

### B.3 Anomalous Cases

The results from Hsu's[1963] analysis confirm that if the parametric coupling matrix is diagonal, which results in a set of uncoupled Mathieu equations, then only main parametric resonances will occur. In the general case of a nonsymmetric coupling matrix, main as well as sum and difference type parametric resonances occur. In the mine hoist system, the parametric coupling matrix is always symmetric, and consequently parametric resonances of the difference type do not arise. Hsu[1963] notes that in the case where repeated natural frequencies of the system occur, or when different combinations of  $i, j, s$  overlay each other, then an anomalous situation arises, where more than one resonance is excited simultaneously, and a more detailed analysis is required. The former case of repeated natural frequencies has been analysed by Nayfeh[1983b] and Tezak et al[1982]. The later case of multi-frequency excitation of a two degree of freedom system is considered by Nayfeh[1983a].

The stability analysis presented considers the stability of the motion in the neighbourhood of the first order expansion. Consequently principal<sup>5</sup> regions of main and combination parametric resonance are considered. This follows Hsu's[1963] argument that although higher order regions may co-exist within the principal parametric regions, and therefore secular terms arise in the higher order expansion, for small parametric excitation the boundary of stability would be dictated by the principal region, and hence the stability of the first order expansion. In the case of the mine hoist system, anomalous conditions almost always arise since the natural frequencies of the catenary are related by integer multiples, or are commensurable. Thus if one harmonic of the excitation tunes to a region of parametric resonance, then other harmonics will simultaneously excite other resonances. These may be main or combination parametric resonances of the summed type. Thus it is possible to develop a number of anomalous cases, depending on the number of harmonics accounted for in the longitudinal excitation, and the number of lateral modes accounted for in the eigenfunction expansion<sup>6</sup>. Three conditions are examined for the mine hoist system, as presented in table B.1.

It is shown that the first two cases are in fact not anomalous, and Hsu's[1963] formulae can be applied directly. However in the last case, a stability criterion is derived which requires explicit solution to determine the span of the region

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<sup>5</sup>For a multi-degree-of-freedom parametrically excited system, with a single excitation frequency  $\Omega$ , regions of instability arise at  $\Omega = \omega_i \pm \omega_j/n$ ; the principal region refers to  $n = 1$ , and is obtained by considering the first order expansion. Secondary regions occur at  $n = 2, 3 \dots$  as defined by the second order and higher expansions.

<sup>6</sup>Although the model is truncated to account for a finite number of lateral modes, the forced longitudinal response represents the complete solution, without modal truncation.

Table B.1: Anomalous regions of parametric resonance

Case	DOF	Harmonics	Resonance Condition
Case 1	2 DOF	$s = 1, 2$	$\omega \approx 2\omega_1, 2\omega \approx 2\omega_2$
Case 2	3 DOF	$s = 1$	$\omega \approx 2\omega_2 \approx \omega_1 + \omega_3$
Case 3	2 DOF	$s = 1, 2, 3$	$\omega \approx \omega_1, 2\omega \approx 2\omega_1, 3\omega \approx \omega_1 + \omega_2$

of instability. Nayfeh[1983a] presented results of an analysis of a two degree of freedom system to multi-frequency excitation. The third case considered here is identical to the fifth case presented by Nayfeh[1983a], where simultaneous main resonance and a summed combination resonance arises due to two different harmonics.

### B.3.1 Case 1: 2 DOF - $s = 1, 2$ $\omega \approx 2\omega_1$

This case represents a two degree of freedom system excited by a periodic function with two harmonic components at a frequency  $\omega$  and  $2\omega$ . It is assumed that  $\omega$  is close to  $2\omega_1$ , and  $\omega_2$  since the natural frequencies of the system are related by integer numbers. Employing a detuning parameter  $\lambda\epsilon$ , the excitation frequency  $\omega$  may be written as:

$$\omega = 2\omega_1 + \lambda\epsilon$$

$$\omega = \omega_2 + \lambda\epsilon$$

or as:

$$\begin{aligned}\omega - \omega_1 &= \omega_1 + \lambda\epsilon \\ 2\omega - \omega_2 &= \omega_2 + 2\lambda\epsilon\end{aligned}\tag{B.15}$$

Transferring small divisor terms in the perturbation part of the solution to the variational part, and expanding the variational part of equation(B.13) for the two degrees of freedom:

$$-\frac{dA_1}{dt}\sin\omega_1 t + \frac{dB_1}{dt}\cos\omega_1 t = -\frac{\epsilon}{2\omega_1} \sum_{s=1}^2 \left\{ \begin{aligned} &A_1 d_{11}^{(s)}(\cos(\omega_1 + s\omega)t + \cos(\omega_1 - s\omega)t) + \\ &B_1 d_{11}^{(s)}(\sin(\omega_1 + s\omega)t + \sin(\omega_1 - s\omega)t) + \\ &A_2 d_{12}^{(s)}(\cos(\omega_2 + s\omega)t + \cos(\omega_2 - s\omega)t) + \\ &B_2 d_{12}^{(s)}(\sin(\omega_2 + s\omega)t + \sin(\omega_2 - s\omega)t) \end{aligned} \right\} \tag{B.16}$$

$$-\frac{dA_2}{dt}\sin\omega_2 t + \frac{dB_2}{dt}\cos\omega_2 t = -\frac{\epsilon}{2\omega_2} \sum_{s=1}^2 \left\{ \begin{aligned} &A_1 d_{21}^{(s)}(\cos(\omega_1 + s\omega)t + \cos(\omega_1 - s\omega)t) + \\ &B_1 d_{21}^{(s)}(\sin(\omega_1 + s\omega)t + \sin(\omega_1 - s\omega)t) + \\ &A_2 d_{22}^{(s)}(\cos(\omega_2 + s\omega)t + \cos(\omega_2 - s\omega)t) + \\ &B_2 d_{22}^{(s)}(\sin(\omega_2 + s\omega)t + \sin(\omega_2 - s\omega)t) \end{aligned} \right\} \tag{B.17}$$

Substituting equations (B.15) into the above equations and retaining terms which would cause secular behaviour of the perturbation solution: ie any terms which result in frequencies close to  $\omega_1$  or  $\omega_2$  in equations (B.16),(B.17) respectively, results in:

$$-\frac{dA_1}{dt}\sin\omega_1 t + \frac{dB_1}{dt}\cos\omega_1 t = -\frac{\epsilon d_{11}^{(1)}}{2\omega_1} \{A_1 \cos(\omega_1 + \epsilon\lambda)t - B_1 \sin(\omega_1 + \epsilon\lambda)t\}$$

$$-\frac{dA_2}{dt}\sin\omega_2 t + \frac{dB_2}{dt}\cos\omega_2 t = -\frac{\epsilon d_{22}^{(2)}}{2\omega_2} \{A_2 \cos(\omega_2 + 2\epsilon\lambda)t - B_2 \sin(\omega_2 + 2\epsilon\lambda)t\}$$

$$\frac{dA_1}{dt}\cos\omega_1 t + \frac{dB_1}{dt}\sin\omega_1 t = 0$$

$$\frac{dA_2}{dt}\cos\omega_2 t + \frac{dB_2}{dt}\sin\omega_2 t = 0$$

These equations may be solved to determine<sup>7</sup>  $\frac{dA_1}{dt}$ ,  $\frac{dA_2}{dt}$ ,  $\frac{dB_1}{dt}$ ,  $\frac{dB_2}{dt}$ . Applying the Kryloff-Bogoliuboff-Van der Pohl averaging technique<sup>8</sup>, and applying the transformation:

$$X_1 = A_1 + iB_1$$

$$X_2 = A_1 - iB_1$$

$$Y_1 = A_2 + iB_2$$

$$Y_2 = A_2 - iB_2$$

Results in four first order equations:

$$\begin{aligned} \frac{dX_1}{dt} &= -\frac{i\epsilon d_{11}}{4\omega_1} X_2 e^{-i\epsilon\lambda t} \\ \frac{dX_2}{dt} &= \frac{i\epsilon d_{11}}{4\omega_1} X_1 e^{i\epsilon\lambda t} \\ \frac{dY_1}{dt} &= -\frac{i\epsilon d_{22}}{4\omega_2} Y_2 e^{-i2\epsilon\lambda t} \\ \frac{dY_2}{dt} &= \frac{i\epsilon d_{22}}{4\omega_2} Y_1 e^{i2\epsilon\lambda t} \end{aligned} \quad (B.18)$$

These equations can be converted into autonomous form<sup>9</sup>, and consequently the stability of the equations can be examined by considering the roots of the characteristic equation.

<sup>7</sup>Where it is implied henceforth that  $d_{11}$  and  $d_{22}$  are the parametric components of the first harmonic and second harmonic respectively.

<sup>8</sup>This is accomplished by substituting  $\Phi_i = \omega_i t$ , and averaging the equations as  $\frac{1}{2\pi} \int_0^{2\pi} \frac{dA_i}{dt} d\Phi_i$  where  $A_i, B_i$  are treated as constants.

<sup>9</sup>The autonomous form is obtained by applying the transformations  $X_1 = \bar{X}_1 e^{-i\frac{\epsilon\lambda}{2}t}$ ,  $X_2 = \bar{X}_2 e^{i\frac{\epsilon\lambda}{2}t}$ ,  $Y_1 = \bar{Y}_1 e^{-i\epsilon\lambda t}$ ,  $Y_2 = \bar{Y}_2 e^{i\epsilon\lambda t}$ .

Assuming a solution of the form:

$$\begin{aligned} X_1 &= \bar{X}_1 e^{pt - i\frac{\epsilon\lambda}{2}t} \\ X_2 &= \bar{X}_2 e^{pt + i\frac{\epsilon\lambda}{2}t} \\ Y_1 &= \bar{Y}_1 e^{qt - i\epsilon\lambda t} \\ Y_2 &= \bar{Y}_2 e^{qt + i\epsilon\lambda t} \end{aligned} \quad (\text{B.19})$$

Substituting equations (B.19) into the equations (B.18), results in:

$$\begin{bmatrix} p - i\frac{\epsilon\lambda}{2} & \frac{i\epsilon d_{11}}{4\omega_1} & 0 & 0 \\ -\frac{i\epsilon d_{11}}{4\omega_1} & p + i\frac{\epsilon\lambda}{2} & 0 & 0 \\ 0 & 0 & q - i\epsilon\lambda & \frac{i\epsilon d_{22}}{4\omega_2} \\ 0 & 0 & -\frac{i\epsilon d_{22}}{4\omega_2} & q + i\epsilon\lambda \end{bmatrix} \begin{Bmatrix} \bar{X}_1 \\ \bar{X}_2 \\ \bar{Y}_1 \\ \bar{Y}_2 \end{Bmatrix} = \begin{Bmatrix} \vdots \\ 0 \\ \vdots \end{Bmatrix} \quad (\text{B.20})$$

The characteristic equation is obtained by setting the determinant of equation (B.20) to zero:

$$(p^2 + \frac{\epsilon^2\lambda^2}{4} - \frac{\epsilon^2 d_{11}^2}{16\omega_1^2})(q^2 + \epsilon^2\lambda^2 - \frac{\epsilon^2 d_{22}^2}{16\omega_2^2}) = 0$$

Thus :

$$p = \pm \frac{\epsilon}{2} \left( \frac{d_{11}^2}{4\omega_1^2} - \lambda^2 \right)^{\frac{1}{2}}$$

$$q = \pm \epsilon \left( \frac{d_{22}^2}{16\omega_2^2} - \lambda^2 \right)^{\frac{1}{2}}$$

In order for the system to be neutrally stable, the roots  $p, q$  of the above equations must be imaginary.

Thus the stability region is given by:

$$\frac{d_{11}^2}{4\omega_1^2} - \lambda^2 < 0$$

$$\frac{d_{22}^2}{16\omega_2^2} - \lambda^2 < 0$$

Substituting  $\epsilon\lambda$  from equation (B.15) into the above equations and simplifying results in the stability intervals:

$$2\omega_1 - \frac{\epsilon|d_{11}|}{2\omega_1} < \omega < 2\omega_1 + \frac{\epsilon|d_{11}|}{2\omega_1}$$

$$\omega_2 - \frac{\epsilon|d_{22}|}{4\omega_2} < \omega < \omega_2 + \frac{\epsilon|d_{22}|}{4\omega_2}$$

It is evident that these are the same as Hsu's[1963] regions, for  $s = 1, 2$ , and thus this case is not an anomalous case, although this would not be evident beforehand.

### B.3.2 Case 2: 3 DOF - $s = 1$ $\omega \approx 2\omega_2$

This case represents a three degree of freedom system excited sinusoidally at a frequency  $\omega$  close to  $2\omega_2$ . In this case a main and combination parametric resonance is excited simultaneously ie.  $\omega \approx 2\omega_2 \approx \omega_1 + \omega_3$ . Employing a detuning parameter  $\lambda\epsilon$ , the excitation frequency  $\omega$  may be written as:

$$\omega = 2\omega_2 + \lambda\epsilon$$

$$\omega = \omega_1 + \omega_3 + \lambda\epsilon$$

or as:

$$\begin{aligned}\omega - \omega_2 &= \omega_2 + \lambda\epsilon \\ \omega - \omega_1 &= \omega_3 + \lambda\epsilon \\ \omega - \omega_3 &= \omega_1 + \lambda\epsilon\end{aligned}$$

Proceeding in the same manner as the preceding section, and retaining secular terms of the perturbation solution in the variational part of the solution results in three equations.

$$-\frac{dA_1}{dt}\sin\omega_1 t + \frac{dB_1}{dt}\cos\omega_1 t = -\frac{\epsilon d_{13}}{2\omega_1} \{A_3\cos(\omega_1 + \epsilon\lambda)t - B_3\sin(\omega_1 + \epsilon\lambda)t\}$$

$$-\frac{dA_2}{dt}\sin\omega_2 t + \frac{dB_2}{dt}\cos\omega_2 t = -\frac{\epsilon d_{22}}{2\omega_2} \{A_2\cos(\omega_1 + 2\epsilon\lambda)t - B_2\sin(\omega_1 + 2\epsilon\lambda)t\}$$

$$-\frac{dA_3}{dt}\sin\omega_3 t + \frac{dB_3}{dt}\cos\omega_3 t = -\frac{\epsilon d_{31}}{2\omega_3} \{A_1\cos(\omega_3 + \epsilon\lambda)t - B_1\sin(\omega_3 + \epsilon\lambda)t\}$$

and:

$$\frac{dA_i}{dt}\cos\omega_i t + \frac{dB_i}{dt}\sin\omega_i t = 0 \quad i = 1, 2, 3$$

The above six equations may be solved to determine  $\frac{dA_i}{dt}$ ,  $\frac{dB_i}{dt}$ . Applying the Kryoloff-Bogoliuboff-Van der Pohl averaging technique, and applying the transformation:

$$\begin{aligned} X_1 &= A_1 + iB_1 \\ X_2 &= A_1 - iB_1 \\ Y_1 &= A_2 + iB_2 \\ Y_2 &= A_2 - iB_2 \\ Z_1 &= A_3 + iB_3 \\ Z_2 &= A_3 - iB_3 \end{aligned}$$

Results in six first order equations:

$$\begin{aligned} \frac{dX_1}{dt} &= -\frac{i\epsilon d_{13}}{4\omega_1} Z_2 e^{-i\epsilon\lambda t} \\ \frac{dX_2}{dt} &= +\frac{i\epsilon d_{13}}{4\omega_1} Z_1 e^{i\epsilon\lambda t} \\ \frac{dY_1}{dt} &= -\frac{i\epsilon d_{22}}{4\omega_2} Y_2 e^{-i\epsilon\lambda t} \\ \frac{dY_2}{dt} &= +\frac{i\epsilon d_{22}}{4\omega_2} Y_1 e^{i\epsilon\lambda t} \\ \frac{dZ_1}{dt} &= -\frac{i\epsilon d_{31}}{4\omega_3} X_2 e^{-i\epsilon\lambda t} \\ \frac{dZ_2}{dt} &= +\frac{i\epsilon d_{31}}{4\omega_3} X_1 e^{i\epsilon\lambda t} \end{aligned} \tag{B.21}$$

Equations (B.21) can be conveniently transformed into two second order autonomous equations:

$$\ddot{X}_1 + i\lambda\epsilon\dot{X}_1 - \frac{\epsilon^2 d_{13}d_{31}}{16\omega_1\omega_3} X_1 = 0 \tag{B.22}$$

$$\ddot{Y}_1 + i\lambda\epsilon\dot{Y}_1 - \frac{\epsilon^2 d_{22}d_{22}}{16\omega_2^2} Y_1 = 0 \tag{B.23}$$

The solutions to equations (B.22),(B.23) are:

$$\begin{aligned} X_1 &= \bar{X}_1 e^{(a-i\frac{\epsilon\lambda}{2})t} \\ Y_1 &= \bar{Y}_1 e^{(b-i\frac{\epsilon\lambda}{2})t} \end{aligned}$$

$$a = \pm \left( \frac{\epsilon^2 \lambda^2}{4} - \frac{\epsilon^2 d_{13} d_{31}}{16 \omega_1 \omega_3} \right)^{\frac{1}{2}}$$

$$b = \pm \left( \frac{\epsilon^2 \lambda^2}{4} - \frac{\epsilon^2 d_{22}^2}{16 \omega_2^2} \right)^{\frac{1}{2}}$$

These solutions are neutrally stable if  $a, b$  are imaginary. Thus the stability region is defined by:

$$\epsilon^2 \lambda^2 > \frac{\epsilon^2 d_{13} d_{31}}{4 \omega_1 \omega_3}$$

$$\epsilon^2 \lambda^2 > \frac{\epsilon^2 d_{22}^2}{4 \omega_2^2}$$

Substituting  $\epsilon \lambda$  from equation (B.21) into the above equations and simplifying, results in the stability intervals:

$$(\omega_1 + \omega_3) - \frac{\epsilon}{2} \sqrt{\frac{d_{13} d_{31}}{\omega_1 \omega_3}} < \omega < (\omega_1 + \omega_3) + \frac{\epsilon}{2} \sqrt{\frac{d_{13} d_{31}}{\omega_1 \omega_3}}$$

$$2\omega_2 - \frac{\epsilon |d_{22}|}{4\omega_2} < \omega < 2\omega_2 + \frac{\epsilon |d_{22}|}{4\omega_2}$$

It is evident that once again these regions are the same as Hsu's[1963] regions, and thus this case is not anomalous. The largest region would dictate the stability of the system, and thus the relative magnitudes of the coefficients are important.

### B.3.3 Case 3: 2 DOF - $s = 1, 2, 3$ $\omega \approx \omega_1$

This case represents a two DOF system excited by a periodic function with three harmonic components at frequencies  $\omega$ ,  $2\omega$ ,  $3\omega$ . In this case a main and combination parametric resonance is excited simultaneously by the second and third harmonic respectively, ie  $2\omega \approx 2\omega_1, 3\omega \approx \omega_1 + \omega_2$ . Employing a detuning parameter  $\lambda\epsilon$ , the excitation frequency  $\omega$  may be written as:

$$\omega = \omega_1 + \lambda\epsilon$$

$$\omega = \frac{\omega_1 + \omega_2}{3} + \lambda\epsilon$$

or as:

$$\begin{aligned} 2\omega - \omega_1 &= \omega_1 + 2\lambda\epsilon \\ 3\omega - \omega_1 &= \omega_2 + 3\lambda\epsilon \\ 3\omega - \omega_2 &= \omega_1 + 3\lambda\epsilon \end{aligned}$$

Proceeding in the same manner as the preceding section, and retaining secular terms of the perturbation solution in the variational part of the solution results in the equations.

$$-\frac{dA_1}{dt}\sin\omega_1 t + \frac{dB_1}{dt}\cos\omega_1 t = -\frac{\epsilon}{2\omega_1} \left\{ \begin{aligned} &d_{11}^{(2)}(A_1\cos(\omega_1 + 2\epsilon\lambda)t - B_1\sin(\omega_1 + 2\epsilon\lambda)t) \\ &+ d_{12}^{(3)}(A_2\cos(\omega_1 + 3\epsilon\lambda)t - B_2\sin(\omega_1 + 3\epsilon\lambda)t) \end{aligned} \right\}$$

$$-\frac{dA_2}{dt}\sin\omega_2 t + \frac{dB_2}{dt}\cos\omega_2 t = -\frac{\epsilon d_{21}^{(3)}}{2\omega_2} \{A_1\cos(\omega_2 + 3\epsilon\lambda)t - B_1\sin(\omega_2 + 3\epsilon\lambda)t\}$$

$$\frac{dA_1}{dt}\cos\omega_1 t + \frac{dB_1}{dt}\sin\omega_1 t = 0$$

$$\frac{dA_2}{dt}\cos\omega_2 t + \frac{dB_2}{dt}\sin\omega_2 t = 0$$

These equations may be solved to determine  $\frac{dA_1}{dt}, \frac{dA_2}{dt}, \frac{dB_1}{dt}, \frac{dB_2}{dt}$ . Applying the Kryloff-Bogoliuboff-Van der Pohl averaging technique, and applying the transformation:

$$\begin{aligned}
X_1 &= A_1 + iB_1 \\
X_2 &= A_1 - iB_1 \\
Y_1 &= A_2 + iB_2 \\
Y_2 &= A_2 - iB_2
\end{aligned}$$

Results in four first order equations:

$$\begin{aligned}
\frac{dX_1}{dt} &= -\frac{i\epsilon}{4\omega_1} \left\{ d_{11}^{(2)} X_2 e^{-i2\epsilon\lambda t} + d_{12}^{(3)} Y_2 e^{-i3\epsilon\lambda t} \right\} \\
\frac{dX_2}{dt} &= +\frac{i\epsilon}{4\omega_1} \left\{ d_{11}^{(2)} X_1 e^{i2\epsilon\lambda t} + d_{12}^{(3)} Y_1 e^{i3\epsilon\lambda t} \right\} \\
\frac{dY_1}{dt} &= -\frac{i\epsilon}{4\omega_2} \left\{ d_{21}^{(3)} X_2 e^{-i3\epsilon\lambda t} \right\} \\
\frac{dY_2}{dt} &= +\frac{i\epsilon}{4\omega_2} \left\{ d_{21}^{(3)} X_1 e^{i3\epsilon\lambda t} \right\}
\end{aligned} \tag{B.24}$$

By applying successive differentiation and substitution, equations (B.24) are converted into two second order equations.

$$\begin{aligned}
\ddot{Y}_1 + i\frac{3\gamma}{2}\dot{Y}_1 - cbY_1 &= -iaexp^{-2i\gamma t}\dot{Y}_2 \\
\ddot{Y}_2 - i\frac{3\gamma}{2}\dot{Y}_2 - cbY_2 &= +iaexp^{2i\gamma t}\dot{Y}_1
\end{aligned} \tag{B.25}$$

where:

$$\begin{aligned}
a &= \frac{\epsilon d_{11}^{(2)}}{4\omega_1} \\
b &= \frac{\epsilon d_{12}^{(3)}}{4\omega_1} \\
c &= \frac{\epsilon d_{21}^{(3)}}{4\omega_2} \\
\gamma &= 2\epsilon\lambda
\end{aligned} \tag{B.26}$$

Assuming a solution of the form:

$$\begin{aligned} Y_1 &= \bar{Y}_1 e^{pt-i\gamma t} \\ Y_2 &= \bar{Y}_2 e^{pt+i\gamma t} \end{aligned} \quad (\text{B.27})$$

Substitution into the equations (B.25) results in:

$$\begin{bmatrix} (p-i\gamma)^2 + i\frac{3\gamma}{2}(p-i\gamma) - cb & ia(p+i\gamma) \\ -ia(p-i\gamma) & (p+i\gamma)^2 - i\frac{3\gamma}{2}(p+i\gamma) - cb \end{bmatrix} \begin{Bmatrix} \bar{Y}_1 \\ \bar{Y}_2 \end{Bmatrix} = \begin{Bmatrix} \vdots \\ 0 \\ \vdots \end{Bmatrix} \quad (\text{B.28})$$

The characteristic equation is obtained by setting the determinant of equation (B.28) to zero:

$$p^4 + \left(\frac{5\gamma^2}{4} - 2cb - a^2\right)p^2 + \left(\frac{\gamma^2}{2} - cb\right)^2 - a^2\gamma^2 = 0$$

thus the roots of  $p^2$  are:

$$p^2 = -\frac{1}{2}\left(\frac{5\gamma^2}{4} - 2cb - a^2\right) \pm \frac{1}{8}\sqrt{(3\gamma^2 + 4a^2)^2 - 16cb(\gamma^2 + 4a^2)} \quad (\text{B.29})$$

Which can be written as:

$$p^2 = -\Gamma \pm \sqrt{\Delta}$$

For neutral stability, the roots of  $p$  must be imaginary. Thus two situations may arise:

- i)  $\Delta = 0$  and  $\Gamma > 0$
- ii)  $\Gamma \pm \sqrt{\Delta} < 0$  and real.

Thus this case is anomalous and does not reduce to Hsu's[1963] form. Equation (B.29) must be solved for various values of  $\lambda\epsilon$  and the boundary of the stability region constructed, such that the roots for  $p$  are always imaginary.

The two limiting cases ie  $a = 0$  and  $b = c = 0$  are considered to illustrate that the stability condition conforms to known results.

Case 1:  $b = c = 0$

This case is equivalent to removing the third harmonic from the excitation, thus Hsu's[1963] result should be obtained for main parametric resonance with  $s = 2$ :

In this case:

$$p^2 = \frac{1}{2}[-\frac{5\gamma^2}{4} + a^2 \pm \frac{1}{4}(3\gamma^2 + 4a^2)]$$

Thus:

$$p^2 = -\gamma^2, -\frac{\gamma^2}{4} + a^2$$

Thus for  $p$  to be imaginary:

$$\gamma^2 > 4a^2$$

$$\longrightarrow \lambda^2 \epsilon^2 > \left(\frac{\epsilon d_{11}^{(2)}}{4\omega_1}\right)^2$$

Substituting for  $\lambda\epsilon$  confirms Hsu's[1963] result for  $s = 2$ .

$$\omega_1 - \frac{\epsilon |d_{11}^{(2)}|}{4\omega_1} < \omega < \omega_1 + \frac{\epsilon |d_{11}^{(2)}|}{4\omega_1}$$

Case 2:  $a = 0$

This case is equivalent to removing the second harmonic from the excitation, thus Hsu's[1963] result should be obtained for combination parametric resonance with  $s = 3$ :

In this case:

$$\Delta = \gamma^2(9\gamma^2 - 16cb)$$

and

$$p^2 = \frac{1}{2}\left[\left(-\frac{5\gamma^2}{4} + 2cb\right) \pm \frac{1}{4}\sqrt{\Delta}\right]$$

It can be shown that the discriminant becomes complex before the root  $p$  becomes complex, thus the stability is governed by the discriminant:

$$\gamma^2 > \frac{16}{9}cb$$

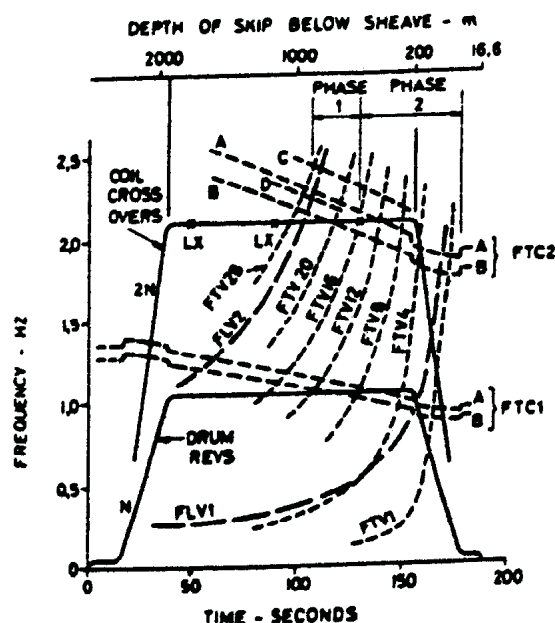
$$\longrightarrow \lambda^2 \epsilon^2 > \left(\frac{\epsilon d_{12}^{(3)} d_{21}^{(3)}}{36\omega_1 \omega_2}\right)$$

Once again substituting for  $\lambda\epsilon$  confirms Hsu's[1963] result for  $s = 3$ .

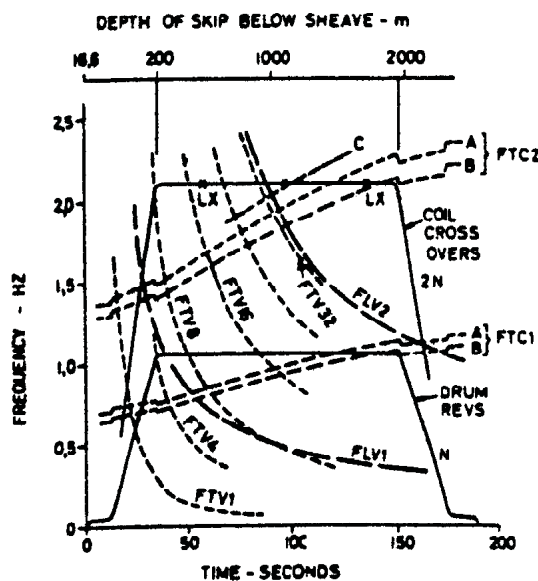
$$\frac{(\omega_1 + \omega_2)}{3} - \frac{\epsilon}{6}\sqrt{\frac{d_{12}^{(3)} d_{21}^{(3)}}{\omega_1 \omega_3}} < \omega < \frac{(\omega_1 + \omega_2)}{3} + \frac{\epsilon}{6}\sqrt{\frac{d_{12}^{(3)} d_{21}^{(3)}}{\omega_1 \omega_3}}$$

## B.4 Kloof Mine Observations

Observations performed at Kloof mine were well documented by Dimitriou and Whillier[1973] in the section *Dynamic behaviour of winding ropes at Kloof*, and will be discussed in light of the present analysis. Figure B.1 is reproduced from their paper, and represents the dynamic characteristics of the Kloof mine, in terms of the uncoupled linear transverse catenary natural frequencies ( $FTC_n$ ), the transverse natural frequencies of the vertical rope ( $FVT_n$ ), and the longitudinal natural frequencies of the vertical rope ( $FVL_n$ ). Figure B.2 represents the lateral and longitudinal natural frequencies calculated according to appendix C. Note that the longitudinal natural frequencies calculated by Dimitriou and Whillier[1973], and presented in figure B.1, treated the vertical rope as if it were fixed end at the sheave. The longitudinal natural frequencies presented in figure B.2 include the catenary length and the inertia of the sheave.



**Fig 3** Variation of frequencies, full ship rising.  $LS = 2.071$  m  
 $FTN$  = Frequency of Transverse vibrations in Catenary  
 $FVN$  = Frequency of Transverse vibrations in Vertical rope  
 $FLVN$  = Frequency of Longitudinal vibrations in Vertical rope  
 $LX$  = Layer cross-over



**Fig 4** Variation of frequencies, empty skip descending.  
 $LS = 1\ 080\ m$

Figure B.1: Dimitriou and Whillier (1973): Kloof Mine shaft dynamic characteristics

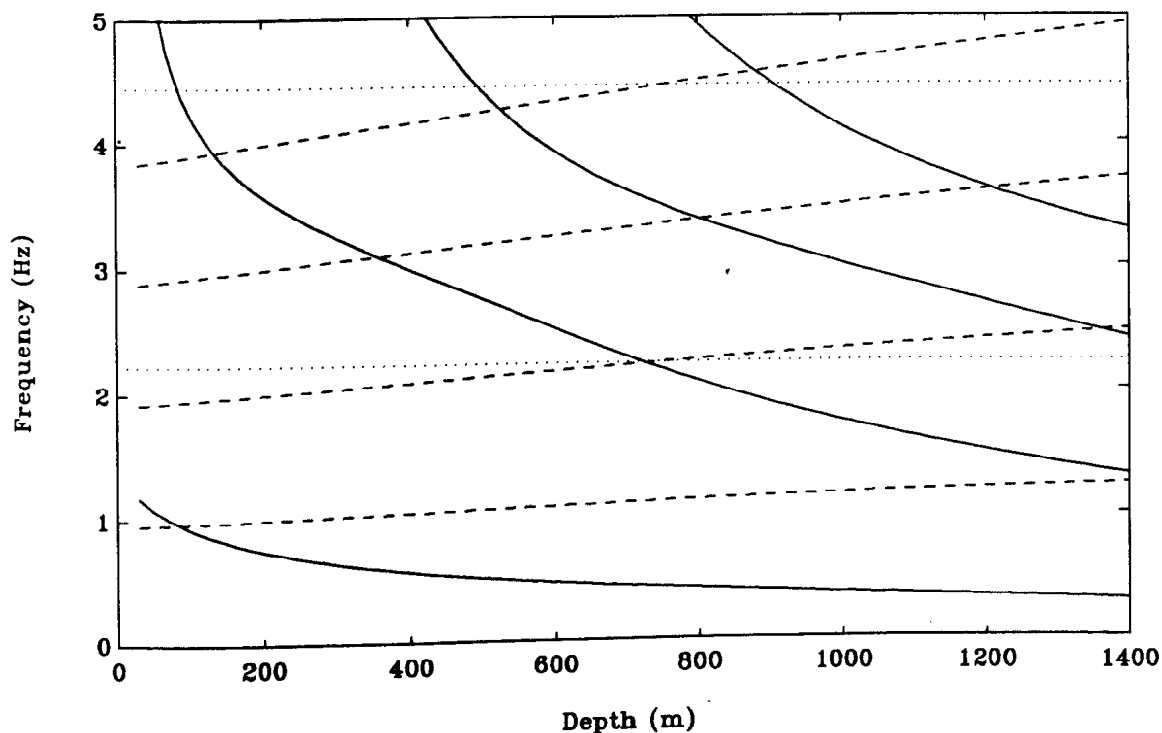


Figure B.2: Ascending Kloof Mine shaft dynamic characteristics

... - Lebus Drum Frequencies, — Longitudinal Frequencies, - - Lateral Catenary Frequencies

Dimitriou and Whillier[1973] made the following observations regarding Kloof.

- (1) Large catenary motion occurs, starting in phase 1, when the skip is approximately 900m below the head sheave during the rising cycle, however large motion is not observed when the skip is lowered.
- (ii) During phase 1, the amplitude increases and the catenaries settle into a clearly defined second mode.
- (iii) During phase 1, the vertical ropes start vibrating transversely at a similar wavelength to that of the catenary. This is not observed when the skip is lowered.
- (iv) On occasions, at the beginning of phase 2, the vibrations develop a large vertical component in excess of 2m. One of the authors believes a first mode pattern was observed in the vertical component of the motion. This motion was termed whip.
- (v) By reducing the speed of winding abruptly at the beginning of phase 1 from 15m/s to 14m/s, the amplitude of the vibrations was significantly reduced.

These observations contain much detail, confirming the potentially complex behaviour of the mine rope system. Considering figure B.1, the authors attributed the resonance and whip behaviour of the catenary to a combination of parametric response due to the longitudinal excitation at the drum, and autoparametric excitation due to parametric response of the lateral modes of the vertical rope, and consequently amplified parametric excitation of the catenary. Dimitriou and Whillier[1973] comment:

*"At Kloof, at the beginning of phase 1, the catenaries are approaching the resonant condition  $FTC2 = 2N$ , due to the z-excitation (out-of-plane lateral excitation) and their amplitude is growing ...". "Therefore, the tension in the catenaries fluctuate with increasing magnitude,  $\epsilon$ , at a frequency  $p$  close to  $4N$ , thus approaching the condition for subharmonic excitation of the vertical ropes."*

*"Another factor, which promotes subharmonic resonance in both the catenaries and vertical ropes, is the harmonic component at  $p = 4N$  of the w-excitation (longitudinal excitation). The magnitude,  $\epsilon$ , of this component is only of the order  $10^{-4}$ . However at the beginning of phase 1,  $\epsilon$  is amplified by resonance of the longitudinal mode of the ropes at  $FLV3 = 4N$ ..". ... "The growth of the transverse vibrations of the vertical ropes provides increased parametric excitation of the catenaries..." "Eventually, the two ropes (catenary and vertical) mutually excite one another....".*

Although this argument was constructed in an ad hoc manner, it succinctly describes the potential interactions which may arise. Dimitriou and Whillier[1973] note that for an appreciable rope length, at least one or more of the higher lateral frequencies of the vertical rope will tune closely to one or more of the natural frequencies of the catenary. In this sense, the system may maintain a state of autoparametric resonance for an appreciable period of the wind.

At the time of executing this phase of the study, it was realised that the aspect of autoparametric excitation via the forced lateral motion of the catenary would be important. However, it was decided that an appreciation of the linear stability of the system in the presence of longitudinal excitation only would represent a beneficial first step.

In the context of the Kloof system, the first and second harmonic of the longitudinal excitation frequency induces main parametric resonance of the first and second catenary modes simultaneously. As noted by Dimitriou and Whillier[1973], the effective amplitude of the parametric excitation depends on the proximity of the harmonic excitations to a condition of longitudinal resonance. It was determined in section B.3.1 that this condition of tuning is not

anomalous, and the size of the region of parametric resonance can be determined directly from Hsu's[1963] results. The resulting regions of instability<sup>10</sup> for main resonance of the first and second catenary modes are illustrated in figure B.3(a),(b) respectively. This figure represents a composite of three levels of longitudinal damping ranging from undamped to a maximum of 1.5% of critical in the first longitudinal mode. The lower part of the figure represents the absolute value of the difference between the longitudinal response at the sheave and winder drum, and is consequently related to the tension fluctuations occurring in the catenary. The parameter epsilon in the upper figure represents a scaling factor applied to the longitudinal excitation at the winder drum. Thus the actual excitation level for the Kloof winder is achieved when epsilon is unity.

It is clear from this figure that the regions of parametric instability are strongly influenced by the degree of longitudinal damping. The peak response in figure B.3(c),(d) relates to resonance of the second and fourth longitudinal modes respectively<sup>11</sup>. Since relative viscous proportional damping has been applied in the model, the modal damping factor applicable to the  $i^{th}$  longitudinal mode is related to that in the first mode by the ratio  $\omega_i/\omega_1$ . Consequently if the first mode is set to have a modal damping factor of 1.5%, the damping factors for the second third and fourth modes would be of the order of 4%, 9% 14% respectively. The assumption of relative proportional damping by industry has led to the conclusion that the longitudinal behaviour of the system can be modelled by considering the fundamental mode only, as response in the higher modes is strongly attenuated. This is clearly evident in figure B.3(c),(d), where the resonant peak is quickly eroded with the inclusion of longitudinal damping. As a result the region of parametric resonance becomes dependent on the base level of excitation applied at the drum, rather than on the longitudinal tuning of the system. Figure B.4 presents a stability plot of the system, where the harmonic balance method proposed by Takahashi[1981a] has been applied to determine the regions of system stability. Identical parameters applied in figure B.3 have been used, and the stability plot represents a composite of three levels of damping. The predicted regions of instability via the perturbation result, and the harmonic balance method are identical. The harmonic balance method is however more general, since it does not require special considera-

<sup>10</sup>The longitudinal excitation amplitudes were calculated for the first and second harmonic of the Lebus excitation frequency in accordance with Appendix A. These are :  $U_1 = 0.2mm$ ,  $U_2 = 0.1mm$

<sup>11</sup>Dimitriou and Whillier neglected the sheave inertia in their calculation of the longitudinal natural frequencies; consequently in their calculation the natural frequency of the third longitudinal mode was higher and resonant at approximately 900m. Once the sheave inertia is accounted for, resonance of the third longitudinal mode is delayed and occurs at approximately 500m below the sheave, whilst the fourth longitudinal mode is resonant at approximately 900m.

tion when dealing with anomalous cases, and depending on the order of the Fourier expansion assumed, can identify the higher order or secondary regions of instability for both main and combination resonance. It is also not limited by the notion of a small parameter.

Although regions of parametric instability do occur, they do not occur over extended lengths of the shaft when  $\epsilon$  is unity. The effect of these regions will be reduced due to the winding speed of the system, and consequently these regions are not viewed as sufficient to warrant the definition of a design criterion. Nevertheless, this exercise was useful in developing an understanding of the system stability and partly confirming the intuitive interpretation of Dimitriou and Whillier[1973] in a formal manner. It is likely that significant longitudinal response would occur due to the coupled lateral motion of the rope. This is a form of autoparametric excitation, where lateral response due to the lateral excitation at the winder drum, promotes longitudinal parametric excitation. This concept provided the basic incentive for developing the stability analysis of the coupled system, where both longitudinal and lateral excitation of the catenary is accounted for, as presented in chapter 4.

Finally, it is noted that this analysis provides evidence of main as well as combination parametric resonances, due to longitudinal excitation. Combination parametric resonance regions do not arise in a string with pinned end conditions, since the lateral eigenfunctions are orthogonal to the longitudinal eigenfunctions. With regard to the influence of rope curvature, Takahashi[1991] considered the regions of parametric resonance of a pinned cable with curvature, under axial excitation at one boundary. Takahashi[1991] reported the existence of regions of additive combination resonance, however, Perkins[1992b] corrected these findings, and demonstrated that only main parametric resonance regions existed, where in addition to being parametrically excited, the system was directly excited due to in-plane curvature coupling. It is surmised that in the present system, since the lateral eigenfunctions are not orthogonal to the longitudinal eigenfunctions, additive combination parametric resonance regions will arise in the presence of external excitation due to the catenary curvature. In practice the catenary curvature is small, and further analysis accounting for catenary curvature was not undertaken.

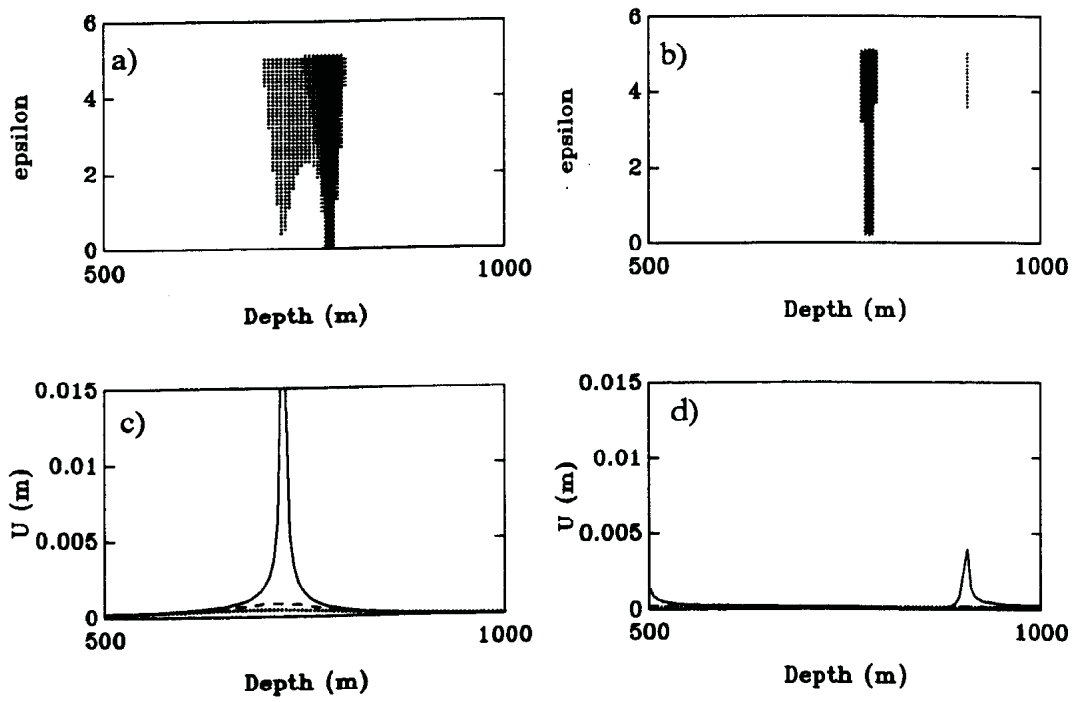


Figure B.3: Main parametric resonance regions - Case 1

(a)  $\Omega = 2\omega_1$

(b)  $2\Omega = 2\omega_2$

$V = 15m/s, \beta = 0.2rad., d = 48mm, R_d = 2.14m.$

—  $\zeta_1 = 0$ , — —  $\zeta_1 = 0.75\%$ , ...  $\zeta_1 = 1.5\%$

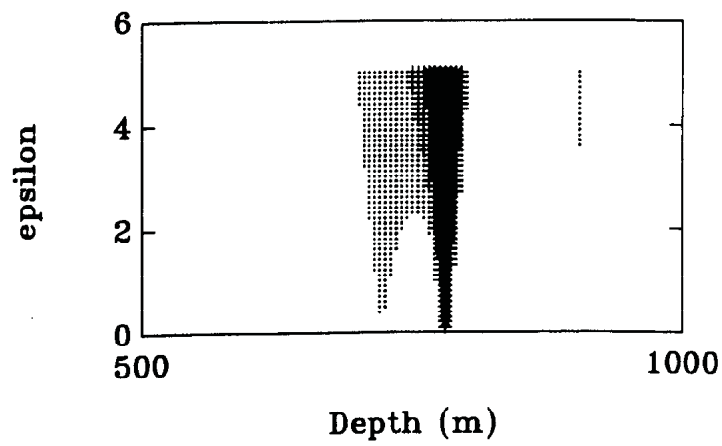


Figure B.4: Parametric resonance regions - Harmonic balance method

$$V = 15m/s, \beta = 0.2rad., d = 48mm, R_d = 2.14m.$$

$$\cdot - \zeta_1 = 0, + - \zeta_1 = 0.75\%, \times - \zeta_1 = 1.5\%$$

# Appendix C

## Linear Longitudinal System Response

The purpose of this appendix is to present the eigenfunctions associated with the longitudinal linear system, in the absence of lateral catenary motion, for use in a normal mode analysis of the non-linear equations of motion. The solution of longitudinal steady state motion of the system, in the presence of relative and general proportional viscous damping, due to an harmonic axial excitation at the winder drum is subsequently presented.

Figure C.1 illustrates the system under consideration, which consists of a rope of material density  $\rho$ , cross sectional area  $A$ , and modulus of elasticity  $E$ . The system is split into two sub-systems denoted by displacement co-ordinates  $u_1(s_1, t)$  and  $u_2(s_2, t)$ . The sub-systems are coupled by a sheave of mass moment of inertia  $I$ . The conveyance is represented by a mass  $M$ , which is suspended at the free end of the second system. The co-ordinates  $s_1, s_2$  represent the position along the rope from the catenary end and the sheave end respectively.

The equations of motion and boundary conditions of the two systems are presented as:

### System 1

$$\frac{\partial^2 u_1}{\partial t^2} = \mu \frac{\partial^3 u_1}{\partial s_1^2 \partial t} + c^2 \frac{\partial^2 u_1}{\partial s_1^2} \quad (\text{C.1})$$

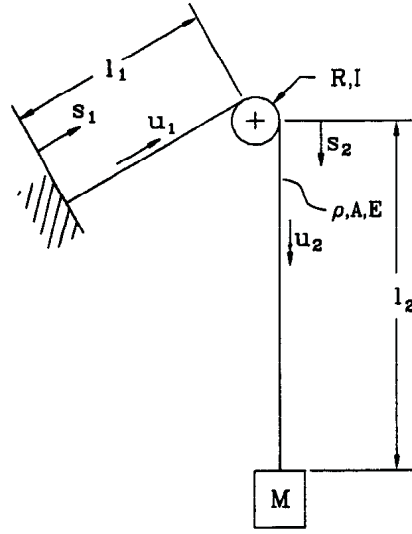


Figure C.1: The longitudinal system

where  $c$  represents the longitudinal wave speed  $c = \sqrt{E/\rho}$ , and  $\mu$  represents the damping coefficient related to the intrinsic dissipation property of the wire rope. This particular form of damping mechanism represents a relative proportional damping model, where the damping action is orthogonal to the normal modes of the undamped system, and proportional to the stiffness properties.

The boundary conditions for the rope are given as:

$$u_1(0, t) = \sum \text{Re}(U_n e^{j\omega t}) \quad (\text{C.2})$$

$$u_1(l_1, t) = u_2(0, t) \quad (\text{C.3})$$

Where  $U_n$  is complex and contains the amplitude and phase of the  $n^{\text{th}}$  harmonic of the excitation applied at the drum due to the Lebus cross over excitation.

### System 2

$$\frac{\partial^2 u_2}{\partial t^2} = \mu \frac{\partial^3 u_2}{\partial s_2^2 \partial t} + c^2 \frac{\partial^2 u_2}{\partial s_2^2} \quad (\text{C.4})$$

The boundary conditions for the rope are given as:

$$u_2(0, t) = u_1(l_1, t) \quad (\text{C.5})$$

$$(AE \frac{\partial u_2}{\partial s_2} + \rho \mu A \frac{\partial^2 u_2}{\partial s_2 \partial t})|_{(l_2, t)} = -M \frac{\partial^2 u_2}{\partial t^2}|_{(l_2, t)} \quad (\text{C.6})$$

### **System Coupling**

The coupling of the catenary system to the vertical section is achieved by ensuring continuity of the longitudinal displacement, as well as a momentum balance across the sheave wheel. These conditions are satisfied by:

$$u_2(0, t) = u_1(l_1, t)$$

$$\begin{aligned} \frac{I}{R^2} \frac{\partial^2 u_1}{\partial t^2}|_{(l_1, t)} &= AE \left\{ \frac{\partial u_2}{\partial s_2}|_{(0, t)} - \frac{\partial u_1}{\partial s_1}|_{(l_1, t)} \right\} \\ &+ \rho A \mu \left\{ \frac{\partial^2 u_2}{\partial s_2 \partial t}|_{(0, t)} - \frac{\partial^2 u_1}{\partial s_1 \partial t}|_{(l_1, t)} \right\} \end{aligned} \quad (\text{C.7})$$

## C.1 Undamped Natural Frequencies

Setting the damping factor  $\mu = 0$ , and the harmonics of the boundary excitation  $U_n = 0$ , the undamped natural frequencies of the system can be determined by assuming.

$$\begin{aligned} u_1(s_1, t) &= [A_1 \cos \gamma s_1 + B_1 \sin \gamma s_1] e^{j\omega t} \\ u_2(s_2, t) &= [A_2 \cos \gamma s_2 + B_2 \sin \gamma s_2] e^{j\omega t} \end{aligned} \quad (C.8)$$

where  $\gamma = \frac{\omega}{c}$  :

From equation (C.2):

$$u_1(0, t) = 0$$

$$\rightarrow A_1 = 0$$

Substituting the expressions (C.8) into equation (C.6) gives:

$$\begin{aligned} AE \frac{du_2}{ds_2} \Big|_{(l_2, t)} &= M\omega^2 u_2 \Big|_{(l_2, t)} \\ \frac{du_2}{ds_2} \Big|_{(l_2, t)} &= \Lambda^2 u_2 \Big|_{(l_2, t)} \end{aligned} \quad (C.9)$$

where  $\Lambda^2 = \frac{M\omega^2}{AE}$ .

Use of relations (C.8) and equation (C.9) gives:

$$A_2[\Lambda^2 \cos \gamma l_2 + \gamma \sin \gamma l_2] + B_2[\Lambda^2 \sin \gamma l_2 - \gamma \cos \gamma l_2] = 0 \quad (C.10)$$

From equation (C.3):

$$u_1|_{(l_1,t)} = u_2|_{(0,t)}$$

which requires that:

$$B_1 \sin \gamma l_1 = A_2 \quad (\text{C.11})$$

Equation (C.7) requires:

$$-\omega^2 \frac{I}{R^2} u_1|_{(l_1,t)} + AE \frac{du_1}{ds_1}|_{(l_1,t)} = AE \frac{du_2}{ds_2}|_{(0,t)}$$

Substituting relations (C.8) into this equation results in:

$$B_1 \left[ \frac{\Gamma^2}{\gamma} \sin \gamma l_1 - \cos \gamma l_1 \right] + B_2 = 0 \quad (\text{C.12})$$

where  $\Gamma^2 = \frac{\omega^2 I}{AE R^2}$

Equations (C.10),(C.11),(C.12) can be written in matrix form as:

$$\begin{bmatrix} \sin \gamma l_1 & -1 & 0 \\ (\frac{\Gamma^2}{\gamma} \sin \gamma l_1 - \cos \gamma l_1) & 0 & 1 \\ 0 & (\Lambda^2 \cos \gamma l_2 + \gamma \sin \gamma l_2) & (\Lambda^2 \sin \gamma l_2 - \gamma \cos \gamma l_2) \end{bmatrix} \begin{Bmatrix} B_1 \\ A_2 \\ B_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

In order for the coefficients  $B_1, A_2, B_2$  to be non-trivial, the determinant must be equal to zero. This defines the frequency equation as:

$$\Delta(\omega) = \sin \gamma l_1 [\Lambda^2 \cos \gamma l_2 + \gamma \sin \gamma l_2] + [\cos \gamma l_1 - \frac{\Gamma^2}{\gamma} \sin \gamma l_1] [\Lambda^2 \sin \gamma l_2 - \gamma \cos \gamma l_2] = 0$$

The roots of the frequency equation determine  $\gamma_i$ , and hence the  $i^{th}$  natural frequency of the system is  $\omega_i = c \gamma_i$ . The  $i^{th}$  eigenfunction coefficients  $B_{1i}, A_{2i}, B_{2i}$

are determined by scaling the coefficients such that  $B_{1_i} = 1$ , and solving for  $A_{2_i}, B_{2_i}$  from equations (C.11), (C.12).

$$B_{1_i} = 1$$

$$A_{2_i} = \sin \gamma_i l_1$$

$$B_{2_i} = \left( -\frac{\Gamma^2}{\gamma_i} \sin \gamma_i l_1 + \cos \gamma_i l_1 \right)$$

## C.2 Forced Longitudinal Damped Response

The damped steady state longitudinal response due to the boundary excitation at the winder drum is determined in the absence of lateral catenary motion.

Since relative proportional viscous damping is included in the equations of motion, the eigenfunctions presented in relations (C.8) are complex. The complex eigenfunctions can be determined by considering the proportionally damped wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \mu \frac{\partial^3 u}{\partial s^2 \partial t} + c^2 \frac{\partial^2 u}{\partial s^2}$$

This equation is variable separable ie:

$$u(s, t) = \phi(s)q(t)$$

and consequently:

$$\phi'' + \gamma^2 \phi = 0$$

$$\ddot{q} + \gamma^2 \mu \dot{q} + \gamma^2 c^2 q = 0$$

Assuming the response is harmonic:

$$q(t) = e^{i\omega t}$$

and solving for  $\gamma$ .

$$\gamma = \frac{\frac{\omega}{c}}{(1 + (\frac{\mu\omega}{c^2})^2)^{1/4}} e^{-i\alpha/2}$$

$$\alpha = \tan^{-1} \frac{\mu\omega}{c^2}$$

Thus  $\phi(s)$  may be presented in the form:

$$\phi(s) = A_1 \cos \gamma s + B_1 \sin \gamma s$$

where  $A_1, B_1, \gamma$  are complex.

Thus conveniently  $\phi(s)$  is of the same form as that determined for the undamped response in relations (C.8), however it is now a complex function.

Applying the boundary conditions stated in equations (C.2),(C.3),(C.6),(C.7), and assuming that the excitation is represented by a single harmonic at frequency  $\omega$ , then the forced response of the rope is given by:

$$u_1(s_1, t) = \phi_1(s_1) e^{i\omega t}$$

$$u_2(s_2, t) = \phi_2(s_2) e^{i\omega t}$$

where:

$$\phi_1(s_1) = A_1 \cos \gamma s_1 + B_1 \sin \gamma s_1$$

$$\phi_2(s_2) = A_2 \cos \gamma s_2 + B_2 \sin \gamma s_2$$

The following matrix equation for the coefficients  $A_1, B_1, A_2, B_2$  may be defined.:

$$[\mathbf{A}] \{x\} = \{b\}$$

where:

$$[\mathbf{A}] =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \cos \gamma l_1 & \sin \gamma l_1 & -1 & 0 \\ \left( \frac{\Gamma^2}{\lambda} \cos \gamma l_1 + \sin \gamma l_1 \right) & \left( \frac{\Gamma^2}{\lambda} \sin \gamma l_1 - \cos \gamma l_1 \right) & 0 & 1 \\ 0 & 0 & (\Lambda^2 \cos \gamma l_2 + \lambda \sin \gamma l_2) & (\Lambda^2 \sin \gamma l_2 - \lambda \cos \gamma l_2) \end{bmatrix}$$

$$\{x\} = \begin{Bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \end{Bmatrix}$$

$$\{b\} = \begin{Bmatrix} U \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

where:

$$\begin{aligned} \Lambda^2 &= \frac{M\omega^2}{AE} & \Gamma^2 &= \frac{I\omega^2}{AER^2} & \gamma &= \frac{\omega}{(1+(\frac{\mu\omega}{c^2})^2)^{1/4}} e^{-i\alpha/2} \\ \alpha &= \tan^{-1} \frac{\omega\mu}{c^2} & \zeta &= \frac{\omega\theta\gamma}{AE} & \lambda &= \gamma + i\zeta \\ \theta &= \rho A\mu & c^2 &= \frac{E}{\rho} \end{aligned}$$

Due to the influence of damping, the longitudinal motion of the system will be phase shifted from the excitation, as expected. This can be demonstrated by considering the catenary motion  $u_1(s_1, t)$ .

$$u_1(s_1, t) = \phi_1(s_1, t)e^{i\omega t}$$

If the applied excitation is of the form  $u_1(0, t) = \text{Re}(Ue^{i\omega t}) = |U|\cos(\omega t + \phi)$ , where  $\omega$  is the excitation frequency, then the response is given by:

$$u_1(s_1, t) = \text{Re}(\phi_1(s_1)e^{i\omega t})$$

$$u_1(s_1, t) = (\text{Re}(\phi_1)\cos\omega t - \text{Im}(\phi_1)\sin\omega t) = A(s_1)\cos(\omega t + \Phi)$$

$$A(s_1, t) = \sqrt{\text{Re}(\phi_1)^2 + \text{Im}(\phi_1)^2}$$

$$\Phi = \tan^{-1}\left(\frac{\text{Im}(\phi_1)}{\text{Re}(\phi_1)}\right)$$

The longitudinal response due to a periodic axial excitation at the winder drum can be determined for each harmonic of the excitation, and consequently the overall response can be determined via superposition of the forced harmonic responses.