Integral Inequalities of Hermite-Hadamard Type and Their Applications

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A thesis submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, South Africa, in fulfilment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

The role of mathematical inequalities in the growth of different branches of mathematics as well as in other areas of science is well recognized in the past several years. The uses of contributions of Newton and Euler in mathematical analysis have resulted in a numerous applications of modern mathematics in physical sciences, engineering and other areas sciences and hence have employed a dominat effect on mathematical inequalities.

Mathematical inequalities play a dynamic role in numerical analysis for approximation of errors in some quadrature rules. Speaking more specifically, the error approximation in quadrature rules such as the mid-point rule, trapezoidal rule and Simpson rule etc. have been investigated extensively and hence, a number of bounds for these quadrature rules in terms of at most second derivative are proven by a number of researchers during the past few years.

The theorey of mathematical inequalities heavily based on theory of convex functions. Actually, the theory of convex functions is very old and its commencement is found to be the end of the nineteenth century. The fundamental contributions of the theory of convex functions can be found in the in the works of O. Hölder [50], O. Stolz [151] and J. Hadamard [48]. At the beginning of the last century J. L. W. V. Jensen [72] first realized the importance convex functions and commenced the symmetric study of the convex functions. In years thereafter this research resulted in the appearance of the theory of convex functions as an independent domain of mathematical analysis.

Although, there are a number of results based on convex function but the most celebrated results about convex functions is the Hermite-Hadamard inequality, due to its rich geometrical significance and many applications in the theory of means and in numerical analysis. A huge number of research articles have been written during the last decade by a number of mathematicians which give new proofs, generalizations, extensions and refitments of the Hermite-Hadamard inequality.

The main intention of this dissertation is to present several Hermite-Hadamard type inequalities when $\left|f^{(n)}\right|^q$, $n \in \mathbb{N}$, $q \geq 1$ belongs to different classes of convex functions

such as convex, s-convex, m-convex, (α, m)-convex, quasi-convex, h-convex, logarithmically convex, s-logarithmically convex, (α, m)-logarithmically convex and geometrically-arithmetically convex functions.

The given results in this dissertation not only generalize and refine a number of results proved for n-times differentiable convex functions but also many interesting refinements inequalities of Hermite-Hadamard type are deduced when $|f'|^q$ or $|f''|^q$, $q \geq 1$ is convex functions, s-convex functions, m-convex functions, (α, m)-convex functions, quasi-convex functions, h-convex functions, logarithmically convex functions, s-logarithmically convex functions, (α, m)-logarithmically convex functions. Applications of the obtained results to special means and new improved error bounds of the quadrature rules such as for the mid-point rule and trapezoidal rule are presented.

Inequalities of Hermite-Hadamard type are also investigated for preinvex, s-preinvex, h-preinvex, quasipreiinvex, logarithmically preinvex functions which are more general phenomena than that of convex, s-convex, h-convex, quasi-convex and logarithmically convex functions respectively. Applications of the results for these classes of functions are given. The research upshots of this thesis make significant contributions in the theory of means and the theory of inequalities.

The results presented in this thesis have been published/submitted in the following papers:

1. **Muhammad Amer Latif** and S. S. Dragomir, New inequalities of Hermite-Hadamard type for *n*-times differentiable convex and concave functions with applications. **(Accepted, Filomat)**

2. **Muhammad Amer Latif**, S. S. Dragomir and E. Momoniat, New inequalities of Hermite-Hadamard type for *n*-times differentiable s-convex functions with applications. **(Analysis Mathematica)**

3. **Muhammad Amer Latif** and S. S. Dragomir, New integral inequalities of Hermite-Hadamard type for *n*-times differentiable s-logarithmically convex functions with applications, **Miskolc Mathematical Notes, Vol. 16 (2015), No. 1, pp. 219–235**.

4. **Muhammad Amer Latif** and S. S. Dragomir, On Hermite-Hadamard type integral inequalities for *n*-times differentiable *s*-logarithmically convex functions with applications to special means. **(Accepted, Applied Mathematics & Information Sciences)**

5. **Muhammad Amer Latif**, S. S. Dragomir and E. Momoniat, On Hermite-Hadamard type integral inequalities for *n*-times differentiable *m*- and (α,m)-logarithmically convex functions. **(Accepted, Filomat)**

6. **Muhammad Amer Latif**, S. S. Dragomir and E. Momoniat, Inequalities of Hermite-Hadamard type for *n*-times differentiable (α,m)-logarithmically convex functions. **(In Review, Bulletin of the Iranian Mathematical Society)**

7. **Muhammad Amer Latif** and S. S. Dragomir, New inequalities of Hermite-Hadamard and Fejer type via preinvexity, **Journal of Computational Analysis and Application, Vol. 19, No. 4, 2015.**

8. **Muhammad Amer Latif,** S. S. Dragomir and E. Momoniat, Some weighted integral inequalities for differentiable h-preinvex functions. **(Accepted, Georgian Mathematical Journal)**

9. **Muhammad Amer Latif** and S. S. Dragomir, Generalization of inequalities of Hermite-Hadamard type for *n*-times differentiable functions through preinvexity. **(Accepted, Georgian Mathematical Journal)**

10. **Muhammad Amer Latif** and S. S. Dragomir, Generalization of Hermite-Hadamard type inequalities for *n*-times differentiable functions which are s-preinvex in the second sense with applications, **Hacettepe Journal of Mathematics and Statistics. 44 (4) (2015) 839-853.**

11. **Muhammad Amer Latif** and S. S. Dragomir, On Hermite-Hadamard type integral inequalities for n-times differentiable log-preinvex functions, **Filomat 29:7 (2015), 1651–1661.**

12. **Muhammad Amer Latif,** S. S. Dragomir and E. Momoniat, Some Fejér type integral inequalities for geometrically-arithmetically-convex functions with applications. **(In Review, Applied Mathematics and Computation)**

13. **Muhammad Amer Latif,** S. S. Dragomir and E. Momoniat, Fejér type integral inequalities related with geometrically-arithmetically-convex functions with applications. (**In Review, Journal of Convex Analysis)**

DECLARATION

I declare that this is my own, unaided work except where due references have been made. It is being submitted for the Degree of Ph.D. to the University of the Witwatersrand, Johannesburg, South Africa. It has not been submitted before for any other degree or examination to any other University.

Muhammad Amer Latif

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(Signature of candidate)

____________________day of ______________________20________________in_____________

DEDICATION

To my parents, all my professors, my wife Rahat, my sons Sulaiman Amer and Abdur Rahaman Amer

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All praises be to ALLAH Whose blessings are never-ending and peace be upon His messenger Muhammad and his honourable family. I am very thankful to the Almighty ALLAH Who blessed me with the opportunity to bring this work in its present form.

During my Ph.D. studies at the University of Witwatersrand, I have interacted with many people who have intensely contributed to my academic career. I am deeply obliged to all of them for their sincere efforts which helped me to bring my ambitions a reality.

A teacher's moral support and encouragement, off course, play very important role in building up the personality of his students. I am greatly indebted to my supervisor Professor S. S. Dragomir for his sincere, continuous help and moral support during my Ph.D. studies at Wits under his kind supervision. He has, literally, not only provided me with his expertise an insight knowledge of the theory of mathematical inequalities but also motivated me how to learn the art of producing new and highly quality results in this subject. Professor Dragomir's contributions in the field of mathematical inequalities have great inspiration to many mathematicians working in this field now and in the years to come. I must say that I have never seen sophisticated, sincere and great mathematician like Professor Dragomir in my life. I would not have been able to complete this work without his friendly, generous and distinguished supervision during the research work and in compilation of this dissertation.

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Muhammad Amer Latif

List of Symbols

The following is the list symbols, notations and abbreviations that are used in this thesis.

$\mathbb{R}$	The set of all real number
$[a,b]$	Real Interval
$L([a,b])$	The space of integrable functions on $[a,b]$
$T(f,\Delta)$	The trapezoidal Rule
$E(f,\Delta)$	Error in Trapezoidal Rule
$\max(\cdot,\cdot)$	Maximum
$\sup(a,b)$	Supremum of a and b
$A(a,b)$	The Arithmetic Mean a and b
$G(a,b)$	The Geometric Mean a and b
$H(a,b)$	The Harmonic Mean a and b
$I(a,b)$	The Identric Mean a and b
$L(a,b)$	The logarithmic Mean a and b
$L_p(a,b)$	The generalized logarithmic Mean a and b
$P_r(a,b)$	The power mean a and b
$B(x,y)$	The Beta function

$B(x, y; z)$	The generalized or incomplete Beta function
${}_1F_2(\alpha, \beta; \gamma; z)$	The hypergeometric function
$SX(h, I)$	The class of h-convex functions
$SV(h, I)$	The class of h-concave functions
K_1^s	The class of s-convex functions in the first sense
K_2^s	The class of s-convex functions in the second sense

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Contents

Chapter 1

Aims, Objectives, Outline

1.1 Introduction

The effectiveness of mathematical inequalities in the enhancement of different branches of mathematics as well as in other areas of science is well recognized in the past several years. The major achievements of mathematical analysis from Newton and Euler to modern applications of mathematics in physical sciences, engineering and other areas have exerted a profound influence on mathematical inequalities. The development of mathematical analysis is critically dependent on the unimpeded flow of the information between the theoretical mathematics looking for applications and mathematicians working in applications who need theory, mathematical models and methods. Twentieth century mathematics has recognized the power of mathematical inequalities which has given rise to a large number of new results and problems and has led to new areas of mathematics.

The specific work "Inequalities" by Hardy, Littlewood and Pólya appeared in 1934 and earned its place as a basic reference for mathematicians. This book is completely devoted to the subject of inequalities and is a useful guide to this exciting field. This work has been accompanied with "Inequalities" by Beckenbach and Bellman written in 1965 and "Analytic Inequalities" by Mitrinović published in 1970, which made considerable contributions to this field. These books provide handy references to the reader wishing to explore the topic in depth and show that the theory of inequalities has been established as a variable field of research. The subject has also received tremendous impetus from outside of mathematics from such diverse fields as mathematical economics, game theory, mathematical programming, control theory, variational methods, operation research, probability and statistics. The theory of inequalities has been recognized as one of the central areas of mathematical analysis throughout the last century and it is a fast growing discipline, with ever-increasing applications in many scientific fields. This growth resulted in the appearance of the theory of inequalities as an independent domain of mathematical analysis.

Over the past decades, theory of inequalities has developed rapidly and unexpected results, along with simpler new proofs for existing results, and, consequently, new vistas for research opened up. In recent years the subject has aroused considerable interest from many mathematicians, and a large number of new results have been investigated in the literature. It is recognized that in general some specific inequalities provide a useful and important device in the development of different branches of mathematics.

The history of convex functions is very long. The beginning the history of the theory of convex functions can be traced back to the end of the nineteenth century. The roots of the theory of convex functions can be found in the fundamental contributions of O. Hölder (1889), O. Stolz (1893) and J. Hadamard (1893). At the beginning of the last century J.L.W.V. Jensen (1905, 1906) first comprehended the eminence and commenced the symmetric investigation of the convex functions. In years thereafter this research resulted in the appearance of the theory of convex functions as an independent domain of mathematical analysis. In recent years, theory of convex functions has been a subject of extensive research and a large number of inequalities for convex functions have appeared which will be discussed in the coming chapters.

1.2 Aims and objectives

The aims and objectives that are accomplished through this research are summarized as follows:

- Firstly, we will establish several new integral identities for differentiable mappings and n-times differentiable mappings defined on an interval of real numbers or defined on an invex subset of real numbers.

- Secondly, by using these identities, mathematical analysis and different forms of convexity such as s-convexity, logarithmic convexity, s-logarithmic convexity, (α,m)-logarithmic convexity, preinvexity, h-preinvexity, logarithmic preinvexity, geometric-arithmetic convexity and using either the geometric symmetry or the arithmetic symmetry of the weight function, a number of new Hermite-Hadamard and Fejér type inequalities are obtained. In order to obtain Hermite-Hadamard type inequalities we have also established many new identities which have played very important role to achieve our results especially for those involving the conditions of logarithmic convexity, s-logarithmic convexity, (α,m)-logarithmic convexity. During our research we have also obtained some results for n-times differentiable mappings satisfying the concavity conditions.

- Thirdly, we have done a comparative study of our results with already existing results. We have also observed through analytical approach that either our results gives better estimates for the error bounds of the difference between the left and the middle terms and right and the middle terms in the Hermite-Hadamard's and Fejér's inequalities or they contain already existing results as special cases.

- Finally, by choosing some functions which satisfy certain conditions stated in our results we have given ample usages of our findings to get some inequalities involving means of real numbers and to some quadrature rules.

1.3 Outline

Chapter 2 deals with some basic definitions and related results concerning the spaces of convex functions, s-convex functions, (α,m)-convex functions, h-convex functions, logarithmic convex functions, s-logarithmic convex functions, (α,m)-logarithmic convex functions, preinvex sets, preinvexity functions, h-preinvex functions, logarithmic preinvex functions. In Chapter 2, some Hermite-Hadamard type inequalities will also be presented so that we can discuss how we have generalized or extended these results in coming chapters. Special means of real numbers are also defined in order to show how we can employ the results of the forthcoming chapters to get inequalities of these means.

Chapter 3 is about Hermite-Hadamard type inequalities for n-times differentiable convex, concave and s-convex functions. In this chapter, by providing a new identity for n-times differentiable mappings and applying the conditions of convexity, concavity, s-convexity and analysis we give new Hermite-Hadamard type inequalities which contain many results proved earlier as a special case for $n = 1$. Moreover, for different values of $n \in \mathbb{N}$, that is is for $n = 1$ and $n = 2$, we have derived many new interesting inequalities. To achieve our main results we have also derived some special definite integrals over the unit interval $[0, 1]$ and $\left[0, \frac{1}{2}\right]$ for the functions $(1-t)^n a^t$ and $t^n a^t$, $a > 0$. We have also provided a plenty of uses of our results to get some inequalities concerning special means of real numbers and to the estimation of error bound in trapezoidal rule. The content of this chapter includes the results of the papers [80] and [81]. The paper [80] has been accepted for publication in **"Filomat"** and the paper [81] has been submitted to **"Analysis Mathematica"** for possible publication.

Chapter 4 covers Hermite-Hadamard type inequalities for n-times differentiable s-logarithmically convex functions. The notion of s-logarithimic convexity is a generalization of the usual convexity. In this chapter we have used some identities for n-times differentiable mappings already exist in literature and identity for n-times differentiable mappings used to establish results of chapter 2. In fact, we have proved inequalities of Hermite-Hadamard type by assuming the s-logarithimic convexity of the nth derivative

$f^{(n)}$ of the mapping f. The results of this chapter are more general as they contain many results as a particular case for $n = 1$ and $n = 2$. We have also given adequate number of results are also given to spectacle applications of our findings to produce inequalities which include special means of real numbers. The results of this chapter are from the references [82] and [83]. The results from [82] have been published in **"Miskolc Mathematical Notes"** and the results of [83] have been accepted for publication in **"Applied Mathematics & Information Sciences"**.

Chapter 5 includes Hermite-Hadamard type inequalities related to another generalization of convexity which is known as (α,m)-logarithmic convexity. In this chapter, we have used the identities utilized in chapter 3 to obtain our results. Actually the assumption of (α,m)-logarithmic convexity on the nth derivative $f^{(n)}$ has resulted in very interesting inequalities which contain many results for particular values of $n \in \mathbb{N}$. In this chapter we have also presented some important definite integrals for the product of monomials t^n, $(1-t)^n$, $n \in \mathbb{N}$ and the exponential function a^t, $a > 0$ over the intervals $[0, 1]$ and $\left[0, \frac{1}{2}\right]$ to obtain the desired results. Sufficient number of results have been given on how to apply these results to special means of real numbers. The results of this chapter are taken from the papers [84] and [85]. These papers are under consideration for possible publication in the **"Filomat"** and **"Bulletin of the Iranian Mathematical Society"** respectively.

Chapter 6 consists of Hermite-Hadamard and Fejér type inequalities involving the assumption that the first derivatives are preinvex functions and h-preinvex functions. Since the criterion of preinvexity a generalization of convexity and h-preinvexity is a further generalization of preinvexity. This reveals that the results of given in this chapter are more general than those proved in the literature. The techniques used to establish are results in the chapter are similar to those used for convex functions but in this paper we have proved some identities for functions defined on an invex set. We have presented ample results which illustrate applications of the obtained results to special means of real numbers. The content of this chapters is taken from the references [86] and [87].

The paper [86] has been published in the **"Journal of Computational Analysis and Application, 19 (4) (2015), 725-739."** and [87] has been accepted for publication in the **"Georgian Mathematical Journal"**.

Chapter 7 provides Hermite-Hadamard type inequalities for n-times differentiable mappings defined on an invex set whose absolute value satisfy the conditions of preinvexity, logarithmic preinvexity and s-preinvexity. Through our results we have also obtained some corrected Hermite-Hadamard type inequalities established for the space of logarithmic preinvex functions. We have provided results of this chapter by using well-known techniques, that is, by proving some new identities, using mathematical analysis and by using the assumptions of preinvexity, logarithmic preinvexity and s-preinvexity on the absolute value of the nth derivative of n-times differentiable mapping f. The results provided in this chapter have been chosen from the papers [88], [89] and [90]. The results of the paper [88] have been accepted for publications in the "Georgian Mathematical Journal", the papers [89] and [90] have been published in the **"Filomat 29:7 (2015), 1651–1661"** and **"Hacettepe Journal of Mathematics and Statistics, 44 (4) (2015), 839- 853"** respectively.

Chapter 8 incorporates some Fejér type inequalities by using the notions of geometric-arithmetic convexity and by using the geometric symmetry of the weight function involved. We have also proved some new error bounds of the difference between the middle and right terms, and between the difference of middle and left terms of the Fejér type inequalities. To this end, we have to prove two new identities by choosing a weight functions which satisfies geometric symmetry condition and a differentiable mapping. At the end of this chapter, some applications to special means of real numbers are stipulated by a selection of appropriate weight function and a differentiable function. The results of this chapter are from the references [91] and [92] of which [91] is under review in **"Applied Mathematics and Computation"** and [92] has been submitted to **"Journal of Convex Analysis"**.

Chapter 9 discusses about the concluding remarks and some new results and extensions to be done as future works.

Chapter 2

Preliminary Definitions and Literature Review

2.1 Introduction

Since the theory of convex functions plays an important role in almost all branches of mathematics, hence it is of worth to describe this notion and related results in the forthcoming paragraphs. At the next step, we will give the other types of convex functions such as quasi-convex, s-convex, m-convex, (α, m)-convex, h-convex, logarithmically convex, s-logarithmically convex, m-logarithmically convex, (α, m)-logarithmically convex, preinvex, prequasiinvex, s-preinvex, m-preinvex, (α, m)-preinvex, h-preinvex, logarithmically preinvex functions and concerned results to these types of convex functions which are in fact generalizations of the usual notion of convex functions.

We will consider the following notations throughout this dissertation

$A(a, b)$ will be usual arithmetic mean of numbers a and b, that is, $A(a, b) = \frac{a+b}{2}$ and $A_\lambda(a, b)$ will be the weighted arithmetic means between the numbers a and b, that is, $A_\lambda(a, b) = \lambda a + (1-\lambda)b$. In a similar way $G(a, b) = \sqrt{ab}$ is the usual geometric mean between a and b and $G_\lambda(a, b) = a^\lambda b^{1-\lambda}$ is the weighted geometric mean of a and b. In this chapter, a complete section is also devoted to special means of real numbers.

2.2 Convex Function

An important mathematical problem is to investigate how function behave under the action of means. The best known case is that of midpoint convex (or Jensen convex) functions, which deals with the arithmetic mean [125, p.6].

J-convex function: A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is convex in Jensen sense or J-convex if

$$f\left(A\left(x_1, x_2\right)\right) \leq A\left(f(x_1), f(x_2)\right) \tag{2.1}$$

for all $x_1, x_2 \in I$.

Convex function: [125, p.6] A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is known to be convex if

$$f(A_\lambda\left(x, y\right)) \leq A_\lambda\left(f(x), f(y)\right) \tag{2.2}$$

for all points x and y in I and $0 \leq \lambda \leq 1$. It is called strictly convex if the inequality (2.2) holds strictly whenever x and y are distinct points and $\lambda \in (0, 1)$. If $-f$ is convex (respectively, strictly convex) then we say that f is concave (respectively, strictly concave). If f is both convex and concave, then f is said to be affine.

In the context of continuity, midpoint convexity means convexity, that is, the following criteria of equivalence of (2.1) and (2.2) given by J. L. W. V. Jensen is valid

Theorem 1 *[125, p.10] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function. Then f is convex iff f is mid-convex, that is,*

$$f\left(A\left(x, y\right)\right) \ \leq A\left(f(x), f(y)\right) \quad \textit{for all } x, y \in I. \tag{2.3}$$

By mathematical induction we can extend the inequality (2.2) to the convex combinations of finitely many points in I. These extensions are known as discrete Jensen inequality and the integral Jensen inequality respectively [125, p.8].

Theorem 2 ***(The discrete case of Jensen's Inequality)*** *A real valued function f defined on an interval I is convex iff for all $x_1, ..., x_n \in I$ and all scalars $p_1, ..., p_n \in [0,1]$ with $P_n = \sum_{i=1}^{n} p_i$ we have*

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i), \tag{2.4}$$

The above inequality is strict if f is strictly convex, all the points x_i are disjoint and all scalars p_i are positive.

Between the space of convex and mid-convex functions there is another space of functions called space of wright convex functions [135, p.7].

Wright Convex function: A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be wright convex if for each $x \leq y$, $z \geq 0$, x, $y + z \in I$, the following inequality is valid

$$f(x + z) - f(x) \leq f(y + z) - f(y). \tag{2.5}$$

There is another sub-space of space of convex functions called space of log-convex functions [140, p.18].

Log(or Multiplicative) Convex function: A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be log-convex if f is positive and log f is convex on I, that is, if f is positive and satisfy

$$f(\alpha x + \beta y) \leq f^{\alpha}(x) f^{\beta}(y) \tag{2.6}$$

for $x, y \in I$, $\alpha, \beta > 0$, $\alpha + \beta = 1$.

The space of convex functions can be characterized in a variety of ways. Here we discuss some regularity and geometric properties [140, p.2]. A function f convex and finite on a closed interval $[a, b]$ is bounded from above by $M = \max\{f(a), f(b)\}$, since for any $\lambda \in [0,1]$, $z = A_\lambda(a,b)$ in the interval,

$$f(z) \leq A_\lambda(f(a), f(b)) \leq \lambda M + (1-\lambda)M = M.$$

It is also bounded from below as we see by writing an arbitrary point in the form $A(a,b)+t$. Then

$$f(A(a,b)) \leq \frac{1}{2} f(A(a,b)+t) + \frac{1}{2} f(A(a,b)-t)$$

or

$$f(A(a,b)+t) \geq 2f(A(a,b)) - f(A(a,b)-t).$$

Using M is upper bound, $-f(A(a,b)-t) \geq -M$, so

$$f(A(a,b)+t) \geq 2f(A(a,b)) - M = m.$$

It is easily seen that a convex function may not be continuous at the boundary points of its domain. On the interior it is not only continuous but it satisfies a stronger condition [140, p.4].

Theorem 3 *If $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is convex, then f satisfies a Lipschitz condition on any closed interval $[a,b]$ contained in the interior of I, that is there is a constant K so that for any two points $x, y \in [a,b]$,*

$$|f(x) - f(y)| \leq K\,|x-y|.$$

Consequently, f is absolutely continuous on $[a,b]$ and continuous on interior of I.

The derivative of a convex function is best studied in terms of the left and right derivatives defined by

$$f'_-(x) = \lim_{y \uparrow x} \frac{f(y)-f(x)}{y-x},\; f'_+(x) = \lim_{y \downarrow x} \frac{f(y)-f(x)}{y-x}. \tag{2.7}$$

The following result for left and right derivatives of the convex functions can be seen in [140, p.5].

Theorem 4 *If $f : I \to \mathbb{R}$ is convex (strictly convex), then $f'_{-}(x)$ and $f'_{+}(x)$ exist and are increasing (strictly increasing) on interior of I.*

The following result about convex functions states that a convex function is differentiable almost everywhere.

Theorem 5 *[140, p.7] If $f : I \to \mathbb{R}$ is convex on the open interval I, then the set E where f' fails to exist is countable. Moreover, f' is continuous on I/E.*

For a twice differentiable function the second derivative test is very useful to check the convexity of a function [140, p.11].

Theorem 6 *Suppose f'' exists on (a, b). Then f is convex iff $f''(x) \geq 0$. And if $f''(x) > 0$ on (a, b), then f is strictly convex on the interval.*

The geometric characterization depends upon the idea of a support line.

Support line for convex function: A function $f : I \to \mathbb{R}$ is convex if and only if there is at least one line of support for f at each $x_0 \in I$, that is, there exists an affine function $A(x) = f(x_0) + m(x - x_0)$ such that $A(x) \leq f(x)$ for every $x \in I$. If f is differentiable at x_0 then this line in unique and this is tangent line.

Theorem 7 *[140, p.12] A function $f : (a, b) \to \mathbb{R}$ is convex iff there is at least one line of support for f at each $x_0 \in (a, b)$.*

The following result is not direct characterization of a convex function but it is closely related to Theorem 7.

Theorem 8 *[140, p.12] Let $f : (a, b) \to \mathbb{R}$ be convex. Then f is differentiable at x_0 iff the line of support for f at x_0 is unique. And in this case,*

$$A(x) = f(x_0) + f'(x_0)(x - x_0)$$

provides this unique support.

Sub-differential (or Generalized derivative): Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function, then sub-differential of f at x, denoted by $\partial f(x)$, is defined as

$$\begin{aligned}\partial f(x) &= \{y \in \mathbb{R} : y \text{ is the slope of a support line for } f \text{ at } x\} \\ &= \{y \in \mathbb{R} : f(u) - f(x) - y\,(u - x) \geq 0,\, u \in I\}.\end{aligned}$$

Thus ∂f is a set-valued function that is single valued and agrees with the ordinary two-sided derivative f' wherever the latter exists. The domain of ∂f (dom ∂f) is the set of all x in I where f has support and range of ∂f is the set of support slopes.

Note: It may be noted that (int I=)$I^\circ \subseteq \operatorname{dom} \partial f \subseteq I$.

There is a connection between a convex function and its sub-differential [125, p.30].

Lemma 9 *Let f be a convex function on an interval I. Then $\partial f(x) \neq \emptyset$ at all interior points of I. Moreover, every function $\omega : I \to \mathbb{R}$ for which $\omega(x) \in \partial f(x)$ whenever $x \in I^\circ$ verifies the double inequality*

$$f'_-(x) \leq \omega(x) \leq f'_+(x),$$

and thus is nondecreasing on I°.

Theorem 10 *Let f be a continuous convex function on an interval I and let $\omega : I \to \mathbb{R}$ be a function such that $\omega(x)$ belong to $\partial f(x)$ for all $x \in I^\circ$. Then*

$$f(z) = \sup\{f(x) + (z - x)\,\omega(x) : x \in I^\circ\} \quad \textit{for all } z \in I.$$

2.3 Quasi-convex functions

It is well-known that notion of quasi-convex functions as given in the definition below, generalizes the notion of convex functions.

Definition 11 *[135, p.7] A function $f : [a, b] \to \mathbb{R}$ is said to to be quasi-convex on $[a, b]$ if*

$$f\left(A_{\lambda}\left(x, y\right)\right) \leq \max\left\{f\left(x\right), f\left(y\right)\right\} \tag{2.8}$$

holds for all $x, y \in [a, b]$ and $0 \leq \lambda \leq 1$.

Evidently, every convex function is a quasi-convex function. The following examples shows that not every quasi-convex function is a convex function.

Example 12 *[63] The function $f : [-2, 2] \to \mathbb{R}$, defined by*

$$f(x) = \begin{cases} 1, & t \in [-2, -1] \\ \\ t^2, & (-1, 2] \end{cases}$$

is not convex on $[-2, 2]$ but it is quasi-convex on $[-2, 2]$.

A quasi-convex function may neither be convex nor continuous, for instance the floor function $f\left(x\right) = \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the greatest integer not greater than x is a monotonic increasing function which is quasi-convex but it is neither convex nor continuous. For further details on convex and quasi-convex functions, we refer the reader to [140].

2.4 s-convex functions

The following two views of s-convexity ($0 < s \leq 1$) of real-valued functions have been discussed by Hudzik and Maligranda in [57].

Definition 13 *[131] A function $f : [0, \infty) \to \mathbb{R}$ is considered to be s-convex in the first sense or Orlicz s-convex if*

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y) \tag{2.9}$$

for all $x, y \in [0,1)$ with $\alpha^s + \beta^s = 1$ for some s with $0 < s \leq 1$. This space of functions is expressed by K_s^1.

Definition 14 *[8] A function $f : [0,\infty) \to \mathbb{R}$ is assumed to be s-convex in the second sense or Breckner s-convex if*

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y) \tag{2.10}$$

for all $x, y \in [0,1)$ with $\alpha + \beta = 1$ and for some fixed s with $0 < s \leq 1$. This space of functions is symbolized by K_s^2.

In [137], Pinheiro, claimed that the space K_s^1 has lack of objectivity. Pinheiro explored this space of s-convex functions and expounded why the s-convexity first sense was abandoned by the literature in the field of inequality. Pinheiro amended the space of s-convexity in the first sense and suggested a geometric interpretation for functions in K_s^1 with some related results. For additional results regarding s-convexity in the first sense see [137].

Also, we note that for $s = 1$, s-convexity (in both senses) reduces to the ordinary convexity of functions defined on $[0,\infty)$.

Hudzik and Maligranda, among others, gave an example of a non-continuous Orlicz s-convex function, which is not Breckner s-convex.

Example 15 *[57] Let $s \in (0,1)$ and a, b, c be real numbers. Define $f : [0,\infty) \to \mathbb{R}$ by*

$$f(u) = \begin{cases} a, & u = 0, \\ \\ bu^s + c, & u > 0, \end{cases}$$

we have the following

(1) if $b \in [0,\infty)$ and $c \in [0,a]$, then $f \in K_s^2$.

(2) if $b \in (0, \infty)$ and $c \in (-\infty, 0)$, then $f \notin K_s^2$.

The condition $\alpha + \beta = 1$ in the definition of K_s^2 can be equivalently replaced by the condition $\alpha + \beta \leq 1$ as explained in the following theorem.

Theorem 16 *[57]Let $f \in K_s^2$. Then the inequality (2.10) is valid for all $u, v \in [0, \infty)$ and , $\alpha, \beta \geq 0$ with $\alpha + \beta \leq 1$ if and only if $f(0) = 0$.*

Some further properties of s-convex functions in the second sense are given in the following theorems.

Theorem 17 *[57]Let $0 < s_1 < s_2 \leq 1$. If $f \in K_{s_2}^2$ and $f(0) = 0$, then $f \in K_{s_1}^2$.*

Theorem 18 *[57]Let f be a nondecreasing function in K_s^2 and g be a nonnegative convex function on $[0, 1)$. Then the composition $f \circ g$ of f with g belongs K_s^2 .*

For more properties and results on s-convex functions in the second sense with regard to geometrical point of view we refer the interested reader to Pinheiro [137].

2.5 m-convex and (α, m)-convex functions

Let b be a positive real number and $[0, b]$ be an interval of the real line $\mathbb{R}$. Let $K(b)$ be the space of all functions $f : [0, b] \to \mathbb{R}$ which are continuous and non-negative on $[0, b]$ with the property $f(0) = 0$.

The space of all functions $f \in K(b)$ which are convex on $[0, b]$ is denoted by $K_C(b)$.

A function $f : [0, b] \to \mathbb{R}$ is starshaped with respect to the origin on $[0, b]$ if

$$f(\lambda x) \leq \lambda f(x),$$

holds for all $x \in [0, b]$ and $0 \leq \lambda \leq 1$.

The space of all functions $f \in K(b)$ which are starshaped with respect to the origin on $[0, b]$ is denoted by $K_S(b)$.

let $K_F(b)$ be the space of all functions $f \in K(b)$ convex in mean on $[0,b]$, that is, the space of all functions $f \in K(b)$ for which $F \in K_C(b)$, where the mean function F of the function $f \in K(b)$ is defined by

$$F(x) = \begin{cases} \frac{1}{x}\int_0^x f(t)dt, & x \in (0,b]\,, \\ \\ 0, & x = 0. \end{cases}$$

In [9], Bruckner and Ostrow, among others, proved that $K_C(b) \subset K_F(b) \subset K_S(b)$.

In [156], G. Toader, defined m-convexity, another intermediate between the usual convexity and starshaped convexity, as follows:

Definition 19 *[156] The function $f : [0,b] \to \mathbb{R}$, $b > 0$, is said to be m-convex, where $m \in [0,1]$, if one has*

$$f(\lambda x + m(1-\lambda)y) \leq \lambda f(x) + m(1-\lambda)f(y) \tag{2.11}$$

for all $x, y \in [0,b]$ and $0 \leq \lambda \leq 1$. If $-f$ is m-convex then f is m-concave.

The space of all m-convex functions on $[0,b]$ for which $f(0) \leq 0$ is denoted by $K_m(b)$. Obviously, for $m = 1$, m-convexity is the standard convexity of functions on $[0,b]$, and for $m = 0$ the concept of starshaped functions.

The following lemmas are well known in literature [156].

Lemma 20 *[156] If f is in the space $K_m(b)$, then it is starshaped.*

Lemma 21 *[156] If f is in the space $K_m(b)$ and $0 < n < m \leq 1$, then f is in the space $K_n(b)$.*

From Lemma 20 and Lemma 21 it follows that

$$K_1(b) \subset K_m(b) \subset K_0(b),$$

whenever $m \in (0,1)$. Note that in the space $K_1(b)$ there are only convex functions $f : [0,b] \to \mathbb{R}$ for which $f(0) \leq 0$, that is, $K_1(b)$ is a proper subspace of the space of convex functions on $[0,b]$. It is interesting to point out that for any $m \in (0,1)$ there are continuous and differentiable functions which are m-convex, but which are not convex in the standard sense [157].

The following theorem reveals the fact that there are continuous and differentiable functions which are m-convex but are not convex in the standard sense.

Theorem 22 *[11] For each $m \in (0,1)$ there is an m-convex polynomial f such that f is not n-convex for any $m < n \leq 1$.*

The following result is a characterization when a differentiable function is m-convex.

Lemma 23 *[11] If f is differentiable, then f is m-convex on $[0,b]$ iff*

$$f(x) \leq mf(y) + f^{'}(x)(x - my)$$

for all $x, y \in [0,b]$.

The notion of m-convexity was further generalized in the following definition:

Definition 24 *[109] The function $f : [0,b] \to \mathbb{R}$, $b > 0$, is said to be (α,m)-convex, where $(\alpha,m) \in [0,1]^2$, if*

$$f(\lambda x + m(1-\lambda)y) \leq \lambda^{\alpha} f(x) + m(1-\lambda^{\alpha})f(y) \tag{2.12}$$

holds or all $x, y \in [0,b]$ and $0 \leq \lambda \leq 1$.

The space of all (α,m)-convex functions on $[0,b]$ for which $f(0) \leq 0$, will be denoted by $K_m^{\alpha}(b)$.

If we take $(\alpha, m) = (1, m)$, it can be easily seen that (α, m)-convexity reduces to m-convexity and for $(\alpha, m) = (1, 1)$, (α, m)-convexity reduces to the notion of usual convexity on $[0, b], b > 0$.

An analogue result of Lemma 23 for the space of (α, m)-convex functions is stated below.

Lemma 25 *If f is differentiable, then f is (α, m)-convex on $[0, b]$ iff*

$$\alpha \left(f(x) - mf\left(y\right)\right) \leq f^{'}(x)(x - my)$$

for all $x, y \in [0, b]$.

2.6 h-convex functions

Let I be an interval in $\mathbb{R}$, then the space of Godunova-Levin functions is defined as follows.

Definition 26 *[47]A function $f : I \rightarrow \mathbb{R}$ is stated to belong to the space of Godunova-Levin functions if f is non-negative and for all $x, y \in I$ and $0 < \lambda < 1$ we have*

$$f(A_{\lambda}\left(x, y\right)) \leq \frac{f(x)}{\lambda} + \frac{f(y)}{1 - \lambda}. \tag{2.13}$$

The space of Godunova-Levin functions is denoted by $Q\left(I\right)$ and was initially described in [47] by Godunova and Levin. Some further properties of it are given in [30], [106] and [107]. It is to be noted that non-negative monotone and non-negative convex functions belong to this space of functions.

The following definition is about the space of P-functions.

Definition 27 *[30] A function $f : I \rightarrow \mathbb{R}$ is said to be P-function, or that f belongs to the space $P(I)$, if f is a non-negative function and for all $x, y \in I$, $0 < \lambda < 1$, we have*

$$f(A_{\lambda}\left(x, y\right)) \leq f(x) + f(y).$$

For further results about the space $P(I)$ we refer the readers to [133].

In the paper [158] a larger space of non-negative functions which contains non-negative convex functions, s-convex in the second sense, Godunova-Levin functions and P-functions was considered. This space of functions is known as space of h-convex functions.

Definition 28 *[158] Let I and J be intervals in $\mathbb{R}$ such that $(0,1) \subseteq J$ and let $h : J \to \mathbb{R}$ be a non-negative function, $h \not\equiv 0$. A non-negative function $f : I \to \mathbb{R}$ is said to be an h-convex function, or that f belongs to the space $SX(h, I)$, if for all $x, y \in I$,$0 < \lambda < 1$ we have*

$$f\left(A_{\lambda}\left(x,y\right)\right) \leq h(\lambda) f(x) + h(1-\lambda) f(y). \tag{2.14}$$

A non-negative function $f : I \to \mathbb{R}$ is said to be an h-concave function, or that f belongs to the space $SV(h, I)$ if the inequality (2.14) is reversed.

Remark 29 *It is easy to observe that if f is a non-negative convex function on I and h a non-negative function such that $h(t) \geq \lambda$ for all $\lambda \in (0,1)$, then*

$$f\left(A_{\lambda}\left(x,y\right)\right) \leq A_{\lambda}\left(f(x), f(y)\right) \leq h(\lambda) f(x) + h(1-\lambda) f(y)$$

for $x, y \in I, 0 < \lambda < 1$,that is $f \in SX(h, I)$. Similarly, If f is a non-negative concave function on I and h a non-negative function such that $h(t) \leq \lambda$ for all $\lambda \in (0,1)$, then $f \in SV(h, I)$.

There exist non-convex functions which are h-convex. The following example illustrates this fact.

Example 30 *Let $h_k(x) = x^k$ where $k < 0$, $x > 0$ and let the function $f : I = [a,b] \to \mathbb{R}$ be defined as*

$$f\left(x\right) = \begin{cases} 1, & x \neq \frac{a+b}{2}, \\ \\ 2^{1-k} & x = \frac{a+b}{2}. \end{cases}$$

Then f is a non-convex function, but it is h_k-convex.

The following results hold.

Proposition 31 *[158] Let h_1, h_2 be non-negative functions defined on an interval J with property $h_2(t) \leq h_1(t)$, $t \in (0,1)$. If $f \in SX(h_2, I)$, then $f \in SX(h_1, I)$. If $f \in SV(h_1, I)$, then $f \in SV(h_2, I)$.*

Proposition 32 *[158] If $f, g \in SX(h, I)$ and $\lambda > 0$, then $f + g, \lambda f \in SX(h, I)$. If $f, g \in SV(h, I)$ and $\lambda > 0$, then $f + g$, $\lambda f \in SV(h, I)$.*

Some very interesting results for h-convex functions can be acquired when the function h in Definition 28 possesses the additional properties given in the definitions below.

Definition 33 *[141] A function $h : J \to \mathbb{R}$ is said to be a super-multiplicative function if*

$$h(xy) \geq h(x)h(y) \tag{2.15}$$

for all $x, y \in J$. If inequality (2.15) is reversed, then h is said to be a sub-multiplicative function. If the equality holds in (2.15), then h is said to be a multiplicative function.

Definition 34 *[141] A function $h : J \to \mathbb{R}$ is exclaimed to be a super-additive function if*

$$h(x+y) \geq h(x) + h(y) \tag{2.16}$$

for all $x, y \in J$, when $x + y \in J$. or all $x, y \in J$. If inequality (2.16) is reversed, then h is said to be a sub-additive function. If the equality holds in (2.16), then h is said to be a additive function.

For more properties on h-convex functions and related results see [158] and [143].

Some further generalizations of logarithmic convexity, s-convexity, m-convexity and (α, m)-convexity are stated in the following definitions.

2.7 s-logarithmically convex functions

Definition 35 *[168] For some $s \in (0,1]$, a positive valued function $f : I \subseteq \mathbb{R} \to (0,\infty)$ is considered to be s-logarithmically convex on I if and only if*

$$f\left(A_{\lambda}\left(x,y\right)\right) \leq \left[f\left(x\right)\right]^{\lambda^{s}} \left[f\left(y\right)\right]^{(1-\lambda)^{s}}$$

holds for all x, $y \in I$ and $0 \leq \lambda \leq 1$.

It is obvious that when $s = 1$ in Definition 35, the s-logarithmically convexity becomes the usual logarithmically convexity.

2.8 m-logarithmically and (α, m)-logarithmically convex functions

Definition 36 *[18] A function $f : [0,b] \to (0,\infty)$ is defined to be m-logarithmically convex if*

$$f\left(\lambda x + m\left(1-\lambda\right)y\right) \leq \left[f\left(x\right)\right]^{\lambda} \left[f\left(y\right)\right]^{m(1-\lambda)}$$

holds for all x, $y \in [0,b]$, $0 \leq \lambda \leq 1$ and $m \in (0,1]$.

Definition 37 *[18] A function $f : [0,b] \to (0,\infty)$ is known to be (α, m)-logarithmically convex if*

$$f\left(\lambda x + m\left(1-\lambda\right)y\right) \leq \left[f\left(x\right)\right]^{\lambda^{\alpha}} \left[f\left(y\right)\right]^{m(1-\lambda^{\alpha})}$$

holds for all x, $y \in [0,b]$, $0 \leq \lambda \leq 1$ and $(\alpha, m) \in (0,1] \times (0,1]$.

It is also obvious that if $m = 1$ in Definition 36 and if $(\alpha, m) = (1,1)$ in Definition 37, the notion of m-logarithmic convexity and (α, m)-logarithmic convexity recapture the notion of usual logarithmic convexity.

2.9 Invex Sets and Preinvex Functions

Convexity plays an influential role in mathematical economics, engineering, management science, and optimization theory. A rapid growth of the research on generalization of convexity has been observed in the past decade. Not long ago, quite a few extensions and generalizations have been considered for convexity. An influential generalization of convex functions is that of preinvex functions pioneered by Weir and Mond [166]. Many researchers have studied the basic properties of the preinvex functions and their role in optimization, variational inequalities and equilibrium problems, see for example the works of Mohn and Neogy [101], Noor [119] and Yang et al. [179].

Definition 38 *[166] A set $K \subseteq \mathbb{R}^n$ is said to be invex if there exists a function $\eta : K \times K \to \mathbb{R}^n$ such that*

$$u + \lambda\eta(v, u) \in K, \ \forall u, v \in K, \lambda \in [0, 1].$$

The invex set K is also called an η-connected set.

Definition 39 *[166] Let K be an invex set with respect to $\eta : K \times K \to \mathbb{R}$. A function $f : K \to \mathbb{R}$ is said to be preinvex with respect to η, if*

$$f(u + \lambda\eta(v, u)) \leq A_\lambda\left(f(v), f(u)\right), \forall u, v \in K, \lambda \in [0, 1].$$

The function f is said to be preincave if and only if $-f$ is preinvex.

It is to be noted that every convex set and convex function are respectively invex set and pre-invex function with respect to the map $\eta\left(v, u\right) = v - u$ but the converse is not true see for instance [166].

Remark 40 *[101] The Definition 38 of an invex set has a clear geometric interpretation. This definition essentially says that there is a path starting from a point u which is* *contained in K. We do not require that the point v should be one of the end points of the path. In case v has to be an end point of the path for every pair of points $u, v \in K$, then $\eta(v,u) = v - u$, and consequently invexity reduces to convexity.*

For further properties of preinvex functions, see [178].

2.10 s-Preinvex Functions

Definition 41 *[96] A function $f : K \to [0, \infty)$ on an invex set $K \subseteq [0, \infty)^n$ is said to be s-preinvex with respect to η, if*

$$f(u + \lambda\eta(v,u)) \leq (1-\lambda)^s f(u) + \lambda^s f(v)$$

holds for all $u, v \in K, \lambda \in [0,1]$ and for some fixed $s \in (0,1]$. The function f is said to be s-preincave if and only if $-f$ is s-preinvex.

2.11 h-Preinvex Functions

Similar to the space of h-convex functions, the space of h-preinvex functions was considered in [108] and [115] independently. This space contains several well-known spaces of functions such as preinvex functions, s-preinvex in the second sense, Godunova Levin preinvex functions (Q-preinvex functions) and P-preinvex functions.

Definition 42 *[108, 115] Let J be a real interval such that $(0,1) \subseteq J$ and let $h : J \to \mathbb{R}$ be a non-negative function with $h \not\equiv 0$. A function $f : K \to \mathbb{R}$ defined on an invex subset K of $\mathbb{R}^n$ is called an h-preinvex with respect to η, if for all $x, y \in K$ and $\lambda \in [0,1]$*

$$f(u + \lambda\eta(v,u)) \leq h(1-\lambda) f(u) + h(\lambda) f(v). \tag{2.17}$$

If the inequality in (2.17) holds in reversed, then f is called h-preincave.

Remark 43 *[108] Note, that every convex function is a h-preinvex function with respect to $\eta(v,u) = v - u$ and $h(\lambda) = \lambda$, $\lambda \in [0,1]$.*

Definition 44 *[166] Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}^n$. A function $f : K \to \mathbb{R}$ is said to be prequasi-invex with respect to η, if*

$$f(u + \lambda\eta(v,u)) \leq \max\{f(u), f(v)\}, \forall u, v \in K, \lambda \in [0,1].$$

Definition 45 *[118] Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}^n$. A function $f : K \to (0,\infty)$ is said to be logarithmic preinvex with respect to η, if*

$$f(u + \lambda\eta(v,u) \leq G_\lambda(f(v), f(u)), \forall u, v \in K, \lambda \in [0,1].$$

It is clear from the arithmetic-geometric mean inequality that if $f : K \to (0,\infty)$ is logarithmic preinvex, we have

$$\begin{aligned} f(u + \lambda\eta(v,u)) &\leq G_\lambda(f(v), f(u)) \leq A_\lambda(f(v), f(u)) \\ &\leq \max\{f(u), f(v)\}, \end{aligned}$$

for all $u, v \in K, \lambda \in [0,1]$.

2.12 Geometrically-Arithmetically Convex Functions

The usual definition of a convex function (of one real variable) depends on the structure of $\mathbb{R}$ as an ordered vector space. As $\mathbb{R}$ is actually an ordered field, it is natural to investigate what happens when addition is replaced by multiplication and the arithmetic mean is replaced by the geometric mean. The characteristic property of the subintervals I of $\mathbb{R}$ is

$$x, y \in I,\ \lambda \in [0,1] \Rightarrow A_\lambda(x,y) \in I,$$

so, in order to draw a parallel in the multiplicative case, we have to restrict to the subintervals J of $(0,1)$ and to use instead the following fact

$$x, y \in I,\ \lambda \in [0,1] \Rightarrow G_\lambda(x,y) \in I.$$

Depending on which type of mean, arithmetic (A), or geometric (G), we consider on the domain and the codomain of definition, we shall encounter one of the following four spaces of functions:

1. (A, A)-convex functions, the usual convex functions

2. (A, G)-convex functions

3. (G, A)-convex functions

4. (G, G)-convex functions.

Here we consider only the (G, A)-convex functions, which are defined as follows.

Definition 46 *[125, p.68] A function $f : I \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ is said to be GA-convex function on I if*

$$f(G_\lambda(x,y)) \leq A_\lambda(f(x), f(y))$$

holds for all x, $y \in I$ and $\lambda \in [0,1]$, where $G_\lambda(x,y) = x^\lambda y^{1-\lambda}$ and $A_\lambda(f(x), f(y)) = \lambda f(x) + (1-\lambda)f(y)$ are respectively the weighted geometric mean of two positive numbers x and y and the weighted arithmetic mean of $f(x)$ and $f(y)$.

The definition of GA-convexity is further generalized as GA-s-convexity in the second sense as follows.

Definition 47 *[65] A function $f : I \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ is said to be s-GA-convex function on I if*

$$f(G_\lambda(x,y)) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds for all x, $y \in I$, $\lambda \in [0,1]$ and for some $s \in (0,1]$.

For the properties of GA-convex functions and GA-s-convex functions, we refer the reader to [58, 65, 181] and the references there in.

Definition 48 *[135, p.138] A function $g : [a,b] \to [0,\infty)$ is said to be symmetric with respect to $\frac{a+b}{2}$ if the equality*

$$g(a+b-x) = g(x)$$

holds for all $x \in [a,b]$.

As parallel in the multiplicative case, we can state the subsequent definition.

Definition 49 *[91] A function $g : [a,b] \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ is said to be geometrically symmetric with respect to $G(a,b)$ if*

$$g\left(\frac{ab}{x}\right) = g(x)$$

holds for all $x \in [a,b]$.

2.13 Mathematical Means

The following definition gives a generalization of the notion for a positive valued function of positive variables.

Definition 50 *[14] A function $M : \mathbb{R}_+^2 \to \mathbb{R}_+$, is called a Mean function if it has the following properties*

1. *Homogeneity: $M(ax, ay) = aM(x,y)$, for all $a > 0$,*
2. *Symmetry : $M(x,y) = M(y,x)$,*
3. *Reflexivity : $M(x,x) = x$,*
4. *Monotonicity: If $x \leq x'$ and $y \leq y'$, then $M(x,y) \leq M(x',y')$,*

5. *intermediacy:* $\min\{x,y\} \le M(x,y) \le \max\{x,y\}$.

We consider the following means for positive real numbers $a,\, b \in \mathbb{R}_+$.

1. The arithmetic mean:
$$A(a,b) = \frac{a+b}{2},$$

2. The geometric mean
$$G(a,b) = \sqrt{ab},$$

3. The harmonic mean:
$$H(a,b) = \frac{2}{\frac{1}{a}+\frac{1}{b}},$$

4. The logarithmic mean:
$$L(a,b) = \frac{b-a}{\ln b - \ln a},\, a \neq b,$$

5. The identric mean
$$I(a,b) = \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}, & a \neq b, \\ a & a = b, \end{cases}$$

6. Generalized log-mean:
$$L_p(a,b) = \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p}, & p \neq 0, -1 \text{ and } a \neq b, \\ \frac{b-a}{\ln b - \ln a}, & p = -1 \text{ and } a \neq b, \\ I(a,b), & p = 0 \text{ and } a \neq b, \\ a, & a = b \end{cases}$$

and

7. The power mean:
$$P_r(a,b) = \left(\frac{a^r + b^r}{2}\right)^{\frac{1}{r}}, \ r \geq 1.$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$, with $L_{-1} = L$ and $L_0 = I$. In particular, we have the inequality $H \leq G \leq L \leq I \leq A$.

The reader may find a comprehensive study on the entire topics of means in [14, 15].

2.14 Hölder's Inequality

In mathematical analysis, Hölder's inequality is an important inequality concerning integrals and is an essential tool to study the L_p spaces. This famous inequality is named after the renowned mathematician Otto Hölder.

Theorem 51 *[50] Let (X, Σ, μ) be an arbitrary measure space and let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that f and g are real- or complex-valued measurable functions defined on X. Then*
$$\|fg\|_1 \leq \|f\|_p \|g\|_q. \tag{2.18}$$
The numbers p and q known to be Hölder conjugates of each other.

Remark 52 *If $p, q \in (1, \infty)$ and $f \in L_p(\mu)$, $g \in L_q(\mu)$, equality holds in Hölder's inequality if and only if $\alpha|f|^p = \beta|g|^q$ a.e. on X for some α, $\beta > 0$.*

Remark 53 *When $p = q = 2$, the inequality (2.18) becomes Cauchy-Schwarz inequality. Hölder's inequality holds even if $\|fg\|_1$ is infinite, the right-hand side also being infinite in that case.*

In Hölder's inequality the following conventions are considered.

- For the Hölder conjugates p and q, $1/\infty$ is understood to be 0.

- If $p, q \in [1, \infty)$, then

$$\|f\|_p = \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} \text{ and } \|g\|_q = \left(\int_X |g|^q \, d\mu\right)^{\frac{1}{q}}.$$

  The expressions $\left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}}$ and $\left(\int_X |g|^q \, d\mu\right)^{\frac{1}{q}}$ can possibly be infinite.

- If $p = \infty$, $\|f\|_\infty$ stands for the essential supremum of f on X give by

$$\|f\|_\infty = \inf \{M \in [0, \infty) : \mu \{x \in X : |f(x)| > M\} = 0\}.$$

  $\|g\|_\infty$ can similarly be defined.

- On the right-hand side of Hölder's inequality, $0 \times \infty$ as well as $\infty \times 0$ means 0. Multiplying $a > 0$ with ∞ gives ∞.

Hölder's Inequality Lebesgue measure

If X is a measurable subset of $\mathbb{R}^n$ with the Lebesgue measure μ, and f and g are real- or complex-valued measurable functions on X, then Hölder inequality can be written as

$$\int_X |fg| \, d\mu \leq \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} \left(\int_X |g|^q \, d\mu\right)^{\frac{1}{q}}.$$

If $X = [a, b]$ and f , g are real-valued measurable functions on $[a, b]$, then

$$\int_a^b |f(x) g(x)| \, dx \leq \left(\int_a^b |f(x)|^p \, dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q \, dx\right)^{\frac{1}{q}}.$$

2.15 Hermite-Hadamard type Inequalities for different spaces of convex functions

It is well known in mathematics literature that if $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex mapping and $a, b \in I$ with $a < b$, then

$$f\left(A\left(a,b\right)\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq A\left(f(a), f(b)\right). \tag{2.19}$$

The inequalities (2.19) hold in reversed direction if f is concave. The double inequality (2.19) was firstly discovered by Ch. Hermite [49] in 1881 in the journal Mathesis but was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result. E. F. Beckenbach [17, p.441], a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard [48] in 1893. Later on, in 1974, D. S. Mitrinović [104] found Hermite's note in Mathesis. This is why, the inequality (2.19) is now commonly referred as the Hermite-Hadamard inequality. Since the discovery of (2.19) in 1883, Hermite-Hadamard inequality (see [135, p.139]) has been considered the most expedient inequality in mathematical analysis. Some of the special inequalities for means can be derived from (2.19) for particular choices of the function f. Moreover, (2.19) provides a necessary and sufficient condition for a function f to be convex in (a, b).

In [44], Fejér gave a weighted generalization of (2.19) while studying trigonometric polynomials. Fejér's original result reads as follows (see [135, p.138]):

Consider the integral $\int_a^b f(x)w(x)dx$ where f is a convex function in the interval (a, b) and g is a positive function in the same interval such that

$$g\left(a+t\right) = g\left(b-t\right), \quad 0 \leq t \leq \frac{1}{2}\left(a+b\right),$$

i.e., $y = g\left(x\right)$ is symmetric curve with respect to the straight line which contains the point $\left(\frac{1}{2}(a+b), 0\right)$ and normal to the x-axis. Then

$$f(A(a,b)) \int_a^b w(x)dx \leq \int_a^b f(x)w(x)dx \leq A(f(a), f(b)) \int_a^b w(x)dx. \tag{2.20}$$

Many mathematicians have tried to answer an important question of acquiring the bounds of the following quantities which appear in (2.19) and in (2.20):

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(A(a,b))\right|, \tag{2.21}$$

$$\left|A(f(a), f(b)) - \frac{1}{b-a}\int_a^b f(x)dx\right|, \tag{2.22}$$

$$\left|\int_a^b f(x)g(x)\,dx - f(A(a,b))\int_a^b g(x)\,dx\right| \tag{2.23}$$

and

$$\left|A(f(a), f(b))\int_a^b g(x)\,dx - \int_a^b f(x)g(x)\,dx\right|. \tag{2.24}$$

In [27], Dragomir and Agarwal proved the following results.

Theorem 54 *[27, Theorem 2.2] Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function and a, $b \in I^\circ$ with $a < b$. If $f' \in L([a,b])$ and $|f'|$ is convex on $[a,b]$, then*

$$\left|A(f(a), f(b)) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{4} A\left(\left|f'(a)\right|, \left|f'(b)\right|\right). \tag{2.25}$$

Theorem 55 *[27, Theorem 2.3] Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function and a, $b \in I^\circ$ with $a < b$. If $f' \in L([a,b])$ and $|f'|^{p/(p-1)}$ is convex on $[a,b]$ for $p > 1$, then*

$$\begin{aligned}
&\left|A(f(a), f(b)) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\qquad \leq \frac{(b-a)}{2(p+1)^{1/p}}\left[A\left(\left|f'(a)\right|^{p/(p-1)}, \left|f'(b)\right|^{p/(p-1)}\right)\right]^{(p-1)/p}.
\end{aligned} \tag{2.26}$$

Pearce and Pečarić [132], improved the inequality (2.26) by proving the following result.

Theorem 56 *[132, Theorem 1] Let $f : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function and a, $b \in I^{\circ}$ with $a < b$. If $f' \in L([a,b])$ and $|f'|^q$ is convex on $[a,b]$ for $q \geq 1$, then*

$$\left| A(f(a), f(b)) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \leq \frac{(b-a)}{4}\left[A\left(\left|f'(a)\right|^q, \left|f'(b)\right|^q \right) \right]^{1/q} \tag{2.27}$$

and

$$\left| f(A(a,b)) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \leq \frac{(b-a)}{4}\left[A\left(\left|f'(a)\right|^q, \left|f'(b)\right|^q \right) \right]^{1/q}. \tag{2.28}$$

Pearce and Pečarić [132] also established the following results by using the concavity condition.

Theorem 57 *[132, Theorem 1] Suppose the assumptions of Theorem 56 are satisfied and $|f'|^q$ is concave on $[a,b]$ for $q \geq 1$. Then*

$$\left| A(f(a), f(b)) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \leq \frac{(b-a)}{4}\left| f'(A(a,b)) \right| \tag{2.29}$$

and

$$\left| f(A(a,b)) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \leq \frac{(b-a)}{4}\left| f'(A(a,b)) \right|. \tag{2.30}$$

As similar result to that of Theorem 54, Kirmici gave the following result which gives bound on (2.21).

Theorem 58 *[74] Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, a, $b \in I^{\circ}$ with $a < b$ and let $q \geq 1$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds*

$$\left| f(A(a,b)) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \leq \frac{(b-a)}{4} A\left(\left|f'(a)\right|, \left|f'(b)\right| \right). \tag{2.31}$$

In [53], Hwang established the following results for convex functions and these results provide a weighted generalization of the results given by (2.28) in Theorem 56.

Theorem 59 *[53] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where $a, b \in I^\circ$ with $a < b$ and let $g : [a, b] \to [0, \infty)$ be continuous positive mapping and symmetric to $\frac{a+b}{2}$. If $\left|f'\right|$ is convex function on $[a, b]$, the following inequality holds:*

$$\left|\int_a^b f(x)g(x)dx - f\left(A\left(a,b\right)\right)\int_a^b g(x)dx\right| \\ \leq (b-a)\,A\left(\left|f'\left(a\right)\right|,\left|f'\left(b\right)\right|\right)\int_0^1 M\left(g;a,b,t\right)dt, \quad (2.32)$$

where $M\left(g;a,b,t\right) = \int_a^{L(a,b,t)} g\left(x\right)dt$ and $L\left(a,b,t\right) = \frac{1+t}{2}a + \frac{1-t}{2}b$.

Theorem 60 *[53] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where $a, b \in I^\circ$ with $a < b$ and let $g : [a, b] \to [0, \infty)$ be continuous positive mapping and symmetric to $\frac{a+b}{2}$ and $q \geq 1$. If $\left|f'\right|^q$ is convex function on $[a, b]$, the following inequality holds:*

$$\left|\int_a^b f(x)g(x)dx - f\left(A\left(a,b\right)\right)\int_a^b g(x)dx\right| \\ \leq (b-a)\left[A\left(\left|f'\left(a\right)\right|^q,\left|f'\left(b\right)\right|^q\right)\right]^{\frac{1}{q}}\int_0^1 M\left(g;a,b,t\right)dt, \quad (2.33)$$

where $M\left(g;a,b,t\right)$ and $L\left(a,b,t\right)$ are defined as in Theorem 59.

Most recently, Hwang [51] established the following results, which generalize the above results of Theorem 56 and Theorem 57, for functions whose nth derivatives in absolute value are convex and concave functions.

Theorem 61 *[51] Suppose $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$, a, $b \in I^\circ$ with $a < b$. If $f^{(n)}$ exists on I°, $f^{(n)} \in L\left([a, b]\right)$ and $\left|f^{(n)}\right|^q$ is convex on $[a, b]$ for $q \geq 1$, $n \in \mathbb{N}$, $n \geq 2$. Then*

$$\left|A\left(f\left(a\right),f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx - \sum_{k=2}^{n-1}\frac{\left(k-1\right)\left(b-a\right)^k}{2\left(k+1\right)!}f^{(k)}(a)\right| \\ \leq \frac{\left(n-1\right)^{1-\frac{1}{q}}\left(b-a\right)^n}{2\left(n+1\right)!}\left[\frac{\left(n^2-2\right)\left|f^{(n)}\left(a\right)\right|^q + n\left|f^{(n)}\left(b\right)\right|^q}{n+2}\right]^{\frac{1}{q}}. \quad (2.34)$$

Theorem 62 *[51] Suppose $f : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$, a, $b \in I^{\circ}$ with $a < b$. If $f^{(n)}$ exists on I°, $f^{(n)} \in L([a,b])$ and $\left|f^{(n)}\right|^q$ is convex on $[a,b]$ for $q \geq 1$, $n \in \mathbb{N}$, $n \geq 1$. Then*

$$\left|\frac{1}{b-a}\sum_{k=0}^{n-1}\frac{\left[1+(-1)^k\right](b-a)^{k+1}}{2^{k+1}(k+1)!}f^{(k)}(A(a,b))-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^n}{2^n(n+1)!}\left[A\left(\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right)\right]^{\frac{1}{q}}. \quad (2.35)$$

Theorem 63 *[51] Suppose $f : I^{\circ} \subset \mathbb{R} \to \mathbb{R}$, a, $b \in I^{\circ}$ with $a < b$. If $f^{(n)}$ exists on I°, $f^{(n)} \in L([a,b])$ and $\left|f^{(n)}\right|^q$ is concave on $[a,b]$, $q \geq 1$. Then*

$$\left|A(f(a),f(b))-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right| \leq \frac{(n-1)(b-a)^n}{2(n+1)!}\left|f^{(n)}\left(\frac{(n^2-2)a+nb}{(n-1)(n+2)}\right)\right| \text{ for } n \geq 2 \quad (2.36)$$

and

$$\left|\frac{1}{b-a}\sum_{k=0}^{n-1}\frac{\left[1+(-1)^k\right](b-a)^{k+1}}{2^{k+1}(k+1)!}f^{(k)}(A(a,b))-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^n}{2^n(n+1)!}\left|f^{(n)}(A(a,b))\right| \text{ for } n \geq 1. \quad (2.37)$$

Following interesting results can be derived directly from Theorem 61, Theorem 62 and Theorem 63 for $n = 2$.

Corollary 64 *[51] Under the assumptions of Theorem 61, for $n = 2$, we have*

$$\left|A(f(a),f(b))-\frac{1}{b-a}\int_a^b f(x)\,dx\right| = \frac{(b-a)^2}{12}\left[A\left(\left|f''(a)\right|^q,\left|f''(b)\right|^q\right)\right]^{\frac{1}{q}}. \quad (2.38)$$

Corollary 65 *[51] Under the assumptions of Theorem 62, for $n = 2$, we have*

$$\left| f\left(A\left(a,b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx \right| \leq \frac{\left(b-a\right)^2}{24}\left[A\left(\left|f''\left(a\right)\right|^q, \left|f''\left(b\right)\right|^q\right)\right]^{\frac{1}{q}}. \tag{2.39}$$

Corollary 66 *[51] Under the assumptions of Theorem 63, for $n = 2$, we have*

$$\left| A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx \right| \leq \frac{\left(b-a\right)^2}{12}\left|f''\left(A\left(a,b\right)\right)\right|. \tag{2.40}$$

It can be noticed that the inequalities (2.38), (2.39) and (2.40) may be better than those given in Theorem 56 and Theorem 57.

As mentioned in the beginning, the notion of convexity has been subject to extensive research and many researchers are either trying to establish different forms of the inequalities (2.19 and (2.20) or to improve the bounds on the quantities given by (2.21)-(2.24).

In this direction, Dragomir and Fitzpatrick [32] proved a variant of (2.19) which holds for s-convex functions in the second sense.

Theorem 67 *[32] Suppose that $f : [0,\infty) \to [0,\infty)$ is an s-convex function in the second sense, where $s \in (0,1)$, a, $b \in [0,\infty)$ with $a < b$. If a, $b \in L\left([a,b]\right)$, then the following inequalities hold*

$$2^{s-1} f\left(A\left(a,b\right)\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1}. \tag{2.41}$$

Kirmaci et al. [77], proved the following results which give bounds on (2.22) by using the assumptions of s-convexity and s-concavity.

Theorem 68 *[77] Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $\left|f'\right|^q$ is s-convex on $[a,b]$, for some fixed*

$s \in (0,1]$ and $q \geq 1$, then the following inequality holds:

$$\left| A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right) dx \right| \\ \leq \frac{(b-a)}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{s+\left(\frac{1}{2}\right)^s}{(s+1)(s+2)}\right]^{\frac{1}{q}}\left[\left|f^{'}\left(a\right)\right|^q + \left|f^{'}\left(b\right)\right|^q\right]^{\frac{1}{q}}. \quad (2.42)$$

Theorem 69 *[77] Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I such that $f^{'} \in L[a,b]$, where $a, b \in I$ with $a < b$. If $\left|f^{'}\right|^q$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $q > 1$, then the following inequality holds:*

$$\left| A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right) dx \right| \leq \frac{(b-a)}{2}\left[\frac{q-1}{2\left(2q-1\right)}\right]^{1-\frac{1}{q}} \\ \times \left[\left(\frac{\left|f^{'}\left(a\right)\right|^q + \left|f^{'}\left(A\left(a,b\right)\right)\right|^q}{s+1}\right)^{\frac{1}{q}} + \left(\frac{\left|f^{'}\left(A\left(a,b\right)\right)\right|^q + \left|f^{'}\left(b\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right] \\ \leq \frac{(b-a)}{2}\left[\left(\left|f^{'}\left(a\right)\right|^q + \left|f^{'}\left(A\left(a,b\right)\right)\right|^q\right)^{\frac{1}{q}} + \left(\left|f^{'}\left(A\left(a,b\right)\right)\right|^q + \left|f^{'}\left(b\right)\right|^q\right)^{\frac{1}{q}}\right]. \quad (2.43)$$

Theorem 70 *[77] Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I such that $f^{'} \in L[a,b]$, where $a, b \in I$ with $a < b$. If $\left|f^{'}\right|^q$ is s-concave on $[a,b]$, for some fixed $s \in (0,1]$ and $q > 1$, then the following inequality holds:*

$$\left| A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right) dx \right| \leq \frac{(b-a)}{2}\left[\frac{q-1}{2\left(2q-1\right)}\right]^{1-\frac{1}{q}} s^{\frac{s-1}{q}} \\ \times \left[\left|f^{'}\left(\frac{3a+b}{2}\right)\right|^q + \left|f^{'}\left(\frac{a+3b}{2}\right)\right|^q\right] \\ \leq \frac{(b-a)}{2}\left[\left|f^{'}\left(\frac{3a+b}{2}\right)\right|^q + \left|f^{'}\left(\frac{a+3b}{2}\right)\right|^q\right]. \quad (2.44)$$

Following results, which give bounds on (2.21) by using the s-convexity and concavity conditions, were proven by Alomari et al. in [3].

Theorem 71 *[77] Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that*

$f' \in L[a,b]$, *where* $a, b \in I$ *with* $a < b$. *If* $|f'|^q$ *is* s*-convex on* $[a,b]$, *for some fixed* $s \in (0,1]$ *and* $q \geq 1$, *then the following inequality holds:*

$$\begin{aligned}
&\left| f(A(a,b)) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \\
&\quad \leq \left(\frac{b-a}{8}\right)\left(\frac{2}{(s+1)(s+2)}\right)^{\frac{1}{q}} \left[\left(\left|f'(a)\right|^q + (s+1)\left|f'(A(a,b))\right|^q \right)^{\frac{1}{q}} \right. \\
&\qquad \left. + \left(\left|f'(b)\right|^q + (s+1)\left|f'(A(a,b))\right|^q \right)^{\frac{1}{q}} \right] \\
&\quad \leq \left(\frac{b-a}{8}\right)\left(\frac{2}{(s+1)(s+2)}\right)^{\frac{1}{q}} \left[\left\{ \left(2^{1-s}+1\right)\left|f'(a)\right|^q + 2^{1-s}\left|f'(a)\right|^q \right\}^{\frac{1}{q}} \right. \\
&\qquad \left. + \left\{ 2^{1-s}\left|f'(a)\right|^q + \left(2^{1-s}+1\right)\left|f'(a)\right|^q \right\}^{\frac{1}{q}} \right]. \qquad (2.45)
\end{aligned}$$

Theorem 72 *[3] Let* $f : I \subset [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I° *such that* $f' \in L[a,b]$, *where* $a, b \in I$ *with* $a < b$. *If* $|f'|^{p/(p-1)}$ *is* s*-convex on* $[a,b]$, *for some fixed* $s \in (0,1]$ *and* $p > 1$, *then the following inequality holds:*

$$\begin{aligned}
&\left| f(A(a,b)) - \frac{1}{b-a}\int_a^b f(x)\,dx \right| \leq \left(\frac{b-a}{4}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}} \\
&\qquad \times \left[\left(\left|f'(a)\right|^q + \left|f'(A(a,b))\right|^q \right)^{\frac{1}{q}} + \left(\left|f'(b)\right|^q + \left|f'(A(a,b))\right|^q \right)^{\frac{1}{q}} \right] \\
&\quad \leq \left(\frac{b-a}{4}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{2}{q}} \left[\left\{ \left(2^{1-s}+s+1\right)\left|f'(a)\right|^q + 2^{1-s}\left|f'(b)\right|^q \right\}^{\frac{1}{q}} \right. \\
&\qquad \left. + \left\{ 2^{1-s}\left|f'(a)\right|^q + \left(2^{1-s}+s+1\right)\left|f'(b)\right|^q \right\}^{\frac{1}{q}} \right], \qquad (2.46)
\end{aligned}$$

where p *is the conjugate of* q, $q = p/(p-1)$.

Theorem 73 *[3] Let* $f : I \subset [0,\infty) \to \mathbb{R}$ *be a differentiable mapping on* I° *such that* $f' \in L[a,b]$, *where* $a, b \in I$ *with* $a < b$. *If* $|f'|^q$ *is concave on* $[a,b]$, *for* $q > 1$, *then the* *following inequality holds:*

$$\left|f\left(A\left(a,b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right| \leq \left(\frac{b-a}{4}\right)\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left[\left|f'\left(\frac{3a+b}{4}\right)\right|+\left|f'\left(\frac{a+3b}{4}\right)\right|\right]. \quad (2.47)$$

In [70], Jiang et al. generalized the results of Theorem 61 and Theorem 62 for s-convex functions.

Theorem 74 *[70] If $f\left(x\right)$ is an n-times differentiable on $[a,b]\subset[0,\infty)$ such that $\left|f^{(n)}\left(x\right)\right|^q$ is s-convex on $[a,b]$ for $q\geq 1$ and $n\geq 2$, then*

$$\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx-\sum_{k=2}^{n-1}\frac{\left(k-1\right)\left(b-a\right)^k}{2\left(k+1\right)!}f^{(k)}(a)\right| \leq \frac{\left(b-a\right)^n}{2n!}\left[P\left|f^{(n)}\left(a\right)\right|^q+Q\left|f^{(n)}\left(b\right)\right|^q\right]^{\frac{1}{q}}, \quad (2.48)$$

where the sum above takes 0 when $n=2$,

$$P=\frac{n\left(n-1\right)+s\left(n-2\right)}{\left(n+s\right)\left(n+s+1\right)}, Q=nB\left(n,s+1\right)-2B\left(n+1,s+1\right)$$

and

$$B\left(x,y\right)=\int_0^1 t^{x-1}\left(1-t\right)^{y-1}dt$$

x, $y>0$ is the Beta function.

Theorem 75 *[70] If $f\left(x\right)$ is an n-times differentiable on $[a,b]\subset[0,\infty)$ such that*

$\left|f^{(n)}(x)\right|^q$ *is* s*-convex on* $[a,b]$ *for* $q \geq 1$ *and* $n \geq 2$*, then*

$$\left|\frac{1}{b-a}\sum_{k=0}^{n-1}\frac{\left[1+(-1)^k\right](b-a)^{k+1}}{2^{k+1}(k+1)!}f^{(k)}(A(a,b))-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq M\left[N\left|f^{(n)}(a)\right|^q+\frac{2}{n+s+1}\left|f^{(n)}(A(a,b))\right|^q+N\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}, \quad (2.49)$$

where

$$M=\frac{1}{2n!}\left(\frac{2}{n+1}\right)^{1-\frac{1}{q}}\left(\frac{b-a}{2}\right)^n \text{ and } B(n+1,s+1).$$

Now we present some results of interest for another generalizations of the convexity, namely, for logarithmic convexity, s-logarithmic convexity, m- and (α,m)-logarithmic convexity.

Gill et al. [46], proved that the following result holds for log-convex functions.

Theorem 76 *[46] Let* f *be a positive, log-convex function on* $[a,b]$*. Then*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leq L(f(a),f(b)). \quad (2.50)$$

For f *a positive log-concave function, the inequality is reversed.*

Xi et al. [168] obtained the following Hermite-Hadamard type inequalities for s-logarithmically convex functions.

Theorem 77 *[168] Let* $f: I \subseteq [0,\infty) \to (0,\infty)$ *be a differentiable function on* I°*,* a*,* $b \in I^\circ$ *with* $a<b$ *and* $f' \in L([a,b])$*. If* $|f(x)|^q$ *for* $q \geq 1$ *is* s*-logarithmically convex on* $[a,b]$ *for some given* $s \in (0,1]$*, then*

$$\left|f(a)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)}{4}\left(\frac{1}{2}\right)^{1-1/q}\left\{3^{(q-1)/q}\left[L_1(\mu,q)\right]^{1/q}+\left[L_2(\mu,q,b)\right]^{1/q}\right\}, \quad (2.51)$$

where

$$L_1(\mu,q) \le \begin{cases} \left|f'(a)f'(b)\right|^{sq/2} F_1(\mu_1), & 0 < \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right| \le 1, \\ \\ \left|f'(a)f'(b)\right|^{q/(2s)} F_1(\mu_2), & 1 \le \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right|, \\ \\ \left|f'(a)f'(b)\right|^{sq/2} F_1(\mu_3), & 0 < \left|f^{(n)}(a)\right| \le 1 < \left|f^{(n)}(b)\right|, \\ \\ \left|f'(a)f'(b)\right|^{q/(2s)} F_1(\mu_4), & 0 < \left|f^{(n)}(b)\right| \le 1 < \left|f^{(n)}(a)\right|, \end{cases}$$

$$L_2(\mu,q,u) \le \begin{cases} \left|f'(u)\right|^{sq/2} F_1(\mu_1), & 0 < \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right| \le 1, \\ \\ \left|f'(u)\right|^{q/(2s)} F_1(\mu_2), & 1 \le \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right|, \\ \\ \left|f'(u)\right|^{sq/2} F_1(\mu_3), & 0 < \left|f^{(n)}(a)\right| \le 1 < \left|f^{(n)}(b)\right|, \\ \\ \left|f'(u)\right|^{q/(2s)} F_1(\mu_4), & 0 < \left|f^{(n)}(b)\right| \le 1 < \left|f^{(n)}(a)\right|, \end{cases}$$

$$F_1(\nu) = \begin{cases} \frac{1}{\ln \nu}\left(2\nu - 1 - \frac{\nu-1}{\ln \nu}\right) & \nu \ne 1, \\ \\ \frac{3}{2} & \nu = 1, \end{cases}$$

$$F_2(\nu) = \begin{cases} \frac{1}{\ln \nu}\left(\nu - \frac{\nu-1}{\ln \nu}\right) & \nu \ne 1, \\ \\ \frac{1}{2} & \nu = 1, \end{cases}$$

and

$$\mu_1 = \left|\frac{f'(a)}{f'(b)}\right|^{sq/2}, \mu_2 = \left|\frac{f'(a)}{f'(b)}\right|^{q/(2s)},$$
$$\mu_3 = \frac{\left|f'(a)\right|^{sq/2}}{\left|f'(b)\right|^{q/(2s)}}, \mu_4 = \frac{\left|f'(a)\right|^{q/(2s)}}{\left|f'(b)\right|^{qs/2}}.$$

Theorem 78 *[168] Under the conditions of Theorem 77, we have*

$$\left|f(b) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ \leq \frac{(b-a)}{4}\left(\frac{1}{2}\right)^{1-1/q}\left\{[L_2(\mu,q,a)]^{1/q} + 3^{(q-1)/q}\left[L_1\left(\mu^{-1},q\right)\right]^{1/q}\right\}, \quad (2.52)$$

where $L_1(\mu,q)$*,* $L_2(\mu,q,u)$*,* $F_1(\nu)$*,* $F_2(\nu)$ *and* μ_i *for* $i = 1, 2, 3, 4$ *are defined as in Theorem 77.*

Theorem 79 *[168] Under the conditions of Theorem 77, we have*

$$\left|A(f(a), f(b)) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ \leq \frac{(b-a)}{4}\left(\frac{1}{2}\right)^{1-1/q}\left\{[L_2(\mu,q,b)]^{1/q} + \left[L_1\left(\mu^{-1},q,a\right)\right]^{1/q}\right\}, \quad (2.53)$$

where $L_1(\mu,q)$*,* $L_2(\mu,q,u)$*,* $F_1(\nu)$*,* $F_2(\nu)$ *and* μ_i *for* $i = 1, 2, 3, 4$ *are defined as in Theorem 77.*

Bai et al. [18] obtained the following Hermite-Hadamard type inequalities for m- and (α, m)-logarithmically convex functions.

Theorem 80 *[18] Let* $I \subseteq [0,\infty)$ *be an open real interval and let* $f : I \to (0,\infty)$ *be a differentiable function on* I *such that* $f' \in L([a,b])$ *for* $0 \leq a < b < \infty$*. If* $\left|f'(x)\right|^q$ *is*

(α, m)-logarithmically convex on $\left[0, \frac{b}{m}\right]$ for $q \in [1, \infty)$, $(\alpha, m) \in (0,1] \times (0,1]$, we have

$$\left|A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx\right| \\ \leq \frac{(b-a)}{2}\left|f'\left(\frac{b}{m}\right)\right|^m \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left[E_1\left(\alpha, m, q\right)\right]^{\frac{1}{q}}, \quad (2.54)$$

where

$$\mu = \frac{\left|f'(a)\right|}{\left|f'\left(\frac{b}{m}\right)\right|^m},\ E_1\left(\alpha, m, q\right) = \begin{cases} \frac{1}{2}, & \mu = 1 \\ F_1\left(\mu, \alpha q\right), & 0 < \mu < 1 \\ F_1\left(\mu, \frac{q}{\alpha}\right), & \mu > 1 \end{cases}$$

and

$$F_1\left(u, v\right) = \frac{1}{v^2\left(\ln u\right)^2}\left[v\left(u^v - 1\right)\ln u - 2\left(u^{v/2} - 1\right)^2\right]$$

for u, $v > 0$, $u \neq 1$.

Corollary 81 *[18] Let $I \subseteq [0, \infty)$ be an open real interval and let $f : I \to (0, \infty)$ be a differentiable function on I such that $f' \in L([a, b])$ for $0 \leq a < b < \infty$. If $\left|f'(x)\right|^q$ is m-logarithmically convex on $\left[0, \frac{b}{m}\right]$ for $q \in [1, \infty)$, $m \in (0, 1]$, we have*

$$\left|A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx\right| \\ \leq \frac{(b-a)}{2}\left|f'\left(\frac{b}{m}\right)\right|^m \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left[E_1\left(1, m, q\right)\right]^{\frac{1}{q}}, \quad (2.55)$$

where $E_1\left(\alpha, m, q\right)$ is defined as in Theorem 80.

Theorem 82 *[18] Let $I \subseteq [0, \infty)$ be an open real interval and let $f : I \to (0, \infty)$ be a differentiable function on I such that $f' \in L([a, b])$ for $0 \leq a < b < \infty$. If $\left|f'(x)\right|^q$ is*

(α, m)-logarithmically convex on $\left[0, \frac{b}{m}\right]$ for $q \in [1, \infty)$, $(\alpha, m) \in (0,1] \times (0,1]$, we have

$$\left| f\left(A\left(a,b\right)\right) - \frac{1}{b-a} \int_a^b f\left(x\right) dx \right| \leq \frac{(b-a)}{4} \left| f'\left(\frac{b}{m}\right) \right|^m \left(\frac{1}{2}\right)^{3-1/q} \left[E_2\left(\alpha, m, q\right)\right], \quad (2.56)$$

where

$$\mu = \frac{\left|f'(a)\right|}{\left|f'\left(\frac{b}{m}\right)\right|^m}, \; E_2\left(\alpha, m, q\right) = \begin{cases} 2\left(\frac{1}{8}\right)^{1/q}, & \mu = 1 \\ \left[F_2\left(\mu, \alpha q\right)\right]^{1/q} + \left[F_3\left(\mu, \alpha q\right)\right]^{1/q}, & 0 < \mu < 1 \\ \left[F_2\left(\mu, \frac{q}{\alpha}\right)\right]^{1/q} + \left[F_3\left(\mu, \frac{q}{\alpha}\right)\right]^{1/q}, & \mu > 1 \end{cases}$$

and

$$F_2\left(u, v\right) = \frac{1}{v^2 \left(\ln u\right)^2} \left[\frac{v}{2} u^{v/2} \ln u - u^{v/2} + 1\right]$$
$$F_3\left(u, v\right) = \frac{1}{v^2 \left(\ln u\right)^2} \left[u^v - \frac{v}{2} u^{v/2} \ln u - u^{v/2}\right]$$

for u, $v > 0$, $u \neq 1$.

Corollary 83 *Let $I \subseteq [0, \infty)$ be an open real interval and let $f : I \to (0, \infty)$ be a differentiable function on I such that $f' \in L([a,b])$ for $0 \leq a < b < \infty$. If $\left|f'(x)\right|^q$ is m-logarithmically convex on $\left[0, \frac{b}{m}\right]$ for $q \in [1, \infty)$, $m \in (0,1]$, we have*

$$\left| f\left(A\left(a,b\right)\right) - \frac{1}{b-a} \int_a^b f\left(x\right) dx \right| \leq \frac{(b-a)}{4} \left| f'\left(\frac{b}{m}\right) \right|^m \left(\frac{1}{2}\right)^{3-1/q} \left[E_2\left(1, m, q\right)\right], \quad (2.57)$$

where $E_2\left(\alpha, m, q\right)$ is defined as in Theorem 80.

Since the notion of invex sets, preinvex functions are considered to be a significant generalizations of convex sets and convex functions respectively, so in this forthcoming paragraph we mention some inequalities of Hermite-Hadamard type for the space of preinvex functions. We also mention inequalities of Hermite-Hadamard type for log-preinvex functions which is a further generalization of preinvex functions.

Noor [118] has obtained the following Hermite-Hadamard inequalities for the preinvex and log-preinvex functions.

Theorem 84 *[118] Let $f:[a,a+\eta(b,a)]\rightarrow(0,\infty)$ be a preinvex function on the interval of the real numbers K° (the interior of K) and a, $b\in K^{\circ}$ with $a<a+\eta(b,a)$. Then the following inequality holds:*

$$\begin{aligned} f\left(A\left(2a,\eta(b,a)\right)\right) &\leq \frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)} f\left(x\right)dx \\ &\leq A\left(f\left(a\right),f\left(a+\eta(b,a)\right)\right)\leq A\left(f\left(a\right),f\left(b\right)\right). \quad (2.58)\end{aligned}$$

Theorem 85 *[118] Let $f:[a,a+\eta(b,a)]\rightarrow(0,\infty)$ be a log-preinvex function. Then*

$$\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)} f\left(x\right)dx \leq \frac{f\left(a\right)-f\left(b\right)}{\log f\left(a\right)-\log f\left(b\right)}. \quad (2.59)$$

Barani et al. [13] presented the following estimates of the difference between the rightmost and the middle terms (in which some preinvex functions are involved.

Theorem 86 *[13] Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:K\times K\rightarrow\mathbb{R}$. Suppose that $f:K\rightarrow\mathbb{R}$ is a differentiable function .Assume $p\in\mathbb{R}$ with $p>1$. If $\left|f'\right|^{\frac{p}{p-1}}$ is preinvex on K then, for every $a,b\in K$ with $\eta\left(b,a\right)\neq0$, then the following inequality holds:*

$$\begin{aligned} &\left|A\left(f(a),f(a+\eta\left(b,a\right))\right)-\frac{1}{\eta\left(b,a\right)}\int_{a}^{a+\eta(b,a)} f(x)dx\right| \\ &\qquad \leq \frac{\left|\eta\left(b,a\right)\right|}{2\left(1+p\right)^{\frac{1}{p}}}\left[A\left(\left|f'\left(a\right)\right|^{\frac{p}{p-1}},\left|f'\left(b\right)\right|^{\frac{p}{p-1}}\right)\right]^{\frac{p-1}{p}}. \quad (2.60)\end{aligned}$$

Theorem 87 *[13]Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:K\times K\rightarrow\mathbb{R}$. Suppose that $f:K\rightarrow\mathbb{R}$ is a differentiable function. If $\left|f'\right|$ is preinvex on K then, for* *every $a, b \in K$ with $\eta(b,a) \neq 0$, the following inequality holds:*

$$\left| A\left(f(a), f(a+\eta(b,a))\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \right| \leq \frac{|\eta(b,a)|}{4} A\left(\left| f'(a) \right|, \left| f'(b) \right| \right). \quad (2.61)$$

Most recently, Li [96] introduced the notion of s-preinvexity and established a inequalitity for this space of functions.

Theorem 88 *[96] Let $f : K = [a, a+\eta(b,a)] \subseteq [0,\infty) \to (0,\infty)$ be a s-preinvex function on the interval of the real numbers K° (the interior of K) and a, $b \in K^\circ$ with $\eta(b,a) > 0$. Then the following inequalitity holds:*

$$2^{s-1} f\left(A\left(2a, \eta(b,a)\right)\right) \leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \leq \frac{f(a)+f(b)}{s+1}. \quad (2.62)$$

For log-preinvex functions, following Hermite-Hadamard type inequalities were also proved in [146].

Theorem 89 *[146] Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If $|f|$ is log-preinvex on K, for every $a, b \in K$ with $\eta(b,a) > 0$, we have the inequality*

$$\left| \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx - f\left(A\left(2a, \eta(b,a)\right)\right) \right| \leq \eta(b,a) \left[\frac{\left(\left|f'(b)\right|\right)^{1/2} - \left(\left|f'(a)\right|\right)^{1/2}}{\log\left(\left|f'(b)\right|\right) - \log\left(\left|f'(a)\right|\right)} \right]^2. \quad (2.63)$$

Theorem 90 *[146] Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If $|f|^q$, $q > 1$, $q \in \mathbb{R}$, is a log-*

preinvex on K, for every $a, b \in K$ with $\eta(b,a) > 0$, we have the inequality

$$\left| f\left(A\left(2a, \eta(b,a)\right)\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\, dx \right| \\ \leq \frac{\eta(b,a)\sqrt{|f'(a)|}}{2^{1/p}(p+1)^{1/p} q^{1/q}} \left[\frac{\left(|f'(b)|\right)^{q/2} - \left(|f'(a)|\right)^{q/2}}{\left(\log\left(|f'(b)|\right) - \log\left(|f'(a)|\right)\right)} \right]^{1/q}, \quad (2.64)$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

In recent years, a number of findings have been seen on Hermite-Hadamard type integral inequalities for GA-convex and for GA-s-convex functions. We cite here some of Hermite-Hadamard type inequalities related to the concept of geometrically-arithmetically convex functions.

Zhang et al. in [183] established the following Hermite-Hadamard type integral inequalities for GA-convex function.

Theorem 91 *[183] Let $f : I \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I^\circ$ with $a < b$ and $f' \in L([a,b])$. If $|f'|^q$ is GA-convex on $[a,b]$ for $q \geq 1$, we have the following inequality:*

$$\left| bf(b) - af(a) - \int_a^b f(x)\, dx \right| \leq \frac{\left[(b-a) A(a,b)\right]^{1-\frac{1}{q}}}{2^{\frac{1}{q}}} \\ \times \left\{ \left[L\left(a^2, b^2\right) - a^2 \right] \left| f'(a) \right|^q + \left[b^2 - L\left(a^2, b^2\right) \right] \left| f'(b) \right|^q \right\}^{\frac{1}{q}}. \quad (2.65)$$

Theorem 92 *[183] Let $f : I \subseteq \mathbb{R}_+ = (0,\infty) \to \mathbb{R}$ be a function differentiable function on I° and $a, b \in I^\circ$ with $a < b$ and $f' \in L([a,b])$. If $|f'|^q$ is GA-convex on $[a,b]$ for $q > 1$, we have the following inequality:*

$$\left| bf(b) - af(a) - \int_a^b f(x)\, dx \right| \\ \leq (\ln b - \ln a) \left[L\left(a^{\frac{2q}{q-1}}, b^{\frac{2q}{q-1}}\right) \right]^{1-\frac{1}{q}} \left[A\left(\left| f'(a) \right|^q \left| f'(b) \right|^q \right) \right]^{\frac{1}{q}}. \quad (2.66)$$

Theorem 93 *[183] Let $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ be a differentiable function on I° and a, $b \in I^\circ$ with $a < b$ and $f' \in L([a, b])$. If $|f'|^q$ is GA-convex on $[a, b]$ for $q \geq 1$, we have the following inequality:*

$$\left| bf(b) - af(a) - \int_a^b f(x)\,dx \right| \leq \frac{(\ln b - \ln a)^{1-\frac{1}{q}}}{(2q)^{\frac{1}{q}}} \left[L\left(a^{\frac{2q}{q-1}}, b^{\frac{2q}{q-1}} \right) \right]^{1-\frac{1}{q}}$$
$$\times \left\{ \left[L\left(a^{2q}, b^{2q}\right) - a^{2q} \right] \left| f'(a) \right|^q + \left[b^{2q} - L\left(a^{2q}, b^{2q}\right) \right] \left| f'(b) \right|^q \right\}^{\frac{1}{q}}. \quad (2.67)$$

Theorem 94 *[183] Let $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ be a differentiable function on I° and a, $b \in I^\circ$ with $a < b$ and $f' \in L([a, b])$. If $|f'|^q$ is GA-convex on $[a, b]$ for $q > 1$ and $2q > p > 0$, then*

$$\left| bf(b) - af(a) - \int_a^b f(x)\,dx \right| \leq \frac{(\ln b - \ln a)^{1-\frac{1}{q}}}{p^{\frac{1}{q}}} \left[L\left(a^{\frac{2q-p}{q-1}}, b^{\frac{2q-p}{q-1}} \right) \right]^{1-\frac{1}{q}}$$
$$\times \left\{ \left[L(a^p, b^p) - a^p \right] \left| f'(a) \right|^q + \left[b^p - L(a^p, b^p) \right] \left| f'(b) \right|^q \right\}^{\frac{1}{q}}. \quad (2.68)$$

İşcan [65], proved the following result for GA-s-convex functions in the second sense.

Theorem 95 *[65] Suppose that $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ is s-GA-convex in the second sense and a, $b \in I$ with $a < b$. If $f \in L([a, b])$, then one has the inequalities:*

$$2^{s-1} f(G(a, b)) \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \leq \frac{f(a) + f(b)}{s + 1}. \quad (2.69)$$

If f in Theorem 95 is GA-convex function, then we get the following Hermite-Hadamard type inequalities.

$$f(G(a, b)) \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \leq A(f(a), f(b)). \quad (2.70)$$

For several results which refine the results of Theorem 91-Theorem 94, we refer the readers to [93] and for the results which provide bounds on

$$\left| f\left(G\left(a,b\right)\right) - \int_a^b \frac{f\left(x\right)}{x} dx \right| \text{ and } \left| A\left(f\left(a\right), f\left(b\right)\right) - \int_a^b \frac{f\left(x\right)}{x} dx \right|$$

please see the recent article of İşcan [65].

It has been observed from the above discussion that the generalization of the convexity resulted in a number of papers on the inequalities (2.19) and (2.20) which provides new proofs, noteworthy extensions, generalizations, refinements, counterparts, new Hermite-Hadamard-type inequalities, Fejér-type inequalities by using different forms of convexity and numerous applications, we refer the readers to [1]-[3], [4]-[5], [10]-[13], [18]-[45], [51]-[55], [58]-[60], [63]-[68], [71], [74]-[79], [96]-[100], [108], [110]-[118], [120]-[123], [128]-[130], [132]-[133], [138]-[139], [141]-[154], [158]-[164], [159]-[162], [167]-[177], [180]-[185] and the references cited therein.

Chapter 3

Inequalities of H-H type for n-times differentiable convex, concave and s-convex functions with applications

The content of this chapter includes the results of the papers "M. A. Latif, S. S. Dragomir, New Inequalities of Hermite-Hadamard type for n-times differentiable convex and concave functions with applications, Filomat. (Accepted)" and "M. A. Latif, S. S. Dragomir and E. Momoniat, New inequalities of Hermite-Hadamard type for n-times differentiable s-convex functions with applications, Analysis Mathematica. (Submitted)".

3.1 Introduction

This chapter is devoted to the study of inequalities of Hermite-Hadamard type using the conditions of convexity, concavity and s-convexity over an interval of the set of real numbers. In this chapter, we establish a new identity for n-times differentiable functions and then by using this identity and mathematical analysis, we present some new inequalities H-H type for functions whose nth derivatives in absolute value belongs to the spaces of convex, concave and s-convex functions. We also derive several new inequalities of H-H type in terms of functions whose first and second derivatives in absolute value are convex, concave and s-convex functions as special cases. The results of this chapter provide refinements of the trapezoidal type inequalities given in Theorem 61 and Theorem 63. We will demonstrate applications of our results to the trapezoidal formula and to get some new inequalities for special means of real numbers as well.

This chapter is organized as follows. In Section 3.3, we first prove a new identity for functions which are n-times differentiable. We then use this identity to get new inequalities of H-H type for n-times differentiable functions which possesses the conditions of convexity and concavity. We prove new inequalities of H-H type for n-times differentiable functions which belongs to the space K_s^2 of s-convex functions in the second sense by applying the identity proved in Section 3.2. These results are further generalizations of the results of Section 3.2. Section 3.4 is dedicated to the applications of results of Section 3.2 and Section 3.3 to well-known trapezoidal rule. Finally, in Section 3.5 we give applications of some of the results of Section 3.2 and Section 3.3 to special means of real numbers.

3.2 New inequalities of H-H type for n-times differentiable convex and concave functions

The following Lemma is essential in establishing our main results in this section:

Lemma 96 *Suppose that $[c,d] \subset \mathbb{R}$ and $f : [c,d] \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on (c,d) for $n \in \mathbb{N}$. If $f^{(n)}$ belongs to the space $L([a,b])$ of all integrable functions over*

$[a,b]$, where $a,\ b\in(c,d)$ with $a<b$, we have the identity

$$\begin{aligned} A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx-\sum_{k=1}^{n-1}\frac{k\left[1+\left(-1\right)^k\right]\left(b-a\right)^k}{2^{k+1}\left(k+1\right)!}f^{(k)}\left(A\left(a,b\right)\right) \\ =\frac{\left(b-a\right)^n}{2^{n+1}n!}\int_0^1\left(1-t\right)^{n-1}\left(n-1+t\right)f^{(n)}\left(A_{\frac{1-t}{2}}\left(a,b\right)\right)dt \\ +\frac{\left(-1\right)^n\left(b-a\right)^n}{2^{n+1}n!}\int_0^1\left(1-t\right)^{n-1}\left(n-1+t\right)f^{(n)}\left(A_{\frac{1-t}{2}}\left(b,a\right)\right)dt, \quad (3.1)\end{aligned}$$

where an empty sum is understood to be nil.

Proof. Suppose

$$I_n=\frac{\left(b-a\right)^n}{2^{n+1}n!}\int_0^1\left(1-t\right)^{n-1}\left(n-1+t\right)f^{(n)}\left(A_{\frac{1-t}{2}}\left(a,b\right)\right)dt$$

and

$$J_n=\frac{\left(-1\right)^n\left(b-a\right)^n}{2^{n+1}n!}\int_0^1\left(1-t\right)^{n-1}\left(n-1+t\right)f^{(n)}\left(A_{\frac{1-t}{2}}\left(b,a\right)\right)dt.$$

For $n=1$, we have

$$I_1=\frac{b-a}{4}\int_0^1 tf'\left(A_{\frac{1-t}{2}}\left(a,b\right)\right)dt$$

and

$$J_1=\frac{\left(-1\right)\left(b-a\right)}{4}\int_0^1 tf'\left(A_{\frac{1-t}{2}}\left(b,a\right)\right)dt.$$

By integration by parts and using the substitution $x=A_{\frac{1-t}{2}}\left(a,b\right)$ for I_1 and $x=A_{\frac{1-t}{2}}\left(b,a\right)$ for J_1, we obtain

$$I_1=\frac{1}{2}f\left(b\right)-\frac{1}{b-a}\int_{\frac{a+b}{2}}^b f\left(x\right)$$

and

$$J_1=\frac{1}{2}f\left(a\right)-\frac{1}{b-a}\int_a^{\frac{a+b}{2}} f\left(x\right).$$

Hence

$$I_1 + J_1 = A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx,$$

which coincides with the L.H.S of (3.1) for $n = 1$.

Similarly for $n = 2$, and using similar arguments as above, we have

$$I_2 + J_2 = A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx$$

which coincides with the L.H.S of (3.1) for $n = 2$.

Suppose (3.1) holds for $n = m - 1 \geq 3$.

Now for $n = m$, we have

$$\begin{aligned}
&\frac{(b-a)^m}{2^{m+1}m!}\int_0^1 (1-t)^{m-1}(m-1+t)\, f^{(m)}\left(A_{\frac{1-t}{2}}(a,b)\right)dt \\
&\qquad + \frac{(-1)^m (b-a)^m}{2^{m+1}m!}\int_0^1 (1-t)^{m-1}(m-1+t)\, f^{(m)}\left(A_{\frac{1-t}{2}}(b,a)\right)dt \\
&\qquad = -\frac{(b-a)^{m-1}(m-1)\left[1+(-1)^{m-1}\right]}{2^m m!} f^{(m-1)}\left(A\left(a,b\right)\right) \\
&\qquad + \frac{(b-a)^{m-1}}{2^m (m-1)!}\int_0^1 (1-t)^{m-2}(m-2+t)\, f^{(m)}\left(A_{\frac{1-t}{2}}(a,b)\right)dt \\
&\qquad + \frac{(-1)^{m-1}(b-a)^{m-1}}{2^m (m-1)!}\int_0^1 (1-t)^{m-2}(m-2+t)\, f^{(m)}\left(A_{\frac{1-t}{2}}(b,a)\right)dt \\
&= A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx - \sum_{k=1}^{m-2}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(A\left(a,b\right)\right) \\
&\qquad - \frac{(b-a)^{m-1}(m-1)\left[1+(-1)^{m-1}\right]}{2^m m!} f^{(m-1)}\left(A\left(a,b\right)\right) \\
&= A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx - \sum_{k=1}^{m-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(A\left(a,b\right)\right).
\end{aligned}$$

This concludes the validity of the lemma. ■

Now we state and prove some new Hermite-Hadamard type inequalities for functions whose nth derivatives in absolute value are convex and concave.

Theorem 97 *Let $[c,d]$ be a real interval and let $f:[c,d]\to\mathbb{R}$ be a function such that $f^{(n)}$ exists on (c,d). Suppose that $f^{(n)}$ belongs to the space $L([a,b])$ of all integrable functions over $[a,b]$ for $n\in\mathbb{N}$, where $a,\, b\in(c,d)$ with $a<b$. If $\left|f^{(n)}\right|^q$ is convex on $[a,b]$ for $1\le q<\infty$, we have the inequality*

$$\begin{aligned}&\left|A(f(a),f(b))-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}(A(a,b))\right|\\&\quad\le\frac{(b-a)^n n^{1-1/q}}{2^{n+\frac{1}{q}+1}(n+1)!}\left\{\left(\frac{(n^2+n-1)\left|f^{(n)}(a)\right|^q+(n^2+3n+1)\left|f^{(n)}(b)\right|^q}{n+2}\right)^{1/q}\right.\\&\qquad\left.+\left(\frac{(n^2+n-1)\left|f^{(n)}(b)\right|^q+(n^2+3n+1)\left|f^{(n)}(a)\right|^q}{n+2}\right)^{1/q}\right\}.\qquad(3.2)\end{aligned}$$

Proof. From Lemma 96, application of the Hölder inequality and the convexity of $\left|f^{(n)}\right|^q$ on $[a,b]$, we have

$$\begin{aligned}&\left|A(f(a),f(b))-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}(A(a,b))\right|\\&\quad\le\frac{(b-a)^n}{2^{n+1}n!}\left(\int_0^1(1-t)^{n-1}(n-1+t)\,dt\right)^{1-1/q}\\&\quad\times\left\{\left(\int_0^1(1-t)^{n-1}(n-1+t)\left[\left(\frac{1-t}{2}\right)\left|f^{(n)}(a)\right|^q+\left(\frac{1+t}{2}\right)\left|f^{(n)}(b)\right|^q\right]dt\right)^{1/q}\right.\\&\quad\left.+\left(\int_0^1(1-t)^{n-1}(n-1+t)\left[\left(\frac{1-t}{2}\right)\left|f^{(n)}(b)\right|^q+\left(\frac{1+t}{2}\right)\left|f^{(n)}(a)\right|^q\right]dt\right)^{1/q}\right\}\\&\quad=\frac{(b-a)^n n^{1-1/q}}{2^{n+\frac{1}{q}+1}(n+1)!}\left\{\left(\frac{n^2+n-1}{n+2}\left|f^{(n)}(a)\right|^q+\frac{n^2+3n+1}{n+2}\left|f^{(n)}(b)\right|^q\right)^{1/q}\right.\\&\qquad\left.+\left(\frac{n^2+n-1}{n+2}\left|f^{(n)}(b)\right|^q+\frac{n^2+3n+1}{n+2}\left|f^{(n)}(a)\right|^q\right)^{1/q}\right\},\end{aligned}$$

which is the desired result. ■

Corollary 98 *If the postulations of Theorem 97 are fulfilled and if $q=1$, we have the*

inequality

$$\left| A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx - \sum_{k=1}^{n-1} \frac{k\left[1+\left(-1\right)^k\right]\left(b-a\right)^k}{2^{k+1}\left(k+1\right)!} f^{(k)}\left(A\left(a,b\right)\right)\right| \\ \leq \frac{n\left(b-a\right)^n}{2^n\left(n+1\right)!} A\left(\left|f^{(n)}\left(a\right)\right|, \left|f^{(n)}\left(b\right)\right|\right). \quad (3.3)$$

Corollary 99 *Under the assumptions of Theorem 97, if $n=1$, we have the inequality*

$$\left| A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx\right| \\ \leq \frac{\left(b-a\right)}{8}\left\{\left(\frac{A\left(\left|f'\left(a\right)\right|^q, 5\left|f'\left(b\right)\right|^q\right)}{3}\right)^{1/q} + \left(\frac{A\left(\left|f'\left(b\right)\right|^q, 5\left|f'\left(a\right)\right|^q\right)}{3}\right)^{1/q}\right\}. \quad (3.4)$$

Corollary 100 *If we take $q=1$ in Corollary 99, we have*

$$\left| A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx\right| \leq \frac{\left(b-a\right)}{4} A\left(\left|f'\left(a\right)\right|, \left|f'\left(b\right)\right|\right). \quad (3.5)$$

Corollary 101 *Let the assumptions of Theorem 97 be fulfilled and if $n=2$, we have*

$$\left| A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx\right| \\ \leq \frac{\left(b-a\right)^2}{24}\left\{\left(\frac{A\left(5\left|f''\left(a\right)\right|^q, 11\left|f''\left(b\right)\right|^q\right)}{8}\right)^{1/q} + \left(\frac{A\left(5\left|f''\left(b\right)\right|^q, 11\left|f''\left(a\right)\right|^q\right)}{8}\right)^{1/q}\right\}. \quad (3.6)$$

Corollary 102 *If $q=1$ in Corollary 101, we have*

$$\left| A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx\right| \leq \frac{\left(b-a\right)^2}{24} A\left(\left|f''\left(a\right)\right|, \left|f''\left(b\right)\right|\right). \quad (3.7)$$

Remark 103 *It is obvious that the inequality (3.7) gives better estimate than that of (2.38) for $q>1$.*

Theorem 104 *Let $[c,d]$ be a real interval and let $f:[c,d]\to\mathbb{R}$ be a function such that $f^{(n)}$ exists on (c,d). Suppose that $f^{(n)}$ belongs to the space $L([a,b])$ of all integrable functions over $[a,b]$ for $n\in\mathbb{N}$, where a, $b\in(c,d)$ with $a<b$. If $\left|f^{(n)}\right|^q$ is convex on $[a,b]$ for $1<q<\infty$, we have the inequality*

$$\begin{aligned}
&\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx-\sum_{k=1}^{n-1}\frac{k\left[1+\left(-1\right)^k\right]\left(b-a\right)^k}{2^{k+1}\left(k+1\right)!}f^{(k)}\left(A\left(a,b\right)\right)\right| \\
&\qquad\leq\frac{\left(b-a\right)^n\left[n^{(2q-1)/(q-1)}-\left(n-1\right)^{(2q-1)/(q-1)}\right]^{1-1/q}}{2^{n+\frac{1}{q}+1}n!}\left(\frac{q-1}{2q-1}\right)^{1-1/q} \\
&\qquad\times\left\{\left[P\left|f^{(n)}\left(a\right)\right|^q+Q\left|f^{(n)}\left(b\right)\right|^q\right]^{1/q}+\left[P\left|f^{(n)}\left(b\right)\right|^q+Q\left|f^{(n)}\left(a\right)\right|^q\right]^{1/q}\right\}, \quad (3.8)
\end{aligned}$$

where

$$P=\frac{1}{nq-q+2}\text{ and }Q=\frac{nq-q+3}{\left(nq-q+1\right)\left(nq-q+2\right)}.$$

Proof. Using Lemma 96, the Hölder inequality and the convexity of $\left|f^{(n)}\right|^q$ on $[a,b]$, we have

$$\begin{aligned}
&\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx-\sum_{k=1}^{n-1}\frac{k\left[1+\left(-1\right)^k\right]\left(b-a\right)^k}{2^{k+1}\left(k+1\right)!}f^{(k)}\left(A\left(a,b\right)\right)\right| \\
&\qquad\leq\frac{\left(b-a\right)^n}{2^{n+1}n!}\left(\int_0^1\left(n-1+t\right)^{q/(q-1)}dt\right)^{1-1/q} \\
&\qquad\times\left\{\left(\int_0^1\left(1-t\right)^{q(n-1)}\left[\left(\frac{1-t}{2}\right)\left|f^{(n)}\left(a\right)\right|^q+\left(\frac{1+t}{2}\right)\left|f^{(n)}\left(b\right)\right|^q\right]dt\right)^{1/q}\right. \\
&\qquad\left.+\left(\int_0^1\left(1-t\right)^{q(n-1)}\left[\left(\frac{1-t}{2}\right)\left|f^{(n)}\left(b\right)\right|^q+\left(\frac{1+t}{2}\right)\left|f^{(n)}\left(a\right)\right|^q\right]dt\right)^{1/q}\right\}. \quad (3.9)
\end{aligned}$$

From (3.9) we get the inequality (3.8), since

$$\int_0^1\left(n-1+t\right)^{q/(q-1)}dt=\left(\frac{q-1}{2q-1}\right)\left[n^{(2q-1)/(q-1)}-\left(n-1\right)^{(2q-1)/(q-1)}\right],$$

$$\int_0^1 (1-t)^{q(n-1)+1}\,dt = \frac{1}{nq-q+2} = P$$

and

$$\int_0^1 (1-t)^{q(n-1)}(1+t)\,dt = \frac{nq-q+3}{(nq-q+1)(nq-q+2)} = Q.$$

■

Corollary 105 *Let the postulations of Theorem 104 be met and if $n=1$, we have the inequality*

$$\begin{aligned}\left|A(f(a),f(b)) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| &\le \frac{(b-a)}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q} \\ &\times\left\{\left(\frac{A\left(\left|f'(a)\right|^q, 3\left|f'(b)\right|^q\right)}{2}\right)^{1/q} + \left(\frac{A\left(\left|f'(b)\right|^q, 3\left|f'(a)\right|^q\right)}{2}\right)^{1/q}\right\}, \quad (3.10)\end{aligned}$$

Corollary 106 *Under the assumptions of Theorem 104, if $n=2$, then*

$$\begin{aligned}\left|A(f(a),f(b)) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| &\le \frac{(b-a)^2\left[2^{(2q-1)/(q-1)}-1\right]^{1-1/q}}{2^{1/q+4}}\left(\frac{q-1}{2q-1}\right)^{1-1/q} \\ &\times\left\{\left[P\left|f''(a)\right|^q + Q\left|f''(b)\right|^q\right]^{1/q} + \left[P\left|f''(b)\right|^q + Q\left|f''(a)\right|^q\right]^{1/q}\right\} \quad (3.11)\end{aligned}$$

holds valid, where

$$P = \frac{1}{q+2} \text{ and } Q = \frac{q+3}{(q+1)(q+2)}.$$

Theorem 107 *Let $[c,d]$ be a real interval and let $f:[c,d]\to\mathbb{R}$ be a function such that $f^{(n)}$ exists on (c,d). Suppose that $f^{(n)}$ belongs to the space $L([a,b])$ of all integrable functions over $[a,b]$ for $n\in\mathbb{N}$, where a, $b\in(c,d)$ with $a<b$. If $\left|f^{(n)}\right|^q$ is convex on $[a,b]$*

for $1 < q < \infty$, *we have the inequality*

$$\left| A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx - \sum_{k=1}^{n-1} \frac{k\left[1+\left(-1\right)^k\right]\left(b-a\right)^k}{2^{k+1}\left(k+1\right)!} f^{(k)}\left(A\left(a,b\right)\right) \right|$$
$$\leq \frac{\left(b-a\right)^n}{2^{n+\frac{1}{q}+1}n!}\left(\frac{q-1}{nq-1}\right)^{1-1/q} \left\{ \left[R\left|f^{(n)}\left(a\right)\right|^q + S\left|f^{(n)}\left(b\right)\right|^q\right]^{1/q} + \left[R\left|f^{(n)}\left(b\right)\right|^q + S\left|f^{(n)}\left(a\right)\right|^q\right]^{1/q} \right\}, \quad (3.12)$$

where

$$R = \frac{n^{q+2} - \left(n+q+1\right)\left(n-1\right)^{q+1}}{\left(q+1\right)\left(q+2\right)}$$

and

$$S = \frac{\left(2q-n+4\right)n^{q+1} - \left(q-n+3\right)\left(n-1\right)^{q+1}}{\left(q+1\right)\left(q+2\right)}.$$

Proof. Using Lemma 96, the Hölder inequality and the convexity of $\left|f^{(n)}\right|^q$ on $[a,b]$, we have

$$\left| A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx - \sum_{k=1}^{n-1} \frac{k\left[1+\left(-1\right)^k\right]\left(b-a\right)^k}{2^{k+1}\left(k+1\right)!} f^{(k)}\left(A\left(a,b\right)\right) \right|$$
$$\leq \frac{\left(b-a\right)^n}{2^{n+1}n!}\left(\int_0^1 \left(1-t\right)^{q(n-1)/(q-1)}dt\right)^{1-1/q}$$
$$\times\left\{ \left(\int_0^1 \left(n-1+t\right)^q\left[\left(\frac{1-t}{2}\right)\left|f^{(n)}\left(a\right)\right|^q + \left(\frac{1+t}{2}\right)\left|f^{(n)}\left(b\right)\right|^q\right]dt\right)^{1/q} + \left(\int_0^1 \left(n-1+t\right)^q\left[\left(\frac{1-t}{2}\right)\left|f^{(n)}\left(b\right)\right|^q + \left(\frac{1+t}{2}\right)\left|f^{(n)}\left(a\right)\right|^q\right]dt\right)^{1/q} \right\}. \quad (3.13)$$

Since

$$\int_0^1 \left(1-t\right)^{q(n-1)/(q-1)}dt = \frac{q-1}{nq-1},$$

$$\int_0^1 \left(n-1+t\right)^q\left(1-t\right)dt = \frac{n^{q+2} - \left(n+q+1\right)\left(n-1\right)^{q+1}}{\left(q+1\right)\left(q+2\right)}$$

and

$$\int_0^1 (n-1+t)^q (1+t)\,dt = \frac{(2q-n+4)\,n^{q+1} - (q-n+3)\,(n-1)^{q+1}}{(q+1)\,(q+2)}$$

and hence (3.12) follows from (3.12). ■

Corollary 108 *If the suppositions of Theorem 107 are matched and if $n=1$, we have the inequality*

$$\begin{aligned} &\left| A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx \right| \\ &\quad \le \frac{(b-a)}{2^{2+1/q}} \left\{ \left[R\left|f'\left(a\right)\right|^q + S\left|f'\left(b\right)\right|^q \right]^{1/q} + \left[R\left|f'\left(b\right)\right|^q + S\left|f'\left(a\right)\right|^q \right]^{1/q} \right\}, \quad (3.14) \end{aligned}$$

$$R = \frac{1}{(q+1)\,(q+2)} \text{ and } S = \frac{2q+3}{(q+1)\,(q+2)}.$$

Corollary 109 *Substitution of $n=2$ in Theorem 107, produces the following result*

$$\begin{aligned} &\left| A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx \right| \le \frac{(b-a)^2}{2^{4+\frac{1}{q}}} \left(\frac{q-1}{2q-1}\right)^{1-1/q} \\ &\quad \times \left\{ \left[R\left|f''\left(a\right)\right|^q + S\left|f''\left(b\right)\right|^q \right]^{1/q} + \left[R\left|f''\left(b\right)\right|^q + S\left|f''\left(a\right)\right|^q \right]^{1/q} \right\}, \quad (3.15) \end{aligned}$$

$$R = \frac{2^{q+2} - (q+3)}{(q+1)\,(q+2)} \text{ and } S = \frac{2^{q+2}-1}{q+2}.$$

Theorem 110 *Let $[c,d]$ be a real interval and let $f:[c,d] \to \mathbb{R}$ be a function such that $f^{(n)}$ exists on (c,d). Suppose that $f^{(n)}$ belongs to the space $L\left([a,b]\right)$ of all integrable functions over $[a,b]$ for $n \in \mathbb{N}$, where $a,\ b \in (c,d)$ with $a<b$. If $\left|f^{(n)}\right|^q$ is convex on $[a,b]$*

for $1 < q < \infty$, *we have the inequality*

$$\left| A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx - \sum_{k=1}^{n-1} \frac{k\left[1+\left(-1\right)^k\right]\left(b-a\right)^k}{2^{k+1}\left(k+1\right)!} f^{(k)}\left(A\left(a,b\right)\right)\right|$$
$$\leq \frac{n^{n+1-1/q}\left(b-a\right)^n}{2^{n+1+2/q}n!}\left[B\left(\frac{1}{n}; \frac{nq-1}{q-1}, \frac{2q-1}{q-1}\right)\right]^{1-1/q}$$
$$\times\left\{\left[\left|f^{(n)}\left(a\right)\right|^q + 3\left|f^{(n)}\left(b\right)\right|^q\right]^{1/q} + \left[3\left|f^{(n)}\left(a\right)\right|^q + \left|f^{(n)}\left(b\right)\right|^q\right]^{1/q}\right\}, \quad (3.16)$$

where

$$B\left(x; \alpha, \beta\right) = \int_0^x t^{\alpha-1}\left(1-t\right)^{1-\beta}dt, 0 \leq x \leq 1, \alpha > 0, \beta > 0$$

is the incomplete beta function.

Proof. Using Lemma 96, the Hölder inequality and the convexity of $\left|f^{(n)}\right|^q$ on $[a,b]$, we have

$$\left| A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx - \sum_{k=1}^{n-1} \frac{k\left[1+\left(-1\right)^k\right]\left(b-a\right)^k}{2^{k+1}\left(k+1\right)!} f^{(k)}\left(A\left(a,b\right)\right)\right|$$
$$\leq \frac{\left(b-a\right)^n}{2^{n+1}n!}\left(\int_0^1 \left(1-t\right)^{q(n-1)/(q-1)}\left(n-1+t\right)^{q/(q-1)}dt\right)^{1-1/q}$$
$$\times\left\{\left(\int_0^1\left[\left(\frac{1-t}{2}\right)\left|f^{(n)}\left(a\right)\right|^q + \left(\frac{1+t}{2}\right)\left|f^{(n)}\left(b\right)\right|^q\right]dt\right)^{1/q}\right.$$
$$\left. + \left(\int_0^1\left[\left(\frac{1-t}{2}\right)\left|f^{(n)}\left(b\right)\right|^q + \left(\frac{1+t}{2}\right)\left|f^{(n)}\left(a\right)\right|^q\right]dt\right)^{1/q}\right\}. \quad (3.17)$$

The inequality (3.16) follows from the inequality (3.17) and using the fact that

$$\int_0^1 \left(1-t\right)^{q(n-1)/(q-1)}\left(n-1+t\right)^{q/(q-1)}dt = n^{\frac{nq+q-1}{q-1}} B\left(\frac{1}{n}; \frac{nq-1}{q-1}, \frac{2q-1}{q-1}\right).$$

This completes the proof of the theorem. ■

Corollary 111 *Suppose the assumptions of Theorem 110 are satisfied and if* $n = 1$, *the* *inequality (3.16) becomes the inequality (3.10), since for $n = 1$, we have*

$$B\left(1; 1, \frac{2q-1}{q-1}\right) = B\left(1, \frac{2q-1}{q-1}\right) = \int_0^1 (1-t)^{\frac{2q-1}{q-1}-1}\, dt = \frac{q-1}{2q-1}.$$

Corollary 112 *Under the assumptions of Theorem 110 and $n = 2$, we have the following inequality*

$$\left|A(f(a), f(b)) - \frac{1}{b-a}\int_a^b f(x)\, dx\right| \leq \frac{(b-a)^2}{2}\left[B\left(\frac{1}{2}; \frac{2q-1}{q-1}, \frac{2q-1}{q-1}\right)\right]^{1-1/q} \times \left\{\left[\frac{A\left(|f''(a)|^q, 3|f''(b)|^q\right)}{4}\right]^{1/q} + \left[\frac{A\left(3|f''(a)|^q, |f''(b)|\right)^q}{4}\right]^{1/q}\right\}, \quad (3.18)$$

where $B(x; \alpha, \beta)$ is as defined in Theorem 110.

A result for concave functions is given in the following theorem.

Theorem 113 *Let $[c, d]$ be a real interval and let $f : [c, d] \to \mathbb{R}$ be a function such that $f^{(n)}$ exists on (c, d). Suppose that $f^{(n)}$ belongs to the space $L([a, b])$ of all integrable functions over $[a, b]$ for $n \in \mathbb{N}$, where a, $b \in (c, d)$ with $a < b$. If $\left|f^{(n)}\right|^q$ is convex on $[a, b]$ for $1 \leq q < \infty$, we have the inequality*

$$\left|A(f(a), f(b)) - \frac{1}{b-a}\int_a^b f(x)\, dx - \sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}(A(a,b))\right| \leq \frac{n(b-a)^n}{2^{n+1}(n+1)!}\left\{\left|f^{(n)}\left(\frac{(n^2+n-1)a + (n^2+3n+1)b}{2n(n+2)}\right)\right| + \left|f^{(n)}\left(\frac{(n^2+n-1)b + (n^2+3n+1)a}{2n(n+2)}\right)\right|\right\}. \quad (3.19)$$

Proof. By the concavity of $\left|f^{(n)}\right|^q$ and the power-mean inequality, we have

$$\left|f^{(n)}(A_\lambda(x,y))\right| \geq A_\lambda\left(\left|f^{(n)}(x)\right|, \left|f^{(n)}(y)\right|\right)$$

which shows that $\left|f^{(n)}\right|$ is also concave.

Hence by using Lemma 96 and the Jensen integral inequality, we have

$$\left|A\left(f\left(a\right), f\left(b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx-\sum_{k=1}^{n-1}\frac{k\left[1+\left(-1\right)^k\right]\left(b-a\right)^k}{2^{k+1}\left(k+1\right)!}f^{(k)}\left(A\left(a,b\right)\right)\right|$$
$$\leq \frac{\left(b-a\right)^n}{2^{n+1}n!}\left(\int_0^1\left(1-t\right)^{n-1}\left(n-1+t\right)dt\right)$$
$$\times\left\{\left|f^{(n)}\left(\frac{\left(1-t\right)^{n-1}\left(n-1+t\right)\left(A_{\frac{1-t}{2}}\left(a,b\right)\right)dt}{\int_0^1\left(1-t\right)^{n-1}\left(n-1+t\right)dt}\right)\right|\right.$$
$$\left.+\left|f^{(n)}\left(\frac{\left(1-t\right)^{n-1}\left(n-1+t\right)\left(A_{\frac{1-t}{2}}\left(b,a\right)\right)dt}{\int_0^1\left(1-t\right)^{n-1}\left(n-1+t\right)dt}\right)\right|\right\}.$$

The above inequality gives the desired result. ■

Corollary 114 *Suppose the assumptions of Theorem 113 are fulfilled and if* $n=1$*, we have the inequality*

$$\left|A\left(f\left(a\right), f\left(b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right|$$
$$\leq\frac{\left(b-a\right)}{8}\left[\left|f'\left(\frac{a+5b}{6}\right)\right|+\left|f'\left(\frac{b+5a}{6}\right)\right|\right]. \quad (3.20)$$

Corollary 115 *Suppose the assumptions of Theorem 113 are fulfilled and if* $n=2$*, we have the inequality*

$$\left|A\left(f\left(a\right), f\left(b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right|$$
$$\leq\frac{\left(b-a\right)^2}{24}\left[\left|f''\left(\frac{5a+11b}{16}\right)\right|+\left|f''\left(\frac{5b+11a}{16}\right)\right|\right]. \quad (3.21)$$

3.3 New inequalities of H-H type for n-times differentiable s-convex functions

We will use Lemma 96 and the following lemma to establish our main results in this section.

Lemma 116 *[136] Let $x \geq 0$, $y \geq 0$, the inequality*

$$(x+y)^{\theta} \leq x^{\theta} + y^{\theta}$$

holds for $0 < \theta \leq 1$ and the inequality

$$(x-y)^{\theta} \leq x^{\theta} - y^{\theta}$$

holds for $\theta \geq 1$.

Now we state and prove some new H-H type inequalities for functions whose nth derivatives in absolute value are s-convex and s-concave.

Theorem 117 *Let $[c,d]$ be an interval in $[0,\infty)$ and let $f : [c,d] \to \mathbb{R}$ be a function such that $f^{(n)}$ exists on (c,d). Suppose that $f^{(n)}$ is in the space $L([a,b])$ of all integrable functions on $[a,b]$, where a, $b \in (c,d)$ with $a < b$, $n \in \mathbb{N}$. If $\left|f^{(n)}\right|^{q}$ is an s-convex function in the second sense over $[a,b]$ for some fixed s, $0 < s \leq 1$ and $1 \leq q < \infty$, we have the*

inequality

$$\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right.$$
$$\left.-\sum_{k=1}^{n-1}\frac{k\left[1+\left(-1\right)^k\right]\left(b-a\right)^k}{2^{k+1}\left(k+1\right)!}f^{(k)}\left(A\left(a,b\right)\right)\right|\leq\frac{\left(b-a\right)^n}{2^{n+s/q+1}n!}\left(\frac{n}{n+1}\right)^{1-1/q}$$
$$\times\left\{\left(\frac{\left(n^2+s\left(n-1\right)\right)\left|f^{(n)}\left(a\right)\right|^q}{\left(n+s\right)\left(n+s+1\right)}+\left(\frac{n}{n+1}+\frac{n\left(n+s\right)}{n+s+1}B\left(s+1,n\right)\right)\left|f^{(n)}\left(b\right)\right|^q\right)^{1/q}\right.$$
$$\left.+\left(\frac{\left(n^2+s\left(n-1\right)\right)\left|f^{(n)}\left(b\right)\right|^q}{\left(n+s\right)\left(n+s+1\right)}+\left(\frac{n}{n+1}+\frac{n\left(n+s\right)}{n+s+1}B\left(s+1,n\right)\right)\left|f^{(n)}\left(a\right)\right|^q\right)^{1/q}\right\}, \tag{3.22}$$

where

$$B\left(\alpha,\beta\right)=\int_0^1 t^{\alpha-1}\left(1-t\right)^{\beta-1}dt,\alpha,\beta>0$$

is the Euler Beta function.

Proof. From Lemma 96, the Hölder inequality and s-convexity of $\left|f^{(n)}\right|^q$ on $[a,b]$, we have

$$\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx-\sum_{k=1}^{n-1}\frac{k\left[1+\left(-1\right)^k\right]\left(b-a\right)^k}{2^{k+1}\left(k+1\right)!}f^{(k)}\left(A\left(a,b\right)\right)\right|$$
$$\leq\frac{\left(b-a\right)^n}{2^{n+s/q+1}n!}\left(\int_0^1\left(1-t\right)^{n-1}\left(n-1+t\right)dt\right)^{1-1/q}$$
$$\times\left\{\left(\int_0^1\left(1-t\right)^{n-1}\left(n-1+t\right)\left[\left(1-t\right)^s\left|f^{(n)}\left(a\right)\right|^q+\left(1+t\right)^s\left|f^{(n)}\left(b\right)\right|^q\right]dt\right)^{1/q}\right.$$
$$\left.+\left(\int_0^1\left(1-t\right)^{n-1}\left(n-1+t\right)\left[\left(1-t\right)^s\left|f^{(n)}\left(b\right)\right|^q+\left(1+t\right)^s\left|f^{(n)}\left(a\right)\right|^q\right]dt\right)^{1/q}\right\}. \tag{3.23}$$

Since

$$\int_0^1\left(1-t\right)^{n-1}\left(n-1+t\right)dt=\frac{n}{n+1},$$

$$\int_0^1 (1-t)^{n-1}(n-1+t)(1-t)^s\,dt = \frac{n^2+s(n-1)}{(n+s)(n+s+1)},$$

by using the property

$$B(x,y+1) = \frac{y}{x+y}B(x,y)$$

of the Euler Beta function and Lemma 116, we have

$$\begin{aligned}
&\int_0^1 (1-t)^{n-1}(n-1+t)(1+t)^s\,dt \\
&= \int_0^1 (1-t)^{n-1}(n-1+t)(1+t^s)\,dt \\
&= \frac{n}{n+1} + nB(s+1,n) - B(s+1,n+1) \\
&= \frac{n}{n+1} + nB(s+1,n) - \frac{n}{n+s+1}B(s+1,n) \\
&= \frac{n}{n+1} + \frac{n(n+s)}{n+s+1}B(s+1,n).
\end{aligned}$$

From the above facts and the inequality (3.23), we get the required inequality (3.22). Hence the proof of the Theorem 117 finishes. ■

The following corollaries are direct consequences of Theorem 117.

Corollary 118 *Under the assumptions of Theorem 117, if $q=1$, we have the inequality*

$$\begin{aligned}
&\left|A(f(a),f(b)) - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}(A(a,b))\right| \\
&\le \frac{(b-a)^n}{2^{n+s+1}n!}\left(\frac{n^2+s(n-1)}{(n+s)(n+s+1)} + \frac{n}{n+1} + \frac{n(n+s)}{n+s+1}B(s+1,n)\right)\left[\left|f^{(n)}(a)\right| + \left|f^{(n)}(b)\right|\right],
\end{aligned} \tag{3.24}$$

where $B(\alpha,\beta)$ is the Euler Beta function defined as in Theorem 117.

Corollary 119 *With the assumptions of Theorem 117 and taking $n=1$, we have the*

inequality

$$\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right|$$
$$\leq\left(b-a\right)\left(\frac{1}{2}\right)^{3+s/q-1/q}\left\{\left[\frac{\left|f'\left(a\right)\right|^q}{\left(1+s\right)\left(2+s\right)}+\left(\frac{1}{2}+\frac{1}{2+s}\right)\left|f'\left(b\right)\right|^q\right]^{1/q}\right.$$
$$\left.+\left[\left(\frac{1}{2}+\frac{1}{2+s}\right)\left|f'\left(a\right)\right|^q+\frac{\left|f'\left(b\right)\right|^q}{\left(1+s\right)\left(2+s\right)}\right]^{1/q}\right\}. \quad (3.25)$$

Corollary 120 *If we take $q=1$ in Corollary 119, we have*

$$\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right|$$
$$\leq\frac{\left(b-a\right)\left(s+3\right)}{s+1}\left(\frac{1}{2}\right)^{2+s}A\left(\left|f'\left(a\right)\right|,\left|f'\left(b\right)\right|\right). \quad (3.26)$$

Corollary 121 *Suppose the assumptions of Theorem 117 are fulfilled and if $n=2$, we have*

$$\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right|\leq\frac{\left(b-a\right)^2}{2^{4+s/q}}\left(\frac{2}{3}\right)^{1-1/q}$$
$$\times\left\{\left[\frac{\left(s+4\right)}{\left(s+2\right)\left(s+3\right)}\left|f''\left(a\right)\right|^q+\left(\frac{2}{3}+\frac{2}{\left(s+1\right)\left(s+3\right)}\right)\left|f''\left(b\right)\right|^q\right]^{1/q}\right.$$
$$\left.+\left[\frac{\left(s+4\right)}{\left(s+2\right)\left(s+3\right)}\left|f''\left(b\right)\right|^q+\left(\frac{2}{3}+\frac{2}{\left(s+1\right)\left(s+3\right)}\right)\left|f''\left(a\right)\right|^q\right]^{1/q}\right\}. \quad (3.27)$$

Corollary 122 *If $q=1$ in Corollary 121, we have*

$$\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right|$$
$$\leq\frac{\left(b-a\right)^2}{2^{3+s}}\left(\frac{2}{3}+\frac{s^2+7s+8}{\left(s+1\right)\left(s+2\right)\left(s+3\right)}\right)A\left(\left|f''\left(a\right)\right|,\left|f''\left(b\right)\right|\right). \quad (3.28)$$

Theorem 123 *Let $[c,d]$ be an interval in $[0,\infty)$ and let $f:[c,d]\to\mathbb{R}$ be a function* *such that $f^{(n)}$ exists on (c,d). Suppose that $f^{(n)}$ is in the space $L([a,b])$ of all integrable functions on $[a,b]$, where a, $b \in (c,d)$ with $a<b$, $n \in \mathbb{N}$. If $\left|f^{(n)}\right|^q$ is an s-convex function in the second sense over $[a,b]$ for some fixed s, $0<s\leq 1$ and $1<q<\infty$, we have the inequality*

$$\left|A(f(a),f(b)) - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}(A(a,b))\right|$$
$$\leq \frac{(b-a)^n\left[n^{(2q-1)/(q-1)}-(n-1)^{(2q-1)/(q-1)}\right]^{1-1/q}}{2^{n+s/q+1}n!}\left(\frac{q-1}{2q-1}\right)^{1-1/q}$$
$$\times\left\{\left[\frac{\left|f^{(n)}(a)\right|^q}{nq-q+s+1}+\left(\frac{1}{nq-q+1}+B(s+1,nq-q+1)\right)\left|f^{(n)}(b)\right|^q\right]^{1/q}\right.$$
$$\left.+\left[\frac{\left|f^{(n)}(b)\right|^q}{nq-q+s+1}+\left(\frac{1}{nq-q+1}+B(s+1,nq-q+1)\right)\left|f^{(n)}(a)\right|^q\right]^{1/q}\right\}. \quad (3.29)$$

where $B(\alpha,\beta)$ is the Euler Beta function defined as in Theorem 117.

Proof. Using Lemma 96, by using the first inequality in Lemma 116, the Hölder inequality and convexity of $\left|f^{(n)}\right|^q$ on $[a,b]$, we have

$$\left|A(f(a),f(b)) - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}(A(a,b))\right|$$
$$\leq \frac{(b-a)^n}{2^{n+s/q+1}n!}\left(\int_0^1 (n-1+t)^{q/(q-1)}\,dt\right)^{1-1/q}$$
$$\times\left\{\left(\int_0^1 (1-t)^{q(n-1)}\left[(1-t)^s\left|f^{(n)}(a)\right|^q+(1+t^s)\left|f^{(n)}(b)\right|^q\right]dt\right)^{1/q}\right.$$
$$\left.+\left(\int_0^1 (1-t)^{q(n-1)}\left[(1-t)^s\left|f^{(n)}(b)\right|^q+(1+t^s)\left|f^{(n)}(a)\right|^q\right]dt\right)^{1/q}\right\}. \quad (3.30)$$

By simple computations, we observe that

$$\int_0^1 (n-1+t)^{q/(q-1)}\,dt = \left(\frac{q-1}{2q-1}\right)\left[n^{(2q-1)/(q-1)} - (n-1)^{(2q-1)/(q-1)}\right],$$

$$\int_0^1 (1-t)^{q(n-1)+s}\,dt = \frac{1}{nq-q+s+1}$$

and

$$\int_0^1 (1-t)^{q(n-1)}\left(1+t^s\right)dt = \frac{1}{nq-q+1} + B\left(s+1, nq-q+1\right).$$

Using the above results in (3.30), we obtain the required result. Hence the inequality in the statement is acquired. ■

Corollary 124 *Suppose the assumptions of Theorem 123 are satisfied and if $n=1$, we have the inequality*

$$\left|A\left(f(a), f(b)\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)}{2^{2+s/q}}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left(\frac{1}{s+1}\right)^{1/q}$$
$$\times\left\{\left[\left|f'(a)\right|^q + (s+2)\left|f'(b)\right|^q\right]^{1/q} + \left[\left|f'(b)\right|^q + (s+2)\left|f'(a)\right|^q\right]^{1/q}\right\}. \quad (3.31)$$

Corollary 125 *Under the assumptions of Theorem 123, if $n=2$, we have the inequality*

$$\left|A\left(f(a), f(b)\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)^2\left[2^{(2q-1)/(q-1)}-1\right]^{1-1/q}}{2^{4+s/q}}\left(\frac{q-1}{2q-1}\right)^{1-1/q}$$
$$\times\left\{\left[\frac{1}{q+s+1}\left|f''(a)\right|^q + \left(\frac{1}{q+1} + B(s+1,q+1)\right)\left|f''(b)\right|^q\right]^{1/q}\right.$$
$$\left. + \left[\frac{1}{q+s+1}\left|f''(b)\right|^q + \left(\frac{1}{q+1} + B(s+1,q+1)\right)\left|f''(a)\right|^q\right]^{1/q}\right\}, \quad (3.32)$$

where $B(\alpha,\beta)$ is the Euler Beta function defined as in Theorem 117.

Theorem 126 *Let $[c,d]$ be an interval in $[0,\infty)$ and let $f:[c,d]\to\mathbb{R}$ be a function such that $f^{(n)}$ exists on (c,d). Suppose that $f^{(n)}$ is in the space $L([a,b])$ of all integrable functions on $[a,b]$, where $a, b\in(c,d)$ with $a<b$, $n\in\mathbb{N}$. If $\left|f^{(n)}\right|^q$ is an s-convex function*

in the second sense over $[a,b]$ for some fixed s, $0 < s \leq 1$ and $1 < q < \infty$, we have the inequality

$$\left| A(f(a), f(b)) - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=1}^{n-1} \frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}(A(a,b)) \right|$$

$$\leq \frac{(b-a)^n}{2^{n+\frac{s}{q}+1} n!} \left(\frac{q-1}{nq-1}\right)^{1-1/q} \left\{ \left[P\left|f^{(n)}(a)\right|^q + Q\left|f^{(n)}(b)\right|^q\right]^{1/q} + \left[P\left|f^{(n)}(b)\right|^q + Q\left|f^{(n)}(a)\right|^q\right]^{1/q} \right\}, \quad (3.33)$$

where

$$P = n^{q+s+1} B\left(\frac{1}{n}; s+1, q+1\right),$$

$$Q = n^q\left(\frac{s+2}{s+1}\right) - \frac{1}{q+1} - B(s+1, q+1),$$

$B(\alpha, \beta)$ is the Euler Beta function defined as in Theorem 117 and

$$B(x; \alpha, \beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\,dt, \alpha, \beta > 0, 0 \leq x \leq 1$$

is the incomplete Beta function.

Proof. Using Lemma 96, the first inequality in Lemma 116, the Hölder inequality and convexity of $\left|f^{(n)}\right|^q$ on $[a,b]$, we have

$$\left| A(f(a), f(b)) - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=1}^{n-1} \frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}(A(a,b)) \right|$$

$$\leq \frac{(b-a)^n}{2^{n+s/q+1} n!} \left(\int_0^1 (1-t)^{q(n-1)/(q-1)}\,dt\right)^{1-1/q}$$

$$\times \left\{ \left(\int_0^1 (n-1+t)^q \left[(1-t)^s\left|f^{(n)}(a)\right|^q + (1+t^s)\left|f^{(n)}(b)\right|^q\right] dt\right)^{1/q} + \left(\int_0^1 (n-1+t)^q \left[(1-t)^s\left|f^{(n)}(b)\right|^q + (1+t^s)\left|f^{(n)}(a)\right|^q\right] dt\right)^{1/q} \right\}. \quad (3.34)$$

By using the second inequality of Lemma 116 and simple computation, it is easy to observe that

$$\int_0^1 (1-t)^{q(n-1)/(q-1)}\,dt = \frac{q-1}{nq-1},$$

$$\begin{aligned}\int_0^1 (n-1+t)^q (1-t)^s\,dt &= n^{q+s+1}\int_0^{\frac{1}{n}} t^s (1-t)^q\,dt \\ &= n^{q+s+1} B\left(\frac{1}{n}; s+1, q+1\right) = P\end{aligned}$$

and

$$\begin{aligned}\int_0^1 (n-1+t)^q (1+t^s)\,dt &\le \int_0^1 (n^q-(1-t)^q)(1+t^s)\,dt \\ &= n^q\left(\frac{s+2}{s+1}\right) - \frac{1}{q+1} - B(s+1,q+1) = Q.\end{aligned}$$

Hence (3.33) follows from (3.34) and using the above results. The proof of the theorem is thus established. ■

Corollary 127 *Suppose the assumptions of Theorem 126 are satisfied and if $n=1$, we have the inequality*

$$\begin{aligned}&\left|A(f(a),f(b)) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)}{2^{2+s/q}} \\ &\times\left\{\left[B(s+1,q+1)\left|f'(a)\right|^q + \left(\frac{s+2}{s+1} - \frac{1}{q+1} - B(s+1,q+1)\right)\left|f'(b)\right|^q\right]^{1/q}\right. \\ &\left.+\left[B(s+1,q+1)\left|f'(b)\right|^q + \left(\frac{s+2}{s+1} - \frac{1}{q+1} - B(s+1,q+1)\right)\left|f'(a)\right|^q\right]^{1/q}\right\}, \end{aligned} \tag{3.35}$$

where $B(\alpha,\beta)$ is the Euler Beta function defined as in Theorem 117.

Corollary 128 *Suppose the assumptions of Theorem 126 are satisfied and if $n=2$, we* *have the inequality*

$$
\begin{aligned}
&\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right| \\
&\quad\leq\frac{\left(b-a\right)^2}{2^{4+\frac{s}{q}}}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left\{\left[2^{q+s+1}B\left(\frac{1}{2};s+1,q+1\right)\left|f''\left(a\right)\right|^q\right.\right. \\
&\quad\left.+\left(2^q\left(\frac{s+2}{s+1}\right)-\frac{1}{q+1}-B\left(s+1,q+1\right)\right)\left|f''\left(b\right)\right|^q\right]^{1/q} \\
&\quad+\left[2^{q+s+1}B\left(\frac{1}{2};s+1,q+1\right)\left|f''\left(b\right)\right|^q\right. \\
&\quad\left.\left.+\left(2^q\left(\frac{s+2}{s+1}\right)-\frac{1}{q+1}-2^{q+s+1}B\left(s+1,q+1\right)\right)\left|f''\left(a\right)\right|^q\right]^{1/q}\right\}, \qquad (3.36)
\end{aligned}
$$

where $B\left(\alpha,\beta\right)$ is the Euler Beta function defined as in Theorem 117 and $B\left(x;\alpha,\beta\right)$ is the incomplete Beta function defined as in Theorem 126.

A different approach results in the following theorem.

Theorem 129 *Let $[c,d]$ be an interval in $[0,\infty)$ and let $f:[c,d]\to\mathbb{R}$ be a function such that $f^{(n)}$ exists on (c,d). Suppose that $f^{(n)}$ is in the space $L\left([a,b]\right)$ of all integrable functions on $[a,b]$, where $a,\,b\in(c,d)$ with $a<b$, $n\in\mathbb{N}$. If $\left|f^{(n)}\right|^q$ is an s-convex function in the second sense over $[a,b]$ for some fixed s, $0<s\leq1$ and $1<q<\infty$, we have the inequality*

$$
\begin{aligned}
&\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx-\sum_{k=1}^{n-1}\frac{k\left[1+\left(-1\right)^k\right]\left(b-a\right)^k}{2^{k+1}\left(k+1\right)!}f^{(k)}\left(A\left(a,b\right)\right)\right| \\
&\quad\leq\frac{n^{n+1-1/q}\left(b-a\right)^n}{2^{n+s/q+1}n!}\left(\frac{1}{s+1}\right)^{1/q}\left[B\left(\frac{1}{n};\frac{nq-1}{q-1},\frac{2q-1}{q-1}\right)\right]^{1-1/q} \\
&\quad\times\left\{\left[\left|f^{(n)}\left(a\right)\right|^q+\left(2^{s+1}-1\right)\left|f^{(n)}\left(b\right)\right|^q\right]^{1/q}\right. \\
&\quad\left.+\left[\left(2^{s+1}-1\right)\left|f^{(n)}\left(a\right)\right|^q+\left|f^{(n)}\left(b\right)\right|^q\right]^{1/q}\right\}, \qquad (3.37)
\end{aligned}
$$

where

$$B(x;\alpha,\beta)=\int_0^x t^{\alpha-1}(1-t)^{1-\beta}\,dt, 0\leq x\leq 1, \alpha>0, \beta>0$$

is the incomplete beta function.

Proof. Using Lemma 96, the Hölder inequality and the convexity of $\left|f^{(n)}\right|^q$ on $[a,b]$, we have

$$\begin{aligned}
&\left|A(f(a),f(b))-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}(A(a,b))\right| \\
&\leq \frac{(b-a)^n}{2^{n+s/q+1}n!}\left(\int_0^1(1-t)^{q(n-1)/(q-1)}(n-1+t)^{q/(q-1)}\,dt\right)^{1-1/q} \\
&\times\left\{\left(\int_0^1\left[(1-t)^s\left|f^{(n)}(a)\right|^q+(1+t)^s\left|f^{(n)}(b)\right|^q\right]dt\right)^{1/q}\right. \\
&\left.+\left(\int_0^1\left[(1-t)^s\left|f^{(n)}(b)\right|^q+(1-t)^s\left|f^{(n)}(a)\right|^q\right]dt\right)^{1/q}\right\}. \quad (3.38)
\end{aligned}$$

By simple computations of integrals, the inequality (3.37) follows from the inequality (3.38) and using the fact that

$$\int_0^1(1-t)^{q(n-1)/(q-1)}(n-1+t)^{q/(q-1)}\,dt=n^{\frac{nq+q-1}{q-1}}B\left(\frac{1}{n};\frac{nq-1}{q-1},\frac{2q-1}{q-1}\right).$$

Hence we get the stated inequality of Theorem 129. ■

Corollary 130 *Suppose the assumptions of Theorem 129 are satisfied and if $n=1$, we have*

$$\begin{aligned}
&\left|A(f(a),f(b))-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{(b-a)}{2^{2+s/q}}\left(\frac{1}{s+1}\right)^{1/q}\left(\frac{q-1}{2q-1}\right)^{1-1/q} \\
&\times\left\{\left[\left|f'(a)\right|^q+\left(2^{s+1}-1\right)\left|f'(b)\right|^q\right]^{1/q}+\left[\left(2^{s+1}-1\right)\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]^{1/q}\right\}. \quad (3.39)
\end{aligned}$$

Proof. Proof follows from the fact that

$$B\left(1;1,\frac{2q-1}{q-1}\right)=B\left(1,\frac{2q-1}{q-1}\right)=\int_0^1(1-t)^{\frac{2q-1}{q-1}-1}\,dt=\frac{q-1}{2q-1}.$$

■

Corollary 131 *Under the assumptions of Theorem 129 and $n=2$, we have the following inequality*

$$\begin{aligned}&\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right|\\&\qquad\leq\frac{\left(b-a\right)^2}{2^{1+s/q+1/q}}\left(\frac{1}{s+1}\right)^{1/q}\left[B\left(\frac{1}{2};\frac{2q-1}{q-1},\frac{2q-1}{q-1}\right)\right]^{1-1/q}\\&\times\left\{\left[\left|f''\left(a\right)\right|^q+\left(2^{s+1}-1\right)\left|f''\left(b\right)\right|^q\right]^{1/q}+\left[\left(2^{s+1}-1\right)\left|f''\left(a\right)\right|^q+\left|f''\left(b\right)\right|^q\right]^{1/q}\right\},\quad(3.40)\end{aligned}$$

where $B\left(\alpha,\beta\right)$ is defined as in Theorem 117.

3.4 Applications to the Trapezoidal Formula

3.4.1 Applications of inequalities for convex functions to the Trapezoidal Formula

Let Δ be a division of the interval $[a,b]$, i.e. $a=x_0<x_1<...<x_{n-1}<x_n=b$, and consider the quadrature formula

$$\int_a^b f(x)dx=T(f,\Delta)+E(f,\Delta),$$

where

$$T(f,\Delta)=\sum_{i=0}^{n-1}\left(x_{i+1}-x_i\right)A\left(f\left(x_i\right),f\left(x_{i+1}\right)\right)$$

is the trapezoidal versions and $E(f,\Delta)$ is the associated error. Here, we derive some error estimates for the trapezoidal formula in terms of absolute values of the second derivative of f which may be better than those already existing in the literature.

Theorem 132 *Let $f:[c,d]\subseteq\mathbb{R}\to\mathbb{R}$ possesses second order derivative f'' in (c,d) such that f'' belongs to the space $L([a,b])$ of all integrable functions over $[a,b]$, where $a,b\in(c,d)$ with $a<b$. If $\left|f''\right|^q$ is convex on $[a,b]$ for $q\geq 1$, then for every division Δ of $[a,b]$, we have*

$$|E(f,\Delta)|\leq\frac{1}{24}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3\left\{\left(\frac{A\left(5\left|f''(x_i)\right|^q,11\left|f''(x_{i+1})\right|^q\right)}{8}\right)^{1/q}\right.$$
$$\left.+\left(\frac{A\left(5\left|f''(x_{i+1})\right|^q,11\left|f''(x_i)\right|^q\right)}{8}\right)^{1/q}\right\}. \tag{3.41}$$

Proof. By applying Corollary 101 on the subinterval $[x_i,x_{i+1}]$ $(i=0,1,\ldots,n-1)$ of the division Δ, we have

$$\begin{aligned}|E(f,\Delta)|&=\left|\sum_{i=0}^{n-1}\left\{(x_{i+1}-x_i)\frac{f(x_i)+f(x_{i+1})}{2}-\int_{x_i}^{x_{i+1}}f(x)\,dx\right\}\right|\\&\leq\sum_{i=0}^{n-1}(x_{i+1}-x_i)\left|\frac{f(x_i)+f(x_{i+1})}{2}-\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}f(x)\,dx\right|\\&\leq\frac{1}{24}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3\left\{\left(\frac{5\left|f''(x_i)\right|^q+11\left|f''(x_{i+1})\right|^q}{16}\right)^{1/q}\right.\\&\qquad\left.+\left(\frac{5\left|f''(x_{i+1})\right|^q+11\left|f''(x_i)\right|^q}{16}\right)^{1/q}\right\}.\end{aligned}\tag{3.42}$$

Hence the result is proved. ■

Corollary 133 *Under the assumptions of Theorem 117, for $q=1$, we have*

$$|E(f,\Delta)|\leq\frac{1}{12}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3A\left(\left|f''(x_i)\right|,\left|f''(x_{i+1})\right|\right). \tag{3.43}$$

Theorem 134 *Let $f : [c,d] \subseteq \mathbb{R} \to \mathbb{R}$ possesses second order derivative in (c,d) such that f'' belongs to the space $L([a,b])$ of all integrable functions over $[a,b]$, where $a,b \in (c,d)$ with $a<b$. If $|f''|^q$ is convex on $[a,b]$ for $q>1$, then for every division Δ of $[a,b]$, we have*

$$\begin{aligned} |E(f,\Delta)| \leq{} & \frac{\left[2^{(2q-1)/(q-1)}-1\right]^{1-1/q}}{(2^{1/q})\cdot 8}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3 \\ & \times\left\{\left[\frac{(q+1)\left|f''(x_i)\right|^q+(q+3)\left|f''(x_{i+1})\right|^q}{(q+1)(q+2)}\right]^{1/q}\right. \\ & \left.+\left[\frac{(q+1)\left|f''(x_{i+1})\right|^q+(q+3)\left|f''(x_i)\right|^q}{(q+1)(q+2)}\right]^{1/q}\right\}. \end{aligned} \tag{3.44}$$

Proof. It follows by applying Corollary 106 on the subinterval $[x_i, x_{i+1}]$ $(i=0,1,\ldots,n-1)$ of the division Δ. ■

Theorem 135 *Let $f : [c,d] \subseteq \mathbb{R} \to \mathbb{R}$ possesses second order derivative f'' in (c,d) such that f'' belongs to the space $L([a,b])$ of all integrable functions over $[a,b]$, where $a,b \in (c,d)$ with $a<b$. If $|f''|^q$ is convex on $[a,b]$ for $q>1$, then for every division Δ of $[a,b]$, we have*

$$\begin{aligned} |E(f,\Delta)| \leq{} & \frac{1}{2^{4+\frac{1}{q}}}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3 \\ & \times\left\{\left[\frac{(2^{q+2}-q-3)\left|f''(x_i)\right|^q+(q+1)(2^{q+2}-1)\left|f''(x_{i+1})\right|^q}{(q+1)(q+2)}\right]^{1/q}\right. \\ & \left.+\left[\frac{(2^{q+2}-q-3)\left|f''(x_{i+1})\right|^q+(q+1)(2^{q+2}-1)\left|f''(x_i)\right|^q}{(q+1)(q+2)}\right]^{1/q}\right\}. \end{aligned} \tag{3.45}$$

Proof. It is a direct consequence of application of Corollary 109 on the subinterval $[x_i, x_{i+1}]$ $(i=0,1,\ldots,n-1)$ of the division Δ. ■

Error bound for trapezoidal formula when $\left|f''\right|^q$ for $q \geq 1$ is concave may be given as follows.

Theorem 136 *Let $f : [c,d] \subseteq \mathbb{R} \to \mathbb{R}$ possesses second order derivative f'' in (c,d) such that f'' belongs to the space $L([a,b])$ of all integrable functions over $[a,b]$, where $a,b \in (c,d)$ with $a < b$. If $\left|f''\right|^q$ is convex on $[a,b]$ for $q \geq 1$, then for every division Δ of $[a,b]$, we have*

$$|E(f,\Delta)| \leq \frac{1}{24}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3\left[\left|f''\left(\frac{5x_i+11x_{i+1}}{16}\right)\right| + \left|f''\left(\frac{5x_{i+1}+11x_i}{16}\right)\right|\right]. \quad (3.46)$$

Proof. It can be proved easily by using Corollary 115 over the subinterval $[x_i, x_{i+1}]$ $(i = 0, 1, \ldots, n-1)$ of the division δ. ■

3.4.2 Applications of inequalities for s-convex functions to the Trapezoidal Formula

Theorem 137 *Let $f : [c,d] \subseteq \mathbb{R} \to \mathbb{R}$ possesses second order derivative f'' in (c,d) such that f'' belongs to the space $L([a,b])$ of all integrable functions over $[a,b]$, where $a,b \in (c,d)$ with $a < b$. If $\left|f''\right|^q$ is an s-convex on $[a,b]$, $q \geq 1$ and for some fixed s, $0 < s \leq 1$, then for every division Δ of $[a,b]$, we have*

$$\begin{aligned}|E(f,\Delta)| \leq{}& \frac{1}{2^{4+s/q}}\left(\frac{2}{3}\right)^{1-1/q}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3 \\ &\times\left\{\left[\frac{(s+4)}{(s+2)(s+3)}\left|f''(x_i)\right|^q + \left(\frac{2}{3}+\frac{2}{(s+1)(s+3)}\right)\left|f''(x_{i+1})\right|^q\right]^{1/q}\right. \\ &\left.+\left[\frac{(s+4)}{(s+2)(s+3)}\left|f''(x_{i+1})\right|^q + \left(\frac{2}{3}+\frac{2}{(s+1)(s+3)}\right)\left|f''(x_i)\right|^q\right]^{1/q}\right\}. \quad (3.47)\end{aligned}$$

Proof. By applying Corollary 121 on the subinterval $[x_i, x_{i+1}]$ $(i = 0, 1, \ldots, n-1)$ of the division Δ, we have

$$\begin{aligned}
|E(f,\Delta)| &= \left|\sum_{i=0}^{n-1}\left\{(x_{i+1}-x_i)\frac{f(x_i)+f(x_{i+1})}{2} - \int_{x_i}^{x_{i+1}} f(x)\,dx\right\}\right| \\
&\leq \sum_{i=0}^{n-1}(x_{i+1}-x_i)\left|\frac{f(x_i)+f(x_{i+1})}{2} - \frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} f(x)\,dx\right| \\
&\leq \frac{1}{2^{4+s/q}}\left(\frac{2}{3}\right)^{1-1/q}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3 \\
&\times\left\{\left[\frac{(s+4)}{(s+2)(s+3)}\left|f''(x_i)\right|^q + \left(\frac{2}{3}+\frac{2}{(s+1)(s+3)}\right)\left|f''(x_{i+1})\right|^q\right]^{1/q}\right. \\
&\left.+\left[\frac{(s+4)}{(s+2)(s+3)}\left|f''(x_{i+1})\right|^q + \left(\frac{2}{3}+\frac{2}{(s+1)(s+3)}\right)\left|f''(x_i)\right|^q\right]^{1/q}\right\}. \quad (3.48)
\end{aligned}$$

■

Corollary 138 *Under the assumptions of Theorem 137, for $q = 1$, we have*

$$\begin{aligned}
|E(f,\Delta)| &\leq \frac{1}{2^{4+s}}\left(\frac{2}{3}+\frac{s^2+7s+8}{(s+1)(s+2)(s+3)}\right) \\
&\times\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3\left[\left|f''(x_i)\right| + \left|f''(x_{i+1})\right|\right]. \quad (3.49)
\end{aligned}$$

Theorem 139 *Let $f : [c,d] \subseteq \mathbb{R} \to \mathbb{R}$ possesses second order derivative in (c,d) such that f'' belongs to the space $L([a,b])$ of all integrable functions over $[a,b]$, where $a, b \in (c,d)$ with $a < b$. If $\left|f''\right|^q$ is an s-convex on $[a,b]$, $q > 1$ and for some fixed s, $0 < s \leq 1$,* *then for every division Δ of $[a,b]$, we have*

$$\begin{aligned}
|E(f,\Delta)| \leq{}& \frac{\left[2^{(2q-1)/(q-1)}-1\right]^{1-1/q}}{2^{4+s/q}}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3 \\
&\times\left\{\left[\frac{1}{q+s+1}\left|f''(x_i)\right|^q+\left(\frac{1}{q+1}+B(s+1,q+1)\right)\left|f''(x_{i+1})\right|^q\right]^{1/q}\right. \\
&\left.+\left[\frac{1}{q+s+1}\left|f''(x_{i+1})\right|^q+\left(\frac{1}{q+1}+B(s+1,q+1)\right)\left|f''(x_i)\right|^q\right]^{1/q}\right\}, \quad (3.50)
\end{aligned}$$

where $B(\alpha,\beta)$ is the Euler Beta function defined as in Theorem 117.

Proof. It follows from Corollary 125. ■

Theorem 140 *Let $f:[c,d]\subseteq\mathbb{R}\to\mathbb{R}$ possesses second order derivative f'' in (c,d) such that f'' belongs to the space $L([a,b])$ of all integrable functions over $[a,b]$, where $a,b\in(c,d)$ with $a<b$. If $\left|f''\right|^q$ is an s-convex on $[a,b]$, $q>1$ and for some fixed s, $0<s\leq 1$, then for every division Δ of $[a,b]$, we have*

$$\begin{aligned}
|E(f,\Delta)| \leq{}& \frac{1}{2^{4+\frac{1}{q}}}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3\left\{\left[2^{q+s+1}B\left(\frac{1}{2};s+1,q+1\right)\left|f''(x_i)\right|^q\right.\right. \\
&\left.+\left(2^q\left(\frac{s+2}{s+1}\right)-\frac{1}{q+1}-B(s+1,q+1)\right)\left|f''(x_{i+1})\right|^q\right]^{1/q} \\
&+\left[2^{q+s+1}B\left(\frac{1}{2};s+1,q+1\right)\left|f''(x_{i+1})\right|^q\right. \\
&\left.\left.+\left(2^q\left(\frac{s+2}{s+1}\right)-\frac{1}{q+1}-2^{q+s+1}B(s+1,q+1)\right)\left|f''(x_i)\right|^q\right]^{1/q}\right\}, \quad (3.51)
\end{aligned}$$

where $B(\alpha,\beta)$ is the Euler Beta functions and $B(x;\alpha,\beta)$ is the incomplete beta function defined as in Theorem 117 and Theorem 126 respectively.

Proof. It is a direct consequence of Corollary 128. ■

Theorem 141 *Let $f:[c,d]\subseteq\mathbb{R}\to\mathbb{R}$ possesses second order derivative f'' in (c,d) such that f'' belongs to the space $L([a,b])$ of all integrable functions over $[a,b]$, where*

$a,b \in (c,d)$ with $a < b$. If $\left|f''\right|^q$ is an s-convex on $[a,b]$, $q > 1$ and for some fixed s, $0 < s \leq 1$, then for every division Δ of $[a,b]$, we have

$$\begin{aligned} |E(f,\Delta)| \leq{} & \frac{1}{2^{1+s/q+1/q}}\left(\frac{1}{s+1}\right)^{1/q}\left[B\left(\frac{1}{2};\frac{2q-1}{q-1},\frac{2q-1}{q-1}\right)\right]^{1-1/q} \\ & \times\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3\left\{\left[\left|f''(x_i)\right|^q+\left(2^{s+1}-1\right)\left|f''(x_{i+1})\right|^q\right]^{1/q}\right. \\ & \left.+\left[\left(2^{s+1}-1\right)\left|f''(x_i)\right|^q+\left|f''(x_{i+1})\right|^q\right]^{1/q}\right\}, \quad (3.52) \end{aligned}$$

where $B(x;\alpha,\beta)$ is the incomplete beta function defined as in Theorem 126.

Proof. The proof follows by using Corollary 131. ■

3.5 Applications to the Special Means

3.5.1 Applications of inequalities for convex functions to the Special Means

Now we apply some of our results to establish some inequalities for special means.

Theorem 142 *If a, $b \in \mathbb{R}_+$, $a < b$ and $p \in \mathbb{N}$, $p > 1$, we have*

$$\left|A\left(a^{p+1},b^{p+1}\right)-L_{p+1}^{p+1}\left(a^{p+1},b^{p+1}\right)\right| \leq \frac{p(p+1)(b-a)^2}{12}A\left(a^{p-1},b^{p-1}\right).$$

Proof. For the function $f(x) = x^{p+1}$, $x \in \mathbb{R}_+$ and $p \in \mathbb{N}$, then $\left|f''(x)\right| = (p+1)px^{p-1}$ is a convex function on $\mathbb{R}_+$. Applying Corollary 102, we obtain the required result. ■

Theorem 143 *If* $a,\ b \in \mathbb{R}_+,\ a < b$ *and* $p,\ q \in \mathbb{N},\ p, q > 1,$ *we have*

$$\begin{aligned}
&\left| A\left(a^{p+1/q}, b^{p+1/q}\right) - L_{p+1/q}^{p+1/q}\left(a^{p+1/q}, b^{p+1/q}\right) \right| \\
&\quad \leq \frac{(b-a)^2 \left[2^{(2q-1)/(q-1)} - 1\right]^{1-1/q}}{16} \left(\frac{q-1}{2q-1}\right)^{1-1/q} \left(n - 1 + \frac{1}{q}\right)\left(n + \frac{1}{q}\right) \\
&\qquad \times \left\{ \left[\frac{A\left(a^{(p-2)q+1}, b^{(p-2)q+1}\right)}{q+2} + \frac{b^{(p-2)q+1}}{(q+1)(q+2)}\right]^{1/q} \right. \\
&\qquad\qquad \left. + \left[\frac{A\left(a^{(p-2)q+1}, b^{(p-2)q+1}\right)}{q+2} + \frac{a^{(p-2)q+1}}{(q+1)(q+2)}\right]^{1/q} \right\}. \qquad (3.53)
\end{aligned}$$

Proof. Let $f(x) = x^{p+1/q},\ x \in \mathbb{R}_+,\ p,\ q \in \mathbb{N},\ p, q > 1$. Then

$$\left|f''(x)\right|^q = \left(p - 1 + \frac{1}{q}\right)^q \left(p + \frac{1}{q}\right)^q x^{(p-2)q+1}$$

is convex on $\mathbb{R}_+$ and the result follows directly from Corollary 106. ■

Theorem 144 *If* $a,\ b \in \mathbb{R}_+,\ a < b$ *and* $p,\ q \in \mathbb{N},\ p, q > 1,$ *we have*

$$\begin{aligned}
&\left| A\left(a^{p+1/q}, b^{p+1/q}\right) - L_{pn+1/q}^{p+1/q}\left(a^{p+1/q}, b^{p+1/q}\right) \right| \\
&\quad \leq \frac{(b-a)^2}{2^{4+1/q}} \left(\frac{q-1}{2q-1}\right)^{1-1/q} \left(p - 1 + \frac{1}{q}\right)\left(p + \frac{1}{q}\right) \\
&\qquad \times \left\{ \left[\frac{2^{q+2} - (q+3)}{(q+1)(q+2)} a^{(p-2)q+1} + \frac{2^{q+2} - 1}{q+2} b^{(p-2)q+1}\right]^{1/q} \right. \\
&\qquad\qquad \left. + \left[\frac{2^{q+2} - (q+3)}{(q+1)(q+2)} b^{(p-2)q+1} + \frac{2^{q+2} - 1}{q+2} a^{(p-2)q+1}\right]^{1/q} \right\} \qquad (3.54)
\end{aligned}$$

Proof. Let $f(x) = x^{p+1/q},\ x \in \mathbb{R}_+,\ p,\ q \in \mathbb{N},\ p, q > 1$. Then

$$\left|f''(x)\right|^q = \left(p - 1 + \frac{1}{q}\right)^q \left(p + \frac{1}{q}\right)^q x^{(p-2)q+1}$$

is convex on $\mathbb{R}_+$ and the result follows directly from Corollary 109. ■

Theorem 145 *For $a,\ b \in \mathbb{R}_+$, $a < b$ and $p,\ q \in \mathbb{N}$, $p, q > 1$ with $p < q$, we have*

$$\begin{aligned} &\left| A\left(a^{p/q^2+2}, b^{p/q+2}\right) - L_{p/q^2+2}^{p/q^2+2}\left(a^{p/q^2+2}, b^{p/q^2+2}\right)\right| \\ &\qquad \leq \frac{(b-a)^2}{12}\left(\frac{p}{q^2}+2\right)\left(\frac{p}{q^2}+1\right) A\left(a^{p/q^2}, b^{p/q^2}\right). \end{aligned} \tag{3.55}$$

Proof. Let $f(x) = x^{p/q^2+2}$, $x \in \mathbb{R}_+$, then $\left|f''(x)\right| = \left(\frac{p}{q^2}+2\right)\left(\frac{p}{q^2}+1\right) x^{p/q^2}$. It is obvious that $\left|f''(x)\right|^q$ is concave on $\mathbb{R}_+$. Applying Corollary 115, we get the desired result. ■

Theorem 146 *If $a,\ b \in \mathbb{R}_+$, $a < b$, we have*

$$\left|H^{-1}(a,b) - L^{-1}(a,b)\right| \leq \frac{(b-a)^2}{6} A\left(a^{-3}, b^{-3}\right) \tag{3.56}$$

Proof. Let $f(x) = \frac{1}{x}$, $x \in \mathbb{R}_+$, then $\left|f''(x)\right| = \frac{2}{x^3}$ is convex on $\mathbb{R}_+$ so applying Corollary 102, we get the inequality (3.56). ■

Remark 147 *Many other interesting inequalities for means can be obtained by applying the other results to the function $f(x) = \frac{1}{x}$, $x \in \mathbb{R}_+$, however the details are left to the interested reader.*

3.5.2 Applications of inequalities for s-convex functions to the Special Means

Now we apply our results from Section 3.3 to give some inequalities for special means.

It is shown in [57] that the function $f : [0, \infty) \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} a, & t = 0, \\ bt^s + c, & t > 0, \end{cases}$$

where $s \in (0,1)$, a, b, $c \in \mathbb{R}$ with $0 \leq c \leq a$, $b \geq 0$, is s-convex on $[0,\infty)$. Hence, for $a = c = 0$, $b \geq 0$, the function $f(t) = bt^s$ is an s-convex function on $[0,\infty)$, where $s \in (0,1)$.

Theorem 148 *For a, $b \in \mathbb{R}_+$, $a < b$, $0 < s < 1$ and $q \in [1,\infty)$, we have*

$$\begin{aligned}
&\left| A\left(a^{s+2}, b^{s+2}\right) - L_{s+2}^{s+2}\left(a^{s+2}, b^{s+2}\right) \right| \\
&\qquad \leq \frac{(b-a)^2}{2^{3+s}} \left(\frac{2}{3}(s+2)(s+1) + \frac{s^2+7s+8}{s+3} \right) A\left(a^s, b^s\right). \quad (3.57)
\end{aligned}$$

Proof. Let $f(x) = x^{s+2}$, $x \in \mathbb{R}_+$. Then $\left|f''(x)\right| = (s+2)(s+1)x^s$ is an s-convex function on $\mathbb{R}_+$. Applying Corollary 122, we obtain the required result. ■

Theorem 149 *For a, $b \in \mathbb{R}_+$, $a < b$ and $q \in (1,\infty)$, we have*

$$\begin{aligned}
&\left| A\left(a^{s/q+1}, b^{s/q+1}\right) - L_{s/q+1}^{s/q+1}\left(a^{s/q+1}, b^{s/q+1}\right) \right| \\
&\qquad \leq \frac{(b-a)}{2^{2+s/q}} \left(\frac{q-1}{2q-1} \right)^{1-1/q} \left(\frac{1}{s+1} \right)^{1/q} (s/q+1) \\
&\qquad \times \left\{ \left[2A\left(a^s, b^s\right) + (s+1)b^s\right]^{1/q} + \left[2A\left(a^s, b^s\right) + (s+1)a^s\right]^{1/q} \right\}. \quad (3.58)
\end{aligned}$$

Proof. Let $f(x) = x^{s/q+1}$, $x \in \mathbb{R}_+$. Then$\left|f'(x)\right|^q = \left[(s/q+1)\right]^q x^s$ is an s-convex function on $\mathbb{R}_+$. Applying Corollary 124, we obtain the required result. ■

Theorem 150 *For a, $b \in \mathbb{R}_+$, $a < b$ and n, $q \in \mathbb{N}$, $n, q > 1$, we have*

$$\begin{aligned}
&\left| A\left(a^{s/q+1}, b^{s/q+1}\right) - L_{s/q+1}^{s/q+1}\left(a^{s/q+1}, b^{s/q+1}\right) \right| \\
&\qquad \leq \frac{(b-a)}{2^{2+s/q-1/q}} \left(\frac{1}{s+1} \right)^{1/q} \left(\frac{q-1}{2q-1} \right)^{1-1/q} (s/q+1) \\
&\qquad \times \left\{ \left[A\left(a^s, b^s\right) + (2^s-1)b^s\right]^{1/q} + \left[(2^s-1)a^s + A\left(a^s, b^s\right)\right]^{1/q} \right\}. \quad (3.59)
\end{aligned}$$

Proof. Let $f(x) = x^{s/q+1}$, $x \in \mathbb{R}_+$. Then$\left|f'(x)\right|^q = \left[(s/q+1)\right]^q x^s$ is an s-convex function on $\mathbb{R}_+$. Applying Corollary 130, we obtain the required result. ■

Remark 151 *Many other interesting inequalities for means can be obtained by applying the other results to some suitable s-convex functions, however the details are left to the interested reader.*

Remark 152 *For $s = 1$, we get some of the results of Section 4.2.*

Chapter 4

H-H type inequalities for n-times differentiable s-logarithmically convex functions with applications to special means

The results of this chapter are from "M. A. Latif, S. S. Dragomir, New integral inequalities of Hermite-Hadamard type for n-times differentiable s-logarithmically convex functions with applications, Miskolc Mathematical Notes. 16 (1) (2015) 219-235." and "M. A. Latif, S. S. Dragomir, On Hermite-Hadamard Type Integral Inequalities for n-times differentiable s-logarithmically convex functions with applications to special means, Applied Mathematics & Information Sciences. (Accepted)".

4.1 Introduction

As mentioned in Chapter 2 that many mathematicians are trying to generalize the special convexity in a number of ways and one of them is the s-logarithmic convexity. In this chapter, we present some new integral inequalities of Hermite-Hadamard type for functions whose nth derivatives in absolute value are s-logarithmically convex. We believe that our results may provide refinements of the results for s-logarithmically convex functions that already exist in literature. The results of of this chapter not only generalize the results from [168] but many other interesting results can be obtained for functions whose first and second derivatives in absolute value are s-logarithmically convex which may be better than those from [168]. In this chapter, we have proved a new identity involving n-times differentiable mappings defined over the set of real numbers, an identity of Chapter 3, an identity for n-times differentiable mappings from [51] and several new auxiliary results to acquire our results. Finally, we exhibit some new inequalities of special means of positive real numbers as applications of the established results in this chapter.

The organization of this chapter is described as follows:

The beginning of Section 4.2 of this chapter contains quite a few identities for n-times differentiable mappings and some auxiliary results involving some special integrals over the interval $[0, 1]$. The other part of Section 4.2 contains integral inequalities of H-H type for functions whose nth derivatives satisfy s-logarithmically convexity conditions, which are new in the field of mathematical inequalities. Section 4.3 of this chapter is also for integral inequalities of H-H type for functions whose nth derivatives satisfy s-logarithmically convexity postulations which further generalize some of the results of Section 4.2. At the end, Section 4.4 contains some inequalities of means of positive real numbers, which arise from applications of some of findings of Section 4.2 and Section 4.3.

4.2 On H-H type integral inequalities for n-times differentiable s-logarithmically convex functions

First we quote and establish some useful lemmas to prove our mains results of this section.

Lemma 153 *[51] Suppose that $f : [c, d] \subseteq \mathbb{R} \to \mathbb{R}$ is a function such that $f^{(n)}$ exists on (c, d), where $n \in \mathbb{N}$ and $n \geq 1$. If $f^{(n)}$ is in the space $L([a, b])$ of all integrable functions* *on* $[a, b]$*, for* $a, b \in (c, d)$ *with* $a < b$*, the equality holds*

$$A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx - \sum_{k=2}^{n-1}\frac{\left(k-1\right)\left(b-a\right)^k}{2\left(k+1\right)!}f^{(k)}(a)$$
$$= \frac{\left(b-a\right)^n}{2n!}\int_0^1 t^{n-1}\left(n-2t\right)f^{(n)}(A_t\left(a,b\right))dt, \quad (4.1)$$

where the sum above takes 0 *when* $n = 1$ *and* $n = 2$*.*

Lemma 154 *Suppose that* $f : [c, d] \subseteq \mathbb{R} \to \mathbb{R}$ *is a function such that* $f^{(n)}$ *exists on* (c, d)*, where* $n \in \mathbb{N}$ *and* $n \geq 1$*. If* $f^{(n)}$ *is in the space* $L\left([a, b]\right)$ *of all integrable functions on* $[a, b]$*, for* $a, b \in (c, d)$ *with* $a < b$*, the equality holds*

$$\sum_{k=0}^{n-1}\frac{\left[\left(-1\right)^k+1\right]\left(b-a\right)^k}{2^{k+1}\left(k+1\right)!}f^{(k)}\left(A\left(a,b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx$$
$$= \frac{\left(-1\right)\left(b-a\right)^n}{n!}\int_0^1 K_n(t)f^{(n)}(A_t\left(a,b\right))dt, \quad (4.2)$$

where

$$K_n(t) := \begin{cases} t^n, & t \in \left[0, \frac{1}{2}\right], \\ \\ \left(t-1\right)^n, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Proof. For $n = 1$, we have

$$\left(-1\right)\left(b-a\right)\int_0^1 K_1(t)f^{(n)}(ta+\left(1-t\right)b)dt$$
$$= -\left(b-a\right)\int_0^{\frac{1}{2}} tf'(ta+\left(1-t\right)b)dt - \left(b-a\right)\int_{\frac{1}{2}}^1 \left(t-1\right)f'(A_t\left(a,b\right))dt$$
$$= f\left(A\left(a,b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx,$$

which is the left hand side of (4.2) for $n = 1$.

Suppose that (4.2) holds for $n = m - 1$, $m > 2$, that is

$$\sum_{k=0}^{m-2} \frac{\left[(-1)^k + 1\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}(A(a,b)) - \frac{1}{b-a}\int_a^b f(x)\,dx$$
$$= \frac{(-1)(b-a)^{m-1}}{(m-1)!}\int_0^1 K_{m-1}(t) f^{(m-1)}(A_t(a,b))dt. \quad (4.3)$$

Now for $n = m$, by integration by parts and using (4.3), we have

$$\begin{aligned}
&\frac{(-1)(b-a)^m}{m!}\int_0^1 K_m(t) f^{(m)}(A_t(a,b))dt \\
&= \frac{\left[(-1)^{m-1}+1\right](b-a)^{m-1}}{2^m m!} f^{(m-1)}(A(a,b)) \\
&+ \frac{(-1)(b-a)^{m-1}}{(m-1)!}\int_0^1 K_{m-1}(t) f^{(m-1)}(A_t(a,b))dt \\
&= \frac{(b-a)^{m-1}\left[(-1)^m+1\right]}{2^m m!} f^{(m-1)}(A(a,b)) \\
&+ \sum_{k=0}^{m-2} \frac{\left[(-1)^{k+1}-1\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}(A(a,b)) - \frac{1}{b-a}\int_a^b f(x)\,dx, \quad (4.4)
\end{aligned}$$

which is the required identity (4.2). This completes the proof of the Lemma. ■

The following useful result will also help us establishing our results.

Lemma 155 *If $\mu > 0$ and $n \in \mathbb{N} \cup \{0\}$, then*

$$E_1(n;\mu) := \int_0^1 t^n \mu^t dt = \begin{cases} \frac{(-1)^{n+1} n!}{(\ln\mu)^{n+1}} + n!\mu \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!(\ln\mu)^{k+1}}, & \mu \neq 1 \\ \\ \frac{1}{n+1}, & \mu = 1. \end{cases} \quad (4.5)$$

Proof. For $\mu = 1$, the result follows immediately.

If $n = 0$, then

$$\int_0^1 \mu^t dt = \frac{1-\mu}{\ln\mu}$$

which is the same when we take $n = 0$ on the R.H.S of (4.5).

Suppose now that $\mu \neq 1$ and $n = 1$, we have

$$\int_0^1 t\mu^t dt = \frac{\mu}{\ln \mu} - \frac{\mu}{(\ln \mu)^2} + \frac{1}{(\ln \mu)^2},$$

and it coincides with the right hand side of (4.5) for $n = 1$.

Suppose (4.5) is true for $n-1$, i.e.

$$\int_0^1 t^{n-1}\mu^t dt = \frac{(-1)^n (n-1)!}{(\ln \mu)^n} + (n-1)!\mu \sum_{k=0}^{n-1} \frac{(-1)^k}{(n-1-k)! (\ln \mu)^{k+1}}. \tag{4.6}$$

Now by integration by parts and using (4.6), we have

$$\begin{aligned}
\int_0^1 t^n \mu^t dt &= \frac{\mu}{\ln \mu} - \frac{n}{\ln \mu} \int_0^1 t^{n-1}\mu^t dt \\
&= \frac{\mu}{\ln \mu} - \frac{n}{\ln \mu} \left[\frac{(-1)^n (n-1)!}{(\ln \mu)^n} + (n-1)!\mu \sum_{k=0}^{n-1} \frac{(-1)^k}{(n-1-k)! (\ln \mu)^{k+1}} \right] \\
&= \frac{\mu}{\ln \mu} + \frac{(-1)^{n+1} n!}{(\ln \mu)^{n+1}} + n!\mu \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{(n-1-k)! (\ln \mu)^{k++2}} \\
&= \frac{n!\mu}{n! \ln \mu} + \frac{(-1)^{n+1} n!}{(\ln \mu)^{n+1}} + n!\mu \sum_{k=1}^{n} \frac{(-1)^k}{(n-k)! (\ln \mu)^{k+1}} \\
&= \frac{(-1)^{n+1} n!}{(\ln \mu)^{n+1}} + n!\mu \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)! (\ln \mu)^{k+1}}.
\end{aligned}$$

The proof of lemma is thus accomplished. ■

Lemma 156 *If $\mu > 0$ and $n \in \mathbb{N} \cup \{0\}$, then*

$$E_2(n;\mu) := \int_0^1 (1-t)^n \mu^t dt = \begin{cases} \frac{n!\mu}{(\ln \mu)^{n+1}} - n! \sum_{k=0}^n \frac{1}{(n-k)!(\ln \mu)^{k+1}}, & \mu \neq 1 \\ \\ \frac{1}{n+1}, & \mu = 1. \end{cases} \tag{4.7}$$

Proof. By making the substitution $t = 1 - u$ in (4.5), in which μ is replaced by $\frac{1}{\mu}$, we get (4.7). ∎

Lemma 157 *If $\mu > 0$ and $n \in \mathbb{N} \cup \{0\}$, then*

$$G_1(n;\mu) := \int_0^{\frac{1}{2}} t^n \mu^t dt = \begin{cases} \frac{(-1)^{n+1} n!}{(\ln \mu)^{n+1}} + n! \mu^{1/2} \sum_{k=0}^{n} \frac{(-1)^k}{2^{n-k}(n-k)!(\ln \mu)^{k+1}}, & \mu \neq 1 \\ \\ \frac{1}{2^{n+1}(n+1)}, & \mu = 1. \end{cases} \tag{4.8}$$

Proof. It follows from Lemma 155 by making use of the substitution $t = \frac{u}{2}$. ∎

Lemma 158 *If $\mu > 0$ and $n \in \mathbb{N} \cup \{0\}$, then*

$$G_2(n;\mu) := \int_{\frac{1}{2}}^{1} (1-t)^n \mu^t dt$$

$$= \begin{cases} \frac{n!\mu}{(\ln \mu)^{n+1}} - n! \mu^{1/2} \sum_{k=0}^{n} \frac{1}{2^{n-k}(n-k)!(\ln \mu)^{k+1}} & \mu \neq 1 \\ \\ \frac{1}{2^{n+1}(n+1)}, & \mu = 1. \end{cases} \tag{4.9}$$

Proof. It follows from Lemma 157 by making the substitution $1-t = u$ and replacing μ by $\frac{1}{\mu}$. ∎

Lemma 159 *[165, 43] For $\alpha > 0$ and $\mu > 0$, we have*

$$I(\alpha;\mu) := \int_0^1 t^{\alpha-1} \mu^t dt = \begin{cases} \mu \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (\ln \mu)^{k-1}}{(\alpha)_k} < \infty, & \mu \neq 1 \\ \\ \frac{1}{\alpha}, & \mu = 1. \end{cases} \tag{4.10}$$

where

$$(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+k-1).$$

Moreover, it holds

$$\left| I\left(\alpha;\mu\right)-\mu\sum_{k=1}^{m}\left(-1\right)^{k-1}\frac{\left(\ln\mu\right)^{k-1}}{\left(\alpha\right)_{k}}\right|\leq\frac{\left|\ln\mu\right|}{\alpha\sqrt{2\pi\left(m-1\right)}}\left(\frac{\left|\ln\mu\right|e}{m-1}\right)^{m-1}.$$

Lemma 160 *[43, 165] For $\alpha>0$ and $\mu>0$, we have*

$$J\left(\alpha;\mu\right):=\int_{0}^{1}\left(1-t\right)^{\alpha-1}\mu^{t}dt=\begin{cases}\sum_{k=1}^{\infty}\frac{\left(\ln\mu\right)^{k-1}}{\left(\alpha\right)_{k}}<\infty, & \mu\neq1,\\ \\ \frac{1}{\alpha}, & \mu=1.\end{cases}\tag{4.11}$$

where

$$\left(\alpha\right)_{k}=\alpha\left(\alpha+1\right)\left(\alpha+2\right)...\left(\alpha+k-1\right).$$

Lemma 161 *If $\mu>0$ and $n\in\mathbb{N}$, then*

$$\begin{aligned}F\left(n;\mu\right)&:=nE_{1}\left(n-1;\mu\right)-2E_{1}\left(n;\mu\right)\\&=\begin{cases}\frac{\left(-1\right)^{n}n!\left[\ln\mu+2\right]}{\left(\ln\mu\right)^{n+1}}-\frac{2\mu}{\ln\mu}-n!\mu\sum_{k=1}^{n}\frac{\left(-1\right)^{k}\left[\ln\mu+2\right]}{\left(n-k\right)!\left(\ln\mu\right)^{k+1}}, & \mu\neq1,\\ \\ \frac{n-1}{n+1}, & \mu=1.\end{cases}\end{aligned}\tag{4.12}$$

Proof. The proof of (4.12) follows directly from Lemma 155. ■

Lemma 162 *If $\mu>0$ and $n\in\mathbb{N}$, then*

$$\begin{aligned}G\left(n;\mu\right)&:=nE_{2}\left(n-1;\mu\right)-E_{2}\left(n;\mu\right)\\&=\begin{cases}\frac{n!\mu\left[\ln\mu-1\right]}{\left(\ln\mu\right)^{n+1}}+\frac{1}{\ln\mu}-n!\sum_{k=1}^{n}\frac{\ln\mu-1}{\left(n-k\right)!\left(\ln\mu\right)^{k+1}}, & \mu\neq1,\\ \\ \frac{n}{n+1}, & \mu=1.\end{cases}\end{aligned}\tag{4.13}$$

Proof. The proof of (4.13) is a direct consequence of Lemma 156. ■

Lemma 163 *For $\alpha > 0$ and $\mu > 0$, we have*

$$H(\alpha;\mu) := nI(\alpha;\mu) - 2I(\alpha+1;\mu) = \begin{cases} \mu \sum_{k=1}^{\infty} \frac{[n(\alpha+k)-2\alpha](-\ln\mu)^{k-1}}{(\alpha)_{k+1}}, & \mu \neq 1, \\ \\ \frac{n(\alpha+1)-2\alpha}{\alpha(\alpha+1)}, & \mu = 1. \end{cases} \tag{4.14}$$

where

$$(\alpha)_{k+1} = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+k)$$

Proof. It can be easily proved by using Lemma 159. ∎

Lemma 164 *For $\alpha > 0$ and $\mu > 0$, we have*

$$K(\alpha;\mu) := nJ(\alpha;\mu) - J(\alpha+1;\mu) = \begin{cases} \sum_{k=1}^{\infty} \frac{[n(\alpha+k)-\alpha](-\ln\mu)^{k-1}}{(\alpha)_{k+1}}, & \mu \neq 1, \\ \\ \frac{n(\alpha+1)-\alpha}{\alpha(\alpha+1)}, & \mu = 1. \end{cases} \tag{4.15}$$

where

$$(\alpha)_{k+1} = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+k)$$

Proof. It can be easily proved by using Lemma 159. ∎

Lemma 165 *[182] Let $0 < \xi \leq 1 \leq \eta$, $0 \leq \lambda \leq 1$ and $0 < s \leq 1$. Then*

$$\xi^{\lambda^s} \leq \xi^{s\lambda} \text{ and } \eta^{\lambda^s} \leq \eta^{s\lambda+1-s}. \tag{4.16}$$

We are now ready to set off our first result.

Theorem 166 *Let $f : [c,d] \subseteq [0,\infty) \to (0,\infty)$ be a function which possess nth derivative $f^{(n)}$ on (c,d). Suppose that $f^{(n)}$ is in the space $L([a,b])$ of all integrable functions on $[a,b]$, where a, $b \in (c,d)$ with $a < b$. If $\left|f^{(n)}\right|^q$ is s-logarithmically convex on $[a,b]$,*

$1 \leq q < \infty$ *and for some fixed* s, $0 < s \leq 1$, $n \in \mathbb{N}$, $n \geq 2$, *the following inequality holds*

$$\left| A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx - \sum_{k=2}^{n-1} \frac{\left(k-1\right)\left(b-a\right)^k}{2\left(k+1\right)!} f^{(k)}(a) \right| \leq \frac{\left(b-a\right)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-1/q} \left|f^{(n)}(a)\right|^{\delta} \left|f^{(n)}(b)\right|^{\theta} \left[F\left(\mu;n\right)\right]^{1/q}, \quad (4.17)$$

where $\mu = \left|\frac{f^{(n)}(a)}{f^{(n)}(b)}\right|^{sq}$, $F\left(\mu;n\right)$ *is defined in Lemma 161 and*

$$\left(\delta,\theta\right) = \begin{cases} \left(0,s\right), & \textit{if } 0 < \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right| \leq 1, \\ \left(1-s,1\right), & \textit{if } 1 \leq \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right|, \\ \left(0,1\right), & \textit{if } 0 < \left|f^{(n)}(a)\right| \leq 1 < \left|f^{(n)}(b)\right|, \\ \left(1-s,s\right), & \textit{if } 0 < \left|f^{(n)}(b)\right| \leq 1 < \left|f^{(n)}(a)\right|. \end{cases}$$

Proof. Suppose $n \geq 2$. By Lemma 153, s-logarithmic convexity of $\left|f^{(n)}\right|^q$ on $[a,b]$ and Hölder inequality, we have

$$\left| A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx - \sum_{k=2}^{n-1} \frac{\left(k-1\right)\left(b-a\right)^k}{2\left(k+1\right)!} f^{(k)}(a) \right| \leq \frac{\left(b-a\right)^n}{2n!}\left(\int_0^1 t^{n-1}\left(n-2t\right)dt\right)^{1-1/q}\left(\int_0^1 t^{n-1}\left(n-2t\right)\left|f^{(n)}(a)\right|^{qt^s}\left|f^{(n)}(b)\right|^{q(1-t)^s}dt\right)^{1/q}. \quad (4.18)$$

If $0 < \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right| \leq 1$, from (4.16) and Lemma 161, we have

$$\begin{aligned} &\int_0^1 t^{n-1}\left(n-2t\right)\left|f^{(n)}(a)\right|^{qt^s}\left|f^{(n)}(b)\right|^{q(1-t)^s}dt \\ &\leq \int_0^1 t^{n-1}\left(n-2t\right)\left|f^{(n)}(a)\right|^{qst}\left|f^{(n)}(b)\right|^{sq(1-t)}dt \\ &= \left|f^{(n)}(b)\right|^{sq}\int_0^1 t^{n-1}\left(n-2t\right)\mu^t dt \\ &= \left|f^{(n)}(b)\right|^{sq}\left[nE_1\left(n-1;\mu\right) - 2E_1\left(n;\mu\right)\right] = \left|f^{(n)}(b)\right|^{sq} F\left(n;\mu\right), \quad (4.19) \end{aligned}$$

where $\mu = \left|\frac{f^{(n)}(a)}{f^{(n)}(b)}\right|^{sq}$.

If $1 \leq \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right|$, from (4.16) and by using Lemma 161, we have

$$\begin{aligned}
&\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(a)\right|^{qt^s}\left|f^{(n)}(b)\right|^{q(1-t)^s} dt \\
&\qquad \leq \left|f^{(n)}(a)\right|^{q(1-s)}\left|f^{(n)}(b)\right|^{q}\int_0^1 t^{n-1}(n-2t)\mu^t dt \\
&\qquad = \left|f^{(n)}(a)\right|^{q(1-s)}\left[nE_1(n-1;\mu)-2E_1(n;\mu)\right] \\
&\qquad\qquad = \left|f^{(n)}(a)\right|^{q(1-s)}\left|f^{(n)}(b)\right|^{q}F(n;\mu). \qquad (4.20)
\end{aligned}$$

If $0 < \left|f^{(n)}(a)\right| \leq 1 \leq \left|f^{(n)}(b)\right|$, from (4.16) and by Lemma 161, we obtain

$$\begin{aligned}
&\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(a)\right|^{qt^s}\left|f^{(n)}(b)\right|^{q(1-t)^s} dt \leq \left|f^{(n)}(b)\right|^{q}\int_0^1 t^{n-1}(n-2t)\mu^t dt \\
&\qquad = \left|f^{(n)}(b)\right|^{q}\left[nE_1(n-1;\mu)-2E_1(n;\mu)\right] = \left|f^{(n)}(b)\right|^{q}F(n;\mu). \qquad (4.21)
\end{aligned}$$

Lastly, if $0 < \left|f^{(n)}(b)\right| \leq 1 \leq \left|f^{(n)}(a)\right|$, from (4.16) and Lemma161, we get that

$$\begin{aligned}
&\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(a)\right|^{qt^s}\left|f^{(n)}(b)\right|^{q(1-t)^s} dt \\
&\qquad \leq \left|f^{(n)}(a)\right|^{q(1-s)}\left|f^{(n)}(b)\right|^{sq}\int_0^1 t^{n-1}(n-2t)\mu^t dt \\
&\qquad = \left|f^{(n)}(a)\right|^{q(1-s)}\left|f^{(n)}(b)\right|^{sq}\left[nE_1(n-1;\mu)-2E_1(n;\mu)\right] \\
&\qquad\qquad = \left|f^{(n)}(b)\right|^{sq}\left|f^{(n)}(a)\right|^{q(1-s)}F(n;\mu). \qquad (4.22)
\end{aligned}$$

Combining (4.18)-(4.22), we observe that the inequality (4.17) holds. Hence the anticipated result is achieved. ■

Corollary 167 *Suppose the assumptions of Theorem 166 are satisfied and if $q = 1$, we* *have the inequality*

$$\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx-\sum_{k=2}^{n-1}\frac{\left(k-1\right)\left(b-a\right)^k}{2\left(k+1\right)!}f^{(k)}(a)\right|$$
$$\leq\frac{\left(b-a\right)^n}{2n!}\left|f^{(n)}(a)\right|^{\delta}\left|f^{(n)}(b)\right|^{\theta}F\left(n;\mu\right), \quad (4.23)$$

where $F(n;\mu)$ is defined in Lemma 161, (δ,θ) and μ are defined in Theorem 166.

Corollary 168 *Under the assumptions of Theorem 193, if $n=2$, we have the inequalities*

$$\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right|$$
$$\leq\frac{\left(b-a\right)^2}{4}\left(\frac{1}{3}\right)^{1-1/q}\left|f''(a)\right|^{\delta}\left|f''(b)\right|^{\theta}\left[F\left(2;\mu\right)\right]^{1/q}, \quad (4.24)$$

where $\mu=\left|\frac{f''(a)}{f''(b)}\right|^{sq}$,

$$\left(\delta,\theta\right)=\begin{cases}\left(0,s\right), & \text{if } 0<\left|f''(a)\right|,\left|f''(b)\right|\leq 1,\\ \left(1-s,1\right), & \text{if } 1\leq\left|f''(a)\right|,\left|f''(b)\right|,\\ \left(0,1\right), & \text{if } 0<\left|f''(a)\right|\leq 1<\left|f''(b)\right|,\\ \left(1-s,s\right), & \text{if } 0<\left|f''(b)\right|\leq 1<\left|f''(a)\right|\end{cases}$$

and

$$F\left(2;\mu\right)=\begin{cases}\frac{2(1+\ln\mu)\ln\mu+4(1-\mu)}{(\ln\mu)^3}, & \mu\neq 1,\\ \\ \frac{1}{3}, & \mu=1.\end{cases}$$

Remark 169 *For $s=1$, one can get very interesting inequalities from (4.17), (4.23) and (4.24) for log-convex functions.*

Theorem 170 *Let $f:[c,d]\subseteq[0,\infty)\to(0,\infty)$ be a function which possess nth derivative $f^{(n)}$ on (c,d). Suppose that $f^{(n)}$ is in the space $L\left([a,b]\right)$ of all integrable functions on*

$[a,b]$, *where* $a,\ b\in(c,d)$ *with* $a<b$. *If* $\left|f^{(n)}\right|^{q}$ *is* s*-logarithmically convex on* $[a,b]$, $1<q<\infty$ *and for some fixed* s, $0<s\leq 1$, $n\in\mathbb{N}$, $n\geq 2$, *the following inequality holds*

$$\begin{aligned}
&\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx-\sum_{k=2}^{n-1}\frac{\left(k-1\right)\left(b-a\right)^{k}}{2\left(k+1\right)!}f^{(k)}(a)\right| \\
&\qquad\leq\frac{\left(b-a\right)^{n}\left[n^{(2q-1)/(q-1)}-\left(n-2\right)^{(2q-1)/(q-1)}\right]^{1-1/q}}{2^{2-1/q}n!} \\
&\qquad\times\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left|f^{(n)}(a)\right|^{\delta}\left|f^{(n)}(b)\right|^{\theta}\left[I\left(\left(n-1\right)q+1;\mu\right)\right]^{1/q}, \quad (4.25)
\end{aligned}$$

where μ *and* (δ,θ) *are defined as in Theorem 166 and* $I\left(\alpha;\mu\right)$ *is defined in Lemma 159.*

Proof. Since $\left|f^{(n)}\right|^{q}$ is s-logarithmically convex on $[a,b]$ for $q\in(1,\infty)$, $s\in(0,1]$, hence from Lemma 153 and the Hölder inequality, we have

$$\begin{aligned}
&\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx-\sum_{k=2}^{n-1}\frac{\left(k-1\right)\left(b-a\right)^{k}}{2\left(k+1\right)!}f^{(k)}(a)\right| \\
&\leq\frac{\left(b-a\right)^{n}}{2n!}\left(\int_{0}^{1}\left(n-2t\right)^{q/(q-1)}dt\right)^{1-1/q}\left(\int_{0}^{1}t^{q(n-1)}\left|f^{(n)}(ta+\left(1-t\right)b\right|^{q}dt\right)^{1/q} \\
&\leq\frac{\left(b-a\right)^{n}}{2^{2-1/q}n!}\left[n^{(2q-1)/(q-1)}-\left(n-2\right)^{(2q-1)/(q-1)}\right]^{1-1/q} \\
&\times\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left(\int_{0}^{1}t^{q(n-1)}\left|f^{(n)}\left(a\right)\right|^{qt^{s}}\left|f^{(n)}\left(b\right)\right|^{q(1-t)^{s}}dt\right)^{1/q}.
\end{aligned}$$

By using (4.16), Lemma 159 and using similar arguments as in proving Theorem 166, we have the inequality (4.25). This completes the proof of the theorem. ■

Corollary 171 *Taking into consideration the hypotheses of Theorem 170 and letting* $n=2$, *we obtain*

$$\begin{aligned}
&\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx\right| \\
&\qquad\leq\frac{\left(b-a\right)^{2}}{2}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left|f''(a)\right|^{\delta}\left|f''(b)\right|^{\theta}\left[I\left(q+1;\mu\right)\right]^{1/q}, \quad (4.26)
\end{aligned}$$

where μ and (δ,θ) are defined in Corollary 168 and

$$I(q+1;\mu)=\begin{cases}\mu\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(\ln\mu)^{k-1}}{(q+1)_k}<\infty, & \mu\neq 1,\\ \\ \frac{1}{q+1}, & \mu=1,\end{cases}$$

$$(q+1)_k=(q+1)(q+2)\cdots(q+k).$$

Corollary 172 *Meeting the assumptions of Theorem 170 for $n=2$ and $s=1$, gives*

$$\left|A(f(a),f(b))-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{2}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left|f''(b)\right|\left[I(q+1;\mu)\right]^{1/q}, \quad (4.27)$$

where $\mu=\left|\frac{f''(a)}{f''(b)}\right|^q$,

$$I(q+1;\mu)=\begin{cases}\mu\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(\ln\mu)^{k-1}}{(q+1)_k}<\infty, & \mu\neq 1,\\ \\ \frac{1}{q+1}, & \mu=1\end{cases}$$

and

$$(q+1)_k=(q+1)(q+2)\cdots(q+k).$$

Now we give some results related to left-side of Hermite-Hadamard's inequality for n-times differentiable s-logarithmically convex functions.

Theorem 173 *Let $f:[c,d]\subseteq[0,\infty)\to(0,\infty)$ be a function which possess nth derivative $f^{(n)}$ on (c,d). Suppose that $f^{(n)}$ is in the space $L([a,b])$ of all integrable functions on $[a,b]$, where $a,\ b\in(c,d)$ with $a<b$. If $\left|f^{(n)}\right|^q$ is s-logarithmically convex on $[a,b]$,*

$1 \leq q < \infty$ *and for some fixed* s, $0 < s \leq 1$, $n \in \mathbb{N}$, $n \geq 1$, *the following inequality holds*

$$\left| \sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1 \right] (b-a)^k}{2^{k+1} (k+1)!} f^{(k)} (A(a,b)) - \frac{1}{b-a} \int_a^b f(x)\, dx \right|$$
$$\leq \frac{(b-a)^n \left| f^{(n)}(a) \right|^{\delta} \left| f^{(n)}(b) \right|^{\theta}}{n! 2^{(n+1)(q-1)/q} (n+1)^{1-1/q}} \left\{ \left[G_1(n;\mu) \right]^{1/q} + \left[G_2(n;\mu) \right]^{1/q} \right\}, \quad (4.28)$$

where $\mu = \left| \frac{f^{(n)}(a)}{f^{(n)}(b)} \right|^{sq}$ *and* (δ, θ) *are defined as in Theorem 166, and* $G_1(n;\mu)$, $G_2(n;\mu)$ *are defined in Lemma 157 and Lemma 158 respectively.*

Proof. Suppose $n \geq 1$. By using Lemma 154, the s-logarithmically convexity of $\left| f^{(n)} \right|$ and the Hölder inequality, we have

$$\left| \sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1 \right] (b-a)^k}{2^{k+1} (k+1)!} f^{(k)} (A(a,b)) - \frac{1}{b-a} \int_a^b f(x)\, dx \right|$$
$$\leq \frac{(b-a)^n}{n!} \left[\left(\int_{\frac{1}{2}}^1 (1-t)^n\, dt \right)^{1-1/q} \left(\int_{\frac{1}{2}}^1 (1-t)^n \left| f^{(n)}(a) \right|^{qt^s} \left| f^{(n)}(b) \right|^{q(1-t)^s} dt \right)^{1/q} \right.$$
$$\left. + \left(\int_0^{\frac{1}{2}} t^n dt \right)^{1-1/q} \left(\int_0^{\frac{1}{2}} t^n \left| f^{(n)}(a) \right|^{qt^s} \left| f^{(n)}(b) \right|^{q(1-t)^s} dt \right)^{1/q} \right].$$

From (4.16), Lemma 157, Lemma 158 and the same reasoning as in proving Theorem 166, we have the required inequality (4.28). This accomplishes the proof of the required inequality. ■

Corollary 174 *If the hypotheses of Theorem 173 are fulfilled and if* $q = 1$, *we have*

$$\left| \sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1 \right] (b-a)^k}{2^{k+1} (k+1)!} f^{(k)} (A(a,b)) - \frac{1}{b-a} \int_a^b f(x)\, dx \right|$$
$$\leq \frac{(b-a)^n \left| f^{(n)}(a) \right|^{\delta} \left| f^{(n)}(b) \right|^{\theta}}{n!} \left\{ G_1(n;\mu) + G_2(n;\mu) \right\}, \quad (4.29)$$

where $\mu = \left|\frac{f^{(n)}(a)}{f^{(n)}(b)}\right|^{s}$, $G_1(n;\mu)$ *and* $G_2(n;\mu)$ *are defined in Lemma 157, Lemma 158 respectively and* (δ,θ) *are defined as in Theorem 166.*

Corollary 175 *Let the postulations of Theorem 173 are met and if* $s = 1$*, we have*

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^{k}+1\right](b-a)^{k}}{2^{k+1}(k+1)!}f^{(k)}(A(a,b)) - \frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)^{n}\left|f^{(n)}(b)\right|}{n!2^{(n+1)(q-1)/q}(n+1)^{1-1/q}}\left\{[G_1(n;\mu)]^{1/q} + [G_2(n;\mu)]^{1/q}\right\}, \quad (4.30)$$

where $\mu = \left|\frac{f^{(n)}(a)}{f^{(n)}(b)}\right|^{q}$ *and* $G_1(n;\mu)$ *and* $G_2(n;\mu)$ *are defined in Lemma 157, Lemma 158 respectively.*

Corollary 176 *According to suppositions of Theorem 173 and letting* $n = 1$*, gives us the following results*

$$\left|f(A(a,b)) - \frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)}{2^{3(1-1/q)}}\left|f'(a)\right|^{\delta}\left|f'(b)\right|^{\theta}\left\{[G_1(1;\mu)]^{1/q} + [G_2(1;\mu)]^{1/q}\right\}, \quad (4.31)$$

where $\mu = \left|\frac{f'(a)}{f'(b)}\right|^{sq}$,

$$G_1(1;\mu) = \begin{cases} \frac{2+\mu^{1/2}(\ln\mu-2)}{2(\ln\mu)^2}, & \mu \neq 1, \\ \\ \frac{1}{8}, & \mu = 1, \end{cases} \qquad G_2(1;\mu) = \begin{cases} \frac{2\mu-\mu^{1/2}(\ln\mu-2)}{2(\ln\mu)^2}, & \mu \neq 1, \\ \\ \frac{1}{8}, & \mu = 1, \end{cases}$$

and

$$(\delta,\theta) = \begin{cases} (0,s), & \text{if } 0 < \left|f'(a)\right|, \left|f'(b)\right| \leq 1, \\ (1-s,1), & \text{if } 1 \leq \left|f'(a)\right|, \left|f'(b)\right|, \\ (0,1), & \text{if } 0 < \left|f'(a)\right| \leq 1 \leq \left|f'(b)\right|, \\ (1-s,s), & \text{if } 0 < \left|f'(b)\right| \leq 1 \leq \left|f'(a)\right|. \end{cases}$$

Theorem 177 *Let $f:[c,d]\subseteq[0,\infty)\to(0,\infty)$ be a function which possess nth derivative $f^{(n)}$ on (c,d). Suppose that $f^{(n)}$ is in the space $L([a,b])$ of all integrable functions on $[a,b]$, where a, $b\in(c,d)$ with $a<b$. If $\left|f^{(n)}\right|^{q}$ is s-logarithmically convex on $[a,b]$, $1<q<\infty$ and for some fixed s, $0<s\leq 1$, $n\in\mathbb{N}$, $n\geq 1$, the following inequality holds*

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^{k}+1\right](b-a)^{k}}{2^{k+1}(k+1)!}f^{(k)}(A(a,b))-\frac{1}{b-a}\int_{a}^{b}f(x)\,dx\right|$$
$$\leq\frac{(b-a)^{n}\left|f^{(n)}(a)\right|^{\delta}\left|f^{(n)}(b)\right|^{\theta}}{2^{n+1/p}(np+1)^{1/p}n!}\left\{\left[G_{1}(0;\mu)\right]^{1/q}+\left[G_{2}(0;\mu)\right]^{1/q}\right\},\quad(4.32)$$

where

$$G_{1}(0;\mu)=\begin{cases}\frac{\mu^{1/2}-1}{\ln\mu}, & \mu\neq 1,\\ \frac{1}{2}, & \mu=1,\end{cases}\qquad G_{2}(0;\mu)=\begin{cases}\frac{\mu-\mu^{1/2}}{\ln\mu}, & \mu\neq 1,\\ \frac{1}{2}, & \mu=1,\end{cases}$$

μ and (δ,θ) are defined in Theorem 166 and p and q are Hölder conjugates of each other.

Proof. From Lemma 154, the Hölder integral inequality and s-logarithmic convexity of $\left|f^{(n)}\right|^{q}$ on $[a,b]$, we have

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^{k}+1\right](b-a)^{k}}{2^{k+1}(k+1)!}f^{(k)}(A(a,b))-\frac{1}{b-a}\int_{a}^{b}f(x)\,dx\right|$$
$$\leq\frac{(b-a)^{n}}{n!}\left[\left(\int_{0}^{\frac{1}{2}}t^{np}dt\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{1}{2}}\left|f^{(n)}(a)\right|^{qt^{s}}\left|f^{(n)}(b)\right|^{q(1-t)^{s}}dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^{1}(1-t)^{np}\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left|f^{(n)}(a)\right|^{qt^{s}}\left|f^{(n)}(b)\right|^{q(1-t)^{s}}dt\right)^{\frac{1}{q}}\right].$$

Using (4.16) and similar arguments as in proving Theorem 166, we get (4.32). The proof of the theorem is completed. ■

Corollary 178 *Under the assumptions of Theorem 177, if $n = 1$, we have the inequality*

$$\left| f\left(A\left(a,b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx \right| \\ \leq \frac{\left(b-a\right)\left|f'\left(a\right)\right|^{\delta}\left|f'\left(b\right)\right|^{\theta}}{2^{1+1/p}\left(p+1\right)^{1/p}}\left\{\left[G_1\left(0;\mu\right)\right]^{1/q} + \left[G_2\left(0;\mu\right)\right]^{1/q}\right\}, \quad (4.33)$$

where $G_1(0;\mu)$ and $G_2(0;\mu)$ are defined in Theorem 177, (δ,θ) and μ are defined as in Corollary 176 and $\frac{1}{p} + \frac{1}{q} = 1$.

Corollary 179 *With the assumptions of Theorem 177, if $s = 1$, we have the inequality*

$$\left|\sum_{k=0}^{n-1}\frac{\left[\left(-1\right)^k + 1\right]\left(b-a\right)^k}{2^{k+1}\left(k+1\right)!}f^{(k)}\left(A\left(a,b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx\right| \\ \leq \frac{\left(b-a\right)^n\left|f^{(n)}\left(b\right)\right|}{2^{n+1/p}\left(np+1\right)^{1/p}n!}\left\{\left[G_1\left(0;\mu\right)\right]^{1/q} + \left[G_2\left(0;\mu\right)\right]^{1/q}\right\}, \quad (4.34)$$

where $G_1(0;\mu)$ and $G_2(0;\mu)$ are defined in Corollary 176, $\mu = \left|\frac{f^{(n)}(a)}{f^{(n)}(b)}\right|^q$ and p and q are Hölder conjugates of each other.

4.3 More on new inequalities of H-H type for n-times differentiable s-logarithmically convex functions

The following Lemma 96 and the lemmas of Section 4.2 are used to establish some new integral inequalities of Hermite-Hadamard type for n-times differentiable s-logarithmically convex functions in this section.

Theorem 180 *Let $f : [c,d] \subseteq [0,\infty) \to (0,\infty)$ be a function which possess nth derivative $f^{(n)}$ on (c,d). Suppose that $f^{(n)}$ is in the space $L([a,b])$ of all integrable functions on $[a,b]$, where a, $b \in (c,d)$ with $a < b$. If $\left|f^{(n)}\right|^q$ is s-logarithmically convex on $[a,b]$,*

$1 \leq q < \infty$ and for some fixed s, $0 < s \leq 1$, $n \in \mathbb{N}$, $n \geq 1$, the following inequality holds

$$\left| A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx - \sum_{k=1}^{n-1} \frac{k\left[1+\left(-1\right)^k\right]\left(b-a\right)^k}{2^{k+1}\left(k+1\right)!} f^{(k)}\left(A\left(a,b\right)\right)\right|$$
$$\leq \frac{\left(b-a\right)^n}{2^{n+1}n!}\left(\frac{n}{n+1}\right)^{1-\frac{1}{q}} \left|f^{(n)}\left(a\right)\right|^{\delta}\left|f^{(n)}\left(b\right)\right|^{\theta}\left\{\left[G\left(n;\mu\right)\right]^{\frac{1}{q}} + \left[G\left(n;\mu^{-1}\right)\right]^{\frac{1}{q}}\right\}, \quad (4.35)$$

where $\mu = \left|\frac{f^{(n)}(b)}{f^{(n)}(a)}\right|^{sq/2}$,

$$\left(\delta,\theta\right) = \begin{cases} \left(s/2, s/2\right), & \text{if } 0 < \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right| \leq 1, \\ \left(1-s/2, 1-s/2\right), & \text{if } 1 \leq \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right|, \\ \left(s/2, 1-s/2\right), & \text{if } 0 < \left|f^{(n)}(a)\right| \leq 1 \leq \left|f^{(n)}(b)\right|, \\ \left(1-s/2, s/2\right) & \text{if } 0 < \left|f^{(n)}(b)\right| \leq 1 \leq \left|f^{(n)}(a)\right|, \end{cases}$$

and $G\left(n;\mu\right)$ is defined in Lemma 162.

Proof. From Lemma 96, the Hölder inequality and using the fact that $\left|f^{(n)}\right|^q$ is s-logarithmically convex on $[a,b]$, we have

$$\left| A\left(f\left(a\right), f\left(b\right)\right) - \sum_{k=1}^{n-1} \frac{k\left[1+\left(-1\right)^k\right]\left(b-a\right)^k}{2^{k+1}\left(k+1\right)!} f^{(k)}\left(A\left(a,b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx\right|$$
$$\leq \frac{\left(b-a\right)^n}{2^{n+1}n!}\left(\int_0^1 \left(1-t\right)^{n-1}\left(n-1+t\right)dt\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left(\int_0^1 \left(1-t\right)^{n-1}\left(n-1+t\right)\left|f^{(n)}\left(a\right)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}\left(b\right)\right|^{q\left(\frac{1+t}{2}\right)^s}dt\right)^{1/q}\right.$$
$$\left.+\left(\int_0^1 \left(1-t\right)^{n-1}\left(n-1+t\right)\left|f^{(n)}\left(b\right)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}\left(a\right)\right|^{q\left(\frac{1+t}{2}\right)^s}dt\right)^{1/q}\right\}. \quad (4.36)$$

When $0 < \left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right| \leq 1$, by using Lemma 162 and (4.16), we have

$$\begin{aligned}
&\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(a)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(b)\right|^{q\left(\frac{1+t}{2}\right)^s}dt \\
&\quad\leq \int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(a)\right|^{sq\left(\frac{1-t}{2}\right)}\left|f^{(n)}(b)\right|^{sq\left(\frac{1+t}{2}\right)}dt \\
&\quad= \left|f^{(n)}(a)f^{(n)}(b)\right|^{sq/2}\int_0^1 (1-t)^{n-1}(n-1+t)\mu^t dt \\
&\quad= \left|f^{(n)}(a)f^{(n)}(b)\right|^{sq/2} G(n;\mu). \qquad (4.37)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(b)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(a)\right|^{q\left(\frac{1+t}{2}\right)^s}dt \\
&\quad\leq \int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(b)\right|^{sq\left(\frac{1-t}{2}\right)}\left|f^{(n)}(a)\right|^{sq\left(\frac{1+t}{2}\right)}dt \\
&\quad= \left|f^{(n)}(a)f^{(n)}(b)\right|^{sq/2}\int_0^1 (1-t)^{n-1}(n-1+t)\mu^{-t} dt \\
&\quad= \left|f^{(n)}(a)f^{(n)}(b)\right|^{sq/2} G\left(n;\mu^{-1}\right). \qquad (4.38)
\end{aligned}$$

When $\left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right| \geq 1$, by using Lemma 162 and (4.16), we have

$$\begin{aligned}
&\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(a)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(b)\right|^{q\left(\frac{1+t}{2}\right)^s}dt \\
&\quad\leq \left|f^{(n)}(a)f^{(n)}(b)\right|^{q(1-s/2)}\int_0^1 (1-t)^{n-1}(n-1+t)\mu^t dt \\
&\quad= \left|f^{(n)}(a)f^{(n)}(b)\right|^{q(1-s/2)} G(n;\mu). \qquad (4.39)
\end{aligned}$$

and

$$\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(b)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(a)\right|^{q\left(\frac{1+t}{2}\right)^s}dt$$
$$\leq \left|f^{(n)}(a)f^{(n)}(b)\right|^{q(1-s/2)}\int_0^1 (1-t)^{n-1}(n-1+t)\mu^{-t}dt$$
$$= \left|f^{(n)}(a)f^{(n)}(b)\right|^{q(1-s/2)}G\left(n;\mu^{-1}\right). \quad (4.40)$$

When $0<\left|f^{(n)}(a)\right|\leq 1\leq\left|f^{(n)}(b)\right|$, by using Lemma 162 and (4.16), we have

$$\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(a)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(b)\right|^{q\left(\frac{1+t}{2}\right)^s}dt$$
$$\leq \left|f^{(n)}(a)\right|^{sq/2}\left|f^{(n)}(b)\right|^{q(1-1/2s)}\int_0^1 (1-t)^{n-1}(n-1+t)\mu^{t}dt$$
$$= \left|f^{(n)}(a)\right|^{sq/2}\left|f^{(n)}(b)\right|^{q(1-s/2)}G(n;\mu). \quad (4.41)$$

and

$$\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(b)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(a)\right|^{q\left(\frac{1+t}{2}\right)^s}dt$$
$$\leq \left|f^{(n)}(a)\right|^{sq/2}\left|f^{(n)}(b)\right|^{q(1-1/2s)}\int_0^1 (1-t)^{n-1}(n-1+t)\mu^{-t}dt$$
$$= \left|f^{(n)}(a)\right|^{q/2s}\left|f^{(n)}(b)\right|^{q(1-s/2)}G\left(n;\mu^{-1}\right). \quad (4.42)$$

When $0<\left|f^{(n)}(b)\right|\leq 1\leq\left|f^{(n)}(a)\right|$, by using Lemma 162 and (4.16), we have

$$\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(a)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(b)\right|^{q\left(\frac{1+t}{2}\right)^s}dt$$
$$\leq \left|f^{(n)}(a)\right|^{q(1-s/2)}\left|f^{(n)}(b)\right|^{sq/2}\int_0^1 (1-t)^{n-1}(n-1+t)\mu^{t}dt$$
$$= \left|f^{(n)}(a)\right|^{q(1-s/2)}\left|f^{(n)}(b)\right|^{sq/2}G(n;\mu). \quad (4.43)$$

and

$$\int_0^1 (1-t)^{n-1}(n-1+t)\left|f^{(n)}(b)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}(a)\right|^{q\left(\frac{1+t}{2}\right)^s}dt$$
$$\leq \left|f^{(n)}(a)\right|^{q(1-s/2)}\left|f^{(n)}(b)\right|^{sq/2}\int_0^1 (1-t)^{n-1}(n-1+t)\mu^{-t}dt$$
$$= \left|f^{(n)}(a)\right|^{q(1-s/2)}\left|f^{(n)}(b)\right|^{sq/2} G\left(n;\mu^{-1}\right), \quad (4.44)$$

where $\mu=\left|\frac{f^{(n)}(b)}{f^{(n)}(a)}\right|^{sq/2}$. A combination of (4.37)-(4.44) into (4.36) gives the desired result. The proof of the theorem is thus accomplished. ■

Corollary 181 *Under the assumptions of Theorem 180, if $q=1$, we have the inequality*

$$\left|A(f(a),f(b))-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}(A(a,b))\right|$$
$$\leq \frac{(b-a)^n}{2^{n+1}n!}\left|f^{(n)}(a)\right|^{\delta}\left|f^{(n)}(b)\right|^{\theta}\left\{G(n;\mu)+G\left(n;\mu^{-1}\right)\right\}, \quad (4.45)$$

where $G(n;\mu)$ is defined in Lemma 162, μ and (δ,θ) are defined in Theorem 180.

Corollary 182 *Taking the assumptions of Theorem 180 into account and letting $n=1$, delivers the following result*

$$\left|A(f(a),f(b))-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq (b-a)\left(\frac{1}{2}\right)^{3-\frac{1}{q}}\left|f'(a)\right|^{\delta}\left|f'(b)\right|^{\theta}\left\{[G(1;\mu)]^{\frac{1}{q}}+\left[G\left(1;\mu^{-1}\right)\right]^{\frac{1}{q}}\right\}, \quad (4.46)$$

where

$$G(1;\mu)=\begin{cases}\frac{\mu(\ln\mu-1)+1}{(\ln\mu)^2}, & \mu\neq 1\\ \frac{1}{2}, & \mu=1\end{cases}, \ \mu=\left|\frac{f'(b)}{f'(a)}\right|^{sq/2}$$

and

$$(\delta,\theta)=\begin{cases}(s/2,s/2), & \text{if } 0<\left|f'(a)\right|,\left|f'(b)\right|\leq 1,\\ (1-s/2,1-s/2), & \text{if } 1\leq\left|f'(a)\right|,\left|f'(b)\right|,\\ (s/2,1-s/2), & \text{if } 0<\left|f'(a)\right|\leq 1\leq\left|f'(b)\right|,\\ (1-s/2,s/2) & \text{if } 0<\left|f'(b)\right|\leq 1\leq\left|f'(a)\right|.\end{cases}$$

Corollary 183 *If we take $q=1$ in Corollary 182, we have*

$$\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right| \leq\left(\frac{b-a}{4}\right)\left|f'\left(a\right)\right|^{\delta}\left|f'\left(b\right)\right|^{\theta}\left\{G\left(1;\mu\right)+G\left(1;\mu^{-1}\right)\right\}, \quad (4.47)$$

where $G\left(1;\mu\right)$, μ and (δ,θ) are defined in Corollary 182.

Corollary 184 *Suppose the assumptions of Theorem 180 are fulfilled and if $n=2$, we have*

$$\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right| \leq\frac{\left(b-a\right)^2}{16}\left(\frac{2}{3}\right)^{1-\frac{1}{q}}\left|f''\left(a\right)\right|^{\delta}\left|f''\left(b\right)\right|^{\theta}\left\{\left[G\left(2;\mu\right)\right]^{\frac{1}{q}}+\left[G\left(2;\mu^{-1}\right)\right]^{\frac{1}{q}}\right\}, \quad (4.48)$$

where

$$G\left(2;\mu\right)=\begin{cases}\frac{2\mu(\ln\mu-1)-(\ln\mu)^2+2}{(\ln\mu)^3}, & \mu\neq 1,\\ \frac{2}{3}, & \mu=1,\end{cases}\ ;\ \mu=\left|\frac{f''(b)}{f''(a)}\right|^{sq/2}$$

and

$$(\delta,\theta)=\begin{cases}(s/2,s/2), & \text{if } 0<\left|f''(a)\right|,\left|f''(b)\right|\leq 1,\\ (1-s/2,1-s/2), & \text{if } 1\leq\left|f''(a)\right|,\left|f''(b)\right|,\\ (s/2,1-s/2), & \text{if } 0<\left|f''(a)\right|\leq 1\leq\left|f''(b)\right|,\\ (1-s/2,s/2) & \text{if } 0<\left|f''(b)\right|\leq 1\leq\left|f''(a)\right|.\end{cases}$$

Corollary 185 *If $q=1$ in Corollary 184, we have*

$$\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx\right| \\ \leq\frac{\left(b-a\right)^{2}}{16}\left|f''\left(a\right)\right|^{\delta}\left|f''\left(b\right)\right|^{\theta}\left\{G\left(2;\mu\right)+G\left(2;\mu^{-1}\right)\right\}, \quad (4.49)$$

where $F_1\left(\nu,2\right)$, μ and $\left(\delta,\theta\right)$ are defined as in Corollary 184.

Theorem 186 *Let $f:[c,d]\subseteq[0,\infty)\rightarrow(0,\infty)$ be a function which possess nth derivative $f^{(n)}$ on (c,d). Suppose that $f^{(n)}$ is in the space $L\left([a,b]\right)$ of all integrable functions on $[a,b]$, where a, $b\in(c,d)$ with $a<b$. If $\left|f^{(n)}\right|^{q}$ is s-logarithmically convex on $[a,b]$, $1<q<\infty$ and for some fixed s, $0<s\leq1$, $n\in\mathbb{N}$, $n\geq1$, the following inequality holds*

$$\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx-\sum_{k=1}^{n-1}\frac{k\left[1+\left(-1\right)^{k}\right]\left(b-a\right)^{k}}{2^{k+1}\left(k+1\right)!}f^{(k)}\left(A\left(a,b\right)\right)\right| \\ \leq\frac{\left(b-a\right)^{n}\left[n^{(2q-1)/(q-1)}-\left(n-1\right)^{(2q-1)/(q-1)}\right]^{1-1/q}}{2^{n+1}n!}\left(\frac{q-1}{2q-1}\right)^{1-1/q} \\ \times\left|f^{(n)}\left(a\right)\right|^{\delta}\left|f^{(n)}\left(b\right)\right|^{\theta}\left\{\left[J\left(\left(n-1\right)q+1;\mu\right)\right]^{\frac{1}{q}}+\left[J\left(\left(n-1\right)q+1;\mu^{-1}\right)\right]^{\frac{1}{q}}\right\}, \quad (4.50)$$

where $J\left(\alpha,\mu\right)$ is defined in Lemma 160, μ and $\left(\delta,\theta\right)$ are defined in Theorem 180.

Proof. Using Lemma 96, the Hölder inequality and the s-logarithmic convexity of $\left|f^{(n)}\right|^q$ on $[a,b]$, we have

$$\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right.$$
$$\left.-\sum_{k=1}^{n-1}\frac{k\left[1+\left(-1\right)^k\right]\left(b-a\right)^k}{2^{k+1}\left(k+1\right)!}f^{(k)}\left(A\left(a,b\right)\right)\right|\leq\frac{\left(b-a\right)^n}{2^{n+1}n!}\left(\int_0^1\left(n-1+t\right)^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left(\int_0^1\left(1-t\right)^{q(n-1)}\left[\left|f^{(n)}\left(a\right)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}\left(b\right)\right|^{q\left(\frac{1+t}{2}\right)^s}\right]dt\right)^{1/q}\right.$$
$$\left.+\left(\int_0^1\left(1-t\right)^{q(n-1)}\left[\left|f^{(n)}\left(b\right)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}\left(a\right)\right|^{q\left(\frac{1+t}{2}\right)^s}\right]dt\right)^{1/q}\right\}. \quad (4.51)$$

The proof follows by using Lemma 165 and by using similar arguments as in proving Theorem 180. ■

Corollary 187 *Under the assumptions of Theorem 186, if $n=1$, we have the inequality*

$$\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right|$$
$$\leq\frac{\left(b-a\right)}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left|f'\left(a\right)\right|^{\delta}\left|f'\left(b\right)\right|^{\theta}\left\{\left[J\left(1;\mu\right)\right]^{\frac{1}{q}}+\left[J\left(1;\mu^{-1}\right)\right]^{\frac{1}{q}}\right\}, \quad (4.52)$$

where

$$J\left(1;\mu\right)=\begin{cases}\sum_{k=1}^{\infty}\frac{\left(\ln\mu\right)^{k-1}}{k!}<\infty, & \mu\neq1,\\ 1, & \mu=1,\end{cases}$$

μ and (δ,θ) is defined as in Corollary 182.

Corollary 188 *Under the assumptions of Theorem 186, if $n = 2$, we have the inequality*

$$\begin{aligned}
&\left| A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx \right| \\
&\quad \leq \frac{\left(b-a\right)^2 \left[2^{(2q-1)/(q-1)} - 1\right]^{1-1/q}}{16} \left(\frac{q-1}{2q-1}\right)^{1-1/q} \\
&\quad \times \left|f''\left(a\right)\right|^{\delta} \left|f''\left(b\right)\right|^{\theta} \left\{ \left[J\left(q+1;\mu\right)\right]^{\frac{1}{q}} + \left[J\left(q+1;\mu^{-1}\right)\right]^{\frac{1}{q}} \right\}, \quad (4.53)
\end{aligned}$$

where

$$J\left(q+1;\mu\right) = \begin{cases} \sum_{k=1}^{\infty} \frac{(\ln\mu)^{k-1}}{(q+1)_k} < \infty, & \mu \neq 1, \\ \frac{1}{q+1}, & \mu = 1, \end{cases}$$

$$\left(q+1\right)_k = \left(q+1\right)\left(q+2\right)...\left(q+k\right),$$

μ and (δ, θ) are defined as in Corollary 184.

Theorem 189 *Let $f : [c, d] \subseteq [0, \infty) \to (0, \infty)$ be a function which possess nth derivative $f^{(n)}$ on (c, d). Suppose that $f^{(n)}$ is in the space $L([a, b])$ of all integrable functions on $[a, b]$, where a, $b \in (c, d)$ with $a < b$. If $\left|f^{(n)}\right|^q$ is s-logarithmically convex on $[a, b]$, $1 < q < \infty$ and for some fixed s, $0 < s \leq 1$, $n \in \mathbb{N}$, $n \geq 1$, the following inequality holds*

$$\begin{aligned}
&\left| A\left(f\left(a\right), f\left(b\right)\right) - \sum_{k=1}^{n-1} \frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(A\left(a,b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx \right| \\
&\quad \leq \frac{n^{n+1-\frac{1}{q}}\left(b-a\right)^n}{2^{n+1}n!} \left[B\left(\frac{1}{n}; \frac{nq-1}{q-1}, \frac{2q-1}{q-1}\right)\right]^{1-\frac{1}{q}} \\
&\quad \times \left|f^{(n)}\left(a\right)\right|^{\delta} \left|f^{(n)}\left(b\right)\right|^{\theta} \left\{ \left[E_2\left(0;\mu\right)\right]^{\frac{1}{q}} + \left[E_2\left(0;\mu^{-1}\right)\right]^{\frac{1}{q}} \right\}, \quad (4.54)
\end{aligned}$$

where

$$E_2\left(0;\mu\right) = \begin{cases} \frac{\mu-1}{\ln\mu}, & \mu \neq 1, \\ 1, & \mu = 1, \end{cases}, \mu = \left|\frac{f^{(n)}(b)}{f^{(n)}(a)}\right|^{sq/2},$$

$$B\left(z;\alpha,\beta\right) = \int_0^z t^{\alpha-1}\left(1-t\right)^{\beta-1}dt, 0 \leq z \leq 1, \alpha > 0, \beta > 0$$

is the incomplete Beta function and (δ, θ) are defined as in Theorem 180.

Proof. Using Lemma 96, the Hölder inequality and the s-logarithmic convexity of $\left|f^{(n)}\right|^q$ on $[a, b]$, we have

$$\begin{aligned}
&\left|A\left(f\left(a\right), f\left(b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx-\sum_{k=1}^{n-1}\frac{k\left[1+\left(-1\right)^k\right]\left(b-a\right)^k}{2^{k+1}\left(k+1\right)!}f^{(k)}\left(A\left(a,b\right)\right)\right| \\
&\qquad\leq\frac{\left(b-a\right)^n}{2^{n+1}n!}\left(\int_0^1\left(1-t\right)^{q(n-1)/(q-1)}\left(n-1+t\right)^{q/(q-1)}dt\right)^{1-1/q} \\
&\qquad\quad\times\left\{\left(\int_0^1\left|f^{(n)}\left(a\right)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}\left(b\right)\right|^{q\left(\frac{1+t}{2}\right)^s}dt\right)^{1/q}\right. \\
&\qquad\qquad\left.+\left(\int_0^1\left|f^{(n)}\left(b\right)\right|^{q\left(\frac{1-t}{2}\right)^s}\left|f^{(n)}\left(a\right)\right|^{q\left(\frac{1+t}{2}\right)^s}dt\right)^{1/q}\right\}. \qquad (4.55)
\end{aligned}$$

By using (4.16) and the fact that

$$\begin{aligned}
&\int_0^1\left(1-t\right)^{q(n-1)/(q-1)}\left(n-1+t\right)^{q/(q-1)}dt \\
&=n^{\frac{nq+q-1}{q-1}}\int_0^{\frac{1}{n}}t^{\frac{(n-1)q}{q-1}}\left(1-t\right)^{\frac{q}{q-1}}dt=n^{\frac{nq+q-1}{q-1}}B\left(\frac{1}{n};\frac{nq-1}{q-1},\frac{2q-1}{q-1}\right),
\end{aligned}$$

we get the required inequality (4.54) from (4.55). ■

Corollary 190 *Considering that suppositions of Theorem 189 are met and letting $n=1$, we have the inequality*

$$\begin{aligned}
&\left|A\left(f\left(a\right), f\left(b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right| \\
&\qquad\leq\frac{\left(b-a\right)}{4}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left|f'\left(a\right)\right|^{\delta}\left|f'\left(b\right)\right|^{\theta}\left\{\left[E_2\left(0;\mu\right)\right]^{\frac{1}{q}}+\left[E_2\left(0;\mu^{-1}\right)\right]^{\frac{1}{q}}\right\}, \qquad (4.56)
\end{aligned}$$

where $E_2(0;\mu)$ is defined in Theorem 189 and μ, (δ, θ) are defined as in Corollary 182.

Corollary 191 *Assuming that the postulations of Theorem 189 are pleased and letting $n = 2$, we obtain the following result*

$$\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx\right|\leq\frac{\left(b-a\right)^{2}}{2^{1+1/q}}\left[B\left(\frac{1}{2};\frac{2q-1}{q-1},\frac{2q-1}{q-1}\right)\right]^{1-\frac{1}{q}}$$
$$\times\left|f''\left(a\right)\right|^{\delta}\left|f''\left(b\right)\right|^{\theta}\left\{\left[E_{2}\left(0;\mu\right)\right]^{\frac{1}{q}}+\left[E_{2}\left(0;\mu^{-1}\right)\right]^{\frac{1}{q}}\right\}, \quad (4.57)$$

where $B\left(z;\alpha,\beta\right)$, $E_{2}\left(0;\mu\right)$ are defined in Theorem 189 and μ, $\left(\delta,\theta\right)$ are defined as in Corollary 184.

Remark 192 *We can get several interesting inequalities for log-convex functions by setting $s = 1$ in the results of this section. However, the details are left to the interested reader.*

4.4 Applications to Special Means

In what follows we will use the above means and the established results of the previous section to obtain some interesting inequalities involving means.

Theorem 193 *Let $0 < a < b \leq 1$, $r < 0$, $r \neq -1, -2$, $s \in (0,1]$ and $q \geq 1$.*

1. *If $r \neq -3$, then*

$$\left|A\left(a^{r+2},b^{r+2}\right)-\left[L_{r+2}\left(a,b\right)\right]^{r+2}\right|$$
$$\leq\frac{\left(b-a\right)^{2}}{4}\left(\frac{1}{3}\right)^{1-1/q}\left|\left(r+2\right)\left(r+1\right)\right|\left[\frac{2G\left(a^{rq(1-s)},b^{rq(1-s)}\right)}{rqs\left(\ln b-\ln a\right)}\right]^{2/q}$$
$$\times\left[A\left(a^{rqs},b^{rqs}\right)-L\left(a^{rqs},b^{rqs}\right)\right]^{1/q}.$$

2. If $r=-3$, then

$$\left|\frac{1}{H(a,b)}-\frac{1}{L(a,b)}\right| \leq \frac{(b-a)^2}{2}\left(\frac{1}{3}\right)^{1-1/q}\left[\frac{2G\left(a^{-3q(1-s)},b^{-3q(1-s)}\right)}{3qs(\ln a-\ln b)}\right]^{2/q}$$
$$\times\left[A\left(a^{-3qs},b^{-3qs}\right)-L\left(a^{-3qs},b^{-3qs}\right)\right]^{1/q}.$$

Proof. Let $f(x)=\frac{x^{r+2}}{(r+2)(r+1)}$ for $0<x\leq 1$. Then $\left|f''(x)\right|=x^r$ and

$$\ln\left|f''(A_\lambda(x,y))\right|^q \leq \lambda^s \ln\left|f''(x)\right|^q+(1-\lambda)^s\ln\left|f''(y)\right|^q$$

for x, $y\in(0,1]$, $\lambda\in[0,1]$, $s\in(0,1]$ and $q\geq 1$. This shows that $\left|f''(x)\right|^q=x^{rq}$ is s-logarithmically convex function on $(0,1]$. Since$\left|f''(a)\right|>\left|f''(b)\right|=b^r\geq 1$, hence

$$\mu=\left|\frac{f''(a)}{f''(b)}\right|^{qs}=\left(\frac{a}{b}\right)^{rqs}$$

and

$$\begin{aligned}
&\left|f''(b)\right|^q\left|f''(a)\right|^{q(1-s)}F_1(\mu,2)\\
&=2a^{rq(1-s)}b^{rq(1-s)}\left[\frac{rqs\left(a^{rqs}+b^{rqs}\right)(\ln a-\ln b)+2\left(b^{rqs}-a^{rqs}\right)}{r^3q^3s^3(\ln a-\ln b)^3}\right]\\
&=\left[\frac{4a^{rq(1-s)}b^{rq(1-s)}}{r^2q^2s^2(\ln a-\ln b)^2}\right]\left[\frac{a^{rqs}+b^{rqs}}{2}-\frac{b^{rqs}-a^{rqs}}{rqs(\ln b-\ln a)}\right]\\
&=\left[\frac{2G\left(a^{rq(1-s)},b^{rq(1-s)}\right)}{rqs(\ln b-\ln a)}\right]^2\left[A\left(a^{rqs},b^{rqs}\right)-L\left(a^{rqs},b^{rqs}\right)\right].
\end{aligned}$$

Substituting the above quantities in Corollary 168, we get the required inequality. ■

Remark 194 *The other results given above may also give very interesting inequalities containing means and the details are left to the interested reader.*

In what follows we will use the above means and the established results of the previous section to obtain some interesting inequalities involving means.

Theorem 195 *Let* $0 < a < b \leq 1$, $r < 0$, $r \neq -1$, $s \in (0, 1]$ *and* $q \geq 1$.

1. *If* $r \neq -2$, *then*

$$\left|A\left(a^{r+1}, b^{r+1}\right) - \left[L_{r+1}(a, b)\right]^{r+1}\right| \leq (b-a)\left(\frac{1}{2}\right)^{3-\frac{1}{q}} |r+1| \left[G\left(a^{r}, b^{r}\right)\right]^{2}$$
$$\times \left\{ a^{-rs}\left[\frac{2\left(1 - b^{-rqs/2} L\left(a^{rqs/2}, b^{rqs/2}\right)\right)}{rqs(\ln b - \ln a)}\right]^{1/q} \right.$$
$$\left. + b^{-rs}\left[\frac{2\left(a^{-rqs/2} L\left(a^{rqs/2}, b^{rqs/2}\right) - 1\right)}{rqs(\ln b - \ln a)}\right]^{1/q} \right\}.$$

2. *If* $r = -2$, *then*

$$\left|\frac{1}{H(a, b)} - \frac{1}{L(a, b)}\right| \leq (b-a)\left(\frac{1}{2}\right)^{3-\frac{1}{q}} \left[G\left(a^{-2}, b^{-2}\right)\right]^{2}$$
$$\times \left\{ a^{2s}\left[\frac{1 - b^{qs} L\left(a^{-qs}, b^{-qs}\right)}{qs(\ln a - \ln b)}\right]^{1/q} + b^{2s}\left[\frac{a^{qs} L\left(a^{-qs}, b^{-qs}\right) - 1}{qs(\ln a - \ln b)}\right]^{1/q} \right\}.$$

Proof. Let $f(x) = \frac{x^{r+1}}{r+1}$ for $0 < x \leq 1$. Then $|f'(x)| = x^{r}$ and

$$\ln\left|f'\left(A_{\lambda}(x, y)\right)\right|^{q} \leq \lambda^{s} \ln\left|f'(x)\right|^{q} + (1-\lambda)^{s} \ln\left|f'(y)\right|^{q}$$

for x, $y \in (0, 1]$, $\lambda \in [0, 1]$, $s \in (0, 1]$ and $q \geq 1$. This shows that $|f'(x)|^{q} = x^{rq}$ is s-logarithmically convex function on $(0, 1]$. Since$|f'(a)| > |f'(b)| = b^{r} \geq 1$, hence

$$\mu = \left|\frac{f'(a)}{f'(b)}\right|^{sq/2} = \left(\frac{b}{a}\right)^{rqs/2}, \ \mu^{-1} = \left(\frac{a}{b}\right)^{rqs/2}$$

and

$$\begin{aligned}
&\left|f'(a)\, f'(b)\right|^{(1-s/2)} \left\{[F_1(\mu, 1)]^{\frac{1}{q}} + \left[F_1\left(\mu^{-1}, 1\right)\right]^{\frac{1}{q}}\right\} \\
&= [G(a^r, b^r)]^2 \left\{ a^{-rs} \left[\frac{2\left(1 - b^{-rqs/2} L\left(a^{rqs/2}, b^{rqs/2}\right)\right)}{rqs(\ln b - \ln a)}\right]^{1/q} \right. \\
&\left. + b^{-rs} \left[\frac{2\left(a^{-rqs/2} L\left(a^{rqs/2}, b^{rqs/2}\right) - 1\right)}{rqs(\ln b - \ln a)}\right]^{1/q} \right\}.
\end{aligned}$$

Substituting the above quantities in Corollary 182, we get the required inequality. ■

Remark 196 *Many interesting inequalities of means of positive real numbers can be obtained by using the other results, however, the details are left to the interested reader.*

Chapter 5

H-H type integral inequalities for n-times differentiable m- and (α,m)-logarithmically convex functions

The results of this chapter are taken from "M. A. Latif, S. S. Dragomir and E. Momoniat, On Hermite-Hadamard type integral inequalities for n-times differentiable m- and (α, m)-logarithmically convex functions, Filomat. (Accepted)" and "M. A. Latif, S. S. Dragomir and E. Momoniat, Inequalities of Hermite-Hadamard type for n-times differentiable (α, m)-logarithmically convex functions, Bulletin of the Iranian Mathematical Society. (Submitted)".

5.1 Introduction

In Chapter 2 it has been discussed that the definition of convex functions plays an important role in the theory of convex analysis and in many other branches of pure and applied mathematics. A number of remarkable and significant results in the theory of inequality hinge on this definition. The definition of convexity has been generalized in diverse ways such as s-convexity, m-convexity, (α, m)-convexity, h-convexity, logarithmic-convexity, s- logarithmic convexity, m-logarithmic convexity, (α, m)-logarithmic convexity and h-logarithmic-convexity but we will focus on the m-logarithmic convexity, (α, m)-logarithmic convexity as generalizations of the convexity to prove our results in this Chapter. In this chapter, we will use identities of Section 3.2 of Chapter 3 and Section 4.2 of Chapter 4 to establish Hermite-Hadamard type inequalities for functions whose nth derivatives are m- and $(\alpha\ ,m)$-logarithmically convex functions. We will also obtain several results for trapezoidal and midpoint type inequalities in terms of first and second derivatives that are m- and $(\alpha\ ,m)$-logarithmically convex functions which are contained in our results as special cases. The results of this chapter also provide corrected version of some Hermite-Hadamard type inequalities proven in [18] and may provide refinements of some results for $(\alpha\ ,m)$-logarithmically convex functions already existing in recent concerned literature of inequalities.

A brief outline of this chapter is as follows:

Section 5.2 of this chapter contains integral inequalities of H-H type obtained by applying the results of Section 4.2 in Chapter 4. Section 5.3 comprehends Hermite-Hadamard type by using an identity given by Lemma 96 of Section 3.2 in Chapter 3 and the results of Section 4.2 in Chapter 4. Section of 5.4 is devoted to inequalities of special means of non-negative real numbers which resulted in as applications of our discoveries.

5.2 On H-H type integral inequalities for n-times differentiable m- and (α,m)-logarithmically convex functions

We will use Lemmas of Section 4.2 of Chapter 4 to establish the results of this section.

Theorem 197 *Suppose that (c, d) is an open real interval such that $[0, \infty) \subseteq (c, d)$ and let $f : (c, d) \to (0, \infty)$ be a function which possesses the nth derivative $f^{(n)}$ on (c, d). Assume that $f^{(n)}$ belongs to the space $L([a, b])$ of all integrable functions on $[a, b]$ for*

$n \in \mathbb{N}$, $n \geq 2$, $0 \leq a < b < \infty$. If $\left|f^{(n)}\right|^q$ is (α, m)-logarithmically convex on $\left[0, \frac{b}{m}\right]$ for $1 \leq q < \infty$, $(\alpha, m) \in (0,1] \times (0,1]$, we have

$$\left|A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx - \sum_{k=2}^{n-1}\frac{\left(k-1\right)\left(b-a\right)^k}{2\left(k+1\right)!}f^{(k)}(a)\right|$$
$$\leq \frac{\left(b-a\right)^n}{2n!}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m\left(\frac{n-1}{n+1}\right)^{1-1/q}\mu^{\theta}\left[F\left(n;\mu^{\alpha q}\right)\right]^{1/q}, \quad (5.1)$$

where $\mu = \frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}$,

$$\theta = \begin{cases} 0, & 0 < \mu \leq 1, \\ 1-\alpha, & \mu > 1, \end{cases}$$

and $F\left(n;\nu\right)$ is defined in Lemma 161.

Proof. Suppose $n \geq 2$ and $a, b \in I$, $0 \leq a < b < \infty$. By (α, m)-logarithmic convexity of $\left|f^{(n)}\right|^q$ on $\left[0, \frac{b}{m}\right]$, Lemma 153 and Hölder inequality, we have

$$\left|A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx - \sum_{k=2}^{n-1}\frac{\left(k-1\right)\left(b-a\right)^k}{2\left(k+1\right)!}f^{(k)}(a)\right|$$
$$\leq \frac{\left(b-a\right)^n\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}{2n!}\left(\int_0^1 t^{n-1}\left(n-2t\right)dt\right)^{1-1/q}\left(\int_0^1 t^{n-1}\left(n-2t\right)\mu^{qt^{\alpha}}dt\right)^{1/q}$$
$$= \frac{\left(b-a\right)^n\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}{2n!}\left(\frac{n-1}{n+1}\right)^{1-1/q}\left(n\int_0^1 t^{n-1}\mu^{qt^{\alpha}}dt - 2\int_0^1 t^n\mu^{qt^{\alpha}}dt\right)^{1/q}, \quad (5.2)$$

where $\mu = \frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}$.
For $\mu = 1$, we have

$$n\int_0^1 t^{n-1}\mu^{qt^{\alpha}}dt - 2\int_0^1 t^n\mu^{qt^{\alpha}}dt = n\int_0^1 t^{n-1}dt - 2\int_0^1 t^n dt = \frac{n-1}{n+1}. \quad (5.3)$$

For $0<\mu<1$, we have from Lemma 165 that $\mu^{qt^{\alpha}}\leq\mu^{\alpha qt}$. Hence by using Lemma 161, we get

$$n\int_{0}^{1}t^{n-1}\mu^{qt^{\alpha}}dt-2\int_{0}^{1}t^{n}\mu^{qt^{\alpha}}dt\leq n\int_{0}^{1}t^{n-1}\mu^{\alpha qt}dt-2\int_{0}^{1}t^{n}\mu^{\alpha qt}dt$$
$$=nE\left(n-1;\mu^{\alpha q}\right)-2E\left(n;\mu^{\alpha q}\right)=F\left(n;\mu^{\alpha q}\right). \quad (5.4)$$

Also for $\mu>1$, we have from Lemma 165 that $\mu^{qt^{\alpha}}\leq\mu^{q\alpha t+q-q\alpha}$. By Lemma 161, we obtain

$$n\int_{0}^{1}t^{n-1}\mu^{qt^{\alpha}}dt-2\int_{0}^{1}t^{n}\mu^{qt^{\alpha}}dt\leq n\int_{0}^{1}t^{n-1}\mu^{q\alpha t+q(1-\alpha)}dt-2\int_{0}^{1}t^{n}\mu^{q\alpha t+q(1-\alpha)}dt$$
$$=\mu^{q(1-\alpha)}\left[nE\left(n-1;\mu^{\alpha q}\right)-2E\left(n;\mu^{\alpha q}\right)\right]=\mu^{q(1-\alpha)}F\left(n;\mu^{\alpha q}\right). \quad (5.5)$$

On combining (5.2)-(5.5) gives the inequality (5.1). Hence the proof of the theorem accomplishes. ■

Corollary 198 *Substitution of $q=1$ in Theorem 197 gives the following outcome*

$$\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx-\sum_{k=2}^{n-1}\frac{\left(k-1\right)\left(b-a\right)^{k}}{2\left(k+1\right)!}f^{(k)}(a)\right|$$
$$\leq\frac{\left(b-a\right)^{n}}{2n!}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^{m}\mu^{\theta}F\left(n;\mu^{\alpha}\right), \quad (5.6)$$

where $F\left(n;v\right)$ is defined in Lemma 161 and θ, μ are defined in Theorem 197.

Corollary 199 *Letting $n=2$ in Theorem 197, yields the the inequality*

$$\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx\right|$$
$$\leq\frac{\left(b-a\right)^{2}}{4}\left|f''\left(\frac{b}{m}\right)\right|^{m}\left(\frac{1}{3}\right)^{1-1/q}\mu^{\theta}\left[F\left(2;\mu^{\alpha q}\right)\right]^{1/q}, \quad (5.7)$$

where θ is defined in Theorem 197, $\mu = \frac{\left|f''(a)\right|}{\left|f''\left(\frac{b}{m}\right)\right|^m}$ *and*

$$F\left(2;\mu^{\alpha q}\right) = \begin{cases} \frac{2\alpha q\left(1+\mu^{\alpha q}\right)\ln\mu + 4\left(1-\mu^{\alpha q}\right)}{(\alpha q)^3(\ln\mu)^3}, & \mu \neq 1, \\ \\ \frac{1}{3}, & \mu = 1. \end{cases}$$

Corollary 200 *Under the assumptions of Theorem 197, if $n = 2$ and $\alpha = 1$, then following inequality holds when the absolute value of the second derivative of f is an m-logarithmically convex function*

$$\left|A\left(f(a), f(b)\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ \leq \frac{(b-a)^2}{4}\left|f''\left(\frac{b}{m}\right)\right|^m \left(\frac{1}{3}\right)^{1-1/q}\left[F\left(2;\mu^q\right)\right]^{1/q}, \quad (5.8)$$

where $\mu = \frac{\left|f''(a)\right|}{\left|f''\left(\frac{b}{m}\right)\right|^m}$ *and*

$$F\left(2;\mu^{q}\right) = \begin{cases} \frac{2q\left(1+\mu^{q}\right)\ln\mu + 4\left(1-\mu^{q}\right)}{q^3(\ln\mu)^3}, & \mu \neq 1, \\ \\ \frac{1}{3}, & \mu = 1. \end{cases}$$

Remark 201 *The inequalities (5.7) and (5.8) are the new inequalities which provide new estimates of the difference between the rightmost and the middle terms in the H-H inequality (2.19) in terms of the second derivatives that are (α, m)-logarithmically and m-logarithmically convex respectively.*

Theorem 202 *Suppose that (c, d) is an open real interval such that $[0, \infty) \subseteq (c, d)$ and let $f : (c, d) \to (0, \infty)$ be a function which possesses the nth derivative $f^{(n)}$ on (c, d). Assume that $f^{(n)}$ belongs to the space $L([a, b])$ of all integrable functions on $[a, b]$ for $n \in \mathbb{N}$, $n \geq 2$, $0 \leq a < b < \infty$. If $\left|f^{(n)}\right|^q$ is (α, m)-logarithmically convex on $\left[0, \frac{b}{m}\right]$ for*

$1 < q < \infty$, $(\alpha, m) \in (0,1] \times (0,1]$, *we have*

$$\left|A(f(a), f(b)) - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|$$
$$\leq \frac{(b-a)^n\left[n^{(2q-1)/(q-1)} - (n-2)^{(2q-1)/(q-1)}\right]^{1-1/q}}{2^{2-1/q}n!}$$
$$\times\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m \mu^\theta\left[I\left(nq-q+1;\mu^{\alpha q}\right)\right]^{1/q}, \quad (5.9)$$

where μ and θ are defined in Theorem 197 and $I(\alpha; v)$ is defined in Lemma 159.

Proof. Since $\left|f^{(n)}\right|^q$ is (α, m)-logarithmically convex on $\left[0, \frac{b}{m}\right]$ for $q \in (1, \infty)$, $(\alpha, m) \in (0,1] \times (0,1]$, hence by using Lemma 153 and Hölder inequality, we have

$$\left|A(f(a), f(b)) - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|$$
$$\leq \frac{(b-a)^n}{2n!}\left(\int_0^1 (n-2t)^{q/(q-1)}\,dt\right)^{1-1/q}\left(\int_0^1 t^{q(n-1)}\left|f^{(n)}(ta+(1-t)\,b\right|^q dt\right)^{1/q}$$
$$\leq \frac{(b-a)^n\left[n^{(2q-1)/(q-1)} - (n-2)^{(2q-1)/(q-1)}\right]^{1-1/q}}{2^{2-1/q}n!}$$
$$\times\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m\left(\int_0^1 t^{q(n-1)}\mu^{qt^\alpha}dt\right)^{1/q}, \quad (5.10)$$

where $\mu = \frac{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}{\left|f^{(n)}(a)\right|}$.

For $\mu = 1$, we have

$$\int_0^1 t^{q(n-1)}\mu^{qt^\alpha}dt = \int_0^1 t^{q(n-1)}dt = \frac{1}{nq-q+1}.$$

For $0 < \mu < 1$, $\mu^{qt^{\alpha}} \leq \mu^{\alpha qt}$ and hence by using Lemma 165, we have

$$\int_0^1 t^{q(n-1)} \mu^{qt^{\alpha}} dt \leq \int_0^1 t^{q(n-1)} \mu^{q\alpha t} dt$$
$$\leq \mu^{\alpha q} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (\alpha q)^{k-1} (\ln \mu)^{k-1}}{(nq-q+1)_k} < \infty.$$

For $\mu > 1$, $\mu^{qt^{\alpha}} \leq \mu^{q\alpha t + q(1-\alpha)}$, and hence by Lemma 165, we obtain

$$\int_0^1 t^{q(n-1)} \mu^{qt^{\alpha}} dt \leq \int_0^1 t^{q(n-1)} \mu^{q\alpha t + q(1-\alpha)} dt$$
$$\leq \mu^{q(1-\alpha)} \left(\mu^{q\alpha} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (q\alpha)^{k-1} (\ln \mu)^{k-1}}{(nq-q+1)_k} \right) < \infty.$$

Thus the inequality (5.9) follows by using the above inequalities in (5.10). The proof of the theorem is thus completed. ■

Corollary 203 *Suppose the conditions of Theorem 202 are met and $n = 2$. Then*

$$\left| A(f(a), f(b)) - \frac{1}{b-a} \int_a^b f(x)\, dx \right|$$
$$\leq \frac{(b-a)^2}{4} \left(\frac{q-1}{2q-1} \right)^{1-1/q} \left| f''\left(\frac{b}{m} \right) \right|^m \mu^{\theta} \left[I\left(q+1; \mu^{\alpha q}\right) \right]^{1/q}, \quad (5.11)$$

where $\mu = \frac{|f''(a)|}{|f''(\frac{b}{m})|^m}$, θ is defined in Theorem 197 and $I(\alpha; v)$ is defined in Lemma 159.

Corollary 204 *If we let $n = 2$, $\alpha = 1$ in Theorem 202 , we get the following useful result*

$$\left| A(f(a), f(b)) - \frac{1}{b-a} \int_a^b f(x)\, dx \right|$$
$$\leq \frac{(b-a)^2}{4} \left(\frac{q-1}{2q-1} \right)^{1-1/q} \left| f''\left(\frac{b}{m} \right) \right|^m \left[I\left(q+1; \mu^{\alpha q}\right) \right]^{1/q}, \quad (5.12)$$

where $\mu = \frac{|f''(a)|}{|f''(\frac{b}{m})|^m}$ and $I(\alpha; v)$ is defined in Lemma 159.

Now we give some results related to left-side of Hermite-Hadamard's inequality for n-times differentiable (α, m)-logarithmically convex functions.

Theorem 205 *Suppose that (c,d) is an open real interval such that $[0,\infty) \subseteq (c,d)$ and let $f:(c,d) \to (0,\infty)$ be a function which possesses the nth derivative $f^{(n)}$ on (c,d). Assume that $f^{(n)}$ belongs to the space $L([a,b])$ of all integrable functions on $[a,b]$ for $n \in \mathbb{N}$, $n \geq 1$, $0 \leq a < b < \infty$. If $\left|f^{(n)}\right|^q$ is (α,m)-logarithmically convex on $\left[0, \frac{b}{m}\right]$ for $1 \leq q < \infty$, $(\alpha,m) \in (0,1] \times (0,1]$, we have*

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}(A(a,b)) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^n \left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}{2^{(n+1)(q-1)/q}(n+1)^{1-1/q} n!} \mu^\theta \left\{\left[G_1(n;\mu^{\alpha q})\right]^{1/q} + \left[G_2(n;\mu^{\alpha q})\right]^{1/q}\right\}, \quad (5.13)$$

where μ, θ are defined in Theorem 197 and $G_1(n;v)$, $G_2(n;v)$ are respectively defined in Lemma 157 and Lemma 158.

Proof. Suppose $n \geq 1$. By using Lemma 154, the (α,m)-logarithmic convexity of $\left|f^{(n)}\right|$ and the Hölder inequality, we have

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}(A(a,b)) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\leq \frac{(b-a)^n}{n!}\left[\int_0^{\frac{1}{2}} t^n \left|f^{(n)}(ta+(1-t)b)\right| dt + \int_{\frac{1}{2}}^1 (1-t)^n \left|f^{(n)}(ta+(1-t)b)\right| dt\right] \\
&\leq \frac{(b-a)^n \left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}{n!}\left[\left(\int_0^{\frac{1}{2}} t^n dt\right)^{1-1/q}\left(\int_0^{\frac{1}{2}} t^n \mu^{qt^\alpha} dt\right)^{1/q}\right. \\
&\quad \left. + \left(\int_{\frac{1}{2}}^1 (1-t)^n dt\right)^{1-1/q}\left(\int_{\frac{1}{2}}^1 (1-t)^n \mu^{qt^\alpha} dt\right)^{1/q}\right] \\
&= \frac{(b-a)^n \left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}{n! 2^{(n+1)(q-1)/q}(n+1)^{1-1/q}}\left[\left(\int_0^{\frac{1}{2}} t^n \mu^{qt^\alpha} dt\right)^{1/q} + \left(\int_{\frac{1}{2}}^1 (1-t)^n \mu^{qt^\alpha} dt\right)^{1/q}\right], \quad (5.14)
\end{aligned}$$

where $\mu = \frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^{m}}$.

For $\mu = 1$, we have

$$\begin{aligned}
&\left(\int_0^{\frac{1}{2}} t^n \mu^{qt^\alpha} dt\right)^{1/q} + \left(\int_{\frac{1}{2}}^1 (1-t)^n \mu^{qt^\alpha} dt\right)^{1/q} \\
&= \left(\int_0^{\frac{1}{2}} t^n dt\right)^{1/q} + \left(\int_{\frac{1}{2}}^1 (1-t)^n dt\right)^{1/q} = \frac{2}{2^{(n+1)/q}(n+1)^{1/q}}.
\end{aligned}$$

For $0 < \mu < 1$, $\mu^{qt^\alpha} \leq \mu^{\alpha qt}$ and hence by using Lemma 157 and Lemma 158, we have

$$\begin{aligned}
&\left(\int_0^{\frac{1}{2}} t^n \mu^{qt^\alpha} dt\right)^{1/q} + \left(\int_{\frac{1}{2}}^1 (1-t)^n \mu^{qt^\alpha} dt\right)^{1/q} \\
&\leq \left(\int_0^{\frac{1}{2}} t^n \mu^{\alpha qt} dt\right)^{1/q} + \left(\int_{\frac{1}{2}}^1 (1-t)^n \mu^{\alpha qt} dt\right)^{1/q} \\
&= \left[G_1\left(n; \mu^{\alpha q}\right)\right]^{1/q} + \left[G_2\left(n; \mu^{\alpha q}\right)\right]^{1/q}.
\end{aligned}$$

For $\mu > 1$, $\mu^{qt^\alpha} \leq \mu^{q\alpha t + q(1-\alpha)}$ and hence by using Lemma 157 and Lemma 158, we have

$$\begin{aligned}
&\left(\int_0^{\frac{1}{2}} t^n \mu^{qt^\alpha} dt\right)^{1/q} + \left(\int_{\frac{1}{2}}^1 (1-t)^n \mu^{qt^\alpha} dt\right)^{1/q} \\
&\leq \mu^{1-\alpha} \left\{\left(\int_0^{\frac{1}{2}} t^n \mu^{\alpha qt} dt\right)^{1/q} + \left(\int_{\frac{1}{2}}^1 (1-t)^n \mu^{\alpha qt} dt\right)^{1/q}\right\} \\
&= \mu^{1-\alpha} \left\{\left[G_1\left(n; \mu^{\alpha q}\right)\right]^{1/q} + \left[G_2\left(n; \mu^{\alpha q}\right)\right]^{1/q}\right\}.
\end{aligned}$$

Hence the inequality (5.13) follows from the above facts when applied to (5.14). ■

Corollary 206 *Suppose the assumptions of Theorem 107 are fulfilled and if $q = 1$, we* *have*

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}(A(a,b))-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)^n\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}{n!}\mu^\theta\left\{G_1(n;\mu^\alpha)+G_2(n;\mu^\alpha)\right\}, \quad (5.15)$$

where μ, θ *are defined in Theorem 197 and* $G_1(n;v)$, $G_2(n;v)$ *are respectively defined in Lemma 157 and Lemma 158.*

Remark 207 *Suppose the conditions of Theorem 205 are satisfied and if* $n=1$, *we get the corrected inequality given in Theorem 82.*

Remark 208 *Suppose the conditions of Theorem 205 are satisfied and if* $n=1$, *we get the corrected inequality given in Corollary 83.*

Corollary 209 *Suppose the assumptions of Theorem 205 hold true and if* $\alpha=1$, *then we get the following inequality when* $\left|f^{(n)}\right|^q$ *is* m*-logarithmically convex on* $\left[0,\frac{b}{m}\right]$ *for* $q\in[1,\infty)$, $m\in(0,1]$*:*

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}(A(a,b))-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)^n\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}{2^{(n+1)(q-1)/q}(n+1)^{1-1/q}n!}\left\{[G_1(n;\mu^q)]^{1/q}+[G_2(n;\mu^q)]^{1/q}\right\}, \quad (5.16)$$

where μ *is defined in Theorem 197 and* $G_1(n;v)$, $G_2(n;v)$ *are respectively defined in Lemma 157 and Lemma 158.*

Theorem 210 *Suppose that* (c,d) *is an open real interval such that* $[0,\infty)\subseteq(c,d)$ *and let* $f:(c,d)\to(0,\infty)$ *be a function which possesses the nth derivative* $f^{(n)}$ *on* (c,d). *Assume that* $f^{(n)}$ *belongs to the space* $L([a,b])$ *of all integrable functions on* $[a,b]$ *for* $n\in\mathbb{N}$, $n\geq 1$, $0\leq a<b<\infty$. *If* $\left|f^{(n)}\right|^q$ *is* (α,m)*-logarithmically convex on* $\left[0,\frac{b}{m}\right]$ *for*

$1 < q < \infty$, $(\alpha, m) \in (0,1] \times (0,1]$, *we have*

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}(A(a,b)) - \frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)^n}{2^{n+1/p}(np+1)^{1/p} n!} \left|f^{(n)}\left(\frac{b}{m}\right)\right|^m \left\{\left[\frac{\alpha q \left(\frac{1}{2}\right)^{\alpha-1} \mu^{q\left(\frac{1}{2}\right)^\alpha}}{\ln \mu}\right]^{1/q} + \left[\frac{\alpha q \mu^q - \alpha q \left(\frac{1}{2}\right)^{\alpha-1} \mu^{q\left(\frac{1}{2}\right)^\alpha}}{\ln \mu}\right]^{1/q}\right\}, \quad (5.17)$$

where p and q are Hölder conjugates of each other and $\mu = \frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}$.

Proof. From Lemma 154, the Hölder integral inequality and (α, m)-logarithmic convexity of $\left|f^{(n)}\right|^q$, we have

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}(A(a,b)) - \frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(\eta(b,a))^n \left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}{n!} \left[\left(\int_0^{\frac{1}{2}} t^{np} dt\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}\right)^{qt^\alpha} dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 (1-t)^{np}\,dt\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 \left(\frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}\right)^{qt^\alpha} dt\right)^{\frac{1}{q}}\right] \quad (5.18)$$

from which the required inequality follows. This completes the proof of the theorem. ■

Corollary 211 *Considering that the assumptions of Theorem 210 are fulfilled and letting*

$n = 1$, *we get the following inequality*

$$\left| f\left(A\left(a,b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx \right| \leq \frac{(b-a)}{2^{1+1/p}\left(p+1\right)^{1/p}}\left| f'\left(\frac{b}{m}\right)\right|^m \left\{ \left[\frac{\alpha q\left(\frac{1}{2}\right)^{\alpha-1}\mu^{q\left(\frac{1}{2}\right)^{\alpha}}}{\ln\mu} \right]^{1/q} + \left[\frac{\alpha q\mu^q - \alpha q\left(\frac{1}{2}\right)^{\alpha-1}\mu^{q\left(\frac{1}{2}\right)^{\alpha}}}{\ln\mu} \right]^{1/q} \right\}, \quad (5.19)$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* $\mu = \frac{\left|f'(a)\right|}{\left|f'\left(\frac{b}{m}\right)\right|^m}$.

Corollary 212 *Suppose that* (c,d) *is an open real interval such that* $[0,\infty) \subseteq (c,d)$ *and let* $f:(c,d)\to(0,\infty)$ *be a function which possesses the nth derivative* $f^{(n)}$ *on* (c,d). *Assume that* $f^{(n)}$ *belongs to the space* $L\left([a,b]\right)$ *of all integrable functions on* $[a,b]$ *for* $n\in\mathbb{N}$, $n\geq 1$, $0\leq a<b<\infty$. *If* $\left|f^{(n)}\right|^q$ *is m-logarithmically convex on* $\left[0,\frac{b}{m}\right]$ *for* $1<q<\infty$, $m\in(0,1]$, *we have*

$$\left| \sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(A\left(a,b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx \right| \leq \frac{(b-a)^n\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}{2^{n+1/p}\left(np+1\right)^{1/p}n!}\left(\frac{q}{\ln\mu}\right)^{1/q}\mu^{1/2}\left\{1+\left(\mu^{q/2}-1\right)^{1/q}\right\}, \quad (5.20)$$

where p *and* q *are Hölder conjugates of each other and* $\mu = \frac{\left|f^{(n)}(a)\right|}{\left|f^{(n)}\left(\frac{b}{m}\right)\right|^m}$.

5.3 Further results on inequalities of H-H type for n-times differentiable m-logarithmically and (α,m)-logarithmically convex functions

We will use the Lemma 96 and Lemma 155-Lemma 165 of Section 4.2 to prove inequalities of Hermite-Hadamard type for n-times differentiable m-logarithmically and (α,m)-logarithmically convex functions in this section, which further generalize the results of Section 5.2.

Theorem 213 *Suppose that (c,d) is an open real interval such that $[0,\infty) \subseteq (c,d)$ and let $f:(c,d) \to (0,\infty)$ be a function which possesses the nth derivative $f^{(n)}$ on (c,d). Assume that $f^{(n)}$ belongs to the space $L([a,b])$ of all integrable functions on $[a,b]$ for $n \in \mathbb{N}$, $n \geq 1$, $0 \leq a < b < \infty$. If $\left|f^{(n)}\right|^q$ is (α,m)-logarithmically convex on $\left[0,\frac{b}{m}\right]$ for $1 \leq q < \infty$, $(\alpha,m) \in (0,1] \times (0,1]$, we have*

$$\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx-\sum_{k=1}^{n-1}\frac{k\left[1+\left(-1\right)^k\right]\left(b-a\right)^k}{2^{k+1}\left(k+1\right)!}f^{(k)}\left(A\left(a,b\right)\right)\right|$$
$$\leq \frac{(b-a)^n}{2^{n+1}n!}\left(\frac{n}{n+1}\right)^{1-\frac{1}{q}}\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m \mu^\theta\left\{\left[G\left(n;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q}+\left[G\left(n;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\}, \tag{5.21}$$

where $G(n;\xi)$ is defined in Lemma 161, $\mu=\frac{f^{(n)}(a)}{\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m}$ and

$$\theta=\begin{cases}\frac{\alpha}{2}, & 0<\mu\leq 1\\ \\ 1-\frac{\alpha}{2}, & \mu>1.\end{cases}$$

Proof. From Lemma 96, the Hölder inequality and using the fact that $\left|f^{(n)}\right|^q$ is (α, m)-logarithmically convex on $\left[0, \frac{b}{m}\right]$, we have

$$\left|A\left(f\left(a\right), f\left(b\right)\right) - \sum_{k=1}^{n-1} \frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(A\left(a,b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx\right|$$
$$\leq \frac{(b-a)^n}{2^{n+1}n!}\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m \left(\int_0^1 (1-t)^{n-1}(n-1+t)\,dt\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{q\left(\frac{1-t}{2}\right)^\alpha}dt\right)^{1/q}\right.$$
$$\left. + \left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{q\left(\frac{1+t}{2}\right)^\alpha}dt\right)^{1/q}\right\}, \quad (5.22)$$

where $\mu = \frac{f^{(n)}(a)}{\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m}$.
It is obvious that

$$\int_0^1 (1-t)^{n-1}(n-1+t)\,dt = \frac{n}{n+1}. \quad (5.23)$$

Also when $0 < \mu \leq 1$, then by using Lemma 165 and Lemma 162, we obtain

$$\left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{q\left(\frac{1-t}{2}\right)^\alpha}dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{q\left(\frac{1+t}{2}\right)^\alpha}dt\right)^{\frac{1}{q}}$$
$$\leq \left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{\alpha q\left(\frac{1-t}{2}\right)}dt\right)^{\frac{1}{q}} + \left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{\alpha q\left(\frac{1+t}{2}\right)}dt\right)^{\frac{1}{q}}$$
$$= \mu^{\frac{\alpha}{2}}\left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{-\frac{\alpha q t}{2}}dt\right)^{\frac{1}{q}} + \mu^{\frac{\alpha}{2}}\left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{\frac{\alpha q t}{2}}dt\right)^{\frac{1}{q}}$$
$$= \mu^{\frac{\alpha}{2}}\left\{\left[G\left(n;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q} + \left[G\left(n;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\}. \quad (5.24)$$

When $\mu > 1$. By using Lemma 165 and Lemma 162, we have

$$\left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{q\left(\frac{1-t}{2}\right)^\alpha}dt\right)^{\frac{1}{q}} + \left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{q\left(\frac{1+t}{2}\right)^\alpha}dt\right)^{\frac{1}{q}}$$
$$\leq \left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{\alpha q\left(\frac{1-t}{2}\right)+q-\alpha q}dt\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{\alpha q\left(\frac{1+t}{2}\right)+q-\alpha q}dt\right)^{\frac{1}{q}}$$
$$= \mu^{1-\frac{\alpha}{2}}\left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{-\frac{\alpha q t}{2}}dt\right)^{\frac{1}{q}} + \mu^{1-\frac{\alpha}{2}}\left(\int_0^1 (1-t)^{n-1}(n-1+t)\,\mu^{\frac{\alpha q t}{2}}dt\right)^{\frac{1}{q}}$$
$$= \mu^{1-\frac{\alpha}{2}}\left\{\left[G\left(n;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q} + \left[G\left(n;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\}. \quad (5.25)$$

A combination of (5.22)-(5.25) gives the desired result. ■

Corollary 214 *Let f be as in Theorem 213 are satisfied and if $q = 1$, we have*

$$\left|A\left(f(a), f(b)\right) - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(A(a,b)\right)\right|$$
$$\leq \frac{(b-a)^n}{2^{n+1}n!}\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m \mu^\theta\left\{G\left(n;\mu^{-\frac{\alpha}{2}}\right) + G\left(n;\mu^{\frac{\alpha}{2}}\right)\right\}, \quad (5.26)$$

where $G(n;\xi)$ is defined in Lemma 162, $\mu = \frac{f^{(n)}(a)}{\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m}$ and θ is defined in Theorem 213.

Corollary 215 *Suppose that the assumptions of Theorem 213 are valid and $n = 1$, we have the inequality*

$$\left|A\left(f(a), f(b)\right) - \frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\leq \frac{(b-a)}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[f'\left(\frac{b}{m}\right)\right]^m \mu^\theta\left\{\left[G\left(1;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q} + \left[G\left(1;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\}, \quad (5.27)$$

where $\mu = \frac{f^{'}(a)}{\left[f^{'}\left(\frac{b}{m}\right)\right]^m}$, θ *is defined in Theorem 213 and*

$$G(1;\xi) = \begin{cases} \frac{1}{\ln\xi}\left[\xi + \frac{1-\xi}{\ln\xi}\right], & \xi \neq 1 \\ \\ \frac{1}{2}, & \xi = 1. \end{cases}$$

Corollary 216 *Corollary 215 with* $q = 1$ *gives the following result*

$$\left|A(f(a), f(b)) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{4}\left[f^{'}\left(\frac{b}{m}\right)\right]^m \mu^\theta \left\{\left[G\left(1;\mu^{-\frac{\alpha}{2}}\right)\right] + \left[G\left(1;\mu^{\frac{\alpha}{2}}\right)\right]\right\}, \quad (5.28)$$

where $G(1;\xi)$ *and* μ *are defined as in Corollary 215 and* θ *is as defined in Theorem 213.*

Corollary 217 *Let the assumptions of Theorem 213 be fulfilled and* $n = 2$*, we have*

$$\left|A(f(a), f(b)) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{16}\left(\frac{2}{3}\right)^{1-\frac{1}{q}}\left[f^{''}\left(\frac{b}{m}\right)\right]^m \mu^\theta \left\{\left[G\left(2;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q} + \left[G\left(2;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\}, \quad (5.29)$$

where $\mu = \frac{f^{''}(a)}{\left[f^{''}\left(\frac{b}{m}\right)\right]^m}$, θ *is as defined in Theorem 213 and*

$$G(2;\xi) = \begin{cases} \frac{2\xi\ln\xi - (\ln\xi)^2 - 2\xi + 2}{(\ln\xi)^3}, & \xi \neq 1, \\ \frac{2}{3}, & \xi = 1. \end{cases}$$

Corollary 218 *If* $q = 1$ *in Corollary 217, we have*

$$\left|A(f(a), f(b)) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{16}\left[f^{''}\left(\frac{b}{m}\right)\right]^m \mu^\theta \left\{G\left(2;\mu^{-\frac{\alpha}{2}}\right) + G\left(2;\mu^{\frac{\alpha}{2}}\right)\right\}, \quad (5.30)$$

where θ is defined in Theorem 213 and μ, $G(2;\xi)$ are defined in Corollary 217.

Theorem 219 *Suppose that (c,d) is an open real interval such that $[0,\infty) \subseteq (c,d)$ and let $f:(c,d) \to (0,\infty)$ be a function which possesses the nth derivative $f^{(n)}$ on (c,d). Assume that $f^{(n)}$ belongs to the space $L([a,b])$ of all integrable functions on $[a,b]$ for $n \in \mathbb{N}$, $n \geq 1$, $0 \leq a < b < \infty$. If $\left|f^{(n)}\right|^q$ is (α,m)-logarithmically convex on $\left[0,\frac{b}{m}\right]$ for $1 < q < \infty$, $(\alpha,m) \in (0,1] \times (0,1]$, we have*

$$\left|A(f(a),f(b)) - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=1}^{n-1} \frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}(A(a,b))\right|$$
$$\leq \frac{(b-a)^n\left[n^{(2q-1)/(q-1)} - (n-1)^{(2q-1)/(q-1)}\right]^{1-\frac{1}{q}}}{2^{n+1}n!}\left(\frac{q-1}{2q-1}\right)^{1/q}\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m \mu^\theta$$
$$\times\left\{\left[J\left(nq-q+1;\mu^{-\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}} + \left[J\left(nq-q+1;\mu^{\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}\right\}, \quad (5.31)$$

where $\mu = \frac{f^{(n)}(a)}{\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m}$, $J(\alpha;\xi)$ is defined in Lemma 160 and θ is defined in Theorem 213.

Proof. Using Lemma 96, the Hölder inequality and the (α,m)-logarithmic convexity of $\left|f^{(n)}\right|^q$ on $\left[0,\frac{b}{m}\right]$, we have

$$\left|A(f(a),f(b)) - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=1}^{n-1} \frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}(A(a,b))\right|$$
$$\leq \frac{(b-a)^n}{2^{n+1}n!}\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m\left(\int_0^1 (n-1+t)^{\frac{q}{q-1}}\,dt\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left(\int_0^1 (1-t)^{q(n-1)}\mu^{q\left(\frac{1-t}{2}\right)^\alpha}dt\right)^{1/q} + \left(\int_0^1 (1-t)^{q(n-1)}\mu^{q\left(\frac{1+t}{2}\right)^\alpha}dt\right)\right\}^{1/q}. \quad (5.32)$$

The proof follows by using similar arguments as in proving Theorem 213, using Lemma 160 and Lemma 165. ■

Corollary 220 *Replacement of $n=1$ in Theorem 219, gives the following interesting*

inequality

$$\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right| \\ \leq \frac{\left(b-a\right)}{4}\left(\frac{q-1}{2q-1}\right)^{1/q}\left[f'\left(\frac{b}{m}\right)\right]^m \mu^\theta\left\{\left[J\left(1;\mu^{-\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}+\left[J\left(1;\mu^{\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}\right\}, \quad (5.33)$$

where $\mu=\frac{f'(a)}{\left[f''\left(\frac{b}{m}\right)\right]^m}$,

$$G\left(1;\xi\right)=\sum_{k=1}^{\infty}\frac{\left(\ln\xi\right)^{k-1}}{k!}<\infty$$

and θ *is defined in Theorem 219.*

Corollary 221 *Let* f *be as in Theorem 219, if* $n=2$*, we have the inequality*

$$\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx\right| \\ \leq \frac{\left(b-a\right)^2\left[2^{(2q-1)/(q-1)}-1\right]^{1-\frac{1}{q}}}{16}\left(\frac{q-1}{2q-1}\right)^{1/q}\left[f''\left(\frac{b}{m}\right)\right]^m\mu^\theta \\ \times\left\{\left[G\left(q+1;\mu^{-\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}+\left[G\left(q+1;\mu^{\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}\right\}, \quad (5.34)$$

where $\mu=\frac{f''(a)}{\left[f''\left(\frac{b}{m}\right)\right]^m}$,

$$G\left(q+1;\xi\right)=\sum_{k=1}^{\infty}\frac{\left(\ln\xi\right)^{k-1}}{\left(q+1\right)_k}<\infty$$

and θ *is defined in Theorem 213.*

Theorem 222 *Suppose that* (c,d) *is an open real interval such that* $[0,\infty)\subseteq(c,d)$ *and let* $f:(c,d)\to(0,\infty)$ *be a function which possesses the nth derivative* $f^{(n)}$ *on* (c,d)*. Assume that* $f^{(n)}$ *belongs to the space* $L([a,b])$ *of all integrable functions on* $[a,b]$ *for* $n\in\mathbb{N}$*,* $n\geq 1$*,* $0\leq a<b<\infty$*. If* $\left|f^{(n)}\right|^q$ *is* (α,m)*-logarithmically convex on* $\left[0,\frac{b}{m}\right]$ *for*

$1 < q < \infty$, $(\alpha, m) \in (0,1] \times (0,1]$, *we have*

$$\left| A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx - \sum_{k=1}^{n-1} \frac{k\left[1+\left(-1\right)^k\right]\left(b-a\right)^k}{2^{k+1}\left(k+1\right)!} f^{(k)}\left(A\left(a,b\right)\right)\right|$$
$$\leq \frac{n^{n+1-\frac{1}{q}}\left(b-a\right)^n}{2^{n+1}n!}\left[B\left(\frac{1}{n};\frac{nq-1}{q-1},\frac{2q-1}{q-1}\right)\right]^{1-\frac{1}{q}}$$
$$\times \mu^{\theta}\left\{\left[E_2\left(0;\mu^{-\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}} + \left[E_2\left(0;\mu^{\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}\right\}, \quad (5.35)$$

where $\mu = \frac{f^{(n)}(a)}{\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m}$,

$$E_2\left(0;\xi\right) = \begin{cases} \frac{\xi-1}{\ln\xi}, & \xi \neq 1 \\ 1, & \xi = 1 \end{cases},$$

$$B\left(z;\alpha,\beta\right) = \int_0^z t^{\alpha-1}\left(1-t\right)^{\beta-1}dt, 0 \leq z \leq 1, \alpha > 0, \beta > 0$$

is the incomplete Beta function and θ *is defined in Theorem 213.*

Proof. Using Lemma 96, the Hölder inequality and the (α, m)-logarithmic convexity of $\left|f^{(n)}\right|^q$ on $\left[0, \frac{b}{m}\right]$, we have

$$\left| A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx - \sum_{k=1}^{n-1} \frac{k\left[1+\left(-1\right)^k\right]\left(b-a\right)^k}{2^{k+1}\left(k+1\right)!} f^{(k)}\left(A\left(a,b\right)\right)\right|$$
$$\leq \frac{\left(b-a\right)^n}{2^{n+1}n!}\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m\left(\int_0^1 \left(1-t\right)^{q(n-1)/(q-1)}\left(n-1+t\right)^{q/(q-1)}dt\right)^{1-1/q}$$
$$\times\left\{\left(\int_0^1 \mu^{q\left(\frac{1-t}{2}\right)^{\alpha}}dt\right)^{1/q} + \left(\int_0^1 \mu^{q\left(\frac{1+t}{2}\right)^{\alpha}}dt\right)^{1/q}\right\}. \quad (5.36)$$

By using Lemma 165, Lemma 156 and the fact that

$$\int_0^1 \left(1-t\right)^{q(n-1)/(q-1)}\left(n-1+t\right)^{q/(q-1)}dt$$
$$= n^{\frac{nq+q-1}{q-1}}\int_0^{\frac{1}{n}} t^{\frac{(n-1)q}{q-1}}\left(1-t\right)^{\frac{q}{q-1}}dt = n^{\frac{nq+q-1}{q-1}}B\left(\frac{1}{n};\frac{nq-1}{q-1},\frac{2q-1}{q-1}\right),$$

we get the required inequality (5.35) from (5.36). ■

Corollary 223 *If the assumptions of Theorem 222 are met and $n = 1$, we have the inequality*

$$\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx\right| \leq \frac{\left(b-a\right)}{4}\left(\frac{2q-1}{q-1}\right)^{1-\frac{1}{q}}\mu^{\theta}\left\{\left[E_{2}\left(0;\mu^{-\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}+\left[E_{2}\left(0;\mu^{\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}\right\}, \quad (5.37)$$

where $\mu = \frac{f'(a)}{\left[f'\left(\frac{b}{m}\right)\right]^{m}}$,

$$E_{2}\left(0;\xi\right)=\begin{cases}\frac{\xi-1}{\ln\xi}, & \xi\neq1\\ 1, & \xi=1\end{cases}$$

and θ are defined as in Theorem 213.

Corollary 224 *Let the hypotheses of Theorem 222 are gratified and $n = 2$, we have the inequality*

$$\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx\right| \leq \frac{\left(b-a\right)^{2}}{2^{1+\frac{1}{q}}}\left[B\left(\frac{1}{2};\frac{2q-1}{q-1},\frac{2q-1}{q-1}\right)\right]^{1-\frac{1}{q}}\mu^{\theta}\left\{\left[E_{2}\left(0;\mu^{-\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}+\left[E_{2}\left(0;\mu^{\frac{\alpha q}{2}}\right)\right]^{\frac{1}{q}}\right\}, \quad (5.38)$$

where $\mu = \frac{f''(a)}{\left[f''\left(\frac{b}{m}\right)\right]^{m}}$,

$$E_{2}\left(0;\xi\right)=\begin{cases}\frac{\xi-1}{\ln\xi}, & \xi\neq1\\ 1, & \xi=1\end{cases},$$

$B(z;a,b)$ is the incomplete Beta function as defined in Theorem 222 and θ is defined as in Theorem 213.

Theorem 225 *Suppose that (c,d) is an open real interval such that $[0,\infty)\subseteq(c,d)$ and let $f:(c,d)\rightarrow(0,\infty)$ be a function which possesses the nth derivative $f^{(n)}$ on (c,d).*

Assume that $f^{(n)}$ belongs to the space $L([a,b])$ of all integrable functions on $[a,b]$ for $n \in \mathbb{N}$, $n \geq 1$, $0 \leq a < b < \infty$. If $\left|f^{(n)}\right|^q$ is (α,m)-logarithmically convex on $\left[0, \frac{b}{m}\right]$ for $1 < q < \infty$, $(\alpha,m) \in (0,1] \times (0,1]$ and $0 \leq r \leq (n-1)q$. Then

$$\left|A(f(a), f(b)) - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=1}^{n-1} \frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}(A(a,b))\right|$$
$$\leq \frac{(b-a)^n}{2^{n+1}n!}\left[\frac{(q-1)\left(n^2q - nr - 2n + r + 1\right)}{(nq-r-1)(nq+q-r-2)}\right]^{1-\frac{1}{q}}\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m \mu^\theta$$
$$\times\left\{\left[K\left(r+1; \mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q} + \left[K\left(r+1; \mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\}. \quad (5.39)$$

$\mu = \frac{f^{(n)}(a)}{\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m}$, θ is defined in Theorem 213 and $K(\alpha;\xi)$ is expressed in Lemma 164.

Proof. From Lemma 96, the Hölder inequality and using the fact that $\left|f^{(n)}\right|^q$ is (α,m)-logarithmically convex on $\left[0, \frac{b}{m}\right]$, we have

$$\left|A(f(a), f(b)) - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=1}^{n-1} \frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}(A(a,b))\right|$$
$$\leq \frac{(b-a)^n}{2^{n+1}n!}\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m \left(\int_0^1 (1-t)^{(nq-q-r)/(q-1)}(n-1+t)\,dt\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left(\int_0^1 (1-t)^r(n-1+t)\,\mu^{q\left(\frac{1-t}{2}\right)^\alpha} dt\right)^{1/q}\right.$$
$$\left. + \left(\int_0^1 (1-t)^r(n-1+t)\,\mu^{q\left(\frac{1+t}{2}\right)^\alpha} dt\right)^{1/q}\right\}, \quad (5.40)$$

The rest of the proof is similar to that of the proof of Theorem 219 by using Lemma 164 and Lemma 165. ■

Corollary 226 *Suppose that the conditions given in Theorem 225 are met and $r = 0$,* *we have*

$$\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx-\sum_{k=1}^{n-1}\frac{k\left[1+\left(-1\right)^k\right]\left(b-a\right)^k}{2^{k+1}\left(k+1\right)!}f^{(k)}\left(A\left(a,b\right)\right)\right|$$
$$\leq\frac{\left(b-a\right)^n}{2^{n+1}n!}\left[\frac{\left(q-1\right)\left(n^2q-2n+1\right)}{\left(nq-1\right)\left(nq+q-2\right)}\right]^{1-\frac{1}{q}}\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m\mu^\theta$$
$$\times\left\{\left[K\left(1;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q}+\left[K\left(1;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\}. \quad (5.41)$$

Corollary 227 *Suppose that the hypothesis given in Theorem 225 are valid and* $r=\left(n-1\right)q$*, we have*

$$\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{b-a}\int_a^b f\left(x\right)dx-\sum_{k=1}^{n-1}\frac{k\left[1+\left(-1\right)^k\right]\left(b-a\right)^k}{2^{k+1}\left(k+1\right)!}f^{(k)}\left(A\left(a,b\right)\right)\right|$$
$$\leq\frac{\left(b-a\right)^n}{2^{n+1}n!}\left[\frac{q-2n+2r+1}{2\left(q-1\right)}\right]^{1-\frac{1}{q}}\left[f^{(n)}\left(\frac{b}{m}\right)\right]^m\mu^\theta$$
$$\times\left\{\left[K\left(\left(n-1\right)q+1;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q}+\left[K\left(\left(n-1\right)q+1;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\}. \quad (5.42)$$

Remark 228 *Several interesting inequalities for* m*-logarithmically convex functions can be obtained by setting* $\alpha=1$ *in the results presented in this section. However, we leave the details to the interested reader.*

Remark 229 *We can get several interesting inequalities for logarithmically convex functions by setting* $\alpha=1$ *and* $m=1$ *in the results proved above. However, the details are left to the interested reader.*

5.4 Applications to Special Means

In what follows we use the means defined in Section 2.13 and determine new and motivating inequalities containing means with the help of our findings of Section 5.3.

Theorem 230 *Let $0 < a < b \leq 1$, $r < 0$, $r \neq -1$ and $q \geq 1$.*

1. *If $r \neq -2$, then*

$$\begin{aligned}
&\left|A\left(a^{r+1}, b^{r+1}\right) - L_{r+1}^{r+1}(a,b)\right| \\
&\leq (b-a)\left(\frac{1}{2}\right)^{3-\frac{2}{q}} |r+1| \left(\frac{1}{qr(\ln b - \ln a)}\right)^{1/q} \\
&\times \left\{ b^{r/2}\left[b^{qr/2} - L\left(a^{qr/2}, b^{qr/2}\right)\right]^{1/q} + a^{r/2}\left[L\left(a^{qr/2}, b^{qr/2}\right) - a^{qr/2}\right]^{1/q} \right\}.
\end{aligned}$$

2. *If $r = -2$, then*

$$\begin{aligned}
&\left|\frac{1}{H(a,b)} - \frac{1}{L(a,b)}\right| \leq (b-a)\left(\frac{1}{2}\right)^{3-\frac{1}{q}} \left(\frac{1}{q(\ln a - \ln b)}\right)^{1/q} \\
&\times \left\{ b^{-1}\left[b^{-q} - L\left(a^{-q}, b^{-q}\right)\right]^{1/q} + a^{-1}\left[L\left(a^{-q}, b^{-q}\right) - a^{-q}\right]^{1/q} \right\}.
\end{aligned}$$

Proof. Let $f(x) = \frac{x^{r+1}}{r+1}$ for $0 < x \leq 1$. Then $\left|f'(x)\right| = x^r$ and

$$\ln\left|f'(A_\lambda(x,y))\right|^q \leq A_\lambda\left(\ln\left|f'(x)\right|^q, \ln\left|f'(y)\right|^q\right)$$

for $0 < x$, $y \leq 1$, $0 \leq \lambda \leq 1$ and $q \geq 1$. This shows that $\left|f'(x)\right|^q = x^{rq}$ is logarithmically convex function on $(0,1]$ so that we have $(\alpha, m) = (1,1)$, $\mu = \left|\frac{f'(a)}{f'(b)}\right|$ and $\theta = \frac{1}{2}$.

Since$\left|f'(a)\right| > \left|f'(b)\right| = b^r \geq 1$, hence

$$\mu = \left|\frac{f'(a)}{f'(b)}\right| = \left(\frac{a}{b}\right)^r$$

and

$$
\begin{aligned}
&\left[f'\left(\frac{b}{m}\right)\right]^{m}\mu^{\theta}\left\{\left[G\left(1;\mu^{-\frac{\alpha q}{2}}\right)\right]^{1/q}+\left[G\left(1;\mu^{\frac{\alpha q}{2}}\right)\right]^{1/q}\right\} \\
&=\sqrt{f'(b)\,f'(a)}\left\{\left[G\left(1;\mu^{-\frac{q}{2}}\right)\right]^{1/q}+\left[G\left(1;\mu^{\frac{q}{2}}\right)\right]^{1/q}\right\} \\
&=\sqrt{a^{r}b^{r}}\left\{\left[G\left(1;\mu^{-\frac{q}{2}}\right)\right]^{1/q}+\left[G\left(1;\mu^{\frac{q}{2}}\right)\right]^{1/q}\right\} \\
&=\left(\frac{2}{qr\left(\ln b-\ln a\right)}\right)^{1/q}\left\{b^{r/2}\left[b^{qr/2}-L\left(a^{qr/2},b^{qr/2}\right)\right]^{1/q}\right. \\
&\left.+a^{r/2}\left[L\left(a^{qr/2},b^{qr/2}\right)-a^{qr/2}\right]^{1/q}\right\}.
\end{aligned}
$$

Substituting the above quantities in Corollary 215, we get the required results. ■

Remark 231 *Many interesting inequalities of means can be obtained from the other results of Section 5.3, however, the details are left to the readers.*

Chapter 6

Inequalities of H-H and Fejér type for preinvex and h-preinvex functions

The content of this chapters is taken from the references "M. A. Latif and S. S. Dragomir, New Inequalities of Hermite-Hadamard and Fejer type via preinvexity, Journal of Computational Analysis and Application. 19 (4) 2015 725-739." and "M. A. Latif, S. S. Dragomir and E. Momoniat, Some weighted integral inequalities for differentiable h-preinvex functions, Georgian Mathematical Journal. (Accepted)".

6.1 Introduction

The main purpose of this chapter is to provide new weighted Hermite-Hadamard type inequalities, Hermite-Hadamard and Fejér type inequalities for functions whose derivatives in absolute value are preinvex and h-preinvex. Since notion of preinvex and h-preinvex functions are further generalizations of convex and h-convex functions, hence the results presented in this chapter provide extensions of those given in earlier works. We will deduce some interesting inequalities from the obtained results when h is super-additive. We will also obtain inequalities of Hermite-Hadamard type for functions whose derivatives in absolute value are s-preinvex as a special case of our established results.

In this chapter, first we establish two weighted identities defined on an open invex subset of set of real numbers and then by the usage of the Hölder integral inequality, the notion of preinvexity and h-preinvexity, we present Fejér type and weighted integral inequalities of Hermite-Hadamard type for functions whose derivatives in absolute value raised to certain powers are preinvex and h-preinvex functions. The results generalize those results proved in a very recent papers [146] and [53]. We give applications of few results to special means of positive real numbers

The organization of this chapter is as follows:

In Section 6.2, a weighted identity for differentiable functions defined on an invex subset of real numbers is proved and then Fejér type inequalities are established by using this identity, the notion of preinvexity and analysis. In Section 6.3, weighted Hermite-Hadamard type inequalities are obtained with the help of a new weighted identity for functions defined on invex subset of real numbers, h-preinvexity and analysis. A number of inequalities Hermite-Hadamard type are also deduced for s-preinvex functions and when h satisfies super-additivity conditions. Applications of some results of Section 6.2 are given in Section 6.3.

6.2 New inequalities of H-H and Fejér type via preinvexity

The following Lemma is essential in establishing our main results in this section:

Lemma 232 *Suppose that $f : K \to \mathbb{R}$ is a is mapping defined on a set K, where K is an open invex interval with respect to $\eta : K \times K \to \mathbb{R}$ of the set of real numbers. Suppose also that $f : K \to \mathbb{R}$ possesses the first order derivative f' on K and that f' belongs to the space $L([a, a + \eta(b, a)])$ of all integrable function over $[a, a + \eta(b, a)]$, where $a\ b \in K$ with $\eta(b, a) > 0$. If $w : [a, a + \eta(b, a)] \to [0, \infty)$ is an integrable mapping, we have the* *identity*

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,w(x)dx-\frac{1}{\eta(b,a)}f\left(A\left(2a,\eta(b,a)\right)\right)\int_a^{a+\eta(b,a)} w(x)dx$$
$$=\eta(b,a)\int_0^1 k(t)f'\left(a+t\eta(b,a)\right)dt, \quad (6.1)$$

where

$$k(t)=\begin{cases}\int_0^t w\left(a+s\eta(b,a)\right)ds, & t\in\left[0,\frac{1}{2}\right)\\ \\ -\int_t^1 w\left(a+s\eta(b,a)\right)ds, & t\in\left[\frac{1}{2},1\right].\end{cases}$$

Proof. We observe that

$$I=\int_0^1 k(t)f'\left(a+t\eta(b,a)\right)dt=\int_0^{\frac{1}{2}}\left(\int_0^t w\left(a+s\eta(b,a)\right)ds\right)f'\left(a+t\eta(b,a)\right)dt$$
$$+\int_{\frac{1}{2}}^1\left(-\int_t^1 w\left(a+s\eta(b,a)\right)ds\right)f'\left(a+t\eta(b,a)\right)dt=I_1+I_2. \quad (6.2)$$

By integration by parts, we get

$$I_1=\left(\int_0^t w\left(a+s\eta(b,a)\right)ds\right)\frac{f\left(a+t\eta(b,a)\right)}{\eta(b,a)}\Bigg|_0^{\frac{1}{2}}$$
$$-\frac{1}{\eta(b,a)}\int_0^{\frac{1}{2}} w\left(a+t\eta(b,a)\right)f\left(a+t\eta(b,a)\right)dt$$
$$=\frac{f\left(A\left(2a,\eta(b,a)\right)\right)}{\eta(b,a)}\int_0^{\frac{1}{2}} w\left(a+t\eta(b,a)\right)dt$$
$$-\frac{1}{\eta(b,a)}\int_0^{\frac{1}{2}} w\left(a+t\eta(b,a)\right)f\left(a+t\eta(b,a)\right)dt. \quad (6.3)$$

Similarly, we also have

$$I_2 = \left(-\int_t^1 w\left(a + s\eta\left(b,a\right)\right) ds \right) \frac{f\left(a + t\eta\left(b,a\right)\right)}{\eta\left(b,a\right)} \Bigg|_{\frac{1}{2}}^{1}$$
$$- \frac{1}{\eta\left(b,a\right)} \int_{\frac{1}{2}}^{1} w\left(a + t\eta\left(b,a\right)\right) f\left(a + t\eta\left(b,a\right)\right) dt$$
$$= \frac{f\left(A\left(2a, \eta\left(b,a\right)\right)\right)}{\eta\left(b,a\right)} \int_{\frac{1}{2}}^{1} w\left(a + t\eta\left(b,a\right)\right) dt$$
$$- \frac{1}{\eta\left(b,a\right)} \int_{\frac{1}{2}}^{1} w\left(a + t\eta\left(b,a\right)\right) f\left(a + t\eta\left(b,a\right)\right) dt. \quad (6.4)$$

From (6.3) and (6.4), we get

$$I = \frac{f\left(A\left(2a, \eta\left(b,a\right)\right)\right)}{\eta\left(b,a\right)} \int_0^1 w\left(a + t\eta\left(b,a\right)\right) dt$$
$$- \frac{1}{\eta\left(b,a\right)} \int_0^1 w\left(a + t\eta\left(b,a\right)\right) f\left(a + t\eta\left(b,a\right)\right) dt.$$

Using the change of variable $x = a + t\eta\left(b,a\right)$ for $t \in [0,1]$ and multiplying both sides by $\eta\left(b,a\right)$, we get (6.1). The lemma is thus established. ■

Remark 233 *If we take $w(x) = 1$, $x \in [a, a + t\eta\left(b,a\right)]$ in Lemma 232, then (6.1) reduces to*

$$\frac{1}{\eta\left(b,a\right)} \int_a^{a+\eta(b,a)} f\left(x\right) dx + f\left(A\left(2a, \eta\left(b,a\right)\right)\right)$$
$$= \eta\left(b,a\right) \int_0^1 k(t) f'\left(a + t\eta\left(b,a\right)\right) dt, \quad (6.5)$$

where

$$k(t) = \begin{cases} t, & t \in \left[0, \frac{1}{2}\right) \\ t - 1, & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

This is one of the results from [146].

Remark 234 *If $\eta(b,a) = b-a$ in Lemma 232, then (6.1) becomes Lemma 2.1 from [144, page 379].*

Now using Lemma 232, we prove our results:

Theorem 235 *Suppose that $f : K \to \mathbb{R}$ is a mapping defined on K, where K is an open invex interval with respect to $\eta : K \times K \to \mathbb{R}$ of the set of real numbers. Assume also that $f : K \to \mathbb{R}$ possesses first order derivative f' over K and that f' is in the space $L([a, a+\eta(b,a)])$ of all integrable functions over $[a, a+\eta(b,a)]$. If $w : [a, a+\eta(b,a)] \to [0,\infty)$ is an integrable mapping symmetric to $A(2a, \eta(b,a))$, where $a, b \in K$ with $\eta(b,a) > 0$ and $|f'|$ is preinvex on K, the following inequality holds*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,w(x)dx - \frac{1}{\eta(b,a)} f(A(2a,\eta(b,a)))\int_a^{a+\eta(b,a)} w(x)dx\right|$$
$$\leq \left(\frac{1}{\eta(b,a)}\int_a^{a+\frac{1}{2}\eta(b,a)} [\eta(b,a) - 2(x-a)]\,w(x)dx\right) A\left(\left|f'(a)\right|, \left|f'(b)\right|\right). \quad (6.6)$$

Proof. From Lemma 232 and the preinvexity of $|f'|$ on K, we have

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,w(x)dx - \frac{1}{\eta(b,a)} f(A(2a,\eta(b,a)))\int_a^{a+\eta(b,a)} w(x)dx\right|$$
$$\leq \eta(b,a)\int_0^{\frac{1}{2}}\left(\int_0^t w(a+s\eta(b,a))\,ds\right)\left[(1-t)\left|f'(a)\right| + t\left|f'(b)\right|\right]dt$$
$$+\,\eta(b,a)\int_{\frac{1}{2}}^1\left(\int_t^1 w(a+s\eta(b,a))\,ds\right)\left[(1-t)\left|f'(a)\right| + t\left|f'(b)\right|\right]dt \quad (6.7)$$

By the change of the order of integration, we have

$$\int_0^{\frac{1}{2}}\int_0^{t} w\left(a+s\eta\left(b,a\right)\right)\left[\left(1-t\right)\left|f'\left(a\right)\right|+t\left|f'\left(b\right)\right|\right]dsdt$$
$$=\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w\left(a+s\eta\left(b,a\right)\right)\left[\left(1-t\right)\left|f'\left(a\right)\right|+t\left|f'\left(b\right)\right|\right]dtds$$
$$=\int_0^{\frac{1}{2}} w\left(a+s\eta\left(b,a\right)\right)\left[\left(\frac{\left(1-s\right)^2}{2}-\frac{1}{8}\right)\left|f'\left(a\right)\right|+\left(\frac{1}{8}-\frac{s^2}{2}\right)\left|f'\left(b\right)\right|\right]ds. \quad (6.8)$$

Using the change of variable $x=a+s\eta\left(b,a\right)$ for $s\in[0,1]$, we have from (6.9) that

$$\int_0^{\frac{1}{2}}\int_0^{t} w\left(a+s\eta\left(b,a\right)\right)\left[\left(1-t\right)\left|f'\left(a\right)\right|+t\left|f'\left(b\right)\right|\right]dsdt$$
$$=\frac{1}{\eta\left(b,a\right)}\left|f'\left(a\right)\right|\int_a^{a+\frac{1}{2}\eta(b,a)}\left(\frac{1}{2}\left(1-\frac{x-a}{\eta\left(b,a\right)}\right)^2-\frac{1}{8}\right)w\left(x\right)dx$$
$$+\frac{1}{\eta\left(b,a\right)}\left|f'\left(b\right)\right|\int_a^{a+\frac{1}{2}\eta(b,a)}\left(\frac{1}{8}-\frac{1}{2}\left(\frac{x-a}{\eta\left(b,a\right)}\right)^2\right)w\left(x\right)dx. \quad (6.9)$$

Similarly by change of order of integration and using the fact that w is symmetric to $a+\frac{1}{2}\eta\left(b,a\right)$, we obtain

$$\int_{\frac{1}{2}}^{1}\int_t^{1} w\left(a+s\eta\left(b,a\right)\right)\left[\left(1-t\right)\left|f'\left(a\right)\right|+t\left|f'\left(b\right)\right|\right]dsdt$$
$$=\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s} w\left(a+\left(1-s\right)\eta\left(b,a\right)\right)\left[\left(1-t\right)\left|f'\left(a\right)\right|+t\left|f'\left(b\right)\right|\right]dtds$$
$$=\frac{1}{\eta\left(b,a\right)}\left|f'\left(a\right)\right|\int_{\frac{1}{2}}^{1}\left(\frac{1}{8}-\frac{1}{2}\left(1-s\right)^2\right)w\left(a+\left(1-s\right)\eta\left(b,a\right)\right)ds$$
$$+\frac{1}{\eta\left(b,a\right)}\left|f'\left(b\right)\right|\int_{\frac{1}{2}}^{1}\left(\frac{s^2}{2}-\frac{1}{8}\right)w\left(a+\left(1-s\right)\eta\left(b,a\right)\right)ds \quad (6.10)$$

By the change of variable $x = a + (1 - s)\,\eta(b, a)$, we get form (6.10) that

$$\int_{\frac{1}{2}}^{1} \int_{t}^{1} w\left(a + s\eta(b, a)\right) \left[(1 - t) \left|f'(a)\right| + t \left|f'(b)\right|\right] dsdt$$
$$= \frac{1}{\eta(b, a)} \left|f'(a)\right| \int_{a}^{a+\frac{1}{2}\eta(b,a)} \left(\frac{1}{8} - \frac{1}{2}\left(\frac{x - a}{\eta(b, a)}\right)^2\right) w(x)\,dx$$
$$+ \frac{1}{\eta(b, a)} \left|f'(b)\right| \int_{a}^{a+\frac{1}{2}\eta(b,a)} \left(\frac{1}{2}\left(1 - \frac{x - a}{\eta(b, a)}\right)^2 - \frac{1}{8}\right) w(x)\,dx. \quad (6.11)$$

Substituting (6.9) and (6.11) in (6.7) and simplifying, we get the inequality (6.6). ■

Corollary 236 *If we take $w(x) = 1$, for $x \in [a, a + \eta(b, a)]$ in Theorem 235, we get*

$$\left|\frac{1}{\eta(b, a)} \int_{a}^{a+\eta(b,a)} f(x)\,dx - f\left(A\left(2a, \eta(b, a)\right)\right)\right| \leq \frac{\eta(b, a)}{8} A\left(\left|f'(a)\right|, \left|f'(b)\right|\right). \quad (6.12)$$

The above corollary is Theorem 5 from [146].

Remark 237 *If $\left|f'\right|$ is convex on $[a, b]$, then $\eta(b, a) = b - a$. Hence from Theorem 235, and using the symmetricity of w about $A(a, b)$, we get Theorem 2.3 from [144, page 380].*

Theorem 238 *Suppose that $f : K \to \mathbb{R}$ is a mapping defined on K, where K is an open invex interval with respect to $\eta : K \times K \to \mathbb{R}$ of the set of real numbers. Assume also that $f : K \to \mathbb{R}$ possesses first order derivative f' over K and that f' is in the space $L([a, a + \eta(b, a)])$ of all integrable functions over $[a, a + \eta(b, a)]$. If $w : [a, a + \eta(b, a)] \to [0, \infty)$ is an integrable mapping symmetric to $A(2a, \eta(b, a))$, where a, $b \in K$ with $\eta(b, a) > 0$ and $\left|f'\right|^q$ is preinvex on K for $q > 1$, the following inequality* *holds*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,w(x)dx-\frac{1}{\eta(b,a)}f\left(A\left(2a,\eta(b,a)\right)\right)\int_a^{a+\eta(b,a)} w(x)dx\right|$$
$$\leq \eta(b,a)\left(\frac{1}{(\eta(b,a))^2}\int_a^{a+\frac{1}{2}\eta(b,a)}\left[\frac{\eta(b,a)}{2}-(x-a)\right]w^p(x)\,dx\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{A\left(2\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)}{12}\right)^{\frac{1}{q}}+\left(\frac{A\left(\left|f'(a)\right|^q,2\left|f'(b)\right|^q\right)}{12}\right)^{\frac{1}{q}}\right], \quad (6.13)$$

where p and q are Hölder conjugates of each other.

Proof. From Lemma 232 and change of order of integration, we get

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,w(x)dx-\frac{1}{\eta(b,a)}f\left(A\left(2a,\eta(b,a)\right)\right)\int_a^{a+\eta(b,a)} w(x)dx\right|$$
$$\leq \eta(b,a)\int_0^{\frac{1}{2}}\left(\int_0^t w\left(a+s\eta(b,a)\right)ds\right)\left|f'\left(a+t\eta(b,a)\right)\right|dt$$
$$+\eta(b,a)\int_{\frac{1}{2}}^1\left(\int_t^1 w\left(a+s\eta(b,a)\right)ds\right)\left|f'\left(a+t\eta(b,a)\right)\right|dt$$
$$=\eta(b,a)\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w\left(a+s\eta(b,a)\right)\left|f'\left(a+t\eta(b,a)\right)\right|dtds$$
$$+\eta(b,a)\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^s w\left(a+s\eta(b,a)\right)\left|f'\left(a+t\eta(b,a)\right)\right|dtds. \quad (6.14)$$

By the Hölder's inequality, we have

$$\eta(b,a)\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w\left(a+s\eta(b,a)\right)\left|f'\left(a+t\eta(b,a)\right)\right|dtds$$
$$\leq \eta(b,a)\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}} w^p\left(a+s\eta(b,a)\right)dtds\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\int_s^{\frac{1}{2}}\left|f'\left(a+t\eta(b,a)\right)\right|^q dtds\right)^{\frac{1}{q}}. \quad (6.15)$$

Since $\left|f'\right|^{q}$ for $q>1$, is preinvex on K, for every $a,b\in K$ and $t\in[0,1]$ we have

$$\left|f'\left(a+t\eta\left(b,a\right)\right)\right|^{q}\leq A_{t}\left(\left|f'\left(b\right)\right|^{q},\left|f'\left(a\right)\right|^{q}\right)$$

hence by solving elementary integrals and using the substitution $x=a+s\eta\left(b,a\right)$, $s\in[0,1]$, we have from (6.15) that

$$\begin{aligned}
&\eta\left(b,a\right)\int_{0}^{\frac{1}{2}}\int_{s}^{\frac{1}{2}}w\left(a+s\eta\left(b,a\right)\right)\left|f'\left(a+t\eta\left(b,a\right)\right)\right|dtds\\
&\leq\eta\left(b,a\right)\left(\int_{0}^{\frac{1}{2}}\int_{s}^{\frac{1}{2}}w^{p}\left(a+s\eta\left(b,a\right)\right)dtds\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{1}{2}}\int_{s}^{\frac{1}{2}}\left[\left(1-t\right)\left|f'\left(a\right)\right|^{q}+t\left|f'\left(b\right)\right|^{q}\right]dtds\right)^{\frac{1}{q}}\\
&\qquad=\eta\left(b,a\right)\left(\frac{1}{\left(\eta\left(b,a\right)\right)^{2}}\int_{a}^{a+\frac{1}{2}\eta\left(b,a\right)}\left[\frac{\eta\left(b,a\right)}{2}-\left(x-a\right)\right]w^{p}\left(x\right)dx\right)^{\frac{1}{p}}\\
&\qquad\qquad\times\left(\frac{A\left(2\left|f'\left(a\right)\right|^{q},\left|f'\left(b\right)\right|^{q}\right)}{12}\right)^{\frac{1}{q}}. \qquad (6.16)
\end{aligned}$$

Analogously, using the symmetricity of w about $a+\frac{1}{2}\eta\left(b,a\right)$, we also have

$$\begin{aligned}
&\eta\left(b,a\right)\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{s}w\left(a+s\eta\left(b,a\right)\right)\left|f'\left(a+t\eta\left(b,a\right)\right)\right|dtds\\
&\qquad\leq\eta\left(b,a\right)\left(\frac{1}{\left(\eta\left(b,a\right)\right)^{2}}\int_{a}^{a+\frac{1}{2}\eta\left(b,a\right)}\left[\frac{\eta\left(b,a\right)}{2}-\left(x-a\right)\right]w^{p}\left(x\right)dx\right)^{\frac{1}{p}}\\
&\qquad\qquad\times\left(\frac{A\left(\left|f'\left(a\right)\right|^{q},2\left|f'\left(b\right)\right|^{q}\right)}{12}\right)^{\frac{1}{q}} \qquad (6.17)
\end{aligned}$$

Using (6.16) and (6.17) in (6.14), we get the required inequality. ■

Corollary 239 *If the conditions of Theorem 238 are satisfied and if* $w(x)=1$, $x\in$

$[a, a+\eta(b,a)]$, we have the following inequality

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx + f(A(2a,\eta(b,a)))\right| \le \eta(b,a)\left(\frac{1}{8}\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{A\left(2\left|f'(a)\right|^q, \left|f'(b)\right|^q\right)}{12}\right)^{\frac{1}{q}} + \left(\frac{A\left(\left|f'(a)\right|^q, 2\left|f'(b)\right|^q\right)}{12}\right)^{\frac{1}{q}}\right], \quad (6.18)$$

where p and q are Hölder conjugates of each other.

Corollary 240 *[144, Theorem 2.5, page 381] Suppose $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a differentiable mapping on I°, a, $b \in I$ with $a < b$. Let $w : [a,b] \to [0,\infty)$ be an integrable mapping symmetric to $\frac{a+b}{2}$ such that $f' \in L([a,b])$. If $\left|f'\right|^q$ is convex for $q > 1$ on $[a,b]$, we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,w(x)dx - \frac{1}{b-a}f(A(a,b))\int_a^b w(x)dx\right|$$
$$\le (b-a)\left(\frac{1}{(b-a)^2}\int_{\frac{a+b}{2}}^b (x - A(a,b))\,w^p(x)\,dx\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{A\left(2\left|f'(a)\right|^q, \left|f'(b)\right|^q\right)}{12}\right)^{\frac{1}{q}} + \left(\frac{A\left(\left|f'(a)\right|^q, 2\left|f'(b)\right|^q\right)}{12}\right)^{\frac{1}{q}}\right], \quad (6.19)$$

where p and q are Hölder conjugates of each other.

Proof. It follows from Theorem 238 by taking $\eta(b,a) = b-a$ and using the symmetry of w about $\frac{a+b}{2}$. ■

For our next results we need the following Lemma:

Lemma 241 *Suppose that $f : K \to \mathbb{R}$ is a is mapping defined on a set K, where K is an open invex interval with respect to $\eta : K \times K \to \mathbb{R}$ of the set of real numbers. Suppose also that $f : K \to \mathbb{R}$ possesses the first order derivative f' on K and that f' belongs to the space $L([a, a+\eta(b,a)])$ of all integrable function over $[a, a+\eta(b,a)]$, where a $b \in K$*

with $\eta(b,a) > 0$. *If* $w : [a, a+\eta(b,a)] \to [0,\infty)$ *is an integrable mapping, we have the following identity*

$$-\frac{A\left(f(a), f(a+\eta(b,a))\right)}{\eta(b,a)} \int_a^{a+\eta(b,a)} w(x)dx + \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\, w(x)dx$$
$$= \frac{\eta(b,a)}{2} \int_0^1 p(t) f'\left(a+t\eta(b,a)\right) dt, \quad (6.20)$$

where

$$p(t) = \int_t^1 w\left(a+s\eta(b,a)\right) ds - \int_0^t w\left(a+s\eta(b,a)\right) ds, \ \ t \in [0,1].$$

Proof. It suffices to note that

$$\begin{aligned} J &= \int_0^1 p(t) f'\left(a+t\eta(b,a)\right) dt \\ &= -\int_0^1 \left(\int_0^t w\left(a+s\eta(b,a)\right) ds\right) f'\left(a+t\eta(b,a)\right) dt \\ &\quad + \int_0^1 \left(\int_t^1 w\left(a+s\eta(b,a)\right) ds\right) f'\left(a+t\eta(b,a)\right) dt = J_1 + J_2 \quad (6.21) \end{aligned}$$

By integration by parts, we get

$$\begin{aligned} J_1 &= -\left.\frac{\left(\int_0^t w\left(a+s\eta(b,a)\right) ds\right) f\left(a+t\eta(b,a)\right)}{\eta(b,a)}\right|_0^1 \\ &\quad + \frac{1}{\eta(b,a)} \int_0^1 w\left(a+t\eta(b,a)\right) f\left(a+t\eta(b,a)\right) dt \\ &= -\frac{f\left(a+\eta(b,a)\right)}{\eta(b,a)} \int_0^1 w\left(a+t\eta(b,a)\right) dt \\ &\quad + \frac{1}{\eta(b,a)} \int_0^1 w\left(a+t\eta(b,a)\right) f\left(a+t\eta(b,a)\right) dt. \quad (6.22) \end{aligned}$$

Similarly. we also have

$$J_2 = -\frac{f(a)}{\eta(b,a)} \int_0^1 w\left(a+t\eta(b,a)\right) dt + \frac{1}{\eta(b,a)} \int_0^1 w\left(a+t\eta(b,a)\right) f\left(a+t\eta(b,a)\right) dt. \quad (6.23)$$

Using (6.22) and (6.23) in (6.21), we obtain

$$J = -\frac{f(a) + f(a + \eta(b,a))}{\eta(b,a)} \int_0^1 w(a + t\eta(b,a))\, dt + \frac{2}{\eta(b,a)} \int_0^1 w(a + t\eta(b,a))\, f(a + t\eta(b,a))\, dt \quad (6.24)$$

By the change of variable $x = a + t\eta(b,a)$ for $t \in [0,1]$ and by multiplying both sides of (6.24) by $\frac{\eta(b,a)}{2}$, we get (6.20). Hence the proof of the lemma is thus accomplished. ■

Remark 242 *If we take $w(x) = 1$, $x \in [a, a + \eta(b,a)]$, we get*

$$-A(f(a), f(a + \eta(b,a)) + \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\, dx = \frac{\eta(b,a)}{2} \int_0^1 (1 - 2t) f'(a + t\eta(b,a))\, dt, \quad (6.25)$$

which is Lemma 2.1 from [13, Page 3].

Theorem 243 *Suppose that $f : K \to \mathbb{R}$ is a mapping defined on K, where K is an open invex interval with respect to $\eta : K \times K \to \mathbb{R}$ of the set of real numbers. Assume also that $f : K \to \mathbb{R}$ possesses first order derivative f' over K and that f' is in the space $L([a, a + \eta(b,a)])$ of all integrable functions over $[a, a + \eta(b,a)]$. If $w : [a, a + \eta(b,a)] \to [0, \infty)$ is an integrable mapping symmetric to $A(2a, \eta(b,a))$, where a, $b \in K$ with $\eta(b,a) > 0$ and $|f'|^q$ is preinvex on K for $q > 1$, the following inequality holds*

$$\left| \frac{A(f(a), f(a + \eta(b,a))}{\eta(b,a)} \int_a^{a+\eta(b,a)} w(x) dx - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\, w(x) dx \right| \leq \frac{1}{2} \left(\int_0^1 g^p(t) dt \right)^{\frac{1}{p}} \left[A\left(\left| f'(a) \right|^q, \left| f'(b) \right|^q \right) \right]^{\frac{1}{q}}, \quad (6.26)$$

where

$$g(t) = \left| \int_{a+t\eta(b,a)}^{a+(1-t)\eta(b,a)} w(x)\, dx \right|, t \in [0,1] \ \ and \ \frac{1}{p} + \frac{1}{q} = 1.$$

Proof. From Lemma 241, we get

$$\left|\frac{A\left(f(a),f(a+\eta\left(b,a\right)\right)}{\eta\left(b,a\right)}\int_{a}^{a+\eta(b,a)}w(x)dx-\frac{1}{\eta\left(b,a\right)}\int_{a}^{a+\eta(b,a)}f\left(x\right)w(x)dx\right|$$
$$\leq\frac{\eta\left(b,a\right)}{2}\int_{0}^{1}\left|\int_{t}^{1}w\left(a+s\eta\left(b,a\right)\right)ds-\int_{0}^{t}w\left(a+s\eta\left(b,a\right)\right)ds\right|\left|f^{'}\left(a+t\eta\left(b,a\right)\right)\right|dt. \tag{6.27}$$

Since w is symmetric to $a+\frac{1}{2}\eta\left(b,a\right)$, we can write

$$\begin{aligned}
&\int_{t}^{1}w\left(a+s\eta\left(b,a\right)\right)ds-\int_{0}^{t}w\left(a+s\eta\left(b,a\right)\right)ds\\
&=\int_{t}^{1}w\left(a+s\eta\left(b,a\right)\right)ds-\int_{0}^{t}w\left(a+\left(1-s\right)\eta\left(b,a\right)\right)ds\\
&=\frac{1}{\eta\left(b,a\right)}\int_{a+t\eta(b,a)}^{a+\eta(b,a)}w\left(x\right)dx+\frac{1}{\eta\left(b,a\right)}\int_{a+\eta(b,a)}^{a+(1-t)\eta(b,a)}w\left(x\right)dx\\
&=\begin{cases}\frac{1}{\eta(b,a)}\int_{a+t\eta(b,a)}^{a+(1-t)\eta(b,a)}w\left(x\right)dx, & t\in\left[0,\frac{1}{2}\right]\\ \\ -\frac{1}{\eta(b,a)}\int_{a+(1-t)\eta(b,a)}^{a+t\eta(b,a)}w\left(x\right)dx, & t\in\left[\frac{1}{2},1\right].\end{cases}
\end{aligned} \tag{6.28}$$

Using (6.28) in (6.27) we obtain

$$\left|\frac{A\left(f(a),f(a+\eta\left(b,a\right)\right)}{\eta\left(b,a\right)}\int_{a}^{a+\eta(b,a)}w(x)dx-\frac{1}{\eta\left(b,a\right)}\int_{a}^{a+\eta(b,a)}f\left(x\right)w(x)dx\right|$$
$$\leq\frac{1}{2}\int_{0}^{1}g(x)\left|f^{'}\left(a+t\eta\left(b,a\right)\right)\right|dt, \tag{6.29}$$

where

$$g(t)=\left|\int_{a+t\eta(b,a)}^{a+(1-t)\eta(b,a)}w\left(x\right)dx\right|,t\in\left[0,1\right].$$

By Hölder's inequality, it follows from (6.29) that

$$\left|\frac{A\left(f(a),f(a+\eta\left(b,a\right)\right)}{\eta\left(b,a\right)}\int_{a}^{a+\eta(b,a)}w(x)dx-\frac{1}{\eta\left(b,a\right)}\int_{a}^{a+\eta(b,a)}f\left(x\right)w(x)dx\right|$$
$$\leq\frac{1}{2}\left(\int_{0}^{1}g^{p}(t)dt\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|f^{'}\left(a+t\eta\left(b,a\right)\right)\right|^{q}dt\right)^{\frac{1}{q}}.\quad(6.30)$$

Since $\left|f^{'}\left(a+t\eta\left(b,a\right)\right)\right|^{q}$ is preinvex on K, for every a, $b\in K$ and $t\in[0,1]$, we have

$$\left|f^{'}\left(a+t\eta\left(b,a\right)\right)\right|^{q}\leq A_{t}\left(\left|f^{'}\left(b\right)\right|^{q},\left|f^{'}\left(a\right)\right|^{q}\right)$$

and hence from (6.30), we get that

$$\left|\frac{A\left(f(a),f(a+\eta\left(b,a\right)\right)}{\eta\left(b,a\right)}\int_{a}^{a+\eta(b,a)}w(x)dx-\frac{1}{\eta\left(b,a\right)}\int_{a}^{a+\eta(b,a)}f\left(x\right)w(x)dx\right|$$
$$\leq\frac{1}{2}\left(\int_{0}^{1}g^{p}(t)dt\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left[(1-t)\left|f^{'}\left(a\right)\right|^{q}+t\left|f^{'}\left(b\right)\right|^{q}\right]dt\right)^{\frac{1}{q}}$$
$$=\frac{1}{2}\left(\int_{0}^{1}g^{p}(t)dt\right)^{\frac{1}{p}}\left(\frac{\left|f^{'}\left(a\right)\right|^{q}+\left|f^{'}\left(b\right)\right|^{q}}{2}\right)^{\frac{1}{q}},\quad(6.31)$$

The inequality (6.31 is the required result. ■

Corollary 244 *If we take $\eta\left(b,a\right)=b-a$ in Theorem 243, we have the inequality*

$$\left|\frac{A\left(f(a),f(b)\right)}{b-a}\int_{a}^{b}w(x)dx-\frac{1}{b-a}\int_{a}^{b}f\left(x\right)w(x)dx\right|$$
$$\leq\frac{1}{2}\left(\int_{0}^{1}g^{p}(t)dt\right)^{\frac{1}{p}}\left(\left[A\left(\left|f^{'}\left(a\right)\right|^{q},\left|f^{'}\left(b\right)\right|^{q}\right)\right]\right)^{\frac{1}{q}},\quad(6.32)$$

where

$$g(t)=\left|\int_{tb+(1-t)a}^{ta+(1-t)b}w\left(x\right)dx\right|,\ t\in[0,1]\ \text{and}\ \frac{1}{p}+\frac{1}{q}=1.$$

The above corollary is Theorem 2.8 from [144, page 383].

Corollary 245 *[13, Theorem 2.2, page 4] Let f be as in Theorem 243 and $w(x) = 1$, $x \in [a, a + \eta(b, a)]$. Then*

$$\left| A\left(f(a), f(a + \eta(b, a))\right) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)\, dx \right| \\ \leq \frac{\eta(b, a)}{2(p+1)^{\frac{1}{p}}} \left[A\left(\left| f'(a) \right|^q, \left| f'(b) \right|^q \right) \right]^{\frac{1}{q}}, \quad (6.33)$$

where p and q are Hölder conjugates of each other.

Proof. It follows from the fact that

$$\int_0^1 g^p(t)dt = \int_0^1 \left(\left| \int_{a+t\eta(b,a)}^{a+(1-t)\eta(b,a)} dx \right|^p \right) dt \\ = (\eta(b, a))^p \int_0^1 |1 - 2t|^p\, dt = \frac{(\eta(b, a))^p}{p+1}.$$

■

Corollary 246 *[27] If the conditions of Theorem 243 are fulfilled and if $w(x) = 1$, $x \in [a, b]$ and $\eta(b, a) = b - a$, then we have the inequality:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}} \left[A\left(\left| f'(a) \right|^q, \left| f'(b) \right|^q \right) \right]^{\frac{1}{q}}, \quad (6.34)$$

where p and q are Hölder conjugates of each other.

Proof. It follows from Corollary 245. ■

6.3 Some weighted integral inequalities for differentiable h-preinvex functions

In this section we present some Fejér type inequalities for more general condition of preinvexity, namely, the h-preinvexity. In order to prove results of this section we need the following lemma.

Lemma 247 *[95] Suppose that $f : K \to \mathbb{R}$ is a is mapping defined on a set K, where K is an open invex interval with respect to $\eta : K \times K \to \mathbb{R}$ of the set of real numbers. Suppose also that $f : K \to \mathbb{R}$ possesses the first order derivative $f^{'}$ on K and that $f^{'}$ belongs to the space $L\left([a, a+\eta(b,a)]\right)$ of all integrable function over $[a, a+\eta(b,a)]$, where a $b \in K$ with $\eta(b,a) > 0$. If $g : [a, a+\eta(b,a)] \to [0,\infty)$ is an integrable mapping, we have the following identity*

$$\begin{aligned}
&\frac{g(a)}{2}\left[f(a)+f(a+\eta(b,a))\right]-g(a+\eta(b,a))\,f\left(A\left(2a,\eta(b,a)\right)\right)\\
&\qquad+\frac{\eta(b,a)}{4}\int_0^1\left[f\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)+f\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right]\\
&\qquad\times\left[g^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)+g^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right]dt\\
&\qquad=\frac{\eta(b,a)}{4}\left\{\int_0^1\left[g\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)-g\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right.\right.\\
&\left.\left.+g(a+\eta(b,a))\right]\times\left[-f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)+f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right]dt\right\}.
\end{aligned} \tag{6.35}$$

Remark 248 *If we take $\eta(b,a) = b-a$, then Lemma 247 reduces to Lemma 2.1 from [53].*

Now using Lemma 247, we shall propose some new upper bounds for the difference between the leftmost and the middle terms of weighted version of the Hadamard type inequality proved in [118] using h-preinvex mappings.

In what follows we use the notations $L^{'}(a,b,t) = a+\left(\frac{1-t}{2}\right)\eta(b,a)$ and $U^{'}(a,b,t) = a+\left(\frac{1+t}{2}\right)\eta(b,a)$ for our convenience.

Theorem 249 *Suppose that $f : K \to \mathbb{R}$ is a mapping defined on K, where K is an open invex interval with respect to $\eta : K \times K \to \mathbb{R}$ of the set of real numbers. Assume also that $f : K \to \mathbb{R}$ possesses first order derivative $f^{'}$ over K and that $f^{'}$*

is in the space $L([a,a+\eta(b,a)])$ *of all integrable functions over* $[a,a+\eta(b,a)]$. *If* $w:[a,a+\eta(b,a)]\to[0,\infty)$ *is an integrable mapping symmetric to* $A(2a,\eta(b,a))$, *where* $a,\, b\in K$ *with* $\eta(b,a)>0$ *and* $\left|f^{'}\right|$ *is h-preinvex on* K, *the following inequality holds*

$$\begin{aligned}
&\left|\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx - f(A(2a,\eta(b,a)))\int_a^{a+\eta(b,a)} w(x)\,dx\right| \\
&\quad \le \eta(b,a)\,A\left(\left|f^{'}(a)\right|,\left|f^{'}(b)\right|\right)\int_0^1\left[h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right]M^{'}(w;a,b,t)\,dt, \quad (6.36)
\end{aligned}$$

where $M^{'}(w;a,b,t)=\int_a^{L^{'}(a,b,t)}w(x)\,dx$ *for all* $t\in[0,1]$.

Proof. Let $g(t)=\int_a^t w(x)\,dx$ for all $t\in[a,a+\eta(b,a)]$ in Lemma 247, we have

$$\begin{aligned}
&- f(A(2a,\eta(b,a)))\int_a^{a+\eta(b,a)} w(x)\,dx \\
&\quad +\frac{\eta(b,a)}{4}\int_0^1\left[f\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)+f\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right] \\
&\quad\quad \times\left[w\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)+w\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right]dt \\
&\quad\quad =\frac{\eta(b,a)}{4}\left\{\int_0^1\left[\int_a^{L^{'}(a,b,t)}w(x)\,dx+\int_{U^{'}(a,b,t)}^{a+\eta(b,a)}w(x)\,dx\right]\right. \\
&\quad\quad \left.\times\left[-f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)+f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right]dt\right\}. \quad (6.37)
\end{aligned}$$

Since $w(x)$ is symmetric to $a+\frac{1}{2}\eta(b,a)$, we have

$$w\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)=w\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right) \quad (6.38)$$

and

$$\int_a^{L^{'}(a,b,t)}w(x)\,dx=\int_{U^{'}(a,b,t)}^{a+\eta(b,a)}w(x)\,dx \quad (6.39)$$

for all $t \in [0,1]$. Hence by using (6.38), we have

$$\begin{aligned}
&\frac{\eta(b,a)}{4}\int_0^1 \left[f\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)+f\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right] \\
&\qquad \times\left[w\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)+w\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right]dt \\
&\qquad =\frac{\eta(b,a)}{2}\int_0^1 f\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)w\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)dt \\
&\qquad +\frac{\eta(b,a)}{2}\int_0^1 f\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)w\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)dt \\
&=\int_a^{a+\frac{1}{2}\eta(b,a)} f(x)\,w(x)\,dx+\int_{a+\frac{1}{2}\eta(b,a)}^{a+\eta(b,a)} f(x)\,w(x)\,dx=\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx. \quad (6.40)
\end{aligned}$$

Using (6.39) and (6.40) in (6.37) and using the h-preinvexity of $\left|f'\right|$ on K, we have

$$\begin{aligned}
&\left|\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx-f(A(2a,\eta(b,a)))\int_a^{a+\eta(b,a)} w(x)\,dx\right| \\
&\le\frac{\eta(b,a)}{2}\int_0^1 M'(w;a,b,t)\left[\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|+\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|\right]dt \\
&\qquad\le\frac{\eta(b,a)}{2}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\int_0^1\left[h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right]M'(w;a,b,t)\,dt \quad (6.41)
\end{aligned}$$

which is the required inequality (6.36). ■

Corollary 250 *In Theorem 249, if we take $w(x)=\frac{1}{\eta(b,a)}$ for all $x\in[a,a+\eta(b,a)]$, then (6.36) becomes the following inequality for h-preinvex functions*

$$\begin{aligned}
&\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx-f(A(2a,\eta(b,a)))\right| \\
&\qquad\le\frac{\eta(b,a)}{2}A\left(\left|f'(a)\right|,\left|f'(b)\right|\right)\int_0^1(1-t)\left[h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right]dt. \quad (6.42)
\end{aligned}$$

Corollary 251 *If $\eta(b,a)=b-a$ in Theorem 249, we have the following inequality for* *h-convex functions*

$$\left|\int_a^b f(x)\,w(x)\,dx - f(A(a,b))\int_a^b w(x)\,dx\right|$$
$$\leq (b-a)\,A\left(\left|f'(a)\right|,\left|f'(b)\right|\right)\int_0^1\left[h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right]M(w;a,b,t)\,dt, \quad (6.43)$$

where $M(w;a,b,t)=\int_a^{L(a,b,t)} w(x)\,dx$ *for all* $t\in[0,1]$ *and* $L(a,b,t)=\frac{1+t}{2}a+\frac{1-t}{2}b$.

Corollary 252 *If* $\eta(b,a)=b-a$ *and* $w(x)=\frac{1}{b-a}$ *for all* $x\in[a,b]$ *in Theorem 249, we get the following inequality for h-convex functions*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f(A(a,b))\right|$$
$$\leq \frac{(b-a)}{2}A\left(\left|f'(a)\right|,\left|f'(b)\right|\right)\int_0^1 (1-t)\left[h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right]dt. \quad (6.44)$$

Corollary 253 *Suppose* $h(t)=t^s$, $s\in(0,1]$ *in Corollary 250, we have the following inequality for s-preinvex functions*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx - f(A(2a,\eta(b,a)))\right|$$
$$\leq \frac{\eta(b,a)\left(2^{s+1}-1\right)}{2^s(s+1)(s+2)}A\left(\left|f'(a)\right|,\left|f'(b)\right|\right). \quad (6.45)$$

Corollary 254 *If* $\eta(b,a)=b-a$ *in Corollary 253, we have the following inequality for s-convex functions*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f(A(a,b))\right|$$
$$\leq \frac{(b-a)\left(2^{s+1}-1\right)}{2^s(s+1)(s+2)}A\left(\left|f'(a)\right|,\left|f'(b)\right|\right). \quad (6.46)$$

Remark 255 *If* $h(t)=t$, *the inequality (6.36) becomes the inequality proved in [95, Theorem 7] and the inequality (6.43) recaptures the inequality (2.32) for convex functions.*

Corollary 256 *Suppose the assumptions of Theorem 249 are satisfied. If h is super-additive, we have the inequality*

$$\left|\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx - f\left(A\left(2a,\eta(b,a)\right)\right)\int_a^{a+\eta(b,a)} w(x)\,dx\right|$$
$$\leq \eta(b,a)\,h(1)\,A\left(\left[\left|f'(a)\right|,\left|f'(b)\right|\right]\right)\int_0^1 M'(w;a,b,t)\,dt. \quad (6.47)$$

Remark 257 *If $\eta(b,a) = b-a$ in Corollary 256, we get a result when $\left|f'\right|$ is h is a super-additive h-convex function.*

Theorem 258 *Suppose that $f : K \to \mathbb{R}$ is a mapping defined on K, where K is an open invex interval with respect to $\eta : K \times K \to \mathbb{R}$ of the set of real numbers. Assume also that $f : K \to \mathbb{R}$ possesses first order derivative f' over K and that f' is in the space $L([a, a+\eta(b,a)])$ of all integrable functions over $[a, a+\eta(b,a)]$. If $w : [a, a+\eta(b,a)] \to [0,\infty)$ is an integrable mapping symmetric to $A(2a,\eta(b,a))$, where a, $b \in K$ with $\eta(b,a) > 0$ and $\left|f'\right|^q$ is h-preinvex on K for $q > 1$, the following inequality holds*

$$\left|\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx - f\left(A\left(2a,\eta(b,a)\right)\right)\int_a^{a+\eta(b,a)} w(x)\,dx\right|$$
$$\leq \eta(b,a)\left[A\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\right]^{\frac{1}{q}}$$
$$\times\left(\int_0^1\left[h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right]dt\right)^{\frac{1}{q}}\left(\int_0^1\left[M'(w;a,b,t)\right]^p dt\right)^{\frac{1}{p}}, \quad (6.48)$$

where p and q are Hölder conjugates of each other and $M'(w;a,b,t)$ is defined as in Theorem 249.

Proof. Continuing from inequality (6.41) in the proof of Theorem 249 and using the Hölder's integral inequality, we have

$$\left|\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx - f(A(2a,\eta(b,a)))\int_a^{a+\eta(b,a)} w(x)\,dx\right|$$
$$\le \frac{\eta(b,a)}{2}\left(\int_0^1 \left[M'(w;a,b,t)\right]^p dt\right)^{\frac{1}{p}}\left[\left(\int_0^1 \left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 \left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right] \quad (6.49)$$

By the power-mean inequality $t^r + s^r < 2^{1-r}(t+s)^r$ for $t>0$, $s>0$ and $r<1$ and by the the h-preinvexity of $\left|f'\right|^q$ on K for $q>1$, we have for every $a,b\in K$ with $\eta(b,a)>0$ the following inequality

$$\left(\int_0^1 \left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}} + \left(\int_0^1 \left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$\le 2^{1-\frac{1}{q}}\left[\int_0^1 \left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt + \int_0^1 \left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right]^{\frac{1}{q}}$$
$$\le 2^{1-\frac{1}{q}}\left[\int_0^1 \left\{h\left(\frac{1+t}{2}\right)\left|f'(a)\right|^q + h\left(\frac{1-t}{2}\right)\left|f'(b)\right|^q\right.\right.$$
$$\left.\left.+h\left(\frac{1-t}{2}\right)\left|f'(a)\right|^q + h\left(\frac{1+t}{2}\right)\left|f'(b)\right|^q\right\} dt\right]^{\frac{1}{q}}$$
$$= 2^{1-\frac{1}{q}}\left[\left|f'(a)\right|^q + \left|f'(b)\right|^q\right]^{\frac{1}{q}}\left(\int_0^1 \left[h\left(\frac{1-t}{2}\right) + h\left(\frac{1+t}{2}\right)\right] dt\right)^{\frac{1}{q}}. \quad (6.50)$$

Using the last inequality (6.50) in (6.49), we get the desired inequality. ■

Corollary 259 *In Theorem 258, if we take* $w(x) = \frac{1}{\eta(b,a)}$ *for all* $x \in [a, a+\eta(b,a)]$ *with*

$\eta(b,a) > 0$, then (6.48) becomes the following inequality

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx - f(A(2a,\eta(b,a)))\right|$$
$$\leq \frac{\eta(b,a)}{2(1+p)^{\frac{1}{p}}}\left(\int_0^1\left[h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right]dt\right)^{\frac{1}{q}}\left[A\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\right]^{\frac{1}{q}}, \quad (6.51)$$

where p and q are Hölder conjugates of each other.

Corollary 260 *If we take $\eta(b,a) = b-a$ in Theorem 258, then (6.48) becomes the following inequality for h-convex functions*

$$\left|\int_a^b f(x)\,w(x)\,dx - f(A(a,b))\int_a^b w(x)\,dx\right|$$
$$\leq \eta(b,a)\left[A\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\right]^{\frac{1}{q}}\left(\int_0^1[M(w;a,b,t)]^p\,dt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\left[h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right]dt\right)^{\frac{1}{q}}, \quad (6.52)$$

where $M(w;a,b,t) = \int_a^{L(a,b,t)} w(x)\,dx$, $L(a,b,t) = \left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b$ for all $t \in [0,1]$ and p and q are Hölder conjugates of each other.

Corollary 261 *Assume that all the conditions of Theorem 258 are satisfied and in addition if h is super-additive, we have the following inequality*

$$\left|\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx - f(A(2a,\eta(b,a)))\int_a^{a+\eta(b,a)} w(x)\,dx\right|$$
$$\leq \eta(b,a)(h(1))^{\frac{1}{q}}\left[A\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\right]^{\frac{1}{q}}\left(\int_0^1\left[M'(w;a,b,t)\right]^p dt\right)^{\frac{1}{p}}, \quad (6.53)$$

where p and q are Hölder conjugates of each other.

Remark 262 *If $\eta(b,a) = b-a$ in Corollary 261, we have the following result when h is* *a super-additive function*

$$\left|\int_a^b f(x)\,w(x)\,dx - f(A(a,b))\int_a^b w(x)\,dx\right|$$
$$\leq (b-a)\,(h(1))^{\frac{1}{q}}\left[A\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\right]^{\frac{1}{q}}\left(\int_0^1 [M(w;a,b,t)]^p\,dt\right)^{\frac{1}{p}}, \quad (6.54)$$

where $M(w;a,b,t)$ is defined in Corollary 260 and $\frac{1}{p}+\frac{1}{q}=1$.

Corollary 263 *Suppose that $h(t)=t^s$, $t\in(0,1]$ in Corollary 259, we have the following result from for s-preinvex functions*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx - f(A(2a,\eta(b,a)))\right|$$
$$\leq \frac{\eta(b,a)}{2^{1-\frac{1}{q}}(1+p)^{\frac{1}{p}}(s+1)^{\frac{1}{q}}}\left[A\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\right]^{\frac{1}{q}}, \quad (6.55)$$

where p and q are Hölder conjugates of each other.

A similar result is stated as follows.

Theorem 264 *Suppose that $f:K\to\mathbb{R}$ is a mapping defined on K, where K is an open invex interval with respect to $\eta:K\times K\to\mathbb{R}$ of the set of real numbers. Assume also that $f:K\to\mathbb{R}$ possesses first order derivative f' over K and that f' is in the space $L([a,a+\eta(b,a)])$ of all integrable functions over $[a,a+\eta(b,a)]$. If $w:[a,a+\eta(b,a)]\to[0,\infty)$ is an integrable mapping symmetric to $A(2a,\eta(b,a))$, where a, $b\in K$ with $\eta(b,a)>0$ and $\left|f'\right|^q$ is h-preinvex on K for $q\geq 1$, the following inequality* *holds*

$$\left|\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx - f(A(2a,\eta(b,a)))\int_a^{a+\eta(b,a)} w(x)\,dx\right|$$
$$\leq \eta(b,a)\left[A\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\right]^{\frac{1}{q}}\left(\int_0^1 M'(w;a,b,t)\,dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 M'(w;a,b,t)\left[h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right]dt\right)^{\frac{1}{q}}, \quad (6.56)$$

where $M'(w;a,b,t)$ is defined as in Theorem 249.

Proof. Continuing from inequality (6.41) in the proof of Theorem 249 and using the well known Hölder's integral inequality, we have

$$\left|\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx - f(A(2a,\eta(b,a)))\int_a^{a+\eta(b,a)} w(x)\,dx\right|$$
$$\leq \frac{\eta(b,a)}{2}\left(\int_0^1 M'(w;a,b,t)\,dt\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left[\int_0^1 M'(w;a,b,t)\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\int_0^1 M'(w;a,b,t)\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right]^{\frac{1}{q}}\right\}. \quad (6.57)$$

By the power-mean inequality $t^r + s^r < 2^{1-r}(t+s)^r$ for $t>0$, $s>0$ and $r<1$, and by the h-preinvexity of $\left|f'\right|^q$ on K for $q>1$, we have for every $a,b\in K$ with $\eta(b,a)>0$ the following inequality

$$\left[\int_0^1 M'(w;a,b,t)\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right]^{\frac{1}{q}}$$
$$+\left[\int_0^1 M'(w;a,b,t)\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right]^{\frac{1}{q}}$$
$$\leq 2\left(\int_0^1 M'(w;a,b,t)\left[h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right]dt\right)^{\frac{1}{q}}\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}. \quad (6.58)$$

Utilizing inequality (6.58) in (6.57), we get the inequality (6.56). ■

Corollary 265 *Suppose that all the assumptions of Theorem 264 are satisfied and if* $w(x)=\frac{1}{\eta(b,a)}$ *for all* $x\in[a,a+\eta(b,a)]$ *with* $\eta(b,a)>0$, *we have the following inequality*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx-f(A(2a,\eta(b,a)))\right|$$
$$\leq \eta(b,a)\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left[A\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\right]^{\frac{1}{q}}$$
$$\times\left[\int_0^1\left(\frac{1-t}{2}\right)\left[h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right]dt\right]^{\frac{1}{q}}. \quad (6.59)$$

Corollary 266 *If we take* $\eta(b,a)=b-a$ *in Theorem 264, we have the following weighted inequality for* h*-convex functions*

$$\left|\int_a^b f(x)\,w(x)\,dx-f(A(a,b))\int_a^b w(x)\,dx\right|$$
$$\leq (b-a)\left[A\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\right]^{\frac{1}{q}}\left(\int_0^1 M(w;a,b,t)\,dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 M(w;a,b,t)\left[h\left(\frac{1-t}{2}\right)+h\left(\frac{1+t}{2}\right)\right]dt\right)^{\frac{1}{q}}, \quad (6.60)$$

where $M(w;a,b,t)=\int_a^{L(a,b,t)} w(x)\,dx$, $L(a,b,t)=\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b$ *for all* $t\in[0,1]$.

Corollary 267 *If $w(x) = \frac{1}{b-a}$ for all $x \in [a,b]$ in Corollary 266, we have the following inequality for h-convex functions*

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - f(A(a,b)) \right| \leq (b-a) \left(\frac{1}{4} \right)^{1-\frac{1}{q}} \left[A\left(\left| f'(a) \right|^q, \left| f'(b) \right|^q \right) \right]^{\frac{1}{q}} \times \left[\int_0^1 \left(\frac{1-t}{2} \right) \left[h\left(\frac{1-t}{2} \right) + h\left(\frac{1+t}{2} \right) \right] dt \right]^{\frac{1}{q}}. \quad (6.61)$$

Remark 268 *If in Corollary 267 we take $h(t) = t$, we have the inequality 2.33 given in Theorem 60.*

Corollary 269 *If the conditions of Theorem 264 are met and h is a super-additive function, the following inequality holds*

$$\left| \int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx - f(A(2a,\eta(b,a))) \int_a^{a+\eta(b,a)} w(x)\,dx \right| \leq \eta(b,a)(h(1))^{\frac{1}{q}} \left[A\left(\left| f'(a) \right|^q, \left| f'(b) \right|^q \right) \right]^{\frac{1}{q}} \int_0^1 M'(w;a,b,t)\,dt, \quad (6.62)$$

where $M'(w;a,b,t)$ is defined as in Theorem 249.

Corollary 270 *If h is super-additive in Corollary 265, we have the inequality*

$$\left| \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx - f(A(2a,\eta(b,a))) \right| \leq \frac{\eta(b,a)(h(1))^{\frac{1}{q}}}{4} \left[A\left(\left| f'(a) \right|^q, \left| f'(b) \right|^q \right) \right]^{\frac{1}{q}}. \quad (6.63)$$

Corollary 271 *When h is super-additive in Corollary 266, we have the inequality*

$$\left| \int_a^b f(x)\,w(x)\,dx - f(A(a,b)) \int_a^b w(x)\,dx \right| \leq (b-a)(h(1))^{\frac{1}{q}} \left[A\left(\left| f'(a) \right|^q, \left| f'(b) \right|^q \right) \right]^{\frac{1}{q}} \int_0^1 M(w;a,b,t)\,dt, \quad (6.64)$$

where $M(w;a,b,t)=\int_a^{L(a,b,t)} w(x)\,dx,\ L(a,b,t)=\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)b$ *for all* $t\in[0,1]$.

Corollary 272 *In Corollary 267, if* h *is super-additive then we have the inequality*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f(A(a,b))\right|\le\frac{(b-a)(h(1))^{\frac{1}{q}}}{4}\left[A\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\right]^{\frac{1}{q}}. \quad (6.65)$$

Corollary 273 *If* $h(t)=t^s$, $t\in(0,1]$ *in Corollary 265, we have the following inequality for* s*-preinvex functions*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx-f(A(2a,\eta(b,a)))\right|$$
$$\le\eta(b,a)\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{2^{s+1}-1}{2^s(s+1)(s+2)}\right)^{\frac{1}{q}}\left[A\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\right]^{\frac{1}{q}}. \quad (6.66)$$

Corollary 274 *If* $\eta(b,a)=b-a$*, we have the following inequality for* s*-convex functions*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f(A(a,b))\right|$$
$$\le(b-a)\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{2^{s+1}-1}{2^s(s+1)(s+2)}\right)^{\frac{1}{q}}\left[A\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\right]^{\frac{1}{q}}. \quad (6.67)$$

6.4 Applications to Special Means

Let a and b be positive real numbers such that $a<b$. Consider the function $M:=M(a,b):[a,a+\eta(b,a)]\times[a,a+\eta(b,a)]\to\mathbb{R}^+$, which is one of the means defined in Section 2.13, therefore we can obtain various inequalities for these means as follows:

Setting $\eta(b,a)=M(b,a)$ in (6.12), (2.19) and (6.33), we obtain the following interesting inequalities involving means:

$$\left|\frac{1}{M(b,a)}\int_a^{a+M(b,a)} f(x)\,dx-f\left(a+\frac{1}{2}M(b,a)\right)\right|\le\frac{M(b,a)}{8}\left[\left|f'(a)\right|+\left|f'(b)\right|\right], \quad (6.68)$$

$$\left|\frac{1}{M(b,a)}\int_a^{a+M(b,a)} f(x)\,dx + f\left(a+\frac{1}{2}M(b,a)\right)\right| \le M(b,a)\left(\frac{1}{8}\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{A\left(2\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)}{12}\right)^{\frac{1}{q}}+\left(\frac{A\left(\left|f'(a)\right|^q,2\left|f'(b)\right|^q\right)}{12}\right)^{\frac{1}{q}}\right] \quad (6.69)$$

and

$$\left|\frac{f(a)+f(a+M(b,a))}{2}-\frac{1}{M(b,a)}\int_a^{a+M(b,a)} f(x)\,dx\right|$$
$$\le \frac{M(b,a)}{2(p+1)^{\frac{1}{p}}}\left[A\left(\left|f'(a)\right|^q,\left|f'(b)\right|^q\right)\right]^{\frac{1}{q}}. \quad (6.70)$$

Letting $M = A$, G, H, P_r, I, L, L_p in (6.35), (6.36) and (6.37), we can get the required inequalities. However, the details are left to the interested reader.

Chapter 7

Inequalities of H-H type for n-times differentiable preinvex, s-preinvex and log-preinvex functions

The results provided in this chapter have been chosen from the papers M. A. Latif and S. S. Dragomir, Generalization of inequalities of Hermite-Hadamard type for n-times differentiable functions through preinvexity, Georgian Mathematical Journal. 23 (1) (2016) 97-104.", "M. A. Latif and S. S. Dragomir, On Hermite-Hadamard type integral inequalities for n-times differentiable log-preinvex functions, Filomat. 29:7 (2015) 1651–1661." and "M. A. Latif and S. S. Dragomir, Generalization of Hermite-Hadamard type inequalities for n-times differentiable functions which are s-preinvex in the second sense with applications, Hacettepe Journal of Mathematics and Statistics. 44 (4) (2015) 839-853.".

7.1 Introduction

Hermite-Hadamard type inequality has been the subject of intensive research in the present era and hence various refinements of the H-H inequalities for the convex functions and its alternative forms have been added in the literature by many researchers. Hwang [51], initiated the research towards Hermite-Hadamard type inequalities for functions whose nth derivatives satisfy the conditions of convexity and concavity. The results of

Hwang have motivated many researchers working in this field, which resulted in a number of Hermite-Hadamard type inequalities in recent years for n-times differentiable mappings with certain assumptions of different kinds of convexity. Here we refer the latest works of Bai el al. [19], Hong et al. [56], Jiang et al. [70], Latif [94], Wang et al. [160], Wnag and Qi [162] for n-times differentiable mappings with the particular hypothesis of convexities.

The main aspiration of the present chapter is to propose new Hermite-Hadamard type inequalities which are connected with the right-side and left-side of Hermite-Hadamard inequality for n-times differentiable preinvex functions, s-preinvex functions and log-prinvex functions which generalize and extend those results established for n-times differentiable preinvex functions and n-times differentiable convex, s-convex and log-convex functions.

This chapter is split in sections in the following manner.

Section 7.2 incorporates more general H-H type inequalities for n-times differentiable mappings. The main source of inspiration of this section are the results given in [162] in which more general inequalities for n-times differentiable convex functions are presented by using a more general identity . We will prove a more general identity for n-times differentiable mappings defined on an invex subset of real numbers analogous to that of given in [162]. The usage of this identity, some special definite integrals, preinvexity and analysis gives us more general results of this section. Section 7.3 provides some inequalities of Hermite-Hadamard type by using two more general identities established for n-times differentiable mappings defined on an invex subset of the set of real numbers and by using standard techniques. The results of this section provide extensions of those results proved in [70]. Applications of our results are given in this section as well. In Section 7.3, inequalities of Hermite-Hadamard type are incorporated for n-times log-prinvex functions. Through our results of this section some corrected inequalities are achieved, which were proved in [146].

7.2 Generalization of inequalities of H-H type for n-times differentiable functions through preinvexity

The following Lemmas play a key role in establishing results in this section:

Lemma 275 *Suppose that $f : K \to \mathbb{R}$ is a mapping defined on K, where K is an open invex interval with respect to $\eta : K \times K \to \mathbb{R}$ of the set of real numbers. Assume that $f^{(n-1)}(x)$ is absolutely continuous on $[a, a + \eta(b, a)]$, a, $b \in K$ with $\eta(b, a) > 0$ and $n \in \mathbb{N}$. If $f^{(n)}(x)$ exists on $[a, a + \eta(b, a)]$, then for λ, $\mu \in \mathbb{R}$ and $t \in [0, 1]$, the following identity holds:*

$$\begin{aligned}
S(t; \lambda, \mu) &\triangleq -\lambda f(a) - (1-\mu) f(a + \eta(b,a)) \\
&\quad + \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx - \sum_{k=0}^{n-1} \frac{(-1)^k (\eta(b,a))^k}{(k+1)!} \left\{ t^k [t - (k+1)\lambda] \right. \\
&\quad \left. - (t-1)^k [t - 1 + (k+1)(1-\mu)] \right\} f^{(k)}(a + t\eta(b,a)) \\
&= \frac{(-1)^{n-1} (\eta(b,a))^n}{n!} \left\{ \int_0^t z^{n-1} (n\lambda - z) f^{(n)}(a + z\eta(b,a))\,dz \right. \\
&\quad \left. + \int_t^1 (z-1)^{n-1} (1 - z - n(1-\mu)) f^{(n)}(a + z\eta(b,a))\,dz \right\}. \qquad (7.1)
\end{aligned}$$

Proof. When $n = 1$, we have by integrating by parts that

$$\begin{aligned}
&\eta(b,a) \left\{ \int_0^t (\lambda - z) f'(a + z\eta(b,a))\,dz + \int_t^1 (\mu - z) f'(a + z\eta(b,a))\,dz \right\} \\
&\quad = -\lambda f(a) - (1-\mu) f(a + \eta(b,a)) \\
&\quad\quad - (\mu - \lambda) f(a + t\eta(b,a)) + \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx. \qquad (7.2)
\end{aligned}$$

Suppose (7.1) is valid for $n = m - 1$, that is

$$
\begin{aligned}
&-\lambda f(a) - (1-\mu) f(a+\eta(b,a)) \\
&\quad + \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx - \sum_{k=0}^{m-2} \frac{(-1)^k (\eta(b,a))^k}{(k+1)!} \Big\{ t^k \left[t - (k+1)\lambda\right] \\
&\qquad - (t-1)^k \left[t - 1 + (k+1)(1-\mu)\right] \Big\} f^{(k)}(a + t\eta(b,a)) \\
&= \frac{(-1)^{m-2} (\eta(b,a))^{m-1}}{(m-1)!} \left\{ \int_0^t z^{m-2} ((m-1)\lambda - z) f^{(m-1)}(a + z\eta(b,a))\,dz \right. \\
&\qquad \left. + \int_t^1 (z-1)^{m-2} (1 - z - (m-1)(1-\mu)) f^{(m-1)}(a + z\eta(b,a))\,dz \right\}. \qquad (7.3)
\end{aligned}
$$

Now for $n = m$, we have by integrating by parts that

$$
\begin{aligned}
&\frac{(-1)^{m-1} (\eta(b,a))^m}{m!} \left\{ \int_0^t z^{m-1} (m\lambda - z) f^{(m)}(a + z\eta(b,a))\,dz \right. \\
&\qquad \left. + \int_t^1 (z-1)^{m-1} (1 - z - m(1-\mu)) f^{(m)}(a + z\eta(b,a))\,dz \right\} \\
&= -\frac{(-1)^{m-1} (\eta(b,a))^{m-1}}{m!} \left\{ \left[t^{m-1}(t - m\lambda) - (t-1)^{m-1}(t - 1 + m(1-\mu)) \right] \right. \\
&\qquad \times f^{(m-1)}(a + t\eta(b,a)) + \frac{(-1)^{m-2} (\eta(b,a))^{m-1}}{(m-1)!} \\
&\qquad \times \left\{ \int_0^t z^{m-2} ((m-1)\lambda - z) f^{(m-1)}(a + z\eta(b,a))\,dz \right. \\
&\qquad \left. + \int_t^1 (z-1)^{m-2} (1 - z - (m-1)(1-\mu)) f^{(m-1)}(a + z\eta(b,a))\,dz \right\} \qquad (7.4)
\end{aligned}
$$

Using (7.3) in (7.4) and simplifying, we get (7.1). ■

Remark 276 *If $\eta(b,a) = b-a$ in Lemma 275. Then*

$$
\begin{aligned}
&-\lambda f(a) - (1-\mu) f(b) + \frac{1}{b-a}\int_a^b f(x)\,dx \\
&\qquad -\sum_{k=0}^{n-1} \frac{(-1)^k (b-a)^k}{(k+1)!} \left\{ t^k \left[t-(k+1)\lambda\right] \right. \\
&\qquad \left. -(t-1)^k \left[t-1+(k+1)(1-\mu)\right] \right\} f^{(k)}(tb+(1-t)a) \\
&= \frac{(-1)^{n-1}(b-a)^n}{n!} \left\{ \int_0^t z^{n-1}(n\lambda - z) f^{(n)}(zb+(1-z)a)\,dz \right. \\
&\qquad \left. + \int_t^1 (z-1)^{n-1}(1-z-n(1-\mu)) f^{(n)}(zb+(1-z)a)\,dz \right\}. \qquad (7.5)
\end{aligned}
$$

Lemma 277 *[160] Let α, $\beta \in \mathbb{R}$, ξ, $c \geq 0$ and $r > -1$. Then*

$$
\begin{aligned}
&\int_0^c u^r |\xi - u|\,du \\
&= \frac{1}{(r+1)(r+2)} \begin{cases} \left[(r+2)\xi - (r+1)c\right] c^{r+1}, & \xi \geq c \\ \\ (r+1)c^{r+2} - (r+2)c^{r+1}\xi + 2\xi^{r+2}, & 0 \leq \xi \leq c \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^c (\alpha u + \beta)|\xi - u|^r\,du = \frac{1}{(r+1)(r+2)} \\
&\times \begin{cases} \left[(r+2)\beta + \alpha\xi\right]\xi^{r+1} - \left[\alpha c(r+1) + \beta(r+2) + \alpha\xi\right](\xi - c)^{r+1}, & \xi \geq c \\ \\ \left[(r+2)\beta + \alpha\xi\right]\xi^{r+1} + \left[\beta(r+2) + \alpha(c+cr+\xi) + \alpha\xi\right](c-\xi)^{r+1}, & 0 \leq \xi \leq c. \end{cases}
\end{aligned}
$$

Theorem 278 *Suppose that $f : K \to \mathbb{R}$ is a mapping defined on K, where K is an open invex interval with respect to $\eta : K \times K \to \mathbb{R}$ of the set of real numbers and that $f^{(n-1)}(x)$ is absolutely continuous on $[a, a+\eta(b,a)]$, a, $b \in K$ with $\eta(b,a) > 0$ and $n \in \mathbb{N}$. If $\left|f^{(n)}\right|^q$ satisfies the condition of preinvexity on K for $q \geq 1$, for all $t \in [0,1]$ and λ,*

$\mu \in [0,1]$, *the following inequality holds*

$$
\begin{aligned}
|S(t;\lambda,\mu)| \leq \frac{(\eta(b,a))^n}{n!} & \Big\{ [A(\lambda,t;n)]^{1-1/q} \\
& \times \left[A(\lambda,t;n) \left|f^{(n)}(a)\right|^q + A(\lambda,t;n+1) \left(\left|f^{(n)}(b)\right|^q - \left|f^{(n)}(a)\right|^q \right) \right]^{1/q} \\
& + [A(1-\mu,1-t;n)]^{1-1/q} \Big[A(1-\mu,1-t;n) \left|f^{(n)}(b)\right|^q \\
& + A(1-\mu,1-t;n+1) \left(\left|f^{(n)}(a)\right|^q - \left|f^{(n)}(b)\right|^q \right) \Big]^{1/q} \Big\}, \quad (7.6)
\end{aligned}
$$

where for $c \geq 0$ *and* $r > -1$

$$
\begin{aligned}
A(\xi,c;r+1) &= \int_0^c u^r |n\xi - u| \, du \\
&= \frac{1}{(r+1)(r+2)} \begin{cases} [n\xi(r+2) - (r+1)c] \, c^{r+1}, & n\xi \geq c \\ \\ (r+1)c^{r+2} - n\xi(r+2)c^{r+1} + 2(n\xi)^{r+2}, & 0 \leq n\xi \leq c. \end{cases}
\end{aligned}
$$

Proof. By Lemma 275, the Hölder's inequality and the preinvexity of $\left|f^{(n)}\right|^q$ on K, $q \geq 1$, $n \in \mathbb{N}$, we have

$$
\begin{aligned}
|S(t;\lambda,\mu)| \leq \frac{(\eta(b,a))^n}{n!} & \left\{ \int_0^t z^{n-1} |n\lambda - z| \left|f^{(n)}(a + z\eta(b,a))\right| dz \right. \\
& \left. + \int_t^1 (z-1)^{n-1} |1 - z - n(1-\mu)| \left|f^{(n)}(a + z\eta(b,a))\right| dz \right\} \\
& \leq \frac{(\eta(b,a))^n}{n!} \left\{ \left(\int_0^t z^{n-1} |n\lambda - z| \, dz \right)^{1-1/q} \right. \\
& \times \left[\int_0^t z^{n-1} |n\lambda - z| \left((1-z) \left|f^{(n)}(a)\right|^q + z \left|f^{(n)}(b)\right|^q \right) dz \right]^{1/q} \\
& + \left(\int_t^1 (1-z)^{n-1} |1 - z - n(1-\mu)| \, dz \right)^{1-1/q} \\
& \left. \times \left[\int_t^1 (1-z)^{n-1} |1 - z - n(1-\mu)| \left((1-z) \left|f^{(n)}(a)\right|^q + z \left|f^{(n)}(b)\right|^q \right) dz \right]^{1/q} \right\}. \quad (7.7)
\end{aligned}
$$

Using Lemma 277, we observe that

$$\int_0^t z^{n-1} |n\lambda - z| \, dz = A(\lambda, t; n),$$

$$\int_t^1 (1-z)^{n-1} |1 - z - n(1-\mu)| \, dz = A(1-\mu, 1-t; n),$$

$$\begin{aligned}
&\int_0^t z^{n-1} |n\lambda - z| \left((1-z) \left|f^{(n)}(a)\right|^q + z \left|f^{(n)}(b)\right|^q\right) dz \\
&= A(\lambda, t; n) \left|f^{(n)}(a)\right|^q + A(\lambda, t; n+1) \left(\left|f^{(n)}(b)\right|^q - \left|f^{(n)}(a)\right|^q\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_t^1 (1-z)^{n-1} |1 - z - n(1-\mu)| \left((1-z) \left|f^{(n)}(a)\right|^q + z \left|f^{(n)}(b)\right|^q\right) dz \\
&= A(1-\mu, 1-t; n) \left|f^{(n)}(b)\right|^q + A(1-\mu, 1-t; n+1) \left(\left|f^{(n)}(a)\right|^q - \left|f^{(n)}(b)\right|^q\right).
\end{aligned}$$

Substituting the above inequalities into (7.7), gives us the desired inequality (7.6). ■

Remark 279 *If we take $n = 1$, $t = \frac{1}{2}$ and $0 \leq \lambda \leq \frac{1}{2} \leq \mu \leq 1$ in Theorem 278, we get the following inequality:*

$$\begin{aligned}
&\left|\lambda f(a) + (1-\mu) f(a + \eta(b,a)) + (\mu - \lambda) f(A(2a, \eta(b,a)))\right. \\
&\qquad\qquad \left. - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x) \, dx\right| \\
&\leq \frac{\eta(b,a)}{24} \left[\left(8 - 9\lambda + 24\lambda^2 - 8\lambda^3 - 21\mu + 24\mu^2 - 8\mu^3\right) \left|f^{(n)}(a)\right|\right. \\
&\qquad\qquad \left. + \left(10 - 3\lambda + 8\lambda^3 - 15\mu + 8\mu^3\right) \left|f^{(n)}(b)\right|\right]. \quad (7.8)
\end{aligned}$$

Theorem 280 *Suppose that $f : K \to \mathbb{R}$ is a mapping defined on K, where K is an open invex interval with respect to $\eta : K \times K \to \mathbb{R}$ of the set of real numbers and that $f^{(n-1)}(x)$ is absolutely continuous on $[a, a + \eta(b,a)]$, a, $b \in K$ with $\eta(b,a) > 0$ and $n \in \mathbb{N}$. If*

$\left|f^{(n)}\right|^q$ *satisfies the condition of preinvexity on* K *for* $q>1$ *and* $q(n-1)\geq r\geq 0$, *for all* $t\in[0,1]$ *and* $\lambda,\ \mu\in[0,1]$, *the following inequality holds*

$$\begin{aligned}
|S(t;\lambda,\mu)| \leq \frac{(\eta(b,a))^n}{n!}&\left\{\left[A\left(\lambda,t;\frac{nq-r-1}{q-1}\right)\right]^{1-1/q}\right.\\
&\times\left[A(\lambda,t;r+1)\left|f^{(n)}(a)\right|^q + A(\lambda,t;r+2)\left(\left|f^{(n)}(b)\right|^q-\left|f^{(n)}(a)\right|^q\right)\right]^{1/q}\\
&+\left[A\left(1-\mu,1-t;\frac{nq-r-1}{q-1}\right)\right]^{1-1/q}\left[A(1-\mu,1-t;r+1)\left|f^{(n)}(b)\right|^q\right.\\
&\left.\left.+A(1-\mu,1-t;r+2)\left(\left|f^{(n)}(a)\right|^q-\left|f^{(n)}(b)\right|^q\right)\right]^{1/q}\right\}, \quad (7.9)
\end{aligned}$$

where $A(\xi,c;r+1)$ *is defined as in Theorem 278,* $c\geq 0$, $r>1$.

Proof. From Lemma 275, Hölder's inequality and the preinvexity of $\left|f^{(n)}\right|^q$ on K, $q>1$ and $q(n-1)\geq r\geq 0$, $n\in\mathbb{N}$, we have

$$\begin{aligned}
|S(t;\lambda,\mu)| \leq \frac{(\eta(b,a))^n}{n!}&\left\{\left(\int_0^t z^{[(n-1)q-r]/(q-1)}|n\lambda-z|\,dz\right)^{1-1/q}\right.\\
&\times\left[\int_0^t z^r|n\lambda-z|\left((1-z)\left|f^{(n)}(a)\right|^q+z\left|f^{(n)}(b)\right|^q\right)dz\right]^{1/q}\\
&+\left(\int_t^1(1-z)^{[(n-1)q-r]/(q-1)}|1-z-n(1-\mu)|\,dz\right)^{1-1/q}\\
&\left.\times\left[\int_t^1(1-z)^r|1-z-n(1-\mu)|\left((1-z)\left|f^{(n)}(a)\right|^q+z\left|f^{(n)}(b)\right|^q\right)dz\right]^{1/q}\right\}. \quad (7.10)
\end{aligned}$$

The rest of the proof is similar to that of the proof of Theorem 278 ■

Corollary 281 *Under the assumptions of Theorem 280*

1. *If $r=0$, we have*

$$
\begin{aligned}
\left|S\left(t;\lambda,\mu\right)\right| \leq \frac{\left(\eta\left(b,a\right)\right)^{n}}{n!}&\left\{\left[A\left(\lambda,t;\frac{nq-1}{q-1}\right)\right]^{1-1/q}\right.\\
&\times\left[A\left(\lambda,t;1\right)\left|f^{(n)}\left(a\right)\right|^{q}+A\left(\lambda,t;2\right)\left(\left|f^{(n)}\left(b\right)\right|^{q}-\left|f^{(n)}\left(a\right)\right|^{q}\right)\right]^{1/q}\\
&+\left[A\left(1-\mu,1-t;\frac{nq-1}{q-1}\right)\right]^{1-1/q}\left[A\left(1-\mu,1-t;1\right)\left|f^{(n)}\left(b\right)\right|^{q}\right.\\
&\left.\left.+A\left(1-\mu,1-t;2\right)\left(\left|f^{(n)}\left(a\right)\right|^{q}-\left|f^{(n)}\left(b\right)\right|^{q}\right)\right]^{1/q}\right\}. \qquad (7.11)
\end{aligned}
$$

2. *If $r=(n-1)\,q$, we get*

$$
\begin{aligned}
&\left|S\left(t;\lambda,\mu\right)\right|\\
&\leq\frac{\left(\eta\left(b,a\right)\right)^{n}}{n!}\left\{\left[A\left(\lambda,t;1\right)\right]^{1-1/q}\left[A\left(\lambda,t;(n-1)\,q+1\right)\left|f^{(n)}\left(a\right)\right|^{q}\right.\right.\\
&\qquad\left.+A\left(\lambda,t;(n-1)\,q+2\right)\left(\left|f^{(n)}\left(b\right)\right|^{q}-\left|f^{(n)}\left(a\right)\right|^{q}\right)^{1/q}\right]\\
&\quad+\left[A\left(1-\mu,1-t;1\right)\right]^{1-1/q}\left[A\left(1-\mu,1-t;(n-1)\,q+1\right)\left|f^{(n)}\left(b\right)\right|^{q}\right.\\
&\qquad\left.\left.+A\left(1-\mu,1-t;(n-1)\,q+2\right)\left(\left|f^{(n)}\left(a\right)\right|^{q}-\left|f^{(n)}\left(b\right)\right|^{q}\right)\right]^{1/q}\right\}. \qquad (7.12)
\end{aligned}
$$

Theorem 282 *Suppose that $f:K\to\mathbb{R}$ is a mapping defined on K, where K is an open invex interval with respect to $\eta:K\times K\to\mathbb{R}$ of the set of real numbers and that $f^{(n-1)}(x)$ is absolutely continuous on $[a,a+\eta\left(b,a\right)]$, a, $b\in K$ with $\eta\left(b,a\right)>0$ and $n\in\mathbb{N}$. If $\left|f^{(n)}\right|^{q}$ satisfies the condition of preinvexity on K for $q>1$, for all $t\in[0,1]$ and λ,*

$\mu \in [0,1]$, the following inequality holds

$$\begin{aligned}
|S(t;\lambda,\mu)| \leq \frac{(\eta(b,a))^n}{n!} & \left\{ \left[B\left(\lambda,0,1,t;\frac{2q-1}{q-1}\right)\right]^{1-1/q} \right. \\
& \times \left[A(0,t;(n-1)q+1)\left|f^{(n)}(b)\right|^q + B(0,-1,1;(n-1)q+1)\left|f^{(n)}(a)\right|^q \right]^{1/q} \\
& + \left[B\left(1-\mu,0,1,1-t;\frac{2q-1}{q-1}\right)\right]^{1-1/q} \\
& \times \left[B(0,-1,1,1-t;(n-1)q+1)\left|f^{(n)}(b)\right|^q \right. \\
& \left. \left. + A(0,1-t;(n-1)q+1)\left|f^{(n)}(a)\right|^q \right]^{1/q} \right\}, \quad (7.13)
\end{aligned}$$

where $A(\xi,c;r+1)$ is defined as in Theorem 278 and

$$B(\xi,\alpha,\beta,c;r+1) = \frac{1}{(r+1)(r+2)} \times \begin{cases} [n\xi\beta + \alpha n\xi](n\xi)^{r+1} \\ -[\alpha c(r+1)+\beta(r+2)+\alpha n\xi](n\xi-c)^{r+1}, & n\xi \geq c \\ [(r+2)\beta + \alpha n\xi](n\xi)^{r+1} \\ +[\beta(r+2)+\alpha(c+cr+n\xi)](c-n\xi)^{r+1}, & 0 \leq n\xi \leq c, \end{cases}$$

$c \geq 0$, $r > 1$, α, $\beta \in \mathbb{R}$.

Proof. Applying Lemma 275, Hölder's inequality and preinvexity of $\left|f^{(n)}\right|^q$ on K, $q>1$, we have

$$\begin{aligned}
|S(t;\lambda,\mu)| \le \frac{(\eta(b,a))^n}{n!}&\left\{\left(\int_0^t |n\lambda - z|^{q/(q-1)}\,dz\right)^{1-1/q}\right.\\
&\times\left[\int_0^t z^{(n-1)q}\left((1-z)\left|f^{(n)}(a)\right|^q + z\left|f^{(n)}(b)\right|^q\right)dz\right]^{1/q}\\
&+\left(\int_t^1 |1-z-n(1-\mu)|^{q/(q-1)}\,dz\right)^{1-1/q}\\
&\left.\times\left[\int_t^1 (1-z)^{(n-1)q}\left((1-z)\left|f^{(n)}(a)\right|^q + z\left|f^{(n)}(b)\right|^q\right)dz\right]^{1/q}\right\}. \quad (7.14)
\end{aligned}$$

Using the similar arguments as that of the proof of Theorem 278, we get the inequality (7.13). ■

Theorem 283 *Suppose that $f: K \to \mathbb{R}$ is a mapping defined on K, where K is an open invex interval with respect to $\eta: K \times K \to \mathbb{R}$ of the set of real numbers and that $f^{(n-1)}(x)$ is absolutely continuous on $[a, a+\eta(b,a)]$, a, $b \in K$ with $\eta(b,a) > 0$ and $n \in \mathbb{N}$. If $\left|f^{(n)}\right|^q$ satisfies the condition of preinvexity on K for $q > 1$, for all $t \in [0,1]$ and λ, $\mu \in [0,1]$, the following inequality holds*

$$\begin{aligned}
|S(t;\lambda,\mu)| \le \frac{(\eta(b,a))^n}{n!}&\left\{\left[B\left(\lambda,0,1,t;\frac{2q-1}{q-1}\right)\right]^{1-1/q}\right.\\
&\times\left[A(0,t;(n-2)q+2)\left|f^{(n)}(b)\right|^q + B(0,-1,1,t;(n-2)q+2)\left|f^{(n)}(a)\right|^q\right]^{1/q}\\
&+\left[B\left(1-\mu,1,0,1-t;\frac{2q-1}{q-1}\right)\right]^{1-1/q}\\
&\left.\times\left[B(0,-1,1,1-t;(n-2)q+2)\left|f^{(n)}(b)\right|^q + A(0,1-t;(n-2)q+2)\left|f^{(n)}(a)\right|^q\right]^{1/q}\right\}, \quad (7.15)
\end{aligned}$$

where $A(\xi,c;r+1)$ and $B(\xi,\alpha,\beta,c;r+1)$ are defined as in Theorem 278 and Theorem 282 respectively, $c \ge 0$, $r > 1$, α, $\beta \in \mathbb{R}$.

Proof. Applying Lemma 275, using Hölder's inequality and preinvexity of $\left|f^{(n)}\right|^q$ on K, $q>1$, results in

$$
\begin{aligned}
|S(t;\lambda,\mu)| \leq \frac{(\eta(b,a))^n}{n!} & \left\{ \left(\int_0^t z\,|n\lambda - z|^{q/(q-1)}\, dz \right)^{1-1/q} \right. \\
& \times \left[\int_0^t z^{(n-2)q+1} \left((1-z)\left|f^{(n)}(a)\right|^q + z\left|f^{(n)}(b)\right|^q \right) dz \right]^{1/q} \\
& + \left(\int_t^1 (1-z)\,|1-z-n(1-\mu)|^{q/(q-1)}\, dz \right)^{1-1/q} \\
& \left. \times \left[\int_t^1 (1-z)^{(n-2)q+1} \left((1-z)\left|f^{(n)}(a)\right|^q + z\left|f^{(n)}(b)\right|^q \right) dz \right]^{1/q} \right\}. \quad (7.16)
\end{aligned}
$$

The rest of the proof is similar to that of the proof of Theorem 278. ■

Theorem 284 *Suppose that $f : K \to \mathbb{R}$ is a mapping defined on K, where K is an open invex interval with respect to $\eta : K \times K \to \mathbb{R}$ of the set of real numbers and that $f^{(n-1)}(x)$ is absolutely continuous on $[a, a+\eta(b,a)]$, a, $b \in K$ with $\eta(b,a) > 0$ and $n \in \mathbb{N}$. If $\left|f^{(n)}\right|^q$ satisfies the condition of preinvexity on K for $q > 1$, for all $t \in [0,1]$ and λ, $\mu \in [0,1]$, the following inequality holds*

$$
\begin{aligned}
|S(t;\lambda,\mu)| \leq & \frac{(\eta(b,a))^n}{n!} \left(\frac{q-1}{nq-1} \right)^{1-1/q} \\
& \times \left\{ t^{(nq-1)/q} \left[B(\lambda,1,0,t;q+1)\left|f^{(n)}(b)\right|^q + B(\lambda,-1,1,t;q+1)\left|f^{(n)}(a)\right|^q \right]^{1/q} \right. \\
& + (1-t)^{(nq-1)/q} \left[B(1-\mu,-1,1,1-t;q+1)\left|f^{(n)}(b)\right|^q \right. \\
& \left. \left. + B(1-\mu,1,0,1-t;q+1)\left|f^{(n)}(a)\right|^q \right]^{1/q} \right\}, \quad (7.17)
\end{aligned}
$$

where $B(\xi,\alpha,\beta,c;r+1)$ is defined as in Theorem 282, $c \geq 0$, $r > 1$, α, $\beta \in \mathbb{R}$.

Proof. Utilizing Lemma 275, using Hölder's inequality and preinvexity of $\left|f^{(n)}\right|^q$ on K, $q>1$, results in

$$
\begin{aligned}
|S(t;\lambda,\mu)| \leq \frac{(\eta(b,a))^n}{n!} & \left\{ \left(\int_0^t z^{(n-1)q/(q-1)} dz \right)^{1-1/q} \right. \\
& \times \left[\int_0^t |n\lambda - z|^q \left((1-z)\left|f^{(n)}(a)\right|^q + z\left|f^{(n)}(b)\right|^q \right) dz \right]^{1/q} \\
& + \left(\int_t^1 (1-z)^{(n-1)q/(q-1)} dz \right)^{1-1/q} \\
\times & \left. \left[\int_t^1 |1-z-n(1-\mu)|^q \left((1-z)\left|f^{(n)}(a)\right|^q + z\left|f^{(n)}(b)\right|^q \right) dz \right]^{1/q} \right\}. \quad (7.18)
\end{aligned}
$$

The rest of the proof is similar to that of the Theorem 278. ■

7.3 Generalization of H-H type inequalities for n-times differentiable functions which are s-preinvex in the second sense

We need the following lemmas to prove results in this section as well as the results of Section 7.5.

Lemma 285 *[94] Suppose that $f: K \to \mathbb{R}$ is a mapping defined over $K \subseteq \mathbb{R}$, where K is an open invex subset with respect to $\eta: K \times K \to \mathbb{R}$ of the set of real numbers. If the nth derivative $f^{(n)}$ exists on K and $f^{(n)}$ is in the space $L([a, a+\eta(b,a)])$ of all integrable functions on $[a, a+\eta(b,a)]$, $a, b \in K$ with $\eta(b,a)>0$ and $n \in \mathbb{N}$, $n \geq 1$, the following* *equality holds:*

$$
\begin{aligned}
&-A\left(f\left(a\right),f\left(a+\eta\left(b,a\right)\right)\right)+\frac{1}{\eta\left(b,a\right)}\int_{a}^{a+\eta(b,a)}f\left(x\right)dx \\
&\qquad\qquad\qquad+\sum_{k=2}^{n-1}\frac{\left(-1\right)^{k}\left(k-1\right)\left(\eta\left(b,a\right)\right)^{k}}{2\left(k+1\right)!}f^{(k)}(a+\eta\left(b,a\right)) \\
&\qquad\qquad\qquad=\frac{\left(-1\right)^{n-1}\left(\eta\left(b,a\right)\right)^{n}}{2n!}\int_{0}^{1}t^{n-1}\left(n-2t\right)f^{(n)}(a+t\eta\left(b,a\right))dt, \quad (7.19)
\end{aligned}
$$

where the sum above takes 0 *when* $n=1$ *and* $n=2$*.*

Lemma 286 *[94] Suppose that* $f:K\to\mathbb{R}$ *is a mapping defined over* $K\subseteq\mathbb{R}$*, where* K *is an open invex subset with respect to* $\eta:K\times K\to\mathbb{R}$ *of the set of real numbers. If the* n*th derivative* $f^{(n)}$ *exists on* K *and* $f^{(n)}$ *is in the space of* $L\left(\left[a,a+\eta\left(b,a\right)\right]\right)$ *all integrable functions on* $\left[a,a+\eta\left(b,a\right)\right]$*,* $a,b\in K$ *with* $\eta\left(b,a\right)>0$ *and* $n\in\mathbb{N}$*,* $n\geq1$*, the following equality holds:*

$$
\begin{aligned}
&\sum_{k=0}^{n-1}\frac{\left[\left(-1\right)^{k}+1\right]\left(\eta\left(b,a\right)\right)^{k}}{2^{k+1}\left(k+1\right)!}f^{(k)}\left(A\left(2a,\eta\left(b,a\right)\right)\right)-\frac{1}{\eta\left(b,a\right)}\int_{a}^{a+\eta(b,a)}f\left(x\right)dx \\
&\qquad\qquad=\frac{\left(-1\right)^{n+1}\left(\eta\left(b,a\right)\right)^{n}}{n!}\int_{0}^{1}K_{n}(t)f^{(n)}(a+t\eta\left(b,a\right))dt, \quad (7.20)
\end{aligned}
$$

where

$$
K_{n}(t):=\begin{cases} t^{n}, & t\in\left[0,\frac{1}{2}\right] \\ \\ \left(t-1\right)^{n}, & t\in\left(\frac{1}{2},1\right] \end{cases}.
$$

We are now ready to present our results.

Theorem 287 *Suppose that* $f:K\to\mathbb{R}$ *is a mapping defined over* K*, where* K *is an open invex subset with respect to* $\eta:K\times K\to\mathbb{R}$ *of the set of non-negative real numbers. Let the* n*th derivative* $f^{(n)}$ *exist on* K *and* $f^{(n)}$ *be in the space* $L\left(\left[a,a+\eta\left(b,a\right)\right]\right)$ *of all integrable functions on* $\left[a,a+\eta\left(b,a\right)\right]$*,* $a,b\in K$ *with* $\eta\left(b,a\right)>0$*. If* $\left|f^{(n)}\right|^{q}$ *for* $q\geq1$*, is*

s-preinvex on K for $n \in \mathbb{N}$, $n \geq 2$, $s \in (0,1]$, the following inequality holds:

$$\begin{aligned} &\left| A\left(f\left(a\right), f\left(a+\eta\left(b,a\right)\right)\right) - \frac{1}{\eta\left(b,a\right)} \int_a^{a+\eta(b,a)} f\left(x\right) dx \right. \\ &\qquad \left. - \sum_{k=2}^{n-1} \frac{\left(-1\right)^k \left(k-1\right) \left(\eta\left(b,a\right)\right)^k}{2\left(k+1\right)!} f^{(k)}(a+\eta\left(b,a\right)) \right| \\ &\qquad \leq \frac{\left(\eta\left(b,a\right)\right)^n}{2n!} \left(\frac{n-1}{n+1}\right)^{1-\frac{1}{q}} \left(Q\left|f^{(n)}(a)\right|^q + P\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}, \end{aligned} \tag{7.21}$$

where

$$P = \frac{n\left(n-1\right) + s\left(n-2\right)}{\left(n+s\right)\left(n+s+1\right)}, Q = nB\left(n, s+1\right) - 2B\left(n+1, s+1\right)$$

and

$$B(x,y) = \int_0^1 t^{x-1}\left(1-t\right)^{y-1}$$

for $x > 0$, $y > 0$ is the Beta function.

Proof. Suppose $n \geq 2$ and $q = 1$. By s-preinvexity of $\left|f^{(n)}\right|$ and lemma 285, we get

$$\begin{aligned} &\left| A\left(f\left(a\right), f\left(a+\eta\left(b,a\right)\right)\right) - \frac{1}{\eta\left(b,a\right)} \int_a^{a+\eta(b,a)} f\left(x\right) dx \right. \\ &\qquad \left. - \sum_{k=2}^{n-1} \frac{\left(-1\right)^k \left(k-1\right) \left(\eta\left(b,a\right)\right)^k}{2\left(k+1\right)!} f^{(k)}(a+\eta\left(b,a\right)) \right| \\ &\qquad \leq \frac{\left(\eta\left(b,a\right)\right)^n}{2n!} \int_0^1 t^{n-1}\left(n-2t\right)\left|f^{(n)}(a+t\eta\left(b,a\right))\right| dt \\ &\qquad \leq \frac{\left(\eta\left(b,a\right)\right)^n}{2n!} \int_0^1 t^{n-1}\left(n-2t\right)\left(\left(1-t\right)^s\left|f^{(n)}(a)\right| + t^s\left|f^{(n)}(b)\right|\right) dt \\ &= \frac{\left(\eta\left(b,a\right)\right)^n}{2n!} \left(\left|f^{(n)}(b)\right| \int_0^1 t^{n+s-1}\left(n-2t\right) dt + \left|f^{(n)}(a)\right| \int_0^1 t^{n-1}\left(n-2t\right)\left(1-t\right)^s dt\right). \end{aligned} \tag{7.22}$$

Since

$$\int_0^1 t^{n+s-1}\left(n-2t\right) dt = \frac{n\left(n-1\right) + s\left(n-2\right)}{\left(n+s\right)\left(n+s+1\right)} = P$$

and

$$\int_0^1 t^{n-1}(n-2t)(1-t)^s\,dt = n\int_0^1 t^{n-1}(1-t)^s\,dt - 2\int_0^1 t^n(1-t)^s\,dt$$
$$= nB(n, s+1) - 2B(n+1, s+1) = Q.$$

Using the above observations in (7.22), we get (7.21). The proof for the case $q = 1$ is complete.

Assume now that $q > 1$, then by the s-preinvexity of $\left|f^{(n)}\right|^q$ on K, lemma 285 and the Hölder's inequality, we have

$$\begin{aligned}
&\Bigg|A\left(f(a), f(a+\eta(b,a))\right) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx \\
&\qquad - \sum_{k=2}^{n-1}\frac{(-1)^k (k-1)(\eta(b,a))^k}{2(k+1)!} f^{(k)}(a+\eta(b,a))\Bigg| \\
&\le \frac{(\eta(b,a))^n}{2n!}\left(\int_0^1 t^{n-1}(n-2t)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{n-1}(n-2t)\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}} \\
&\qquad \le \frac{(\eta(b,a))^n}{2n!}\left(\int_0^1 t^{n-1}(n-2t)\,dt\right)^{1-\frac{1}{q}} \\
&\times\left(\left|f^{(n)}(b)\right|^q\int_0^1 t^{n+s-1}(n-2t)\,dt + \left|f^{(n)}(a)\right|^q\int_0^1 t^{n-1}(n-2t)(1-t)^s\right)^{\frac{1}{q}}
\end{aligned} \tag{7.23}$$

which is the inequality (7.21). Hence the proof of the theorem is complete. ■

Remark 288 *If in Theorem 287, we take $s = 1$, we get Theorem 8 from [94].*

Corollary 289 *Let f be as in Theorem 287 and $n = 2$, we have*

$$\begin{aligned}
&\left|A\left(f(a), f(a+\eta(b,a))\right) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \\
&\qquad \le \frac{(\eta(b,a))^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{\left|f''(a)\right|^q + \left|f''(b)\right|^q}{(s+3)(s+2)}\right)^{\frac{1}{q}}.
\end{aligned} \tag{7.24}$$

Remark 290 *In Corollary 289 $s = 1$, we get a result proved in Corollary 1 from [94].*

Remark 291 *In Theorem 287, replacement of $\eta(b, a) = b - a$ gives a result proved in Theorem 1.1 from [70].*

Theorem 292 *Suppose that $f : K \to \mathbb{R}$ is a mapping defined over K, where K is an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ of the set of non-negative real numbers. Let the nth derivative $f^{(n)}$ exist on K and $f^{(n)}$ be in the space $L([a, a + \eta(b, a)])$ of all integrable functions on $[a, a + \eta(b, a)]$, $a, b \in K$ with $\eta(b, a) > 0$. If $\left|f^{(n)}\right|^q$ for $q \geq 1$, is s-preinvex on K for $n \in \mathbb{N}$, $n \geq 2$, $s \in (0, 1]$, the following inequality holds:*

$$\begin{aligned}
&\Bigg| A\left(f(a), f(a + \eta(b, a))\right) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)\, dx \\
&\qquad - \sum_{k=2}^{n-1} \frac{(-1)^k (k-1) (\eta(b, a))^k}{2(k+1)!} f^{(k)}(a + \eta(b, a)) \Bigg| \\
&\qquad\qquad \leq \frac{(\eta(b, a))^n (n-1)^{1-1/q}}{2n!} \\
&\qquad \times \Bigg[\left\{ nB(nq - q + 1, s + 1) - 2B(nq - q + 2, s + 1) \right\} \left|f^{(n)}(a)\right|^q \\
&\qquad\qquad \left\{ + \frac{n}{nq - q + s + 1} - \frac{2}{nq - q + s + 2} \right\} \left|f^{(n)}(b)\right|^{q1/q} \Bigg]^{1/q}, \qquad (7.25)
\end{aligned}$$

where

$$B(x, y) = \int_0^1 t^{x-1} (1 - t)^{y-1}$$

for $x > 0$, $y > 0$ is the Beta function.

Proof. The case when $q = 1$ is easy to prove so we assume that $q > 1$. By making use of Lemma 285, the Hölder inequality and the s-preinvexity of $\left|f^{(n)}\right|^q$, we have

$$\left|A\left(f\left(a\right), f\left(a+\eta\left(b,a\right)\right)\right) - \frac{1}{\eta\left(b,a\right)}\int_a^{a+\eta(b,a)} f\left(x\right)dx \right.$$
$$\left. -\sum_{k=2}^{n-1}\frac{\left(-1\right)^k\left(k-1\right)\left(\eta\left(b,a\right)\right)^k}{2\left(k+1\right)!}f^{(k)}(a+\eta\left(b,a\right))\right| \le \frac{\left(\eta\left(b,a\right)\right)^n}{2n!}\left(\int_0^1\left(n-2t\right)dt\right)^{1-1/q}$$
$$\times\int_0^1 t^{q(n-1)}\left(n-2t\right)\left(\left(1-t\right)^s\left|f^{(n)}(a)\right|^q + t^s\left|f^{(n)}(b)\right|^q\right)dt = \frac{\left(\eta\left(b,a\right)\right)^n\left(n-1\right)^{1-1/q}}{2n!}$$
$$\times\left[\left\{nB\left(nq-q+1,s+1\right)-2B\left(nq-q+2,s+1\right)\right\}\left|f^{(n)}(a)\right|^q\right.$$
$$\left.+\left\{\frac{n}{nq-q+s+1}-\frac{2}{nq-q+s+2}\right\}\left|f^{(n)}(b)\right|^{q1/q}\right]^{1/q}. \quad (7.26)$$

The proof is thus complete. ■

Corollary 293 *Let f be as stated in Theorem 292 and $n = 2$. Then*

$$\left|A\left(f\left(a\right), f\left(a+\eta\left(b,a\right)\right)\right) - \frac{1}{\eta\left(b,a\right)}\int_a^{a+\eta(b,a)} f\left(x\right)dx\right|$$
$$\le \frac{\left(\eta\left(b,a\right)\right)^2}{2^{2-1/q}}\times\left[\left\{B\left(q+1,s+1\right)-B\left(q+2,s+1\right)\right\}\left|f''(a)\right|^q\right.$$
$$\left.+\frac{\left|f''(b)\right|^q}{\left(q+s+1\right)\left(q+s+2\right)}\right]^{1/q}, \quad (7.27)$$

where $B(x, y)$, $x, y > 0$ is the Beta function.

Corollary 294 *If we take $q = 1$ and $s = 1$ in Corollary 293, then*

$$\left|A\left(f\left(a\right), f\left(a+\eta\left(b,a\right)\right)\right) - \frac{1}{\eta\left(b,a\right)}\int_a^{a+\eta(b,a)} f\left(x\right)dx\right|$$
$$\le \frac{\left(\eta\left(b,a\right)\right)^2}{24}\left[\left|f''(a)\right|^q + \left|f''(b)\right|\right]. \quad (7.28)$$

Remark 295 *For $\eta(b,a) = b - a$, we obtain new bounds of the difference between the middle and right side of Hermite-Hadamard's inequalities (2.19) in terms of second order derivatives.*

Now we give some results related to left-side of Hermite-Hadamard's inequality for n-times differentiable s-preinvex functions.

Theorem 296 *Suppose that $f : K \to \mathbb{R}$ is a mapping defined over K, where K is an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ of the set of non-negative real numbers. Let the nth derivative $f^{(n)}$ exist on K and $f^{(n)}$ be in the space $L([a, a + \eta(b,a)])$ of all integrable functions on $[a, a + \eta(b,a)]$, $a, b \in K$ with $\eta(b,a) > 0$. If $\left|f^{(n)}\right|^q$ for $q > 1$, is s-preinvex on K for $n \in \mathbb{N}$, $n \geq 1$, $s \in (0,1]$, the following inequality holds:*

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](\eta(b,a))^k}{2^{k+1}(k+1)!} f^{(k)}(A(2a, \eta(b,a))) \right.$$
$$\left. - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \right| \leq \frac{(\eta(b,a))^n}{2^n n!\,(np+1)^{\frac{1}{p}}} \left[\frac{\left|f^{(n)}(a)\right|^q + \left|f^{(n)}(b)\right|^q}{s+1}\right]^{\frac{1}{q}}, \quad (7.29)$$

where p and q are Hölder conjugates of each other.

Proof. Suppose $n \geq 1$. By using lemma 286 and the s-preinvexity of $\left|f^{(n)}\right|^q$ on K for $n \in \mathbb{N}$, $n \geq 1$, $q > 1$, we have

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](\eta(b,a))^k}{2^{k+1}(k+1)!} f^{(k)}(A(2a, \eta(b,a))) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx\right| \\
&\qquad \leq \frac{(\eta(b,a))^n}{n!} \int_0^1 |K_n(t)| \left|f^{(n)}(a + t\eta(b,a))\right| dt \\
&\qquad \leq \frac{(\eta(b,a))^n}{n!} \left(\int_0^1 |K_n(t)|^p\right)^{\frac{1}{p}} \left(\int_0^1 \left|f^{(n)}(a + t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}. \quad (7.30)
\end{aligned}$$

Since

$$\int_0^1 |K_n(t)|^p = \int_0^{\frac{1}{2}} t^{np} dt + \int_{\frac{1}{2}}^1 (1-t)^{np}\,dt = \frac{1}{2^{np}(np+1)}$$

and

$$\int_0^1 \left|f^{(n)}(a+t\eta(b,a))\right|^q dt \leq \int_0^1 (1-t)^s \left|f^{(n)}(a)\right|^q dt + \int_0^1 t^s \left|f^{(n)}(b)\right|^q dt$$
$$= \frac{\left|f^{(n)}(a)\right|^q + \left|f^{(n)}(b)\right|^q}{s+1}$$

By application of the above observations in (7.30), we get the desired inequality (7.29). This completes the proof of the theorem. ■

Corollary 297 *Let f be as in Theorem 296 and $n = 2$, we obtain the following inequality:*

$$\left|f\left(A\left(2a, \eta(b,a)\right)\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\, dx\right|$$
$$\leq \frac{(\eta(b,a))^2}{8(2p+1)^{\frac{1}{p}}} \left[\frac{\left|f''(a)\right|^q + \left|f''(b)\right|^q}{s+1}\right]^{\frac{1}{q}}, \quad (7.31)$$

where p and q are Hölder conjugates of each other.

Corollary 298 *In Corollary 297, if we take $s = 1$, then one gets the following result:*

$$\left|f\left(A\left(2a, \eta(b,a)\right)\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\, dx\right|$$
$$\leq \frac{(\eta(b,a))^2}{8(2p+1)^{\frac{1}{p}}} \left[\frac{\left|f''(a)\right|^q + \left|f''(b)\right|^q}{2}\right]^{\frac{1}{q}}, \quad (7.32)$$

where p and q are Hölder conjugates of each other.

Corollary 299 *In Theorem 296, if $\eta(b,a) = b - a$, we have the following inequality:*

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(A(a,b)\right) - \frac{1}{b-a}\int_a^b f(x)\, dx\right|$$
$$\leq \frac{(b-a)^n}{2^n n!\,(np+1)^{\frac{1}{p}}} \left[\frac{\left|f^{(n)}(a)\right|^q + \left|f^{(n)}(b)\right|^q}{s+1}\right]^{\frac{1}{q}}, \quad (7.33)$$

where p and q are Hölder conjugates of each other.

A different approach leads us to the following result:

Theorem 300 *Suppose that $f : K \to \mathbb{R}$ is a mapping defined over K, where K is an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ of the set of non-negative real numbers. Let the nth derivative $f^{(n)}$ exist on K and $f^{(n)}$ be in the space $L\left([a, a + \eta(b, a)]\right)$ of all integrable functions on $[a, a + \eta(b, a)]$, $a, b \in K$ with $\eta(b, a) > 0$. If $\left|f^{(n)}\right|^q$ for $q > 1$, is s-preinvex on K for $n \in \mathbb{N}$, $n \geq 1$, $s \in (0, 1]$, the following inequality holds:*

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right] (\eta(b, a))^k}{2^{k+1} (k+1)!} f^{(k)} \left(A\left(2a, \eta(b, a)\right)\right) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)\, dx\right|$$
$$\leq \frac{(\eta(b, a))^n}{2^{n+\frac{1}{p}} (np+1)^{\frac{1}{p}} n!} \left[\left(\frac{(2^{s+1} - 1)\left|f^{(n)}(a)\right|^q + \left|f^{(n)}(b)\right|^q}{2^{s+1}(s+1)}\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\frac{\left|f^{(n)}(a)\right|^q + (2^{s+1} - 1)\left|f^{(n)}(b)\right|^q}{2^{s+1}(s+1)}\right)^{\frac{1}{q}}\right], \quad (7.34)$$

where p and q are Hölder conjugates of each other.

Proof. From lemma 286 and the power-mean integral inequality, we have

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right] (\eta(b, a))^k}{2^{k+1} (k+1)!} f^{(k)} \left(A\left(2a, \eta(b, a)\right)\right) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)\, dx\right|$$
$$\leq \frac{(\eta(b, a))^n}{n!} \left[\left(\int_0^{\frac{1}{2}} t^{np} dt\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left|f^{(n)}(a + t\eta(b, a))\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\int_{\frac{1}{2}}^1 (1-t)^{np} dt\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 \left|f^{(n)}(a + t\eta(b, a))\right|^q dt\right)^{\frac{1}{q}}\right]. \quad (7.35)$$

Since $\left|f^{(n)}\right|^q$ is s-preinvex on K in the second sense for $n \in \mathbb{N}$, $n \geq 1$, $q \in \mathbb{R}$, $q > 1$ and $s \in (0,1]$. Hence for every $a, b \in K$ with $\eta(b,a) > 0$, we have

$$\int_0^{\frac{1}{2}} t^n \left|f^{(n)}(a + t\eta(b,a))\right|^q dt \leq \left|f^{(n)}(a)\right|^q \int_0^{\frac{1}{2}} (1-t)^s \, dt + \left|f^{(n)}(b)\right|^q \int_0^{\frac{1}{2}} t^s dt$$
$$= \frac{2^{s+1}-1}{2^{s+1}(s+1)} \left|f^{(n)}(a)\right|^q + \frac{1}{2^{s+1}(s+1)} \left|f^{(n)}(b)\right|^q \quad (7.36)$$

and

$$\int_0^{\frac{1}{2}} \left|f^{(n)}(a + t\eta(b,a))\right|^q dt \leq \left|f^{(n)}(a)\right|^q \int_{\frac{1}{2}}^1 (1-t)^s \, dt + \left|f^{(n)}(b)\right|^q \int_{\frac{1}{2}}^1 t^s dt$$
$$= \frac{1}{2^{s+1}(s+1)} \left|f^{(n)}(a)\right|^q + \frac{2^{s+1}-1}{2^{s+1}(s+1)} \left|f^{(n)}(b)\right|^q. \quad (7.37)$$

Also

$$\int_0^{\frac{1}{2}} t^{np} dt = \int_{\frac{1}{2}}^1 (1-t)^{np} \, dt = \frac{1}{2^{np+1}(np+1)}. \quad (7.38)$$

Using (7.36), (7.37) and (7.38) in (7.35), we get the required inequality (7.34). ■

Remark 301 *For $s = 1$, Theorem 300 becomes Theorem 11 from [94].*

Corollary 302 *For $s = 1$ and $n = 2$, we get the following inequality from [94]:*

$$\left| f\left(A\left(2a, \eta(b,a)\right)\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\, dx \right|$$
$$\leq \frac{(\eta(b,a))^2}{8 \cdot 2^{\frac{1}{p}} (2p+1)^{\frac{1}{p}}} \left[\left(\frac{3\left|f''(a)\right|^q + \left|f''(b)\right|^q}{8} \right)^{\frac{1}{q}} + \left(\frac{\left|f''(a)\right|^q + 3\left|f''(b)\right|^q}{8} \right)^{\frac{1}{q}} \right], \quad (7.39)$$

where p and q are Hölder conjugates of each other.

Theorem 303 *Suppose that $f : K \to \mathbb{R}$ is a mapping defined over K, where K is an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ of the set of non-negative real numbers. Let the nth derivative $f^{(n)}$ exist on K and $f^{(n)}$ be in the space $L([a, a + \eta(b,a)])$ of all* *integrable functions on* $[a, a+\eta(b,a)]$, $a, b \in K$ *with* $\eta(b,a) > 0$. *If* $\left|f^{(n)}\right|^q$ *for* $q \geq 1$, *is* s*-preinvex on* K *for* $n \in \mathbb{N}$, $n \geq 1$, $s \in (0,1]$, *the following inequality holds:*

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right](\eta(b,a))^k}{2^{k+1}(k+1)!} f^{(k)}(A(2a, \eta(b,a))) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx\right|$$
$$\leq \frac{(\eta(b,a))^n (n+1)^{\frac{1}{q}}}{2^{(n+1)\left(1-\frac{1}{q}\right)}(n+1)!} \left[\left(L\left|f^{(n)}(a)\right|^q + M\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}} + \left(M\left|f^{(n)}(a)\right|^q + N\left|f^{(n)}(b)\right|^q\right)^{\frac{1}{q}}\right], \quad (7.40)$$

where

$$L = B\left(\frac{1}{2}; n+1, s+1\right), M = \frac{1}{2^{n+s+1}(n+s+1)},$$
$$N = B(s+1, n+1) - B\left(\frac{1}{2}; s+1, n+1\right)$$

and

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}$$

for $x > 0$, $y > 0$ *is the Beta function, and*

$$B(z; x, y) = \int_0^z t^{x-1}(1-t)^{y-1}$$

is the generalization of the Beta function $B(x,y)$.

Proof. It is not difficult to see that (7.40) holds true for $q=1$. Suppose that $q>1$. From lemma 286 and the Hölder's integral inequality, we have

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](\eta(b,a))^k}{2^{k+1}(k+1)!}f^{(k)}(A(2a,\eta(b,a)))-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right|$$

$$\leq\frac{(\eta(b,a))^n}{n!}\left[\left(\int_0^{\frac{1}{2}}t^n dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}t^n\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^1(1-t)^n dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1(1-t)^n\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right]. \quad (7.41)$$

Since $\left|f^{(n)}\right|^q$ is s-preinvex on K in the second sense for $n\in\mathbb{N}$, $n\geq1$, $q\in\mathbb{R}$, $q\geq1$ and $s\in(0,1]$. Hence for every $a,b\in K$ with $\eta(b,a)>0$, we have

$$\int_0^{\frac{1}{2}}t^n\left|f^{(n)}(a+t\eta(b,a))\right|^q dt\leq\left|f^{(n)}(a)\right|^q\int_0^{\frac{1}{2}}t^n(1-t)^s dt+\left|f^{(n)}(b)\right|^q\int_0^{\frac{1}{2}}t^{n+s}dt$$

$$=B\left(\frac{1}{2};n+1,s+1\right)\left|f^{(n)}(a)\right|^q+\frac{\left|f^{(n)}(b)\right|^q}{2^{n+s+1}(n+s+1)} \quad (7.42)$$

and

$$\int_0^{\frac{1}{2}}(1-t)^n\left|f^{(n)}(a+t\eta(b,a))\right|^q dt$$

$$\leq\left|f^{(n)}(a)\right|^q\int_{\frac{1}{2}}^1(1-t)^{n+s}dt+\left|f^{(n)}(b)\right|^q\int_{\frac{1}{2}}^1 t^s(1-t)^n dt$$

$$=\frac{\left|f^{(n)}(a)\right|^q}{2^{n+s+1}(n+s+1)}+\left[B(s+1,n+1)-B\left(\frac{1}{2};s+1,n+1\right)\right]\left|f^{(n)}(b)\right|^q. \quad (7.43)$$

Using (7.42), (7.43) and

$$\int_0^{\frac{1}{2}}t^{np}dt=\int_{\frac{1}{2}}^1(1-t)^{np}dt=\frac{1}{2^{np+1}(np+1)}$$

in (7.41), we get the required inequality (7.40). ■

Corollary 304 *If we choose $n = 2$ and $s = 1$ in the Theorem 303, we get the following inequality:*

$$\left| f\left(A\left(2a, \eta\left(b,a\right)\right)\right) - \frac{1}{\eta\left(b,a\right)} \int_a^{a+\eta(b,a)} f\left(x\right) dx \right|$$
$$\leq \frac{\left(\eta\left(b,a\right)\right)^2 \cdot 3^{\frac{1}{q}-1}}{2^{4-\frac{3}{q}}} \left[\left(\frac{5\left|f''\left(a\right)\right|^q + 3\left|f''\left(b\right)\right|^q}{192} \right)^{\frac{1}{q}} + \left(\frac{3\left|f''\left(a\right)\right|^q + 5\left|f''\left(b\right)\right|^q}{192} \right)^{\frac{1}{q}} \right]. \quad (7.44)$$

Proof. Since for $n = 2$ and $s = 1$, $L = B\left(\frac{1}{2}; 3, 2\right) = \frac{5}{192}$, $M = \frac{1}{64}$ and $N = B\left(2,3\right) - B\left(\frac{1}{2}; 2, 3\right) = \frac{5}{192}$ and hence proof follows. ■

7.4 Applications to Special Means

Now, let a and b be positive real numbers such that $a < b$. Consider the function $M := M(a,b) : [a, a + \eta(b,a)] \times [a, a + \eta(b,a)] \to \mathbb{R}^+$, which is one of the means defined in Section 2.13, therefore we can obtain several inequalities for these means as follows:

Setting $\eta(b,a) = M(b,a)$ in (7.24), (7.31), one can obtain the following interesting inequalities involving means:

$$\left| \frac{f\left(a\right) + f\left(a + M(b,a)\right)}{2} - \frac{1}{M(b,a)} \int_a^{a+M(b,a)} f\left(x\right) dx \right|$$
$$\leq \frac{\left(M(b,a)\right)^2}{2} \left(\frac{1}{6}\right)^{1-\frac{1}{q}} \left(\frac{\left|f''\left(a\right)\right|^q + \left|f''\left(b\right)\right|^q}{\left(s+3\right)\left(s+2\right)} \right)^{\frac{1}{q}}. \quad (7.45)$$

$$\left| f\left(a + \frac{1}{2}M(b,a)\right) - \frac{1}{M(b,a)} \int_a^{a+M(b,a)} f\left(x\right) dx \right|$$
$$\leq \frac{\left(M(b,a)\right)^2}{8\left(2p+1\right)^{\frac{1}{p}}} \left[\frac{\left|f''\left(a\right)\right|^q + \left|f''\left(b\right)\right|^q}{s+1} \right]^{\frac{1}{q}}, \quad (7.46)$$

Letting $M = A,\ G,\ H,\ P_r,\ I,\ L,\ L_p$ in (7.45) and in (7.46), we get the inequalities involving means for a particular choice of a twice differentiable s-preinvex function f, and the details are left to the interested reader.

7.5 On H-H type integral inequalities for n-times differentiable log-preinvex functions

In this section we provide some Hermite-Hadamard type inequalities for n-times differentiable log-preinvex functions.

Theorem 305 *Suppose that $f : K \to \mathbb{R}$ is a mapping defined over K, where K is an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ of the set of real numbers. Let the nth derivative $f^{(n)}$ exist on K and $f^{(n)}$ be in the space $L([a, a + \eta(b, a)])$ of all integrable functions on $[a, a + \eta(b, a)]$, $a, b \in K$ with $\eta(b, a) > 0$. If $\left|f^{(n)}\right|^q$ for $q \geq 1$, is* log*-preinvex on K for $n \in \mathbb{N}$, $n \geq 2$, the following inequality holds:*

$$\left| A(f(a), f(a + \eta(b, a))) - \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)\, dx \right.$$
$$\left. - \sum_{k=2}^{n-1} \frac{(-1)^k (k-1)(\eta(b, a))^k}{2(k+1)!} f^{(k)}(a + \eta(b, a)) \right|$$
$$\leq \frac{(\eta(b, a))^n}{2n!} \left(\frac{n-1}{n+1}\right)^{1-1/q} [\kappa(n, q; a, b)]^{1/q}, \quad (7.47)$$

where

$$\kappa(n, q; a, b) = \frac{(-1)^n n! \left\{q\left[\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right)\right] + 2\right\} \left|f^{(n)}(a)\right|^q}{q^{n+1}\left[\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right)\right]^{n+1}}$$
$$- \frac{2\left|f^{(n)}(b)\right|^q}{q\left[\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right)\right]}$$
$$- n!\left|f^{(n)}(b)\right|^q \sum_{k=1}^{n} \frac{(-1)^k \left\{q\left[\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right)\right] + 2\right\}}{q^{k+1}(n-k)!\left[\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right)\right]^{k+1}}.$$

Proof. Suppose $n \geq 2$. Let $a, b \in K$. Since K is an invex set with respect to η, for every $t \in [0,1]$ we have $a + t\eta(b,a) \in K$. By log-preinvexity of $\left|f^{(n)}\right|^q$, Lemma 285 and Hölder inequality, we have

$$\begin{aligned}
&\left| A\left(f\left(a\right), f\left(a+\eta\left(b,a\right)\right)\right) - \frac{1}{\eta\left(b,a\right)} \int_a^{a+\eta(b,a)} f\left(x\right) dx \right. \\
&\qquad \left. - \sum_{k=2}^{n-1} \frac{\left(-1\right)^k \left(k-1\right) \left(\eta\left(b,a\right)\right)^k}{2\left(k+1\right)!} f^{(k)}(a+\eta\left(b,a\right)) \right| \leq \frac{\left(\eta\left(b,a\right)\right)^n}{2n!} \\
&\quad \times \left(\int_0^1 t^{n-1} \left(n-2t\right) dt \right)^{1-1/q} \left(\int_0^1 t^{n-1} \left(n-2t\right) \left|f^{(n)}(a+t\eta\left(b,a\right))\right|^q dt \right)^{1/q} \\
&\leq \frac{\left(\eta\left(b,a\right)\right)^n}{2n!} \left(\frac{n-1}{n+1}\right)^{1-1/q} \left(\int_0^1 t^{n-1} \left(n-2t\right) \left(\left(\left|f^{(n)}(a)\right|\right)^{q(1-t)} \left(\left|f^{(n)}(b)\right|\right)^{qt} \right) dt \right)^{1/q} \\
&\quad = \frac{\left(\eta\left(b,a\right)\right)^n \left|f^{(n)}(a)\right|}{2n!} \left(\frac{n-1}{n+1}\right)^{1-1/q} \left(n \int_0^1 t^{n-1} \mu^t dt - 2 \int_0^1 t^n \mu^t dt \right)^{1/q}, \quad (7.48)
\end{aligned}$$

where $\mu = \frac{\left|f^{(n)}(b)\right|^q}{\left|f^{(n)}(a)\right|^q} \neq 1$.

By Lemma 155, we have

$$\begin{aligned}
&n \int_0^1 t^{n-1} \mu^t dt - 2 \int_0^1 t^n \mu^t dt = \frac{\left(-1\right)^n n!}{\left(\ln \mu\right)^n} \\
&\quad + n!\mu \sum_{k=1}^{n} \frac{\left(-1\right)^{k-1}}{\left(n-k\right)! \left(\ln \mu\right)^k} - \frac{2\left(-1\right)^{n+1} n!}{\left(\ln \mu\right)^{n+1}} - 2n!\mu \sum_{k=0}^{n} \frac{\left(-1\right)^k}{\left(n-k\right)! \left(\ln \mu\right)^{k+1}} \\
&\qquad = \frac{\left(-1\right)^n n! \left[\ln \mu + 2\right]}{\left(\ln \mu\right)^{n+1}} - \frac{2\mu}{\ln \mu} - n!\mu \sum_{k=1}^{n} \frac{\left(-1\right)^k \left[\ln \mu + 2\right]}{\left(n-k\right)! \left(\ln \mu\right)^{k+1}}. \quad (7.49)
\end{aligned}$$

Applying (7.49) in (7.48) and replacing $\frac{\left|f^{(n)}(b)\right|^q}{\left|f^{(n)}(a)\right|^q} \neq 1$, we get the desired inequality (7.47).

■

Corollary 306 *Let f be as in Theorem 305 and if $q = 1$, the following inequality holds*

$$\left| A\left(f\left(a\right), f\left(a+\eta\left(b,a\right)\right)\right) - \frac{1}{\eta\left(b,a\right)} \int_a^{a+\eta(b,a)} f\left(x\right) dx \right.$$
$$\left. - \sum_{k=2}^{n-1} \frac{\left(-1\right)^k \left(k-1\right) \left(\eta\left(b,a\right)\right)^k}{2\left(k+1\right)!} f^{(k)}(a+\eta\left(b,a\right)) \right| \le \frac{\left(\eta\left(b,a\right)\right)^n}{2n!} \kappa\left(n,1;a,b\right), \quad (7.50)$$

where

$$\kappa\left(n,1;a,b\right) = \frac{\left(-1\right)^n n! \left\{ \left[\ln\left(\left|f^{(n)}\left(b\right)\right|\right) - \ln\left(\left|f^{(n)}\left(a\right)\right|\right)\right] + 2 \right\} \left|f^{(n)}\left(a\right)\right|^q}{\left[\ln\left(\left|f^{(n)}\left(b\right)\right|\right) - \ln\left(\left|f^{(n)}\left(a\right)\right|\right)\right]^{n+1}}$$
$$- \frac{2\left|f^{(n)}\left(b\right)\right|^q}{\left[\ln\left(\left|f^{(n)}\left(b\right)\right|\right) - \ln\left(\left|f^{(n)}\left(a\right)\right|\right)\right]}$$
$$- n! \left|f^{(n)}\left(b\right)\right|^q \sum_{k=1}^{n} \frac{\left(-1\right)^k \left\{ \left[\ln\left(\left|f^{(n)}\left(b\right)\right|\right) - \ln\left(\left|f^{(n)}\left(a\right)\right|\right)\right] + 2 \right\}}{\left(n-k\right)! \left[\ln\left(\left|f^{(n)}\left(b\right)\right|\right) - \ln\left(\left|f^{(n)}\left(a\right)\right|\right)\right]^{k+1}}.$$

Corollary 307 *Let f be as in Theorem 305 and $n = 2$, the following inequality is valid*

$$\left| A\left(f\left(a\right), f\left(a+\eta\left(b,a\right)\right)\right) - \frac{1}{\eta\left(b,a\right)} \int_a^{a+\eta(b,a)} f\left(x\right) dx \right|$$
$$\le \frac{\left(\eta\left(b,a\right)\right)^2}{4} \left(\frac{1}{3}\right)^{1-1/q} \left[\kappa\left(2,q;a,b\right)\right]^{1/q}, \quad (7.51)$$

where

$$\kappa\left(2,q;a,b\right) = \frac{2\left\{ q\left[\ln\left(\left|f''\left(b\right)\right|\right) - \ln\left(\left|f''\left(a\right)\right|\right)\right] + 2 \right\} \left|f''\left(a\right)\right|^q}{q^3 \left[\ln\left(\left|f''\left(b\right)\right|\right) - \ln\left(\left|f''\left(a\right)\right|\right)\right]^3}$$
$$+ \frac{2\left\{ q\left[\ln\left(\left|f''\left(b\right)\right|\right) - \ln\left(\left|f''\left(a\right)\right|\right)\right] - 2 \right\} \left|f''\left(b\right)\right|^q}{q^3 \left[\ln\left(\left|f''\left(b\right)\right|\right) - \ln\left(\left|f''\left(a\right)\right|\right)\right]^3}.$$

Corollary 308 *Let f be as in Theorem 305 and $n = 2$, $q = 1$, the following inequality holds*

$$\left| A\left(f\left(a\right), f\left(a+\eta\left(b,a\right)\right)\right) - \frac{1}{\eta\left(b,a\right)} \int_a^{a+\eta(b,a)} f\left(x\right) dx \right| \le \frac{\left(\eta\left(b,a\right)\right)^2}{4} \kappa\left(2,1;a,b\right), \quad (7.52)$$

where

$$\kappa(2,1;a,b) = \frac{2\left\{\left[\ln\left(\left|f''(b)\right|\right) - \ln\left(\left|f''(a)\right|\right)\right] + 2\right\}\left|f''(a)\right|}{\left[\ln\left(\left|f''(b)\right|\right) - \ln\left(\left|f''(a)\right|\right)\right]^3} + \frac{2\left\{\left[\ln\left(\left|f''(b)\right|\right) - \ln\left(\left|f''(a)\right|\right)\right] - 2\right\}\left|f''(b)\right|}{\left[\ln\left(\left|f''(b)\right|\right) - \ln\left(\left|f''(a)\right|\right)\right]^3}.$$

Remark 309 *If $\eta(b,a) = b - a$ in the inequalities (7.51) and (7.52), one can get inequalities for the bounds of the difference between middle and the right most terms in the Hermite-Hadamard inequalities (2.19) in terms of second order derivatives for log-convex functions.*

Theorem 310 *Suppose that $f : K \to \mathbb{R}$ is a mapping defined over K, where K is an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ of the set of real numbers. Let the nth derivative $f^{(n)}$ exist on K and $f^{(n)}$ be in the space $L([a, a+\eta(b,a)])$ of all integrable functions on $[a, a+\eta(b,a)]$, $a, b \in K$ with $\eta(b,a) > 0$. If $\left|f^{(n)}\right|^q$ for $q > 1$, is* log*-preinvex on K for $n \in \mathbb{N}$, $n \geq 2$, the following inequality holds:*

$$\begin{aligned} &\left| A\left(f(a), f(a+\eta(b,a))\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \right. \\ &\qquad \left. - \sum_{k=2}^{n-1} \frac{(-1)^k (k-1)(\eta(b,a))^k}{2(k+1)!} f^{(k)}(a+\eta(b,a)) \right| \\ &\leq \frac{(\eta(b,a))^n \left[n^{(2q-1)/(q-1)} - (n-2)^{(2q-1)/(q-1)}\right]^{1-1/q} \left|f^{(n)}(b)\right|}{2^{2-1/q} n!} \\ &\quad \times \left(\frac{q-1}{2q-1}\right)^{1-1/q} \left(\sum_{k=1}^{\infty} (-q)^{k-1} \frac{\left[\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right)\right]^{k-1}}{(q(n-1)+1)_k}\right)^{1/q}. \end{aligned} \tag{7.53}$$

Proof. By log-preinvexity of $\left|f^{(n)}\right|^q$, Lemma 285 and Hölder inequality, we have

$$\begin{aligned}
&\left|A\left(f\left(a\right), f\left(a+\eta\left(b,a\right)\right)\right) - \frac{1}{\eta\left(b,a\right)}\int_a^{a+\eta(b,a)} f\left(x\right)dx \right.\\
&\left. - \sum_{k=2}^{n-1}\frac{\left(-1\right)^k\left(k-1\right)\left(\eta\left(b,a\right)\right)^k}{2\left(k+1\right)!}f^{(k)}(a+\eta\left(b,a\right))\right| \le \frac{\left(\eta\left(b,a\right)\right)^n}{2n!}\\
&\times\left(\int_0^1\left(n-2t\right)^{q/(q-1)}dt\right)^{1-1/q}\left(\int_0^1 t^{q(n-1)}\left|f^{(n)}(a+t\eta\left(b,a\right))\right|^q dt\right)^{1/q}\\
&\le \frac{\left(\eta\left(b,a\right)\right)^n\left[n^{(2q-1)/(q-1)}-\left(n-2\right)^{(2q-1)/(q-1)}\right]^{1-1/q}}{2^{2-1/q}n!}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\\
&\times\left(\int_0^1 t^{q(n-1)}\left(\left(\left|f^{(n)}(a)\right|\right)^{q(1-t)}\left(\left|f^{(n)}(b)\right|\right)^{qt}\right)dt\right)^{1/q}\\
&= \frac{\left(\eta\left(b,a\right)\right)^n\left[n^{(2q-1)/(q-1)}-\left(n-2\right)^{(2q-1)/(q-1)}\right]^{1-1/q}\left|f^{(n)}(a)\right|}{2^{2-1/q}n!}\\
&\times\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left(\int_0^1 t^{q(n-1)}\mu^t dt\right)^{1/q}, \quad (7.54)
\end{aligned}$$

where $\mu = \frac{\left|f^{(n)}(b)\right|^q}{\left|f^{(n)}(a)\right|^q} \neq 1$. Applying Lemma 159 to the integral in the inequality (7.54) and simplifying, we get the required inequality (7.53). ■

Corollary 311 *Let f be as in Theorem 310 and $n=2$. Then*

$$\begin{aligned}
&\left|A\left(f\left(a\right), f\left(a+\eta\left(b,a\right)\right)\right) - \frac{1}{\eta\left(b,a\right)}\int_a^{a+\eta(b,a)} f\left(x\right)dx\right| \le \frac{\left(\eta\left(b,a\right)\right)^2\left|f''(b)\right|}{2}\\
&\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left(\sum_{k=1}^{\infty}\frac{\left(-q\right)^{k-1}\left[\ln\left(\left|f''(b)\right|\right)-\ln\left(\left|f''(a)\right|\right)\right]^{k-1}}{\left(q+1\right)_k}\right)^{1/q}. \quad (7.55)
\end{aligned}$$

Corollary 312 *If $\eta(b,a) = b - a$ in Corollary 311, we have*

$$\left|A(f(a), f(b)) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2 |f''(b)|}{2}\left(\frac{q-1}{2q-1}\right)^{1-1/q}$$
$$\times\left(\sum_{k=1}^{\infty}\frac{(-q)^{k-1}\left[\ln(|f''(b)|) - \ln(|f''(a)|)\right]^{k-1}}{(q+1)_k}\right)^{1/q}. \quad (7.56)$$

Some results related to left-side of Hermite-Hadamard's inequality for n-times differentiable log-preinvex functions are given in the following theorems.

Theorem 313 *Suppose that $f : K \to \mathbb{R}$ is a mapping defined over K, where K is an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ of the set of real numbers. Let the nth derivative $f^{(n)}$ exist on K and $f^{(n)}$ be in the space $L([a, a+\eta(b,a)])$ of all integrable functions on $[a, a+\eta(b,a)]$, $a, b \in K$ with $\eta(b,a) > 0$. If $\left|f^{(n)}\right|^q$ for $q \geq 1$, is* log-*preinvex on K for $n \in \mathbb{N}$, $n \geq 1$, the following inequality holds:*

$$\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k + 1\right](\eta(b,a))^k}{2^{k+1}(k+1)!}f^{(k)}(A(2a, \eta(b,a)))\right.$$
$$\left. - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right|$$
$$\leq \frac{(\eta(b,a))^n \left|f^{(n)}(a)\right|}{2^{(n+1)(q-1)/q}(n+1)^{1-1/q}(n!)^{1-1/q}}\left\{[\kappa_1(n,q;a,b)]^{1/q} + [\kappa_2(n,q;a,b)]^{1/q}\right\}, \quad (7.57)$$

where

$$\kappa_1(n,q;a,b) = \frac{(-1)^{n+1}}{q^{n+1}\left(\ln(|f^{(n)}(b)|) - \ln(|f^{(n)}(a)|)\right)^{n+1}}$$
$$+\left(\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{q/2}\sum_{k=0}^{n}\frac{(-1)^k}{q^{k+1}2^{n-k}(n-k)!\left(\ln(|f^{(n)}(b)|) - \ln(|f^{(n)}(a)|)\right)^{k+1}}$$

and

$$\kappa_2(n,q;a,b) = \frac{\left|f^{(n)}(b)\right|^q}{q^{n+1}\left(\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)\right)^{n+1}\left|f^{(n)}(a)\right|^q} - \left(\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{q/2}\sum_{k=0}^{n}\frac{1}{q^{k+1}2^{n-k}(n-k)!\left(\ln\left(\left|f^{(n)}(b)\right|\right)-\ln\left(\left|f^{(n)}(a)\right|\right)\right)^{k+1}}.$$

Proof. Suppose $n \geq 1$. By using Lemma 286 and the log-preinvexity of $\left|f^{(n)}\right|$ on K for $n \in \mathbb{N}$, $n \geq 1$, we have

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](\eta(b,a))^k}{2^{k+1}(k+1)!}f^{(k)}(A(2a,\eta(b,a))) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right| \\
&\quad\leq \frac{(\eta(b,a))^n}{n!}\left[\int_0^{\frac{1}{2}}t^n\left|f^{(n)}(a+t\eta(b,a))\right|dt + \int_{\frac{1}{2}}^{1}(1-t)^n\left|f^{(n)}(a+t\eta(b,a))\right|dt\right] \\
&\qquad\leq \frac{(\eta(b,a))^n\left|f^{(n)}(a)\right|}{n!}\left[\left(\int_0^{\frac{1}{2}}t^n dt\right)^{1-1/q}\left(\int_0^{\frac{1}{2}}t^n\mu^t dt\right)^{1/q}\right. \\
&\qquad\qquad\left.+\left(\int_{\frac{1}{2}}^{1}(1-t)^n dt\right)^{1-1/q}\left(\int_{\frac{1}{2}}^{1}(1-t)^n\mu^t dt\right)^{1/q}\right], \quad (7.58)
\end{aligned}$$

where $\mu = \frac{\left|f^{(n)}(b)\right|^q}{\left|f^{(n)}(a)\right|^q} \neq 1$. Applying Lemma 157 and Lemma 158 to the integrals in the inequality (7.58) and replacing $\mu = \frac{\left|f^{(n)}(b)\right|^q}{\left|f^{(n)}(a)\right|^q} \neq 1$, we get the desired inequality (7.57). ■

Corollary 314 *Let f be as defined in Theorem 313 and $q = 1$. Then*

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1}\frac{\left[(-1)^k+1\right](\eta(b,a))^k}{2^{k+1}(k+1)!}f^{(k)}(A(2a,\eta(b,a)))\right. \\
&\left.-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right| \leq (\eta(b,a))^n\left|f^{(n)}(a)\right|\left\{\kappa_1(n,1;a,b)+\kappa_2(n,1;a,b)\right\}, \quad (7.59)
\end{aligned}$$

where

$$\kappa_1(n,1;a,b) = \frac{(-1)^{n+1}}{\left(\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right)\right)^{n+1}} + \left(\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{1/2} \sum_{k=0}^{n} \frac{(-1)^k}{2^{n-k}(n-k)!\left(\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right)\right)^{k+1}}$$

and

$$\kappa_2(n,1;a,b) = \frac{\left|f^{(n)}(b)\right|}{\left(\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right)\right)^{n+1}\left|f^{(n)}(a)\right|} - \left(\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{1/2} \sum_{k=0}^{n} \frac{1}{2^{n-k}(n-k)!\left(\ln\left(\left|f^{(n)}(b)\right|\right) - \ln\left(\left|f^{(n)}(a)\right|\right)\right)^{k+1}}.$$

Corollary 315 *[146] If we take $n = 1$ in Corollary 314, we have*

$$\left| f\left(A\left(2a, \eta(b,a)\right)\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx \right| \leq \eta(b,a)\left[\frac{\left(\left|f'(b)\right|\right)^{1/2} - \left(\left|f'(a)\right|\right)^{1/2}}{\ln\left(\left|f'(b)\right|\right) - \ln\left(\left|f'(a)\right|\right)}\right]^2. \quad (7.60)$$

Corollary 316 *[146] If $\eta(b,a) = b - a$ in Corollary 315, we have*

$$\left| f\left(A(a,b)\right) - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \leq (b-a)\left[\frac{\left(\left|f'(b)\right|\right)^{1/2} - \left(\left|f'(a)\right|\right)^{1/2}}{\ln\left(\left|f'(b)\right|\right) - \ln\left(\left|f'(a)\right|\right)}\right]^2. \quad (7.61)$$

Theorem 317 *Suppose that $f : K \to \mathbb{R}$ is a mapping defined over K, where K is an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ of the set of real numbers. Let the nth derivative $f^{(n)}$ exist on K and $f^{(n)}$ be in the space $L([a, a + \eta(b,a)])$ of all integrable functions on $[a, a + \eta(b,a)]$, $a, b \in K$ with $\eta(b,a) > 0$. If $\left|f^{(n)}\right|^q$ for $q > 1$, is* log-*preinvex* *on K for $n \in \mathbb{N}$, $n \geq 1$, the following inequality holds:*

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right] (\eta(b,a))^k}{2^{k+1}(k+1)!} f^{(k)}(A(2a, \eta(b,a))) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx\right|$$
$$\leq \frac{(\eta(b,a))^n \left[\sqrt{|f^{(n)}(a)|} + \sqrt{|f^{(n)}(b)|}\right]}{2^{n+1/p}(np+1)^{1/p} q^{1/q} n!} \left[\frac{\left(\left|f^{(n)}(b)\right|\right)^{q/2} - \left(\left|f^{(n)}(a)\right|\right)^{q/2}}{\left(\ln\left(|f^{(n)}(b)|\right) - \ln\left(|f^{(n)}(a)|\right)\right)}\right]^{1/q}, \quad (7.62)$$

where p and q are Hölder conjugates of each other.

Proof. From Lemma 286, the Hölder integral inequality and log-preinvexity of $\left|f^{(n)}\right|^q$, we have

$$\left|\sum_{k=0}^{n-1} \frac{\left[(-1)^k + 1\right] (\eta(b,a))^k}{2^{k+1}(k+1)!} f^{(k)}(A(2a, \eta(b,a))) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx\right|$$
$$\leq \frac{(\eta(b,a))^n \left|f^{(n)}(a)\right|}{n!} \left[\left(\int_0^{\frac{1}{2}} t^{np} dt\right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left(\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{qt} dt\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\int_{\frac{1}{2}}^1 (1-t)^{np} dt\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 \left(\frac{\left|f^{(n)}(b)\right|}{\left|f^{(n)}(a)\right|}\right)^{qt} dt\right)^{\frac{1}{q}}\right]$$
$$= \frac{(\eta(b,a))^n \left[\sqrt{|f^{(n)}(a)|} + \sqrt{|f^{(n)}(b)|}\right]}{2^{n+1/p}(np+1)^{1/p} q^{1/q} n!} \left[\frac{\left(\left|f^{(n)}(b)\right|\right)^{q/2} - \left(\left|f^{(n)}(a)\right|\right)^{q/2}}{\ln\left(|f^{(n)}(b)|\right) - \ln\left(|f^{(n)}(a)|\right)}\right]^{1/q}. \quad (7.63)$$

The above is the required inequality. ■

Corollary 318 *Let f be defined as in Theorem 317. If $n = 1$, the following inequality holds*

$$\left|f(A(2a, \eta(b,a))) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx\right|$$
$$\leq \frac{\eta(b,a) \left[\sqrt{|f'(a)|} + \sqrt{|f'(b)|}\right]}{2^{1+1/p}(p+1)^{1/p} q^{1/q}} \left[\frac{\left(\left|f'(b)\right|\right)^{q/2} - \left(\left|f'(a)\right|\right)^{q/2}}{\left(\ln\left(|f'(b)|\right) - \ln\left(|f'(a)|\right)\right)}\right]^{1/q}, \quad (7.64)$$

where p and q are Hölder conjugates of each other.

Corollary 319 *If we take $\eta(b,a) = b - a$ in (7.64), we get the inequality:*

$$\left| f\left(A\left(a,b\right)\right) - \frac{1}{b-a}\int_a^b f\left(x\right)dx \right| \leq \frac{\left(b-a\right)\left[\sqrt{\left|f'(a)\right|}+\sqrt{\left|f'(b)\right|}\right]}{2^{1+1/p}\left(p+1\right)^{1/p}q^{1/q}}\left[\frac{\left(\left|f'(b)\right|\right)^{q/2}-\left(\left|f'(a)\right|\right)^{q/2}}{\left(\log\left(\left|f'(b)\right|\right)-\log\left(\left|f'(a)\right|\right)\right)}\right]^{1/q}. \quad (7.65)$$

Remark 320 *Inequalities (7.64) and (7.65) are the corrected inequalities that are given in Theorem 90 and its related corollary from [146].*

Chapter 8

Fejér type inequalities for GA-convex functions with applications

The results of this chapter are from the "M. A. Latif, S. S. Dragomir and E. Momoniat, Some Fejér type integral inequalities for geometrically-arithmetically-convex functions with applications, Applied Mathematics and Computation. (Submitted)" and "M. A. Latif, S. S. Dragomir and E. Momoniat, Fejér Type integral inequalities related with geometrically-arithmetically-convex functions with applications, Journal of Convex Analysis. (Submitted)".

8.1 Introduction

It has been revealed in Section 2.12 of Chapter 2 that $\mathbb{R}$ is an ordered field, hence it is natural to investigate what happens when addition is replaced by multiplication and the arithmetic mean is replaced by the geometric mean. This fact leads us to the following four spaces of functions:

1. (A, A)-convex functions, the usual convex functions
2. (A, G)-convex functions

3. (G, A)-convex functions

4. (G, G)-convex functions.

In this chapter, we will contemplate the case of GA-convex functions to substantiate our results. In this chapter, we will introduce a new notion of geometrically symmetric functions and by using this notion we prove weighted generalizations of (2.70), which can be named as Fejér type integral inequalities for (2.70). It can also be noted that the Hermite-Hadamard type inequalities proved in [65], [93] and [183] for GA-convex and s-GA-convex functions seem not to be enough in this field. The foremost motivation of the present chapter is to provide further extensions of the results associated with the weighted generalizations of (2.70). In this chapter two new weighted identities involving a differentiable mapping and a geometrically symmetric functions are proved. The applications of these two identities and GA-convexity gives us the results in this chapter, which are more general and better estimates for the difference between the right most and the middle terms of the weighted version of (2.70).

This chapter is organized in the following approach:

In Section 8.2, we prove Fejér type integral inequalities which is a weighted generalization of (2.70). We establish several new integral inequalities for the right part of the weighted version of the inequality (2.70) by using a new identity involving a differentiable mapping and a geometrically symmetric mapping, the notion of GA-convexity and some auxiliary results. These results not only provide weighted generalization of the results from [65], [93] connected with the right part of (2.70) but also give refinements of those results from [65] for particular choice of the geometrically symmetric weight functions.

In Section 8.3, we will prove another weighted integral identity for the left part of the inequality weighted version of the inequality (2.70) involving a differentiable mapping and a geometrically symmetric function. We use this identity, the geometrical-arithmetical convexity and some auxiliary results to obtain some new Fejér type integral inequalities related with the left part of the weighted version of the inequality (2.70). The results of

Section 8.3 provide refinements and a weighted version of the results given in [65] for the left part of the inequality (2.70).

Section 8.4 is devoted for some applications of our results of Section 8.2 and Section 8.3 to produce some new inequalities of special means of positive real numbers.

8.2 Some Fejér type integral inequalities for GA-convex functions with applications

Throughout this chapter we consider $G_{(1-t)/2}(a,b) = a^{(1-t)/2}b^{(1+t)/2}$ and $G_{(1-t)/2}(b,a) = a^{(1+t)/2}b^{(1-t)/2}$. The Beta function and the integral form of the hypergeometric function are defined as follows to be used in the sequel of the chapter

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt, \alpha > 0, \beta > 0$$

and

$${}_2F_1(\alpha,\beta;\gamma;z) = \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\,dt$$

for $|z| < 1, \gamma > \beta > 0$.

Theorem 321 *Suppose that $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ is a GA-convex function on $[a,b]$ and that $g : [a,b] \to [0,\infty)$ is a mapping which owns the property of continuity, positivity and geometric symmetry with respect to $G(a,b)$. Then*

$$f(G(a,b))\int_a^b \frac{g(x)}{x}dx \le \int_a^b \frac{f(x)g(x)}{x}dx \le A(f(a),f(b))\int_a^b \frac{g(x)}{x}dx. \tag{8.1}$$

Proof. By the GA-convexity of f, we have

$$\begin{aligned} f(G(a,b))\int_0^1 g\left(a^{1-t}b^t\right)dt &\le \int_0^1 \left[\frac{1}{2}f\left(a^tb^{1-t}\right) + \frac{1}{2}f\left(a^{1-t}b^t\right)\right] g\left(a^{1-t}b^t\right)dt \\ &= \frac{1}{2}\int_0^1 f\left(a^tb^{1-t}\right)g\left(a^{1-t}b^t\right)dt + \frac{1}{2}\int_0^1 f\left(a^{1-t}b^t\right)g\left(a^{1-t}b^t\right)dt. \end{aligned} \tag{8.2}$$

By geometrical symmetry of g with respect to $G(a,b)$, we also have

$$\int_0^1 f\left(a^t b^{1-t}\right) g\left(a^{1-t} b^t\right) dt = \int_0^1 f\left(a^t b^{1-t}\right) g\left(a^t b^{1-t}\right) dt. \tag{8.3}$$

Hence by using (8.2) in (8.3) and by the change of variables $x = a^t b^{1-t}$ and $y = a^{1-t} b^t$, we obtain

$$\begin{aligned}
&\frac{f(G(a,b))}{\ln b - \ln a} \int_a^b \frac{g(x)}{x} dx \\
&\quad \leq \frac{1}{2} \int_0^1 f\left(a^t b^{1-t}\right) g\left(a^t b^{1-t}\right) dt + \frac{1}{2} \int_0^1 f\left(a^{1-t} b^t\right) g\left(a^{1-t} b^t\right) dt \\
&\quad = \frac{1}{2(\ln b - \ln a)} \int_a^b \frac{f(x) g(x)}{x} dx + \frac{1}{2(\ln b - \ln a)} \int_a^b \frac{f(y) g(y)}{y} dy \\
&\quad = \frac{1}{2(\ln b - \ln a)} \int_a^b \frac{f(x) g(x)}{x} dx + \frac{1}{2(\ln b - \ln a)} \int_a^b \frac{f(x) g(x)}{x} dx \\
&\quad = \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x) g(x)}{x} dx.
\end{aligned} \tag{8.4}$$

By the GA-convexity on $[a,b]$ and geometrical symmetry of g with respect to $G(a,b)$, we have

$$f\left(a^{1-t} b^t\right) g\left(a^{1-t} b^t\right) \leq [(1-t) f(a) + t f(b)] g\left(a^{1-t} b^t\right) \tag{8.5}$$

and

$$f\left(a^t b^{1-t}\right) g\left(a^t b^{1-t}\right) \leq [(1-t) f(b) + t f(a)] g\left(a^{1-t} b^t\right). \tag{8.6}$$

Adding (8.5) and (8.6) and integrating with respect to t over $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 f\left(a^t b^{1-t}\right) g\left(a^t b^{1-t}\right) dt + \int_0^1 f\left(a^{1-t} b^t\right) g\left(a^{1-t} b^t\right) dt \\
&\qquad \leq [f(a) + f(b)] \int_0^1 g\left(a^{1-t} b^t\right) dt.
\end{aligned} \tag{8.7}$$

By the change of variables $x = a^t b^{1-t}$ and $y = a^{1-t} b^t$ in (8.7), we get

$$\frac{1}{\ln b - \ln a} \int_a^b \frac{f(x) g(x)}{x} dx \leq \frac{[f(a) + f(b)]}{2(\ln b - \ln a)} \int_a^b \frac{g(x)}{x} dx. \tag{8.8}$$

Combining the inequalities (8.4) and (8.8), we get the required result. ■

With the intention of proving our results a weighted integral identity has to be developed in the following lemma which plays an imperative role in establishing our foremost results.

Lemma 322 *Let $f : [c,d] \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a function which possesses the first order derivative f' on (c,d), where a, $b \in (c,d)$ with $a < b$ and f' is in the space $L([a,b])$ of all integrable functions over $[a,b]$. If $g : [a,b] \to [0,\infty)$ is a mapping which owns the property of continuity, positivity and geometric symmetry with respect to $G(a,b)$, the following equality holds*

$$\begin{aligned} & A(f(a), f(b)) \int_a^b \frac{g(x)}{x} dx - \int_a^b \frac{f(x) g(x)}{x} dx \\ & \qquad = \frac{\ln b - \ln a}{4} \int_0^1 \left(\int_{G_{(1-t)/2}(b,a)}^{G_{(1-t)/2}(a,b)} \frac{g(x)}{x} dx \right) \\ & \qquad \times \left[G_{\frac{1-t}{2}}(a,b) f'\left(G_{\frac{1-t}{2}}(a,b)\right) - G_{\frac{1-t}{2}}(b,a) f'\left(G_{\frac{1-t}{2}}(b,a)\right) \right] dt. \end{aligned} \tag{8.9}$$

Proof. Let

$$I_1 = \int_0^1 \left(\int_{G_{\frac{1-t}{2}}(b,a)}^{G_{\frac{1-t}{2}}(a,b)} \frac{g(x)}{x} dx \right) G_{\frac{1-t}{2}}(a,b) f'\left(G_{\frac{1-t}{2}}(a,b)\right) dt$$

and

$$I_2 = \int_0^1 \left(\int_{G_{\frac{1-t}{2}}(b,a)}^{G_{\frac{1-t}{2}}(a,b)} \frac{g(x)}{x} dx \right) G_{\frac{1-t}{2}}(b,a) f'\left(G_{\frac{1-t}{2}}(b,a)\right) dt.$$

Since $g : [a, b] \to [0, \infty)$ is geometrically symmetric to $G(a, b)$, hence $g\left(G_{\frac{1-t}{2}}(a, b)\right) = g\left(G_{\frac{1-t}{2}}(b, a)\right)$ for all $t \in [0, 1]$. By this, we have

$$\begin{aligned}
I_1 &= \int_0^1 \left(\int_{G_{\frac{1-t}{2}}(b,a)}^{G_{\frac{1-t}{2}}(a,b)} \frac{g(x)}{x} dx \right) G_{\frac{1-t}{2}}(a, b) f'\left(G_{\frac{1-t}{2}}(a, b)\right) dt \\
&= \frac{2}{\ln b - \ln a} \int_0^1 \left(\int_{G_{\frac{1-t}{2}}(b,a)}^{G_{\frac{1-t}{2}}(a,b)} \frac{g(x)}{x} dx \right) d\left[f\left(G_{\frac{1-t}{2}}(a, b)\right)\right] \\
&= \frac{2}{\ln b - \ln a} \left(\int_{G_{\frac{1-t}{2}}(b,a)}^{G_{\frac{1-t}{2}}(a,b)} \frac{g(x)}{x} dx \right) f\left(G_{\frac{1-t}{2}}(a, b)\right)\Bigg|_0^1 \\
&\quad - \int_0^1 \left[g\left(G_{\frac{1-t}{2}}(a, b)\right) + g\left(G_{\frac{1-t}{2}}(b, a)\right)\right] f\left(G_{\frac{1-t}{2}}(a, b)\right) dt \\
&= \frac{2f(b)}{\ln b - \ln a} \int_a^b \frac{g(x)}{x} dx - 2\int_0^1 g\left(G_{\frac{1-t}{2}}(a, b)\right) f\left(G_{\frac{1-t}{2}}(a, b)\right) dt \\
&= \frac{2f(b)}{\ln b - \ln a} \int_a^b \frac{g(x)}{x} dx - \frac{4}{\ln b - \ln a} \int_{\sqrt{ab}}^b \frac{g(x) f(x)}{x} dx. \quad (8.10)
\end{aligned}$$

Analogously, we have

$$-I_2 = \frac{2f(a)}{\ln b - \ln a} \int_a^b \frac{g(x)}{x} dx - \frac{4}{\ln b - \ln a} \int_a^{\sqrt{ab}} \frac{g(x) f(x)}{x} dx. \quad (8.11)$$

Adding (8.10) and (8.11) and multiplying the result by $\frac{\ln b - \ln a}{4}$, we get the required identity. ■

Lemma 323 *For $u, v > 0$, we have*

$$\zeta(u, v) \triangleq \int_0^1 u^{(1-t)/2} v^{(1+t)/2} dt = \begin{cases} \sqrt{v} L\left(\sqrt{u}, \sqrt{v}\right), & u \neq v, \\ \\ u, & u = v, \end{cases}$$

$$\Psi(u,v) \triangleq \frac{1}{2}\int_0^1 t u^{(1-t)/2} v^{(1+t)/2} dt = \begin{cases} \frac{v-\sqrt{v}L\left(\sqrt{u},\sqrt{v}\right)}{\ln v - \ln u}, & u \neq v \\ \\ \frac{1}{4}u, & u = v, \end{cases}$$

and

$$\Phi(u,v) \triangleq \frac{1}{2}\int_0^1 t^2 u^{(1-t)/2} v^{(1+t)/2} dt = \begin{cases} \frac{4\sqrt{v}L\left(\sqrt{u},\sqrt{v}\right) - 4v + v(\ln v - \ln u)}{(\ln v - \ln u)^2}, & u \neq v \\ \\ \frac{1}{6}u, & u = v. \end{cases}$$

Proof. The proof follows from a straightforward computation. ■

Fejér type inequalities for GA-convex functions, which provide weighted generalization of some of the results established in recent literature concerning GA-convex functions are ready to be proved now.

Theorem 324 *Let $f : [c,d] \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a function which possesses the first order derivative f' on (c,d), where $a,\, b \in (c,d)$ with $a < b$ and f' is in the space $L([a,b])$ of all integrable functions over $[a,b]$. If $g : [a,b] \to [0,\infty)$ is a mapping which owns the property of continuity, positivity and geometric symmetry with respect to $G(a,b)$ and $\left|f'\right|^q$ is in the space of GA-convex on $[a,b]$ for $q \geq 1$, then the following inequality holds*

$$\begin{aligned} &\left| A(f(a), f(b)) \int_a^b \frac{g(x)}{x} dx - \int_a^b \frac{f(x) g(x)}{x} dx \right| \\ &\leq \frac{(\ln b - \ln a)^2}{2^{1+1/q}} \|g\|_\infty \Big\{ [\Psi(a,b)]^{1-1/q} \\ &\times \left([\Psi(a,b) - \Phi(a,b)] \left|f'(a)\right|^q + [\Psi(a,b) + \Phi(a,b)] \left|f'(b)\right|^q \right)^{1/q} \\ &+ [\Psi(b,a)]^{1-1/q} \Big([\Psi(b,a) + \Phi(b,a)] \left|f'(a)\right|^q \\ &+ [\Psi(b,a) - \Phi(b,a)] \left|f'(b)\right|^q \Big)^{1/q} \Big\}, \end{aligned} \tag{8.12}$$

where $\|g\|_\infty = \sup_{x\in[a,b]} g(x) < \infty$.

Proof. From Lemma 322 and Hölder's inequality , we have

$$\begin{aligned}
&\left|A\left(f\left(a\right),f\left(b\right)\right)\int_a^b\frac{g\left(x\right)}{x}dx-\int_a^b\frac{f\left(x\right)g\left(x\right)}{x}dx\right| \\
&\leq\frac{\ln b-\ln a}{4}\int_0^1\left(\int_{G_{\frac{1-t}{2}}(b,a)}^{G_{\frac{1-t}{2}}(a,b)}\frac{g\left(x\right)}{x}dx\right) \\
&\times\left[G_{\frac{1-t}{2}}\left(a,b\right)\left|f'\left(G_{\frac{1-t}{2}}\left(a,b\right)\right)\right|+G_{\frac{1-t}{2}}\left(b,a\right)\left|f'\left(G_{\frac{1-t}{2}}\left(b,a\right)\right)\right|\right]dt \\
&\leq\frac{\left(\ln b-\ln a\right)^2}{4}\left\|g\right\|_\infty\int_0^1\left[tG_{\frac{1-t}{2}}\left(a,b\right)\left|f'\left(G_{\frac{1-t}{2}}\left(a,b\right)\right)\right|\right. \\
&\left.+tG_{\frac{1-t}{2}}\left(b,a\right)f'\left(G_{\frac{1-t}{2}}\left(b,a\right)\right)\right]dt\leq\frac{\left(\ln b-\ln a\right)^2}{4}\left\|g\right\|_\infty \\
&\times\left\{\left(\int_0^1tG_{\frac{1-t}{2}}\left(a,b\right)dt\right)^{1-\frac{1}{q}}\left(\int_0^1tG_{\frac{1-t}{2}}\left(a,b\right)\left|f'\left(G_{\frac{1-t}{2}}\left(a,b\right)\right)\right|^qdt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1tG_{\frac{1-t}{2}}\left(b,a\right)dt\right)^{1-\frac{1}{q}}\left(\int_0^1tG_{\frac{1-t}{2}}\left(b,a\right)\left|f'\left(G_{\frac{1-t}{2}}\left(b,a\right)\right)\right|^qdt\right)^{\frac{1}{q}}\right\}. \quad (8.13)
\end{aligned}$$

By the GA-convexity of $\left|f'\right|^q$ on $[a,b]$ for $q\geq 1$ and by using Lemma 323, we have

$$\begin{aligned}
&\int_0^1tG_{\frac{1-t}{2}}\left(a,b\right)\left|f'\left(G_{\frac{1-t}{2}}\left(a,b\right)\right)\right|^q \\
&\leq\left|f'\left(a\right)\right|^q\int_0^1t\left(\frac{1-t}{2}\right)a^{(1-t)/2}b^{(1+t)/2}dt+\left|f'\left(b\right)\right|^q\int_0^1t\left(\frac{1+t}{2}\right)a^{(1-t)/2}b^{(1+t)/2}dt \\
&=\left[\Psi\left(a,b\right)-\Phi\left(a,b\right)\right]\left|f'\left(a\right)\right|^q+\left[\Psi\left(a,b\right)+\Phi\left(a,b\right)\right]\left|f'\left(b\right)\right|^q \quad (8.14)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1tG_{\frac{1-t}{2}}\left(b,a\right)\left|f'\left(G_{\frac{1-t}{2}}\left(b,a\right)\right)\right|^q \\
&\leq\left|f'\left(a\right)\right|^q\int_0^1t\left(\frac{1+t}{2}\right)a^{(1+t)/2}b^{(1-t)/2}dt+\left|f'\left(b\right)\right|^q\int_0^1t\left(\frac{1-t}{2}\right)a^{(1+t)/2}b^{(1-t)/2}dt \\
&=\left[\Psi\left(b,a\right)+\Phi\left(b,a\right)\right]\left|f'\left(a\right)\right|^q+\left[\Psi\left(b,a\right)-\Phi\left(b,a\right)\right]\left|f'\left(b\right)\right|^q. \quad (8.15)
\end{aligned}$$

Using (8.14) and (8.15) in (8.13), we get the required result. ■

Corollary 325 *Let f be as in Theorem 324 and $q = 1$, then the following inequality holds*

$$\begin{aligned}
&\left| A\left(f\left(a\right), f\left(b\right)\right) \int_a^b \frac{g\left(x\right)}{x} dx - \int_a^b \frac{f\left(x\right) g\left(x\right)}{x} dx \right| \\
&\leq \frac{\left(\ln b - \ln a\right)^2}{4} \left\|g\right\|_\infty \left\{ \left[\Psi\left(a,b\right) + \Psi\left(b,a\right) - \Phi\left(a,b\right) + \Phi\left(b,a\right)\right] \left|f'\left(a\right)\right| \right. \\
&\left. + \left[\Psi\left(a,b\right) + \Psi\left(b,a\right) + \Phi\left(a,b\right) - \Phi\left(b,a\right)\right] \left|f'\left(b\right)\right| \right\}, \quad (8.16)
\end{aligned}$$

where $\left\|g\right\|_\infty = \sup_{x \in [a,b]} g\left(x\right) < \infty$.

Corollary 326 *If* $g\left(x\right) = \frac{1}{\ln b - \ln a}$, *for all* $x \in [a, b]$ *in Theorem 324, then*

$$\begin{aligned}
&\left| A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f\left(x\right)}{x} dx \right| \\
&\leq \frac{\left(\ln b - \ln a\right)}{2^{1+1/q}} \left\{ \left[\Psi\left(a,b\right)\right]^{1-1/q} \left(\left[\Psi\left(a,b\right) - \Phi\left(a,b\right)\right] \left|f'\left(a\right)\right|^q \right. \right. \\
&\left. + \left[\Psi\left(a,b\right) + \Phi\left(a,b\right)\right] \left|f'\left(b\right)\right|^q \right)^{1/q} + \left[\Psi\left(b,a\right)\right]^{1-1/q} \\
&\left. \times \left(\left[\Psi\left(b,a\right) + \Phi\left(b,a\right)\right] \left|f'\left(a\right)\right|^q + \left[\Psi\left(b,a\right) - \Phi\left(b,a\right)\right] \left|f'\left(b\right)\right|^q \right)^{1/q} \right\}. \quad (8.17)
\end{aligned}$$

Corollary 327 *If $q = 1$ in Corollary 326, then we get the following inequality*

$$\begin{aligned}
&\left| A\left(f\left(a\right), f\left(b\right)\right) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f\left(x\right)}{x} dx \right| \\
&\leq \frac{\left(\ln b - \ln a\right)}{4} \left\{ \left[\Psi\left(a,b\right) + \Psi\left(b,a\right) + \Phi\left(b,a\right) - \Phi\left(a,b\right)\right] \left|f'\left(a\right)\right| \right. \\
&\left. + \left[\Psi\left(b,a\right) + \Psi\left(a,b\right) + \Phi\left(a,b\right) - \Phi\left(b,a\right)\right] \left|f'\left(b\right)\right| \right\}. \quad (8.18)
\end{aligned}$$

Theorem 328 *Let $f : [c, d] \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a function which possesses the first order derivative f' on (c, d), where $a, b \in (c, d)$ with $a < b$ and f' is in the space $L\left([a, b]\right)$ of all integrable functions over $[a, b]$. If $g : [a, b] \to [0, \infty)$ is a mapping which owns the property of continuity, positivity and geometric symmetry with respect to $G\left(a, b\right)$ and $\left|f'\right|^q$ is in* *the space of GA-convex on $[a,b]$ for $q>1$, then the following inequality holds*

$$\begin{aligned}
&\left|A\left(f\left(a\right),f\left(b\right)\right)\int_a^b\frac{g\left(x\right)}{x}dx-\int_a^b\frac{f\left(x\right)g\left(x\right)}{x}dx\right| \\
&\leq\frac{\left(\ln b-\ln a\right)^{2-1/q}\left\|g\right\|_\infty}{4\cdot q^{1/q}}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left\{b^{1/2}\left(\left[L\left(a^{q/2},b^{q/2}\right)-a^{q/2}\right]\left|f'\left(a\right)\right|^q\right.\right. \\
&\quad\left.+\left[2b^{q/2}-a^{q/2}-L\left(a^{q/2},b^{q/2}\right)\right]\left|f'\left(b\right)\right|^q\right)^{1/q} \\
&\quad+a^{1/2}\left(\left[L\left(a^{q/2},b^{q/2}\right)+b^{q/2}-2a^{q/2}\right]\left|f'\left(a\right)\right|^q\right. \\
&\quad\left.\left.+\left[b^{q/2}-L\left(a^{q/2},b^{q/2}\right)\right]\left|f'\left(b\right)\right|^q\right)^{1/q}\right\}, \quad (8.19)
\end{aligned}$$

where $\left\|g\right\|_\infty=\sup_{x\in[a,b]}g\left(x\right)<\infty$.

Proof. From Lemma 322 and Hölder's inequality , we have

$$\begin{aligned}
&\left|A\left(f\left(a\right),f\left(b\right)\right)\int_a^b\frac{g\left(x\right)}{x}dx-\int_a^b\frac{f\left(x\right)g\left(x\right)}{x}dx\right| \\
&\leq\frac{\ln b-\ln a}{4}\int_0^1\left(\int_{G_{\frac{1-t}{2}}(b,a)}^{G_{\frac{1-t}{2}}(a,b)}\frac{g\left(x\right)}{x}dx\right) \\
&\times\left[G_{\frac{1-t}{2}}\left(a,b\right)\left|f'\left(G_{\frac{1-t}{2}}\left(a,b\right)\right)\right|+G_{\frac{1-t}{2}}\left(b,a\right)\left|f'\left(G_{\frac{1-t}{2}}\left(b,a\right)\right)\right|\right]dt \\
&\leq\frac{\left(\ln b-\ln a\right)^2}{4}\left\|g\right\|_\infty\int_0^1\left[tG_{\frac{1-t}{2}}\left(a,b\right)\left|f'\left(G_{\frac{1-t}{2}}\left(a,b\right)\right)\right|\right. \\
&\left.+tG_{\frac{1-t}{2}}\left(b,a\right)f'\left(G_{\frac{1-t}{2}}\left(b,a\right)\right)\right]dt\leq\frac{\left(\ln b-\ln a\right)^2}{4}\left\|g\right\|_\infty\left(\int_0^1t^{q/(q-1)}dt\right)^{1-1/q} \\
&\times\left\{\left(\int_0^1\left[G_{\frac{1-t}{2}}\left(a,b\right)\right]^q\left|f'\left(G_{\frac{1-t}{2}}\left(a,b\right)\right)\right|^qdt\right)^{1/q}\right. \\
&\left.+\left(\int_0^1\left[G_{\frac{1-t}{2}}\left(b,a\right)\right]^q\left|f'\left(G_{\frac{1-t}{2}}\left(b,a\right)\right)\right|^qdt\right)^{1/q}\right\}. \quad (8.20)
\end{aligned}$$

Since

$$\int_0^1 \left[G_{\frac{1-t}{2}}(a,b)\right]^q \left|f'\left(G_{\frac{1-t}{2}}(a,b)\right)\right|^q = \int_0^1 a^{q(1-t)/2} b^{q(1+t)/2} \left|f'\left(a^{(1-t)/2} b^{(1+t)/2}\right)\right|$$
$$\leq \left|f'(a)\right|^q \int_0^1 \left(\frac{1-t}{2}\right) a^{q(1-t)/2} b^{q(1+t)/2} dt + \left|f'(b)\right|^q \int_0^1 \left(\frac{1+t}{2}\right) a^{q(1-t)/2} b^{q(1+t)/2} dt$$
$$= \frac{b^{q/2}\left[L\left(a^{q/2}, b^{q/2}\right) - a^{q/2}\right]}{q(\ln b - \ln a)} \left|f'(a)\right|^q$$
$$+ \frac{b^{q/2}\left[2b^{q/2} - a^{q/2} - L\left(a^{q/2}, b^{q/2}\right)\right]}{q(\ln b - \ln a)} \left|f'(b)\right|^q \quad (8.21)$$

and

$$\int_0^1 \left[G_{\frac{1-t}{2}}(b,a)\right]^q \left|f'\left(G_{\frac{1-t}{2}}(b,a)\right)\right|^q = \int_0^1 a^{q(1+t)/2} b^{q(1-t)/2} \left|f'\left(a^{(1+t)/2} b^{(1-t)/2}\right)\right|$$
$$\leq \left|f'(a)\right|^q \int_0^1 \left(\frac{1+t}{2}\right) a^{q(1+t)/2} b^{q(1-t)/2} dt + \left|f'(b)\right|^q \int_0^1 \left(\frac{1-t}{2}\right) a^{q(1+t)/2} b^{q(1-t)/2} dt$$
$$= \frac{a^{q/2}\left[L\left(a^{q/2}, b^{q/2}\right) + b^{q/2} - 2a^{q/2}\right]}{q(\ln b - \ln a)} \left|f'(a)\right|^q$$
$$+ \frac{a^{q/2}\left[b^{q/2} - L\left(a^{q/2}, b^{q/2}\right)\right]}{q(\ln b - \ln a)} \left|f'(b)\right|^q. \quad (8.22)$$

The inequality (8.19) can be proved by applying (8.22) and (8.23) in (8.20). ■

Corollary 329 *Let f be as in Theorem 328 and $g(x) = \frac{1}{\ln b - \ln a}$ for all $x \in [a,b]$, then the following inequality holds*

$$\left|A(f(a), f(b)) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx\right|$$
$$\leq \frac{(\ln b - \ln a)^{1-1/q}}{4 \cdot q^{1/q}} \left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} \left\{ b^{1/2} \left(\left[L\left(a^{q/2}, b^{q/2}\right) - a^{q/2}\right] \left|f'(a)\right|^q \right.\right.$$
$$\left. + \left[2b^{q/2} - a^{q/2} - L\left(a^{q/2}, b^{q/2}\right)\right] \left|f'(b)\right|^q \right)^{1/q}$$
$$+ a^{1/2} \left(\left[L\left(a^{q/2}, b^{q/2}\right) + b^{q/2} - 2a^{q/2}\right] \left|f'(a)\right|^q \right.$$
$$\left.\left. + \left[b^{q/2} - L\left(a^{q/2}, b^{q/2}\right)\right] \left|f'(b)\right|^q \right)^{1/q} \right\}. \quad (8.23)$$

Theorem 330 *Let $f:[c,d]\subseteq\mathbb{R}_{+}\to\mathbb{R}$ be a function which possesses the first order derivative f' on (c,d), where $a,b\in(c,d)$ with $a<b$ and f' is in the space $L([a,b])$ of all integrable functions over $[a,b]$. If $g:[a,b]\to[0,\infty)$ is a mapping which owns the property of continuity, positivity and geometric symmetry with respect to $G(a,b)$ and $|f'|^{q}$ is in the space of GA-convex on $[a,b]$ for $q>1$, then the following inequality holds*

$$\begin{aligned}
&\left|A(f(a),f(b))\int_a^b\frac{g(x)}{x}dx-\int_a^b\frac{f(x)g(x)}{x}dx\right| \\
&\quad\le\frac{(\ln b-\ln a)^{2-1/q}\|g\|_\infty}{2(4q)^{1/q}}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left\{[L(a^q,b^q)-a^q]\left|f'(a)\right|^q\right. \\
&\qquad\left.+[b^q-L(a^q,b^q)]\left|f'(b)\right|^q\right\}^{1/q}, \quad (8.24)
\end{aligned}$$

where $\|g\|_\infty=\sup_{x\in[a,b]}g(x)<\infty$.

Proof. From Lemma 322 and Hölder's inequality , we have

$$\begin{aligned}
&\left|A(f(a),f(b))\int_a^b\frac{g(x)}{x}dx-\int_a^b\frac{f(x)g(x)}{x}dx\right| \\
&\quad\le\frac{\ln b-\ln a}{4}\int_0^1\left(\int_{G_{\frac{1-t}{2}}(b,a)}^{G_{\frac{1-t}{2}}(a,b)}\frac{g(x)}{x}dx\right)\left[G_{\frac{1-t}{2}}(a,b)\left|f'\left(G_{\frac{1-t}{2}}(a,b)\right)\right|\right. \\
&\qquad\left.+G_{\frac{1-t}{2}}(b,a)\left|f'\left(G_{\frac{1-t}{2}}(b,a)\right)\right|\right]dt\le\frac{(\ln b-\ln a)^2}{4}\|g\|_\infty \\
&\quad\times\int_0^1\left[tG_{\frac{1-t}{2}}(a,b)\left|f'\left(G_{\frac{1-t}{2}}(a,b)\right)\right|+tG_{\frac{1-t}{2}}(b,a)f'\left(G_{\frac{1-t}{2}}(b,a)\right)\right]dt \\
&\qquad\le\frac{(\ln b-\ln a)^2}{4}\|g\|_\infty\left(\int_0^1t^{q/(q-1)}dt\right)^{1-1/q} \\
&\qquad\times\left\{\left(\int_0^1\left[G_{\frac{1-t}{2}}(a,b)\right]^q\left|f'\left(G_{\frac{1-t}{2}}(a,b)\right)\right|^q dt\right)^{1/q}\right. \\
&\qquad\left.+\left(\int_0^1\left[G_{\frac{1-t}{2}}(b,a)\right]^q\left|f'\left(G_{\frac{1-t}{2}}(b,a)\right)\right|^q dt\right)^{1/q}\right\}. \quad (8.25)
\end{aligned}$$

By the power-mean inequality ($a^r + b^r \leq 2^{1-r}(a+b)^r$ for $a>0, b>0$ and $r<1$), we have

$$\begin{aligned}
&\left(\int_0^1 \left[G_{\frac{1-t}{2}}(a,b)\right]^q \left|f'\left(G_{\frac{1-t}{2}}(a,b)\right)\right|^q dt\right)^{1/q} \\
&\qquad + \left(\int_0^1 \left[G_{\frac{1-t}{2}}(b,a)\right]^q \left|f'\left(G_{\frac{1-t}{2}}(b,a)\right)\right|^q dt\right)^{1/q} \\
&\qquad \leq 2^{1-1/q}\left(\int_0^1 \left[G_{\frac{1-t}{2}}(a,b)\right]^q \left|f'\left(G_{\frac{1-t}{2}}(a,b)\right)\right|^q dt\right. \\
&\qquad\qquad \left. + \int_0^1 \left[G_{\frac{1-t}{2}}(b,a)\right]^q \left|f'\left(G_{\frac{1-t}{2}}(b,a)\right)\right|^q dt\right)^{1/q}. \quad (8.26)
\end{aligned}$$

Since $\left|f'\right|^q$ is GA-convex on $[a,b]$ for $q>1$

$$\begin{aligned}
&\int_0^1 \left[G_{\frac{1-t}{2}}(a,b)\right]^q \left|f'\left(G_{\frac{1-t}{2}}(a,b)\right)\right|^q dt + \int_0^1 \left[G_{\frac{1-t}{2}}(b,a)\right]^q \left|f'\left(G_{\frac{1-t}{2}}(b,a)\right)\right|^q dt \\
&\quad \leq \left|f'(a)\right|^q \left[\int_0^1 \left(\frac{1-t}{2}\right) a^{q(1-t)/2} b^{q(1+t)} dt + \int_0^1 \left(\frac{1+t}{2}\right) a^{q(1+t)/2} b^{q(1-t)} dt\right] \\
&\quad + \left|f'(b)\right|^q \left[\int_0^1 \left(\frac{1+t}{2}\right) a^{q(1-t)/2} b^{q(1+t)} dt + \int_0^1 \left(\frac{1-t}{2}\right) a^{q(1+t)/2} b^{q(1-t)} dt\right] \\
&\qquad = \left[\frac{2L(a^q,b^q)-2a^q}{q(\ln b - \ln a)}\right]\left|f'(a)\right|^q + \left[\frac{2b^q - 2L(a^q,b^q)}{q(\ln b - \ln a)}\right]\left|f'(b)\right|^q. \quad (8.27)
\end{aligned}$$

Using (8.27) in (8.26), we get

$$\begin{aligned}
&\left(\int_0^1 \left[G_{\frac{1-t}{2}}(a,b)\right]^q \left|f'\left(G_{\frac{1-t}{2}}(a,b)\right)\right|^q dt\right)^{1/q} \\
&\qquad + \left(\int_0^1 \left[G_{\frac{1-t}{2}}(b,a)\right]^q \left|f'\left(G_{\frac{1-t}{2}}(b,a)\right)\right|^q dt\right)^{1/q} \\
&\quad \leq 2^{1-2/q}\left(\left[\frac{L(a^q,b^q)-a^q}{q(\ln b - \ln a)}\right]\left|f'(a)\right|^q + \left[\frac{b^q - L(a^q,b^q)}{q(\ln b - \ln a)}\right]\left|f'(b)\right|^q\right)^{1/q}. \quad (8.28)
\end{aligned}$$

Applying (8.28) in (8.25), we obtain the required inequality (8.24). ■

Corollary 331 *If the assumptions of Theorem 330 are satisfied and if* $g(x) = \frac{1}{\ln b - \ln a}$

for all $x \in [a,b]$, then the following inequality holds

$$\left|A\left(f\left(a\right),f\left(b\right)\right)-\frac{1}{\ln b-\ln a}\int_a^b\frac{f\left(x\right)}{x}dx\right|$$
$$\le\frac{\left(\ln b-\ln a\right)^{1-1/q}}{2\left(4q\right)^{1/q}}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left\{\left[L\left(a^q,b^q\right)-a^q\right]\left|f'\left(a\right)\right|^q\right.$$
$$\left.+\left[b^q-L\left(a^q,b^q\right)\right]\left|f'\left(b\right)\right|^q\right\}^{1/q}. \quad (8.29)$$

Theorem 332 *Let $f:[c,d]\subseteq\mathbb{R}_+\to\mathbb{R}$ be a function which possesses the first order derivative f' on (c,d), where $a,b\in(c,d)$ with $a<b$ and f' is in the space $L([a,b])$ of all integrable functions over $[a,b]$. If $g:[a,b]\to[0,\infty)$ is a mapping which owns the property of continuity, positivity and geometric symmetry with respect to $G(a,b)$ and $\left|f'\right|^q$ is in the space of GA-convex on $[a,b]$ for $q>1$, then the following inequality holds*

$$\left|A\left(f\left(a\right),f\left(b\right)\right)\int_a^b\frac{g\left(x\right)}{x}dx-\int_a^b\frac{f\left(x\right)g\left(x\right)}{x}dx\right|$$
$$\le\frac{\left(\ln b-\ln a\right)^2\left[L\left(a^{q/[2(q-1)]},b^{q/[2(q-1)]}\right)\right]^{1-1/q}\left\|g\right\|_\infty}{8}$$
$$\times\left\{\left(b^{1/2}\left[B\left(q+1,q+1\right)\right]^{1/q}+a^{1/2}\left[{}_2F_1\left(-q,q+1;q+2;-1\right)\cdot\frac{1}{q+1}\right]^{1/q}\right)\left|f'\left(a\right)\right|\right.$$
$$\left.+\left(a^{1/2}\left[B\left(q+1,q+1\right)\right]^{1/q}+b^{1/2}\left[{}_2F_1\left(-q,q+1;q+2;-1\right)\cdot\frac{1}{q+1}\right]^{1/q}\right)\left|f'\left(b\right)\right|\right\},$$
$$(8.30)$$

where $\left\|g\right\|_\infty=\sup_{x\in[a,b]}g\left(x\right)<\infty$ and ${}_2F_1\left(\cdot,\cdot;\cdot;\cdot\right)$ is the hypergeometric function.

Proof. From Lemma 322 and the GA-convexity of $\left|f'\right|$ on $[a,b]$, we have

$$\left|A\left(f(a),f(b)\right)\int_a^b \frac{g(x)}{x}dx-\int_a^b \frac{f(x)g(x)}{x}dx\right|$$
$$\leq \frac{\ln b-\ln a}{4}\int_0^1\left(\int_{G_{\frac{1-t}{2}}(b,a)}^{G_{\frac{1-t}{2}}(a,b)}\frac{g(x)}{x}dx\right)\left[G_{\frac{1-t}{2}}(a,b)\left|f'\left(G_{\frac{1-t}{2}}(a,b)\right)\right|\right.$$
$$\left.+G_{\frac{1-t}{2}}(b,a)\left|f'\left(G_{\frac{1-t}{2}}(b,a)\right)\right|\right]dt\leq\frac{(\ln b-\ln a)^2}{4}\|g\|_\infty$$
$$\times\int_0^1\left[tG_{\frac{1-t}{2}}(a,b)\left|f'\left(G_{\frac{1-t}{2}}(a,b)\right)\right|+tG_{\frac{1-t}{2}}(b,a)f'\left(G_{\frac{1-t}{2}}(b,a)\right)\right]dt$$
$$\leq\frac{(\ln b-\ln a)^2}{4}\|g\|_\infty\left\{\int_0^1 a^{(1-t)/2}b^{(1+t)/2}\left[t\left(\frac{1-t}{2}\right)\left|f'(a)\right|\right.\right.$$
$$\left.+t\left(\frac{1+t}{2}\right)\left|f'(b)\right|\right]dt+\int_0^1 a^{(1+t)/2}b^{(1-t)/2}\left[t\left(\frac{1+t}{2}\right)\left|f'(a)\right|\right.$$
$$\left.\left.+t\left(\frac{1-t}{2}\right)\left|f'(b)\right|\right]dt\right\}. \quad (8.31)$$

Using Hölder integral inequality, we have

$$\int_0^1 a^{(1-t)/2}b^{(1+t)/2}\left[t\left(\frac{1-t}{2}\right)\left|f'(a)\right|+t\left(\frac{1+t}{2}\right)\left|f'(b)\right|\right]dt$$
$$\leq\left(\int_0^1 a^{q(1-t)/[2(q-1)]}b^{q(1+t)/[2(q-1)]}dt\right)^{1-1/q}$$
$$\times\left\{\left[\int_0^1 t^q\left(\frac{1-t}{2}\right)^q dt\right]^{1/q}\left|f'(a)\right|+\left[\int_0^1 t^q\left(\frac{1+t}{2}\right)^q dt\right]^{1/q}\left|f'(b)\right|\right\}$$
$$=\frac{1}{2}\left[b^{q/[2(q-1)]}L\left(a^{q/[2(q-1)]},b^{q/[2(q-1)]}\right)\right]^{1-1/q}\left\{\left[B(q+1,q+1)\right]^{1/q}\left|f'(a)\right|\right.$$
$$\left.+\left[{}_2F_1(-q,q+1;q+2;-1)\cdot\frac{1}{q+1}\right]^{1/q}\left|f'(b)\right|\right\}. \quad (8.32)$$

Similarly, we have

$$\begin{aligned}
&\int_0^1 a^{(1+t)/2}b^{(1-t)/2}\left[t\left(\frac{1+t}{2}\right)\left|f'(a)\right|+t\left(\frac{1-t}{2}\right)\left|f'(b)\right|\right]dt \\
&\qquad\leq\left(\int_0^1 a^{q(1+t)/[2(q-1)]}b^{q(1-t)/[2(q-1)]}dt\right)^{1-1/q} \\
&\qquad\times\left\{\left[\int_0^1 t^q\left(\frac{1+t}{2}\right)^q dt\right]^{1/q}\left|f'(a)\right|+\left[\int_0^1 t^q\left(\frac{1-t}{2}\right)^q dt\right]^{1/q}\left|f'(b)\right|\right\} \\
&\qquad=\frac{1}{2}\left[a^{q/[2(q-1)]}L\left(a^{q/[2(q-1)]},b^{q/[2(q-1)]}\right)\right]^{1-1/q}\left\{\left[B\left(q+1,q+1\right)\right]^{1/q}\left|f'(b)\right|\right. \\
&\qquad\qquad\left.+\left[{}_2F_1\left(-q,q+1;q+2;-1\right)\cdot\frac{1}{q+1}\right]^{1/q}\left|f'(a)\right|\right\}. \qquad (8.33)
\end{aligned}$$

Using (8.32) and (8.33) in (8.31), we obtain the required inequality (8.30). ■

Corollary 333 *Under the assumptions of Theorem 332, if* $g(x)=\frac{1}{\ln b-\ln a}$ *for all* $x\in[a,b]$*, then the following inequality holds*

$$\begin{aligned}
&\left|A\left(f(a),f(b)\right)-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\right| \\
&\qquad\leq\frac{(\ln b-\ln a)\left[L\left(a^{q/[2(q-1)]},b^{q/[2(q-1)]}\right)\right]^{1-1/q}}{2} \\
&\times\left\{\left(b^{1/2}\left[B\left(q+1,q+1\right)\right]^{1/q}+a^{1/2}\left[{}_2F_1\left(-q,q+1;q+2;-1\right)\cdot\frac{1}{q+1}\right]^{1/q}\right)\left|f'(a)\right|\right. \\
&+\left.\left(a^{1/2}\left[B\left(q+1,q+1\right)\right]^{1/q}+b^{1/2}\left[{}_2F_1\left(-q,q+1;q+2;-1\right)\cdot\frac{1}{q+1}\right]^{1/q}\right)\left|f'(b)\right|\right\}, \qquad (8.34)
\end{aligned}$$

where ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ *is the hypergeometric function.*

8.3 Fejér type integral inequalities related with GA-convex functions with applications

The following weighted integral identity plays a key role in establishing results in this section.

Lemma 334 *Let $f : [c,d] \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a function which possesses the first order derivative f' on (c,d), where a, $b \in (c,d)$ with $a < b$ and f' is in the space $L([a,b])$ of all integrable functions over $[a,b]$. If $g : [a,b] \to [0,\infty)$ is a mapping which owns the property of continuity, positivity and geometric symmetry with respect to $G(a,b)$, the following equality holds*

$$\begin{aligned} f(G(a,b)) \int_a^b \frac{g(x)}{x} dx - \int_a^b \frac{f(x) g(x)}{x} dx \\ = \frac{\ln b - \ln a}{2} \int_0^1 \left(\int_a^{G_{\frac{1-t}{2}}(b,a)} \frac{g(x)}{x} dx \right) \\ \times \left[G_{\frac{1-t}{2}}(b,a) f'\left(G_{\frac{1-t}{2}}(b,a)\right) - G_{\frac{1-t}{2}}(a,b) f'\left(G_{\frac{1-t}{2}}(a,b)\right) \right] dt. \quad (8.35) \end{aligned}$$

Proof. Let

$$I_1 = \int_0^1 \left(\int_a^{G_{\frac{1-t}{2}}(b,a)} \frac{g(x)}{x} dx \right) G_{\frac{1-t}{2}}(b,a) f'\left(G_{\frac{1-t}{2}}(b,a)\right) dt$$

and

$$I_2 = \int_0^1 \left(\int_a^{G_{\frac{1-t}{2}}(b,a)} \frac{g(x)}{x} dx \right) G_{\frac{1-t}{2}}(a,b) f'\left(G_{\frac{1-t}{2}}(a,b)\right) dt.$$

Since $g : [a,b] \to [0,\infty)$ is geometrically symmetric to $G(a,b)$, hence $g\left(G_{\frac{1-t}{2}}(a,b)\right) = g\left(G_{\frac{1-t}{2}}(b,a)\right)$ gives

$$\int_a^{G_{\frac{1-t}{2}}(b,a)} \frac{g(x)}{x} dx = \int_{G_{\frac{1-t}{2}}(a,b)}^b \frac{g(x)}{x} dx$$

for all $t \in [0,1]$.

Now we observe that

$$\begin{aligned}
I_1 &= \int_0^1 \left(\int_a^{G_{\frac{1-t}{2}}(b,a)} \frac{g(x)}{x} dx \right) G_{\frac{1-t}{2}}(b,a) f'\left(G_{\frac{1-t}{2}}(b,a) \right) dt \\
&= -\frac{2}{\ln b - \ln a} \int_0^1 \left(\int_a^{G_{\frac{1-t}{2}}(b,a)} \frac{g(x)}{x} dx \right) d\left[f\left(G_{\frac{1-t}{2}}(b,a) \right) \right] \\
&= -\frac{2}{\ln b - \ln a} \left(\int_a^{G_{\frac{1-t}{2}}(b,a)} \frac{g(x)}{x} dx \right) f\left(G_{\frac{1-t}{2}}(b,a) \right) \Bigg|_0^1 \\
&\quad - \int_0^1 g\left(G_{\frac{1-t}{2}}(b,a) \right) f\left(G_{\frac{1-t}{2}}(b,a) \right) dt \\
&= \frac{2f(G(a,b))}{\ln b - \ln a} \int_a^{\sqrt{ab}} \frac{g(x)}{x} dx - \int_0^1 g\left(G_{\frac{1-t}{2}}(b,a) \right) f\left(G_{\frac{1-t}{2}}(b,a) \right) dt \\
&= \frac{2f(G(a,b))}{\ln b - \ln a} \int_a^{\sqrt{ab}} \frac{g(x)}{x} dx - \frac{2}{\ln b - \ln a} \int_a^{\sqrt{ab}} \frac{g(x) f(x)}{x} dx. \qquad (8.36)
\end{aligned}$$

Similarly, we have

$$-I_2 = \frac{2f(G(a,b))}{\ln b - \ln a} \int_{\sqrt{ab}}^b \frac{g(x)}{x} dx - \frac{2}{\ln b - \ln a} \int_{\sqrt{ab}}^b \frac{g(x) f(x)}{x} dx. \qquad (8.37)$$

Adding (8.36) and (8.37) and multiplying the result by $\frac{\ln b - \ln a}{2}$, we get the required identity. This completes the proof of the Lemma. ■

We now establish new Fejér type inequalities for GA-convex functions, which provide weighted generalization of some of the results established in recent literature concerning GA-convex functions.

Theorem 335 *Let $f : [c,d] \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a function which possesses the first order derivative f' on (c,d), where $a, b \in (c,d)$ with $a < b$ and f' is in the space $L([a,b])$ of all integrable functions over $[a,b]$. If $g : [a,b] \to [0,\infty)$ is a mapping which owns the property of continuity, positivity and geometric symmetry with respect to $G(a,b)$ and $|f'|^q$ is in* *the space of GA-convex on $[a,b]$ for $q \geq 1$, then the following inequality holds*

$$\begin{aligned}
&\left| f\left(G\left(a,b\right)\right) \int_a^b \frac{g\left(x\right)}{x} dx - \int_a^b \frac{f\left(x\right) g\left(x\right)}{x} dx \right| \\
&\qquad \leq \frac{\left(\ln b - \ln a\right)^2}{4} \left\|g\right\|_\infty \left\{ \left[\zeta\left(a,b\right) - 2\Psi\left(a,b\right)\right]^{1-1/q} \right. \\
&\quad \times \left(\left[\zeta\left(a,b\right) + 2\Phi\left(a,b\right)\right] A\left(\left|f'\left(a\right)\right|^q, \left|f'\left(b\right)\right|^q \right) - 2\Psi\left(a,b\right) \left|f'\left(a\right)\right|^q \right)^{1/q} \\
&\qquad + \left[\zeta\left(b,a\right) - 2\Psi\left(b,a\right)\right]^{1-1/q} \left(\left[\zeta\left(b,a\right) + 2\Phi\left(b,a\right)\right] \right. \\
&\qquad\qquad \left. \left. \times A\left(\left|f'\left(a\right)\right|^q, \left|f'\left(b\right)\right|^q \right) - 2\Psi\left(b,a\right) \left|f'\left(b\right)\right|^q \right)^{1/q} \right\}, \qquad (8.38)
\end{aligned}$$

where $\left\|g\right\|_\infty = \sup_{x\in[a,b]} g\left(x\right) < \infty$ *and* $\zeta\left(\cdot,\cdot\right)$, $\Psi\left(\cdot,\cdot\right)$, $\Phi\left(\cdot,\cdot\right)$ *are defined in Lemma 323.*

Proof. From Lemma 334 and Hölder's inequality , we have

$$\begin{aligned}
&\left| f\left(G\left(a,b\right)\right) \int_a^b \frac{g\left(x\right)}{x} dx - \int_a^b \frac{f\left(x\right) g\left(x\right)}{x} dx \right| \\
&\qquad \leq \frac{\ln b - \ln a}{2} \int_0^1 \left(\int_a^{G_{\frac{1-t}{2}}\left(b,a\right)} \frac{g\left(x\right)}{x} dx \right) \left[G_{\frac{1-t}{2}}\left(a,b\right) \left| f'\left(G_{\frac{1-t}{2}}\left(a,b\right) \right) \right| \right. \\
&\qquad\quad \left. + G_{\frac{1-t}{2}}\left(b,a\right) \left| f'\left(G_{\frac{1-t}{2}}\left(b,a\right) \right) \right| \right] dt \leq \frac{\left(\ln b - \ln a\right)^2}{4} \left\|g\right\|_\infty \\
&\quad \times \int_0^1 \left[\left(1-t\right) G_{\frac{1-t}{2}}\left(a,b\right) \left| f'\left(G_{\frac{1-t}{2}}\left(a,b\right) \right) \right| + \left(1-t\right) G_{\frac{1-t}{2}}\left(b,a\right) f'\left(G_{\frac{1-t}{2}}\left(b,a\right) \right) \right] dt \\
&\qquad \leq \frac{\left(\ln b - \ln a\right)^2}{4} \left\|g\right\|_\infty \left\{ \left(\int_0^1 \left(1-t\right) G_{\frac{1-t}{2}}\left(a,b\right) dt \right)^{1-1/q} \right. \\
&\quad \times \left(\int_0^1 \left(1-t\right) G_{\frac{1-t}{2}}\left(a,b\right) \left| f'\left(G_{\frac{1-t}{2}}\left(a,b\right) \right) \right|^q dt \right)^{1/q} + \left(\int_0^1 \left(1-t\right) G_{\frac{1-t}{2}}\left(b,a\right) dt \right)^{1-1/q} \\
&\qquad\qquad \left. \times \left(\int_0^1 \left(1-t\right) G_{\frac{1-t}{2}}\left(b,a\right) \left| f'\left(G_{\frac{1-t}{2}}\left(b,a\right) \right) \right|^q dt \right)^{1/q} \right\}. \qquad (8.39)
\end{aligned}$$

By the GA-convexity of $\left|f'\right|^q$ on $[a,b]$ for $q \geq 1$ and by using Lemma 323, we have

$$\begin{aligned}
\int_0^1 (1-t)\, G_{\frac{1-t}{2}}(a,b) \left|f'\left(G_{\frac{1-t}{2}}(a,b)\right)\right|^q &\leq \left|f'(a)\right|^q \int_0^1 (1-t)\left(\frac{1-t}{2}\right) a^{(1-t)/2} b^{(1+t)/2} dt \\
&\quad + \left|f'(b)\right|^q \int_0^1 (1-t)\left(\frac{1+t}{2}\right) a^{(1-t)/2} b^{(1+t)/2} dt \\
&= \left[\frac{1}{2}\zeta(a,b) - 2\Psi(a,b) + \Phi(a,b)\right]\left|f'(a)\right|^q + \left[\frac{1}{2}\zeta(a,b) + \Phi(a,b)\right]\left|f'(b)\right|^q \quad (8.40)
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 (1-t)\, G_{\frac{1-t}{2}}(b,a) \left|f'\left(G_{\frac{1-t}{2}}(b,a)\right)\right|^q &\leq \left|f'(a)\right|^q \int_0^1 (1-t)\left(\frac{1+t}{2}\right) a^{(1+t)/2} b^{(1-t)/2} dt \\
&\quad + \left|f'(b)\right|^q \int_0^1 (1-t)\left(\frac{1-t}{2}\right) a^{(1+t)/2} b^{(1-t)/2} dt \\
&= \left[\frac{1}{2}\zeta(b,a) + \Phi(b,a)\right]\left|f'(a)\right|^q + \left[\frac{1}{2}\zeta(b,a) - 2\Psi(b,a) + \Phi(b,a)\right]\left|f'(b)\right|^q. \quad (8.41)
\end{aligned}$$

Using (8.40) and (8.41) in (8.39), we get the required result. ■

Corollary 336 *Let f be as in Theorem 335 and $q = 1$, then the following inequality holds*

$$\begin{aligned}
&\left|f(G(a,b)) \int_a^b \frac{g(x)}{x} dx - \int_a^b \frac{f(x)\, g(x)}{x} dx\right| \\
&\quad \leq 2\|g\|_\infty \left\{\left[L(a,b) - \sqrt{a}A\left(\sqrt{a},\sqrt{b}\right) - \frac{1}{4}a(\ln b - \ln a)\right]\left|f'(a)\right|\right. \\
&\qquad \left. + \left[L(a,b) - \sqrt{b}A\left(\sqrt{a},\sqrt{b}\right) + \frac{1}{4}b(\ln b - \ln a)\right]\left|f'(b)\right|\right\}, \quad (8.42)
\end{aligned}$$

where $\|g\|_\infty = \sup_{x\in[a,b]} g(x) < \infty$.

Corollary 337 *If $g(x) = \frac{1}{\ln b - \ln a}$, for all $x \in [a,b]$ in Theorem 335, then*

$$\begin{aligned}
&\left| f\left(G\left(a,b\right)\right) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f\left(x\right)}{x} dx \right| \\
&\leq \frac{\left(\ln b - \ln a\right)}{4} \left\{ \left[\zeta\left(a,b\right) - 2\Psi\left(a,b\right)\right]^{1-1/q} \left(\left[\zeta\left(a,b\right) + 2\Phi\left(a,b\right)\right] A\left(\left|f'\left(a\right)\right|^q, \left|f'\left(b\right)\right|^q \right) \right.\right. \\
&\qquad \left. -2\Psi\left(a,b\right) \left|f'\left(a\right)\right|^q \right)^{1/q} + \left[\zeta\left(b,a\right) - 2\Psi\left(b,a\right)\right]^{1-1/q} \\
&\quad \left. \times \left(\left[\zeta\left(b,a\right) + 2\Phi\left(b,a\right)\right] A\left(\left|f'\left(a\right)\right|^q, \left|f'\left(b\right)\right|^q \right) - 2\Psi\left(b,a\right) \left|f'\left(b\right)\right|^q \right)^{1/q} \right\}, \quad (8.43)
\end{aligned}$$

where $\zeta(\cdot,\cdot)$, $\Psi(\cdot,\cdot)$ and $\Phi(\cdot,\cdot)$ are defined in Lemma 323.

Corollary 338 *If $q = 1$ in Corollary 337, then we get the following inequality*

$$\begin{aligned}
&\left| f\left(G\left(a,b\right)\right) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f\left(x\right)}{x} dx \right| \\
&\leq \frac{2}{\ln b - \ln a} \left\{ \left[L\left(a,b\right) - \sqrt{a} A\left(\sqrt{a}, \sqrt{b}\right) - \frac{1}{4} a \left(\ln b - \ln a\right) \right] \left|f'\left(a\right)\right| \right. \\
&\qquad \left. + \left[L\left(a,b\right) - \sqrt{b} A\left(\sqrt{a}, \sqrt{b}\right) + \frac{1}{4} b \left(\ln b - \ln a\right) \right] \left|f'\left(b\right)\right| \right\}. \quad (8.44)
\end{aligned}$$

Theorem 339 *Let $f : [c,d] \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a function which possesses the first order derivative f' on (c,d), where $a, b \in (c,d)$ with $a < b$ and f' is in the space $L([a,b])$ of all integrable functions over $[a,b]$. If $g : [a,b] \to [0,\infty)$ is a mapping which owns the property of continuity, positivity and geometric symmetry with respect to $G(a,b)$ and $\left|f'\right|^q$ is in* *the space of GA-convex on $[a,b]$ for $q>1$, then the following inequality holds*

$$\begin{aligned}
&\left|f\left(G\left(a,b\right)\right)\int_a^b\frac{g\left(x\right)}{x}dx-\int_a^b\frac{f\left(x\right)g\left(x\right)}{x}dx\right| \\
&\quad\le\frac{\left(\ln b-\ln a\right)^{2-1/q}\left\|g\right\|_\infty}{4\cdot q^{1/q}}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left\{b^{1/2}\left(\left[L\left(a^{q/2},b^{q/2}\right)-a^{q/2}\right]\left|f'\left(a\right)\right|^q\right.\right. \\
&\qquad\left.+\left[2b^{q/2}-a^{q/2}-L\left(a^{q/2},b^{q/2}\right)\right]\left|f'\left(b\right)\right|^q\right)^{1/q} \\
&\qquad+a^{1/2}\left(\left[L\left(a^{q/2},b^{q/2}\right)+b^{q/2}-2a^{q/2}\right]\left|f'\left(a\right)\right|^q\right. \\
&\qquad\qquad\left.\left.+\left[b^{q/2}-L\left(a^{q/2},b^{q/2}\right)\right]\left|f'\left(b\right)\right|^q\right)^{1/q}\right\}, \quad (8.45)
\end{aligned}$$

where $\left\|g\right\|_\infty=\sup_{x\in[a,b]}g\left(x\right)<\infty$.

Proof. From Lemma 334 and Hölder's inequality , we have

$$\begin{aligned}
&\left|f\left(G\left(a,b\right)\right)\int_a^b\frac{g\left(x\right)}{x}dx-\int_a^b\frac{f\left(x\right)g\left(x\right)}{x}dx\right| \\
&\quad\le\frac{\ln b-\ln a}{2}\int_0^1\left(\int_a^{G_{\frac{1-t}{2}}(b,a)}\frac{g\left(x\right)}{x}dx\right)\left[G_{\frac{1-t}{2}}\left(a,b\right)\left|f'\left(G_{\frac{1-t}{2}}\left(a,b\right)\right)\right|\right. \\
&\qquad\left.+G_{\frac{1-t}{2}}\left(b,a\right)\left|f'\left(G_{\frac{1-t}{2}}\left(b,a\right)\right)\right|\right]dt\le\frac{\left(\ln b-\ln a\right)^2\left\|g\right\|_\infty}{4} \\
&\times\int_0^1\left[\left(1-t\right)G_{\frac{1-t}{2}}\left(a,b\right)\left|f'\left(G_{\frac{1-t}{2}}\left(a,b\right)\right)\right|+\left(1-t\right)G_{\frac{1-t}{2}}\left(b,a\right)f'\left(G_{\frac{1-t}{2}}\left(b,a\right)\right)\right]dt \\
&\quad\le\frac{\left(\ln b-\ln a\right)^2\left\|g\right\|_\infty}{4}\left(\int_0^1\left(1-t\right)^{q/(q-1)}dt\right)^{1-1/q} \\
&\quad\times\left\{\left(\int_0^1\left[G_{\frac{1-t}{2}}\left(a,b\right)\right]^q\left|f'\left(G_{\frac{1-t}{2}}\left(a,b\right)\right)\right|^q dt\right)^{1/q}\right. \\
&\qquad\left.+\left(\int_0^1\left[G_{\frac{1-t}{2}}\left(b,a\right)\right]^q\left|f'\left(G_{\frac{1-t}{2}}\left(b,a\right)\right)\right|^q dt\right)^{1/q}\right\}. \quad (8.46)
\end{aligned}$$

Since

$$\int_0^1 \left[G_{\frac{1-t}{2}}(a,b)\right]^q \left|f'\left(G_{\frac{1-t}{2}}(a,b)\right)\right|^q = \int_0^1 a^{[q(1-t)]/2} b^{[q(1+t)]/2} \left|f'\left(a^{(1-t)/2} b^{(1+t)/2}\right)\right|$$
$$\leq \left|f'(a)\right|^q \int_0^1 \left(\frac{1-t}{2}\right) a^{[q(1-t)]/2} b^{[q(1+t)]/2} dt$$
$$+ \left|f'(b)\right|^q \int_0^1 \left(\frac{1+t}{2}\right) a^{[q(1-t)]/2} b^{[q(1+t)]/2} dt$$
$$= \frac{b^{q/2}\left[L\left(a^{q/2}, b^{q/2}\right) - a^{q/2}\right]}{q(\ln b - \ln a)} \left|f'(a)\right|^q$$
$$+ \frac{b^{q/2}\left[2b^{q/2} - a^{q/2} - L\left(a^{q/2}, b^{q/2}\right)\right]}{q(\ln b - \ln a)} \left|f'(b)\right|^q \quad (8.47)$$

and

$$\int_0^1 \left[G_{\frac{1-t}{2}}(b,a)\right]^q \left|f'\left(G_{\frac{1-t}{2}}(b,a)\right)\right|^q = \int_0^1 a^{[q(1+t)]/2} b^{q(1-t)/2} \left|f'\left(a^{(1+t)/2} b^{(1-t)/2}\right)\right|$$
$$\leq \left|f'(a)\right|^q \int_0^1 \left(\frac{1+t}{2}\right) a^{[q(1+t)]/2} b^{[q(1-t)]/2} dt$$
$$+ \left|f'(b)\right|^q \int_0^1 \left(\frac{1-t}{2}\right) a^{[q(1+t)]/2} b^{[q(1-t)]/2} dt$$
$$= \frac{a^{q/2}\left[L\left(a^{q/2}, b^{q/2}\right) + b^{q/2} - 2a^{q/2}\right]}{q(\ln b - \ln a)} \left|f'(a)\right|^q$$
$$+ \frac{a^{q/2}\left[b^{q/2} - L\left(a^{q/2}, b^{q/2}\right)\right]}{q(\ln b - \ln a)} \left|f'(b)\right|^q. \quad (8.48)$$

The inequality (8.45) is proved by applying (8.47) and (8.47) in (8.46). ■

Corollary 340 *Let f be as in Theorem 339 and if $g(x) = \frac{1}{\ln b - \ln a}$ for all $x \in [a,b]$, then* *the following inequality holds*

$$\left|f\left(G\left(a,b\right)\right)-\frac{1}{\ln b-\ln a}\int_a^b\frac{f\left(x\right)}{x}dx\right|$$
$$\leq\frac{\left(\ln b-\ln a\right)^{1-1/q}}{4\cdot q^{1/q}}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left\{b^{1/2}\left(\left[L\left(a^{q/2},b^{q/2}\right)-a^{q/2}\right]\left|f^{'}\left(a\right)\right|^q\right.\right.$$
$$\left.+\left[2b^{q/2}-a^{q/2}-L\left(a^{q/2},b^{q/2}\right)\right]\left|f^{'}\left(b\right)\right|^q\right)^{1/q}$$
$$+a^{1/2}\left(\left[L\left(a^{q/2},b^{q/2}\right)+b^{q/2}-2a^{q/2}\right]\left|f^{'}\left(a\right)\right|^q\right.$$
$$\left.\left.+\left[b^{q/2}-L\left(a^{q/2},b^{q/2}\right)\right]\left|f^{'}\left(b\right)\right|^q\right)^{1/q}\right\}. \quad (8.49)$$

Theorem 341 *Let $f:[c,d]\subseteq\mathbb{R}_+\to\mathbb{R}$ be a function which possesses the first order derivative $f^{'}$ on (c,d), where $a,\, b\in(c,d)$ with $a<b$ and $f^{'}$ is in the space $L\left([a,b]\right)$ of all integrable functions over $[a,b]$. If $g:[a,b]\to[0,\infty)$ is a mapping which owns the property of continuity, positivity and geometric symmetry with respect to $G\left(a,b\right)$ and $\left|f^{'}\right|^q$ is in the space of GA-convex on $[a,b]$ for $q>1$, then the following inequality holds*

$$\left|f\left(G\left(a,b\right)\right)\int_a^b\frac{g\left(x\right)}{x}dx-\int_a^b\frac{f\left(x\right)g\left(x\right)}{x}dx\right|$$
$$\leq\frac{\left(\ln b-\ln a\right)^{2-1/q}}{2\cdot\left(4q\right)^{1/q}}\left\|g\right\|_\infty\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left\{\left[L\left(a^q,b^q\right)-a^q\right]\left|f^{'}\left(a\right)\right|^q\right.$$
$$\left.+\left[b^q-L\left(a^q,b^q\right)\right]\left|f^{'}\left(b\right)\right|^q\right\}^{1/q}, \quad (8.50)$$

where $\left\|g\right\|_\infty=\sup_{x\in[a,b]}g\left(x\right)<\infty$.

Proof. From Lemma 334 and Hölder's inequality, we have

$$\begin{aligned}
&\left|f\left(G\left(a,b\right)\right)\int_a^b \frac{g\left(x\right)}{x}dx-\int_a^b \frac{f\left(x\right)g\left(x\right)}{x}dx\right| \\
&\quad\leq \frac{\ln b-\ln a}{2}\int_0^1\left(\int_a^{G_{\frac{1-t}{2}}(b,a)}\frac{g\left(x\right)}{x}dx\right)\left[G_{\frac{1-t}{2}}\left(a,b\right)\left|f'\left(G_{\frac{1-t}{2}}\left(a,b\right)\right)\right|\right. \\
&\qquad\left.+G_{\frac{1-t}{2}}\left(b,a\right)\left|f'\left(G_{\frac{1-t}{2}}\left(b,a\right)\right)\right|\right]dt\leq\frac{\left(\ln b-\ln a\right)^2}{4}\left\|g\right\|_\infty \\
&\times\int_0^1\left[\left(1-t\right)G_{\frac{1-t}{2}}\left(a,b\right)\left|f'\left(G_{\frac{1-t}{2}}\left(a,b\right)\right)\right|+\left(1-t\right)G_{\frac{1-t}{2}}\left(b,a\right)f'\left(G_{\frac{1-t}{2}}\left(b,a\right)\right)\right]dt \\
&\quad\leq\frac{\left(\ln b-\ln a\right)^2}{4}\left\|g\right\|_\infty\left(\int_0^1\left(1-t\right)^{q/(q-1)}dt\right)^{1-1/q} \\
&\quad\times\left\{\left(\int_0^1\left[G_{\frac{1-t}{2}}\left(a,b\right)\right]^q\left|f'\left(G_{\frac{1-t}{2}}\left(a,b\right)\right)\right|^q dt\right)^{1/q}\right. \\
&\qquad\left.+\left(\int_0^1\left[G_{\frac{1-t}{2}}\left(b,a\right)\right]^q\left|f'\left(G_{\frac{1-t}{2}}\left(b,a\right)\right)\right|^q dt\right)^{1/q}\right\}. \qquad (8.51)
\end{aligned}$$

By the power-mean inequality ($a^r+b^r\leq 2^{1-r}\left(a+b\right)^r$ for $a>0,b>0$ and $r<1$), we have

$$\begin{aligned}
&\left(\int_0^1\left[G_{\frac{1-t}{2}}\left(a,b\right)\right]^q\left|f'\left(G_{\frac{1-t}{2}}\left(a,b\right)\right)\right|^q dt\right)^{1/q} \\
&\qquad+\left(\int_0^1\left[G_{\frac{1-t}{2}}\left(b,a\right)\right]^q\left|f'\left(G_{\frac{1-t}{2}}\left(b,a\right)\right)\right|^q dt\right)^{1/q} \\
&\quad\leq 2^{1-1/q}\left(\int_0^1\left[G_{\frac{1-t}{2}}\left(a,b\right)\right]^q\left|f'\left(G_{\frac{1-t}{2}}\left(a,b\right)\right)\right|^q dt\right. \\
&\qquad\left.+\int_0^1\left[G_{\frac{1-t}{2}}\left(b,a\right)\right]^q\left|f'\left(G_{\frac{1-t}{2}}\left(b,a\right)\right)\right|^q dt\right)^{1/q}. \qquad (8.52)
\end{aligned}$$

Since $\left|f'\right|^q$ is GA-convex on $[a,b]$ for $q>1$

$$\begin{aligned}
&\int_0^1 \left[G_{\frac{1-t}{2}}(a,b)\right]^q \left|f'\left(G_{\frac{1-t}{2}}(a,b)\right)\right|^q dt + \int_0^1 \left[G_{\frac{1-t}{2}}(b,a)\right]^q \left|f'\left(G_{\frac{1-t}{2}}(b,a)\right)\right|^q dt \\
&\quad \le \left|f'(a)\right|^q \left[\int_0^1 \left(\frac{1-t}{2}\right) a^{[q(1-t)]/2} b^{q(1+t)} dt + \int_0^1 \left(\frac{1+t}{2}\right) a^{[q(1+t)]/2} b^{q(1-t)} dt\right] \\
&\quad + \left|f'(b)\right|^q \left[\int_0^1 \left(\frac{1+t}{2}\right) a^{[q(1-t)]/2} b^{q(1+t)} dt + \int_0^1 \left(\frac{1-t}{2}\right) a^{[q(1+t)]/2} b^{q(1-t)} dt\right] \\
&\quad = \left[\frac{2L(a^q,b^q)-2a^q}{q(\ln b-\ln a)}\right]\left|f'(a)\right|^q + \left[\frac{2b^q-2L(a^q,b^q)}{q(\ln b-\ln a)}\right]\left|f'(b)\right|^q. \quad (8.53)
\end{aligned}$$

Using (8.53) in (8.52), we get

$$\begin{aligned}
&\left(\int_0^1 \left[G_{\frac{1-t}{2}}(a,b)\right]^q \left|f'\left(G_{\frac{1-t}{2}}(a,b)\right)\right|^q dt\right)^{1/q} \\
&\quad + \left(\int_0^1 \left[G_{\frac{1-t}{2}}(b,a)\right]^q \left|f'\left(G_{\frac{1-t}{2}}(b,a)\right)\right|^q dt\right)^{1/q} \\
&\quad \le 2^{1-2/q}\left(\left[\frac{L(a^q,b^q)-a^q}{q(\ln b-\ln a)}\right]\left|f'(a)\right|^q + \left[\frac{b^q-L(a^q,b^q)}{q(\ln b-\ln a)}\right]\left|f'(b)\right|^q\right)^{1/q}. \quad (8.54)
\end{aligned}$$

Applying (8.54) in (8.51), we obtain the required inequality (8.50). ■

Corollary 342 *Let f be as in Theorem 341 and if $g(x)=\frac{1}{\ln b-\ln a}$ for all $x\in[a,b]$, then the following inequality holds*

$$\begin{aligned}
&\left|f(G(a,b)) - \frac{1}{\ln b-\ln a}\int_a^b \frac{f(x)}{x}dx\right| \\
&\quad \le \frac{(\ln b-\ln a)^{1-1/q}}{2\cdot(4q)^{1/q}}\|g\|_\infty \left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} \left\{\left[L(a^q,b^q)-a^q\right]\left|f'(a)\right|^q\right. \\
&\quad \left. + \left[b^q-L(a^q,b^q)\right]\left|f'(b)\right|^q\right\}^{1/q}. \quad (8.55)
\end{aligned}$$

Theorem 343 *Let $f:[c,d]\subseteq\mathbb{R}_+\to\mathbb{R}$ be a function which possesses the first order derivative f' on (c,d), where $a,\,b\in(c,d)$ with $a<b$ and f' is in the space $L([a,b])$ of all integrable functions over $[a,b]$. If $g:[a,b]\to[0,\infty)$ is a mapping which owns the property*

of continuity, positivity and geometric symmetry with respect to $G(a,b)$ *and* $\left|f'\right|^q$ *is in the space of GA-convex on* $[a,b]$ *for* $q>1$*, then the following inequality holds*

$$\begin{aligned}
&\left|f(G(a,b))\int_a^b \frac{g(x)}{x}dx - \int_a^b \frac{f(x)g(x)}{x}dx\right| \\
&\qquad \le \frac{(\ln b-\ln a)^2\left[L\left(a^{q/[2(q-1)]}, b^{q/[2(q-1)]}\right)\right]^{1-1/q}\|g\|_\infty}{8} \\
&\quad \times\left\{\left(b^{1/2}\left(\frac{1}{2q+1}\right)^{1/q} + a^{1/2}\left[{}_2F_1(-q,1;q+2;-1)\cdot\frac{1}{q+1}\right]^{1/q}\right)\left|f'(a)\right|\right. \\
&\quad \left.+\left(a^{1/2}\left(\frac{1}{2q+1}\right)^{1/q} + b^{1/2}\left[{}_2F_1(-q,1;q+2;-1)\cdot\frac{1}{q+1}\right]^{1/q}\right)\left|f'(b)\right|\right\}, \quad (8.56)
\end{aligned}$$

where $\|g\|_\infty = \sup_{x\in[a,b]} g(x) < \infty$ *and* ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ *is the hypergeometric function.*

Proof. From Lemma 334 and the GA-convexity of $\left|f'\right|$ on $[a,b]$, we have

$$\begin{aligned}
&\left|f(G(a,b))\int_a^b \frac{g(x)}{x}dx - \int_a^b \frac{f(x)g(x)}{x}dx\right| \\
&\qquad \le \frac{\ln b-\ln a}{2}\int_0^1\left(\int_a^{G_{\frac{1-t}{2}}(b,a)}\frac{g(x)}{x}dx\right)\left[G_{\frac{1-t}{2}}(a,b)\left|f'\left(G_{\frac{1-t}{2}}(a,b)\right)\right|\right. \\
&\qquad\quad \left.+G_{\frac{1-t}{2}}(b,a)\left|f'\left(G_{\frac{1-t}{2}}(b,a)\right)\right|\right]dt \le \frac{(\ln b-\ln a)^2}{4}\|g\|_\infty \\
&\int_0^1\left[(1-t)G_{\frac{1-t}{2}}(a,b)\left|f'\left(G_{\frac{1-t}{2}}(a,b)\right)\right| + (1-t)G_{\frac{1-t}{2}}(b,a)f'\left(G_{\frac{1-t}{2}}(b,a)\right)\right]dt \\
&\qquad \le \frac{(\ln b-\ln a)^2}{4}\|g\|_\infty\left\{\int_0^1 a^{(1-t)/2}b^{(1+t)/2}\left[(1-t)\left(\frac{1-t}{2}\right)\left|f'(a)\right|\right.\right. \\
&\qquad\quad \left.+(1-t)\left(\frac{1+t}{2}\right)\left|f'(b)\right|\right]dt + \int_0^1 a^{(1+t)/2}b^{(1-t)/2} \\
&\qquad\quad \left.\times\left[(1-t)\left(\frac{1+t}{2}\right)\left|f'(a)\right| + (1-t)\left(\frac{1-t}{2}\right)\left|f'(b)\right|\right]dt\right\}. \quad (8.57)
\end{aligned}$$

Using Hölder integral inequality, we have

$$\begin{aligned}
&\int_0^1 a^{(1-t)/2}b^{(1+t)/2}\left[(1-t)\left(\frac{1-t}{2}\right)\left|f'(a)\right|+(1-t)\left(\frac{1+t}{2}\right)\left|f'(b)\right|\right]dt \\
&\qquad \leq \left(\int_0^1 a^{[q(1-t)]/[2(q-1)]}b^{[q(1+t)]/[2(q-1)]}dt\right)^{1-1/q} \\
&\times\left\{\left[\int_0^1 (1-t)^q\left(\frac{1-t}{2}\right)^q dt\right]^{1/q}\left|f'(a)\right|+\left[\int_0^1 (1-t)^q\left(\frac{1+t}{2}\right)^q dt\right]^{1/q}\left|f'(b)\right|\right\} \\
&\qquad = \frac{1}{2}\left[b^{q/[2(q-1)]}L\left(a^{q/[2(q-1)]},b^{q/[2(q-1)]}\right)\right]^{1-1/q}\left\{\left(\frac{1}{2q+1}\right)^{1/q}\left|f'(a)\right|\right. \\
&\qquad\qquad \left.+\left[{}_2F_1(-q,1;q+2;-1)\cdot\frac{1}{q+1}\right]^{1/q}\left|f'(b)\right|\right\}. \qquad (8.58)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_0^1 a^{(1+t)/2}b^{(1-t)/2}\left[t\left(\frac{1+t}{2}\right)\left|f'(a)\right|+t\left(\frac{1-t}{2}\right)\left|f'(b)\right|\right]dt \\
&\qquad \leq \left(\int_0^1 a^{[q(1+t)]/[2(q-1)]}b^{[q(1-t)]/[2(q-1)]}dt\right)^{1-1/q} \\
&\times\left\{\left[\int_0^1 (1-t)^q\left(\frac{1+t}{2}\right)^q dt\right]^{1/q}\left|f'(a)\right|+\left[\int_0^1 (1-t)^q\left(\frac{1-t}{2}\right)^q dt\right]^{1/q}\left|f'(b)\right|\right\} \\
&\qquad = \frac{1}{2}\left[a^{q/[2(q-1)]}L\left(a^{q/[2(q-1)]},b^{q/[2(q-1)]}\right)\right]^{1-1/q}\left\{\left(\frac{1}{2q+1}\right)^{1/q}\left|f'(b)\right|\right. \\
&\qquad\qquad \left.+\left[{}_2F_1(-q,1;q+2;-1)\cdot\frac{1}{q+1}\right]^{1/q}\left|f'(a)\right|\right\}. \qquad (8.59)
\end{aligned}$$

Using (8.58) and (8.59) in (8.57), we obtain the required inequality (8.56). ■

Corollary 344 *Let f be as in Theorem 343 and if $g(x)=\frac{1}{\ln b-\ln a}$ for all $x\in[a,b]$. Then* *the following inequality holds*

$$\left| f\left(G\left(a,b\right)\right) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f\left(x\right)}{x}dx \right|$$
$$\leq \frac{\left(\ln b - \ln a\right)\left[L\left(a^{q/[2(q-1)]}, b^{q/[2(q-1)]}\right)\right]^{1-1/q}}{2}$$
$$\times \left\{ \left(b^{1/2}\left(\frac{1}{2q+1}\right)^{1/q} + a^{1/2}\left[{}_2F_1\left(-q,1;q+2;-1\right)\cdot\frac{1}{q+1}\right]^{1/q} \right)\left|f'\left(a\right)\right| \right.$$
$$\left. + \left(a^{1/2}\left(\frac{1}{2q+1}\right)^{1/q} + b^{1/2}\left[{}_2F_1\left(-q,1;q+2;-1\right)\cdot\frac{1}{q+1}\right]^{1/q} \right)\left|f'\left(b\right)\right| \right\}, \quad (8.60)$$

where ${}_2F_1\left(\cdot,\cdot;\cdot;\cdot\right)$ *is the hypergeometric function.*

8.4 Applications to Special Means

In this section we apply some of the established inequalities of Fejér type involving the product of a geometrically-arithmetically convex function and a geometrically symmetric function of Section 8.2 and Section 8.3 to construct inequalities for special means of positive real numbers.

Now let $f(x) = x^r$ for $x > 0$, $r \in \mathbb{R}$ with $r \neq 0$. Then

$$\left|f'\left(x^\lambda y^{1-\lambda}\right)\right|^q = |r|^q\left[x^{q(r-1)}\right]^\lambda\left[y^{q(r-1)}\right]^{1-\lambda} \leq |r|^q\left[\lambda x^{q(r-1)} + \left(1-\lambda\right)y^{q(r-1)}\right]$$

for $\lambda \in [0,1]$, x, $y > 0$ and $q \geq 1$. That is $\left|f'\left(x\right)\right|^q = |r|^q x^{q(r-1)}$ is geometrically-arithmetically convex on $[a,b]$ for $q \geq 1$ and $r \neq 1$, where a, $b > 0$.

Let $g : [a,b] \to \mathbb{R}_0$ be defined as

$$g\left(x\right) = \left(\frac{x}{\sqrt{ab}} - \frac{\sqrt{ab}}{x}\right)^2, x \in \left[a,b\right].$$

It is obvious that

$$g\left(\frac{ab}{x}\right)=g\left(x\right)$$

for all $x\in[a,b]$. Hence

$$g\left(x\right)=\left(\frac{x}{\sqrt{ab}}-\frac{\sqrt{ab}}{x}\right)^{2},x\in[a,b]$$

is geometrically symmetric with respect to $x=G\left(a,b\right)$.

Now we apply our results to get new inequalities of special means of positive real numbers.

Theorem 345 *Let $0<a<b$, $r\in\mathbb{R}\backslash\left\{-2,0,1,2\right\}$ and $q\geq 1$. Then*

$$\begin{aligned}
&\left|\frac{2A\left(a^{r},b^{r}\right)L\left(a^{2},b^{2}\right)-L\left(a^{r+2},b^{r+2}\right)}{\left[G\left(a,b\right)\right]^{2}}\right.\\
&\qquad\left.-\left[G\left(a,b\right)\right]^{2}L\left(a^{r-2},b^{r-2}\right)+2L\left(a^{r},b^{r}\right)-2A\left(a^{r},b^{r}\right)\right|\\
&\qquad\leq\frac{\left|r\right|\left(b-a\right)^{2}}{2^{1+1/q}G\left(a,b\right)L\left(a,b\right)}\left\{\left[\Psi\left(a,b\right)\right]^{1-1/q}\right.\\
&\times\left[2\Psi\left(a,b\right)A\left(a^{q(r-1)},b^{q(r-1)}\right)+\Phi\left(a,b\right)\left(b^{q(r-1)}-a^{q(r-1)}\right)\right]^{1/q}+\left[\Psi\left(b,a\right)\right]^{1-1/q}\\
&\qquad\left.\times\left[2\Psi\left(b,a\right)A\left(a^{q(r-1)},b^{q(r-1)}\right)+\Phi\left(b,a\right)\left(a^{q(r-1)}-b^{q(r-1)}\right)\right]^{1/q}\right\},\qquad(8.61)
\end{aligned}$$

where $\Psi\left(\cdot,\cdot\right)$ and $\Phi\left(\cdot,\cdot\right)$ are defined as in Lemma 323.

Proof. Applying Theorem 324 to the functions

$$f(x)=x^{r}\text{ for }x>0,\ r\in\mathbb{R}\backslash\left\{-2,0,1,2\right\}$$

and

$$g\left(x\right)=\left(\frac{x}{\sqrt{ab}}-\frac{\sqrt{ab}}{x}\right)^{2},x\in[a,b]$$

we get the desired result. ■

Corollary 346 *Let f be as in Theorem 345 and if $r=-1$, the the following inequality holds*

$$\begin{aligned}&\left|\frac{2L\left(a^2,b^2\right)\left[H\left(a,b\right)\right]^{-1}+L\left(a,b\right)}{\left[G\left(a,b\right)\right]^2}-\left[G\left(a,b\right)\right]^2L\left(a^{-3},b^{-3}\right)-2\left[H\left(a,b\right)\right]^{-1}\right|\\&\leq\frac{\left(b-a\right)^2}{2^{1+1/q}G\left(a,b\right)L\left(a,b\right)}\left\{\left[\Psi\left(a,b\right)\right]^{1-1/q}\right.\\&\times\left[2\Psi\left(a,b\right)A\left(a^{-2q},b^{-2q}\right)+\Phi\left(a,b\right)\left(b^{-2q}-a^{-2q}\right)\right]^{1/q}\\&\left.+\left[\Psi\left(b,a\right)\right]^{1-1/q}\left[2\Psi\left(b,a\right)A\left(a^{-2q},b^{-2q}\right)+\Phi\left(b,a\right)\left(a^{-2q}-b^{-2q}\right)\right]^{1/q}\right\},\qquad(8.62)\end{aligned}$$

where $\Psi\left(\cdot,\cdot\right)$ and $\Phi\left(\cdot,\cdot\right)$ are defined as in Lemma 323.

Corollary 347 *Let f be as in Theorem 345, the following inequality holds true for $q=1$*

$$\begin{aligned}&\left|\frac{2A\left(a^r,b^r\right)L\left(a^2,b^2\right)-L\left(a^{r+2},b^{r+2}\right)}{\left[G\left(a,b\right)\right]^2}\right.\\&\left.-\left[G\left(a,b\right)\right]^2L\left(a^{r-2},b^{r-2}\right)+2L\left(a^r,b^r\right)-2A\left(a^r,b^r\right)\right|\\&\leq\frac{\left|r\right|\left(b-a\right)^2}{4G\left(a,b\right)L\left(a,b\right)}\left\{L\left(\sqrt{a},\sqrt{b}\right)A\left(a^{r-1},b^{r-1}\right)\right.\\&\times\left[2A\left(\sqrt{a},\sqrt{b}\right)-L\left(\sqrt{a},\sqrt{b}\right)\right]+2\left(r-1\right)\left(L\left(\sqrt{a},\sqrt{b}\right)\right.\\&\left.\left.\times\left[L\left(\sqrt{a},\sqrt{b}\right)-2A\left(\sqrt{a},\sqrt{b}\right)\right]+A\left(a,b\right)\right)L\left(a^{r-1},b^{r-1}\right)\right\}.\qquad(8.63)\end{aligned}$$

Corollary 348 *If we take $r=-1$ in Corollary 347, then the following inequality holds valid*

$$\begin{aligned}&\left|\frac{2L\left(a^2,b^2\right)\left[H\left(a,b\right)\right]^{-1}+L\left(a,b\right)}{\left[G\left(a,b\right)\right]^2}-\left[G\left(a,b\right)\right]^2L\left(a^{-3},b^{-3}\right)-2\left[H\left(a,b\right)\right]^{-1}\right|\\&\leq\frac{\left(b-a\right)^2}{4G\left(a,b\right)L\left(a,b\right)}\left\{L\left(\sqrt{a},\sqrt{b}\right)A\left(a^{-2},b^{-2}\right)\right.\\&\times\left[2A\left(\sqrt{a},\sqrt{b}\right)-L\left(\sqrt{a},\sqrt{b}\right)\right]-4\left(L\left(\sqrt{a},\sqrt{b}\right)\right.\\&\left.\left.\times\left[2A\left(\sqrt{a},\sqrt{b}\right)-L\left(\sqrt{a},\sqrt{b}\right)\right]-A\left(a,b\right)\right)L\left(a^{-2},b^{-2}\right)\right\}.\qquad(8.64)\end{aligned}$$

Theorem 349 *Let $0 < a < b$, $r \in \mathbb{R} \backslash \{-2, 0, 1, 2\}$ and $q > 1$. Then*

$$\begin{aligned}
&\left| \frac{2A(a^r, b^r) L(a^2, b^2) - L(a^{r+2}, b^{r+2})}{[G(a,b)]^2} \right. \\
&\qquad \left. - [G(a,b)]^2 L\left(a^{r-2}, b^{r-2}\right) + 2L(a^r, b^r) - 2A(a^r, b^r) \right| \\
&\leq \frac{(b-a)^{2-1/q} |r|}{4 \cdot q^{1/q} G(a,b) [L(a,b)]^{1-1/q}} \left(\frac{2q-1}{q-1}\right)^{1-\frac{1}{q}} \left\{ b^{1/2} \left(\left[L\left(a^{q/2}, b^{q/2}\right) - a^{q/2} \right] a^{q(r-1)} \right. \right. \\
&\qquad + \left. \left[2b^{q/2} - a^{q/2} - L\left(a^{q/2}, b^{q/2}\right) \right] b^{q(r-1)} \right)^{1/q} \\
&\qquad + a^{1/2} \left(\left[L\left(a^{q/2}, b^{q/2}\right) + b^{q/2} - 2a^{q/2} \right] a^{q(r-1)} \right. \\
&\qquad \left. \left. + \left[b^{q/2} - L\left(a^{q/2}, b^{q/2}\right) \right] b^{q(r-1)} \right)^{1/q} \right\}. \qquad (8.65)
\end{aligned}$$

Proof. Applying Theorem 328 to the functions

$$f(x) = x^r \text{ for } x > 0, r \in \mathbb{R} \backslash \{-2, 0, 1, 2\}$$

and

$$g(x) = \left(\frac{x}{\sqrt{ab}} - \frac{\sqrt{ab}}{x} \right)^2, x \in [a, b]$$

we get the desired result. ■

Corollary 350 *Suppose the assumptions of Theorem 349 are fulfilled and if $r = -1$, the following inequality holds true*

$$\begin{aligned}
&\left| \frac{2[H(a,b)]^{-1} L(a^2, b^2) + L(a,b)}{[G(a,b)]^2} - [G(a,b)]^2 L\left(a^{-3}, b^{-3}\right) - 2[H(a,b)]^{-1} \right| \\
&\leq \frac{(b-a)^{2-1/q}}{4 \cdot q^{1/q} G(a,b) [L(a,b)]^{1-1/q}} \left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} \left\{ b^{1/2} \left(\left[L\left(a^{q/2}, b^{q/2}\right) - a^{q/2} \right] a^{-2q} \right. \right. \\
&\qquad + \left. \left[2b^{q/2} - a^{q/2} - L\left(a^{q/2}, b^{q/2}\right) \right] b^{-2q} \right)^{1/q} \\
&\qquad + a^{1/2} \left(\left[L\left(a^{q/2}, b^{q/2}\right) + b^{q/2} - 2a^{q/2} \right] a^{-2q} \right. \\
&\qquad \left. \left. + \left[b^{q/2} - L\left(a^{q/2}, b^{q/2}\right) \right] b^{-2q} \right)^{1/q} \right\}. \qquad (8.66)
\end{aligned}$$

Theorem 351 *Let* $0 < a < b$, $r \in \mathbb{R} \backslash \{-2, 0, 1, 2\}$ *and* $q > 1$. *Then*

$$\begin{aligned}
&\left| \frac{2A(a^r, b^r) L(a^2, b^2) - L(a^{r+2}, b^{r+2})}{[G(a,b)]^2} \right. \\
&\qquad \left. - [G(a,b)]^2 L\left(a^{r-2}, b^{r-2}\right) + 2L(a^r, b^r) - 2A(a^r, b^r) \right| \\
&\qquad\qquad \leq \frac{|r|(b-a)^2}{2^{2/q+1} G(a,b) L(a,b)} \left(\frac{q-1}{2q-1} \right)^{1-\frac{1}{q}} \\
&\qquad\qquad \times \left[rL(a^{qr}, b^{qr}) - (r-1) L\left(a^{q(r-1)}, b^{q(r-1)}\right) L(a^q, b^q) \right]^{1/q}. \quad (8.67)
\end{aligned}$$

Proof. Applying Theorem 330 to the functions

$$f(x) = x^r \text{ for } x > 0, r \in \mathbb{R} \backslash \{-2, 0, 1, 2\}$$

and

$$g(x) = \left(\frac{x}{\sqrt{ab}} - \frac{\sqrt{ab}}{x} \right)^2, x \in [a, b]$$

we get the desired result. ■

Corollary 352 *Suppose the assumptions of Theorem 351 are satisfied and if* $r = -1$, *then the following inequality holds valid*

$$\begin{aligned}
&\left| \frac{2[H(a,b)]^{-1} L(a^2, b^2) + L(a,b)}{[G(a,b)]^2} - [G(a,b)]^2 L\left(a^{-3}, b^{-3}\right) - 2[H(a,b)]^{-1} \right| \\
&\qquad\qquad \leq \frac{(b-a)^2}{2^{2/q+1} G(a,b) L(a,b)} \left(\frac{q-1}{2q-1} \right)^{1-\frac{1}{q}} \\
&\qquad\qquad\qquad \times \left[2L\left(a^{-2q}, b^{-2q}\right) L\left(a^{-1}, b^{-1}\right) - L\left(a^{-q}, b^{-q}\right) \right]^{1/q}. \quad (8.68)
\end{aligned}$$

Applications of our results of Section 8.3 are given in the following theorems.

Theorem 353 *Let $0 < a < b$, $r \in \mathbb{R} \backslash \{-2, 0, 1, 2\}$ and $q \geq 1$. Then*

$$\begin{aligned}
&\Big| 2G^{r-2}(a,b)\, L\left(a^2, b^2\right) + 2L\left(a^r, b^r\right) - 2G^r(a,b) \\
&\qquad - G^2(a,b)\, L\left(a^{r-2}, b^{r-2}\right) + \frac{L\left(a^{r+2}, b^{r+2}\right)}{G^2(a,b)} \Big| \\
&\leq \frac{(b-a)|r|}{2G(a,b)} \left\{ \left[\sqrt{b} L\left(\sqrt{a}, \sqrt{b}\right) - G(a,b)\right]^{1-1/q} \right. \\
&\times \left(\left[\frac{4\sqrt{b} L\left(\sqrt{a}, \sqrt{b}\right) - 4b}{\ln b - \ln a} - G(a,b) + 2b \right] A\left(a^{q(r-1)}, b^{q(r-1)}\right) \right. \\
&\left. - \left[b - \sqrt{b} L\left(\sqrt{a}, \sqrt{b}\right)\right] a^{q(r-1)} \right)^{1/q} + \left[G(a,b) - \sqrt{a} L\left(\sqrt{a}, \sqrt{b}\right)\right]^{1-1/q} \\
&\times \left(\left[\frac{4\sqrt{a} L\left(\sqrt{a}, \sqrt{b}\right) - 4a}{\ln b - \ln a} + G(a,b) - 2a \right] A\left(a^{q(r-1)}, b^{q(r-1)}\right) \right. \\
&\left. \left. + \left[a - \sqrt{a} L\left(\sqrt{a}, \sqrt{b}\right)\right] b^{q(r-1)} \right)^{1/q} \right\}. \qquad (8.69)
\end{aligned}$$

Proof. Applying Theorem 335 to the functions

$$f(x) = x^r \text{ for } x > 0,\ r \in \mathbb{R} \backslash \{-2, 0, 1, 2\}$$

and

$$g(x) = \left(\frac{x}{\sqrt{ab}} - \frac{\sqrt{ab}}{x} \right)^2, x \in [a, b]$$

we get the desired result. ■

Corollary 354 *Suppose the assumptions of Theorem 335 are satisfied and if $r = -1$,* *then the following inequality holds*

$$\begin{aligned}
&\left|\frac{2L\left(a^{2},b^{2}\right)-2G^{2}\left(a,b\right)+L\left(a,b\right)G\left(a,b\right)}{G^{3}\left(a,b\right)}-G^{2}\left(a,b\right)L\left(a^{-3},b^{-3}\right)+2L\left(a^{-1},b^{-1}\right)\right| \\
&\leq\frac{\left(b-a\right)}{2G\left(a,b\right)}\left\{\left[\sqrt{b}L\left(\sqrt{a},\sqrt{b}\right)-G\left(a,b\right)\right]^{1-1/q}\right. \\
&\times\left(\left[\frac{4\sqrt{b}L\left(\sqrt{a},\sqrt{b}\right)-4b}{\ln b-\ln a}-G\left(a,b\right)+2b\right]A\left(a^{-2q},b^{-2q}\right)\right. \\
&\left.-\left[b-\sqrt{b}L\left(\sqrt{a},\sqrt{b}\right)\right]a^{-2q}\right)^{1/q}+\left[G\left(a,b\right)-\sqrt{a}L\left(\sqrt{a},\sqrt{b}\right)\right]^{1-1/q} \\
&\times\left(\left[\frac{4\sqrt{a}L\left(\sqrt{a},\sqrt{b}\right)-4a}{\ln b-\ln a}+G\left(a,b\right)-2a\right]A\left(a^{-2q},b^{-2q}\right)\right. \\
&\left.\left.+\left[a-\sqrt{a}L\left(\sqrt{a},\sqrt{b}\right)\right]b^{-2q}\right)^{1/q}\right\}. \qquad (8.70)
\end{aligned}$$

Corollary 355 *Under the assumptions of Theorem 335, then the following inequality holds true for $q=1$*

$$\begin{aligned}
&\left|2G^{r-2}\left(a,b\right)L\left(a^{2},b^{2}\right)+2L\left(a^{r},b^{r}\right)-2G^{r}\left(a,b\right)\right. \\
&\left.-G^{2}\left(a,b\right)L\left(a^{r-2},b^{r-2}\right)+\frac{L\left(a^{r+2},b^{r+2}\right)}{G^{2}\left(a,b\right)}\right| \\
&\leq\frac{\left(b-a\right)\left|r\right|}{2G\left(a,b\right)}\left[\frac{8A\left(\sqrt{a},\sqrt{b}\right)L\left(\sqrt{a},\sqrt{b}\right)-8A\left(a,b\right)}{\ln b-\ln a}+2\left(b-a\right)\right. \\
&\left.\left(r-2\right)\left(b-a\right)G^{2}\left(a,b\right)L_{r-3}^{r-3}\left(a,b\right)-\left(r-\frac{3}{2}\right)\left(b-a\right)G\left(a,b\right)L_{r-\frac{5}{2}}^{r-\frac{5}{2}}\left(a,b\right)\right]. \qquad (8.71)
\end{aligned}$$

Corollary 356 *If we take $r=-1$ in Corollary 355, then the following inequality holds* *valid*

$$\left|\frac{2L\left(a^2,b^2\right)-2G^2\left(a,b\right)+L\left(a,b\right)G\left(a,b\right)}{G^3\left(a,b\right)}-G^2\left(a,b\right)L\left(a^{-3},b^{-3}\right)+2L\left(a^{-1},b^{-1}\right)\right|$$
$$\leq\frac{(b-a)}{2}\left[\frac{8A\left(\sqrt{a},\sqrt{b}\right)L\left(\sqrt{a},\sqrt{b}\right)-8A\left(a,b\right)}{\left(\ln b-\ln a\right)G\left(a,b\right)}+\frac{2\left(b-a\right)}{G\left(a,b\right)}\right.$$
$$\left.-3\left(b-a\right)G\left(a,b\right)L_{-4}^{-4}\left(a,b\right)+\frac{5}{2}\left(b-a\right)L_{\frac{7}{2}}^{\frac{7}{2}}\left(a,b\right)\right].\quad(8.72)$$

Theorem 357 *Let* $0<a<b$*,* $r\in\mathbb{R}\backslash\{-2,0,1,2\}$ *and* $q>1$*. Then*

$$\left|2G^{r-2}\left(a,b\right)L\left(a^2,b^2\right)+2L\left(a^r,b^r\right)-2G^r\left(a,b\right)\right.$$
$$\left.-G^2\left(a,b\right)L\left(a^{r-2},b^{r-2}\right)+\frac{L\left(a^{r+2},b^{r+2}\right)}{G^2\left(a,b\right)}\right|$$
$$\leq\frac{\left(\ln b-\ln a\right)^{1-1/q}\left(b-a\right)|r|}{4\cdot q^{1/q}G\left(a,b\right)}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left\{b^{1/2}\left(\left[L\left(a^{q/2},b^{q/2}\right)-a^{q/2}\right]a^{q(r-1)}\right.\right.$$
$$\left.+\left[2b^{q/2}-a^{q/2}-L\left(a^{q/2},b^{q/2}\right)\right]b^{q(r-1)}\right)^{1/q}$$
$$+a^{1/2}\left(\left[L\left(a^{q/2},b^{q/2}\right)+b^{q/2}-2a^{q/2}\right]a^{q(r-1)}\right.$$
$$\left.\left.+\left[b^{q/2}-L\left(a^{q/2},b^{q/2}\right)\right]b^{q(r-1)}\right)^{1/q}\right\}.\quad(8.73)$$

Proof. Applying Theorem 339 to the functions

$$f(x)=x^r\text{ for }x>0,r\in\mathbb{R}\backslash\{-2,0,1,2\}$$

and

$$g\left(x\right)=\left(\frac{x}{\sqrt{ab}}-\frac{\sqrt{ab}}{x}\right)^2,x\in[a,b]$$

we get the desired result. ■

Corollary 358 *Suppose the assumptions of Theorem 357 are fulfilled and if* $r=-1$*,* *then the following inequality holds true*

$$\left|\frac{2L\left(a^{2},b^{2}\right)-2G^{2}\left(a,b\right)+L\left(a,b\right)G\left(a,b\right)}{G^{3}\left(a,b\right)}-G^{2}\left(a,b\right)L\left(a^{-3},b^{-3}\right)+2L\left(a^{-1},b^{-1}\right)\right|$$
$$\leq\frac{\left(\ln b-\ln a\right)^{1-1/q}\left(b-a\right)}{4\cdot q^{1/q}G\left(a,b\right)}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left\{b^{1/2}\left(\left[L\left(a^{q/2},b^{q/2}\right)-a^{q/2}\right]a^{-2q}\right.\right.$$
$$\left.+\left[2b^{q/2}-a^{q/2}-L\left(a^{q/2},b^{q/2}\right)\right]b^{-2q}\right)^{1/q}$$
$$\left.+a^{1/2}\left(\left[L\left(a^{q/2},b^{q/2}\right)+b^{q/2}-2a^{q/2}\right]a^{-2q}+\left[b^{q/2}-L\left(a^{q/2},b^{q/2}\right)\right]b^{-2q}\right)^{1/q}\right\}. \quad (8.74)$$

Theorem 359 *Let* $0<a<b$*,* $r\in\mathbb{R}\backslash\{-2,0,1,2\}$ *and* $q>1$*. Then*

$$\left|2G^{r-2}\left(a,b\right)L\left(a^{2},b^{2}\right)+2L\left(a^{r},b^{r}\right)-2G^{r}\left(a,b\right)\right.$$
$$\left.-G^{2}\left(a,b\right)L\left(a^{r-2},b^{r-2}\right)+\frac{L\left(a^{r+2},b^{r+2}\right)}{G^{2}\left(a,b\right)}\right|\leq\frac{\left(b-a\right)^{2}\left|r\right|}{2^{2/q+1}L\left(a,b\right)G\left(a,b\right)}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}$$
$$\times\left\{rL\left(a^{qr},b^{qr}\right)-\left(r-1\right)L\left(a^{q(r-1)},b^{q(r-1)}\right)L\left(a^{q},b^{q}\right)\right\}^{1/q}. \quad (8.75)$$

Proof. Applying Theorem 341 to the functions

$$f(x)=x^{r}\text{ for }x>0,r\in\mathbb{R}\backslash\{-2,0,1,2\}$$

and

$$g\left(x\right)=\left(\frac{x}{\sqrt{ab}}-\frac{\sqrt{ab}}{x}\right)^{2},x\in\left[a,b\right]$$

we get the desired result. ■

Corollary 360 *Suppose the assumptions of Theorem 359 are satisfied and if* $r=-1$*,* *then the following inequality holds valid*

$$\left|\frac{2L\left(a^{2},b^{2}\right)-2G^{2}\left(a,b\right)+L\left(a,b\right)G\left(a,b\right)}{G^{3}\left(a,b\right)}-G^{2}\left(a,b\right)L\left(a^{-3},b^{-3}\right)+2L\left(a^{-1},b^{-1}\right)\right|$$
$$\leq\frac{\left(b-a\right)^{2}}{2^{2/q+1}G\left(a,b\right)L\left(a,b\right)}\left(\frac{2q-1}{q-1}\right)^{1-\frac{1}{q}}$$
$$\times\left[2L\left(a^{-2q},b^{-2q}\right)L\left(a^{-1},b^{-1}\right)-L\left(a^{-q},b^{-q}\right)\right]^{1/q}. \quad (8.76)$$

Chapter 9

Concluding Remarks and Future Work

9.1 Conclusion

Since the discovery of the Hermite-Hadamard inequality (2.19), many mathematicians are trying to refine, generalize or to prove numerous forms of this inequality (these types of inequalities are known as Hermite-Hadamard type inequalities) in different settings by using various types of generalizations of the usual convexity. This dissertation is, therefore, a step towards this direction. In Chapter 3, we have derived some inequalities of Hermite Hadamard type by establishing a more general identity for functions that possesses nth derivative on interior of an interval of real numbers, by using the Hölder's inequality and the assumptions that the mappings $\left|f^{(n)}\right|^q$, $q \geq 1$ are convex, concave or s-concave. The results of Chapter 3, certainly, provide refinements of those results proved in [51] and [70], and we can get many interesting results for the cases when $n = 1$ and $n = 2$.

In a similar fashion, we have discussed Hermite-Hadamard type inequalities for n-times differentiable s-logarithmic, m-logarithmic and (α, m)-logarithmic convex functions in Chapter 4 and Chapter 5, which have extended and generalized several results related to this topic from [18] and [168]. It is interesting to that in Chapter 4, we have evaluated the integrals of the forms $\int_0^1 t^n \mu^t dt$, $\int_0^1 (1-t)^n \mu^t dt$, $\int_0^{\frac{1}{2}} t^n \mu^t dt$ and $\int_{\frac{1}{2}}^1 (1-t)^n \mu^t dt$, $\mu > 0$ and $n \in \mathbb{N}$, which helped us obtaining the results analytically. We have also acquired some corrected inequalities from our results, in particular for $n = 1$ as a special case.

It has been remarked in Chapter 2 that the preinvexity has been seen to be a significant generalizations of the usual convexity. The further generalizations of preinvexity is the h-preinvexity, hence the results of Chapter 6 contains more general weighted Hermite-Hadamard type and Fejér type inequalities. Indeed, the results provide noteworthy generalizations of the results given in [53] and [143].

Chapter 7 describes several extensions and broader view of those Hermite-Hadamard type inequalities proved in [70, 146] and [162] for convex, s-convex, s-preinvex, and log-preinvex functions. Through our findings, we have obtained some corrected Hermite-Hadamard type inequalities established for log-preinvex functions.

Chapter 8 is for Fejér type results using geometric-arithmetic convexity and geometric symmetry, which give weighted generalizations and refinements of a number of results recently proven in this field of study. Applications of our several results to trapezoidal rule and to special means of real numbers have also been given in each chapter of this dissertation.

9.2 Future Work

Most recently, mathematicians are trying to obtain q-analogues of mathematical objects by using q-calculus. In latest articles, Noor et al. [123], Sudsutad et al. [150] and Tariboon and Ntouyas [152] have attempted to prove some integral inequalities using quantum calculus.

The following q-analogue of the inequality (2.19) is proved by Tariboon and Ntouyas.

Theorem 361 *[152] Let $f : J = [a, b] \to \mathbb{R}$ be a continuous function on J and $0 < q < 1$.*

Then we have

$$f\left(A\left(a,b\right)\right)\leq\frac{1}{b-a}\int_{a}^{b}f\left(t\right)\ {}_{a}d_{q}t\leq\frac{qf\left(a\right)+f\left(b\right)}{1+q}. \tag{9.1}$$

Noor et al. [123] and Sudsutad et al. [150] have proved the following result independently.

Lemma 362 *Let $f:J:=[a,b]\rightarrow\mathbb{R}$ be a continuous function and $0<q<1$. If ${}_{a}D_{q}f$ is an integrable function on J° (the interior of J), then the following equality holds:*

$$\begin{aligned}&\frac{1}{b-a}\int_{a}^{b}f\left(s\right)\ {}_{a}d_{q}s-\frac{qf\left(a\right)+f\left(b\right)}{1+q}\\&\qquad=\frac{q\left(b-a\right)}{1+q}\int_{0}^{1}\left(1-\left(1+q\right)s\right)_{a}D_{q}f\left(sb+\left(1-s\right)a\right)\ {}_{0}d_{q}s. \end{aligned}\tag{9.2}$$

By using Lemma 362, the following quantum estimates for the right-side of (9.1) are proved in [123] and [150] by assuming that the q-derivative $\left|{}_{a}D_{q}f\right|^{r}$, $r\geq1$ is convex and quasi-convex.

Theorem 363 *[123, 150] Let $f:J=[a,b]\rightarrow\mathbb{R}$ be a q-differentiable function on J° with ${}_{a}D_{q}f$ be continuous and integrable on J, where $0<q<1$. If $\left|{}_{a}D_{q}f\right|^{r}$, $r\geq1$ is convex function, then*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_{a}^{b}f\left(t\right)\ {}_{a}d_{q}t-\frac{qf\left(a\right)+f\left(b\right)}{1+q}\right|\leq\frac{q\left(b-a\right)}{1+q}\left(\frac{q\left(2+q+q^{3}\right)}{\left(1+q\right)^{3}}\right)^{1-\frac{1}{r}}\\&\quad\times\left[\frac{q\left(1+4q+q^{2}\right)}{\left(1+q+q^{2}\right)\left(1+q\right)^{3}}\left|{}_{a}D_{q}f\left(a\right)\right|^{r}+\frac{q\left(1+3q^{2}+2q^{3}\right)}{\left(1+q+q^{2}\right)\left(1+q\right)^{3}}\left|{}_{a}D_{q}f\left(b\right)\right|^{r}\right]^{\frac{1}{r}}. \end{aligned}\tag{9.3}$$

Theorem 364 *[123] Let $f:J=[a,b]\rightarrow\mathbb{R}$ be a q-differentiable function on J° with ${}_{a}D_{q}f$ be continuous and integrable on J, where $0<q<1$. If $\left|{}_{a}D_{q}f\right|^{r}$, $r\geq1$ is convex* *function, where* $p,\ r>1$, $\frac{1}{p}+\frac{1}{r}=1$, *then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\ {}_ad_qt-\frac{qf(a)+f(b)}{1+q}\right|\leq\frac{q(b-a)}{1+q}\left(\frac{q(2+q+q^3)}{(1+q)^3}\right)^{\frac{1}{p}}$$
$$\times\left[\frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3}\left|{}_aD_qf(a)\right|^r+\frac{q(1+3q^2+2q^3)}{(1+q+q^2)(1+q)^3}\left|{}_aD_qf(b)\right|^r\right]^{\frac{1}{r}}. \quad (9.4)$$

Theorem 365 *[123] Let* $f:J=[a,b]\to\mathbb{R}$ *be a* q*-differentiable function on* J° *with* ${}_aD_qf$ *be continuous and integrable on* J*, where* $0<q<1$*. If* $\left|{}_aD_qf\right|^r$, $r\geq 1$ *is quasi convex function, where* $p,\ r>1$, $\frac{1}{p}+\frac{1}{r}=1$, *then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\ {}_ad_qt-\frac{qf(a)+f(b)}{1+q}\right|$$
$$\leq\frac{q(b-a)}{1+q}\left(\frac{q(2+q+q^3)}{(1+q)^3}\right)^{\frac{1}{p}}\left(\frac{q(2+q+q^3)}{(1+q)^3}\sup\left\{\left|{}_aD_qf(a)\right|,\left|{}_aD_qf(b)\right|\right\}\right)^{\frac{1}{r}}. \quad (9.5)$$

Theorem 366 *[123] Let* $f:J=[a,b]\to\mathbb{R}$ *be a* q*-differentiable function on* J° *with* ${}_aD_qf$ *be continuous and integrable on* J*, where* $0<q<1$*. If* $\left|{}_aD_qf\right|^r$, $r\geq 1$ *is quasi convex function, where* $p,\ r>1$, *then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\ {}_ad_qt-\frac{qf(a)+f(b)}{1+q}\right|$$
$$\leq\frac{q^2(b-a)(2+q+q^3)}{(1+q)^4}\left(\sup\left\{\left|{}_aD_qf(a)\right|,\left|{}_aD_qf(b)\right|\right\}\right)^{\frac{1}{r}}. \quad (9.6)$$

Similar results with the assumption of preinvexity on the q-derivative $\left|{}_aD_qf\right|^r$, $r\geq 1$ can be found in [124]. The readers are suggested to see [150] and [152] for further results on q-integral inequalities for convex functions. Another interesting area which has not developed so far is that of establishing Hermite-Hadamard and Fejér type inequalities using harmonic convexity. Moreover, the results presented in Chapter 8 of this thesis can also be extended to functions of two or more variables. These areas of research have not so much expanded yet, we therefore, encourage the researchers to go into details to do further research in these areas.

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