

https://dspace.org/

Branefused black hole operators

Robert De Mello Koch; Minkyoo Kim; Seok Kim; Jehyun Lee; Siyul Lee

SPRINGER
https://hdl.handle.net/10539/46014
WOS

Downloaded from a DSpace repository. https://dspace.org/

Published for SISSA by Springer

Received: April 17, 2025
Accepted: June 16, 2025
Published: July 22, 2025

Brane-fused black hole operators

Robert de Mello Koch[a,b] **Minkyoo Kim**[c] **Seok Kim**[d] **Jehyun Lee**[d] **and Siyul Lee**[e]

[a] *School of Science, Huzhou University, Huzhou 313000, China*
[b] *School of Physics and Mandelstam Institute for Theoretical Physics, University of the Witwatersrand, Wits, 2050, South Africa*
[c] *Center for Quantum Spacetime (CQUeST), Sogang University, Seoul 04107, Korea*
[d] *Department of Physics and Astronomy & Center for Theoretical Physics, Seoul National University, Seoul 08826, Korea*
[e] *Instituut voor Theoretische Fysica, KU Leuven, Celestijnenlaan 200D, 3001 Leuven, Belgium*

E-mail: robert@zjhu.edu.cn, mimkim80@gmail.com, seokkimseok@gmail.com, ljs9125@snu.ac.kr, siyul.lee@kuleuven.be

Abstract: We construct infinitely many new $\frac{1}{16}$-BPS cohomologies of the 4d maximal super-Yang-Mills theory and interpret them as a black hole wrapped by dual giant graviton hairs. Since the black hole inside a dual giant feels the RR 5-form flux reduced by one unit, its microstate should essentially be an SU$(N-1)$ cohomology. However, due to the fortuitous nature of the black hole microstates, promoting an SU$(N-1)$ black hole state to SU(N) generally fails to yield a cohomology. We show at $N = 3$ that suitable fusion products with the dual giants yield cohomologies. The core black hole size is probed by the minimal size of the dual giant which can wrap it. We also discuss two types of large black hole hairs: large conformal descendants of gravitons and large dual giants. We prove that any SU(N) black hole cohomology admits infinitely many hairs of the first type.

Keywords: AdS-CFT Correspondence, Black Holes in String Theory, Supersymmetric Gauge Theory

ArXiv ePrint: 2412.08695

Open Access, © The Authors.
Article funded by SCOAP^3.

https://doi.org/10.1007/JHEP07(2025)216

Contents

1 Introduction **1**

2 Black hole cohomologies with dual giant hairs **4**
- 2.1 General methods for the construction 7
- 2.2 A tower of dual giant hairs 10
- 2.3 No cohomologies below the threshold of the tower 18

3 More hairy black hole cohomologies **19**

4 General comments on two types of hairs **23**

5 Conclusion and remarks **28**

A Trace relations of SU(3) gravitons **29**
- A.1 The fundamental trace relations 29
- A.2 The bases for the non-fundamental trace relations 32

1 Introduction

Recently, there have been developments in constructing cohomologies for the $\frac{1}{16}$-BPS local operators of the $\mathcal{N} = 4$ Yang-Mills theory [1–9]. A goal of this program is to better understand the microstates of BPS black holes [10–13] from the CFT dual. While the correct microstate structure will be visible in the strongly coupled QFT, the studies have been focused on the 1-loop BPS states expecting non-renormalization of their spectrum [1, 14]. Also, while the semi-classical gravity emerges at large N, progress has been made in the 'quantum' regimes at low N. Somewhat surprisingly, certain spectral aspects of the black holes reminiscent of the BPS black hole hairs were observed even at small N [3].

In this paper, we construct the representatives of new black hole cohomologies in the maximal super-Yang-Mills theory with the gauge group SU(3) and provide their 'gravity dual' interpretation. Our interpretation is related to the novel AdS black holes [15] (see also [16]) with the dual giant graviton hairs [17–19]. As we explain below, $N = 3$ is the minimal value which admits such a hair structure in the local microstate operators. Although this work and [15] discuss the opposite extremes, $N = 3$ vs. ∞, their qualitative similarities lead us to suspect that similar cohomologies exist at all N and interpolate the two.

The $\frac{1}{16}$-BPS operators O are annihilated by a pair of mutually Hermitian conjugate supercharges Q and $Q^\dagger$: $[Q, O\} = 0$, $[Q^\dagger, O\} = 0$. Since these supercharges are nilpotent, the BPS operators are in 1-to-1 correspondence to the cohomology classes of Q. The cohomology classes are defined by Q-closed operators with the identification of two such operators differing by a Q-exact term: $O \sim O + [Q, \Lambda\}$. So a representative of a cohomology class may differ from the BPS operator by a Q-exact term: it only gives the information on the spectrum of BPS

JHEP07(2025)216

operators. In this paper, we consider the cohomologies with the supercharge Q of the classical interacting theory, which are in 1-to-1 map to the BPS states of the 1-loop Hamiltonian.

One should distinguish the two types of cohomologies: the graviton and the black hole types. We follow the recent characterizations [9] for the $\mathcal{N}=4$ Yang-Mills theory.[1] Consider an infinite sequence $\{O_N\}$ of gauge invariant operators defined by fixing the 'shape' but taking the fields appearing in the operator to be $N\times N$ traceless matrices. (The operator shape is fixed by specifying the fields and how they enter the multi-trace structure, like $\mathrm{tr}(XY)\mathrm{tr}(Z\psi_1)$, $\mathrm{tr}(Z^2 f)$.) With the operator shape fixed, the classical scaling dimension as well as the spins and the R-charges are the same for all operators in the sequence. If O_N is Q-closed, so are $O_{N'}$ with $N'<N$. This is because the $N\times N$ matrix equation $[Q,O_N\}=0$ with the classical supercharge Q reduces to the $N'\times N'$ matrix equation $[Q,O_{N'}\}=0$ if one consistently turns off all the fields except the $N'\times N'$ block-diagonal elements. So if any operator in the sequence $\{O_N\}$ is Q-closed, the operators are Q-closed for all $N\leq N_*$ for some integer N_*. If $N_*=\infty$, i.e. if all the operators in the sequence are Q-closed, we say they represent graviton cohomologies. If a finite N_* exists, we say they represent the black hole cohomologies. The name graviton for the former type is clear: increasing N with the fixed shape, and thus fixed energy E, O_N will eventually describe BPS states at $E\ll N$ and should belong to the low energy supergraviton spectrum. O_N in this sequence at finite N (or $E\sim N$) represent generalizations of the supergravitons, the giant graviton states. On the other hand, calling the latter black holes is a somewhat rough notion at this moment. The latter type will certainly contain black hole states, while it is not clear whether there may be operators which neither deserve to be called gravitons nor black holes.[2] However, we shall use these intuitive terminologies in this paper. In [9], the former/latter types are called the monotone/fortuitous cohomologies, respectively.

The graviton cohomologies are Q-closed without using trace relations. Construction of the graviton cohomologies reduces to that of the *single trace* cohomologies, which are completely understood (e.g. appendix C.3 of [21]). The general graviton cohomologies can be represented by multiplying these single-trace cohomologies.[3]

On the other hand, the black hole cohomologies are Q-closed by trace relations, and there is no systematic way to generate all black hole cohomologies. Instead, we shall use the 'ansatz' devised in [8] to construct new Q-closed operators of the black hole type, using a class of known trace relations. One should also show that the constructed Q-closed operator is not Q-exact. There is no known general method for this other than to list all the Q-exact operators with given quantum numbers and show that none of their linear combinations reproduces the operator under consideration. It is in general a very difficult computation. However, in some non-generic cases one can identify a particular term in the operator which obstructs the Q-exactness, in which case it becomes easier to show that the operator does not belong to the trivial cohomology class. The new cohomologies constructed in this paper will naturally have such obstruction terms, as we explain below and in section 2.

[1] The criteria here apply only to the $\mathcal{N}=4$ theory with the SU(N) gauge group, without baryons. More intrinsically, the graviton and baryon states can be constructed by quantizing the (super-)moduli space.

[2] Although not in the microstate picture, conical defects contribute to the Euclidean gravitational path integral in AdS_3, which are neither gravitons nor black holes [20]. We thank Masamichi Miyaji for this comment.

[3] These multi-trace representatives are overcomplete. This is because, although they are Q-closed without using trace relations, some linear combinations can become Q-exact due to trace relations.

JHEP07(2025)216

Black holes in AdS may be dressed by the hairs outside the event horizon. Recently, two types of hairs were found for the charged rotating AdS black holes. For simplicity, consider the black holes in $AdS_5 \times S^5$ carrying two equal angular momenta $J \equiv J_1 = J_2$ and three equal electric charges $R \equiv R_1 = R_2 = R_3$. If J is too large, the black hole admits particle hairs that carry large angular momenta [22–24]. If R is too large, electrically charged hairs are admitted in the form of the large dual giant gravitons wrapping the core black hole [15]. Both types of hairy black holes may reach the BPS limit $E = 3R + 2J$, implying the existence of hairy BPS black holes and their cohomologies.

Since one can distinguish (to certain extent) the graviton/black hole type cohomologies at finite N as reviewed above, we can also try to identify the hairy cohomologies at finite N. The hairs of the first kind with large J have been noticed even for $N = 2$ [3]. It was proposed that such states belong to the cohomologies admitting product representatives. That is, the product of a black hole cohomology and a graviton cohomology represents another cohomology. This product often becomes Q-exact if the graviton part does not carry large enough J, while it represents a non-trivial new cohomology when the graviton carries large enough J. See section 4 for general discussions on such hairy cohomologies with large J.

In this paper, we study the cohomologies for BPS black holes wrapped by the dual giant graviton hairs. The physical aspects of these hairy microstates are quite different from those of the previous paragraph, e.g. they defy the product representatives as we explain now.

The dual giant graviton is a D3-brane extended in the global AdS_5. It backreacts to the background by reducing the RR flux by one unit inside it [19]. So any object inside the dual giant is essentially that of the SU$(N-1)$ gauge theory. If a black hole is wrapped by a dual giant graviton, its microstate should be morally that of the SU$(N-1)$ theory. Of course the whole system including the dual giant hair should be described covariantly in the SU(N) theory, but there should be a sense in which the shape of the SU$(N-1)$ core cohomology can be identified. However, due to the fortuitous nature of the black hole cohomology, SU$(N-1)$ black hole cohomologies fail to be the SU(N) cohomologies if $N_* = N - 1$ (which will be the case in our examples). That is, although the operator O_{N-1} satisfies $[Q, O_{N-1}\} = 0$ thanks to SU$(N-1)$ trace relations, the operator O_N of the same shape may not, i.e. $[Q, O_N\} \neq 0$. So the hairy cohomologies involving the dual giants do not admit simple product representatives. Instead, the gauge orientations of the dual giant part and the black hole part should be entangled for the entire operator to be Q-closed, forming a *fusion product*.

$N = 3$ is the minimal value for which the SU$(N-1)$ theory is non-trivial. We shall find Q-closed operators which are fusion products of the black hole O_3 and the dual giant graviton G_3 in the SU(3) theory. O_3 is the SU(3) uplift of an SU(2) black hole cohomology O_2. Unlike O_2, O_3 is not Q-closed in general. G_3 is a (dual giant) graviton cohomology of the SU(3) theory. The whole operator is made Q-closed by suitably entangling the gauge orientations of the two parts. To show that these operators are not Q-exact, we consider the terms which contain only the SU(2) $\times$ U(1) block-diagonal elements of the fields. Among those block-diagonal terms, we identify a term of the form $O_2 G_1$, a product of the SU(2) black hole cohomology O_2 and the U(1) graviton cohomology G_1. Since both O_2 and G_1 represent non-trivial cohomologies and also since the action of Q does not mix the SU(2) and U(1) blocks, this term is not Q-exact. This obstructs the full operator from being Q-exact. This proof also hints towards the bulk nature of the operator, the SU$(N-1)$ core inside the D-brane and the U(1) part on the brane.

JHEP07(2025)216

Field	R_m	R_{m+1}	R_{m+2}	J_1+J_2	J_1-J_2	j
ϕ^m	1	0	0	0	0	2
ψ_m	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	0	4
f	0	0	0	2	0	6

Table 1. Quantum numbers of the seven BPS elementary fields in the BMN subsector. $m = 1, 2, 3$ and the subscripts for R are written modulo 3.

We will show that these obstruction terms exist only when G_3 is larger than a critical size. We interpret this as the brane probing the size of the core black hole, i.e. the brane should be larger than the black hole to wrap it.

The fact that the cohomology remains non-trivial in the $\mathrm{SU}(N-1) \times \mathrm{U}(1)$ sector implies that the corresponding microstate is not a typical $\mathrm{SU}(N)$ black hole state, but essentially that of an $\mathrm{SU}(N-1)$ theory supplemented by the U(1) brane degrees of freedom. Typical black hole cohomologies would not have such obstruction terms within the block-diagonal elements.

The rest of this paper is organized as follows. In section 2, building on the ansatz of [8], we construct Q-closed operators in which the dual giant graviton part G_3 involves n scalars. For $n = 1, 2$, we show that no such operators contain obstruction terms against the Q-exactness whereas for all $n \geq 3$, we find non-trivial cohomologies. In section 3, we find more general hairy black hole cohomologies using this method. In section 4, we make general discussions on the two types of hairy cohomologies at arbitrary N, in particular proving the existence of infinitely many hairy cohomologies of the product form. We conclude in section 5 with remarks.

2 Black hole cohomologies with dual giant hairs

In this section, we construct novel black hole cohomologies in the 4d $\mathcal{N} = 4$ Yang-Mills theory with gauge group SU(3). They transform in the irreps $[n \geq 3, 0]$ of the $\mathrm{SU}(3)_R \subset \mathrm{SU}(4)_R$ R-symmetry, and we interpret these as the black hole wrapped by the dual giant hair. We start by clarifying notations before we present our strategy and results in subsequent subsections.

The $\mathcal{N} = 4$ Yang-Mills theory has 16 Poincaré supercharges Q^i_α, $\overline{Q}_{i\dot{\alpha}}$, where $i = 1, \cdots, 4$ is the fundamental index for the $\mathrm{SO}(6) \sim \mathrm{SU}(4)_R$ R-symmetry group and $\alpha = \pm$, $\dot{\alpha} = \dot{\pm}$ are doublet indices for the SO(4) rotation. It also has 16 conformal supercharges, which are Hermitian conjugate to the Poincaré supercharges in the radial quantization. We are interested in local operators annihilated by $Q \equiv Q^4_-$ and its conjugate $Q^\dagger$, thus $\frac{1}{16}$-BPS. Their scaling dimensions E saturate the BPS bound

$$E \geq R_1 + R_2 + R_3 + J_1 + J_2 \,, \tag{2.1}$$

where R_m are the three Cartans of the $\mathrm{SU}(4)_R$ and J_i are the two Cartans of the SO(4) rotation.

We shall consider a 1d truncation of this problem, which was referred to as the BMN subsector in [3]. (See also [1, 2], which considered the same truncation without explicitly relating it to the BMN matrix theory, and also [25] for further studies.) With this truncation and at the 1-loop level, BPS operators are constructed using seven elementary fields: 3 components ϕ^m of the scalar, 3 components ψ_m of the fermion, and one f of the field strength,

JHEP07(2025)216

j	F_0	F_1	F_2	F_3	F_4	F_{exc}	B_1	B_2	B_3	B_{exc}
24	$[0,0]$									
26										
28										
30	$[0,0]$	$[3,0]$								
32		$[4,0]$								
34		$[5,0]$					$[3,1]$			
36	$[0,0]$	$[6,0]$					$[4,1]$			$[3,0]$
38		$[7,0]$				$[1,0]$	$[5,1]$			
40		$[8,0]$	$[5,0]$		$[3,1]$		$[6,1]$			
42	$[0,0]$	$[9,0]$	$[6,0]$		$[4,1]$		$[7,1]$			$[1,1]$
44		$[10,0]$	$[7,0]$		$[5,1]$		$[8,1]$	$[5,1]$		
46		$[11,0]$	$[8,0]$		$[6,1]$	$[2,0]$	$[9,1]$	$[6,1]$		$[5,0]$
48	$[0,0]$	$[12,0]$	$[9,0]$		$[7,1]$	$[3,0]$	$[10,1]$	$[7,1]$		$[4,1]$
50		$[13,0]$	$[10,0]$	$[7,0]$	$[8,1]$		$[11,1]$	$[8,1]$		$[4,0]$
52		$[14,0]$	$[11,0]$	$[8,0]$	$[9,1]$	$[2,0]$	$[12,1]$	$[9,1]$		$[3,1]$
54		$[15,0]$	$[12,0]$	$[9,0]$	$[10,1]$	$[4,1]$	$[13,1]$	$[10,1]$	$[7,1]$	

Table 2. $\text{SU}(3)_R$ Dynkin labels of fermionic/bosonic black hole cohomologies in the SU(3) theory, after factoring out the SU(1|3) descendants and the conjectured graviton hairs.

as summarized in table 1. $m = 1, 2, 3$ is the (anti-)fundamental index for $\text{SU}(3)_R \subset \text{SU}(4)_R$ that commutes with Q and $R_1 + R_2 + R_3$. The charges of an operator are simply the sum of those of the constituent elementary fields. In table 1, we also listed the values of the charge

$$j \equiv 2(R_1 + R_2 + R_3) + 3(J_1 + J_2)\,. \tag{2.2}$$

We will often call it the order, originating from the practice of series expanding the index as $\mathcal{I}(t) = \sum_j a_j t^j$ with $j = 0, 1, 2, \cdots$.

As explained in the introduction, the BPS operators are in one-to-one correspondence with cohomology classes with respect to the supercharge Q. We shall consider the cohomologies at the 1-loop level, i.e. in the interacting classical field theory. Q acts on the elementary fields as

$$Q\phi^m = 0\,, \qquad Q\psi_m = -\frac{i}{2}\epsilon_{mnp}[\phi^n, \phi^p]\,, \qquad Qf = -i[\phi^m, \psi_m]\,. \tag{2.3}$$

The goal of this section is to find representatives of new cohomologies in the SU(3) gauge theory. Our constructions will be guided by the index of the SU(3) black hole cohomologies computed in [8]. The black hole cohomology index takes the form of

$$\mathcal{I} = \mathcal{I}_{s.c.d.} \cdot \mathcal{I}_{w_2} \cdot \sum_i (-1)^{F_i} \chi(\mathbf{R}_i)\; t^{j_i}\,. \tag{2.4}$$

The last factor stands for 'core' black hole primary cohomologies: they are grouped by irreducible representations $\mathbf{R}_i$ of $\text{SU}(3)_R$, algebraically represented by its character $\chi(\mathbf{R}_i)$, and graded according to their order j and their boson/fermion statistics. The first factor $\mathcal{I}_{s.c.d.}$ is the character of the SU(1|3) multiplet which factors out the contributions from the

JHEP07(2025)216

superconformal descendants. Moreover, it was assumed empirically that w_2 type gravitons (which will be defined shortly) can multiply the core black hole cohomologies to yield new black hole cohomologies, whose index is the second factor $\mathcal{I}_{w_2}$: see [8] for the details. Table 2 presents the black hole cohomology index (2.4) by listing the $\mathrm{SU}(3)_R$ irreps $\mathbf{R}_i$ for core black hole cohomologies for each order j, and for bosons (B) and fermions (F) separately. We grouped the states into series of towers along the columns of the table, and left out states that do not fit in any of the series to the exceptional columns F_{exc} and B_{exc}. Whereas the classification into towers and the factoring out of $\mathcal{I}_{w_2}$ have all been based on observation of patterns, results of this paper will justify and give physical interpretations to some of these structures.

In this section we discuss the tower F_1. For every $n \geq 3$, the index predicts fermionic cohomologies at the order

$$j \equiv 2(R_1 + R_2 + R_3) + 3(J_1 + J_2) = 24 + 2n\,, \tag{2.5}$$

in the $[n, 0]$ representation of $\mathrm{SU}(3)_R$. We would like to interpret these as the core SU(2) black hole cohomology with $j = 24$ wrapped by a dual giant graviton hair made of n scalar fields. This is a viable proposal at least according to the quantum numbers. The smallest black hole cohomology in the SU(2) theory has $j = 24$, more concretely $R_1 = R_2 = R_3 = \frac{3}{2}$, $J_1 = J_2 = \frac{5}{2}$ and it is a singlet under $\mathrm{SU}(3)_R$ [1]. The dual giant hair is made of n symmetrized scalars, therefore transforms in the $[n, 0]$ representation of $\mathrm{SU}(3)_R$ and carries quantum numbers $R_1 + R_2 + R_3 = n$, $J_1 = J_2 = 0$ and $j = 2n$. All quantum numbers of the SU(2) black hole cohomology and the dual giant hair add up to those of the tower F_1 in table 2.

The 'core' black hole cohomology in the SU(2) theory can be represented by [3]

$$\mathcal{O}^2_{24} = \epsilon^{pqr} v^m{}_q v^n{}_r \mathrm{tr}(\psi_{(m}\psi_n\psi_{p)})\,, \tag{2.6}$$

where $v^m{}_n = \mathrm{tr}(\phi^m\psi_n) - \frac{1}{3}\delta^m_n \mathrm{tr}(\phi^p\psi_p)$ is a single-trace graviton operator. Throughout this paper, we denote black hole cohomologies in the $\mathrm{SU}(N)$ theory at order j as $\mathcal{O}^N_j$. Note that $\mathcal{O}^2_{24}$ contains 2 scalars and 5 fermions, and no f. In the SU(2) theory, it is Q-closed by SU(2) trace relations. The right hand side of (2.6) also defines an operator in the SU(3) gauge theory (which we still denote by $\mathcal{O}^2_{24}$ although it is an SU(3) operator) in terms of the fields ϕ^m, ψ_m that are now 3×3 matrices. However, this operator is not Q-closed because the relevant SU(2) trace relations do not hold in the SU(3) theory. [5] computed the anomalous dimensions of operators in this sector at $N > 2$, indeed finding that all of them are non-BPS.

Instead, the hairy black hole cohomology in the SU(3) theory that we want to construct contains n scalars in addition to $\mathcal{O}^2_{24}$, so it contains $n+2$ scalars and 5 fermions. We also want it to be in the $[n, 0]$ representation of $\mathrm{SU}(3)_R$. At first sight, one may simply try multiplying $\mathcal{O}^2_{24}$ by some gauge-invariant graviton operator that involves n symmetrized scalars. However, such a product operator is not Q-closed. Since Q annihilates scalars, it only acts on the $\mathcal{O}^2_{24}$ part of the product. So Q acting on the product will be the same graviton operator times $Q\mathcal{O}^2_{24}$, which would typically be non-zero in the SU(3) theory. Therefore we need a more elaborate way to dissolve the n symmetrized scalars in the black hole operator $\mathcal{O}^2_{24}$. Roughly speaking, we want to insert n scalar matrices $\phi^{(a_1}, \phi^{a_2}, \cdots, \phi^{a_n)}$ suitably inside the multi-trace structure of $\mathcal{O}^2_{24}$ to form a fusion product.

JHEP07(2025)216

In the rest of this section, we implement this idea in a systematic framework to construct black hole cohomologies that correspond to $n \geq 3$, accounting for the tower F_1. We also show that it fails to detect any cohomology at $n = 1, 2$, as anticipated by the index.

2.1 General methods for the construction

In this subsection, we outline our method for constructing the hairy black hole cohomologies. We will first review the ansatz devised in [8] to construct Q-closed non-graviton operators. Then we will explain how some of the operators can be proved to be not Q-exact. This proof is inspired by the physical picture of the black hole wrapped by a dual giant, and applies for some operators that admit such an interpretation.

2.1.1 The ansatz for Q-closed operators

Recall the classification of the graviton versus the black hole cohomologies: an operator is of black hole type if it is Q-closed only by trace relations, which do not hold for arbitrarily large N. The ansatz devised in [8] leverages this property to construct a Q-closed operator from trace relations between graviton operators, which have been obtained as a byproduct of computing the finite N graviton cohomology index. This ansatz does not construct all black hole type Q-closed operators, but it has proved to be successful in constructing a useful class of such operators. In fact, the representative (2.6) of the smallest SU(2) black hole cohomology was found along this strategy, as well as the smallest SU(3) black hole cohomology [8].

Let us collectively denote the single trace graviton operators by $\{g_I\}$. This set has been classified in [21]: the symmetrized scalars $u_n \equiv \mathrm{tr}(\phi^{(i_1} \cdots \phi^{i_n)})$ and their PSU(1, 2|3) descendants. At finite N, only a subset of those with $n \leq N$ need to be used [3]. In the BMN subsector of the SU(3) theory, the complete list is given by

$$
\begin{aligned}
&u_2{}^{ij} \equiv \mathrm{tr}\left(\phi^{(i}\phi^{j)}\right), && u_3{}^{ijk} \equiv \mathrm{tr}\left(\phi^{(i}\phi^{j}\phi^{k)}\right), \\
&v_2{}^{i}{}_{j} \equiv \mathrm{tr}\left(\phi^{i}\psi_j\right) - \tfrac{1}{3}\delta^i_j \mathrm{tr}(\phi^a\psi_a), && v_3{}^{ij}{}_{k} \equiv \mathrm{tr}\left(\phi^{(i}\phi^{j)}\psi_k\right) - \tfrac{1}{4}\delta^i_k \mathrm{tr}\left(\phi^{(j}\phi^{a)}\psi_a\right) - \tfrac{1}{4}\delta^j_k \mathrm{tr}\left(\phi^{(i}\phi^{a)}\psi_a\right), \\
&w_2{}^{i} \equiv \mathrm{tr}\left(f\phi^i + \tfrac{1}{2}\epsilon^{ia_1a_2}\psi_{a_1}\psi_{a_2}\right), && w_3{}^{ij} \equiv \mathrm{tr}\left(f\phi^{(i}\phi^{j)} + \epsilon^{a_1a_2(i}\phi^{j)}\psi_{a_1}\psi_{a_2}\right).
\end{aligned}
\tag{2.7}
$$

We will often omit the subscripts 2 and 3 when it can be inferred from the number of indices.

The space of all graviton type operators is generated by the single trace gravitons (2.7) as their Fock space, i.e. all multi-graviton operators are polynomials of g_I's. However, these polynomials are redundant in that some polynomials are Q-exact, thus belong to trivial cohomologies. A Q-exact polynomial represents a cohomological relation between the monomials in g_I's. All the relations can be obtained from a finite number of fundamental relations which we write as $R_a(g_I) = Qr_a$, where $R_a(g_I)$ is a Q-exact polynomial in g_I and r_a is the corresponding local gauge-invariant operator. Each r_a is, unlike R_a, not a graviton type operator. There is an ambiguity in r_a of adding any Q-closed operator, but we make convenient choices of r_a as listed in appendix A. All our choices are such that r_a become zero under the restriction of the elementary fields into diagonal traceless matrices. Here note that the graviton cohomologies originate from the symmetrized scalars, so they are faithfully

JHEP07(2025)216

represented by the eigenvalues of the elementary fields [3, 8]. All r_a's being zero under this restriction illustrates that they are not graviton type.

The exhaustive list of the fundamental relations R_a between u_2, u_3, v_2 and v_3 (while excluding w_2 and w_3 because we do not need them in this work) is presented in (A.2) We also present in (A.3) r_a's corresponding to some of the R_a's that are used throughout this paper.

Each fundamental relation $R_a(g_I)$ can be multiplied by any polynomials of gravitons, say $f_a(g_I)$, resulting in non-fundamental relations at higher orders. Certain linear combinations of the non-fundamental relations may vanish identically, yielding relations of relations:

$$\sum_a f_a(g_I) R_a(g_I) = 0\,. \tag{2.8}$$

Whenever there is a relation of relations (2.8), one can consider the operator

$$\sum_a f_a(g_I) r_a\,, \tag{2.9}$$

where R_a's have been replaced with their corresponding r_a's such that $Qr_a = R_a$. Because all gravitons are annihilated by Q, Q acting on (2.9) is precisely (2.8), which equals zero. So (2.9) is an operator that is Q-closed by trace relations, and it is not a graviton type operator. Therefore, it serves as an ansatz for the black hole cohomologies.

2.1.2 A hairy proof of non-Q-exactness

To construct a nontrivial cohomology, it remains to prove that the black hole type Q-closed operator obtained using the ansatz is not Q-exact. In other words, we need to exclude Q-exact operators among those obtained by the ansatz, or combinations thereof. In general, this involves extremely heavy computations because there is no generally applicable method other than to rule out by brute force all Q-exact operators in the target charge sector. This is a huge linear problem that has been the computational bottleneck against finding more black hole cohomologies in [8]. However, for some operators that we present in this work, it can be proved relatively easily that they cannot be Q-exact.

As mentioned in the Introduction, the SU(3) black hole operator with dual giant hair that we seek after should contain a core SU(2) black hole in some sense. With this insight, consider restricting gauge orientations of all fields to be included in the $\mathrm{SU}(2)\times\mathrm{U}(1)$ subgroup of SU(3). That is, we write all elementary fields in the basis of Gell-Mann matrices and turn off $4,\cdots,7^{th}$ components. Equivalently, we write all fields in the $2+1$ block diagonal form, turning off the off-block-diagonal components. In addition, we also turn off the 8th component, or the 1×1 block, for the fermions and for f. That is,

$$\begin{aligned}
\phi^m &= \phi^m{}_1 T^1 + \phi^m{}_2 T^2 + \phi^m{}_3 T^3 + \sqrt{3}\hat{\phi}^m T^8\,,\\
\psi_m &= \psi_{m1} T^1 + \psi_{m2} T^2 + \psi_{m3} T^3\,,\\
f &= f_1 T^1 + f_2 T^2 + f_3 T^3\,,
\end{aligned} \tag{2.10}$$

or in matrix form,

$$\phi^m = \begin{pmatrix} \phi^m|_{\mathrm{SU}(2)} & 0 \\ 0 & 0 \end{pmatrix} + \hat{\phi}^m \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \psi_m = \begin{pmatrix} \psi_m|_{\mathrm{SU}(2)} & 0 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} f|_{\mathrm{SU}(2)} & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.11}$$

JHEP07(2025)216

Note that we denote the U(1) component with a hat, and we will often abuse notation by omitting the $|_{\mathrm{SU}(2)}$ sign. It should be clear that if a field appears in the same expression as the hatted letters, it indicates the SU(2) block. Although they are not needed for the present work, one may turn on the U(1) components $\hat{\psi}_m$ and $\hat{f}$ if needed.

Let $\mathcal{O}^3[R]$ be a gauge-invariant operator in the SU(3) theory in an irrep R of the $\mathrm{SU}(3)_R$ global symmetry. Restricting each field in the SU(2) × U(1) block diagonal form (2.10), $\mathcal{O}^3[R]$ is expanded into many terms that contain different numbers of the SU(2) components and the U(1) components. It can be understood as the SU(2) gauge-invariant operators with the abelian U(1) components acting as numerical coefficients. For instance, (note that $\mathrm{tr}[\phi^k]_{\mathrm{SU}(2)} = 0$)

$$\mathrm{tr}[\phi^{(i}\phi^j\phi^{k)}] \xrightarrow{\mathrm{SU}(2)\times\mathrm{U}(1)} \mathrm{tr}[\phi^{(i}\phi^j\phi^{k)}]_{\mathrm{SU}(2)} + 3\hat{\phi}^{(i}\mathrm{tr}[\phi^j\phi^{k)}]_{\mathrm{SU}(2)} - 6\hat{\phi}^{(i}\hat{\phi}^j\hat{\phi}^{k)}\,. \tag{2.12}$$

Meanwhile, since $\mathrm{SU}(3)_R$ is a global symmetry that is orthogonal to the gauge groups, such an expansion in terms of products between U(1) and SU(2) operators can be organized according to representations of each part under $\mathrm{SU}(3)_R$. Schematically,

$$\mathcal{O}^3[R] \xrightarrow{\mathrm{SU}(2)\times\mathrm{U}(1)} \sum_{R_1\otimes R_2\supset R} \hat{\mathcal{O}}[R_1]\mathcal{O}^2[R_2]\,, \tag{2.13}$$

where the summation is over all irreps R_1 and R_2 of $\mathrm{SU}(3)_R$ such that irrep decomposition of $R_1 \otimes R_2$ contains R. On the right hand side, we imply the R part of the Clebsch-Gordan decomposition of the tensor product $R_1 \otimes R_2$. $\mathcal{O}^2[R_2]$ is a gauge-invariant operator in the SU(2) theory, i.e. a multi-trace operator consisting of the SU(2) fields, while $\hat{\mathcal{O}}[R_1]$ is simply a polynomial in $\hat{\phi}$ that multiplies $\mathcal{O}^2[R_2]$ like a numerical coefficient.

Our goal is to prove that $\mathcal{O}^3[R]$ is not Q-exact using results for the simpler operators $\mathcal{O}^2[R_2]$. To do so, we first need to ensure that the Q-exactness property is preserved under the SU(2) × U(1) restriction. Suppose that $\mathcal{O}^3[R]$ is Q-exact, so that $\mathcal{O}^3[R] = Q\mathcal{O}^3[R]$. What we need is that the action of Q and the SU(2) × U(1) restriction commute with each other:

$$Q\ (\mathrm{SU}(2)\times\mathrm{U}(1) \text{ restriction of } \mathcal{O}^3[R]) = \mathrm{SU}(2)\times\mathrm{U}(1) \text{ restriction of } \mathcal{O}^3[R].$$

To show this, recall the rule (2.3) for the action of Q. First, take the SU(2) × U(1) restriction *after* the action of Q by substituting each field on the right hand sides of (2.3) as in (2.10). Each equation in (2.3) becomes

$$Q\Psi = \begin{pmatrix} Q\Psi|_{\mathrm{SU}(2)} & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.14}$$

where Ψ collectively denotes any fields, and $Q\Psi|_{\mathrm{SU}(2)}$ indicates the action of Q (2.3) understood as all fields being 2×2 matrices. Note that the U(1) components do not survive the commutators. On the other hand, now take the SU(2) × U(1) restriction *before* the action of Q by making the restriction (2.10) on the left hand sides of (2.3). Then they become

$$Q\Psi = \begin{pmatrix} Q\Psi|_{\mathrm{SU}(2)} & 0 \\ 0 & Q\Psi|_{\mathrm{U}(1)} = 0 \end{pmatrix}. \tag{2.15}$$

Agreement between (2.14) and (2.15) completes the desired proof.

JHEP07(2025)216

Applying the logic of the previous paragraph, if $\mathcal{O}^3[R]$ is Q-exact, then the r.h.s. of (2.13) which is the SU(2) × U(1) restriction of $\mathcal{O}^3[R]$, must also be Q-exact. Meanwhile, the global symmetry $\mathrm{SU}(3)_R$ by definition commutes with Q, so Q acting on a definite irrep of $\mathrm{SU}(3)_R$ does not change the representation. So if the r.h.s. of (2.13) is Q-exact, every term in the sum must be Q-exact separately. Finally, note that Q does not act on the U(1) components. Thus, the SU(2) operator $\mathcal{O}^2[R_2]$ for each term in the sum must be Q-exact. In short,

$$\begin{aligned} & Q\mathcal{O}^3[R] \xrightarrow{\mathrm{SU}(2)\times\mathrm{U}(1)} \sum_{R_1\otimes R_2\supset R} \hat{\mathcal{O}}[R_1]\mathcal{O}^2[R_2] \\ \Rightarrow \qquad & \mathcal{O}^3[R] \xrightarrow{\mathrm{SU}(2)\times\mathrm{U}(1)} \sum_{R_1\otimes R_2\supset R} \hat{\mathcal{O}}[R_1]\mathcal{O}^2[R_2]\,, \quad \text{where} \quad \mathcal{O}^2[R_2] = Q\mathcal{O}^2[R_2]\,. \end{aligned} \tag{2.16}$$

As a result, for $\mathcal{O}^3[R]$ to be Q-exact, every SU(2) gauge-invariant operators $\mathcal{O}^2[R_2]$ that appears non-trivially in the sum (2.13) must separately be Q-exact. Equivalently, if any of $\mathcal{O}^2[R_2]$ in the sum is not Q-exact, then the SU(3) operator $\mathcal{O}^3[R]$ cannot be Q-exact either.

In the rest of this paper, we will use this method to prove that some Q-closed operators that we construct are not Q-exact. In fact, we will find that the smallest SU(2) black hole cohomology $\mathcal{O}^2_{24}$ of (2.6) serves as $\mathcal{O}^2[R_2]$ that obstructs the Q-exactness of the SU(3) operators. This also allows the interpretation of the SU(3) operators as the core SU(2) black hole microstate wrapped by dual giant hairs, because its SU(2) × U(1) restriction (2.13) crystallizes in the SU(2) black hole cohomology.

2.2 A tower of dual giant hairs

In this subsection, we construct new black hole cohomologies $\mathcal{O}^3_{24+2n}$ $(n = 3, 4, \cdots)$ whose quantum numbers correspond to those in the infinite F_1 tower of table 2. They have charge $j = 24 + 2n$, transform under the representation $[n, 0]$ of $\mathrm{SU}(3)_R$, and are fermions.

We start with the first cohomology: $n = 3$. Following the method outlined in 2.1, we first construct Q-closed ansätze (2.9) for the black hole cohomology. In a hope to find the picture of the core SU(2) black hole (2 scalars, 5 fermions) wrapped by the dual giant hair (3 scalars), we look for operators that contain 5 scalars and 5 fermions. Since Q converts one fermion into two scalars, we need to find relations (2.8) between the trace relations that contain 7 scalars and 4 fermions. Furthermore, since we look for operators that transform under the representation $[n, 0]$, we consider a set of the relations (2.8) that transform as an $[n, 0]$ tensor. The relations with different field contents or different tensor structures cannot have linear relations between them, so we can consistently restrict as such.

In (A.2), we listed all fundamental trace relations $R_a(g_I)$ of gravitons in the SU(3) theory that contain the letters ϕ^m and ψ_m only. Each R_a has definite numbers of scalars and fermions, which restricts the graviton polynomial $f_a(g_I)$ that may multiply R_a to enter the desired relation of relations. As an illustration, let us take the example of $R^{(0,2)}_{14}[1, 3]$, which contains 3 scalars and 2 fermions and transforms under the $[1, 3]$ representation of $\mathrm{SU}(3)_R$. We need to add 4 scalars and 2 fermions by means of $f_a(g_I)$. There are two options: $u_2v_2v_2$ and v_3v_3.

R_a as well as the gravitons $u_{2,3}$ and $v_{2,3}$ all belong to definite irreps of $\mathrm{SU}(3)_R$. We need to contract the tensor indices of $u_2v_2v_2$ or v_3v_3 and $R^{(0,2)}_{14}$ appropriately to produce a tensor in the $[3, 0]$ representation. A group theory argument can be used to determine the number of

JHEP07(2025)216

independent contractions. First consider $u_2v_2v_2R_{14}^{(0,2)}$. Note that two identical fermions v_2 in the $[1,1]$ irrep are multiplied. So only the anti-symmetric part survives in their tensor product:

$$([1,1]\otimes[1,1])_A=[3,0]+[0,3]+[1,1]\,. \tag{2.17}$$

Then 7 independent $[3,0]$ tensors are obtained by decomposing the product $u_2v_2v_2R_{14}^{(0,2)}$:

$$[2,0]\otimes([1,1]\otimes[1,1])_A\otimes[1,3]=\cdots+(7\times[3,0])+\cdots\,. \tag{2.18}$$

Similarly, there are 4 independent $[3,0]$ tensors in the product $v_3v_3R_{14}^{(0,2)}$:

$$([2,1]\otimes[2,1])_A\otimes[1,3]=\cdots+(4\times[3,0])+\cdots\,. \tag{2.19}$$

In the previous paragraph, we illustrated that there are 7 and 4 $[3,0]$ tensors in the products $u_2v_2v_2R_{14}^{(0,2)}$ and $v_3v_3R_{14}^{(0,2)}$, respectively. Repeating the argument above for every fundamental trace relation R_a, we find that there are 39 non-fundamental trace relations in the irrep $[3,0]$ that span the desired trace relations. We list the 39 basis tensors in (A.4), labeling them generically as s_a with $a=1,\cdots,39$.

Before knowing that the fundamental trace relations R's themselves are polynomials of gravitons, all 39 trace relations s_a are independent. However, as polynomials of gravitons, one can find linear relations (2.8) between the 39 relations. As each s_a is a $[3,0]$ tensor with 3 symmetrized fundamental indices, a relation requires that the same linear combination between s_a's vanishes for all 10 independent components of $[3,0]$. We find that there are 14 relations, and we present one of them (chosen with hindsight):

$$\begin{aligned}0=&24s_1-\frac{16}{15}s_3+\frac{4}{15}s_4-\frac{32}{15}s_5+16s_9+85s_{10}+5s_{11}-40s_{12}-\frac{5}{3}s_{13}+\frac{5}{3}s_{14}-40s_{15}\\&+\frac{125}{7}s_{17}+31s_{18}+\frac{24}{7}s_{19}+\frac{543}{7}s_{20}+\frac{8}{5}s_{22}+\frac{16}{15}s_{23}+\frac{251}{7}s_{30}-s_{31}-9s_{32}+\frac{174}{7}s_{33}-\frac{17}{2}s_{34}.\end{aligned} \tag{2.20}$$

Every such relation leads to a Q-closed operator (2.9) by evaluating the same combination with R_a replaced by r_a.

Now with the 14 ansätze for the black hole cohomology, one should verify if any of these are not Q-exact. Instead of determining if each ansatz is Q-exact or not, which will not be computationally viable, we apply the strategy of section 2.1.2 to see if we can prove that any one of the ansätze is not Q-exact.

As explained in section 2.1.2 in a generic context, the key is to restrict a target ansatz $\mathcal{O}^3[3,0]$ into $\mathrm{SU}(2)\times\mathrm{U}(1)$ block diagonal fields (2.10) and expand as (2.13). Among the expansion, we choose to focus on terms with $R_1=[3,0]$. Then, R_2 can only be the irreps whose tensor product with $R_1=[3,0]$ contains $[3,0]$. The answer is that $R_2=[0,0]$, $[1,1]$, $[2,2]$ and $[3,3]$ are possible. Below, we present a rather algebraic route towards this answer in part to make the abstract arguments of section 2.1.2 more tangible.

Suppose that we have an operator $\mathcal{O}^{ijk}$ in the $[3,0]$ representation of $\mathrm{SU}(3)_R$, and restrict its field contents as in (2.10). Equivalent to focusing on the part of (2.13) with $R_1=[3,0]$ is to collect terms cubic in $\hat{\phi}^m$. (Recall that scalars are the only U(1) parts that we keep in our restriction.) As these are bosonic U(1) components, products of three factors of $\hat{\phi}^m$

JHEP07(2025)216

can only appear with their indices symmetrized, thus in the $[3,0]$ representation. Without any loss of generality, we can organize such terms as

$$\mathcal{O}^{ijk} \xrightarrow{\mathrm{SU}(2)\times\mathrm{U}(1)} \hat{\phi}^a\hat{\phi}^b\hat{\phi}^c\tilde{\mathcal{T}}^{(ijk)}_{(abc)}\,. \tag{2.21}$$

$\tilde{\mathcal{T}}^{(ijk)}_{(abc)}$ is not a tensor with an $\mathrm{SU}(3)_R$ irrep because it has non-zero traces, e.g. $\tilde{\mathcal{T}}^{(ijk)}_{(ibc)} \neq 0$. Instead, it should be understood as a product $[3,0]\otimes[0,3]$. By subtracting the traces of $\tilde{\mathcal{T}}$, the traces of the traces, and its trace, one can write $\tilde{\mathcal{T}}$ in terms of four traceless tensors $\mathcal{T}$'s:

$$\tilde{\mathcal{T}}^{(ijk)}_{(abc)} = \mathcal{T}^{(ijk)}_{(abc)} + \delta^i_a\mathcal{T}^{(jk)}_{(bc)} + \delta^{(ij)}_{(ab)}\mathcal{T}^k_c + \delta^{(ijk)}_{(abc)}\mathcal{T}\,. \tag{2.22}$$

Now each $\mathcal{T}$ on the right hand side is traceless, e.g. $\mathcal{T}^{(ijk)}_{(ibc)} = 0$, and is symmetric under permutations both within upper and lower indices as explicitly marked. Thus they transform in irreps $[3,3]$, $[2,2]$, $[1,1]$ and $[0,0]$ of $\mathrm{SU}(3)_R$. (2.22) is an algebraic way of writing $[3,0]\otimes[0,3] = [3,3]\oplus[2,2]\oplus[1,1]\oplus[0,0]$. As a result, (2.21) becomes

$$\mathcal{O}^{ijk} \xrightarrow{\mathrm{SU}(2)\times\mathrm{U}(1)} \hat{\phi}^i\hat{\phi}^j\hat{\phi}^k\mathcal{T} + \hat{\phi}^a\hat{\phi}^{(i}\hat{\phi}^j\mathcal{T}^{k)}_a + \hat{\phi}^a\hat{\phi}^b\hat{\phi}^{(i}\mathcal{T}^{jk)}_{ab} + \hat{\phi}^a\hat{\phi}^b\hat{\phi}^c\mathcal{T}^{ijk}_{abc}\,. \tag{2.23}$$

This is an algebraic realization of (2.13) for $R=[3,0]$ and $R_1=[3,0]$, for which R_2 runs over $[0,0]$, $[1,1]$, $[2,2]$ and $[3,3]$.

Now recall from section 2.1.2 that if any of the four SU(2) operators $\mathcal{T}$, $\mathcal{T}^k_c$, $\mathcal{T}^{jk}_{bc}$, $\mathcal{T}^{ijk}_{abc}$ in (2.23) is not Q-exact, then the SU(3) operator $\mathcal{O}^{ijk}$ cannot be Q-exact regardless of all other terms. Using this argument, we expand all 14 Q-closed operators as in (2.23), and in particular find the corresponding SU(2) operator $\mathcal{T}$ which is an $\mathrm{SU}(3)_R$ singlet. There exist linear combinations of the 14 Q-closed operators for which their corresponding $\mathcal{T}$ are non-zero multiples of the smallest black hole cohomology in the SU(2) theory, namely $\mathcal{O}^2_{24}$ in (2.6). One such operator is what corresponds to (2.20), (named according to the convention $\mathcal{O}^N_j$)

$$\begin{aligned}
(\mathcal{O}^3_{30})^{ijk} =\ & 24\epsilon^{a_1a_2(i}\epsilon^{j|b_1b_2}v^c{}_d v^{|k)}{}_{a_1}v^d{}_{b_1}(r^{(0,2)}_{12})_{a_2b_2c} - \frac{16}{15}\epsilon^{a_1a_2(i}v^{j|}{}_b v^b{}_c v^c{}_{a_1}(r^{(0,2)}_{12})^{|k)}_{a_2} \\
& + \frac{4}{15}\epsilon^{a_1a_2a_3}v^b{}_{a_1}v^{(i}{}_{a_2}v^j{}_{a_3}(r^{(0,2)}_{12})^{k)}_b - \frac{32}{15}\epsilon^{a_1a_2(i}v^j{}_b v^{k)}{}_{a_1}v^b{}_c(r^{(0,2)}_{12})^c_{a_2} + 16\epsilon^{a_1a_2(i}v^{j|}{}_d v^b{}_{a_1}v^d{}_c(r^{(0,2)}_{12})^{|k)c}_{a_2b} \\
& + 85\epsilon^{a_1a_2(i|}u^{bc}v^d{}_{a_1}v^{|j}{}_b(r^{(0,3)}_{14})^{k)}_{a_2cd} + 5\epsilon^{a_1a_2(i|}u^{bc}v^d{}_b v^{|j}{}_{a_1}(r^{(0,3)}_{14})^{k)}_{a_2cd} - 40\epsilon^{a_1a_2(i}u^{j|b}v^c{}_{a_1}v^d{}_b(r^{(0,3)}_{14})^{|k)}_{a_2cd} \\
& - \frac{5}{3}\epsilon^{a_1a_2(i|}u^{cd}v^b{}_{a_1}v^{|j}{}_b(r^{(0,3)}_{14})^{k)}_{a_2cd} + \frac{5}{3}\epsilon^{a_1a_2(i}u^{j|c}v^b{}_{a_1}v^d{}_b(r^{(0,3)}_{14})^{|k)}_{a_2cd} - 40\epsilon^{a_1a_2(i}u^{j|b}v^c{}_{a_1}v^{|k)}{}_d(r^{(0,3)}_{14})^d_{a_2bc} \\
& + \frac{125}{7}\epsilon^{a_1a_2(i}v^{j|b}{}_{a_1}v^{cd}{}_b(r^{(0,3)}_{14})^{|k)}_{a_2cd} + 31\epsilon^{a_1a_2(i}v^{bc}{}_{a_1}v^{d|j}{}_b(r^{(0,3)}_{14})^{k)}_{a_2cd} + \frac{24}{7}\epsilon^{a_1a_2(i}v^{jk)}{}_b v^{cd}{}_{a_1}(r^{(0,3)}_{14})^b_{a_2cd} \\
& + \frac{543}{7}\epsilon^{a_1a_2a_3}v^{b(i}{}_{a_1}v^{j|c}{}_{a_2}(r^{(0,3)}_{14})^{|k)}_{a_3bc} + \frac{8}{5}\epsilon^{a_1a_2(i|}v^b{}_{a_1}v^{|jk)}{}_b(r^{(0,2)}_{16})_{a_2} + \frac{16}{15}\epsilon^{a_1a_2(i}v^j{}_{a_1}v^{k)b}{}_{a_2}(r^{(0,3)}_{16})_b \\
& + \frac{251}{7}\epsilon^{a_1a_2a_3}v^{b_1}{}_{a_1}v^{b_2(i}{}_{a_2}(r^{(0,3)}_{16})^{jk)}_{a_3b_1b_2} - \epsilon^{a_1a_2a_3}v^{(i|}{}_{a_1}v^{b_1b_2}{}_{a_2}(r^{(0,3)}_{16})^{|jk)}_{a_3b_1b_2} - 9\epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{bc}{}_d(r^{(0,3)}_{16})^{d|k)}_{a_2bc} \\
& + \frac{174}{7}\epsilon^{a_1a_2(i|}v^b{}_{a_1}v^{c|j}{}_d(r^{(0,3)}_{16})^{k)d}_{a_2bc} - \frac{17}{2}\epsilon^{a_1a_2(i|}v^b{}_{a_1}v^{cd}{}_{a_2}(r^{(0,3)}_{16})^{|jk)}_{bcd}\,.
\end{aligned} \tag{2.24}$$

The operators r's are related to the fundamental trace relations R with same labels by $Qr=R$, whose expressions are listed in (A.3). Restricting this operator into the $\mathrm{SU}(2)\times\mathrm{U}(1)$ form,

JHEP07(2025)216

collecting terms cubic in $\hat{\phi}$ and expanding as in (2.23), the corresponding SU(2) operator $\mathcal{T}$ is

$$\mathcal{T} = -\frac{648}{5}\mathcal{O}_{24}^2 \,. \tag{2.25}$$

The fact that $\mathcal{O}_{24}^2$ is not Q-exact completes the proof that (2.24) is a nontrivial cohomology at the order $j = 30$ in the SU(3) gauge theory.

One also finds that the cohomology represented by (2.24) is not graviton-like. An abstract argument is that we used trace relations to show that (2.24) is Q-closed. Another practical check is to restrict all the elementary fields to diagonals. It has been explained that graviton operators can be faithfully represented by the eigenvalues of the elementary fields [3]. However, as explained in section 2.1 we have chosen the operators r_a's to be all zero upon the diagonal restriction. So the operator $\mathcal{O}_{30}^3$, which is a linear combination of r_a's, is also zero after the restriction. This proves that $\mathcal{O}_{30}^3$ is either trivial or black hole type cohomology, between which we know that the latter is the case. Below in this paper, similar arguments will be implicitly assumed whenever we claim that certain operators are not graviton-like. Note also that this argument applies to each independent SU(2) term in the SU(2) × U(1) decomposition (2.13) of the non-graviton operator: if $\mathcal{O}^{ijk}$ in (2.23) is a non-graviton cohomology, then each $\mathcal{T}$ on the right hand side must be non-graviton as well.

Let us make a remark on the other 13 combinations of Q-closed operators. For all Q-closed operators, the only possible non-Q-exact operator among $\mathcal{T}$, $\mathcal{T}_c^k$, $\mathcal{T}_{bc}^{jk}$ and $\mathcal{T}_{abc}^{ijk}$ in their decomposition (2.23) is $\mathcal{T}$, and $\mathcal{T}$ is cohomologous to $\mathcal{O}_{24}^2$ if not trivial. This is because all these operators are Q-closed SU(2) operators at $j = 24$ and not of the graviton type. [1] has shown that $\mathcal{O}_{24}^2$ is the unique cohomology with these conditions. Therefore, it is always possible to find 13 linear combinations of the 14 operators whose corresponding $\mathcal{T}$ operator is trivial. While our arguments can prove that an operator is not Q-exact, the $\mathcal{T}$ part being zero does not prove that it is Q-exact. Thus, we have not proved that (2.24) is the unique Q-cohomology class within our ansatz. However, supported by the black hole cohomology index summarized in table 2 we conjecture that (2.24) indeed represents the unique cohomology at $j = 30$.

We now repeat all steps of the previous subsection more arduously to find a black hole cohomology at $j = 32$, with the $SU(3)_R$ representation $[4, 0]$. This accounts for the second entry in the tower F_1 of table 2. As the logical steps are identical to the previous subsection, we minimize the details to reach the result quickly.

We look for a cohomology that contains 6 scalars and 5 fermions, so we first build non-fundamental trace relations that contain 8 scalars and 4 fermions, and that belongs to the $[4, 0]$ representation of $SU(3)_R$. By group theory argument, we find that there are 107 independent tensors at $[4, 0]$ that span these relations. We list all of them in (A.5).

Among the relations (A.5), 57 are independent and there are 50 linear relations, yielding 50 Q-closed operators. As our proof based on the SU(2) × U(1) restriction can only determine if there is at least one non-Q-exact operator, we only present one operator that we will shortly prove to be not Q-exact:

$$(\mathcal{O}_{32}^3)^{ijkl} =$$

$$\begin{aligned}
&162576\epsilon^{a_1a_2(i}\epsilon^{j|a_3a_4}v^{|k}{}_{a_1}v^{l)}{}_{a_5}v^{a_5a_6}{}_{a_3}(r_{12}^{(0,1)})_{a_2a_4a_6} - 3240\epsilon^{a_1a_2(i}\epsilon^{j|a_3a_4}v^{|k|}{}_{a_1}v^{a_5}{}_{a_2}v^{|l)a_6}{}_{a_3}(r_{12}^{(0,1)})_{a_4a_5a_6} \\
&+19080\epsilon^{a_1a_2(i}\epsilon^{j|a_3a_4}v^{|k|}{}_{a_1}v^{a_5}{}_{a_3}v^{|l)a_6}{}_{a_2}(r_{12}^{(0,1)})_{a_4a_5a_6} - 183096\epsilon^{a_1a_2(i}\epsilon^{j|a_3a_4}v^{|k|}{}_{a_1}v^{a_5}{}_{a_6}v^{|l)a_6}{}_{a_3}(r_{12}^{(0,1)})_{a_2a_4a_5}
\end{aligned}$$

JHEP07(2025)216

$$
\begin{aligned}
&-283500\epsilon^{a_1a_2(i}\epsilon^{j|a_3a_4}v^{a_5}{}_{a_1}v^{a_6}{}_{a_2}v^{|kl)}{}_{a_3}(r_{12}^{(0,1)})_{a_4a_5a_6}+126000\epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k}{}_{a_2}v^{l)a_3}{}_{a_4}(r_{12}^{(0,1)})^{a_4}_{a_3}\\
&-16800\epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k}{}_{a_3}v^{l)a_3}{}_{a_4}(r_{12}^{(0,1)})^{a_4}_{a_2}+75600\epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k}{}_{a_4}v^{l)a_3}{}_{a_2}(r_{12}^{(0,1)})^{a_4}_{a_3}\\
&-46200\epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_4}v^{|kl)}{}_{a_2}(r_{12}^{(0,1)})^{a_4}_{a_3}-21000\epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_4}v^{|kl)}{}_{a_3}(r_{12}^{(0,1)})^{a_4}_{a_2}\\
&+75600\epsilon^{a_1a_2(i}v^{j|}{}_{a_4}v^{a_3}{}_{a_1}v^{|kl)}{}_{a_2}(r_{12}^{(0,1)})^{a_4}_{a_3}+75600\epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_4}v^{|k|a_4}{}_{a_2}(r_{12}^{(0,1)})^{|l)}_{a_3}\\
&-37800\epsilon^{a_1a_2(i|}v^{a_3}{}_{a_1}v^{a_4}{}_{a_2}v^{|jk}{}_{a_3}(r_{12}^{(0,1)})^{l)}_{a_4}-14000\epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k|}{}_{a_2}v^{a_3a_4}{}_{a_5}(r_{12}^{(0,1)})^{|l)a_5}_{a_3a_4}\\
&+14000\epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_4}v^{|kl)}{}_{a_5}(r_{12}^{(0,1)})^{a_4a_5}_{a_2a_3}-28000\epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_5}v^{a_4|k}{}_{a_2}(r_{12}^{(0,1)})^{l)a_5}_{a_3a_4}\\
&+14000\epsilon^{a_1a_2(i|}v^{a_3}{}_{a_1}v^{a_4}{}_{a_5}v^{|jk}{}_{a_2}(r_{12}^{(0,1)})^{l)a_5}_{a_3a_4}+31500\epsilon^{a_1a_2a_3}v^{(i}{}_{a_1}v^{j}{}_{a_2}v^{k}{}_{a_3}(r_{14}^{(0,1)})^{l)}\\
&-31500\epsilon^{a_1a_2(i|}\epsilon^{a_3a_4|j}v^{k}{}_{a_1}v^{l)}{}_{a_2}v^{a_5}{}_{a_3}(r_{14}^{(0,1)})_{a_4a_5}+315000\epsilon^{a_1a_2(i|}u^{a_3a_4a_5}v^{|j}{}_{a_1}v^{k}{}_{a_2}(r_{14}^{(0,2)})^{l)}_{a_3a_4a_5}\\
&-1055700\epsilon^{a_1a_2(i}u^{j|a_3}v^{|k|}{}_{a_1}v^{a_4a_5}{}_{a_2}(r_{14}^{(0,2)})^{|l)}_{a_3a_4a_5}+363660\epsilon^{a_1a_2(i}u^{j|a_3}v^{|k|}{}_{a_1}v^{a_4a_5}{}_{a_3}(r_{14}^{(0,2)})^{|l)}_{a_2a_4a_5}\\
&-596400\epsilon^{a_1a_2(i}u^{j|a_3}v^{|k|}{}_{a_4}v^{a_4a_5}{}_{a_1}(r_{14}^{(0,2)})^{|l)}_{a_2a_3a_5}+1905300\epsilon^{a_1a_2(i}u^{j|a_3}v^{a_4}{}_{a_1}v^{a_5|k}{}_{a_2}(r_{14}^{(0,2)})^{l)}_{a_3a_4a_5}\\
&+890400\epsilon^{a_1a_2(i}u^{j|a_3}v^{a_4}{}_{a_1}v^{a_5|k}{}_{a_3}(r_{14}^{(0,2)})^{l)}_{a_2a_4a_5}+82500\epsilon^{a_1a_2(i}u^{j|a_3}v^{a_4}{}_{a_5}v^{a_5|k}{}_{a_1}(r_{14}^{(0,2)})^{l)}_{a_2a_3a_4}\\
&+2288100\epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j}{}_{a_1}v^{k|a_5}{}_{a_2}(r_{14}^{(0,2)})^{|l)}_{a_3a_4a_5}-247560\epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j}{}_{a_1}v^{k|a_5}{}_{a_3}(r_{14}^{(0,2)})^{|l)}_{a_2a_4a_5}\\
&+1506900\epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j}{}_{a_5}v^{k|a_5}{}_{a_1}(r_{14}^{(0,2)})^{|l)}_{a_2a_3a_4}-3441300\epsilon^{a_1a_2(i|}u^{a_3a_4}v^{a_5}{}_{a_1}v^{|jk}{}_{a_2}(r_{14}^{(0,2)})^{l)}_{a_3a_4a_5}\\
&-4275300\epsilon^{a_1a_2(i|}u^{a_3a_4}v^{a_5}{}_{a_1}v^{|jk}{}_{a_3}(r_{14}^{(0,2)})^{l)}_{a_2a_4a_5}+30360\epsilon^{a_1a_2(i}u^{jk}v^{l)}{}_{a_1}v^{a_3a_4}{}_{a_5}(r_{14}^{(0,2)})^{a_5}_{a_2a_3a_4}\\
&-134400\epsilon^{a_1a_2(i}u^{jk|}v^{a_3}{}_{a_1}v^{|l)a_4}{}_{a_5}(r_{14}^{(0,2)})^{a_5}_{a_2a_3a_4}+340680\epsilon^{a_1a_2(i}u^{j|a_3}v^{|k}{}_{a_1}v^{l)a_4}{}_{a_5}(r_{14}^{(0,2)})^{a_5}_{a_2a_3a_4}\\
&+1010400\epsilon^{a_1a_2(i}u^{j|a_3}v^{a_4}{}_{a_1}v^{|kl)}{}_{a_5}(r_{14}^{(0,2)})^{a_5}_{a_2a_3a_4}-414240\epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j}{}_{a_1}v^{kl)}{}_{a_5}(r_{14}^{(0,2)})^{a_5}_{a_2a_3a_4}\\
&+48000\epsilon^{a_1a_2(i}v^{jk|}{}_{a_1}v^{a_3a_4}{}_{a_5}(r_{16}^{(0,2)})^{|l)a_5}_{a_2a_3a_4}-58800\epsilon^{a_1a_2(i}v^{jk|}{}_{a_5}v^{a_3a_4}{}_{a_1}(r_{16}^{(0,2)})^{|l)a_5}_{a_2a_3a_4}\\
&-369600\epsilon^{a_1a_2(i}v^{j|a_3}{}_{a_1}v^{a_4|k}{}_{a_5}(r_{16}^{(0,2)})^{l)a_5}_{a_2a_3a_4}+33600\epsilon^{a_1a_2(i}v^{j|a_3}{}_{a_1}v^{a_4a_5}{}_{a_2}(r_{16}^{(0,2)})^{|kl)}_{a_3a_4a_5}\\
&+12250\epsilon^{a_1a_2(i}u^{jk|}v^{a_3}{}_{a_1}v^{a_4}{}_{a_5}(r_{16}^{(0,2)})^{|l)a_5}_{a_2a_3a_4}-615850\epsilon^{a_1a_2(i}u^{j|a_3}v^{|k|}{}_{a_1}v^{a_4}{}_{a_5}(r_{16}^{(0,2)})^{|l)a_5}_{a_2a_3a_4}\\
&+369250\epsilon^{a_1a_2(i}u^{j|a_3}v^{|k|}{}_{a_5}v^{a_4}{}_{a_1}(r_{16}^{(0,2)})^{|l)a_5}_{a_2a_3a_4}+184625\epsilon^{a_1a_2(i}u^{j|a_3}v^{a_4}{}_{a_1}v^{a_5}{}_{a_2}(r_{16}^{(0,2)})^{|kl)}_{a_3a_4a_5}\\
&-504800\epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j}{}_{a_1}v^{k}{}_{a_5}(r_{16}^{(0,2)})^{l)a_5}_{a_2a_3a_4}-1085950\epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j|}{}_{a_1}v^{a_5}{}_{a_2}(r_{16}^{(0,2)})^{|kl)}_{a_3a_4a_5}\\
&+701100\epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j|}{}_{a_1}v^{a_5}{}_{a_3}(r_{16}^{(0,2)})^{|kl)}_{a_2a_4a_5}\,.
\end{aligned}
\tag{2.26}
$$

By the SU(2) × U(1) restriction of the fields, this operator can be expanded as in (2.13). This time we focus on terms with $R_1 = [4, 0]$. In other words, we collect terms that are quartic in the U(1) components $\hat{\phi}^m$. The analogous expansion to (2.23) is

$$
\mathcal{O}^{ijkl}\xrightarrow{\mathrm{SU(2)\times U(1)}}\hat{\phi}^i\hat{\phi}^j\hat{\phi}^k\hat{\phi}^l\mathcal{T}+\hat{\phi}^a\hat{\phi}^{(i}\hat{\phi}^j\hat{\phi}^k\mathcal{T}_a^{l)}+\hat{\phi}^a\hat{\phi}^b\hat{\phi}^{(i}\hat{\phi}^j\mathcal{T}_{ab}^{kl)}+\hat{\phi}^a\hat{\phi}^b\hat{\phi}^c\hat{\phi}^{(i}\mathcal{T}_{abc}^{jkl)}+\hat{\phi}^a\hat{\phi}^b\hat{\phi}^c\hat{\phi}^d\mathcal{T}_{abcd}^{ijkl}. \tag{2.27}
$$

Restricting (2.26) into the SU(2) × U(1) form, collecting terms quartic in $\hat{\phi}$ and expanding as in (2.27), the corresponding SU(2) operator $\mathcal{T}$ that is an $\mathrm{SU}(3)_R$ singlet is a non-trivial multiple of the SU(2) black hole cohomology operator $\mathcal{O}_{24}^2$ (2.6):

$$
\mathcal{T} = -1020600\ \mathcal{O}_{24}^2\,. \tag{2.28}
$$

This completes the proof that (2.26) is a black hole cohomology at the order $j = 32$ in the SU(3) gauge theory, and we conjecture its uniqueness.

We might repeat the entire procedure for $n = 5, 6, \cdots$. Instead, we present a constructive method to obtain the infinite tower of cohomologies for all $n \geq 5$, based on two cohomologies $\mathcal{O}_{30}^3$ and $\mathcal{O}_{32}^3$ already constructed. The cohomologies obtained using this method should be

JHEP07(2025)216

contained in the ansatz of section 2.1. The key idea is to multiply the two cohomologies by the u-type gravitons. This method will be used in the next section as well.

In the SU(3) gauge theory, the u-type single graviton operators are

$$u_2{}^{ij} \equiv \mathrm{tr}\left(\phi^{(i}\phi^{j)}\right) , \qquad u_3{}^{ijk} \equiv \mathrm{tr}\left(\phi^{(i}\phi^{j}\phi^{k)}\right) , \tag{2.29}$$

under the $[2,0]$ and $[3,0]$ representations of $\mathrm{SU}(3)_R$, respectively. Let us consider the simplest example, the product between u_2 and $\mathcal{O}^3_{30}$. The general case will follow in a straightforward manner. As $[2,0]\otimes[3,0]=[5,0]\oplus[3,1]\oplus[1,2]$, the product is decomposed into three irreps:

$$\begin{aligned}
(u_2\mathcal{O}^3_{30})^{ijklm} &= u_2{}^{(ij}(\mathcal{O}^3_{30})^{klm)} , \\
(u_2\mathcal{O}^3_{30})^{ijk}_{l} &= \epsilon_{lab}u_2{}^{a(i}(\mathcal{O}^3_{30})^{jk)b} , \\
(u_2\mathcal{O}^3_{30})^{i}_{jk} &= \epsilon_{jab}\epsilon_{kcd}u_2{}^{ac}(\mathcal{O}^3_{30})^{ibd} .
\end{aligned} \tag{2.30}$$

Since Q annihilates both u_2 and $\mathcal{O}^3_{30}$, it is clear that all three operators are Q-closed.

Now we shall employ the strategy of section 2.1.2 to determine whether we can prove if these products are not Q-exact. For each of the product operators in (2.30), we perform the $\mathrm{SU}(2)\times\mathrm{U}(1)$ restriction as in (2.10) and collect, this time, terms at the $\hat\phi^5$ order. There are two possibilities for such terms: $(2,3)$ or $(0,5)$ of the $\hat\phi$ factors may originate from $(u_2,\mathcal{O}^3_{30})$, respectively. Note from (2.29) that u_2 contains only 2 scalars, and that it is impossible to extract only one factor of $\hat\phi$ from u_2 because $\mathrm{tr}[\phi^i]_{\mathrm{SU}(2)}=0$,

$$\mathrm{tr}[\phi^{(i}\phi^{j)}] \xrightarrow{\mathrm{SU}(2)\times\mathrm{U}(1)} \mathrm{tr}[\phi^{(i}\phi^{j)}]_{\mathrm{SU}(2)} + 6\hat\phi^i\hat\phi^j . \tag{2.31}$$

On the other hand, terms cubic in $\hat\phi$ under the $\mathrm{SU}(2)\times\mathrm{U}(1)$ restriction of $\mathcal{O}^3_{30}$ have been formally written down in (2.23), among which $\mathcal{T}=-\frac{648}{5}\mathcal{O}^2_{24}$ (2.28). The other $\mathcal{T}$'s with the $[1,1]$, $[2,2]$, $[3,3]$ representations are all Q-exact following a two-step logic. First, they cannot be graviton operators because the operator on the left is a non-graviton operator, see discussion below (2.25). Second, the only non-graviton cohomology of the SU(2) theory at the order t^{24} is the $\mathrm{SU}(3)_R$-singlet $\mathcal{O}^2_{24}$ [1], which belongs to the singlet $\mathcal{T}$. Terms quintic in $\hat\phi$ under the $\mathrm{SU}(2)\times\mathrm{U}(1)$ restriction of $\mathcal{O}^3_{30}$ can be collected in a form similar to (2.23). However, all SU(2) blocks that appear on the right hand side will be Q-exact for a similar reason: there is no SU(2) black hole cohomology at the order t^{20} [1].

Collecting both possibilities for the numbers of $\hat\phi$ factors, we have

$$\begin{aligned}
(u_2\mathcal{O}^3_{30})^{ijklm} &\xrightarrow{\mathrm{SU}(2)\times\mathrm{U}(1)} \hat\phi^i\hat\phi^j\hat\phi^k\hat\phi^l\hat\phi^m\cdot 6\mathcal{T} + (Q\text{-exact}) , \\
(u_2\mathcal{O}^3_{30})^{ijk}_{l} &\xrightarrow{\mathrm{SU}(2)\times\mathrm{U}(1)} \epsilon_{lab}\hat\phi^a\hat\phi^i\hat\phi^j\hat\phi^k\hat\phi^b\cdot 6\mathcal{T} + (Q\text{-exact}) = 0 + (Q\text{-exact}) , \\
(u_2\mathcal{O}^3_{30})^{i}_{jk} &\xrightarrow{\mathrm{SU}(2)\times\mathrm{U}(1)} \epsilon_{jab}\epsilon_{kcd}\hat\phi^a\hat\phi^c\hat\phi^i\hat\phi^b\hat\phi^d\cdot 6\mathcal{T} + (Q\text{-exact}) = 0 + (Q\text{-exact}) ,
\end{aligned} \tag{2.32}$$

where we put all Q-exact terms under the rug. For the latter two, the obstruction terms from $\mathcal{T}\propto\mathcal{O}^2_{24}$ vanish due to the antisymmetry of the ϵ-tensor.

(2.32) are specific applications of the general principle (2.13) on the operators with $R=[5,0]$, $[3,1]$ and $[1,2]$. We are particularly interested in the term on the right hand side with $R_1=[5,0]$ and $R_2=[0,0]$, i.e. quintic in $\hat\phi$ and the SU(2) block being the $\mathrm{SU}(3)_R$ singlet,

JHEP07(2025)216

which is the only option for being not Q-exact. For the first operator with $R = [5,0]$, (2.32) shows that the corresponding SU(2) block $\mathcal{O}^2[0,0]$ is a non-zero multiple of $\mathcal{T} = -\frac{648}{5}\mathcal{O}^2_{24}$, possibly up to addition of Q-exact terms. Therefore, this term obstructs the $[5,0]$ operator $(u_2\mathcal{O}^3_{30})^{ijklm}$ from being Q-exact. In contrast, for the latter two operators with $R = [3,1]$ and $[1,2]$, $R_1 = [5,0]$ and $R_2 = [0,0]$ are not allowed in the summation in (2.13) since $[5,0] \otimes [0,0]$ clearly does not contain $[3,1]$ or $[1,2]$. This fact is naturally reflected in (2.32) as the corresponding terms being inevitably zero by the tensor structure. Therefore, the SU(2) black hole cohomology $\mathcal{O}^2_{24}$ does not obstruct the $[3,1]$ operator $(u_2\mathcal{O}^3_{30})^{ijk}_l$ and the $[1,2]$ operator $(u_2\mathcal{O}^3_{30})^i_{jk}$ from being Q-exact.

So we have proved that among the products between u_2 and $\mathcal{O}^3_{30}$, the totally symmetric part with the representation $[5,0]$ is not Q-exact. We propose that this represents the third black hole cohomology in the tower F_1.

Aiming for $n \geq 6$, it is straightforward to extend this proof into products between arbitrary numbers of u_2 and u_3, and either $\mathcal{O}^3_{30}$ or $\mathcal{O}^3_{32}$. Consider any such product where the total number of scalars from the u-type gravitons is $n-k \geq 2$, and they multiply $\mathcal{O}^3_{24+2k}$ with either $k = 3$ or 4. Among the tensor product between arbitrary numbers of $[2,0]$ and $[3,0]$ (representing the u-type gravitons) and $[3,0]$ or $[4,0]$ (representing the black hole cohomology), there will be exactly one piece of $[n,0]$ represented by the totally symmetric product

$$(u\cdots u\mathcal{O}^3_{24+2k})^{i_1\cdots i_n} = u^{(i_1\cdots}\cdots u^{\cdots i_{n-k}}(\mathcal{O}^3_{24+2k})^{i_{n-k+1}\cdots i_n)}\,, \tag{2.33}$$

and all other irreps in the tensor product will follow by partly contracting the $\mathrm{SU}(3)_R$ indices of $u\cdots u\mathcal{O}^3_{24+2k}$ with ϵ-tensors, as exemplified in (2.30).

Then, we perform the $\mathrm{SU}(2)\times\mathrm{U}(1)$ restriction (2.10) and collect terms at the $\hat{\phi}^n$ order. Again, there are two possibilities on the origin of the $\hat{\phi}$ factors: $(n-k,k)$ or $(n-k-2,k+2)$ of the $\hat{\phi}$ factors may originate from $(u\cdots u, \mathcal{O}^3_{24+2k})$, respectively. Note that the two parts carry at maximum $n-k$ and $k+2$ factors of scalars respectively, and it is impossible to extract all but one scalars as the U(1) component from the gravitons, because for both u_2 and u_3, this will leave only one traceless field inside an SU(2) trace. However, the non-Q-exact SU(2) block cannot arise from the second possibility $(n-k-2,k+2)$, because there is no SU(2) black hole cohomology at the order $j = 20$ that may appear as the SU(2) block in the $\hat{\phi}^{k+2}$ term in the $\mathrm{SU}(2)\times\mathrm{U}(1)$ restriction of $\mathcal{O}^3_{24+2k}$. So $(n-k,k)$ being the only possibility, every scalar in the u-type gravitons must contribute by its U(1) component and k factors of $\hat{\phi}$ are extracted from $\mathcal{O}^3_{24+2k}$. The relevant expansion is

$$u^{(i_1\cdots}\cdots u^{\cdots i_{n-k})} \xrightarrow{\mathrm{SU}(2)\times\mathrm{U}(1)} \hat{\phi}^{i_1}\cdots\hat{\phi}^{i_{n-k}}\; C + \cdots\,, \tag{2.34}$$

where C is a non-zero integer, and the omitted terms $\cdots$ will at best contribute to Q-exact terms in the $\mathrm{SU}(2)\times\mathrm{U}(1)$ restriction of (2.33). As for $\mathcal{O}^3_{24+2k}$, one uses either (2.23) or (2.27).

Combining the expansions for each part, we have

$$(u\cdots u\mathcal{O}^3_{24+2k})^{i_1\cdots i_n} \xrightarrow{\mathrm{SU}(2)\times\mathrm{U}(1)} \hat{\phi}^{i_1}\cdots\hat{\phi}^{i_n}\; C\mathcal{T} + (Q\text{-exact})\,, \tag{2.35}$$

where $\mathcal{T} = -\frac{648}{5}\mathcal{O}^2_{24}$ (2.25) if $k = 3$ and $\mathcal{T} = -\frac{216}{125}\mathcal{O}^2_{24}$ (2.28) if $k = 4$. On the other hand, for all other irreps from the product of the u-type gravitons and $\mathcal{O}^3_{24+2k}$ with at least one

JHEP07(2025)216

lower index, the corresponding term vanishes because the contractions with ϵ-tensors will annihilate the symmetric product $\hat{\phi}^{i_1}\cdots\hat{\phi}^{i_n}$.

Looking at (2.35) as a specific case of (2.13) for $R = R_1 = [n, 0]$, it shows that the term with $R_2 = [0, 0]$ is not Q-exact because it is proportional to the SU(2) black hole cohomology. Thus, (2.33) defines a black hole cohomology of the SU(3) theory at the order t^{24+2n}, under the $[n, 0]$ representation of $\mathrm{SU}(3)_R$, where $n-k$ can be any integer $n-k \geq 2$, and $k = 3$ or 4.

For any $n \geq 6$, (2.33) defines the cohomology non-uniquely as the product of u-type gravitons and $\mathcal{O}^3_{30}$ or $\mathcal{O}^3_{32}$. For example, the totally symmetric products $(u_3)^{(i_1i_2i_3}(\mathcal{O}^3_{30})^{i_4i_5i_6)}$ and $(u_2)^{(i_1i_2}(\mathcal{O}^3_{32})^{i_3i_4i_5i_6)}$ both define a $[6, 0]$ cohomology at the order t^{36}, and for higher orders the apparent multiplicity grows further. Moreover, had we directly extended the procedure we used for $n = 3$ and 4 into higher n using relations of relations, we could have obtained cohomologies that may apparently look different from (2.33), potentially adding to the multiplicity at every order. However, the actual multiplicity will depend on how many combinations of them are Q-exact. It is beyond the scope of the present work to fully answer this question. As the $\mathrm{SU}(3)_R$-singlet SU(2) block $\mathcal{T}$ is always a multiple of $\mathcal{O}^2_{24}$, our method is only able to determine at each order if there is at least one black hole cohomology. So (2.33) constructs at least one SU(3) black hole cohomology at the order t^{24+2n} and under the $\mathrm{SU}(3)_R$ representation $[n, 0]$ for all $n \geq 5$, in addition to the cases $n = 3$ and 4 found earlier in this subsection. They would constitute an infinite series of SU(3) black hole cohomologies whose overall charges j and the $\mathrm{SU}(3)_R$ representations match with the tower F_1 detected by the index, as shown in table 2.

Inspired by the gravitational picture of the dual giant hairs at large N [15, 23], it is tempting to conjecture that the multiplicity at each n is exactly 1. The basic idea on the bulk side is that the u type gravitons carrying no spin have their wavefunctions peaked at the central region of AdS, which in the presence of the core black hole does not exist due to the event horizon. But collectively exciting these gravitons as a large enough spherical dual giant, its radial location can be outside the event horizon, providing a hair. Since large part of the apparent multiplicities in the previous paragraph comes from the multigraviton degeneracies in the absence of black holes, we expect that the core black hole operator will render most of them Q-exact. Proving that these multiplicities of hairy cohomologies are all 1, one would be showing that this emergent bulk picture is realized even at $N = 3$. We again emphasize that we already have strong evidence for this from the index, as table 2 shows no hint of such large multiplicities beyond the tower F_1.

If it is indeed true that the hair at given n (within our ansatz) is unique, one may still think that the 'graviton part' of (2.33) does not really look like dual giant gravitons that are commonly known in the literature. For simplicity, let us take all the $[n, 0]$ indices to be $i_1 = \cdots = i_n = 3$. This case is supposed to represent a half-BPS dual giant graviton made of n scalars $\phi^3 \equiv Z$ wrapping the core black hole microstate $\mathcal{O}^2_{24}$. In [26], the symmetric Schur polynomials are expected to represent half-BPS dual giant gravitons in the absence of core black holes. On the other hand, in (2.33), $n-k$ of Z's are grouped into monomials of $u_2 = \mathrm{tr}(Z^2)$ and $u_3 = \mathrm{tr}(Z^3)$, while k of them are fused with $\mathcal{O}^2_{24}$ to form $\mathcal{O}^3_{24+2k}$. The latter fused part has to do with the presence of black holes. Apart from this difference, symmetric Schur polynomials are highly nontrivial combinations of multitrace operators while (2.33) is almost in a definite multitrace basis. These features look very different

JHEP07(2025)216

from each other. Here we note that the Schur basis of [26] is relevant as the physical BPS states (in the free limit), while (2.33) are merely representatives of cohomologies. It may be possible to cleverly add Q-exact terms to (2.33) and make the graviton part look more like the Schur polynomial for $n \gg 1$.

2.3 No cohomologies below the threshold of the tower

We have found the SU(3) black hole cohomologies at order $j = 24 + 2n$ in the representation $[n, 0]$ for all $n \geq 3$. From the gravity dual (barring the difference between $N = 3$ and ∞), the presence of an infinite tower is natural because the dual giant wrapping the core black hole can be indefinitely large with larger electric charge R. On the opposite end, the dual giant graviton should be larger than a certain size to wrap a definite core black hole. It would also be natural to imagine so for a definite black hole microstate. At this point, note that table 2 manifests a curious feature, that the cohomologies only start to appear at $n = 3$. So we are naturally led to ask what our general methods yield at $n = 1$ and 2. We show in this subsection that indeed the methods of our section 2.1 fail to produce new hairy cohomologies at $n = 1, 2$. Our methods either yield no Q-closed operators at all ($n = 1$), or only yield a Q-closed operator with no SU(2) × U(1) term which obstructs the Q-exactness ($n = 2$). The failure to find cohomologies with too small hairs at $n \leq 2$ adds support to our attributing the tower F_1 to the black holes with dual giant hair. We also claim that $n = 3$ indicates the minimal size of the dual giant graviton to wrap the core black hole, roughly equivalent to the size of the core black hole itself.

Let us proceed similarly to section 2.2, to find a black hole cohomology for $n = 1$, i.e. at order $j = 26$, with 3 scalars and 5 fermions, under the representation $[1, 0]$. There are only three ways to multiply the fundamental trace relations (A.2) by gravitons to achieve the desired field contents and the representation. They are

$$\begin{aligned}
(s_{26,1})^i &= \epsilon^{a_1 a_2 a_3} v^{a_4}{}_{a_1} v^{a_5}{}_{a_2} (R^{(0,2)}_{14})^i_{a_3 a_4 a_5} \,, \\
(s_{26,2})^i &= v^i{}_a (R^{(0,3)}_{20})^a \,, \\
(s_{26,3})^i &= (R^{(0,4)}_{26})^i \,.
\end{aligned} \tag{2.36}$$

However, the three operators are independent of each other. (It is trivial by definition of the fundamental trace relation that $s_{26,3}$ is independent from the other two.) So there is no relation of relations (2.8), and thus no Q-closed operator that can be written under our ansatz.

We turn to the case $n = 2$, i.e. at order $j = 28$, with 4 scalars and 5 fermions, under the representation $[2, 0]$. There are 13 tensors that span the non-fundamental trace relations with the desired field contents in the representation $[2, 0]$:

$$\begin{aligned}
(s_{28,1})^{ij} &= \epsilon^{a_1 a_2 (i|} v^{a_3}{}_{a_5} v^{a_4 a_5}{}_{a_1} (R^{(0,2)}_{14})^{|j)}_{a_2 a_3 a_4} \,, \\
(s_{28,2})^{ij} &= \epsilon^{a_1 a_2 a_3} v^{a_4}{}_{a_1} v^{a_5 (i}{}_{a_2} (R^{(0,2)}_{14})^{j)}_{a_3 a_4 a_5} \,, \\
(s_{28,3})^{ij} &= \epsilon^{a_1 a_2 (i|} v^{a_3}{}_{a_5} v^{|j) a_4}{}_{a_1} (R^{(0,2)}_{14})^{a_5}_{a_2 a_3 a_4} \,, \\
(s_{28,4})^{ij} &= \epsilon^{a_1 a_2 (i} v^{j)}{}_{a_1} v^{a_3 a_4}{}_{a_5} (R^{(0,2)}_{14})^{a_5}_{a_2 a_3 a_4} \,, \\
(s_{28,5})^{ij} &= \epsilon^{a_1 a_2 (i} v^{j)}{}_{a_5} v^{a_3 a_4}{}_{a_1} (R^{(0,2)}_{14})^{a_5}_{a_2 a_3 a_4} \,,
\end{aligned}$$

JHEP07(2025)216

$$
\begin{aligned}
(s_{28,6})^{ij} &= \epsilon^{a_1a_2(i}v^{j)}{}_{a_3}v^{a_3}{}_{a_1}(R^{(0,2)}_{16})_{a_2}\,,\\
(s_{28,7})^{ij} &= \epsilon^{a_1a_2(i}v^{j)}{}_{a_1}v^{a_3}{}_{a_2}(R^{(0,2)}_{16})_{a_3}\,,\\
(s_{28,8})^{ij} &= \epsilon^{a_1a_2(i|}v^{a_3}{}_{a_4}v^{a_4}{}_{a_1}(R^{(0,2)}_{16})^{|j)}_{a_2a_3}\,,\\
(s_{28,9})^{ij} &= \epsilon^{a_1a_2a_3}v^{a_4}{}_{a_1}v^{(i}{}_{a_2}(R^{(0,2)}_{16})^{j)}_{a_3a_4}\,,\\
(s_{28,10})^{ij} &= \epsilon^{a_1a_2(i}v^{j)}{}_{a_1}v^{a_3}{}_{a_4}(R^{(0,2)}_{16})^{a_4}_{a_2a_3}\,,\\
(s_{28,11})^{ij} &= \epsilon^{a_1a_2a_3}v^{a_4}{}_{a_1}v^{a_5}{}_{a_2}(R^{(0,2)}_{16})^{ij}_{a_3a_4a_5}\,,\\
(s_{28,12})^{ij} &= v^{ij}{}_a(R^{(0,3)}_{20})^a\,,\\
(s_{28,13})^{ij} &= v^{(i}{}_a(R^{(0,3)}_{22})^{j)a}\,.
\end{aligned}
\tag{2.37}
$$

There is one linear combination of these that vanishes,

$$
\begin{aligned}
&-30s_{28,1}+30s_{28,3}+\frac{15}{4}s_{28,4}-15s_{28,5}-\frac{7}{20}s_{28,6}+\frac{21}{80}s_{28,7}\\
&+\frac{49}{36}s_{28,8}-\frac{7}{12}s_{28,9}+\frac{7}{24}s_{28,10}+s_{28,11}=0\,,
\end{aligned}
\tag{2.38}
$$

which provides a Q-closed operator via (2.9).

By the $\mathrm{SU}(2)\times\mathrm{U}(1)$ restriction, we expand this operator as in (2.13), focusing on terms with $R_1=[2,0]$. The analogous expansion to (2.23) is

$$
\mathcal{O}^{ij}\xrightarrow{\mathrm{SU}(2)\times\mathrm{U}(1)}\hat{\phi}^i\hat{\phi}^j\mathcal{T}+\hat{\phi}^a\hat{\phi}^{(i}\mathcal{T}^{j)}_a+\hat{\phi}^a\hat{\phi}^b\mathcal{T}^{ij}_{ab}\,.
\tag{2.39}
$$

Extracting the singlet part $\mathcal{T}$ from the Q-closed operator, we get

$$
\mathcal{T}=0\,.
\tag{2.40}
$$

Therefore, the only Q-closed operator that is constructed by our ansatz cannot be proved to be not Q-exact with the method of section 2.1.2. Although this does not prove that it is Q-exact, we find it suggestive that both our method and the index (table 2) predict no non-trivial cohomologies in this charge sector. We emphasize that (2.40) is not as trivial as it seems. If one makes $\mathrm{SU}(2)\times\mathrm{U}(1)$ restrictions of individual $s_{28,a}$'s, many of the corresponding $\mathcal{T}$ parts are nonzero and proportional to the $\mathrm{SU}(2)$ cohomology $\mathcal{O}^2_{24}$. However, they all intricately cancel in the prescribed linear combination (2.38). So our method fails to produce new cohomologies nontrivially.

3 More hairy black hole cohomologies

In this section, we consider the hairy black hole cohomologies obtained by multiplying the w-type gravitons to the cohomologies in the F_1 tower of table 2. We show that products with the w_2 gravitons always lead to new cohomologies, partially justifying hairs speculated in [8]. Then we comment on possibilities of new cohomologies by further dressing with the w_3 graviton hair.

We start by summarizing the results from the previous section. For every integer $n\geq 3$, there exists a black hole cohomology $(\mathcal{O}^3_{24+2n})^{i_1\cdots i_n}$ at order $24+2n$ in the $[n,0]$ representation

JHEP07(2025)216

of $\mathrm{SU}(3)_R$. For each n, upon the $\mathrm{SU}(2)\times\mathrm{U}(1)$ restriction of the fields and decomposing as in (2.23), the singlet part $\mathcal{T}$ that corresponds to $\mathcal{O}^2[R_2]$ in (2.13) with $R_1=[n,0]$ and $R_2=[0,0]$ is a non-zero multiple of the SU(2) black hole cohomology $\mathcal{O}^2_{24}$, and other $\mathcal{T}$'s with non-trivial representations under $\mathrm{SU}(3)_R$ are Q-exact. In short,

$$(\mathcal{O}^3_{24+2n})^{i_1\cdots i_n}\xrightarrow{\mathrm{SU}(2)\times\mathrm{U}(1)}\hat\phi^{i_1}\cdots\hat\phi^{i_n}\ C_n\mathcal{O}^2_{24}+(Q\text{-exact})\,.\qquad(n\geq 3,\ C_n\neq 0)\tag{3.1}$$

First consider the product $w_2\mathcal{O}^3_{24+2n}$, where we recall that ${w_2}^i\equiv\mathrm{tr}\left(f\phi^i+\frac{1}{2}\epsilon^{ia_1a_2}\psi_{a_1}\psi_{a_2}\right)$. The product decomposes into two irreps,

$$[1,0]\otimes[n,0]=[n+1,0]\oplus[n-1,1]\,.\tag{3.2}$$

The explicit forms of the operators are

$$\begin{aligned}(w_2\mathcal{O}^3_{24+2n})^{i_1\cdots i_{n+1}}&=w^{(i_1}(\mathcal{O}^3_{24+2n})^{i_2\cdots i_{n+1})}\,,\\(w_2\mathcal{O}^3_{24+2n})^{i_1\cdots i_{n-1}}_j&=\epsilon_{jab}w^a(\mathcal{O}^3_{24+2n})^{bi_1\cdots i_{n-1}}\,.\end{aligned}\tag{3.3}$$

Both operators in (3.3) are automatically Q-closed, because both w_2 and $\mathcal{O}^3_{24+2n}$ separately are.

Now let us try to determine if they are not Q-exact, using the method of section 2.1.2. We perform the $\mathrm{SU}(2)\times\mathrm{U}(1)$ restriction of the fields (2.10) and collect terms that contain n factors of the U(1) components $\hat\phi$. No factor of $\hat\phi$ can be extracted from ${w_2}^i\equiv\mathrm{tr}\left(f\phi^i+\frac{1}{2}\epsilon^{ia_1a_2}\psi_{a_1}\psi_{a_2}\right)$, because if one extracts $\hat\phi$ from the only ϕ inside the trace, it will leave the traceless field f alone in the SU(2) trace. So the w_2 graviton must be entirely SU(2), and all n factors of $\hat\phi$ must come from $\mathcal{O}^3_{24+2n}$. The latter is summarized in (3.1), so we can easily write the $\hat\phi^n$ terms in the $\mathrm{SU}(2)\times\mathrm{U}(1)$ restriction of the two operators in (3.3):

$$\begin{aligned}(w_2\mathcal{O}^3_{24+2n})^{i_1\cdots i_{n+1}}&\xrightarrow{\mathrm{SU}(2)\times\mathrm{U}(1)}\hat\phi^{(i_2}\cdots\hat\phi^{i_{n+1}}\ C_nw^{i_1)}\mathcal{O}^2_{24}+(Q\text{-exact})\,,\\(w_2\mathcal{O}^3_{24+2n})^{i_1\cdots i_{n-1}}_j&\xrightarrow{\mathrm{SU}(2)\times\mathrm{U}(1)}\epsilon_{jab}\hat\phi^b\hat\phi^{i_1}\cdots\hat\phi^{i_{n-1}}\ C_nw^a\mathcal{O}^2_{24}+(Q\text{-exact})\,.\end{aligned}\tag{3.4}$$

(3.4) are specific applications of the general principle (2.13) on the operators with $R=[n+1,0]$ and $[n-1,1]$. We focus on the term with $R_1=[n,0]$ and $R_2=[1,0]$ on the right hand side. $R_1\otimes R_2=[n,0]\otimes[1,0]$ contains both $[n+1,0]$ and $[n-1,1]$, so this term is allowed for both operators. The advantage of this choice of R_1 and R_2 will become clear shortly. (3.4) shows that for both operators, the corresponding SU(2) block $\mathcal{O}^2[1,0]$ is a non-zero multiple of $w^i\mathcal{O}^2_{24}$. Meanwhile, the SU(2) operator $w^i\mathcal{O}^2_{24}$ has been proved to be a black hole cohomology in [3], so it is not Q-exact. Therefore, the terms proportional to $w^i\mathcal{O}^2_{24}$ obstruct the operators on the left hand sides from being Q-exact, proving that the entire product between w_2 and $\mathcal{O}^3_{24+2n}$, which consists of irreps $[n+1,0]$ and $[n-1,1]$, will be the new black hole cohomologies.

This argument can be repeated for products with arbitrary number of w_2 gravitons. Consider the product between k w_2 gravitons and $\mathcal{O}^3_{24+2n}$. The representation of the product would be $[1,0]_S^{\otimes k}\otimes[n,0]=[k,0]\otimes[n,0]$. $[1,0]_S^{\otimes k}$ is the symmetric product of k $[1,0]$'s because w_2 are identical bosons, which is $[k,0]$. We write the product operator schematically as

$$(w_2)^k\mathcal{O}^3_{24+2n}\,,\tag{3.5}$$

neglecting the tensor structure. In other words, (3.5) collectively denotes all possible irreps that arise from the tensor product $[k,0]\otimes[n,0]$.

JHEP07(2025)216

We then perform the SU(2) × U(1) restriction and collect terms in $\hat{\phi}^n$. Because no factor of $\hat{\phi}$ can be extracted from the w_2 gravitons, we can use (3.1) directly to write the SU(2) × U(1) restriction of (3.5):

$$(w_2)^k \mathcal{O}^3_{24+2n} \xrightarrow{\text{SU(2)}\times\text{U(1)}} \hat{\phi}^n \; C_n (w_2)^k \mathcal{O}^2_{24} + (Q\text{-exact})\,. \tag{3.6}$$

It is again written schematically neglecting the tensor structure, but the tensor structure on the left between $[k,0]$ of $(w_2)^k$ and $[n,0]$ of $\mathcal{O}^3_{24+2n}$ is identical to that on the right between $[k,0]$ of $(w_2)^k$ and $[n,0]$ of $\hat{\phi}^n$.

(3.6) is a specific application of the general principle (2.13) on the operator $(w_2)^k \mathcal{O}^3_{24+2n}$ with $\text{SU}(3)_R$ irrep R that is contained in $[k,0]\otimes[n,0]$. We are focusing on the term with $R_1 = [n,0]$ and $R_2 = [k,0]$ on the right hand side, which is clearly allowed because R is contained in $R_1 \otimes R_2$ by definition. (3.6) shows that the SU(2) block $\mathcal{O}^2[R_2]$ for this term is a non-zero multiple of $(w_2)^k \mathcal{O}^2_{24}$. Meanwhile, all SU(2) operators $(w_2)^k \mathcal{O}^2_{24}$ have been conjectured to be black hole cohomologies, i.e. not Q-exact, in [3]. While it was shown only for $k = 1$, it was conjectured for all $k > 1$ based on the index over black hole cohomologies which exhibits a geometric series that can be interpreted as being freely generated by the w_2 gravitons. It therefore acts as the obstruction term against the Q-exactness of all product operators $(w_2)^k \mathcal{O}^3_{24+2n}$.

This partially justifies the scenario of [8] that the SU(3) black hole cohomologies would be freely dressed by the w_2 gravitons hairs. This scenario was established rather empirically, based on i) the observation in [3] that the w_2 graviton can dress the black hole cohomologies to yield new black hole cohomologies in the SU(2) theory, and ii) that the SU(3) index simplifies significantly when the geometric series with respect to the w_2 gravitons is factored out. Table 2 is the result of the factorization. Here we have shown that the black hole cohomologies in the F_1 tower can be dressed by any number of the w_2 graviton hair.

Finally, we comment on the w_3 graviton hair. Products with the w_3 gravitons will only allow us to speculate, as opposed to guarantee, new hairy black hole cohomologies.

Consider the product between a single w_3 graviton and $\mathcal{O}^3_{24+2n}$. They transform in the irreps $[2,0]$ and $[n,0]$ of $\text{SU}(3)_R$, respectively. The product decomposes into three irreps,

$$[2,0]\otimes[n,0] = [n+2,0]\oplus[n,1]\oplus[n-2,2]\,. \tag{3.7}$$

The explicit forms of operators are

$$\begin{aligned}
(w_3\mathcal{O}^3_{24+2n})^{i_1\cdots i_{n+2}} &= w^{(i_1 i_2}(\mathcal{O}^3_{24+2n})^{i_3\cdots i_{n+2})}\,,\\
(w_3\mathcal{O}^3_{24+2n})^{i_1\cdots i_n}_{j} &= \epsilon_{ja_1a_2} w^{a_1(i_1}(\mathcal{O}^3_{24+2n})^{i_2\cdots i_n)a_2}\,,\\
(w_3\mathcal{O}^3_{24+2n})^{i_1\cdots i_{n-2}}_{jk} &= \epsilon_{a_1a_2(j}\epsilon_{k)b_1b_2} w^{a_1b_1}(\mathcal{O}^3_{24+2n})^{a_2b_2i_1\cdots i_{n-2}}\,.
\end{aligned} \tag{3.8}$$

All three operators are automatically Q-closed, because both w_3 and $\mathcal{O}^3_{24+2n}$ separately are.

Now we study if they are not Q-exact, using the method of section 2.1.2. We perform the SU(2) × U(1) restriction of the fields (2.10) and collect the terms that contain $n+1$ factors of $\hat{\phi}$. The one added factor of $\hat{\phi}$ is because one and only one factor of $\hat{\phi}$ can be extracted from $w_3 = \text{tr}\left(f\phi^{(i}\phi^{j)} + \epsilon^{a_1a_2(i}\phi^{j)}\psi_{a_1}\psi_{a_2}\right)$. w_3 contains at most two scalars, but it is impossible to

JHEP07(2025)216

extract two scalar U(1) components from $\mathrm{tr}[f\phi^{(i}\phi^{j)}]$ because it will leave the SU(2) block of the traceless f alone inside the SU(2) trace. It is also impossible to extract no $\hat{\phi}$ from w_3, because the entire w_3 graviton operator will then be restricted to the SU(2) block, which is zero. Thus, terms with $n+1$ factors of $\hat{\phi}$ can only be obtained by extracting one $\hat{\phi}$ from w_3 and n from $\mathcal{O}^3_{24+2n}$. Under the $\mathrm{SU}(2)\times\mathrm{U}(1)$ restriction of w_3, terms that contain one factor of $\hat{\phi}$ is

$$w_3^{ij} \to 2\hat{\phi}^{(i} w_2^{j)} \,. \tag{3.9}$$

On the other hand, the $\hat{\phi}^n$ terms in the $\mathrm{SU}(2)\times\mathrm{U}(1)$ restriction of $\mathcal{O}^3_{24+2n}$ was already given in (3.1). Combining the two, the $\hat{\phi}^{n+1}$ terms from the three operators in (3.8) are

$$\begin{aligned}
w^{(i_1 i_2}(\mathcal{O}^3_{24+2n})^{i_3\cdots i_{n+2})} &\xrightarrow{\mathrm{SU}(2)\times\mathrm{U}(1)} 2C_n\hat{\phi}^{(i_1\cdots i_{n+1}}\, w^{i_{n+2})}\mathcal{O}^2_{24} + (Q\text{-exact})\,, \\
\epsilon_{ja_1a_2} w^{a_1(i_1}(\mathcal{O}^3_{24+2n})^{i_2\cdots i_n)a_2} &\xrightarrow{\mathrm{SU}(2)\times\mathrm{U}(1)} -C_n\epsilon_{ja_1a_2}\hat{\phi}^{(i_1\cdots i_n)a_1}\, w^{a_2}\mathcal{O}^2_{24} + (Q\text{-exact})\,, \\
\epsilon_{a_1a_2(j}\epsilon_{k)b_1b_2} w^{a_1b_1}(\mathcal{O}^3_{24+2n})^{a_2b_2i_1\cdots i_{n-2}} &\xrightarrow{\mathrm{SU}(2)\times\mathrm{U}(1)} (Q\text{-exact})\,.
\end{aligned} \tag{3.10}$$

(3.10) are specific applications of the general principle (2.13) on the operators with $R=[n+2,0]$, $[n,1]$ and $[n-2,2]$, and we are focusing on the term with $R_1=[n+1,0]$ and $R_2=[1,0]$ on the right hand side. For the first two operators, the SU(2) block $\mathcal{O}^2[R_2]\propto w_2\mathcal{O}^2_{24}$ obstructs the operators on the left hand sides from being Q-exact. In contrast, this term cannot exist for the third operator, because $[n+1,0]\otimes[1,0]$ does not contain $[n-2,2]$. So there is no obstruction against the Q-exactness of the third operator.

This argument for a single factor of w_3 can be generalized to an arbitrary number of them, analogously to the factors of w_2. Consider the product between k w_3 gravitons and $\mathcal{O}^3_{24+2n}$. The $\mathrm{SU}(3)_R$ representation of the product will be $[2,0]_S^{\otimes k}\otimes[n,0]$, where $[2,0]_S^{\otimes k}$ is the symmetric product of k $[2,0]$'s. We write the product operator schematically as

$$(w_3)^k\mathcal{O}^3_{24+2n}\,, \tag{3.11}$$

neglecting the tensor structure. In other words, (3.11) collectively denotes all possible irreps that arise from the tensor product $[2,0]_S^{\otimes k}\otimes[n,0]$.

We then perform the $\mathrm{SU}(2)\times\mathrm{U}(1)$ restriction and collect terms at the $\hat{\phi}^{n+k}$ order, anticipating each w_3 to contribute one factor of $\hat{\phi}$. We aim to find the SU(2) operator $(w_2)^k\mathcal{O}^2_{24}$ as the obstruction term against the Q-exactness. Combining (3.9) and (3.1),

$$(w_3)^k\mathcal{O}^3_{24+2n} \xrightarrow{\mathrm{SU}(2)\times\mathrm{U}(1)} \hat{\phi}^{n+k}\; C'_n(w_2)^k\mathcal{O}^2_{24} + (Q\text{-exact})\,, \tag{3.12}$$

where C'_n is the product of C_n and the numerical factor that arises from the tensor structure, such as 2 and -1 for the first two of (3.10). It is again written schematically neglecting the tensor structure, but it is not as trivial as the case for w_2, so let us comment on it. Each $[2,0]_{w_3}$ on the l.h.s. is decomposed on the r.h.s. to a symmetric product $[1,0]_{\hat{\phi}}\otimes_S[1,0]_{w_2}$, see (3.9). Of the product between $(w_3)^k$ and $\mathcal{O}^3_{24+2n}$ on the l.h.s., it is only the part where the k factors of $[1,0]_{\hat{\phi}}$ and $[n,0]_{\mathcal{O}^3_{24+2n}}$ are multiplied symmetrically into $[n+k,0]$ that appears as $\hat{\phi}^{n+k}$ on the r.h.s. Then, since the k factors of $[1,0]_{\hat{\phi}}$ are symmetrized, the k factors of $[1,0]_{w_2}$ must be symmetrized among themselves as well. This is what appears as $(w_2)^k\mathcal{O}^2_{24}$ on the

JHEP07(2025)216

r.h.s. To summarize, among the operator $(w_3)^k\mathcal{O}^3_{24+2n}$ on the l.h.s. that collectively denotes many irreps that appear in $[2,0]_S^{\otimes k}\otimes[n,0] = ([1,0]_{\hat{\phi}}\otimes_S[1,0]_{w_2})_S^{\otimes k}\otimes[n,0]$, only the part where k factors of $[1,0]_{\hat{\phi}}$ and $[n,0]_{\mathcal{O}^3_{24+2n}}$ are symmetrized into $[n+k,0]$ and k factors of $[1,0]_{w_2}$ are symmetrized into $[k,0]$ admit the term proportional to $(w_2)^k\mathcal{O}^2_{24}$ as displayed on the r.h.s.

Provided that such a term on the r.h.s. is present, this obstructs the operator on the l.h.s. from being Q-exact. However, it is too quick to declare that we have found new cohomologies. Recall the product operator $(w_2)^k\mathcal{O}^3_{24+2(n+k)}$ and the proof for its non-Q-exactness presented earlier in this section, but with the value of n shifted. Its representation is $[1,0]_S^{\otimes k}\otimes[n+k,0] = [k,0]\otimes[n+k,0]$. Reflecting on the previous paragraph, this is precisely the representation of the subset of $(w_3)^k\mathcal{O}^3_{24+2n}$ that admits the obstruction term, and the obstruction terms for the two operators are identical.

As we have commented towards the end of section 2.2 on $\mathcal{O}^3_{24+2n}$ with $n\geq 6$, our method guarantees only one nontrivial cohomology if many Q-closed operators share the common obstruction term. So we cannot conclude that the black hole cohomology with the w_3 graviton hair $(w_3)^k\mathcal{O}^3_{24+2n}$ yields any new black hole cohomology in addition to those with the w_2 hair $(w_2)^k\mathcal{O}^3_{24+2(n+k)}$. On the other hand, we cannot conclude that they are not cohomologous either. With pending definiteness, we briefly comment on possible scenarios.

Taking the most conservative viewpoint that we only take into account the guaranteed black hole cohomologies,[4] the w_3 hair does not add any new cohomology.

Now we explore more progressive viewpoints. First consider the dual giant hair cohomologies in the F_1 tower dressed by one factor of the w_3 graviton hair. Among the product $w_3\mathcal{O}^3_{24+2n}$ that transforms under $[2,0]\otimes[n,0] = [n+2,0]\oplus[n,1]\oplus[n-2,2]$, those that have the $w_2\mathcal{O}^2_{24}$ obstruction term correspond to $[1,0]\otimes[n+1,0] = [n+2,0]\oplus[n,1]$. Note that $n\geq 3$, and that $w_3\mathcal{O}^3_{24+2n}$ has the order $j = 34+2n$. Thus, if $w_3\mathcal{O}^3_{24+2n}$ belongs to the different cohomology class from $w_2\mathcal{O}^3_{24+2(n+1)}$, then they will nicely account for the F_2 and F_4 towers in table 2. Second and finally, consider the dual giant hair cohomologies in the F_1 tower dressed by two factors of the w_3 graviton hair. Among the product $(w_3)^2\mathcal{O}^3_{24+2n}$, those that have the $(w_2)^2\mathcal{O}^2_{24}$ obstruction term correspond to $[2,0]\otimes[n+2,0] = [n+4,0]\oplus[n+2,1]\oplus[n,2]$. If they are not cohomologous to the corresponding parts of $(w_2)^2\mathcal{O}^3_{24+2(n+2)}$, they will account for towers that start with $[7,0]$, $[5,1]$ and $[3,2]$ at $j=50$. From table 2, we note that the first such tower nicely accounts for the series F_3, while the other two have not been detected by the index. We feel that some of the product cohomologies discussed in this paragraph (if not all) have chances to be new cohomologies, independent of the w_2 hairs, accounting for some towers in table 2. Since there are no independent obstruction terms against the Q-exactness in the $\mathrm{SU}(2)\times\mathrm{U}(1)$ restriction, it will require a different method to prove or disprove this.

4 General comments on two types of hairs

In section 2, we studied novel hairy black hole cohomologies which are fusion products of core black hole microstates and the dual giant graviton hairs. On the other hand, more traditional hairs given by the point particles in AdS and their quantum wave functions

[4] This was our viewpoint of section 2.2, where we advocated a single F_1 tower of dual giant hairs despite possible multiplicities for $n\geq 6$. In this case, we were also strongly guided by the gravity results of [15, 23].

JHEP07(2025)216

have been discussed in [3, 8] and in section 3. Their cohomologies were suggested to admit product representatives of the form (black hole) × (graviton). Concrete examples of both hairy cohomologies are available only at small N. In this section, we argue the existence of the product hairy cohomologies at general N, and comment on related works in the literature.

The product cohomologies were suggested to represent the microstates of hairy black holes that have fast rotating point particle hairs outside the horizon.[5] To be definite, consider the cohomology represented by the product operator

$$\mathcal{O} \cdot \partial_1^{j_1} \partial_2^{j_2} \mathrm{tr}(X^2)\,, \tag{4.1}$$

where $\mathcal{O}$ is a fixed black hole cohomology, $\partial_{1,2}$ are the two holomorphic derivatives on $\mathbb{R}^4$ that commute with Q, and $\partial_1^{j_1} \partial_2^{j_2} \mathrm{tr}(X^2)$ is a conformal descendant of the primary graviton operator $\mathrm{tr}(X^2)$ at the origin of $\mathbb{R}^4$, where $X \equiv \phi^1$ in the notation of this paper. (4.1) is Q-closed because both factors separately are. The goal of this section is to discuss the (non)-Q-exactness of (4.1) for general N. The discussions in this section will hold for more general graviton operators replacing $\mathrm{tr}(X^2)$. Also, most of the arguments (before the gravity dual analysis) will hold for any $\mathcal{O}$ that represents a nontrivial cohomology, either of graviton or of black hole type.

We first show a modest claim, that it is impossible for (4.1) to be Q-exact for all $j_{1,2} \geq 0$. To prove this statement, let us assume otherwise, that they are all Q-exact. One can equivalently state that the operator

$$\mathcal{O} \cdot \mathrm{tr}(X(z)X(z)) \tag{4.2}$$

is Q-exact for any $z = (z_1, z_2) \equiv z_\alpha$, where $X(z)$ is the holomorphic field operator defined in terms of the letters at the origin by

$$X(z) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} (z_1 D_1 + z_2 D_2)^n X(0) = \sum_{n=0}^{\infty} \frac{1}{n!} (z_\alpha D_\alpha)^n X(0)\,. \tag{4.3}$$

(4.2) for a nonzero z_α is Taylor expanded into (4.1).

In terms of holomorphic fields, the Q-transformation takes the following local form [27, 28],

$$\begin{aligned} Q\phi^m(z) &= [\lambda(z), \phi^m(z)]\,, & Q\psi_m(z) &= \epsilon_{mnp}[\phi^n(z), \phi^p(z)] + \{\lambda(z), \psi_m(z)\}\,, \\ Qf(z) &= [\phi^m(z), \psi_m(z)] + [\lambda(z), f(z)]\,, & Q\lambda(z) &= \{\lambda(z), \lambda(z)\}\,, \end{aligned} \tag{4.4}$$

where we have included the gaugino λ,

$$\begin{aligned} \phi^m(z) &= \sum_{n=0}^{\infty} \frac{1}{n!} (z_\alpha D_\alpha)^n \phi^m(0)\,, & \psi_m(z) &= \sum_{n=0}^{\infty} \frac{1}{n!} (z_\alpha D_\alpha)^n \psi_m(0)\,, \\ f(z) &= \sum_{n=0}^{\infty} \frac{1}{n!} (z_\alpha D_\alpha)^n f(0)\,, & \lambda(z) &= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (z_\alpha D_\alpha)^n (z_\beta \lambda_\beta(0))\,. \end{aligned} \tag{4.5}$$

In the holomorphic field basis, the supercharge Q acts 'locally' at given z, i.e. it transforms a local operator at z into a local operator at the same z.

[5] More specifically, they can be interpreted as the BPS limits of the 'gray galaxies' [22–24].

JHEP07(2025)216

Now we come back to the assumption (to be disproved) that (4.2) for any z is Q-exact, say $Q\Lambda$. Since $\mathcal{O}$ and the graviton part $\mathrm{tr}(X(z)X(z))$ are not Q-exact by themselves but only via a trace relation, There must be a term in Λ on which Q acts and produces a letter $X(z)$ and another letter at the same z: note from (4.4) that Q always produces two letters at the same location. Then by a trace relation, $X(z)$ enters the graviton part while the other enters $\mathcal{O}$. So $\mathcal{O}$ must contain a local field at z. However, the local field at $z \neq 0$ is a sum of infinitely many letters at the origin, with indefinitely many derivatives. As $\mathcal{O}$ is an operator of fixed shape defined at the origin, it cannot have a term with infinitely many letters, and thus cannot contain the local field at $z \neq 0$. This proves that (4.2) cannot be Q-exact, and it follows that (4.1) cannot be Q-exact for all $j_{1,2} \geq 0$. Therefore, *some* conformal descendant graviton hair should be allowed, admitting the product representative (4.1).

We can slightly generalize the argument to show that, for a given black hole cohomology $\mathcal{O}$, an infinite number of the products with conformal descendant gravitons (4.1) must be not Q-exact. Let us assume the opposite, that all but a finite number $n \geq 1$ of products (4.1) are Q-exact. Then, among the Taylor expansion at the origin of the operator (4.2), up to n of the expansion terms will be not Q-exact. However, consider a more general class of operators

$$\mathcal{O} \cdot \left[\mathrm{tr}(X(z)^2) + \sum_{i=1}^{n} a_i \mathrm{tr}(X(z_i)^2) \right] , \tag{4.6}$$

for some z and z_i $(i = 1, \cdots, n)$ that are all different. Now consider the Taylor expansion of (4.6) at the origin. In general, it will be possible to tune n coefficients a_i's so that the coefficients for the n non-Q-exact expansion terms vanish. For these suitably chosen values of a_i's for given z and z_i, (4.6) must be Q-exact by trace relations. However, repeating the previous argument, $\mathcal{O}$ must contain local fields at some of z and z_i's, which is impossible for the black hole cohomology with fixed shape defined at the origin. So we conclude that, even after excluding any finite subset of operators in (4.1), there still exist non-Q-exact operators in the remaining set. A consequence is that infinitely many operators of the form (4.1) are not Q-exact.[6]

This is still weaker than what we conjecture to be true. We expect that, if (4.1) for some values $(j_1, j_2) = (j_1^{(0)}, j_2^{(0)})$ of $j_{1,2}$ is not Q-exact, then the same is true for all $j_1 \geq j_1^{(0)}$, $j_2 \geq j_2^{(0)}$. Namely, we conjecture that (4.1) is nontrivial beyond certain lower bound. We leave either its proof or disproof as a future work.

The conjecture of the previous paragraph is partly motivated by [3], which studied the index for the cohomologies at $N = 2$ and also the perturbative black hole hair at $N = \infty$. Here we discuss the latter studies in more detail. From the gravity dual perspective, our conjecture is natural because the size of the core black hole is fixed by $\mathcal{O}$ and the wavefunction of the graviton will be farther away from the black hole as we increase $j_{1,2}$, making it easier to be a hair outside the horizon. For simplicity, consider the AdS_5 BPS black hole with macroscopic angular momenta $J_1 = J_2 \equiv J$ and charges $R_1 = R_2 = R_3 \equiv R$ [10], whose microstate is $\mathcal{O}$. Such BPS black hole solutions are labeled by one parameter, which we call

[6]The argument of this paragraph does not extend to the case with $n = \infty$. For instance, one can construct a local operator at $z = 0$ by an infinite linear combination $\Psi(0) = \oint \frac{dz}{z} \Psi(z)$. If the graviton part consists of local fields at $z = 0$ only, there are known examples in which (4.6) can be Q-exact: see appendix D of [3].

JHEP07(2025)216

q. The charges carried by this black hole are given by

$$R = \frac{N^2}{2}\left(\frac{q}{\ell^2} + \frac{q^2}{2\ell^4}\right), \quad J = \frac{N^2}{2}\left(\frac{3q^2}{2\ell^4} + \frac{q^3}{\ell^6}\right), \tag{4.7}$$

where ℓ is the AdS radius, satisfying a charge relation

$$J(R) = -\frac{N^2}{2} - 3R + \frac{(N^2+4R)^{\frac{3}{2}}}{2N}. \tag{4.8}$$

It was noted [3] that the lower bound on j_1, j_2 for the hair to exist is related to the point particle trajectory (or the peak of its quantum wavefunction) staying outside the event horizon. The black hole hair for the bulk field dual to $\mathrm{tr}(X^2)$ is shown to exist for $j_1 + j_2 > \frac{4q}{\ell^2}$ [3].[7] For later use, we note that in the large charge regime $q \gg \ell^2$, the bound for the BPS hair to exist can be written as

$$j \equiv j_1 + j_2 > 2\left(\frac{2J}{N^2}\right)^{\frac{1}{3}} \tag{4.9}$$

in terms of the macroscopic charge.

Now we reconsider the bound (4.9) by combining the QFT and gravity insights. If the black hole microstate $\mathcal{O}$ carries angular momentum J, it may come either from derivatives or from the intrinsic spin carried by the fields. At very large charges $R \gg N^2$ and $J \gg N^2$, the charge relation (4.8) for the large black holes is approximated by

$$J(R) \sim \frac{4R^{\frac{3}{2}}}{N}. \tag{4.10}$$

In this 'Cardy' limit, one expects that the angular momentum J come almost entirely from the derivatives. This is because in the Cardy limit, the QFT dual derivation of the free energy yields divergence due to unsuppressed numbers of derivatives [29–33]. So the numbers J_1, J_2 of derivatives carried by the black hole microstate operator $\mathcal{O}$ are approximately given by

$$J_1 \approx J_2 = J \sim \frac{4R^{\frac{3}{2}}}{N}. \tag{4.11}$$

At least for the order of magnitude estimate, the number of letters can be taken to be proportional to R unless $\mathcal{O}$ is dominantly made by the field strength f, since all other elementary fields carry $O(1)$ R-charges. Therefore, distributing the J derivatives to $\sim R$ letters, each letter carries roughly the following number of derivatives,

$$\frac{J}{R} \sim \frac{J^{\frac{1}{3}}}{N^{\frac{2}{3}}}, \tag{4.12}$$

[7] [3] also studied the linearized BPS equation for a scalar ϕ in AdS_5 dual to the operator $\mathrm{tr}(X^2)$, valid when its amplitude is small. Such solutions were found when $j_1 + j_2 > \frac{4q}{\ell^2}$, while otherwise the solution for ϕ diverges at the horizon and invalidates the linearized approximation assumed. Even for the finite solutions at $j_1 + j_2 > \frac{4q}{\ell^2}$, there is an issue of higher derivative singularities: we conjecture that they are acceptable singularities in string theory, partly because similar results are obtained from the probe particle analysis [3]. We thank Shiraz Minwalla for raising the issue of singularities to us.

JHEP07(2025)216

for a typical large black hole microstate. So the bound (4.9) for the angular momentum of the BPS hair obtained from the gravity side can be interpreted as the letters in the graviton $\partial^j \mathrm{tr}(X^2)$ carrying more derivatives than the typical letters in the black hole microstate $\mathcal{O}$ do. In other words, the parton momentum distributions inside the graviton and the black hole should be separated to admit the nontrivial product hairs. It will be interesting to make a field theoretic analysis of the allowed product cohomologies focusing on the parton momentum distributions.

Related to the product cohomologies with a large number of derivatives, we also note that the anomalous dimensions of such product operators have been studied in the literature: see [34, 35]. Namely, the operator product expansion of $O_1(0)O_2(x)$ has been discussed (in the Lorentzian signature), from which one can study the anomalous dimensions of operators of the form $O_1(0)\partial^j O_2(0)$. For the composite operator with smallest dimension, the anomalous dimension at large j is suppressed as

$$E - E_{\mathrm{BPS}} = cj^{-\tau} + \cdots , \tag{4.13}$$

with a certain number c and a negative exponent $-\tau$. Although this does not say which operators are exactly BPS (even for the operators with vanishing leading coefficient c), perhaps one can combine the intuitions of these papers and our findings in this section to prove stronger statements.

On the other hand, for the new types of hairy cohomologies explored in this paper which defy the product representatives, we have little to say at general N at this moment. However, like the product operators at large j with small anomalous dimensions quoted in the previous paragraph, one may be able to show from QFT that the anomalous dimensions are small or zero when the dual giant size n (as in the $[n, 0]$ representation in section 2) is sufficiently large. From the gravity perspective, there is a common feature between the product operators at large j and the dual giant hairy operators at large n. Since we consider the core black hole operator with definite size, it occupies a definite finite region around the center of AdS. For the product hair, large j indicates that the hair is very far away from the black hole, implying that the mutual back-reaction is parametrically small. This would be the gravity dual interpretation of (4.13). As for the dual giant hair, large R-charge indicates that the dual giant radius is parametrically large, again implying that the mutual back-reaction is small. In fact, this was the key idea for constructing the approximate hairy black hole solutions in [15, 22, 23].

As for the black hole microstates with large dual giant hairs, we consider them as the BPS states on $S^3 \times \mathbb{R}$. Exciting one large dual giant graviton would approximately correspond to exciting the large classical solution in the U(1) Cartan of SU(N). This is effectively like going deep into the Coulomb branch so that the gauge symmetry is broken to U(1) $\times$ SU($N-1$). In this description, morally, the $1 \times (N-1)$ and the $(N-1) \times 1$ off-diagonal modes would be made very heavy by the U(1) Coulomb branch VEV, acquiring mass proportional to $\sqrt{n}$. Exciting the large U(1) part to realize the large dual giant hair and then exciting the BPS black hole state in the SU($N-1$) part, we may systematically study the quantum back-reactions by the heavy off-diagonal modes. This may perhaps shed more light on the SU(2) $\times$ U(1) terms identified in section 2 versus the rest, where the former terms obstructed

JHEP07(2025)216

the full SU(3) operator from being Q-exact. For concrete studies in this setup, it would be helpful to use the large charge effective field theory on the moduli space [36] (see also [37]). Perhaps we may be able to derive a large charge asymptotic formula for the small anomalous dimensions of the hairy black hole state, analogous to (4.13). The SU($N-1$) core black hole part may be studied from the classical BPS solutions using the setup explored in [27, 38].

5 Conclusion and remarks

We constructed infinitely many new $\frac{1}{16}$-BPS cohomologies in the SU(3) maximal super-Yang-Mills theory and interpreted them as the finite N versions of the black hole microstates containing dual giant graviton hairs. Our constructions explain the tower of black hole states F_1 appearing in the SU(3) index, table 2. We also partly justified the product hairs conjectured in [8], by showing that the product between the gravitons w_2 and the cohomologies in the F_1 tower are not Q-exact. Based on our methods, it may be possible to construct the cohomologies in the fermionic towers F_2, F_3, F_3 and the bosonic towers B_1, B_2, B_3. We illustrated the possibilities for the fermionic towers $F_{2,3,4}$ in section 3. As for the fermionic cohomologies in the tower F_0, we may need different ideas, perhaps generalizing the ansatz of [3]. We also discussed the general structures of hairy black hole cohomologies of two types, the point particle hair and the dual giant graviton hair. In particular, we proved the existence of infinitely many cohomologies with rotating point particle hairs at general N.

We used a novel method to prove that the Q-closed operators constructed are not Q-exact. We showed that the terms in the operators made of the SU($N-1$) $\times$ U(1) block diagonal components cannot be Q-exact, using our knowledge on the SU($N-1$) cohomologies. This setup is motivated from the fact that the objects inside a dual giant graviton essentially looks like those of the SU($N-1$) theory. So this proof also clarifies the bulk interpretation as the black hole microstate wrapped by a dual giant graviton. Although this is quite intuitive, it would be more desirable to find an SU(N) covariant characterization of the brane wrapping the core black hole. Since the two parts of the operator have their gauge orientations entangled to each other, the characterization would hopefully be purely group theoretic. We expect it will help us visualize the emergent bulk physics, say diagnosing the relative radial locations of the event horizon and the dual giant. We hope to come back to this question in the near future.

In this paper we studied how a black hole microstate can be dressed by hairs, and discussed in the bulk how the size of the black hole can be probed by the hairs. However, this is perhaps a secondary question in the cohomology program for the BPS black hole microstates. A more central and challenging question would be the structures of the 'core' black hole operators. In the SU(2) theory, an infinite series of the core black hole operators was found explicitly [3], but the well-organized nature of these operators is unlike what we would expect from the 'typical' black hole microstates. In the SU(3) theory, one black hole cohomology without an apparent hair structure was found in [8]. However, to study the typical black hole states more extensively, one needs a broader framework than the ansatz used in this paper. We note that the large N and large charge index of these cohomologies were studied in the literature (see [29, 39, 40] and the references therein), which reproduces the Bekenstein-Hawking entropy of the BPS black holes. These results ensure that there should be a large number of black hole cohomologies at weak coupling.

JHEP07(2025)216

We remark one way to extend the ansatz of [8]. Let us denote the graviton operators by $C_0 = \{g_I\}$, the level-0 cohomologies. As we reviewed in section 2.1.1, the ansatz for the Q-closed non-graviton operators (2.9) is based on the relations of relations (2.8) between the gravitons, C_0. The ansatz leads to a subset of the non-graviton cohomologies, which we denote by C_1, the level-1 cohomologies. Then, one can find cohomologous relations (Q-exact polynomials) between $C_0 \cup C_1$, not just between the gravitons C_0. Such new trace relations $R_a^{(1)} = Qr_a^{(1)}$ at level 1 will extend the ansatz for the Q-closed operators and possibly yield new cohomologies, that we denote by C_2. This procedure can be repeated recursively to construct level-n cohomologies C_n for all $n \geq 0$. It will be interesting to know if all cohomologies can be constructed from $C_0 \cup C_1 \cup C_2 \cup \cdots$, or if not, how much can be covered by this recursive construction. The level n may also have a physical interpretation, perhaps the complexity of the black hole microstates. There may be simpler models than the $\mathcal{N} = 4$ Yang-Mills theory where these questions can be answered more concretely. (For instance, see [41] for a recent study of fortuitous cohomologies in the supersymmetric SYK models.)

Another promising approach to uncovering new cohomologies associated with trace relations involves utilizing the restricted Schur polynomial basis [42, 43] (see also [44, 45] for closely related constructions and [46] for a pedagogical review). The restricted Schur polynomials exploit the permutation symmetry that exchanges identical fields, to construct the basis. Each element of this basis is labeled by a collection of Young diagrams which label irreducible representations of the permutation group which swaps fields in the composite operator. A key advantage of this basis is its systematic treatment of trace relations. Specifically, the complete set of trace relations are captured by the statement that operators labelled by Young diagrams with more than N rows vanish. It is interesting to see if this approach is able to construct new Q-closed operators.

Acknowledgments

We thank Yiming Chen, Jaehyeok Choi, Sunjin Choi, Eunwoo Lee, Shiraz Minwalla and Masamichi Miyaji for helpful discussions. This work is supported in part by a start up research fund of Huzhou University, a Zhejiang Province talent award, a Changjiang Scholar award (RdMK), NRF grants 2023R1A2C200536011, 2022R1I1A1A01069032, 2020R1A6A1A03047877 (CQUeST) (MK), NRF grant 2021R1A2C2012350 (SK, JL) and FWO project G094523N (SL).

A Trace relations of SU(3) gravitons

In this appendix, we present the expressions for the graviton trace relations that are referred to throughout the body of this paper.

A.1 The fundamental trace relations

We list the fundamental trace relations between the SU(3) gravitons that only involve the scalars ϕ^m and the fermions ψ_m. These relations are referred to as $R_a(g_I)$ in section 2.1.1, representing polynomials of single-trace gravitons g_I that are Q-exact. These are fundamental in the sense that they generate all such polynomials. In short, they generate all cohomologous relations between the SU(3) multi-graviton operators that only consist of scalars and fermions.

JHEP07(2025)216

The single trace graviton operators only involving scalars and fermions are given by

$$
\begin{aligned}
&u_2{}^{ij} \equiv \mathrm{tr}\left(\phi^{(i}\phi^{j)}\right), && u_3{}^{ijk} \equiv \mathrm{tr}\left(\phi^{(i}\phi^{j}\phi^{k)}\right), \\
&v_2{}^{i}{}_{j} \equiv \mathrm{tr}\left(\phi^{i}\psi_{j}\right) - \frac{1}{3}\delta^{i}_{j}\mathrm{tr}(\phi^{a}\psi_{a}), && v_3{}^{ij}{}_{k} \equiv \mathrm{tr}\left(\phi^{(i}\phi^{j)}\psi_{k}\right) - \frac{1}{4}\delta^{i}_{k}\mathrm{tr}\left(\phi^{(j}\phi^{a)}\psi_{a}\right) - \frac{1}{4}\delta^{j}_{k}\mathrm{tr}\left(\phi^{(i}\phi^{a)}\psi_{a}\right),
\end{aligned}
\tag{A.1}
$$

The complete list of fundamental trace relations of the gravitons listed above are [8]

$$
\begin{aligned}
&t^{10}[1,2](u_2u_3) && : (R^{(0,0)}_{10})^{i}_{jk} && = \epsilon_{a_1a_2(j}\epsilon_{k)b_1b_2}u^{a_1b_1}u^{ia_2b_2} \\
&t^{12}[0,0](u_2u_2u_2) && : (R^{(0,0)}_{12}) && = \epsilon_{a_1a_2a_3}\epsilon_{b_1b_2b_3}u^{a_1b_1}u^{a_2b_2}u^{a_3b_3} \\
&t^{12}[2,2](u_2u_2u_2,u_3u_3) && : (R^{(0,0)}_{12})^{ij}_{kl} && = \epsilon_{a_1a_2(k}\epsilon_{l)b_1b_2}\left(u^{a_1b_1}u^{a_2b_2}u^{ij} + 6u^{a_1b_1(i}u^{j)a_2b_2}\right) \\
&t^{12}[0,3](u_2v_3) && : (R^{(0,1)}_{12})_{ijk} && = \epsilon_{(i|a_1a_2}\epsilon_{|j|b_1b_2}u^{a_1b_1}v^{a_2b_2}{}_{|k)} \\
&t^{12}[1,1](u_2v_3,u_3v_2) && : (R^{(0,1)}_{12})^{i}_{j} && = \epsilon_{ja_1a_2}\left(4u^{a_1b}v^{ia_2}{}_{b} + 3u^{ia_1b}v^{a_2}{}_{b}\right) \\
&t^{12}[2,2](u_2v_3,u_3v_2) && : (R^{(0,1)}_{12})^{ij}_{kl} && = \epsilon_{a_1a_2(k}\left(u^{a_1(i}v^{j)a_2}{}_{l)} + u^{ija_1}v^{a_2}{}_{l)}\right) \\
&t^{14}[1,0](u_2u_2v_2) && : (R^{(0,1)}_{14})^{i} && = \epsilon_{a_1a_2a_3}u^{ia_1}u^{ba_2}v^{a_3}{}_{b} \\
&t^{14}[0,2](u_2u_2v_2,u_3v_3) && : (R^{(0,1)}_{14})_{ij} && = \epsilon_{a_1a_2(i|}\left(\epsilon_{b_1b_2b_3}u^{a_1b_1}u^{a_2b_2}v^{b_3}{}_{|j)} - 2\epsilon_{|j)b_1b_2}u^{a_1b_1c}v^{a_2b_2}{}_{c}\right) \\
&t^{14}[2,1](u_2u_2v_2,u_3v_3) && : (R^{(0,1)}_{14})^{ij}_{k} && = \epsilon_{ka_1a_2}\left(3u^{(a_1b}u^{ij)}v^{a_2}{}_{b} + 4u^{a_1b}u^{a_2(i}v^{j)}{}_{b} + 24u^{a_1b(i}v^{j)a_2}{}_{b}\right) \\
&t^{14}[1,3](u_2u_2v_2,u_3v_3) && : (R^{(0,1)}_{14})^{i}_{jkl} && = \epsilon_{(j|a_1a_2}\epsilon_{|k|b_1b_2}\left(u^{a_1b_1}u^{a_2b_2}v^{i}{}_{|l)} + 6u^{ia_1b_1}v^{a_2b_2}{}_{|l)}\right) \\
&t^{14}[3,2](u_2u_2v_2,u_3v_3) && : (R^{(0,1)}_{14})^{ijk}_{lm} && = \epsilon_{a_1a_2(l}\left(u^{(a_1i}u^{jk)}v^{a_2}{}_{m)} + 6u^{a_1(ij}v^{k)a_2}{}_{m)}\right) \\
&t^{14}[1,3](v_2v_3) && : (R^{(0,2)}_{14})^{i}_{jkl} && = \epsilon_{a_1a_2(j}v^{a_1}{}_{k}v^{ia_2}{}_{l)} \\
&t^{16}[0,1](u_2v_2v_2,v_3v_3) && : (R^{(0,2)}_{16})_{i} && = \epsilon_{ia_1a_2}\left(12u^{bc}v^{a_1}{}_{b}v^{a_2}{}_{c} + 13u^{a_1b}v^{a_2}{}_{c}v^{c}{}_{b} + 12v^{a_1b}{}_{c}v^{a_2c}{}_{b}\right) \\
&t^{16}[1,2](u_2v_2v_2,v_3v_3) && : (R^{(0,2)}_{16})^{i}_{jk} && = \epsilon_{a_1a_2(j}\left(3u^{ib}v^{a_1}{}_{k)}v^{a_2}{}_{b} - 7u^{ia_1}v^{b}{}_{k)}v^{a_2}{}_{b} + 6u^{a_1b}v^{i}{}_{k)}v^{a_2}{}_{b} + 24v^{a_1b}{}_{k)}v^{ia_2}{}_{b}\right) \\
&t^{16}[2,3](u_2v_2v_2,v_3v_3) && : (R^{(0,2)}_{16})^{ij}_{klm} && = \epsilon_{a_1a_2(k}\left(u^{a_1(i}v^{j)}{}_{l}v^{a_2}{}_{m)} + 3v^{a_1(i}{}_{l}v^{j)a_2}{}_{m)}\right) \\
&t^{18}[0,0](u_3v_2v_2) && : (R^{(0,2)}_{18}) && = \epsilon_{a_1a_2a_3}u^{a_1bc}v^{a_2}{}_{b}v^{a_3}{}_{c} \\
&t^{20}[1,0](v_2v_2v_3) && : (R^{(0,3)}_{20})^{i} && = 2v^{a}{}_{c}v^{b}{}_{a}v^{ic}{}_{b} - 3v^{i}{}_{a}v^{c}{}_{b}v^{ab}{}_{c} \\
&t^{22}[2,0](u_2v_2v_2v_2) && : (R^{(0,3)}_{22})^{ij} && = u^{ij}v^{a}{}_{b}v^{b}{}_{c}v^{c}{}_{a} - 3u^{a(i}v^{j)}{}_{b}v^{b}{}_{c}v^{c}{}_{a} + 3u^{ab}v^{(i}{}_{a}v^{j)}{}_{c}v^{c}{}_{b} \\
&t^{24}[0,0](u_2v_2v_2v_3) && : (R^{(0,3)}_{24}) && = \epsilon_{a_1a_2a_3}u^{a_1b}v^{a_2}{}_{b}v^{a_3c}{}_{d}v^{d}{}_{c} \\
&t^{26}[1,0](v_2v_2v_2v_3) && : (R^{(0,4)}_{26})^{i} && = v^{i}{}_{a}v^{a}{}_{b}v^{d}{}_{c}v^{bc}{}_{d} \\
&t^{30}[0,0](v_2v_2v_2v_2v_2) && : (R^{(0,5)}_{30}) && = v^{a}{}_{b}v^{b}{}_{c}v^{c}{}_{d}v^{d}{}_{e}v^{e}{}_{a} \\
&t^{30}[3,0](v_2v_2v_2v_2v_2) && : (R^{(0,5)}_{30})^{ijk} && = \epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k)}{}_{a_2}v^{b}{}_{c}v^{c}{}_{d}v^{d}{}_{b}.
\end{aligned}
\tag{A.2}
$$

JHEP07(2025)216

The superscripts (n_f, n_ψ) indicate the numbers of f and ψ in the term with the largest number of f. Note that we only consider examples with $n_f = 0$. On the left, we also indicated the types of graviton monomials that appear in the polynomial.

As these graviton trace relations stand for Q-exact graviton polynomials, each of them can be written as $R = Qr$. Below we present the explicit expressions for the relevant r's. Note that there is an ambiguity in choosing r given R. We take advantage of this ambiguity to write all r's such that they vanish if all fields are diagonal. This ensures that r's are non-graviton operators, see also discussion below (2.25).

$$
\begin{aligned}
(r_{10}^{(0,1)})^i_{jk} =& -2\epsilon_{a_1a_2(j}\mathrm{tr}\left(\phi^{a_1}\phi^{a_2}\phi^i\psi_{k)}\right),\\
r_{12}^{(0,1)} =& \epsilon_{a_1a_2a_3}\left[6\mathrm{tr}\left(\psi_b\phi^{a_1}\right)\mathrm{tr}\left(\phi^b\phi^{a_2}\phi^{a_3}\right)-\mathrm{tr}\left(\psi_b\phi^{a_1}\phi^{a_2}\right)\mathrm{tr}\left(\phi^b\phi^{a_3}\right)\right]\\
&-3\epsilon_{a_1a_2a_3}\Big[\mathrm{tr}\left(\psi_b\phi^b\phi^{a_1}\phi^{a_2}\phi^{a_3}\right)+\mathrm{tr}\left(\psi_b\phi^{a_1}\phi^b\phi^{a_2}\phi^{a_3}\right)+\mathrm{tr}\left(\psi_b\phi^{a_1}\phi^{a_2}\phi^b\phi^{a_3}\right)\\
&\qquad+\mathrm{tr}\left(\psi_b\phi^{a_1}\phi^{a_2}\phi^{a_3}\phi^b\right)\Big],\\
(r_{12}^{(0,1)})^{ij}_{kl} =& -2\epsilon_{a_1a_2(k}\left[\mathrm{tr}\left(\psi_{l)}\phi^{(i}\phi^{j)}\phi^{a_1}\phi^{a_2}\right)+7\mathrm{tr}\left(\psi_{l)}\phi^{(i|}\phi^{a_1}\phi^{|j)}\phi^{a_2}\right)\right],\\
(r_{12}^{(0,2)})_{ijk} =& \frac{1}{2}\epsilon_{a_1a_2(i}\mathrm{tr}\left(\phi^{a_1}\psi_j\phi^{a_2}\psi_{k)}\right),\\
(r_{12}^{(0,2)})^i_j =& 6\mathrm{tr}\left(\phi^{(i}\phi^{a)}\psi_{(a}\psi_{j)}\right)-5\mathrm{tr}\left(\phi^{[i}\psi_a\phi^{a]}\psi_j\right),\\
(r_{12}^{(0,2)})^{ij}_{kl} =& \mathrm{tr}\left(\phi^{(i}\phi^{j)}\psi_{(k}\psi_{l)}\right),\\
(r_{14}^{(0,2)})^i =& 3\mathrm{tr}\left(\phi^i\psi_{a_1}\phi^{a_1}\phi^{a_2}\psi_{a_2}\right)+2\mathrm{tr}\left(\phi^i\phi^{a_1}\right)\mathrm{tr}\left(\phi^{a_2}\psi_{(a_1}\psi_{a_2)}\right)\\
&-6\mathrm{tr}\left(\phi^i\psi_{a_1}\right)\mathrm{tr}\left(\phi^{[a_1}\phi^{a_2]}\psi_{a_2}\right)-\mathrm{tr}\left(\phi^i\psi_{a_1}\psi_{a_2}\right)\mathrm{tr}\left(\phi^{a_1}\phi^{a_2}\right),\\
(r_{14}^{(0,2)})_{ij} =& \frac{5}{9}\epsilon_{a_1a_2a_3}\left[2\mathrm{tr}\left(\psi_{(i}\psi_{j)}\phi^{a_1}\phi^{a_2}\phi^{a_3}\right)+\mathrm{tr}\left(\psi_{(i}\phi^{a_1}\psi_{j)}\phi^{a_2}\phi^{a_3}\right)\right]\\
&+\epsilon_{a_1a_2(i}\left[\mathrm{tr}\left(\psi_{j)}\psi_{a_3}\phi^{a_1}\phi^{a_2}\phi^{a_3}\right)+\mathrm{tr}\left(\psi_{j)}\psi_{a_3}\phi^{a_1}\phi^{a_3}\phi^{a_2}\right)+\mathrm{tr}\left(\psi_{j)}\psi_{a_3}\phi^{a_3}\phi^{a_1}\phi^{a_2}\right)\right]\\
&+\epsilon_{a_1a_2(i}\left[\mathrm{tr}\left(\psi_{j)}\phi^{a_1}\psi_{a_3}\phi^{a_2}\phi^{a_3}\right)+\mathrm{tr}\left(\psi_{j)}\phi^{a_1}\psi_{a_3}\phi^{a_3}\phi^{a_2}\right)+\mathrm{tr}\left(\psi_{j)}\phi^{a_3}\psi_{a_3}\phi^{a_1}\phi^{a_2}\right)\right]\\
&+\epsilon_{a_1a_2(i}\left[\mathrm{tr}\left(\psi_{j)}\phi^{a_1}\phi^{a_2}\psi_{a_3}\phi^{a_3}\right)+\mathrm{tr}\left(\psi_{j)}\phi^{a_1}\phi^{a_3}\psi_{a_3}\phi^{a_2}\right)+\mathrm{tr}\left(\psi_{j)}\phi^{a_3}\phi^{a_1}\psi_{a_3}\phi^{a_2}\right)\right]\\
&+\epsilon_{a_1a_2(i}\left[\mathrm{tr}\left(\psi_{j)}\phi^{a_1}\phi^{a_2}\phi^{a_3}\psi_{a_3}\right)+\mathrm{tr}\left(\psi_{j)}\phi^{a_1}\phi^{a_3}\phi^{a_2}\psi_{a_3}\right)+\mathrm{tr}\left(\psi_{j)}\phi^{a_3}\phi^{a_1}\phi^{a_2}\psi_{a_3}\right)\right]\\
&-\frac{1}{3}\epsilon_{a_1a_2(i}\Big[5\mathrm{tr}\left(\psi_{j)}\phi^{a_1}\phi^{a_2}\right)\mathrm{tr}\left(\psi_{a_3}\phi^{a_3}\right)+2\mathrm{tr}\left(\psi_{j)}\phi^{(a_1}\phi^{a_3)}\right)\mathrm{tr}\left(\psi_{a_3}\phi^{a_2}\right)\\
&\qquad-2\mathrm{tr}\left(\psi_{j)}\phi^{a_2}\right)\mathrm{tr}\left(\psi_{a_3}\phi^{(a_1}\phi^{a_3)}\right)\Big],\\
(r_{14}^{(0,2)})^{ij}_k =& 12\mathrm{tr}\left(\phi^{(i}\phi^a\phi^{j)}\psi_{(a}\psi_{k)}\right)+12\mathrm{tr}\left(\phi^{(i|}\phi^a\phi^{|j)}\psi_{(a}\psi_{k)}\right)+54\mathrm{tr}\left(\phi^{(i}\phi^j\psi_{(a}\phi^{a)}\psi_{k)}\right)\\
&-36\mathrm{tr}\left(\phi^{(i}\phi^{j)}\psi_{(a}\phi^a\psi_{k)}\right),\\
(r_{14}^{(0,2)})^i_{jkl} =& 2\epsilon_{a_1a_2(j}\left[\mathrm{tr}\left(\phi^i\phi^{a_1}\phi^{a_2}\psi_k\psi_{l)}\right)+3\mathrm{tr}\left(\phi^i\phi^{a_1}\psi_k\phi^{a_2}\psi_{l)}\right)-2\mathrm{tr}\left(\phi^i\psi_k\phi^{a_1}\phi^{a_2}\psi_{l)}\right)\right],\\
(r_{14}^{(0,2)})^{ijk}_{lm} =& 4\mathrm{tr}\left(\phi^{(i}\phi^j\phi^{k)}\psi_{(l}\psi_{m)}\right)+3\mathrm{tr}\left(\phi^{(i}\phi^j\psi_{(l}\phi^{k)}\psi_{m)}\right)
\end{aligned}
$$

JHEP07(2025)216

$$
\begin{aligned}
(r_{14}^{(0,3)})^{i}_{jkl} =& -\frac{1}{2}\mathrm{tr}\left(\phi^{i}\psi_{(j}\psi_{k}\psi_{l)}\right),\\
(r_{16}^{(0,3)})_{i} =& \frac{39}{4}\mathrm{tr}\left(\psi_{i}\{\psi_{b_1}\psi_{b_2},\phi^{b_1}\phi^{b_2}\}\right)+2\mathrm{tr}\left(\psi_{i}\psi_{b_1}\phi^{b_1}\psi_{b_2}\phi^{b_2}\right)-\frac{61}{4}\mathrm{tr}\left(\psi_{i}\psi_{b_1}\phi^{b_2}\psi_{b_2}\phi^{b_1}\right)\\
&+\frac{97}{4}\mathrm{tr}\left(\psi_{i}\phi^{b_1}\psi_{b_1}\psi_{b_2}\phi^{b_2}\right)-\frac{41}{4}\mathrm{tr}\left(\psi_{i}\phi^{b_2}\psi_{b_1}\psi_{b_2}\phi^{b_1}\right)-5\mathrm{tr}\left(\psi_{i}\psi_{b_1}\phi^{b_1}\phi^{b_2}\psi_{b_2}\right)\\
&-\frac{25}{2}\mathrm{tr}\left(\psi_{i}\psi_{b_1}\phi^{b_2}\phi^{b_1}\psi_{b_2}\right)+2\mathrm{tr}\left(\psi_{i}\phi^{b_1}\psi_{b_1}\phi^{b_2}\psi_{b_2}\right)-\frac{61}{4}\mathrm{tr}\left(\psi_{i}\phi^{b_2}\psi_{b_1}\phi^{b_1}\psi_{b_2}\right)\\
&-\frac{11}{4}\mathrm{tr}\left(\phi^{b_1}\phi^{b_2}\right)\mathrm{tr}\left(\psi_{i}\psi_{b_1}\psi_{b_2}\right)-\frac{27}{2}\mathrm{tr}\left(\psi_{b_1}\psi_{b_2}\right)\mathrm{tr}\left(\psi_{i}\phi^{b_1}\phi^{b_2}\right)+\frac{29}{4}\mathrm{tr}\left(\phi^{b_2}\psi_{b_2}\right)\mathrm{tr}\left(\psi_{i}[\psi_{b_1},\phi^{b_1}]\right),\\
(r_{16}^{(0,3)})^{i}_{jk} =& 2\mathrm{tr}\left(\psi_{(j}\psi_{k)}\psi_{b}\phi^{b}\phi^{i}\right)-4\mathrm{tr}\left(\psi_{(j}\psi_{k)}\psi_{b}\phi^{i}\phi^{b}\right)-\mathrm{tr}\left(\psi_{(j|}\psi_{b}\psi_{|k)}\{\phi^{b},\phi^{i}\}\right)-4\mathrm{tr}\left(\psi_{(j}\psi_{k)}\phi^{(b}\psi_{b}\phi^{i)}\right)\\
&+7\mathrm{tr}\left(\psi_{(j|}\{\psi_{b},\phi^{b}\}\psi_{|k)}\phi^{i}\right)-11\mathrm{tr}\left(\psi_{(j|}\{\psi_{b},\phi^{i}\}\psi_{|k)}\phi^{b}\right)-4\mathrm{tr}\left(\psi_{(j}\psi_{k)}\phi^{b}\phi^{i}\psi_{b}\right)\\
&+2\mathrm{tr}\left(\psi_{(j}\psi_{k)}\phi^{i}\phi^{b}\psi_{b}\right)+3\mathrm{tr}\left(\psi_{(j|}\psi_{b}\right)\mathrm{tr}\left(\psi_{|k)}[\phi^{b},\phi^{i}]\right)+6\mathrm{tr}\left(\psi_{(j}\phi^{[b}\right)\mathrm{tr}\left(\{\psi_{k)},\psi_{b}\}\phi^{i]}\right),\\
(r_{16}^{(0,3)})^{ij}_{klm} =& -\mathrm{tr}(\phi^{(i}\phi^{j)}\psi_{(k}\psi_{l}\psi_{m)})-2\mathrm{tr}(\phi^{(i}\psi_{(k}\phi^{j)}\psi_{l}\psi_{m)}).
\end{aligned}
\tag{A.3}
$$

All these expressions except for $r_{14}^{(0,2)}[3,2]$ and $r_{16}^{(0,3)}[2,3]$ were brought from [8]. Note that we only listed r's for $j \leq 16$. Although r's with higher j ($j \geq 18$) are included in our ansatz for constructing some Q-closed operators in section 2.2, the above r's are sufficient to show the existence of the cohomology.

A.2 The bases for the non-fundamental trace relations

In this subsection, we list the bases for the non-fundamental graviton trace relations whose field contents and the $\mathrm{SU}(3)_R$ representation are relevant to constructing the new cohomologies in section 2.2.

We first list all 39 tensors that span the relations, that trasform under the $[3,0]$ representation and contain 7 scalars and 4 fermions. Linear relations between these relations (e.g. (2.20)) correspond to the relations of relations (2.8), which lead to the ansatz (2.9) for the Q-closed operator with the $[3,0]$ representation and $j=30$ as relevant for the $n=3$ case of section 2.2. We label them with a simplified subscript $s_{1,\cdots,39}$.

$$
\begin{aligned}
s_1 &= \epsilon^{a_1a_2(i}\epsilon^{j|b_1b_2}v^{c}{}_{d}v^{|k)}{}_{a_1}v^{d}{}_{b_1}(R_{12}^{(0,1)})_{a_2b_2c}\,, & s_2 &= \epsilon^{a_1a_2(i}\epsilon^{j|b_1b_2}v^{|k)}{}_{d}v^{c}{}_{a_1}v^{d}{}_{b_1}(R_{12}^{(0,1)})_{a_2b_2c}\,,\\
s_3 &= \epsilon^{a_1a_2(i}v^{j|}{}_{b}v^{b}{}_{c}v^{c}{}_{a_1}(R_{12}^{(0,1)})^{|k)}_{a_2}\,, & s_4 &= \epsilon^{a_1a_2a_3}v^{b}{}_{a_1}v^{(i}{}_{a_2}v^{j}{}_{a_3}(R_{12}^{(0,1)})^{k)}_{b}\,,\\
s_5 &= \epsilon^{a_1a_2(i}v^{j}{}_{b}v^{k)}{}_{a_1}v^{b}{}_{c}(R_{12}^{(0,1)})^{c}_{a_2}\,, & s_6 &= \epsilon^{a_1a_2(i|}v^{b}{}_{c}v^{c}{}_{d}v^{d}{}_{a_1}(R_{12}^{(0,1)})^{|jk)}_{a_2b}\,,\\
s_7 &= \epsilon^{a_1a_2a_3}v^{b}{}_{a_1}v^{c}{}_{a_2}v^{(i}{}_{a_3}(R_{12}^{(0,1)})^{jk)}_{bc}\,, & s_8 &= \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{b}{}_{d}v^{d}{}_{c}(R_{12}^{(0,1)})^{|k)c}_{a_2b}\,,\\
s_9 &= \epsilon^{a_1a_2(i}v^{j|}{}_{d}v^{b}{}_{a_1}v^{d}{}_{c}(R_{12}^{(0,1)})^{|k)c}_{a_2b}\,, & s_{10} &= \epsilon^{a_1a_2(i|}u^{bc}v^{d}{}_{a_1}v^{|j}{}_{b}(R_{14}^{(0,2)})^{k)}_{a_2cd}\,,\\
s_{11} &= \epsilon^{a_1a_2(i|}u^{bc}v^{d}{}_{b}v^{|j}{}_{a_1}(R_{14}^{(0,2)})^{k)}_{a_2cd}\,, & s_{12} &= \epsilon^{a_1a_2(i}u^{j|b}v^{c}{}_{a_1}v^{d}{}_{b}(R_{14}^{(0,2)})^{|k)}_{a_2cd}\,,\\
s_{13} &= \epsilon^{a_1a_2(i|}u^{cd}v^{b}{}_{a_1}v^{|j}{}_{b}(R_{14}^{(0,2)})^{k)}_{a_2cd}\,, & s_{14} &= \epsilon^{a_1a_2(i}u^{j|c}v^{b}{}_{a_1}v^{d}{}_{b}(R_{14}^{(0,2)})^{|k)}_{a_2cd}\,,\\
s_{15} &= \epsilon^{a_1a_2(i}u^{j|b}v^{c}{}_{a_1}v^{|k)}{}_{d}(R_{14}^{(0,2)})^{d}_{a_2bc}\,, & s_{16} &= \epsilon^{a_1a_2a_3}u^{(ij|}v^{c}{}_{a_1}v^{d}{}_{a_2}(R_{14}^{(0,2)})^{|k)}_{a_3cd}\,,\\
s_{17} &= \epsilon^{a_1a_2(i}v^{j|b}{}_{a_1}v^{cd}{}_{b}(R_{14}^{(0,2)})^{|k)}_{a_2cd}\,, & s_{18} &= \epsilon^{a_1a_2(i}v^{bc}{}_{a_1}v^{d|j}{}_{b}(R_{14}^{(0,2)})^{k)}_{a_2cd}\,,\\
s_{19} &= \epsilon^{a_1a_2(i}v^{jk)}{}_{b}v^{cd}{}_{a_1}(R_{14}^{(0,2)})^{b}_{a_2cd}\,, & s_{20} &= \epsilon^{a_1a_2a_3}v^{b(i}{}_{a_1}v^{j|c}{}_{a_2}(R_{14}^{(0,2)})^{|k)}_{a_3bc}\,,
\end{aligned}
$$

JHEP07(2025)216

$$
\begin{aligned}
s_{21} &= \epsilon^{a_1a_2(i}v^{j}{}_{b}v^{k)b}{}_{a_1}(R^{(0,2)}_{16})_{a_2}\,, & s_{22} &= \epsilon^{a_1a_2(i|}v^{b}{}_{a_1}v^{|jk)}{}_{b}(R^{(0,2)}_{16})_{a_2}\,,\\
s_{23} &= \epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k)b}{}_{a_2}(R^{(0,2)}_{16})_{b}\,, & s_{24} &= \epsilon^{a_1a_2(i}v^{j|}{}_{b}v^{bc}{}_{a_1}(R^{(0,2)}_{16})^{|k)}_{a_2c}\,,\\
s_{25} &= \epsilon^{a_1a_2(i|}v^{c}{}_{b}v^{|j|b}{}_{a_1}(R^{(0,2)}_{16})^{|k)}_{a_2c}\,, & s_{26} &= \epsilon^{a_1a_2(i|}v^{b}{}_{a_1}v^{|j|c}{}_{b}(R^{(0,2)}_{16})^{|k)}_{a_2c}\,,\\
s_{27} &= \epsilon^{a_1a_2b}v^{c}{}_{a_1}v^{(ij}{}_{a_2}(R^{(0,2)}_{16})^{k)}_{bc}\,, & s_{28} &= \epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k)b}{}_{c}(R^{(0,2)}_{16})^{c}_{a_2b}\,,\\
s_{29} &= \epsilon^{a_1a_2(i|}v^{b}{}_{a_1}v^{|jk)}{}_{c}(R^{(0,2)}_{16})^{c}_{a_2b}\,, & s_{30} &= \epsilon^{a_1a_2a_3}v^{b_1}{}_{a_1}v^{b_2(i}{}_{a_2}(R^{(0,2)}_{16})^{jk)}_{a_3b_1b_2}\,,\\
s_{31} &= \epsilon^{a_1a_2a_3}v^{(i|}{}_{a_1}v^{b_1b_2}{}_{a_2}(R^{(0,2)}_{16})^{|jk)}_{a_3b_1b_2}\,, & s_{32} &= \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{bc}{}_{d}(R^{(0,2)}_{16})^{d|k)}_{a_2bc}\\
s_{33} &= \epsilon^{a_1a_2(i|}v^{b}{}_{a_1}v^{c|j}{}_{d}(R^{(0,2)}_{16})^{k)d}_{a_2bc} & s_{34} &= \epsilon^{a_1a_2(i|}v^{b}{}_{a_1}v^{cd}{}_{a_2}(R^{(0,2)}_{16})^{|jk)}_{bcd}\\
s_{35} &= \epsilon^{ab(i}v^{j}{}_{a}v^{k)}{}_{b}R^{(0,2)}_{18}\,, & s_{36} &= u^{a(i}v^{j}{}_{a}(R^{(0,3)}_{20})^{k)}\,,\\
s_{37} &= u^{(ij}v^{k)}{}_{a}(R^{(0,3)}_{20})^{a}\,, & s_{38} &= v^{(ij}{}_{a}(R^{(0,3)}_{22})^{k)a}\,,\\
s_{39} &= u^{(ij}(R^{(0,4)}_{26})^{k)}\,. & & & & (A.4)
\end{aligned}
$$

Among these 39 trace relations, 25 are independent and there are 14 relations of relations, among which is (2.20). They provide 14 operators which are Q-closed by trace relations. One of these is presented in (2.24), and we prove subsequently that it is not Q-exact and thus represents a black hole cohomology.

Similarly, we now list all 107 tensors that span the relations, that trasform under the $[4,0]$ representation and contain 8 scalars and 4 fermions. Linear relations between these relations correspond to (2.8), which lead to the ansatz (2.9) for the Q-closed operator with the $[4,0]$ representation and $j=32$ as relevant for the $n=4$ case of section 2.2. We label them with a simplified subscript $s_{1,\cdots,107}$.

$$
\begin{aligned}
s_1 &= \epsilon^{a_1a_2(i}\epsilon^{j|a_3a_4}v^{|k}{}_{a_1}v^{l)}{}_{a_2}v^{a_5a_6}{}_{a_3}(R^{(0,1)}_{12})_{a_4a_5a_6}\,, & s_2 &= \epsilon^{a_1a_2(i}\epsilon^{j|a_3a_4}v^{|k}{}_{a_1}v^{l)}{}_{a_5}v^{a_5a_6}{}_{a_3}(R^{(0,1)}_{12})_{a_2a_4a_6}\,,\\
s_3 &= \epsilon^{a_1a_2(i}\epsilon^{j|a_3a_4}v^{|k|}{}_{a_1}v^{a_5}{}_{a_2}v^{|l)a_6}{}_{a_3}(R^{(0,1)}_{12})_{a_4a_5a_6}\,, & s_4 &= \epsilon^{a_1a_2(i}\epsilon^{j|a_3a_4}v^{|k|}{}_{a_1}v^{a_5}{}_{a_3}v^{|l)a_6}{}_{a_2}(R^{(0,1)}_{12})_{a_4a_5a_6}\,,\\
s_5 &= \epsilon^{a_1a_2(i}\epsilon^{j|a_3a_4}v^{|k|}{}_{a_1}v^{a_5}{}_{a_6}v^{|l)a_6}{}_{a_3}(R^{(0,1)}_{12})_{a_2a_4a_5}\,, & s_6 &= \epsilon^{a_1a_2(i}\epsilon^{j|a_3a_4}v^{|k|}{}_{a_6}v^{a_5}{}_{a_1}v^{|l)a_6}{}_{a_3}(R^{(0,1)}_{12})_{a_2a_4a_5}\,,\\
s_7 &= \epsilon^{a_1a_2(i}\epsilon^{j|a_3a_4}v^{a_5}{}_{a_1}v^{a_6}{}_{a_2}v^{|kl)}{}_{a_3}(R^{(0,1)}_{12})_{a_4a_5a_6}\,, & s_8 &= \epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k}{}_{a_2}v^{l)a_3}{}_{a_4}(R^{(0,1)}_{12})^{a_4}_{a_3}\,,\\
s_9 &= \epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k}{}_{a_3}v^{l)a_3}{}_{a_4}(R^{(0,1)}_{12})^{a_4}_{a_2}\,, & s_{10} &= \epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k}{}_{a_4}v^{l)a_3}{}_{a_2}(R^{(0,1)}_{12})^{a_4}_{a_3}\,,\\
s_{11} &= \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_2}v^{|kl)}{}_{a_4}(R^{(0,1)}_{12})^{a_4}_{a_3}\,, & s_{12} &= \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_4}v^{|kl)}{}_{a_2}(R^{(0,1)}_{12})^{a_4}_{a_3}\,,\\
s_{13} &= \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_4}v^{|kl)}{}_{a_3}(R^{(0,1)}_{12})^{a_4}_{a_2}\,, & s_{14} &= \epsilon^{a_1a_2(i}v^{j|}{}_{a_4}v^{a_3}{}_{a_1}v^{|kl)}{}_{a_2}(R^{(0,1)}_{12})^{a_4}_{a_3}\,,\\
s_{15} &= \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_2}v^{|k|a_4}{}_{a_3}(R^{(0,1)}_{12})^{|l)}_{a_4}\,, & s_{16} &= \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_4}v^{|k|a_4}{}_{a_2}(R^{(0,1)}_{12})^{|l)}_{a_3}\,,\\
s_{17} &= \epsilon^{a_1a_2(i|}v^{a_3}{}_{a_1}v^{a_4}{}_{a_2}v^{|jk}{}_{a_3}(R^{(0,1)}_{12})^{l)}_{a_4}\,, & s_{18} &= \epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k}{}_{a_4}v^{l)a_3}{}_{a_5}(R^{(0,1)}_{12})^{a_4a_5}_{a_2a_3}\,,\\
s_{19} &= \epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k|}{}_{a_2}v^{a_3a_4}{}_{a_5}(R^{(0,1)}_{12})^{|l)a_5}_{a_3a_4}\,, & s_{20} &= \epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k|}{}_{a_5}v^{a_3a_4}{}_{a_2}(R^{(0,1)}_{12})^{|l)a_5}_{a_3a_4}\,,\\
s_{21} &= \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_4}v^{|kl)}{}_{a_5}(R^{(0,1)}_{12})^{a_4a_5}_{a_2a_3}\,, & s_{22} &= \epsilon^{a_1a_2(i}v^{j|}{}_{a_4}v^{a_3}{}_{a_1}v^{|kl)}{}_{a_5}(R^{(0,1)}_{12})^{a_4a_5}_{a_2a_3}\,,\\
s_{23} &= \epsilon^{a_1a_2(i}v^{j|}{}_{a_4}v^{a_3}{}_{a_5}v^{|kl)}{}_{a_1}(R^{(0,1)}_{12})^{a_4a_5}_{a_2a_3}\,, & s_{24} &= \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_2}v^{a_4|k}{}_{a_5}(R^{(0,1)}_{12})^{l)a_5}_{a_3a_4}\,,\\
s_{25} &= \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_4}v^{a_4|k}{}_{a_5}(R^{(0,1)}_{12})^{l)a_5}_{a_2a_3}\,, & s_{26} &= \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_5}v^{a_4|k}{}_{a_2}(R^{(0,1)}_{12})^{l)a_5}_{a_3a_4}\,,\\
s_{27} &= \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_5}v^{a_4|k}{}_{a_3}(R^{(0,1)}_{12})^{l)a_5}_{a_2a_4}\,, & s_{28} &= \epsilon^{a_1a_2(i}v^{j|}{}_{a_5}v^{a_3}{}_{a_1}v^{a_4|k}{}_{a_2}(R^{(0,1)}_{12})^{l)a_5}_{a_3a_4}\,,\\
s_{29} &= \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_2}v^{a_4a_5}{}_{a_3}(R^{(0,1)}_{12})^{kl)}_{a_4a_5}\,, & s_{30} &= \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_4}v^{a_4a_5}{}_{a_2}(R^{(0,1)}_{12})^{kl)}_{a_3a_5}\,,\\
s_{31} &= \epsilon^{a_1a_2(i|}v^{a_3}{}_{a_1}v^{a_4}{}_{a_2}v^{|jk}{}_{a_5}(R^{(0,1)}_{12})^{l)a_5}_{a_3a_4}\,, & s_{32} &= \epsilon^{a_1a_2(i|}v^{a_3}{}_{a_1}v^{a_4}{}_{a_5}v^{|jk}{}_{a_2}(R^{(0,1)}_{12})^{l)a_5}_{a_3a_4}\,,\\
s_{33} &= \epsilon^{a_1a_2(i|}v^{a_3}{}_{a_1}v^{a_4}{}_{a_5}v^{|jk}{}_{a_4}(R^{(0,1)}_{12})^{l)a_5}_{a_2a_3}\,, & s_{34} &= \epsilon^{a_1a_2(i|}v^{a_3}{}_{a_1}v^{a_4}{}_{a_2}v^{a_5|j}{}_{a_3}(R^{(0,1)}_{12})^{kl)}_{a_4a_5}\,,\\
s_{35} &= \epsilon^{a_1a_2a_3}v^{(i}{}_{a_1}v^{j}{}_{a_2}v^{k}{}_{a_3}(R^{(0,1)}_{14})^{l)}\,, & s_{36} &= \epsilon^{a_1a_2(i|}\epsilon^{a_3a_4|j}v^{k}{}_{a_1}v^{l)}{}_{a_2}v^{a_5}{}_{a_3}(R^{(0,1)}_{14})_{a_4a_5}\,,
\end{aligned}
$$

JHEP07(2025)216

$$
\begin{aligned}
s_{37} &= \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_2}v^{a_4}{}_{a_3}(R_{14}^{(0,1)})^{|kl)}_{a_4}, \\
s_{38} &= \epsilon^{a_1a_2(i}v^{j|}{}_{a_3}v^{a_3}{}_{a_1}v^{a_4}{}_{a_2}(R_{14}^{(0,1)})^{|kl)}_{a_4}, \\
s_{39} &= \epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{k|}{}_{a_2}v^{a_3}{}_{a_4}(R_{14}^{(0,1)})^{|l)a_4}_{a_3}, \\
s_{40} &= \epsilon^{a_1a_2(i|}\epsilon^{a_3a_4|j}v^{k|}{}_{a_1}v^{a_5}{}_{a_2}v^{a_6}{}_{a_3}(R_{14}^{(0,1)})^{|l)}_{a_4a_5a_6}, \\
s_{41} &= \epsilon^{a_1a_2(i|}\epsilon^{a_3a_4|j}v^{k|}{}_{a_1}v^{a_5}{}_{a_3}v^{a_6}{}_{a_5}(R_{14}^{(0,1)})^{|l)}_{a_2a_4a_6}, \\
s_{42} &= \epsilon^{a_1a_2(i|}v^{a_3}{}_{a_1}v^{a_4}{}_{a_2}v^{a_5}{}_{a_3}(R_{14}^{(0,1)})^{|jkl)}_{a_4a_5}, \\
s_{43} &= \epsilon^{a_1a_2(i|}v^{a_3}{}_{a_1}v^{a_4}{}_{a_3}v^{a_5}{}_{a_4}(R_{14}^{(0,1)})^{|jkl)}_{a_2a_5}, \\
s_{44} &= \epsilon^{a_1a_2(i}v^{j|}{}_{a_1}v^{a_3}{}_{a_2}v^{a_4}{}_{a_5}(R_{14}^{(0,1)})^{|kl)a_5}_{a_3a_4}, \\
s_{45} &= \epsilon^{a_1a_2(i}v^{j|}{}_{a_5}v^{a_3}{}_{a_1}v^{a_4}{}_{a_3}(R_{14}^{(0,1)})^{|kl)a_5}_{a_2a_4}, \\
s_{46} &= \epsilon^{a_1a_2(i}u^{jk|a_3}v^{a_4}{}_{a_1}v^{a_5}{}_{a_2}(R_{14}^{(0,2)})^{|l)}_{a_3a_4a_5}, \\
s_{47} &= \epsilon^{a_1a_2(i}u^{jk|a_3}v^{a_4}{}_{a_1}v^{a_5}{}_{a_3}(R_{14}^{(0,2)})^{|l)}_{a_2a_4a_5}, \\
s_{48} &= \epsilon^{a_1a_2(i}u^{jk|a_3}v^{a_4}{}_{a_1}v^{a_5}{}_{a_4}(R_{14}^{(0,2)})^{|l)}_{a_2a_3a_5}, \\
s_{49} &= \epsilon^{a_1a_2(i}u^{j|a_3a_4}v^{|k|}{}_{a_1}v^{a_5}{}_{a_2}(R_{14}^{(0,2)})^{|l)}_{a_3a_4a_5}, \\
s_{50} &= \epsilon^{a_1a_2(i}u^{j|a_3a_4}v^{|k|}{}_{a_1}v^{a_5}{}_{a_3}(R_{14}^{(0,2)})^{|l)}_{a_2a_4a_5}, \\
s_{51} &= \epsilon^{a_1a_2(i}u^{j|a_3a_4}v^{|k|}{}_{a_5}v^{a_5}{}_{a_1}(R_{14}^{(0,2)})^{|l)}_{a_2a_3a_4}, \\
s_{52} &= \epsilon^{a_1a_2(i|}u^{a_3a_4a_5}v^{|j}{}_{a_1}v^{k}{}_{a_2}(R_{14}^{(0,2)})^{l)}_{a_3a_4a_5}, \\
s_{53} &= \epsilon^{a_1a_2(i}u^{jk|a_3}v^{|l)}{}_{a_5}v^{a_4}{}_{a_1}(R_{14}^{(0,2)})^{a_5}_{a_2a_3a_4}, \\
s_{54} &= \epsilon^{a_1a_2(i}u^{jk|}v^{a_3}{}_{a_1}v^{a_4a_5}{}_{a_2}(R_{14}^{(0,2)})^{|l)}_{a_3a_4a_5}, \\
s_{55} &= \epsilon^{a_1a_2(i}u^{jk|}v^{a_3}{}_{a_1}v^{a_4a_5}{}_{a_3}(R_{14}^{(0,2)})^{|l)}_{a_2a_4a_5}, \\
s_{56} &= \epsilon^{a_1a_2(i}u^{jk|}v^{a_3}{}_{a_4}v^{a_4a_5}{}_{a_1}(R_{14}^{(0,2)})^{|l)}_{a_2a_3a_5}, \\
s_{57} &= \epsilon^{a_1a_2(i}u^{j|a_3}v^{|k|}{}_{a_1}v^{a_4a_5}{}_{a_2}(R_{14}^{(0,2)})^{|l)}_{a_3a_4a_5}, \\
s_{58} &= \epsilon^{a_1a_2(i}u^{j|a_3}v^{|k|}{}_{a_1}v^{a_4a_5}{}_{a_3}(R_{14}^{(0,2)})^{|l)}_{a_2a_4a_5}, \\
s_{59} &= \epsilon^{a_1a_2(i}u^{j|a_3}v^{|k|}{}_{a_4}v^{a_4a_5}{}_{a_1}(R_{14}^{(0,2)})^{|l)}_{a_2a_3a_5}, \\
s_{60} &= \epsilon^{a_1a_2(i}u^{j|a_3}v^{a_4}{}_{a_1}v^{a_5|k}{}_{a_2}(R_{14}^{(0,2)})^{l)}_{a_3a_4a_5}, \\
s_{61} &= \epsilon^{a_1a_2(i}u^{j|a_3}v^{a_4}{}_{a_1}v^{a_5|k}{}_{a_3}(R_{14}^{(0,2)})^{l)}_{a_2a_4a_5}, \\
s_{62} &= \epsilon^{a_1a_2(i}u^{j|a_3}v^{a_4}{}_{a_5}v^{a_5|k}{}_{a_1}(R_{14}^{(0,2)})^{l)}_{a_2a_3a_4}, \\
s_{63} &= \epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j}{}_{a_1}v^{k|a_5}{}_{a_2}(R_{14}^{(0,2)})^{|l)}_{a_3a_4a_5}, \\
s_{64} &= \epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j}{}_{a_1}v^{k|a_5}{}_{a_3}(R_{14}^{(0,2)})^{|l)}_{a_2a_4a_5}, \\
s_{65} &= \epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j}{}_{a_5}v^{k|a_5}{}_{a_1}(R_{14}^{(0,2)})^{|l)}_{a_2a_3a_4}, \\
s_{66} &= \epsilon^{a_1a_2(i|}u^{a_3a_4}v^{a_5}{}_{a_1}v^{|jk}{}_{a_2}(R_{14}^{(0,2)})^{l)}_{a_3a_4a_5}, \\
s_{67} &= \epsilon^{a_1a_2(i|}u^{a_3a_4}v^{a_5}{}_{a_1}v^{|jk}{}_{a_3}(R_{14}^{(0,2)})^{l)}_{a_2a_4a_5}, \\
s_{68} &= \epsilon^{a_1a_2(i}u^{jk}v^{l)}{}_{a_1}v^{a_3a_4}{}_{a_5}(R_{14}^{(0,2)})^{a_5}_{a_2a_3a_4}, \\
s_{69} &= \epsilon^{a_1a_2(i}u^{jk|}v^{a_3}{}_{a_1}v^{|l)a_4}{}_{a_5}(R_{14}^{(0,2)})^{a_5}_{a_2a_3a_4}, \\
s_{70} &= \epsilon^{a_1a_2(i}u^{j|a_3}v^{|k}{}_{a_1}v^{l)a_4}{}_{a_5}(R_{14}^{(0,2)})^{a_5}_{a_2a_3a_4}, \\
s_{71} &= \epsilon^{a_1a_2(i}u^{j|a_3}v^{a_4}{}_{a_1}v^{|kl)}{}_{a_5}(R_{14}^{(0,2)})^{a_5}_{a_2a_3a_4}, \\
s_{72} &= \epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j}{}_{a_1}v^{kl)}{}_{a_5}(R_{14}^{(0,2)})^{a_5}_{a_2a_3a_4}, \\
s_{73} &= \epsilon^{a_1a_2(i}v^{jk|}{}_{a_1}v^{a_3a_4}{}_{a_5}(R_{16}^{(0,2)})^{|l)a_5}_{a_2a_3a_4}, \\
s_{74} &= \epsilon^{a_1a_2(i}v^{jk|}{}_{a_5}v^{a_3a_4}{}_{a_1}(R_{16}^{(0,2)})^{|l)a_5}_{a_2a_3a_4}, \\
s_{75} &= \epsilon^{a_1a_2(i}v^{j|a_3}{}_{a_1}v^{a_4|k}{}_{a_5}(R_{16}^{(0,2)})^{l)a_5}_{a_2a_3a_4}, \\
s_{76} &= \epsilon^{a_1a_2(i}v^{j|a_3}{}_{a_1}v^{a_4a_5}{}_{a_2}(R_{16}^{(0,2)})^{|kl)}_{a_3a_4a_5}, \\
s_{77} &= \epsilon^{a_1a_2a_3}v^{(ij}{}_{a_1}v^{k|b}{}_{a_2}(R_{16}^{(0,2)})^{|l)}_{a_3b}, \\
s_{78} &= \epsilon^{a_1a_2(i}v^{jk}{}_{a_1}v^{l)c}{}_{b}(R_{16}^{(0,2)})^{b}_{a_2c}, \\
s_{79} &= \epsilon^{a_1a_2(i}v^{jk|}{}_{a_1}v^{bc}{}_{a_2}(R_{16}^{(0,2)})^{|l)}_{bc}, \\
s_{80} &= \epsilon^{a_1a_2(i}v^{j|b}{}_{a_1}v^{c|k}{}_{a_2}(R_{16}^{(0,2)})^{|l)}_{bc}, \\
s_{81} &= \epsilon^{a_1a_2a_3}v^{(ij}{}_{a_1}v^{kl)}{}_{a_2}(R_{16}^{(0,2)})_{a_3}, \\
s_{82} &= \epsilon^{a_1a_2(i}v^{jk}{}_{a_1}v^{l)b}{}_{a_2}(R_{16}^{(0,2)})_{b}, \\
s_{83} &= \epsilon^{a_1a_2(i}u^{jk|}v^{a_3}{}_{a_1}v^{a_4}{}_{a_5}(R_{16}^{(0,2)})^{|l)a_5}_{a_2a_3a_4}, \\
s_{84} &= \epsilon^{a_1a_2(i}u^{j|a_3}v^{|k|}{}_{a_1}v^{a_4}{}_{a_5}(R_{16}^{(0,2)})^{|l)a_5}_{a_2a_3a_4}, \\
s_{85} &= \epsilon^{a_1a_2(i}u^{j|a_3}v^{|k|}{}_{a_5}v^{a_4}{}_{a_1}(R_{16}^{(0,2)})^{|l)a_5}_{a_2a_3a_4}, \\
s_{86} &= \epsilon^{a_1a_2(i}u^{j|a_3}v^{a_4}{}_{a_1}v^{a_5}{}_{a_2}(R_{16}^{(0,2)})^{|kl)}_{a_3a_4a_5}, \\
s_{87} &= \epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j}{}_{a_1}v^{k}{}_{a_5}(R_{16}^{(0,2)})^{l)a_5}_{a_2a_3a_4}, \\
s_{88} &= \epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j|}{}_{a_1}v^{a_5}{}_{a_2}(R_{16}^{(0,2)})^{|kl)}_{a_3a_4a_5}, \\
s_{89} &= \epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j|}{}_{a_1}v^{a_5}{}_{a_3}(R_{16}^{(0,2)})^{|kl)}_{a_2a_4a_5}, \\
s_{90} &= \epsilon^{a_1a_2(i}u^{jk}v^{l)}{}_{a_1}v^{a_3}{}_{a_4}(R_{16}^{(0,2)})^{a_4}_{a_2a_3}, \\
s_{91} &= \epsilon^{a_1a_2(i}u^{jk}v^{l)}{}_{a_4}v^{a_3}{}_{a_1}(R_{16}^{(0,2)})^{a_4}_{a_2a_3}, \\
s_{92} &= \epsilon^{a_1a_2(i}u^{jk|}v^{a_3}{}_{a_1}v^{a_4}{}_{a_2}(R_{16}^{(0,2)})^{|l)}_{a_3a_4}, \\
s_{93} &= \epsilon^{a_1a_2(i}u^{j|a_3}v^{|k}{}_{a_1}v^{l)}{}_{a_4}(R_{16}^{(0,2)})^{a_4}_{a_2a_3}, \\
s_{94} &= \epsilon^{a_1a_2(i}u^{j|a_3}v^{|k|}{}_{a_1}v^{a_4}{}_{a_2}(R_{16}^{(0,2)})^{|l)}_{a_3a_4}, \\
s_{95} &= \epsilon^{a_1a_2(i}u^{j|a_3}v^{|k|}{}_{a_1}v^{a_4}{}_{a_3}(R_{16}^{(0,2)})^{|l)}_{a_2a_4}, \\
s_{96} &= \epsilon^{a_1a_2(i|}u^{a_3a_4}v^{|j}{}_{a_1}v^{k}{}_{a_2}(R_{16}^{(0,2)})^{l)}_{a_3a_4}, \\
s_{97} &= \epsilon^{a_1a_2a_3}u^{(ij}v^{k}{}_{a_1}v^{l)}{}_{a_2}(R_{16}^{(0,2)})_{a_3}, \\
s_{98} &= \epsilon^{a_1a_2(i}u^{jk}v^{l)}{}_{a_1}v^{b}{}_{a_2}(R_{16}^{(0,2)})_{b}, \\
s_{99} &= \epsilon^{a_1a_2(i}u^{j|b}v^{|k}{}_{a_1}v^{l)}{}_{a_2}(R_{16}^{(0,2)})_{b}, \\
s_{100} &= \epsilon^{a_1a_2(i}v^{j}{}_{a_1}v^{kl)}{}_{a_2}(R_{18}^{(0,2)}), \\
s_{101} &= u^{a(i}v^{jk}{}_{a}(R_{20}^{(0,3)})^{l)}, \\
s_{102} &= u^{(ij}v^{kl)}{}_{a}(R_{20}^{(0,3)})^{a}, \\
s_{103} &= u^{a(ij}v^{k}{}_{a}(R_{20}^{(0,3)})^{l)}, \\
s_{104} &= u^{(ijk}v^{l)}{}_{a}(R_{20}^{(0,3)})^{a}, \\
s_{105} &= u^{a(i}v^{j}{}_{a}(R_{22}^{(0,3)})^{kl)}, \\
s_{106} &= u^{(ij}v^{k}{}_{a}(R_{22}^{(0,3)})^{l)a}, \\
s_{107} &= u^{(ijk}(R_{26}^{(0,4)})^{l)}, \qquad\qquad\qquad\

Among these 107 graviton trace relations, 57 are independent and there are 50 relations of relations. They provide 50 operators which are Q-closed by trace relations. One of these is presented in (2.26), and we prove subsequently that it is not Q-exact and thus represents a black hole cohomology.

Data Availability Statement. This article has no associated data or the data will not be deposited.

Code Availability Statement. This article has no associated code or the code will not be deposited.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License (CC-BY4.0), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

JHEP07(2025)216

References

[1] C.-M. Chang and Y.-H. Lin, *Words to describe a black hole*, *JHEP* **02** (2023) 109 [arXiv:2209.06728] [INSPIRE].

[2] S. Choi, S. Kim, E. Lee and J. Park, *The shape of non-graviton operators for SU(2)*, *JHEP* **09** (2024) 029 [arXiv:2209.12696] [INSPIRE].

[3] S. Choi et al., *Towards quantum black hole microstates*, *JHEP* **11** (2023) 175 [*Erratum ibid.* **03** (2025) 091] [arXiv:2304.10155] [INSPIRE].

[4] C.-M. Chang, L. Feng, Y.-H. Lin and Y.-X. Tao, *Decoding stringy near-supersymmetric black holes*, *SciPost Phys.* **16** (2024) 109 [arXiv:2306.04673] [INSPIRE].

[5] K. Budzik, H. Murali and P. Vieira, *Following Black Hole States*, arXiv:2306.04693 [INSPIRE].

[6] K. Budzik et al., *Semi-chiral operators in 4d $\mathcal{N} = 1$ gauge theories*, *JHEP* **05** (2024) 245 [arXiv:2306.01039] [INSPIRE].

[7] C.-M. Chang, Y.-H. Lin and J. Wu, *On $\frac{1}{8}$-BPS black holes and the chiral algebra of $\mathcal{N} = 4$ SYM*, *Adv. Theor. Math. Phys.* **28** (2024) 2431 [arXiv:2310.20086] [INSPIRE].

[8] J. Choi et al., *Finite N black hole cohomologies*, *JHEP* **12** (2024) 029 [arXiv:2312.16443] [INSPIRE].

[9] C.-M. Chang and Y.-H. Lin, *Holographic covering and the fortuity of black holes*, arXiv:2402.10129 [INSPIRE].

[10] J.B. Gutowski and H.S. Reall, *Supersymmetric AdS_5 black holes*, *JHEP* **02** (2004) 006 [hep-th/0401042] [INSPIRE].

[11] J.B. Gutowski and H.S. Reall, *General supersymmetric AdS_5 black holes*, *JHEP* **04** (2004) 048 [hep-th/0401129] [INSPIRE].

[12] Z.-W. Chong, M. Cvetic, H. Lu and C.N. Pope, *General non-extremal rotating black holes in minimal five-dimensional gauged supergravity*, *Phys. Rev. Lett.* **95** (2005) 161301 [hep-th/0506029] [INSPIRE].

[13] H.K. Kunduri, J. Lucietti and H.S. Reall, *Supersymmetric multi-charge AdS_5 black holes*, *JHEP* **04** (2006) 036 [hep-th/0601156] [INSPIRE].

[14] S. Minwalla, *Supersymmetric States in $\mathcal{N} = 4$ Yang Mills*, talk given at *Strings 2006* Beijingn, China, June (2006).

[15] S. Choi et al., *Dual dressed black holes as the end point of the charged superradiant instability in $\mathcal{N} = 4$ Yang Mills*, *SciPost Phys.* **18** (2025) 137 [arXiv:2409.18178] [INSPIRE].

[16] O. Henriksson, C. Hoyos and N. Jokela, *Novel color superconducting phases of $\mathcal{N} = 4$ super Yang-Mills at strong coupling*, *JHEP* **09** (2019) 088 [arXiv:1907.01562] [INSPIRE].

[17] J. McGreevy, L. Susskind and N. Toumbas, *Invasion of the giant gravitons from Anti-de Sitter space*, *JHEP* **06** (2000) 008 [hep-th/0003075] [INSPIRE].

[18] M.T. Grisaru, R.C. Myers and O. Tafjord, *SUSY and goliath*, *JHEP* **08** (2000) 040 [hep-th/0008015] [INSPIRE].

[19] A. Hashimoto, S. Hirano and N. Itzhaki, *Large branes in AdS and their field theory dual*, *JHEP* **08** (2000) 051 [hep-th/0008016] [INSPIRE].

[20] N. Benjamin, S. Collier and A. Maloney, *Pure Gravity and Conical Defects*, *JHEP* **09** (2020) 034 [arXiv:2004.14428] [INSPIRE].

[21] J. Kinney, J.M. Maldacena, S. Minwalla and S. Raju, *An Index for 4 dimensional super conformal theories*, *Commun. Math. Phys.* **275** (2007) 209 [hep-th/0510251] [INSPIRE].

[22] S. Kim et al., *'Grey Galaxies' as an endpoint of the Kerr-AdS superradiant instability*, *JHEP* **11** (2023) 024 [arXiv:2305.08922] [INSPIRE].

[23] S. Choi, D. Jain, S. Kim, G. Kwon, V. Krishna, E. Lee, S. Minwalla, C. Patel and K. Sharma, *Supersymmetric Grey Galaxies and Revolving Black Holes*, to appear.

[24] K. Bajaj et al., *Grey Galaxies in AdS_5*, arXiv:2412.06904 [INSPIRE].

[25] C.-M. Chang, *Witten index of BMN matrix quantum mechanics*, arXiv:2404.18442 [INSPIRE].

[26] S. Corley, A. Jevicki and S. Ramgoolam, *Exact correlators of giant gravitons from dual $N = 4$ SYM theory*, *Adv. Theor. Math. Phys.* **5** (2002) 809 [hep-th/0111222] [INSPIRE].

[27] L. Grant, P.A. Grassi, S. Kim and S. Minwalla, *Comments on 1/16 BPS Quantum States and Classical Configurations*, *JHEP* **05** (2008) 049 [arXiv:0803.4183] [INSPIRE].

[28] C.-M. Chang and X. Yin, *1/16 BPS states in $\mathcal{N} = 4$ super-Yang-Mills theory*, *Phys. Rev. D* **88** (2013) 106005 [arXiv:1305.6314] [INSPIRE].

[29] S. Choi, J. Kim, S. Kim and J. Nahmgoong, *Large AdS black holes from QFT*, arXiv:1810.12067 [INSPIRE].

[30] A. Arabi Ardehali, *Cardy-like asymptotics of the 4d $\mathcal{N} = 4$ index and AdS_5 blackholes*, *JHEP* **06** (2019) 134 [arXiv:1902.06619] [INSPIRE].

[31] M. Honda, *Quantum Black Hole Entropy from 4d Supersymmetric Cardy formula*, *Phys. Rev. D* **100** (2019) 026008 [arXiv:1901.08091] [INSPIRE].

[32] J. Kim, S. Kim and J. Song, *A 4d $\mathcal{N} = 1$ Cardy Formula*, *JHEP* **01** (2021) 025 [arXiv:1904.03455] [INSPIRE].

[33] A. Cabo-Bizet, D. Cassani, D. Martelli and S. Murthy, *The asymptotic growth of states of the 4d $\mathcal{N} = 1$ superconformal index*, *JHEP* **08** (2019) 120 [arXiv:1904.05865] [INSPIRE].

[34] L.F. Alday and J.M. Maldacena, *Comments on operators with large spin*, *JHEP* **11** (2007) 019 [arXiv:0708.0672] [INSPIRE].

JHEP07(2025)216

[35] Z. Komargodski and A. Zhiboedov, *Convexity and Liberation at Large Spin*, *JHEP* **11** (2013) 140 [arXiv:1212.4103] [INSPIRE].

[36] G. Cuomo, L. Rastelli and A. Sharon, *Moduli spaces in CFT: large charge operators*, *JHEP* **09** (2024) 185 [arXiv:2406.19441] [INSPIRE].

[37] V. Ivanovskiy et al., *Vacuum Condensates on the Coulomb Branch*, arXiv:2405.19043 [INSPIRE].

[38] S. Baiguera, T. Harmark and Y. Lei, *The Panorama of Spin Matrix theory*, *JHEP* **04** (2023) 075 [arXiv:2211.16519] [INSPIRE].

[39] A. Cabo-Bizet, D. Cassani, D. Martelli and S. Murthy, *Microscopic origin of the Bekenstein-Hawking entropy of supersymmetric AdS_5 black holes*, *JHEP* **10** (2019) 062 [arXiv:1810.11442] [INSPIRE].

[40] F. Benini and E. Milan, *Black Holes in 4D $\mathcal{N} = 4$ Super-Yang-Mills Field Theory*, *Phys. Rev. X* **10** (2020) 021037 [arXiv:1812.09613] [INSPIRE].

[41] C.-M. Chang, Y. Chen, B.S. Sia and Z. Yang, *Fortuity in SYK Models*, arXiv:2412.06902 [INSPIRE].

[42] R. Bhattacharyya, S. Collins and R. de Mello Koch, *Exact Multi-Matrix Correlators*, *JHEP* **03** (2008) 044 [arXiv:0801.2061] [INSPIRE].

[43] R. Bhattacharyya, R. de Mello Koch and M. Stephanou, *Exact Multi-Restricted Schur Polynomial Correlators*, *JHEP* **06** (2008) 101 [arXiv:0805.3025] [INSPIRE].

[44] T.W. Brown, P.J. Heslop and S. Ramgoolam, *Diagonal multi-matrix correlators and BPS operators in $N = 4$ SYM*, *JHEP* **02** (2008) 030 [arXiv:0711.0176] [INSPIRE].

[45] T.W. Brown, P.J. Heslop and S. Ramgoolam, *Diagonal free field matrix correlators, global symmetries and giant gravitons*, *JHEP* **04** (2009) 089 [arXiv:0806.1911] [INSPIRE].

[46] R. de Mello Koch, M. Kim and A.L. Mahu, *A pedagogical introduction to restricted Schur polynomials with applications to heavy operators*, *Int. J. Mod. Phys. A* **39** (2024) 2430003 [arXiv:2409.15751] [INSPIRE].

JHEP07(2025)216