems share characteristics of both Change and Compare problems. There is implied action on one of two sets, but a comparison is also invoked' (p56). In the observation made above as a result of the empirical data from the learner generated examples in Cycle 3, there are shared characteristics between the change and collection problems – and defining which problem type a particular word problems belongs to required careful consideration of the choice of words and structure of the problem. Using the static-dynamic distinction a problem is a change problem if there is a dynamic action involved. However in the static situation of parts of a whole being missing, there can be an implied action (in the past or future) which is brought to mind, and the notion of static versus dynamic stories then depends on the temporal frame being considered. In this way the distinctions between the problem types come to be somewhat blurred. This is perhaps not surprising given that all of the problem types refer to different ways of expressing the same additive relationship, and how the relationship is described can relate to a current state, or the recent past or near future, but may also invoke an implied change in the more distant past or future.

There were very few collection stories generated in the collective example space, suggesting that collection problems should be included in future teaching plans. All of the collection problems were of the part unknown sub type.

Table 55: Compare stories

Compare	Frequency	Examples
Compare (difference unknown)	3	How many more than is 10 than 5? I have 9 pen and my friend have 7. How many more do I have Than my [incomplete] I have 16 pencils and my friend have 10 pencils. How many pencils do my friend need?

There were very few compare problem types all of which were in the part unknown sub type. The reference example for the compare problem included in the intervention was of the part unknown sub-type. That very few learners brought a compare problem to mind, even when prompted to make use of the term 'more than' in their story, suggests that the learners in this class were not yet seeing this problem type as a dimension of possible variation within their personal example space for additive relations.

Incoherent story telling

There were four learners (all from the support group) who were not yet able to accurately draw a whole-part-part diagram and tell a story which cohered with the number triple that they were given: S3 who had missed school and returned on the day of the story-telling task; S6 who did not finish the story-telling task; S7 who had problems with coherently completing the syntax models for her number triple; and S6 who required teacher intervention to record his thinking.

Learner S3 was in the support group, and had been absent for four consecutive days just prior to the final lesson of the intervention. In his attempt on Main Task 7, he told and wrote two incoherent change decrease stories: 'I have 6 cats and 1 cat ran away. 1 cat is missing', 'I have 5 lions. 1 lion is missing? How many is missing?'. The first story offered an incoherent number triple 6-1-1 and neglected to pose a question, and the second story neglected to pose an appropriate question.

Learner S6 was able to correctly complete a whole-part-part diagram using the number tripe 5-3-2, and also completed a family of number sentences for this relationship correctly. She did not finish the story telling task writing only 'I have 5 cards 3'.

Learner S7 attempted to work with the number tripe 8-5-3, however she wrote the 8 as the whole, 5 as the bigger part, and completed the other part with 7. Her family of number sentences included similarly incoherent offerings: 8 = 5 + [13], 8 = 2 + [5], and the two correct number sentences of 8 – 5 = [3] and 8 – 3 = [5]. The stories S7 offered were: 'I have 8 stickers. I get lost'; 'I have 5 cats. 2 cats ran away'; and 'I have 8 books 3 books get lost. How many more books are There?'. Her last example was a change decrease problem type and may be considered to cohere with her two correct number sentences for the 8-5-3 number triple. This story not included in the 57 coherent stories of the collective example space, as S7 was unable to coherent complete the whole-part-part diagram and family of number sentences. There was some evidence of her connecting one of her stories to the two correct number sentences she produced.

When working on Main Task 7 learner S4 worked with the number triple 5-4-1, however he positioned 5 as the whole, 4 as a part, and did not complete his whole-part-part diagram to include 1 as the other part. His family of number sentences as incoherent as he offered 5 = 4 + [2], 5 = [4] + 4, 5 = 4 – 5, and 5 – [4] = 4. Aware that S4 struggled to focus generally and with writing tasks in particular, I spent some time working orally with him during the lesson. After being asked to show me 5 beads (which he counted in ones), and asked to make the parts in his diagram, he moved 4 beads and correctly said that the other part would be 1. He said he wanted to tell a story about a dog, and asked me how to write dog. I wrote 'dog' in his book, and moved on to work with another child in the class. While I was away S4 wrote this story: 'I have 5 dog my mom bring 4 dog my mom bring did my mom bring I have howe'. When I returned to work with S4 again, it appeared that he was trying to tell a change increase story: starting with 5 dogs and 4 more being brought by his mother, but was not able to maintain focus on his narrative to write it down. I offered him the bead string to model his story. He counted in ones to reach five, then counted in ones for another four. In total he now had 9 beads. I asked him how many altogether and he counted in ones (counting all) to reach 9. When asked to tell me his story he orally re-stated his problem as follows: My mom bring 5 dogs. My daddy did bring 4 dogs. How many are there left?'. (which I wrote down for him in his book). This narration revealed that S4 he could tell his change increase story giving the start and the change, but his narrative broken down when posing the question. Rather than ask 'how many are there altogether', he asked 'how many are there left?' He seemed to have a conflict between his change increase story (starting with 5 and a change of 4 more dogs), and a change decrease question (starting with 5 and wondering how many are left) which would cohere with this whole-part-part diagram and number triple 5-4-1. S4 did not yet seem able to pose a question and distinguish between a change increase and change decrease problem contexts. Working with me and using the bead string, S4 was able to complete a whole-part-part diagram for his dog story, correctly identifying 9 as the whole, and 5 and 4 as the two parts. (I drew the whole-part-part structure, and he told me which numbers to put into the whole, and in each part, while referring to the bead string of 9 partitioned into 5 and 4). Once I had pointed that asking 'how many are left?' means one part has been taken away and the other part stays behind (is not taken) which is what is left; and 'how many altogether' means the two parts are put together or joined, he attempted a new story orally. He said: 'I have 5 dogs. My brother take-away 4 dogs. How many are left?' I wrote this story down for him. This final story was a coherent change increase story for his 5-4-1 number triple. It was not included in the 57 coherent stories of the collective example space, as he produced the story, only after interaction with me during the lesson. This may indicate that lack of fluency with oral language was reflected in his difficulty with creating stories. In particular it highlights a potential difficulty with the dual meaning of the word 'left', with learners expected to infer which meaning from the context: 'left' as a relative position (left-right); and 'left' as a contrast to being gone or taken (left behind-taken away). The apparent efficacy of the pedagogic intervention (at least for immediately following the teacher interjection) may also suggests that oral fluencies can be supported with teacher attention to word meanings.

These examples provide evidence that a minority of learners in the class were not yet able to coherently tell additive relations stories. A few continued to struggle with the syntax models for expressing an additive relation, one learner worked slowly and meticulously but did not complete the story telling task; another had difficulties focusing and recording his thoughts in writing. That all of these learners were from the support group, resonates with the finding that the intervention did not seem to hold as much promise for the support group (compared to the core and extension groups). For some support group learners the distinction between change increase and change decrease contexts, and their symbolisation was not yet secure. This – together with the absence of other foundational competencies listed above – made learning relating to the more difficult static problems types (collection and compare) seemingly out of reach for the support learners at this point in their development.

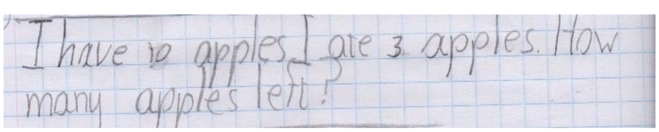
Local personal example spaces

Turning from the collective example space of the stories told across the whole class in response to the story-telling task, I now consider what may be inferred about the local personal example spaces of the learners in the class (who were telling coherent stories). I consider the extent to which there was evidence of learners using the type of word problem as a dimension of possible variation across their three story examples. While three stories is clearly not the full extent of a learners' personal example space, creating all three examples within the same problem type, may suggest that the chosen problem type may be dominant for this learner (and that greater exposure to other reference examples may be necessary).

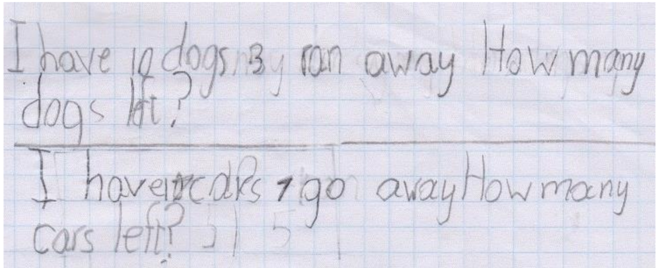
To demonstrate how the personal example space for each learner was inferred from their work on this story telling task, I contrast the examples generated by Retabile (case study learner from the core group) to Learner C1 (also from the core group).

Retabile generated three stories for the same number triple (10-7-3), but adopted the same change (decrease) situation in all three of them:

Figure 94: Retabile telling stories for 10-3-7



I have 10 apples. I ate 3 apples. How many apples left?



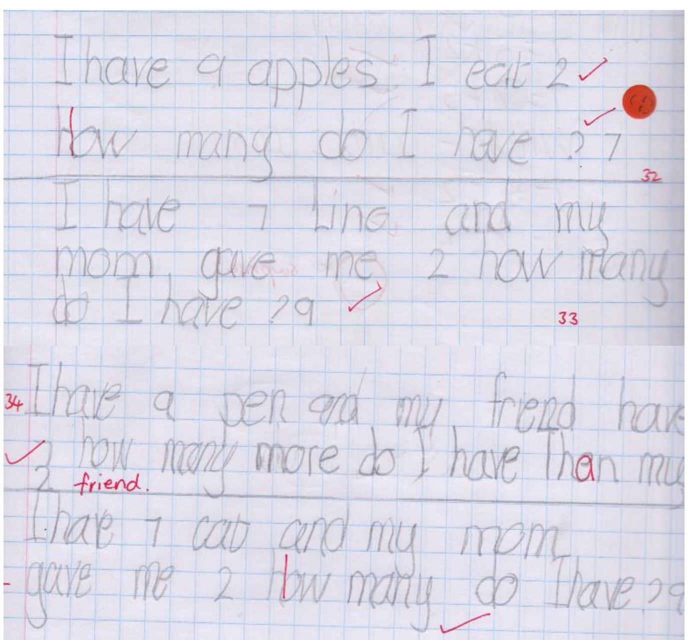
I have 10 dogs. 3 ran away. How many dogs left?

I have 10 cars. 7 go away. How many cars left?

As directed Retabile kept the numbers invariant. She varied the characters in her story and the verbs relating to removal (apples being eaten, dogs running away, and cars going away). Her question was kept invariant with the structure 'How many 'characters' left?' She did not follow the instruction to make use of the word 'more' in one of her stories. It seems that for Retabile the change decrease problem type was a key reference example. The word problem types as defined in the conventional example space did not yet seem to be possible dimensions of variation for her. She did vary the numbers slightly from the second story to the third, which may suggest awareness of the relationship that if 10 – 3 = 7, then 10 – 7 = 3.

The three stories told by Learner C1 three stories included change (increase) word problems alongside change (decrease) stories. C1 worked with the number triple 9-7-2 and brought to mind two stories of change. The first related to apples being eaten, and the second to 'lino' (sic) (possible 'lion') being given.

Figure 95: Learner C1 telling stories for 9-7-2



I have 9 apples. I eat 2. How many do I have?

I have 7 lino (sic) my mom give me 2. how many do I have?

I have 9 pen and my friend have 7. How many more do I have Then my [incomplete]

I have 7 cat and my mom gave me 2 how many do I have?

His third story was in the compare category (although this was not completed) where he had 9 pens and his friend had 7. His fourth story was a repeat of his second story which was of the change (increase) type, however he changed 'lino' to cat. When presented with a number triple C1 could bring to mind two different change contexts, and position this alongside a compare situation. Learner C1 seems to have a simultaneous awareness of the two change situations (change increase and change decrease). The dimension of possible variation relating to problem types in the local personal example space for C1 included change (increase), change (decrease) and compare problem types.

The above contrast between Retabile and C1 shows that while some learners (like Retabile) told all of their stories within the same problem type, other learners (like C1) told stories which spanned different problem types. It was therefore appropriate to provide an overview of the variation in the three stories evident from the coherent stories generated by the learners in the class. I considered the type of stories told by each learner, and whether or not the learners were telling all their stories as a single problem type, or were able to make use of more than one story type.

Figure 96: Number of different types of coherent stories told by learners

Number of different story types	Number of learners	Types of stories told	Number of learners
1 story-type	10	Change (decrease) only	7
		Change (increase) only	2
		Compare only	1
2 story-types	10	Change (decrease) and change (increase)	6
		Change (decrease) and collection	2
		Change (decrease) and compare	1
		Change (increase) and collection	1
3 story types	2	Change (decrease), change (increase) and compare	2

For ten of the learners, all of their stories were of the same problem type, which for most (7) learners the stories told were of the change (decrease) problem type. Two learners only told change (increase) stories, and one learner only told compare stories. The dominance of the change story narrative, where there is a start, something happens and then there is a result was evident with most learners who only told one story type using this problem type. Bringing to mind the static situations of the collection and compare problem types seemed to be more difficult, with fewer learners telling stories about these static situations. Of the 10 learners who told two story types, the majority of them (6 learners) told change stories making use of change (decrease) and change (increase) contexts. These dynamic change stories figured in the narratives told by 21 of the 22 learners (only one learner told only compare stories). However, there is evidence that after the intervention, the collective example space included some examples from the static word problem types, as compare or collection problem types were brought to mind by 7 (just less than one third) of the learners. The intervention therefore seems to hold potential – both in terms of supporting learners to solve a wider range of word problem types (evident in their shifts in written test attainment), as well as in terms of some of the learners bringing the static problem situations to mind.

Summary of findings

For ease of reference, this section offers a consolidation of the findings in response to the question: *'what evidence of learning gains (in relation to the learning goals), if any, is seen as a result of the teaching intervention?'*

Learning Goal 1: Problem solving

In relation to Learning Goal 1: Problem solving, a simple marking framework (comprising a total of 7 marks for correctness of solutions for the 7 test items) was applied to the written test attainment. Comparisons between the pre-test, post-test and delayed post-test were made which considered only whether or not the learner correctly solved the items in the written test. The findings from this analysis pertaining to the whole class were as follows:

1. The mean result for cycle 2 learners (n=26) improved by + 14 percentage points from pre-test (mean = 3.5 marks, 50%, SD = 1.9) to post-test (mean = 4.5 marks, 64%, SD = 2.2 marks) and improved further by another + 9 percentage points for the delayed post-test (mean = 5.1 marks, 73% marks, SD = 1.7 marks) when the simple marking framework with a total of 7 marks for correctness of solutions was applied.
2. Considering on the correctness of solutions, the *compare (matching) problem* was consistently found to be easier to solve that *the compare (disjoint set)* problem, which was the most difficult problem to solve in all three tests (pre-test; post-test and delayed post-test).
3. More learners correctly solved the *collection problem* in the post-test and the delayed post-test, compared to the pre-test, suggesting general transfer of problem solving approaches to an unfamiliar problem type.
4. Considering the changes in facility for each word problem item, there was the greatest difference in facility (indicated the biggest learning gain) for the *compare (disjoint set)* word problem from delayed pre-test to the delayed post-test.

When the attainment in written tests (again considering the simple marking framework for correctness of solution) was disaggregated according to the learner groups adopted in the small group sessions of the intervention, the following findings emerged:

5. For all three groups (support core and extension), the *compare (matching)* problem was consistently found to be easier to solve that the *compare (disjoint set)* problem in all three tests (pre-test; post-test and delayed post-test).
6. For the support group learners, the facility of the *collection problem* declined from pre-test to post-test; revealing that support learners were not yet transferring problem solving approaches to an unfamiliar problem type; however they were better able to solve this problem by the delayed-post-test.
7. For the core and extension group learners, the facility of the *collection problem* increased from pre-test to post-test, revealing that core learners were transferring problem solving approaches to an unfamiliar problem type; however there was a decline in facility for this question by the delayed post-test (while remaining higher than the pre-test facility).

Learning Goal 1: Problem solving and Learning Goal 2: Representations

The expanded marking framework resulted in a total possible test score of 22 marks. Analysis of attainment in the pre-test and post-test using this expanded framework revealed the following:

8. In cycle 3 (n = 26), the mean improved but the standard deviation increased. The mean for the pre-test was 8.9 (SD = 5.1); the mean for the post-test was 12.5 (SD = 6.3), indicating a learning gain of + 16 percentage points.
9. In cycle 3 (n = 26), the mean for Learning Goal 1: Problem solving (with a possible total of 10 marks) improved from 4.7 (SD = 2.4) to 6.4 (SD = 3.1), indicating a learning gain of + 17 percentage points (with a wider distribution of marks however, indicated by the increase in standard deviation).
10. In cycle 3 (n = 26), the mean for Learning Goal 2: Representations (with a possible total of 12 marks) improved from 4.3 (SD = 2.9) to 6.1 (SD = 3.4), indicating a learning gain of + 15 percentage points.
11. Each of the criteria in the expanded marking framework were met by more learners by the post-test. The coherent prompted number line was the criterion that was most seldom met, and this showed the smallest change in facility from pre-test to post-test, while the biggest change in facility were evident for the criteria relating to the partition problem (multiple solution problem).

In addition this expanded marking framework was also applied to the Cycle 2 written test data, and the Cycle 3 delayed post-test data so that comparisons between the two cycles could be made. When this was undertaken, as a result of the format changes in the tests across the cycles, the criterion for a prompted number line was removed, resulting in a total possible test score of 18 marks.

12. There were improved mean results for the whole class in Cycles 2 and 3 from pre-test to post-test. In Cycle 3 there was a further improvement in the mean result of the delayed post-test with a smaller standard deviation. This implies that there were positive learning gains in both cycles.
13. In cycle 3 (n = 26), using the expanded marking framework (excluding the prompted number lines, with 18 possible marks) the mean for the pre-test was 8.1 (SD = 4.5); the mean for the post-test was 11.4 (SD = 5.5); and the mean for the delayed post-test was 13.4 (SD = 3.4). This is an average learning gain of +29 percentage points. T-tests established that the differences in means were significant.
14. Both cycle 2 (n = 22) and cycle 3 (n = 26) revealed that there were learning gains from pre-test to post-test, with average learning gains for +16 percentage points +29 percentage points respectively (this used expanded marking framework with a possible mark allocation of 18).
15. For the learners in cycle 2 (n = 22) and in cycle 3 (n = 26), the *compare (matching)* problem was consistently found to be easier than the *compare (disjoint set)* in all the written tests.
16. For the learners in cycle 2 (n = 22) and in cycle 3 (n = 26), the facility of the *collection problem* increased over time, revealing that on average the learners were transferring problem solving approaches to an unfamiliar problem type.

Learning Goal 2: Representations

For Learning Goal 2: Representations, all of the representations developed by the Cycle 3 learners in response to the written test items were analysed. First the self-selected representations were distinguished as coherent and incoherent models, which resulted in the following findings across the collective example space of self-selected representations:

17. In cycle 3 the number of coherent models self-selected or prompted by learners in the pre-test (121 coherent models) increased in the post-test (172 coherent models), and this was maintained in the delayed post-test (174 coherent models). Chi-squared tests showed that this was statistically significant ($p<0.05$).
18. The number of incoherent models self-selected or prompted by learners in the pre-test (106 incoherent models) decreased by the post-test (87 incoherent models), and decreased further by the delayed post-test (34 incoherent models), which Chi-squared tests showed were statistically significant ($p<0.05$).
19. The core and extension groups produced more coherent models of representations and fewer incoherent models of representations in the written test, by the post-test compared to the pre-test. For the support group, the reverse was the case; although by the delayed post-test this group were producing more coherent models, and fewer incoherent models.

Next the representations (both self-selected and prompted) were classified by representation type in terms of line, group and syntax models:

20. The observed increase in coherent models of representation by the whole class in Cycle 3, was reflected in each of the types of representations (line, group and syntax) where increases in coherence, and decreases in incoherence were found to be statistically significant using Chi-squared tests ($p<0.05$) for all three types of representations.
21. While learners coherent use of syntax models improved substantially (there were three times as many coherent syntax models in the post-test, compared to the pre-test) with related declines in incoherent syntax models over time, there was evidence that learners continued to be insecure with the line model representation where incoherent line models persisted in the post-test stage (there was only a 20% increase in coherent line models).

The collective example space of representations were then coded with further more detailed sub-categories relating to the group, line, and syntax models, and the following findings emerged:

22. Learners self-selecting to use group models in problem solving tended to draw and act on ones, and did not make use of group-wise actions. There was limited evidence (3 examples) of learners using group-wise arrangements or actions on group models in the post-test; revealing that group-wise actions are possible; however this was not a statistically significant change.
23. There was evidence of some learners shifting from structured number lines, to using empty number lines (although this was not significant); and of learners shifting from hops in ones to jumps of more than one (which was more likely to be done on empty rather than structured number lines). Compared to the group model representations, there was more evidence of group wise actions on the number lines.
24. There were 36 examples of the whole-part-part syntax model (introduced through the intervention task design), being adopted by learners. The majority (61%) of these whole-part-

part diagrams were scaled to reflect awareness of the measurement scale in the diagram, while a sizable proportion of the whole-part-part diagrams (39%) were unscaled.

25. Learners made more use of the standard form of number sentences in pre-tests, and with greater use of the alternative form of number sentences evident in the post-test. This was a statistically significant change ($p<0.05$).

Finally the collective example space of representations was coded in terms of calculation actions (evident across the three representation types). The following findings emerged:

26. There was a decline in *take-away actions* and an increase in *partitioning* and *difference/ matching* actions from pre-test to post-test. The Chi-squared test on this data revealed that the changes in categories relating to calculation actions was significant ($p<0.05$).
27. There was evidence of learners using a *take-away action* by crossing out on a group model, or hopping or jumping back from a starting number on a number line (in roughly equal proportions in both pre-test and post-test). There was a slight decline in the take-away action on the line model from pre-test to post-test. There were a minority of examples using a take-away action by drawing, erasing and redrawing on a group model.
28. Learners made greater use of *partitioning actions* in the post-test (12 examples of partitioning), compared to the post-test (26 examples of partitioning). In both pre and post-test there were about 5 examples of arranging objects/ symbols in a left-right partition using a group model (this was the dominant partitioning action in the pre-test). By the post-test there were 10 examples of learners using the symbolic whole-part-part structure
29. Learners made use *matching* and *difference actions* in both the pre-test and the post-test. By the post-test there were many more examples (three times as many as the pre-test) of 1:1 matching actions on group models, and many more examples (4 times as many as the pre-test) for difference actions on line models.

Learning Goal 3: Story telling

From Cycle 1 and Cycle 2 intervention it was established that learners were unfamiliar with using story-telling to pose and explain word problems. Such tasks required teacher intervention, and could not be independently completed by learners prior to the intervention. Evidence of Mpho learning to engage in story-telling tasks during the pre-interview, which was typical of the interview engagement of learners, support this finding for cycle 3 as well.

The whole-class activity on the story telling task was analysed by reviewing the learner activity on the final task in the lesson intervention (Main Task 7: Learners generating examples). I first considered the collective example space distinguishing coherent and incoherent stories narrated by the learners:

30. The majority of learners (22 of the 28 learners) in Cycle 3 were able to narrate an additive relation word problem story which was *coherent* and which connected with the syntax model representation of the additive relation. 57 coherent stories were narrated and comprised the collective example space of coherent stories;

31. In contrast, a minority of learners (6 of the 28 learners) were not yet able to tell coherent and connected additive relations word problem stories. These learners had different reasons for their absences of learning: They were either absent for some of the intervention, did not complete the task, or were not able to correctly depict a number triple using syntax models from this they could then tell coherent stories, or were not able to record their thoughts in written format. All of these learners were from the support group, adding further evidence that the intervention was less effective for this group of learners.

The collective example space of coherent stories was then analysed for the types of additive relation word problems that the learners generated:

32. In Cycle 3 (and consistent with the findings in Cycles 1 and 2) the dynamic *change* problem type dominated the collective example space, when children were given opportunities to tell their own stories. The static problem types (*collection* and *compare*) were less commonly brought to mind by the learners.
33. Within the *change* problem type stories narrated by learners, the *change decrease* problem type was more common than the *change increase* problem types. There was evidence of a few learners telling change unknown problem types; but most learners narrated change result unknown stories.
34. Evidence emerged of learners having difficulties with using the term 'more' as part of a question in two ways: firstly to denote an action of increasing (in a *change increase* situation) 'how many <u>more</u> are needed to reach a target?'; and secondly as a comparison (in a *compare* situation) 'How many <u>more</u> are here, <u>than</u> there?'
35. While many of the learners were able to solve the static *collection* and *compare* problems (as evident in the analysis of written tests) in the story telling tasks, they were not yet bringing these problems types to mind as part of the local collective example space, when asked to generate examples of additive relations word problems.

The examples that individual learners generated in response to the story-telling task, was then analysed for what could be inferred about their local personal example spaces:

36. Examination of the local individual example spaces of learners revealed that *change* problem types (describing problem situations that were dynamic or where the action was explicit) featured in the local personal example spaces for all (except one) learner in the class.
37. For 9 of the learners, the only problem type they brought to mind was either *change increase* or *change decrease*.
38. There was however evidence of (12) learners being able to narrate more than one type of additive relation story for the same additive relation; suggesting that problem types were starting to figure as dimensions of possible variation for these learners.

Looking across the above findings, there was evidence of learning gains from before the intervention to after the intervention. These learning gains were for each of the three Learning Goals. Analysis of the learning gains by learner group revealed that the core and extension groups showed higher gains than the support group. Detailed analysis of each learning goal, revealed areas that learners continued to find difficult, as well as some common misconceptions which arose in their learning activity. The extent to which these observed learning gains may be attributed to the teaching intervention is discussed in Chapter 7. To make these inferences the findings are discussed drawing on expectations emerging from the literature, as well as the evidence of the implementation of the intervention and detailed analysis of learning by the three case study learners.

CHAPTER 7: Interpreting the learning gains - a focus on three case study learners

This chapter bring together the findings relating to learning gains for the whole class, together with the findings relating to implementation of the intervention to account for the learning gains in relation to three case study learners. It answered the question *'what evidence of learning gains (in relation to the learning goals), if any, are seen as a result of the teaching intervention* ?' for the three case study learners. This question is addressed in terms of the three interrelated learning goals (problem solving, representations and storytelling) for Mpho from the support group (the child with the highest shift in attainment from the lower attainment levels), for Retabile from the core group (the child with the highest shift in attainment), and Gavril from the extension group (the child with the lowest shift in attainment from the higher attainment levels).

The chapter opens with a short description of each case study learner. This is followed by a discussion of learning gains (and absences of learning gains) in terms of the three interrelated learning goals: LG 1: Solving Problems, LG 2: Representations and LG 3: Story telling for these case study learners. In this chapter I exemplify evidence of learning gains, and, where relevant, contrast this with evidence of lack of learning, or learning that was different to the expected learning. As such I make best case selections to depict evidence of learning gains, as well as telling case selections to illustrate the absence of learning gains for particular learners. I discuss each of the learning goals in turn, drawing on illustrative vignettes from across the three case study learners.

Following a synopsis of the learning gains for each case study learner, the theoretical distinctions relating to problem solving, representations and stories within a narrative approach to mathematics is foregrounded, with the learners and tasks more backgrounded. The focus then shifts back to learners and items in summary sections at the end of the chapter.

The three case study learners

In this section I briefly sketch a profile of each case study learner and their position in the class in terms of ability groups and overall attainment in the written tests. This provides the back ground context to each case study learner whose activity is then discussed in relation to each of the learning goals in the sections below.

Mpho (largest learning gain in the support group)

Mpho was the learner with the greatest shift in attainment from the pre-test to the post-test in the lower attainment support group. At the time of the intervention, Mpho was a 9- year old boy with Sesotho as his home language. He had repeated Grade 1, and so was automatically promoted to Grade

2 (as he was already on year older than his age cohort). He was allocated to the support group in the class as he was not able to solve any of the problem in the pre-test. His written test attainment improved in the post-test, with further improvement evident by the delayed post-test (with the same trends evident using both the simple and the expanded marking frameworks). Mpho worked hard during the intervention completing and obtaining feedback on 52 individual work cards.

Figure 97: Mpho's written test results (simple marking framework)

	Q1 Change result unknown	Q2 Compare (matching)	Q3 Collection	Q4 Compare (disjoint set)	Q5: Partition problem	Q6: Bare calculation 21-6 =	Q7: Bare calculation 23-18 =	
Marks	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**7.0**
Pre-test	0	0	0	0	0	0	0	0
Post-test	1	1	0	0	0	1	0	3
Delayed post-test	1	0	1	1	1	1	0	5

This quantitative depiction shows how Mpho's attainment shifted by test item. Mpho was not able to answer any of the questions correctly in the written pre-test. He showed attainment improvements for the change and compare (matching) word problems. In the post-test he was able to generate two possible options for the partitions problem. He was able to correctly calculate the bare calculation that lent itself to a take-away strategy, however his working for the calculation that lent itself to a difference strategy was not correct. Mpho was not able to transfer his problem solving skills to the collection problem (which was not included in the HLT). He was also not able to solve the compare (disjoint set) problem which had been omitted from the ALT of the support group. By the delayed post-test, Mpho was able to solve the collection and the compare word problems.

Table 56: Mpho's written test results (expanded marking framework)

	Marks	**Pre-test (18)**	**Post-test (18)**	**Delayed post-test (18)**
Change word problem	**4**	2	4	4
Compare (matching) word problem	**4**	0	3	1
Collection word problem	**4**	0	0	4
Compare word problem	**4**	0	2	3
Partitions problem	**2**	0	0	0
Total marks	**18**	**2**	**9**	**12**

This reveals that Mpho improved further by the delayed post-test, maintaining his learning gains and demonstrating the extension of his ability to solve collection and compare word problems. The problem solving learning gains, from before the intervention to after it (evident from pre-test to post-test), for Mpho related to the change, compare (matching) and partitions word problem, and a bare calculation involving an efficient take-away strategy. He retained these learning gains and showed further improvements in the delayed post-test.

Mpho's representations were more coherent by the post-test and this was retained in the delayed post-test.

Table 57: Mpho's representations

	Coherent	**Incoherent**
Pre-test	3	7
Post-test	7	3
Delayed post-test	7	3

Mpho made greater use of line and syntax models after the intervention. He continued to produce 2 incoherent line models in the pre-test.

Table 58: Mpho's type of representations

	Group		**Line**		**Syntax**	
	Coherent	Incoherent	Coherent	Incoherent	Coherent	Incoherent
Pre-test	1	1	2	6	0	0
Post-test	0	0	2	2	5	1
Delayed post-test	2	2	1	1	4	0

Mpho's coherent representation became more structured as he produced more coherent syntax models after the intervention. Syntax models were completely absent from his pre-test activity.

There were learning gains in Mpho's ability to use story-telling to pose additive relations word problems when comparing his pre-interview to the post-interview and his work on Main task 7: Learner generated examples. In both interview settings Mpho made use of change type contexts. In the pre-interview Mpho required support and teaching modelling to demonstrate how to tell a story for a number sentence, and was then able to replicate this for himself. His learning gain was not in relation to increasing his word problem example space to include compare and collection word problem types but <u>within</u> the change word problem category, he shifted from considering change decrease (change unknown) contexts, to including change increase (change unknown) situations. This finding for Mpho was in line with the actual learning trajectory for the support group, which differed from the hypothesised learning trajectory, with compare (disjoint set) problems not included in small group sessions.

Table 59: Mpho telling additive relation stories

Mpho	**Before intervention**	After intervention
Change (decrease)	Tell a story for 10 – 7 = …(pre-interview) There was 10 cars. 7 was broken. How much is left? There was 20 cars 10 broke-ted, there was 10 left. There was 10 hearts, 7…10 hearts, 10 heart, 4 choc…10 chocolates, 7 was eaten. Is 3. Tell a story to explain 7 – … = 4 to a friend (pre-interview) It's like, there's no number in the middle. There's a…there's a 7 minus hhhmmm equals 4.	Tell a story for 10 – 7 = …(pre-interview) Mpho have 10 sweets, my friend take aways 7 sweets, how many Mpho have? My friend have 14…13 sweets. He take away 7 sweets. 15, 16 sweets, is 1, 2, 3, 4, 5, 6. My friend have 17 sweets, he take away 7, 1, 2, 3, 4, 5, 6, 7 [counting backwards on number line] **Tell a story to explain 7 – … = 4 to a friend (post-interview)** 7…take away "hhhmmm" equals 4. [How would I find what mmmm is?] You must use a number line of [as in or] a…counters of like…sticks. **Write three stories about 6-4-2 (Main task 7, Day 10)** I have 6 eggs. My friend breaks 2. How many are left?
Change (increase)		**Task 7: Write three stories about 6-4-2. One of you stories must use the words 'more than'** I have 2 cars. My mom bring more. I have 6 cats now. How many cats did my mom bring? I have 2 car. My dad bring more. I have 6 cars now. How many car did my mom bring?
Compare disjoint set		

Retabile (largest learning gain in the core group)

Retabile represents an example of the greatest shift in attainment from the pre-test to the post-test. At the time of the intervention Retabile was an 8 year-old girl with isiXhosa as her home language. She was the correct age for her cohort. She was allocated to the core group as she solved 3 of the problems in the pre-test. By her post-test she could solve all the problems. Retabile worked consistently during the intervention, completing and receiving feedback on 25 work cards.

Table 60: Retabile's written test results (simple marking framework)

	Q1 Change result unknown	Q2 Compare (matching)	Q3 Collection	Q4 Compare (disjoint set)	Q5: Partition problem	Q6: Bare calculation 21-6 =	Q7: Bare calculati on 23-18 =	
Marks	1	1	1	1	1	1	1	7
Pre-test	1	1	1	0	0	0	0	3
Post-test	1	1	1	1	1	1	1	7

Delayed post-test	1	1	1	1	1	1	1	7

This quantitative depiction shows how Retabile's attainment shifted by question. In the pre-test Retabile was already able to solve compare (matching) and collection word problems. She showed attainment improvements for the change and compare (disjoint set) word problems. She was able to work systematically and to show a complete solution for the partitions problem. Her bare calculations improved. By the delayed post-test, Retabile maintained post-test results, again solving all of the problems.

Table 61: Retabile's written test results (expanded marking framework)

	Available marks	**Pre-test (18)**	**Post-test (18)**	**Delayed post-test (18)**
Change word problem	**4**	3	4	4
Compare (matching) word problem	**4**	4	4	4
Collection word problem	**4**	4	4	4
Compare word problem	**4**	2	4	4
Partitions problem	**2**	0	2	1.5
Total marks	**18**	**13**	**18**	**17.5**

The problem solving learning gains (evident from pre-test to post-test) for Retabile related to being able to express a change problem situation using a number sentence, solving a compare (disjoint set) word problem, working systematically and offering a complete solution using syntax models for the partition problem and accurately completing subtraction calculations. She retained these learning gains in the delayed post-test.

Retabile produced more coherent representations after the intervention.

Table 62: Retabile's representations

	Coherent	**Incoherent**
Pre-test	6	4
Post-test	14	2
Delayed post-test	2	0

She made greater use of syntax models and her line models became more coherent (although she still had two incoherent line models in the post-test).

Table 63: Retabile's type of representations

	Group		Line		Syntax	
	Coherent	Incoherent	Coherent	Incoherent	Coherent	Incoherent
Pre-test	4	0	2	4	0	0
Post-test	3	0	3	2	8	0
Delayed post-test	2	0	0	0	0	0

Retabile's coherent representations were more structured as she made more use of syntax models. It was notable that she continued to use iconic and indexical representations (where she acted on ones and not groups) alongside her syntax models.

In the interviews, Retabile expanded her example space from take-away contexts only, to take away contexts as well as a compare (reach a target) problem, and two compare (disjoint set) problems: one using the comparative language of 'more than' and another using the comparative language of 'less than'. Although she was able to solve a compare problem, when posing compare problems the comparative language was however not yet secure and she required teacher prompting initially to re-voice these compare stories.

Table 64: Retabile telling additive relation stories

Retabile	**Before intervention**	After intervention
Change (decrease)	**Task: Tell a story for 10 – 7 = …** Ten people are in the bus And seven people come out. And then three people is left. [Can you tell me another story for ten minus 7 maybe a bit harder this time?] There are …there are ten …There are ten tops…And bottles… [What happens to the ten bottles?] …There are ten bottles. There were ten bottles somebody took them [How many did they take?] Seven. They took seven [And then what happened in the story?] There were three left Tell a story to explain 7 - … = 4 to a friend Seven Seven minus [Retabile writes 3] [How do you know it is 3?] I counted. 6,5,4. And then it was 3	**Task: Tell a story for 10 – 7 = …** I got 10 marbles. 7 marbles. 7 marbles… I gave my brother 7 marbles. I…How many marbles do I have? [Can you tell me another story that needs 10 minus 7?] I have 10 apples. I eat 7 apples. How many apples I have left? **Tell a story to explain 7 - … = 4 to a friend** Seven minus… Seven minus… Seven… Seven minus … Seven minus three Seven minus three is equal to four/ [Have you got a picture that you could draw or a story you could tell or diagram that would help someone to understand that?] [Retabile draws a 4.3.4 whole-part-part diagram] I saw 7 butterflies. Three fly away. How many… how many butterflies…how many butterflies are left?

		Seven… Seven sweets…I gave my friend three sweets. I have 7 sweets…I give … I give away… [Following teacher support, Retabile re-tells with change unknown] I have seven sweets. I give some to my friend. I have … I have four sweets left. Write three stories about 10-7-3 (Main task 7, Day 10) I have 10 apples. I ate 3 apples. How many apples left? I have 10 dogs. 3 ran away. How many dogs left? I have 10 cars. 7 go away. How many cars left?
Change (increase)		**Task: Tell a story for 10 – 7 = … (post-interview)** [Can you tell me a sticker story?] Ten and 7. I have 7 stickers. How many more stickers do I need to get 10 stickers? [And another story?] I have… I have seven. I have ssss. I have seven cars. My mommy…my mom gave me ssss... three cars. How many cars do I have together?
Compare disjoint set		**Task: Tell a story for 10 – 7 = … (post-interview)** [Can you tell a 'how many more' story?] I have 10 cars. Teacher Nicky have 7 cars. You have ten and I have seven [How…] How many…Cars. How many more do I have than Teacher Nicky? How many more cars do I have than teacher Nicky. I have seven dogs. Teacher Nicky have ten dogs. How many more does Teacher Nicky have than me? [Can you tell a 'how many less' story?] I have seven cats. Teacher Nicky have ten cats. How many …how many less do I have than teacher Nicky?

Gavril (lack of learning gains in the extension group)

Gavril was 8 years old at the time of the intervention. He is a boy with a local Rwandan language as his home language. He was the correct age for his cohort. He was allocated to the extension group as his attainment in the pre-test was 88%. He attained 100% for the post-test. Gavril represents an example (from the 12 selected learners for interviews) of the smallest shift in attainment from the pre-test to the post-test, in the upper attainment levels. Gavril only completed 18 individual work cards. He was rewarded with 27 stickers.

The following presents a visual depiction of the Gavril's shifts in attainment evident from the pre-test to the post-test.

Table 65: Gavril's written test results (simple marking framework)

	Q1 Change result unknown	Q2 Compare (matching)	Q3 Collection	Q4 Compare (disjoint set)	Q5: Partition problem	Q6: Bare calculation 21-6 =	Q7: Bare calculation 23-18 =	
Marks	1	1	1	1	1	1	1	
Pre-test	1	1	1	1	1	0	1	6
Post-test	1	1	1	1	1	1	1	7
Delayed post-test	1	0	1	1	1	0	1	5

This quantitative depiction shows how Gavril's attainment shifted by question. Gavril was already able to solve all of the word problems. He made an error in the pre-test when working on the bare calculation. He was able to work systematically and to show a complete solution for the partitions problem. By the post-test Gavril was able to successfully complete all of the questions, obtaining 100%. By the delayed post-test, Gavril made calculation errors with the compare (matching) word problem and in one of the bare calculation questions.

Table 66: Gavril's written test results (expended marking framework)

	Marks	**Pre-test (18)**	**Post-test (18)**	**Delayed post-test (18)**
Change word problem	**4**	4	4	4
Compare (matching) word problem	**4**	4	4	2
Collection word problem	**4**	4	4	4
Compare word problem	**4**	4	4	3.5
Partitions problem	**2**	2	2	2
Total marks	**18**	**18**	**18**	**15.5**

There were no problem solving learning gains (evident from pre-test to post-test) for Gavril. Working off a high base where he only made a bare calculation error in the pre-test, Gavril obtained full marks for the post-test. He made two calculation errors in the delayed post-test and as a result his attainment declined.

In terms of representations, Gavril produced more incoherent representations by the post-test.

Table 67: Gavril's representations

	Coherent	Incoherent
Pre-test	9	2
Post-test	8	4
Delayed post-test	5	1

Mpho struggled to represent additive relations on a number line and this was evident in the pre-test, and ongoing into the post-test.

Table 68: Gavril's type of representations

	Group		Line		Syntax	
	Coherent	Incoherent	Coherent	Incoherent	Coherent	Incoherent
Pre-test	1	0	4	2	4	0
Post-test	0	0	2	4	6	0
Delayed post-test	0	1	0	0	5	0

Considering the stories that Gavril narrated during the interviews, his example space for word problems remained dominated by change actions. He was only able to bring change type stories to mind, and despite being offered direct prompts to shift away from these situations, his concept of subtraction remained fixed with a 'take-away' action. That Gavril was also unable to shift his example space to include compare type problems was however surprising and is discussed below.

Table 69: Gavril telling additive relation stories

Gavril	**Before intervention**	After intervention
Change (decrease)	**Task: Tell a story for 10 – 7 = …** There is 10, um, toy cars…and 7 toy cars. 7 take away…must I do a minus or write take away? [What's happening in your story? There are 10 toy cars, and 7 toy cars, then what happens?] Must try too…the 10 and the 7 must try to make a take away answer. [So what's a story that's going to be a take away story?] There is 10 toy cars, and 7 toy cars. 10 take away 7 equals question mark, question mark. Answer is then, ja 3. **Tell a story to explain 7 - … = 4 to a friend.** I need 7 to take away something to make 4. [Gavril raises 5 and 2 fingers, lowers 3, then writes down answer of 3]	**Task: Tell a story for 10 – 7 = …** I have 10 cars. 7 ran away. How many do I have left? 7….oh….oh ja. I got 7 cars. Tim have 10 cars. We mix it up. Then I take 10 he takes 7…yo!...yo. [So you have 10 cars and Tim has 7 cars. And then what question can we ask?] And then altogether makes 17. [Can we have a story about 10 take away 7 if you've got 10 cars and Tim's got 7 cars?] Oh, equals 3. [What question would you ask?] I would ask…mmmm. I would ask….How many do they have left? I have 7…mmm…ja…I have 10 cars … hmm … 10 stickers … I give Anele 7. How many do I have left? **Tell a story to explain 7 - … = 4 to a friend** Uh…7 take away mmm equals 4. [How do you find out what mmm is?] By putting a number or…or….by putting … um … a number…3.

		[How did you find number 3?] Cause I had 7, I took away 3. **Task 7: Write three stories about 18-8-10 (Main task 7, Day 10)** I have 18 cats. 10 ran away. How many do I have left? I have 18 tops. 8 are missing. How many do I have left? I have 10 lions. 8 ran away. How many do I have left?
Change (increase)		
Compare disjoint set		

Qualitative learning gains for three case study learners for each learning goal

The three case study learners are now discussed qualitatively with reference to each of the learning goals. Detailed case study write ups for each learner are included in Annexure 1. These annexures provide the full evidence of learner activity for each case study learner on each of the main tasks of the intervention and in the pre- and post-assessment tasks. Examples of both learning gains (best-case examples) and absences of learning gains (telling case examples) evident for the case study learners are presented as short vignettes which draw on the written work from the tests as well as during the intervention, as well as their interview responses. 'Vignette 1: Mpho learning to tell stories' was presented above (on page 246) to illustrate how learners were inducted into problem posing tasks. Vignettes 2 – 10, presented below, were selected in order to provide a balance across the learning goals, and across the case study learners to include both telling and best case illustrations.

Table 70: Overview of illustrative vignettes against the learning goals and by case study learner

Learning goal	Mpho	Retabile	Gavril
Learning goal 1: Problem solving	**Vignette 3:** Telling case of Mpho solving the partitions problem	**Vignette 2:** Best case of Retabile solving the partitions problem	
Learning goal 2: Representations	**Vignette 4:** Best case of Gavril shifting from iconic to 'more structured' indexical representations	**Vignette 5**: Best case of Retabile using 'more varied' representations	**Vignette 6:** Best case of Gavril shifting from a take-away to a difference strategy on a number line **Vignette 7:** Telling case of Gavril's misconceptions with using a number line
Learning goal 3: Storytelling	**Vignette 8:** Best case of Mpho generating his own additive relations stories	**Vignette 9**: Best case of Retabile telling compare (disjoint set) stories	**Vignette 10:** Telling case of Gavril's reluctance / inability to tell compare (disjoint set problems)

Learning goal 1: Solving problems

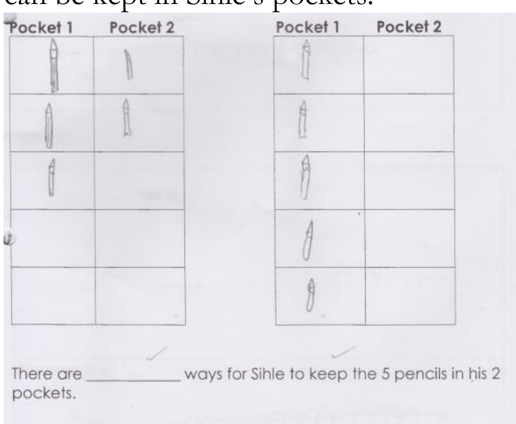
In this section I present a quantitative analysis of learning gains by discussing two vignettes: A best case example from Retabile and a telling case example from Mpho relating to their activity on the partitions problem. As Gavril was able to solve this problem prior to the intervention, and repeated this after the intervention, there was no learning gain for evident for him and so I have not included his work on this problem. I contrast the way Retabile approached the partitions word problem (as a best case example of problem solving learning gains), to the way Mpho approached the same tasks (as a telling case example of partial learning gains).

Vignette 2: Best case of Retabile solving the partitions problem

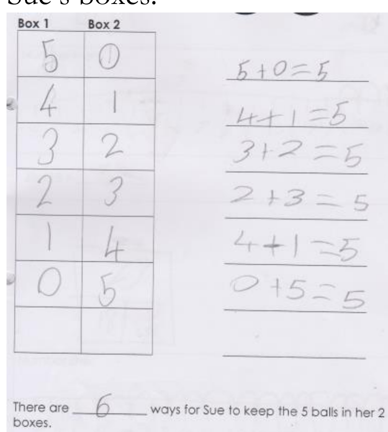
Retabile's activity on the partitions problem represents a best case example. She was able to move from identifying two partitions where she used iconic representations, to systematically working with number symbols and related number sentences to find a complete solution to this problem.

Figure 98: Retabile's partition problem: pre-test and post-test

Sihle has 5 pencils. He keeps them in his two pockets. Show all the ways that the pencils can be kept in Sihle's pockets.



Sue has 5 balls. She keeps them in two boxes. Show all the ways that the balls can be kept in Sue's boxes.



In the pre-test Retabile draw illustrations of pencils to depict the partition problem situation. She seemed to provide two possible options: 3 pencils in pocket 1 and 2 pencils in pocket 2; and 5 pencils in pocket 1 and 0 in pocket 2. She did not answer how many ways there were in total. The layout of the question seemed to confuse her, and she did not establish a complete solution.

In the post-test Retabile worked with number symbols to depict the partition problem situation. She provided a full set of options, working systematically from 5 to 0. She also wrote accompanying number sentences in the form whole = part + part. She correctly answered that there were 6 possible ways to arrange the balls. In the delayed post-test, Retabile was again able to offer a systematic and complete solution to this problem. She recorded her number sentences using a 'part + part = whole' structure.

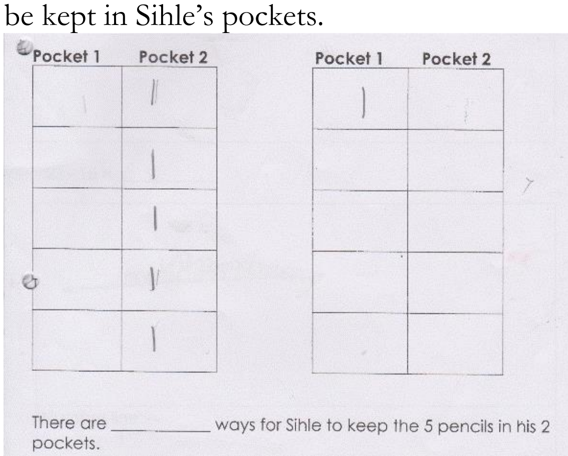
This vignette reveals that Retabile's learning gains for the partition problem related to each of the elements in focus in the marking framework: she found all of the possible partitions and represented these using a syntax model, her solution was complete, her work was systematic, and she identified how many partitions there were in total. This learning gain may be associated with Retabile's engagement on Main task 2: Partitions problem during the intervention. Being in the core group Retabile was able to re-tell and enact a similar version of this problem (using five monkeys and two trees). She was able to retell the story for different numbers of monkeys (she worked on 4 monkeys), and was provided with opportunity to record her solution to this problem for different numbers of monkeys (she completed a solution for 7 monkeys). Retabile experienced this problem in numerous whole-class teaching sessions where increasingly structured representations were used (shifting from iconic whole-part-part diagrams to symbolic whole-part-part diagrams and related number sentences). She was also present when the monkey and trees context for this problem was shifted (through teacher narration) to a golden books and boxes context.

Vignette 3: Telling case of Mpho solving the partitions problem

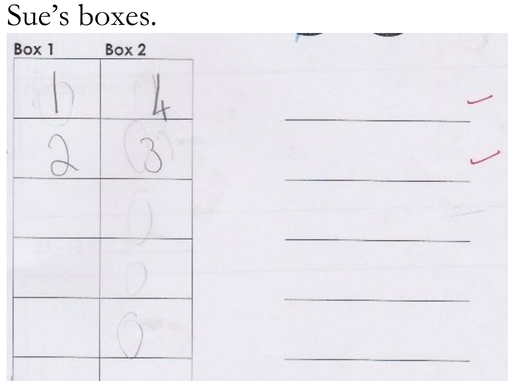
Mpho's activity on the partitions problems represents a telling case example.

Figure 99: Mpho's partition problems pre-test and post-test

Sihle has 5 pencils. He keeps them in his two pockets. Show all the ways that the pencils can be kept in Sihle's pockets.



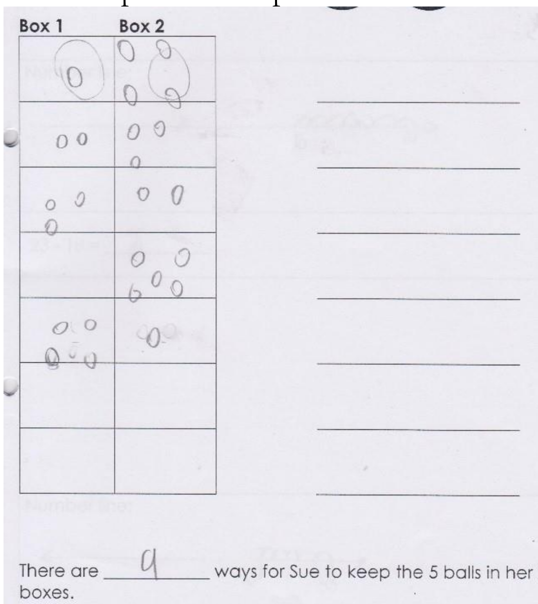
Sue has 5 balls. She keeps them in two boxes. Show all the ways that the balls can be kept in Sue's boxes.



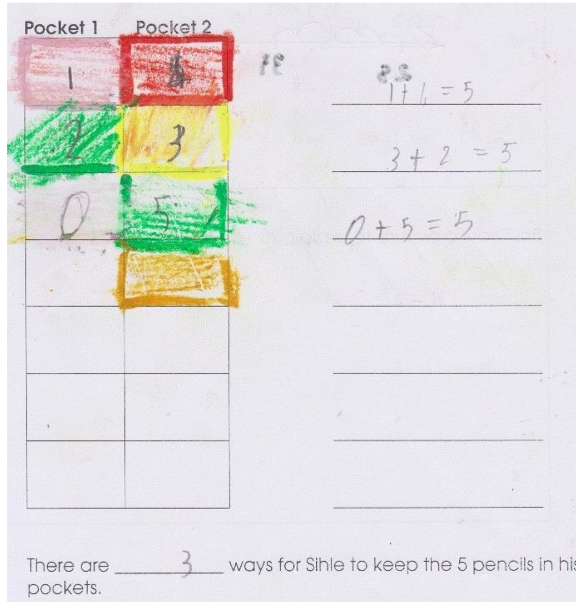
In the pre-test Mpho drew 5 lines (depicting 5 pencils) in pocket 2, and 1 line in pocket 1. He used an indexical representation for an incorrect partition (5-1). This may also be interpreted as a correct 0-5 partition, and an incomplete 1-0 partition. In the post-test, Mpho made use of number symbols and created two correct partitions: 1-4 and 2-3. He did not write down any number sentences.

Figure 100: Mpho's partition problems post interview and delayed post-test

Sihle has 5 pencils. He keeps them in his two pockets. Show all the ways that the pencils can be kept in Sihle's pockets.



Sue has 5 balls. She keeps them in two boxes. Show all the ways that the balls can be kept in Sue's boxes.



In the post interview Mpho was able to identify 5 correct partitions. He made use of indexical representations, which made use of a group pattern (using dice patterns). He incorrectly answered the question of how many partitions there were by counting his non-zero partitions to reach a solution of 9. He did not write any number sentences (syntax model) to support his working. In the delayed post-test he identified 3 correct partitions and supported these with number sentences. This time he correctly answered how many ways he had found (indicating 3).

This vignette reveals that Mpho's learning gains in relation to the partition problem were partial. He certainly demonstrated his problem solving processes better in a 1:1 interview setting than he was able to in a written test situation. His learning gains were partial in that he found some and not all of the possible partitions and represented these at times using syntax models and at times using group models. His solution was neither complete, nor systematic. He attempted to identify how many ways there were in total, but this was not yet secure for him (at post-interview stage). This partial learning gain is attributed to Mpho being in the support group, where his actual engagement on Main task 2: Partitions problem during the intervention was limited and differed from the hypothesised learning trajectory. Being in the support group Mpho was not provided with small group opportunities to re-tell and enact a similar version of this problem (using five monkeys and two trees). He was not provided with opportunities to retell the story for different numbers of monkeys, and nor was he given opportunity to work record his solution to this problem for different numbers of monkeys. Like Retabile, Mpho experienced this problem in numerous of the whole-class teaching sessions where increasingly structured representations were used (shifting from iconic whole-part-part diagrams to symbolic whole-part-part diagrams and related number sentences). He was also present when the monkey and trees context for this problem was shifted (through teacher narration) to a golden books and boxes context. However his individual engagement with task was not sufficient to enable him to solve this problem completely. In variation theory terms, for such learning gains to be evident, individual learner engagement on this task varying some key attributes such as the number of monkeys and the problem contexts (as provided to Retabile and Gavril) would be required.

Learning goal 2: Representations

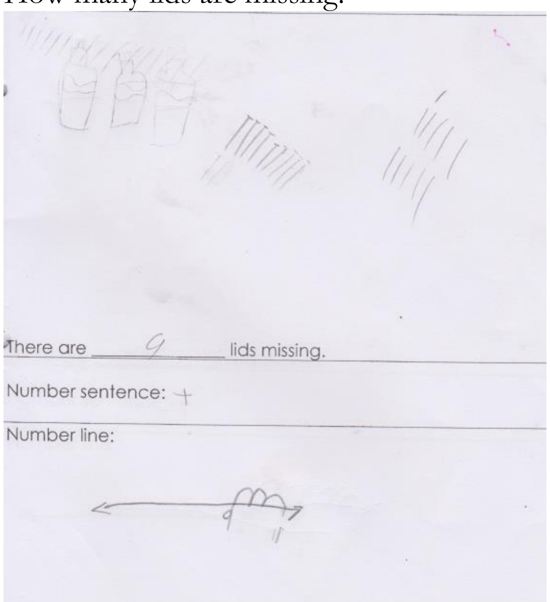
In this section I contrast Mpho's activity on the compare (matching) word problem with Retabile's work on the change problem.

Vignette 4: Best case of Mpho shifting from iconic to more structured indexical representations

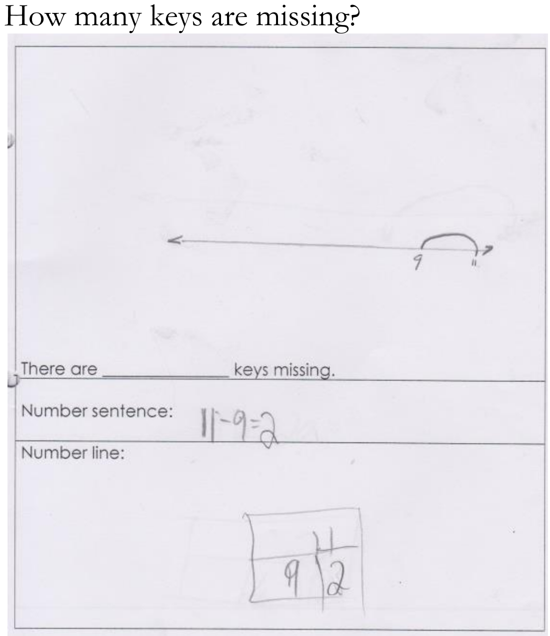
I use Mpho's compare (matching) word problem to provide a best case example of the evidence of his shift away from using iconic and indexical representations (where each element in the image was countable in ones) towards a more structured representations.

Figure 101: Mpho's compare (matching) problem pre-test and post-test

There are 11 bottles but only 9 lids.
How many lids are missing?



There are 11 locks but only 9 keys.
How many keys are missing?



In the pre-test Mpho started to draw an iconic representation of the problem context, sketching three bottles. He then drew an indexical representation using tally marks next to this, and then attempted his solution again (he tried to erase this prior work, although the markings remain clear enough to see on his script). He had two other indexical representations of his work. He sketched a group of 11 lines, where he used a group by five structure. He sketched another group of 9 lines, each of which had a dot drawn above it. It is assumed that the lines represented bottles, and the dots represented lids. He drew two lines below the nine lines with their dots. These two bottles, did not have dots above them. From this drawing, it seems as if Mpho understood the problem context, and was able to solve it diagrammatically using indexical representations. However when writing this down symbolically, Mpho completed the answer sentence by writing ‘9’: There are 9 lids missing. This may indicate that Mpho did not understand the meaning of the word ‘missing’. If the question was ‘How many lids?’ (ignoring ‘are missing’) it would be correct to answer: ‘There are 9 lids’. This may be considered sensible if this question was considered to be a reading comprehension question. Mpho did not write a number sentence, and wrote only the plus sign: ‘+’. For his number line he depicted 11 on an empty number line, and showed 3 hops backwards to 9.

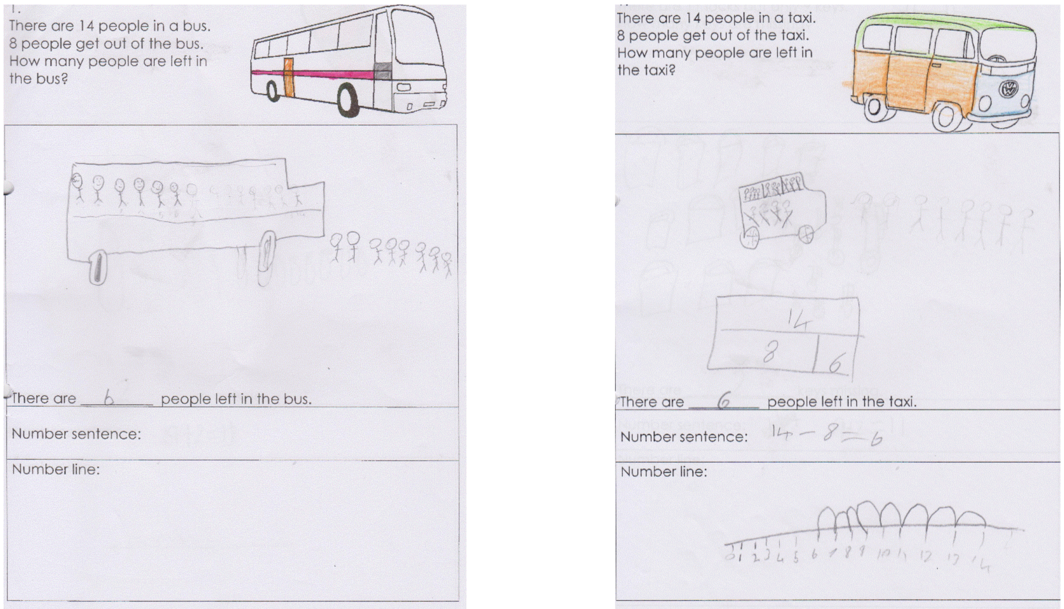
In the post-test Mpho was able to correctly solve this problem. Although he did not write the symbolic answer to complete the answer sentence, he depicted his solution using three symbolic representations: An empty number line where he drew a jump from 9 to 11 (or vice versa); a whole-part-part diagram where he correctly partitioned 11 into 9 and 2 and labelled the larger part 9, and a number sentence: ‘11 – 9 = 2’.

The learning gain in this vignette related to his ‘more structured’ use of representations and improved ability to record his solutions symbolically rather than his improved problem solving (from the pre-test he seemed able to solve the problem from his indexical representation). By the post-test he was able to make use of the empty number line showing hops in ones, write a number sentence, and represent this as a whole-part-part diagram. In this compare (matching) question Mpho could model the problem situation appropriately. These syntax models denote a shift towards more structured representations. This finding is attributed to Mpho’s engagement with Enabling task 11: syntax model fluencies. These syntax model tasks were in focus during the whole class sessions from Day 4 to Day 9 of the intervention. They also figured in the individual work cards which Mpho completed. Mpho completed and gained feedback on 10 syntax model fluency cards (of the 52 cards that he completed during the intervention). Through this learner activity he was supported to shift his representations to be more structured, and he gained confidence in using the syntax model representations though the opportunities provided to practice this using individual work cards. This related to Feature 6.3: Facilitating opportunities for learners to practice and receive feedback on pre-requisite fluencies.

Vignette 5: Best case of Retabile using ‘more varied’ representations

The following best case example from Retabile depicts the shift to more flexible use of a wider range of representations. This is related to Feature 4,1: Flexible movement between representations where sense-making was primary, was encouraged. This example also denotes a shift towards more structured representations.

Figure 102: Retabile's change problem (pre-test and post-test)



In both tests Retabile provided the correct answer using a number symbol, and an iconic depiction of the problem situation using ones. In the post-test she also wrote an appropriate number sentence. She made use of a number line representation on which she required 0 as reference point, depicted hops back in ones and made use of a take away calculation strategy.

The learning gains evidence in this example are reflected in the quantitative marking framework as although Retabile was able to solve the problem in both tests, her marks improved in the post-test as she was able to express the additive relationship using a number sentence. The qualitative learning gains relate to the increased variety in the types of representations she used. While in the pre-test she only used an iconic representation, in the post-test she continued to use the iconic representation but added in syntax and line models alongside this. Retabile was encouraged during the intervention to draw what made sense to her. She was also supported to make use of line models and syntax models, referred to as Enabling task C: Line models and Enabling task D: Syntax models. Retabile completed and received feedback on 25 individual work cards during the intervention. Two of these work cards related to line models, and her activity in the core group work on Days 4-8 of the intervention included line model tasks. Retabile completed and received feedback on 5 individual word cards relating to syntax models, which were also in focus during whole-class sessions from Day 4-9, and in the core group work on Day 8.

Line model representations

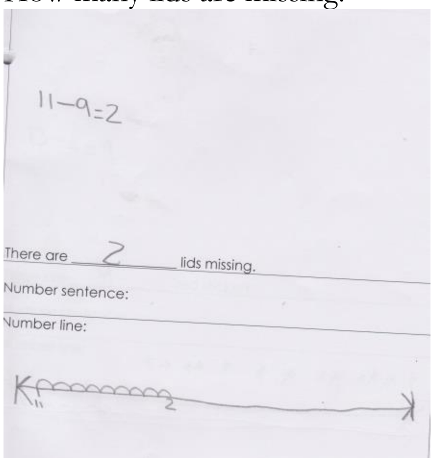
To discuss the learning gains relating to line model representations, I exemplify first with a best-case example of the shift from a take-away strategy to a difference strategy on a number line using a best-case example from Gavril. I then discuss Gavril's learning gains in relation to the use of the line model, as his work included telling cases where there were some shifts towards better aware and more structured use of the line model, coupled with evidence of some lingering misconceptions.

Vignette 6: Best case of Gavril shifting from a take-away to difference strategy on a number line

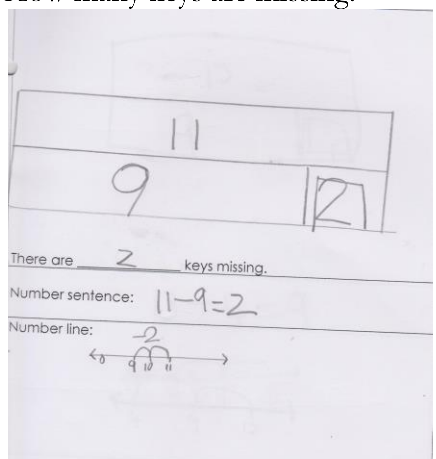
The following is Gavril's responses to the compare (matching) problem.

Figure 103: Gavril's compare (matching) problem pre-test and post-test

There are 11 bottles but only 9 lids.
How many lids are missing?



There are 11 locks but only 9 keys.
How many keys are missing?



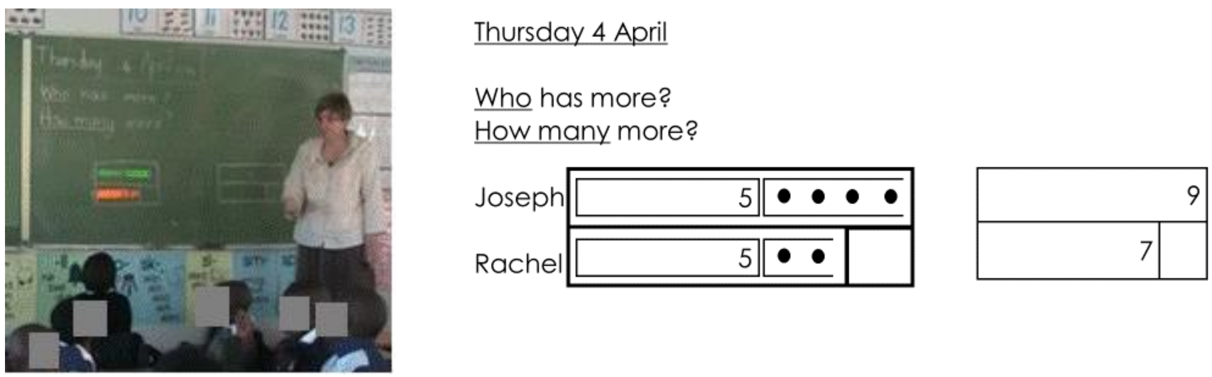
In the pre-test Gavril made use of a number sentence (syntax model) and correctly answered the question. He drew an empty number line and reversed the numbers (with the larger number (11) placed on the left, and the smaller number (2) on the right. He correctly depicted a take-away strategy showing 9 hops back from 11 to reach 2. However given the numbers in this problem, this strategy was not efficient.

In the post-test Gavril took care to draw a whole-part-part diagram (syntax model) using a ruler, and depicting the rectangle for the 9 as bigger than the rectangle for the 2. He revealed his awareness of the unknown, drawing a box around the 2. He completed the answer sentence correctly and provided a number sentence. When working on a line model representation, he made use of an empty number line, with an efficient strategy of two hops back from 11 to reach 9. He labelled his hop back using a label '-2'.

For this question although there were no learning gains evident in the quantitative marking of the questions, there were learning gains evident in the qualitative analysis: Gavril indicated his awareness of the unknown, he made use of a whole-part-part diagram, and he shifted from an inefficient take-away strategy on the line model, to an efficient difference strategy. It is inferred that the learning gain relating to awareness of the unknown seemed to be associated with the enabling tasks, as there was only one teaching episode where I made use of a box to depict an unknown during a whole class discussion.

On Day 8 of the intervention there was a whole class discussion on a compare (disjoint set) problem where the problem involved two learners: Joseph and Rebecca who were comparing how many stickers they had: 'Joseph has 7 stickers. Rebecca has 9 stickers. How many more stickers does Rebecca have than Joseph?' An indexical whole-part-part diagram had been created through Joseph and Rebecca sticking five strips onto the blackboard. I then drew a rectangle around Joseph's stickers, and around Rebecca's stickers to create a whole-part-part diagram, where the '2 more for Rebecca to have the same as Joseph' was left empty. I replicated this w-p-p diagram on the board, and invited 3 learners to come and each fill in the whole and the two parts. Volunteer learners were able to write 9 for the whole, 7 for the bigger part, and 2 for the smaller part:

Figure 104: Learners completing a whole-part-part diagram



I then invited learners to give me number sentences for this w-p-p diagram and wrote down what learners offered:

> L: 9 equals 7 plus 2 [T writes 9 = 7 + 2]
> L: 9 equals 2 plus 7 [T writes 9 = 2 + 7]
> T: That's my whole is equal to the part plus the part. Can somebody give me minus?
> L: Nine minus 7 equals 2 [T writes 9 – 7 = 2]
> T: And another one
> L: Nine minus 2 equals 7 [T writes 9 – 2 = 7]
> T: Fantastic

I then attempted to draw learners' attention back to the questions, and to bring the question (the unknown) into focus within the symbolic syntactical representations. To do so I introduced a symbolic notation of using a box (or a number in a box) to depict an unknown. I drew boxes around all the twos (the unknown in the story). I modelled finishing a problem with an answer sentence, by asking learner to restate and then answer the question:

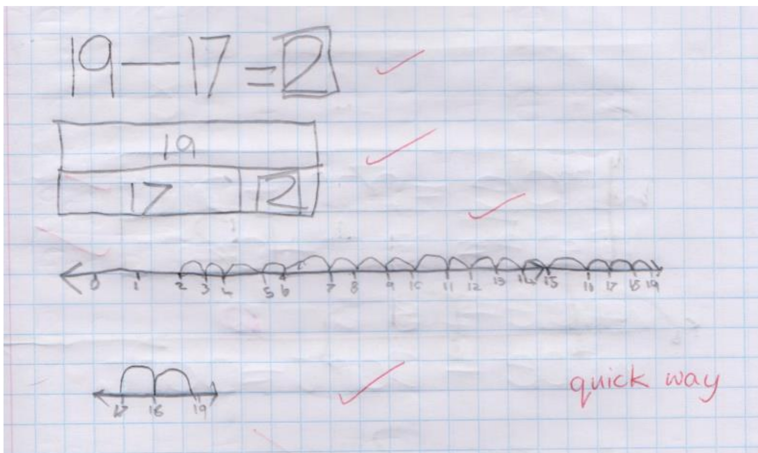
> T: Gavril what was the question?
> Gavril: How many more does Joseph have than Rebecca?
> T: Lovely: How many more does Joseph have than Rebecca? Here was Joseph, here was Rebecca [gestures to w-p-p diagram] And what didn't we know? What number didn't we know?
> Ls: 2
> T: Which number didn't we know?
> Ls: 2

> T: So the two was the number we had to find. We didn't know what this was [covers the 2 in the w-p-p diagram] and we had to try and work it out. So in our number sentence, I am showing that that's the one I didn't know [T draws squares around the 2s in all the number sentences]. That was what we were trying to find.

This was the only point in the lesson intervention where I introduced the formal notation of using a 'box' to depict an unknown while working on a problem. The box notation however also appeared on most of the enabling tasks relating to the syntax models.

Gavril's learning gain relating to the difference strategy is attributed to his engagement with the small group tasks on Main task 3, on Day 8 of the intervention. In this session Gavril calculated 14 – 8 using a whole-part-part diagram and a take-away strategy counting back 8 in hops of ones from 14 to reach 6. He then again used a whole-part-part diagram and contrasted a take-away strategy to a difference strategy for 19 -17, as shown:

Figure 105: Extract of Gavril's small group work on Day 8



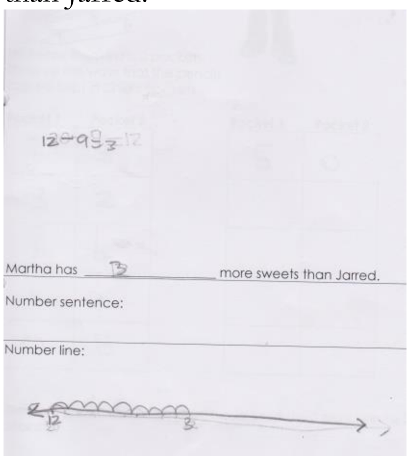
This was followed by his accurate work on 23 – 20 = …, 27 – 27 = … where he repeated the use of the whole-part-part diagram and used only difference strategies on the number lines. It is notable that at this point, he was drawing attention to the unknown and marking these with a box (most other learners did not notice to use the box to denote the unknown).

Vignette 7: Telling case of Gavril's difficulties with using a number line

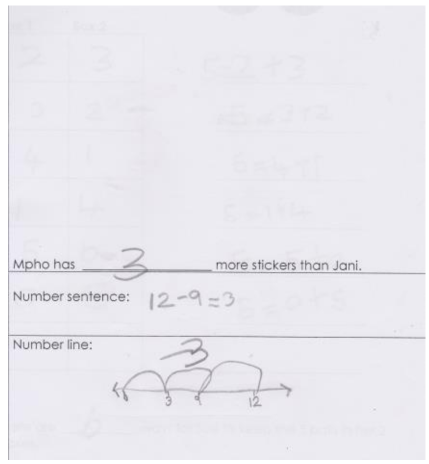
However in Gavril's work there was evidence of confusion in how he used the line model representation. Qualitative analysis of Gavril's line model representations reveals three things. Firstly in the pre-test he was not yet secure with positioning smaller numbers on the left and bigger number on the right and frequently reversed his number lines. This difficulty was corrected by the post-test where all of his lines displayed the correct 'smallest on the left to biggest on the right' orientation. Secondly in the pre-test he only depicted actions using counting in ones (hops). By the post-test, he was making use of group-wise actions on the number lines making use of jumps of more than one. Thirdly, having worked on two calculation strategies for the number line (take-away and difference) during the intervention, by the post-test he was not yet securely depicting one of the parts in the whole-part-part additive relation as the action on the number line (a size of the jump forwards or backwards). Instead he was positioning all three members of the whole-part-part number triple as numbers located on the number line. He had not yet noticed that at times the members of the number triple are numbers positioned in sequence on a line, and at times these numbers refer to a measurement (the size of a jump from one number to the next).

Figure 106: Gavril's compare (disjoint set) pre-test and post-test

Jarred has 9 sweets. Martha has 12 sweets. How many more sweets does Martha have than Jarred?



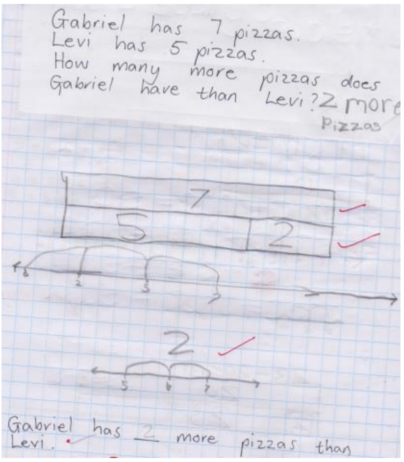
Janie has 9 stickers. Mpho has 12 stickers. How many more stickers does Mpho have than Jani?



In the pre-test Gavril chose to use a syntax model for his working, writing a number sentence. He correctly completed the answer sentence. He sketched an empty number line showing 9 hops back from 12 to reach 3, although he reversed the numbers (by positioning 12 on the left of 3).

In the post-test Gavril correctly answered the question and chose not to show any working. He wrote a number sentences and completed the answer sentence. His confusion with regard to the number line model was again revealed when he incorrectly depicted 3, 9 and 12 as points on the line and arranged these from smallest to biggest, from left to right. He seemed to be aware that he needed to denote a '-3' action, but did not know how to depict this. He created 3 jumps: from 0 to 3, from 3 to 9 and from 9 to 12 labelling these as -3. It was clear that he was not yet noticing that the 3 referred to 3 hops, or a jump of 3, and was positioning 3 as a number on the line. This misconception was evident in Gavril's written work when he completed the 'how many more problem' related to Main task 6: Compare (disjoint set) problem on Day 9 of the intervention:

Figure 107: Gavril's compare (disjoint set) problem (Day 9)



Gavril correctly used a whole-part-part diagram, but his first attempt at a line model involved positioning 0, 2, 5 and 7 on the number-line and showing group-wise jumps between these numbers. This was corrected during the session where he was directed to position the two numbers onto the line, and show the 2 as hops from one number to reach the other. This intervention was however not sufficient to overcome the misconception (it was probably too directed on how to correct the error, without insufficient attention to his comparing the two representations, and then repeating this comparison with other another number triple). The same misconception was then repeated in the post-test.

This telling case points to the need to contrast the two ways that numbers are used in the line model: firstly number are used to create the number sequence (depicted as numbers below the line); secondly numbers can be used to indicate the measurement from one number to another (depicted as a hop or jump from one number to the next, and shown above the line). In the former case numbers are *objects* on the line; in the latter case numbers label the size of an *action* on the line. In future design cycles, this distinction should be brought into the intervention task design.

Learning goal 3: Story telling

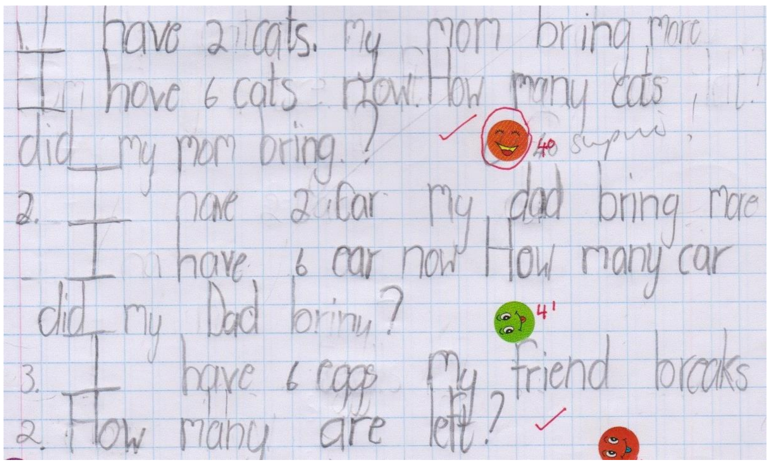
As learner activity with story-telling tasks was not evident in written tests, I provide detail of evidence of learning gains in relation to this learning goal for each of the three case study learners. As story-telling tasks were not included in the written tests I use several vignettes in this section to exemplify the case study learners' activity on story-telling tasks. I present each case study learner in turn and select vignettes to exemplify their engagement on story telling tasks. For each case study learner I open the story telling discussion with their activity on Main task 7: Learner generated examples where they were expected to tell three additive relations word problem for a number triple, and use the word 'more' in one of these stories. For Mpho I include a discussion on how he told stories to explain the number sentence 10 – 7 = … in the interviews and his engagement on the generalised story tasks. For both Retabile and Gavril I include vignettes on their activity on the tasks relating to telling compare (disjoint set) problems, as well as their engagement on the generalised problem tasks Here I contrast

Retabile's telling of compare (disjoint set) type problems, to Gavril's refusal or inability to bring compare (disjoint set) stories to mind (despite his being able to solve problems of this type).

Vignette 8: Best case of Mpho generating his own additive relations stories

Mpho's activity with Main task 6: Learners' generating examples is revealing of his personal example space for additive relation word problems. He worked with the number triple 6-4-2 and correctly specialised a whole-part part diagram and related family of number sentences for this. He told the following as his three stories for his 6-4-2 additive relation:

Figure 108: Mpho's stories for 6-4-2



I have 2 cats. My mom bring more. I have 6 cats now. How many cats did my mom bring?

I have 2 car. My dad bring more. I have 6 cars now. How many car did my mom bring?

I have 6 eggs. My friend breaks 2. How many are left?

As directed, Mpho kept the numbers invariant across his three stories. He varied the characters (cats, cars, and eggs) and varied the verb from the first two stories (where the verb used was 'bring') for the third story (where he used the verb 'break'). He followed the instruction to include an example that used the term 'more'. However he did not use the comparative language of 'than'. He chose to make use of change contexts for all three of his stories. In his first two stories his use of 'more' related to an action of increasing (bringing more), and he generated change increase stories with the change unknown. He also posed a change decrease story with a result unknown in the context of broken eggs. It seemed that for Mpho the change story type was dominant. He was able to solve both change decrease and change increase problems, when these stories were presented to him, and he seemed to only be able to bring change problems to mind when asked to pose word problems.

Mpho's example space for additive relations expanded to include two kinds of change word problems: change increase and change decrease. He had not been able to solve or pose either of these problem types prior to the intervention. The inclusion of both change increase and change decrease problems into his personal potential example space is important, as it reveals that by the end of the intervention he was aware of change increase (change unknown) problems. He confidently posed change increase (change unknown) problems making use of the term 'more' to describe a process of increasing. His activity when explaining a subtraction number sentence provides further evidence of this learning gain.

Mpho's response to the task of tell a story for 10 – 7 = …, was presented as 'Vignette 1: Mpho learning to tell stories' (on page 246). He first offered 'ten minus seven equals three'. After hearing a teacher presented story about balls being kicked over a fence, he offered 'There was 10 cars. 7 was broken. How much is left?'. His story was a mixture of the collection type problem where two states are in focus: broken and not broken, however his question referred to a change type problem context where there had a change which meant some things 'were left'.

In the pre-interview, with the subsequent line of questioning for Mpho to tell harder stories, it became apparent that the numbers in the story were the key dimension of possible variation for his personal example space as he retold his cars story with bigger numbers: 'There was 20 cars 10 broke-ted, there was 10 left.'[27] In his retelling of the story, he chose numbers which he probably knew as a known fact (10 plus 10 is 20, or 20 minus 10 is 10). This was inferred as he did not need to calculate the solution using either fingers or drawings (which he had required previously for his calculation of 10 minus 7). The solution to his problem (10 cars) seemed to come to mind spontaneously for him. He neglected to pose the question, leaving this implicit, and rather answered it. When I again constrained the numbers for Mpho, he was able to tell a new story for 10 – 7 = …, and again made use of a change context: 'There was 10 hearts, 7…10 hearts, 10 heart, 4 choc…10 chocolates, 7 was eaten. Is 3.' Mpho's story was not completely coherent, but the overall meaning of what he was trying to communicate was evident. He seemed to imagine 10 chocolate hearts where 7 were eaten. This was confirmed when I restated his story and modelled the process of posing a question 'So we've got 10 chocolate hearts…10 chocolate hearts. 7 were eaten. How many were left?' and Mpho nodded in agreement. Again Mpho neglected to pose the question, but offered the solution: 'is 3'. This revealed that he could vary the change problem context from 'broken cars' to 'eaten chocolates'. He did not offer another problem type. All his variations were change decrease (result unknown), with a possible vague awareness of collection problem types in his personal potential example space. This suggested that his personal example space may have been limited to change decrease and collection problem types.

In the post-interview Mpho was more fluent in being able to offer stories for a symbolic number sentence, and he no longer required an example and teacher prompting to support his narration. He offered his story for 10 – 7 = … as: 'Mpho have 10 sweets. My friend take aways 7 sweets. How many Mpho have?' Mpho spontaneously used a change problem context involving a friend who took sweets away from him. By the post-test he was still showing difficulties with the verbs in English grammar ('take aways' not 'takes away', 'have' not 'has', 'have' not 'does have'); however his story was mathematically coherent and he posed a question rather than providing the solution. When asked to tell another one that a bit harder using 10 minus 7, he again chose to vary the numbers used in the problem: 'My friend have 70 sweets'. Mpho continued to associate 'harder' with bigger numbers. He

[27] Again the difficulties Mpho has as a second language speaker of English was evident with the past tense of an irregular verb, where an auxiliary verb was missing, and the verb was treated as if it was a regular verb ('broke-ted' instead of 'were broken').

maintained the context of sweets taken away by a friend, and increased the number range considerably (again making use of change problem context). As a result I directed him to constrain the numbers to 10 and 7. With the direction to make the problem harder, he attempted to change '10 – 7 = …' to a change decrease (start unknown) number sentence: '… - 7 = 10'.

> Mpho: My friend have 14…13 sweets, he take away 7 sweets, 15, 16 sweets[reaches for structured number line, counting back in ones from 16] , is 1, 2, 3, 4, 5, 6. My friend have 17 sweets, he take away 7, [using structured number line, counting back in ones from 17] 1, 2, 3, 4, 5, 6, 7
> T: And you get to?
> Mpho: 10
> (Mpho post-interview)

The story inferred from his varying the starting number (14,13 then 15, 16), as well as his actions which showed how we was solving this problem, was that the story he was trying to pose was: 'My friend has some sweets. He takes away 7 sweets and is left with 10. How many did he have to start with?' However '…- 7 = 10' was not a known fact calculation for Mpho. This was perhaps why he considered this to be a 'harder' story. He seemed to be trying to find a solution to this by trial and error. He tried 14, then 13 and needed a structured number line to guide his calculation. At first he just looked at the structured number line, and later he touched it showing jumps backwards in ones, to confirm his calculation process. He tried 15, then 16 before settling on 17 as his starting number. His strategy of using a number line, making use of a forwards count (1,2,3,4) while moving his finger backwards along the structured number line led to the correct solution of <u>17</u> – 7 = 10. This interaction revealed that what was in focus for Mpho was the structure of the number sentence. He varied the story (maintain the re-prompted constraint to only use 10 and 7) by changing the structure of number sentence. In so doing he did not reveal whether his personal example space included problem types other than change type problems (as the question intended). However he did reveal that his example space now included change problems where the position of the unknown was varied.

The above vignette provides evidence of several small learning gains for Mpho in relation to learning goal 3: Storytelling. Firstly in the pre-interview, Mpho did not know how to use storytelling to pose a word problem for a number sentence, but by the post interview he was fluently able to tell stories and pose questions appropriately. Secondly in the pre-interview it was also clear that he conceptualised subtraction using a change context, with a take-away calculation strategy. With teacher support, he posed change decrease (result unknown) problems. In the post-interview he was able to independently pose change decrease (result unknown) problems. With teacher support to constrain the numbers, he was also able to tell a change decrease (start unknown) story. However this problem situation was less secure for him, and he neglected to pose the question. This marked a shift from subtraction number sentences being interpreted as models for change decrease (result unknown) problem situations, to subtraction number sentences also being a model for change decrease (start unknown). Together with the evidence discussed above of Mpho generating change increase (change unknown) problems, this provided evidence that Mpho's additive relations word problem example space had expanded to include varying the position of the unkown, although it remained limited to change problem types.

Vignette 9: Best case of Retabile telling compare (disjoint set) problems

There were also learning gains evident for Retabile in terms of her use of storytelling. Retabile wrote the following stories for the number triple 10-3-7 when working on Main task 7: Learner generated examples:

> I have 10 apples. I ate 3 apples. How many apples are left?
> I have 10 dogs. 3 ran away. How many are left?
> I have 10 cars. 7 go away. How many are left?

In this activity Retabile revealed that change (decrease) contexts are dominant for her as all three of her stories had the same change decrease (result unknown) structure. She failed to apply the tasks constraint to use 'more than' in one of her stories.

In the pre-interview Retabile did not tell any compare problem type stories. In the post-interview also, these were not spontaneously told by Retabile. However I probed to see whether she was able to tell a compare (disjoint set) story where she made use of the words 'more than'. She was less fluent in recounting this (compared to the other stories she told during the post-interview) and required some teacher support to invoke a context of two disjoint sets:

> T: I want to see if you can use 'more than' [in a story].
> R: More than
> T: What if you have 10 and I have 7?
> R: I have 10 cars. Teacher Nicky have 7 cars. [… long pause][28]
> T: Now you have got to ask the question. It is quite a tricky question hey? [… long pause]. Lets tell your story again (pointing at the 10-7-3 whole-part-part diagram diagram). You have 10 cars. Teacher Nicky has seven cars…
> R: Yes…
> T: What question can we ask?
> R: You have ten and I have seven [Hides her face in her hands…long pause]
> T: How…
> R: How many…
> T: Good. How many…
> R: Cars.
> T: Mmm
> R: How many more do I have than Teacher Nicky?
> *(Retabile post-interview)*

This reveals that recounting a 'compare story' was not yet fluent for Retabile. She required teacher prompting to imagine that the comparison was between two disjoint sets (her and teacher Nicky). But she then spontaneously introduced a problem context of cars being compared in the two sets. It was difficult for her to pose the question, and needed a teacher prompt to start her off. Once I had suggested 'how …', she then was slowly able to formulate the appropriate question with phrases being re-voiced for her by the teacher, which seemed to reassure her to continue. Finally she was able to restate the question and introduce the problem situation of cars, which she had introduced (and then neglected) as she tried to formulate the question. Difficulties with posing compare questions was consistently evident within the intervention experience, where this language seemed new to learners.

[28] I note the grammatically incorrect use of 'have' for 'has'. However as this is plain English grammar, and not specific to the mathematics I have not attended to this kind of error (which is common amongst second language speakers of English).

Many learners needed prompts (at times spoken, and at times written) to articulate the compare questions.

Retabile gained confidence through this post-interview process, and when prompted to tell another story like 'her car story', was able to do so fluently: 'I have seven dogs. Teacher Nicky have ten dogs. How many more does Teacher Nicky have than me?' This gave me some evidence that Retabile was now able to tell compare stories, and I wondered if this extended to comparative situations where the words 'less than' were in focus. I asked 'can we make one that's got a question: 'How many less?' ' and wrote the word less on a piece of paper. Retabile then told this story: 'I have seven cats. Teacher Nicky have ten cats. How many …how many less do I have than teacher Nicky?'. She did this fluently, and spontaneously changed the problem context from dogs to cats. She correctly reversed the word order in the sentence for herself and teacher Nicky, for the 'less than' comparison to make sense. This provided evidence that although Retabile was unsure at first, when supported, she was able to demonstrate her ability to tell compare stories in the post-interview situation.

This learning gain is attributed to Retabile's engagement during the small group work on Main task 6: Compare (disjoint set) and Day 7 and Day 8, and during whole class teaching sessions on Day 8 and Day 9, and the teacher support offered in the interview. In the video recordings of the lesson, Retabile was seen to struggle at first to re-tell and compare type story with her partner (Rebecca). In the core group session on Day 7 learners pairs had been given compare type problems (involving themselves as the characters) which they worked on together to solve. They managed to draw a whole-part-part diagram and write a number sentence to solve the compare problem. Finally they were expected to re-tell and explain a new compare story. Retabile worked with Rebecca and chose to compare 8 stickers to 2 stickers, however they drew the following whole-part-part diagram.

Figure 109: Retabile and Rebecca's whole –part-part diagram comparing 8 to 2.

10	
8	2

The learners incorrectly combined the two amounts to reach the total. When attempting to re-tell the compare type story their conceptual difficulties with what they had done were surfaced:

> T: Girls are you ready to tell your story. Let's listen quietly as there are others who can go back to their desks if they want to.
> Rebecca: I have 8 stickers. Retabile 2 stickers. How many does …Retabile
> Retabile: How many more does …
> Rebecca: How many more does Refilwe need … How many more does
> Retabile: not need
> T: Ok I am interested in that story. Does one of you have 8 and other have 2?
> T: Or have you got 10 and 8 or 10 and 2. So who has got the more stickers here?
> Retabile: Rebecca
> T: Rebecca. How many has Rebecca got?
> Rebecca: 8
> T: And you [Retabile] have:
> Retabile: 2

T: 2. But then you are not comparing. Are you seeing how many you have got altogether? We want a story about how many more…
Rebecca: Retabile has … [looking back at the w-p-p]…[long pause]
T: I will give you girls a little bit of time to work that out.
(Day 7 lesson transcript, core group work)

Retabile was aware of that she needed to pose a question using 'How many more does', but was unsure how to do this (perhaps as she was conflicted by the incorrect whole-part-part diagram). Rebecca overlaid a change increase (reach a target) problem onto the compare situation. She abandoned or ignored the compare context, and imposed a context of needing more. This may be a result of the whole being 10 (where in the sticker stories, 10 was the target), the whole-part-part diagram seemed to bring this story to mind for her. Retabile then seemed to bring in a collection problem context, contrasting needing to not needing with a set. At this point it was clear that the pair were not aligned in the story they were trying to tell so I tried to establish whether they were comparing 8 to 10, or 2 to 10. I was aware that the referent in the compare story ought to be the whole in the whole-part-part diagram, and was trying to establish whether the whole was being used appropriately. With this clarification it became clear that Rebecca and Retabile were attempting to compare 8 to 2, but that they had positioned both sets as parts (and so were no longer able to compare them). The learners clarified which numbers they were using and I offered specific feedback on what had gone wrong and then provided them with more time to trying and resolve their conflict. I returned to the two girls later in the session, having heard some other stories:

T: Can you two tell a story yet? I don't want to know how many stars you have altogether, I want to know if you have got 10 and you have got 8: who has got more? Who has more, and how many more? Do you want to try again lets see:
Rebecca: I have 10 stars. Retabile have 8 [inaudible]
Retabile: How many more Rebecca have than me?
T: Well done. How many more does Rebecca have than...
Retabile: Than me
T: Excellent good story do you want to try that one more time?
Retabile: I have..Rebecca have 10 stickers. I have 8 stickers. How many more have than me?
How many more does Rebecca have than me?
(Day 7 lesson transcript, core group work)

The above provides evidence of Retabile learning to tell compare stories during the intervention. She and Rebecca opted to revise their story, rather than revise their whole-part-part diagram. Retabile correctly compared 10 to 8, and was able to pose a question using the phrase ' how many more'.

On Day 9 Retabile independently solved a compare type problem. She worked on the task: 'Retabile has 5 dogs. Kate has 3 dogs. How many more dogs does Retabile have than Kate?'. She correctly drew a 5-3-2 whole-part-part diagram, and wrote the answer sentence: 'Retabile has 2 more dogs than Kate'. This provides an indication of the way in which Retabile engaged with the compare type problem tasks in the intervention, and it is inferred that this activity led to her learning gain in being able to solve compare type problems, and, with some teacher support, being able to tell compare-types stories.

This learning gain of now being able to work with compare type problems, was evident again in Retabile's activity with the generalised problem tasks. In both interviews Retabile spontaneously specialised the generalized problems and introduced whole numbers which she could work with to replace the unknowns.

> T: I have some apples. You take some of my apples. How can you work out how many I have left?
> Retabile: I have some apples. [pause] How many apples?
> T: How many apples? Do you want to tell me how many?
> Retabile: Six?
> T: Six. And then what happens?
> Retabile: Take away….four
> T: Yes
> Retabile: I have two
> T: Two. So you made the story: I have six apples. You took away….
> Retabile: Four
> T: Four. And how many are left?
> Retabile: Two
> (Retabile, pre-interview)

Initially Retabile was unsure whether she could specialise, and asked to be given the number of apples. But she quickly specialised introducing six and four. She fluently solved this process to model the generalised change context.

Retabile repeated this process in the post-interview, choosing the same number triple 6-4-2 which seemed to be a known fact for her. There were no shifts in learning evident for the generalised change problem. She seemed slightly more confident in the post-interview when she knew that specialising was permitted. Both before and after the intervention Retabile specialized and modelled a process of solving the change problem.

Retabile provides a best case example of learner activity on the generalised compare word problem task. In the pre-interview she was able to specialise the problem and demonstrate understanding of who had more. However she was not able to invoke a 1:1 matching action or an appropriate mathematical relationship for this problem. By the post-interview she invoked a 1:1 matching action to solve the problem.

In the pre-interview Retabile's responded as follows to the task of 'I have some sweets. You have some sweets. You have more sweets than me. How can you work out how many more sweets you have?':

> Retabile: I have got 3. You have got 7.
> T: Ok so you have got 3 and I have got 7.
> [Long pause…]
> So how many more have I got?
> [Long pause…]
> Will it help if we have a look? [T reaches for counters] You said you have got?
> Retabile: Three [T gives her 3 counters]
> T: And I have got?
> Retabile: Seven
> T: Seven [T takes out 7 counters]. How many more have I got?
> [Long pause…] Do you know how to work out 'how many more'?
> Retabile: No. [She shakes her head]
> (Retabile pre-interview)

This shows that Retabile knew that was happening in the problem situation (that she had 3, and I had more with 7), but she did not know how to compare these amounts and respond to the question: 'How many more do I have?'

By the post-interview the process of comparison seems no longer to be a problem for her. She correctly re-voiced the generalised question, specialising it to involve 10 and 6 sweets, posed an appropriate question, and quickly provided an answer:

> Retabile: I have 10 sweets. You have 6 sweets.
> T: OK
> Retabile: How …many… more... do I have…?
> T: Good. How many more do you have?
> Retabile: Two
> T: Two more?...Can you show me a diagram?
> Retabile: Yes
> [Retabile draws wpp with 10 as whole, and 6 as part and 2 as part]
> [Long pause, appears concerned]
> T: What's gone wrong there?
> [Long silence then she corrects 2, and replaces with 4].
> (Retabile post-interview)

Retabile was able to provide an answer to the questions 'How many more do I have?', and conceptualised this diagrammatically as a whole-part-part diagram. However she was incorrect in her calculation that 10 compared to 6, was 2. She corrected this only after teacher intervention for her to find the error. However I think this provides evidence that Retabile shifted in her knowledge of how to compare disjoint sets (she correctly positioned and labelled the one set as a whole, and the other as part as the unknown), although she calculated the unknown incorrectly and had to revisit this calculation.

Vignette 10: Telling case of Gavril's reluctance/ inability to tell compare (disjoint set) problems

In Gavril's post-interview it was clear that he was reluctant and/or unable to tell compare (disjoint set problems), despite being able to solve these problems (as evident in both his pre- and post-tests). Despite prompts and attempts to shift Gavril away from change word problem contexts towards using comparative language, he remained stuck with change contexts.

I prompted Gavril to tell a sticker story, hoping that this would provoke him to use the comparative language of 'more'. He ignored the suggestion to use stickers, and chose to refer to cars, which he then changed back to stickers. He imposed a change decrease action onto this story: 'I have 7…mmm…ja…I have 10 cars…hmm…10 stickers…I give Anele 7. How many do I have left?' It is notable that all the stories Gavril used to explain the subtraction number sentence made use of change actions. Later in the post-interview I again tried to get Gavril to bring comparison stories to mind:

> T: So Gavril I want to see, can you tell me a story where the question is how many more do you have than me? Can you tell me a story about that? How many more do you have than me? You don't have to write it you can just tell me.
> Gavril: Oh!…you have 30 stickers, I have…Oh, yo…ja…you have 30 stickers I take 20. How many do….hoe!....how many you got?

(Gavril post-interview)

Although Gavril could solve compare (matching) and compare (disjoint set) problems, he was not yet able to bring these to mind and retell these stories. So his reference narratives remain at change level, although his problem solving and representations show that he is able to deal with compare situations.

The above suggests that additional time was required during the intervention to shift children away from dominant take-away problem situations, towards making use of compare contexts in their problem posing activities. A task such as: 'Read this story (a compare (disjoint set) problem). Tell another story like this one', may support this process. In future intervention designs, additional time for such tasks should be trialled to test whether this supports more children to be able to pose compare (disjoint set) problems (and whether this impacts on how many children can solve such problems).

Gavril was able to show some learning gains and to use a 1:1 matching action for compare situations in the generalised change and compare problems in the interviews. Gavril's response to the generalised change (decrease) task was typical of all the learners interviewed (and similar to Retabile's response reported above). Gavril was able to specialise, and chose a known fact additive relation (10-5-5) and he invoked a change decrease action of removal (take-away). There were no learning gains evident for this task – as at both pre-interview and post-interview stages, as at both points in time he was able to specialise and demonstrate his ability to work with change decrease (result unknown) problem contexts.

Learning gains were however evident for the generalised compare (disjoint set) task. Gavril provides a best-case example of learner activity on this task. In the pre-interview it was clear that Gavril was not sure of how to solve a general compare (disjoint set) problem:

T: I've got some sweets you've got some but you've got more than me. How do you work out how many more than me you have?
Gavril: I have 20, and you've got 10, then, whooo, must I take away or must I plus?
T: Mmm, it's quite tricky hey? Think about it. I've got…you said you've got 20 and I've got 10, you've got more than me right? How many more have you got than me?
Gavril: Yo!
T: Is that idea of "more" a bit tricky? Shall we try it with a smaller number first?
(Gavril pre-interview)

I introduced concrete counters, giving Gavril 6 bottles tops and keeping 4 bottle tops in front of me. He said that he had six, and that I had four. With this intervention – and without Gavril arranging the counters in any way – he said 'ooooh', and was then independently able to return to his calculation involving 20 and 10:

Gavril: I got 20, you got 10, you need 10 more.
T: Well done, so I need 10 more to be the same as you. Okay, good job. And with the 4 and the 6, how many more was it?
Gavril: Two
T: Two. Well done.
(Gavril pre-interview)

After his initial uncertainty, Gavril was able to specialise the general compare (disjoint set problem), making use of 20 and 10 demonstrating awareness of the notion of 20 being more than 10. He then asked whether he needed to 'take-away' or 'plus'.

In the post-interview Gavril was able to respond to this question fluently. He once again specialised using a known fact relationship of 20-10-10. He did not need to use a 1:1 matching action as he was aware of the additive relationship, and how to use it in a comparison context:

> T: I have some sweets and you have some sweets. You have more sweets than me. How do we work out how many sweets you have? How many more sweets you have?
> Gavril: Ok…I have 10 and…mmm…I have 20 and you have 10. Oooh. So…
> T: So how many more have you got than me?
> Gavril: Ooooo 10.
> T: You've got 10? Oh, how did you work that out?
> Gavril: Because 10 plus 10 is 20. And take away another 10 is still 10.
> (Gavril post-interview)

This interaction demonstrated his awareness of the equivalence of 'part + part = whole', and 'whole – part = part'. He was no longer concerned as to whether it was 'plus' or 'take away' as he knew: '10 plus 10 is 20. And take away another 10 is still 10'. This was taken as evidence of learning gain where Gavril became aware of the relationship between addition and subtraction. He was no longer focused on which operation was appropriate, as he flexibly moved between the equivalent expressions of 'part + part = whole' and whole – part = part'.

Synopsis of learning gains for each case study learner

The actual learning trajectory of each of the case study learners was unique. This depended both on their actual starting point as well as their activity in class as they engaged in the assigned tasks. For Retabile and Gavril the opportunities to learn closely matched the hypothesised learning trajectory. For Mpho the opportunities to learn did not include the same focus on the partition problem and the compare (disjoint set problem) and this is seen to impact on his learning gains. In this section I provide a tabulated summary of the learning gains or absences for each of the case study learners.

Table 71: Mpho's learning trajectory

Learning goals	Mpho's starting point (pre-test and pre-interview)	Mpho's end point (post-test and post-interview)	Learning gain
LG 1: Problem solving			
Change problems	Not yet able to solve change increase (result unknown), or decrease problems, with number range to 10	Able to solve change and compare (matching) word problems, partial correct response to partition problems	✓ Yes
Collection problems	Not yet able to solve collection problems	Not yet able to solve collection problems	× No
Compare problems	Not yet able to solve compare problems	Not yet able to solve compare problems	× No
LG 2: Representations			
Group model	Some iconic grouping using a dice pattern evident in pre-interview	Grouping actions evident in iconic representations Grouping actions evident with jumps on number line Grouping action evident when acting on bead string	✓ Yes ✓ Yes ✓ Yes
	No evidence of take-away action in ones	Take-away action in ones secure	✓ Yes
	Unaware of subtracting as difference using 1:1 matching action	Unaware of subtracting as difference using 1:1 matching action	× No
Line model	Empty number line, using count all and counting on in ones	Empty number line, using counting on and jumps of groups more than one	✓ Yes
	Smaller on left and bigger on right secure	Smaller on left and bigger on right secure	× No
	Take-away calculation strategy not yet secure	Take-away calculation strategy secure	✓ Yes
	Unaware of difference calculation strategy	Used difference calculation strategy to solve compare (matching) word problem	✓ Yes
Syntax model	No evidence of number sentences. Wrote only operation symbols + or -	Writes number sentences in form part + part = whole and whole - part = part	✓ Yes
	Not yet familiar with family of equivalent number sentences	Fluently completes family of equivalent number sentences	✓ Yes
	Not yet familiar with whole-part-part diagram	Draws whole-part-part diagrams to solve word problems and to explain relationship between number sentences	✓ Yes
LG 3: Story telling			
Problem posing	No previous experience of generating their own examples of word problems	Fluently tells decrease (result unknown) and change increase (change unknown) stories.	✓ Yes
Explaining problems	No previous experience of telling stories to explain word problems	Explains solutions to problems and explains number sentences using change situations	✓ Yes
Proportion of learning gains to absences of learning		13 learning gains, 4 absences	76%

Table 72: Retabile's learning trajectory

Learning goals	Retabile's starting point (pre-test and pre-interview)	Retabile's end point (post-test and post-interview)	Learning gain
LG 1: Problem solving			
Change problems	Able to solve change decrease problems (result unknown), with number range to 20, but not able to express relationship using number sentence	Able to solve change decrease problems (result unknown), with number range to 20, and able to express relationship using a number sentence and whole-part-part diagram	✓ Yes
Collection problems	Able to solve collection problems, with number range to 20	Able to solve collection problems, with number range to 20	✓ Yes
Compare problems	Not yet able to solve compare problems	Able to solve compare problems using 1:1 matching, whole-part-part diagram and number sentence	✓ Yes
Partition problems	Identifies two partitions using iconic representations	Identifies all partitions working systematically and completely using number sentences	✓ Yes
LG 2: Representations			
Group model	Works in ones, no evidence of grouping actions	Works in ones, no evidence of grouping actions	× No
	Take-away action in ones	Take-away action in ones	× No
	Unaware of subtracting as difference using 1:1 matching action	Uses difference strategy for compare problem	✓ Yes
Line model	Structured number with reference point of 1, shows action of hops in ones using count on or back in ones	Structured number with reference point of 0, shows action of hops in ones using count on or back in ones	× No
	Smaller on left and bigger on right secure	Smaller on left and bigger on right secure	× No
	Take-away calculation strategy secure	Take-away calculation strategy secure	× No
	Unaware of difference calculation strategy	Difference calculation strategy used for compare word problem but not for 23 – 18 bare calculation	✓ Yes
Syntax model	Secure with reading and writing number sentences in the form part + part = whole and whole – part = part but not yet seeing relationship between them	Recognised relationship between number sentences and depicts the common relationship using a whole-part-part diagram.	✓ Yes
	Not yet familiar with whole = part + part and part = whole – part forms of number sentences	Uses number sentences in the form of whole=part + part in partition problem. Fluently completes family of equivalent number sentences.	✓ Yes
	Not yet familiar with whole-part-part diagram	Draws whole-part-part diagrams to solve word problems and to explain relationship between number sentences	✓ Yes
LG 3: Story telling			
Problem posing	No previous experience of generating their own examples of word problems	Fluently tells change decrease (result unknown), change increase (change unknown) word problems. With prompting is able to tell compare word problems using both 'more than' and 'less than'.	✓ Yes
Explaining problems	No previous experience of telling stories to explain word problems	Explains word problems using whole-part-part diagrams and generating stories for explaining (using change decrease result unknown contexts)	✓ Yes
Proportion of learning gains to absences of learning		12 learning gains, 5 absences	71%

Table 73: Gavril's learning trajectory

Learning goals	Gavril's starting point (pre-test and pre-interview)	Gavril's end point (post-test and post-interview)	Learning gain
LG 1: Problem solving			
Change problems	Able to solve change problems	Able to solve change problems	× No
Collection problems	Able to solve collection problems	Able to solve collection problems	× No
Compare (matching) problems	Able to solve compare (matching) problems	Able to solve compare (matching) problems	× No
Compare problems	Able to solve compare (disjoint set) problems	Able to solve compare (disjoint set) problems	× No
Partition problems	Able to solve partition problem working systematically and completely	Able to solve partition problem working systematically and completely	× No
LG 2: Representations			
Group model	Works in ones, no evidence of grouping actions	Still works in ones, but evidence of acting on groups with the bead string, and jumps of more than one shown on the number line	✓ Yes
	Take-away action in ones	Take-away action in ones	× No
	Unaware of subtracting as difference using 1:1 matching action	Uses difference strategy for compare problem	✓ Yes
Line model	Empty number line, shows action of hops in ones using count on or back in ones	Empty number line, shows action of hops in ones using count on or back in ones	× No
	Smaller on left and bigger on right not yet secure	Smaller on left and bigger on right secure	✓ Yes
	Take-away calculation strategy secure	Take-away calculation strategy secure	× No
	Unaware of difference calculation strategy	Difference calculation strategy used for compare word problem and evidence of some line model confusion	✓ Yes
Syntax model	Secure with reading and writing number sentences in the form part + part = whole and whole – part = part but not yet seeing relationship between them	Recognised relationship between number sentences and depicts the common relationship using a whole-part-part diagram.	✓ Yes
	Not yet familiar with whole = part + part and part = whole – part forms of number sentences	Uses number sentences in the form of whole=part + part in partition problem. Fluently completes family of equivalent number sentences.	✓ Yes
	Not yet familiar with whole-part-part diagram	Draws whole-part-part diagrams to solve word problems and to explain relationship between number sentences	✓ Yes
LG 3: Story telling			
Problem posing	No previous experience of generating their own examples of word problems	Fluently tells change decrease (result unknown), word problems. Imposes change decrease action of take-away even when prompted to use possible compare contexts.	✓ Yes
Explaining problems	No previous experience of telling stories to explain word problems	Explains word problems using whole-part-part diagrams and generating stories for explaining (using change decrease result unknown contexts)	✓ Yes
Proportion of learning gains to absences of learning		9 learning gains: 8 absences of learning	53%

The tables above reveal that qualitative learning gains were evident for each of the case study learners. While Retabile and Mpho represent best-case examples, Gavril represents a telling case of a learner in the upper attainment levels who came into the intervention with much of what was to be learnt already secure. Although Gavril was already able to solve all of the word problems, there was evidence of Gavril learning in relation to his use of representations.

This chapter has provided the qualitative detail relating to learning gains for each of the case study learners. This evidence, together with the findings in the previous chapter relating to learning gains evident for the whole class, and each ability group, lead to my claim that the intervention shows promise in supporting children's learning in relation to additive relations word problems. By using vignettes of specific learning gains for particular children, evidence of how this learning may have come about through the learner activity in the lesson intervention was presented alongside the particular learning gain.

CHAPTER 8: Conclusion

The third intervention cycle of this design experiment has shown promise in the particular local context of the focal school. The promising features have implications both theoretically and practically relating to the teaching and learning of additive relations word problems in particular and mathematics in general. The implications of the study are discussed firstly in relation to researching the efficacy of a narrative approach to mathematics learning with young children and secondly in relation to the ontological innovations which emerged from this study.

In order to discuss the implications of the study for improving the teaching of additive relations, it is worth first revisiting the research aims and questions. In the context of a paucity of research in South African Foundation Phase mathematics, this study set out to research a narrative teaching approach for a specific mathematics topic: additive relation word problems. In particular the teaching approach focused on 'compare' type word problems where the numbers of objects in two disjoined sets are compared. The following was the primary research question: To what extent do young children's example space of additive relations expand to include compare type word problems?

At the heart of the study therefore, was a question relating to the efficacy of a teaching strategy in a particular classroom-based design experiment. Efficacy of teaching was measured by looking for evidence of learning gains related to three interrelated learning goals: Learning goal 1: Solve a range of additive relation word problem types; Learning goal 2: Flexibly use a range of representations to pose and explain word problems; and Learning goal 3: Tell stories to pose and explain word problems. The following subsidiary questions broke down the primary research question:

1. How was the intervention designed; how was the design refined over multiple research cycles; and how did the third cycle intervention play out in this particular local context?
2. What evidence of learning gains (in relation to the learning goals), if any, are seen as a result of the teaching intervention:
    - What evidence of learning to solve a range of additive relation word problems (LG 1) is seen during and following the intervention?
    - How did children make their thinking visible by using narrative in this intervention, particularly with regard to:
        - Flexibly using a range of representations to pose and explain word problems (LG 2); and
        - Telling stories to pose and explain word problems (LG 3).

In this concluding chapter I first offer a summary response to the primary research question where the efficacy of the intervention was found to hold promise. I consider the theoretical implications of this finding in terms of how narrative can be used as a cognitive and pedagogic strategy in the mathematics learning and teaching of young children. I then discuss the theoretical implications for improving the teaching and learning of additive relations word problems, and compare type problems in particular.

Two contrasting approaches to additive relations word problems which figure independently in the mathematics education literature, were brought alongside each other within a single intervention design. Particular patterns of children's reasoning about additive relations word problems were documented from the South African ELL children in this design experiment, and new actions and contrasts relating to additive relations were brought into focus. In so doing I focus on ontological innovations within the additive relations domain which were either inserted into the intervention task design and thought to support learning, or were common misconceptions of patterns of learner activity noticed within this design experiment. I suggest some implications for the study for curriculum developers and teacher educators relating to how to better support teachers in approaching the additive relations word problem domain. To close this chapter, and this thesis, I suggest areas for possible further research in the light of its findings.

Synopsis of the study findings

In this section I summarise the response to the primary research question: *To what extent do young children's example space of additive relations expand to include compare type word problems?*

Both the second and the third cycle intervention showed promise in expanding young children's example space for additive relations word problems. In both cycles the mean results improved from pre-test to post-test. The gains evident immediately after the intervention were retained in a delayed post-test administered for the third cycle which showed further improvements in the mean with a reduced standard deviation. The improvements evident in the delayed post-test are not attributed solely to the intervention, but demonstrate that maturation and further teaching and learning further enabled improved learner attainment in the written test. These trends were evident when using a simple marking framework attending only to correctness of answers and an expanded marking framework which included consideration for correctness of solution; coherence of self-selected representations showing progress towards problem solving; correct partitioning; systematic working as well as coherence relating to prompted number lines; prompted number sentences and other self-selected representations of the same problem situation. The effect sizes of the shifts in means from pre-test to post-test was 0.7 (medium) in both cycles, while the effect size of shifts in the mean from pre-test to delayed post-test was 1.3 (large). T-tests established that these shifts in means were statistically significant.

The third cycle intervention showed more promise in expanding children's examples spaces to include compare type word problems, than the cycle 2 intervention did. Considering the changes in facility of the word problem test items there was the greatest difference in facility between the pre-test and post-test result for the compare (disjoint set) problem in the third cycle. This suggested that the intervention held promise in expanding learners' example space for additive relations word problems to include the compare type word problem.

The third intervention cycle also showed promise in relation to learning gains in terms of representations. There were statistically significant shifts from coherent to incoherent representations in the learners' self- selected models. These shifts toward greater coherence were evident across group, line and syntax representations. While learners use coherent use of syntax models increased substantially, there was evidence that learners were less secure with coherent line models. This finding – of reluctance to use, and lack of fluency when using, a number line - coheres with the empirical evidence from small-scale research in South African contexts of a Grade 2 and a Grade 4 class analysed in Takane, Tshesane and Askew (in press), and of a Grade 4 class analysed in Wasserman (2015).

Within representation types there was evidence of learners shifting towards more structured use of these representations. Considering the calculation actions evident across representation types there was greater use of 1:1 matching and partitioning actions by the post-test; and declines in the use of take-away actions.

In contrast to the finding of Takane in a South African isiZulu grade 2 class that 'when the learners were asked to come up with their own stories, they offered non-mathematical word problems' (Takane, Tshesane et al. in press), the majority of learners in the third cycle of this design experiment were able to narrate a coherent additive relations word problem story which was connected to a syntax model representation. Most learners told stories relating to change problem type, with fewer stories in the collection and compare problem categories. There was evidence that 12 learners were able to narrate more than one type of additive relations story.

The learning gains for the extension and core groups were greater than the learning gains for the support group. The core group showed the greatest learning gains, suggesting that the intervention was most successful in 'raising the middle' of the class. Analysis of the intervention implementation against the planned intervention design from the perspective of teacher in terms of task design revealed that the hypothetical learning trajectory for the core and extension groups closely matched the actual learning trajectory. This was not the case for the support group where a lower than expected starting point resulted in endogenous design changes to the learning trajectory for this group of learners. The teaching adjustments made to the actual learning trajectory in response to observed learner activity on the tasks resulted in the support group having less opportunity to learn about compare (disjoint set) word problems.

As was expected there were variations in the learning gains for particular learners. These depended both on their intellectual and social starting points and their learner activity in relation to the tasks. The differences in this regard have been exemplified with three case study learners in the previous chapter and will not be repeated here. The learner activity on the main and enabling tasks have provided further insight into the successive patterns of learner thinking, which may be used to further improve the task design by anticipating common learner misconceptions.

Research limitations

While acknowledging the finding of substantial learning gains, I also note the limitations of this study. Each intervention cycle was undertaken in a tightly defined timeframe of ten consecutive days of schooling. This was imposed as a result the need to focus attention on the intervention design inputs so that the learners could be assessed before and after the intervention, while limiting other factors which may have impacted on learning (such as maturation, or mathematics learning through other learner activity in mathematics, and engagement with the normal classroom teacher). As a result of the tight timeframe constraints, some opportunities which arose in the course of the intervention, could not be exploited. By way of example, when learners started exploring the possibilities arising from three partitions (of 3 trees in the partition problem), this could not be extended. The additional time spent on the partition problem, resulted in less time being spent on the Main task 3: Learner generated examples of change problems. The limited timeframe impacted negatively on the support group of the third cycle intervention. More time was spent on pre-requisite fluencies with less time available for activity on the variations of the partition problem and the compare (disjoint set) word problems. This was thought to impact on the learning gains for this group. In future design cycles, additional time should be allocated for an intense intervention (15 lessons), or additional lessons should be planned for a spread out over a term in a more naturalistic setting. Further the tight timeframes meant the representations introduced were only consolidated in relation to the content domain of additive relations. The line, group and syntax models are relevant representations in several content domains in mathematics, but their use could not be developed in more varied contexts during this tightly constrained intervention.

Reflecting back on the intervention and considering the nature of my own self-reflection and the type of feedback obtained from critical peer-review I felt that in many cases during the lesson intervention my teaching was too directed towards which representation ought to be used, and did not give sufficient wait-time to learners to come up with their own representations. Several of my reflective comments include critical reflections like 'I should have let them work more on their own'. Takane, Tshesane et al. (in press) have noted the importance of this kind of emergent 'horizontal modeling' in their Grade 2 excerpts of a word problems focused intervention. This research is therefore limited by an inevitable gap between theory (a hoped for) and practice (the in-the moment decision making required in a classroom) relating to the teacher role. In future intervention cycles greater attention ought to placed on making use of wait time (Tobin 1986), and slowing down the teaching process where a cycle of thinking, marking, re-marking, rehearsing and then communicating (Mason and Spence 1999) may be an important addition to the theoretical framework.

This research was also limited in that while its overall findings suggest that intervention holds promise, there were different findings for the support group learners where learning gains were far more modest.

This suggests that a different kind of intervention, or greater attention to task design for support group learners would be required. In several points in my critical self-reflective notes I noticed a tendency to deviate from the theoretical features when working with the support group, noting that 'my work with this group tends to become procedural, with more teacher talk and less time given for learner activity and learner talk'. I think that this is in important observation as it reveals that under the stress of in-the-moment decision making during the intervention, I was not only unable to follow the HTL in terms of task selection and sequencing but also deviated more from the theoretical features when working with this group. Their lack of learning evident in the findings section in Chapter 6, seem to be related to their lack of foundational competencies to some extent; but there was clear evidence that they had less opportunity to engage with the tasks (some tasks were omitted completely), and that their engagement with tasks, was at times not of the same quality of teacher-lead approaches as key theoretical features were less evident for this group.

My own language proficiency impacted on what could be researched with these ELLs. Not speaking Sesotho or isiXhosa was a research limitation. The problem contexts used in English could have been explored in the home languages of the learners. In so doing, the home languages of the learners would have been used as a resource. In future intervention cycles, tasks which encourage peer-to-peer use of particular home language to tell stories, and then re-tell then in English should be included. These tasks would benefit by the facilitation of a teacher who is fluent in these languages and English.

Finally there was a methodological limitation in relation to the analysis of test data. Using two marking frameworks for assessment of written tests meant that it was possible to conduct inter-rater reliability testing on the expanded version of the framework. Such tests are used to assess the degree to which different raters/observers five consistent estimates of the same phenomenon. Such tests were not included in the study, and ought to be included in future design cycles.

Research implications

The research implications are discussed first in terms of how the study has extended the theoretical framing of a particular pedagogical approach to mathematics teaching and learning. The focus here is at a broad level of the emerging characteristics of 'a narrative approach' to mathematics learning which have been sharpened through this study. The research implications are then discussed in terms of the ontological innovations which emerge from this study and which pertain to the more detailed level of additive relations word problems. Finally some implications are discussed which may contribute towards better guidance and support for Foundation Phase teachers approaching the teaching of additive relations word problems.

What are the implications of this study for the efficacy of a narrative approach to teaching mathematics?

This section focuses first on the implications of this study. It then turns to situate this study in relation to a previous design experiment study seeking to highlight how this study is a theoretical extension of pedagogical approach to mathematics learning.

Implication 1: This study shows that when appropriately supported young ELL learners in a South African township school, can expand their additive relations word problem example space in a relatively short timeframe

This design experiment reveals that when adequately supported with careful task design and effort in monitoring and responding to learner activity, Grade 2 ELL children in a township *can* improve their additive relations problem solving, in a relatively short time frame. The majority of the learners in the third cycle of this design experiment were able to solve compare type problems at the end of the 10-day intervention. These learners were also able to produce evidence of movements towards more structured representations, and towards better learner explanation and problem posing using storytelling. This study therefore mirrors other emerging studies in South Africa that have shown that substantial learning gains are possible in short term interventions focused on early number in the Foundation Phase (Takane (PhD in process); Morrison (PhD in process); Graven 2012 and Wasserman 2015).

The approach trialled in this study offers a window into the 'possible' in ELL South African township contexts. I emphasize that the approach described should be viewed as an indication of what might be possible, rather than prescriptive of an approach that should be adopted by all teachers. It is not expected that the third cycle intervention could be easily replicated in a normative South African township classroom. One of the main reasons for this assertion is that the intervention approach was intense (concentrated into 10 consecutive days), involved a significant deviation from the common pedagogical approaches in South African classrooms and was highly demanding of the lead teacher. Nevertheless what has been achieved in an intervention of short duration, may now be trialled as researched tasks and suggested teaching approaches that could be tested in other similar contexts over longer intervention periods (over several terms, or an academic year). Attention would need to be given to supporting the lead teacher to adopt the pedagogical approaches articulated in the theoretical framework (and it is anticipated that such a support process, would be of research interest and would further deepen and clarify the theoretical framework).

Implication 2: Trialling an innovative pedagogic approach in a challenging context, necessitates attending to frameworks for action which are particular to the local context

Considering the promising results shown in relation to learning gains in this design experiment, there seems be potential in using narrative in mathematics education in more challenging contexts. However the adaptations required for the intervention from cycle 2 to cycle 3, which led to the inclusion of a framework for action are important considerations in this more challenging context. This design experiment revealed that the approach adopted could not simply be imposed onto a new local classroom contexts, and that careful planning and consideration for the existing classroom culture was required. This led to tighter planning relating to training behaviour (Feature 5), and being explicit in the general pedagogic style (Feature 6).

Implication 3: This study has extended the theoretical foundation underlying a narrative approach to teaching and learning mathematics with young children

This study denotes a small step towards further refining a pedagogical approach to mathematics teaching and learning with young children. As such it offers a theoretical extension to a small existing research base on the promising impacts of the use of narrative in mathematics with young children. I previously researched the potential of using narrative as a pedagogic strategy through a design experiment focused on the use of narrative for teaching parity to young (5-6 year old) children and found promising results in less challenging schooling contexts in a state school in England, and private school in South Africa(Roberts and Stylianides 2013). The approach adopted in the 'Telling and illustrating stories of parity' study has now been researched again with three important changes: The narrative in mathematics approach was used with slightly older children (8-9 year olds), the mathematics topic was changed from parity to additive relations word problems, and this intervention was undertaken in a more challenging implementation context of a public primary school in a South African township, where the majority of learners were ELLs. The study of 'Telling and illustrating additive relations stories' has also yielded promising results for learning. This strengthens the argument that focusing attention on how narrative can be used as pedagogic and cognitive strategy in the teaching of mathematics with young learners seems to have potential. This research work is small-scale, embryonic and exploratory. Nevertheless the findings of this design experiment, when taken together with the findings from the previous design experiment, suggest that the 'narrative approach' adopted in both experiments, shows promise in supporting the learning of young children in different local contexts and concerning different mathematical topics.

This study has helped to extend the theoretical foundations of what is meant by a 'narrative approach' as the theoretical features of the narrative approach are now situated within a broader theoretical framework of orienting theories, domain specific instructional theories, and related frameworks for action. There are a few possible defining characteristics for a narrative approach which have been more clearly defined in relation to the use of storytelling and representations as entangled elements of a narrative approach.

The first possible defining characteristic of a narrative approach is that both logical scientific knowing and narrative/paradigmatic knowing are to be attended to simultaneously. In this study both kinds of knowing were viewed as necessarily and simultaneously present in additive relations word problems (Feature 3.1). By focusing on additive relations word problems, the narrative knowing was necessarily present (by the nature of mathematics word problems as expressed in natural language, and adopting elements of the narrative form as a series of connected events). In the previous design experiment paradigmatic knowing was imposed onto the content specific domain of parity, as it was not necessarily present in the parity content domain.

The second possible defining characteristic which emerges is that a narrative approach uses storytelling both as a pedagogic strategy to motivate learning and encourage sense making (Feature 3.2) and as a cognitive strategy which draws on the human powers of imagining and expressing (Feature 3.3). In both design experiments the lead teacher selected and carefully planned the reference stories which would be introduced into the classroom community. In this regard storytelling was invoked by the teacher to harness the imagination and interest of the learners. But in both studies, there was also a deliberate shift from storytelling by the teacher, to storytelling by learners. This theoretical feature was present and discussed in both design experiments on narrative in mathematics, but it was refined and described in greater detail in this thesis.

The third possible defining characteristic of the narrative approach, adopted in both design experiments, was encouraging flexible movement between representations where sense making is primary (Feature 4.1). Through this design experiment, the way in which representations are conceptualised to support children in making their thinking visible has been further refined. The recognition that secure use of a particular representations takes time, and over time, representations should be reified (Feature 4.2) was introduced. Attending to this aspect meant that there was careful planning and attention to how representations were to be introduced and slowly developed over time. The process of how to induct young learners into using symbolic formal notation which starts from their informal notations has been carefully considered. In this process shifts in attention are required in both directions – from concrete objects to symbolic objects (and vice versa) and both processes of specialising and generalising are necessary (Feature 4.3). This is a theoretical component which saw substantial changes in the teacher-lead approach to signification. Further enhancements in adopting more of an emergent approach to symbolising and communicating, while engaging in the social performance within the class community ought to be sought in future design cycles.

Finally, and related to the above, was the introduction of a teaching-learning trajectory from counting to calculating where increasingly structured representations are expected (Feature 4.4). These representations make use of line, group and syntax models, and within each representation there are possibilities for increasing structure.

What are the theoretical implications for teaching and learning additive relations word problems?

There are several theoretical extensions which emerged through the dialectical process of drawing on education literature relating to additive relations word problems, and reflecting on the empirical data emerging from each intervention cycle in this design experiment.

Implication 4: Research findings relating to learner activity on additive relations word problems conducted in other contexts have been researched with South African township classes

There are several findings emerging from this design experiment which support findings relating to learner activity of additive relations word problems in other parts of the world. While these research findings have been largely accepted as likely to be generalizable to other groups of children; it was not known whether the findings from the CGI, VDC and RGH studies (all conducted in the Global North) would hold with South African ELL learners. There are several of the international research findings which have now been established to hold for the learners in this design experiment.

Considering the sub-types of each additive relations word problem type, the compare (disjoint) set problem was consistently found to be the most difficult for learners to solve. This confirms the international research finding that compare problems are most difficult, which has been established across various contexts.

Prior research has contrasted dynamic/action problem situations (change increase and change decrease problem types) with static situations (collection and compare problem types). The action described in the dynamic situations relates to actions of removal or joining, and the collection and compare problems are described as having no action. Carpenter et al. (1999) referred to the process of 'matching' as a direct modelling option when solving compare (disjoint set) problems. The empirical results indicated that inserting attention to 1-1 matching actions was found to be useful to helping learners to deal with static compare situations. This point – relating to the pedagogical implications of approaching additive relations word problems - is not widely made in the literature. For young children unfamiliar with comparing, they need to bring to mind the 1:1 matching action as a possible process to impose onto an apparently static situation. The '1:1 matching action' is an action intuitively performed by experts when comparing which is done without detailed attention to what this mental action entails. Teachers can support learners to bring to mind a 1:1 matching action, by inserting compare (matching) problem situations into their task design sequences.

This design experiment provided empirical data of young South African learners trying to make sense of compare-type problems. The tendency of many ELLs in this design experiment to respond to compare word problems like *'Mahlodi has 12 sweets. Moeketsi has 8 sweets. How many more sweets does Mahlodi have than Moeketsi?'* with: *'Mahlodi has 12 sweets',* has now been documented. The empirical evidence of young learners working with compare problems suggested that that problem solving process breaks down at the point of constructing a situation model, and not at the point of considering an appropriate mathematical model. As observed by Verchaffel and De Corte in the Flemish context, it appears that these learners have not 'sufficiently mastered the semantic schemata for understanding the problems and for relating them to already available solution strategies' (Verschaffel and De Corte 1993). The task design from this design experiment suggested that engaging learners in narrative processes where they are expected to model the problem situations and then retell and vary the word problems, to become fluent in using the sematic schemata may assist them to become more experienced and better able to make sense of compare type problems. This finding contradicts the advice offered in official South African government documentation, where having identified low attainment in word problems solving in the diagnostic report of the annual national assessments, the following 'remedial measures' were recommended:

> Procedures of answering word problems should be taught. These include: reading with understanding, underlining the key words and information, translating certain words into mathematics language, identifying the appropriate mathematics operation, and constructing a mathematics sentence. (DBE 2012, p.110)

The key-word approach has been challenged theoretically in other contexts; and this design experiment provides evidence of an alternative approach which – within this local context – has supported learners to be better able to solve word problems after a relative short intervention period. This has been done by attending to the learners' dual need to develop their English language proficiency by expecting them to orally re-tell, and vary word problems, as well as to generate word problems of their own through problem posing activities.

Implication 5: A partition word problem can be used as an entry into the conventional example space for single answer additive relations word problems

In defining the theoretical framework for this design experiment, three aspects of additive relations which figure in the mathematics education literature – a counting pathway, a measurement pathway and a structural approach - were discussed. Most of the work from the United States of America on the conventional example space for additive relations word problems has taken place within a 'counting pathway approach' and although multiple solution word problems and approaches to early algebra figure in this landscape, I have not found much research which seeks to bridge the different approaches. I am aware of the Venkat, Ekdahl (2014) intervention which dealt with part-part-whole relations, taking a structural approach and introducing structural representations into three Grade 3 classrooms in South Africa. As with the task design for this design experiment, a partition problem was used as a starting task in Venkat, Ekdahl et al. (2014) to induct children into thinking flexibly about the structure of additive relations using syntax models (whereas the whole to be partitioned was 9 in the Grade 3 context, the whole was 5 in this Grade 2 study). In this design experiment indexical whole-part-part diagrams where the act of partitioning was concretely experienced through tearing indexical 5 –strips, which were then shifted over time to include symbolic whole-part-part diagrams and related number sentences were used. The representations initially introduced through learner activity on the partition problem deliberately guided learners to adopt a structural approach (syntax models) which could then be built upon and applied as cross cutting features of all additive relations problem types.

Further the decision to use the same syntax models involving a whole-part-part diagram depicting length and using this for all additive relations was thought to be important as these representations would remain relevant when the constraint on whole numbers was relaxed to include continuous measurement contexts. Cross-cutting syntax models were selected as these can be applied to both whole number and rational number contexts.

Implication 6: A change increase (equalise) problem type can be introduced as a precursor to the compare problem types

Other research has established that the equalise problem within the change increase problem type, share characteristics of both the change and compare problem types (Carpenter, Fennema et al. 1999). Verschaffel and De Corte (1993) had found that re-wording the compare problem to make the 1:1 matching action more explicit helped young and inexperienced problem solvers to make sense of this most difficult problem type. Re-wording a compare problem to be: 'Joe has 8 marbles, Tom has 5 marbles, *what must Tom do to get the same as Joe?*' made the action implicit in the static compare situation more explicit.

In this design experiment a change increase (equalise) problem was introduced into the intervention design. This drew on the above mentioned research, but the wording was deliberately varied in order to support ELL learners to make sense of the different specialised mathematical uses of the term 'more'. As a result the equalise problem was posed as: 'Joe has 8 marbles, Tom has 5 marbles. *How many more marbles does Tom need to have the same as Joe?*' The problem context was also changed from Joe, Tom (and their marbles), to be directly relevant to children and their particular local classroom context where they were incentivised to collect ten stickers. They were repeatedly asked (and frequently engaged on their own in the problem solving activity of): how many more stickers they needed to reach a target of ten (or multiples of ten stickers). This innovation, introduced into the third intervention cycle, was thought to be important – not just because the change increase problem made the implicit action of comparison (a change increase action to equalise) more explicit; but also because of the various ways in which the term 'more' is used in English language. These ELLs required experiences of expecting to make sense of, and construct their own, narratives making use of the various specialised uses of the term 'more': including 'this is <u>more</u>, that is less' (descriptive adjective), and '1 <u>more</u> than 6 is 7' (comparative adjectives), 'I need <u>more</u> stickers'; which seem to be precursors to use of the term more in the comparative phrase ('how many <u>more</u> is 7 is than 6?). Learners were also encouraged to use the term 'more', and pose their own questions in English making use of the term 'more' in different ways. Many learners in this design experiment continued to have difficulties with posing questions using the term more, suggesting that this was an area where they needed specific and continued support to successfully use the specialised mathematics register in English.

Implication 7: A compare (matching) problem type can be introduced as a precurser to the compare (disjoint set) problem type

With the 1:1 matching action identified as an object worthy of study, how best to induct the novice comparer into using a 1:1 matching action was therefore also worth attending to. The approach used in this design experiment, which showed positive learning gains (for the core and extension learners who were provided adequate opportunity to engage with the tasks that required this process), was the insertion of a 'compare (matching)' problem type as a sub-type within the compare problem types identified in the operational approach. Such a compare (matching) problem was not a new category of additive relation word problem. It had been alluded to peripherally by Carpenter, Fennema et al. (1999) when they wrote: 'We can make problem easier for children by making the action or relationships in the problem as clear as possible' (p. 11), and later when they assert that 'If the context of wording of compare problems provides cues for matching, direct modellers can solve them' (p. 27). What was done in this design experiment was to extend this peripheral advice, and explicitly focus pedagogical attention on a compare problem type where a 1:1 matching action was suggested by the problem context. Such a problem was then inserted into the task design, and used as part of the assessment tools. This problem type was labelled as a compare (matching) problem (see main task 5: Compare (matching)). Matching situations are problem contexts where each element of one set have to fit together with each element of another set. The matching situations used in this design experiment were locks fitting with keys, bowls fitting with lids, bowls fitting with spoons. The compare problem type category was therefore split into two categories: compare (matching) and compare (disjoint) set problems. The compare (matching) problem types were problem situations where a 1:1 matching action was implicit in the problem situation. Learners in this design experiment – in both cycle 2 and cycle 3 – were more successful in solving the compare (matching) word problem, than they with solving the compare (disjoint set) problem. This finding was consistent across cycle 2 and cycle 3 and across time (pre-test; post-test and delayed post-test). This suggests that the compare (matching) word problem ought to be considered distinct from the compare (disjoint set) problem types (although both relate to a static situation of comparison). In this design experiment learners could draw on their paradigmatic knowing of compare (matching) problem situation, and seemed to be more likely to invoke a 1:1 matching action in compare (disjoint set) situations. The insertion of the compare (matching) problem just prior to the experience of the compare (disjoint set) problem, seemed to support the learners' sense making for the compare problem types.

Implication 8: A teaching-learning trajectory which considers shifts <u>within</u> modes of representations where learning gains relating to increasingly structured representations are in focus can be used to interpret and support young learners' representations

Another theoretical implication emerging from this design experiment relates to the way in which young children's representations were coded and interpreted. A framework to support teachers' interpretation of learners' representations when engaging with whole number additive relation tasks is proposed. It builds on previous South African research on a conceptual framework for the specialization of modes of representation in early grade mathematics. Using a combination of empirical data of learners' representations of additive relations and literature, the adapted framework is exemplified focusing on shifts within modes of representation which denote a move from counting to calculating.

I argue that working *within* a particular mode of representation, makes it possible to discern shifts from counting to calculating when attention is focused on the <u>structure</u> of, and <u>actions</u> on, representations.

There are several important features to which I have attended when interpreting learner's representations. The *modes of representation*, as defined by Ensor et al. (2009), are useful distinctions and I consider these to be the *first dimension* of possible variation when interpreting learners' representations. The other three dimensions of possible variation are dimensions for interpreting representations within a particular mode of representation. The *second dimension is the arrangement* of elements within a representation, referring to the spatial positioning of elements in relation to each other. The *third dimension is the group-wise* depictions evident in a child's representation. The way children group items, and act on their groups, suggests shifts of attention between considering each object as a member of a group and acting on the group itself. In the context of early grade learners' work on additive relations, these groups are commonly single objects (one's), pair-wise, 5-wise or 10-wise groups. Together the arrangement of objects, and the group-wise depictions may be considered to define the *structure* of the representation. The learners' *actions (the fourth dimension)* made visible by their gestures or markings are significant because they depict change or movement in their representations. Changes children make to representations depict a process and offer insights into the chronology of the development of their representations. The curricular and literature based push for moves towards more symbolic representations tend to ignore the time dimension, however the data from this study is a reminder that expansions and progressions within modes of representation are necessary; and that such processes are gradual over time.

The following table provides a summary of the conceptual framework for the analysis showing the four dimensions of possible variation and their related range of permissible variation.

Table 74: Dimensions of possible variation in young children's representations of additive relations

Dimension	**Range of permissible variation**				
Representation mode	Concrete	Iconic	Indexical	Symbolic	Syntactical
Representation types	Group		Line		Syntax
Structure: Arrangement	None	Horizontal linear	Left-right partition	Top-bottom partition	Array
Structure: Group-wise	1s	2s	5s	10s	Other
Action	Enclosing	Gesturing	Erasing or moving	Depiction of change	Other

How this framework can be adopted for interpreting the representations young children in relation to their additive relations example space, with examples drawn from learner work, is described in detail in Roberts (2015).

What are implications of this study for practitioners?

At the practitioner level, this classroom-based design experiment aimed to research and innovate the approaches to teaching additive relation word problems with a view to include compare type problems.

The implications of this research are humble, in that some recommendations emerge about possible approaches to the teaching of additive relations word problems, which may now be implemented with other classes in the focal schools that are partners in the development project of which this study was a part. Some readers of the thesis may recognise the local context described as Foundation Phase South African township classrooms where the majority of learners are ELLs and notice similarities with their own local contexts. In such cases, there are implications emerging from this study which could lead to further adaptation and trial for efficacy in these similar contexts.

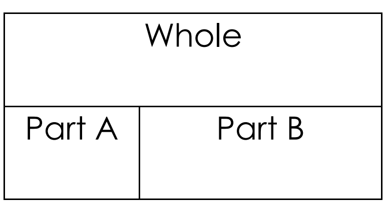
Implication 9: Main and enabling tasks which have been researched and found to yield promising results in a particular South African local context, can now be trialled with other learners

At the practitioner level I offer the main tasks designed to support children to make use of narrative in expanding understanding of additive relations to the practitioner community for further trialling and development. The experiences of the cycle 3 intervention have suggested further slight improvements on tasks. I offer two refinements to indicate the type of improvements anticipated. Firstly additional time should be allocated to the intervention, so that Main task 3: Learner generated examples of change situations figures in the task sequence as originally intended. Secondly a final task to allow learners to classify and build on each other's stories should be included in the intervention design.

Implication 10: A family of equivalent number sentences (expressing the same additive relation) is a mathematical object that is worthy of attention

Learners in this design experiment were expected to securely read and write the family of equivalent number sentences that all express the same additive relationship. It was expected that it would become a known fact that the following number sentences all belong to the same family:

Figure 110: General syntax model



Part A + Part B = Whole
Whole = Part A + Part B
Whole – Part B = Part A
Part A = Whole - Part B

Part B + Part A = Whole
Whole = Part B + Part A
Whole – Part A = Part B
Part B = Whole – Part A

In this design experiment it was expected that over time, number triples would become familiar known facts to learners, especially involving 5 and 10, and the learners should be able to generate all of the equivalent number sentences for a particular number triple. For example for the number triple 5-3-2 learners would know that $5 = 3 + 2$, $5 = 2 + 3$, $5 - 3 = 2$ and $5 - 2 = 3$ (and that reversing the left and right sides of the equals sign or 'same as' sign generates four more family members for this number triple). This was developed over time, and while some learners were fluently able to work with all 8 members of the family, for other learners only 2 or 4 members were attended to. This family of number sentences (with eight possible number sentences) therefore becomes an object worthy of study. When children find one family member, the whole-part-part diagram, and all of the other members of the family should become available to them.

This is not a new observation in the mathematics literature. It is a position strongly supported by the advocates of the structural approach. The findings from this design experiment support the view that the mathematical object of the family of eight equivalent number sentences to describe an additive relation supports flexible problem solving in relation to additive relation word problems. All eight of the number sentences describe the same relationship. Any of these eight number sentences could be used to solve an additive relation word problem.

Implication 11: The ELL learners in this study revealed a common misconception relating to how to answer compare type problems. A line of questioning, developed in response of this common misconception, seems to hold some promise in scaffolding learners to make sense of the compare (disjoint set) problem

In this design experiment, there was evidence collected that many ELLs responded to compare word problems like *'Mahlodi has 12 sweets. Moeketsi has 8 sweets. How many more sweets does Mahlodi have than Moeketsi?'* with: *'Mahlodi has 12 sweets'*. In the design experiment, it seemed to help to rather ask three related questions:

1. Who has more sweets?
2. How many sweets does Mahlodi have?
3. How many more sweets does Mahlodi have than Moeketsi?

By asking all three questions, it was assumed that learners would consider each question to be distinct (and be unlikely to answer question 2 and question 3, with identical responses). The first question was posed to see if a learner understood the idea of 'more'. The second question was posed to allow learners to show that they knew how many Mahlodi had. By the third question, it was expected that learners would not say how many Mahlodi had (as was their common misconception when asked only question 3). It was expected that they would now focus on the 'how many more' part of the compare question. By using a 1:1 matching action the 3 more sweets can be visible to learners. This is also seen when sketching a whole-part-part diagram, where the bigger amount (Moeketsi has more) is the whole, and this must be compared to the smaller amount (Mahlodi has less), which is a part. Further it was found many ELL learners in this design experiment found it easier to solve the compare problem if it was posed as: *'How many more does Moeketsi need to have the same as Mahlodi?'*. This phrasing of the question seemed to be more closely aligned to the change increase problem type, and to make the action in the problem more explicit. These components arose as contingency responses to what was observed of learner activity on the compare tasks. They are offered as possible scaffolds to support ELL learners to make sense of the compare problem type. They are intended as temporary scaffolds which will fade over time (or may be brought to mind by the learner, rather than invoked by the teacher).

What are the broader implication for improving curriculum and/or guidelines for teachers in South Africa?

The findings of this design experiment have been promising in the local context of the focal school. Should the intervention task design be found to yield similar results in other South African Foundation Phase contexts, when implemented by teachers other than the researcher, then it may be appropriate to use the research findings to improve the guidance provided to Foundation phase teachers (in curriculum documentation and through professional development offerings). Roberts (in press) offers a critique of how additive relations word problems are presented in the South African Foundation curriculum and proposes that clarification note is developed to better support teacher and teacher educators with setting additive relation word problem tasks and tracking learner experiences with solving word problems.

The findings from this design experiment are not yet sufficiently robust, to generalise to a national level. The synthesis of literature and theoretical framework applied in this design experiment may however support a process of improving the design and range of tasks selected for working with learners on additive relationships.

Implications for further research

There is a specific research interest which emerges was a result of the findings of this study.

Implication 12: Research a further intervention cycle in a more naturalistic setting with a practicing teacher leading the intervention

Given that the third cycle intervention shows promise in supporting learners to expand their additive relations word problem example space to compare type problems, a revised intervention design (which takes into account the analysis of the third cycle) should be trialled in a more naturalistic setting, where a practicing teacher is supported by the researcher. This further intervention cycle would have a two-fold purpose. Firstly it would establish whether the task design, proposed teacher guidelines and related theoretical framework is sufficiently robust to be undertaken by another teacher and yield similarly positive results in terms of learning gains. Secondly it would shift attention away from the researcher as the lead teacher, and onto the researcher as a mentor for a practicing teacher. This would help to document the kind of skills and support practicing teachers require to adopt a narrative approach to this difficult to teacher topic.

Conclusion

This thesis offers a picture of what might be possible in South African early grade classrooms. Being set in the challenging context of a full service school in a poor township community, learning gains in relation to additive relations word problems have been researched. An intervention making use of a narrative approach to mathematics learning which was of relatively short duration (10 lessons) has yielded promise – particularly for raising attainment of the middle. The approach adopted differs from the approaches advocated for in the South African curriculum where a 'key word' strategy approach to word problem is advocated for and structural representations of whole-part-part diagrams and related families of additive relation number sentences are absent in policy documentation. This approach recognises counting as a starting point, but deliberately approaches additive relations structurally and is mindful of the continuous measurement contexts for visualing additive relations. The design experiment method with its iterative design cycles – that allow for engagement with literature, while critical reflecting on empirical data emerging from a particular local context - has been found to yield several implications at theoretical, practical and research levels. Thinking carefully and purposefully about starting points, tasks and their sequencing, pedagogical and orienting frameworks; and then measuring (and refining the measurement of) learning gains repeatedly with different groups of learners in the same local context has been productive both for learning to teach and for teaching for learning.

References

Adler, J. (2002). "Lessons From and In Curriculum Reform Across Contexts?" The Mathematics Educator **12**(2): 205.

Anghileri, J. (2000). Teaching number sense. London, Continuum.

Anghileri, J. (2005). Chapter 5: Making Sense of Symbols. Children's Mathematical Thinking in the Primary Years: Perspectives Children's Learning J. Anghileri. London, Continuum International Publishing Group**:** 74-91.

Askew, M. (2012). Transforming Primary Mathematics. New York, Routledge.

Askew, M. (2013). "Big ides in primary mathematics: Issues and directions." Perspective **31**(3): 6-18.

Askew, M. and M. Brown (2003). How do we teach children to be numerate? A professional user review of UK research undertaken for the British Educational Research Association. London, BERA.

Back, J., J. Piggott, et al. (2010). "Telling Stories: the place of narrative in teaching and learning mathematics." Mathematics Teaching, Association for Teachers of Mathematics **218**.

Barmby, P., D. Bolden, et al. (2014). Understanding and Enriching Problem Solving in Primary Mathematics. Northwich, Critical Publishing.

Bass, H. (2015). Quantities, numbers, number names and the real number line. Primary mathematics study on whole numbers: ICMI Study 23, Macau, China, International Commission on Mathematics Instruction (ICMI).

Bastable, V. and D. Shifter (2008). Classroom Stories: Examples of Elementary Students Engaged in Early Algebra. Algebra in the Early Grades. J. Kaput, D. Carraher and M. Blanton. New York, Taylor and Francis**:** 165-184.

Bell, A. (1993). "Principles for the design of teaching." Educational Studies in Mathematics **24**: 5-34.

Bless, C. and R. Kathuria (1993). Fundamentals of social statistics: An African perspective. Johannesburg, Juta.

Boaler, J. (2010). The Elephant in the Classroom. Helping Children Learn & Love Maths. . London, Souvenir Press.

Boaler, J. (2014). Ability grouping in mathematics classrooms. Encyclopedia of Mathematics Education. Lerman. London, Springer**:** 1-4.

Bransford, J., A. Brown, et al. (2000). How people learn: Brain, mind, experience,and school. Washington, DC, National Academy Press.

Brombacher and Associates. (2015). "Brombacher and Associates website." from www.brombacher.ac.za.

Brown, A. L. (1992). "Design experiments: Theoretical and Methodological Challenges in Creating Complex Interventions in Classroom Settings." Journal of the Learning Sciences **2**: 141-178.

Bruner, J. (1996). The Culture of Education. Cambridge, MA, Harvard University Press.

Bruner, J. (2003). Making Stories, Harvard University Press.

Burkhardt, H. and A. Schoenfeld (2003). "Improving Educational Research: Toward a more useful, more influential and better-funded enterprise." Educational Research **32**(9): 3-14.

Butterworth, B. (2015). Low Numeracy: From brain to education. The Twenty-third ICMI Study:Primary Mathematics Study on Whole Numbers, Macau, China.

Carnoy, M., L. Chisholm, et al., Eds. (2012). The Low Achievement Trap : Comparing Schooling in Botswana and South Africa. Pretoria, HSRC.

Carpenter, T., E. Fennema, et al. (1999). Children's Mathematics: Cognitively Guided Instruction. Portsmouth, Heineman.

Carpenter, T., M. L. Franke, et al. (2003). Thinking Mathematically: Integrating Arithmetic and Algebra in Elementary School. Portsmouth, Heinemann.

Carpenter, T., J. Hiebert, et al. (1997). Making Sense: Teaching and Learning Mathematics with Understanding Portsmouth, Heineman.

Carpenter, T., J. Hiebert, et al. (1981). "Problem Structure and First-Grade Children's Initial Solution Processes for Simple Addition and Subtraction Problems." Journal for Research in Mathematics Education **12**(1 (Jan 1981)): 27-39.

Carpenter, T. and J. Moser (1984). "The Acquisition of Addition and Subtraction Concepts in Grades One through Three." Journal for Research in Mathematics Education **15**(3 (May, 1984)): 179-202.

Carpenter, T., J. Moser, et al. (1988). "Representation of Addition and Subtraction Word Problems." Journal for Research in Mathematics Education **19**(4): 345-357.

Chapman, O. (2006). "Classroom Practices for Context of Mathematics Word Problems." Educational Studies in Mathematics(2): 211-230.

Clements, D. and J. Sarama (2009). Learning and Teaching Early Math: The Learning Trajectories Approach. New York, Routledge.

Cobb, P., J. Confrey, et al. (2003). "Design Experiments in Educational Research." Educational Researcher **February**.

Cobb, P., K. Gravemeijer, et al. (1997). Mathematitizing and Symbolising: The emergence of chains of signification in one first-grade classroom. Situated Cognition: Social, Semiotic, and Psychological Perspectives. D. Kirshner and J. A. Whitson. Mahaw New Jersey, Lawrence Erlbaum Associates.

Cobb, P., T. Wood, et al. (1991). "Assessment of a Problem-Centered Second-Grade Mathematics Project." Journal for Research in Mathematics Education **22**(1 (January 1991)): 3-29.

Cohen, J. (1988). Statistical Power Analysis for the Behavioral Sciences, L. Erlbaum Associates.

Crocker, L. (2006). Introduction of measurement theory. Handbook of complementary methods in education research. J. L. Green, G. Camilli and P. B. Elmore. Mahwah, NJ, Lawrence Erlbaum Associates**:** 371-384.

Cronbach, L. J. (1951). "Coefficient alpha and the internal structure of tests." Psychometrika **16**: 297-334.

Dalvit, L., S. Murray, et al. (2009). "Deconstructing language myths: which languages of learning and teaching in South Africa?" Journal of Education **46**: 33-56.

Davis, B. (1996). Teaching Mathematics: Toward a Sound Alternative. New York, Routledge.

Department of Education (DOE) (1997). Language in Education Policy. D. o. E. (DOE). Pretoria, Government of South Africa.

Department: Basic Education Republic of South Africa (2011). Curriculum and Assessment Policy Standards Grades 1-3: Mathematics. CAPS. Pretoria**:** 502.

Department: Basic Education Republic of South Africa (2012). Annual National Assessments 2011 Diagnostic Report Foundation Phase: Mathematics and Home Language. Pretoria.

Department: Basic Education Republic of South Africa (2013). Annual National Assessments 2012 Diagnostic Report Foundation Phase: Mathematics and Home Language. Pretoria.

Department: Basic Education Republic of South Africa (2015). Annual National Assessments 2013 Diagnostic Report Foundation Phase: Mathematics and Home Language. Pretoria.

Design-Based Research Collaborative (2003). "Design-based research: An emerging paradigm for educational inquiry." Educational Researcher **32**(1): 5-8.

Dewey, J. (1933). How We Think: A Restatement of the Relation of Reflective Thinking in the Education Process. London, D.C. Heath and Co.

Dienes, Z. P. (1963). An experimental study od mathematics learning. London, Huchinson.

DiSessa, A. and P. Cobb (2004). "Ontological Innovation and the Role of Theory in Design Experiments." The Journal of the Learning Sciences **13**(1): 77-103.

Dreyfus, T. (2014). Abstraction in Mathematics Education. Encyclopedia of Mathematics Education. S. Lerman. London, Springer**:** 5-7.

Duval, R. (1999). Representation, vision and vizualisation: Cognitive functions in mathematical thinking. Proceedings os the Twenty-first Annual meeting of the North American Chapter of the Internations Group for the Psychology of Mathematics Education, Columbus, Ohio, ERIC Clearninghouse for Science, Mathematics and Environmental Education

Dweck, C. (1986). "Motivational Processes Affecting Learning " American Psychologist **41**(10): 1040-1048.

Egan, K. (1989). Teaching as Story Telling: An alternative approach to teaching and curriculum in the elementary school. Chicago, University of Chicago Press.

Egan, K. (2002). Getting it wrong from the beginning: Our progessivist inheritance from Herbert Spencer, John Dewy, and Jean Piaget. New Haven, Yale University Press.

Eisner, E. W. (1993). "Forms of understanding and the future of educational research." Educational Researcher **22**(7 (October 1993)): 5-11.

Ensor, P., T. Dunne, et al. (2002). "Textbooks, teaching and learning in primary mathematics classrooms." African journal of research in SMT education **6**: 21-35.

Ensor, P., U. Hoadley, et al. (2009). "Specialising pedagogic text and time in Foundation Phase numeracy classrooms." Journal of Education **47**(2009): 5-30.

Feigenson, L., S. Dehaene, et al. (2004). "Core systems of number." Trends in Cognitive Sciences **8** (7): 307-314.

Fleisch, B. (2008). Primary education in crisis: Why South African schoolchildren underachieve in reading and mathematics. Global, Juta.

Gattegno, C. (1964). For the teaching of elementary mathematics. New York, Cuisenaire Company of America.

Gattegno, C. (1973). Common Sense of Teaching Mathematics. New york, Educational Solutions.

Gelman, R. and C. R. Gallistel (1978, 1986). The child's understanding of number. Cambridge, Massachusetts, Harvard University Press.

Glaser, B. (1965). "The Constant Comparative Method of Qualitative Analysis' in Social Problems." Social problems **12**(4): 436-445.

Gravemeijer, K. (1994). Developing realistic mathematics education. Utrecht, Freudenthal Institute.

Gravemeijer, K. (1997). "Solving word problems: A case of modelling?" Learning and Instruction **1**(4): 389=397.

Graven, M. and E. Heyd-Metzuyanim (2014). "Exploring the Limitations and Possibilities of Researching Mathematical Dispositions of Learners with low Literacy Levels." Scientia in educatione **5**(1): 20-35.

Graven, M. (2012). "Accessing and assessing young learners mathematical dispositions." South African Journal of Childhood Education **2**(1): 49-62.

Graven, M., H. Venkat , et al. (2013). "Place value without number sense: Exploring the need for mental mathematical skills assessment within the Annual National Assessments " South African Journal of Childhood Education **3**(2): 131-143.

Gray, E. and D. Tall (1994). "Duality, ambiguity and flexibility: a proceptual view of simple arithmetic." Journal of Research in Mathematics Education **26**: 115-141.

Greer, B. (1997). "Modelling reality in mathematics classrooms: The case of word problems." Learning and Instruction **7**(4): 293-307.

Henning, E. and R. Ragpot (2014). "Pre-school children's bridge to symbolic knowledge: first literature framework for a learning and cognition lab at a South African university." South African Journal of Psychology: 1-10.

Hoadley, U. (2007). The reproduction of social class inequalities through mathematics pedagogies in South African primary schools. *Journal of Curriculum Studies, 39*(6), 679-706.

Hoadley, U. (2012). "What do we know about teaching and learning in South African primary schools?,." Education As Change **16**(2): 187-202.

Holt (1964). How children fail. London, Pitman.

Howie, S. J. (2004). "A national assessment in mathematics within an international comparative assessment." Perspectives in Education **2**(2): 149-162.

Kaput, J., Carraher, D., & Banton, M. (2008). *Algebra in the Early Grades*: Routledge.

Kaur, B. (2015). The model method - A tool for representing and visualising relationships. Primary mathematics study on whole numbers: ICMI Study 23, Macau, China, International Commission on Mathematics Instruction (ICMI).

Kelly, A. E. E. (2003). Theme issue on the role of design in educational research [Special issue]. Educational Researcher. **32**.

Kilpatrick, J., J. Swafford, et al. (2001). Adding it up: Helping children learn mathematics. Washington DC, National Academy Press.

Leikin, R. and S. Dinur (2007). "Teacher flexibility in mathematical discussion." Journal of Mathematical Behavior **26**: 328-347.

Lerman, S. (2000). The social turn in mathematics education research. International perspectives on mathematics education. . B. J. Westport, Ablex**:** 19–44.

Levin, J. and J. Nolan (1996). Principals of classroom management: A professional decision-making model. London, Ally & Bacon.

Li, S. (1999). "Does practice make perfect?" For the Learning of Mathematics **19**(3): 33-35.

Mabeba, M. Z. and E. Prinsloo (2000). "Perceptions of discipline and ensuing discipline problems in secondary education." South African Journal of Education **20**.

Marais, P. and C. Meier (2010). "Disruptive behaviour in the Foundation Phase of schooling." South African Journal of Education **30**: 41-57.

Marton, F., & Booth, S. (1997). *Learning and Awareness*. Mahwah, New Jersey: Erlbaum.

Mason, J. (1989). "Mathematical Abstraction Seen as a Delicate Shift of Attention." For the Learning of Mathematics **9**(2): 2-8.

Mason, J. (2004). Designing and using mathematical tasks. Milton Keynes, The Open University.

Mason, J. (2007). Making Use of Children's Powers to Produce Algebraic Thinking. Algebra in the Early Grades. J. Kaput, D. Carraher and M. Banton, Routledge**:** 57-94.

Mason, J. and M. Spence (1999). "Beyond mere knowledge of mathematics: The importance of knowing-to act in the moment." Educational Studies in Mathematics **38**(1-3): 135-161.

Mitchell, J. (1984). Typicality and the case study. Ethnographic Research: A Guide to Conduct. R. F. Ellen. New York, Academic Press**:** 238-241.

Mitchell, M. and J. Jolley (2012). Research design explained. Belmont, CA, Cengage Learning,.

Moodley, V. (forthcoming). An Investigation into the feasibiity of the positive behaviour support model for Limpopo's primary schools (Grade R-3): Preliminary Findings. Making Education Inclusive. E. Walton and S. Moonsamy. Newton on Tyme, Cambridge Scholar Publlishers.

Neuman, D. (1987). The origin of arithmetic skills: A phenomenographic approach, Universitatis Gothenburgensis.

Piaget, J. (1970). Genetic Epistemology. New York, Norton.

Polya, G. (1945). How to solve it. London, Penguin Books.

Riley, M., J. Greeno, et al. (1983). Development of children's problem solving ability in arithmetic. The development of mathematical thinking. H. Ginsburg. New York, Academic Press.

Roberts, N., & Stylianides, A. J. (2013). Telling and illustrating stories of parity: a classroom-based design experiment on young children's use of narrative in mathematics. *ZDM Mathematics Education*, 453-467.

Roberts, N. (2015). Interpreting children's representations of whole number additive relations in the early grades. ICMI Study 23: Primary Mathematics Study on Whole Numbers, Macau, China.
Roberts, N (in press) Additive Relations Word Problems in the South African Curriculum and Assessment Policy Standard (CAPS) at Foundation Phase, African Journal of Research in Mathematics Science and Technology
Runesson, U. (2005). "Beyond discourse and interaction. Variation: a critical aspect for teaching and learning mathematics." Cambridge Journal of Education **35**(1): 69-87.
Runesson, U. and A. Kullberg (2010). Learning from variation. Differences in learners' ways of experiencing differences. The Sourcebook on Nordic Research in Mathematics Education. B. Sriraman, C. Bergsten, S. Goodchildet al.
Sasman, M. C., L. Linchevski, et al. (1999). The influence of different representations on children's generalisation thinking processes. Seventh Annual Conference of the Southern African Association for Research in Mathematics and Science Education, Harare, Zimbabwe.
Savard, A., A. Polotskaia, et al. (2013). Tasks to promote holistic flexible reasoning about simple additive structures. ICMI Study 22: Task design in Mathematics Education.
Schiro, M. S. (2004). Oral story telling and teaching mathematics: Pedagogical and Multicultural Perspectives. London, Sage Publications.
Schmittau, J. (2004). Cultural Historical Theory and Mathematics Education Vygotsky's Educational Theory in Cultural Context. A. G. Kozulin, A. B. V. and M. S. Cambridge, Cambridge University Press**:** 225-245.
Schoenfeld, A. H. (1991). On mathematics as sense-making: An informal attack on the unfortunate divorce of formal and informal mathematics. J. F. Voss, D. N. Perkins and S. J. W. Hillsdale, NJ, Erlbaum.
Schoenfeld, A. H. (1992). Learning to think mathematically:problem solving, metacognition, and sense making in mathematics. Handbook of research on mathematics teaching and learning. D. A. Grows. New York, Macmillan**:** 334-370.
Schoenfeld, A. H. (2006). Design Experiments. Handbook of Complementary Methods in Educational Research. J. L. Green, G. Camilli, P. B. Elmore, A. Skukauskaite and E. Grace. Washinton, D.C, Americal Educational Research Association**:** 103-205.
Schollar, E. (2008). Final Report: The primary mathematics research project 2004-2007. Towards evidence-based educational development in South Africa: Mimeo.
Sepeng, P. (2014). "Use of Common-sense Knowledge, Language and Reality in Mathematics Word Problem Solving." African Journal of Research in Mathematics, Science and Technology Education **18**(1): 14-24.
Setati, M. (2002). "Researching Mathematics Education and Language in Multilingual South Africa." The Mathematics Educator **12**(2): 6-20.
Sfard, A. (2008). Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing. New York, Cambridge University Press.
Silver, E. A. and J. P. Smith (1980). Think of a related problem. Problem Solving in School Mathematics. S. Krulik and R. E. Reys. Reston, Virginia, NCTM**:** 146-156.
Simon, A. M., Ed. (1997). Developing new models of mathematics teaching: An imperative for research on mathematics teacher development. Mathematics teachers in transition. Maywah, NJ, Erlbaum.
Simon, M. (1995). "Reconstructing mathematics pedagogy from a constructivist perspective." Journal for Research in Mathematics Education **26**: 114-145.
Simon, M. A. and R. Tzur (2004). "Explicating the Role of Mathematical Tasks in Conceptual Learning: An Elaboration of the Hypothetical Learning Trajectory." Mathematical thinking and learning **6**(2): 91–104.
Spaull, N. (2013). "Poverty & Privilege: Primary School Inequality in South Africa." International Journal of Educational Development **33**(2013): 436-447.

Stigler, J. and J. Hiebert (1998). "Teaching is a Cultural Activity." American Educator **22**(4): 4-11.
Stylianides, A. J. and G. Sylianides (2014). "Impacting positively on students' mathematical problem solving belifs: An instructional intervention of short duration." The Journal of Mathematical Behaviour **33**(2014): 8-29.
Stylianides, G. J. and A. J. Stylianides (2014). "The role of instructional engineering in reducing the uncertainties of ambitious teaching." Cognition and Instruction **32**(4): 374-415.
Tabak, I. (2004). "Reconstructing Context: Negotiating the tension between exogenous and endogneous educational design." Educational Psychologist **39**(4): 225-233.
Tahta, D. (1972). A Boolean Anthology: Selected Writings of Mary Boole on Mathematics Education. Derby, Association of Teachers of Mathematics.
Takane, T., H. Tshesane, et al. (in press). From theory to practice: Challenges in adopting pedagogies of mathematizing in South Africa. Improving primary mathematics education, teaching and learning: Research for development in resource constrained contexts. H. Venkat and M. Graven, Palgrave.
Tall, D. O. (2004). "Building theories: Three Worlds of Mathematics." For the Learning of Mathematics **24**(1): 29-32.
Taylor, N. and P. Vinjevold, Eds. (1999). Getting learning right: report of the President's Education Initiative Research Project. Johannesburg, Joint Education Trust.
Thompson, P. (1985). Experience, problem solving, and learning mathematics: considerations in developing mathematics curricula. . Learning and teaching mathematical problem solving: multiple research perspectives. S. EA. Hillsdale, Erlbaun: 189-236.
Tobin, K. (1986). "Effects of teacher wait time on discourse characteristics in mathematics and language arts classes." American Educational Research Journal **23**(2): 191-200.
Treffers, A. (2008). Grade 1 (and 2) - Calculation up to 20. Children learn mathematics: A learning teaching trajectory with intermediate attainment targets for calculation with whole numbers in primary school. M. Van den Heuvel-Panhuizen. Rotterdam, Netherlands, Sense Publishers: 43-60.
Treffers, A. (2008). Kindergarten 1 and 2 - Growing number sense. Children learn mathematics: A learning teaching trajectory with intermediate attainment targets for calculation with whole numbers in primary school. M. Van den Heuvel-Panhuizen. Rotterdam, Netherlands, Sense Publishers: 31-42.
Treffers, A. and K. Buys (2008). Grade 2 (and 3) - Calculation up to 100. Children learn mathematics: A learning teaching trajectory with intermediate attainment targets for calculation with whole numbers in primary school. M. Van den Heuvel-Panhuizen. Rotterdam, Netherlands, Sense Publishers: 43-60.
Tshesane, H. (2014). Exploring Grade 4 learners' use of models and strategies for solving addition and subtraction problems (unpublished master's dissertation). Johannesburg, University of the Witwatersrand.
van den Heuvel-Panhuizen, C. Kühne, et al. (2012). Learning Pathway for Number in the Early Grades. Northlands, Macmillan.
van den Heuvel-Panhuizen, M., Ed. (2008). Children learn mathematics: A learning teaching trajectory with intermediate attainment targets for calculation with whole numbers in primary school. Rotterdam, Netherlands, Sense Publishers.
Venkat, H., J. Adler, et al. (2009). "Mathematics and science education research, policy and practice in South Africa: What are the relationships?" African Journal of Research in MST Education **Special Issue**: 5-27.
Venkat, H., A.-l. Ekdahl, et al. (2014). Connection and simultaneity: Analysing South African G3 part-part-whole teaching. 38th Conference of the International Group for the Psychology of Mathematics Education and the 36th Conference of the North American Chapter of the Psychology of Mathematics Education, Vancouver, Canada.

Venkat, H. and D. Naidoo (2012). "Analyzing coherence for conceptual learning in a Grade 2 numeracy lesson." Education As Change **16**(1): 21-33.

Verschaffel, L. (1994). "Using Retelling Data to Study Elementary School Children's Representations and Solutions of Compare Problems." Journal for Research in Mathematics Education **25**(2): 141-165.

Verschaffel, L. (2002). Taking the modeling perspective seriously at the elementary level: Promises and pitfalls. PME 26, University of East Anglia, Norwich.

Verschaffel, L. and E. De Corte (1993). "A Decade of Research On Word Problem Solving in Leuven: Theoretical, Methodological, and Practical Outcomes." Educational Psychology Review **5**(3, European Educational Psychology): 239-256.

Verschaffel, L., F. Depaepe, et al. (2014). Word Problems in Mathematics Education. Encyclopedia of Mathematics Education. S. Lerman. London, Springer: 356-358.

Vygotsky, L. (1965). Thought and language (E. Hanfmann & G. Vakar, Translation). Cambridge, MA, MIT Press.

Walton, E. (2015). Working towards education for all in Gauteng. Twenty years of education transformation in Gauteng 1994 to 2014: An Independent Review. F. Maringe and M. Prew. Somerset West, African Minds for the Gauteng Department of Education: 210-227.

Wasserman, A. (2015). Investigating a mathematics recover program for assessment and intervention with groups of Grade 4 learners. Master of Education, Rhodes UNiversity.

Watson, A. and J. Mason (1998). "Questions and prompts for mathematical thinking: Challenging questions to tease out structures and concepts of mathematics."

Watson, A. and J. Mason (2005). Mathematics as a Constructive Activity: Learners Generating Examples. New York, Routledge.

Weitz, M., & Venkat, H. (2013). Assessing early number learning: how useful is the Annual National Assessment in Numeracy? *Perspectives in Education: Primary mathematics: addressing the crisis, 33*(3), 49-65.

Wheatley, G. (1992). "The role of reflection in mathematics learning." Educational Studies in Mathematics **23**(5): 529-541.

Wright, R. J., Stanger, G., Stafford, A. K., & Martland, J. (2006). *Teaching Number in the Classroom with 4-8 year-olds*. London: Sage.

Yin, R. (2006). Case study methods. Handbook of Complementary Methods in Education Research 3rd edition. J. L. Green, G. Camilli, P. B. Elmore and P. B. Elmore: 111-123.

Zazkis, R., & Liljedahl, P. (2009). *Teaching mathematics as storytelling*. Rotterdam: Sense Publishers.