APPENDICES

Appendix 2.1

Properties of limits of functions (Larson, Hostetler & Edwards, 1994)

Limits when *x* tends to *c*.

Let *b* and *c* be real numbers, let *n* be a positive integer, and let *f* and *g* be functions with the following limits: $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = K$.

1. Scalar multiple: $\lim_{x \to c} [b\, f(x)] = bL$
2. Sum or difference: $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$
3. Product: $\lim_{x \to c} [f(x)g(x)] = LK$
4. Quotient: $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{K}$, provided $K \neq 0$
5. Power: $\lim_{x \to c} [f(x)]^n = L^n$ (p. 71)

The limit of a function involving a radical

Let *n* be a positive integer. The following limit is valid for all *c* if *n* is odd, and is valid for $c > 0$ if *n* is even. $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$. (Larson et al., 1994: 73)

The limit of a composite function

If *f* and *g* are functions such that $\lim_{x \to c} g(x) = L$ and $\lim_{x \to c} f(x) = f(L)$, then $\lim_{x \to c} f(g(x)) = f(L)$. (1994: 73)

The Squeeze Theorem

If $h(x) \leq f(x) \leq g(x)$ for all *x* in an open interval containing *c*, except possibly at *c* itself, and if $\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$, then $\lim_{x \to c} f(x)$ exists and is equal to *L*. (1994: 80)

Similar properties are also stated for infinite limits.

Let *c* and *L* be real numbers, and let *f* and *g* be functions such that $\lim_{x \to c} f(x) = \infty$ and $\lim_{x \to c} g(x) = L$.

1. Sum or difference: $\lim_{x \to c} [f(x) \pm g(x)] = \infty$
2. Product: $\lim_{x \to c} [f(x)g(x)] = \infty, \quad L > 0$

   $\lim_{x \to c} [f(x)g(x)] = -\infty, \quad L < 0$
3. Quotient: $\lim_{x \to c} \frac{g(x)}{f(x)} = 0$

Similar properties hold for one-sided limits and for functions for which the limit of $f(x)$ as *x* approaches *c* is $-\infty$. (1994: 100)

For limits at infinity, Larson et al. indicate that they have similar properties. (1994: 198)

Appendix 2.2

The Mozambican Secondary School syllabus on limits of functions

UNIDADE IV
LIMITES E CONTINUIDADE DE FUNÇÕES
OBJECTIVOS ESPECÍFICOS

O aluno deve ser capaz de:

- explicar a noção de limite de uma função;
- definir limite de uma função *f(x)* quando $x \to a$ sendo a um valor finito e quando $x \to \infty$;
- determinar o limite de uma função nos dois casos indicados no objectivo anterior;
- explicar e aplicar as regras das operações com limites de funções;
- identificar as formas indeterminadas te limites de funções; levantar as indeterminações;
- calcular limites laterais;
- identificar, justificar e aplicar os limites notáveis: $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$; $\lim\limits_{x\to 0}\frac{senx}{x} = 1$; $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e$
- explicar e definir função contínua num ponto e função contínua num intervalo
- identificar uma função contínua dado o seu gráfico;
- determinar se uma função é contínua, dada a sua expressão analítica.

CONTEÚDO

1. Definição de limite duma função; função infinitamente pequena e infinitamente grande

2. Propriedades dos limites de funções

3. Limites notáveis; aplicações ao cálculo de outros limites; indeterminações

4. Continuidade: definição; limites laterais; propriedades; operações sobre funções contínuas

UNIT IV
LIMITS AND CONTINUITY OF FUNCTIONS
SPECIFIC OBJECTIVES

The student must be able to:

- explain the notion of limit of a function;
- define the limit of a function *f(x)* when $x \to a$, where a is a finite value and when $x \to \infty$;
- determine the limit of a function in the two cases indicated in the previous objective;
- explain and apply the operating rules for limits of functions;
- identify indeterminate forms of limits of functions; to handle these indeterminate forms;
- calculate one-sided limits;
- identify, justify and apply the special limits: $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$; $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$; $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e$
- explain and define continuous function in one point and in an interval;
- identify a continuous function given by its graph;
- determine whether a function given by its analytic expression is continuous or not.

CONTENTS

1. Definition of limit of a function; infinitely small and infinitely big functions

2. Limits of functions' properties

3. Special limits; application to other limits' calculations

4. Continuity: definition; one-sided limits; operations with continuous functions

ORIENTAÇÕES METODOLÓGICAS

O professor deve tentar primeiro dar uma ideia intuitiva deste conceito e a seguir tentar construir com base nesta ideia inicial, a definição propriamente. Não tem qualquer sentido exigir que os alunos memorizem a definição sem entenderem o que estão a dizer! Para chegar à definição, pode-se tomar uma função qualquer, constituir uma sucessão de valores de x tendente para um certo valor a e verificar que a esta sucessão corresponde uma sucessão de valores de f(x) tendente para um certo b. Neste caso, b será o limite de f(x) quando x tender para a.

Não está previsto que se faça a demonstração dos limites notáveis. No entanto, é razoável que se dê alguma explicação/justificação para que não se crie a impressão de que estes resultados aparecem por acaso.

Para o limite notável igual a *e*, pode-se levar os alunos a calcularem sucessivamente valores de $\left(1+\frac{1}{x}\right)^x$, dando a x valores cada vez maiores (por exemplo 1, 10, 100, 1000, 10000, 100000, etc.), eles verificarão que os resultados se aproximarão de 2,718…= e.

Para o limite notável $\frac{senx}{x}$ quando x tende para zero, pode-se verificar no gráfico que muito perto de zero, as funções $y = senx$ (numerador) e $y = x$ (denominador) são praticamente coincidentes, isto é, são iguais; logo, o quociente entre “quantidades iguais” é igual a 1.

METHODOLOGICAL GUIDELINES

The teacher must first try to give an intuitive idea of this concept and then attempt to construct upon this initial idea, the definition itself. It does not make sense to ask that the students memorise the definition without understanding what they are saying! In order to reach the definition, it is possible to use any function, to constitute a sequence of values for x that goes to a certain value a, and to verify that there is a corresponding sequence of values for $f(x)$ that goes to b. In this case, b will be the limit of $f(x)$ when x goes to a.

Special limits are not expected to be demonstrated. Nevertheless, it is reasonable that some explanation/justification be given, in order to avoid the idea that these results happen by chance.

For the special limit that is equal to e, the students can be asked to calculate successive values of $\left(1+\frac{1}{x}\right)^x$, with increasing values of x (for example 1, 10, 100, 1000, 10000, 100000, etc.), and they will see that the results will approach 2,718…= e.

For the special limit of $\frac{\sin x}{x}$ when x goes to zero, it can be seen on the graph that, very close to zero, the functions $y = \sin x$ (numerator) and $y = x$ (denominator) are almost coincident, that is, are equal; therefore, the quotient between “equal quantities” is equal to 1.

Appendix 2.3

Techniques to be taught in Mozambican secondary schools

The solution to the example given is presented for each kind of task. The solutions of T_{1B}, T_{1C}, T_{1D} and T_{2A} (second example) come from the correction guide distributed by the Ministry of Education. For the limits T_{2A} (first example), T_{2B} and T_{4A}, I did not have this guide, and I present the way first year university students usually solve these tasks. The solution of T_{1A} comes from Worksheet 2.

T_{1A}*: Limit of a continuous function*

$$\lim_{x\to -1}|x+5| = \left|\lim_{x\to -1}(x+5)\right| = |-1+5| = 4$$

T_{1B}*: Indeterminate form of a rational function when* $x \to a$

This kind of task is solved by factorisation and cancellation.

$$\lim_{x\to 1}\frac{x^3-3x^2+2x}{x^2-4x+3} = \lim_{x\to 1}\frac{(x-1)(x^2-2x)}{(x-1)(x-3)} = \lim_{x\to 1}\frac{x^2-2x}{x-3} = \frac{1}{2}$$

T_{1C}*: Indeterminate form involving square roots*

These tasks are solved using the rationalisation and cancellation technique.

$$\lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{x-a} = \lim_{x\to a}\frac{\left(\sqrt{x}-\sqrt{a}\right)\left(\sqrt{x}+\sqrt{a}\right)}{(x-a)\left(\sqrt{x}+\sqrt{a}\right)} = \lim_{x\to a}\frac{x-a}{(x-a)\left(\sqrt{x}+\sqrt{a}\right)} = \lim_{x\to a}\frac{1}{\sqrt{x}+\sqrt{a}}$$
$$= \frac{1}{2\sqrt{a}}$$

T_{1D}: *Indeterminate form involving a trigonometric function*

This kind of task is usually solved using the special trigonometric limit $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

$$\lim_{x\to 0}\frac{\sin 5x}{\sin 3x} = \left[\frac{0}{0}\right] = \lim_{x\to 0}\frac{\sin 5x}{5x}.\frac{3x}{\sin 3x}.\frac{5}{3} = 1.1.\frac{5}{3} = \frac{5}{3}$$

T_{2A}*: Limit of a polynomial, a rational or an irrational function when* $x \to \infty$

For a polynomial, the task is solved by taking out the highest factor of *x* as common factor.

$$\lim_{x\to +\infty}\left(-3x^3+4x-1\right) = \lim_{x\to +\infty} x^3\left(-3+\frac{4}{x^2}-\frac{1}{x^3}\right) = +\infty.(-3) = -\infty$$

For a rational function, these tasks are also solved by factorisation and cancellation.

$$\lim_{x\to\infty}\frac{x^2-1}{x^2-x}=\lim_{x\to\infty}\frac{x^2\left(1-\frac{1}{x^2}\right)}{x^2\left(1-\frac{1}{x}\right)}=\lim_{x\to\infty}\frac{1-\frac{1}{x^2}}{1-\frac{1}{x}}=1$$

The same techniques are used for functions involving radicals.

T$_{2B}$: *Indeterminate form such as* $\left[1^{\infty}\right]$

These tasks are solved using the special limit $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$.

$$\lim_{n\to\infty}\left(1-\frac{1}{2n}\right)^n=\left[1^{\infty}\right]=\lim_{n\to\infty}\left(1-\frac{1}{2n}\right)^{-2n.\left(\frac{1}{-2n}\right).n}=e^{\lim\limits_{n\to\infty}-\frac{n}{2n}}=e^{-\frac{1}{2}}=\frac{\sqrt{e}}{e}$$

T$_3$: *Determine the limit of a function from the graph*

I do not have the correction guide for this examination.

T$_{4A}$: *Discussion of the continuity of a function using its analytical expression*

a) The domain of the function is $D(f)=IR\setminus\{-3,3\}$. Therefore, f is not continuous for $x=-3$. $\lim_{x\to-3}f(x)=\infty$. There is a vertical asymptote.

b) $\lim_{x\to3}f(x)=\lim_{x\to3}\frac{x^2-2x-3}{x^2-9}=\lim_{x\to3}\frac{(x+1)(x-3)}{(x+3)(x-3)}=\lim_{x\to3}\frac{x+1}{x+3}=\frac{4}{6}=\frac{2}{3}$

$f(3)=m+4$

For continuity, $\lim_{x\to3}f(x)=f(3)$, that is $m+4=\frac{2}{3}\Rightarrow m=\frac{2}{3}-4=-\frac{10}{3}$

T$_{4B}$: *Discussion of the continuity of a function using its graph*

a)
- $f(0)=2$
- $\lim_{x\to0^-}f(x)=1$
- $\lim_{x\to0^+}f(x)=2$
- $f'(-2)$ does not exist

b) $x=0$

According to the syllabus and the tasks in the examinations, teachers are expected to teach these techniques to solve the kinds of algebraic tasks described above. The technique used to read limits from the graph is not explicit.

Appendix 2.4

Worksheet 1

ESCOLA SECUNDÁRIA JOSINA MACHEL

FICHA DE EXERCÍCIOS

LIMITE DE FUNÇÕES E CONTINUIDADE

I

CALCULE CADA UM DOS SEGUINTES LIMITES

( ) SOLUÇÃO

1. $\lim_{x\to\infty} \frac{x^3-x-1}{2x^3+3x^2}$ $\left(\frac{1}{2}\right)$
2. $\lim_{x\to\infty} \frac{3x^4+1}{-2x+1}$ (∞)
3. $\lim_{x\to\infty} \frac{3}{x^3+1}$ (0)
4. $\lim_{x\to\infty} \frac{2x^2+1}{10.000}$ (∞)
5. $\lim_{x\to\infty} \frac{(3x+1)^2(2x+3)^3}{2x^5+3}$ (36)
6. $\lim_{x\to\infty} \frac{(x+4)^7(2x+1)^{10}}{x^{17}+4x+1}$ (1024)
7. $\lim_{x\to\infty} \frac{(4x^2+1)^2(3x^3+2)^3}{(x^3+5x^2+3)^{10}}$ (0)
8. $\lim_{x\to\infty} \frac{2x-1}{\sqrt{4x^2-1}-\sqrt{x^2-1}} = \left(\frac{1}{2}\right)$
9. $\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x+1}$ (1)
10. $\lim_{x\to\infty} \frac{2x+3}{x+\sqrt{x}}$ (2)
11. $\lim_{x\to\infty} \frac{x^2}{10+x\sqrt{x}}$ (∞)
12. $\lim_{x\to\infty} \frac{\sqrt{x+\sqrt{x+3}}}{\sqrt{x+1}}$ (1)
13. $\lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}}$ (1)
14. $\lim_{x\to\infty} \left(\sqrt{x^2+x}-\sqrt{x^2+1}\right)$ $\left(\frac{1}{2}\right)$
15. $\lim_{x\to-\infty} \left(\sqrt{1-x}+x\right)$ (∞)
16. $\lim_{x\to+\infty} \left(\sqrt{x+3}-\sqrt{x}\right)$ (0)
17. $\lim_{x\to+\infty} \left(\sqrt[3]{x+1}-x\right)$ (∞)
18. $\lim_{x\to\infty} \left(x+\sqrt[3]{1-x^3}\right)$ (0)
19. $\lim_{x\to+\infty} \left[x\left(\sqrt{x^2+6}-x\right)\right]$ $(3$
20. $\lim_{x\to+\infty} \left[x\left(\sqrt{x^2-5x+6}-x\right)\right]$ $\left(-\frac{5}{2}\right.$
21. $\lim_{x\to+\infty} \left[\sqrt{(x+a)(x+b)}-x\right]$ $\left(\frac{a+}{2}\right.$
22. $\lim_{x\to+\infty} \left[\sqrt{x(x+a)}-x\right] = \frac{a}{2}$
23. $\lim_{x\to+\infty} \left(\sqrt{x+a}-\sqrt{x}\right) = (0)$

II

1. $\lim_{x\to 2} (3x^2+5)$ (17)
2. $\lim_{x\to 2} \frac{x^2-4}{x^2+x-6}$ $\left(\frac{4}{5}\right)$
3. $\lim_{x\to 1} \frac{x^2+1}{3x+2}$ $\left(\frac{2}{5}\right)$
4. $\lim_{x\to 3} \frac{x^2-3x}{x^2-5x+6}$ (3)
5. $\lim_{x\to 3} \frac{x^2-9}{x-3}$ (6)
6. $\lim_{x\to 3} \frac{x^2-25}{x^2+2x-15}$ $(-\infty)$
7. $\lim_{x\to -1} \frac{x^3+x^2-4x-4}{x^2+4x+3}$ $\left(-\frac{3}{2}\right)$
8. $\lim_{x\to 1} \frac{x^3-6x^2+11x-6}{x^3+x^2-5x+3}$ (∞)
9. $\lim_{x\to 2} \frac{x^3-5x^2+8x-4}{x^3-3x^2+4}$ $\left(\frac{1}{3}\right)$
10. $\lim_{x\to -1} \frac{x^3+1}{x+1}$ (3)
11. $\lim_{x\to 1} \frac{x^4-1}{-x^3+1}$ $\left(-\frac{4}{3}\right)$
12. $\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1}$ (2)
13. $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$ $\left(\frac{1}{6}\right)$
14. $\lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a}$ $\left(\frac{1}{2\sqrt{a}}\right)$
15. $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$ $\left(\frac{1}{2}\right)$
16. $\lim_{x\to -3} \frac{1-\sqrt{x+4}}{x+3}$ $\left(-\frac{1}{2}\right)$
17. $\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3}$ $\left(\frac{1}{4}\right)$
18. $\lim_{x\to 1} \frac{\sqrt{x+1}-\sqrt{2}}{\sqrt{2x+2}-2}$ $\left(\frac{1}{\sqrt{2}}\right)$
19. $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{\sqrt[3]{x+1}-1}$ $\left(\frac{3}{2}\right)$
20. $\lim_{x\to 1} \frac{\sqrt{x}-1}{\sqrt[3]{x}-1}$ $\left(\frac{3}{2}\right)$
21. $\lim_{x\to 64} \frac{\sqrt{x}-8}{\sqrt[3]{x}-4}$ (3)
24. $\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{\sqrt[4]{x}-1}$ $\left(\frac{4}{3}\right)$
25. $\lim_{x\to \infty} \frac{\sqrt{3x+3}-\sqrt{3x}}{\sqrt{x+2}-\sqrt{3x-2}}$ (∞)
26. $\lim_{x\to 3} \frac{x\sqrt{x}-3\sqrt{3}}{\sqrt{x}-\sqrt{3}}$ (9)
27. $\lim_{x\to 1} \log_2 \frac{\sqrt{x}-1}{x-1}$ (-1)
28. $\lim_{x\to 3} \frac{x-3}{\sqrt{3x}-x}$ (-2)
29. $\lim_{x\to 1} \frac{\sqrt{x}-1}{x^2-1}$ $\left(\frac{1}{4}\right)$
30. $\lim_{x\to 2} \frac{\sqrt{x}-\sqrt{2}}{x-2}$ $\left(\frac{1}{2\sqrt{2}}\right)$
31. $\lim_{x\to 4} \frac{3-\sqrt{5+x}}{1-\sqrt{5-x}}$ $\left(-\frac{1}{3}\right)$
32. $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$ (1)
33. $\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$ $(x>0)$ $\left(\frac{1}{\ }\right)$

II (cont.)

(34) $\lim\limits_{h\to 0} \dfrac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h}$ $\left(\dfrac{1}{3\sqrt[3]{x^2}}\right)$

(35) $\lim\limits_{x\to 3} \dfrac{\sqrt{x^2-2x+6}-\sqrt{x^2+2x-6}}{x^2-4x+3}$ $\left(-\dfrac{1}{3}\right)$

(36) $\lim\limits_{x\to 1} \dfrac{\sqrt[3]{x^2}-2\sqrt[3]{x}+1}{(x-1)^2}$ $\left(\dfrac{1}{9}\right)$

(37) $\lim\limits_{x\to 7} \dfrac{2-\sqrt{x-3}}{x^2-49}$ $\left(-\dfrac{1}{56}\right)$

(38) $\lim\limits_{x\to 8} \dfrac{x-8}{\sqrt[3]{x}-2}$ (12)

(39) $\lim\limits_{x\to 2} \dfrac{x^2-4}{x^2-3x+2}$ (4)

(40) $\lim\limits_{h\to 0} \dfrac{(x+h)^3-x^3}{h}$ $(3x^2)$

(8) $\lim\limits_{x\to 0} \dfrac{\operatorname{sen}^2 x}{x\sqrt{x}}$ (0)

(9) $\lim\limits_{x\to 0} \dfrac{\operatorname{sen}\frac{x}{2}}{8x}$ $\left(\dfrac{1}{16}\right)$

(10) $\lim\limits_{x\to 0} \dfrac{\operatorname{sen}x-\operatorname{tg}x}{x}$ (0)

(11) $\lim\limits_{x\to 0} \dfrac{\operatorname{sen}4x}{\operatorname{tg}2x}$ (2)

(12) $\lim\limits_{x\to 0} \dfrac{\operatorname{sen}2x-\operatorname{sen}x}{x\cos x}$ (1)

(13) $\lim\limits_{x\to 0} \dfrac{\operatorname{sen}3x-\operatorname{sen}x}{\operatorname{sen}5x}$ $\left(\dfrac{2}{5}\right)$

(14) $\lim\limits_{x\to 0} \dfrac{x+\operatorname{sen}3x}{4x-\operatorname{tg}2x}$ (2)

(15) $\lim\limits_{x\to 0} \dfrac{\sqrt{2x+1}-\sqrt{x+1}}{\operatorname{sen}x}$ $\left(\dfrac{1}{2}\right)$

III

(1) $\lim\limits_{x\to 2} \dfrac{\operatorname{sen}x}{x}$ $\left(\dfrac{1}{2}\operatorname{sen}2\right)$

(2) $\lim\limits_{x\to 0} \dfrac{\operatorname{sen}3x}{x}$ (3)

(3) $\lim\limits_{x\to 0} \dfrac{\operatorname{sen}5x}{\operatorname{sen}2x}$ $\left(\dfrac{5}{2}\right)$

(4) $\lim\limits_{x\to 0} x\operatorname{sen}\dfrac{\pi}{x}$ (π)

(5) $\lim\limits_{x\to 0} \dfrac{1-\sqrt{\cos x}}{x^2}$ $\left(\dfrac{1}{4}\right)$

(6) $\lim\limits_{x\to \frac{\pi}{4}} \dfrac{\operatorname{sen}x-\cos x}{1-\operatorname{tg}x}$ $\left(-\dfrac{1}{\sqrt{2}}\right)$

(7) $\lim\limits_{x\to -1} \dfrac{\operatorname{sen}(x+1)}{x^2-3x-4}$ $\left(-\dfrac{1}{5}\right)$

IV

(1) $\lim\limits_{x\to \infty} \left(\dfrac{x}{x-1}\right)^x$ (e^{-1})

(2) $\lim\limits_{x\to \infty} \left(\dfrac{x^2-4}{x^2+4}\right)^{\frac{x-1}{x+1}}$ (1)

(3) $\lim\limits_{x\to 0} (1+\operatorname{sen}x)^{\frac{1}{x}}$ (e)

(4) $\lim\limits_{x\to 0} \sqrt[x]{5^{\operatorname{sen}x}}$ (5)

(5) $\lim\limits_{x\to 0} \left(\dfrac{\operatorname{sen}2x}{x}\right)^{1+x}$

(6) $\lim\limits_{x\to \infty} x[\ln(x+1)-\ln x]$ (1)

(7) $\lim\limits_{x\to 0} \dfrac{\log(1+10x)}{x}$ $(10\log e)$

V

① CALCULE $\lim\limits_{x\to+\infty} f(x)$ E $\lim\limits_{x\to-\infty} f(x)$ NOS SEGUINTES CASOS:

a) $f(x) = \dfrac{4}{3^x}$ (sol: $0; \infty$)

b) $f(x) = 5^{x-1} - 5$ (sol: $\infty; -5$)

c) $f(x) = 5^{1-x} - 5$ sol: $(-5; \infty)$

d) $f(x) = \dfrac{x-1}{x}$ $(1, 1)$

e) $f(x) = 2 + \dfrac{1}{\sqrt{x}}$ (sol: 2; N.E.)

f) $f(x) = \left(\dfrac{1}{2}\right)^x + 2$ (sol: $2; \infty$)

g) $f(x) = \dfrac{3x+1}{12x+13}$ $\left(\dfrac{1}{4}; \dfrac{1}{4}\right)$

② CALCULE OS LIMITES LATERAIS:

a) $\lim\limits_{x\to-4^+} \sqrt[3]{\dfrac{1}{x+4}} = +\infty$; $\lim\limits_{x\to-4^-} \sqrt[3]{\dfrac{1}{x+4}} = -\infty$

b) $\lim\limits_{x\to0^+} \dfrac{|\operatorname{tg} x|}{x} = 1$; $\lim\limits_{x\to0^-} \dfrac{|\operatorname{tg} x|}{x} = -1$

c) $\lim\limits_{x\to0^+} \dfrac{1}{1+2^{\frac{1}{x}}} = 0$; $\lim\limits_{x\to0^-} \dfrac{1}{1+2^{\frac{1}{x}}} = 1$

d) $\lim\limits_{x\to0^+} a^x = 1$; $\lim\limits_{x\to0^-} a^x = 1$

e) $\lim\limits_{x\to1^+} \dfrac{1}{x-1} = +\infty$; $\lim\limits_{x\to1^-} \dfrac{1}{x-1} = -\infty$

③ ESTUDE A CONTINUIDADE DAS SEGUINTES FUNÇÕES E CLASSIFIQUE O(S) PONTO(S) DE DESCONTINUIDADE

a) $f(x) = \begin{cases} 5x+1, & \text{se } x \geq 3 \\ \dfrac{x+2}{x^2-6x+5}, & \text{se } x < 3 \end{cases}$

b) $f(x) = \begin{cases} x^2+2; & \text{se } x \geq 0 \\ x+2 & \text{se } x < 0 \end{cases}$

c) $f(x) = \begin{cases} x+3; & \text{se } x < 2 \\ 7 & \text{se } x = 2 \\ \dfrac{x^2+x-6}{x-2}; & \text{se } x > 2 \end{cases}$

d) $f(x) = \begin{cases} \dfrac{x^2-1}{x-1} & \text{se } x < 1 \\ 2 & \text{se } x = 1 \\ 2x & \text{se } x > 1 \end{cases}$

e) $f(x) = \begin{cases} \dfrac{x^2-1}{x+1} & \text{se } x < -1 \\ -2 & \text{se } x = -1 \\ -2x & \text{se } x > -1 \end{cases}$

f) $f(x) = \begin{cases} \dfrac{(x^2-1)(x^2-4)}{(x+2)(x^2-3x+2)}; & x \neq \{-2, 1, 2 \\ x+1 & x \in \{-2; 1, \end{cases}$

g) $f(x) = \dfrac{x^2-4}{x-2}$; $x \neq 2$

Appendix 2.5

Worksheet 2

13

repetido p. 13 = p12

C. *Exercícios propostos*

1. Calcule os limites seguintes;

a) $\lim\limits_{x\to1}\frac{(x^2-3)(4x-2)}{5-x^2}$ b) $\lim\limits_{x\to2}\frac{3x}{2-x}$ c) $\lim\limits_{x\to0}\frac{3x}{3-2x^2}$

2. Considere a função, assim definida:

$$f(x)=\begin{cases}2x+1, & se\ \ x<1\\ 3, & se\ \ x=1\\ x^2-1, & se\ \ x>1\end{cases}$$

a) Calcule $\lim\limits_{x\to1-0}f(x)$ e $\lim\limits_{x\to1+0}f(x)$

b) Existe $\lim\limits_{x\to1}f(x)$? Justifique.

c) Calcule f(1)

3. Considere as funções assim definidas:

$f(x)=\begin{cases}2x+3, & se\ x<-2\\ x+4, & se\ x\geq-2\end{cases}$ e $g(x)=\begin{cases}x^2-1, & se\ x<-2\\ x+2, & se\ x\geq-2\end{cases}$

a) Mostre as funções f e g não tem limite quando $x\to-2$

b) Determine (f + g) (x)

c) Calcule $\lim\limits_{x\to-2}(f+g)(x)$

4. Calcule os seguintes limites:

a) $\lim\limits_{x\to-\infty}(3x^3+5x^2-1)$ b) $\lim\limits_{x\to+\infty}(5x^3-3x^2+5)$

c) $\lim\limits_{x\to3}\frac{\frac{2}{x-3}}{\frac{1}{x^2-3x}}$ d) $\lim\limits_{x\to1}\frac{\frac{x+3}{x^2 1-1}}{(x-2)\ (x^2-4x+3)}$ e) $\lim\limits_{x\to3}\left(\frac{1-}{x-3}-\frac{1}{x^2-7x+12}\right)$

f) $\lim\limits_{x\to0}\left(2x\sqrt{\frac{dx+1}{x}}\right)$ g) $\lim\limits_{x\to\pi+0}(x+2\pi)$ h) $\lim\limits_{x\to-\infty}\log_3\left(\frac{3x^3+5x^2+1}{x^3-x^2+1}\right)$

i) $\lim\limits_{x\to+\infty}\log_{\frac{1}{3}}\frac{x^3+2x+1}{9x^3+1}$ *k)* $\lim\limits_{x\to+\infty}\frac{1}{3^{x-2}}$ *l)* $\lim\limits_{x\to1}\frac{1}{x-1}\left(\frac{1}{x+3}-\frac{2}{3x+5}\right)$

m) $\lim\limits_{x\to-\infty}\frac{(2x+1)\ (x+4)}{(3x-1)\ (x+5)}$

5. Calcule

$a)\ \lim_{x\to 5}\frac{x^2-4x-5}{x^2-8x+15}$ $b)\ \lim_{x\to 2}\frac{x^3-8}{x^2-5x+6}$ $c)\ \lim_{x\to 1}\frac{x^3-x^2+5x-5}{3-3x}$ $d)\ \lim_{x\to 1}\frac{x^3+x^2-5x+3}{x^3-x^2-x+1}$

$e)\ \lim_{x\to 1}\frac{x^6-1}{1-x^4}$ $f)\ \lim_{x\to -3}\frac{x^2+7x+12}{x^2-9}$

6. Calcula os limites das seguintes funções quando $x\to 0$

a. $f(x)=\dfrac{x^2-2x}{3x^2+x}$

b. $f(x)=\dfrac{2x^4-3x^3+4x^2}{5x^2-6x^3+7x^5}$

c. $f(x)=\dfrac{x^3+3x}{x^3-x^2}$

d. $f(x)=\dfrac{(x-a)^3}{x^3-a^3}$

e. $f(x)=\dfrac{(x+a)^3-a^3}{x}$

7. Calcule os seguintes limites

$a)\ \lim_{x\to 4}\frac{x-4}{\sqrt{x}-2};$

$b)\ \lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2-x}}{3x};$

$c)\ \lim_{x\to -3}\frac{2x+6}{\sqrt{x-4}-\sqrt{2x+7}};$

$d)\ \lim_{x\to 2}\frac{\sqrt{x+2}-2}{\sqrt{3-x}-1}$

$e)\ \lim_{x\to 1}\frac{\sqrt{x+3}-2}{\sqrt{2x-1}+1}$

$f)\ \lim_{x\to -1}\frac{\sqrt{x^2-3x}-2}{\sqrt{2x+3}-1};$

$g)\ \lim_{x\to 8}\frac{\sqrt[3]{x}-2}{x-8};$

$h)\ \lim_{x\to 0}\frac{1-\sqrt[3]{1-x}}{x};$

$i)\ \lim_{x\to 1}\frac{x^2-3x+2}{\sqrt[3]{9-x}-2}$

$j)\ \lim_{x\to 1}\frac{\sqrt[4]{x}-1}{\sqrt{x}-1}$

$k)\ \lim_{x\to 0}\frac{x-\sqrt{x}}{3\sqrt{x}-x};$

$l)\ \lim_{x\to 1}\left(\frac{1}{1-x}-\frac{3}{1-x^3}\right);$

$m)\ \lim_{x\to 1}\frac{x^m-1}{x^4-1};\ n\in IN$

$n)\ \lim_{x\to 0}\frac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1}$ *use as substituição* $:1+x=y^6$

$o)\ \lim_{x\to 4}\frac{1-\sqrt{x^2-15}}{x-4}$

$p)\ \lim_{x\to 0}\frac{x}{1-\sqrt{1-x}}$

8. Calcula $\lim\limits_{x\to\infty} f(x)$ se:

$a)\ f(x)=\dfrac{5-2x+7x^2}{2x^2+6x-8}$

$b)\ f(x)=\dfrac{-x^3-x+14}{4x^3+5x^2-1}$

$c)\ f(x)=\dfrac{2x^6-6x^2}{3x^5-5x^3}$

$d)\ f(x)=\dfrac{1991x}{x^2-2x}$

$e)\ f(x)=\dfrac{(4x+2)^2(5x-3)^2}{x^5+72}$

$f)\ f(x)=\dfrac{(x-2)^{70}(2x+3)^{30}}{(4x-1)^{15}(5-x)^{85}}$

9. Seja $f(x)=\dfrac{ax^3+x^2-4x+5}{bx^3-x^2+2x-6}$ *calcule* $\lim\limits_{x\to\infty} f(x)$ *se*:

$a)\ a\neq 0 e b\neq 0$

$b)\ a\neq 0 e b=0$

$c)\ a=0 e b\neq 0$

$d)\ a=b=0$

10. Determine $a\in IR$ *se*:

$a)\ \lim\limits_{x\to\infty}\dfrac{ax^3-1}{2x^3+7x^2}=-4$

$b)\ \lim\limits_{x\to\infty}\dfrac{ax^3-2x^3+a^3}{2x^3+ax^2+5a}=a$

11. Calcule

$a)\ \lim\limits_{x\to+\infty}\dfrac{|4x|}{x+16}$

$b)\ \lim\limits_{x\to-\infty}\dfrac{|4x|}{x+16}$

$c)\ \lim\limits_{x\to+\infty}\dfrac{|x|+2x}{3x+1}$

$d)\ \lim\limits_{x\to-\infty}\dfrac{|x|+2x}{3x+1}$

$e)\ \lim\limits_{x\to+\infty}\dfrac{\sqrt{x^2}}{2x-5}$

$f)\ \lim\limits_{x\to-\infty}\dfrac{\sqrt{x^2}}{2x-5}$

$g)\ \lim\limits_{x\to+\infty}\dfrac{\sqrt[6]{x}+\sqrt[3]{x}}{2-\sqrt[4]{x}}$

$h)\ \lim\limits_{x\to+\infty}\dfrac{2x^3-3x+1}{\sqrt{x^6+3x}}$

$i)\ \lim\limits_{x\to+\infty}=\left(\sqrt{x^2+2x}-\sqrt{x+1}\right)$

$j)\ \lim\limits_{x\to+\infty}=\left(\sqrt{x^2+2x-1}-\sqrt{x^2+5x}\right)$

12. Determine o limite das expressões seguintes, quando $x \to 0$

$a)\ \dfrac{tg\,x}{2x}$ $b)\ \dfrac{4x}{sen\,5x}$ $c)\ \dfrac{sen\,2x}{3x+1}$

$d)\ \dfrac{sen\,2x - sen\,x}{2x}$ $e)\ \dfrac{sen(a+x) - sen\,a}{x}$ $f)\ \dfrac{sen\,x \cdot sen\,4x}{x^2}$

$g)\ \dfrac{sen\,2x}{\sqrt{x+9}-3}$ $h)\ \dfrac{sen\,x - sen\,5x}{sen\,2x - sen\,4x}$ $i)\ \dfrac{1-\cos 2x}{2x^2}$

$j)\ \dfrac{sen\,x - tg\,x}{x^3}$

13. Calcule os seguintes limites:

$a)\ \lim\limits_{x\to\infty}\left(1-\dfrac{1}{x}\right)^x;$

$b)\ \lim\limits_{x\to\infty}\{x[\ln(x+1)-\ln x]\};$

$c)\ \lim\limits_{x\to\infty}\left(\dfrac{2x+3}{2x+1}\right)^{x+1}$

$d)\ \lim\limits_{x\to\infty}\left(1+\dfrac{3}{x}\right)^{2x+5}$

$e)\ \lim\limits_{x\to\infty}\left(1-\dfrac{1}{x}\right)^{-3}$

$f)\ \lim\limits_{x\to\infty}\left(\dfrac{x}{x+1}\right)^{x+6}$

$g)\ \lim\limits_{x\to\infty}\left(\dfrac{x+5}{x+2}\right)^{2x};$

$h)\ \lim\limits_{x\to\infty}\left(\dfrac{x^2+1}{x^2-1}\right)^{x^2}$

$m)\ \lim\limits_{x\to 0}\left(\dfrac{k-x}{k}\right)^{-\frac{k}{x}}$

$n)\ \lim\limits_{x\to 0}(1+sen\,x)^{\frac{1}{x}}$

$o)\ \lim\limits_{x\to 0}\dfrac{\ln(1+mx)}{x}$

$p)\ \lim\limits_{x\to 0}(1+sen\ 2x)^{\frac{1}{sen\,5x}}$

$q)\ \lim\limits_{x\to 0}(2-\cos x)^{-sen^{-2}x}$

$r)\ \lim\limits_{x\to 2}\left(\dfrac{1}{2}x\right)^{(x-2)^{-2}}$

$s)\ \lim\limits_{x\to 0}\dfrac{\ln\cos x}{x^2}$

Fim

Appendix 2.6

Kinds of tasks in the reference mathematical organisation

MO_1

T_1 CALCULATE THE LIMIT OF A FUNCTION WHEN $x \to a$

T_{1A} Limit of a continuous function

T_{1B} Indeterminate form of a rational function when $x \to a$

T_{1C} Indeterminate form involving square roots

T_{1D} Indeterminate form involving a trigonometric function

T_2 CALCULATE THE LIMIT OF A FUNCTION WHEN $x \to \pm\infty$

T_{2A} Limit of a polynomial when $x \to \pm\infty$

T_{2B} Indeterminate form of a rational or irrational function when $x \to \pm\infty$

T_{2C} Indeterminate form such as 1^{∞}

T_3 DETERMINE THE LIMIT OF A FUNCTION FROM THE GRAPH

T_4 STUDY THE CONTINUITY OF A FUNCTION

T_{4A} Discussion of the continuity of a function using its analytical expression

T_{4B} Discussion of the continuity of a function using its graph

T_5: APPLY LIMITS IN MATHEMATICS OR IN OTHER DISCIPLINES

MO_2

T_{6A}: Prove the existence (or non-existence) of the limit of a function f as $x \to a$, where a is a real number

T_{6B}: Prove the existence (or non-existence) of the limit of a function f as $x \to \pm\infty$

T_7: Prove the existence (or non-existence) of one-sided limits for certain kinds of functions (such as monotonic functions)

T_{8A}: Prove the properties used to justify the way certain limits of functions are calculated when $x \to a$, where a is a real number

T_{8B}: Prove the properties used to justify the way certain limits of functions are calculated when $x \to \pm\infty$

T_9: Prove continuity of a function

Appendix 2.7

Classification of tasks in the textbooks

Demidovitch

	TASKS OF MO_1	167
T_{1B}	191-198; 266, 269, 270, 293 a), 298, 301 a), c); 302	16
T_{1C}	199-210, 293 b), c); 300; 301 b)	16
T_{1D}	216-240; 268; 293 d), e); 296, 297, 299	31
T_{2A}	170 a) to d); 171-180 (sequences); 181-190; 211-215; 264, 265, 267; 303 a) to d)	37
T_{2B}	241-263	23
T_{4A}	313-315; 316 a) to f); 317-335	28
T_5	271-287; 291, 292, 294, 295	21
	TASKS OF MO_2	18
T_{1A}	168; 169 a); 288-289	4
T_{1B}	166-167; 169 b) and c); 290	5
T_{2A}	304-312	9

Piscounov

	NUMBER OF TASKS OF MO_1	60
T_{1A}	1-3	3
T_{1B}	11-20	10
T_{1C}	21-26; 60	7
T_{1D}	31-40; 46-47; 49; 52; 56; 61, 62	17
T_{2A}	4-10; 27-30; 53	12
T_{2B}	41-45; 48; 50-51	8
T_{4A}	57-59	3
T_5		
	NUMBER OF TASKS OF MO_2	
T_{1A}		
T_{1B}		
T_4		

Two tasks were not classified. They are $\lim_{n\to\infty} n\left[a^{\frac{1}{n}}-1\right]$ and $\lim_{x\to 0}\frac{e^{\alpha x}-e^{\beta x}}{x}$ (n° 55).

Appendix 6.1

Letter to teachers

Dear colleague,

I am a lecturer at Eduardo Mondlane University and I am looking for volunteers to participate in a research project in Mathematics Education. This project aims to investigate the evolution of pre-service and in-service teachers' conceptions on limits of functions, through their participation in a research group[1].

Two pre-service, two in-service teachers and myself will constitute a research group. The two pre-service teachers will be graduate students of the Pedagogical University and the two in-service teachers will be Master Degree students of the Faculty of Education. All of them are expected to explore one aspect of limits of functions, among several issues that I will select for them. They also are expected to write their dissertation about this topic during one academic year, the last year of their degree, and I shall be their supervisor.

If you join the group, you will also have to participate in periodical seminars, where you will explain your research (methods, findings, difficulties) to the other members of the group, get feedback from them, and discuss the research progress of your colleagues.

As my concern is to analyse your ideas about limits of functions and its teaching and learning, as well as the evolution of these ideas during the year, I will interview each member of the group three times: at the beginning, in the middle, and at the end of their research process. These interviews will be audio-taped, and the seminars will be video-taped. I will transcribe all the tapes. Tapes, videos and transcripts will be kept in a safe place.

In all transcriptions as well as in my writing (reports, PhD dissertation and articles), I will use pseudonyms instead of your name, in order to protect your identity. The issue of confidentiality will be discussed during the first meeting of the group, and can be discussed again whenever one of the members of the groups feels, either individually or with the whole group. What is to be done with the research material will be discussed at the end of our work together.

I believe that your participation in the research group will be a good experience for you, and will facilitate the research for your dissertation. You are welcome to ask any question for a better understanding of the project.

---

[1] At the beginning of the research, I wanted to compare the evolution of knowledge of teachers who already taught limits and teachers who never taught this topic. This was not possible because only one of the teacher had experience in teaching limits.

Appendix 6.2

PARTICIPATION FORM TO THE RESERCH GROUP ABOUT LIMITS OF FUNCTIONS

FULL NAME:

BIRTH DATE:

PROVINCE OF BIRTH:

STUDIES

SECONDARY SCHOOL

YEARS	GRADES	SCHOOL

HIGHER STUDIES

YEARS	COURSE/YEAR	INSTITUTION

EXPERIENCE IN TEACHING MATHEMATICS

TOPIC THAT YOU WOULD LIKE TO INVESTIGATE (PREFRENTIAL ORDER)

1	
2	
3	

List of research topics

- The history of the limit concept and its implications for teaching;
- The use of different settings and models to teach the limit concept;
- Alternative ways of introducing the limit concept in schools;
- Applications of limits of functions in mathematics and in other sciences;
- Basic repertoire for teaching limits of functions in schools;
- Students' conceptions and difficulties when learning the limit concept.

Appendix 6.3

Guidelines for teachers' first interview

Introduction

As I explained in the meeting where I presented my research project to you and your colleagues, the aim of this first interview is:

- To analyse how you have been in contact with the concept "limits of functions" through several institutions, for instance as a student at secondary school, as a student at university, as a teacher at secondary school some years later (only for experienced teachers).
- To understand your personal ideas about the teaching and learning of limits of functions at secondary school level.

You will be interviewed again during the process of our work together, and at the end of it, in order to analyse with you the evolution of your ideas about this concept and its teaching.

I will use a tape recorder and will transcribe the interview to analyse it. The interview will be confidential. Tapes and transcriptions will be kept carefully and the name of the interviewee will be changed.

Do you have any questions before we start the interview?

1. **Personal relation to the limit concept**

At secondary school as a student

Where did you study as a secondary school student? When?

Do you remember the first time you met the limit concept?

Where was it? At secondary school? Which grade?

Can you remember how the teacher introduced this concept? Did s/he use numerical values? How did you use numerical values? Did s/he give a definition? Did s/he give one of these four definitions [*showing Sheet 1*]? Did s/he use graphs? How?

What kind of tasks were you asked to solve? Can you remember what difficulties you faced when solving these tasks?

Did you use a textbook? Which one? Did you use this textbook [*showing the former Mozambican textbook for pre-university level*]? How did you use it?

Can you remember what you understood of this concept at secondary school? What were your feelings about it? Did you understand what the applications of this concept were?

At university as a student

Where did you study after secondary school? In which subjects did you meet the limit concept again? How did the teacher introduce it? Can you see any differences between what you did at secondary school and what you did at university related to this concept?

Did you understand the concept and its applications at university better? How? Why?

Did you understand the ε-δ definition? When?

Did you use any textbook? Which one?

At secondary school as a teacher [*only for experienced teachers*]

Did you teach limits of functions at school? Which grade? When?

How did you introduce the concept? What kind of tasks did you give to your students?

What kind of material did you use to prepare your lessons? Did you work with colleagues? How? Why?

Did you see any difference between the way you taught limits of functions at school and the way you had been taught as a secondary school student?

Did you learn something about "limits of functions" while preparing your lessons or teaching at school? What did you learn?

2. Teaching limits of functions at secondary school

In Mozambican secondary schools, limits are usually introduced through sequences. What do you think about this way of approaching limits?

What other ways of approaching limits do you think could be used at this level? Which one do you think more appropriate to secondary school level?

Which definition should be taught in secondary schools [s*howing Sheet 1*]? At what level do you think that a formal definition should be taught? Why?

What kind of task do you think should be given to the students in secondary schools?

(i) Finding the limit by calculating values like these examples [s*howing Sheet 2*]? Why? What advantages do you see in these tasks?

(ii) Linking the limit to the graph, like these examples [s*howing Sheet 3*]? Which ones? Why? Which difficulties do you think students would face?

Do you think that these tasks for calculating limits could be used at secondary school level [s*howing Sheet 4*]? What technique should be used for each of them? In you opinion, what other tasks for calculating limits should be given?

Do you think that tasks linking calculations and graphs, like these ones, should be used [s*howing Sheet 5*]? Why?

In your opinion, what kind of applications of the limit concept should be taught at school?

What kind of applications of the limit concept should be taught at university?

Do you think it useful teaching limits of functions in secondary schools? Why? How do you think students will use this concept later, during their studies at university for example? In which subjects? In which areas?

How would you explain what "limits of functions" are to somebody who doesn't know mathematics, for example a teacher of Portuguese language? What kind of metaphor or analogy would you use?

Do you think that "limits of functions" play a special role in the teaching and learning of mathematics at school?

Which difficulties do you think students meet when learning the limit concept? How do you explain these difficulties?

Debriefing

Summary of the main points learned from the interview. Is there anything else you would like to bring up?

Sheet 1

Definition 1

$\lim_{x \to a} f(x) = L$ means that $f(x)$ approaches L as x approaches a

Definition 2

We say that f (a real-valued function defined on a domain D) has limit L as x tends to a if for each sequence $x_1, x_2, x_3 \dots x_n, \dots$ (belonging to D, different from a) converging to a, the sequence $f(x_n)$ converges to L.

We write $\lim_{x \to a} f(x) = L$

Definition 3 (Weierstrass definition)

$\lim_{x \to a} f(x) = L$ means that, given $\varepsilon > 0$ we can find a number $\delta > 0$ such as if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$

Definition 4 (Topological definition)

If $f : A \to Y$ is a function from a subset A of a topological space X to a topological space Y and x_0 is an adherence point of A, we say that y in Y is the limit of f at x_0 if for every neighbourhood V of y there exists a neighbourhood U of x_0 such that $f(A \cap U) \subset V$

Sheet 2

The function f is defined by $f(x) = \dfrac{1}{x-2}$. Fill the table and indicate the corresponding limit.

a.

x	10	100	1000	10000	100000	1000000	10000000
$f(x)$							

$\lim_{x \to +\infty} f(x) =$

b.

x	2,1	2,01	2,001	2,0001	2,00001	2,000001	2,0000001
$f(x)$							

$\lim_{x \to 2} f(x) =$

Sheet 3

1. Can you read on the graph the limit of the function when *x* goes to infinite?

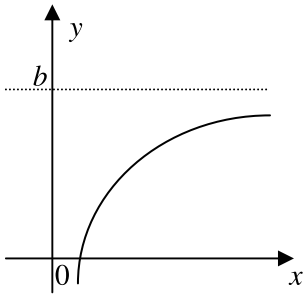


Figure 1

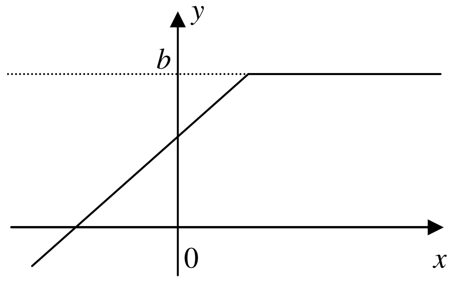


Figure 2

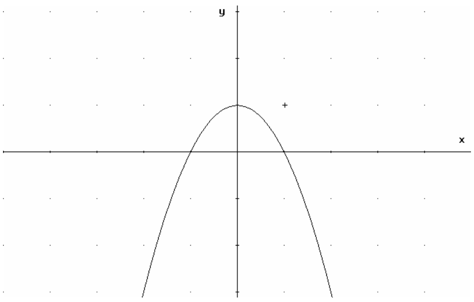


Figure 3

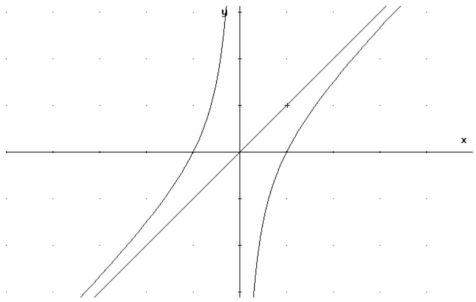


Figure 4

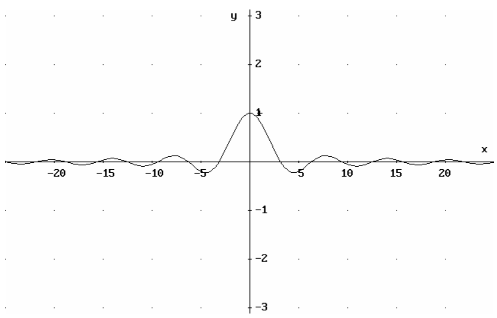
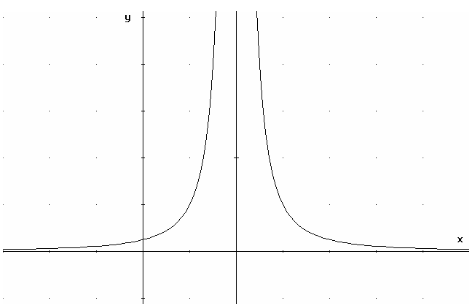


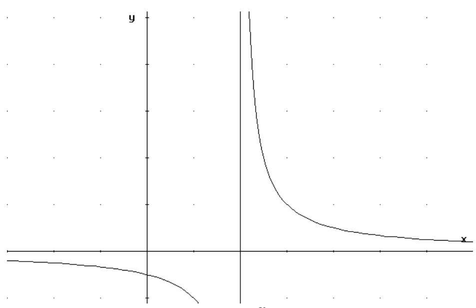
Figure 5

2. Can you read on the graph the limit of the function when *x* goes to *a*?



a

Figure 1



a

Figure 2

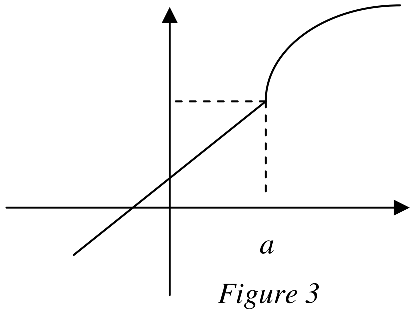


Figure 3

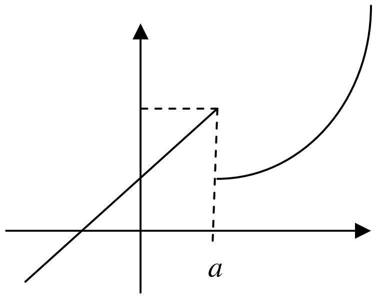


Figure 4

3. Read the domain and the limits of the following functions on its graph.

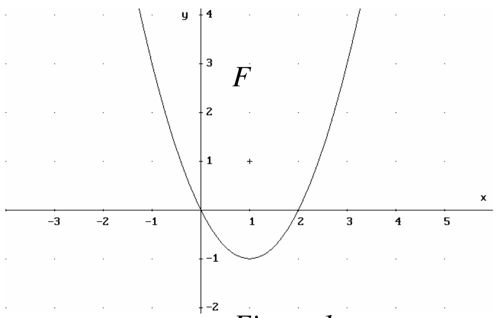


Figure 1

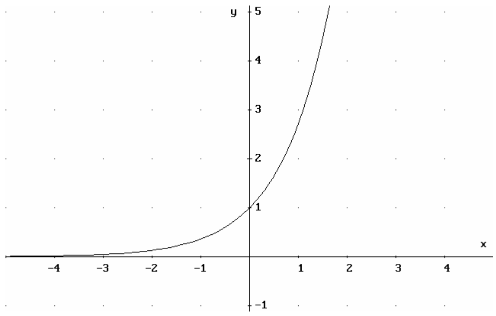


Figure 2

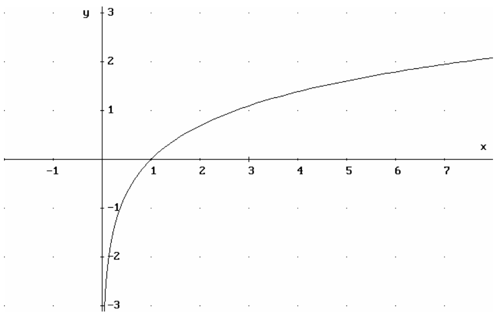


Figure 3

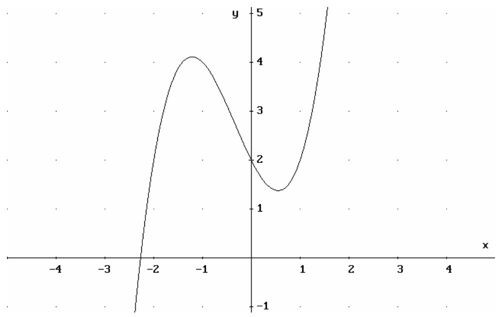


Figure 4

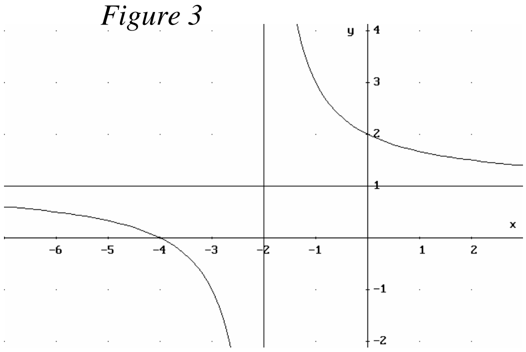


Figure 5

4. The graph of the function f has two asymptotes $x = -1$ e $y = -2$.

Sketch two possible graphs for *f*, corresponding to different positions of the graph in relation to its asymptotes.

5. Sketch a possible graph for the function f using the following limits:

a. $\lim_{x\to-\infty} f(x) = 1^+$

$\lim_{x\to+\infty} f(x) = 1^-$

$\lim_{x\to 2^-} f(x) = +\infty$

$\lim_{x\to 2^+} f(x) = -\infty$

b. $\lim_{x\to-\infty} f(x) = 1^-$

$\lim_{x\to+\infty} f(x) = 1^+$

$\lim_{x\to 2^-} f(x) = -\infty$

$\lim_{x\to 2^+} f(x) = -\infty$

c. $\lim_{x\to-\infty} f(x) = +\infty$

$\lim_{x\to+\infty} f(x) = +\infty$

$\lim_{x\to 0^-} f(x) = -\infty$

$\lim_{x\to 0^+} f(x) = +\infty$

Sheet 4

Calculate the following limits:

a. $\lim_{x\to\infty}\left(-x^3 + 3x - 5\right)$

b. $\lim_{x\to\infty}\dfrac{x^2-3}{x^2+x}$

c. $\lim_{x\to-5}\dfrac{x^2-25}{x^2+2x-15}$

d. $\lim_{x\to a}\dfrac{\sqrt{x}-\sqrt{a}}{x-a}$

e. $\lim_{x\to 0}\dfrac{\sin 3x}{\sin 5x}$

f. $\lim_{x\to\infty}\left(\dfrac{x+2}{x+1}\right)^{2x-3}$

g. $\lim_{x\to\infty} f(x)$, where $f(x) = \begin{cases} \dfrac{1}{x}, \text{ if } x \text{ is a rational number} \\ 0, \text{ if } x \text{ is a irrational number} \end{cases}$

Sheet 5

For each of the following functions:

(i) Determine the domain and the limits;

(ii) Sketch a possible graph using these limits.

a. $f(x) = \dfrac{x-5}{x}$

b. $f(x) = \dfrac{3x-1}{2-x}$

c. $f(x) = x^2 - 3x$

d. $f(x) = e^{2x}$

e. $f(x) = \ln(2x-1)$

Appendix 6.4

Timetable

	Abel	Bernardo	Mateus	David	Ernesto	Frederico
First Interview	07/08/2002	07/08/2002	24/09/2002	27/09/2002	07/10/2002	21/10/2002
1st Seminar	18/11/2002	18/11/2002	18/11/2002	18/11/2002	18/11/2002	18/11/2002
2nd Seminar	08/03/2003		08/03/2003	08/03/2003	08/03/2003	08/03/2003
3rd Seminar	12/04/2003		12/04/2003	12/04/2003	12/04/2003	12/04/2003
Second Interview	23/04/2003		24/04/2003	25/04/2003	17/04/2003	30/04/2003
4th Seminar	10/05/2003		10/05/2003	10/05/2003	10/05/2003	10/05/2003
5th Seminar	31/05/2003		31/05/2003	31/05/2003	31/05/2003	31/05/2003
6th Seminar	21/06/2003		21/06/2003	21/06/2003	21/06/2003	21/06/2003
7th Seminar	13/09/2003		13/09/2003	13/09/2003	13/09/2003	13/09/2003
8th Seminar	11/10/2003		11/10/2003	11/10/2003	11/10/2003	11/10/2003
9th Seminar	21/11/2003		21/11/2003	21/11/2003		21/11/2003
10th Seminar	12/12/2003		12/12/2003	12/12/2003		12/12/2003
11th Seminar	21/02/2004		21/02/2004	21/02/2004		21/02/2004
12th Seminar	06/03/2004		06/03/2004	06/03/2004		06/03/2004
Presentation			26/12/2003			22/12/2003
Third Interview			18/03/2004			11/03/2004
13th Seminar	03/04/2004		03/04/2004	03/04/2004		03/04/2004
Third Interview	13/12/2004			23/09/2004		
Presentation	06/09/2005			09/11/2004		

Appendix 6.5

Guidelines for teachers' second interview

1st Part

1. **Essential features**

What are, in your opinion, the key ideas of the limit concept?

Do you think that the ε-δ definition reflects these ideas?

2. **Different settings and models**

In which settings do you think that the limit concept can be studied? Can you explain? Can you give examples?

Which representations do you think are more important/suitable for studying this concept at secondary school level? Why?

Do you know several ways of telling that the limit of $f(x)$ is b when x goes to a? Which ones?

3. **Alternative ways of approaching**

Can you explain how the limit concept is usually introduced in Mozambican secondary schools?

What do you think of this approach?

Do you know other ways of approaching this concept? Which ones?

Which approach do you think would be more suitable at secondary school level?

4. **Strength of the concept**

Do you think that the limit concept plays a special role in mathematics? For the learning of mathematics? Can you explain?

In your point of view, what other mathematical concepts have strong links with limits?

5. **Applications of the concept**

Do you know some applications of the limit concept in mathematics? In other sciences? Can you explain them?

Do you think some of them could be used at school? Which ones? Why?

6. **Basic repertoire**

Do you know what kinds of tasks about limits are usually given at secondary school level in Mozambique? Can you give some examples? Do you think that these tasks are suitable at that level? Why?

What other tasks could be used in secondary schools? Can you give some examples? How do you think that these tasks would help students understand this concept?

7. **Students' conceptions and difficulties**

Do you know which difficulties secondary school students usually face when studying the limit concept? In your opinion, why do they face these difficulties? What could we do to help them construct a "good" limit concept?

2nd Part

Do you think that your knowledge and ideas about limits of functions evolved since the beginning of the research? How? What did you learn?

Can you explain what you have learnt:

- During the 1st interview;
- Working alone on your research;
- During the supervision sessions;
- During the seminars;
- Working with other(s) teacher(s) in the group out of the seminars.

Appendix 6.6

Guidelines for teachers' third interview

Introduction

As I explained during the 11th seminar, the aims of this interview are to:

- Find out how your knowledge about limits of functions evolved since the beginning of your research;
- Analyse your current ideas about the teaching of limits in Mozambican secondary schools;
- Analyse the role in the evolution of your knowledge and ideas of each activity: your personal research, the supervision sessions, the seminars, the first and second interviews, and your final presentation.

Do you have any questions?

1. Knowledge related to the five topics.

What have you learnt since the beginning of your research? [*I will let the teacher present his ideas in his own way. I will only ask a few questions to be sure that we cover all research topics*]

- History of the limit concept;
- Several settings and registers in which this concept can be studied;
- Different ways of approaching the concept;
- Applications of the limit concept in mathematics and in other sciences;
- Students' conceptions and difficulties when learning this concept.

At this point do you feel that you are well prepared for teaching limits in a secondary school ?

2. Teaching limits in Mozambican Secondary Schools

- If you had to teach limits, how would you organise this teaching?
- Would you teach a definition? Which one? (Sheet 1). During the first interview, I showed you some tasks about limits using different representations. Which tasks would you now use in your teaching? (Sheets 2 a 5) Would you use other tasks? Which ones?

3. The contribution of each activity to the whole process

How do you think that your knowledge and ideas about teaching limits of functions evolved since the beginning of the research?

Can you explain what you learnt during each of these activities:

- Working alone on your research;
- During the supervision sessions;
- During the interviews;
- During the seminars [*analysing more specifically the interaction with other colleagues*];
- Working with other(s) teacher(s) of the group outside of the seminars;
- During your presentation.

What has changed for you during this process (as a teacher, as a person?)

4. Essential features

During the 1st and 2nd interviews, I asked you the question: How would you explain what "limits of functions" are to somebody who doesn't know mathematics, for example a teacher of Portuguese language? What kind of metaphor or analogy would you use? Do you remember your answers? How would you answer this question now?

Appendix 6.7

Content of the seminars

Seminar	Date	Content
1	18/11/2002	Introduction of the members of the group Each teacher presents his topic, his project, and what he has done so far Forms of collaboration between members of the group
2	08/03/2003	Each teacher presents his work in progress and gets feedback from the group Issues that need to be discussed by the group (formal definition) Forms of collaboration between members of the group
3	12/04/2003	Discussion about the ε-δ definition of limits of functions Preparation for the second interview
4	10/05/2003	Each teacher presents his work in progress, and gets feedback from the group Suggestions made during the second interviews for the seminars
5	31/05/2003	Discussion of settings and registers in which limits of functions can be studied Presentation of references in a scientific work
6	21/06/2003	Each teacher presents his work in progress and gets feedback from the group
7	13/09/2003	Each teacher presents his work in progress and gets feedback from the group
8	11/10/2003	Each teacher presents his work in progress and gets feedback from the group
9	21/11/2003	Discussion of David's worksheet about applications of limits in other sciences Discussion of Frederico's chapter II "The Greeks' ideas" Discussion of Mateus's chapters II "Theoretical framework" and IV "Studying limits in different settings and models"
10	10/12/2003	Discussion of David's work about "Application of limits to continuity and derivatives" Discussion of Abel's chapter I "Introduction of the limit concept in Mozambican schools" Discussion of my paper about "The evolution of secondary school Mozambican teachers' knowledge about the ε-δ definition of limits of functions"[2]
11	21/02/2004	Discussion of Abel's chapter II "Literature review" Discussion of David's work "Application of limits for sketching the graph of a function and to the derivatives"
12	06/03/2004	Discussion of Abel's chapter V "Alternative ways for introducing the limit concept" Discussion of David's work "Application of limits for studying a function and to integrals"
13	03/04/2004	Discussion of:David's work about ""Application of limits for sketching the graph of a function" Discussion of Abel's work about "Interviews analysis"

[2] Danielle Huillet. Concepções de Professores do Ensino Secundário Geral acerca da Definição ε-δ do Conceito Limites de Funções. III Seminário de Investigação da UEM. 4-6 de Novembro de 2003. Maputo

Appendix 6.8

Codes for Interviews' Analysis

EF **E**ssential **F**eatures

- **EF-D** **D**ynamic point of view
- **EF-S** **S**tatic point of view
- **EF-O** **O**perational point of view

DSM **D**ifferent **S**ettings and **M**odels

- **DSM-A** **A**lgebraic setting
  - **DSM-A-P** **P**olynomials
  - **DSM-A-R** **R**ational functions
  - **DSM-A-I** **I**rrational functions
  - **DSM-A-T** **T**rigonometric functions
  - **DSM-A-E** **E**xponential functions
- **DSM-C** **C**inematic setting
- **DSM-Fo** **Fo**rmal setting
- **DSM-Fu** **Fu**nctional setting
- **DSM-Ge** **Ge**ometrical setting
- **DSM-Gr** **Gr**aphical setting
  - **DSM-Gr-R** **R**eading a limit from a graph
  - **DSM-Gr-S** **S**ketching a graph using limits
- **DSM-Nu** **Nu**merical setting
- **DSM-NL** **N**atural **L**anguage setting
- **DSM-T** **T**opological setting

AWI **A**lternative **W**ays of **I**ntroducing

- **AWI-NV** Through **N**umerical **V**alues
- **AWI-TL** Through the **T**angent **L**ine problem
- **AWI-V** Through problems of **V**elocity

SC **S**trength of the **C**oncept

- **SC-L** **L**inks with other concepts (functions, infinity, etc.)
- **SC-BC** **B**asic **C**oncept for defining derivative and integral concepts
- **SC-AS** **A**pplication to other **S**ciences

BR **B**asic **R**epertoire

- **BR-N** **N**umerical tasks
- **BR-D** Application of the ε-δ **D**efinition tasks

BR-G **G**raphical tasks

BR-Nu **Nu**merical tasks

SCD **S**tudents **C**onceptions and **D**ifficulties

SCD-A **A**lgebraic difficulties

SCD-A- P **P**olynomials

SCD-A- R **R**ational functions

SCD-A- I **I**rrational functions

SCD-A- T **T**rigonometric functions

SCD-A- E **E**xponential functions

SCD-C **C**onceptual difficulties

SCD-Gr **Gr**aphical difficulties

SCD-Gr-R Difficulties for **R**eading graphs

SCD-Gr-S Difficulties for **S**ketching graphs

SCD-NL **N**atural **L**anguage difficulties

SCD-Nu **Nu**merical difficulties

SCD-S **S**ymbolic difficulties

O **O**ther Issues

Appendix 6.9

Analysis of Teacher X's First Interview

1. Interview settings

2. Studies and professional experience (lines)

3. Trajectory of his relation to limits of functions

1st contact Institution 1 (lines)

2nd contact Institution 2 (lines)

3rd contact Institution 3 (lines)

4. Knowledge about limits

4.1. Essential features

4.2. Different settings and models

Algebraic setting

Formal setting

Functional setting

Graphical setting

Numerical setting

Natural language setting

4.3. Alternative ways of introducing limits

4.4. Strength of the concept

4.5. Basic repertoire

4.6. Students conceptions and difficulties

As a teacher

As a student: the research

Confidence

5. Teaching limits in Mozambican secondary schools

6. Identity

7. Other issues

8. Conclusion

Appendix 6.10

Comparison table first interview – The first encounter with the limit concept

Questions: In Mozambican secondary schools, limits are usually introduced through sequences. What do you think about this way of approaching limits?

What other ways of approaching limits do you think that could be used in schools? Which one do you think more appropriate to secondary school level?

Abel	Mateus	David	Frederico
He says that the introduction used in secondary schools is not very efficient, but he does not seem to know alternatives. It can be the reason why he chose the topic "Alternative Ways of Introducing Limits", but he seems to feel very insecure about it. A: ... *Now if I will succeed in finding these other alternatives* I: *Hum, hum, hum* A: *I don't know!*	He knows how to approach limits starting with sequences, as stated in the Mozambican syllabus, but thinks that teachers should use the graphical setting more. He also suggests a more geometrical approach but did not explain what "geometrical" means for him. It seems that he is also speaking about the graphical setting. *Maybe if during the introduction of the limit concept, it was more using this method … of tables and, maybe geometrical method as well (…) Hum, geometrical, to see more or less using a graph what we call limit*	We did not focus specifically on this topic but, speaking about secondary school, he states that teachers always introduce limit in the same way.	When Frederico taught limits at the Agricultural School, he introduced the concept through numerical sequences (cf 2nd contact). At the end of the interview, he says that visualising limits through graphs could help students understand this concept.

All teachers know the way limits of functions are usually introduced in Mozambican secondary schools, according to the syllabus. None of them presented any alternative to this introduction. At the end of the interview, two of them (Mateus and Frederico) speak about using more graphs, maybe influenced by my tasks.

Appendix 6.11

Comparison table third interview – The first encounter with the limit concept

	Abel	Mateus	David	Frederico
1st Interview	He says that the introduction used in secondary schools is not very efficient, but he does not seem to know alternatives. It can be the reason why he chose the topic “Alternative Ways of Introducing Limits”, but he seems to feel very insecure about it. A: ... *Now if I will succeed in finding these other alternatives* I: *Hum, hum, hum* A: *I don’t know!*	He knows how to approach limits starting with sequences, as stated in the Mozambican syllabus, but thinks that teachers should use the graphical setting more. He also suggests a more geometrical approach but did not explain what “geometrical” means for him. It seems that he is also speaking about the graphical setting. *Maybe if during the introduction of the limit concept, it was more using this method … of tables and, maybe geometrical method as well (…) Hum, geometrical, to see more or less using a graph what we call limit*	We did not focus specifically on this topic but, speaking about secondary school, he states that teachers always introduce limit in the same way.	When Frederico taught limits at the Agricultural School, he introduced the concept through numerical sequences (cf 2nd contact). At the end of the interview, he says that visualising limits through graphs could help students understand this concept.
3rd Interview	As it was the subject of his research, Abel learnt a lot about different ways of introducing limits at school. He explains how he planed his intervention in a classroom and that he feels very disappointed with the results.	We did not focus specifically on this issue. Mateus speaks about how he would introduce this concept at school beginning with everyday examples (a moving train, a boundary). He would then use a numerical register, followed by a graphical	David remembers that Abel spoke about introducing limits through the instantaneous velocity but it is not so clear for him. As a teacher he would use the numerical register in the first place, and then he would give the definition, because any concept must have its	We do not speak specifically about this issue but he states that it is necessary to use several alternatives and methods for students to get an idea of the limit concept. He says that, through the history of the concept, he was able to see that this topic must be handled carefully.

	He says that, before doing this research, he didn't know that there were other alternatives than the one he used at school: he used to give the definition, the calculation rules, and tasks. Now he gives more importance to the graph. When I ask him how he would teach limits at secondary school, he began worrying about the syllabus and the time.	interpretation. He would try to get the students to calculate limits and then use them to sketch the graph of the function.	definition, even if the students will not understand it.	He suggests that the study of limits should begin in Grade 11 instead of Grade 12. He also states that he would use a more graphical method to introduce limits When I ask him how he would teach limits at school, he says that he would begin with figures, intuitive ideas coming from the history of the concept. Then he would work in the numerical and graphical settings. He also says that he would teach the definition using increments, confusing this with the definition of the derivative. He would then teach the ε-δ definition, but without any application.

1st Interview: All teachers know the way limits of functions are usually introduced in Mozambican secondary schools, according to the syllabus. None of them presented any alternative to this introduction. At the end of the interview, two of them (Mateus and Frederico) speak about using more graphs, maybe influenced by my tasks.

3rd Interview: The analysis of the third interview shows clearly that all teachers learnt about other ways of introducing the limit concept in schools, and are willing to try to apply some of them.

Abel, who was researching this topic, obviously learnt several ways of introducing the limit concept. In his dissertation he presented different possible introductions of this concept: They are: (i) Introduction through the tangent line (geometrical setting); (ii) Introduction through a graph (graphical setting); (iii) Introduction through a rational function (numerical and graphical settings); (iv) Introduction through a sequence (numerical and graphical settings); (v) Introduction through instantaneous velocity (cinematic setting); (vi) Introduction using several settings (numerical, graphical and formal setting).

He experimented with the introduction of limits in a classroom, using the graphical register with computers. Nevertheless, the results were not as good as he was expecting and he is quite disappointed with that. When asked how he would introduce this concept at school, he is the only one who worried with the syllabus and with the time allocated for this topic. In fact, out of the 4 teachers, he is the only one who taught limits in Grade 12.

Mateus would begin with everyday examples and then use the numerical setting, followed by a graphical interpretation. He would try to lead students to use the results of their calculation to sketch the graph of the function.

David didn't understand very well the way Abel introduced limits trough instantaneous velocity. As a teacher he would use the numerical setting and then give the definition, even knowing that the students will not understand it. He would also use the graphical register.

Frederico was able to see, through the history of the limit concept (his own topic) that the limit concept must be handled very carefully. He would use a more graphical method to introduce this concept. He would also teach the formal definition, but without any application.

At the beginning of the research, the teachers only knew the way to introduce the limit which is usual in Mozambican institutions: the definition, the rules and tasks to calculate limits. They now are aware that it is not the only way to introduce limits, even if they don't have a deep understanding of all the alternatives presented by Abel in his dissertation. They also seem willing to use other alternatives, giving emphasis to the graphical and/or numerical setting. David and Frederico would also teach the formal definition, although they know that the students will not understand it. I can see two interpretations for this fact:

The first one is the difficulty that teachers have in breaking the rules of the institutional relation. The numerical and graphical representations are mentioned in the syllabus, even if they are not used in practice. Extending the use of these two registers cannot be considered as withdrawing too much from the syllabus. But the formal definition is also part of the syllabus. Not to teach the definition can be considered as acting "against" the syllabus, which represents authority.

The second interpretation is a mathematical one: a concept must have a definition.

These two interpretations are illustrated by the following quotation from David's third interview: "The definition? Yes I would give it! Because it is a concept, and any concept must be defined … but, in my own particular case, I think that I would use it very little".

See 13th seminar: discussion between David (standing up for the numerical register) and Mateus and Frederico (advocating that the graphical register was more important).

Appendix 6.12

Categories

First Encounter

FE-MK1	The teacher only knows the way the first encounter with limits is organized in Mozambican secondary schools according to the syllabus.
FE-MK2	The teacher knows of other ways to organize the first encounter with limits of functions and is able to explain at least one of them.
FE-T1	The teacher does not challenge the way the first encounter with limits of functions is organized in Mozambican secondary schools according to the syllabus.
FE-T2	The teacher does not challenge the way the first encounter with limits of functions is organized in Mozambican secondary schools but is aware of students' difficulties.
FE-T3	The teacher knows the way the first encounter with limits of functions is organized in Mozambican secondary schools, is aware of students' difficulties and suggests some change in the institutional relation.
FE-T4	The teacher explains how he would organize the first encounter in schools in a different way (new institutional relation).
FE-T5	The teacher explains how he would organize the first encounter within a new institutional relation and presents strong arguments to defend his ideas.
FE-T6	The teacher explains how he would organize the first encounter within a new institutional relation, presents strong arguments to defend his ideas, figures out possible problems and explain how to avoid them.

Social justification

SJ-MK1	The teacher does not acknowledge the importance of the limit concept.
SJ-MK2	The teacher knows of few applications of limits in mathematics and physics.
SJ-MK3	The teacher knows of several applications of limits in mathematics.
SJ-MK4	The teacher knows of several applications of limits in mathematics and in other sciences.
SJ-T1	The teacher would not show the importance of the limit concept to students (personal relation= Secondary School institutional relation).
SJ-T2	The teacher is willing to explain the importance of the limit concept to students.
SJ-T3	The teacher is willing to use tasks that show the importance of the limit concept.

Essential features

EF-MK1	The teacher speaks about limits using mainly one of its features.
EF-MK2	The teacher speaks about limits using mainly two features, but is not aware of the different features.
EF-MK3	The teacher speaks about limits using three features, but is not aware of the different features.
EF-MK4	The teacher knows that the limit concept can be seen from different points of view.
EF-T1	The teacher would teach limits mainly from an operational point of view.
EF-T2	The teacher is willing to show students other features of the limit concept.

Graphical register

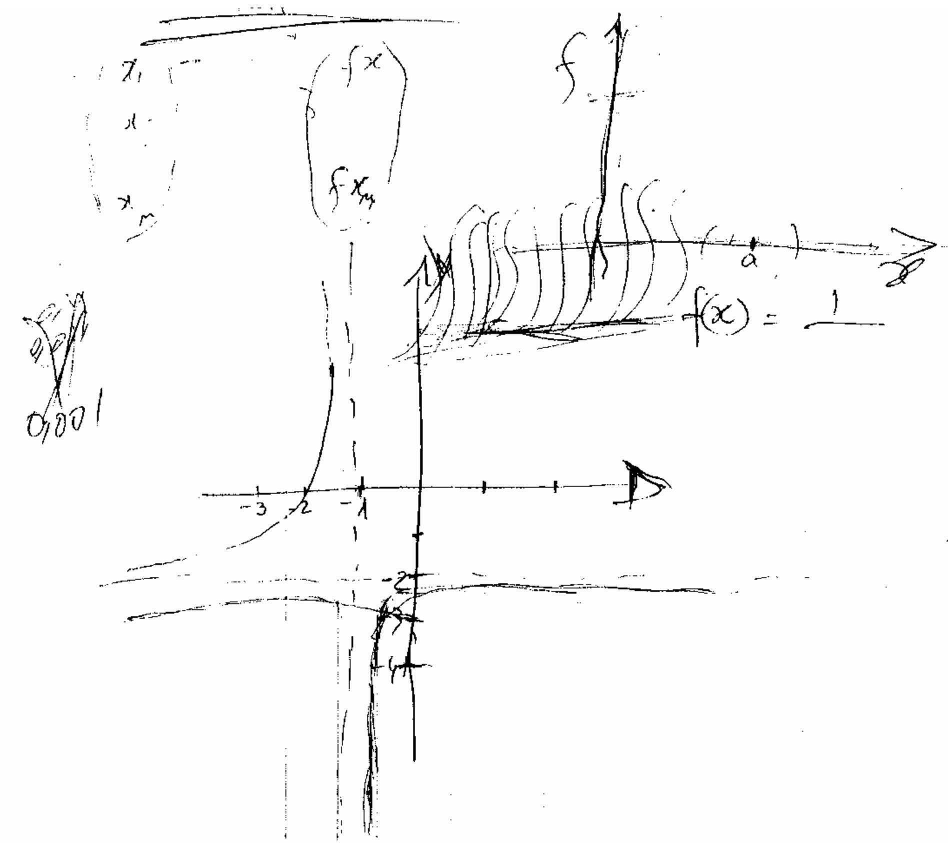
GRR1	The teacher is not able to read any limit from the graphs.
GRR2	The teacher is able to read some limits along a vertical or an horizontal asymptote (when the graph does not cross the asymptote).
GRR3	The teacher is able to read limits along a vertical or an horizontal asymptote (when the graph does not cross the asymptote), and infinite limits at infinity ($x \to \infty$, $y \to \infty$).
GRR4	The teacher is able to read limits along a vertical or an horizontal asymptote (even when the graph crosses the asymptote), and infinite limits at infinity ($x \to \infty$, $y \to \infty$).
GRR5	The teacher is able to read most limits but still faces small difficulties.
GRR6	The teacher is able to read all kinds of limits.
GRS1	The teacher is not able to sketch any graph using limits or asymptotes.
GRS2	The teacher is not able to indicate any limit on axes. He is able to sketch a standard graph having two asymptotes, one vertical and one horizontal.
GRS3	The teacher is able to indicate limits along a vertical or an horizontal asymptote as a whole branch. He does not acknowledge that producing several branches may generate a graph that is not a function.
GRS4	The teacher is able to indicate limits along a vertical or an horizontal asymptote as a whole branch. He acknowledges that the produced graph does not represent a function.
GRS5	The teacher is able to indicate limits along a vertical or an horizontal asymptote as a local behaviour.
GRS6	The teacher is able to indicate any kind of limit on axes.
GR-T1	The teacher would not use graphs when teaching limits.
GR-T2	The teacher acknowledges the importance of the graphical register in the teaching of limits.
GR-T3	The teacher acknowledges the importance of the graphical register and explains how he would use it or articulate it with other registers.

Definition

D-MK1	The teacher is not able to write correctly the ε-δ definition.
D-MK2	The teacher writes correctly the ε-δ definition, but is not sure about it.
D-MK3	The teacher is sure about the correct ε-δ definition, but is not able to explain it.
D-MK4	The teacher is sure about the correct ε-δ definition, and is able to explain it.
D-T1	The teacher would teach the ε-δ definition
D-T2	The teacher acknowledges students' difficulties in understanding the definition. As a consequence he would teach the ε-δ definition without applications.
D-T3	The teacher acknowledges students' difficulties in understanding the definition. He is inclined not to teach it but is not sure about that.
D-T4	The teacher acknowledges students' difficulties in understanding the definition and, as a consequence, would not teach the ε-δ definition.

Appendix 7.1

Abel's graph during the first interview



A. 1st interview Annex 1 I1A/1

Appendix 9.1

Limits in the everyday language (From Collins COBUILD Dictionary)

1. A **limit** is the greatest amount, extent, or degree of something that is possible.
2. **A limit** of a particular kind is the largest or smallest amount of something such as time or money that is allowed because of a rule, law, or decision.
3. The **limit** of an area is its boundary or edge.
4. **The limits of** a situation are the facts involved in it which make only some actions or results possible.
5. I you **limit** something, your prevent it from becoming greater than a particular amount or degree.
6. 6. If you **limit yourself** to something, or if someone or something **limits** you, the number of things that you have or do is reduced.
7. If something **is limited to** a particular place or group of people, it exists only in that place, or is had or done only by that group.
8. age limit, limited
9. If an area or a place is **off limits**, you are not allowed to go there.
10. If someone is **over the limit**, they have drunk more alcohol than they are legally allowed to when driving a vehicle.
11. If you say **the sky is the limit**, you mean that there is nothing to prevent someone or something from being very successful.
12. If you add **within limits** to a statement, you mean that it is true or applies only when talking about reasonable or normal situations.

Appendix 9.2

Classification of teachers’ knowledge about essential features

		Everyday concept	Mathematical concept
Abel	**Previous (I1)**	*It’s not coming to me at this moment*	Approach (4 times) It doesn’t get there (4 times) The values that tend to ***More dynamic***
	Previous as seen at the end (I3)		It was calculations ***Operational***
	Final (I3)	Wall (twice) Boundary A place (twice) You get there A mark I can’t go further than the wall The end ***More static, but also dynamic***	x approaches a fixed number fixed value (twice) approaches ***Static and dynamic***
Mateus	**Previous (I1)**	Boundary (3 times) Order [Instruction] (twice) All stay close to me ***Static***	Number A value which approaches Limit of function is a value An approximate value Values close to a certain value Close value ***Static, dynamic and operational***
	Previous as seen at the end (I3)		I had an idea of limits as a static concept limits only as calculating numbers ***Operational***
	Final (I3)	Moving train Boundary (twice) Border (3 times), area near the boarder ***Static and dynamic***	*He did not focus on this point*
David	**Previous (I1)**	*He did not focus on this point*	A sequence Repetition (3 times) ***Static (?)Dynamic (?)***
	Previous as seen at the end (I3)		Limits were only calculations They were calculations where ... we solved ***Operational***
	Final (I3)	Landmarks, They can be reached or not ***More static, also dynamic***	Concept of approaching ***Dynamic***
Frederico	**Previous (I1)**	The limit of freedom Rules, regulations You cannot decide limitation You cannot, must not exceed At that time, I want everybody at home ***Static***	*He did not focus on this point*

Appendix 11.1

Third seminar – 12 April 2003

1. Description of the seminar

To begin, I explained the main aim of the seminar: to analyse the ε-δ definition, because I noticed that during the first interviews most teachers had difficulties with this definition. The second point was to prepare the second interview (S3/16-33). Then I asked the teachers whether they had already reflected on the definition (S3/33-35).

Abel volunteered himself, went to the blackboard and tried to explain the definition. He wrote the analytical expression of a function $f(x) = \frac{(2x+1)(x-1)}{(x-1)}$ and explained that, in the numerical register, we can choose values for *x* in a neighbourhood of 1, which tend to one from the left or from the right. He drew a graph, wrote 1-δ and 1+δ on the *x*-axis, and ε+1 and ε-1 on the *y*-axis (S3/47-64). Then he said:

Então desses valores, vamos fazer corresponder, euh, valor de δ, um menos δ neste caso, e temos aqui um mais … δ. Esses valores têm os seus correspondentes deste lado, ε … 1 aqui menos ε, e, e este valor à direita é um, um mais ε [escreve ε+1, ε-1]. E aqui temos o ε que nós consideramos entretanto, euh quanto é? 3! (S3/64-67)	So these values will correspond to, er, δ-value, one minus δ in this case, and we have here one plus …δ. These values have their correspondents on this side, ε … one here minus ε, and, and this value on the right is one, one plus ε … [he writes ε+1, ε-1]. And here we have ε that we consider as, er how much? 3!

Then he changed and wrote ε+3 and ε-3.

From that quote, we can see that Abel is not really sure about the graphical interpretation of this limit. He indicated correctly the interval]1-δ, 1+δ[on the *x*-axis, but indicated a wrong interval on the *y*-axis: on the first place]ε-1, ε+1[, and then]ε-3, ε+3[, both intervals centred in ε.

I then told him that it should be 3- ε and 3+ ε (S3/71). He changed again and then tried to explain.

Aqui nesses valores, vamos ter nossos valores correspondentes de ε+3, aqui ε-3, valores correspondentes 1-δ, vamos ter ε-3, e 1+δ, vamos ter ε+3. [...] Vamos considerar que, para qualquer que seja o ε maior, qualquer ε maior que zero, existe sempre … um δ, que é também maior que zero, nós podemos escrever aqui … maior que zero tal que … o valor absoluto, porque tenho 1-δ e o outro 1+δ. Nós encontramos 1-δ, depois encontramos 1+δ, nós determinamos valor absoluto dentro desses valores, aqueles outros. Portanto, euh, um … menos, portanto temos aqui,	Here for these values, we will have our corresponding values of ε+3, here ε-3, corresponding values 1-δ, we will have ε-3, and 1+δ, we will have ε+3. [...] We will consider that for any ε greater, any ε greater than zero, it always exists … δ, which is also greater than zero, we can write here … greater than zero such that … the absolute value, because I have 1-δ and the other 1+δ. We find 1-δ, then we find 1+δ, we determine the absolute values for these values, the other ones. Therefore, er, one … minus, so we have here, we considered 1–δ such that, er, we will attribute to *a*,

consideramos 1-δ tal que, euh, vamos atribuir a *a*, não é, mas podemos deixar como um menos, euh, deve ser 1- δ, não é, o correspondente é ε-3. Vamos ter … aqui, euh, este também vai ser menor que δ, significa que o limite aqui, os correspondentes aqui ε+3, ε-3, estão a ver este módulo [escreve no quadro] correspondente é o raio 3 … portanto esse vai ser menor que … ε. Dali podemos dizer que, nós podemos tornar esse número tão próximo de um, por mais que seja, vamos ter os valores correspondentes aqui ε+3 ou ε-3 … Não sei se tenho que apresentar a tangente mas é a deslocação, é esse movimento que nós, é essa aproximação, é essa deslocação que nós, euh, atribuímos aos valores (S3/73-89).

you know, but we can leave one minus, er, it must be 1–δ, you know, the corresponding is ε-3. We will have … here, er, the corresponding here ε+3, ε-3, you see the absolute value [writing on the blackboard] corresponding to the radius 3 … therefore this will be less than … ε. Then we can say that, we can make this number as close as one, we will have the corresponding values here ε+3 or ε-3 … I don't know whether I should draw the tangent line, but the motion, it is this movement that we, it is this approach, it is this motion that we, er, attribute to the values.

Here again the explanation is not very clear. Abel seems to be explaining to his colleagues as he used to explain to his students. However, as he had stopped teaching for five years, he probably did not remember exactly this explanation.

He wrote the following definition on the blackboard:

$$\forall \varepsilon > 0 \quad \exists \delta > 0 : |1-a| < \delta \Rightarrow |f(x)-a| < \varepsilon$$

A discussion arose about this definition.

E: Abel, euh, eu só queria perguntar acerca de, do segundo, segundo módulo! Eu acho que deve ser $f(x)-b<\varepsilon$! Naquele caso o *b* é 3? (S3/93-94)

E: Abel, er, I would like to ask you about, the second, the second modulus! I think that it should be $f(x)-b<\varepsilon$! In that case *b* is 3?

Abel changed again the definition and wrote:

$$\forall \varepsilon > 0 \quad \exists \delta > 0 : |x-1| < \delta \Rightarrow |f(x)-3| < \varepsilon .$$

Then Mateus asked for the meaning of ε and δ (S3/102/112). Abel tried to explain.

A: Ah, hum, portanto o ponto δ é este aqui, esses valores que eu faço diminuir, eu posso ter 1 aqui - δ, mais um outro valor $1+\delta_1$, δ_1, depois aqui outro $1+\delta_2$, δ_3, δ, são valores que eu vou atribuindo, e vou ter os valores correspondentes, também para ε, euh, aqui um, resultado ε_1+3, ε_2+3, por ai adiante, os valores correspondentes de, de um à direita. Posso ter também … euh, eu considero que $1-\delta$, portanto $\delta_1, 1-\delta_1, 1-\delta_2$, $1-\delta_3$ e por ai adiante. Quanto mais próximo, eu terei também os valores correspondentes $\varepsilon_1, \varepsilon_2$ por ai adiante. Portanto desculpa mas eu estava a

A: Ah, hum, therefore δ, this point δ here, these decreasing values, I can have here 1 here -δ, then another value $1+\delta_1$, δ_1, then here another one $1+\delta_2$, δ_3, δ, they are values that I can choose, and I will have the corresponding values, for ε as well, er, here one, result ε_1+3, ε_2+3, and so on, the corresponding values from, from one to the right. I can also have … er, I consider $1-\delta$, therefore $\delta_1, 1-\delta_1, 1-\delta_2, 1-\delta_3$ and so on. When closer, I will also have the corresponding values $\varepsilon_1, \varepsilon_2$ and so on. So, sorry but I was thinking, er, that we had a value up to one, but we don't have it! … Now, I understand this as … these values I am pensar, hum, ter um valor até 1 e não há! … Agora, eu entendo isso como... esses valores que eu procuro aproximar cada vez mais a um! … E depois ter os valores correspondentes! … Não sei, enquanto ao ε iríamos dizer que, euh, é … é um comprimento! É uma deslocação! (S3/114-123)

trying to approach them more and more to one! And then the corresponding values! … I don't know, for ε I would say that, er, it is … it is a length! It is a motion!

Once again Abel's explanations are not very clear. He did not really explain the meaning of ε and δ, but rather how to use different values for them. Furthermore, while he indicated correctly the values 1-δ and 1+δ, centred in 1, he indicated the other values as ε+3, as if they were centred in ε. It is the same error that he had committed at the beginning when he wrote ε+1 and ε-1, as well as ε+3 and ε-3.

Then Ernesto[3], who was still participating in the study at that time, raised the question of the dependence between the two variables.

E: Há uma dependência, quer dizer há uma dependência de, de duas variáveis, que é temos a variável *x* que é independente	E: There is a dependence, I mean the two variables are dependent, we have the *x*-variable which is independent
A: Sim, a variável independente	A: Yes, the independent variable
E: E o *y* que é dependente de *x*	E: and *y* that depends on *x*
A: Sim	A: Yes
E: Então na definição diz que, para qualquer ε … tem que existir um δ	E: Then the definition says that, for any ε … it should exist some δ
A: Sim	A: Yes
E: então, euh, tal que o módulo de *x* menos, menos 1	E: then, er, in such a way that the modulus of *x* minus, minus 1
A: Sim	A: Yes
E: tenha que ser menor que δ! Mas do jeito como a definição começa, eu entendo que o, o δ é um valor que resulta é da escolha do ε! Porque a gente escolhe o ε, que é qualquer …	E: should be less than δ! But from the way the definition begins, I understand that δ is a value that results from the choice of ε! Because if we choose ε, which is any value …
A: Sim	A: Yes
E: O, o ε é qualquer! A gente escolhe! Que não é um valor de *x*! Então desse ε, encontraremos um δ correspondente. Agora, euh, aqui no, assim acho que estamos a adicionar, quando diz que é módulo de *x*-1 menor que δ. Parece-me que estamos a adicionar o valor, a distância no eixo do *x*, com o valor do *y*! (S3/134-159)	E: ε has any value! We choose it! Which is not an *x*-value! Then from this ε, we will find a corresponding δ. Now, er, in that way I think that we are adding, when it says that modulus of *x*-1 is less than δ. It seems that we are adding the value, the distance on the *x*-axis, with the *y*-value!

This is a very interesting remark, because Ernesto pointed out one of the difficulties students face for understanding the ε-δ definition. There is an inversion of the steps: the chosen radius ε

[3] Ernesto passed away in November 2003, without concluding his dissertation and being interviewed for the third time

is related to the dependent variable *y*, while the radius δ, which depends on ε, relates to the independent variable *x*. This difficulty had been evidenced by Courant and Robbins (1978), quoted by Fischbein (1994) in the case of limits of sequences.

> There is a definite psychological difficulty in grasping this precise definition of limit. Our intuition suggests a "dynamic" idea of a limit as the result of the process of "motion": We move on through the row of integers 1, 2, 3, … *n*, … and then observe the behavior of the sequence a_n. We feel that the approach $a_n \to a$ should be observable. But this "natural" attitude is not capable of clear mathematical formulation. To arrive at a precise definition we must *reverse* the order of steps; instead of looking at the independent variable *n* and then at the dependent variable a_n, we must base our definition on what we have to do if we wish actually to check the statement $a_n \to a$. In such a procedure, we must first choose an arbitrarily small margin around *a* and then determine whether we can meet this condition by taking the independent variable *n* sufficiently large. Then, by giving symbolic names, ε and *N*, to the phrases "arbitrarily small margin" and "sufficiently large *n*" we are led to the precise definition of limits (1993: 238)Then Ernesto went to the blackboard and changed the values in the graphical representation, indicating *a* and δ on the *x*-axis,
> and $f(a+\delta)$, $f(a)$, $f(a-\delta)$, and *ε* on the *y*-axis (S3/164-196).

He explained his dilemma: do we choose δ or ε (arbitrarily small)?

E: Agora, a minha questão é esta: segundo a definição, para qualquer ε que possamos escolher	E: Then my question is the following: according to the definition, for any ε that we choose
A: Hum	A: Hum
E: e segundo o que está aqui nós escolhemos primeiro	E: and according to what is here we first choose
A: Ah, escolhemos é o δ!	A: Ah, we choose δ!
E: é o δ!	E: It's δ!
A: Porque tem esses valores	A: Because we have these values
E: e então esta distância é resultado	E: and then this distance is the result
A: do valor de δ!	A: of the δ-value!
E: Sim, escolhemos o δ.	E: Yes, we choose δ.
A: Hum, hum	A: Hum, hum
E: De tal maneira que, acho que talvez é isto, só que, porque tem aparecido habitualmente no, nos livros, são esses livros eu acho que neste caso teremos que ver... porque realmente o que tem aparecido nos livros é "para qualquer ε "	E: In such a way that, maybe it's right but, because what is usually in, in books, I think that we should see these books … because in fact what can be found in books is "for any ε "

A: Ah, sim, nós estávamos a, a escolher para o δ os valores da tangente!	A: Ah, yes, we were choosing for δ the values of the tangent!
E: Então [inaudível] escolhe-se o δ.	E: Then [inaudible] we choose δ.
A: o δ	A: δ
E: Então, euh, entretanto é esta a questão que queria, queria apresentar, o problema do, dos ε e dos δ (S3/198-230).	E: Then, er, this is the question that I wanted, that I wanted to ask, the problem of, of ε and δ-values.

This quote gives evidence that these two teachers were not sure about the definition. Ernesto knew that, according to textbooks, we should choose ε (arbitrarily small). From Abel's explanation, it seems that we choose δ, whereas the definition that he wrote began with $\forall\varepsilon > 0$. In fact, Abel confirmed that he first chose δ.

Then Ernesto tried to explain better his difficulties, speaking about sequences, neighbourhood, one-sided limits, but in a rather confusing way (S3/244-263). The following discussion was also confusing, several teachers speaking at the same time (S3/267-272). I tried to re-focus: what do we choose?

I: Agora eu pergunto: "afinal o que é que a gente escolhe? É o ε ou é o δ? Ou seja, é um intervalo, é uma vizinhança de *f*(*a*) ou uma vizinhança de *a*?" Porque acho que não está claro! Necessita explicações!	I: Then now I ask the question: "what do we choose? ε or δ? I mean, is it an interval, a neighbourhood of *f*(*a*) or a neighbourhood of *a*?" Because I think that it is not clear! We need to clarify!
E: Bom, euh, aquilo que	E: Well, er, what
I: Nas suas explicações, Abel disse que escolhia o δ e que a partir do δ escolhido	I: In his explanations, Abel said that he chose δ and from the chosen δ
E: o ε	E: ε
I: ia descobrir o ε. A definição diz o contrário, como fez notar o Ernesto, não é! Então? (S3/274-84)	I: he would find ε. The definition says the opposite, as pointed out by Ernesto, isn't it? So?

Abel said that he chose δ but, according to the written definition, we should choose ε! (S3/286-89). He then changed his definition on the blackboard:

$$\forall\delta > 0 \quad \exists\varepsilon > 0 : |x-1| < \delta \Rightarrow |f(x)-3| < \varepsilon$$

Ernesto referred to handbooks stating that δ depends on ε.

E: talvez o problema seja dado, de, de algumas literaturas, porque se for para o caso de, de literaturas, de autores russos	E: maybe the problem comes from, from some books, because in the case of, of books, by Russian authors
A: Hum	A: Hum
E: Eles até tentam especificar na própria definição, que o δ é definido por ε	E: They try to specify in the definition itself that δ is defined by ε

A: Sim

A: Yes

E: e neste caso, eu acho que escolheu é o ε, e o δ é que é resultante da escolha, da escolha do ε. Só que, euh, o que tenho reparado já em, em autores por exemplo portugueses e brasileiros, talvez seja a tentativa de facilitar um bocado a compreensão da própria definição, até em alguns casos já não são leccionados (S3/315-327).

E: and in that case, I think that you chose ε, and that δ is the result of that choice. But, er, what I noticed in, in authors for example Portuguese and Brazilian authors, maybe trying to facilitate the understanding of the definition, in some cases they don't teach it.

Ernesto, as an experienced teacher, is now suggesting a new question: is it worth to teach this definition in secondary schools? Nobody took up this new question.

After a silence, Abel explained that each Δx had its corresponding Δy (S3/331-37). I then asked which definition was correct: the first one or the second one? They laughed. Mateus stated that we have to choose ε.

I: Agora, qual é a definição correcta? É a primeira ou é a segunda? …. [riem-se] Qualquer que seja ε existe δ ou qualquer que seja δ existe ε?

I: Then, which one is the correct definition? The first one or the second one? … [They laugh] For any ε it exists δ, or for any δ it exists ε?

M: Eu, como disse o Ernesto, queria dizer, eu por exemplo tenho mais, bom a partir de, da 4ª até hoje, tenho o, a primeira definição! Qualquer ε, então escolhe-se o ε! Não, não é o δ que se escolhe! (S3/341-46)

M: As Ernesto said, I wanted to say that, in my case, well, since Grade 4 up to now, I have the first definition! Any ε, then we choose ε! No, we don't choose δ!

Mateus was the only one who seemed to be sure about the definition. I suggested an experiment with the function chosen by Abel (S3/348), but nobody took up this proposal. The discussion went on about what should be chosen, ε or δ, and what should be associated, ε with x or with y? Mateus insisted that we must choose ε, but associating ε with the variable x (S3/350-82; 406-16). David said that $x - 1$ should be less that δ, not ε (S3/386). Frederico also referred to the textbooks.

F: Não, eu estava a dizer que, a forma de utilização de, das letras não tem importância que, o que importa considerarmos o ε no eixo dos, do, das abcissas e o δ no eixo do, de, das ordenadas. Mas nos livros que eu vi, o ε sempre aparece no eixo das ordenadas e δ é que aparece no eixo das

F: No, I was saying that, the way we use the letters is not important, what is important is considering ε on the x-axis and δ on the y-axis. But in the books that I read, ε always appears on the y-axis and δ on the axis of

A: abcissas

A: the x-axis

F: abcissas (S3/459-66).

F: the x-axis

Frederico was also matching ε with x and δ with y. Since the teachers were not reaching any conclusion, I suggested a numerical task, using the function $f(x) = x + 1$ in a neighbourhood of $x = 1$ (S3/487-505). Abel, who was still at the blackboard, did not understand the task (S3/507-14). Mateus told him to choose some ε-value and suggested $\varepsilon = 0.1$, which Abel finally wrote on the blackboard (S3/516-50). I then suggest that we look for the corresponding δ-value (S3/554).

After many hesitations and suggestions by Ernesto and Mateus, Abel wrote on the blackboard $|x-1| < 0.1$ (S3/556-613). He then faced difficulties in solving the inequality $|x-1| < 0.1$. David suggested that he used the definition of an absolute value (S3/632), Mateus gave the result: between 0.9 and 1.1 (S3/634). Then Abel wrote on the blackboard (S3/634):

$$\begin{aligned}&\vee |x-1| < -0{,}1\\ &\vee\, x-1 < -0{,}1\\ &x > 0{,}9\end{aligned}$$

In Mozambican institutions, students usually solve this kind of inequalities by separating into two inequalities: $x-1 < 0.1 \wedge x-1 > -0.1$. This way of solving inequalities is confusing, because of the use of logic symbols, which can differ depending on the kind of inequality, as shown in the following examples: $|x-1| < 0.1 \Leftrightarrow x-1 < 0.1 \wedge x-1 > -0.1$, but $|x-1| > 0.1 \Leftrightarrow x-1 < -0.1 \vee x-1 > 0.1$. While the first inequality is solved using the conjunction symbol $\wedge$, the second is solved using the disjunction symbol $\vee$. Students who memorise the solution without understanding often mix-up the two solutions. It seems that Abel is also trying to solve the inequality using his memory rather than his understanding of it. For this reason he missed the first part of the solution, he wrongly used a disjunction symbol, and wrongly wrote the second part $|x-1| < -0.1$ instead of $x-1 > -0.1$. He then made an error in the solution of the inequality (he changed < for >) probably to obtain the result that Mateus indicated: $x > 0.9$. All these errors show a poor understanding of the solution of inequalities.

I indicated the errors and explained how to solve this kind of double inequality, without separating into two inequalities: $|x-1| < 0.1 \Leftrightarrow -0.1 < x-1 < 0.1 \Leftrightarrow 0.9 < x < 1.1$ (S3/648-698).

As the ε-δ definition had been wrongly applied, using the ε-value as it was δ, I decided to go the blackboard. I corrected the definition (S3/710-730), and asked for the limit of the function when *x* tends to 1 (S3/741). Mateus said that it was 2 (S3/743), and I concluded that we can only use the definition to show that the limit is really 2 (S3/745-47). The teachers worked individually or in pairs for some time, without reaching any conclusion (S3/747-760). As we had already spent a lot of time, I went to the blackboard and explained: how to use the definition to find the δ-value and how to interpret the result on a graph (S3/764-784).

2. Teachers' participation

Abel

At the beginning of the seminar, Abel took the position of an experienced teacher, willing to explain his colleagues the ε-δ definition. He correctly wrote the definition but, when challenged by his colleagues' questions, changed it according to his colleagues' suggestions, showing that he was not sure of this definition. He wrote three definitions:

$\forall \varepsilon > 0 \;\; \exists \delta > 0 : |1-a| < \delta \Rightarrow |f(x) - a| < \varepsilon$ [a instead of *f*(*a*) in the second module]

$\forall \varepsilon > 0 \;\; \exists \delta > 0 : |x-1| < \delta \Rightarrow |f(x) - 3| < \varepsilon$ [correct definition]

$\forall \delta > 0 \;\; \exists \varepsilon > 0 : |x-1| < \delta \Rightarrow |f(x) - 3| < \varepsilon$ [changing the roles of ε and δ]

This shows that his knowledge of the definition was weak. He probably memorized it but without a profound understanding, as he will confirm during the 2nd and 3rd interviews.

He was also unable to give a clear graphical interpretation of this limit.

During the discussion on whether we have to choose ε or δ (arbitrarily small), he wrongly stated that we choose δ.

Abel also faced difficulties when working in the numerical register: he was not able to choose a ε-value, and was not able to solve the inequality $|x-1| < 0.1$. When solving this inequality he made several errors:

- He wrote $|x-1| < -0.1$, which does not make sense in that case.
- He used incorrectly the symbol ∨.
- He changed the symbol of the inequality $\begin{array}{l} \vee\, x - 1 < -0{,}1 \\ x > 0{,}9 \end{array}$.

It seems that Abel was trying to solve the inequality using his memory instead of his understanding of the solution. As a matter of fact, he told me during the 2nd interview that he was embarrassed about the errors that he had made during the seminar. However he had exposed himself by going to the blackboard to explain the definition and solve the tasks.

As a conclusion, we can say that Abel memorized the definition but did not understand it. Furthermore, he showed a weak knowledge of basic mathematical concepts.

Mateus

Mateus participated quite a lot in the discussion during the third seminar. At the beginning of the seminar, he asked for the meaning of the symbols ε and δ, showing that he was not afraid of showing his difficulties. During the discussion on whether we have to choose ε or δ, he stated that we have to choose ε, showing that he was sure of this knowledge. He immediately understood the task when I suggested a shift to the numerical register, and gave Abel orientations to solve the task on the blackboard. He then gave Abel instructions to solve the inequality $|x-1| < 0.1$. He was also the first one to answer when I asked for the limit of the function when x tends to one.

To sum up, at the beginning of the seminar Mateus knew the ε–δ definition, but was not able to interpret it. He was aware of his difficulties and freely presented them to his colleagues. He also showed that he had a good general mathematical knowledge.

Ernesto

Like Mateus, Ernesto participated very actively during the seminar. He detected the errors in Abel's first definition, and pointed out the main difficulty of the definition: the inversion of the steps (δ, related to the independent variable, depends on ε, related to the dependent variable y). He also pointed out the contradiction between the definition as stated in books and Abel's explanation. As a consequence he asked the question which provoked the main discussion: do we choose ε or δ?

All these interventions show that Ernesto knew the ε-δ definition. He probably memorised it because he had to teach it, but did not understand it. However he was able to reflect on it and acknowledged that Abel's explanation did not correspond to the written definition.

David

David did not participate much in the discussion during the seminar. His interventions were the following:

- When discussing whether to choose ε or δ, he told Abel that x - a should be less than δ (S3/299-301).
- He repeated this information later on (S3/386), showing that he was sure of this knowledge.

- He tried to help Abel to solve the inequality, telling him to use the definition of an absolute value (S3/632).
- He agreed when I told Abel that the sign of the inequality should not change (S3/662) and told him that “or” was wrong.

Só que poderia ser $x - 1$ módulo menor que 0,1. Não tem que existir o “ou” (S3/668).	But it could be module of $x - 1$ less than 0.1. “or” should not exist.

To sum up, David’s interventions were always right. We can surmise that, as the youngest of the group and confronted with experienced teachers such as Abel, Ernesto and Frederico, he did not feel comfortable in giving his own ideas.

Frederico

The only intervention by Frederico was during the discussion about the choice of ε or δ. He stated:

Não, eu estava a dizer que, a forma de utilização de, das letras não tem importância que, o que importa considerarmos o ε no eixo dos, do, das abcissas e o δ no eixo do, de, das ordenadas. Mas nos livros que eu vi, o ε sempre aparece no eixo das ordenadas e δ é que aparece no eixo das abcissas (S3/459-66)	No, I was saying that the way the, the letters are used is not important, what is important is to consider ε on the x-axis and δ on the, the, the y-axis. But in the book I read, ε always appears on the y-axis and it is δ that appears on the x-axis

This quote shows that Frederico did not have a good knowledge of the definition.

Overview

From the discussion during the seminar, the following conclusions about the teachers’ knowledge of the definition can be drawn:

- All five teachers memorised the definition during their studies, without understanding it.
- Two of the teachers, Abel and Ernesto, taught this definition at school. As a consequence they had to reflect on it.
- Abel was convinced that he understood the definition, and willing to explain it to his colleagues. However his knowledge was not stable and he made several errors, both when writing down the definition and in his explanations.
- Ernesto had a good knowledge of the terms of the definition, but not a conceptual understanding of it; he was able to point out the difficulties that he faced in interpreting it.

- Like Ernesto, Mateus knew the definition but was aware that he did not understand it. He was willing to try to understand and asked the question about the meaning of ε and δ.
- As he stated during the first interview, David was aware that he did not understand the definition. We can surmise that this is why he did not speak much during the seminar. Nevertheless, all his interventions were right.
- Frederico did not participate much because he did not have a good knowledge of the definition.
- Furthermore, the discussion showed some teachers' difficulties in working with other basic mathematical concepts, such as inequalities and absolute values.

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