THE LANGUAGE OF SETS

By

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"SETS" is a word that is spreading all over the world and the question always arises "to what end?" So let me make it clear that the language of sets is purely a language; it is a way of presenting mathematics in which we can see a common language in the explanations that we give. It does not mean that you are going to change any of the ways in which you teach your concepts, but that you will use the language of sets in order to clarify and simplify your work in arithmetic, algebra and geometry. You will still teach calculus the same way; you will teach the other things the same way, but you will probably discuss it and present it in the manner of sets if you feel that it is appropriate for what you want to do.

Also, it not only simplifies, but brings certain of the concepts into sharper focus than perhaps the old system did. But do not run away with the idea that here is something new, that the end product is going to be something entirely different.

What is a set? A set is a collection of things, ideas, movements that you can define; it is made up of a selection of members, and each member of the set is easily definable. All through your teaching career, you have been teaching this; you have been teaching the set of prime numbers, the set of even numbers, the set of odd numbers, the set of square numbers; whenever you have been looking at a collection of these things, you have actually been dealing with sets. It was Georg Cantor of course who brought this to light when he was thinking about sets of points in the circumference of a circle. Everything is made up of different sets. Some of these have a particular description and a particular name. Even in English you have heard of a gaggle of geese—gaggle is a name for a set. What we called collective nouns were actually sets. Anything that we can well define, where we can define each member, can be called a set. The jive is a collection of movements in a dance—that too is a set. We have also the idea of triangles as a set of figures with certain characteristic properties; this is a set. Some of them have special names and some of the sets have no special name; the last four houses in the street is a collection, but has no particular name. The objects that are in your pockets at the moment, figures half of whose perimeter is curved—these are things to which no name is given, but they are still sets. There are two ways of describing a set—we can either list the members or we can give a description of them.

Here are some examples of sets and we always denote our sets by these braces:— { }

A {John, honesty, 5, cabbage}
B {46, 48, 50}
C {Red, Blue, Yellow}
D {All whole numbers $>$ 17}

A is a set; the members are John, honesty, 5 and cabbage, but we cannot find any description for it.

Set B is the set of even numbers between 45 and 51.

When we come to set C, it is the set—red, blue, yellow. There we have listed them; if we want to describe them, they will be the primary colours.

Set D is the set of all whole numbers greater than 17; there we have given a description. Can we list them? You can try but it is going to take a long time because the whole numbers that are greater than 17 are an infinite series. Therefore, the description will be far better than the listing.

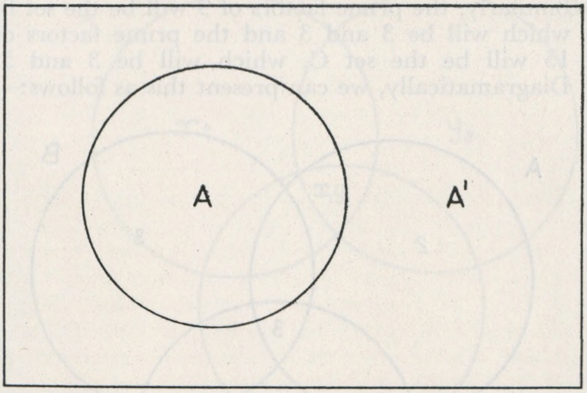
So here we have the idea of how, when we are looking at sets, we either list them or we define them.

I would like to give you a few definitions before I put them into practice.

Equal sets are sets that have the same elements though not necessarily in the same order. For example, Set A could be $\{10, -7, 4\}$ and set B can be $\{4, 10, -7\}$. They have the same members, the same number of members and therefore they are equal. Another example is set A which is $\{1^2, 2^2, 3^2, 4^2\}$ and set B which is $\{\sqrt{1}, \sqrt{256}, \sqrt{16}, \sqrt{81}\}$. It is interesting that those two sets are equal; in set B $\sqrt{256}$ is 16, and I have got 4^2 in set A. When we have sets which have the same elements, though not necessarily in the same order, we say they are equal.

The biggest set is the *universal set*. For example, all the ladies in this room is one particular set and all the gentlemen in this room is another set; this implies we have got two sets. There must be another set in which they are all involved and that is what we call the universal set. In mathematics it is always the universal set that is of prime importance, because when we are doing graphical work we have to consider whether we are using the universal set of natural numbers, the universal set of rational numbers or the universal set of real numbers. This universal set must always be given if you are going to do any particular work in the graphical line.

The universal set is represented by $\square$ or the letter U. May I give you an example? Take, for instance, all the children in your school—that would be a universal set. Divide them into two categories—the ones who wear spectacles and the ones who do not. Those who wear glasses would be set A; all the others (those who do not wear glasses) are known as the *complement* and are classed as A′. Therefore, A + A′ make up the whole of the universal set. We can represent it thus:



We can, of course, have sets that have nothing to do with each other; there is nothing actually in common between them. Here is set A {1, 2, 3} and here is set B {4, 5, 6}. These are two sets which have nothing in common and nothing to do with each other and they are known as *disjoint sets*.

An *empty set* is a set that does not satisfy the conditions at all. An example is the set of two-digit even primes. The answer is that there are none—the set is empty. "All in front of me now who have played for England in the Cup Final" is an empty set.

So we have equal sets, universal sets, empty sets and disjoint sets.

Suppose I name {1, 2, 3, 4} as set A; each element or member is a member of the set and we say that 2 is a member of set A or 4 is a member of set A. We write this as follows:—

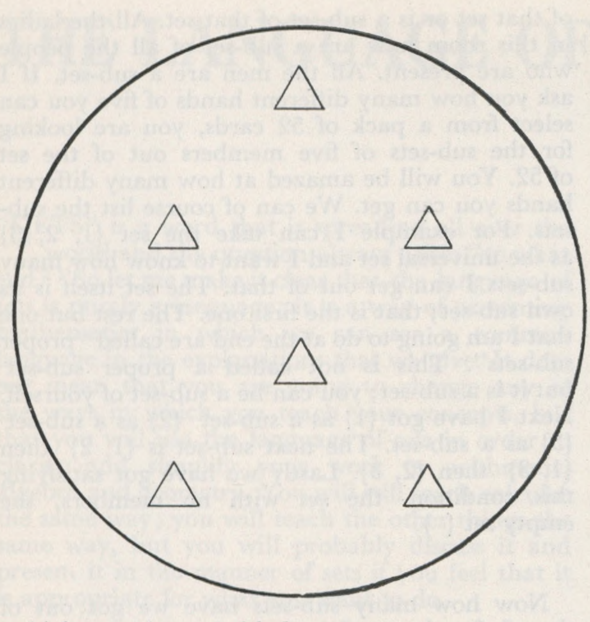
$$2 \in A : 4 \in A$$

But it is interesting that in this set are also contained other sets or "*sub-sets*"; if I said that set B is 2 and 3, you can see that set B is part of that set or is a sub-set of that set. All the ladies in this room now are a sub-set of all the people who are present. All the men are a sub-set. If I ask you how many different hands of five you can select from a pack of 52 cards, you are looking for the sub-sets of five members out of the set of 52. You will be amazed at how many different hands you can get. We can of course list the sub-sets. For example I can take the set {1, 2, 3} as the universal set and I want to know how many sub-sets I can get out of that. The set itself is its own sub-set; that is the first one. The rest but one that I am going to do at the end are called "proper sub-sets". This is not called a proper sub-set but it is a sub-set; you can be a sub-set of yourself. Next I have got {1} as a sub-set {2} as a sub-set {3} as a sub-set. The next sub-set is {1, 2} then {1, 3} then {2, 3}. Lastly we have got satisfying this condition the set with no members, the empty set { }.

Now how many sub-sets have we got out of there? 8 sub-sets. We have 3 members in our universal set and we have 2^3 sub-sets. If we had had 4 members in the universal set, we should have had 2^4 and so the answer is that in all our sets, we can get 2^n sub-sets out of them. If you ever ask your children to write down the sub-sets of some particular thing, you have not got to work out the answer; you have only to say 2 to the power of however many elements are contained and that is the answer; so if they have not found that number, you tell them to go on working and looking for the others.

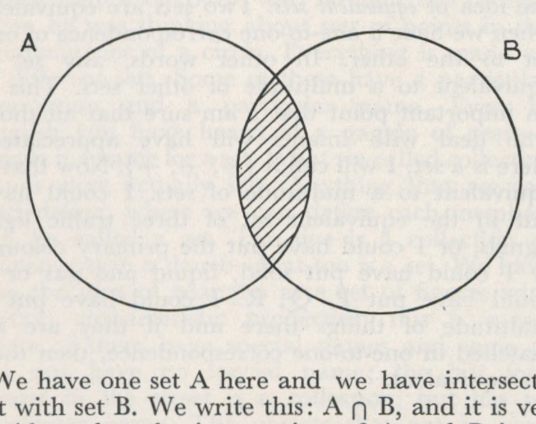
So we have the idea of universal sets, of disjoint or empty sets, we have the idea of sub-sets and now we have probably the greatest idea of all, the idea of *equivalent sets*. Two sets are equivalent when we have a one-to-one correspondence of one set to the other. In other words, any set is equivalent to a multitude of other sets. This is an important point that I am sure that all those who deal with infants will have appreciated. Here is a set, I will call it {+, +, +}. Now that is equivalent to a multitude of sets. I could have put in the equivalent set of three traffic light signals, or I could have put the primary colours, or I could have put solid, liquid and gas or I could have put P, Q, R. I could have put a multitude of things there and if they are all matched in one-to-one correspondence, then they are equivalent.

Here then is the basis of cardinal number, that if we have got this set (and of course, this is done with children, only they use concrete objects)—



which we have said by a one-to-one correspondence has a shorthand sign of "6", then any other set that can be matched in a one-to-one correspondence with this will take this sign—6. When we talk about bringing up children with the idea of fiveness and fourness and threeness, all we are doing is matching sets in a one-to-one correspondence and where they completely match and none remain extra, they are identical or equivalent. If this is so, then we have the definition of cardinal number—the cardinal number is a class of equivalent sets, such that all sets that are equivalent will be given the same cardinal number.

Let us now proceed to the *operation of sets*. The first operation and probably the most important is the *intersection of sets*.

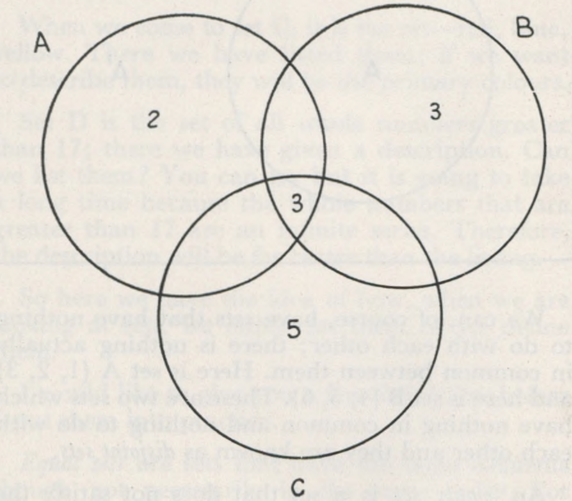


We have one set A here and we have intersected it with set B. We write this: $A \cap B$, and it is very evident that the intersection of A and B is the shaded part in the diagram. It is easy to see that A intersected with B is the same as B intersected with A and we write this $A \cap B = B \cap A$.

Here we have the commutative law coming out in the matter of sets. This kind of work is regularly done in schools. Take for example all the boys in the football team and all those in the cricket team. We have 11 in the football team and we have 11 in the cricket team, but it does not follow that there are 22 altogether because there may be some who are in both the football team and the cricket team. This is the kind of thing that is done in the infant school.

The most important point about this set language is that number through set language can have its greatest force in the infant school.

Taking this a step further, let us look at the numbers 6, 9 and 15. We are going to split them up into their prime factors. The prime factors of 6 will be the set A, which will be 2 and 3. Similarly, the prime factors of 9 will be the set B, which will be 3 and 3 and the prime factors of 15 will be the set C, which will be 3 and 5. Diagramatically, we can present this as follows:—



From this diagram, I am sure you will see that A intersected with B and then intersected with C is the same as A intersected with B with C. In other words, we have got the associative law coming out in this. What do you think is going to go in the very centre of the diagram? In that section there is the intersection of set A, set B and set C. What have we there? We have got a 3 in sets A, B and C. What then have I got left? In set B I have a 3 left. In set C I have a 5, and in set A a 2. If we multiply those together, we shall have the LCM, 90.

Now let us look for the HCF of 6, 9 and 15. Have we anything now that we can put into the middle. We have the 3. Now in set A, we have the 2 and 3; in set B, the 3 and another 3 and in set C we have the 3 and a 5. Thus here we have the HCF of 3. From that intersection, which is the same thing, we can work out HCF's and things like that.

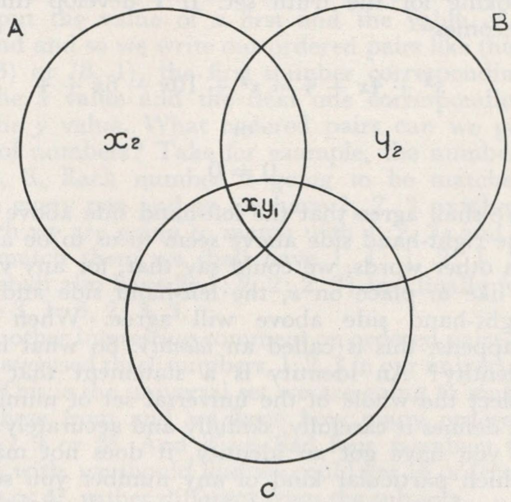
We can do exactly the same with algebra if we wish. Here is an illustration:—

$$x^2y = \{x_1, x_2, y_1\}$$

$$xy^2 = \{x_1, y_1, y_2\}$$

$$xy = \{x_1, y_1\}$$

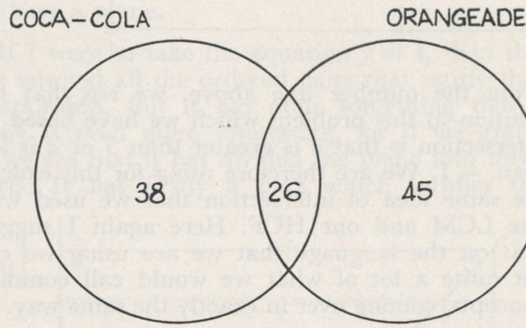
Here we put our three given algebraic expressions into the form of sets. Let us now try, as before, to put these three sets into one of our diagrams, which are called "Venn Diagrams":—



By doing an analysis similar to the previous one, we find that the HCF is xy, and the LCM x^2y^2. So this kind of intersection is a very powerful way to bring out such things as LCM and HCF.

These diagrams also bring out other kinds of information. Remember a Venn diagram is not anything mathematical, but is something that tries to make the development and the ensuing mathematical thought clearer.

Let us take another example. 114 people were interviewed. They were asked whether they drank Coca Cola or orangeade. 64 drank Coca Cola out of the 114 that were interviewed. 71 drank orangeade and 26 drank both. Now let us see what information we can get out of this. First of all we have the people who drank Coca Cola, then we have the people who drank orangeade. 26 drank both. It is evident that the 26 who drank both are in effect the intersection of those two sets of people. Now let us have a look at the Coca Cola people. 26 are already in, 64 drank it altogether and so we can say 38 drank Coca Cola only. Now when we come to the orangeade, we have got 26 already in and 71 drank orangeade, so according to that, there were 45 people who drank orangeade only. Let us now arrange these facts on a Venn diagram:—

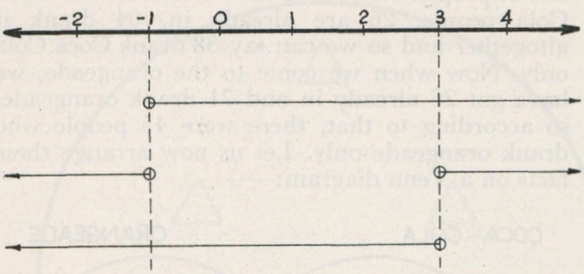


Now if you add them all up together, how many have we got? 109. Then there are 109 in all those, but how many people were interviewed? 114. So 5 of them did not drink either. Thus we have analysed the information given to us and we have come to the conclusion that 5 of them did not drink either.

I have said previously that it is no good thinking about equals unless we are going to think about "greater than" and "less than". Perhaps one of the most recent innovations is that children are dealing with inequalities. This equation for example:

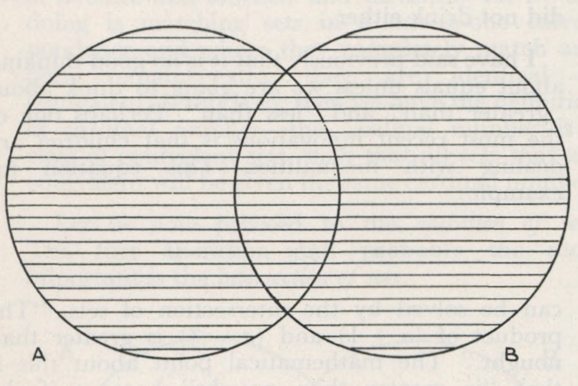
$$(a + 1)(a - 3) > 0$$

can be solved by the intersection of sets. "The product of $(a + 1)$ and $(a - 3)$ is greater than nought." The mathematical point about this is that "is greater than nought", because if the product of two things is greater than nought, then they must either both be positive or they must both be negative because the product of two positives is a positive and the product of two negatives is a positive. $(a + 1)$ is greater than nought and $(a - 3)$ is greater than nought, so we have got our two positives; or $(a + 1)$ is less than nought and $(a - 3)$ is less than nought, so we have got our two negatives. Now let us solve these. To make them positive, a must be greater than -1 in the first bracket and a must be greater than 3 in the second bracket. To make the two brackets negative, a must be less than -1 and less than $+3$. Now if I put these facts on our number line, this is how we shall represent the facts we have just deduced:—



From the number line above, we see that the solution to this problem which we have based on intersection is that a is greater than 3 or a is less than -1. We are therefore using for this exactly the same idea of intersection that we used with our LCM and our HCF. Here again I suggest that on the language that we are using we can get quite a lot of what we would call common concepts, coming over in exactly the same way.

The next point is the *union of sets*. We have seen above that sets are intersected when there are common elements to both. The definition of *union* of sets is that the union between A and B occurs when we have in the set the elements which are common to A or common to B or both.



For example, the shaded part in the above sketch is the union of sets A and B. You can see again that it is the same as B united with A.

Here is the intersection sign, $\cap$, which in some books is called "cap" and here is the union sign, $\cup$, which in some books is known as "cup". If I wanted to illustrate the union of sets, I would probably do it in this way.

Set A $\{1, 2, 3\}$ and set B $\{3, 4, 5\}$. The union of set A with set B is the elements that are common to A or common to B or both, which are 1, 2, 3, 4, 5, so that would be the union of A and B. We write this: $A \cup B = \{1, 2, 3, 4, 5\}$. The intersection of A and B is the elements which are in both A and in B. There is only one, 3, and so the intersection of them is the set $\{3\}$. We write this again, $A \cap B = \{3\}$.

It is considered that it is the *language* that is the vital thing in sets. May I stress again that, as I see it, it is not that you have got to go and teach things any differently, it is that you may use this kind of language to join and bind them up, although there may be some people who will now teach LCM and HCF on Venn diagrams. I also said that it was useful for making definitions clearer. For example, I think the definition of cardinal number is clearer on this particular thing and we might have a look at this—

$$(x + 2)^2 = x\,(x + 10) - 6(x - \tfrac{2}{3})$$

There is an open sentence for which we are looking for the truth set. If I develop this, it becomes—

$$x^2 + 4x + 4 = x^2 + 10x - 6x + 4$$

or

$$0 = 0$$

We shall agree that the left-hand side above and the right-hand side above seem to us to be alike. In other words, we could say that, for any value I like to place on x, the left-hand side and the right-hand side above will agree. When this happens, this is called an *identity*. So what is an identity? An identity is a statement that can select the whole of the universal set of numbers. It defines it carefully, skilfully and accurately and if you have got an identity, it does not matter which particular kind of any number you select out of any universe at all, you will make the statement true.

As another example, let us take—

$$(x + 5)^2 = x(x + 10) + 26$$

This becomes:

$$x^2 + 10x + 25 = x^2 + 10x + 26$$

$$\text{i.e. } 0 = 1$$

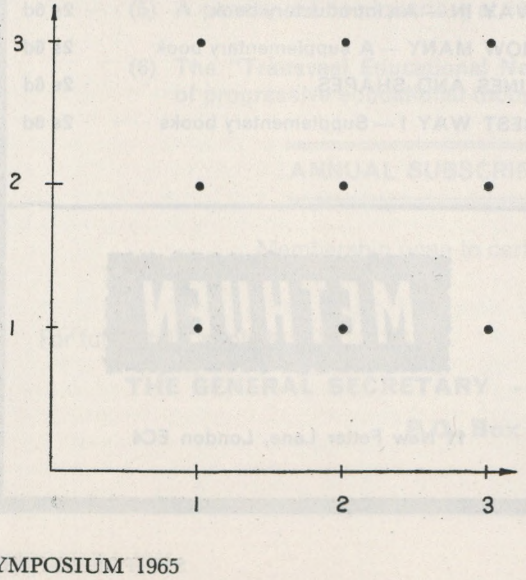
This, of course, is absurd. You cannot take *any* number that will make this second statement true and therefore that is the *nul set*.

In an identity, we use the universal set and in the second example, we are using the nul set. In between comes what you know as the equation. An equation is a statement where we imply that the solution set that we are after is neither the universal set nor empty. It is possible that if you use this language as we have used it here then the child will be able to see quite clearly the difference between an identity and an equation. There may be many who say "Well we shall not be teaching this kind of stuff up to this standard." My answer is that the people in the secondary schools should know what is going on in the infant schools and that the infant teachers also should know what is going on in other parts of the school. I do not know where primary mathematics ends and secondary starts. This particular language of sets is offered in order that we may be able to clarify things throughout.

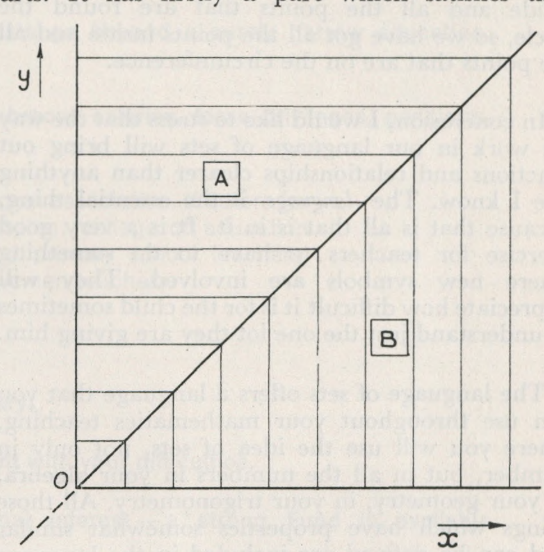
When we come to consider ordered pairs, here I think sets give us rather a good description. Why are they called ordered? If I were to take the numbers 1 and 3, then the pair (1, 3) is not the same as the pair (3, 1). The order is the important thing, you see. In our Cartesian work, we put the value of x first and the value of y second and so we write our ordered pairs like this: (2, 3) or (8, 1), the first number corresponding to the x value and the next one corresponding to the y value. What ordered pairs can we get out of numbers? Take for example, the numbers 1, 2, 3. Each number is going to be matched with every one and so we have 1, 2, 3 numbers which we are going to match with 1, 2, 3; and if we match them we shall have 1, 1; 1, 2; 1, 3. We shall also have 2, 1; 2, 2; 2, 3; and finally we have 3, 1; 3, 2; 3, 3.

Another interesting comment on ordered pairs—we have used three numbers, 1, 2, 3 in our universal set. This is the universal set we are going to make all these from and we have how many ordered pairs? 9 or 3^2. And if we had four members to work with, we should find we could get 16 ordered pairs or 4^2, rather different from the sub-sets.

If I put these 9 pairs on to a graph, then I would have 9 points, because this was my universal set. Had I made the universal set of the set of real numbers, then I would have had as my graph the entire plane. This is the difference between making a universal set a set of those nine numbers, as against making the universal set the set of all real numbers. The important point in all this work is "What is your universal set?" When we only have a few points, we set up a lattice; when it is a question of all real numbers, it is then that we have a plane.



If I were to take the equation $y = x$, then this has selected all the ordered pairs that satisfy that relationship. But it has done something that I think is even more interesting, for if we graph it, we see that it has divided the plane into three parts. It has firstly a part which satisfies the



condition $y = x$, i.e. a straight line, then we have the part "A" in which $y > x$ and finally, we have the part "B" where $y < x$.

I am going to ask you, whatever you do, to try to work in these three different kinds of things; not only should you have a graph where $y = x$, but you should also have a graph where y is *greater* than x, and where y is *less* than x. We do not work this trichotomy law nearly as much as we should.

Finally, we come to the question of locus. A locus is a set of points, and only those points that satisfy a given condition—which may be an equation. The given condition may be an inequality with two variables. For example, the locus that satisfies this equation: $x^2 + y^2 = 16$. I do not think we should ever leave it at that with our pupils and I suppose you have gathered what I would have done in my school if we were working on this kind of thing. We should do $x^2 + y^2 > 16$; $x^2 + y^2 < 16$ and for an extra bit of fun: $x^2 + y^2 \leqq 16$. What are we going to do about this? $x^2 + y^2 = 16$—you will have to accept that it is a circle with the radius of 4 units. Now let us take a line that cuts the circumference (which is a set of points). Our first relationship is that $x^2 + y^2$ should be equal to 16. The locus of such a point is a circle whose radius is 4 units. Our second relationship is "it is greater than 16", so if it is greater than, it is a point outside the circumference of the circle. The next one is "it is less than 16", so it is a point inside the circle. Why not do them all at once? And when we come to the special one, "it is less than or equal to", then I hope you can see that the solution to that is the universal set of all real numbers that is inside and all the points that are round the circle, so we have got all the points inside and all the points that are on the circumference.

In conclusion, I would like to stress that the way we work in our language of sets will bring out functions and relationships clearer than anything else I know. The *language* is the essential thing, because that is all that is in it. It is a very good exercise for teachers to have to do something where new symbols are involved. They will appreciate how difficult it is for the child sometimes to understand just the one lot they are giving him.

The language of sets offers a language that you can use throughout your mathematics teaching, where you will use the idea of sets, not only in number, but in all the numbers in your algebra, in your geometry, in your trigonometry. All those things which have properties somewhat similar and can be defined are included in the language of sets.