STRUCTURE SENSE:

WORKING WITH BRACKETS IN ALGEBRAIC EXPRESSIONS

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9304121N

A RESEARCH REPORT SUBMITTED TO THE FACULTY OF EDUCATION, UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG, IN FULFILMENT OF THE REQUIREMENTS OF THE DEGREE OF MASTER OF EDUCATION

in the

The Faculty of Education

at the

University of Witwatersrand

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JOHANNESBURG 2022

DECLARATION

I, Nadia Mahomed Theba, declare that this research report is my own unaided work and all references used have been appropriately cited. It is being submitted for the degree of Master of Education at the University of the Witwatersrand, Johannesburg. It has not been submitted before, for any degree or examination, at any other University. The ethics clearance number received from the school of ethics for this research project is Protocol Number 2020ECE177M.

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9 November 2022

ABSTRACT

The interpretation of brackets in algebraic expressions depends on a learner's understanding of the role that brackets play in different types of structures. Working from a Vygotskian perspective, this study investigated how learners interpret brackets as a tool and a sign, by investigating and analysing the surface and systemic structural errors in typical/atypical and simple/complex structures. Fifty-eight Grade 10 learners from under-resourced schools in Johannesburg wrote a test designed for Grade 9 learners. This study focused on their responses to five test items. While it was expected that Grade 10 learners would demonstrate competency and structure sense in dealing with Grade 9-level expressions, the error analysis revealed that this was not the case. Findings suggest that: 1) more brackets in typical expressions encourage surface structure sense; 2) better surface structure sense is displayed in atypical expressions with brackets that include fewer elements; 3) surface structure sense is displayed on typical and atypical structures if brackets indicate multiplication, rather than subtraction; 4) systemic structure sense is evident on simple structures when brackets signify one operation, such as multiplication or subtraction; 5) systemic structure sense appears to be affected by the positioning of elements in simple structures; and 6) systemic structure sense is less evident in complex structures which involve more than one operation of multiplication. The study concludes that the participating Grade 10 learners are not yet able to cope with algebraic structures at Grade 9 level.

ACKNOWLEDGEMENTS

I am deeply indebted to my supervisor, Professor Craig Pournara for his continuous support and guidance throughout my studies. I am grateful for his constant encouragement and patience during challenging stages in the study.

To my daughter, Amber, who traversed the challenging journey of this research report, I am grateful for the strength she exhibited and shared when I needed it most.

To my mother Farida, who assisted in my daily routines which allowed me time to spend on completing this research report.

A very special thank you to my dear friend and fellow student, Sheldon, for our long discussions, in helping me develop the ideas involved in the research report. You were always just a phone call away and were always passionate about our conversations.

I am grateful to Geeta for her advice on academic aspects in the development of this research report.

To Asma, who helped me with the technical aspects of computer skills to create tables and figures for my research report.

Finally, to my dear friend Neville, thank you for encouraging me to persevere through the demanding phases in the study.

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LIST OF ABBREVIATIONS/ACRONYMS

CAPS – Curriculum Assessment Policy Statement

PG1 – Performance Group 1

PG2 – Performance Group 2

PG3 – Performance Group 3

TIMSS – Trends in Mathematics and Science Study

WMCS – Wits Maths Connect School

Surface Structure Error Codes

BBUT - Binomial in Brackets not recognised as 2 Unlike Terms

BG2F - Brackets Not Recognised as a Grouping Tool that Separates 2 Factors

BG3F - Brackets Not Recognised as a Grouping Tool that Separates 3 Factors

BGS - Brackets Not Recognised as a Grouping Tool for Distributive Law and Separation Tool for the Operation of Subtraction

BSS - Brackets Not Recognised as a Separation Tool for Subtraction

Systemic Structure Error Codes

DLNC - Distributive Law Not Applied Correctly

SNC - Subtraction Not Applied Correctly

Algebraic Error Codes

BMSN - Brackets Treated as Multiplication Where it Should Not be Applied

BNMS - Brackets Not Treated as Multiplication Where it Should be Applied

IOO - Incorrect Order of Operations Using Multiplication

LC - Lack of Closure (inappropriate conjoining)

OATO - Operation on Adjacent Terms Only

P2L - Error in Product of Terms Consisting of 2 Letters

PC - Error in Product of a Monomial and a Constant

PLN - Error in Product of Terms Consisting of Letters and Numbers

S - Error with Sign

SS - No Operation of Subtraction where it Should be Applied

CHAPTER 1 – INTRODUCTION

1.1 Introduction

As a mathematics teacher of Grades 9 and 10 for two decades, I have frequently encountered learners making algebraic errors, particularly in expressions involving brackets, for example in $(x+2)^2$ and $4(x-5)$. In the past, my approach for addressing these errors involved re-emphasising appropriate algebraic procedures to encourage correct solutions from learners. For example, an appropriate algebraic procedure to simply $4(x-5)$ is to multiply 4 with both terms in brackets to reach the answer $4x-20$; but I had minimal long-term success. Success was often limited to the specific lesson when the re-emphasising occurred. I was not always able to identify the underlying factors that led to my learners' persistent errors in algebra. Furthermore, the prescriptive nature (time and content) of the Curriculum Assessment Policy Statement (CAPS) (Department of Basic Education, 2011) implemented by the Department of Basic Education (DBE), leaves little room for deviation from the curriculum. This means that my re-emphasising approach was simply putting a 'band-aid' on the problem without recognising the underlying factors that lead to these difficulties with algebraic structures.

Research has identified various reasons for the poor performance in mathematics by South African learners. These include factors such as learning deficits and insufficient learning opportunities within different quintile schools (Spaull & Kotze, 2015; Van den Berg & Gustafsson, 2019); language barriers (Adler, 2001; Howie, 2012; Schafer, 2010; Vorster, 2008) and challenges in the learning of mathematics such as algebra (Makonye & Khanyile, 2015; Pournara, Hodgen, Sanders, & Adler, 2016), which contribute to the low achievement levels of South African learners in mathematics. In comparison to other countries, South Africa is one of the lowest performing countries as evidenced by international studies on mathematics such as the Trends in Mathematics and Science Study (TIMSS) (Reddy & Arends, 2016) which is an international study that assesses the mathematics and science knowledge of learners in Grades 5 and 8, administered every four years.

My study is focused on the algebraic performance of a small random sample of Grade 10 learners from 11 disadvantaged schools in Johannesburg. I focus on the difficulties that are evident in learners' interpretation of brackets in algebraic expressions and how inappropriate and incorrect interpretations of brackets result in errors. I analyse the errors made by the learners in algebraic expressions by addressing the notion of structure sense.

1.2 Algebra and its Challenges

Algebra is a branch of mathematics that involves symbolising numerical relationships and using rules to manipulate symbols in structures (Kieran, 1992). The abstract nature of mathematics, especially algebra, renders mathematics challenging for many learners (Carraher, Schliemann, & Brizuela, 2000), due to various underlying factors. One underlying factor is the progression from arithmetic to algebra (Carraher, Schliemann, & Brizuela, 2000; Warren, 2003) where mathematical structures that are represented by numbers develop towards mathematical structures represented by letters (Warren, 2003). Another factor is the perception of structure and relationships within algebraic structures (Kieran, 1989). Relationships in structures are evident in the arrangement of elements such as variables, numbers, mathematical symbols and brackets. The manner in which elements are organised in structures, results in different relationships. For example, "the elements 6, 8 and 48 can be structured into $6 \times 8 = 48$ arrangement but not into $6 \times 48 = 8$" (Venkat, Askew, Watson, & Mason, 2019, p. 14). Arithmetic does not merely consist of the calculation of symbols and numbers by using specific operations, rather relations and meanings exist between these numbers and symbols which are important for the understanding of the abstract nature of algebra (Booth, 1988; Filloy & Rojano, 1989).

Simplification using structural operations on arithmetic expressions differs from algebraic expressions (Warren, 2003). When learners move from arithmetic to algebra, previous numerical relationships which are represented as symbolic relationships require different types of structural operations on symbols (Kieran, 1992). For example, in the arithmetic expression $9 - (3 - 1)$, the grouped numerical term in brackets $(3 - 1)$is the first order of operation. The result of this operation is then subtracted from the monomial 9 resulting in a purely numerical solution of 7. In an algebraic expression with a similar arrangement of elements and brackets, such as $9 - (a - b)$, illustrates the subtraction of grouped terms in brackets $(a - b)$ from the monomial 9. A structural operation of subtraction is used on each term in brackets resulting in an algebraic solution of $9 - a + b$.

Warren (2003) suggests that learners experience challenges with structures in algebra because of difficulties experienced with structural aspects in arithmetic. In arithmetic, four operations of addition, subtraction, multiplication and division are used by applying basic properties such as commutativity, associativity and distributivity (Kieran, 1992). For example, if learners experience challenges in using distribution in arithmetic, such as in $3(7 + 4) = 3(7) + 3(4)$, and if not addressed and rectified, it could lead to challenges with the structural operation of the distribution of multiplication over addition in algebra, such as in $2(a + b) = 2a + 2b$. Therefore, an understanding of arithmetic can assist in an understanding of algebra.

Learners have a perception that the "essence of algebra" (Kieran, 1992, p. 390) is about memorising different rules and processes occurring in algebraic structures (Kieran, 1992). However, difficulties in algebra may be the result of learners not recognising the relationship between elements in algebraic structures, in particular the arrangement of brackets. From a Vygotskian point of view, I argue that the role of brackets can be regarded as a 'tool and sign' (Vygotsky & Cole, 1978) which I discuss in Section 2.5. If brackets are not interpreted correctly, it could result in errors. For example, brackets in expressions can be regarded as a tool in $6 - (3 + x)$ which groups $(3 + x)$; and as a sign for subtraction of $(3 + x)$ from 6. In my teaching experience, the presence of brackets in expressions is frequently a mental trigger for learners to use multiplication, despite brackets signalling an operation other than multiplication. Such misinterpretations of brackets are evident in this study, for example in $3a - (b + a)$, learners saw brackets grouping $b + a$ and used multiplication inappropriately, to obtain $-3a^2b$. Although learners may be able to correctly use operations of multiplication and subtraction independently, it is being able to know which operation to use which usually results from the correct interpretation of brackets in algebraic structures.

1.3 Rationale for Undertaking this Study

The rationale for undertaking this study is to investigate Grade 10 learners' misinterpretations of brackets in algebraic structures such as algebraic expressions. Initially, in my research, I was drawn to identifying the different types of procedural errors made by learners. However, as I proceeded with the research, aspects of structure sense surfaced (from other studies), such as the ability to use appropriate operations depending on the arrangement of elements in an expression. These aspects identified possible deep-rooted misinterpretations that learners in this study may have on different types of algebraic expressions such as not recognising the relationship of brackets as a tool, and brackets as a signifier for operations.

This study aims to address structure sense of Grade 10 learners based on test performance by considering learners' interpretations of brackets in algebraic expressions and errors that occur from misinterpretations. I use error analyses to assess the structure sense of algebraic expressions with brackets, and not to identify the errors that are made. Researchers (Borasi, 1994; Makonye & Khanyile, 2015; Gardee, 2015) propound that an awareness of errors that can be made by learners, can be utilised to anticipate the kinds of errors that learners could make in a learning environment, which could assist to improve on teaching. The findings from this study provide some clarity on the structural errors made by learners.

1.4 Structure and Structure Sense in Algebraic Expressions

Structure and structure sense are key aspects in this study. In the late '80s, Kieran (1989, p34) defined the structure of an algebraic expression as being made up of its "surface structure and systemic structure". Surface structure refers to the arrangement of elements in an expression. Systemic structure refers to the properties of the operations which can be used to manipulate the expression into equivalent structures. I will elaborate on Kieran's definition of structure in Section 2.3. Due to different interpretations of the notion of structure in the last three decades, which I will discuss in Section 2.3, no single definition of structure has been formulated. Instead, the notion of structure sense has proved to be more useful for this study.

The ability to recognise the structure of expressions and choose appropriate types of manipulations to transform expressions into equivalent structures is considered as "structure sense" (Linchevski & Livneh, 1999, p. 191). Various interpretations of structure sense from key researchers are provided in Section 2.3 (Hoch & Dreyfus, 2007; Novotna & Hoch, 2008; Schüler-Meyer, 2017). In this study, I draw on the context of structure sense as used by Hoch and Dreyfus (2006), incorporated with Kieran's (1989) definition of structure. For example, structure sense is evident if a learner is able to recognise the surface structure of $2(x+6)$ as the arrangement of brackets as a grouping tool that distinguishes between two factors i.e. 2 and $(x+6)$, and is able to use the systemic structure of applying the distributive law correctly to obtain the equivalent structure of $2x+12$.

I also consider surface structure of expressions in terms of typical/atypical structures, and the systemic structure in terms of simple/complex structures. Typical structures are expressions that are more commonly encountered in textbooks and assessments, whereas atypical structures are not frequently encountered. Simple structures refer to one type of operation required in an expression to obtain equivalent structures, such as applying subtraction or multiplication only. Complex structures involve more than one type of operation required to obtain equivalent structures, such as using both subtraction and multiplication, or applying several uses of the same operation by using different orders of operations. The aspects of structure and structure sense in terms of typical/atypical and simple/complex structures will be elaborated on in Section 2.7. Analyses of errors involving classifications of structures as being typical/atypical and simple/complex will enable me to reflect on whether such classifications are useful. This will be elaborated on in Section 5.8.

In this study, I define structure sense as: *the ability to: 1) recognise the surface and systemic structures of typical and atypical algebraic expressions containing brackets and involving simple and complex operations; and 2) to use appropriate manipulations to obtain equivalent expressions*.

1.5 Selection of Test Items

The data for this study is drawn from learners' responses to a test designed by the Wits Maths Connect Secondary (WMCS) project in 2018. The test was originally intended for Grade 9 learners hence it deals with Grade 8 and 9 mathematics concepts. Further details will be given in Section 3.3.

These test items were not specifically designed for my study, therefore I was restricted in identifying items that best suited my focus on structure sense of algebraic expressions with brackets. The items that best suited my study are five algebraic expressions which learners were required to simplify:

Item 1: $(a+b)b$;

Item 2: $3a-(b+a)$;

Item 3: $2a(a-4)-8$;

Item 4: $(2x+1)(x+4)$ and

Item 5: $2(x+3)^2$.

These items primarily involve the structural operation of distribution, except in $3a - (b + a)$ which involves the structural operation of subtraction. This limitation in the range of expressions will be discussed in greater detail in Section 3.10.1.

1.6 Research Questions

The over-arching research question for this study is:

What errors do Grade 10 learners make when simplifying algebraic expressions with brackets?

Sub-questions

1. To what extent do learners demonstrate surface structure sense in the interpretation of typical and atypical structures of algebraic expressions with brackets?
2. To what extent do learners demonstrate systemic structure sense in the interpretation of simple and complex structures of algebraic expressions with brackets?
3. What algebraic errors result from misinterpretations of surface and systemic structures?

1.7 The Development of Error Codes

There are surprising similarities between the manner in which learners should shift from an operational understanding to a structural understanding of algebraic symbols, and my development of error codes in this study. I started the coding process using operational error codes but these slowly evolved into more sophisticated structural error codes. The discussion that follows explains the process that I undertook.

Initially, I coded operational errors that learners made in simplifying items. For example, the incorrect response to $(a + b)b$ as $a + b^2$, illustrates an operational error by not multiplying b with the first term *a* in brackets which I coded as *distribution with second term only*. The codes were, however, inadequate for this study because they did not specify misinterpretations of brackets. For example, the error in $a + b^2$, suggests the misinterpretation of brackets not being recognised as a signal for multiplication of both terms in brackets.

After numerous discussions with my supervisor, I realised that I was investigating the symptoms and not the underlying factors of misinterpretations of brackets. In the presentation of my research proposal, I indicated that exploring the concepts of *structure* and *structure sense* in the context of algebra and considering different interpretations of these aspects would be part of the journey of the study. I therefore considered researching structure sense in other algebraic topics (Gunnarsson, Sönnerhed & Hennel, 2016; (Hoch & Dreyfus, 2004; Herscovics & Linchevski, 1994; Liebenberg, Sasman, & Olivier, 1999; Gunnarsson, Sönnerhed, & Hernell, 2016) which encouraged me to view errors from a structural point of view. This consequently led to the development of structural error codes. In addition, I generated further codes for algebraic errors that resulted from structural misinterpretations, which supported my decision for using specific structural error codes.

To illustrate the development of surface and systemic structure error codes, I use a learner's incorrect response to item $3a - (b + a)$ (Figure 1.1), which should be correctly simplified as $3a - b - a$.

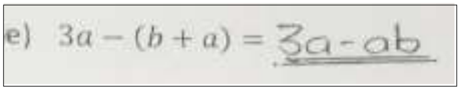


Figure 1.1: Learner response

I initially coded the error as *combining letters with letters*, which indicated the inappropriate conjoining of terms b and a in brackets as ab. However, the incorrect response suggests the misinterpretation of brackets not being recognised as a grouping tool for two unlike terms. Therefore, I recoded the error as 'binomial in brackets is not recognised as two unlike terms' which is more relevant to surface structure sense. This misinterpretation of brackets led to an algebraic error coded as *lack of closure*, which relates to the inappropriate conjoining of $(b + a)$.

1.8 Distribution, the Distributive Property, and the Distributive Law

In this study, the terms *distributive property, distributive law* and *distribution* are used when referring to the product of terms involving brackets. Furthermore, I do not use the terms/phrases *expanding* or *expansion of brackets* or *removing brackets* which are frequently used in many local and international textbooks (Stewart, Redlin, & Watson, 2006, p. 25); nor the colloquial expression *getting rid of brackets* which is commonly used in classrooms. Rather, I refer to brackets as being 'not required' or 'not evident' after manipulation has taken place on brackets in expressions. For example, in $(x - 2)(x + 2)$, the structural operation of distribution is applied to manipulate the structure as $x^2 - 4$, illustrating that brackets are not required in the expression after being manipulated.

1.9 Conclusion

In this chapter, I have identified some challenges that learners face with algebra that can lead to poor performance. I explicated my rationale for undertaking this study which I related to structure and structure sense of algebraic expressions. I provided my own definition of structure sense, after drawing on ideas of key researchers, which forms the core of this study. I introduced classifications of structures as being typical or atypical, and simple or complex. I identified the items selected for this study and provided the research questions.

Chapter 2 will provide the theoretical framework based on Vygotsky's sociocultural theory. The conceptual framework, based on the notion of structure sense, will be linked to brackets as a tool and sign from a Vygotskian point of view. In the literature review, I will report on the findings of selected research that is relevant to my study. I will provide my definition of structure sense which incorporates typical/atypical and simple/complex structures.

In Chapter 3, the research design and methodologies will be discussed, followed by a detailed explanation of the classification of the test items used in this study. The sampling of participants and selection of performance groups will be given. Data analysis procedures will be reviewed. I will also consider ethical aspects and will explain the limitations of my study.

In Chapter 4, I will explain how my definition of structure sense was used to code structural and algebraic errors, using a two-level coding process on incorrect responses. I will draw on examples of learners' incorrect responses to describe errors.

Chapter 5 will present an analysis of the data obtained for the study in two phases. The first phase will focus on the analyses of structural and algebraic errors in misinterpretations of brackets in incorrect responses. The second phase will focus on the analyses of errors in individual items in relation to my classification of typical/atypical and simple/complex structures.

In Chapter 6, conclusions will be drawn based on key findings in relation to my research questions. Limitations and reflections of the study will be provided.

CHAPTER 2 – THEORETICAL FRAMEWORK AND LITERATURE REVIEW

2.1 Introduction

In this chapter, I focus on three main aspects. Firstly, I explain my theoretical framework which is based on Vygotsky's sociocultural theory, with particular attention to the notions of tool and sign in a mathematics culture. Secondly, I elaborate on the notions of structure and structure sense as they appear in the literature, and I argue that structure sense is a more useful notion for this study. I relate brackets to Vygotskian tools and signs and link these to interpretations of structure and structure sense in algebraic expressions. Thirdly, I review literature which investigates learner errors as a result of misinterpretations of brackets in algebraic structures and how this relates to learners' structure sense. Drawing on these three aspects, I define structure sense as used in this study, showing how it includes aspects of brackets as a tool and a sign, in typical/atypical and simple/complex structures.

2.2 Sociocultural Theory in a Mathematics Culture

The theoretical framework in this study uses Vygotsky's sociocultural perspective which is based on the premise that human activity should be understood according to cultural settings. Vygotsky (1978) argued that human development begins in social and cultural settings by considering factors such as social interactions and language (Vygotsky & Cole, 1978). Furthermore, he argued that cultural tools and signs affect children's mental structures and processes.

Cultural settings include the culture of schools, comprising staff, learners and stakeholders which includes rules, values and attitudes of the school. Within the school culture, mathematics culture becomes evident through textbooks, teachers, and interactions between teachers and peers. Schüler-Meyer (2017) explained how algebraic culture is evident in classrooms through resources and the teacher, and it is accessed by learners through their personal interpretations of algebra.

Algebraic expressions are part of the algebraic culture that learners engage in by using their own perceptions. For example, when multiplying variables in algebra, learners access their previous interpretations of multiplication of numbers in arithmetic. Teachers assist learners to access such prior knowledge of multiplication through teacher-learner and learner-learner interactions. Furthermore, teaching resources can connect learners' prior knowledge with current work through examples in textbooks and activities on hand.

Mathematics has its own cultural language consisting of objects and symbols. An object can be regarded as a tool in mathematics, which has its own purposes (Pimm, 2002). A symbol can be regarded as a sign, which has its own meanings in different mathematical contexts (Pimm, 2002). For example, in the context of plotting points on a Cartesian plane, a set of co-ordinates $(x\,;y)$ are represented using brackets. As an object, brackets group the co-ordinate points, and as a symbol, brackets signify that *x* represents the independent variable, and *y* represents the dependent variable.

Tools and signs affect human behaviour by contributing to the learning and development of individuals within cultures (Vygotsky & Cole, 1978). A cultural tool is externally oriented which leads to changes in an object, and a cultural sign is internally driven and thus does not make a physical change on an object. For example, the 'equal to' symbol (=) in algebra is a mathematical tool used to show that two quantities such as in $3x + 6 = x - 2$ are equated. As a mathematical sign, the symbol (=) signifies solving the unknown value to make the equation true. In algebraic expressions, brackets can also be considered as tools and signs, which I discuss in Section 2.5. In addition to being able to interpret tools and signs, another key aspect is being able to interpret structures in algebra.

2.3 Views of Mathematical Structure in Literature

Researchers interpret the notion of structure differently in relation to algebraic expressions. To explain the concept of structure, I focus on four key interpretations by Hoch and Dreyfus (2004); Kieran (1989); Linchevski and Vinner (1990); as well as Novotna and Hoch (2008).

An algebraic structure relates to a system that consists of elements such as numbers, variables, brackets, and operations. Kieran (1989, p. 34) defined structure as being made up of its "surface structure" and "systemic structure". Surface structure refers to "the given form or arrangement of the terms and operations, subject – when arranged sequentially – to the constraints of the order of operations" (Kieran, 1989, p. 34). This refers to the arrangement of elements in the expression. For example, in $4(x + 3) - 2$, x is added to 3 in brackets, which is then multiplied by 4, the result of which is subtracted by 2. Systemic structure refers to the "properties of operations, such as commutativity and associativity, and to the relationships between the operations, such as distributivity” (Kieran, 1989, p. 34). This refers to the properties of the operations which can manipulate the expression into equivalent structures. For example, $4(x+3)-2$ can be manipulated into equivalent expressions of $4x+12-2$ and $4x+10$.

Linchevski and Vinner (1990, p. 86) spoke of “hidden structures” instead of surface and systemic structures for two reasons. Firstly, they argued that although algebraic expressions may look different such as in $4(x+3)-2$ and as in $4(y)-2$, certain elements are hidden by other elements, which can still be viewed as the same surface structure. In $4(x+3)-2$, the bracketed binomial $(x+3)$ is a compound object which is hidden as the single object y in $4y-2$. The surface structure in both expressions could then be regarded as 4 multiplied by an object, then subtract 2. Linchevski and Vinner (1990) argued that it would be easier to operate on a single object than a compound object, such as y as a single object in $4y-2$ instead of $(x+3)$ as a compound object in $4(x+3)-2$. Being able to see and maintain the structure of the single object in an expression instead of the hidden compound object could be useful to perform algebraic operations. Secondly, they argued that Kieran’s definition of systemic structure was too broad because it indicated that an infinite number of equivalent expressions could be expressed. For example, $4(x+3)-2$ can be manipulated as $4x+12-2$ or $4x+10$. However, in a given context, only one preferred expression is required such as $4x+10$, showing complete simplification, which is a requirement in Grade 9. They argued that hidden algebraic structures encourage the preferred equivalent forms required in the context (Linchevski & Vinner, 1990).

Hoch and Dreyfus (2004) shared a similar view to the original definition of structure offered by Kieran. They view structure as an entity consisting of components and the connections that exist between these components. This is related to Kieran’s definition of algebraic structures by considering its “external appearance” instead of the surface structure and “internal order” instead of the systemic structure. An external appearance refers to what can be seen in a given expression. An internal order is determined by the relationships between the components in a structure. The external appearance can be transformed into an internal order. For example, the external appearance of the expression $x(x-3)-15$ conceals the typical structure of a trinomial, which when transformed as $x^2-3x-15$ reveals its internal order. Hoch and Dreyfus (2004) extended their definition of structure, by claiming that two expressions may have a different arrangement of elements, but that the expressions can be equivalent. They claimed that equivalent expressions are different interpretations of the same structure. For example, if the value of $x=2$ is substituted into $x(x-3)-15$ and $x^2-3x-15$ then both expressions equal to -17. Therefore, these expressions are equivalent and are different interpretations of the structure.

Novotna and Hoch (2008, p. 95) used the notion of hidden structure by placing emphasis on the “substitution principle” which means that if a compound term replaces a single term, then the structure remains the same. Using factorisation of expressions in their study, a structure such as $81 - x^2$ has the same structure in $(x-3)^4 - (x+3)^4$ because they both illustrate the difference of two squares – the first in simplest form and the second in a compound form.

The idea of structure is difficult to define, therefore structure sense, is more productive to use in this study. I use structure according to Kieran’s original definition as surface and systemic structure, which is included in my definition of structure sense (Section 2.8).

2.4 Structure Sense

The conceptual framework in this study is built on the premise of “structure sense”, a phrase coined by Linchevski and Livneh (1999, p. 191). As discussed in Chapter 1, structure sense is being able to recognise the structure of expressions and to use appropriate operations to transform expressions into equivalent structures. Linchevski and Livneh (1999) claimed that lack of structure sense is one of the reasons contributing to not understanding algebraic structures. However, they were criticised by other researchers. For example, Hoch (2003) and Novotna and Hoch (2008) argued that the study by Linchevski and Livneh investigated the order of operations used on arithmetical expressions by learners who just began learning about algebra and not on algebraic expressions. Furthermore, they argued that Linchevski and Livneh did not provide any further information on what structure sense meant for algebraic structures which is usually learned after arithmetic expressions. Hoch (2003), at that time, believed that there was no agreement or consensus on the definition of structure, and therefore there was no clear definition of structure sense. The phrase ‘structure sense’, however, provided new opportunities for researchers to investigate and understand how algebraic structures were perceived by learners.

Structure sense was then defined as a learner’s ability to recognise algebraic structures and, in the context of the given structure, to use appropriate previously learned operations (Hoch, 2003). This was illustrated by learners’ attempts to solve $(x^2 - 4x)^2 - x^2 + 4x = 6$, where learner responses suggested that brackets must be simplified first in order to solve the equation. Hoch (2003) stated that structure sense was not evident because learners did not recognise the possibility of operating on $(x^2 - 4x)^2$ as a single entity by using substitution of the single entity, replacing the compound term in brackets, which would make the equation simpler to operate on.

A later development of the definition of structure sense was being able to use a combination of previously learnt practices and techniques appropriately by considering the features of a structure in order to manipulate expressions (Hoch & Dreyfus, 2004). Therefore, a feature of structure sense is to know which interpretation of a structure is required. Hoch and Dreyfus (2004) claimed that equivalent expressions are different interpretations of the same structure and knowing which interpretation to use is structure sense. This involves "looking before doing" (Hoch & Dreyfus, 2004, p. 54). They argued that the inclusion of brackets in an expression positively affected a learner's structure sense. For example, if brackets were included in a structure, learners were more likely to correctly interpret and use appropriate operations than in a structure that did not include brackets. Furthermore, they claimed that structure sense involved being able to interpret and manipulate a structure in a given context, which was dependent on experience (Hoch & Dreyfus, 2004). For example, it will be obvious to a learner who has structure sense to simplify an expression by collecting the like terms $3x$ and $3x$ in $2x^2 - 3x - 3x + 9$ as $2x^2 - 6x + 9$.

A more detailed definition of structure sense was developed by Hoch and Dreyfus (2006) in three parts. Firstly, they stated that structure sense is the ability to recognise a familiar structure in a simple form such as the difference of two squares in $x^2 - y^2$. Secondly, structure sense is being able to recognise complex forms as familiar structures such as the difference of two squares in $(x+2)^2 - (x-3)^2$. Thirdly, structure sense is the ability to select suitable manipulations of structures such as factorising the difference of two squares of

$$(x+2)^2 - (x-3)^2 \text{ as } \big((x+2)+(x-3)\big)\big((x+2)-(x-3)\big).$$

Novotna and Hoch (2008) subsequently adapted this definition of structure in their study of high school and university algebra to determine if learners were able to use structure sense. In their study they emphasised that a significant feature of structure sense is the "substitution principle" (Novotna & Hoch, 2008, p. 95) where learners were able to use compound terms as single objects so that structures became more familiar. They suggested that being able to manipulate expressions is a result of number and symbol sense, by having insight into picking up incorrect answers with numbers and being able to identify appropriate operations in arithmetic.

The next section relates to the Vygotskian concept of signs and tools linked to algebraic expressions in a mathematical culture, followed by the role of brackets in typical and atypical structures.

2.5 Brackets as a Tool and a Sign

In this study, tools and signs apply to the role of brackets in algebraic expressions which affect the interpretation of the structure. As a tool, brackets change the physical arrangement of the expression; and as a sign, brackets signify the appropriate operation to be used in a particular context to obtain equivalent structures. For example, in $-3(x+6)$, brackets are a grouping tool for x and 6, and brackets signify multiplication of the monomial -3 by $x+6$. If the brackets are repositioned in the expression by using the same elements such as in $(-3x)+6$, then brackets are a tool by grouping -3 and x. Brackets then signify the addition of a constant (6) with a negative monomial $-3x$.

The connection between tools and signs may encourage higher mental functioning which is referred to as internalisation (Vygotsky & Cole, 1978). Internalisation indicates the development resulting from converting external processes to internal processes. In algebraic structures, if brackets are interpreted as performing a dual role of a tool and sign, it may enable learners to interpret the structure of an expression, thereby encouraging structure sense. For example, structure sense is evident if brackets are firstly interpreted as a tool in the surface structure of $4(x+2)$ that separates two factors 4 and $(x+2)$. Secondly, if brackets are interpreted as a sign in the systemic structure for the multiplication of both terms of the binomial (x and 2), by the monomial (4).

If learners do not interpret the role of brackets in expressions, this could lead to errors and slips (Olivier, 1986). Errors are regular and systematic incorrect applications (Gardee, 2015) influenced by the educational process, teachers, peers, and the individuals themselves (Olivier, 1986). Slips occur due to careless application by both beginners and experts (Gardee, 2015). In this study, the focus is on errors in learner responses, although some responses suggest slips. For example, in $2a(a-4)-8$, a learner's response of $2a^2-8a$ shows the correct interpretation of the brackets in multiplying the monomial $2a$ by the binomial $a-4$, but the constant -8 is not included in the final answer. I regard this as a slip because the elements in the expression that need to be manipulated, are correctly interpreted by the learner. However, the constant which does not require any manipulation, is not included in the answer.

Errors may be the result of valid but incorrect efforts to adapt prior knowledge to new concepts (Matz, 1980). Furthermore, errors occur because of misinterpretations of symbols which may have more than one meaning in different contexts. This is especially evident in mathematics because the same symbol can have diverse meanings (Pimm, 2002), such as in algebraic expressions where brackets can have different meanings. For example, the structure of multiplication of two numerical binomials $(5-8)(4+2)$, shows brackets as a tool that groups two sets of numbers, and as a sign suggesting using the correct order of operations, i.e., subtraction of $(5-8)$, addition with $(4+2)$ then multiplication of $(-3)(6)$. Previous knowledge on numerical binomials cannot be applied on new concepts. For example, in the multiplication of two binomials with variables $(x-8)(x+2)$, brackets are a grouping tool for two factors, and as a sign for the multiplication of two binomials. Olivier (1986) stressed the importance of understanding why learners make mistakes, emphasising that appropriate action can be taken to address difficulties when there is cognisance of this.

2.6 Structure Sense - Learners' Interpretations of Brackets

Studies on structure sense were undertaken by researchers Hoch and Dreyfus (2004); Gunnarsson, et al., (2016); Hoch and Dreyfus, (2005); as well as Schüler-Meyer, (2017), where learners' interpretations of brackets in arithmetic and algebraic structures were examined. In this section, I discuss literature that illustrates the interpretation of brackets through the operational use of brackets indicating the order of operations; inclusion of mental brackets; how compound terms within brackets are operated on; and the application of the distributive law. Furthermore, I discuss research evidence that suggests that more brackets in an expression supports attention to structure sense (Hoch & Dreyfus, 2004), and the polar viewpoint that the inclusion of brackets in an expression does not encourage structure sense (Gunnarsson, et al., 2016). I also discuss the findings of Papadopoulos and Gunnarsson (2018) who investigated learners' use of mental brackets in expressions which affects the given structure of expressions; and Hoch and Dreyfus (2005) who investigated the challenges in recognising familiar structures in a compound form. Finally, I review a study that suggests possible ways to develop learners' structure sense of expressions by focusing on the distributive law in unfamiliar expressions by Schüler-Meyer (2017).

Hoch and Dreyfus (2004) investigated the effect of brackets on structure sense of expressions of Grade 11 learners. For example, in the equation $\frac{1}{4}-\frac{x}{x-1}-x=7+\frac{1}{4}-\frac{x}{x-1}$ which does not include brackets, the lowest percentage of learners at 6,3% demonstrated structure sense. In the equation $\frac{1}{4}-\frac{x}{x-1}-x=5+\left(\frac{1}{4}-\frac{x}{x-1}\right)$ consisting of one bracket, a higher percentage of learners demonstrated structure sense in finding the solution. However, it was the equation $\left(\frac{1}{4}-\frac{x}{x-1}\right)-x=6+\left(\frac{1}{4}-\frac{x}{x-1}\right)$ featuring two pairs of two brackets that saw the highest percentage of learners who illustrated structure sense. Their findings revealed that when brackets were present in algebraic equations, it presented a better opportunity for learners to focus their attention towards structure sense by identifying 'like terms' and assisted in breaking up a long sequence of symbols.

This is in contrast to a study conducted by Gunnarsson, et. al. (2016) who investigated learners' adoption of the rules for the order of operations in expressions with brackets and expressions without brackets. Expressions consisted of multiplication with either addition or subtraction in the form of $a \pm b \times c$, for example $5+3.2$ which consists of no brackets and $5+(3.2)$ which consists of an emphasising bracket. The findings showed that in expressions with no brackets such as in $5+3\,.\,2$, two of the most frequent types of answers included sequential left to right operations and rules of the order of operations. However, in expressions that included emphasising brackets such as in $5+(3.2)$, the researchers found that brackets in expressions did not improve on learners' performance in the learning of order of operations. Expressions that included brackets showed a greater incidence of errors compared to expressions with no brackets, the latter in which learners were more likely to use the correct order of operations. The findings therefore suggest that the presence of brackets in an expression did not always provide an opportunity for learners to focus their attention towards structure sense, because the order of operations was not always correctly used.

Papadopoulos and Gunnarsson (2018) investigated how Grade 6 learners used mental brackets in arithmetic expressions, by identifying if a relationship existed between the arrangement of an expression and the way that learners perceived it. For example, the arrangement of $\frac{12}{4}+2$ can be perceived as $(12 \div 4)+2$ which illustrates the presence of brackets as a grouping tool to maintain the initial structure of the expression. Similarly, the arrangement of $\frac{12+2.3}{3}$ can be perceived as $(12+2 \times 3) \div 3$ which illustrates the structural role of the brackets for the expressions but also the order of operations within the brackets. Their findings showed that when these expressions were expressed horizontally by learners, the use of mental brackets was not in accordance with the initial structure of the expressions. For example, in the initial expression $\frac{20}{\frac{4}{2}}$ the horizontal response of a learner was "$20{:}\,4{:}\,2 = 20{:}\,2 = 10$". The researchers claimed that although the correct answer of 10 was obtained, the process followed by the learner was incorrect. In another example, $\frac{12}{4+2} = 4 + 2 = 6\ \ 12/6 = 2$, illustrated a separate calculation for the denominator, but a correct answer was obtained. Learners were able to see the mental brackets as a grouping tool for the numerator and denominator. However, by not showing brackets as a grouping tool, the structure of the expression changed.

Hoch and Dreyfus (2005) investigated the application of algebraic techniques of factoring in unfamiliar contexts. When given the expression $x^4 - y^4$, most learners used the correct factoring technique. In $(x-3)^4 - (x+3)^4$ which represents the same structure of the difference of two squares, only minimal responses showed correct factoring techniques. They concluded that learners did not see the relationship between the expression and the structure. Therefore, learners did not see the structure of a standard difference of two squares of $x^2 - y^2$ involving single entities in an unfamiliar structure such as $(x-3)^4 - (x+3)^4$ which involves compound entities. The researchers questioned if structure sense could be taught, and if so, what process it would follow.

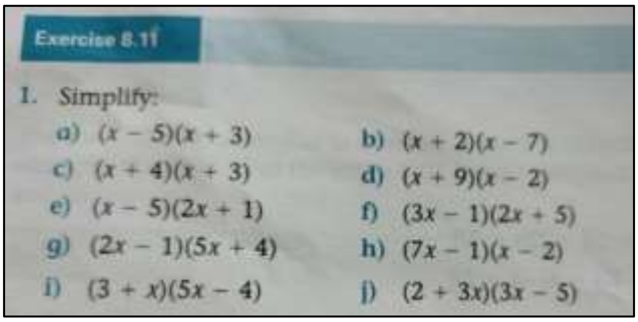
Teaching for the development of structure sense was evident in a study by Schüler-Meyer (2017) which was based on a project which assisted learners to develop their structure sense for the distributive law in unfamiliar expressions. Learners were taught to transform expressions consisting of addition, subtraction and compound terms through "guiding examples and structural mappings" (Schuler-Meyer, 2017, p. 17). Guiding examples refer to using simple expressions that have the same structures as the complex expressions, as a guide to develop structure sense. For example, to use the distributive law in the unfamiliar structure of $(b+c)(-2)$, the expression $2a(b+c)$ was used as a guiding example. This assisted learners in seeing that the structure of the two expressions was the same, but the order of the monomial and binomial was different. Structural mapping is being able to use appropriate operations on manipulated expressions to determine if the initial expression will be obtained. For example, in the factoring of $5ab + 5ac + 5cd$, learners used the distributive law as structural mappings on the factorised expression $5a(b + c + d)$ to determine if the original expression would be obtained. Therefore, the study suggested that learner's structure sense in applying the distributive law in expressions could be developed by using appropriate methods.

One of my recommendations in this study (Section 6.8.2) includes this approach by Schüler-Meyer, in the hope that some evidence is provided that could help teachers to assist learners to develop structure sense.

2.7 Typical/Atypical and Simple/Complex Structures

I define the terms typical and atypical and simple and complex structures in the context of this study. Defining these terms is important to understand the coding of errors involving interpretations of brackets (Section 4.2) and for the analysis of errors (Sections 5.4 and 5.6).

Typical and *atypical structures* refer to *surface structure*. *Surface structure* refers to the arrangement of elements such as letters, numbers and brackets in an algebraic expression. T*ypical* structures, therefore, refer to the arrangement of elements in expressions that are frequently encountered in Grade 9 textbooks and in accordance with the Grade 9 CAPS document (Department of Basic Education, 2011). *Typical* structures include multiplication of two factors such as $(x-5)(x+3)$ (Figure 2.1); the addition and subtraction of grouped terms in brackets such as $-3x-(x-2)$ (Figure 2.2); and a monomial multiplied by a squared bracket such as $2(x-1)^2$ (Figure 2.3).

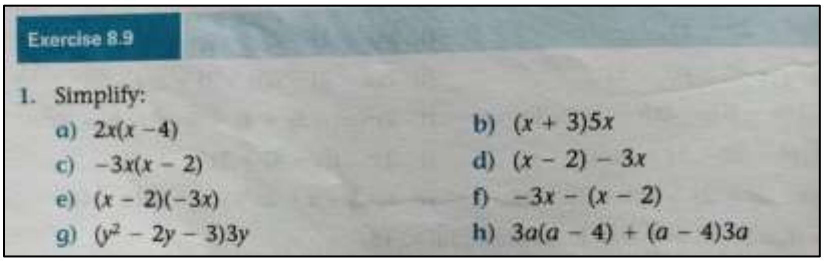


Exercise 8.11

1. Simplify:

a) $(x-5)(x+3)$ b) $(x+2)(x-7)$
c) $(x+4)(x+3)$ d) $(x+9)(x-2)$
e) $(x-5)(2x+1)$ f) $(3x-1)(2x+5)$
g) $(2x-1)(5x+4)$ h) $(7x-1)(x-2)$
i) $(3+x)(5x-4)$ j) $(2+3x)(3x-5)$

Figure 2.1: Extract of learner exercise (Rhodes-Houghton, et al., 2013, p. 112)



Exercise 8.9

1. Simplify:

a) $2x(x-4)$ b) $(x+3)5x$
c) $-3x(x-2)$ d) $(x-2)-3x$
e) $(x-2)(-3x)$ f) $-3x-(x-2)$
g) $(y^2-2y-3)3y$ h) $3a(a-4)+(a-4)3a$

Figure 2.2: Extract of learner exercise (Rhodes-Houghton, et al., 2013, p. 110)

Exercise 8.18

1. Simplify:

a) $2(x-1)^2$	b) $3(x+2)(x-1)$	c) $-(x+3)^2$
d) $-5(x-2)(x+2)$	e) $7(x+10)(x-5)$	f) $-(x-5)(x+5)$
g) $-2(x+8)(x-4)$	h) $-3(x-2)^2$	i) $4(x-1)(x+7)$
j) $2(x-9)(x+9)$	k) $2(2x-1)^2$	l $3(2x+3)^2$
m) $-(2x-5)^2$	n) $-2(2x+3)(2x-1)$	o) $4(2x+y)(2x+y)$

Figure 2.3: Extract of learner exercise (Rhodes-Houghton, et al., 2013, p. 118)

Atypical structures refer to an arrangement of elements in expressions that are less familiar or recognisable in expressions, and which are not common in Grade 9 textbooks and assessments. For example, in $(2x-3)x$, a monomial (x) that is positioned after the bracketed binomial $(2x-3)$, is not commonly encountered in Grade 9. This is evident in the learning requirements of Grade 9 (Department of Basic Education, 2011) (Figure 2.4), where priority is given to structures with monomials positioned before grouped binomials.

2.3 **Algebraic expressions**	- Determine the squares, cubes, square roots and cube roots of single algebraic terms or like algebraic terms - Determine the numerical value of algebraic expressions by substitution • Extend the above algebraic manipulations to include:	**Examples of expanding and simplifying expressions** a) Simplify: $-3(x^3+2x^2-x)-x^2(3x+1)$ [multiply integer and monomial by polynomial] b) Determine/expand: $(x+2)(x-3)$ [multiply binomial by binomial] c) Determine/expand: $(x+2)(x-2)$ [multiply binomial by binomial] d) Determine/expand: $(x+3)^2$ [multiply out a perfect square] e) Simplify: $2(x-3)^2-3(x+1)(2x-5)$ [multiple calculations involving product of binomials]

Figure 2.4: Extract of examples of expanding and simplifying algebraic expressions from CAPS document (Department of Basic Education, 2011, p. 131)

Another atypical structure is the multiplication of a grouped *binomial* by a *trinomial* such as $(5a+2)(25a^2-10a+4)$. This type of structure is typical in Grade 10 prescribed books (Figure 2.5). In Grade 9, however, typical structures rather include multiplication of a *monomial* by a *trinomial* such as $-3(x^3+2x^2-x)$ (Figure 2.4).

h) $(5p-6pq+2q^2)(2p^2+3q^2)$

i) $(5a+2)(25a^2-10a+4)$

j) $(4p^2+14p+49)(2p-7)$

Figure 2.5: Extract of learner exercise (Pike, et al., 2011, p. 12)

Simple and *complex* structures refer to *systemic structure*. *Systemic structure* involves the properties of operations and the relationship between the operations in expressions. *Simple* structures, therefore, refer to expressions that require one type of operation such as subtraction of a grouped binomial from a monomial in $-3x - (x - 2)$ (Figure 2.2) or multiplication of a monomial and binomial in $2x(x - 4)$ (Figure 2.2). *Complex* structures refer to expressions that require more than one type of operation, such as a combination of multiplication and subtraction in $a - (a + 5)(a - 5)$. In $2(x - 9)(x + 9)$ (Figure 2.3), a specific order of multiplication is required, which involves being able to recognise the relationships between the grouped binomials in brackets and prioritising multiplication of the two binomials to obtain $2(x^2 - 9)$ before distribution with the monomial is applied.

2.8 Conclusion

In this chapter, I discussed the notions of structure and structure sense and how these relate to the interpretation of brackets in algebraic expressions and equations. Drawing on Vygotsky's sociocultural theory, I considered the interpretation of brackets in algebraic structures as making sense of brackets as tools and signs; and the errors that can result from misinterpretations of brackets. I incorporated aspects of Kieran's (1989) definition of structure and Hoch and Dreyfus' (2006) definition of structure sense to develop a definition of structure sense that is relevant to this study. As indicated in Chapter 1, I define structure sense as: *the ability 1) to recognise the surface and systemic structures of typical and atypical algebraic expressions containing brackets and involving simple and complex operations; and 2) to use appropriate manipulations to obtain equivalent expressions*. My definition of structure sense in relation to the role of brackets in algebraic structures, is illustrated in Figure 2.6

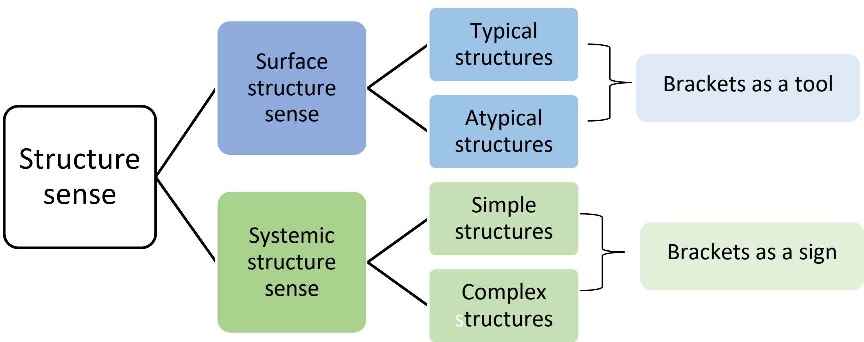


Figure 2.6: My definition of structure sense

A detailed discussion was provided on classifying typical/atypical structures and simple/complex structures which will inform the development of codes in Chapter 4 and the data analysis in Chapter 5.

CHAPTER 3 – RESEARCH DESIGN AND METHODOLOGY

3.1 Introduction

This chapter focuses on the design and methodology that guided this research. I discuss the qualitative approach that I used. I provide a background to the testing instrument that was used in the WMCS Project which is important to understand the rationale for my selection of the five items. I then classify the five items as typical or atypical and simple or complex structures. Random sampling of the participants is explained, followed by the rationale for dividing the sample group of 58 learners into three performance groups. I explain the modus operandi for my analysis as well as the processes involved in preparing the data used for the analysis, by discussing the difference in the coding of items used by WMCS and the coding used in this study. Credibility, transferability, confirmability and dependability are discussed to demonstrate the trustworthiness of the study. In the denouement of the chapter, I discuss the ethical considerations and limitations of the study.

3.2 Research Design and Methodology

A research design is the framework of using different research methods and techniques that are appropriate to analyse data, resulting in credible findings to the research questions (McMillan & Schumacher, 2010). This study employed a qualitative approach and obtained quantified results. Qualitative research involves obtaining data in its natural setting (McMillan & Schumacher, 2010), which in this case refers to acquiring data from actual learner responses to test items. The findings, which are largely dependent on codes and coding, are represented quantifiably in the form of numbers and percentages. I then analysed and coded errors in learner responses, culminating in the identification of trends which pointed to little or no structure sense of algebraic expressions.

This study makes use of secondary data which refers to data collected by another party and not by the researcher (McMillan & Schumacher, 2010). The data was drawn from the WMCS' project archives and I had no involvement in the data collection. The data consists of 58 Grade 10 learners' responses to five test items. The tests were administered during the third quarter of 2018. By that time of the year, it is an expectation that Grade 10 learners would demonstrate a good grasp of algebraic content, particularly since the test content focused on Grade 8 and 9 algebraic concepts. However, this was not the case as will be shown through the analysis.

3.3 Background of the Testing Instrument

My study is located in a larger research and development project conducted by WMCS to investigate the extent to which the participation of teachers in the WMCS' professional development (PD) programme impacted on their Grade 9 learners' performance in mathematics. Grade 10 learners, who were taught by the participating teachers, were also tested to obtain a large enough sample of teachers in the PD course. The items in the test covered a limited content of mathematics which included operations with integers, algebraic expressions, exponents, functions, equations, and numeric patterns. The testing instrument was not adjusted to include items specifically for Grade 10, rather the same Grade 9 testing instrument was used for the Grade 10 learners. The WMCS study used a pre and post-test in a quasi-experimental design. I analysed a selection of items from the post-test which was administered in the third quarter of 2018 to the Grade 10 learners. For the purpose of this study, I refer to the post-test as the test or testing instrument.

3.4 Classification of Test Items

I selected five algebraic expressions from the test to be used as part of this study which differ in the arrangement of elements and the types of operations required. The five test items were classified as being either *typical* or *atypical* and *simple* or *complex* structures based on Grade 9 skills as per the CAPS document (DBE, 2011) (discussed in Section 2.7). I classified typical/atypical structures based on the role of brackets as a tool in the surface structure. I classified simple/complex structures based on the role of brackets as a sign in the systemic structure. It was important to have items with different classifications as it provided an opportunity to analyse errors in items with different arrangements of elements and on different levels of complexity.

The five items that were selected are:

Item 1: $(a+b)b$;

Item 2: $3a-(b+a)$;

Item 3: $2a(a-4)-8$;

Item 4: $(2x + 1)(x + 4)$; and

Item 5: $2(x + 3)^2$.

The classification of the five items as being either *typical/atypical* and *simple/complex* structures is represented in Table 3.1.

Table 3.1: Classification of items

	Typical	Atypical
Simple	$(2x + 1)(x + 4)$ $3a - (b + a)$	$(a + b)b$ $2a(a - 4) - 8$
Complex	$2(x + 3)^2$	

3.4.1 Typical and simple structures

The items $(2x + 1)(x + 4)$ and $3a - (b + a)$ are typical and simple structures. The surface structure of $(2x + 1)(x + 4)$ shows the arrangement of brackets as a grouping tool of two factors. In $3a - (b + a)$ brackets are a grouping tool for $(b + a)$ and a separation tool for the subtraction of the two terms $(b + a)$ from $3a$. These typical structures represent types of expressions routinely encountered in Grade 9 prescribed school textbooks as indicated in Figure 2.1 and Figure 2.2.

The systemic structures are simple because only one type of operation is required in each item. This is evident in $(2x + 1)(x + 4)$, requiring the systemic process of multiplication on each term in both brackets. Similarly, $3a - (b + a)$ requires the same operation of subtraction of $3a$ with both terms in the brackets.

3.4.2 Atypical and simple structures

The items $(a + b)b$ and $2a(a - 4) - 8$ are atypical and simple structures. The surface structures are atypical because the elements in the items are arranged in an unfamiliar arrangement (discussed in Section 2.7). In $(a + b)b$, the positioning of the monomial b is positioned after the grouped binomial $(a + b)$, in contrast to a more typical structure such as $b(a + b)$ where the monomial is placed before the bracket. Item $2a(a - 4) - 8$ indicates a typical structure of $2a(a - 4)$ involving the monomial $2a$ before the grouped binomial $(a - 4)$. However, it is the constant -8 which is positioned after the grouped binomial that makes the item atypical.

The items are considered as simple structures because brackets signify one type of operation, involving the multiplication of a monomial and a binomial. Item $(a + b)b$ requires the monomial b to be multiplied with both terms in brackets. In $2a(a - 4) - 8$, multiplication is required of monomial $2a$ by the binomial. In $2a(a - 4) - 8$, although the constant (-8) is situated after the bracket, no further operation is required on the constant after distribution is applied. This suggests that an expression such as $2a(a - 4)$, which can be considered a simple structure involves the same process of multiplication in $2a(a - 4) - 8$. However, if the expression was changed to $2a(a - 4)8$, it would involve two instances of multiplication, making it a complex structure.

3.4.3 Typical and complex structures

The item $2(x + 3)^2$ is a typical and complex structure. The surface structure is typical because the arrangement of the monomial 2 before the squared binomial $(x + 3)^2$ is evident in expressions which are encountered in Grade 9 as indicated in Figure 2.4.

An expression such as $2(x + 3)$ may be a simple structure involving multiplication of two factors. However, the item in this study $2(x + 3)^2$ illustrates the squaring of a binomial in brackets, arranged after the monomial which implies three factors $2(x + 3)(x + 3)$. Working from left to right in this item may result in errors. Therefore, it requires understanding how the structure works with squaring of a binomial $(x + 3)^2$, and prioritising exponentiation over multiplication and addition, before the distribution of the monomial 2 with the terms in brackets, is applied.

3.4.4 Atypical and complex structures

There are no items from the testing instrument involving brackets that can be classified as atypical and complex.

In summary, the grouping of items has resulted in an unequal number of expressions for each type of classification. In the design of the WMCS test, the test developers focused on content from Grades 8 and 9, and intentionally avoided items that I would have classified as atypical and complex. Therefore, in the testing instrument there are more typical items than atypical items. The three typical items $(2x + 1)(x + 4)$, $3a - (b + a)$ and $2(x + 3)^2$ are more likely to appear in prescribed textbooks and assessments compared to the two atypical items $(a + b)b$ and $2a(a - 4) - 8$.

The imbalance in item types can be considered a limitation of this study, however this was a fait accompli as the test was administered by WCMS in 2018 and I was not involved in the design of the items. Future research on this topic should pay careful attention to item design in relation to the classification used in this study, by ensuring the same number of items for each set of classification, which I elaborate on in Section 6.11. This increases the potential for a more complete and balanced analysis of learners' structure sense with regard to interpretation of brackets in different algebraic expressions.

3.5 Sampling

This study involves Grade 10 learners from 11 secondary schools in Gauteng. The learners in the participating schools hail from indigent and working-class communities. In order to garner an understanding of learners' structure sense of algebraic expressions, I decided to obtain a smaller sample of learners, which would allow me to analyse the interpretation of brackets in algebraic expressions in greater detail. Using random sampling, I obtained an initial sample of 155 learners who met the criteria of providing answers to the five selected items of the test, albeit correct or incorrect answers. Random sampling is an unbiased sampling method where participants are randomly selected from a population, allowing each member of the population to have an equal chance of being selected (Clark & Creswell, 2008; McMillan & Schumacher, 2010). This method was appropriate because it allowed for a range of learners, from the 11 different schools, to be selected.

The advantage of random sampling is that the selected group had the greatest potential to represent the population as a whole, thereby enabling me to make claims beyond the sample. However, during the error analysis process, a proliferation of unusual and varying types of responses were evident (Figure 3.1) and (Figure 3.2) making it difficult to identify common trends and errors.

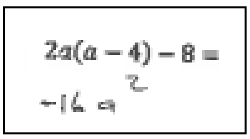


Figure 3.1: Learner response

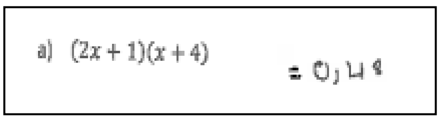


Figure 3.2:Learner response

Such responses were most evident in scripts of learners scoring under 40% for the overall test. Therefore, I decided to focus on learners' scripts scoring above 40%. This reduced the range of errors and enabled me to focus on responses that could be coded, rather than grapple with responses where I could not fathom learners' reasoning based on their written responses.

3.6 Performance Groups

Learners who scored above 40% were divided into three performance groups: Performance Group 1 (PG1), Performance Group 2 (PG2) and Performance Group 3 (PG3), indicated in Table 3.2.

Table 3.2: Performance groups

Performance group	Mark range	Number of learners	Number of errors
1	70% – 100%	25	21
2	50% - 69%	17	31
3	40% - 49%	16	48

An explanation of how these groups were constituted and my rationale for the varying group sizes follows. First, I identified the 25 highest scoring scripts amongst the existing 155 which had already been randomly selected based on the criteria of answering the selected five items in the test. I chose 25 because this number can represent a class in a school and it is sufficiently large to see possible trends in the data. The lowest scoring script of 71% in PG1, produced a mark range of 70-100% yielding a total of 21 errors for the five selected items.

Thereafter, I identified the scripts with scores ranging from 50% to 69%, which comprised a total of 17 learners. Although the number of learners in PG2 was smaller than PG1, there were 31 incorrect responses in PG2, which was higher than the 21 incorrect responses in PG1. Therefore, I decided not to include learners with scores lower than 50% in this performance group.

Finally, scripts with scores ranging from 40% to 49% were identified, constituting a total of 16 learners with 47 incorrect responses. This decision was suitable because the number of learners in PG3 (16) is similar to PG2 (17). Furthermore, 48 incorrect responses in PG3 are similar to the combined number of incorrect responses of PG1 and PG2 with 52 incorrect responses, despite the mark range of PG3 being only 9%. Additional details regarding the number of incorrect responses for each item of each performance group is provided in Appendix A.

The mark range of PG1, PG2 and PG3 consisting of 70%-100%, 50%-69% and 40%-49% respectively, is not an issue of concern in this study, rather the number of learners in each group is important. Twenty-five learners in PG1 is what I intended. The numbers of learners in PG2 and PG3 was an advantage because the numbers of 17 and 16 learners in each respective group is relatively close to 25. In the selection process, the number of scripts from any school ranged between 1 and 20, implying a spread of scripts across every school (instead of one or two schools dominating the selection) with a median number of six learners. I used the median because 20 is an outlier in this distribution.

3.7 Preparation for Data Analysis

Working with the responses by using the actual learner scripts was time-consuming. Therefore, I decided to record all the incorrect responses onto an MS Excel spreadsheet for easier access and coding errors during the analysis. First, I considered the spreadsheet that was developed by the WMCS project members that coded learner responses. A code is used to attach meaning to a certain portion of data which is assigned by the researcher for the purposes of a particular study (McMillan & Schumacher, 2010). Correct final answers were coded by WMCS project members as "1", denoted by the colour green, and incorrect final answers were coded as "2", denoted by the colour red (see Table 3.3).

Table 3.3: Classification of correct and incorrect responses by WMCS project members

Schoo	Class	Lr coc	5c	5e	5g	7a	7b
T02	1001	032	1	2	1	1	2
T02	1001	003	2	2	2	2	2

Using the data from this spreadsheet, I was able to locate the desired learner scripts (photocopied), and transcribed all responses onto a spreadsheet, as indicated in Table 3.4. This spreadsheet was used to identify and code errors as indicated in Section 4.2.

Table 3.4: Responses from learners' test scripts

Schoo	Class coc	Lr coc	5c	(a+b)b	5e	3a-(b+a)	5g	2a(a-4)-8	7a	$(2x+1)(x+4)$	7b	$2(x+3)^2$
T02	1001	032	1	$ab+b^2$	2	$-3ab-3a^2$	1	$2a^2$- 8a - 8	1	$2x^2+8x+x+4$=$2x^2+9x+4$	2	$2(x+3)(x+3)=2x^2+12x+8$
T02	1001	003	2	$ab+b^2=a+b^3$	2	$3a-b-a$	2	$2a^2$-8a-8 = - 6a - 8	2	$2x^2$+8x+x+4 $2x^2$=9x+4 x=$\frac{4}{7}$	2	

To determine the accuracy of these entered responses, I rechecked the transcribed responses. During the transfer of responses onto my spreadsheet, I became aware of assigned codes to learner responses that were classified as incorrect according to the original WMCS coding, but would be correct for the purpose of my study. The initial coding of the WMCS project did not serve the same purpose in my study. For example, in Figure 3.3:

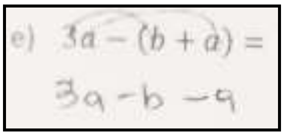


Figure 3.3: Learner response coded as incorrect by WMCS project members

a learner's response $3a - b - a$ shows appropriate operations to obtain an equivalent structure of $3a - (b + a)$. The response was coded as incorrect by the WMCS project members (see Table 3.4) because the learner did not simplify the item completely as $2a - b$. In my study, because I stopped coding after brackets were no longer evident in the item, the learner's response illustrates the correct interpretation of brackets and was not coded as an error.

In other instances, items were coded as incorrect by WMCS project members even though correct responses were provided which could be attributed to human error. Both such instances compelled me to recheck the original coding of all responses which yielded under 5% errors in coding. This was discussed with my supervisor who advised that corrected and appropriate codes be assigned to errors.

Errors were coded according to my definition of structure sense provided in Section 2.8 by examining learners' interpretation of brackets in algebraic expressions in relation to surface and systemic structure sense. I assigned error codes to incorrect responses in a particular manner, which is detailed in Chapter 4.

3.8 Trustworthiness

To establish the strength and relevance of my claims, I considered the trustworthiness of my study in relation to four aspects, viz, credibility, dependability, confirmability and transferability in a qualitative sense.

Credibility refers to the notion of internal consistency and thoroughness of an investigation (Morrow, 2005). In order to achieve internal consistency, it is necessary to provide an explanation of theoretical constructs and the relevance that such theory has on the data in a study (Lincoln & Guba, 1985; Gasson, 2004). Credibility is evident in this study because I used numerous studies involving structure sense to compare my methods of analyses of learners' responses. For example, the method that I used to investigate the structure of sense on expressions with different number of brackets, followed a similar method that Hoch and Dreyfus (2004) used to investigate the effect of brackets on structure sense of expressions in their study. A constant comparison to methods and findings in other studies is important because it assists in identifying any biases or misinterpretations of the researcher (Gasson, 2005).

Transferability is the extent to which a researcher can generalise the findings of a study in a particular context to other similar contexts (Lincoln & Guba, 1985). For example, the data used in this study suggests that learners experienced difficulty in recognising compound structures as single entities (as discussed in Section 5.4.1) which bears a similar finding to a study by Hoch and Dreyfus (2004). Transferability also means that the methods used in this study can be applied to future studies in other contexts (Lincoln & Guba, 1985). I provided a detailed account of the sampling and data analysis which can be replicated by future researchers in other studies. For example, using the specific methods of analysing and coding of the incorrect interpretation of brackets in expressions in this study, can be replicated in a future study involving the interpretation of brackets in algebraic equations such as $(x + 3)^2 = 0$.

Confirmability refers to obtaining findings in a study that addresses what is being researched, and not bringing in any beliefs of the researcher (Lincoln & Guba, 1985; Morrow, 2005). This implies that the researcher must be non-biased throughout the investigation and analyses, but to rather show integrity in all procedures. A researcher can constantly question him/herself during each process to ensure that a non-biased approach is maintained. I exercised confirmability in this study by using an appropriate sampling method to obtain suitable participants in each performance group, and I followed a systematic approach in the coding of errors. This systematic approach was strictly followed by coding structural errors as the first level of coding, followed by the second level of coding of algebraic errors, as discussed in Section 4.5. This procedure ensured an objective analysis of the types of learner errors made. These procedures ensured that I arrived at particular findings in the most objective manner.

Dependability involves using consistent analysis techniques to ensure obtaining authentic results (Gasson, 2004). This implies that procedures used in the analysis must be repeatable and transparent. In this study, I identified errors in learner responses by repeating the same procedure of identifying the type of structural error, followed by identifying the type of algebraic error. Furthermore, I was systematic in not coding any response that did not include a bracket, because responses without brackets was beyond the scope of this study. In the discussion of the types of structural and algebraic errors in incorrect responses, I was transparent and explicit in the reasons for the code allocation by providing examples to justify my analysis.

A study becomes more trustworthy if the four aspects of credibility, dependability, confirmability and transferability are met which I ensured was used throughout the analysis.

3.9 Ethical Considerations

Research ethics refers to being respectful to and minimising harm to others. The data for this project was obtained by WMCS in 2018, and ethical clearance from the University of the Witwatersrand (Protocol number: H17/0101) and the Gauteng Department of Education (Protocol Number: M2017/400AA) was obtained for the project. Consent for the data that was used by WMCS was obtained from both the school and the participants' parents. In this study, although I did not have any direct dealings with human subjects, I maintained the same obligations of being sensitive and ensuring privacy towards the participants and data obtained from the testing instrument. I obtained the ethics clearance from the University of Witwatersrand with Protocol Number 2020ECE177M before commencing the study.

To ensure that the privacy of research participants is protected, three aspects are necessary: "anonymity, confidentiality and the suitable storing of data" (McMillan & Schumacher, 2010, p. 121). I conducted the research by taking into account these ethical considerations. The aspects of anonymity and confidentiality, which were dealt with by the WMCS project during the clearance application, are evident in this study. Anonymity was ensured as neither the names of participants nor schools are mentioned. For example, I assigned allocated learner codes to learners' scripts (Appendix A), which I used in the analysis of incorrect responses. I ensured confidentiality by only using the allocated learner codes in the study, ensuring the privacy of information and participants. Suitable and secure storage of data was necessary after I obtained copies of test scripts of the participating learners. To ensure the safety and confidentiality of these scripts, I stored and locked the scripts safely at my place of residence.

3.10 Limitations for the Analysis of Data

The two significant limitations of this study are the type of structures of the test items and absence of interviews with participants.

3.10.1 Type of structures

The test items which were part of the WMCS project are not necessarily the types of structures that I would have chosen for my study, as it lacked diversity of algebraic structures and choice of elements within these structures.

Lack of types of structures. The pre-designed testing instrument consisted mainly of expressions that involved the structural operation of distribution and subtraction. The limited type of operations prevented me from having access to a wider range of expressions that included brackets in algebraic structures that might have been more suitable for my study. Therefore, I was constrained in making a selection from only those items that were part of the testing instrument resulting in five suitable expressions. For example, the item $\boldsymbol{2(x+3)^2}$ involves recognising the surface structure of three factors and applying the distributive property in the correct order of operations. There were no other items in the test involving the squaring of a binomial. For example, had an expression such as $\boldsymbol{(x-4)^2}$ been included, I would have been able to investigate if the expression was interpreted as a structure involving two factors and/or if the systemic structural operations were used correctly. If additional types of structures were included in the test, an opportunity to compare two or more similar structures could have provided for a more accurate analysis of how brackets were interpreted.

Furthermore, the number of items in this study, according to my classification of typical/atypical and simple/complex as indicated in Table 3.1, was unbalanced. There were two items each in typical/simple and atypical/simple cells, one item is typical/complex but no item is atypical/complex. Therefore, I was unable to analyse how such atypical/complex structures would be interpreted by learners.

Choice of letters and numbers. The elements of structures involving letters and numbers limited my analysis of structure sense of expressions. For example, in the learner response $\boldsymbol{(a+b)b = ab + b^2}$, I was uncertain if the terms in the brackets were multiplied with each other to obtain *ab*, or if *a* was multiplied by *b* positioned after the brackets. In the testing instrument, if the letter *b* after the bracket was changed to another letter, for example *c*, then $\boldsymbol{(a+b)c}$ would promote a clearer analysis of learners' interpretations of the structure. Similarly, in the response $\boldsymbol{(2x+1)(x+4) = 3x + 8x + x + 4}$, I was uncertain if the third and fourth terms $\boldsymbol{x \text{ and } +4}$ were obtained by correctly distributing *1* with $\boldsymbol{(x+4)}$, or if the terms in the bracketed binomial $\boldsymbol{(x+4)}$ were not operated on and rewritten in the response without the brackets. In this case, had the *1* in the first bracket been changed to another number such as 5, it would have been less challenging to analyse the interpretation of the brackets.

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3.10.2 Lack of interviews with participants

Lack of interviews with participants during the WMCS project is a limitation to this study. The presence of the participating learners was not possible as the research was undertaken in 2018, therefore no interviews could be conducted to discuss learners' responses to the test items. Having access to such interviews, could have assisted in the analysis of structures, by obtaining information about how and why errors were made by learners in different structures. For example, in $(a+b)b = ab + 2b$, the method used to obtain the incorrect response could have been explained by learners about whether $(a+b)$ was multiplied by each other to obtain ab or if a was multiplied by *b* positioned after the bracket. This study focuses on the errors that are evident in the responses on a cognitive level, which are interpreted based on my own experience and knowledge of the subject, as well as on other studies conducted in this field.

3.11 Conclusion

It was important to divide the sample group into three performance groups, which was essential in my analysis of structural errors in expressions. The sampling process and performance groups are indicated in Figure 3.4

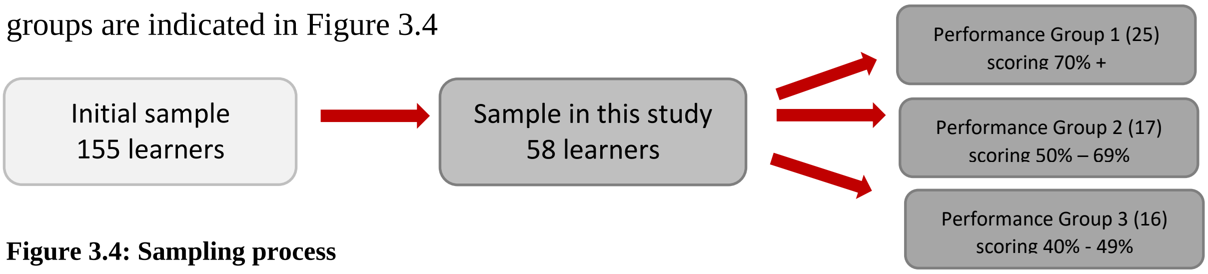


Figure 3.4: Sampling process

The coding of errors involved the interpretation of brackets as a tool and sign in typical/atypical and simple/complex items, by considering surface and systemic structure sense of learners. I was systematic in the coding of errors by using a two-level coding process which is explained in detail in Section 4.5.

CHAPTER 4 – ERROR CODES AND CODING

4.1 Introduction

This chapter serves as a precursor for the data analysis in Chapter 5. I commence by explaining how my definition of structure sense was used in the coding of errors. I provide tables consisting of surface structure, systemic structure and algebraic error codes that I used in the analysis of errors. A brief description of each type of error is given, and errors are explained by applying them to learners' incorrect responses. Examples of algebraic errors are not given because such errors fall within the ambit of structural errors which is discussed in Section 5.3.3. Thereafter, I describe the two-level coding process that I used to code errors resulting from learners' incorrect interpretations. In the first level of coding, incorrect responses are coded as either a surface or systemic structure error. Surface structure errors relate to the incorrect interpretation of brackets as a tool, and systemic structure errors refers to the incorrect or inappropriate operations used between brackets and other elements in expressions. The second level involves allocating algebraic error codes that support surface and systemic structure errors.

4.2 Coding Errors

During the preparation for the data analysis, I developed a spreadsheet of all learner responses that I could utilise for easier access to responses (indicated in Section 3.7 and in Table 3.4). Using these transcribed responses, I identified errors from incorrect responses in accordance with my definition of structure sense as *the ability 1) to recognise the surface and systemic structures of typical and atypical algebraic expressions containing of brackets involving simple and complex operations, and 2) to use appropriate manipulations to obtain equivalent expressions*. Error codes were developed following this process of identification.

In order to illustrate the coding process and the need for codes, I begin with a typical example containing a learner's response to the item $3a - (b + a)$.

$3a - (b + a)$	line 1
$= 3a - ba$	line 2
$= 3b$	line 3

The learner's response in line 2 suggests that he/she did not recognise the binomial in brackets as two unlike terms and conjoined the terms to obtain ba. Line 3 shows further errors. Seemingly the learner focused on "a subtract a" (which eliminates a) and then conjoined 3 and b without paying attention to the subtraction. So there are at least two errors in lines 2 and 3. Line 2 does not include brackets, meaning that the incorrect response that follows in line 3 does not involve the interpretation of brackets. Therefore, I stopped coding if errors did not relate to the interpretation of brackets, which in this case coding of errors ceased at line 2. Hence, the possibility exists that learners may actually have made more errors in their responses than what has been reported in this study. I report only on errors that relate to brackets.

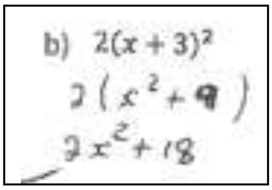
4.3 Surface Structure Errors

Surface structure errors refer to the misinterpretation of brackets as a tool in typical/atypical structures. As discussed in Section 2.7, typical structures involve the role of brackets in familiar expressions such as in Grade 9 textbooks and assessments, and atypical structures involve the role of brackets in expressions that may be part of Grade 9 learning skills but which are not regularly assessed. Table 4.1 lists and explains the codes that I used to identify and describe the surface structure errors, and the pertinent items.

Table 4.1: Surface structure error codes and applicable items

Error code	Description of error code	The role of brackets as a tool	Applicable to items				
			$(\boldsymbol{a}+\boldsymbol{b})\boldsymbol{b}$	$\mathbf{3}\boldsymbol{a}-(\boldsymbol{b}+\boldsymbol{a})$	$\mathbf{2}\boldsymbol{a}(\boldsymbol{a}-\mathbf{4})-\mathbf{8}$	$(\mathbf{2}\boldsymbol{x}+\mathbf{1})(\boldsymbol{x}+\mathbf{4})$	$\mathbf{2}(\boldsymbol{x}+\mathbf{3})^{2}$
BG3F	**B**rackets not recognised as a **G**rouping tool that separates **3** **F**actors	Brackets separate 3 factors, monomial (2) and two binomials $(x+3)$ and $(x+3)$					✓
BSS	**B**rackets not recognised as a **S**eparation tool for **S**ubtraction	Brackets separate the subtraction of $(b+a)$ from $3a$; and 8 from $2a(a-4)$		✓	✓		
BGS	**B**rackets not recognised as a **G**rouping tool for distributive law and **S**eparation tool for the operation of subtraction	Brackets separate 2 factors-monomial (2) and binomial $(a-4)$; and brackets separate the subtraction of 8 from $2a(a-4)$			✓		
BBUT	**B**inomial in **B**rackets not recognised as 2 **U**nlike **T**erms	Binomial $(b+a)$ not recognised as unlike terms in brackets		✓			
BG2F	**B**rackets not recognised as a **G**rouping tool that separates **2** **F**actors	Brackets separate 2 factors – monomial b and binomial $(a+b)$; and two binomials $(2x+1)$ and $(x+4)$	✓			✓	

To illustrate the coding process for surface structure errors, I use an example containing a learner's response to $2(x+3)^2$ as indicated in Figure 4.1, and which is also included in Table 4.2.



line 1

line 2

line 3

Figure 4.1: Learner response

The arrangement of the brackets in $2(x+3)^2$ is a grouping tool that separates three factors i.e. 2; $(x+3)$ and $(x+3)$. The learner's response in line 2 indicates that each term (x and 3) in the brackets has been inappropriately and independently squared as x^2+9. This error implies that the surface structure of brackets was not recognised as a grouping tool of two binomials $(x+3)(x+3)$, in which multiplication is required on both terms in both brackets. This structural error was coded as *brackets not recognised as a grouping tool for 3 factors*. The resulting algebraic error was *not treating brackets as multiplication where it should be applied*. As a consequence of the error in line 2, an incorrect response in line 3 was anticipated, albeit a correct interpretation of the role of brackets as a tool according to the learner's answer in line 2. Line 3 therefore has not been coded further. Table 4.2 provides examples of learners' incorrect responses that illustrate errors in surface structure sense.

Table 4.2: Examples of incorrect responses involving errors in surface structure

Error code	Applicable items	Incorrect response	Indicators	Implies
BG3F	$2(x+3)^2$	$2(x^2+9)$ $=2x^2+18$	Independently squaring terms in brackets as x^2+9	Lack of recognition that squared brackets indicates two binomials $(x+3)(x+3)$
BSS	$3a-(b+a)$	$-3ab+3a^2$	Multiplication of monomial $3a$ by binomial $(b+a)$	Lack of recognition of brackets that separates the subtraction of a compound term $(b+a)$ from a single term $3a$
	$2a(a-4)-8$	$2a^2-8a$	Omitting the constant -8	Lack of recognition of brackets that separate the subtraction of a single term -8 from a compound term $2a(a-4)$
BGS	$2a(a-4)-8$	$(2a^2-8a)-8$ (line 2) $=16a^2-64a$ (line 3)	After correctly applying distribution, brackets are retained in line 2 $(2a^2-8a)$ when brackets are no longer required. Further distribution of multiplication over subtraction is evident in line 3 as $16a^2-64a$	Lack of recognition of brackets that separates the subtraction of a constant -8 from a compound term $2a(a-4)$ which is then multiplied to obtain $2a^2-8a$
BBUT	$3a-(b+a)$	$3a-ba$	Inappropriately conjoining b and $+a$ as ba	Lack of recognition of two unlike terms grouped in brackets
BG2F	$(a+b)b$	$(a+b)$	No operation performed on monomial b	Lack of recognition of multiplication of monomial b by binomial $(a+b)$
	$(2x+1)(x+4)$	$3x+4$	Incorrect operations applied to both terms of binomials	Lack of recognition of multiplication of both terms of both binomial $(2x+1)$ and $(x+4)$

4.3 Systemic Structure Errors

Systemic structure errors refer to the misinterpretation of brackets as a sign, by using incorrect or inappropriate operations on simple/complex expressions. Simple structures involve one type of operation between elements; and complex structures require multiple types of operations between elements in expressions (Section 2.7). Of the five items, $3a - (b + a)$ is the only item that involves the primary operation of subtraction. In $2a(a - 4) - 8$, although the surface structure shows the subtraction of a compound term and a constant, the systemic structure does not involve an operation of subtraction. Therefore, *subtraction not used correctly* does not apply to $2a(a - 4) - 8$. The remaining items involve the distributive law as the main operation. Table 4.3 lists and explains the codes allocated to systemic structure errors and the relevant items.

Table 4.3: Systemic structure error codes and applicable items

Error code	**Description of error code**	**The role of brackets as a sign**	Applicable to items				
			$(a+b)b$	$3a-(b+a)$	$2a(a-4)-8$	$(2x+1)(x+4)$	$2(x+3)^2$
DLNC	**D**istributive **L**aw **N**ot applied **C**orrectly	Brackets signify multiplication of $(b + a)$ by b; $2a$ by $(a - 4)$; $(2x + 1)$ by $(x + 4)$; and $(x + 3)$ by $(x + 3)$ as the first operation followed by multiplication by 2	✓		✓	✓	✓
SNC	**S**ubtraction **N**ot applied **C**orrectly	Brackets signify subtraction of $(b + a)$ from $3a$		✓			

To illustrate the coding process for systemic structure errors, I use an example containing a learner's response to $(a + b)b$ Figure 2.4 which is also included in Table 4.4.

c) $(a + b)b = a + b^{2}$.

Figure 4.2: Learner response

The role of brackets in $(a + b)b$ signals multiplication of the monomial b by the binomial $(a + b)$. The response indicated no multiplication on a by the second term b in the brackets. Although brackets were correctly interpreted as a sign for multiplication, the distributive law was only applied partially. Therefore systemic structure sense was not evident, and the error was coded as *distributive law not used correctly (DLNC).* The resulting algebraic error was *distributive law not used correctly.* Table 4.4 provides examples of learners' incorrect responses that illustrated lack of systemic structure sense.

Table 4.4: Examples of incorrect responses involving errors in systemic structure sense

Error code	Applicable items	Incorrect response	Indicators	Implies
DLNC	$(a+b)b$	$a+b^2$	No multiplication performed on a by monomial b	Partially correct application of distributive law
	$2a(a-4)-8$	$2a^2-8-8$	Incorrect product of second term in brackets (-8)	Partially correct application of distributive law
	$(2x+1)(x+4)$	$3x+8x+x+4$	Incorrect product $(3x)$ from using first terms of binomials.	Partially correct application of distributive law
	$2(x+3)^2$	$2x+6+x+3$	Monomial 2 is multiplied into binomial $x+3$ to obtain $2x+6$, as the first order of multiplication	Multiplication is not used in the correct order for distributive law of three factors
SNC	$3a-(b+a))$	$3a-b+a$	No change in sign of second term in brackets $+a$ after using subtraction	Subtraction is not used correctly.

4.4 Algebraic Error Codes

The misinterpretation of brackets in surface and systemic structure of the expression, leads to algebraic errors. A list of algebraic error codes and applicable items are provided in Table 4.5.

Table 4.5: Algebraic errors codes and applicable items

Algebraic error code	**Description of error code**	**Applicable to items**				
		$(a+b)b$	$\mathbf{3a-(b+a))}$	$\mathbf{2a(a-4)-8}$	$\mathbf{(2x+1)(x+4)}$	$\mathbf{2(x+3)^2}$
IOO	**I**ncorrect **O**rder of **O**perations using multiplication					✓
BMSN	**B**rackets treated as **M**ultiplication where it **S**hould **N**ot be applied		✓	✓		
OATO	**O**peration on **A**djacent **T**erms **O**nly	✓	✓	✓		
PC	Error in **P**roduct of a monomial and **C**onstant					✓
BNMS	**B**rackets **N**ot treated as **M**ultiplication where it **S**hould be applied	✓		✓	✓	✓
LC	**L**ack of **C**losure (inappropriate conjoining)	✓	✓			
SS	No operation of **S**ubtraction where it **S**hould be applied		✓	✓		
PLN	Error in **P**roduct of terms consisting of **L**etters and **N**umbers			✓	✓	✓
S	Error with **S**ign		✓	✓		✓
P2L	Error in **P**roduct of terms consisting of **2 L**etters	✓				

Examples and explanations of algebraic errors are indicated in Table 4.6.

Table 4.6: Examples of incorrect responses involving algebraic errors

Error code	Applicable items	Incorrect response	Indicators	Implies
IOO	$2(x+3)^2$	$(2x+6)^2$	The monomial 2 is multiplied by the binomial $(x+3)$ as the first order of operations to obtain $2x+6$	Incorrect correct order of operations
BMSN	$3a-(b+a)$	$-3ab-3a^2$	Inappropriately multiplying the binomial $(b+a)$ by the monomial $3a$	Not treating the structure as subtraction of simple and compound terms
	$2a(a-4)-8$	$=(2a^2-8a)-8$ (line 2) $=16a^2-32$ (line 3)	Inappropriately multiplying both terms of the binomial $(2a^2-8a)$ by the constant -8 (see line 3)	Brackets treated as multiplication when it should not be used
OATO	$(a+b)b$	$a+b^2$	Multiplication of the second term of binomial b by the adjacent monomial b. No operation on the first term (a) in brackets which is positioned further from monomial b	Operations on adjacent terms only
	$3a-(b+a)$	$3a-b+a$	Subtraction applied to term b which is adjacent to the minus symbol. No change in sign of $+a$ which is positioned further from minus symbol.	Operations on adjacent terms only
	$2a(a-4)-8$	$3a-12$	Operation on adjacent terms $2a$ added to a to obtain $3a$; and -4 added with -8 to obtain -12	Operations on adjacent terms only
PC	$2(x+3)^2$	$2x^2+6x+6x+81$	Operation on monomial and constant is 9^2 instead of 2×9	Incorrect product of constant
BNMS	$(a+b)b$	$(a+b)$	Omitting b in answer	No multiplication on binomial when it should be applied
	$2a(a-4)-8$	$3a-12$	Operation on terms with letters $2a$ and a to obtain $3a$, and addition of constants -4 and -8 to obtain -12	Using operations other than multiplication on the monomial and binomial in brackets
	$(2x+1)(x+4)$	$3x+4$	Operations on terms with letters $2x$ and x to obtain $3x$, and on constants $+1$ and $+4$ to obtain 4	No multiplication on the two binomials when it should be applied
	$2(x+3)^2$	$2(x^2+9)$	Independently squaring the terms in brackets to obtain x^2+9	No multiplication of all terms in brackets

Error code	Applicable items	Incorrect response	Indicators	Implies
LC	$(a+b)b$	ab^2	Inappropriately conjoining a and b in brackets, and multiplying by monomial b	Operating on brackets first (Gunnarsson, et al., 2016) and lack of closure
	$3a-(b+a)$	$3a-ba$	Inappropriate conjoining of unlike terms b and a in brackets	Operating on brackets first leading to lack of closure
SS	$3a-(b+a)$	$3ab+4a$	Multiplication of $3a$ with the first term in brackets b to obtain $3ab$; and addition of $3a$ with the second term in brackets a to obtain $4a$	No operation of subtraction is used when it is required
	$2a(a-4)-8$	$2a^2-8a$	Omitting the constant (-8) in answer	Negated constant is not included in answer
PLN	$2a(a-4)-8$	$2a^2-8-8$	No inclusion of letter a in the product of monomial $2a$ and constant -4.	Incorrect product of a monomial consisting of letter and a number, and the constant in brackets
	$(2x+1)(x+4)$	$2x^2+4x+x+4$	Incorrect product of term with letter $2x$ and constant 4 to obtain $4x$	Incorrect product of a terms consisting of a number and letter, and a constant
	$2(x+3)^2$	$2(x^2+6x+6x+9)$	Incorrect product of 3 and x, in both brackets to obtain $6x$ in both instances	Incorrect products of terms containing numbers and letters
S	$3a-(b+a)$	$3a-b+a$	The sign of a remains positive after using subtraction on terms in brackets.	No operation on second term in brackets
	$2a(a-4)-8$	$2a^2-8a+8$	The sign of the constant changes from -8 to $+8$	Inappropriately changing the sign of the constant
	$2(x+3)^2$	$2(x+3)(x-3)$	The sign of binomial in second bracket is changed to $-$ symbol	Relating the squaring of binomial as products resulting in the difference of two squares
P2L	$(a+b)b$	$ab+2b$	b and b are added to obtain $2b$	The incorrect product of terms consisting of two letters

Algebraic errors in incorrect responses are a result of the misinterpretation of surface and systemic structures. Examples of algebraic errors are therefore not discussed separately.

4.5 Process of Coding

I coded errors using two levels of coding. The first level distinguishes between surface structure and systemic structure errors. The second level focuses on algebraic errors. The flow chart (Figure 4.2) illustrates the processes of coding both structural and algebraic errors.

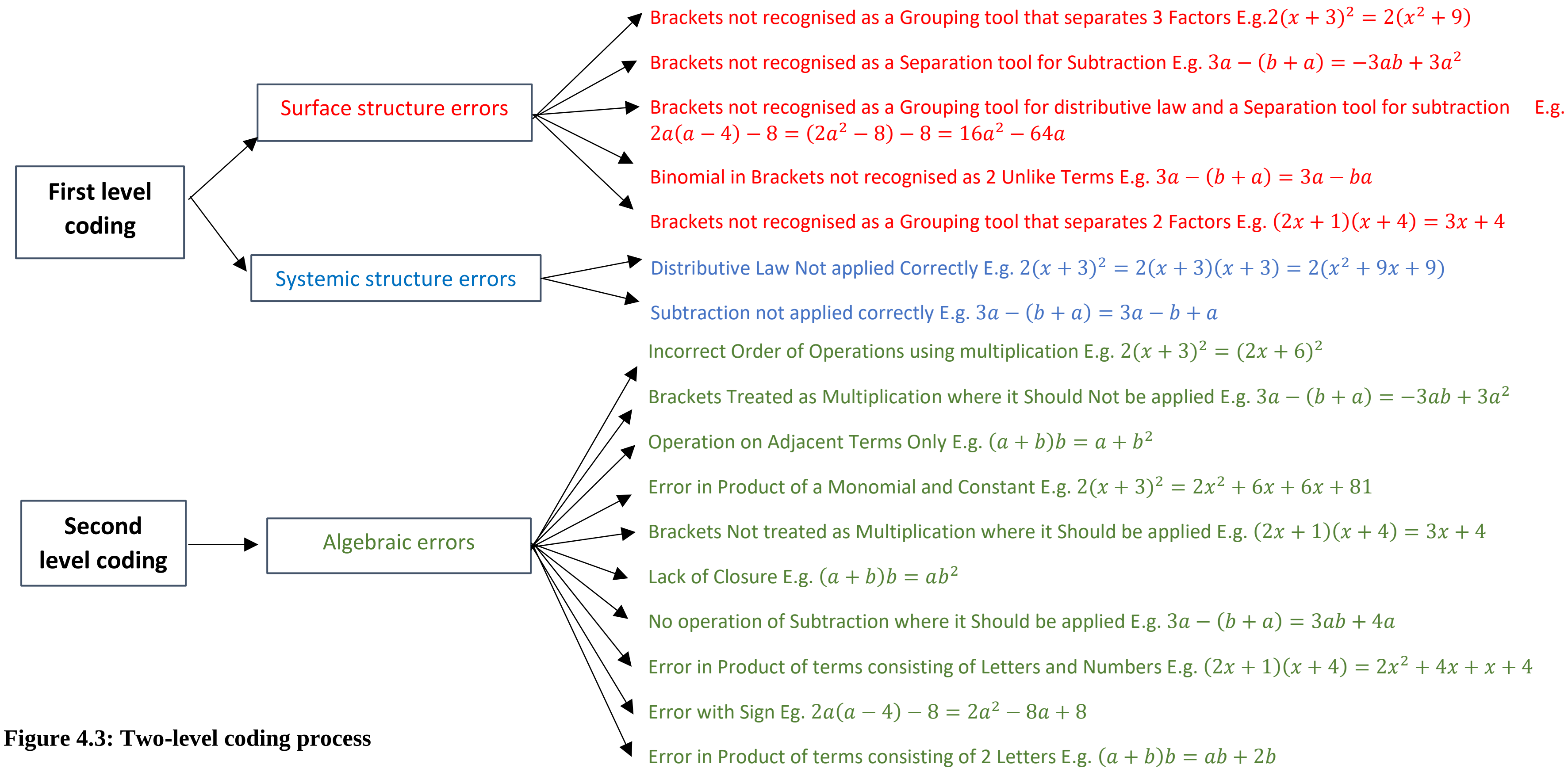


Figure 4.3: Two-level coding process

The first level of coding involved the allocation of either surface structure or systemic structure error codes. Surface structure errors were coded by examining how learners interpreted brackets as a tool in the arrangement of elements in items, such as a grouping tool for 3 factors. Systemic structure errors were coded by examining how learners interpreted brackets as a signifier for operations, such as multiplication of monomials and binomials. I coded surface structure errors before systemic structure errors, because the arrangement of elements in expressions is usually considered before operating on elements in expressions. If there were errors with surface structure features, then systemic structure errors were not considered because systemically the responses would have been incorrect. If incorrect responses showed a correct interpretation of surface structure, then I examined the incorrect responses for systemic structure errors.

Once structural errors were coded, the first level of coding was complete. I then applied the second level of coding by identifying algebraic errors. For every incorrect response related to surface structure, algebraic errors were evident and were allocated an algebraic error code. Although the coding of algebraic errors is independent of structure sense, the allocation of algebraic error codes supported the coding for surface structure errors by focusing on procedural or operational errors made in the manipulation of structures.

I provide a list of surface structure errors (Table 4.7) and systemic structure errors (Table 4.8) leading to algebraic errors.

Table 4.7: Surface structure errors leading to algebraic errors

Surface structure errors	**Algebraic errors**
Brackets not recognised as a grouping tool that separates 3 factors	Brackets not treated as multiplication where it should be applied Incorrect Order of Operations using multiplication
Brackets not recognised as a separation tool for subtraction	Brackets treated as multiplication where it should not be applied No operation of subtraction where it should be applied Lack of closure
Brackets not recognised as a grouping tool for distributive law and a separation tool for subtraction	Brackets treated as multiplication where it should not be applied Operate on adjacent terms only
Binomial in brackets not recognised as two unlike terms	Brackets treated as multiplication where it should not be applied Lack of closure
Brackets not recognised as a grouping tool that separates 2 factors	Brackets not treated as multiplication where it should be applied

Table 4.8: Systemic structure errors leading to algebraic errors

Systemic structure errors	Algebraic errors
Distributive Law not applied correctly	Incorrect order of operations using multiplication Operate on adjacent terms only Error in product of monomial and constant Lack of closure Error in product of terms consisting of letters and numbers
Subtraction not applied correctly	Brackets treated as multiplication where it should not be applied

To illustrate the first and second levels of coding of surface structure errors, I use two learner responses (Table 4.9).

Table 4.9: Coding of surface structure errors and algebraic errors

Incorrect response	**First level of coding: structural errors**		**Second level of coding**
	Surface structure errors	**Systemic structure errors**	**Algebraic errors**
$2(x^2+9)$ $=2x^2+18$	BG3F		BNMS
$3a-(b+a)$ $=-3ab+3a^2$	BSS		BMNS

On the first level, I coded the surface structure error of the first learner response $2(x+3)^2 = 2(x^2+9) = 2x^2+18$ (Table 4.9) as brackets *not recognised as a grouping tool that separates 3 factors (BG3F)*. Because the learner interpreted the surface structure incorrectly, the interpretation of systemic structure would consequently be incorrect. Therefore, I did not code for systemic structure errors. The algebraic error was coded as *brackets not treated as multiplication where it should be (BNMS)* because there was no multiplication of the inner and outer terms of $(x+3)(x+3)$.

On the first level of coding, the second response $3a-(b+a) = -3ab+3a^2$ (Table 4.9) illustrated a surface structure error of *brackets not recognised as a separation tool for subtraction (BSS)*. The incorrect response suggested that the minus symbol was not regarded as an operation. Rather, it was used to negate monomial $3a$. Furthermore, brackets were interpreted as a sign for inappropriate multiplication of both terms $a+b$ in brackets. I coded the algebraic error as *brackets treated as multiplication where it should not be applied (BMNS)*.

Two examples of incorrect responses are given in Table 4.10 which illustrate the first and second levels of coding systemic structure errors.

Table 4.10: Coding of systemic structure errors and algebraic errors

Incorrect response	First level of coding: structural errors		Second level of coding
	Surface structure errors	**Systemic structure errors**	**Algebraic errors**
$2(x+3)^2$ $=2(x+3)(x+3)$ $=2(x^2+9x+9)$ $=2x^2+18x+18$		DLNC	PLN
$3a-(b+a)$ $=3a(-b+a)$ $=-3ab+3a^2$		SNC	BMSN

The first example $2(x+3)^2$ as $2(x+3)(x+3)=2(x^2+9x+9)$ illustrated correct interpretation of surface structure as indicated by $2(x+3)(x+3)$. An error was evident in $2(x^2+9x+9)$ by incorrectly multiplying the inner and outer terms of the grouped binomials $(x+3)(x+3)$ to obtain $9x$ instead of *6x*. On the first level of coding, I allocated *distributive law not applied correctly (DLNC)*. This systemic structure error led to the algebraic *error with the product of terms that consist of a letter and a number (PLN).*

In the second example $3a-(b+a)=3a(-b+a)=-3ab+3a^2$, I coded the systemic structure error on the first level of coding, as *subtraction not used correctly (SNC)*. This error was evident because only the first term in brackets (b) was negated. Brackets were retained in the learner response when brackets were no longer required, which signified distribution. Therefore the algebraic error of *brackets treated as multiplication when it should not be applied (BMNS)* was the second level of coding.

In some instances, two algebraic error codes were allocated to an incorrect response because two errors were simultaneously evident. I provide one example that illustrates two algebraic errors (Table 4.11).

Table 4.11: Coding for two algebraic errors

Incorrect response	First level of coding: structural errors		Second level of coding: algebraic errors
	Surface structure errors	Systemic structure errors	Algebraic errors
$3a - (b + a) = 3b$	BSS		SS LC

The incorrect response $3a - (b + a) = 3b$ was coded as *brackets not used a separation tool for subtraction* (BSS*)*. This error led to two algebraic errors. The first error showed *no operation of subtraction where it should be applied* (SS) on the terms $b + a$, and the second error was *lack of closure* (LC) by inappropriately conjoining terms to obtain $3b$.

More examples of coding surface structure errors are given on Appendix B.

4.6 Conclusion

The process of coding errors needed to be systematic and as accurate as possible, because the analysis of structure sense on the five items was based on the number of errors for structural and algebraic errors. Using the first level of coding, I ensured that every incorrect response was coded once for structural errors. On the second level of coding, every structural coded response was allocated at least one algebraic error code.

CHAPTER 5 – DATA ANALYSIS

5.1 Introduction

In this chapter, I investigate learners' structure sense of expressions consisting of brackets by analysing errors in two phases. In the first phase, I investigate and analyse the errors made in surface structure and systemic structure in the three performance groups. In the second phase of analysis, I investigate structural errors of each item used in this study. In the first phase I identify and analyse the most prevalent errors in surface and systemic structure, by making reference to algebraic errors in incorrect responses. This analysis is important because it addresses the research questions in this study. In the second phase of my analysis, I analyse errors by considering items that were classified as typical/atypical and simple/complex structures. This analysis is salient as it provides evidence of the extent to which the task which I had undertaken of classification of expressions is useful in characterising learners' performances.

PHASE 1 OF ANALYSIS

The first phase involved analysing the seven types of structural errors which were independent of each other. Five surface structure errors related to misinterpretation of brackets as a tool in relation to other elements in the items. There were two systemic structure errors related to the incorrect operations of the brackets as a signifier in the items.

5.2 Percentages of Structural and Algebraic Errors

In order to make fair comparisons of the errors between the three performance groups, it was important to obtain percentages, by taking into account two aspects. Firstly, performance groups consisted of different numbers of leaners (indicated in Table 5.1).

Table 5.1: Number of learners in each performance group

Performance group	Mark range	Number of learners
1	70% – 100%	25
2	50% - 69%	17
3	40% - 49%	16

Secondly, errors did not have the same frequency based on the structure of the items that were used in the test. Therefore, certain errors may have occurred more frequently than others because of the nature of the items. For example, in systemic structure errors, there were four items that involved the distributive property and only one item that related to subtraction indicated in Table 4.3. This implied that there was potential for more incorrect responses to be generated for *distributive law not used correctly* (DLNC). Therefore, one would have expected coding for DLNC to appear more often than coding for *subtraction not applied correctly* (SNC).

Table 5.2 displays the numbers and the calculated percentages of incorrect responses of each item, which I subdivided into three performance groups. The number of incorrect responses shows a transition from qualitative coding to quantitative representation as frequencies and percentages. Percentages were rounded off to one decimal place to allow for ease of comparison between the groups.

Table 5.2: Numbers and percentages of incorrect responses for structural errors

PG 1: above 70%	Surface structure Errors					Systemic structure errors		Algebraic errors									
Item	**BG3F**	**BSS**	**BGS**	**BBUT**	**BG2F**	**DLNC**	**SNC**	**IOO**	**BMSN**	**OATO**	**PC**	**BNMS**	**LC**	**SS**	**PLN**	**S**	**P2L**
$(a+b)b$						3				2							1
$3a-(b+a)$		4					2		2	1				3		1	
$2a(a-4)-8$			2			1			2							1	
$(2x+1)(x+4)$						1									1		
$2(x+3)^2$	6					2		3			1	6					
Number of incorrect responses	6	4	2	0	0	7	2	3	4	3	1	6	0	3	1	2	1
Percentage of incorrect responses	10.3	3.4	3.4	0.0	0.0	3.0	3.4	5.2	3.4	1.7	1.7	2.6	0.0	2.6	0.6	1.1	1.7

PG 2: 50-69%	Surface structure errors					Systemic structure errors		Algebraic errors									
Item	**BG3F**	**BSS**	**BGS**	**BBUT**	**BG2F**	**DLNC**	**SNC**	**IOO**	**BMSN**	**OATO**	**PC**	**BNMS**	**LC**	**SS**	**PLN**	**S**	**P2L**
$(a+b)b$						3				3			2				
$3a-(b+a)$		6		1			1		7				1				
$2a(a-4)-8$		1	3			1			1	1		2			1	1	
$(2x+1)(x+4)$					1	1						1					
$2(x+3)^2$	7					6		2			4	6			1	2	
Number of incorrect responses	7	7	3	1	1	11	1	2	8	4	4	9	3	0	2	3	0
Percentage of incorrect responses	12.1	6.0	5.2	1.7	0.9	4.7	1.7	3.4	6.9	5.4	6.9	3.9	2.6	0.0	1.1	1.7	0.0

PG 3: 40-49%	Surface structure errors					Systemic structure errors		Algebraic errors									
Item	**BG3F**	**BSS**	**BGS**	**BBUT**	**BG2F**	**DLNC**	**SNC**	**IOO**	**BMSN**	**OATO**	**PC**	**BNMS**	**LC**	**SS**	**PLN**	**S**	**P2L**
$(a+b)b$					1	9				8		1	3				1
$3a-(b+a)$		6		2			3		6	2			2	3		2	
$2a(a-4)-8$		2	3			1			1	2		2		1	1	1	
$(2x+1)(x+4)$					2	4						2			3		
$2(x+3)^2$	3					12		8			3	4			3		
Number of incorrect responses	3	8	3	2	3	26	3	8	7	12	3	9	5	4	7	3	1
Percentage of incorrect responses	5.2	6.9	5.2	3.4	2.6	11.2	5.2	13.8	6.0	6.9	5.2	3.9	4.3	3.4	4.0	1.7	1.7

Total number of incorrect responses	**16**	**19**	**8**	**3**	**4**	**44**	**6**	**13**	**19**	**19**	**8**	**24**	**8**	**7**	**10**	**8**	**2**
Total percentage of incorrect responses	**27.6**	**16.4**	**13.8**	**5.2**	**3.4**	**19.0**	**10.3**	**22.4**	**16.4**	**14.0**	**13.8**	**10.3**	**6.9**	**6.0**	**5.7**	**4.6**	**3.4**

The process of generating percentages was of importance, therefore I describe it in detail. The formulae to calculate the percentage of incorrect responses took into account both the number of incorrect responses and the potential number of responses for the error. For example, DLNC in PG1 was calculated as $\frac{7}{232} \times 100\% = 3\%$. The coding for DLNC was applicable to four items, therefore 232 was calculated as 58×4 in which 58 indicated the total number of the sample group responses and 4 indicated the number of items that pertained to this error. In PG1 there was a total of seven incorrect responses. Similarly, SNC in PG1 was calculated as $\frac{2}{58} \times 100\% = 2\%$ since there was only one item applicable to this error and there were two incorrect responses in PG1 for this error.

I calculated the percentages of incorrect responses for each structural and algebraic error using this process, by taking into account the number of incorrect responses, and the number of potential responses based on the number of items that each error related to. I have attempted to capture both the actual prevalence of the incorrect responses based on the number of learners, and the potential number of responses for each error by adjusting the value of the denominator for percentage calculations. In this way, percentages could be compared across the different performance groups. The calculated percentages of errors are independent from each other, because the types of errors relevant to certain items do not relate to errors in other items. This implies that if percentages of incorrect responses of errors are added, the total percentage will not be 100.

The percentages of surface and systemic structure errors are represented as a stacked bar graph (Figure 5.1). Using a stacked bar graph, instead of a bar graph, is beneficial as it is easier to compare the results of each performance group across different errors. Furthermore, a stacked bar graph, displays fewer bars compared to a bar graph. Having more bars in a graph may prove challenging when making comparisons between performance groups across different errors. The stacked bar graphs in Figure 5.1display the relationship that each performance group had on the total percentage of incorrect responses for each error.

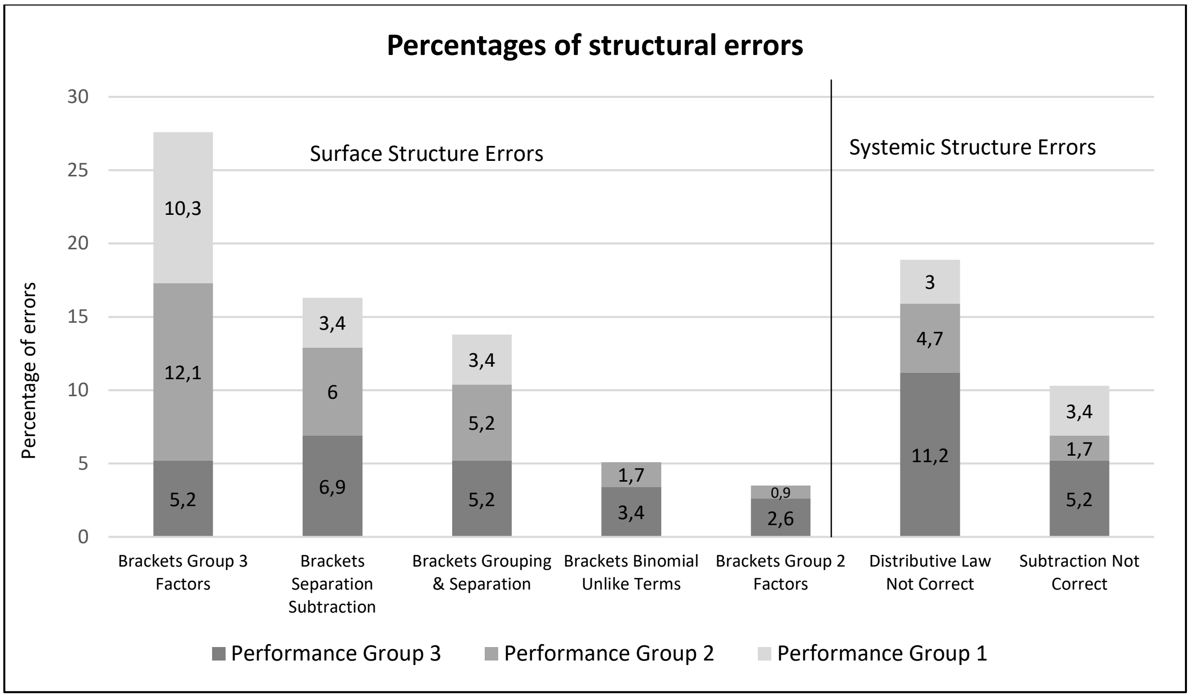


Figure 5.1: Percentages of structural errors

5.3 Results of Structural and Algebraic Errors

5.3.1 Results of surface structure errors

The most prevalent errors related to surface structure sense are presented in Table 5.3.

Table 5.3: Prevalent surface structure errors

Surface structure errors	Percentage of errors
Brackets not recognised as a grouping tool that separates 3 factors	27.6%
Brackets not recognised as a separation tool for subtraction	16.4%
Brackets not recognised as a grouping tool for distributive law and a separation tool for the operation of subtraction	13.8%

The highest percentage of incorrect responses for surface structure errors (27.6%) was *brackets not recognised as a grouping tool for 3 factors* (BG3F), which coincidently was the highest percentage of structural errors. Although BG3F only related to one item $2(x+3)^2$, it showed the highest percentage of incorrect responses. Percentages were highest for PG1 and PG2 at 10,3% and 12,1% respectively (Figure 5.1). This was the only error with a higher percentage in PG1 compared to PG3.

The second highest percentage of incorrect responses for surface structure errors (16.4%) was *not recognising brackets as a separation tool for the operation of subtraction.* Results showed that the highest proportion of incorrect responses for this error was found in PG3, and the lowest in PG1 (Figure 5.1).

The third highest percentage of incorrect responses for surface structure errors (13,8%) was *not recognising brackets as a grouping tool for distributive law and a separation tool for the operation of subtraction.* Most frequent incorrect responses were evident in PG2 and PG3 with 5,2% in each group (Figure 5.1).

Incorrect responses in the remaining two surface structure errors i.e., *binomial in brackets not recognised as two unlike terms* and *brackets not recognised as a grouping tool that separates 2 factors* were less than 10%. I do not discuss errors with frequencies less than 10%.

5.3.2 Results of systemic structure errors

The results of both systemic structure errors were above 10%, as indicated in Table 5.4.

Table 5.4: Prevalent systemic structure errors

Systemic structure errors	Percentage of errors
Distributive law not used correctly	19.0%
Subtraction not used correctly	10.3%

The highest percentage of incorrect responses for systemic structure errors at 19% was *distributive law not used correctly* (DLNC). The highest percentage was in PG3 at 11,2% and the lowest in PG1 at 3% (Figure 5.1).

The second highest percentage for systemic structure errors (10.3%) was *subtraction not used correctly* (SNC). Results showed that the highest proportion of incorrect responses was found in PG3, and the lowest in PG2 (Figure 5.1). Although there were four items included in DLNC, I took this into consideration when calculating the percentages as indicated in Section 5.2, thereby facilitating a fair comparison between DLNC and SNC. Therefore, the high percentage of incorrect responses was not the result of DLNC being applicable to four items.

In summary, the illustrated percentages of structural errors revealed that the lowest performance group had *higher* percentages of incorrect responses. This was evident for all errors, except for BG3F, which showed the *lowest* percentage in PG3. This low percentage was evidenced because there was a low percentage of incorrect responses for surface structure for PG3 but a much higher percentage in the systemic structure. Furthermore, the percentage of incorrect responses in PG3 for each error appeared to be almost equivalent to the two stronger performing groups (PG1 and PG2) combined. Finally, it could be seen that incorrect responses were evident in PG3 or PG2 for every type of structural error but not for PG1. In PG1 there were no incorrect responses for *binomial in brackets not recognised as two unlike terms* and *brackets not recognised a grouping tool for 2 factors* (Figure 5.1).

5.3.3 Results of algebraic errors

The results of algebraic errors presented a range of percentages from 22.4% to 3.4% showing most prevalent incorrect responses in five algebraic errors (indicated in Table 5.5).

Table 5.5: Prevalent algebraic errors

Algebraic errors	Percentage of errors
Incorrect order of operations using multiplication	22.4%
Brackets treated as multiplication where it should be applied	16.4%
Operating on adjacent terms only	14.0%
Error in product of a monomial and constant	13.8%
Brackets not treated as multiplication where it should be applied	10.3%

The remaining algebraic errors, i.e. *lack of closure; no operation of subtraction where it should be applied; error in product of terms consisting of letters and numbers; error with sign;* and *error in product of terms consisting of 2 letters* were less than 10% and are not discussed.

The percentage of algebraic errors is represented as a stacked bar graph (Figure 5.2).

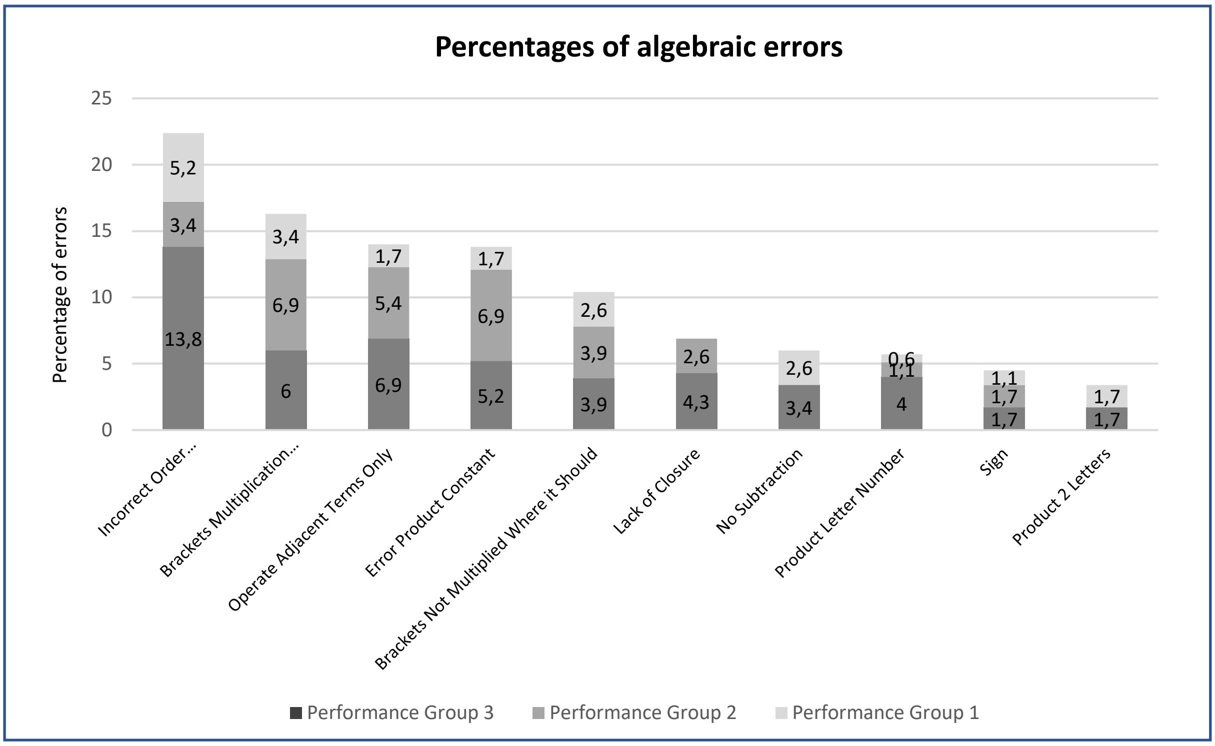


Figure 5.2: Percentages of algebraic errors

The focus of this study is structure sense of expressions, and not specifically on occurring algebraic errors. Therefore, I do not provide a detailed analysis of the percentages of algebraic errors. A discussion of algebraic errors is given in the discussion of surface and systemic structure errors in the next section.

5.4 Analysis of Structural and Algebraic Errors

I analyse the most prevalent misinterpretations of brackets in surface and systemic structure. Analyses of algebraic errors are included within the structural error discussion because it is the misinterpretation of the underlying structure of the given expressions that leads to particular algebraic errors.

5.4.1 Analysis of surface structure errors

I analyse common misinterpretations of brackets in the surface structure by considering the arrangement of brackets to other elements in expressions. Surface structure sense in this study is being able to recognise the role of brackets as a tool, by 'looking' at an expression (Hoch & Dreyfus, 2004). Errors in surface structure sense therefore implies that learners did not recognise the role of brackets as a grouping tool or separation tool in expressions.

Brackets not recognised as a grouping tool that separates 3 factors in $2(x+3)^2$.

The most common misinterpretation was *not recognising brackets as a grouping tool that separates three factors*. In $2(x+3)^2$ the brackets group the compound object $(x+3)^2$ as a single entity which can be interpreted as two factors $(x+3)(x+3)$ indicated by the square on the bracket. Therefore, the brackets group the compound structure $2(x+3)^2$ as three factors $2(x+3)(x+3)$. Surface structure sense is recognising the relationship of brackets as a grouping tool for the multiplication of the monomial by two binomials. Errors in surface structure sense elicited algebraic errors by *brackets not treated as multiplication where it should be applied* (BNMS) and the *incorrect order of operations using multiplication* (IOO). I provide examples of common misinterpretations of brackets related to these algebraic errors.

The most common incorrect response in all performance groups involved *brackets not treated as multiplication where it should be applied* by inappropriately squaring each term in brackets to obtain $2(x^2+9)$. Such misinterpretations are referred to as linear extrapolation errors (Matz, 1980, as cited in Olivier, 1986, p. 8). Such responses suggested that learners overgeneralised the distributive property and did not recognise that $(x+3)^2$ meant that the terms in brackets must be multiplied by each other to obtain a product of two binomials.

Another common incorrect response $(2x+6)^2$ involved two algebraic errors of BNMS and IOO. Incorrect responses were sometimes allocated more than one algebraic error code because of the different errors in one response (discussed in Section 4.5). The expression $(2x+6)^2$ firstly suggested no multiplication of the two binomials indicating BNMS. Secondly, 2 was multiplied into the bracket as the first order of operations using an incorrect order of multiplication indicating IOO. The structure of $(2x+6)^2$ ultimately means $(2x+6)(2x+6)$ which is not an equivalent structure of $2(x+3)^2$, and would therefore be systemically incorrect.

Other incorrect responses such as $2x^2+11$ and $2x^2+12$ suggested that the expression was not recognised as three factors, by obtaining constants 11 and 12 respectively. I was uncertain which operations were used by learners to obtain the constants. The response $2x^2+11$ suggested that 3 in $2(x+3)^2$ was squared to obtain 9 and then added to the monomial 2 to obtain 11. The response $2x^2+12$ suggested that 3 was multiplied by 2 instead of being squared, and then multiplied with the monomial 2 to obtain 12. Therefore, brackets were *treated as multiplication where they should have been applied.*

The high percentage of incorrect responses to $2(x+3)^2$ begs the question of whether learners in this study are able to recognise the relationship of three single entities which are the three factors $2(x+3)(x+3)$, if the entities are in a compound structure of $2(x+3)^2$. A similar finding is evident in a study by Hoch and Dreyfus (2004) (discussed in Chapter 2). In their study, the researchers suggested that learners experienced difficulty in recognising the relationship between elements, if single entities that are squared such as x^2 are presented as compound structures such as in $(x+2)^2$.

In summary, $2(x+3)^2$ illustrates only one set of brackets in the surface structure, yet the squared brackets in the item denotes two brackets. In another item, brackets as a grouping tool for two factors is evident in $(2x+1)(x+4)$. This item has two brackets in its surface structure compared to one bracket in $2(x+3)^2$. The percentage of surface structure errors in $(2x+1)(x+4)$ was considerably lower than $2(x+3)^2$. This implies that if a structure has more brackets such as $2(x+3)(x+3)$, then better structure sense could have been demonstrated. A similar finding in a study by Hoch and Dreyfus (2004) (Section 2.6) was that if a structure consisted of more brackets it would result in better structure sense. The comparison of the percentage of the two items $2(x+3)^2$ and $(2x+1)(x+4)$ therefore suggests errors in surface structure sense in an expression that consists of less brackets, thereby supporting the argument of Hoch and Dreyfus (2004).

Brackets not recognised as a separation tool for subtraction in $\mathbf{3a-(b+a)}$ and $\mathbf{2a(a-4)-8}$.

Another common misinterpretation was *not recognising brackets as a separation tool for subtraction*. In $3a-(b+a)$, surface structure sense is evident if subtraction is recognised as the primary operation of a compound term $(b+a)$ from a single entity $3a$. Errors in surface structure sense was misinterpreting the relationship between elements resulting in algebraic errors of *treating brackets as multiplication where it should not be used (*BMNS*).*

The most common incorrect responses in all performance groups illustrated inappropriately multiplying $3a$ into the brackets in various ways to obtain answers such as

$-3ab+3a^2$; $3ab-3a^2$; $3ab+3a^2$; and $-3ab-3a^2$.

The responses suggested that learners incorrectly interpreted brackets to mean multiplication. Incorrect responses illustrated that $3a$ was multiplied simultaneously with both terms in brackets, and that $3a$ was negated in some instances. This resulted in the signs of some products being negative according to rules of multiplication with negative and positive integers. Learners did not interpret the structure as two separate terms indicated by the minus symbol but regarded brackets to be operated on first as multiplication. Gunnarsson, et. al.'s (2016) study suggested that learners perceived expressions differently when brackets were included in expressions. In their study, which I discussed in Section 2.6, learners did not always follow the rules of the order of operations when brackets were included in items, and learners assumed that brackets must be operated on first. Similarly, in this study, there was scant regard for the minus symbol in $3a - (b + a)$. Rather, the incorrect responses suggested that brackets were operated on first by using multiplication.

In $2a(a - 4) - 8$, errors in structure sense were evident when the minus symbol between the compound term $2a(a - 4)$ and the single term 8 was not recognised as an operation of subtraction. This resulted in an algebraic error of *no operation of subtraction where it should be applied*. Only one incorrect response pertaining to $2a(a - 4) - 8$ for this type of error was evident, therefore my discussion focuses primarily on $3a - (b + a)$.

Brackets not recognised as a grouping tool for distributive law and a separation tool for subtraction in $\boldsymbol{2a(a - 4) - 8}$.

The misinterpretation of the dual role of brackets as a grouping tool for the distributive law and as a separation tool for subtraction, is evident in $2a(a - 4) - 8$. Surface structure sense is evident if the constant -8 positioned after the bracket is recognised as a separate term from the preceding compound structure $2a(a - 4)$. Errors in structure sense led to algebraic errors by *treating brackets as multiplication when it should not be used* (BMSN), and *operating on adjacent terms only* (OATO).

A common misinterpretation suggested that the constant -8 was not recognised as a separate term from the compound structure $a(a - 4)$. Responses illustrated that the monomial before the bracket was correctly distributed over the terms in brackets to obtain $(2a^2 - 8a) - 8$, and the constant -8 was then used further as multiplication on the terms in brackets where multiplication should not be used. For example, incorrect responses such as $16a^2 - 64a$ suggested that -8 was multiplied by both $2a^2 - 8a$. In a study, Gunnarsson et al., (2016) argued that the insertion of brackets in a complex structure changed learners' perceptions of the subtraction of the constant, to one where the constant must be multiplied with the bracket

(Section 2.6). Similarly, in $2a(a-4)-8$, the structure is multifaceted in that brackets are a dual tool which resulted in a misinterpreting the role of brackets.

Other misinterpretations related to the positioning of the brackets in the middle of the expression $2a(a-4)-8$. Errors in surface structure sense were evident when the positioning of the brackets was misinterpreted as having to be operated on by adjacent terms, that led to algebraic errors. For example, the response $3a-12$ illustrated that adjacent terms were operated on, such as grouping like terms. Terms consisting of the same letters, and of constants were operated on, as if mental brackets on $(2a+a)$ and $(-4-8)$ were used. Grouping of like terms and constants are taught in Grade 8 using different types of structures. For example an expression such as $2a+a-4-8$ can be simplified as $3a-12$. The incorrect responses in this study suggested an incorrect interpretation of the positioning of brackets by grouping like terms.

In summation I have found that despite using Grade 9 algebraic structures, the Grade 10 learners did not interpret brackets correctly, thereby illustrating little surface structure sense. Expressions containing hidden structures such as $(x+3)^2$ were not easily recognised as two binomials and were operated on inappropriately. Furthermore, structures which include a minus symbol before or after brackets, were not interpreted as subtraction of terms. Rather multiplication was associated with brackets, by using the distributive law when it should not be applied.

5.4.2 Analysis of systemic structure errors

I analyse common misinterpretations of brackets in the systemic structure by considering properties and relationships between operations in structures. Systemic structure sense is being able to manipulate expressions by using appropriate structural operations, referred to as 'doing' within structures (Hoch & Dreyfus, 2004). Errors involving systemic structure sense therefore implies that learners did not recognise the role of brackets as a sign for operations.

Distributive law not used correctly in $2(x+3)^2$; $(a+b)b$; $2a(a-4)-8$ and $(2x+)(x+4)$

The most common misinterpretation of brackets was *not using the distributive law correctly.* In $2(x+3)^2$, systemic structure sense is evident if the correct order of multiplication is used on the two brackets $(x + 3)(x + 3)$, followed by multiplication by *2*. A common error involved the *incorrect order of operations with multiplication* (IOO) to obtain incorrect responses such as $2x + 6 + 2x + 6$ and $(2x + 6)(2x + 6)$. Such answers suggested that the monomial 2 before the brackets was inappropriately distributed to both terms of both brackets. Similarly responses such as $2x + 6 + x + 3$ suggested that the monomial 2 was distributed to the first bracket. However, there was no further multiplication with terms in the second bracket which implied IOO by working from left to right.

Furthermore, responses such as $2(x^2 + 3x + 3x + 9) = 2x^2 + 6x + 6x + 9$, suggest correct interpretation of systemic structure, but errors were evident in *product of a monomial and constant*. This was illustrated by correctly using distribution to obtain $2(x^2 + 3x + 3x + 9)$. However, when *2* was distributed with the terms in brackets, there was no multiplication on the constant 9. Similarly, in $2(x + 3)(x + 3) = 2(x^2 + 3x + 3x + 6)$, the constants 3 in the two brackets $(x + 3)(x + 3)$ were inappropriately added to obtain 6. Such responses suggest slips because systemic structure sense was demonstrated in the multiplication of all terms except for multiplication of constants.

In $(a + b)b$, systemic structure sense was demonstrated if brackets were recognised as a sign to multiply both terms in brackets with the monomial after brackets to obtain $ab + b^2$. A common misinterpretation was *operating on adjacent terms only*, such as in $a + b^2$. In such responses, the distributive law was applied partially correct, by only multiplying terms with the same letters (*b* and *b*) which are arranged adjacent to each other. Similarly, the response ab^2 suggested the partially correct application of the distributive law by only multiplying the same letters (*b* and *b)*, but no multiplication on a. A further algebraic error in ab^2 was *lack of closure* in the inappropriate conjoining of terms. Finally, frequent responses such as $ab + 2b$, indicated operations on both terms (*a* and *b*) by *b*, but the algebraic error was inappropriately adding b and b to obtain $2b$, instead of multiplying.

In $(2x + 1)(x + 4)$, systemic structure sense was evident if terms in both brackets were multiplied by each other to obtain $2x^2 + 8x + x + 4$. Trends of using the correct process of distribution but obtaining incorrect products was evident in responses such as $2x + 8x + x + 4$ and $3x + 8x + x + 4$. Such responses illustrated incorrect products of $2x$ and $3x$ respectively when multiplying the first terms in the two brackets $2x$ by x. Other responses such as $2x^2 + 4x + x + 4$ illustrated an incorrect product of $4x$ when multiplying $2x$ by 4.

In $2a(a-4)-8$, systemic structure sense was evident if brackets were recognised as a sign to multiply the monomial (*2a*) with both terms in brackets to obtain $2a^2-8a-8$, and that no further operation was required on the constant (–8). Incorrect responses such as $2a^2+8a-8$ suggested the correct interpretation of brackets. However, *errors with sign* were evident when the minus symbol of $-8a$ was inappropriately changed to $+8a$.

Subtraction not used correctly in $3\boldsymbol{a}-(\boldsymbol{b}+\boldsymbol{a})$

In $3a-(b+a)$ systemic structure sense was evident if both terms in brackets (b and a) were subtracted from $3a$ to obtain $3a-b-a$. Misinterpretations involving systemic structure sense were evident when incomplete operations of subtraction were performed on terms in brackets. Prevalent responses such as $3a-b+a$ indicated that subtraction was only applied to b in brackets, by *operating on adjacent terms only*. This error was illustrated by the change in sign of the first term $-b$ in brackets, but no change in sign of the second term $+a$. In other responses such as $3a(-b-a)=-3ab-3a^2$, the correct change in signs of terms was illustrated as $(-b-a)$ after being operated on. However, the retention of brackets changed the structure of the item, which seemed to encourage distribution of $3a$ with $(-b-a)$ by *treating brackets as multiplication when it should not be applied* to obtain $-3ab-3a^2$.

In summary, the most challenging structural operation that affected systemic structure sense involved *distributive law not used correctly*. Responses illustrated that the operation to multiply was recognised, however, the process of multiplication was not used correctly. This was most evident by using the *incorrect order of operations using multiplication* and *operating on adjacent terms only*.

PHASE 2 OF ANALYSIS

The second phase of the analysis considers my initial classification of items being typical/atypical and simple/complex (Section 3.4). The classification of items was based on Grade 9 learning requirement in the CAPS document (Department of Basic Education, 2011) and on learner exercises in prescribed school textbooks. Typical and atypical structures refer to the surface structure of expressions. Typical expressions are structures with arrangements of elements which are included in Grade 9 learning requirements according to CAPS and are more likely to appear in school textbooks and assessments. Atypical structures refer to the arrangement of elements in expressions that are unfamiliar in assessments. Simple and complex structures refer to the systemic structure of expressions. Simple structures require one type of structural operation and complex structures require more than one type of structural operation in an expression.

5.5 Percentages of Errors in Test Items

In order to make objective comparisons of errors in items between the performance groups, it was necessary to use percentages because the group sizes were different. Table 5.5 displays the numbers and calculated percentages of incorrect responses of items which have been subdivided into the three performance groups. The order of the items in Table 5.6 is in accordance to the testing instrument, and does not affect the analysis in any way.

Table 5.6: Numbers and percentages of incorrect responses in each test item

PG 1: 70% and above					
Surface structure errors	$(a+b)b$	$3a-(b+a)$	$2a(a-4)-8$	$(2x+1)(x+4)$	$2(x+3)^2$
BG3F					6
BSS		4			
BGS			2		
BBUT					
BG2F					
Number of incorrect responses in PG1	**0**	**4**	**2**	**0**	**6**
% of incorrect responses in PG1	**0.0**	**6.9**	**3.4**	**0.0**	**10.3**
Systemic structure errors					
DLNC	3		1	1	2
SNC		2			
Number of incorrect responses in PG1	**3**	**2**	**1**	**1**	**2**
% of incorrect responses in PG1	**5.2**	**3.4**	**1.7**	**1.7**	**3.4**

PG 2: 50-69%					
Surface structure errors	$(a+b)b$	$3a-(b+a)$	$2a(a-4)-8$	$(2x+1)(x+4)$	$2(x+3)^2$
BG3F					7
BSS		6	1		
BGS			3		
BBUT		1			
BG2F				1	
Number of incorrect responses in PG2	**0**	**7**	**4**	**1**	**7**
% of incorrect responses in PG2	**0.0**	**12.1**	**6.9**	**1.7**	**12.1**
Systemic Structure Errors					
DLNC	3		1	1	6
SNC		1			
Number of incorrect responses in PG2	**3**	**1**	**1**	**1**	**6**
% of incorrect responses in PG2	**5.2**	**1.7**	**1.7**	**1.7**	**10.3**

PG 3: 40-49%					
Surface structure errors	$(a+b)b$	$3a-(b+a)$	$2a(a-4)-8$	$(2x+1)(x+4)$	$2(x+3)^2$
BG3F					3
BSS		6	2		
BGS			3		
BBUT		2			
BG2F	1			2	
Number of errors in PG3	**1**	**8**	**5**	**2**	**3**
% of errors in PG3	**1.7**	**13.8**	**8.6**	**3.4**	**5.2**
Systemic structure errors					
DLNC	9		1	4	12
SNC		3			
Number of incorrect responses in PG3	**9**	**3**	**1**	**4**	**12**
% of incorrect responses	**15.5**	**5.2**	**1.7**	**6.9**	**20.7**

	$(a+b)b$	$3a-(b+a)$	$2a(a-4)-8$	$(2x+1)(x+4)$	$2(x+3)^2$
Total number of incorrect responses for surface structure	**1**	**19**	**11**	**3**	**16**
Total percentage of incorrect responses of surface structure	**1.7**	**32.8**	**19.0**	**5.2**	**27.6**
Total number of incorrect responses of systemic structure	**15**	**6**	**3**	**6**	**20**
Total percentage of incorrect responses of systemic structure	**25.9**	**10.3**	**5.2**	**10.3**	**34.5**
TOTAL NUMBER OF INCORRECT RESPONSES	**16**	**25**	**14**	**9**	**36**
TOTAL PERCENTAGE OF INCORRECT RESPONSES	**27.6**	**43.1**	**24.1**	**15.5**	**62.1**

A description of the process of generating percentages ensues. The formulae to calculate the percentage of errors considered the numbers of incorrect responses and the potential number of responses for each item. For example, DLNC for $(a+b)b$ in PG3 was calculated as $\frac{9}{58} \times 100\% = 15{,}5\%$, in which nine incorrect responses were evident. Since the sample contained 58 learners, there were 58 expected responses to each item and thus this number (58) was used to calculate percentages. Percentages were rounded off to one decimal place. Percentages were calculated for incorrect responses to individual items, which were independent of other items. This means that the total percentage of errors does not equal to 100.

The total percentage of incorrect responses for surface and systemic structure errors (Table 5.6) was obtained by adding the percentages of incorrect responses of each performance group. For example, the percentage of incorrect responses based on surface structure of $2a(a-4)-8$ was 19% which was calculated as 3.4% + 6.9% + 8.6%. The percentage of incorrect responses based on systemic structure of $2a(a-4)-8$ is 5.4% which was calculated as 1.7% + 1.7% + 1.7%.

Finally, I calculated the total percentage of structural errors of each item, by adding the percentages of surface structure errors and systemic structure errors. For example, the total percentage of structural errors for $2a(a-4)-8$ was calculated as $19\% + 5.2\% = 24.1\%$. These percentages were useful to identify which items presented the highest and lowest percentages of incorrect responses suggesting the most or least challenging structures for the sample group.

A stacked bar graph, illustrated in Figure 5.3, displays the percentages of incorrect responses of surface and systemic structure for each item.

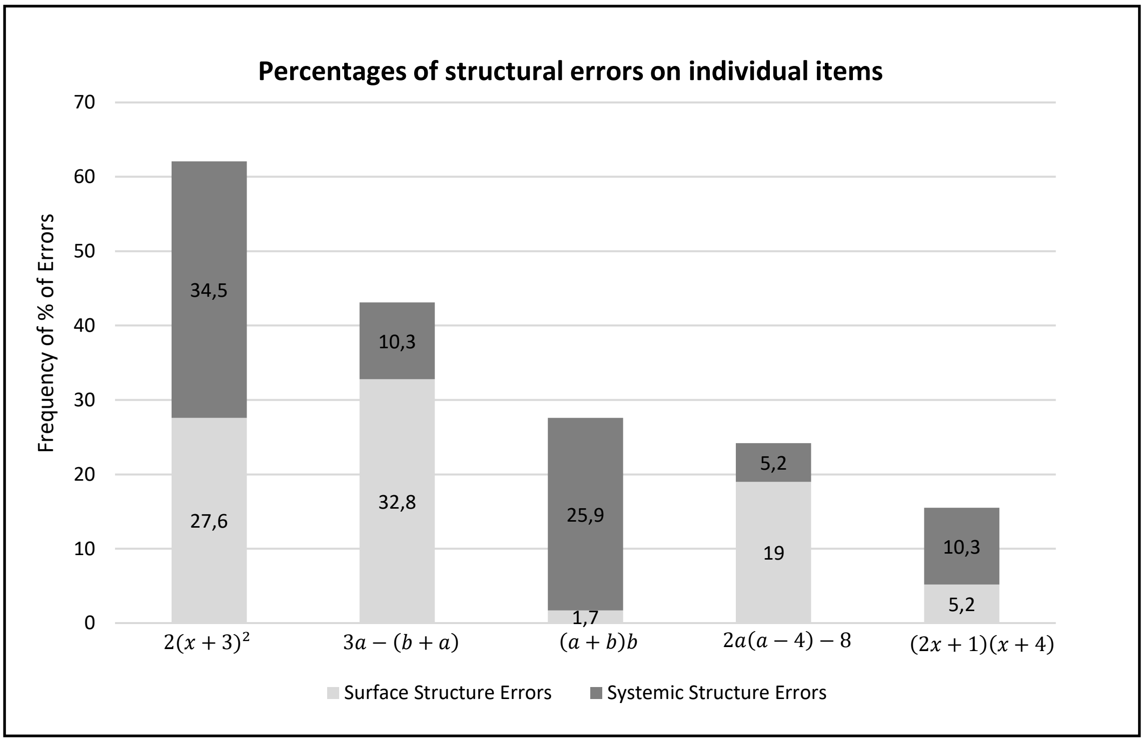


Figure 5.3: Percentages of structural errors of test items

An analysis of the percentages of structural errors is given in Section 5.4.

5.6 Percentage of Errors of Test Items vs Classification of Test Items

Items used in this study were classified as being typical/atypical and simple/complex structures (Section 3.4), as indicated in Table 5.7.

Table 5.7: Classification of test items

	Typical	Atypical
Simple	$(2x+1)(x+4)$ $3a-(b+a)$	$(a+b)b$ $2a(a-4)-8$
Complex	$2(x+3)^2$	

5.6.1 Item: $\boldsymbol{2(x+3)}^2$

I classified $2(x+3)^2$ as a typical and complex structure by considering the surface and systemic structure respectively. Incorrect responses showed the highest percentage of errors in both surface structure (27,6%) and systemic structure (34,5%) compared to other items (Figure 5.3).

The high percentage of surface structure errors could cast doubt on whether this item was indeed, typical. Errors in surface structure sense were more evident in PG1 and PG2 with more than 10% incorrect responses (see Table 5.6), implying that learners in the higher performance groups did not recognise brackets as a grouping tool of three factors The arrangement of elements of $2(x+3)^2$ may be challenging to interpret in Grade 9 because such structures are only taught in Grade 9. In Grade 10, however, the compound entity of grouping three factors is encountered within other compound structures, for example, $3(x-4)^2+(x+2)(x-2)$. This implies that not only should Grade 9 learners be familiar with $2(x+3)^2$, but that Grade 10 learners should be able to simplify this type of expression within a compound structure. Therefore, the high percentage of errors on this item indicated that Grade 10 learners within this sample experienced challenges in interpreting brackets as a grouping tool of three factors, which is not indicative of Grade 10 learners generally in South Africa.

The results of the highest percentage of systemic structure errors validated the complex nature of this item, which involved multiple operations and in using the correct order of operations. The percentage of surface structure errors in PG3 was less than 10%. However in this group most errors were evident in systemic structure which displayed the incorrect use of the distributive law. Systemic structure errors in PG2 were also more than 10%.

5.6.2 Item: $3a - (b + a)$

The item was classified as typical in the surface structure and simple in the systemic structure. Errors displayed the second highest percentage in the surface structure of 32,8%. There were 10,3% for the systemic structure which was slightly above the 10% margin.

My classification of the item being typical was challenged by the poor performance of learners in the interpretation of the surface structure. Learners from PG2 and PG3 showed the highest percentage of surface structure errors at 12,1% and 13,8% respectively (see Table 5.6). The high percentages indicated that lower performing groups had difficulty in interpreting the role of the minus symbol outside a grouped binomial. Structures involving subtraction (and addition) of single and compound entities are readily encountered in Grade 9 which should be recognisable to learners, and therefore such structures should be routine for Grade 10 learners. However, Grade 10 learners in this sample displayed challenges in recognising the subtraction of two terms.

The classification of being a simple structure was confirmed by the low percentage of systemic structure errors in each performance group. Although the overall percentage of errors was greater than 10%, percentages of errors in each group are below 10% (Table 5.6). The low percentages of errors implied that learners had a good understanding of using subtraction on the elements.

5.6.3 Item: $(a + b)b$

I classified $(a + b)b$ as an atypical and a simple structure by considering its surface and systemic structure respectively. The typical nature of $(a + b)b$ was justified not only by the low percentage of errors in each performance group but that the overall percentage of incorrect responses was below 10%.

The item displayed a high percentage of systemic structure errors at 25.9%. Distribution on the monomial and binomial involves one type of structural operation, rendering such a structure as simple in Grade 9. Multiplying a monomial and binomial is evident from Grade 8 learning skills, therefore it was surprising that the Grade 10 learners experienced difficulty in operating on the brackets. On closer examination, most incorrect responses were evident in PG3. The percentages of errors in PG1 and PG2 were less than 10% (see Table 5.5) which suggested that learners from PG1 and PG2 displayed systemic structure sense. If the positioning of the elements was changed to $b(a+b)$, there would be a greater likelihood for a higher percentage of correct operations in a more typical arrangement of elements. Therefore, despite the high percentage of errors for $(a+b)b$, its structure could still be classified as simple.

5.5.4 Item: $\boldsymbol{2a(a-4)-8}$

I classified $2a(a-4)-8$ as being atypical in view of its surface structure and simple in view of its systemic structure. There were 19% surface structure errors and less than 10% for systemic structure.

The atypical classification of the item was based on the positioning of the constant after the brackets. In Grade 9, such arrangements are not common, but in Grade 10, learners should be familiar with constants being placed in any position in an expression. The high percentage of errors of Grade 10 learners in this study suggested that learners had challenges in interpreting structures that included constants that were placed outside brackets, and the classification of $2a(a-4)-8$ being atypical was justified.

The low percentage of incorrect responses in systemic structure implied that learners were able to interpret brackets as a sign for multiplication, and justified the simple nature of the item.

5.6.5 Item: $\boldsymbol{(2x+1)(x+4)}$

The classification of $(2x+1)(x+4)$ as typical and simple was validated by the low percentage of errors in each performance group for both surface and systemic structures. The item also showed lowest percentage of structural errors at 15,5%. Errors in surface structure (5,2%) were significantly less than systemic structure (10,3%) as indicated in Table 5.6. This implied that learners recognised familiar structures involving two brackets where brackets are a grouping tool for two factors.

A good sense of surface structure was displayed in the higher performing groups, having no incorrect responses in PG1 and only one incorrect response in PG2.

Systemic structure errors were just above the margin of 10%. Although a low percentage of systemic structure errors was evident in each performance group, PG3 showed the most incorrect responses at 6,9%. It is interesting to note that the incorrect responses involving the operations on the brackets were partial errors or slips in multiplication. For example, incorrect responses such as $2x^2 + 4x + x + 4$ and $2x + 8x + x + 4$ both indicated the use of the correct and appropriate operations. However, there was a partial error of $4x$ instead of $8x$ in the first response, and a slip with $2x$ instead of $2x^2$ in the second response. This indicated that learners recognised brackets as a sign to multiply two binomials but made algebraic errors when using multiplication. Learners were able to operate on two binomials which was most likely due to the focus of such structures involving the multiplication of two binomials in Grade 9 (see Figure 2.2).

5.7 Final Analysis of Test Items

Percentages of structural errors of the items are arranged from highest to lowest in Table 5.8.

Table 5.8: Percentages of structural errors in test items

	$\mathbf{2(x+3)^2}$	$\mathbf{3a-(b+a)}$	$\mathbf{(a+b)b}$	$\mathbf{2a(a-4)-8}$	$\mathbf{(2x+1)(x+4)}$
% of surface structure errors	27.6	32.8	1.7	19.0	5.2
% of systemic structure errors	34.5	10.2	25.9	5.2	10.3
% of structural errors	**62.1**	**43.1**	**27.6**	**24.2**	**15.5**

In summary, $2(x + 3)^2$ and $3a - (b + a)$ illustrated the two highest percentages of structural errors, displaying high percentages of errors in *both* surface and systemic structures. Furthermore, each item displayed *either* the highest surface structure or systemic structure. In $2(x + 3)^2$, the highest structural error at 62,1% was evident in the surface structure, and in $3a - (b + a)$ the highest structural error at 2,8% was evident in the systemic structure. The remaining three items, $(a + b)b$; $2a(a - 4) - 8$ and $(2x + 1)(x + 4)$, consisted of percentages greater than 10% in *either* surface or systemic structure. In $(a + b)b$ and $(2x + 1)(x + 4)$, systemic structure errors were higher than 10%. In $2a(a - 4) - 8$ the surface structure is higher than 10%.

It is concerning that $2(x + 3)^2$ as a typical structure in Grade 10 had such a high percentage of errors, suggesting that the learners in this study were not able to operate on expressions that are of a Grade 9 level. An impediment is the compressed nature of this item, as a three-factor expression. In $(2x+1)(x+4)$, learners were able to interpret the surface structure and use appropriate operations to simplify. The question is, that if the item $2(x+3)^2$ was presented as $2(x+3)((x+3)$, would the interpretation of the surface structure be improved, and would the distributive law be correctly applied?

The items $3a-(b+a)$ and $2a(a-4)-8$ both showed high percentages of incorrect responses in the interpretation of it surface structure. In these items, the minus symbol is positioned either before or after the brackets, which signifies subtraction. Incorrect responses illustrated that the distributive law still took place instead of subtraction. This implied that the Grade 10 learners in this study had difficulty in recognising that subtraction could take place in expressions that consist of brackets, instead of only using multiplication.

5.8 Summary of Findings

I used percentages of incorrect responses to compare items and performance groups. Percentage calculations for incorrect responses used a denominator that integrated the number of items and the total possible responses. The findings revealed that the most prevalent surface structure error was *brackets not recognised as a grouping tool of 3 factors* which applies to $2(x+3)^2$. The second highest percentage of surface structure errors was evident in *brackets not recognised as a grouping tool for the operation of subtraction* in $3a-(b+a)$. Responses suggested that learners did not interpret the minus symbol between a compound and single term as subtraction but rather as multiplication. This type of misinterpretation was also evident in $2a(a-4)-8$, which includes subtraction of a single term from a compound term.

In the analysis of systemic structure, the highest percentage of errors was evident in *not using the distributive law correctly* (DLNC). The percentage of these incorrect responses was almost double the incorrect responses in subtraction not used correctly (indicated in Table 5.2).

Following the error analysis, I reflected on my classification of items as typical/atypical and simple/complex structures. Learner performance supported my classification for two of the five items, i.e. $(2x+1)(x+4)$ and $2a(a-4)-8$ as simple typical and simple atypical respectively. Based on learners' poor performance on the remaining three items, they could all be reclassified. Firstly $3a-(b+a)$ and $2(x+3)^2$ could be re-classified as atypical. Secondly, $(a+b)b$ could be reclassified as complex instead of simple. Despite these findings, I choose not to re-classify the three items according to learner performance. Items $2(x+3)^2$

and $3a - (b + a)$ are familiar structures based on Grade 9 learning requirements, therefore they remain typical structures according to my definition of surface structure. Item $(a + b)b$ requires one operation of multiplication, and remains a simple structure according to my definition of systemic structure. In order to account for learner performance on the items, I would need to extend my classification to incorporate levels of difficulty. This is beyond the scope of this research report.

CHAPTER 6 – IS THE STRUCTURE SENSE OF GRADE 10 LEARNERS IN ALGEBRA WHAT IT SHOULD BE?

6.1 Introduction

The purpose of this study was to investigate Grade 10 learners' structure sense of algebraic expressions using five items from a testing instrument designed for Grade 9 learners by the Wits Maths Connect Secondary project. Working with test responses, I focused on learners' interpretation of brackets in typical/atypical and simple/complex structures on five test items. The test items were:

Item 1: $(a+b)b$;

Item 2: $3a-(b+a)$;

Item 3: $2a(a-4)-8$;

Item 4: $(2x+1)(x+4)$; and

Item 5: $2(x+3)^2$.

Brackets can be interpreted differently depending on the arrangement of elements in a structure. I used *structure* according to Kieran's original definition as surface and systemic structure. Brackets can be interpreted as a tool and as a sign (Vygotsky, 1978). In an algebraic setting, brackets as a tool relates to the surface structure of an expression. For example, brackets group two factors $(x-5)(x+2)$ for distributive law. Brackets as a sign have relevance for systemic structures. For example, brackets signify specific operations such as multiplication of $(x-5)$ and $(x+2)$. A more detailed discussion of brackets as a tool and a sign can be found in Section 2.5.

6.2 Structure Sense

The notion of structure is however, difficult to define, as discussed in Section 2.3. I therefore chose to use *structure sense* in this study by defining structure sense as *the ability 1) to recognise the surface and systemic structures of typical and atypical algebraic expressions containing brackets and involving simple and complex operations; and 2) to use appropriate manipulations to obtain equivalent expressions*. This definition incorporated aspects from Kieran's (1989) definition of structure as discussed in Section 2.3 and Hoch and Dreyfus'

(2006) definition of structure sense discussed in Section 2.4. My definition of structure sense in relation to the role of brackets in algebraic structures is illustrated in Figure 6.1.

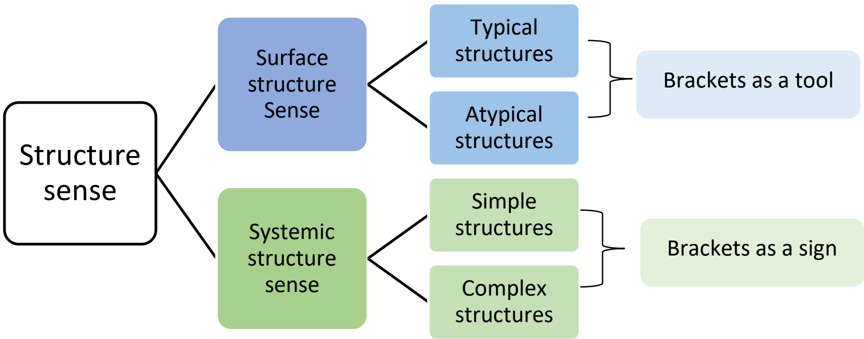


Figure 6.1: My definition of structure sense

6.3 Research Questions

My study was framed by the following research question:

What errors do Grade 10 learners make when simplifying algebraic expressions with brackets?

This research question was subdivided into three questions:

To what extent do learners demonstrate surface structure sense in the interpretation of typical and atypical structures of algebraic expressions with brackets?

To what extent do learners demonstrate systemic structure sense in the interpretation of simple and complex structures of algebraic expressions with brackets?

What algebraic errors result from misinterpretations of surface and systemic structures?

6.4 Sample for the Study

I reduced my initial sample of 155 learners to 58 learners to eliminate the wide range of unexpected and “unrecognisable” responses from learners who scored poorly on the test (see Section 3.5.) The reduced sample was then divided into three performance groups, as explained in Section 3.6. Performance Group 1 included 25 learners scoring 70% and above in the test, Performance Group 2 included 17 learners scoring from 50% to 69%, Performance Group 3 consisted of 16 learners scoring from 40% to 49%. The sampling process was illustrated in Figure 3.4.

6.5 Process of Coding Errors

In the analysis of errors, a qualitative approach was utilised, whereby error codes were allocated to incorrect responses. I coded only those errors which related to the misinterpretation of brackets. Once brackets were omitted from a learners' response, I ignored further errors and stopped coding (discussed in Section 4.2).A two-level coding process was used to code incorrect responses as firstly either surface or systemic structure errors and secondly as algebraic errors, (Figure 6.2).

Learners' errors suggested either a lack of surface structure sense or a lack of systemic structure sense. The first level of coding was aimed at *either* surface structure errors by considering brackets as a tool, *or* at systemic structure errors by considering brackets as a sign. The second level of coding was focused on algebraic errors which were a result of misinterpretation of brackets in structures. A detailed description of the error codes and applicable items is provided in Figure 6.2.

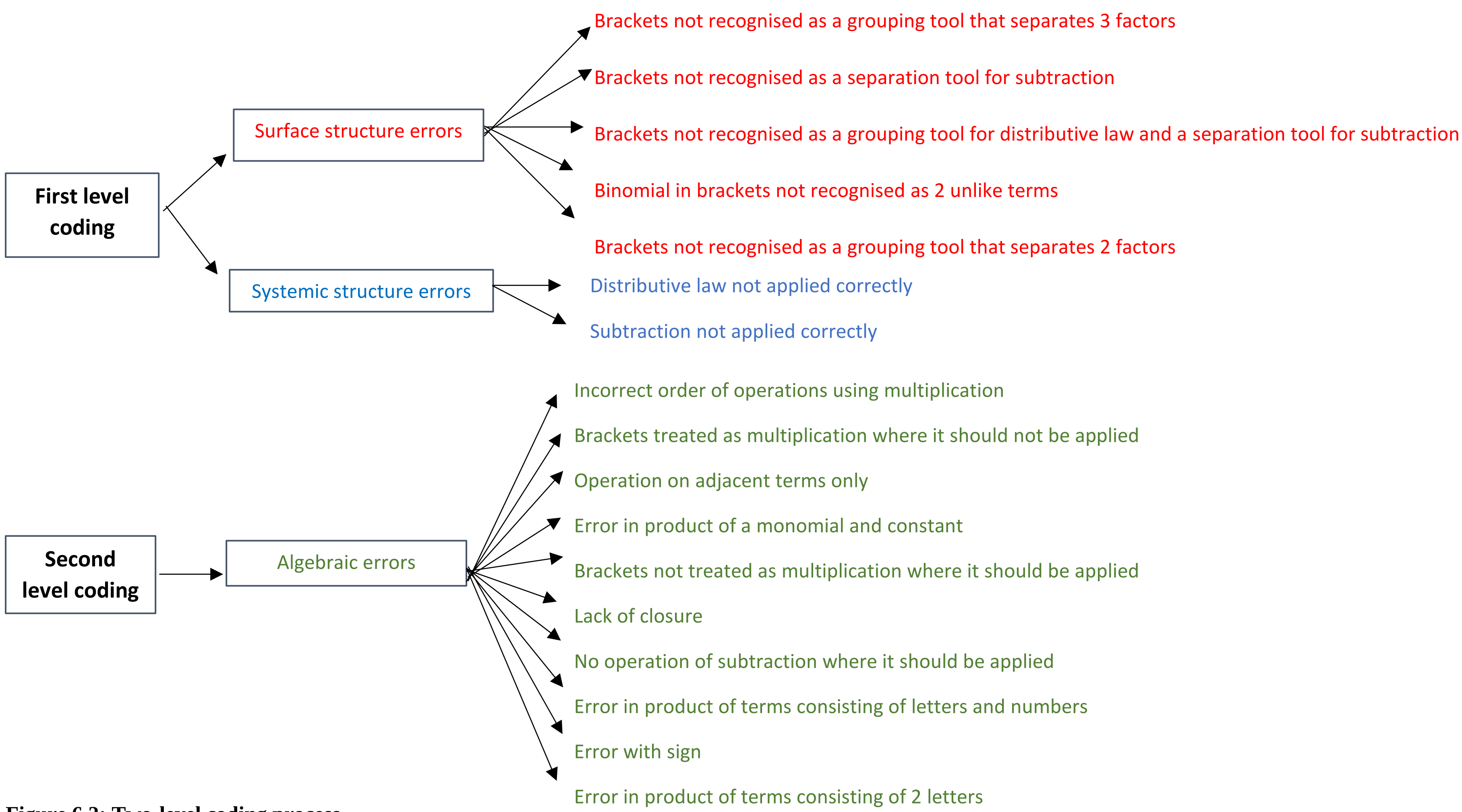


Figure 6.2: Two-level coding process

6.5 Qualitative Approach to Quantitative Results

The results of my analysis were presented in a quantified manner. Using the numbers of incorrect responses of each structural error, I was able to calculate percentages of errors. These percentages were used in two phases of the analysis. In the first phase, I determined the percentages of seven structural errors (consisting of five surface structure errors and two systemic structure errors), and ten algebraic errors given in Section 5.2. In the second phase of the analysis, I determined the percentages of structural errors of each test item, details of which are provided in Section 5.5.

The findings of the first phase of the data analysis, involving the percentages of the most prevalent structural errors, are presented in Figure 6.3. Percentages of the most prevalent algebraic errors are presented in Figure 6.4.

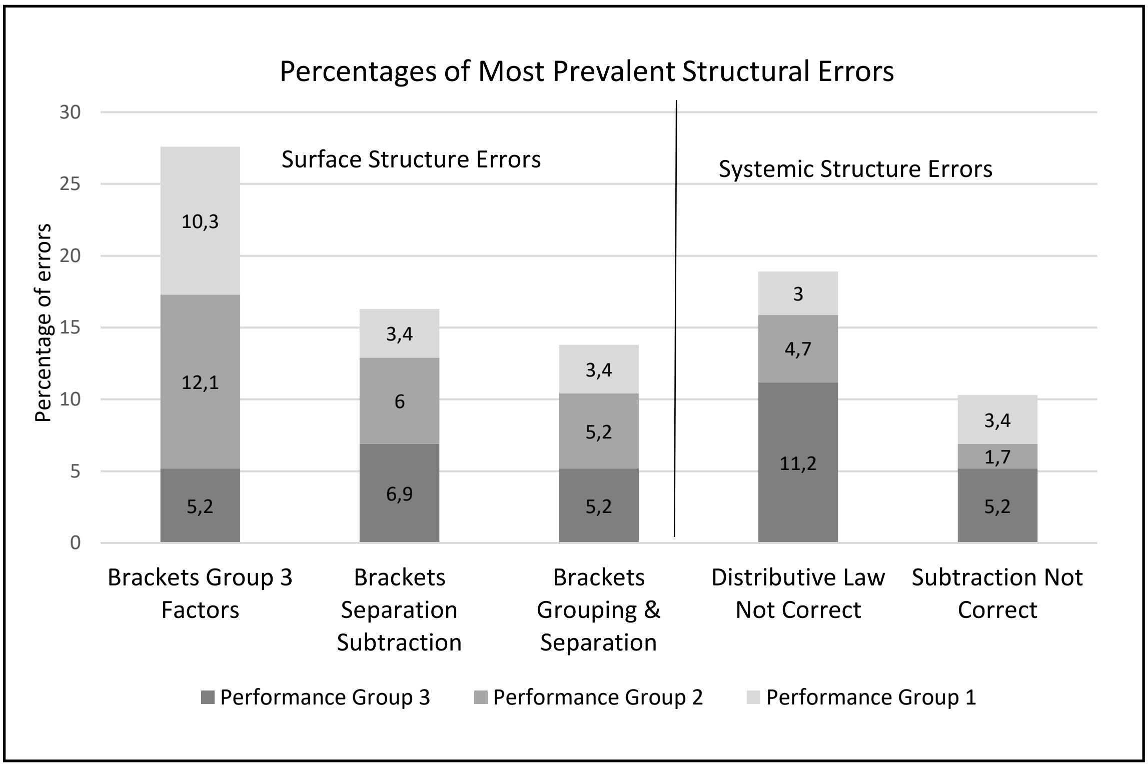


Figure 6.3: Percentages of most prevalent structural errors

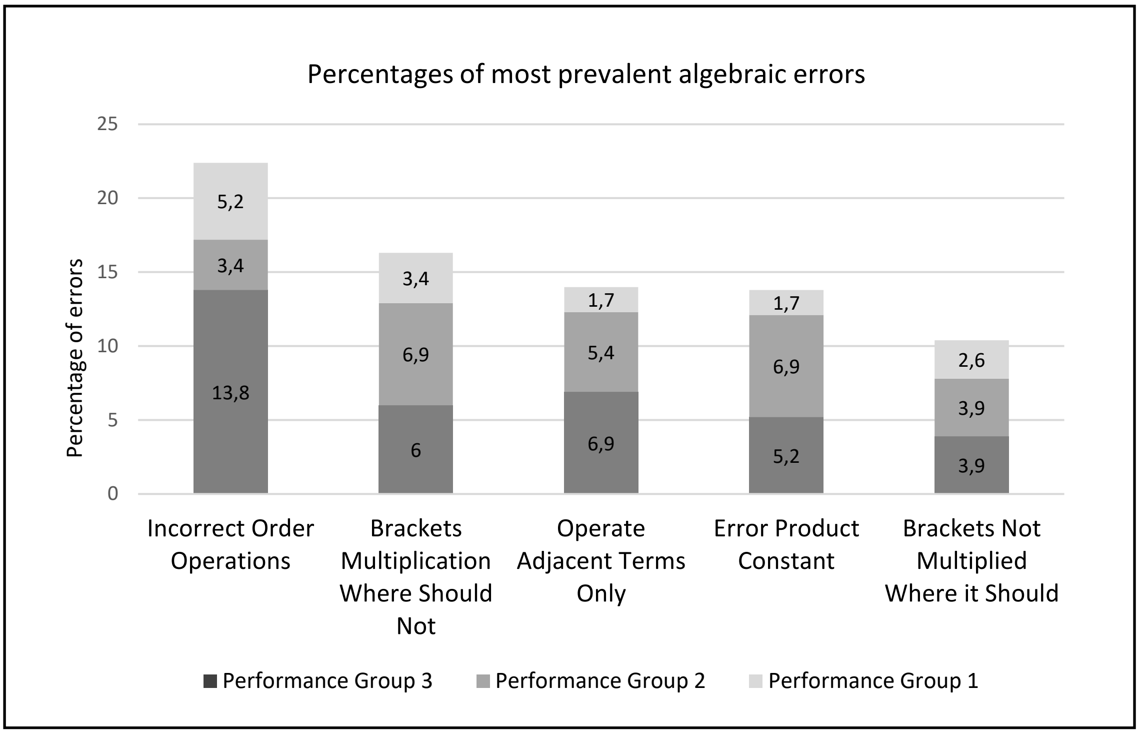


Figure 6.4: Percentages of most prevalent algebraic errors

The most prevalent surface structure error was *not recognising brackets as a grouping tool that separates three factors* (BG3F) in $2(x+3)^2$. This surface structure error typically led to algebraic errors of *brackets not treated as multiplication where it should be applied* such as in $2x^2-11$; and the *incorrect order of operations using multiplication* (IOO) such as $(2x+6)^2$. Examples of incorrect responses relevant to BG3F were given in Section 5.4.1. The highest percentage of errors was evident in the higher performance groups. I anticipated that there would have been more errors in the lowest performance group. However, Performance Group 3 showed more errors in the systemic structure of $2(x+3)^2$ which explained the low percentage of surface structure errors.

The second highest percentage of surface structure errors involved *brackets not recognised as a separation tool for subtraction*, that led to prevalent algebraic errors of *treating brackets as multiplication where it should not be used in* $3a-(b+a)$, for example $-3ab-3a^2$. Examples of learner's incorrect responses were provided in Section 5.4.1 which illustrated that the compound term $(b+a)$ was not subtracted from the single term $3a$. Most incorrect responses were evident in the lower performing groups.

The third highest percentage of surface structure errors shows that *brackets were not treated as a grouping tool for distributive law and a separation tool for subtraction* in $2a(a-4)-8$.

These misinterpretations led to common algebraic errors of *treating brackets as multiplication when it should not be applied* such as $8(2a^2 - 4)$ and *operating on adjacent terms only* such as $3a - 12$. Examples of learners' incorrect responses were provided in Section 5.4.1. These errors were, however, less than 10% in each performance group.

Both systemic structure errors had frequencies above 10%. The most prevalent systemic structure error was *distributive law not used correctly* (DLNC), which relates to four items, i.e. $2(x+3)^2$; $(a+b)b$; $2a(a-4)-8$; $(2x+1)(x+4)$. The single item $3a-(b+a)$ involved the systemic structure error of *subtraction not used correctly*.

Common incorrect responses of *distributive law not used correctly* involved the *incorrect order of operations with multiplication*, for example $2(x+3)^2 = 2(x+3)(x+3) = 2x+6+2x+6$; *errors in product of monomial and constant*, for example

$$2(x+3)^2 = 2(x^2+6x+9) = 2x^2+12x+81$$, and

operating on adjacent terms only, for example $(a+b)b = a+b^2$. Additional examples of incorrect responses were given in Section 5.4.2. Most incorrect responses were found in Performance Group 3 and these were almost double the combined incorrect responses in the two higher performing groups.

The remaining systemic structure error and which was also the lowest structural error was *subtraction not used correctly*. Prevalent algebraic errors related to *operating on adjacent terms only* for example $3a - b + a$ and *treating brackets as multiplication when it should not be applied* for example $3a(-b-a) = 3ab - 3a^2$. In the two higher performing groups, the percentage of systemic structure errors was less than 10%. The percentages of incorrect responses in Performance Group 3 was almost double that of the higher performance groups, following a similar trend to the percentage of errors in DLNC.

The findings of the second phase considered the percentages of structural errors on the individual items are presented in Figure 6.5.

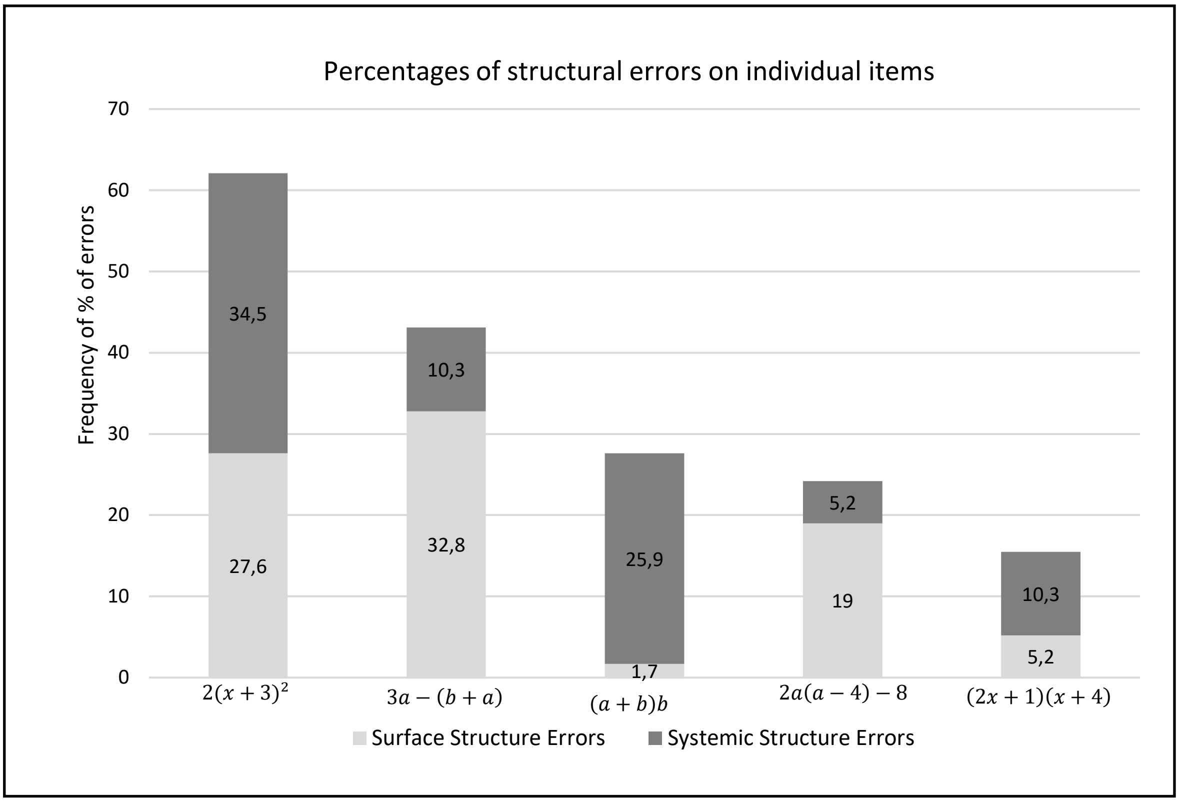


Figure 6.5: Percentages of structural errors on individual items

The highest percentages of structural errors occurred in $2(x+3)^2$ and $3a-(b+a)$. Furthermore, incorrect responses occurring in $2(x+3)^2$ illustrated the highest percentage of surface structure errors, and incorrect responses in $3a-(b+a)$ illustrated the highest percentage of systemic structure errors. There was a considerably higher percentage of incorrect responses related to systemic structure in $(a+b)b$ than surface structure errors. More surface structure errors occurred in $2a(a-4)-8$. Finally, a higher percentage of systemic structure errors occurred in $(2x+1)(x+4)$ compared to surface structure errors.

I analysed the percentages of structural errors in these items in relation to my initial classification of items as being typical/atypical and simple/complex. See Section 5.8 for a reflection on the classification of the test items.

6.6 Results of Analysis

I used the quantified results to provide answers to the research questions.

6.6.1 To what extent do learners demonstrate surface structure sense in the interpretation of typical and atypical structures of algebraic expressions with brackets?

A key finding is that learners demonstrate surface structure sense when expressions include more brackets in *typical* structures. Typical structures involving brackets as a grouping tool that separates factors, were $2(x+3)^2$and $(2x+1)(x+4)$. The percentage of surface structure errors for $2(x+3)^2$ was 27.6% which was considerably higher than $(2x+1)(x+4)$ at 5.2% (Figure 6.5). The high percentage of errors for $2(x+3)^2$ suggested that learners did not recognise the compound term $(x+3)^2$ as two factors $(x+3)(x+3)$. This resulted in the lack of recognition of three factors of $2(x+3)(x+3)$ in a structure that included fewer brackets. In comparison to $(2x+1)(x+4)$, which also involves two factors, the surface structure displayed two pairs of brackets. The two brackets were interpreted more efficiently by learners evidenced in the low percentage of surface structure errors. Therefore, the conclusion drawn is that more brackets in typical expressions encourage surface structure sense. My conclusion supports Hoch and Dreyfus' (2004) finding that more brackets in an algebraic expression result in better structure sense.

Another finding is that surface structure sense is demonstrated in the multiplication of a monomial and binomial in *atypical* structures which include fewer elements. This is evident in the comparison of two *atypical* structures $(a+b)b$ and $2a(a-4)-8$ which involve brackets as a grouping tool for the distributive law. The percentage of surface structure errors for $(a+b)b$ of 1.7% was substantially lower than $2a(a-4)-8$ at 19.0% (Figure 6.5). The low percentage of surface structure errors for $(a+b)b$ suggested that despite the monomial being positioned after the brackets, multiplication of a monomial and a binomial was correctly interpreted. In $2a(a-4)-8$, the monomial positioned before the brackets requires the same type of operation as in $(a+b)b$. Therefore, I anticipated that responses would display similar levels of surface structure sense. An additional element is the constant -8 after the brackets, which presented challenges for learners. Incorrect responses in $2a(a-4)-8$ suggested that learners attempted to operate on the constant -8 with the bracket, for example by multiplying 8 or -8 into the bracket. Hence, I concluded that surface structure sense is more evident in atypical expressions with brackets that include fewer elements.

A further finding is that errors in surface structure sense occur in both *typical* and *atypical* structures when brackets are a grouping tool for subtraction. Structures involving brackets as a separation tool for subtraction are the *typical* structure of $3a-(b+a)$ and the *atypical* structure of $2a(a-4)-8$. There was a high percentage of incorrect responses for both items $3a-(b+a)$ at 32.8% and $2a(a-4)-8$ at 19%, displayed in Figure 6.5. The incorrect responses suggested that the minus symbol in $2a(a-4)-8$ and $3a-(b-a)$ was not regarded as subtraction. Instead, learners multiplied (into the bracket) suggesting that learners treated the brackets as a signal to multiply. Therefore, I concluded that better surface structure sense is displayed on typical and atypical structures if brackets in expressions are a tool for multiplication, but not subtraction.

6.6.2 To what extent do learners demonstrate systemic structure sense in the interpretation of simple and complex structures of algebraic expressions with brackets?

A key finding is that systemic structure sense is displayed in expressions that require one type of operation in the *simple* structures, for example of $2a(a-4)-8$; $3a-(b+a)$ and $(2x+1)(x+4)$. Percentages of errors were relatively low at 5.2% for $2a(a-4)-8$, and 10.3% for both $3a-(b+a)$ and $(2x+1)(x+4)$ (Figure 6.5). In $2a(a-4)-8$ and $(2x+1)(x+4)$, incorrect responses illustrated that the distributive law was only applied partially correctly, which suggested that learners correctly interpret brackets as a signal to multiply but then make errors, for example $(2x+1)(x+4)=2x^2+x+4$. In $3a-(b+a)$, subtraction was applied partially correctly. Learners operated on the term closest to $3a$ which suggested they interpreted the brackets as a signal to subtract but algebraic errors are evident. The conclusion drawn is that systemic structure sense is displayed on simple structures when brackets signify one type of operation such as multiplication or subtraction.

Systemic structure sense depends on the arrangement of elements in expressions. In simple structures such as $2a(a-4)-8$; $(2x+1)(x+4)$; and $3a-(b+a)$, the arrangement of elements was more familiar (see Section 2.7). However, there were partial errors in applying the distributive law. In the simple structure $(a+b)b$, the positioning of the monomial b after the bracketed binomial $(a+b)$ was more challenging for learners with 25.9% incorrect responses (Figure 6.5). It is, therefore, likely that systemic structure sense would be more evident in simple structures that include elements in familiar arrangements but to a lesser extent with unfamiliar arrangements.

Another finding is that systemic structure sense is less evident in the complex expression with brackets. The complex structure of $2(x+3)^2$ requires independent operations involving the distributive law. The high percentage of 34.5% (Figure 6.5) suggested that having three factors in the item was challenging because learners did not use the correct order of multiplication. The items in this study included only one complex items. However, based on the high percentages of errors of the item, it is likely that other complex structures, involving numerous operations, might also reflect less evidence of systemic structure sense.

6.6.3 What algebraic errors result from misinterpretations of surface and systemic structures?

The most prevalent algebraic error is the *incorrect order of operations using multiplication*. This error resulted from *either* not recognising three factors in $2(x+3)^2$ such as in $(2x+6)^2$, or not recognising that the squared bracket indicates two binomials such as in $2(x^2+9)$.

The second most prevalent is *brackets treated as multiplication where it should not be applied*. This was evident in the two items involved subtraction, i.e. $2a(a-4)-8$ and $3a-(b+a)$. Responses indicate that brackets were inappropriately used as multiplication such as in $2a(a-4)-8=2a^2+32$ and $3a-(b+a)=3ab-3a^2$, the incorrect reasoning that 'brackets mean multiply'.

Another prevalent error is to *operate on adjacent terms only*. This error was evident in 1) $(a+b)b=a+b^2$ which suggested that the monomial *b after* the binomial was multiplied by the adjacent term in brackets *b*; 2) $3a-(b+a)=3a-b+a$ which indicated that learners only subtracted the first term (*b*) in the brackets; and 3) $2a(a-4)-8=2a^2-12$ in which the positioning of the brackets in the middle of the expression appeared to trigger some learners to place mental brackets around like terms with letters and around the constants; suggesting that $2a$ and a are multiplied to obtain $2a^2$, and -8 and -4 are added to obtain -12.

Finally, a high percentage of errors involving *products with constants* is evident. Incorrect responses suggest slips rather than errors because the products of all other terms in $2(x+3)^2$ were correct. For example, $2(x+3)^2=2(x+3)(x+3)=2(x^2+3x+3x+6)$, and $2(x+3)^2=2(x^2+6x+9)=2x^2+12x+81$ both displayed correct multiplication of terms except with the constants to obtain *6* and *81* respectively. It is disconcerting that a high percentage, with 13.8% incorrect responses, involved products with constants because this skill is learnt in Grade 8.

6.7 Concluding Remarks on Structural and Algebraic Errors

Before conducting the error analysis, I assumed that most structural and algebraic errors would be made by learners in the lowest performance group. Although this assumption was confirmed by the overall number of errors made on each item by the lowest group, evidence showed that structural and systemic errors were also prevalent in the higher performance groups. For example, percentages of errors of the two higher performance groups in *not recognising brackets as a grouping tool that separates 3 facto*rs in the surface structure was higher than the lowest performing group. Furthermore, percentages of errors in Performance Group 2 in *brackets treated as multiplication where it should not be used in the systemic structure* are higher than Performance Group 3 (discussed in Section 6.5).

There were three observations made regarding the surface structure of expressions. Firstly, I found that more brackets in typical expressions encouraged surface structure sense such as in $(2x+1)(x+4)$ compared to $2(x+3)^2$ which consists of fewer brackets (discussed in Section 6.6.1). Secondly, better surface structure sense was displayed in atypical expressions with brackets that include fewer elements such as in $(a+b)b$, compared to $2a(a-4)-8$ which consists of more elements (discussed in Section 6.6.1). Finally, surface structure sense is displayed on typical and atypical structures if brackets in expressions are a tool for multiplication, instead of subtraction such as $2a(a-4)-8$ and $3a-(b+a)$ (discussed in Section 6.6.1).

Regarding the systemic structure of expressions with brackets, systemic structure sense was evident on simple structures when brackets signify one operation, such as multiplication in $2a(a-4)-8$ and $(2x+1)(x+4)$ or subtraction in $3a-(b+a)$ (discussed in Section 6.6.2). However, systemic structure sense seemed to be affected by the positioning of elements in the simple structure $(a+b)b$, which illustrated more incorrect responses in applying the distributive law correctly. Finally, systemic structure sense was less evident in complex structures (elaborated on in Section 6.6.2) which involved more than one operation of multiplication. Prevailing algebraic errors are a result of: firstly the number of brackets and multiple operations in expressions. For example, the *incorrect order of operations with multiplication* in $2(x+3)^2$ seemed to be a result of only one pair of brackets in the item, but which signified multiple operations of distribution. Secondly, learners had the perception that 'brackets mean multiply' when multiplication was not necessarily the primary operation. For example, in $2a(a-4)-8$ and $3a-(b+a)$, the operation of subtraction was not exhibited in learners' responses because brackets were incorrectly treated as multiplication. Thirdly, the positioning of elements in expressions resulted in algebraic errors. For example, *operating on adjacent terms only* in $(a+b)b$ and $3a-(b+a)$. The positioning of the brackets in the middle of the expression in $2a(a-4)-8$ appeared to trigger mental brackets on adjacent terms. Finally, slips were evident when using algebraic operations as noted above.

6.8 Reflections on the Study

I focus my reflections on two aspects of this study. Firstly, I reflect on the development of the structural and algebraic error codes and secondly, I reflect on the usefulness of analysing incorrect responses for my own teaching.

6.8.1 Generating suitable error codes

In order to analyse structural errors in algebraic expressions, I needed to develop appropriate error codes that addressed the focus of this study. This was a challenge and it took a number of attempts before I finalised the most appropriate codes. Initially the codes did not focus on misinterpretations of structure but rather on whether learners used correct and/or inappropriate operations to simplify the expressions, for example *subtraction of first term only* in $3a-(b+a)=3a-b+a$. I constantly rejected the codes because I felt that the codes were not addressing the aim of the study.

My perseverance at getting the codes "to work" encouraged me to re-explore aspects of structure and structure sense. This led to the development of entirely new codes which proved to be a significant part of my learning on this research journey. The new codes dealt with structure 'head on', which identified possible underlying factors for errors. For example, *subtraction not used correctly* in $3a-(b+a)=3a-b+a$.

In retrospect, the development of suitable codes followed a discussion with my supervisor about the errors that learners were making. It was in my explanation of the errors and how learners were possibly looking at items that led to my 'eureka' moment of suitable structural error codes. My explanation to my supervisor focused on the *type* of errors that learners made for example *subtraction not used correctly*, instead of me *identifying* incorrectly simplified elements such as *subtraction of first term only*.

6.8.2 Reflection on my own teaching

Firstly, this study has been illuminating for my own teaching of mathematics to Grade 9 and 10 learners. I now question learners more regularly after they make errors, as a precursor to understanding and attempting to correct learners' methods and their thinking. This provides greater opportunity to address learners' underlying incorrect thinking. For example, one of my learners responded to a question on the multiplication of two binomials (Figure 6.6) as:

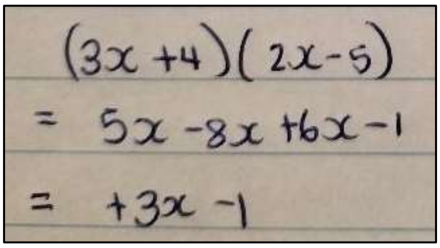


Figure 6.6: Learner response

When I asked the learner to explain how she got her answer, she explained that she used the FOIL (First Outer Inner Last) method to multiply the two binomials. She explained that she "put together" $3x$ and $2x$ to "get" $5x$. Then she "put together" the outer terms $3x$ and -5 to "get" $-8x$, and the inner terms $+4$ and $2x$ to "get" $+6x$. The last step involved multiplying $+4$ and -5 to "get" -1. I subsequently was able to explain to her that she used the correct sequence of operating on the terms by using the FOIL method, however she used a combination of operations instead of just multiplication on the terms. I demonstrated that she added the first terms as "like terms" to obtain $5x$. The outer and inner terms were conjoined by adding the numbers of the terms, and that she used rules of multiplying negative and positive integers. I further explained that the last term was obtained by adding the constants as negative and positive integers. The learner defended the operations that she used, which she based on previous concepts that were taught to her in "putting together like terms" in algebraic expressions. I reminded her the brackets in *this* expression signified multiplication, and that that she was supposed to multiply each term in the first bracket with the terms in the second brackets as $3x(2x-5)+4(2x-5)$.

Such discussions have helped me understand the types of errors made by learners, thereby enabling me to develop deeper insights and to anticipate the type of errors that can be made by learners. I realised that although the method of algebraic procedures such as FOIL, need to be emphasised, I should also focus on how expressions are operated on, such as using multiplication instead of addition. Learners' errors are based on valid efforts to handle questions in mathematics (Olivier, 1986). This means that such mistakes are derived from previously learned information but are misinterpreted in different types of questions (Olivier, 1986). For example, my learner interpreted the expression by using previously learned concepts of adding like terms in expressions. She misinterpreted the role of brackets that signified multiplication.

Secondly, Hoch and Dreyfus (2004, p.54) suggested that teachers must emphasise "looking before doing" when teaching high school algebra (Section 2.4). Learners must be taught to look at the arrangement of the elements such as the brackets in $(3x + 4)(2x - 5)$, and interpret the role of the brackets in the surface structure that groups two factors. Once the surface structure is identified, then only can the appropriate operations be used, such as multiplying the two binomials. This can encourage learners to focus on the elements that are contained in the structure before dealing with the relevant procedures. I have adopted the strategy of encouraging learners to look at structures before operating, so that brackets can be interpreted as a tool and then as a sign. Therefore, in my teaching, I now emphasise *what* is the arrangement of an expression before *how* to operate on the expression.

Finally research studies are valuable in making recommendations for teaching, based on the findings. Schüler-Meyer (2017) proposed a process model to help develop structure sense for learners through guiding examples and structural mappings (discussed in Section 2.6). Guiding examples relate to using simple expressions that have the same structures as the complex expressions, as a guide to develop structure sense. Structural mapping is using appropriate operations on manipulated expressions to determine if the initial expression will be obtained. In this way, structure sense, in applying the distributive law in expressions, can be developed by using appropriate methods. For example, in my teaching of complex products, I now use guiding examples such as $(d + c)(d - c)$ on a complex structure such as $(a - b + c)(a - b - c)$. I place mental brackets around the compound term $(a - b)$ in $(a - b + c)(a - b - c)$ to obtain $\big((a - b) + c\big)\big((a - b) - c\big)$. The compound term $(a - b)$ represents d in the guiding example $(d + c)(d - c)$. In this way, the method of simplifying $(d + c)(d - c) = d^2 - c^2$ can be applied to $(a - b + c)(a - b - c)$ to obtain $(a - b)^2 - c^2$.

As a teacher, I now do not assume that learners can automatically develop structure sense of expressions in the progression of grades but rather that I should play an active role in teaching skills to develop structure sense.

6.9 Limitations of the Test Items

I conducted secondary analysis on learners' responses to test items designed by WMCS. The test instrument was designed for the purpose of testing learners on a range of mathematical concepts and procedures, hence was limited in the number of algebraic expressions dealing with brackets and in the range of algebraic structures available. When I began this research, I was not focusing on the issues of structure so I could not predict the extent of the limitations. The limited variety of the expressions meant that I was unable to investigate all aspects of structure sense in relation to my classification of typical/atypical and simple/complex structures. For example, there was only one item that contained a squared bracket $2(x+3)^2$ and which implied three factors. For my study, ideally there needed to be an item with a squared binomial only such as $(x+5)^2$. This structure could have been useful to investigate how learners interpreted the squared bracket in a two-factor expression.

The choice of elements within the items was also restrictive in investigating how learners considered relationship of brackets to other elements. The structure of $(2x+1)(x+4)$ includes terms that resulted in ambiguous learner responses. For example, in a learner's incorrect response $2x^2+4x+x+4$, I was uncertain how $x+4$ in the answer was obtained. Operations could have included either correctly multiplying 1 in the first bracket by both terms in the second bracket, or the learner could have merely rewritten the terms in the second brackets without using any operation. I therefore interpreted the learner response by assuming that he/she multiplied 1 in the first bracket by both terms $(x+4)$ in the second bracket because the response illustrated that $2x$ in the first bracket was multiplied with x in the second bracket. An expression that included different constants such as $(2x+3)(x+5)$ could have been more valuable to investigate how learners interpreted brackets in multiplying two binomials.

6.10 Recommendations for Further Research

The items used in this study did not allow me to pursue Kieran's (1989) definition of structure to its fullest and to provide a detailed analysis of the definition. The type of structures in the test focused mainly on the distributive property and only one structure on subtraction in $3a-(b+a)$. Although $2a(a-4)-8$ shows subtraction of a constant, there is no operation of subtraction involved, therefore I did not regard subtraction as the focus in this item. The items did not address other aspects of structure such as commutativity and associativity.

As presented in Chapter 5, if I were to redesign the items, I would use the matrix of typical/atypical and simple/complex structures to generate expressions and include at least three items in each cell, so that a more comprehensive analysis could be done. My proposed collection of items is indicated in the matrix (Table 6.1) with possible comparison combinations.

Table 6.1: Collection of items for future testing

	Typical	Atypical
Simple	$3x(x-4)$ $3x-(x+4)$ $(x+3)(x-4)$	$(x-4)3x$ $(x+4)-3x$ $(5-x)(x-3)$
Complex	$-4(x+2)^2$ $3(x-5)(x+5)$ $(x-3)^2+4$	$(x+4)^3$ $(x-5)(x-5)(x-5)$ $4(x+6)(4-x)$

Comparison 1

$3x(x-4)$ and $3x-(x+4)$

A comparison could be made in learner responses between interpreting typical and simple structures in multiplying a monomial by binomial $3x(x-4)$, and subtraction of a binomial from a monomial $3x-(x+4)$.

Comparison 2

$(x-4)-3x$ and $(x+4)3x$

Similarly, a comparison of responses could be made on atypical simple structures involving multiplication of a monomial by a binomial in $(x-4)3x$, and subtraction of a monomial from a binomial in $(x+4)-3x$.

For a more comprehensive analysis, *Comparison 1* and *Comparison 2* items could be evaluated against each other.

Comparison 3

$(x+3)(x-4)$ and $(5-x)(x-3)$

In order to evaluate the structure sense of the simple operations on the multiplication of two binomials, responses to a typical structure such as $(x+3)(x-4)$ could be compared with responses to $(5-x)(x-3)$, which illustrates binomials consisting of two minus symbols and a change in the order of the terms in brackets.

Comparison 4

$-4(x+2)^2$; $3(x-5)(x+5)$ and $(x-3)^2+4$

The structures in the typical complex matrix include a negative monomial multiplied by two identical binomials in $-4(x+2)^2$; a positive monomial multiplied by two binomials with different signs in $3(x-5)(x+5)$; and multiplication of two identical binomials added to a monomial $(x-3)^2+4$. The types of structures indicate different arrangements of elements affecting brackets as a tool, and as signifiers for different operations. This could allow for a comparison of responses in these different structures.

Comparison 5

$(x+3)^3$, $(x-5)(x-5)(x-5)$ and $4(x+6)(4-x)$

Evaluation of responses in atypical and complex structures could include the multiplication of three factors such as three identical binomials in a compound form $(x+4)^3$ which contains one pair of brackets in the surface structure; three identical binomials with three pairs of brackets in $(x-5)(x-5)(x-5)$; and a monomial with two binomials in $4(x+6)(4-x)$, where elements are arranged in an unfamiliar order in brackets.

For future research on structure sense in algebraic expressions, my recommendation to researchers is to use the same number of expressions in simple and complex structures and in familiar and unfamiliar contexts. This could result in a more comprehensive analysis of errors involving the interpretation of brackets in different types of structures. The expressions and comparisons in Table 6.1 serve as an exemplar.

Furthermore, the arrangement of the elements in these structures should be considered so that every aspect of structure can be investigated. The mathematical procedures involved in these expressions should address all operations such as addition, subtraction, multiplication and division using both single and compound terms, to investigate how such structures are interpreted.

6.11 Level of Structure Sense of Grade 10 Learners

The level of structure sense of the Grade 10 learners in this study, based on items designed for Grade 9, is concerning. The findings show that learners experienced challenges with the structure sense of expressions that should be familiar in Grade 9. The data was collected towards the end of the year, which meant that both Grade 9 and 10 learners should have been able to cope with such structures. A disturbing observation is that most of the items for Grade 10 learners should be regarded as being typical and simple. However, this was not evident in the results. In summary, the Grade 10 learners who participated in the WCMS study experienced challenges despite the concepts being covered in Grade 9 syllabus as stipulated by national requirements, encapsulated in CAPS (Department of Basic Education, 2011). I conclude that the level of structure sense of the Grade 10 learners is not on par to the stipulated norms.

Similar trends of poor performance have been seen in international studies where learners in higher grades, participate in assessments for a lower grade (Reddy & Arends, 2016). The testing of Grade 9 learners in the TIMMS study for Grade 8 is indicative of poor performance. In South Africa despite this underperformance, learners are still promoted to the next grade without mastering knowledge in that grade which leads to grade attainment instead of grade quality (Spaull & Kotze, 2015). This implies that Grade 9 learners will be promoted to the Further Education and Training (FET) phase despite mathematical concepts not being grasped. This can result in further mathematical inadequacies when the learners are taught new concepts but have not gained an understanding of previous concepts.

The mathematics performance of learners in South Africa needs to be addressed. This is especially important when learners progress to Grade 10 and *choose* Mathematics as a subject for Grade 12. Algebraic skills such as those explored in this study are essential in all topics and the development of a learner's algebraic structure sense is vital for mathematical proficiency at FET level. In this study, I have identified some trends in the misinterpretation of brackets that limit structure sense of algebraic expressions. This research hopes to provide an understanding of structure sense that would help to attenuate the apprehension with which algebraic equations are tackled. I recommend more research to be conducted to identify further trends that limit structure sense in other mathematical areas. The identification of such trends can be used to develop strategies to improve on learners' mathematical knowledge and skills.

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Appendices

Appendix A

Performance Group 1: Number and percentages of incorrect responses to test items

Learner Code	Test score	$(a+b)b$	$3a-(b+a)$	$2a(a-4)-8$	$(2x+1)(x+4)$	$2(x+3)^2$
1	40					
2	39					
3	38			1		
4	38					
5	38		1			
6	38					
7	36					1
8	36					
9	35			1		
10	35					1
11	34				1	1
12	34	1				1
13	34	1				
14	33	1	1			
15	33					
16	33					
17	33		1	1		
18	33		1			1
19	33		1			1
20	33					
21	33					1
22	32					1
23	32					
24	32					
25	32		1			
Number of incorrect responses		**3**	**6**	**3**	**1**	**8**
% of Incorrect Responses		**12%**	**24%**	**12%**	**4%**	**32%**

Performance Group 2: Number and percentages of incorrect responses to test items

Learner Code	**Test score**	$(a+b)b$	$3a-(b+a)$	$2a(a-4)-8$	$(2x+1)(x+4)$	$2(x+3)^2$
26	31					
27	31	1				1
28	30			1		1
29	30				1	1
30	29		1			
31	28		1			1
32	27					1
33	26		1			1
34	26		1	1		1
35	26			1		1
36	25		1			1
37	25					1
38	25		1			
39	25			1		1
41	24	1	1			
40	23					1
42	23	1	1	1	1	1
Number of incorrect responses		3	8	5	2	13
% of Incorrect Responses		18%	47%	29%	12%	76%

Performance Group 3: Number and percentages of incorrect responses to test items

Learner Code	**Test score**	$(a+b)b$	$3a-(b+a)$	$2a(a-4)-8$	$(2x+1)(x+4)$	$2(x+3)^2$
43	22	1			1	1
44	21					1
45	21	1	1			1
46	21	1	1	1		1
47	21		1		1	1
48	21					1
49	21	1		1	1	1
50	20	1		1		1
51	20	1				1
52	20	1	1			1
53	20		1	1		1
54	19	1	1			
55	19	1	1	1	1	1
56	19		1			1
57	18		1	1	1	1
58	18	1	1		1	1
Number of incorrect responses		**10**	**10**	**6**	**6**	**15**
% of Incorrect Responses		**63%**	**63%**	**38%**	**38%**	**94%**

Appendix B

Test Item: $(\boldsymbol{a}+\boldsymbol{b})\boldsymbol{b}$

Learner Response	First Level of Coding		Second Level of Coding
	Surface Structure Error	**Systemic Structure Error**	**Algebraic Error**
$a+b$	BG2F		BNMS
$a+b^2$		DLNC	OATO
ab^2		DLNC	OATO / LC
$a+2b$		DLNC	P2L

Test Item: $\boldsymbol{3a}-(\boldsymbol{b}+\boldsymbol{a})$

Learner Response	First Level of Coding		Second Level of Coding
	Surface Structure Error	**Systemic Structure Error**	**Algebraic Error**
$-3ab+3a^2$	BSS		BMSN
$3ab+3a^2$	BSS		BMSN
$-3ab-3a^2$	BSS		BMSN
$3ab+3a^2$	BSS		BMSN
$3a^2-b$	BSS		BMSN
$3a-ba$	BBUT		LC
$3b$	BSS		SS/LC
$3ab+4a$	BSS		BMSN/SS
$4a+b$	BSS		SS
$3a(-a-b)$		SNC	BMSN
$3a-b+a$		SNC	OATO/S

Test Item: $2a(a-4)-8$

Learner Response	First Level of Coding		Second Level of Coding
	Surface Structure Error	Systemic Structure Error	Algebraic Error
$(2a^2-8a)-8$ $=16a^2-64a$	BGS		BMSN
$(2a^2-8a)-8$ $=16a^2-32$	BGS		BMSN
$(2a^2-4)-8$ $=8(2a^2-4)$ $=16a^2-32$	BGS		BMSN
$3a-12$	BGS		BNMS/OATO
$-16a(a-4)$ $=-16a^2$	BGS		BMSN
$2a^2-8a$	BSS		SS
$2a^2-8a+8$	BGS		S
$2a^2-8-8$		DLNC	PLN

Item: $(2x+1)(x+4)$

Learner Response	First Level of Coding		Second Level of Coding
	Surface Structure Error	Systemic Structure Error	Algebraic Error
$3x+4$	BG2F		BNMS
$2x+x+1+4$	BG2F		BNMS
$2x^2+x+4$		DLNC	BNMS
$2x^2+4x+x+4$		DLNC	PLN
$2x+8x+x+4$		DLNC	PLN
$3x+8x+x+4$		DLNC	PLN

Item: $\mathbf{2(x+3)^2}$

Learner Response	**First Level of Coding**		**Second Level of Coding**
	Surface Structure Error	**Systemic Structure Error**	**Algebraic Error**
$(2x+6)^2 =$ $(2x+6)(2x+6)$	BG3F		IOO
$(2x+6)^2 =$ $4x^2+36$	BG3F		IOO/BNMS
$2x+12$	BG3F		BNMS
$2x+11$	BG3F		BNMS
$2(x^2+9)$	BG3F		BNMS
$2x+6+2x+6$		DLNC	IOO
$(2x+6)(2x+6)$ $=4x^2+36$		DLNC	IOO
$2x+3x+3x+9$		DLNC	IOO
$2x+6+x+3$		DLNC	IOO
$2x+6+3x+9$		DLNC	IOO
$2(x+3)(x+3)$ $=2x+6$		DLNC	BNMS
$2(x^2+3x+3x+9)$ $=2x^2+6x+6x+81$		DLNC	PC
$2(x+3)(x+3)$ $=2(x^2+9x+9)$		DLNC	PLN
$2(x+3)(x+3)$ $=2x^2+12x+9$		DLNC	PC
$2(x+3)(x+3)$ $=2(x^2+3x+3x+9)$ $=2x^2+9x+9$		DLNC	PLN/PC
$2(x^2+6x+6x+9)$		DLNC	PLN
$2(x+3)(x+3)$ $=2(x^2+6x+6)$		DLNC	PC
$2(x+3)(x-3)$		DLNC	S

Appendix C

WITS SCHOOL OF EDUCATION

UNIVERSITY OF THE WITWATERSRAND JOHANNESBURG

SCHOOL OF EDUCATION ETHICS COMMITTEE

CONSTITUTED UNDER THE UNIVERSITY HUMAN RESEARCH ETHICS COMMITTEE (NON-MEDICAL)

CLEARANCE CERTIFICATE	**PROTOCOL NUMBER: 2020ECE177M**
PROJECT TITLE	Grade 10 learners' use of brackets in algebraic expressions
INVESTIGATOR	**NADIA MAHOMED THEBA**
SCHOOL/DEPARTMENT OF INVESTIGATOR	WITS SCHOOL OF EDUCATION
DATE CONSIDERED	16 November 2020
DECISION OF THE COMMITTEE	Approved unconditionally
EXPIRY DATE	Date of submission of the project report

ISSUE DATE OF CERTIFICATE 20 November 2020 **CHAIRPERSON** (Dr Paul Goldschagg)

cc: Supervisor: Prof Craig Pournara

DECLARATION OF INVESTIGATOR

To be completed in duplicate and **ONE COPY** emailed to the Ethics Office: Matsie.Mabeta@wits.ac.za .

I fully understand the conditions under which I am authorized to carry out the abovementioned research and I guarantee to ensure compliance with these conditions. Should any departure be contemplated from the research procedure as approved I/we undertake to resubmit the protocol to the Committee.

Signature

31 December 2020.

Date

PLEASE QUOTE THE PROTOCOL NUMBER ON ALL ENQUIRIES

Appendix D

UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG

Amendments to Enrolment

Undergraduate/Postgraduate

Please return this form to the Faculty of Humanities

SECTION 1: CURRENT DETAILS (All applicants must complete this section)													
Last Name/Surname (as per Identity Document)	Mahomed Theba												
First Name	Nadia												
Level of Study	Masters in Education												
Person Number/Student Number	9	3	0	4	1	2	1	N					
Date of Birth	0	6	J	A	N	1	9	7	5				
	Day		Month (e.g. Dec)			Year							
South African ID Number/Passport Number	7	5	0	1	0	6	0	0	8	3	0	8	6

SECTION 2: CHANGE OF PROGRAMME		
Programme	Old:	New:
Attendance	Full-Time:	Part-Time:
	From:	To:

SECTION 3: AMENDMENTS TO UNIT ENROLMENT

i) Course you want to drop

Course Code	Course/Topic Name	Unit Class	Term	Approved (signature) by Course Coordinator

ii) Courses you want to register for

Course Code	Course/Topic Name	Unit Class	Term	Approved (signature) by Course Coordinator

Please note that if you are on financial aid and you want to add a unit/units to your enrolment in July, you will be personally liable for the fees of these units. International students need to obtain a new clearance certificate from the International Office if you are adding courses.

UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG

Requests for:

i) Extension of time for submission of research proposal	From	To
ii) Extension of time for submission of research for examination	From 15 / 03 / 2022	To 30 / 06 / 2022
iii) Extension of time for submission of research for graduation (ETD)	From	To
iv) Give details of any previous extensions	From	To
v)* Registration to be put in abeyance	From	To
vi) Give details if your registration has been put in abeyance before	From	To
vii) Title of research to be changed		
From : **Making sense of structure of brackets in algebraic expressions**		
To : **Structure sense: Working with brackets in algebraic expressions**		
vii) Change of Supervisor	From	To

*This applied to MA by research and PhD students ONLY. Please supply letter from employer to indicate when enrolment commenced

SECTION 5: MOTIVATION FOR POSTGRADUATE REQUESTS	
Motivation from Candidate	Due to increased school demands at the start of this year, I did not have sufficient time to work on my thesis in January and February.
Student's Signature	
Motivation from Supervisor	I support the extension. Ms Theba has worked consistently on her study including during the December break. I also support the change in title. It foregrounds the new focus that has come from the analysis
Supervisor's Signature	

PG Coordinator/HOS Signature

Date Submitted to Faculty	0	8	M	A	R	2	0	2	2

FOR OFFICE USE ONLY	
Signature of Student:	
Date Submitted:	
Name & Signature of Faculty	